Explorations in Quantum Gravity: Holography and AdS Black Holes

by

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Abstract

This thesis explores a series of topics in quantum gravity with a focus on the quantum nature of AdS black holes via the AdS/CFT correspondence. In Part I, we examine various holographic approaches to AdS black hole entropy, including (i) from the gravity solution via the Bekenstein Hawking formula, (ii) from the Kerr/CFT correspondence and (iii) from the boundary conformal field theory. We explore these methods with the gravitational implementation of the field theory Cardy-like limit, recently used in the successful microstate countings of AdS black hole entropy in various dimensions. We then consider a deviation from the extremal regime and focus on computing the Bekenstein-Hawking entropy of near-extremal asymptotically AdS_4 electrically charged rotating black holes using three different methods, yielding a unique and universal expression for the entropy.

In Part II, we explore the quantum nature of black holes via the logarithmic corrections to the entropy of AdS black holes in four dimensions. With a focus on AdS_4 solutions in minimal $\mathcal{N} = 2$ gauged supergravity, we show that for extremal black holes the logarithmic correction computed in the near horizon geometry agrees with the result in the full geometry up to zero mode contributions, thus clarifying where the quantum degrees of freedom lie in AdS spacetimes. In contrast to flat space, we observe that the logarithmic correction for supersymmetric black holes can be non-topological in AdS as it is controlled by additional four-derivative terms other than the Euler density.

We also study supersymmetric, rotating, asymptotically AdS_5 black holes and black strings. On the gravity side, we take the near-horizon limit and apply the Kerr/CFT correspondence whose associated charged Cardy formula describes the degeneracy of states at subleading order and determines the logarithmic correction to the entropy, which precisely matches the entropy up to subleading order from the field theory, by the superconformal index and the refined topologically twisted index of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, respectively.

Chapter 1

Introduction

1.1 Motivation

The theory encompassing gravitational physics and quantum physics - quantum gravity - continues to be the most important open issue in the foundations of fundamental physics. Quantum gravity becomes crucial in understanding the first few moments after the Big Bang as well as black holes, where quantum gravitational effects cannot be ignored.

A black hole is a region of a spacetime with an event horizon, beyond which events that take place there cannot classically affect an observer outside this boundary. The event horizon, or horizon for short, is sometimes called the point of no return. Over the past several decades, black holes have been tested extensively with the findings of The Laser Interferometer Gravitational-Wave Observatory (LIGO) and the Event Horizon Telescope (EHT). Black holes have also been in the forefront of theoretical research and provide a theoretical laboratory to test ideas and find new insights into quantum gravity.

There are two crucial ideas that we will discuss in this thesis, that have driven our progress in the field of quantum gravity and are both intimately related to black holes. The first key insight is the successful matching of the microscopic black hole entropy to the macroscopic Bekenstein-Hawking entropy [1]. There are several fundamental ideas to digest here so let us step back to understand what this all means.

Our story starts in the second half of the twentieth century when black hole physics was put on center stage (see for example [2–10]) and notably when Bekenstein conjectured that black holes have a well defined entropy proportional to the area of the horizon. His work on black hole mechanics was followed by Hawking who build on Bekenstein's proposal. In 1974, Hawking discovered that black holes emit thermal radiation corresponding to a temperature, which we now call the Hawking temperature, fixing the constant of proportionality between the entropy and the horizon area.

Indeed, black holes have their own set of analogous thermodynamic laws, as described below.

- The zeroth law: The surface gravity κ is constant over the horizon of a stationary black hole.
- The first law: The change in the energy dE of a stationary black hole is related to the change in the area dA, angular momentum dJ, and electric charge dQ

$$dE = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ. \tag{1.1.1}$$

- The second law: The change in the horizon area is non-decreasing over time.

We can derive the entropy formula for black holes from the first law relating the energy, entropy and temperature. The physical temperature of the black hole, called the Hawking temperature is proportional to the surface gravity

$$T_H = \frac{\kappa}{2\pi}, \tag{1.1.2}$$

while the Bekenstein-Hawking entropy takes the form

$$S^{(1)}_{\mathrm{BH}} = \frac{A}{4G}, \tag{1.1.3}$$

where G is Newton's constant and A is the area of the event horizon. We have set the speed of light c and k_{B} the Boltzmann constant to one, $c = \hbar = k_{\mathrm{B}} = 1$ and will do so for the remainder of this thesis.

The entropy in (1.1.3) is a macroscopic interpretation and is completely universal – at leading order – and is valid for any kind of black hole, regardless of dimension, field content or really anything else. We may then speculate about the microscopic or quantum origin of the entropy via a statistical mechanical approach. This would look something like

$$S^{(2)}_{\mathrm{BH}} = \log \Omega, \tag{1.1.4}$$

where Ω is the number of microstates of the theory. It was not until the mid 1990's when Strominger and Vafa successfully accounted for the microscopic counting of the entropy for a class of five-dimensional black holes [1]. Their analysis was followed by many similar computations for other black holes in different configurations. More importantly, the resolution of the black hole entropy has shed light in our understanding of the quantum properties of black holes and has set the stage for further developments in quantum aspects of black holes.

The second key insight stems from the holographic principle and is widely considered to be a window into the full understanding of quantum gravity. The canonical example in high energy physics of holography is the AdS/CFT correspondence [11]. The principle states that a theory describing gravity is related to a quantum field theory. The gravity theory is typically called "the bulk" and the quantum theory is called "the boundary theory," because it lives on the boundary of the gravity theory and therefore, it is one dimension less than the gravity theory. Like a dictionary, physical concepts from gravity can be translated into physical concepts in the quantum theory.

So what kind of gravity theories have quantum duals? According to the AdS/CFT correspondence, string theories in certain asymptotically Anti-de Sitter (AdS) spacetimes correspond to dual conformal field theories (CFT) in one less dimension.

Let us take a closer look at the gravity side of the correspondence. Global AdS_{d+1} spacetimes are maximally symmetric spacetimes with a negative cosmological constant. We may consider an isometric embedding in Minkowski spacetime in $d+2$ dimensions, where the extra dimension is time-like

$$(X^0, X^1, \ldots, X^d, X^{d+1}) \in \mathbb{R}^{d,2}, \tag{1.1.5}$$

$$\bar{\eta} = \mathrm{diag}(-, +, +, \ldots, +, -). \tag{1.1.6}$$

The Minkowski spacetime takes the form

$$\mathrm{d}s^2 = -\left(\mathrm{d}X^0\right)^2 + \left(\mathrm{d}X^1\right)^2 + \cdots + \left(\mathrm{d}X^d\right)^2 - \left(\mathrm{d}X^{d+1}\right)^2 \equiv \bar{\eta}_{MN}\, \mathrm{d}X^M\, \mathrm{d}X^N, \tag{1.1.7}$$

and AdS_{d+1} is given by the hypersurface

$$\bar{\eta}_{MN} X^M X^N = -\left(X^0\right)^2 + \sum_{i=1}^{d} \left(X^i\right)^2 - \left(X^{d+1}\right)^2 = -L^2, \tag{1.1.8}$$

where L is the radius of curvature of AdS. If we consider the parametrization

$$\begin{cases} X^0 = \alpha \cosh\rho \cos\tau, & \\ X^{d+1} = \alpha \cosh\rho \sin\tau, & \\ X^i = \alpha \sinh\rho\, \mu_i, \qquad \sum_i \mu_i^2 = 1, \qquad i = 1, \ldots, d, & \end{cases} \tag{1.1.9}$$

where μ_i corresponds to a S^{d-1} sphere, then the metric of AdS_{d+1} takes the form

$$ds^2 = \alpha^2 \left(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_{n-2}^2\right), \tag{1.1.10}$$

for $\tau \in [0, 2\pi]$ and $\rho \in \mathbb{R}^+$. Note that (1.1.10) are called the global coordinates of AdS_{d+1} as each point in the spacetime is accounted for only once. The second parametrization that is often used is given by

$$\begin{aligned} X^0 &= \frac{L^2}{2r}\left(1 + \frac{r^2}{L^4}\left(\vec{x}^2 - t^2 + L^2\right)\right), \\ X^i &= \frac{rx^i}{L} \quad \text{for } i \in \{1, \dots, d-1\}, \\ X^d &= \frac{L^2}{2r}\left(1 + \frac{r^2}{L^4}\left(\vec{x}^2 - t^2 - L^2\right)\right), \\ X^{d+1} &= \frac{rt}{L}, \end{aligned} \tag{1.1.11}$$

where

$$t \in \mathbb{R}, \qquad \vec{x} = \left(x^1, \dots, x^{d-1}\right) \in \mathbb{R}^{d-1}, \qquad r \in \mathbb{R}_+. \tag{1.1.12}$$

This parametrization is called the Poincare patch. Unlike global coordinates, the Poincare patch coordinates only cover one-half of the AdS spacetime with metric

$$\mathrm{d}s^2 = \frac{L^2}{r^2}\,\mathrm{d}r^2 + \frac{r^2}{L^2}\left(-\mathrm{d}t^2 + \mathrm{d}\vec{x}^2\right) \equiv \frac{L^2}{r^2}\,\mathrm{d}r^2 + \frac{r^2}{L^2}\left(\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu\right), \tag{1.1.13}$$

where

$$\mu = 0, \dots, d, \qquad x^0 = t \qquad \eta_{\mu\nu} = \mathrm{diag}(-1, +1, \dots, +1). \tag{1.1.14}$$

We often refer to asymptotically AdS spacetimes as spacetimes that, for large radial distance, asymptote to a metric of the form (1.1.10). For example, we can consider a black hole living in AdS spacetime and this is usually referred to as an AdS black hole.

The canonical example of holography is $\mathrm{AdS}_5/\mathrm{CFT}_4$ and is the duality between $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $SU(N)$ and Yang-Mills coupling constant g_{YM} and type IIB superstring theory with string length $l_s^2 = \alpha'$ and coupling constant g_s on $\mathrm{AdS}_5 \times S^5$ with radius of curvature L and N units of $F_{(5)}$ on S^5. In the AdS/CFT correspondence, we typically denote the "AdS" side as the gravity solution, which will mainly be asymptotically AdS black hole solutions in this thesis, while the "CFT" side is the quantum field theory under consideration.

What makes holography so favorable is its computational accessibility. Many of the puzzles presented by gravity can now be reformulated in terms of a quantum field theory and vice versa. For this reason, the AdS/CFT correspondence has not only provided a computational path in quantum gravity but also in fluid dynamics, superconductivity and quantum chromodynamics, connecting a wide range of topics. For example, the fluid/gravity correspondence relates the dynamics of the Einstein equations to that of relativistic Navier-

Stokes equations.

In this thesis, we are particularly interested in how holography is connected with the first key insight regarding black hole thermodynamics. As we mentioned, the microscopic counting of the black hole entropy was first successfully computed by Strominger and Vafa. The caveat is that only asymptotically flat black holes have been rigorously studied in this regard. Since we have the tools in holography to study AdS black holes, we can compute the microscopic counting of the entropy via the conformal field theory. The first attempt at studying the canonical example of $\text{AdS}_5 \times S^5$ dual to $\mathcal{N} = 4$ super Yang-Mills was unfortunately fruitless [12]. It was not until more than a decade later that the issues were resolved [13–15] by considering complex chemical potentials.

In this thesis, we build on the work of [13–15] and we consider AdS black hole solutions with known field theory duals to understand the nature of the microscopic origin of the entropy from a different perspective.

Let us recap what we know so far. We can compute the macroscopic entropy corresponding to the Bekenstein-Hawking entropy $S^{(1)}_{\text{BH}}$ and we know from the AdS/CFT correspondence that the microscopic counting of the entropy $S^{(2)}_{\text{BH}}$ can be found via the dual field theory. We consider yet an additional approach to the entropy via the Kerr/CFT correspondence [16], where we find an impressive match between the properties of black holes and the universal properties of two-dimensional CFTs.

To understand the definition of extremality, we consider the simpler example of a general non-rotating $(d+1)$-dimensional black hole solution of the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2_{d-1}, \tag{1.1.15}$$

where $f(r)$ is a function of the radial coordinate r as well as the mass and charges of the black hole. For example, in the Schwarzschild solution $f(r) = 1 - 2GM/r$. The horizon of the black hole r_+ satisfies $f(r_+) = 0$. For a black hole with additional charges, there are three possibilities depending on the discriminant Δ of $f(r)$:

- naked singularity for $\Delta < 0$,
- extremal for $\Delta = 0$,
- non-extremal for $\Delta > 0$.

In the case of the non-extremal solution, there are two horizons, the inner horizon r_- and the outer horizon r_+, where $r_+ > r_-$ while extremality is obtained when both horizons coincide $r_- = r_+$. For this thesis, we mainly focus on the extremal case for one specific reason. That is, all known extremal black hole solutions develop an AdS_2 factor in the near-horizon geometry. From a geometric point of view, extremal black holes have a throat-geometry, as shown in Figure 1.1. In the case of non-zero angular momentum, we instead find that the near-horizon contains a circle, or several circles, fibered over AdS_2 and from the Kerr/CFT correspondence can be dual to an effective two dimensional CFT model. Therefore, the microscopic degrees of freedom of the black hole can be accounted for via the Cardy formula. We can then utilize the Kerr/CFT correspondence to reproduce the microscopic counting of the entropy $S_{\mathrm{BH}}^{(3)}$ for asymptotically AdS black holes.

Figure 1.2 outlines the overarching goal of the first part of this thesis. More importantly, the three distinct holographic approaches to the entropy have shown to be valid at the extremal limit as well as the near-extremal limit for a class of rotating, electrically charged asymptotically AdS black holes in diverse dimensions.

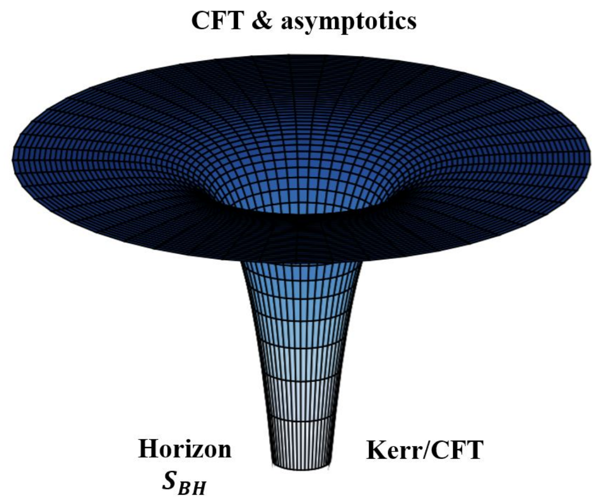

Figure 1.1: The horizon and the asymptotic regions.

Indeed, the black hole entropy at leading order is given by the Bekenstein-Hawking entropy. We now shift our focus to the subleading quantum corrections, taking the form

$$S = \frac{A}{4G} + c \log \frac{A}{G} + \dots, \tag{1.1.16}$$

where c is a constant to be determined. With our current scope of knowledge in quantum gravity, we do not know the full form of the entropy. Instead, one particular quantum correction that we are able to probe are the logarithmic corrections to the entropy. Unlike the area formula, the logarithmic corrections are sensitive to the quantum gravity under consideration. Therefore, these types of corrections provide a unique testing ground for any proposed ultraviolet complete theory of gravity. There has been active exploration on these types of corrections in asymptotically flat spacetimes, both in the extremal and non-extremal regime, mainly pushed by Sen and collaborators. However, there remains a large unexplored territory for these corrections in AdS spacetimes. This is the goal of the second part of this thesis.

We follow two distinct approaches. The first is the heat kernel, which has been widely used in literature for these types of entropy corrections. Our work paves the way for field theoretic computations, which can verify our results. Returning to Figure 1.2, we may speculate to what extent our holographic approaches continue to hold at subleading order. Therefore, our second method is to extend the Kerr/CFT correspondence to compute the logarithmic corrections to asymptotically AdS spacetimes.

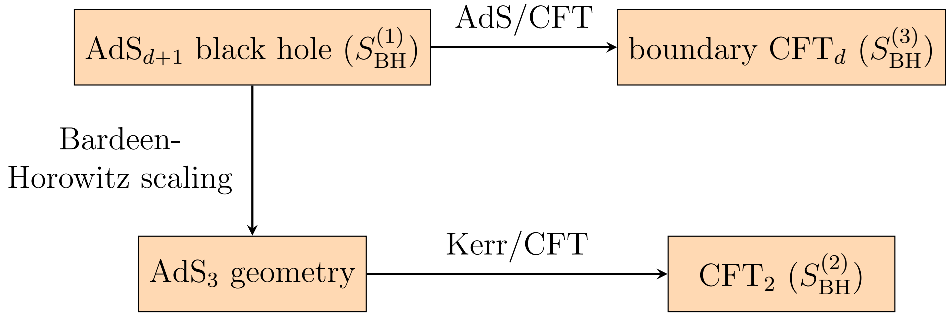

Figure 1.2: The asymptotically AdS black hole entropy can be computed in three different ways ($S_{\rm BH}^{(i)}$), and have found to give one universal result for the entropy. This is valid for both BPS black holes and near-extremal black holes.

1.2 Overview of Thesis

This thesis explores topics in holography and black holes in the pursuit of further advancing our understanding of quantum gravity. The work presented is based on articles written with my advisor Professor Leopoldo Pando Zayas and collaborators. In the following, we give an overview of the structure of the remaining parts of the thesis.

Chapter 2:

In this chapter, we explore the Cardy-like limit, which has played a role in extracting the microscopic counting of black hole entropy in AdS spacetimes via known dual field theories. The Cardy-like limit

$$|\omega_i| \ll 1\,, \tag{1.2.1}$$

corresponds to small angular velocity with respect to the other parameters on the field theory side, as shown in Table 1.1. Our main motivation is to provide a physical interpretation of the Cardy-like limit from the gravity point of view. For a class of electrically charged, rotating, asymptotically AdS black holes, we find that the *gravitational Cardy limit* has the following universal form

$$|a_i g| \to 1, \tag{1.2.2}$$

Dimension of CFT	ω	Δ	J	Q	Entropy Function
$d = 3$	ϵ	1	$1/\epsilon^2$	$1/\epsilon$	$1/\epsilon$
$d = 4$	ϵ	1	$1/\epsilon^3$	$1/\epsilon^2$	$1/\epsilon^2$
$d = 5$	ϵ	1	$1/\epsilon^3$	$1/\epsilon^2$	$1/\epsilon^2$
$d = 6$	ϵ	1	$1/\epsilon^4$	$1/\epsilon^3$	$1/\epsilon^3$

Table 1.1: Scaling of conserved quantum numbers in various field theory dimensions.

where a_i roughly characterizes the angular momenta in units of the inverse radius of AdS, g. As shown in Figure 1.2, we impose the gravitational Cardy limit while also zooming into the near-horizon of the black hole. Near the horizon of these black holes, there exists a three-dimensional AdS subgeometry – $U(1)$ fibered over AdS_2 – which, from holography, corresponds to a two-dimensional quantum theory. This two-dimensional quantum theory successfully accounts for the microscopic black hole entropy via the Cardy formula, which utilizes the universality of the asymptotics of the density of states. The near-horizon limit paired with the gravitational Cardy like limit, reduces the number of $U(1)$ fibers to one, and in effect, we extract only the minimal amount of information to compute the black hole entropy. Therefore, we may reach the same entropy via the three holographic approaches, the gravity AdS_{d+1} solution, the CFT_2 derived from the near-horizon geometry, as well as the CFT_d.

This chapter is based on:

M. David, J. Nian and L. A. Pando Zayas, "Gravitational Cardy Limit and AdS Black Hole Entropy,", JHEP **11** (2020), 041 doi:10.1007/JHEP11(2020)041 [arXiv:2005.10251 [hep-th]].

Chapter 3:

Most of the previous work on the successful matching of the microscopic and macroscopic black hole entropy has assumed supersymmetry, and one natural question that may arise is whether or not the entropy matching is successful when the BPS bound – the intersection of supersymmetry and extremality – is relaxed. This chapter focuses on the near-extremal regime of electrically charged, rotating AdS_4 black holes by exploring the holographic approaches to the entropy both on the gravitational and field theory side. More explicitly, we consider

1. the expansion of the non-extremal AdS_4 black hole solution around the BPS solution,
2. the near-extremal Kerr-Newman-AdS/CFT correspondence from the near-horizon CFT_2,

3. and the microstate counting via AdS/CFT correspondence from the boundary 3d superconformal ABJM theory at small temperature,

as graphically shown in Figure 1.2. Note that we do not assume the Cardy-like limit for any of these cases. In general, regardless whether we take a gravity or field theoretic approach, the entropy takes on one universal expression

$$S_{BH} = S_* + \delta S = S_* + \left(\frac{C}{T_H}\right)_* T_H \,, \tag{1.2.3}$$

where S_* denotes the electrically charged rotating AdS_4 black hole entropy in the BPS limit, while $(C/T_H)_*$ stands for the heat capacity in the BPS limit. This result has several profound consequences. First, we have shown that the Kerr/CFT correspondence [16–18], originally posed for extremal black holes, can also be valid in the near-extremal regime. Second, the universality of the black hole entropy suggests that Fig. 1.2 is not only valid in the supersymmetric regime but also the near-extremal regime. It is an open question to understand the non-supersymmetric regime, especially from the field theory point of view. Finally, our results, which are partially derived from the Kerr/CFT correspondence, suggests that there is a connection between the CFT_d and CFT_2 through some RG flow across dimensions.

This chapter is based on:

M. David and J. Nian, "Universal Entropy and Hawking Radiation of Near-Extremal AdS_4 Black Holes," JHEP **04** (2021), 256 doi:10.1007/JHEP04(2021)256 [arXiv:2009.12370 [hep-th]].

Chapter 4:

While the logarithmic corrections to the Bekenstein Hawking entropy have been well studied for asymptotically flat backgrounds [19–23], these subleading terms in asymptotically AdS remain to be well explored. In this chapter, we study minimal $\mathcal{N} = 2$ gauged supergravity, focusing on the logarithmic corrections to the entropy of asymptotically AdS black holes in four dimensions. Our main approach is via the heat kernel, which we apply to both non-extremal and extremal black holes. A careful limiting procedure as we take the temperature to zero shows that the logarithmic corrections obtained in the non-extremal regime agrees with that of the extremal regime. Moreover, we find that in the BPS case, the logarithmic corrections are topological as they depend on the parameters of the black hole. This is in direct contrast with the results in the asymptotically flat case.

This chapter is based on:

M. David, V. Godet, Z. Liu and L. A. P. Zayas,
'Non-topological logarithmic corrections in minimal gauged supergravity
[arXiv:2112.09444 [hep-th]].

Chapter 5:

As was mentioned in chapter 4, investigating the logarithmic corrections to the black hole entropy via the heat kernel has been quite effective, and we could ask if a similar approach can be taken for black objects in AdS_5. However, probing the logarithmic term via the heat kernel proves to be insufficient for five dimensions. This stems from the fact that the heat kernel coefficient corresponding to the logarithmic term is zero for odd dimensions. Therefore, we consider computing the quantum corrections to the black hole via the Kerr/CFT correspondence, by evaluating the density of states about the saddle point.

On the field theory side, the subleading corrections correspond to the large N expansion of the superconformal index and the refined topologically twisted index of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory for the black hole and black string, respectively. Upon converting from the grand canonical to the microcanonical ensemble, the result derived from the Cardy forumla precisely matches that of the index on the CFT_4.

This chapter is based on:

M. David, A. González Lezcano, J. Nian, and L. A. Pando Zayas,
Logarithmic Corrections to the Entropy of Rotating Back Holes and Black Strings in AdS_5
[arXiv:2106.09730 [hep-th]].

Part I

Holographic Approaches to Black Hole Entropy

Chapter 2

The Gravitational Cardy Limit

2.1 Asymptotically AdS_5 Black Holes

In this section, we consider the asymptotically AdS_5 black holes and the corresponding gravitational Cardy limit. We will demonstrate that the black hole entropy can be computed in various ways as shown in Fig. 1.2, and that the other thermodynamic quantities scale in the gravitational Cardy limit precisely as in the field theory approach following Table 1.1.

2.1.1 AdS_5 Black Hole Solution

In this subsection, we first review the non-extremal asymptotically AdS_5 black hole solution found in [24] with degenerate electric charges $Q_1 = Q_2 = Q_3 = Q$ and two angular momenta $J_{1,2}$, and then take the BPS limit to obtain its supersymmetric version.

The non-extremal asymptotically AdS_5 black hole background was found in [24] as a solution to the equations of motion of the 5d minimal gauged supergravity in the Boyer-Lindquist coordinates $x^\mu = (t,\, r,\, \theta,\, \phi,\, \psi)$. The metric and the gauge field of the black hole solution are given by

$$
\begin{aligned}
ds^2 &= -\frac{\Delta_\theta\left[(1+g^2r^2)\rho^2 dt + 2q\nu\right]\, dt}{\Xi_a \Xi_b \rho^2} + \frac{2q\nu\omega}{\rho^2} + \frac{f}{\rho^4}\left(\frac{\Delta_\theta dt}{\Xi_a\Xi_b} - \omega\right)^2 + \frac{\rho^2 dr^2}{\Delta_r} + \frac{\rho^2 d\theta^2}{\Delta_\theta} \\
&\quad + \frac{r^2+a^2}{\Xi_a}\sin^2\theta\, d\phi^2 + \frac{r^2+b^2}{\Xi_b}\cos^2\theta\, d\psi^2\,, \qquad (2.1.1)
\end{aligned}
$$

$$
A = \frac{\sqrt{3}\, q}{\rho^2}\left(\frac{\Delta_\theta\, dt}{\Xi_a \Xi_b} - \omega\right) + \alpha_5\, dt\,, \qquad (2.1.2)
$$

where

$$
\begin{aligned}
\nu &\equiv b\sin^2\theta\, d\phi + a\cos^2\theta\, d\psi\,, \\
\omega &\equiv a\sin^2\theta\,\frac{d\phi}{\Xi_a} + b\cos^2\theta\,\frac{d\psi}{\Xi_b}\,, \\
\Delta_\theta &\equiv 1 - a^2 g^2\cos^2\theta - b^2 g^2\sin^2\theta\,, \\
\Delta_r &\equiv \frac{(r^2+a^2)(r^2+b^2)(1+g^2r^2)+q^2+2abq}{r^2} - 2m\,, \\
\rho^2 &\equiv r^2 + a^2\cos^2\theta + b^2\sin^2\theta\,, \\
\Xi_a &\equiv 1 - a^2g^2\,, \\
\Xi_b &\equiv 1 - b^2g^2\,, \\
f &\equiv 2m\rho^2 - q^2 + 2abqg^2\rho^2\,,
\end{aligned}
\tag{2.1.3}
$$

and $\alpha_5\, dt$ is a pure gauge term with α_5 a constant. These black hole solutions are characterized by four independent parameters (a, b, m, q). The thermodynamical quantities, including the mass E, the temperature T and the entropy S, can all be expressed in terms of these independent parameters. The other physical quantities, such as the electric charge Q, the electric potential Δ, the angular momenta $J_{1,2}$ and the angular velocities $\Omega_{1,2}$ can similarly be written in terms of the four independent parameters. For example, the gravitational angular velocities $\Omega_{1,2}$ and the temperature T are given by

$$
\begin{aligned}
\Omega_1 &= \frac{a(r_+^2+b^2)(1+g^2r_+^2)+bq}{(r_+^2+a^2)(r_+^2+b^2)+abq}\,, \\
\Omega_2 &= \frac{b(r_+^2+a^2)(1+g^2r_+^2)+aq}{(r_+^2+a^2)(r_+^2+b^2)+abq}\,, \\
T &= \frac{r_+^4\Big[1+g^2(2r_+^2+a^2+b^2)\Big]-(ab+q)^2}{2\pi r_+\Big[(r_+^2+a^2)(r_+^2+b^2)+abq\Big]}\,,
\end{aligned}
\tag{2.1.4}
$$

where r_+ denotes the position of the outer horizon given by the largest root of Δ_r in (4.1.32).

As carefully discussed in [13], it is crucial to make the following important distinctions of these solutions, in the broader context when complex potentials are allowed. The extremal black hole solution is characterized by the appearance of a double root in $\Delta_r = 0$, while the BPS black hole solution is obtained by solving the supersymmetry equations.

The BPS limit is a special limit in the parameter space, such that the backgrounds in this limit are both extremal and supersymmetric. For the class of AdS_5 black hole solutions

(4.1.30), the BPS limit corresponds to the following condition

$$q = \frac{m}{1 + ag + bg} \,. \tag{2.1.5}$$

Moreover, to prevent unphysical naked closed timelike curves (CTC), it is shown in [24] that the BPS solutions should further satisfy the constraint

$$m = \frac{1}{g}(a+b)(1+ag)(1+bg)(1+ag+bg) \,. \tag{2.1.6}$$

Hence, in the BPS limit only two of the four parameters (a, b, m, q) are independent, which can be chosen to be (a, b). The special case $a = b$ corresponds to the supersymmetric AdS_5 black hole solutions found by Gutowski and Reall [25]. In the BPS limit, the outer horizon r_+ coincides with the inner horizon at r_0

$$r_0^2 = \frac{a + b + abg}{g} \,, \tag{2.1.7}$$

and the black hole entropy S_*, the electric charge Q_* and the angular momenta $J_{1,2}^*$ have the following expressions in terms of (a, b)

$$\begin{aligned} S_* &= \frac{\pi^2 (a+b)\sqrt{a+b+abg}}{2g^{3/2}(1-ag)(1-bg)} \,, \\ Q_* &= \frac{\pi(a+b)}{4g(1-ag)(1-bg)} \,, \\ J_1^* &= \frac{\pi(a+b)(2a+b+abg)}{4g(1-ag)^2(1-bg)} \,, \\ J_2^* &= \frac{\pi(a+b)(a+2b+abg)}{4g(1-ag)(1-bg)^2} \,, \end{aligned} \tag{2.1.8}$$

where the entropy S_* is computed from the Bekenstein-Hawking entropy formula

$$S_{BH} = \frac{A}{4G_N} \,, \tag{2.1.9}$$

a quarter of the horizon area in units of Planck length. Using the expressions (2.1.8), we can also rewrite the black hole entropy as a function of Q and $J_{1,2}$

$$S_{BH} = 2\pi \sqrt{\frac{3Q^2}{g^2} - \frac{\pi}{4g^3}(J_1 + J_2)} \,. \tag{2.1.10}$$

The AdS/CFT dictionary helps translate the parameters of the AdS_5 black holes to quantities in $\mathcal{N}=4$ SYM

$$\frac{1}{2}N^2 = \frac{\pi}{4G_N}\ell_5^3, \tag{2.1.11}$$

with $\ell_5 = g^{-1}$ denoting the AdS$_5$ radius. We can rewrite the expression (2.1.10) of the AdS$_5$ black hole entropy (in the unit $G=1$)

$$S_{BH} = 2\pi\sqrt{\frac{3Q^2}{g^2} - \frac{N^2}{2}(J_1+J_2)}\,. \tag{2.1.12}$$

This expression has recently been extracted directly from the boundary CFT in [13–15] with further clarifying field theory work presented in [26–35]. We show below that this boundary CFT result can also be obtained from a particular near-horizon Cardy formula.

2.1.2 Gravitational Cardy Limit

The Cardy-like limit for the $\mathcal{N}=4$ SYM index was defined in [14]. This limit has been discussed in the context of $\mathcal{N}=4$ SYM also in [26, 27, 34]. In the more general context of $\mathcal{N}=1$ superconformal theories, it has been discussed in [28–30]. A key ingredient in the limit is the regime

$$|\omega_i| \ll 1\,, \quad \Delta_I \sim \mathcal{O}(1)\,, \qquad (i=1,\,2;\; I=1,\,2,\,3)\,. \tag{2.1.13}$$

Using the relation found in [13, 33]

$$\mathrm{Re}(\omega_i) = \frac{\partial \Omega_i}{\partial T}\bigg|_{T=0}\,, \quad \mathrm{Re}(\Delta_I) = \frac{\partial \Phi_I}{\partial T}\bigg|_{T=0}\,, \tag{2.1.14}$$

we can express the Cardy-like limit (2.1.13) in terms of quantities in the dual gravity theory, such that

$$\left|\left(\frac{\partial \Omega_i}{\partial T}\right)_{T=0}\right| \ll 1\,, \quad \frac{\partial \Phi_I}{\partial T}\bigg|_{T=0} \sim \mathcal{O}(1)\,, \tag{2.1.15}$$

with $i=1,\,2$ and $I=1,\,2,\,3$. Using the expressions of the thermodynamic quantities (2.1.4), we obtain for the asymptotically AdS$_5$ BPS black holes,

$$\begin{aligned}
\frac{\partial \Omega_1}{\partial T}\bigg|_{\mathrm{BPS}} &= \lim_{T\to 0}\frac{\Omega_1-\Omega_1^*}{T} = \frac{2\pi(-1+ag)}{3g}\sqrt{\frac{1+ag+bg}{ab}}\,,\\
\frac{\partial \Omega_2}{\partial T}\bigg|_{\mathrm{BPS}} &= \lim_{T\to 0}\frac{\Omega_2-\Omega_2^*}{T} = \frac{2\pi(-1+bg)}{3g}\sqrt{\frac{1+ag+bg}{ab}}\,,
\end{aligned} \tag{2.1.16}$$

where $\Omega^*_{1,2}$ are the values of $\Omega_{1,2}$ in the BPS limit. From the expressions of $\frac{\partial \Omega_i}{\partial T}\big|_{\text{BPS}}$ $(i = 1, 2)$, we conclude that for asymptotically AdS_5 BPS black holes, the gravitational Cardy limit corresponds to the special limit of the parameters on the gravity side

$$a \to \frac{1}{g}\,, \quad b \to \frac{1}{g}\,. \tag{2.1.17}$$

For later convenience, we parameterize a and b as

$$a = \frac{1}{g} - \epsilon\,, \quad b = \frac{1}{g} - \epsilon\,. \tag{2.1.18}$$

For this case, ϵ has the dimension of length. Taking the gravitational Cardy limit (2.1.18) for the parameters into account, the BPS thermodynamic quantities (2.1.8) become

$$\begin{aligned} S_* &= \frac{\sqrt{3}\pi^2}{g^5\epsilon^2} + \mathcal{O}(\epsilon^{-1})\,, \\ Q_* &= \frac{\pi}{2g^4\epsilon^2} + \mathcal{O}(\epsilon^{-1})\,, \\ J_1^* &= \frac{2\pi}{g^6\epsilon^3} + \mathcal{O}(\epsilon^{-2})\,, \\ J_2^* &= \frac{2\pi}{g^6\epsilon^3} + \mathcal{O}(\epsilon^{-2})\,, \end{aligned} \tag{2.1.19}$$

which are precisely the scalings of the field theory results [14, 36].

2.1.3 Black Hole Solution in the Near-Horizon + Gravitational Cardy Limit

In the previous subsection, we have obtained the gravitational Cardy limit for the parameters on the gravity side. In this subsection, we discuss how the near-horizon metric changes in this limit as well as clarify other ingredients.

The asymptotically AdS_5 metric (4.1.30) can be written in the following equivalent form, which is more convenient for the discussions in this subsection,

$$ds^2 = -\frac{\Delta_r \Delta_\theta r^2 \sin^2(2\theta)}{4\Xi_a^2 \Xi_b^2 B_\phi B_\psi} dt^2 + \rho^2 \left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\right) + B_\psi (d\psi + v_1 d\phi + v_2 dt)^2 + B_\phi (d\phi + v_3 dt)^2\,, \tag{2.1.20}$$

where

$$\begin{aligned} B_\phi &\equiv \frac{g_{33}\, g_{44} - g_{34}^2}{g_{44}}\,, \qquad B_\psi \equiv g_{44}\,, \\ v_1 \equiv \frac{g_{34}}{g_{44}}\,, &\quad v_2 \equiv \frac{g_{04}}{g_{44}}\,, \quad v_3 \equiv \frac{g_{04}\, g_{34} - g_{03}\, g_{44}}{g_{34}^2 - g_{33}\, g_{44}}\,, \end{aligned} \tag{2.1.21}$$

with the non-vanishing components of the metric (4.1.30) explicitly in the coordinates $(t, r, \theta, \varphi, \psi)$

$$\begin{aligned} g_{00} &= -\frac{\Delta_\theta (1+g^2r^2)}{\Xi_a \Xi_b} + \frac{\Delta_\theta^2 (2m\rho^2 - q^2 + 2abqg^2\rho^2)}{\rho^4 \Xi_a^2 \Xi_b^2}\,, \\ g_{03} = g_{30} &= -\frac{\Delta_\theta \Big[a(2m\rho^2 - q^2) + bq\rho^2(1+a^2g^2) \Big] \sin^2\theta}{\rho^4 \Xi_a^2 \Xi_b}\,, \\ g_{04} = g_{40} &= -\frac{\Delta_\theta \Big[b(2m\rho^2 - q^2) + aq\rho^2(1+b^2g^2) \Big] \cos^2\theta}{\rho^4 \Xi_b^2 \Xi_a}\,, \\ g_{11} &= \frac{\rho^2}{\Delta_r}\,, \qquad\qquad g_{22} = \frac{\rho^2}{\Delta_\theta}\,, \\ g_{33} &= \frac{(r^2+a^2)\sin^2\theta}{\Xi_a} + \frac{a\Big[a(2m\rho^2 - q^2) + 2bq\rho^2 \Big] \sin^4\theta}{\rho^4 \Xi_a^2}\,, \\ g_{44} &= \frac{(r^2+b^2)\cos^2\theta}{\Xi_b} + \frac{b\Big[b(2m\rho^2 - q^2) + 2aq\rho^2 \Big] \cos^4\theta}{\rho^4 \Xi_b^2}\,, \\ g_{34} = g_{43} &= \frac{\Big[ab(2m\rho^2 - q^2) + (a^2+b^2)q\rho^2 \Big] \sin^2\theta \cos^2\theta}{\rho^4 \Xi_a \Xi_b}\,. \end{aligned} \tag{2.1.22}$$

A central element in our approach is a near-horizon limit following the prescription of Bardeen and Horowitz [37] to zoom into a near-horizon region, and at the same time we move to a rotating frame by implementing the following coordinate change

$$r \to r_0 + \lambda\, \widetilde{r}\,, \quad t \to \frac{\widetilde{t}}{\lambda}\,, \quad \phi \to \widetilde{\phi} + g\frac{\widetilde{t}}{\lambda}\,, \quad \psi \to \widetilde{\psi} + g\frac{\widetilde{t}}{\lambda}\,. \tag{2.1.23}$$

Taking $\lambda \to 0$ brings us to a particular near-horizon region of the AdS_5 BPS black holes given by the following metric in the coordinates $(\widetilde{t}, \widetilde{r}, \theta, \widetilde{\phi}, \widetilde{\psi})$

$$
\begin{aligned}
ds^2 = & -\frac{2g(1+5ag)}{a(1+ag)^2}\,\widetilde{r}^2\, d\widetilde{t}^2 + \frac{a}{2g(1+5ag)}\,\frac{d\widetilde{r}^2}{\widetilde{r}^2} + \Lambda_{\mathrm{AdS}_5}(\theta)\left[d\widetilde{\phi} + \frac{3g(1-ag)}{(1+ag)\sqrt{a\left(a+\frac{2}{g}\right)}}\,\widetilde{r}\, d\widetilde{t}\right]^2 \\
& + \frac{a\Big(4-ag+3ag\cos(2\theta)\Big)\cos^2\theta}{2g(1-ag)^2}\left[d\widetilde{\psi} + \frac{6ag\sin^2\theta}{4-ag+3ag\cos(2\theta)}d\widetilde{\phi} + V(\theta)\,\widetilde{r}\, d\widetilde{t}\right]^2 \\
& + \frac{2a}{g(1-ag)}\, d\theta^2\,,
\end{aligned} \tag{2.1.24}
$$

where

$$
\Lambda_{\mathrm{AdS}_5}(\theta) \equiv \frac{4a(2+ag)\sin^2\theta}{g(1-ag)\Big(4-ag+3ag\cos(2\theta)\Big)}\,, \tag{2.1.25}
$$

$$
V(\theta) \equiv \frac{6g^2(1-ag)\sqrt{a\left(a+\frac{2}{g}\right)}}{a(1+ag)\left(4-ag+3ag\cos(2\theta)\right)}\,, \tag{2.1.26}
$$

and for simplicity, we have set $a = b$, in consistency with the gravitational Cardy limit (2.1.18) that will be imposed later. For some special values of θ, the metric (2.1.24) has the topology of two $U(1)$ circles fibered over the AdS_2 parametrized by $(\tilde{t}, \tilde{r})$, as pointed out in [17, 18].

After a further change of coordinates

$$
\tau \equiv \frac{2g(1+5ag)}{a(1+ag)}\,\widetilde{t}\,, \tag{2.1.27}
$$

we can bring the metric (2.1.24) into the form

$$
\begin{aligned}
ds^2 = & \frac{a}{2g(1+5ag)}\left[-\widetilde{r}^2\, d\tau^2 + \frac{d\widetilde{r}^2}{\widetilde{r}^2}\right] + \Lambda_{\mathrm{AdS}_5}(\theta)\left[d\widetilde{\phi} + \frac{3a(1-ag)}{2(1+5ag)\sqrt{a\left(a+\frac{2}{g}\right)}}\,\widetilde{r}\, d\tau\right]^2 \\
& + \frac{a\Big(4-ag+3ag\cos(2\theta)\Big)\cos^2\theta}{2g(1-ag)^2}\left[d\widetilde{\psi} + \frac{6ag\sin^2\theta}{4-ag+3ag\cos(2\theta)}d\widetilde{\phi} + \widetilde{V}(\theta)\,\widetilde{r}\, d\tau\right]^2 \\
& + \frac{2a}{g(1-ag)}\, d\theta^2\,,
\end{aligned} \tag{2.1.28}
$$

where

$$\widetilde{V}(\theta) \equiv \frac{3g(1-ag)\sqrt{a\left(a+\frac{2}{g}\right)}}{(1+5ag)\left(4-ag+3ag\cos(2\theta)\right)}\,. \tag{2.1.29}$$

In both U(1) fibrations, the coefficients in front of $\widetilde{r}\,d\tau$ are proportional to $\partial_T\Omega$ (2.1.16) with $a=b$. Hence, according to the relation (2.1.14), ω_i from $\mathcal{N}=4$ SYM indeed play the role of angular velocities in the metric (2.1.28), and the Cardy-like limit from the field theory side means the angular velocities slow down on some $U(1)$ circles in the near-horizon metric (2.1.28).

In Appendix A.1.1, we verify explicitly that the resulting background is a solution of the 5d minimal gauged supergravity equations of motion. This statement holds for arbitrary values of $a=b$. Up to this point, our approach is completely rigorous and verifying the equations of motion explicitly provides a powerful seal of approval. However, to flesh out the scaling properties of the solution, in what follows we implement the gravitational Cardy limit in the space of parameters which further simplifies the geometry.

We apply the gravitational Cardy limit (2.1.18) to the metric (2.1.28) and keep the leading orders in ϵ, which leads to

$$\begin{aligned} ds^2 = &\frac{1}{12g^2}\left[-\widetilde{r}^2\,d\tau^2+\frac{d\widetilde{r}^2}{\widetilde{r}^2}\right]-\frac{2}{g^3\epsilon}\,d\theta^2-\frac{4\sin^2(\theta)\,\epsilon}{g^3(1+\cos(2\theta))}\left[\frac{1}{\epsilon}d\widetilde{\phi}-\frac{g}{4\sqrt{3}}\,\widetilde{r}\,d\tau\right]^2\\ &+\frac{3\cos^4(\theta)}{g^4}\left[\frac{1}{\epsilon}d\widetilde{\psi}+\frac{2\sin^2(\theta)}{\epsilon\,(1+\cos(2\theta))}d\widetilde{\phi}-\frac{g\sec^2(\theta)}{4\sqrt{3}}\,\widetilde{r}\,d\tau\right]^2\,. \end{aligned} \tag{2.1.30}$$

From this metric, we can see that in the gravitational Cardy limit $\epsilon\to 0$ only one U(1) circle remains non-trivially fibered over AdS_2. We have only assumed that ϵ is small without strictly taking the limit $\epsilon\to 0$, and the near-horizon metric will approximate to AdS_3, as ϵ becomes smaller. However, since the two initial U(1) fibrations give the same result of the black hole entropy according to the Cardy formula and the extreme black hole/CFT correspondence [18], the remaining U(1) is enough to compute the AdS_5 black hole entropy. We will demonstrate this point in the next subsection. To summarize, the gravitational Cardy limit simplifies the near-horizon geometry but keeps the minimal amount of information for computing the black hole entropy.

Let us finish by warning the potentially puzzled reader. The analysis above, surrounding equation (2.1.30), is local and has the sole intention of clarifying the geometry of the gravitational Cardy limit. If bothered by this last limiting procedure it is possible to step back and derive all the quantities from the safer background obtained in equation (2.1.28). However, without this gravitational Cardy limit the connection to the field theory approach would be very tenuous.

2.1.4 Black Hole Entropy from Cardy Formula

In the previous subsection, we showed that a warped AdS_3 geometry appears in the near-horizon region of asymptotically AdS_5 BPS black holes in the gravitational Cardy limit. This circumstance permits the use of ideas presented in [16], which lead to the identification of a Virasoro algebra as the asymptotic symmetries in the near-horizon geometry and, subsequently, to a microscopic description of the black hole entropy via the Cardy formula.

Let us briefly review how the Virasoro algebra emerges as the algebra of asymptotic symmetries of the near-horizon region of the extremal Kerr black hole [16] (see also [38]). Recall that the asymptotic symmetry group is the group of all allowed diffeomorphisms modulo trivial ones where allowed diffeomorphisms are defined as those that preserve certain boundary conditions of the asymptotic metric. The starting element in determining the algebra of asymptotic symmetries is, therefore, to consider diffeomorphims generated by vectors of the form

$$\zeta_\epsilon = \epsilon(\phi)\frac{\partial}{\partial\phi} - r\,\epsilon'(\phi)\frac{\partial}{\partial r}\,, \tag{2.1.31}$$

where $\epsilon(\phi)$ is a function periodic in ϕ. For simplicity we can choose to be $\epsilon(\phi) = -e^{-in\phi}$, and consequently obtain the mode expansion of ζ_ϵ as

$$\zeta_{(n)} = -e^{-in\phi}\frac{\partial}{\partial\phi} - inre^{-in\phi}\frac{\partial}{\partial r}\,, \tag{2.1.32}$$

which satisfies a centreless Virasoro algebra

$$i[\zeta_{(m)},\,\zeta_{(n)}] = (m-n)\zeta_{(m+n)}\,. \tag{2.1.33}$$

The charge associated with the diffeomorphis ζ_ϵ is given by an integral over the boundary of a spatial slice $\partial\Sigma$

$$Q_\zeta = \frac{1}{8\pi G}\int_{\partial\Sigma} k_\zeta\,, \tag{2.1.34}$$

where k_ζ is a 2-form defined for a general perturbation $h_{\mu\nu}$ around the background metric $g_{\mu\nu}$

$$\begin{aligned} k_\zeta[h,\,g] \equiv -\frac{1}{4}\epsilon_{\alpha\beta\mu\nu}\Big[&\zeta^\nu D^\mu h - \zeta^\nu D_\sigma h^{\mu\sigma} + \zeta_\sigma D^\nu h^{\mu\sigma} + \frac{1}{2}hD^\nu\zeta^\mu - h^{\nu\sigma}D_\sigma\zeta^\mu \\ &+ \frac{1}{2}h^{\sigma\nu}(D^\mu\zeta_\sigma + D_\sigma\zeta^\mu)\Big]dx^\alpha\wedge dx^\beta\,, \end{aligned} \tag{2.1.35}$$

with $h \equiv h_{\alpha\beta} g^{\alpha\beta}$. The Dirac bracket of the charges is

$$\{Q_{\zeta_{(m)}},\, Q_{\zeta_{(n)}}\} = Q_{[\zeta_{(m)},\, \zeta_{(n)}]} + \frac{1}{8\pi G} \int_{\partial\Sigma} k_\zeta[\mathcal{L}_\zeta g,\, g]\,, \tag{2.1.36}$$

where $\mathcal{L}_\zeta$ denotes the Lie derivative with respect to ζ

$$\mathcal{L}_\zeta g_{\mu\nu} \equiv \zeta^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu} \partial_\mu \zeta^\rho + g_{\mu\rho} \partial_\nu \zeta^\rho\,. \tag{2.1.37}$$

The mode expansion of the Dirac bracket (2.1.36) leads to a Virasora algebra

$$[L_m,\, L_n] = (m-n) L_{m+n} + \frac{1}{12} c_L \left(m^3 + \alpha m\right) \delta_{m+n,\,0}\,, \tag{2.1.38}$$

where c_L can be obtained from the integral

$$\frac{1}{8\pi G} \int_{\partial\Sigma} k_{\zeta_{(m)}}[\mathcal{L}_{\zeta_{(n)}} g,\, g] = -\frac{i}{12} c_L \left(m^3 + \alpha m\right) \delta_{m+n,\,0}\,, \tag{2.1.39}$$

and α is an irrelevant constant.

To compute the black hole entropy using the Cardy formula, we still need the Frolov-Thorne temperature, which can be obtained in the following way. The quantum fields on the background (4.1.30) can be expanded in the modes $e^{-i\omega t + im\phi}$. After taking the scaling (4.1.37), these modes become

$$e^{-i\omega t + im\phi} = e^{-i\omega \frac{\tilde{t}}{\lambda} + im\left(\tilde{\phi} + g\frac{\tilde{t}}{\lambda}\right)} = e^{-i\left(\frac{\omega}{\lambda} - \frac{mg}{\lambda}\right)\tilde{t} + im\tilde{\phi}} \equiv e^{-in_R \tilde{t} + in_L \tilde{\phi}}\,, \tag{2.1.40}$$

from which we can read off the left-moving and the right-moving mode numbers

$$n_L \equiv m\,, \quad n_R \equiv \frac{\omega - mg}{\lambda}\,. \tag{2.1.41}$$

The Boltzmann factor is

$$e^{-\frac{\omega - m\Omega}{T_H}} = e^{-\frac{n_L}{T_L} - \frac{n_R}{T_R}}\,, \tag{2.1.42}$$

where T_H is the Hawking temperature, and $T_{L,R}$ are the left-moving and the right-moving Frolov-Thorne temperatures. Combining (2.1.41) and (2.1.42), we obtain the near-extremal Frolov-Thorne temperatures

$$T_L = \frac{T_H}{g - \Omega}\,, \quad T_R = \frac{T_H}{\lambda}\,. \tag{2.1.43}$$

The values for the extremal AdS_5 black holes can be obtained by taking the extremal limit ($T_H \to 0$).

In order to apply the technique described above, we need to first transform the AdS_2 Poincaré coordinates $(\widetilde{r}, \tau)$ in the metric (2.1.28) to global coordinates $(\hat{r}, \hat{t})$

$$g\,\widetilde{r} = \hat{r} + \sqrt{1+\hat{r}^2}\cos(\hat{t})\,, \quad g^{-1}\,\tau = \frac{\sqrt{1+\hat{r}^2}\sin(\hat{t})}{\hat{r} + \sqrt{1+\hat{r}^2}\cos(\hat{t})}\,, \tag{2.1.44}$$

which leads to

$$\begin{aligned} -\widetilde{r}^2\,d\tau^2 + \frac{d\widetilde{r}^2}{\widetilde{r}^2} &= -(1+\hat{r}^2)\,d\hat{t}^2 + \frac{d\hat{r}^2}{1+\hat{r}^2}\,,\\ \widetilde{r}\,d\tau &= \hat{r}\,d\hat{t} + d\gamma\,, \end{aligned} \tag{2.1.45}$$

where

$$\gamma \equiv \log\left(\frac{1+\sqrt{1+\hat{r}^2}\sin(\hat{t})}{\cos(\hat{t}) + \hat{r}\sin(\hat{t})}\right)\,. \tag{2.1.46}$$

Consequently, the near-horizon metric (2.1.28) of the AdS_5 BPS black holes can be written as

$$\begin{aligned} ds^2 = {}& \frac{a}{2g(1+5ag)}\left[-(1+\hat{r}^2)\,d\hat{t}^2 + \frac{d\hat{r}^2}{1+\hat{r}^2}\right] + \Lambda_{\mathrm{AdS}_5}(\theta)\left[d\hat{\phi} + \frac{3a(1-ag)}{2(1+5ag)\sqrt{a\left(a+\frac{2}{g}\right)}}\,\hat{r}\,d\hat{t}\right]^2 \\ & + \frac{a\Big(4-ag+3ag\cos(2\theta)\Big)\cos^2\theta}{2g(1-ag)^2}\left[d\hat{\psi} + \frac{6ag\sin^2\theta}{4-ag+3ag\cos(2\theta)}d\hat{\phi} + \widetilde{V}(\theta)\,\hat{r}\,d\hat{t}\right]^2 \\ & + \frac{2a}{g(1-ag)}\,d\theta^2\,, \end{aligned} \tag{2.1.47}$$

where

$$\hat{\phi} \equiv \widetilde{\phi} + \frac{3a(1-ag)\gamma}{2(1+5ag)\sqrt{a\left(a+\frac{2}{g}\right)}}\,, \quad \hat{\psi} \equiv \widetilde{\psi} + \frac{3a(1-ag)\gamma}{2(1+5ag)\sqrt{a\left(a+\frac{2}{g}\right)}}\,. \tag{2.1.48}$$

Applying the formalism reviewed in this subsection, we can compute the central charge and the extremal Frolov-Thorne temperature in the near-horizon region of the asymptotically AdS_5 BPS black hole solutions (2.1.47)

$$c_L = \frac{9\pi a^2}{g(1-ag)(1+5ag)}\,, \tag{2.1.49}$$

$$T_L = \frac{1+5ag}{3a(1-ag)\pi}\sqrt{a\left(a+\frac{2}{g}\right)}\,. \tag{2.1.50}$$

The BPS black hole entropy in this case is then given by the Cardy formula

$$S_{BH} = \frac{\pi^2}{3}c_L T_L = \frac{\pi^2 a\sqrt{2a+a^2 g}}{g^{3/2}(1-ag)^2}\,, \tag{2.1.51}$$

which is the same as the result from gravity (2.1.8) with $a = b$. In fact, we can also apply the formalism discussed in this subsection to the near-horizon metric in the gravitational Cardy limit (2.1.30), which can be recast into the global coordinates

$$\begin{aligned} ds^2 = {} & \frac{1}{12g^2}\left[-(1+\hat{r}^2)\,d\hat{t}^2 + \frac{d\hat{r}^2}{1+\hat{r}^2}\right] - \frac{2}{g^3\epsilon}\,d\theta^2 - \frac{4\sin^2(\theta)\,\epsilon}{g^3(1+\cos(2\theta))}\left[\frac{1}{\epsilon}d\hat{\phi} - \frac{g}{4\sqrt{3}}\,\hat{r}\,d\hat{t}\right]^2 \\ & + \frac{3\cos^4(\theta)}{g^4}\left[\frac{1}{\epsilon}d\hat{\psi} + \frac{2\sin^2(\theta)}{\epsilon\,(1+\cos(2\theta))}d\hat{\phi} - \frac{g\sec^2(\theta)}{4\sqrt{3}}\,\hat{r}\,d\hat{t}\right]^2 . \end{aligned} \tag{2.1.52}$$

The corresponding central charge and the extremal Frolov-Thorne temperature are

$$c_L = \frac{3\pi}{2g^4\epsilon}\,, \qquad T_L = \frac{2\sqrt{3}}{\pi g\epsilon}\,. \tag{2.1.53}$$

The black hole entropy can obtained from the Cardy formula

$$S_{BH} = \frac{\pi^2}{3}c_L T_L = \frac{\sqrt{3}\pi^2}{g^5\epsilon^2}\,, \tag{2.1.54}$$

which exactly matches the gravity result in the gravitational Cardy limit (2.1.19).

2.1.5 Comparison with Results from Boundary CFT

The asymptotically AdS_5 BPS black hole entropy can also be obtained from the boundary $\mathcal{N} = 4$ SYM by extremizing an entropy function [13–15] originally motivated in [39] and more recently studied in [40]. One can first compute the free energy in the large-N limit using the partition function via localization or the 4d superconformal index. The entropy function is then defined as the Legendre transform of the free energy in the large-N limit

$$S(\Delta_I,\,\omega) = \frac{N^2}{2}\frac{\Delta_1\Delta_2\Delta_3}{\omega_1\omega_2} + \sum_{I=1}^{3} Q_I\Delta_I + \sum_{i=1}^{2} J_i\omega_i - \Lambda\left(\sum_I \Delta_I - \sum_i \omega - 2\pi i\right)\,. \tag{2.1.55}$$

In the Cardy-like limit (2.1.13)

$$\omega \sim \epsilon\,,\quad \Delta_I \sim \mathcal{O}(1)\,, \tag{2.1.56}$$

we can read off from the entropy function (2.1.55)

$$S \sim \frac{1}{\epsilon^2}\,,\quad J \sim \frac{1}{\epsilon^3}\,,\quad Q_I \sim \frac{1}{\epsilon^2}\,, \tag{2.1.57}$$

which have been summarized in Table 1.1.

The electric charges Q_I and the angular momenta J_i are real, while the chemical potentials Δ_I and the angular velocities ω_i can be complex, and so can the entropy function S. By requiring that the black hole entropy S_{BH} be real after extremizing the entropy function S, we obtain one more constraint on Q_I and J_i. More precisely, the asymptotically AdS_5 black hole entropy is given by [13–15]

$$S_{BH} = 2\pi\sqrt{Q_1Q_2 + Q_2Q_3 + Q_3Q_1 - \frac{N^2}{2}(J_1 + J_2)}\,, \tag{2.1.58}$$

subject to the constraint

$$\begin{aligned}&\left(Q_1 + Q_2 + Q_3 + \frac{N^2}{2}\right)\left(Q_1Q_2 + Q_2Q_3 + Q_3Q_1 - \frac{N^2}{2}(J_1 + J_2)\right)\\ &- \left(Q_1Q_2Q_3 + \frac{N^2}{2}J_1J_2\right) = 0\,,\end{aligned} \tag{2.1.59}$$

which is a consequence of the reality condition on the black hole entropy S_{BH}.

For the AdS_5 black hole solutions in [24], the electric charges are degenerate, i.e. $Q_1 = Q_2 = Q_3 = Q$. Hence, for this class of black hole solutions in the BPS limit, the black hole entropy becomes

$$S_{BH} = 2\pi\sqrt{3Q^2 - \frac{N^2}{2}(J_1 + J_2)}\,. \tag{2.1.60}$$

This is exactly the same as the result from the horizon area (2.1.12) in the unit $g = 1$, and the one from the Cardy formula (2.1.51). The constraint (2.1.59) for this degenerate case becomes

$$\left(3Q + \frac{N^2}{2}\right)\left(3Q^2 - \frac{N^2}{2}(J_1 + J_2)\right) = Q^3 + \frac{N^2}{2}J_1 J_2\,, \tag{2.1.61}$$

which is also consistent with the thermodynamic quantities from the gravity side (2.1.8).

2.2 Asymptotically AdS_4 Black Holes

In this section, we consider the asymptotically AdS_4 black holes and the corresponding gravitational Cardy limit. Similar to the AdS_5 case, we demonstrate that the AdS_4 black hole entropy can be computed in various ways as shown in Fig. 1.2, and the other thermodynamic quantities scale correspondingly (see Table 1.1) in the gravitational Cardy limit.

2.2.1 AdS_4 Black Hole Solution

The non-extremal rotating, electrically charged asymptotically AdS_4 black hole solution with gauge group $U(1) \times U(1)$ in 4d $\mathcal{N} = 4$ gauged supergravity was constructed in [41]. The solution is characterized by four parameters $(a, m, \delta_1, \delta_2)$. The metric, the scalars and the gauge fields are given by

$$ds^2 = -\frac{\Delta_r}{W}\left(dt - \frac{a\sin^2\theta}{\Xi}d\phi\right)^2 + W\left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\right) + \frac{\Delta_\theta \sin^2\theta}{W}\left[a\,dt - \frac{r_1 r_2 + a^2}{\Xi}d\phi\right]^2\,, \tag{2.2.1}$$

$$\begin{aligned}
e^{\varphi_1} &= \frac{r_1^2 + a^2\cos^2\theta}{W}\,, \qquad \chi_1 = \frac{a(r_2 - r_1)\cos\theta}{r_1^2 + a^2\cos^2\theta}\,,\\
A_1 &= \frac{2\sqrt{2}m\sinh(\delta_1)\cosh(\delta_1)\,r_2}{W}\left(dt - \frac{a\sin^2\theta}{\Xi}d\phi\right) + \alpha_{41}\,dt\,,\\
A_2 &= \frac{2\sqrt{2}m\sinh(\delta_2)\cosh(\delta_2)\,r_1}{W}\left(dt - \frac{a\sin^2\theta}{\Xi}d\phi\right) + \alpha_{42}\,dt\,,
\end{aligned} \tag{2.2.2}$$

where

$$\begin{aligned}
r_i &\equiv r + 2m\sinh^2(\delta_i)\,, \quad (i = 1, 2)\\
\Delta_r &\equiv r^2 + a^2 - 2mr + g^2 r_1 r_2(r_1 r_2 + a^2)\,,\\
\Delta_\theta &\equiv 1 - g^2 a^2\cos^2\theta\,,\\
W &\equiv r_1 r_2 + a^2\cos^2\theta\,,\\
\Xi &\equiv 1 - a^2 g^2\,,
\end{aligned} \tag{2.2.3}$$

and $g \equiv \ell_4^{-1}$ is the inverse of the AdS_4 radius. Note that we have added pure gauge terms to the two gauge fields where α_{41} and α_{42} are constant. The metric (2.2.1) can also be written in the following equivalent expression, which is more convenient for later discussions,

$$ds^2 = -\frac{\Delta_r \Delta_\theta}{B\Xi^2} dt^2 + B\sin^2\theta (d\phi + f\, dt)^2 + W\left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\right), \tag{2.2.4}$$

with

$$\begin{aligned} B &\equiv \frac{(a^2 + r_1 r_2)^2 \Delta_\theta - a^2 \sin^2(\theta)\, \Delta_r}{W\Xi^2}\,, \\ f &\equiv \frac{a\Xi\,(\Delta_r - \Delta_\theta(a^2 + r_1 r_2))}{\Delta_\theta (a^2 + r_1 r_2)^2 - a^2 \Delta_r \sin^2\theta}\,. \end{aligned} \tag{2.2.5}$$

The non-extremal asymptotically AdS_4 black holes with four degenerate electric charges ($Q_1 = Q_2$, $Q_3 = Q_4$) and one angular momentum J have been found in [42], which are characterized by four parameters $(a, m, \delta_1, \delta_2)$. The BPS limit imposes a condition

$$e^{2\delta_1 + 2\delta_2} = 1 + \frac{2}{ag}\,. \tag{2.2.6}$$

For the black hole solution to have a regular horizon, we impose an additional constraint

$$(mg)^2 = \frac{\cosh^2(\delta_1 + \delta_2)}{e^{\delta_1 + \delta_2} \sinh^3(\delta_1 + \delta_2) \sinh(2\delta_1) \sinh(2\delta_2)}\,. \tag{2.2.7}$$

The two conditions (2.2.6) and (3.1.4) in [42] have typos, which have been corrected in [43,44], see also [45]. With these constraints, there are only two independent parameters for asymptotically AdS_4 BPS black holes, which we choose to be (δ_1, δ_2) for convenience. In the BPS limit, the position of the outer horizon is

$$r_+ = \frac{2m \sinh(\delta_1) \sinh(\delta_2)}{\cosh(\delta_1 + \delta_2)}\,, \tag{2.2.8}$$

which coincides with the inner horizon.

The physical quantities of non-extremal AdS_4 black holes can also be solved as functions of $(a, m, \delta_1, \delta_2)$. In particular, the gravitational angular velocity Ω and the temperature T are given by

$$\Omega = \frac{a(1 + g^2 r_1 r_2)}{r_1 r_2 + a^2}\,, \quad T = \frac{\Delta_r'}{4\pi(r_1 r_2 + a^2)}\,. \tag{2.2.9}$$

Moreover, the other thermodynamic quantities of asymptotically AdS_4 black holes are [42]

$$\begin{aligned} S &= \frac{\pi(r_1 r_2 + a^2)}{\Xi}\,, \\ J &= \frac{ma}{2\Xi^2}\left(\cosh(2\delta_1) + \cosh(2\delta_2)\right)\,, \\ Q_1 = Q_2 &= \frac{m}{4\Xi}\sinh(2\delta_1)\,, \\ Q_3 = Q_4 &= \frac{m}{4\Xi}\sinh(2\delta_2)\,. \end{aligned} \tag{2.2.10}$$

2.2.2 Gravitational Cardy Limit

The Cardy-like limit for the 3d ABJM theory was defined in [46, 47],

$$|\omega| \ll 1\,, \quad \Delta_I \sim \mathcal{O}(1)\,, \qquad (I = 1, \cdots, 4)\,. \tag{2.2.11}$$

Using the relations found in [44]

$$\omega = -\lim_{T\to 0}\frac{\Omega - \Omega^*}{T}\,, \quad \Delta_I = -\lim_{T\to 0}\frac{\Phi_I - \Phi_I^*}{T}\,, \tag{2.2.12}$$

with $\Omega^* = g$ and $\Phi_I^* = 1$ denoting the BPS values of Ω and Φ_I, we can find the gravitational counterpart of the Cardy-like limit (2.2.11)

$$\left|\left(\frac{\partial \Omega}{\partial T}\right)_{T=0}\right| \ll 1\,, \quad \left.\frac{\partial \Phi_I}{\partial T}\right|_{T=0} \sim \mathcal{O}(1)\,. \tag{2.2.13}$$

Hence, we obtain for the near-extremal AdS_4 black holes

$$\left.\frac{\partial \Omega}{\partial T}\right|_{\text{BPS}} = \lim_{T\to 0}\frac{\Omega - \Omega_*}{T} = -\frac{\pi e^{\frac{5}{2}(\delta_1+\delta_2)}\left(\coth(\delta_1+\delta_2) - 2\right)\sqrt{\sinh(2\delta_1)\sinh(2\delta_2)}}{\left(\coth(\delta_1+\delta_2)+1\right)\sqrt{\sinh(\delta_1+\delta_2)}\cosh(\delta_1-\delta_2)}\,. \tag{2.2.14}$$

This expression has several roots

$$\delta_1 = 0\,, \quad \delta_2 = 0\,, \quad \delta_1 + \delta_2 = \text{arccoth}(2)\,. \tag{2.2.15}$$

However, $\delta_1 = 0$ and $\delta_2 = 0$ are unphysical, because according to (2.2.8), $\delta_1 = 0$ or $\delta_2 = 0$ will cause $r_+ \to 0$. Hence, we conclude that the gravitational Cardy limit for asymptotically AdS_4 BPS black holes is

$$\delta_1 + \delta_2 \to \text{arccoth}(2)\,. \tag{2.2.16}$$

Equivalently, this can be written in terms of the other parameters as

$$ag \to 1. \tag{2.2.17}$$

We introduce a small parameter ϵ to denote small deviations from this limit, i.e.,

$$\delta_1 + \delta_2 = \operatorname{arccoth}(2) + \epsilon\,. \tag{2.2.18}$$

For this case ϵ is dimensionless. Imposing first the BPS constraint (2.2.6) and the regularity condition (3.1.4) near the horizon, and then taking the gravitational Cardy limit (2.2.18), we obtain the thermodynamic quantities (3.1.7) to the leading order in ϵ

$$\begin{aligned}
S_* &= \frac{\pi}{3g^2\epsilon} + \mathcal{O}(1)\,,\\
J_* &= \frac{\cosh\left(2\delta_1 - \frac{1}{2}\log(3)\right)}{9g^2\epsilon^2\sqrt{2\sinh(4\delta_1) - 5\sinh^2(2\delta_1)}} + \mathcal{O}(\epsilon^{-1})\,,\\
Q_1^* = Q_2^* &= \frac{1}{4g\epsilon\sqrt{6\tanh(\delta_1) + 6\coth(\delta_1) - 15}} + \mathcal{O}(1)\,,\\
Q_3^* = Q_4^* &= \frac{\sqrt{2\tanh(\delta_1) + 2\coth(\delta_1) - 5}}{12\sqrt{3}\,g\epsilon} + \mathcal{O}(1)\,,
\end{aligned} \tag{2.2.19}$$

which are consistent with [36, 46] and the Cardy-like limit on the field theory side (2.2.11)

$$\omega_* \sim \epsilon\,, \quad \Delta_{I*} \sim \mathcal{O}(1)\,. \tag{2.2.20}$$

2.2.3 Black Hole Solution in the Near-Horizon + Gravitational Cardy Limit

In the previous subsection, we have obtained the gravitational Cardy limit for the parameters on the gravity side. In this subsection, we discuss how the near-horizon metric changes when taking the gravitational Cardy limit. In Appendix A.1.2, we verify explicitly that the resulting background is a solution of the 4d gauged supergravity equations of motion. In the following, we implement the gravitational Cardy limit in the space of parameters, which further simplifies the geometry.

For the asymptotically AdS_4 black hole metric (2.2.4), we perform a near-horizon scaling similar to the AdS_5 case (4.1.37)

$$r \to r_* + \lambda\,\widetilde{r}\,, \quad t \to \frac{\widetilde{t}}{\lambda}\,, \quad \phi \to \widetilde{\phi} - g\big[\coth(\delta_1 + \delta_2) - 2\big]\frac{\widetilde{t}}{\lambda}\,. \tag{2.2.21}$$

Furthermore, we take the gravitational Cardy limit (2.2.18) and keep only the leading orders in ϵ. The metric (2.2.4) finally becomes

$$
\begin{aligned}
ds^2 = & -\frac{g^2\left(9-18\,e^{4\delta_1}+e^{8\delta_1}\right)(3+\cos(2\theta))}{3\left(9-10\,e^{4\delta_1}+e^{8\delta_1}\right)}\,\widetilde{r}^2 d\widetilde{t}^2+\frac{3\,(3+\cos(2\theta))\left(e^{4\delta_1}-1\right)\left(e^{4\delta_1}-9\right)}{16g^2\left(9-18e^{4\delta_1}+e^{8\delta_1}\right)}\frac{d\widetilde{r}^2}{\widetilde{r}^2} \\
& +\frac{3+\cos(2\theta)}{2g^2\sin^2\theta}\,d\theta^2+\frac{2\sin^4\theta}{9g^2\,(3+\cos(2\theta))}\left[\frac{d\widetilde{\phi}}{\epsilon}+\frac{\sqrt{3}g^2\cosh\left(2\,\delta_1-\frac{1}{2}\log(3)\right)}{\sqrt{\sinh(2\delta_1)\sinh\left(\log(3)-2\delta_1\right)}}\widetilde{r}\,d\widetilde{t}\right]^2 \\
& +\mathcal{O}(\epsilon)\,.
\end{aligned}
\tag{2.2.22}
$$

Defining

$$
\tau\equiv\frac{4g^2\left(9-5\cosh(4\,\delta_1)+4\sinh(4\,\delta_1)\right)^2}{3\left(5-5\cosh(4\,\delta_1)+4\sinh(4\,\delta_1)\right)}\,\widetilde{t}\,,
\tag{2.2.23}
$$

we can rewrite the metric (2.2.22) as

$$
\begin{aligned}
ds^2 = & \frac{3\,(3+\cos(2\theta))\left(e^{4\delta_1}-1\right)\left(e^{4\delta_1}-9\right)}{16g^2\left(9-18e^{4\delta_1}+e^{8\delta_1}\right)}\left[-\,\widetilde{r}^2 d\tau^2+\frac{d\widetilde{r}^2}{\widetilde{r}^2}\right]+\frac{3+\cos(2\theta)}{2g^2\sin^2\theta}\,d\theta^2 \\
& +\frac{2\sin^4\theta}{9g^2\,(3+\cos(2\theta))}\left[\frac{d\widetilde{\phi}}{\epsilon}+V(\delta_1)\widetilde{r}\,d\tau\right]^2+\mathcal{O}(\epsilon)\,,
\end{aligned}
\tag{2.2.24}
$$

where

$$
V(\delta_1)\equiv\frac{9\cosh\left(2\,\delta_1-\frac{1}{2}\log(3)\right)\,\left(5-5\cosh(4\,\delta_1)+4\sinh(4\,\delta_1)\right)}{2\sqrt{10-6\cosh\left(4\,\delta_1-\log(3)\right)}\left(9-5\cosh(4\,\delta_1)+4\sinh(4\,\delta_1)\right)}\,.
\tag{2.2.25}
$$

2.2.4 Black Hole Entropy from Cardy Formula

For the asymptotically AdS_4 black holes discussed in this section, we apply the Cardy formula to the near-horizon metric only after taking the gravitational Cardy limit. More explicitly, we first rewrite the metric (2.2.24) from the Poincaré coordinates ($\widetilde{r}$, τ) to the global coordinates ($\hat{r}$, $\hat{t}$) using the relations (2.1.44) - (2.1.46). Consequently, the near-horizon metric in the gravitational Cardy limit (2.2.24) becomes

$$
\begin{aligned}
ds^2 = & \frac{3\,(3+\cos(2\theta))\left(e^{4\delta_1}-1\right)\left(e^{4\delta_1}-9\right)}{16g^2\left(9-18e^{4\delta_1}+e^{8\delta_1}\right)}\left[-\,(1+\hat{r}^2)d\hat{t}^2+\frac{d\hat{r}^2}{1+\hat{r}^2}\right]+\frac{3+\cos(2\theta)}{2g^2\sin^2\theta}\,d\theta^2 \\
& +\frac{2\sin^4\theta}{9g^2\,(3+\cos(2\theta))}\left[\frac{d\hat{\psi}}{\epsilon}+V(\delta_1)\hat{r}\,d\hat{t}\right]^2+\mathcal{O}(\epsilon)\,,
\end{aligned}
\tag{2.2.26}
$$

where $\hat{t}$, $\hat{r}$ and γ are defined in (2.1.44) and (2.1.46), while

$$
\hat{\psi}\equiv\hat{\phi}+V(\delta_1)\gamma\epsilon\,.
\tag{2.2.27}
$$

Applying the same formalism in Subsection 2.1.4, we obtain the central charge and the extremal Frolov-Thorne temperature in the near-horizon region of the asymptotically AdS_4 BPS black holes,

$$
\begin{aligned}
c_L &= \frac{3\sqrt{\frac{3}{2}}\, e^{2\delta_1}(3+e^{4\delta_1})\sqrt{5+4\sinh(4\,\delta_1)-5\cosh(4\,\delta_1)}}{g^2(18\,e^{4\,\delta_1}-e^{8\,\delta_1}-9)}\,,\\
T_L &= \frac{9+4\sinh(4\,\delta_1)-5\cosh(4\,\delta_1)}{18\pi\epsilon\sinh(\delta_1)\cosh(\delta_1)\cosh\left(2\,\delta_1-\frac{1}{2}\log(3)\right)\sqrt{2\tanh(\delta_1)+2\coth(\delta_1)-5}}\,.
\end{aligned}
\tag{2.2.28}
$$

Using the Cardy formula, we can compute the black hole entropy of the asymptotically AdS_4 BPS black holes:

$$
S_{BH} = \frac{\pi^2}{3} c_L T_L = \frac{\pi}{3g^2\epsilon}\,, \tag{2.2.29}
$$

which is the same as the black hole entropy in the gravitational Cardy limit (2.2.19) from the gravity side.

2.2.5 Comparison with Results from Boundary CFT

The asymptotically AdS_4 BPS black hole entropy can also be obtained from the boundary 3d ABJM theory by extremizing an entropy function [46,47], which has also been studied in [40]. One can first compute the free energy in the large-N limit using the 3d superconformal index or the partition function via localization. The entropy function is then defined as a Legendre transform of the free energy in the large-N limit

$$
S(\Delta_I,\,\omega) = \frac{2\sqrt{2}\,i\,k^{\frac{1}{2}}N^{\frac{3}{2}}}{3}\frac{\sqrt{\Delta_1\Delta_2\Delta_3\Delta_4}}{\omega} - 2\omega J - \sum_I \Delta_I Q_I - \Lambda\left(\sum_I \Delta_I - 2\omega + 2\pi i\right)\,. \tag{2.2.30}
$$

In the Cardy-like limit (2.2.11)

$$
\omega \sim \epsilon\,,\quad \Delta_I \sim \mathcal{O}(1)\,, \tag{2.2.31}
$$

we can read off from the entropy function (2.2.30)

$$
S \sim \frac{1}{\epsilon}\,,\quad J \sim \frac{1}{\epsilon^2}\,,\quad Q_I \sim \frac{1}{\epsilon}\,, \tag{2.2.32}
$$

which have been summarized in Table 1.1.

Similar to the AdS_5 case, for AdS_4 the electric charges Q_I and the angular momentum J are real, while the chemical potentials Δ_I and the angular velocity ω can be complex, and so can the entropy function S. By requiring that the black hole entropy S_{BH} to be real after extremizing the entropy function S, we obtain one more constraint on Q_I and J. More precisely, for the degenerate case with $Q_1 = Q_3$, $Q_2 = Q_4$ and one angular momentum J, the asymptotically AdS_4 black hole entropy is given by [44, 46, 47]

$$S_{BH} = \frac{2\pi}{3}\sqrt{\frac{9Q_1Q_2(Q_1+Q_2)-2kJN^3}{Q_1+Q_2}}\,, \tag{2.2.33}$$

subject to the constraint

$$2kJ^2N^3 + 2kJN^3(Q_1+Q_2) - 9Q_1Q_2(Q_1+Q_2)^2 = 0\,, \tag{2.2.34}$$

which is a consequence of the reality condition on the black hole entropy S_{BH}.

If we identify the field theory parameters with the ones on the gravity side in the following way [44, 48]

$$\frac{1}{G} = \frac{2\sqrt{2}}{3}g^2k^{\frac{1}{2}}N^{\frac{3}{2}}\,, \quad Q_{BH} = \frac{g}{2}Q\,, \quad J_{BH} = J\,, \tag{2.2.35}$$

we can rewrite the black hole entropy (2.2.33) and the angular momentum as

$$S_{BH} = \frac{\pi}{g^2G}\frac{J_{BH}}{\left(\frac{2}{g}Q_{BH,1}+\frac{2}{g}Q_{BH,2}\right)}\,, \tag{2.2.36}$$

$$J_{BH} = \frac{1}{2}\left(\frac{2}{g}Q_{BH,1}+\frac{2}{g}Q_{BH,2}\right)\left(-1+\sqrt{1+16g^4G^2\frac{2Q_{BH,1}}{g}\frac{2Q_{BH,2}}{g}}\right)\,, \tag{2.2.37}$$

which are consistent with both the thermodynamic quantities on the gravity side (3.1.7) (2.2.19) and the black hole entropy in the gravitational Cardy limit from the Cardy formula (2.2.29).

2.3 Asymptotically AdS_7 Black Holes

In this section, we consider asymptotically AdS_7 black holes and the corresponding gravitational Cardy limit. Similar to the previous sections, we demonstrate that the AdS_7 black hole entropy can be computed in various ways as shown in Fig. 1.2, and the other thermodynamic quantities scale correspondingly in gravitational Cardy limit. For completeness, we discuss two asymptotically AdS_7 black hole solutions in the literature: a special case with all equal charges and all equal angular momenta in Subsection 2.3.1 and a more general case with two equal charges and three independent angular momenta in Subsection 2.3.2.

2.3.1 A Special Case

In this subsection, we consider the gravitational Cardy limit of a special class of non-extremal asymptotically AdS$_7$ black holes discussed in [49].

AdS$_7$ Black Hole Solution

The solutions $\mathcal{M}_7 \times S^4$ to 11d gauged supergravity, with $\mathcal{M}_7$ denoting an asymptotically AdS$_7$ black hole, have the isometry group $SO(2,6) \times SO(5)$. Hence, this class of solutions has three angular momenta from the Cartan of the maximal compact subgroup $SO(6) \subset SO(2,6)$ and two electric charges from the Cartan of $SO(5)$. The most generic solution has not been constructed in the literature so far. Instead, the solutions with two charges and three equal angular momenta were found in [50], while the ones with two equal charges and three angular momenta were found in [51]. As the intersection of these two classes, the solution with two equal charges $Q_1 = Q_2$ and three equal angular momenta $J_1 = J_2 = J_3$ has been considered in [49].

For this special solution, the metric of the asymptotically AdS$_7$ black hole part is given by

$$ds_7^2 = H^{\frac{2}{5}} \left[-\frac{Y}{f_1 \Xi_-^2} dt^2 + \frac{\Xi\, \rho^6}{Y} dr^2 + \frac{f_1}{H^2\, \Xi^2\, \rho^4} \left(\sigma - \frac{2 f_2}{f_1} dt \right)^2 + \frac{\rho^2}{\Xi}\, ds^2_{\mathbb{CP}^2} \right], \tag{2.3.1}$$

where

$$\sigma \equiv d\chi + \frac{1}{2} l_3 \sin^2\xi\,, \tag{2.3.2}$$

$$ds^2_{\mathbb{CP}^2} = d\xi^2 + \frac{1}{4} \sin^2\xi\, (l_1^2 + l_2^2) + \frac{1}{4} l_3^2 \sin^2\xi \cos^2\xi\,, \tag{2.3.3}$$

with (l_1, l_2, l_3) denoting the left-invariant 1-forms of $SU(2)$, which can be explicitly chosen to be [52]

$$\begin{aligned} l_1 &= \sin\psi\, d\theta - \cos\psi \sin\theta\, d\phi\,, \\ l_2 &= \cos\psi\, d\theta + \sin\psi \sin\theta\, d\phi\,, \\ l_3 &= -(d\psi + \cos\theta\, d\phi)\,, \end{aligned} \tag{2.3.4}$$

$$0 \leqslant \theta \leqslant \pi\,, \quad 0 \leqslant \phi \leqslant 2\pi\,, \quad 0 \leqslant \psi \leqslant 4\pi\,. \tag{2.3.5}$$

Moreover, the asymptotically AdS$_7$ black hole solution also contains two 1-forms, one 3-form and two scalars, which are given by

$$
\begin{aligned}
A^1_{(1)} &= A^2_{(1)} = A_{(1)} = \frac{m\sinh(\delta)\cosh(\delta)}{\rho^4\,\Xi H}(dt - a\sigma) + \frac{\alpha_{70}}{\Xi_-}dt\,,\\
A_{(3)} &= \frac{ma\sinh^2(\delta)}{\rho^2\Xi\,\Xi_-}\,\sigma\wedge\,d\sigma + \alpha_{71}\,dt\wedge d\theta\wedge d\psi + \alpha_{72}\,dt\wedge d\xi\wedge d\phi + \alpha_{73}\,dt\wedge d\xi\wedge d\psi\,,\\
X_1 &= X_2 = H^{-1/5}\,,
\end{aligned}
\tag{2.3.6}
$$

where we have added some pure gauge terms to the potentials, and

$$
\begin{aligned}
\rho^2 &\equiv \Xi\, r^2\,,\\
H &\equiv 1 + \frac{2m\sinh^2(\delta)}{\rho^4}\,,\\
f_1 &\equiv \Xi\,\rho^6 H^2 - \frac{\big[2\Xi_+ ma\sinh^2(\delta)\big]^2}{\rho^4} + 2ma^2\big[\Xi_+^2 + \cosh^2(\delta)\,(1-\Xi_+^2)\big]\,,\\
f_2 &\equiv -\frac{g\,\Xi_+\rho^6\,H^2}{2} + ma\cosh^2(\delta)\,,\\
Y &\equiv g^2\rho^8 H^2 + \Xi\,\rho^6 - 2m\rho^2\left[a^2g^2\cosh^2(\delta) + \Xi\right] + 2ma^2\left[\Xi_+^2 + \cosh^2(\delta)\,(1-\Xi_+^2)\right]\,,\\
\Xi_\pm &\equiv 1 \pm ag\,,\quad \Xi \equiv 1 - a^2g^2\,,
\end{aligned}
\tag{2.3.7}
$$

with $g = \ell_7^{-1}$ denoting the inverse of the AdS_7 radius. The thermodynamic quantities of the black hole have the following expressions

$$
\begin{aligned}
T &= \frac{\partial_r Y}{4\pi g\,\rho^3\sqrt{\Xi f_1}}\,,\\
S &= \frac{\pi^3\rho^2\sqrt{f_1}}{4G_N\Xi^3}\,,\\
\Omega &= -\frac{1}{g}\left(g + \frac{2f_2\,\Xi_-}{f_1}\right)\,,\\
\Phi &= \frac{4m\sinh(\delta)\cosh(\delta)}{\rho^4\,\Xi H}\left(\Xi_- - \frac{2af_2\Xi_-}{f_1}\right)\,,\\
E &= \frac{m\pi^2}{32G_N g\Xi^4}\Big[12(ag+1)^2\,(ag(ag+2)-1) - 4\cosh^2(\delta)\left(3a^4g^4 + 12a^3g^3 + 11a^2g^2 - 8\right)\Big]\,,\\
J &= -\frac{ma\pi^2}{16G_N\Xi^4}\Big[4ag(ag+1)^2 - 4\cosh^2(\delta)\left(a^3g^3 + 2a^2g^2 + ag - 1\right)\Big]\,,\\
Q &= \frac{m\pi^2\sinh(\delta)\cosh(\delta)}{4G_N g\Xi^3}\,.
\end{aligned}
\tag{2.3.8}
$$

The expression of the temperature T has three roots $r_\pm$ and r_0, all of which coincide at the extremality. As we can see, the solution depends only on three parameters (a, m, δ). As shown in [49], the BPS condition and the absence of naked closed timelike curves (CTCs) require that

$$e^{2\delta} = 1 - \frac{2}{3ag}\,, \quad m = \frac{128\, e^{2\delta}(3\, e^{2\delta} - 1)^3}{729\, g^4 (e^{2\delta} + 1)^2 (e^{2\delta} - 1)^6}\,. \tag{2.3.9}$$

Hence, there is only one independent parameter in the BPS limit, which we choose to be δ. In addition, all three roots of T, i.e. $r_\pm$ and r_0, coincide in the BPS limit, and its value is

$$r_*^2 = \frac{16}{3g^2(3\, e^{2\delta} - 5)(e^{2\delta} + 1)}\,. \tag{2.3.10}$$

The thermodynamic quantities in the BPS limit become

$$\begin{aligned}
T_* &= 0\,,\\
S_* &= \frac{32\pi^3\sqrt{9\, e^{2\delta} - 7}}{3\sqrt{3}\, G_N g^5 (3\, e^{2\delta} - 5)^3\, (e^{2\delta} + 1)^{3/2}}\,,\\
\Omega_* &= 1\,,\\
\Phi_* &= 2\,,\\
E_* &= \frac{16\pi^2(18\, e^{6\delta} - 21\, e^{4\delta} + 7)}{3\, G_N g^5 (3\, e^{2\delta} - 5)^4 (e^{2\delta} + 1)^2}\,,\\
J_* &= \frac{16\pi^2(9\, e^{4\delta} + 18\, e^{2\delta} - 23)}{9\, G_N g^5 (3\, e^{2\delta} - 5)^4 (e^{2\delta} + 1)^2}\,,\\
Q_* &= \frac{8\pi^2(3\, e^{6\delta} - 5\, e^{4\delta} - 3\, e^{2\delta} + 5)}{G_N g^5 (3\, e^{2\delta} - 5)^4 (e^{2\delta} + 1)^2}\,.
\end{aligned} \tag{2.3.11}$$

Gravitational Cardy Limit

The Cardy-like limit for the 6d $\mathcal{N} = (2, 0)$ theory was defined in [53], which for the special solution with three equal angular momenta is

$$|\omega| \ll 1\,, \quad \Delta \sim \mathcal{O}(1)\,. \tag{2.3.12}$$

Using the following relations found in [49]

$$\omega = \frac{1}{T}(\Omega - \Omega_*)\,, \quad \phi = \frac{1}{T}(\Phi - \Phi_*)\,, \tag{2.3.13}$$

we obtain the corresponding limit as

$$\left| \left(\frac{\partial \Omega}{\partial T} \right)_{T=0} \right| \ll 1 \,, \quad \left. \frac{\partial \Phi}{\partial T} \right|_{T=0} \sim \mathcal{O}(1) \,. \tag{2.3.14}$$

Using the relation (2.3.13), we can express $\left(\frac{\partial \Omega}{\partial T}\right)_*$ in terms of the paramter δ. The explicit form is not very elucidating, but we do find a root to the equation $\left(\frac{\partial \Omega}{\partial T}\right)_* = 0$, which is

$$\delta_* = \frac{1}{2} \log \left(\frac{5}{3} \right) \,. \tag{2.3.15}$$

Hence, the gravitational Cardy limit for the class of asymptotically AdS_7 BPS black holes (2.3.1) is

$$\delta \to \frac{1}{2} \log \left(\frac{5}{3} \right) \,. \tag{2.3.16}$$

Note that this is equivalent to

$$ag \to -1, \tag{2.3.17}$$

as in the other black hole solutions. We can introduce a small parameter to denote the deviation from this limit, i.e.,

$$\delta = \frac{1}{2} \log \left(\frac{5}{3} \right) + \epsilon \,. \tag{2.3.18}$$

For this case ϵ is dimensionless. Expanding in ϵ, we find the BPS thermodynamic quantities (2.3.11) in the gravitational Cardy limit (2.3.18) to the leading order

$$\begin{aligned}
T_* &= 0 \,, \\
S_* &= \frac{\pi^3}{250\, G_N g^5 \epsilon^3} + \mathcal{O}(\epsilon^{-2}) \,, \\
\Omega_* &= 1 \,, \\
\Phi_* &= 2 \,, \\
E_* &= \frac{3\pi^2}{1250\, G_N g^5 \epsilon^4} + \mathcal{O}(\epsilon^{-3}) \,, \\
J_* &= \frac{\pi^2}{1250\, G_N g^5 \epsilon^4} + \mathcal{O}(\epsilon^{-3}) \,, \\
Q_* &= \frac{\pi^2}{500\, G_N g^5 \epsilon^3} + \mathcal{O}(\epsilon^{-2}) \,,
\end{aligned} \tag{2.3.19}$$

which are consistent with [36, 53] and the Cardy-like limit on the field theory side (2.3.12)

$$\omega_* \sim \epsilon\,, \quad \Delta_* \sim \mathcal{O}(1)\,. \tag{2.3.20}$$

Black Hole Solution in the Near-Horizon + Gravitational Cardy Limit

In the previous subsection, we have obtained the gravitational Cardy limit for the parameters on the gravity side. In this subsection, we discuss how the near-horizon metric changes when taking the gravitational Cardy limit. In Appendix A.1.3, we verify explicitly that the resulting background is a solution of the 7d gauged supergravity equations of motion. In the following, we implement the gravitational Cardy limit in the space of parameter which further simplifies the geometry.

We can apply the following scaling near the horizon r_* (2.3.10) to the BPS AdS$_7$ black hole metric (2.3.1)

$$r \to r_* + \lambda\,\widetilde{r}\,, \quad t \to \frac{\widetilde{t}}{\lambda}\,, \quad \chi \to \widetilde{\chi} - \frac{6g\sinh(\delta)}{\cosh(\delta) + 2\sinh(\delta)}\frac{\widetilde{t}}{\lambda}\,, \tag{2.3.21}$$

with $\lambda \to 0$. In addition, taking the gravitational Cardy limit (2.3.18), we obtain the near-horizon metric to the leading order in ϵ

$$\begin{aligned} ds^2 =& -10\,g^2\,2^{2/5}\,\epsilon\,\widetilde{r}^2\,d\widetilde{t}^2 + \frac{1}{16\,g^2\,2^{3/5}}\frac{d\widetilde{r}^2}{\widetilde{r}^2} \\ &+ \frac{2^{2/5}}{25\,g^2\epsilon^2}\left(d\widetilde{\chi} + \frac{1}{2}l_3\sin^2\xi - 5\sqrt{5}\,g^2\epsilon^{3/2}\,\widetilde{r}\,d\widetilde{t}\right)^2 + \frac{2^{2/5}}{5g^2\epsilon}\,ds^2_{\mathbb{CP}^2}\,. \end{aligned} \tag{2.3.22}$$

Defining

$$\tau \equiv 8\sqrt{5}\,g^2\sqrt{\epsilon}\,\widetilde{t}\,, \tag{2.3.23}$$

we can rewrite the near-horizon metric in the gravitational Cardy limit (2.3.22) as follows

$$ds^2 = \frac{1}{16\,g^2\,2^{3/5}}\left[-\widetilde{r}^2\,d\tau^2 + \frac{d\widetilde{r}^2}{\widetilde{r}^2}\right] + \frac{2^{2/5}}{25\,g^2\epsilon^2}\left(d\widetilde{\chi} + \frac{1}{2}l_3\sin^2\xi - \frac{5\epsilon}{8}\,\widetilde{r}\,d\tau\right)^2 + \frac{2^{2/5}}{5g^2\epsilon}\,ds^2_{\mathbb{CP}^2}\,. \tag{2.3.24}$$

Black Hole Entropy from Cardy Formula

For the asymptotically AdS$_7$ black holes discussed in this section, we apply the Cardy formula to the near-horizon metric only after taking the gravitational Cardy limit. More explicitly, we first rewrite the metric (2.3.24) from the Poincaré coordinates ($\widetilde{r}$, τ) to the global coordinates ($\hat{r}$, $\hat{t}$) using the relations (2.1.44) - (2.1.46). Consequently, the near-horizon metric in the gravitational Cardy limit (2.3.24) becomes

$$ds^2 = \frac{1}{16\, g^2\, 2^{3/5}}\left[-(1+\hat{r}^2)\, d\hat{t}^2 + \frac{d\hat{r}^2}{1+\hat{r}^2}\right] + \frac{2^{2/5}}{25\, g^2 \epsilon^2}\left(d\hat{\chi} + \frac{1}{2} l_3 \sin^2\xi - \frac{5\epsilon}{8}\, \hat{r}\, d\hat{t}\right)^2 + \frac{2^{2/5}}{5 g^2 \epsilon}\, ds^2_{\mathbb{CP}^2}\,, \tag{2.3.25}$$

where $\hat{t}$, $\hat{r}$ and γ are defined in (2.1.44) and (2.1.46), while

$$\hat{\chi} \equiv \widetilde{\chi} - \frac{5\epsilon}{8}\, \gamma\,. \tag{2.3.26}$$

Applying the same formalism in Subsection 2.1.4, we obtain the central charge and the extremal Frolov-Thorne temperature in the near-horizon region of the asymptotically AdS_7 BPS black holes as follows

$$c_L = \frac{3\pi^2}{200\, g^5 \epsilon^2}\,, \quad T_L = \frac{4}{5\pi\epsilon}\,. \tag{2.3.27}$$

Using the Cardy formula, we can compute the black hole entropy of the asymptotically AdS_7 BPS black holes

$$S_{BH} = \frac{\pi^2}{3} c_L T_L = \frac{\pi^3}{250\, g^5 \epsilon^3}\,, \tag{2.3.28}$$

which is the same as the black hole entropy in the gravitational Cardy limit (2.3.19) from the gravity side in the unit $G_N = 1$.

Comparison with Results from Boundary CFT

The asymptotically AdS_7 BPS black hole entropy can also be obtained from the boundary 6d $(2,0)$ theory by extremizing an entropy function [14, 49, 53] originally motivated in [54] and more recently studied in [40]. We can first compute the free energy in the large-N limit using the background field method on S^5, the partition function via localization or the 6d superconformal index. The entropy function is then defined as a Legendre transform of the free energy in the large-N limit

$$S(\Delta_I,\, \omega_i) = -\frac{N^3}{24}\frac{\Delta_1^2 \Delta_2^2}{\omega_1 \omega_2 \omega_3} + \sum_{I=1}^{2} Q_I \Delta_I + \sum_{i=1}^{3} J_i \omega_i - \Lambda\left(\sum_{I=1}^{2} \Delta_I - \sum_{i=1}^{3} \omega_i - 2\pi i\right)\,. \tag{2.3.29}$$

In the Cardy-like limit (2.3.12)

$$\omega \sim \epsilon\,, \quad \Delta_I \sim \mathcal{O}(1)\,, \tag{2.3.30}$$

we can read off from the entropy function (2.3.29)

$$S \sim \frac{1}{\epsilon^3}\,, \quad J \sim \frac{1}{\epsilon^4}\,, \quad Q_I \sim \frac{1}{\epsilon^3}\,, \tag{2.3.31}$$

which have been summarized in Table 1.1.

Similar to $\mathrm{AdS}_{4,5}$, for AdS_7 the electric charges Q_I and the angular momenta J_i are real, while the chemical potentials Δ_I and the angular velocities ω_i can be complex, and so can the entropy function S. By requiring that the black hole entropy S_{BH} to be real after extremizing the entropy function S, we obtain one more constraint on Q_I and J_i. More precisely, the most general case with two charges (Q_1, Q_2) and three angular momenta (J_1, J_2, J_3) was discussed in [14,53], while the degenerate case with $Q_1 = Q_2$ and $J_1 = J_2 = J_3$ was discussed in [49]. For the most general case, the asymptotically AdS_7 black hole entropy is [14,53]

$$S_{BH} = 2\pi \sqrt{\frac{3(Q_1^2 Q_2 + Q_1 Q_2^2) - N^3(J_1 J_2 + J_2 J_3 + J_3 J_1)}{3(Q_1 + Q_2) - N^3}}\,, \tag{2.3.32}$$

subject to the constraint

$$\begin{aligned} &\frac{3(Q_1^2 Q_2 + Q_1 Q_2^2) - N^3(J_1 J_2 + J_2 J_3 + J_3 J_1)}{3(Q_1 + Q_2) - N^3} \\ &= \left[\frac{N^3}{3}(J_1 + J_2 + J_3) + \frac{Q_1^2 + Q_2^2}{2} + 2Q_1 Q_2\right] \\ &\quad \times \left[1 - \sqrt{1 - \frac{\frac{2}{3}N^3 J_1 J_2 J_3 + Q_1^2 Q_2^2}{\left(\frac{N^3}{3}(J_1 + J_2 + J_3) + \frac{Q_1^2 + Q_2^2}{2} + 2Q_1 Q_2\right)^2}}\,\right]\,, \end{aligned} \tag{2.3.33}$$

which is a consequence of the reality condition on the black hole entropy S_{BH}.

We apply the general result to the special case $Q_1 = Q_2 = Q$ and $J_1 = J_2 = J_3 = J$,

$$S_{BH} = 2\pi \sqrt{\frac{6Q^3 - 3N^3 J^2}{6Q - N^3}}\,, \tag{2.3.34}$$

with the constraint

$$\frac{6Q^3 - 3N^3 J^2}{6Q - N^3} = \left(N^3 J + 3Q^2\right) \cdot \left[1 - \sqrt{1 - \frac{\frac{2}{3}N^3 J^3 + Q^4}{N^3 J + 3Q^2}}\,\right]\,, \tag{2.3.35}$$

which are consistent with both the thermodynamic quantities on the gravity side (2.3.11) (2.3.19) and the black hole entropy in the gravitational Cardy limit from the Cardy formula

(2.3.28) under the $\mathrm{AdS}_7/\mathrm{CFT}_6$ dictionary of parameters [14, 53]

$$G_N = \frac{3\pi^2}{16\, g^5 N^3}\,. \tag{2.3.36}$$

2.3.2 More General Case

In the previous section, we have discussed a special solution of asymptotically AdS_7 black holes with two equal charges $Q_1 = Q_2$ and three equal angular momenta $J_1 = J_2 = J_3$. In this subsection, we consider a more general solution with two equal charges $Q_1 = Q_2$ and three independent angular momenta (J_1, J_2, J_3), which was first introduced in [51].

AdS_7 Black Hole Solution

The metric for this class of asymptotically AdS_7 black holes is

$$\begin{aligned}
ds^2 = H^{2/5}\Bigg[&\frac{(r^2+y^2)(r^2+z^2)}{R}dr^2 + \frac{(r^2+y^2)(y^2-z^2)}{Y}dy^2 + \frac{(r^2+z^2)(z^2-y^2)}{Z}dz^2 \\
&- \frac{R}{H^2(r^2+y^2)(r^2+z^2)}\mathcal{A}^2 \\
&+ \frac{Y}{(r^2+y^2)(y^2-z^2)}\left(dt' + (z^2-r^2)d\psi_1 - r^2z^2d\psi_2 - \frac{q}{H(r^2+y^2)(r^2+z^2)}\mathcal{A}\right)^2 \\
&+ \frac{Z}{(r^2+z^2)(z^2-y^2)}\left(dt' + (y^2-r^2)d\psi_1 - r^2y^2d\psi_2 - \frac{q}{H(r^2+y^2)(r^2+z^2)}\mathcal{A}\right)^2 \\
&+ \frac{a_1^2\,a_2^2\,a_3^2}{r^2y^2z^2}\Bigg(dt' + (y^2+z^2-r^2)d\psi_1 + (y^2z^2-r^2y^2-r^2z^2)d\psi_2 - r^2y^2z^2d\psi_3 \\
&\qquad - \frac{q}{H(r^2+y^2)(r^2+z^2)}\left(1+\frac{gy^2z^2}{a_1a_2a_3}\right)\mathcal{A}\Bigg)^2\Bigg],
\end{aligned} \tag{2.3.37}$$

while the 1-form, the 2-form, the 3-form and the scalar are

$$
\begin{aligned}
A_{(1)} &= \frac{2m\sinh(\delta)\cosh(\delta)}{H(r^2+y^2)(r^2+z^2)}\mathcal{A}\,,\\
A_{(2)} &= \frac{q}{H(r^2+y^2)(r^2+z^2)}\mathcal{A}\wedge\\
&\quad \Big[dt' + \sum_i a_i^2(g^2 dt' + d\psi_1 + \sum_{i<j} a_i^2 a_j^2(g^2 d\psi_1 + d\psi_2) + a_1^2\, a_2^2\, a_3^2(g^2 d\psi_2 + d\psi_3)\\
&\qquad - g^2(y^2+z^2)dt' - g^2 y^2 z^2 d\psi_1 + a_1\, a_2\, a_3\Big(d\psi_1 + (y^2+z^2)d\psi_2 + y^2 z^2 d\psi_3\Big)\Big]\,,\\
A_{(3)} &= q a_1 a_2 a_3\Big[d\psi_1 + (y^2+z^2)d\psi_2 + y^2 z^2 d\psi_3\Big]\\
&\quad \wedge\left(\frac{1}{(r^2+y^2)z}dz\wedge(d\psi_1 + y^2 d\psi_2) + \frac{1}{(r^2+z^2)y}dy\wedge(d\psi_1 + z^2 d\psi_2)\right)\\
&\quad - qg\mathcal{A}\wedge\left(\frac{z}{r^2+y^2}dz\wedge(d\psi_1 + y^2 d\psi_2) + \frac{y}{r^2+z^2}dy\wedge(d\psi_1 + z^2 d\psi_2)\right)\,,\\
X &= H^{-1/5}\,,
\end{aligned}
\tag{2.3.38}
$$

where

$$
\begin{aligned}
R &\equiv \frac{1+g^2r^2}{r^2}\prod_{i=1}^{3}(r^2+a_i^2) + qg^2(2r^2 + a_1^2 + a_2^2 + a_3^2) - \frac{2qga_1a_2a_3}{r^2} + \frac{q^2g^2}{r^2} - 2m\,,\\
Y &\equiv \frac{1-g^2y^2}{y^2}\prod_{i=1}^{3}(a_i^2 - y^2)\,,\\
Z &\equiv \frac{1-g^2z^2}{z^2}\prod_{i=1}^{3}(a_i^2 - z^2)\,,\\
\mathcal{A} &\equiv dt' + (y^2+z^2)d\psi_1 + y^2 z^2 d\psi_2\,,\\
H &\equiv 1 + \frac{q}{(r^2+y^2)(r^2+z^2)}\,,\\
q &\equiv 2m\sinh^2(\delta)\,.
\end{aligned}
\tag{2.3.39}
$$

It can be shown that after the change of coordinates

$$
\begin{aligned}
t &= t' + (a_1^2 + a_2^2 + a_3^2)\psi_1 + (a_1^2\, a_2^2 + a_2^2\, a_3^2 + a_3^2\, a_1^2)\psi_2 + a_1^2\, a_2^2\, a_3^2\, \psi_3\,,\\
\frac{\phi_i}{a_i} &= g^2 t' + \psi_1 + \sum_{j\neq i} a_j^2(g^2\psi_1 + \psi_2) + \prod_{j\neq i} a_j^2(g^2\psi_2 + \psi_3)\,,
\end{aligned}
\tag{2.3.40}
$$

the metric (2.3.37) can be written in an equivalent form

$$
\begin{aligned}
ds^2 = H^{2/5}\Bigg[&\frac{(r^2+y^2)(r^2+z^2)}{R}dr^2 + \frac{(r^2+y^2)(y^2-z^2)}{Y}dy^2 + \frac{(r^2+z^2)(z^2-y^2)}{Z}dz^2 \\
&- \frac{r^2y^2z^2RYZ}{H^2\prod_{i<j}(a_i^2-a_j^2)^2B_1B_2B_3}dt^2 + B_3(d\phi_3 + v_{32}d\phi_2 + v_{31}d\phi_1 + v_{30}dt)^2 \\
&+ B_2(d\phi_2 + v_{21}d\phi_1 + v_{20}dt)^2 + B_1(d\phi_1 + v_{10}dt)^2\Bigg]\,, \qquad (2.3.41)
\end{aligned}
$$

where B_1, B_2, B_3, v_{10}, v_{20}, v_{21}, v_{30}, v_{31} and v_{32} can be determined by comparing (2.3.41) with (2.3.37). We can see that in the gravitational Cardy limit B_1 and B_2 are subleading compared to B_3. Hence, qualitatively the term $B_3(d\phi_3 + v_{32}d\phi_2 + v_{31}d\phi_1 + v_{30}dt)^2$ in the metric forms the only $U(1)$ circle fibered over AdS_2 in the gravitational Cardy limit of the near-horizon solution, similar to the other cases in this chapter. However, because the explicit expressions of these coefficients are lengthy and not very illuminating, we do not list them here.

The thermodynamic quantities can be expressed as

$$
\begin{aligned}
E &= \frac{\pi^2}{8\Xi_1\Xi_2\Xi_3}\left[\sum_i \frac{2m}{\Xi_i} - m + \frac{5q}{2} + \frac{q}{2}\sum_i\left(\sum_{j\neq i}\frac{2\Xi_j}{\Xi_i} - \Xi_i - \frac{2(1+2a_1a_2a_3g^3)}{\Xi_i}\right)\right]\,,\\
T &= \frac{(1+g^2r_+^2)r_+^2\sum_i\prod_{j\neq i}(r_+^2+a_j^2) - \prod_i(r_+^2+a_i^2) + 2q(g^2r_+^4+ga_1a_2a_3) - q^2g^2}{2\pi r_+\left[(r_+^2+a_1^2)(r_+^2+a_2^2)(r_+^2+a_3^2) + q(r_+^2 - a_1a_2a_3g)\right]}\,,\\
S &= \frac{\pi^3\left[(r_+^2+a_1^2)(r_+^2+a_2^2)(r_+^2+a_3^2) + q(r_+^2 - a_1a_2a_3g)\right]}{4\Xi_1\Xi_2\Xi_3r_+}\,,\\
\Omega_i &= \frac{a_i\left[(1+g^2r_+^2)\prod_{j\neq i}(r_+^2+a_j^2) + qg^2r_+^2\right] - q\prod_{j\neq i}a_jg}{(r_+^2+a_1^2)(r_+^2+a_2^2)(r_+^2+a_3^2) + q(r_+^2 - a_1a_2a_3g)}\,, \qquad (2.3.42)\\
J_i &= \frac{\pi^2 m\left[a_i\cosh^2(\delta) - g\sinh^2(\delta)\left(\prod_{j\neq i}a_j + a_i\sum_{j\neq i}a_j^2g + a_1a_2a_3a_ig^2\right)\right]}{4\Xi_1\Xi_2\Xi_3\Xi_i}\,,\\
\Phi &= \frac{2m\sinh(\delta)\cosh(\delta)\,r_+^2}{(r_+^2+a_1^2)(r_+^2+a_2^2)(r_+^2+a_3^2) + q(r_+^2 - a_1a_2a_3g)}\,,\\
Q &= \frac{\pi^2 m\sinh(\delta)\cosh(\delta)}{\Xi_1\Xi_2\Xi_3}\,,
\end{aligned}
$$

where r_+ denotes the position of the outer horizon, and

$$
\Xi_i \equiv 1 - a_i^2g^2\,,\quad \Xi_{i\pm} \equiv 1 \pm a_ig\,,\quad (i = 1,\,2,\,3)\,. \qquad (2.3.43)
$$

This class of asymptotically AdS_7 black hole solutions is characterized by five parameters

$(m,\, \delta,\, a_1,\, a_2,\, a_3)$. The BPS limit for this class of solutions is

$$e^{2\delta} = 1 - \frac{2}{(a_1 + a_2 + a_3)g}\,, \tag{2.3.44}$$

while the naked closed timelike curves (CTCs) can be avoided by requiring an additional condition

$$q = -\frac{\Xi_{1-}\Xi_{2-}\Xi_{3-}(a_1 + a_2)(a_2 + a_3)(a_3 + a_1)}{(1 - a_1 g - a_2 g - a_3 g)^2 g}\,. \tag{2.3.45}$$

Hence, only three parameters are independent, which we can choose to be $(a_1,\, a_2,\, a_3)$. In the BPS limit, the thermodynamic quantities can be simplified as follows

$$\begin{aligned}
E &= -\frac{\pi^2 \prod_{k<l}(a_k + a_l)\Big[\sum_i \Xi_i + \sum_{i<j} \Xi_i \Xi_j - (1 + a_1 a_2 a_3 g^3)(2 + \sum_i a_i g + \sum_{i<j} a_i a_j g^2)\Big]}{8\, \Xi_{1+}^2 \Xi_{2+}^2 \Xi_{3+}^2 (1 - a_1 g - a_2 g - a_3 g)^2 g}\,,\\
T &= 0\,, \qquad \Omega_i = -g\,, \qquad \Phi = 1\,,\\
S &= -\frac{\pi^3 (a_1 + a_2)(a_2 + a_3)(a_3 + a_1)(a_1 a_2 + a_2 a_3 + a_3 a_1 - a_1 a_2 a_3 g)}{4\Xi_{1+}\Xi_{2+}\Xi_{3+}(1 - a_1 g - a_2 g - a_3 g)^2 g r_0}\,,\\
J_i &= -\frac{\pi^2 (a_1 + a_2)(a_2 + a_3)(a_3 + a_1)\Big[a_i - (a_i^2 + 2a_i \sum_{j\neq i} a_j + \prod_{j\neq i} a_j) g + a_1 a_2 a_3 g^2\Big]}{8\Xi_{1+}\Xi_{2+}\Xi_{3+}\Xi_{i+}(1 - a_1 g - a_2 g - a_3 g)^2 g}\,,\\
Q &= -\frac{\pi^2 (a_1 + a_2)(a_2 + a_3)(a_3 + a_1)}{2\Xi_{1+}\Xi_{2+}\Xi_{3+}(1 - a_1 g - a_2 g - a_3 g) g}\,,
\end{aligned} \tag{2.3.46}$$

where

$$r_0 = \sqrt{\frac{a_1 a_2 + a_2 a_3 + a_3 a_1 - a_1 a_2 a_3 g}{1 - a_1 g - a_2 g - a_3 g}}\,. \tag{2.3.47}$$

Gravitational Cardy Limit

Similar to Subsection 2.3.1, for the more general AdS$_7$ solution with three independent angular momenta, we can translate the Cardy limit for the 6d $\mathcal{N} = (2,0)$ theory defined in [53]

$$|\omega_i| \ll 1\,, \quad \Delta \sim \mathcal{O}(1)\,, \quad (i = 1,\, 2,\, 3) \tag{2.3.48}$$

into the gravitational Cardy-like limit for this class of asymptotically AdS$_7$ black holes

$$\left| \left(\frac{\partial \Omega_i}{\partial T}\right)_{T=0} \right| \ll 1\,, \quad \frac{\partial \Phi}{\partial T}\bigg|_{T=0} \sim \mathcal{O}(1)\,. \tag{2.3.49}$$

A choice of the parameters $(a_1,\, a_2,\, a_3)$ that satisfies the limit (2.3.49) is

$$a_1 = a_2 = a_3 = -\frac{1}{g}\,. \tag{2.3.50}$$

As in the other black hole solutions, this can be summarized as

$$a_i g \to -1. \tag{2.3.51}$$

We can introduce a small parameter ϵ to denote the deviation from this limit, i.e.,

$$a_i = -\frac{1}{g} + \epsilon\,, \quad (i = 1,\, 2,\, 3)\,, \tag{2.3.52}$$

or in a more refined way

$$a_1 = -\frac{1}{g} + \epsilon\,, \quad a_2 = -\frac{1}{g} + \epsilon + \eta_1\,, \quad a_3 = -\frac{1}{g} + \epsilon + \eta_2\,, \quad (\eta_1,\, \eta_2 \ll \epsilon)\,. \tag{2.3.53}$$

Expanding in ϵ, after expanding in η_1 and η_2, we find the BPS thermodynamic quantities (2.3.46) in the gravitational Cardy limit (2.3.52) to the leading order

$$\begin{aligned} S_* &= \frac{\pi^3}{2g^8\epsilon^3} + \mathcal{O}(\epsilon^{-2})\,, \\ J_i^* &= -\frac{\pi^2}{2g^9\epsilon^4} + \mathcal{O}(\epsilon^{-3})\,, \\ Q_* &= \frac{\pi^2}{g^7\epsilon^3} + \mathcal{O}(\epsilon^{-2})\,, \end{aligned} \tag{2.3.54}$$

which are consistent with [36, 53] and the Cardy-like limit on the field theory side (2.3.48)

$$\omega_{i*} \sim \epsilon\,, \quad \Delta_* \sim \mathcal{O}(1)\,. \tag{2.3.55}$$

Black Hole Solution in the Near-Horizon Limit

In this subsection, we consider the near-horizon limit of the asymptotically AdS_7 black hole metric. As mentioned in Subsection 2.3.2, we should in principle take the Cardy limit of the near horizon solution (2.3.41). Applying the refined gravitational Cardy limit (2.3.53), we find that B_1 and B_2 are subleading compared to B_3. Therefore, in the near-horizon limit we obtain an AdS_3 geometry, just like the other cases. However, in practice the expressions of the coefficients are lengthy, so we consider an alternative near-horizon metric discussed in [18]. That is, the metric (2.3.37) can be expressed in an equivalent form

$$
\begin{aligned}
ds^2 = H^{2/5}\Bigg[&\frac{(r^2+y^2)(r^2+z^2)}{R}dr^2 + \frac{(r^2+y^2)(y^2-z^2)}{Y}dy^2 + \frac{(r^2+z^2)(z^2-y^2)}{Z}dz^2 \\
&+ \frac{Y}{(r^2+y^2)(y^2-z^2)}\left(dt - \sum_{i=1}^{3}\frac{(r^2+a_i^2)\gamma_i}{a_i^2-y^2}\frac{d\hat{\phi}_i}{\delta_i} - \frac{q\mathcal{A}}{H(r^2+y^2)(r^2+z^2)}\right)^2 \\
&+ \frac{Z}{(r^2+z^2)(z^2-y^2)}\left(dt - \sum_{i=1}^{3}\frac{(r^2+a_i^2)\gamma_i}{a_i^2-z^2}\frac{d\hat{\phi}_i}{\delta_i} - \frac{q\mathcal{A}}{H(r^2+y^2)(r^2+z^2)}\right)^2 \\
&+ \frac{a_1^2\,a_2^2\,a_3^2}{r^2y^2z^2}\left(dt - \sum_{i=1}^{3}\frac{(r^2+a_i^2)\gamma_i}{a_i^2}\frac{d\hat{\phi}_i}{\delta_i} - \frac{q\mathcal{A}}{H(r^2+y^2)(r^2+z^2)}\left(1+\frac{gy^2z^2}{a_1a_2a_3}\right)\right)^2\Bigg],
\end{aligned}
\tag{2.3.56}
$$

where we have used the changes of coordinates (2.3.40) and

$$
\begin{aligned}
&\hat{\phi}_i \equiv \phi_i - a_i g^2 t\,, \quad (i=1,\,2,\,3)\,, \\
&\gamma_i \equiv a_i^2(a_i^2-y^2)(a_i^2-z^2)\,, \\
&\delta_i \equiv a_i(1-a_i^2g^2)\prod_{j\neq i}(a_i^2-a_j^2)\,.
\end{aligned}
\tag{2.3.57}
$$

Applying the following scaling to the new metric (2.3.56)

$$
r \to r_0(1+\lambda\rho)\,, \quad \phi \to \widetilde{\phi}_i + \frac{\Omega_i^0}{2\pi T_H'^0 r_0\lambda}\,\widetilde{t}, \quad t \to \frac{\widetilde{t}}{2\pi T_H'^0 r_0 \lambda}\,,
\tag{2.3.58}
$$

which is slightly different from the original Bardeen-Horowitz scaling [37], we obtain the near-horizon geometry in the limit $\lambda \to 0$

$$\begin{aligned} ds^2 = H_0^{2/5} \Bigg[& \frac{U_0}{V}\left(-\rho^2 d\widetilde{t}^2 + \frac{d\rho^2}{\rho^2}\right) + \frac{(r_0^2+y^2)(y^2-z^2)}{Y} dy^2 + \frac{(r_0^2+z^2)(z^2-y^2)}{Z} dz^2 \\ & + \frac{Y}{(r_0^2+y^2)(y^2-z^2)}\left(\frac{2r_0(r_0^2+z^2)}{V}\rho\, d\widetilde{t} + \sum_{i=1}^{3}\frac{(r_0^2+a_i^2)\gamma_i}{a_i^2-y^2}\frac{d\widetilde{\phi}_i}{\delta_i} + \frac{q\widetilde{A}}{H_0 U_0}\right)^2 \\ & + \frac{Z}{(r_0^2+z^2)(z^2-y^2)}\left(\frac{2r_0(r_0^2+y^2)}{V}\rho\, d\widetilde{t} + \sum_{i=1}^{3}\frac{(r_0^2+a_i^2)\gamma_i}{a_i^2-z^2}\frac{d\widetilde{\phi}_i}{\delta_i} + \frac{q\widetilde{A}}{H_0 U_0}\right)^2 \\ & + \frac{a_1^2a_2^2a_3^2}{r_0^2 y^2 z^2}\left(\frac{2}{V r_0}\left(U_0 - \frac{qgy^2z^2}{a_1a_2a_3}\right)\rho\, d\widetilde{t} + \sum_{i=1}^{3}\frac{(r_0^2+a_i^2)\gamma_i}{a_i^2}\frac{d\widetilde{\phi}_i}{\delta_i}\right. \\ & \left. + \frac{q\widetilde{A}}{H_0U_0}\left(1+\frac{gy^2z^2}{a_1a_2a_3}\right)\right)^2\Bigg], \end{aligned} \tag{2.3.59}$$

where

$$\begin{aligned} U &\equiv (r^2+y^2)(r^2+z^2), \\ \gamma_i &\equiv a_i^2(a_i^2-y^2)(a_i^2-z^2), \\ \delta_i &\equiv \Xi_i a_i \prod_{j\neq i}(a_i^2-a_j^2), \\ U_0 &\equiv U\Big|_{r=r_0} = (r_0^2+y^2)(r_0^2+z^2), \\ H_0 &\equiv H\Big|_{r=r_0} = 1 + \frac{q}{(r_0^2+y^2)(r_0^2+z^2)}, \\ V &\equiv 6r_0^2 + \sum_{i=1}^{3}a_i^2 + \frac{3(a_1a_2a_3-qg)^2}{r_0^4} + g^2\left[15r_0^4 + 6r_0^2\sum_{i=1}^{3}a_i^2 + \sum_{1\leqslant i<j\leqslant 3}a_i^2a_j^2 + 2q\right], \\ \widetilde{A} &\equiv -\frac{2r_0(2r_0^2+y^2+z^2)}{V}\rho\, d\widetilde{t} - \sum_{i=1}^{3}\gamma_i\frac{d\widetilde{\phi}_i}{\delta_i}. \end{aligned} \tag{2.3.60}$$

Taking the refined gravitational Cardy limit (2.3.53), we can see that two of the three $U(1)$ circles in the near-horizon metric (2.3.59) become degenerate. However, the remaining two $U(1)$ circles are still of the same order in the gravitational Cardy limit. This is expected, because we should take the gravitational Cardy limit of the near-horizon of the metric (2.3.41) instead of (2.3.56), in order to have only one $U(1)$ circle fibered over AdS_2 in the near-horizon plus gravitational Cardy limit. Nevertheless, the gravitational Cardy limit reduces some redundant $U(1)$ circles, while keeping the essential information for the near-horizon Virasoro algebra.

Black Hole Entropy from Cardy Formula

We can apply the formalism described in Subsection 2.1.4. The central charge and the extremal Frolov-Thorne temperature in the near-horizon region of the asymptotically AdS_7 BPS black holes (2.3.59) were obtained in [18]. In the refined gravitational Cardy limit (2.3.53), the results are

$$c_L = \frac{48\pi^2}{g^9 \epsilon^2 V}\,, \quad T_L = \frac{gV}{32\pi\epsilon}\,. \tag{2.3.61}$$

Hence, the black hole entropy from the Cardy formula in the gravitational Cardy limit is

$$S_{BH} = \frac{\pi^2}{3} c_L T_L = \frac{\pi^3}{2g^8 \epsilon^3}\,, \tag{2.3.62}$$

which is exactly the same as the result from the gravity solution (2.3.54).

Comparison with Results from Boundary CFT

As we discussed in Subsection 2.3.1, for the asymptotically AdS_7 black holes with general charges $(Q_1,\, Q_2)$ and angular momenta $(J_1,\, J_2,\, J_3)$, the entropy can be obtained from the boundary 6d $(2,0)$ theory [14, 53], and the results are summarized in (2.3.32) subject to the constraint (2.3.33).

We have discussed a degenerate case in Subsection 2.3.1 with $Q_1 = Q_2$ and $J_1 = J_2 = J_3$. In this subsection, we have seen another degenerate case with $Q_1 = Q_2 = Q$ and $(J_1,\, J_2,\, J_3)$, which consequently has the black hole entropy

$$S_{BH} = 2\pi \sqrt{\frac{6Q^3 - N^3 (J_1 J_2 + J_2 J_3 + J_3 J_1)}{6Q - N^3}}\,, \tag{2.3.63}$$

subject to the constraint

$$\begin{aligned} & \frac{6Q^3 - N^3 (J_1 J_2 + J_2 J_3 + J_3 J_1)}{6Q - N^3} \\ & = \left[\frac{N^3}{3} (J_1 + J_2 + J_3) + 3Q^2 \right] \cdot \left[1 - \sqrt{1 - \frac{\frac{2}{3} N^3 J_1 J_2 J_3 + Q^4}{\left(\frac{N^3}{3} (J_1 + J_2 + J_3) + 3Q^2 \right)^2}} \right]\,, \end{aligned} \tag{2.3.64}$$

which are consistent with both the thermodynamic quantities on the gravity side (2.3.46) (2.3.54) and the black hole entropy in the gravitational Cardy limit from the Cardy formula

(2.3.62) under the $\mathrm{AdS}_7/\mathrm{CFT}_6$ dictionary of parameters [14, 53]

$$G_N = \frac{3\pi^2}{16\, g^5 N^3}\,. \tag{2.3.65}$$

2.4 Asymptotically AdS_6 Black Holes

In this section, we consider the asymptotically AdS_6 black holes and the corresponding gravitational Cardy limit. Similar to the other cases, we demonstrate that the AdS_6 black hole entropy can be computed in various ways as shown in Fig. 1.2, and the other thermodynamic quantities scale correspondingly in the gravitational Cardy limit.

2.4.1 AdS_6 Black Hole Solution

In this subsection, we discuss the near-horizon plus Cardy limit of the non-extremal asymptotically AdS_6 black holes constructed in [55], which are solutions to 6d $\mathcal{N} = 4$ $SU(2)$ gauged supergravity.

The bosonic part of this class of solution is given by the metric, a scalar, a 1-form potential and a 2-form potential. The metric is

$$\begin{aligned} ds^2 = H^{1/2}\Bigg[& \frac{(r^2+y^2)(r^2+z^2)}{R}dr^2 + \frac{(r^2+y^2)(y^2-z^2)}{Y}dy^2 + \frac{(r^2+z^2)(z^2-y^2)}{Z}dz^2 \\ & - \frac{R}{H^2(r^2+y^2)(r^2+z^2)}\mathcal{A}^2 \\ & + \frac{Y}{(r^2+y^2)(y^2-z^2)}\left(dt' + (z^2-r^2)d\psi_1 - r^2 z^2 d\psi_2 - \frac{qr\mathcal{A}}{H(r^2+y^2)(r^2+z^2)}\right)^2 \\ & + \frac{Z}{(r^2+z^2)(z^2-y^2)}\left(dt' + (y^2-r^2)d\psi_1 - r^2 y^2 d\psi_2 - \frac{qr\mathcal{A}}{H(r^2+y^2)(r^2+z^2)}\right)^2 \Bigg], \end{aligned} \tag{2.4.1}$$

while the 1-form potential, the 2-form potential and the scalar are

$$
\begin{aligned}
A_{(1)} &= \frac{2mr\sinh(\delta)\cosh(\delta)}{H(r^2+y^2)(r^2+z^2)}\mathcal{A} + \frac{\alpha_6}{dt}\,, \\
A_{(2)} &= \frac{q}{H(r^2+y^2)^2(r^2+z^2)^2}\Bigg[-\frac{yz\left(2r(2r^2+y^2+z^2)+q\right)}{H} dr\wedge\mathcal{A} \\
&\qquad + z\left((r^2+z^2)(r^2-y^2)+qr\right)dy \\
&\qquad\qquad \wedge\left(dt' + (z^2-r^2)d\psi_1 - r^2z^2 d\psi_2 - \frac{qr\mathcal{A}}{H(r^2+y^2)(r^2+z^2)}\right) \\
&\qquad + y\left((r^2+y^2)(r^2-z^2)+qr\right)dz \\
&\qquad\qquad \wedge\left(dt' + (y^2-r^2)d\psi_1 - r^2y^2 d\psi_2 - \frac{qr\mathcal{A}}{H(r^2+y^2)(r^2+z^2)}\right)\Bigg]\,, \\
X &= H^{-1/4}\,,
\end{aligned}
\tag{2.4.2}
$$

where

$$
\begin{aligned}
t' &\equiv \frac{t}{\Xi_a\Xi_b} - \frac{a^4\phi_1}{\Xi_a\, a(a^2-b^2)} - \frac{b^4\phi_2}{\Xi_b\, b(b^2-a^2)}\,, \\
\psi_1 &\equiv -\frac{g^2t}{\Xi_a\Xi_b} + \frac{a^2\phi_1}{\Xi_a\, a(a^2-b^2)} + \frac{b^2\phi_2}{\Xi_b\, b(b^2-a^2)}\,, \\
\psi_2 &\equiv \frac{g^4t}{\Xi_a\Xi_b} - \frac{\phi_1}{\Xi_a\, a(a^2-b^2)} - \frac{\phi_2}{\Xi_b\, b(b^2-a^2)}\,,
\end{aligned}
\tag{2.4.3}
$$

and

$$
\begin{aligned}
R &\equiv (r^2+a^2)(r^2+b^2) + g^2\left[r(r^2+a^2)+q\right]\left[r(r^2+b^2)+q\right] - 2mr\,, \\
Y &\equiv -(1-g^2y^2)(a^2-y^2)(b^2-y^2)\,, \\
Z &\equiv -(1-g^2z^2)(a^2-z^2)(b^2-z^2)\,, \\
H &\equiv 1 + \frac{qr}{(r^2+y^2)(r^2+z^2)}\,, \\
\mathcal{A} &\equiv dt' + (y^2+z^2)d\psi_1 + y^2z^2 d\psi_2\,, \\
q &\equiv 2m\sinh^2(\delta)\,, \qquad \Xi_a \equiv 1-a^2g^2\,, \qquad \Xi_b \equiv 1-b^2g^2\,.
\end{aligned}
\tag{2.4.4}
$$

Note that we have added a pure gauge term to the 1-form potential. It was shown in [55] that the metric (2.4.1) can be written in an equivalent form

$$ds^2 = H^{1/2}\Bigg[\frac{(r^2+y^2)(r^2+z^2)}{R}dr^2 + \frac{(r^2+y^2)(y^2-z^2)}{Y}dy^2 + \frac{(r^2+z^2)(z^2-y^2)}{Z}dz^2 \\ + \frac{RYZ}{H^2\Xi_a^2\,\Xi_b^2\,a^2b^2(a^2-b^2)^2B_1B_2}dt^2 + B_2(d\phi_2 + v_{21}\,d\phi_1 + v_{20}\,dt)^2 \\ + B_1(d\phi_1 + v_{10}dt)^2\Bigg], \tag{2.4.5}$$

where B_1, B_2, v_{10}, v_{20} and v_{21} can be determined by comparing (2.4.5) with (2.4.1). Because the explicit expressions of these coefficients are lengthy and not very illuminating, we do not list them here. Moreover, we notice a sign error in [55] for the term $\sim dt^2$ in (2.4.5).

The thermodynamic quantities can be expressed as

$$\begin{aligned}
E &= \frac{\pi}{3\Xi_a\Xi_b}\left[2m\left(\frac{1}{\Xi_a}+\frac{1}{\Xi_b}\right) + q\left(1+\frac{\Xi_a}{\Xi_b}+\frac{\Xi_b}{\Xi_a}\right)\right],\\
S &= \frac{2\pi^2\left[(r_+^2+a^2)(r_+^2+b^2)+qr_+\right]}{3\,\Xi_a\,\Xi_b},\\
T &= \frac{2r_+^2(1+g^2r_+^2)(2r_+^2+a^2+b^2) - (1-g^2r_+^2)(r_+^2+a^2)(r_+^2+b^2) + 4qg^2r_+^3 - q^2g^2}{4\pi r_+\left[(r_+^2+a^2)(r_+^2+b^2)+qr_+\right]},\\
J_1 &= \frac{2\pi ma(1+\Xi_b\sinh^2(\delta))}{3\,\Xi_a^2\,\Xi_b}, \qquad J_2 = \frac{2\pi mb(1+\Xi_a\sinh^2(\delta))}{3\,\Xi_b^2\,\Xi_a},\\
\Omega_1 &= \frac{a\left[(1+g^2r_+^2)(r_+^2+b^2)+qg^2r_+\right]}{(r_+^2+a^2)(r_+^2+b^2)+qr_+}, \qquad \Omega_2 = \frac{b\left[(1+g^2r_+^2)(r_+^2+a^2)+qg^2r_+\right]}{(r_+^2+a^2)(r_+^2+b^2)+qr_+},\\
Q &= \frac{2\pi m\sinh(\delta)\cosh(\delta)}{\Xi_a\Xi_b}, \qquad \Phi = \frac{2mr_+\sinh(\delta)\cosh(\delta)}{(r_+^2+a^2)(r_+^2+b^2)+qr_+}.
\end{aligned} \tag{2.4.6}$$

This class of asymptotically AdS_6 black hole solutions is characterized by four parameters $(m,\,\delta,\,a,\,b)$. The BPS limit can be obtained by imposing the following condition

$$e^{2\delta} = 1 + \frac{2}{(a+b)g}. \tag{2.4.7}$$

The absence of the naked closed timelike curves (CTCs) for these supersymmetric black holes requires an additional condition

$$q = \frac{\Xi_{a+}\Xi_{b+}(a+b)r_+}{(1+ag+bg)g}, \tag{2.4.8}$$

where in the BPS limit

$$r_+ \equiv \sqrt{\frac{ab}{1+ag+bg}}\,,\quad \Xi_{a+} \equiv 1+ag\,,\quad \Xi_{b+} \equiv 1+bg\,. \tag{2.4.9}$$

2.4.2 Gravitational Cardy Limit

The Cardy-like limit for the 5d SCFT was defined in [56]

$$|\omega_i| \ll 1\,,\quad \Delta \sim \mathcal{O}(1)\,,\qquad (i=1,\,2)\,. \tag{2.4.10}$$

Using the following relations [44]

$$\omega_i = -\lim_{T\to 0}\frac{\Omega_i - \Omega_i^*}{T}\,,\quad \Delta = -\lim_{T\to 0}\frac{\Phi-\Phi^*}{T}\,, \tag{2.4.11}$$

with $\Omega_i^* = g$ and $\Phi^* = 1$ denoting the BPS values of Ω_i and Φ, we can find the gravitational counterpart of the Cardy-like limit (2.4.10)

$$\left|\left(\frac{\partial\Omega_i}{\partial T}\right)_{T=0}\right| \ll 1\,,\quad \frac{\partial\Phi}{\partial T}\bigg|_{T=0} \sim \mathcal{O}(1)\,. \tag{2.4.12}$$

The equations $\left(\frac{\partial\Omega_i}{\partial T}\right)_* = 0$ have the roots

$$a = \frac{1}{g}\quad\text{and}\quad b=\frac{1}{g}\,. \tag{2.4.13}$$

Hence, the gravitational Cardy limit for the class of asymptotically AdS_6 black holes (2.4.1) is

$$a \to \frac{1}{g}\quad\text{and}\quad b\to\frac{1}{g}\,. \tag{2.4.14}$$

Similar to the black hole solutions in the previous sections, we have

$$a_i g \to 1, \tag{2.4.15}$$

where $a_i = \{a,b\}$. We can introduce small parameters to denote the deviations from this limit, i.e.,

$$a = \frac{1}{g}+\epsilon\,,\quad b=\frac{1}{g}+\epsilon+\eta\,,\quad \text{with } 0\neq\eta\ll\epsilon\,. \tag{2.4.16}$$

Expanding in ϵ after expanding in η, we find the thermodynamic quantities (2.4.6) in the

BPS and gravitational Cardy limit (2.4.16) to the leading order

$$\begin{aligned} S_* &= \frac{4\pi^2}{9g^6\epsilon^2} + \mathcal{O}(\epsilon^{-1})\,, \\ J_1^* &= -\frac{8\pi}{9\sqrt{3}g^7\epsilon^3} + \mathcal{O}(\epsilon^{-2})\,, \\ J_2^* &= -\frac{8\pi}{9\sqrt{3}g^7\epsilon^3} + \mathcal{O}(\epsilon^{-2})\,, \\ Q^* &= \frac{2\pi}{\sqrt{3}g^5\epsilon^2} + \mathcal{O}(\epsilon^{-1})\,, \end{aligned} \tag{2.4.17}$$

which are consistent with [36, 56] and the Cardy-like limit on the field theory side (2.4.10)

$$\omega_i^* \sim \epsilon\,, \quad \Delta_* \sim \mathcal{O}(1)\,. \tag{2.4.18}$$

2.4.3 Black Hole Solution in the Near-Horizon + Gravitational Cardy Limit

In the previous subsection, we have obtained the gravitational Cardy limit for the parameters on the gravity side. In this subsection, we discuss how the near-horizon metric changes when taking the gravitational Cardy limit. In Appendix A.1.4, we verify explicitly that the resulting background is a solution of the 6d gauged supergravity equations of motion. In the following, we implement the gravitational Cardy limit in the space of parameters, which further simplifies the geometry.

We apply the following scaling near the horizon r_+ (2.4.9) to the asymptotically AdS_6 black hole metric (2.4.5) in the BPS limit

$$r \to r_+ + \lambda \widetilde{r}\,, \quad t \to \frac{\widetilde{t}}{\lambda}\,, \quad \phi_1 \to \widetilde{\phi}_1 + g\frac{\widetilde{t}}{\lambda}\,, \quad \phi_2 \to \widetilde{\phi}_2 + g\frac{\widetilde{t}}{\lambda}\,, \tag{2.4.19}$$

with $\lambda \to 0$, and then take the AdS_6 gravitational Cardy limit (2.4.16). To the leading order in ϵ and η, the metric becomes

$$
\begin{aligned}
ds^2 = &-\frac{\sqrt{3}g^2}{4}\sqrt{(1+3g^2y^2)(1+3g^2z^2)\left[3+3g^4y^2z^2+g^2(y^2+z^2)\right]}\,\widetilde{r}^2\,d\widetilde{t}^2 \\
&+(1+3g^2y^2)(1+3g^2z^2)H_*(y,\,z)\frac{d\widetilde{r}^2}{144g^2\widetilde{r}^2} \\
&+\frac{g^2(1+3g^2y^2)(z^2-y^2)}{3(1-g^2y^2)^3}H_*(y,\,z)\,dy^2+\frac{g^2(1+3g^2z^2)(z^2-y^2)}{3(g^2z^2-1)^3}H_*(y,\,z)\,dz^2 \\
&+\frac{4(1-g^2y^2)^2(1-g^2z^2)^2\left[z^2+y^2(1+2g^2z^2)\right]}{3g^4\left[3+3g^4y^2z^2+g^2(y^2+z^2)\right]^2\epsilon^2\eta^2}H_*(y,\,z)\left(d\widetilde{\phi}_1-d\widetilde{\phi}_2\right)^2 \\
&+\frac{(1-g^2y^2)(g^2z^2-1)(1+3g^2y^2)(1+3g^2z^2)}{12g^6(y^2+z^2+2g^2y^2z^2)\epsilon^2}H_*(y,\,z)\left(d\widetilde{\phi}_1-\frac{\sqrt{3}}{2}g^3\epsilon\widetilde{r}d\widetilde{t}\right)^2,
\end{aligned}
\tag{2.4.20}
$$

where

$$
H_*(y,\,z)\equiv\sqrt{1+\frac{8}{(1+3g^2y^2)(1+3g^2z^2)}}. \tag{2.4.21}
$$

Defining

$$
\tau\equiv 6g^2\widetilde{t},\qquad \chi\equiv\frac{\widetilde{\phi}_1-\widetilde{\phi}_2}{g\,\eta}, \tag{2.4.22}
$$

we can rewrite the metric (2.4.20) as

$$
\begin{aligned}
ds^2 = H_*(y,\,z)\Bigg[&\frac{(1+3g^2y^2)(1+3g^2z^2)}{144g^2}\left(-\widetilde{r}^2\,d\tau^2+\frac{d\widetilde{r}^2}{\widetilde{r}^2}\right) \\
&+\frac{g^2(1+3g^2y^2)(z^2-y^2)}{3(1-g^2y^2)^3}\,dy^2+\frac{g^2(1+3g^2z^2)(z^2-y^2)}{3(g^2z^2-1)^3}\,dz^2 \\
&+\frac{4(1-g^2y^2)^2(1-g^2z^2)^2\left[z^2+y^2(1+2g^2z^2)\right]}{3g^2\left[3+3g^4y^2z^2+g^2(y^2+z^2)\right]^2\epsilon^2}d\chi^2 \\
&+\frac{(1-g^2y^2)(g^2z^2-1)(1+3g^2y^2)(1+3g^2z^2)}{12g^6(y^2+z^2+2g^2y^2z^2)\epsilon^2}\left(d\widetilde{\phi}_1-\frac{\sqrt{3}}{12}g\epsilon\widetilde{r}d\tau\right)^2\Bigg].
\end{aligned}
\tag{2.4.23}
$$

2.4.4 Black Hole Entropy from Cardy Formula

For the asymptotically AdS_6 black holes discussed in this section, we apply the Cardy formula to the near-horizon metric only after taking the gravitational Cardy limit. More explicitly, we first rewrite the metric (2.4.23) from Poincaré coordinates $(\widetilde{r},\,\tau)$ to global coordinates $(\hat{r},\,\hat{t})$ using the relations (2.1.44) - (2.1.46). Consequently, the near-horizon metric in the gravitational Cardy limit (2.4.23) becomes

$$
\begin{aligned}
ds^2 = H_*(y,\, z)\bigg[& \frac{(1+3g^2y^2)(1+3g^2z^2)}{144g^2}\left(-(1+\hat{r}^2)\,d\hat{t}^2 + \frac{d\hat{r}^2}{1+\hat{r}^2}\right) \\
& + \frac{g^2(1+3g^2y^2)(z^2-y^2)}{3(1-g^2y^2)^3}\,dy^2 + \frac{g^2(z^2-y^2)(1+3g^2z^2)}{3(g^2z^2-1)^3}\,dz^2 \\
& + \frac{4(1-g^2y^2)^2(1-g^2z^2)^2\left[z^2+y^2(1+2g^2z^2)\right]}{3g^2\left[3+3g^4y^2z^2+g^2(y^2+z^2)\right]^2\epsilon^2}d\chi^2 \\
& + \frac{(1-g^2y^2)(g^2z^2-1)(1+3g^2y^2)(1+3g^2z^2)}{12g^6(y^2+z^2+2g^2y^2z^2)\epsilon^2}\left(d\hat{\psi} - \frac{\sqrt{3}}{12}g\epsilon\hat{r}d\hat{t}\right)^2\bigg],
\end{aligned}
\tag{2.4.24}
$$

where

$$
\hat{\psi} \equiv \widetilde{\phi}_1 - \frac{\sqrt{3}}{12}g\epsilon\gamma\,. \tag{2.4.25}
$$

Applying the same formalism in Subsection 2.1.4 and choosing appropriate ranges of y and z, we obtain the central charge and the extremal Frolov-Thorne temperature in the near-horizon region of the asymptotically AdS_6 BPS black holes as follows:

$$
c_L = \frac{5\pi}{3\sqrt{3}g^5\epsilon}\,, \quad T_L = \frac{4\sqrt{3}}{5\pi g\epsilon}\,. \tag{2.4.26}
$$

Using the Cardy formula, we can compute the black hole entropy of the asymptotically AdS_6 BPS black holes:

$$
S_{BH} = \frac{\pi^2}{3}c_L T_L = \frac{4\pi^2}{9g^6\epsilon^2}\,, \tag{2.4.27}
$$

which is the same as the black hole entropy in the gravitational Cardy limit (2.4.17) from the gravity side.

2.4.5 Comparison with Results from Boundary CFT

For the asymptotically AdS_6 BPS black holes, it was shown in [44, 56, 57] that the entropies of these black holes can be obtained from the boundary 5d $\mathcal{N}=1$ superconformal field theories by extremizing an entropy function, which has also been studied in [40]. We can first compute the free energy in the large-N limit using the 5d superconformal index. The entropy function is then defined as a Legendre transform of the free energy in the large-N limit

$$
S(\Delta_I,\, \omega_i) = -\frac{i\pi}{81g^4G}\frac{\Delta^3}{\omega_1\omega_2} + Q\Delta + \sum_{i=1}^{2}J_i\omega_i + \Lambda\left(\Delta - \sum_{i=1}^{2}\omega_i - 2\pi i\right)\,. \tag{2.4.28}
$$

In the Cardy-like limit (2.4.10)

$$\omega \sim \epsilon\,, \quad \Delta_I \sim \mathcal{O}(1)\,, \tag{2.4.29}$$

we can read off from the entropy function (2.4.28)

$$S \sim \frac{1}{\epsilon^2}\,, \quad J \sim \frac{1}{\epsilon^3}\,, \quad Q_I \sim \frac{1}{\epsilon^2}\,, \tag{2.4.30}$$

which have been summarized in Table 1.1.

Similar to $AdS_{4,5,7}$, for AdS_6 the electric charge Q and the angular momenta J_i are real, while the chemical potential Δ and the angular velocities ω_i can be complex, and so can the entropy function S. By requiring that the black hole entropy S_{BH} to be real after extremizing the entropy function S, we obtain one more constraint on Q and J_i. More precisely, the asymptotically AdS_6 black hole entropy and the corresponding constraint are given implicitly by the following two relations [44, 56]

$$S_{BH}^3 - \frac{2\pi^2}{3g^4G_N}S_{BH}^2 - 12\pi^2\left(\frac{Q}{3g}\right)^2 S_{BH} + \frac{8\pi^4}{3g^4G_N}J_1J_2 = 0\,, \tag{2.4.31}$$

$$\frac{Q}{3g}\,S_{BH}^2 + \frac{2\pi^2}{9g^4G_N}(J_1+J_2)S_{BH} - \frac{4\pi^2}{3}\left(\frac{Q}{3g}\right)^3 = 0\,, \tag{2.4.32}$$

which are consistent with both the thermodynamic quantities on the gravity side (2.4.6) and (2.4.17) as well as the black hole entropy in the gravitational Cardy limit from the Cardy formula (2.4.27) under the AdS_6/CFT_5 dictionary of parameters [44, 56]

$$\frac{1}{g^4G_N} \sim N^{5/2}\,. \tag{2.4.33}$$

2.5 Discussion

In this chapter, we have discussed the near-horizon gravitational Cardy limit of asymptotically $AdS_{4,5,6,7}$ black holes. The gravitational Cardy limit can be written universally as $|a_ig| \to 1$, where a_i are parametrize angular momenta in units of the inverse AdS radius, g, for all the black hole solutions we analyzed. As we have explicitly shown in these examples, the gravitational Cardy limit leads to an AdS_3 geometry near the horizon and is effectively an additional limit on the independent parameters of the black hole solutions. The macroscopic Bekenstein-Hawking entropy of asymptotically AdS black holes has recently been given a microscopic foundation using the dual boundary $CFT_{3,4,5,6}$. Our work relies on a near-horizon

AdS_3 geometry and we provide an effective microscopic description via the CFT_2 Cardy formula obtained from the algebra of asymptotic symmetries.

It is instructive to point out various analogies with the previous instance when string theory answered explicitly the problem of microstate counting for black hole entropy. In the mid 90's, Strominger and Vafa [1] used the full machinery of D-brane technology to provide a microscopic description of the Bekenstein-Hawking entropy of a class of asymptotically flat black holes. Viewing the D-brane description as the UV complete description of gravity, the analogy with the current developments is that the microscopic description of the entropy of AdS_{d+1} black holes in terms of field theory degrees of freedom in the dual CFT_d boundary theory is the UV complete description. After the UV complete description of the 90's, Strominger went on to provide a universal description [58], based only on the near-horizon symmetries exploiting the AdS_3 near-horizon region and the asymptotic symmetry algebra computation of Brown and Henneaux [59]. Similar symmetry-based approaches were shown to apply to a wide variety of black holes by Carlip [60]. The results presented in [17,18] and in this manuscript show that we can understand the entropy of asymptotically AdS black holes based only on near-horizon symmetries via the Kerr/CFT correspondence.

The satisfying aspect of this point of view resides in the separation-of-scales principle. Such a universal feature of gravity as the Bekenstein-Hawking entropy formula can certainly be explained using UV complete formulations of quantum gravity but must also be understood without recourse to the existence of such a UV complete theory and could be determined strictly from low energy symmetry principles.

The point of view advocated in this chapter leads to a number of interesting questions some of which we now describe. It would be interesting to understand the field theory counterpart of the locally AdS_3 near-horizon region that arises from the Bardeen-Horowitz limit plus the gravitational Cardy limit. It clearly suggests the existence of an effective CFT_2 which we have used to microscopically compute the entropy but whose further details we do not know. Some aspects of this effective CFT_2 were studied in [61,62] for the AdS_5 and the AdS_4 black holes, but it required going away from extremality. In the bigger picture described above, understanding how this effective CFT_2 embeds in the boundary CFT_d is the dual to finding the UV complete description of the gravitational theory living near the horizon – a worthy challenge. Along these lines, in this manuscript, we have only discussed the asymptotically AdS black holes in the BPS limit, hence at zero temperature. It would be interesting to extend the discussion to near-extremal asymptotically $AdS_{4,6,7}$ black holes and to reproduce the Bekenstein-Hawking entropy formula from a near-horizon Cardy formula. When higher-derivative terms are included in the gravity theory, the black hole entropy does not obey the area law. It was shown in [63] that the central charge of the near-horizon asymptotic Virasoro symmetry also gets modified in the gravity with higher-derivative terms, while the Frolov-Thorne temperature and the Cardy formula still hold. Other higher-derivative aspects of AdS_4 black holes were recently considered [64, 65]. A tantalizing property of higher-derivative corrections in AdS_5 black holes was recently reported in [66], which showed that the leading α'-correction is absent in the BPS limit. This suggests that the central charge of the near-horizon asymptotic Virasoro symmetry remains the same in this case.

There is another line of attack that is worth sketching. Recall that the original setup for Cardy-like limits is 2d CFT. In this case, one simply has a formula for CFT_2 on $S^1 \times S^1$ which effectively relates the high energy and low energy degrees of freedom. It is fair to think of this relation as a UV/IR relation with the important characteristic of being controlled by the anomaly, c. Similar formulas have been developed in higher dimensions by Di Pietro and Komargodski in [67] and further clarified in [26,68,69]. In particular, in four dimensions they found an effective description of theories in $S^1 \times M^3$ whose effective action is controlled by anomaly coefficients. A similar analysis has been rigorously performed for a set of six-dimensional theories [14, 53, 70]. More closely related to the questions we addressed in this chapter is the recent work of Seok Kim and collaborators who have used an effective low energy action approach to find the leading term in the entropy function for the Cardy-like limit, first in $\mathcal{N} = 4$ SYM as well as in the 6d $\mathcal{N} = (2, 0)$ SCFT living on N M5-branes [14], and later for a more generic 4d $\mathcal{N} = 1$ situation [28]. These developments point to the possibility that the Cardy-like limit may be understood as the leading term in an effective field theory expansion. Although for these cases in the BPS limit the Cardy-like free energy has been derived from the effective quantum field theory approach, higher order corrections as well as finite temperatures should be taken into account to go beyond the leading order in the BPS limit. It would be quite interesting to explore such possibilities on the field theory side and, ultimately, connect it with a more standard hydrodynamics approach on the gravity side [71, 72].

Finally, it would be nice to develop what seems like a more natural AdS_2 or SYK approach to the entropy of extremal AdS black holes as described in Fig. 1.2. Some interesting work along this direction was performed in [73] for AdS_5 and more recently in [74] for AdS_4. Finding the connection between the AdS_2 and AdS_3 low energy descriptions in more details is an interesting problem.

Chapter 3

The Near Extremal Regime

3.1 Near-Extremal AdS_4 Black Hole Entropy

Returning to the AdS_4 black hole solution of [41], we further examine the parameter space with a emphasis on extremality. The extremal black hole solutions are achieved when the function $\Delta_r(r)$ has a double root, or equivalently when the discriminant of $\Delta_r(r)$ vanishes, which can be viewed as an equation for m. We can solve when the discriminant is zero and obtain the extremal value of m as a function of a and $\delta_{1,2}$, i.e.,

$$m = m_{\text{ext}}(a,\, \delta_1,\, \delta_2)\,. \tag{3.1.1}$$

Since this computation is straightforward, we omit the lengthy expression of $m_{\text{ext}}(a,\, \delta_1,\, \delta_2)$. In this case, for $m < m_{\text{ext}}$ the function $\Delta_r(r)$ has two different real roots corresponding to the outer and the inner horizons of a non-extremal black hole. For $m > m_{\text{ext}}$ the function $\Delta_r(r)$ does not have real roots, which implies that the solution has a naked singularity instead of a black hole. We would like to emphasize that the asymptotically AdS_4 Kerr-Newman black holes have also been discussed in [75]. However, the supersymmetric solutions considered [75] have $\frac{1}{2}$-BPS supersymmetry instead of $\frac{1}{4}$-BPS supersymmetry discussed in [41, 42], which makes some features of the black holes different.

Before moving on, we want to emphasize the parameter space we are exploring. First, a BPS black hole is both supersymmetric and extremal. Hence, it satisfies both the supersymmetric condition

$$e^{2\delta_1 + 2\delta_2} = 1 + \frac{2}{ag}\,, \tag{3.1.2}$$

or equivalently,

$$a = a_0\,, \qquad \text{with} \quad a_0 \equiv \frac{2}{g\,(e^{2\delta_1 + 2\delta_2} - 1)}\,, \tag{3.1.3}$$

and the extremal condition (3.1.1). Under the supersymmetric condition (3.1.3), the extremal condition (3.1.1) is equivalent to

$$(mg)^2 = \frac{\cosh^2(\delta_1+\delta_2)}{e^{\delta_1+\delta_2}\sinh^3(\delta_1+\delta_2)\sinh(2\delta_1)\sinh(2\delta_2)}\,, \tag{3.1.4}$$

which can also be obtained by requiring the black hole solution to have a regular horizon. The two conditions (3.1.3) and (3.1.4) in [42] contain typos, which have been corrected in [43,44] and also [45]. With these two constraints, there are only two independent parameters for asymptotically AdS_4 electrically charged rotating BPS black holes. To illustrate the relations of the parameters, we plot in Fig. 3.1 the codimension-1 supersymmetric surface defined by (3.1.3) together with the codimension-1 extremal surface defined by (3.1.1) in the parameter space $(m, a, \delta_1, \delta_2)$, where for simplicity we set $\delta_2 = \delta_1$ and $L = 1$. The intersection of these codimension-1 surfaces is a codimension-2 surface corresponding to the BPS solutions.

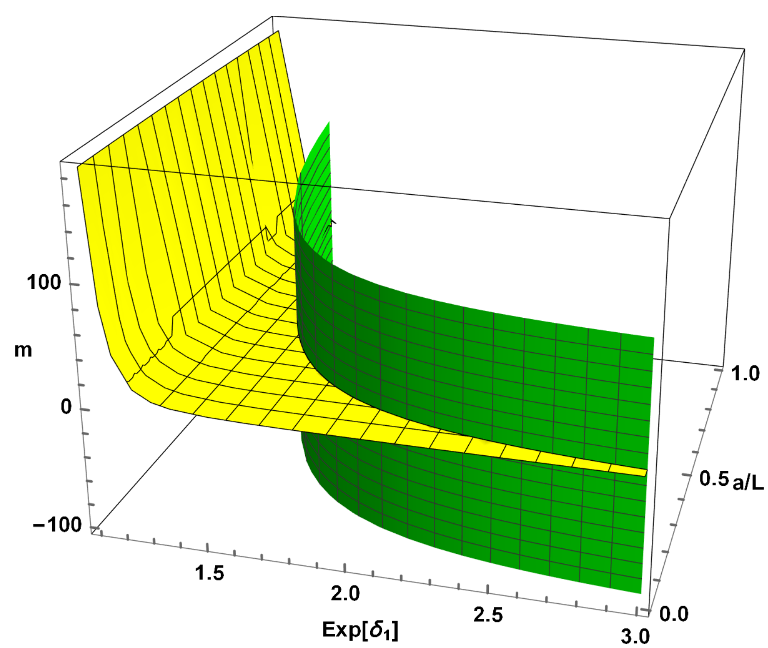

Figure 3.1: The extremal surface (yellow) and the supersymmetric surface (green)

We now collect useful properties of the black hole, including the position of the outer horizon in the BPS limit

$$r_0 = \frac{2m\sinh(\delta_1)\sinh(\delta_2)}{\cosh(\delta_1+\delta_2)}\,, \tag{3.1.5}$$

which coincides with the BPS inner horizon. For the thermodynamic quantities of the non-extremal asymptotically AdS_4 black holes, the gravitational angular velocity Ω and the temperature T_H are given by

$$\Omega = \frac{a(1+g^2 r_1 r_2)}{r_1 r_2 + a^2}\,, \quad T_H = \frac{\Delta_r'}{4\pi(r_1 r_2 + a^2)}\,, \tag{3.1.6}$$

which are evaluated at the outer horizon r_+. The other thermodynamic quantities are [42]

$$\begin{aligned} S &= \frac{\pi(r_1 r_2 + a^2)}{\Xi}\,, \\ J &= \frac{ma}{2\Xi^2}\left(\cosh(2\delta_1) + \cosh(2\delta_2)\right)\,, \\ Q_1 = Q_3 &= \frac{m}{4\Xi}\sinh(2\delta_1)\,, \\ Q_2 = Q_4 &= \frac{m}{4\Xi}\sinh(2\delta_2)\,. \end{aligned} \tag{3.1.7}$$

3.1.1 Near-Extremal AdS$_4$ Black Hole Entropy from Gravity Solution

The asymptotically AdS$_4$ black hole solutions discussed in the previous subsection are in general non-extremal. Since our focus is on near-extremality, we perturb the BPS black hole solution. More precisely, we expand the non-extremal AdS$_4$ black hole solutions around the BPS solution by turning on a small temperature.

We shall do this by studying the parameter space. Before imposing the constraints (3.1.3) and (3.1.4), there are 4 parameters that characterize the black hole solution, and we interchange one of these parameters, a, with the outer horizon r_+, where r_+ is the biggest root of the equation $\Delta_r(r_+) = 0$, i.e.,

$$r_+^2 + a^2 - 2mr_+ + g^2 \prod_{i=1}^{2}\left(r_+ + 2m\sinh^2(\delta_i)\right)\left[\prod_{i=1}^{2}\left(r_+ + 2m\sinh^2(\delta_i)\right) + a^2\right] = 0\,. \tag{3.1.8}$$

There are two reasons why we make this change. The first is pragmatic: this simplifies the algebra significantly. The second is that the outer horizon r_+ plays a clear role in the nAttractor mechanism [76], which will also be relevant for the discussions later in this subsection. Now, we use (3.1.8) to solve for the parameter a in terms of r_+, and the 4 independent parameters for the non-extremal AdS$_4$ black hole solutions are $(r_+, m, \delta_1, \delta_2)$. Correspondingly, there are 4 independent physical quantities (T_H, J, Q_1, Q_2), where we have set $Q_1 = Q_3$ and $Q_2 = Q_4$ as in (3.1.7). Without loss of generality, we further set $\delta_2 = \delta_1$, and therefore $Q_1 = Q_2$, to simplify the discussion.

The black hole entropy in (3.1.7) is valid for any temperature, including small temperature. This is achieved by expanding around the BPS value of the entropy, leading to the expression

$$S = S_* + \left(\frac{C}{T_H}\right)_* T_H + \mathcal{O}(T_H^2)\,, \tag{3.1.9}$$

where S_* denotes the AdS_4 black hole entropy (3.1.7) in the BPS limit

$$S_* = \frac{2\pi}{g^2\,(e^{4\,\delta_1} - 3)}\,, \tag{3.1.10}$$

while C is the heat capacity which is linear in T_H, and $\left(\frac{C}{T_H}\right)_*$ is evaluated in the BPS limit. Computing $\left(\frac{C}{T_H}\right)_*$ is straightforward

$$\left(\frac{C}{T_H}\right)_* = \left(\frac{dS}{dT_H}\right)_* = \left(\frac{\partial S}{\partial r_+}\right)_* \left(\frac{\partial r_+}{\partial T_H}\right)_* + \left(\frac{\partial S}{\partial m}\right)_* \left(\frac{\partial m}{\partial T_H}\right)_* + \left(\frac{\partial S}{\partial \delta_1}\right)_* \left(\frac{\partial \delta_1}{\partial T_H}\right)_*\,, \tag{3.1.11}$$

where $\frac{\partial r_+}{\partial T_H}$, $\frac{\partial m}{\partial T_H}$ and $\frac{\partial \delta_1}{\partial T_H}$ can be obtained by inverting the matrix

$$\frac{\partial(T_H, J, Q_1)}{\partial(r_+, m, \delta_1)}\,. \tag{3.1.12}$$

Once the dust settles, the result is

$$\left(\frac{C}{T_H}\right)_* = \frac{8\sqrt{2}\pi^2\left(e^{4\,\delta_1} - 1\right)^{\frac{3}{2}}}{g^3\,(e^{4\,\delta_1} - 3)\,(e^{8\,\delta_1} + 10\,e^{4\,\delta_1} - 7)}\,. \tag{3.1.13}$$

We comment that this result can also be obtained by only varying S with respect to r_+, i.e.,

$$\left(\frac{C}{T_H}\right)_* = \left(\frac{\partial S}{\partial r_+}\right)_* \left(\frac{\partial r_+}{\partial T_H}\right)_*\,. \tag{3.1.14}$$

This is similar to the AdS_5 case discussed in [33], which is related to the nAttractor mechanism [76]. This hints that the nAttractor mechanism extends to other dimensions.

3.1.2 AdS_4 Black Hole Solution in the Near-Horizon Limit

In this subsection, we consider near-extremal asymptotically AdS_4 black holes close to the $\frac{1}{4}$-BPS solutions by introducing a small positive temperature T, and discuss the corresponding metrics.

It was discussed in [75] that for asymptotically AdS_4 black holes from the extremal case to non-extremal configurations corresponds to perturbing the parameter m from its extremal value m_{ext}. As we can see from Fig. 3.1, when perturbing around the AdS_4 $\frac{1}{4}$-BPS black holes, we can deviate from the extremal surface but still stay in the supersymmetric surface by imposing the supersymmetric condition (3.1.3). Meanwhile, we expand the parameter m around its BPS value given by (3.1.4) with a small dimensionless parameter λ [77] corresponding to near-extremal AdS_4 black hole solutions, i.e.,

$$m = m_0(1 + \lambda^2 \widetilde{m}) \,, \tag{3.1.15}$$

where

$$m_0 \equiv \frac{\cosh(\delta_1 + \delta_2)}{g\, e^{(\delta_1+\delta_2)/2} \sinh^{3/2}(\delta_1 + \delta_2) \sqrt{\sinh(2\delta_1)\sinh(2\delta_2)}} \,. \tag{3.1.16}$$

A similar limit was also used in [78] to study the near-BPS black holes and compared with other limits in [13]. To summarize, near-extremal AdS_4 black holes can be achieved by perturbing the parameter m around m_{ext} while keeping the other parameters fixed. This is made explicit in this chapter by imposing the near-extremal condition (3.1.15) with the parameter a fixed by the supersymmetric condition (3.1.3).

Moreover, we perform a near-horizon scaling to the asymptotically AdS_4 black hole metric (2.2.4), which was first introduced by Bardeen and Horowitz in [37] and extensively studied [79] for the BPS AdS_4 black holes

$$r \to r_0 + \lambda\, \widetilde{r} \,, \quad t \to \frac{\widetilde{t}}{\lambda} \,, \quad \phi \to \widetilde{\phi} - g\big[\coth(2\delta_1) - 2\big] \frac{\widetilde{t}}{\lambda} \,. \tag{3.1.17}$$

In principle, for near-extremal black holes we should consider the near-horizon scaling $r \to r_+ + \lambda\, \widetilde{r}$. However, the near-extremal condition (3.1.15) implies that r_+ and r_0 only differ by a constant of order λ. Hence, we can absorb that constant into $\widetilde{r}$ and still take $r \to r_0 + \lambda\, \widetilde{r}$ in the near-horizon scaling. This kind of near-horizon scaling for near-extremal black holes has been used in [80]. To summarize, we impose the near-horizon scaling (3.1.17) together with the condition conditions (3.1.15) and (3.1.3).

Taking the limit $\lambda \to 0$, the metric (2.2.4) becomes

$$\begin{aligned} ds^2 = &- \frac{\left(e^{8\,\delta_1} + 10\, e^{4\,\delta_1} - 7\right)\,\left(e^{4\,\delta_1} + \cos(2\theta)\right)}{2\,(e^{4\,\delta_1}+1)^2}\, g^2\, \widetilde{r}^2\, d\widetilde{t}^2 + \frac{2\,\left(e^{4\,\delta_1} + \cos(2\theta)\right)}{g^2\,\left(e^{8\,\delta_1} + 10\, e^{4\,\delta_1} - 7\right)}\, \frac{d\widetilde{r}^2}{\widetilde{r}^2} \\ &+ \frac{2\,\left(e^{4\,\delta_1} + \cos(2\theta)\right)}{g^2\,\left(e^{8\,\delta_1} - 2\, e^{4\,\delta_1} - 1 - 2\cos(2\,\theta)\right)}\, d\theta^2 \\ &+ \Lambda_{\mathrm{AdS}_4}(\theta) \left[d\widetilde{\phi} + \frac{g^2\, e^{\delta_1}\,(e^{4\,\delta_1} - 3)\sqrt{\mathrm{csch}(2\,\delta_1)}\,\mathrm{sech}(2\,\delta_1)}{1 + \coth(2\,\delta_1)}\, \widetilde{r}\, d\widetilde{t} \right]^2 \,, \end{aligned} \tag{3.1.18}$$

where

$$\Lambda_{\mathrm{AdS}_4}(\theta) \equiv \frac{2\,\left(e^{8\,\delta_1} - 2\,e^{4\,\delta_1} - 1 - 2\cos(2\,\theta)\right)\,\sin^2(\theta)}{g^2\,(e^{4\,\delta_1}-3)^2\,\left(e^{4\,\delta_1}+\cos(2\theta)\right)}\,. \tag{3.1.19}$$

From the near-horizon, we are now in a position to extract the necessary details to compute the entropy via the Kerr/CFT correspondence. As we shall see, there are several methods to compute the central charges, which requires a rewriting of the near-horizon geometry in different coordinate systems. To make things clearer, we summarize each of these different expressions of the near-horizon metric. The change of coordinates

$$\tau \equiv \frac{g^2\,\left(e^{8\,\delta_1}+10\,e^{4\,\delta_1}-7\right)}{2(e^{4\delta_1}+1)}\,\widetilde{t}\,, \quad \rho \equiv \widetilde{r}\,, \tag{3.1.20}$$

allows us to write (3.1.18) in Poincaré coodinates

$$\begin{aligned} ds^2 = {} & \frac{2\,\left(e^{4\,\delta_1}+\cos(2\theta)\right)}{g^2\,\left(e^{8\,\delta_1}+10\,e^{4\,\delta_1}-7\right)}\left(-\rho^2\,d\tau^2+\frac{d\rho^2}{\rho^2}\right)+\frac{2\,\left(e^{4\,\delta_1}+\cos(2\theta)\right)}{g^2\,\left(e^{8\,\delta_1}-2\,e^{4\,\delta_1}-1-2\cos(2\,\theta)\right)}\,d\theta^2 \\ & + \Lambda_{\mathrm{AdS}_4}(\theta)\left[d\widetilde{\phi}+\frac{2\left(e^{8\,\delta_1}-4\,e^{4\,\delta_1}+3\right)\sqrt{\mathrm{csch}(2\,\delta_1)}}{e^{\delta_1}\left(e^{8\,\delta_1}+10\,e^{4\,\delta_1}-7\right)}\,\rho\,d\tau\right]^2\,. \end{aligned} \tag{3.1.21}$$

Therefore, it is clear that the near-horizon scaling we applied to the metric leaves us with a circle fibered over AdS_2, yielding a warped AdS_3 geometry. We now see that the near-horizon metric in Poincaré coodinates (3.1.21) is in the standard form

$$ds^2 = f_0(\theta)\left(-\rho^2\,d\tau^2+\frac{d\rho^2}{\rho^2}\right)+f_\theta(\theta)\,d\theta^2+\gamma_{ij}(\theta)\,\left(dx^i+k^i\rho\,d\tau\right)\,\left(dx^j+k^j\rho\,d\tau\right)\,, \tag{3.1.22}$$

with $x^i \in \{\widetilde{\phi}\}$ for the AdS_4 case, and the coefficients $f_0(\theta)$, $f_\theta(\theta)$, k^i and $\gamma_{ij}(\theta)$ are functions of θ in general.

Now, we transform the Poincaré coordinates $(\tau,\rho,\theta,\widetilde{\phi})$ in the metric (3.1.21) to the global coordinates $(\hat{t},\hat{r},\theta,\hat{\phi})$ using the following relations

$$g\,\rho = \hat{r}+\sqrt{1+\hat{r}^2}\cos(\hat{t})\,, \quad g^{-1}\,\tau = \frac{\sqrt{1+\hat{r}^2}\sin(\hat{t})}{\hat{r}+\sqrt{1+\hat{r}^2}\cos(\hat{t})}\,, \tag{3.1.23}$$

which leads to

$$\begin{aligned} -\rho^2\,d\tau^2+\frac{d\rho^2}{\rho^2} &= -(1+\hat{r}^2)\,d\hat{t}^2+\frac{d\hat{r}^2}{1+\hat{r}^2}\,, \\ \rho\,d\tau &= \hat{r}\,d\hat{t}+d\kappa\,, \end{aligned} \tag{3.1.24}$$

where

$$\kappa \equiv \log\left(\frac{1+\sqrt{1+\hat{r}^2}\,\sin(\hat{t})}{\cos(\hat{t})+\hat{r}\sin(\hat{t})}\right). \tag{3.1.25}$$

Consequently, the metric (3.1.21) can be rewritten as

$$\begin{aligned} ds^2 =\, & \frac{2\,\left(e^{4\,\delta_1}+\cos(2\theta)\right)}{g^2\,\left(e^{8\,\delta_1}+10\,e^{4\,\delta_1}-7\right)}\left[-(1+\hat{r}^2)\,d\hat{t}^2+\frac{d\hat{r}^2}{1+\hat{r}^2}\right]+\frac{2\,\left(e^{4\,\delta_1}+\cos(2\theta)\right)}{g^2\,\left(e^{8\,\delta_1}-2\,e^{4\,\delta_1}-1-2\cos(2\,\theta)\right)}\,d\theta^2 \\ & +\Lambda_{\mathrm{AdS}_4}(\theta)\left[d\hat{\phi}+\frac{2\left(e^{8\,\delta_1}-4\,e^{4\,\delta_1}+3\right)\sqrt{\mathrm{csch}(2\,\delta_1)}}{e^{\delta_1}\left(e^{8\,\delta_1}+10\,e^{4\,\delta_1}-7\right)}\,\hat{r}\,d\hat{t}\right]^2, \end{aligned} \tag{3.1.26}$$

where

$$\hat{\phi} \equiv \widetilde{\phi}+\frac{2\left(e^{8\,\delta_1}-4\,e^{4\,\delta_1}+3\right)\sqrt{\mathrm{csch}(2\,\delta_1)}}{e^{\delta_1}\left(e^{8\,\delta_1}+10\,e^{4\,\delta_1}-7\right)}\,\kappa\,. \tag{3.1.27}$$

Besides the near-horizon scaling (3.1.17), we can also apply a light-cone scaling in the near-horizon region [77]

$$x^+ \equiv \epsilon\left(\phi+\frac{e^{4\,\delta_1}-3}{e^{4\,\delta_1}-1}\,gt\right),\quad x^- \equiv \phi-\frac{e^{4\,\delta_1}-3}{e^{4\,\delta_1}-1}\,gt\,, \tag{3.1.28}$$

and then consider the following near-horizon scaling in the light-cone coordinates

$$r \to r_0+\epsilon\,\widetilde{r}\,,\quad t \to \frac{e^{4\,\delta_1}-1}{e^{4\,\delta_1}-3}\,\frac{x^+-\epsilon x^-}{2g\epsilon}\,,\quad \phi \to \frac{x^++\epsilon x^-}{2\epsilon}\,. \tag{3.1.29}$$

Together with the condition (3.1.15) and taking the limit $\epsilon \to 0$, we obtain the near-horizon metric for the AdS_4 near-extremal black holes in the coordinates $(x^+,\,\widetilde{r},\,\theta,\,x^-)$

$$\begin{aligned} ds^2 =\, & -\frac{\left(e^{4\,\delta_1}-1\right)^2\left(e^{8\,\delta_1}+10\,e^{4\,\delta_1}-7\right)\left(e^{4\,\delta_1}+\cos(2\theta)\right)}{8\,\left(e^{8\,\delta_1}-2\,e^{4\,\delta_1}-3\right)^2}\,\widetilde{r}^2\,dx^{+2}+\frac{2\,\left(e^{4\,\delta_1}+\cos(2\theta)\right)}{g^2\left(e^{8\,\delta_1}+10\,e^{4\,\delta_1}-7\right)}\,\frac{d\widetilde{r}^2}{\widetilde{r}^2} \\ & +\frac{2\,\left(e^{4\,\delta_1}+\cos(2\theta)\right)}{g^2\,\left(e^{8\,\delta_1}-2\,e^{4\,\delta_1}-1-2\cos(2\,\theta)\right)}\,d\theta^2+\Lambda_{\mathrm{AdS}_4}(\theta)\left[dx^-+\frac{e^{\delta_1}\,\mathrm{sech}(2\,\delta_1)}{\mathrm{csch}(2\,\delta_1)^{3/2}}\,g\,\widetilde{r}\,dx^+\right]^2, \end{aligned} \tag{3.1.30}$$

where $\Lambda_{\mathrm{AdS}_4}(\theta)$ is the same as (3.1.19). Introducing some new coordinates

$$\hat{x}^+ \equiv \frac{g\left(e^{4\,\delta_1}-1\right)\left(e^{8\,\delta_1}+10\,e^{4\,\delta_1}-7\right)}{4\,\left(e^{8\,\delta_1}-2\,e^{4\,\delta_1}-3\right)}\,x^+\,,\quad \hat{\rho} \equiv \widetilde{r}\,,\quad \hat{x}^- \equiv x^-\,, \tag{3.1.31}$$

we can rewrite the near-horizon metric in the light-cone coordinates (3.1.30) as

$$ds^2 = \frac{2\,\left(e^{4\,\delta_1}+\cos(2\theta)\right)}{g^2\,(e^{8\,\delta_1}+10\,e^{4\,\delta_1}-7)}\left[-\hat{\rho}^2\,d\hat{x}^{+2}+\frac{d\hat{\rho}^2}{\hat{\rho}^2}\right]+\frac{2\,\left(e^{4\,\delta_1}+\cos(2\theta)\right)}{g^2\,\left(e^{8\,\delta_1}-2\,e^{4\,\delta_1}-1-2\cos(2\,\theta)\right)}\,d\theta^2$$
$$+\Lambda_{\mathrm{AdS}_4}(\theta)\left[d\hat{x}^- + \frac{2\left(e^{8\,\delta_1}-4\,e^{4\,\delta_1}+3\right)\sqrt{\mathrm{csch}(2\,\delta_1)}}{e^{\delta_1}\,(e^{8\,\delta_1}+10\,e^{4\,\delta_1}-7)}\,\hat{\rho}\,d\hat{x}^+\right]^2 . \tag{3.1.32}$$

We see that the metric (3.1.32) is in the standard form

$$ds^2 = f_0(\theta)\left(-\hat{\rho}^2\,d\hat{x}^{+2}+\frac{d\hat{\rho}^2}{\hat{\rho}^2}\right)+f_\theta(\theta)\,d\theta^2+\gamma_{ij}(\theta)\,\left(dx^i+k^i\hat{\rho}\,d\hat{x}^+\right)\,\left(dx^j+k^j\hat{\rho}\,d\hat{x}^+\right)\,, \tag{3.1.33}$$

with $x^i \in \{\hat{x}^-\}$ for the AdS_4 case, and k^i, $f_0(\theta)$, $f_\theta(\theta)$ and $\gamma_{ij}(\theta)$ remain the same as (3.1.22). To summarize, we now have several different expressions for the near-horizon metric in Poincaré and global coordinates. This is useful when we utilize the near-extremal Kerr/CFT correspondence.

3.1.3 Near-Extremal AdS_4 Black Hole Entropy from Cardy Formula

After obtaining the various expressions of the near-horizon metric of the asymptotically AdS_4 black holes, we are now ready to compute the central charges and the Frolov-Thorne temperatures using the near-extremal Kerr/CFT correspondence as well as hidden conformal symmetry of the near-horizon geometry to find the AdS_4 black hole entropy in the near-extremal limit. For the left central charge c_L and the right central charge c_R, there are two different ways for computing each of them, depending on which coordinate system we choose. We summarize each of these diverse approaches as a consistency check on our computation as well as to keep things self-contained.

The Kerr/CFT correspondence was originally posed for asymptotically flat extremal Kerr black holes [16] and was later shown to also be valid for asymptotically AdS black holes [17, 18]. For the near-extremal case, [77, 80] initiated some progress and we extend those results here by computing the entropies of near-extremal AdS_4 black holes via the Cardy formula. Before further exploring the near-extremal case, let us take a step back and recall how the Kerr/CFT correspondence works. The basic idea is the following. Taking the Bardeen-Horowitz near-horizon scaling [37], the near-horizon geometry of an asymptotically flat or asymptotically AdS extremal black hole contains $U(1)$ cycles fibered on AdS_2. The near-horizon asymptotic symmetries are characterized by diffeomorphims generated by the vectors

$$\zeta_\epsilon = \epsilon(\phi)\frac{\partial}{\partial\phi} - r\,\epsilon'(\phi)\frac{\partial}{\partial r}\,. \tag{3.1.34}$$

The mode expansion of a diffeomorphism generating vector ζ is

$$\zeta_{(n)} = -e^{-in\tilde{\phi}}\frac{\partial}{\partial\tilde{\phi}} - inre^{-in\tilde{\phi}}\frac{\partial}{\partial\hat{r}}\,. \tag{3.1.35}$$

We can define a 2-form k_ζ for a general perturbation $h_{\mu\nu}$ around the background metric $g_{\mu\nu}$ as

$$\begin{aligned} k_\zeta[h,\,g] \equiv -\frac{1}{4}\epsilon_{\alpha\beta\mu\nu}\Big[&\zeta^\nu D^\mu h - \zeta^\nu D_\sigma h^{\mu\sigma} + \zeta_\sigma D^\nu h^{\mu\sigma} + \frac{1}{2}hD^\nu\zeta^\mu - h^{\nu\sigma}D_\sigma\zeta^\mu \\ &+ \frac{1}{2}h^{\sigma\nu}(D^\mu\zeta_\sigma + D_\sigma\zeta^\mu)\Big]dx^\alpha\wedge dx^\beta\,. \end{aligned} \tag{3.1.36}$$

We also define the Lie derivative with respect to ζ, denoted by $\mathcal{L}_\zeta$, as

$$\mathcal{L}_\zeta g_{\mu\nu} \equiv \zeta^\rho\partial_\rho g_{\mu\nu} + g_{\rho\nu}\partial_\mu\zeta^\rho + g_{\mu\rho}\partial_\nu\zeta^\rho\,. \tag{3.1.37}$$

The left central charge c_L of the near-horizon Virasoro algebra can be computed using the Kerr/CFT correspondence in two slightly different ways. For the first method, the central charge can be computed using the following integral [16–18]

$$\frac{1}{8\pi G}\int_{\partial\Sigma} k_{\zeta_{(m)}}[\mathcal{L}_{\zeta_{(n)}}g,\,g] = -\frac{i}{12}c_L\,(m^3+\alpha m)\,\delta_{m+n,\,0}\,, \tag{3.1.38}$$

where g denotes the near-horizon metric of the near-extremal AdS_4 black hole in global coordinates (3.1.26). An explicit evaluation of (3.1.38) shows that

$$c_L = \frac{24\sqrt{2\,(e^{4\,\delta_1}-1)}}{g^2\,(e^{8\delta_1}+10\,e^{4\,\delta_1}-7)}\,. \tag{3.1.39}$$

The other way of computing c_L is to evaluate the following integral [77]

$$\frac{1}{8\pi G_N}\int_{\partial\Sigma} k_{\xi_n}\,[\mathcal{L}_{\xi_m}\bar{g},\,\bar{g}] = \delta_{n+m,\,0}\,n^3\frac{c_L}{12}\,, \tag{3.1.40}$$

where $\bar{g}$ denotes the standard form (3.1.22) of the near-horizon metric of the near-extremal AdS_4 black hole in Poincaré coordinates (3.1.21). More precisely, we obtain with the unit $G_N = 1$

$$c_L = \frac{3k_{\tilde{\phi}}}{G_N}\int_0^\pi d\theta\,\sqrt{\mathrm{Det}\,(\gamma_{ij}(\theta))\;f_\theta(\theta)} = \frac{24\sqrt{2\,(e^{4\,\delta_1}-1)}}{g^2\,(e^{8\delta_1}+10\,e^{4\,\delta_1}-7)}\,, \tag{3.1.41}$$

which matches exactly the result of c_L (3.1.39) from the first approach.

The right central charge c_R can also be obtained in two different ways. Although we describe the two methods, we prefer one method over the other because of its robustness. The first approach is to compute the quasi-local charge [77, 81, 82] using the standard form of the near-horizon metric of the near-extremal AdS$_4$ black hole in Poincaré coordinates (3.1.22), which is given by the integral

$$\frac{c_R}{12} = \frac{1}{8\pi G_N} \int dx^i \, d\theta \, \frac{k_i k_j \gamma_{ij}(\theta) \sqrt{\text{Det}\,(\gamma_{ij}(\theta))\; f_\theta(\theta)}}{2\Lambda_0 f_0(\theta)} \,, \tag{3.1.42}$$

where $f_0(\theta)$, $f_\theta(\theta)$, $\gamma_{ij}(\theta)$ and k_i are defined in (3.1.22), and the parameter Λ_0 denotes a UV cutoff in r. This approach has been used to compute the right central charge c_R for near-extremal AdS$_5$ black holes [61]. For the four-dimensional case, the integral (3.1.42) can be applied to the near-horizon metric (3.1.21) in Poincaré coordinates to compute c_R. However, the result is not very illuminating due to the unfixed cutoff Λ_0.

To compute c_R, we choose a more concrete approach using light-cone coordinates as introduced in [77]. More precisely, a scale-covariant right central charge $c_R^{(cov)}$ can be computed from the near-horizon metric (3.1.32) by using

$$c_R^{(cov)} = 3k_- \, \epsilon \int_0^\pi d\theta \sqrt{\text{Det}\,(\gamma_{ij}(\theta))\; f_\theta(\theta)} = \frac{24\,\epsilon \sqrt{2\,(e^{4\,\delta_1} - 1)}}{g^2\,(e^{8\delta_1} + 10\,e^{4\,\delta_1} - 7)} \,, \tag{3.1.43}$$

where the factors $\gamma_{ij}(\theta)$, $f_\theta(\theta)$ and k_- are defined in (3.1.33). Like in [77], we can define a scale-invariant right central charge $c_R \equiv c_R^{(cov)}/\epsilon$, which in this case is

$$c_R = \frac{24\sqrt{2\,(e^{4\,\delta_1} - 1)}}{g^2\,(e^{8\delta_1} + 10\,e^{4\,\delta_1} - 7)} \,. \tag{3.1.44}$$

We see that the result is exactly the same as the left central charge computed in (3.1.39) and (3.1.41). To summarize, the explicit expression for the integral changes slightly depending on the coordinate system, and we have shown that all the results do indeed lead to the same central charge.

Now that we have taken care of the central charges, and have consistently gotten that $c_L = c_R$, the final ingredient is the Frolov-Thorne temperatures T_L and T_R. We have seen in [79] that for the BPS case $T_R = 0$. For the near-extremal case, T_L can still be computed in the same way discussed in [79], and its value remains the same as the BPS case, as it is unaffected by whether we impose the condition (3.1.15). Therefore, we find

$$T_L = \frac{e^{\delta_1}\left(e^{8\,\delta_1} + 10\,e^{4\,\delta_1} - 7\right)\sqrt{\sinh(2\,\delta_1)}}{4\pi\left(e^{8\,\delta_1} - 4\,e^{4\,\delta_1} + 3\right)}\,. \tag{3.1.45}$$

On the other hand, T_R is proportional to the physical Hawking temperature T_H. To find the exact expression of T_R, we apply the technique of hidden conformal symmetry. This method was first introduced in [83], and later generalized to many different cases. The basic idea is to define a set of near-horizon conformal coordinates and corresponding locally-defined vector fields with $SU(2,\mathbb{R})$ Lie algebra, such that the wave equation of an uncharged massless scalar field becomes the quadratic Casimir of the $SU(2,\mathbb{R})$ Lie algebra. In this way, we can fix the Frolov-Thorne temperatures $T_{L,R}$ and the mode numbers $N_{L,R}$ for non-extremal black holes.

In particular, [80] has considered the hidden conformal symmetry of an AdS$_4$ black hole close to the solutions discussed in this chapter. We can apply the same technique by first expanding Δ_r

$$\Delta_r = k(r-r_+)(r-r_s) + \mathcal{O}\left((r-r_+)^3\right)\,, \tag{3.1.46}$$

where k and r_s can be read off from the Taylor expansion to quadratic order in $(r-r_+)$. Based on hidden conformal symmetry [80], the right temperature is

$$T_R = \frac{k(r_+ - r_s)}{4\pi a \Xi}\,. \tag{3.1.47}$$

Comparing with the Hawking temperature T_H given by (3.1.6), we find that

$$T_R = \frac{a^2 + r_1^2}{a - a^3 g^2}\,T_H\,. \tag{3.1.48}$$

We also find that the expression obtained using hidden conformal symmetry for T_L is (3.1.45) as expected. Using the Cardy formula, we obtain the near-extremal AdS$_4$ black hole entropy

$$\begin{aligned} S &= \frac{\pi^2}{3}c_L T_L + \frac{\pi^2}{3}c_R T_R \\ &= S_* + \left(\frac{C}{T_H}\right)_* T_H\,, \\ &= \frac{2\pi}{g^2\left(e^{4\,\delta_1} - 3\right)} + \frac{8\sqrt{2}\pi^2\left(e^{4\,\delta_1} - 1\right)^{\frac{3}{2}}}{g^3\left(e^{4\,\delta_1} - 3\right)\left(e^{8\,\delta_1} + 10\,e^{4\,\delta_1} - 7\right)}\,T_H\,, \end{aligned} \tag{3.1.49}$$

where the BPS entropy S_* is

$$S_* = \frac{\pi^2}{3}c_L T_L\,, \tag{3.1.50}$$

while the near-extremal correction to the black hole entropy is

$$\delta S = \frac{\pi^2}{3} c_R T_R \equiv \left(\frac{C}{T_H}\right)_* T_H\,. \tag{3.1.51}$$

We see that this result from the near-horizon CFT_2 and the Cardy formula is exactly the same as the results from the gravity side ((3.1.9), (3.1.10) and (3.1.13)).

3.1.4 Near-Extremal AdS_4 Black Hole Entropy from Boundary CFT

What remains is the computation of the near-extremal entropy from the boundary CFT. In the BPS limit, the AdS_4 black hole entropy can be obtained by extremizing an entropy function, which was derived by the superconformal index or supersymmetric localization of the 3d ABJM theory on the boundary of electrically charged rotating AdS_4 BPS black holes [46, 47]. More precisely, the BPS entropy function is

$$S(\widetilde{\Delta}_I, \widetilde{\omega}) = -\frac{4\sqrt{2}\, i\, k^{\frac{1}{2}} N^{\frac{3}{2}}}{3} \frac{\sqrt{\widetilde{\Delta}_1 \widetilde{\Delta}_2 \widetilde{\Delta}_3 \widetilde{\Delta}_4}}{\widetilde{\omega}} + \widetilde{\omega} J + \sum_I \widetilde{\Delta}_I Q_I + \Lambda \left(\sum_I \widetilde{\Delta}_I - \widetilde{\omega} - 2\pi i\right)\,, \tag{3.1.52}$$

where $\widetilde{\Delta}_I$ are chemical potentials corresponding to the electric charges Q_I, and $\widetilde{\omega}$ is the angular velocity. To extremize the entropy function (3.1.52), we solve the equations

$$\frac{\partial S}{\partial \widetilde{\Delta}_I} = 0\,, \quad \frac{\partial S}{\partial \widetilde{\omega}} = 0\,, \tag{3.1.53}$$

which can be expressed explicitly as

$$Q_I + \Lambda = \frac{4\sqrt{2}\, i\, k^{\frac{1}{2}} N^{\frac{3}{2}}}{3} \frac{\sqrt{\widetilde{\Delta}_1 \widetilde{\Delta}_2 \widetilde{\Delta}_3 \widetilde{\Delta}_4}}{2\widetilde{\Delta}_I \widetilde{\omega}}\,, \tag{3.1.54}$$

$$J - \Lambda = -\frac{4\sqrt{2}\, i\, k^{\frac{1}{2}} N^{\frac{3}{2}}}{3} \frac{\sqrt{\widetilde{\Delta}_1 \widetilde{\Delta}_2 \widetilde{\Delta}_3 \widetilde{\Delta}_4}}{\widetilde{\omega}^2}\,. \tag{3.1.55}$$

Substituting these equations back into the entropy function (3.1.52), we obtain

$$S = -2\pi i \Lambda\,. \tag{3.1.56}$$

Moreover, the equations (3.1.54) and (3.1.55) can be combined into one equation:

$$Q_1Q_2Q_3Q_4 + \Lambda\left(\sum_{I<J<K} Q_IQ_JQ_K\right) + \Lambda^2\left(\sum_{I<J} Q_IQ_J\right) + \Lambda^3\left(\sum_I Q_I\right) + \Lambda^4$$
$$= -\frac{2}{9}kN^3(\Lambda^2 - 2\Lambda J + J^2)\,, \tag{3.1.57}$$

which can be written more compactly as

$$\Lambda^4 + A\,\Lambda^3 + B\,\Lambda^2 + C\,\Lambda + D = 0\,, \tag{3.1.58}$$

with the real-valued coefficients

$$\begin{aligned} A &= \sum_{I=1}^{4} Q_I\,, \\ B &= \sum_{I<J} Q_IQ_J + \frac{2}{9}kN^3\,, \\ C &= \sum_{I<J<K} Q_IQ_JQ_K - \frac{4}{9}kN^3J\,, \\ D &= Q_1Q_2Q_3Q_4 + \frac{2}{9}kN^3J^2\,. \end{aligned} \tag{3.1.59}$$

In order to obtain a real-valued black hole entropy, the expression (3.1.56) implies that Λ should have a purely imaginary root. Since (3.1.58) is a quartic equation of Λ with real coefficients, the imaginary roots should come in pairs. Consequently, (3.1.58) can be factorized as

$$(\Lambda^2 + \alpha)(\Lambda^2 + \beta\,\Lambda + \mu) = \Lambda^4 + \beta\,\Lambda^3 + (\alpha + \mu)\,\Lambda^2 + \alpha\beta\,\Lambda + \alpha\mu\,. \tag{3.1.60}$$

Comparing (3.1.60) with (3.1.58), we find

$$A = \beta\,,\quad B = \alpha + \mu\,,\quad C = \alpha\beta\,,\quad D = \alpha\mu\,, \tag{3.1.61}$$

or equivalently,

$$\alpha = \frac{C}{A}\,,\quad \beta = A\,,\quad \mu = B - \frac{C}{A} = \frac{AD}{C}\,. \tag{3.1.62}$$

According to (3.1.56), the imaginary root $\Lambda = i\sqrt{\alpha} = i\sqrt{\frac{C}{A}}$ leads to the real-valued AdS$_4$ BPS black hole entropy

$$S_{BH}^{*} = 2\pi\sqrt{\frac{Q_1Q_2Q_3 + Q_1Q_2Q_4 + Q_1Q_3Q_4 + Q_2Q_3Q_4 - \frac{4}{9}kN^3J}{Q_1 + Q_2 + Q_3 + Q_4}}\,. \tag{3.1.63}$$

For the special case $Q_1 = Q_3$, $Q_2 = Q_4$, the expression above becomes

$$S_{BH}^{*} = \frac{2\pi}{3}\sqrt{\frac{9Q_1Q_2(Q_1 + Q_2) - 2kJN^3}{Q_1 + Q_2}}\,. \tag{3.1.64}$$

After imposing the identifications of parameters introduced in [44, 46, 47]

$$Q_{BH,I} = \frac{g}{2}Q_I\,, \quad J_{BH} = J\,, \qquad I \in \{1, \cdots, 4\} \tag{3.1.65}$$

and using an entry from the AdS/CFT dictionary

$$\frac{1}{G_N} = \frac{2\sqrt{2}}{3}g^2k^{\frac{1}{2}}N^{\frac{3}{2}}\,, \tag{3.1.66}$$

we can rewrite the BPS black hole entropy (3.1.64) as

$$S_{BH}^{*} = \frac{\pi}{g^2G}\frac{J_{BH}}{\left(\frac{2}{g}Q_{BH,1} + \frac{2}{g}Q_{BH,2}\right)}\,, \tag{3.1.67}$$

which can be subsequently written in terms of the free parameters (δ_1, δ_2) on the gravity side in the BPS limit. For the special case $\delta_1 = \delta_2$, the BPS black hole entropy obtained from the boundary CFT is

$$S_{BH}^{*} = \frac{2\pi}{g^2\left(e^{4\,\delta_1} - 3\right)}\,, \tag{3.1.68}$$

which is exactly the same as the BPS result from the gravity side (3.1.10) and the one from the near-horizon Kerr/CFT correspondence (3.1.50).

In addition to the black hole entropy, the electric charges Q_I's and the angular momentum J should also satisfy a constraint, which originates from the consistency of two expressions of μ in (3.1.62), i.e.,

$$B - \frac{C}{A} - \frac{AD}{C} = 0\,. \tag{3.1.69}$$

More explicitly, for the special case $Q_1 = Q_3$, $Q_2 = Q_4$ the constraint is

$$\frac{2}{9}kN^3 + (Q_1+Q_2)^2 + \frac{2kJN^3}{9(Q_1+Q_2)} + \frac{2kJN^3\left[Q_1Q_2 + J(Q_1+Q_2)\right]}{2kJN^3 - 9Q_1Q_2(Q_1+Q_2)} = 0\,. \tag{3.1.70}$$

We emphasize that the constraint is not unique. A constraint multiplied by a constant or some regular function of Q_I and J can produce new constraints. For later convenience, we define

$$h \equiv \frac{J^2}{4g^5(Q_1+Q_2)^2}\left[\frac{2}{9}kN^3 + (Q_1+Q_2)^2 + \frac{2kJN^3}{9(Q_1+Q_2)} + \frac{2kJN^3\left[Q_1Q_2 + J(Q_1+Q_2)\right]}{2kJN^3 - 9Q_1Q_2(Q_1+Q_2)}\right], \tag{3.1.71}$$

whose BPS value will be called h_*, and

$$h_* = 0 \tag{3.1.72}$$

is one of the BPS constraints. So far we have only considered the BPS black holes from the boundary CFT in this subsection. To extend the BPS results to the near-extremal case, similar to the AdS_5 case discussed in [61], we generalize the quartic equation (3.1.58) from the BPS limit to the near-extremal case by perturbing Λ and h as

$$(\Lambda+\delta\Lambda)^4 + A\,(\Lambda+\delta\Lambda)^3 + B\,(\Lambda+\delta\Lambda)^2 + C\,(\Lambda+\delta\Lambda) + D + (h_* + \delta h) = 0\,, \tag{3.1.73}$$

which at the order $\mathcal{O}(\delta\Lambda)$ is

$$(4\Lambda^3 + 3A\,\Lambda^2 + 2B\,\Lambda + C)\,\delta\Lambda + \delta h = 0\,. \tag{3.1.74}$$

For the root $\Lambda = i\sqrt{\frac{C}{A}}$, which has led to the BPS black hole entropy, we can solve (3.1.74) and obtain

$$\delta\Lambda = \frac{\delta h}{2C - 2i\sqrt{\frac{C}{A}}\left(B - 2\frac{C}{A}\right)}\,. \tag{3.1.75}$$

Based on (3.1.56), the correction to the BPS black hole entropy is

$$\delta S = -2\pi i\,\delta\Lambda\,. \tag{3.1.76}$$

Hence, only the imaginary part of $\delta\Lambda$ will contribute to the real part of δS. If we assume that δh is purely imaginary, then

$$\begin{aligned}\mathrm{Im}(\delta\Lambda) &= \delta h\,\mathrm{Re}\left[\frac{1}{2C - 2i\sqrt{\frac{C}{A}}\left(B - 2\frac{C}{A}\right)}\right]\\ &= \frac{2C\,\delta h}{\left[2C - 2i\sqrt{\frac{C}{A}}\left(B - 2\frac{C}{A}\right)\right]\left[2C + 2i\sqrt{\frac{C}{A}}\left(B - 2\frac{C}{A}\right)\right]}\\ &= \frac{\delta h}{2C + \frac{2}{A}\left(B - 2\frac{C}{A}\right)^2}\,. \end{aligned} \tag{3.1.77}$$

Therefore, for real-valued δS we have

$$\delta S = -2\pi i\,\mathrm{Im}(\delta\Lambda) = \frac{-\pi i\,\delta h}{C + \frac{1}{A}\left(B - 2\frac{C}{A}\right)^2}\,. \tag{3.1.78}$$

We view δh as a small change of h from its BPS value, i.e.,

$$\delta h = h - h_* = h\,. \tag{3.1.79}$$

We can compute δh by

$$\delta h = \frac{\partial h}{\partial Q_I}\,\delta Q_I + \frac{\partial h}{\partial J}\,\delta J\,, \tag{3.1.80}$$

with the transformations similar to the AdS_5 case [61]

$$\delta Q_I = \eta Q_I\,, \quad \delta J_i = \eta J_i\,. \tag{3.1.81}$$

For the near-extremal case, we relate the transformation parameter η with the temperature change

$$2\pi i\,\delta T_H = 2\eta\,, \tag{3.1.82}$$

where $\delta T_H = T_H - T_H^* = T_H$. Now, we apply (3.1.80) to the explicit choice of h given by (3.1.71). In the unit $G_N = 1$, the near-extremal correction to the BPS entropy for the special case $\delta_1 = \delta_2$ becomes

$$\delta S = \frac{8\sqrt{2}\pi^2\left(e^{4\,\delta_1} - 1\right)^{\frac{3}{2}}}{g^3\left(e^{4\,\delta_1} - 3\right)\left(e^{8\,\delta_1} + 10\,e^{4\,\delta_1} - 7\right)}\,T_H \equiv \left(\frac{C}{T_H}\right)_*\,T_H\,. \tag{3.1.83}$$

Combining the BPS black hole entropy from the boundary CFT (3.1.68) and the near-extremal correction (3.1.83), we obtain the near-extremal AdS_4 black hole entropy from the boundary CFT

$$
\begin{aligned}
S_{BH} &= S_{BH}^{*} + \delta S \\
&= \frac{2\pi}{g^2\left(e^{4\,\delta_1}-3\right)} + \frac{8\sqrt{2}\pi^2\left(e^{4\,\delta_1}-1\right)^{\frac{3}{2}}}{g^3\left(e^{4\,\delta_1}-3\right)\left(e^{8\,\delta_1}+10\,e^{4\,\delta_1}-7\right)}\,T_H \\
&\equiv S_* + \left(\frac{C}{T_H}\right)_* T_H\,,
\end{aligned}
\tag{3.1.84}
$$

which matches perfectly with the results from gravity solution ((3.1.9), (3.1.10) and (3.1.13)) and from the near-horizon Kerr/CFT correspondence ((3.1.49), (3.1.50) and (3.1.51)).

3.2 Hawking Radiation and Near-Extremal AdS_4 Black Hole

In Sec. 3.1, we have derived the near-extremal AdS_4 black hole entropy using three different approaches and obtained one universal result. In particular, the approach of the near-horizon Kerr/CFT correspondence shows that there exists a near-horizon CFT_2, which accounts for the low-energy spectrum of the black hole microstates.

As we have seen in Subsection 3.1.3, the near-extremal black hole entropy can be decomposed into the contributions from the left and the right sectors of the near-horizon CFT_2. The expression from the canonical ensemble is

$$
S_{BH} = \frac{\pi^2}{3} c_L T_L + \frac{\pi^2}{3} c_R T_R\,,
\tag{3.2.1}
$$

which has been discussed extensively for the asymptotically flat black holes [84–91], while the expression from the microcanonical ensemble is [92]

$$
S_{BH} = 2\pi\sqrt{\frac{c_L N_L}{6}} + 2\pi\sqrt{\frac{c_R N_R}{6}}\,,
\tag{3.2.2}
$$

where N_L and N_R are the left and the right mode numbers, respectively. Comparing the expressions (3.2.1) and (3.2.2), we find that the temperatures $T_{L,R}$ can be related to the mode numbers $N_{L,R}$

$$
T_L = \frac{1}{\pi}\sqrt{\frac{6N_L}{c_L}}\,,\quad T_R = \frac{1}{\pi}\sqrt{\frac{6N_R}{c_R}}\,.
\tag{3.2.3}
$$

The explicit expressions of N_L and N_R for near-extremal AdS_4 black holes considered in this work with $\delta_1 = \delta_2$ are

$$
\begin{aligned}
N_L &= \frac{e^{8\,\delta_1} + 10\, e^{4\,\delta_1} - 7}{4\sqrt{2} g^2 \left(e^{4\,\delta_1} - 3\right)^2 \sqrt{e^{4\,\delta_1} - 1}}\,, \\
N_R &= \frac{4\sqrt{2}\pi^2 \left(e^{4\,\delta_1} - 1\right)^{\frac{5}{2}} T_H^2}{g^4 \left(e^{4\,\delta_1} - 3\right)^2 \left(e^{8\,\delta_1} + 10\, e^{4\,\delta_1} - 7\right)}\,.
\end{aligned}
\tag{3.2.4}
$$

Suppose that the left and the right mode numbers in the BPS limit are N_L^* and N_R^* respectively, where

$$
N_R^* = 0\,. \tag{3.2.5}
$$

As discussed in [61, 93], for the near-extremal case the left-moving and the right-moving modes become

$$
\begin{aligned}
N_L &= N_L^* + \delta N_L \approx N_L^*\,, \\
N_R &= N_R^* + \delta N_R = \delta N_R\,,
\end{aligned}
\tag{3.2.6}
$$

with $\delta N_L = \delta N_R \ll N_L^*$. If we assume that the right modes obey a canonical ensemble, then the partition function of the right sector can be written as

$$
Z_R = \sum_{N_R} q^{N_R} d(N_R) = \sum_{N_R} q^{N_R} e^{S_R} = \sum_{N_R} q^{N_R} e^{2\pi\sqrt{c_R N_R/6}}\,. \tag{3.2.7}
$$

We evaluate this partition function using a saddle-point approximation with respect to N_R, and the result is

$$
\delta N_R = N_R = q\frac{\partial}{\partial q}\log Z_R \approx \frac{c_R \pi^2}{6\,\left(\log(q)\right)^2}\,, \quad \text{with} \quad \log(q) < 0\,. \tag{3.2.8}
$$

The occupation number in the right sector is given by Bose-Einstein statistics

$$
\rho_R(k_0) = \frac{q^n}{1 - q^n} = \frac{e^{-\frac{k_0}{T_R}}}{1 - e^{-\frac{k_0}{T_R}}}\,, \tag{3.2.9}
$$

where n is the momentum quantum number of the mode moving in the time circle for the near-horizon region of AdS_4 black holes. From (3.2.8) we can solve for q in terms of $\delta N_R = N_R$, and then combining it with (3.2.9) we obtain

$$
T_R = \frac{k_0}{\pi n}\sqrt{6\,\frac{\delta N_R}{c_R}} = \frac{1}{\pi}\sqrt{\frac{6\,N_R}{c_R}}\,, \tag{3.2.10}
$$

where we used $k_0 = n$. A similar expression holds for T_L, i.e.,

$$T_L = \frac{1}{\pi}\sqrt{\frac{6\,N_L}{c_L}}\,. \tag{3.2.11}$$

We see that (3.2.10) and (3.2.11) are completely consistent with (3.2.3). In the limit $k_0 \sim T_R \ll T_L$, the occupation number in the left sector can be approximated as

$$\rho_L(k_0) = \frac{e^{-\frac{k_0}{T_L}}}{1 - e^{-\frac{k_0}{T_L}}} \approx \frac{T_L}{k_0} = \frac{1}{\pi k_0}\sqrt{\frac{6\,N_L}{c_L}}\,. \tag{3.2.12}$$

According to [61, 93], Hawking radiation can be formulated as a scattering process of left and right modes in the near-horizon CFT_2. Therefore, we can evaluate the Hawking radiation rate for near-extremal AdS_4 black holes based on the analyses above

$$d\Gamma \sim \frac{d^4k}{k_0}\frac{1}{p_0^L\,p_0^R}|\mathcal{A}|^2\,c_L\,\rho_L(k_0)\,\rho_R(k_0)\,, \tag{3.2.13}$$

where the central charge c_L provides the degrees of freedom for a given momentum quantum number n, and $\mathcal{A}$ is the disc amplitude of strings depending on details of the near-horizon CFT_2. From (3.2.12) we see that

$$c_L\,\rho_L(k_0) \sim S_L \propto (\text{horizon area})\,. \tag{3.2.14}$$

Consequently, the Hawking radiation rate becomes

$$d\Gamma \sim (\text{horizon area}) \cdot \frac{e^{-\frac{k_0}{T_R}}}{1 - e^{-\frac{k_0}{T_R}}} d^4k\,, \tag{3.2.15}$$

which implies that the radiation spectrum is thermal and governed by a temperature T_R proportional to the Hawking temperature T_H. Therefore, we have found a microscopic formalism of Hawking radiation in the near-horizon CFT_2. According to this picture, the scattering of modes is unitary; hence there is no information loss during the Hawking radiation process.

Since the boundary CFT can exactly reproduce the near-extremal black hole entropy (3.2.1), this microscopic formalism of Hawking radiation can in principle be embedded in higher-dimensional boundary CFT, which is the 3d superconformal ABJM theory for AdS_4 black holes.

Like the AdS_5 case discussed in [61], we have not taken into account the global structure of AdS space. Particularly, due to the conformal boundary of AdS space, once the radiation reaches the boundary, it will bounce back and head towards the black hole. Therefore, our current model provides a microscopic description for the Hawking radiation immediately after creation. We leave the full evolution of Hawking radiation for future work.

3.3 Discussion

We have studied the electrically charged rotating AdS_4 black holes in the near-extremal limit. Moreover, by studying the parameter space we have successfully defined a way to approach near-extremal supersymmetric black holes. We have then computed the entropy using three different approaches: (i) from the gravity solution, (ii) from the near-horizon CFT_2 via the Kerr/CFT correspondence and (iii) from the boundary CFT via the AdS/CFT correspondence. Remarkably, these three results match precisely, giving us a universal and unique expression for the entropy in the near-extremal limit. This supports the near-extremal microstate counting in the boundary CFT and in the near-horizon CFT_2. We also have shown that the extension of the Kerr/CFT correspondence, originally posed for extremality, to near-extremal black holes is valid. Using the results of near-extremal black hole entropy, we provide a microscopic description of Hawking radiation, and qualitatively show that unitarity and information are preserved during the Hawking radiation process.

The success of this work provides motivation to further study near-extremality in other dimensions and indeed show that the three diverse entropy computations lead to one universal entropy. Besides the near-extremal AdS_5 black holes discussed in [61] and the AdS_4 case discussed in this chapter, we can also consider the known AdS_6 and AdS_7 [94] black hole solutions. Similar results from different approaches listed in Fig. 1.2 are expected. Moreover, the unifying picture Fig. 1.2 can potentially be valid beyond the Bekenstein-Hawking entropy. Hence, it would be interesting to study the subleading corrections to the Bekenstein-Hawking entropy and see if the different approaches still provide a unique expression for the entropy, in the same spirit of [48, 95–104]. A recent work [65] shows that Sen's classical entropy function formalism [105] can be applied to asymptotically AdS_4 black holes to capture higher derivative corrections to the Bekenstein-Hawking entropy, which complements the methods in Fig. 1.2.

Besides the microstate counting of black holes, a more interesting question is how to use field theory techniques to study dynamical process in black hole physics. For instance, Hawking radiation on asymptotically AdS black holes has been studied within the framework of AdS/CFT correspondence previously in [106, 107]. Some recent progress has been made for microscopic description of Hawking-Page transition [108]. Another related problem is to reproduce the Page curve in the black hole evaporation process [109–111], which has been studied in the framework of 2d JT gravity coupled to a 2d bath CFT [112–115]. Our approach in [61] and in this chapter provides another powerful framework of studying these problems. In order to do that, however, we have to first carefully study the Hawking radiation at a later time in the dual boundary field theory and in the near-horizon CFT_2 to resolve the issues from the global property of AdS space. We hope to refine our microscopic models and study these more physical problems in the near future.

Part II

Logarithmic Corrections to AdS Black Hole Entropy

Chapter 4

Five Dimensional AdS Black Objects

4.1 AdS$_5$ Black Holes

4.1.1 The Superconformal Index and Black Hole Entropy

An efficient way to count $\frac{1}{16}$-BPS states in $\mathcal{N}=4$ SYM is to consider the theory on $S^1 \times S^3$ and evaluate the superconformal index (SCI) [12, 116]:

$$\mathcal{I}(\tau;\Delta) = \mathrm{Tr}\left[(-1)^F\, e^{-\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}} v_a^{Q_a} p^{J_1+\frac{r}{2}} q^{J_2+\frac{r}{2}}\right], \tag{4.1.1}$$

where β is the circumference of S^1, and F is the fermionic operator, while $Q_{a=1,2,3}$ are flavor charges with associated fugacities $v_a = e^{2\pi i \Delta_a}$. With r we denote the R-charge. The fugacities $p = e^{2\pi i\tau}$ and $q = e^{2\pi i \sigma}$ are associated to the angular momenta $J_{1,2}$ of S^3, and the combinations $J_{1,2} + \frac{r}{2}$ commute with the supercharge $\mathcal{Q}$. In what follows we set $\tau = \sigma$ for simplicity. Note that the counting of states that the SCI offers should be seen as performed in the grand-canonical ensemble, since we are keeping fixed the values of chemical potentials while summing over all possible charges.

According to the AdS/CFT correspondence, $SU(N)$ $\mathcal{N}=4$ SYM is dual to type IIB supergravity on AdS$_5 \times S^5$, in which one can find supersymmetric black hole solutions that are asymptotically AdS, rotating and electrically charged. Remarkably, in recent years plenty of evidences have been gathered indicating that $\mathcal{I}(\tau;\Delta)$ captures the entropy of such black holes [13–15] (see [30–32, 68, 69, 117, 118] for further developments and [103, 119] for a more complete list of references).

The SCI can be written as a contour integral over the holonomies of the gauge group

[120, 121]:

$$
\begin{aligned}
\mathcal{I}\left(\tau;\Delta\right) &= \kappa_N \int_0^1 \prod_{\mu=1}^{N-1} du_\mu \mathcal{Z}\left(u;\Delta,\tau\right) \,, \\
\mathcal{Z}\left(u;\Delta,\tau\right) &= \frac{\prod_{a=1}^{3}\prod_{i\neq j}\widetilde{\Gamma}\left(u_{ij}+\Delta_a;\tau\right)}{\prod_{i\neq j}\widetilde{\Gamma}\left(u_{ij};\tau\right)} \,, \\
\kappa_N &= \frac{(p;p)_\infty^{N-1}\,(q;q)_\infty^{N-1}}{N!} \prod_{a=1}^{3}\left(\widetilde{\Gamma}(\Delta_a;\tau)\right)^{N-1} \,,
\end{aligned}
\tag{4.1.2}
$$

where $(\cdot\,;\cdot)_\infty$ is the Pochhammer symbol, and $\widetilde{\Gamma}(u;\tau)$ is the elliptic Gamma function defined both in Appendix C.1. There are two main approaches to evaluate the N-dimensional integral over the holonomies of the gauge group representing the SCI. The first approach relies on a direct application of the residue theorem, and yields what is known in the literature as the Bethe-Ansatz method. The second approach implements a saddle-point evaluation of the integral.

The Bethe-Ansatz Approach

The location of the poles of (4.1.2) is given by the solutions to the set of equations:

$$
Q_k(\hat{u};\Delta,\tau) = 1, \ \ \forall \ k = 1,\cdots,N \,,
\tag{4.1.3}
$$

where

$$
Q_k(u;\Delta,\tau) = e^{2\pi i\lambda} \prod_{l=1(\neq k)}^{N} \prod_{a=1}^{3} \frac{\theta_1(-u_{kl}+\Delta_a;\tau)}{\theta_1(u_{kl}+\Delta_a;\tau)}
\tag{4.1.4}
$$

are the Bethe-Ansatz operators and the values $\hat{u}$ satisfying (4.1.3) are called Bethe-Ansatz solutions. We then define $\mathrm{BA} = \{\hat{u} \mid (4.1.3) \text{ is satisfied}\}$. With λ we have denoted a Lagrange multiplier implementing the $SU(N)$ constraint on the holonomies $\sum_{i=1}^{N} u_i \in \mathbb{Z}$, and $\theta_1(u;\tau)$ is the elliptic theta function defined in Appendix C.1. Upon direct application of the residue theorem, $\mathcal{I}(\tau,\Delta)$ can be rewritten in terms of a discrete sum as:

$$
\begin{aligned}
\mathcal{I}(\tau;\Delta) &= \kappa_N \sum_{\hat{u}\in\mathrm{BA}} \mathcal{Z}(\hat{u};\Delta,\tau) H(\hat{u};\Delta,\tau)^{-1} \,, \\
H\left(\hat{u};\Delta,\tau\right) &= \det\left[\frac{1}{2\pi i}\frac{\partial\left(Q_1,\cdots,Q_N\right)}{\partial\left(u_1,\cdots,u_{N-1},\lambda\right)}\right] .
\end{aligned}
\tag{4.1.5}
$$

Let us emphasize that (4.1.5) is not the full story, since the application of the residue theorem required the poles to be isolated, and there is enough evidence by now [122, 123] that this is not the case generically. We shall focus only on the contributions coming from isolated poles (see [124] for more detailed discussions on this point). A set of solutions to the equations (4.1.3) was found in [125] and it is given by:

$$
\begin{aligned}
&u_i = u_{\hat{j},\hat{k}} = \bar{u} + \frac{\hat{j}}{m} + \frac{\hat{k}}{n}\left(\tau + \frac{r}{m}\right) \,, \\
&\hat{j} = 0, \cdots, m-1\,, \qquad \hat{k} = 0, \cdots, n-1\,, \\
&r = 0, \cdots, n-1\,,
\end{aligned}
\tag{4.1.6}
$$

where $N = mn$, hence, each set $\{u_i\}$ in (4.1.6) can be labeled by the numbers $\{m, n, r\}$. These solutions to the Bethe-Ansatz equations for the SCI were, in fact, inspired by the set of solutions found in [125] for the Bethe-Ansatz equations associated to the topologically twisted index. We will discuss that case in Sec. 4.2.1. However, in the large-N limit, it was possible to argue that the configuration corresponding to $\{1, N, 0\}$ contributed dominantly to the SCI. We shall refer to the $\{1, N, 0\}$ solution as the "basic" solution, namely

$$
\hat{u}_{\text{basic}} = \left\{ u_i = \bar{u} + \frac{i}{N}\tau \,\middle|\, i = 1, 2, \cdots, N-1 \right\} \bigcup \{u_N = \bar{u}\} \,, \tag{4.1.7}
$$

where $\bar{u}$ is determined as

$$
N\bar{u} + \frac{N(N-1)}{2N}\tau \in \mathbb{Z}\,. \tag{4.1.8}
$$

The parameter $\bar{u}$ enforces the $SU(N)$ constraint $\sum_{i=1}^{N} u_i \in \mathbb{Z}$ on the holonomies, which, together with the periodicity properties of $\mathcal{I}(\tau;\Delta)$ allows us to obtain:

$$
\bar{u} = \frac{k}{N} - \frac{N-1}{2N}\tau\,, \qquad k = 1, \cdots, N-1\,, \tag{4.1.9}
$$

each of which contributes identically to the $\mathcal{I}(\tau;\Delta)$, we therefore have that, in the appropriate regime of chemical potentials:

$$
\log \mathcal{I}(\tau;\Delta)\big|_{\text{Basic BA}} = -\frac{i\pi(N^2-1)}{\tau^2}\Delta_1\Delta_2\Delta_3 + \log N \, + \mathcal{O}(N^0). \tag{4.1.10}
$$

We see that the $\log N$ in (4.1.10) has a purely combinatorial origin, whose precise form is quite insensitive to details about the theory in which $\mathcal{I}(\tau;\Delta)$ is being evaluated.

The Saddle Point Approach and the Cardy-Like Expansion

Let us further reinforce the idea that the logarithmic correction to the SCI has a combinatorial origin. To do so we briefly reproduce here the saddle-point evaluation of (4.1.2) implemented in [103].

The strict Cardy-like limit

By the strict Cardy-like limit we mean that we keep only the most divergent term in a $\tau \to 0$ expansion[1]. The study of the strict Cardy-like limit was the subject of several works [14, 26–30], and the main idea is to rewrite (4.1.2) in the following way:

$$\mathcal{I}(\tau;\Delta) = \kappa_N \int \prod_{\mu=1}^{N-1} du_\mu \exp\left(\frac{1}{\tau^2} S_{\text{eff}}(u;\Delta,\tau)\right) , \tag{4.1.11}$$

where $S_{\text{eff}}(u;\Delta,\tau)$ is appropriately defined such that $\mathcal{Z}(u;\Delta,\tau)$ in (4.1.2) is recovered. The $\frac{1}{\tau^2}$ factor can be used as a large control parameter to apply the saddle-point method in the strict Cardy-like limit. We are exploiting the fact that we already know the leading contribution in such limit is precisely of the order $\mathcal{O}\left(\frac{1}{\tau^2}\right)$. The saddle-point equations have the form:

$$\frac{\partial}{\partial u_\mu} S_{\text{eff}}(u;\Delta,\tau)\bigg|_{u=\text{saddle}} = 0\,, \qquad (\mu = 1,\cdots,N-1)\,. \tag{4.1.12}$$

The set with all identical holonomies, namely $u_i = u_j$ for all $i,j \in \{1,\cdots,N\}$ [14, 26] is one of the most well-known solutions to (4.1.12). The effective action at this saddle point successfully counted the dual AdS_5 black hole microstates [14].

There are N distinct sets of identical holonomies satisfying the $SU(N)$ constraint $\sum_{i=1}^N u_i \in \mathbb{Z}$, namely

$$u^{(m)} = \left\{ u_j^{(m)} = \frac{m}{N} \,\middle|\, j = 1,\cdots,N \right\} , \qquad (m = 0,1,\cdots,N-1)\,. \tag{4.1.13}$$

Within the appropriate range of chemical potentials, the saddle points (4.1.13) yield the following effective action:

$$\frac{1}{\tau^2}\sum_{m=0}^{N-1} S_{\text{eff}}(u^{(m)};\Delta,\tau) = \exp\left(-\frac{i\pi(N^2-1)}{\tau^2}\Delta_1\Delta_2\Delta_3 + \log N + \mathcal{O}(-1/|\tau|)\right) . \tag{4.1.14}$$

[1] More refined limits were considered in [126]. The authors discussed the limit where $q = e^{2\pi i\tau}$ approaches roots of unit.

From (4.1.13) and (4.1.14) we see that the logarithmic correction has its origin in the multiplicity of the saddle points. This result remains true even for more generic $\mathcal{N}=1$ toric quiver gauge theories, as emphasized in [103], which renders the $\log N$ correction a quite robust one. Note that we have not made use of the large-N limit here, therefore, provided that we remain at small values of τ, (4.1.14) holds for finite N (Evidence in favor of this has been given in [122]).

The Cardy-like expansion

With the Bethe-Ansatz approach we have learned that even for generic values of τ, in the large-N limit, the $\log N$ is the same and arises from degeneracies of Bethe-Ansatz solutions. At this point we have shown that also for finite N, in the strict Cardy-like limit, the $\log N$ has a combinatorial origin.

We now proceed to include subleading corrections in inverse powers of τ and show that, indeed, the $\log N$ remains unchanged. This is an important step, since it helps us build an intuition that we later import to a different situation, namely the refined topologically twisted index, where we have only access to the strict Cardy-like limit and argue about the possibility of the combinatorial nature of $\log N$ to remain true as we depart from this limit. We then focus on the effective action evaluated near the leading saddle-point solution (4.1.13). Following [103], we make the Ansatz for saddle-point solutions in the finite Cardy-like expansion,

$$u^{(m)} = \left\{ u_j^{(m)} = \frac{m}{N} + v_j\tau \;\middle|\; v_j \sim \mathcal{O}(|\tau|^0),\ \sum_{j=1}^{N} v_j = 0 \right\}, \quad (m = 0, 1, \cdots, N-1)\,, \quad (4.1.15)$$

and evaluate the effective action around this Ansatz. For a suitable choice of chemical potentials, the following expression was obtained:

$$\log \mathcal{I}(\tau;\Delta) = -\frac{i\pi(N^2-1)}{\tau^2}\Delta_1\Delta_2\Delta_3 + \log N \, + \mathcal{O}(e^{-1/|\tau|})\,. \quad (4.1.16)$$

The exponentially suppressed correction comes from the asymptotic expansion of the building blocks of the effective action, namely elliptic Gamma functions and Pochhammer symbols (see Appendix C.1.1). An important aspect about (4.1.16) is that it includes all power-like corrections in τ, and rather remarkably, it is a series that truncates at the leading order. A prominent role in the technical evaluation of (4.1.16) was the cancellation of the $\mathcal{O}(\tau^0)$ contribution which was given in terms of the effective action of a matrix model of $SU(N)$

level $k = N$ Chern-Simons theory on S^3 (see [103] for a more detailed discussion).

What seems like a rather technical step when analyzed from the strictly mathematical perspective of looking at the asymptotic behavior of $S_{\text{eff}}(u;\Delta,\tau)$, becomes very natural when viewed from an effective field theory perspective. Such analysis was carried out in [127], where the Cardy-like (small τ) expansion was shown to geometrically correspond to shrinking the S^1 circle, thus leading to an effective field theory on S^3 organized in inverse powers of the circumference of S^1.

In particular, a careful treatment of the Kaluza-Klein reduction on S^1 yields a result compatible with (4.1.16), where the $\log N$ is associated to degeneracies of vacua. This effective field theory approach clarifies the organization of the index in inverse powers of $|\tau|$ and further confirms the logarithmic term as certain degeneracy of vacua [126, 127]. Specifically, the effective field theory approach allows to establish the existence of a minimum of $S_{\text{eff}}(u;\Delta,\tau)$ at $u = 0$, which spontaneously breaks the one-form symmetry $\mathbb{Z}_N$ of the 4d $\mathcal{N} = 4$ SYM theory. The fact that $u = 0$ spontaneously breaks $\mathbb{Z}_N$ implies the existence of exactly $N - 1$ additional local minima which contribute equally to the index, hence the $\log N$ correction to the logarithm of the SCI.

Summarizing, the logarithmic correction to the logarithm of the SCI, which we refer to as $\Delta\log\mathcal{I}_{\text{CFT}_4}$, has been shown to be robust. In [103] it was originally obtained using two different approaches to evaluate the index: the saddle-point approximation and the Bethe-Ansatz approach. In the latter approach, the logarithmic term appears as the degeneracy of the Bethe-Ansatz solutions. The same logarithmic contribution was also shown to persist for a large class of $\mathcal{N} = 1$ superconformal field theories. The form of the logarithmic correction was further confirmed in [128], which provides an interpretation for certain exponentially suppressed terms. In [129], the logarithmic corrections were extended to other gauge groups and the results were shown to be compatible with the $SU(N)$ analysis.

The black hole entropy is extracted from the SCI by implementing an inverse Laplace transformation, which yields the degeneracy of a state with given energy and charges. In the regime of large charges, we can reduce the inverse Laplace transformation to a Legendre transformation using the saddle point approximation. This is tantamount to changing from a grand-canonical ensemble to a microcanonical one. At the leading order in N, the two ways of approaching the entropy should be equivalent. However, when studying subleading structures we have to be more careful, since the very process of going from one ensemble to the other could modify the subleading corrections we are trying to probe. To be more specific, let us call ΔS_{CFT_4} the subleading logarithmic correction to the black hole entropy. Then we expect that in general $\Delta S_{\text{CFT}_4} = \Delta\log\mathcal{I}_{\text{CFT}_4} +$ (corrections from changing ensemble). Let us now study more carefully the contribution coming from the change of ensemble.

4.1.2 The Logarithmic Correction Associated to Changing Ensemble

We denote $\mathcal{I}_{\text{GC}}$ as the index computeded in the grand-canonical ensemble and $\mathcal{I}_{\text{MC}}$ as the index in the microcanonical ensemble, i.e., the index for fixed values of the charges. We consider D chemical potentials μ_I ($I = 1, \cdots, D$) satisfying the constraint,

$$\sum_{I=1}^{D} c_I \mu_I = n_0, \tag{4.1.17}$$

where $c_I = 1$ for μ_I associated to electric charges and $c_I = -1$ for μ_I associated to angular momenta. We implement the inverse Laplace transform which takes us from the grand-canonical ensemble to the microcanonical ensemble

$$\mathcal{I}_{\text{MC}} = \int d^D\mu \, d\Lambda \exp\left\{\log \mathcal{I}_{\text{GC}} - \sum_{I=1}^{D} Q_I \mu_I - \Lambda\left(\sum_{I=1}^{D} c_I \mu_I - n_0\right)\right\}, \tag{4.1.18}$$

where Λ is the Lagrange multiplier associated to the constraint (4.1.17). Note that we have already considered the case of equal angular momenta when computing the index in the grand-canonical ensemble. Otherwise, we would have also needed an additional Lagrange multiplier accounting for the constraint among rotations. We know the logarithmic corrections in the grand-canonical ensemble takes the form

$$\log \mathcal{I}_{\text{GC}} = \log \mathcal{I}_{\text{GC}}^{(\text{leading})} + \log N. \tag{4.1.19}$$

Imposing (4.1.19), we find that the index in the microcanonical ensemble takes the form

$$\mathcal{I}_{\text{MC}} = N \int d^D\mu \, d\Lambda \exp\left[\log \mathcal{I}_{\text{GC}}^{(\text{leading})} - \sum_{I=1}^{D} Q_I \mu_I - \Lambda\left(\sum_{I=1}^{D} c_I \mu_I - n_0\right)\right]. \tag{4.1.20}$$

We are now ready to implement the saddle point method, keeping the subleading logarithmic corrections associated to the one-loop determinant. The saddle point equations are given as

$$\begin{aligned}
\frac{\partial}{\partial \mu_I}\left[\log \mathcal{I}_{\text{GC}}^{(\text{leading})} - \sum_{I=1}^{D} Q_I \mu_I - \Lambda\left(\sum_{I=1}^{D} c_I \mu_I - n_0\right)\right] &= 0, \\
\frac{\partial}{\partial \Lambda}\left[\log \mathcal{I}_{\text{GC}}^{(\text{leading})} - \sum_{I=1}^{D} Q_I \mu_I - \Lambda\left(\sum_{I=1}^{D} c_I \mu_I - n_0\right)\right] &= 0,
\end{aligned} \tag{4.1.21}$$

which leads to

$$\begin{aligned}
&\frac{\partial \log \mathcal{I}_{\rm GC}^{(\rm leading)}}{\partial \mu_I} = Q_I + c_I \Lambda, \\
&\sum_{I=1}^{D} c_I \mu_I = n_0.
\end{aligned} \tag{4.1.22}$$

A very important property of $\log \mathcal{I}_{\rm GC}^{(\rm leading)}$ is its homogeneity of degree one in the chemical potentials. This implies the following crucial relation

$$\log \mathcal{I}_{\rm GC}^{(\rm leading)} = \sum_{I=1}^{D} \mu_I \frac{\partial \log \mathcal{I}_{\rm GC}^{(\rm leading)}}{\partial \mu_I}. \tag{4.1.23}$$

Evaluating at the saddle point values, we obtain

$$\log \mathcal{I}_{\rm GC}^{\star(\rm leading)} = \sum_{I=1}^{D} \mu_I^\star \left(Q_I + c_I \Lambda\right), \tag{4.1.24}$$

such that the saddle point imposed on (4.1.20) yields

$$\begin{aligned}
\mathcal{I}_{\rm MC} &\approx N \exp\left\{\log \mathcal{I}_{\rm GC}^{\star(\rm leading)} - \sum_{I=1}^{D} Q_I \mu_I^\star - \Lambda\left(\sum_{I=1}^{D} c_I \mu_I^\star - n_0\right) - \frac{1}{2}\log\det\left(H\right)\right\} \\
&= N \exp\left\{\sum_{I=1}^{D} \mu_I^\star \left(Q_I + c_I\Lambda\right) - \sum_{I=1}^{D} Q_I \mu_I^\star - \Lambda\left(\sum_{I=1}^{D} c_I \mu_I^\star - n_0\right) - \frac{1}{2}\log\det\left(H\right)\right\} \\
&= N e^{n_0 \Lambda - \frac{1}{2}\log\det(H)}.
\end{aligned} \tag{4.1.25}$$

The Hessian H has the form

$$H = \begin{pmatrix}
\frac{\partial^2 \log \mathcal{I}_{\rm GC}^{(\rm leading)}}{\partial \mu_1^2} & \cdots & \frac{\partial^2 \log \mathcal{I}_{\rm GC}^{(\rm leading)}}{\partial \mu_1 \partial \mu_D} & \frac{\partial^2 \log \mathcal{I}_{\rm GC}^{(\rm leading)}}{\partial \mu_1 \partial \Lambda} \\
. & . & . & . \\
. & . & . & . \\
\frac{\partial^2 \log \mathcal{I}_{\rm GC}^{(\rm leading)}}{\partial \mu_D \partial \mu_1} & \cdots & \frac{\partial^2 \log \mathcal{I}_{\rm GC}^{(\rm leading)}}{\partial \mu_D^2} & \frac{\partial^2 \log \mathcal{I}_{\rm GC}^{(\rm leading)}}{\partial \mu_D \partial \Lambda} \\
\frac{\partial^2 \log \mathcal{I}_{\rm GC}^{(\rm leading)}}{\partial \Lambda \partial \mu_1} & \cdots & \frac{\partial^2 \log \mathcal{I}_{\rm GC}^{(\rm leading)}}{\partial \Lambda \partial \mu_D} & \frac{\partial^2 \log \mathcal{I}_{\rm GC}^{(\rm leading)}}{\partial \Lambda^2}
\end{pmatrix}. \tag{4.1.26}$$

Since $\log \mathcal{I}_{\rm GC}^{(\rm leading)}$ is a homogeneous function of degree one, the chemical potentials can appear either in the numerator or the denominator in a way that the second derivative terms appearing along the diagonal of H vanish when μ_I appears in the denominator. To keep track of this we define a list of numbers $\{\delta_1, \cdots, \delta_D\}$ such that δ_I vanishes when μ_I is in the numerator of $\log \mathcal{I}_{\text{GC}}^{(\text{leading})}$ and it is equal to one otherwise. This implies the following scaling of H

$$\det H \sim \det \begin{pmatrix} \mathcal{O}(N^2)\delta_1 & \cdots & \mathcal{O}(N^2) & c_1 \\ . & . & . & . \\ . & . & . & . \\ \mathcal{O}(N^2) & \cdots & \mathcal{O}(N^2)\delta_D & c_D \\ c_1 & \cdots & c_D & 0 \end{pmatrix} \sim \mathcal{O}(N^{2(D-1)}). \tag{4.1.27}$$

Defining $D = d + 1$, where d is the number of independent chemical potentials, the index computed in the microcanonical ensemble up to logarithmic corrections takes the form

$$\log \mathcal{I}_{\text{MC}} \approx n_0 \Lambda + (1 - d) \log N. \tag{4.1.28}$$

Since the chemical potentials are constrained by having equal angular momenta as well as the BPS condition, we have $d = 3$ and therefore the logarithmic correction in the microcanonical ensemble is

$$\Delta S_{\text{CFT}_4} = -2 \log N. \tag{4.1.29}$$

We expect this 4-dimensional result to match with the subleading correction coming from the 2-dimensional Cardy formula.

4.1.3 Black Hole, Its Entropy and Near-Horizon Limit

The non-extremal asymptotically AdS_5 black hole background was found in [24]. In the Boyer-Lindquist coordinates $x^\mu = (t,\, r,\, \theta,\, \phi,\, \psi)$, the metric and the gauge field are given by [2]

$$\begin{aligned} ds^2 &= -\frac{\left[(1 + g^2 r^2)\rho^2 dt + 2q\nu\right] dt}{\Xi \rho^2} + \frac{2q}{\rho^2 \Xi}\nu^2 + \frac{f}{\rho^4 \Xi^2}\left(dt - \nu\right)^2 \\ &\quad + \frac{\rho^2 dr^2}{\Delta_r} + \frac{\rho^2}{\Xi}\left(d\theta^2 + \sin^2\theta \, d\phi^2 + \cos^2\theta \, d\psi^2\right), \end{aligned} \tag{4.1.30}$$

$$A = \frac{\sqrt{3}\, q}{\rho^2 \Xi}\left(dt - \nu\right), \tag{4.1.31}$$

[2] For simplicity, we consider the black hole with equal angular momenta $J_1 = J_2$ and equal electric charges $Q_1 = Q_2 = Q_3$.

where

$$\begin{aligned}
\nu &\equiv a\,(\sin^2\theta\, d\phi + \cos^2\theta\, d\psi)\,, \quad \Xi \equiv 1 - a^2 g^2\,,\\
\Delta_r &\equiv \frac{(r^2+a^2)^2(1+g^2r^2) + q^2 + 2a^2 q}{r^2} - 2m\,, \\
\rho^2 &\equiv r^2 + a^2, \quad f \equiv 2m\rho^2 - q^2 + 2a^2 q g^2 \rho^2\,.
\end{aligned} \tag{4.1.32}$$

These black hole solutions are characterized by three independent parameters (m, a, q), and g is the inverse radius of AdS_5.

We are ultimately interested in exploring the black hole solution for the parameter space satisfying supersymmetry and extremality, i.e. BPS. The supersymmetric limit corresponds to

$$q = \frac{m}{1+2ag}\,. \tag{4.1.33}$$

However, this is not enough to ensure physical solutions and therefore we must also consider an additional constraint to prevent naked closed timelike curves, which in the BPS limit takes the form

$$m = \frac{2a(1+ag)^2(1+2ag)}{g}\,. \tag{4.1.34}$$

Extremality occurs when the inner horizon and the outer horizon coincide, which for our solution gives the double root

$$r_0^2 = \frac{a(2+ag)}{g}\,. \tag{4.1.35}$$

The macroscopic Bekenstein-Hawking entropy for the supersymmetric black hole, computed as a quarter of the area of the horizon (in units of $G_N = 1$), is

$$S_{\mathrm{BH}} = \frac{\pi^2 a^{3/2}\sqrt{2+ag}}{g^{3/2}(1-ag)^2} = 2\pi\sqrt{\frac{3Q^2}{g^2} - \frac{\pi}{2g^3}J}\,, \tag{4.1.36}$$

where we have written it explicitly in terms of the electric charge, Q, and the angular momentum, J. The remarkable achievement of [13–15] was to obtain this expression for the black hole entropy as the Legendre transform of the leading N^2-part of the SCI (4.1.16), thus providing it with a microscopic explanation.

Given that the AdS/CFT correspondence geometrizes RG flow in the radial direction, it is convenient to consider zooming into a near-horizon region (IR), r_0, while assuming a co-rotating frame:

$$r \to r_0 + \lambda \widetilde{r}\,, \quad t \to \frac{\widetilde{t}}{\lambda}\,, \quad \phi \to \widetilde{\phi} + g\frac{\widetilde{t}}{\lambda}\,, \quad \psi \to \widetilde{\psi} + g\frac{\widetilde{t}}{\lambda}\,, \tag{4.1.37}$$

where we have also imposed both (4.1.33) and (4.1.34). Taking $\lambda \to 0$ brings us to a near-horizon region of the AdS_5 BPS black hole:

$$\begin{aligned} ds^2 = {}& \alpha_1 \left[-\widetilde{r}^2\, d\tau^2 + \frac{d\widetilde{r}^2}{\widetilde{r}^2} \right] + \Lambda_1(\theta) \left[d\widetilde{\phi} + \alpha_2\, \widetilde{r}\, d\tau \right]^2 \\ & + \Lambda_2(\theta) \left[d\widetilde{\psi} + \beta_1(\theta) d\widetilde{\phi} + \beta_2(\theta)\, \widetilde{r}\, d\tau \right]^2 + \alpha_3\, d\theta^2\,, \end{aligned} \tag{4.1.38}$$

where

$$\begin{aligned} \alpha_1 &= \frac{a}{2g(1+5ag)}\,, \\ \alpha_2 &= \frac{3a(1-ag)}{2(1+5ag)\sqrt{a\left(a+\frac{2}{g}\right)}}\,, \\ \alpha_3 &= \frac{2a}{g(1-ag)}\,, \\ \Lambda_1(\theta) &= \frac{4a(2+ag)\sin^2\theta}{g(1-ag)(4-ag+3ag\cos(2\theta))}\,, \\ \Lambda_2(\theta) &= \frac{a(4-ag+3ag\cos(2\theta))\cos^2\theta}{2g(1-ag)^2}\,, \\ \beta_1(\theta) &= \frac{6ag\sin^2\theta}{4-ag+3ag\cos(2\theta)}\,, \\ \beta_2(\theta) &= \frac{3g(1-ag)\sqrt{a\left(a+\frac{2}{g}\right)}}{(1+5ag)(4-ag+3ag\cos(2\theta))}\,. \end{aligned} \tag{4.1.39}$$

It is in the near-horizon limit at extremality where we find that the near-horizon geometry is locally a $U(1)^2$-bundle over AdS_2. The asymptotic symmetries of this space can be studied via the Kerr/CFT correspondence, which associates to each $U(1)$-fiber in (4.1.38) a central charge and an effective temperature in the CFT_2. We can apply the Kerr/CFT correspondence to either $U(1)$, and the results of the black hole entropy from the Cardy formula are the same [18, 79].

4.1.4 Kerr/CFT Correspondence and Charged Cardy Formula

Let us briefly review the Cardy formula which determines the degeneracy of states in a CFT_2. We are interested in its application up to and including the logarithmic corrections to the degeneracy of states, with constraints among the charges and chemical potentials. We consider the partition function of a CFT_2 with n global $U(1)$ symmetries expressed in the grand-canonical ensemble

$$Z(\tau, \bar{\tau}, \vec{\mu}) = \mathrm{Tr}\, e^{2\pi i \tau L_0 - 2\pi i \bar{\tau} \bar{L}_0 + 2\pi i \mu_i P^i}\,, \tag{4.1.40}$$

where P^i are the conserved charges of the global $U(1)$'s, and μ_i are the corresponding chemical potentials. One particular property of the CFT_2 with conserved currents is that under modular transformations

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d}\,, \quad \mu_i \to \mu_i' = \frac{\mu_i}{c\tau + d}\,, \quad i = 1, \cdots, n, \tag{4.1.41}$$

the partition function transforms as (in a special choice of normalization)

$$Z\left(\tau', \vec{\mu}'\right) = e^{-2\pi i \left(\frac{c\mu^2}{c\tau + d}\right)} Z(\tau, \vec{\mu})\,. \tag{4.1.42}$$

Therefore, the modular invariance of the partition function requires

$$Z(\tau, \bar{\tau}, \vec{\mu}) = e^{-\frac{2\pi i \mu^2}{\tau}}\, Z\left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}, \frac{\vec{\mu}}{\tau}\right)\,, \tag{4.1.43}$$

where $\mu^2 \equiv \mu_i \mu_j k^{ij}$ with k^{ij} denoting the matrix of the Kac-Moody levels of the $U(1)$ currents. The modular invariance (4.1.43) implies that for small τ

$$Z(\tau, \bar{\tau}, \vec{\mu}) \approx e^{-\frac{2\pi i \mu^2}{\tau}}\, e^{-\frac{2\pi i E_L^v}{\tau} + \frac{2\pi i E_R^v}{\bar{\tau}} + \frac{2\pi i \mu_i p_v^i}{\tau}}\,, \tag{4.1.44}$$

where E_L^v, E_R^v and p_v^i are the lowest eigenvalues of L_0, $\bar{L}_0$ and P^i respectively. Moreover, we take E_L^v, E_R^v to be negative, and $p_v^i = 0$, corresponding to an electrically neutral vacuum.

Let us take a moment to understand the charges p_i of the theory, which include the angular momenta p_1, p_2 and the electric charges p_3, p_4, p_5, originally coming from the AdS_5 black hole solution. Particularly, for the 5d BPS black hole of interest, such charges obey a linear constraint of the generic form

$$\sum_{i=1}^{5} b^i p_i = M, \tag{4.1.45}$$

where b^i are some constant coefficients and M is related to the mass of the black hole. Therefore, (4.1.45) implements the BPS conditions (4.1.33) and (4.1.34). Since we are considering $p^i \sim \mathcal{O}(N^0)$, it can be seen from (4.1.45) that $M \sim N^0$.

In the grand-canonical ensemble, we fix chemical potentials and admit all values of charges. We consider a linear constraint among chemical potentials

$$\sum_{i=1}^{5} s^i \mu_i = C\,, \tag{4.1.46}$$

where C is a constant of the order $\mathcal{O}(N^0)$. The constraint (4.1.17) is a special case of (4.1.46). As we are going to see, this leads to the result that in terms of the scaling of N, $s^i \sim k_{ii}^{-1}$. Moreover, in order to compare to the CFT$_4$ with equal angular momenta, we consider an additional constraint of the form

$$\sum_{i=1}^{2} \alpha^i \mu_i = 0, \tag{4.1.47}$$

of which the constraint $\mu_1 = \mu_2$ is a special case. To clarify how we use these constraints to derive the logarithmic corrections, we carefully change to the microcanonical ensemble by integrating over chemical potentials while respecting the constraints (4.1.46) and (4.1.47). The density of states $\rho(\tau, \bar{\tau}, \vec{\mu})$ can be expressed as the inverse Fourier transform of $Z(\tau, \bar{\tau}, \vec{\mu})$

$$\rho(E_L, E_R, \vec{p}) = \int d\tau\, d\bar{\tau}\, d^n\mu\, d\lambda_1\, d\lambda_2\, \exp\left[2\pi i S(\mu, \tau, \bar{\tau}) + 2\pi i \lambda_1 \left(\sum_{i=1}^{5} s^i \mu_i - C\right) + 2\pi i \lambda_2 \sum_{i=1}^{2} \alpha^i \mu_i\right], \tag{4.1.48}$$

$$S(\mu, \tau, \bar{\tau}) = -\frac{\mu^2}{\tau} - \frac{E_L^v}{\tau} + \frac{E_R^v}{\bar{\tau}} + \frac{\mu_i p_v^i}{\tau} - \tau E_L + \bar{\tau} E_R - \mu_i p^i\,, \tag{4.1.49}$$

where E_L, E_R and p^i are the eigenvalues of L_0, $\bar{L}_0$ and P^i, respectively, and n denotes the number of independent chemical potentials. Before we proceed, we take a moment to discuss the scaling of the various expressions and parameters involved. This is a crucial step in understanding which terms contribute to the subleading corrections of the entropy. The modular parameters are order-1 parameters: $\tau, \bar{\tau} \sim \mathcal{O}(N^0)$. Similarly, we take $p^i \sim N^0$ and

$C \sim N^0$. From (4.1.49), we find that $\mu^2 \sim \mu_i \, p^i \, \tau$, which solving for the scaling of μ_i gives

$$\mu_i \sim p^j k_{ij} \tau \quad \Rightarrow \quad \mu_i \sim \sum_j p^j k_{ij} \sim (s^i)^{-1}, \tag{4.1.50}$$

where we have made the summation over the indices explicit, to make it clear that the highest order in the summation should be the scaling of μ_i and s^i. Likewise, $E_L \sim E_R \sim E_L^v \sim E_R^v \sim \mu^2 \sim \sum_{i,j} p^i p^j k_{ij}$.

Therefore, we have related the different parameters to the matrix k_{ij}, where the scaling can be found via the Kac-Moody levels. There are two types of levels that we are interested in. The Kac-Moody level from the $SU(2)$ rotation, i.e. k^{11} or k^{22}, is proportional to the central charge c [130], which is of the order of Newton's constant $G^{-1} \sim N^2$. The Kac-Moody levels from $U(1)$ gauge symmetries, i.e. k^{ii} $(i > 2)$, are proportional to N^{-2} [131, 132]. For the BPS AdS_5 black hole, the N-dependences of various factors are

$$E_L^v, E_R^v \sim N^2\,, \quad 4E_L - \mathcal{P}^2, E_R \sim N^2\,, \quad k^{11}, k^{22} \sim N^2\,, \quad k^{33}, k^{44}, k^{55} \sim N^{-2}\,, \tag{4.1.51}$$

which implies that $k_{11}, k_{22} \sim N^{-2}$ and $k_{33}, k_{44}, k_{55} \sim N^2$. Moreover, this also implies that

$$s^1 = s^2 \sim N^2, \quad s^3 = s^4 = s^5 \sim N^{-2}, \tag{4.1.52}$$

or likewise $s_1 = s_2 \sim N^0$, $s_3 = s_4 = s_5 \sim N^0$. Due to the definition of μ_i and p_i as $\mu^2 = \mu_i \mu_j k^{ij}$ and $\mathcal{P}^2 \equiv p_i p_j k^{ij}$, we lower and raise the indices of μ_i, p_i and s_i with k_{ij}. However, for α_i we do not need to raise indices with k_{ij}, as α_1 and α_2 are the net scaling because the right hand side of (4.1.47) is zero.

With these scalings in mind, we can now proceed to compute the saddle point. From (4.1.48) and (4.1.49) let us define

$$\widetilde{S}(\mu,\, \tau,\, \bar{\tau}) \equiv S(\mu,\, \tau,\, \bar{\tau}) + \lambda_1 \left(\sum_{i=1}^{5} s^i \mu_i - C \right) + \lambda_2 \sum_{i=1}^{2} \alpha^i \mu_i \,. \tag{4.1.53}$$

The equations for the fixed points have the form

$$i = 1,\, 2: \quad \frac{\partial \widetilde{S}}{\partial \mu^i} = -2\frac{k_{ij}\mu^j}{\tau} + \frac{(p_v)_i}{\tau} - p_i + \lambda_1 s_i + \lambda_2 \alpha_i = 0\,, \tag{4.1.54a}$$

$$i = 3,\, 4,\, 5: \quad \frac{\partial \widetilde{S}}{\partial \mu^i} = -2\frac{k_{ij}\mu^j}{\tau} + \frac{(p_v)_i}{\tau} - p_i + \lambda_1 s_i = 0\,, \tag{4.1.54b}$$

$$\frac{\partial \widetilde{S}}{\partial \tau} = \frac{\mu^2}{\tau^2} + \frac{E_L^v}{\tau^2} - \frac{\mu_i p_v^i}{\tau^2} - E_L = 0\,, \tag{4.1.54c}$$

$$\frac{\partial \widetilde{S}}{\partial \bar{\tau}} = -\frac{E_R^v}{\bar{\tau}^2} + E_R = 0\,, \tag{4.1.54d}$$

$$\frac{\partial \widetilde{S}}{\partial \lambda_1} = \sum_{i=1}^{5} s^i \mu_i - C = 0\,, \tag{4.1.54e}$$

$$\frac{\partial \widetilde{S}}{\partial \lambda_2} = \sum_{i=1}^{2} \alpha^i \mu_i = 0\,. \tag{4.1.54f}$$

We define the values of the saddle to be $(\mu_i)_0$, τ_0 and $\bar{\tau}_0$, such that (4.1.54a) gives

$$(\mu_i)_0 = \begin{cases} \frac{1}{2}k_{ij}\left(p_v^j - p^j\tau_0 + \lambda_1 s^j \tau_0 + \lambda_2 \alpha^j \tau_0\right), & i = 1, 2\,, \\ \frac{1}{2}k_{ij}\left(p_v^j - p^j\tau_0 + \lambda_1 s^j \tau_0\right), & i = 3, 4, 5\,, \end{cases} \tag{4.1.55}$$

where we can redefine p^i by shifting it as follows

$$\widetilde{p}^i \equiv \begin{cases} p^i - \lambda_1 s^i - \lambda_2 \alpha^i\,, & i = 1, 2\,, \\ p^i - \lambda_1 s^i\,, & i = 3, 4, 5\,. \end{cases} \tag{4.1.56}$$

Therefore, we rewrite

$$(\mu_i)_0 = \frac{1}{2}k_{ij}\left(p_v^j - \widetilde{p}^j \tau_0\right)\,, \tag{4.1.57}$$

with τ_0 satisfying

$$\tau_0^2 E_L = \mu_0^2 + E_L^v - (\mu_i)_0 p_v^i\,. \tag{4.1.58}$$

Using (4.1.57), we find that

$$\mu_0^2 = \frac{1}{4}\left[k_{ij}p_v^i p_v^j + \tau_0^2 k_{ij}\widetilde{p}^i \widetilde{p}^j\right] - \frac{1}{2}k_{ij}\widetilde{p}^i p_v^j \tau_0 = \frac{1}{4}k_{ij}\tau_0^2 \widetilde{p}^i \widetilde{p}^j\,, \tag{4.1.59}$$

where $k_{im}k^{il} = \delta_m^l$, and in the second equality we have assumed that the vacuum is electrically neutral, i.e. $p_v^i = 0$. Inserting (4.1.59) in (4.1.58), we obtain

$$\tau_0 = \pm i\sqrt{\frac{4(-E_L^v)}{4E_L - k_{ij}p^i p^j}}\,. \tag{4.1.60}$$

The saddle point for $\bar{\tau}$ trivially is $\bar{\tau}_0 = \pm i\sqrt{\frac{-E_R^v}{E_R}}$. Consequently, $(\mu_i)_0$ given by (4.1.57) now has the form

$$(\mu_i)_0 = -\frac{1}{2}k_{ij}\widetilde{p}^j\tau_0 = \mp i k_{ij}\widetilde{p}^j\sqrt{\frac{-E_L^v}{4E_L - k_{ij}\widetilde{p}^i\widetilde{p}^j}}\,. \tag{4.1.61}$$

Imposing (4.1.54f), we find

$$\sum_{i=1}^{2}\alpha^i(\mu_i)_0 = -k_{ij}\alpha^i\widetilde{p}^j\sqrt{\frac{E_L^v}{4E_L - k_{ij}\widetilde{p}^i\widetilde{p}^j}} = 0 \quad \Rightarrow \quad \alpha^1 k_{11}\widetilde{p}^1 + \alpha^2 k_{22}\widetilde{p}^2 = 0\,. \tag{4.1.62}$$

Choosing the normalization of the Kac-Moody levels such that $k_{11} = k_{22}$, we can solve for the Lagrange multiplier λ_2

$$\begin{aligned} &\alpha^1 k_{11}\widetilde{p}^1 + \alpha^2 k_{22}\widetilde{p}^2 = 0\,,\\ \Rightarrow \quad & k_{11}\left(\alpha^1(p^1 - \lambda_1 s^1 - \lambda_2\alpha^1) + \alpha^2(p^2 - \lambda_1 s^2 - \lambda_2\alpha^1)\right) = 0\,,\\ \Rightarrow \quad & \alpha^1 p^1 + \alpha^2 p^2 - \lambda_1(\alpha^1 s^2 + \alpha^2 s^2) - \lambda_2((\alpha^1)^2 + (\alpha^2)^2) = 0\,. \end{aligned} \tag{4.1.63}$$

We now set $\alpha^1 = -\alpha^2$, since both p^1 and p^2 should have the same scaling. Therefore, we obtain

$$\alpha_1, \alpha_2 \sim 1, \tag{4.1.64}$$

and

$$\alpha^1 \sim \alpha^2 \sim 1\,, \tag{4.1.65}$$

where we do not need to raise indices with k_{ij} here, as α_1 and α_2 are the net scaling, and the right hand side of (4.1.47) is zero. Moreover, $s^1 = s^2$ as they correspond to the equal angular momenta. We then find from (4.1.63) that

$$\begin{aligned} &\alpha^1(p^1 - p^2) = 2(\alpha^1)^2\lambda_2\,,\\ \Rightarrow \quad & \lambda_2 = \frac{p^1 - p^2}{2\alpha^1}\,, \end{aligned} \tag{4.1.66}$$

which vanish for $p^1 = p^2$. This implies that λ_2 does not affect the logarithmic corrections to the entropy for the case of equal angular momenta, as its contribution to the determinant of the Hessian matrix is of the order $\mathcal{O}(N^0)$. We are also interested in the scaling of λ_1. From (4.1.54e), we find

$$\sum_{i}^{5} s^i(\mu_i)_0 = -k_{ij}s^i\widetilde{p}^j\sqrt{\frac{E_L^v}{4E_L - k_{ij}\widetilde{p}^i\widetilde{p}^j}} = C\,. \tag{4.1.67}$$

As we expect the scaling to remain the same for any arbitrary values of the charges, we consider a special case $p^i = p^i_v = 0$ and find from (4.1.56) and (4.1.67) that

$$\begin{aligned} \lambda_1\sum_{i=1}^{5} k_{ii}s^i s^i &= C\sqrt{\frac{4E_L - \lambda_1^2\sum_{i=1}^5 k_{ii}s^is^i}{E_L^v}}\,,\\ \Rightarrow\quad \lambda_1^2\left(\sum_{i=1}^{5} k_{ii}s^is^i\right)^2 &= C^2\left(\frac{4E_L}{E_L^v} - \frac{\lambda_1^2\sum_{i=1}^5 k_{ii}s^is^i}{E_L^v}\right)\,,\\ \Rightarrow\quad \lambda_1^2\left[\left(\sum_{i=1}^{5} k_{ii}s^is^i\right)^2 + C^2\frac{\sum_{i=1}^5 k_{ii}s^is^i}{E_L^v}\right] &= 4C^2\frac{E_L}{E_L^v}\,. \end{aligned} \tag{4.1.68}$$

Let us now discuss the scalings of each of these terms. Given (4.1.51) and (4.1.52), we have at the leading order $E_L \sim E_L^v \sim N^2$, $C \sim N^0$ and $k_{ii}s^is^i \sim N^2$ and therefore

$$\lambda_1 \sim N^{-2}. \tag{4.1.69}$$

The leading order value of the degeneracy is obtained by evaluating the action at the saddle point values, which gives

$$\log\rho_0 = \sqrt{E_L^v\left(4E_L - k_{ij}\widetilde{p}^i\widetilde{p}^j\right)} + 2\sqrt{E_RE_R^v}\,. \tag{4.1.70}$$

We would like to comment on the scaling with respect to N in (4.1.70). At the leading order, $\lambda_1 s^i \sim \mathcal{O}(N^{-2})$ which implies that $\mathcal{O}(\widetilde{p}^i) \sim \mathcal{O}(p^i)$. Therefore, (4.1.70) coincides with the leading order of the degeneracy with $\widetilde{p}^i$ replaced by p^i. This is important as we can see that the constraint we imposed only affects the subleading order of the entropy.

Note that we have more than one saddle points, namely one for each choice of signs in the values of $\tau_0, \bar{\tau}_0, (\mu_i)_0$. However, one saddle dominates over the others as $S(\mu,\tau,\bar{\tau})$ is exponentially suppressed. To see this explicitly, let us take

$$\tau_0 = i\epsilon_\tau \sqrt{\frac{4(-E_L^v)}{4E_L - k_{ij}p^i p^j}}\,, \qquad \bar{\tau}_0 = i\epsilon_{\bar{\tau}} \sqrt{\frac{-E_R^v}{E_R}}\,, \qquad \mu_{i,0} = -i\epsilon_\tau k_{ij}\, p^j \sqrt{\frac{-E_L^v}{4E_L - k_{ij}p^i p^j}}\,, \tag{4.1.71}$$

where ϵ_τ and $\epsilon_{\bar{\tau}}$ take on values of ± 1. Then, under the constraints imposed by the Lagrange multipliers, the density of states (4.1.48) can be approximated by the saddle points

$$\begin{aligned}
\rho_0 &= \sum_{\epsilon_\tau, \epsilon_{\bar{\tau}} = \pm 1} \exp\left(2\pi i S(\mu, \tau, \bar{\tau})\right) \\
&= \sum_{\epsilon_\tau, \epsilon_{\bar{\tau}} = \pm 1} \exp\left\{2\pi i \left[-\frac{i\epsilon_\tau}{2}\frac{\mathcal{P}^2\sqrt{-E_L^v}}{\sqrt{4E_L - \mathcal{P}^2}} - i\epsilon_\tau(-E_L^v)\sqrt{\frac{4E_L - \mathcal{P}^2}{4(-E_L^v)}} + i\epsilon_{\bar{\tau}}(-E_R^v)\sqrt{\frac{E_R}{-E_R^v}} \right.\right. \\
&\qquad\qquad \left.\left. -i\epsilon_\tau E_L\sqrt{\frac{-4E_L^v}{4E_L - \mathcal{P}^2}} + i\epsilon_{\bar{\tau}} E_R\sqrt{\frac{-E_R^v}{E_R}} - (-i\epsilon_\tau)\mathcal{P}^2\sqrt{\frac{-E_L^v}{4E_L - \mathcal{P}^2}}\right]\right\} \\
&= \sum_{\epsilon_\tau, \epsilon_{\bar{\tau}} = \pm 1} \exp\left[2\pi\left(\epsilon_\tau\sqrt{-E_L^v(4E_L - \mathcal{P}^2)} - \epsilon_{\bar{\tau}}\sqrt{-4E_R^v E_R}\right)\right].
\end{aligned} \tag{4.1.72}$$

If we now select the combination of $\epsilon_\tau = 1$ and $\epsilon_{\bar{\tau}} = -1$, which maximizes the exponent in (4.1.72), we can write

$$\rho_0 \sim \exp\left[2\pi\left(\sqrt{-E_L^v(4E_L - \mathcal{P}^2)} + \sqrt{-4E_R^v E_R}\right)\right] + \ldots\,, \tag{4.1.73}$$

where the dots denote the exponentially suppressed terms of subleading non-logarithmic order. To summarize, given the behavior of Z, the degeneracy ρ can be determined using the saddle-point approximation with the dominant saddle at

$$\tau_0 = \sqrt{\frac{4E_L^v}{4E_L - \mathcal{P}^2}}\,, \quad \bar{\tau}_0 = -\sqrt{\frac{E_R^v}{E_R}}\,, \quad \mu_{i,0} = -k_{ij}\, p^j \sqrt{\frac{E_L^v}{4E_L - \mathcal{P}^2}}\,, \tag{4.1.74}$$

where k_{ij} is the inverse matrix of k^{ij}, and $\mathcal{P}^2 \equiv p^i p^j k_{ij}$. Note that the saddle-point values τ_0 and $(\mu_i)_0$ are parametrically small, as $\mathcal{P}^2 \gg |E_L|$, which is reminiscent of the 4d Cardy limit originally used in [13, 14] and recently clarified in [126, 127]. Moreover, the Cardy limit in 2d, which assumes that the levels of the theory is much larger than the Casimir energy, is compatible with the Cardy limit in 4d, which focuses on small chemical potentials and large charges, as they both address the high energy states of the theory and in our particular case address the entropy of extremal black holes.

At the saddle (4.1.74), the density of states ρ reaches its extremum ρ_0, and the corresponding entropy is

$$S(\mu_0, \tau_0, \bar{\tau}_0) = \log \rho_0 \approx 2\pi\sqrt{-E_L^v(4E_L - \mathcal{P}^2)} + 2\pi\sqrt{-E_R^v(4E_R)}\,. \tag{4.1.75}$$

This expression is also called the charged Cardy formula in [133], which implies a microcanonical ensemble of black hole microstates. If we apply $E_L^v = E_R^v = -c/24$, and define the temperatures $T_{L,R}$ through

$$E_L - \frac{\mathcal{P}^2}{4} = \frac{\pi^2}{6}cT_L^2\,, \quad E_R = \frac{\pi^2}{6}cT_R^2\,, \tag{4.1.76}$$

we can rewrite the entropy (4.1.75) as

$$S = \frac{\pi^2}{3}cT_L + \frac{\pi^2}{3}cT_R\,, \tag{4.1.77}$$

where T_R is proportional to the physical Hawking temperature T_H. This formula coincides, at the leading order, with the canonical ensemble version of the charged Cardy formula, and has been successfully used in a variety of cases [16, 83, 134–136]. However, we emphasize that obtaining (4.1.77) did not involve a change of ensemble, as we merely re-identified certain combinations.

From the near-horizon CFT_2 and the Kerr/CFT correspondence we know that for the BPS AdS_5 black hole

$$c_L = \frac{9\pi a^2}{G_N g(1-ag)(1+5ag)} = \frac{18N^2(ag)^2}{(1-ag)(1+5ag)}\,, \tag{4.1.78}$$

$$T_L = \frac{1+5ag}{3a(1-ag)\pi}\sqrt{a\left(a+\frac{2}{g}\right)}\,, \tag{4.1.79}$$

where we have used the $\mathrm{AdS}_5/\mathrm{CFT}_4$ dictionary $\frac{1}{2}N^2 = \frac{\pi}{4G_N}\ell_5^3 = \frac{\pi}{4G_N g^3}$. Note that both c_L and T_L are dimensionless. Consequently, the BPS AdS_5 black hole entropy at the leading order in N is given by the Cardy formula

$$S_{\mathrm{CFT}_2} = \frac{\pi^2}{3}c_L T_L = \frac{2N^2\pi(ag)^{3/2}\sqrt{2+ag}}{(1-ag)^2}\,. \tag{4.1.80}$$

This near-horizon CFT_2 result matches the macroscopic Bekenstein-Hawking entropy of the black hole (4.1.36), as shown in [17, 18, 79].

4.1.5 Logarithmic Corrections from Near-Horizon CFT$_2$

To derive the logarithmic corrections to the black hole entropy from the near-horizon CFT$_2$, we evaluate the Cardy formula beyond its leading saddle-point value by including its Gaussian correction. Namely, we consider a logarithmic correction $\Delta S_{\mathrm{CFT}_2}$ obtained from expanding τ, $\bar{\tau}$ and $\vec{\mu}$ to the quadratic order around the saddle point given by (4.1.74). The result is

$$\Delta S_{\mathrm{CFT}_2} = -\frac{1}{2}\log\frac{\det\mathcal{A}}{(2\pi)^{n+2}}\,, \tag{4.1.81}$$

where $\mathcal{A}$ is the Hessian of the exponent in the integrand of (4.1.48) around the saddle point (4.1.74), and has the form $\mathcal{A}_{\mu\nu} = \frac{\partial^2 S}{\partial x^\mu \partial x^\nu}$, where $x^\mu = \{\tau, \bar{\tau}, \lambda, \mu^{i=1,\cdots,n}\}$, whose only non-trivial elements in the presence of constraints are

$$\begin{aligned}
\frac{\partial^2 \widetilde{S}}{\partial\tau\partial\mu^i} &= 2\frac{k_{ij}(\mu^j)_0}{\tau_0},\\
\frac{\partial^2 \widetilde{S}}{\partial\tau^2} &= -\frac{2}{\tau_0^3}\left(k_{ij}(\mu^i)_0(\mu^j)_0 + E_L^v\right),\\
\frac{\partial^2 \widetilde{S}}{\partial\bar{\tau}^2} &= 2\frac{E_R^v}{\bar{\tau}_0^3},\\
\frac{\partial^2 \widetilde{S}}{\partial\lambda_1\partial\mu^i} &= s_i, \quad (i = 1, \cdots, 5)\\
\frac{\partial^2 \widetilde{S}}{\partial\lambda_2\partial\mu^i} &= \alpha_i, \quad (i = 1, 2)\\
\frac{\partial^2 \widetilde{S}}{\partial\mu^i\partial\mu^j} &= -2\frac{k_{ij}}{\tau_0}.
\end{aligned} \tag{4.1.82}$$

We see that k_{11} and k_{22} come from the $SU(2)$ rotation, which corresponds to the angular momenta, while k^{ii} $(i > 2)$ come from the $U(1)$ gauge symmetries. At the subleading order the Hessian takes the form

$$\det\mathcal{A} = \frac{(2\pi)^{n+2}}{16}\left(-E_L^v\right)^{-\frac{n+1}{2}}\left(4E_L - \mathcal{P}^2\right)^{\frac{n+3}{2}}\left(-E_R^v\right)^{-\frac{1}{2}}\left(4E_R\right)^{\frac{3}{2}}\det(H)\,, \tag{4.1.83}$$

where

$$H \sim \left(\begin{array}{c|c} k_{ij} & \mathcal{S}^T \\ \hline \mathcal{S} & 0 \end{array}\right), \tag{4.1.84}$$

and

$$\mathcal{S} = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 \\ \alpha_1 & \alpha_2 & 0 & 0 & 0 \end{pmatrix}. \tag{4.1.85}$$

Note that this result is different than in [131], as we have considered two linear constraints on the chemical potentials.

For supersymmetric extremal (BPS) black holes, one of the Frolov-Thorne temperatures T_R vanishes, as it is proportional to the Hawking temperature, and only the left sector contributes to the black hole entropy. Consequently, (4.1.83) for BPS black holes becomes

$$(\det \mathcal{A})_{\text{BPS}} = \frac{(2\pi)^{n+2}}{16} (-E_L^v)^{-\frac{n+1}{2}} (4E_L - \mathcal{P}^2)^{\frac{n+3}{2}} \det(H)\,, \tag{4.1.86}$$

where

$$-E_L^v = \frac{c}{24}\,, \quad 4E_L - \mathcal{P}^2 = \frac{2\pi^2}{3} c T_L^2\,. \tag{4.1.87}$$

With the scalings in (4.1.51), (4.1.52) and (4.1.64), the Hessian takes on the N-dependence

$$\det H \sim N^2, \tag{4.1.88}$$

such that

$$(\det \mathcal{A})_{\text{AdS}_5 \text{ Black Hole}} \sim (N^2)^{-\frac{n+1}{2}} (N^2)^{\frac{n+3}{2}} \left(N^2\right) = N^4\,. \tag{4.1.89}$$

Note that the result is independent of n since the scaling of E_L^v and $4E_L - \mathcal{P}^2$ are equal. Therefore, the logarithmic correction to the leading-order BPS AdS_5 black hole entropy (4.1.80) is

$$\Delta S_{\text{CFT}_2} = -\frac{1}{2} \log \frac{\det \mathcal{A}}{(2\pi)^{n+2}} = -2 \log N + \mathcal{O}(1)\,, \tag{4.1.90}$$

which precisely agrees with ΔS_{CFT_4} in (4.1.29).

4.2 AdS_5 Black Strings

4.2.1 AdS_5 Black String Entropy from Boundary $\mathcal{N} = 4$ SYM

A rotating AdS_5 black string solution in gauged supergravity has been discussed in [133, 137], where it was shown that its leading-order entropy can be obtained from the refined topologically twisted index of $\mathcal{N} = 4$ SYM on $S^2 \times T^2$.

The topologically twisted index of $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ is defined as the supersymmetric index of the theory on $T^2 \times S^2$ with a topological twist on S^2 [138, 139], its Hamiltonian interpretation being

$$Z(p_a,\Delta_a) = (-1)^F e^{2\pi i\tau\,\{\mathcal{Q},\mathcal{Q}^\dagger\}} e^{i\Delta_a J_a}\,. \tag{4.2.1}$$

The topologically twisted index depends on a set of chemical potentials, Δ_a, for the generators of flavor symmetries ($a=1,2,3$), a modular parameter of the torus τ and magnetic fluxes p_a. The topologically twisted index of $\mathcal{N}=4$ SYM with gauge group $SU(N)$ admits a presentation as an integral over the space of holonomies in the following way:

$$\begin{aligned} Z(p_a,\Delta_a) &= \frac{1}{N!}\sum_m \oint_{\mathcal{C}} \prod_{\mu=1}^{N-1} \left(du_\mu \eta(q)^2\right) \mathcal{Z}_{TT}(u,\Delta_a,\tau,p_a)\,,\\ \mathcal{Z}_{TT}(u,\Delta_a,\tau,p_a) &= \prod_{i,j=1}^{N}\left[\frac{\theta_1(u_{ij};\tau)}{i\eta(q)}\prod_{a=1}^{3}\left(\frac{i\eta(q)}{\theta_1(u_{ij}+\Delta_a;\tau)}\right)^{m_{ij}-p_a+1}\right]\,, \end{aligned} \tag{4.2.2}$$

where $\eta(q)$ is the Dedekind eta function that we define in Appendix C.1. We can evaluate (4.2.2) as the sum over residues [139] which takes the following explicit form:

$$Z(p_a,\Delta_a) = \eta(q)^{2(N-1)} \sum_{\hat{u}\in\mathrm{BA}} \prod_{i,j=1}^{N}\left[\prod_{a=1}^{3}\left(\frac{\theta_1(u_{ij};\tau)}{\theta_1(u_{ij}+\Delta_a;\tau)}\right)^{1-p_a}\right] H^{-1}(\hat{u},\Delta,\tau), \tag{4.2.3}$$

where, analogously to the SCI discussed in Sec. 4.1.1, BA stands for the set of solutions to the Bethe-Ansatz equations (4.1.3), and $H(\hat{u},\Delta,\tau)$ is the Jacobian defined in (4.1.5). The location of a set of such residues was found in [125] and have the form given by (4.1.6) labeled by $\{u_i\}$ with integers $\{m,n,r\}$.

In fact, the set of solutions found in [125] inspired the evaluation of the SCI carried in [15], where the $\{u_i\}$ are also organized according to equation (4.1.6), however, in the large-N limit, it was possible to argue that the configuration corresponding to $\{1,N,0\}$ was dominant. For fixed $\{m,n,r\}$, it is possible to count how many values of $\bar{u}$ give non-equivalent contributions to topologically twisted index (by non-equivalent we mean, those which are not identified by periodicity $u\sim u+1$ or $u\sim u+\tau$). Once again, imposing the $SU(N)$ constraint we find that:

$$\begin{aligned} \bar{u} &= \frac{k}{N} - \frac{1}{2N}\left[n(m-1)+m(n-1)\left(\tau+\frac{r}{m}\right)\right]\,,\\ k &= 0,1,\cdots,N-1\,, \end{aligned} \tag{4.2.4}$$

which reduces to (4.1.9) for $\{m,n,r\}=\{1,N,0\}$. We then conclude that there is a degeneracy factor of N for each $\{m,n,r\}$ configuration contributing to the topologically twisted index.

To argue that there is no other contribution of the same order that spoils the value of the coefficient of $\log N$ would require a more detailed study of the large-N behavior of the topologically twisted index, which has been studied recently in [124] at the leading order in N. A systematic study of subleading corrections to the topologically twisted index still remains an open problem. It is, however, very tempting to conjecture that indeed, there is no contribution other than the one originated from degeneracy of Bethe-Ansatz solutions and, consequently, the coefficient of $\log N$ is 1 also for the topologically twisted index in the grand-canonical ensemble.

One can further refine the topologically twisted index by adding a rotation on S^2 [138]. This will modify the integral expression (4.2.2) through the appropriate fugacities associated to the rotation on S^2, namely $\xi = e^{2\pi i\omega}$. To be concrete, we would have:

$$
\begin{aligned}
Z(p_a,\Delta_a)_{\text{refined}} &= \frac{1}{N!}\sum_m \oint_{\mathcal{C}} \prod_{\mu=1}^{N-1}\left(du_\mu \eta(q)^2\right)\mathcal{Z}_{TTref}(u,\Delta_a,\tau,p_a)\,,\\
\mathcal{Z}_{TTref}(u,\Delta_a,\tau,p_a) &= \prod_{i,j=1}^{N}\left[\frac{\theta_1\left(u_{ij}+2\omega j;\tau\right)}{i\eta(q)}\prod_{a=1}^{3}\left(\frac{i\eta(q)}{\theta_1\left(u_{ij}+\Delta_a+2\omega j;\tau\right)}\right)^{m_{ij}-p_a+1}\right].
\end{aligned}
\tag{4.2.5}
$$

The refined topologically twisted index has been studied, in the strict Cardy-like limit, in [137], where the correction due to the refinement could be factored out in the following way:

$$
Z(p_a,\Delta_a)_{\text{refined}}\Big|_{\tau\to 0} = Z(p_a,\Delta_a)Z_\omega\,, \tag{4.2.6}
$$

where $Z(p_a,\Delta_a)$ is the unrefined topologically twisted index, and Z_ω is the correction associated to the refinement. The explicit form of Z_ω is irrelevant to us, while only the fact that it is independent on u, p_a and Δ_a will be important. To the best of our knowledge, the direct application of the Bethe-Ansatz approach to the refined topologically twisted index has not been performed yet. However, we can exploit the fact that in the Cardy-like limit there is a simple connection to the unrefined index, namely (4.2.6), and based on the intuition we have gained by studying the SCI, to argue that the combinatorial origin of $\log N$ corrections is still there at small τ, therefore we do not expect it to go away as we depart from the Cardy-like limit.

As we have discussed in the AdS_5 black hole case, the logarithmic correction to the entropy can be seen as essentially arising from the degeneracy of dominant Bethe-Ansatz solutions to the appropriate partition function of the boundary $\mathcal{N}=4$ SYM. As in the case of the SCI, the logarithmic correction we compute for the topologically twisted index is in the grand-canonical ensemble. However, since we find that the result matches that of the microcanonical ensemble, we conjecture that there are no additional logarithmic contributions associated to the change of ensembles. Therefore, for the BPS rotating AdS_5 black string considered in [133, 137], the logarithmic correction to $\log Z^{(\text{leading})}(p_a, \Delta_a)$ can be obtained from the degeneracy of dominant residues contributing to the topologically twisted index of $\mathcal{N} = 4$ SYM, i.e.

$$\Delta \log Z(p_a, \Delta_a) = \log N \, . \tag{4.2.7}$$

This result has the same origin (in the Bethe-Ansatz treatment [15]) as in the SCI, and we expect a similar robustness as the logarithmic correction to the AdS_5 black hole.

Since $\log Z^{(\text{leading})}(p_a, \Delta_a) \sim N^2$ and it is homogeneous of degree one in the chemical potentials, it is possible to apply the result of Sec. 4.1.2 to conclude that the logarithmic correction has an additional contribution from the change of ensemble which again takes the form $-d \log N$, where $d = 3$ is the number of independent chemical potentials for the rotating AdS_5 black string. We then conclude that

$$\Delta S_{\text{CFT}_4} = (1-d)\log N + \mathcal{O}(1) = -2 \log N + \mathcal{O}(1). \tag{4.2.8}$$

4.2.2 AdS_5 Black String Entropy from Near-Horizon CFT_2

The near-horizon geometry of the rotating AdS_5 black string solution is [133]

$$\begin{aligned} ds^2 = & \frac{(-\mathcal{M}\,\Pi)^{2/3}}{\Theta^2}\left[-r^2\,d\tau^2 + \frac{dr^2}{r^2} + \frac{\mathcal{W}\Theta^2}{\mathcal{M}^2}\left(dy - \frac{\mathcal{M}}{\Theta\sqrt{\mathcal{W}}}\,r d\tau\right)^2\right] \\ & + \frac{(-\mathcal{M})^{2/3}}{\Pi^{1/3}}\left[d\theta^2 + \sin^2\theta\left(d\varphi + \frac{\mathcal{J}}{\mathcal{M}}\,dy\right)^2\right], \end{aligned} \tag{4.2.9}$$

where $\mathcal{M} \equiv -p_1 p_2 p_3$ is the product of magnetic charges, and $\mathcal{J}$ is the angular momentum, while

$$\begin{aligned} \Theta &\equiv (p^1)^2 + (p^2)^2 + (p^3)^2 - 2(p^1p^2 + p^1p^3 + p^2p^3)\,, \\ \Pi &\equiv (-p^1 + p^2 + p^3)(p^1 - p^2 + p^3)(p^1 + p^2 - p^3)\,, \\ \mathcal{W} &\equiv \frac{-4q_0 p^1p^2p^3 - \mathcal{J}^2}{\Theta}\,, \end{aligned} \tag{4.2.10}$$

with q_0 denoting the momentum added along the black string direction. Using the standard Kerr/CFT correspondence, we obtain the central charge of the near-horizon CFT_2

$$c_L = \frac{6\mathcal{M}}{G_4 \Theta}\,. \tag{4.2.11}$$

This central charge was found in [133] as a Brown-Henneaux central charge [59].

To compute the black string entropy using the Cardy formula, we still need the Frolov-Thorne temperature, which can be computed from the standard formalism for the Kerr/CFT correspondence [140]

$$T_L = \frac{\sqrt{\mathcal{W}}\,\Theta}{2\pi \mathcal{M}}\,. \tag{4.2.12}$$

Therefore, the Cardy formula leads to the rotating AdS_5 black string (BS) entropy

$$S_{\mathrm{BS}} = \frac{\pi^2}{3} c_L T_L = \frac{\pi \sqrt{\mathcal{W}}}{G_4}\,, \tag{4.2.13}$$

which is the same as the leading-order rotating AdS_5 black string entropy [133, 137].

For the logarithmic correction to the AdS_5 black string entropy from the near-horizon CFT_2, we apply the same technique as the AdS_5 black hole case. As mentioned in [133, 137], the rotating AdS_5 black string solution has one angular momentum and three electric charges. Similar to the BPS AdS_5 black hole case,

$$c \sim N^2\,, \quad k^{11} \sim N^2\,, \quad k^{ii} \sim N^{-2} \quad (i = 2, 3, 4)\,, \tag{4.2.14}$$

where we take $n = 4$ in the general formula (4.1.86) due to the following reason. Three $U(1)$ electric charges have three corresponding chemical potentials Δ_a subject to a constraint, hence there are only two indepedent $U(1)$ electric charges. The angular momentum $\mathcal{J}$ appearing in the second line of (4.2.9) can be viewed as an additional $U(1)$, which can be treated in the same way as a $U(1)$ electric charge [21], while in the first line of (4.2.9) there is actually another angular momentum hidden in the BTZ part of the metric. Hence, from the near-horizon region of the rotating AdS_5 black string there are still one angular momentum and three $U(1)$ charges (including $\mathcal{J}$), which are independent of each other. Unlike the AdS_5 black hole case where we can choose one of the two angular momenta, for the rotating AdS_5 black string the way of counting near-horizon symmetries is unique.

The reasoning of Sec. 4.1.5 can be followed in its entirety except that from the start there are only 4 chemical potentials, one conjugate to angular momentum and three conjugate to electric charges, obeying one constraint. This is in contrast with the AdS_5 black hole with 5 chemical potentials, two conjugate to angular momenta and three conjugate to electric charges. The scalings of the Kac-Moody levels and other parameters are the same. Moreover, only one Lagrange multiplier is needed, λ_1, and we find that $\det H \sim N^2$, as in the case of the AdS_5 black hole with the same final result as in (4.1.90). Consequently,

$$(\det \mathcal{A})_{\mathrm{AdS}_5\ \mathrm{Black\ String}} \sim (N^2)^{-\frac{n+1}{2}} (N^2)^{\frac{n+3}{2}} \left(N^2\right) = N^4\,, \tag{4.2.15}$$

and the logarithmic correction to the leading-order AdS_5 black string entropy (4.2.13) is

$$\Delta S_{\mathrm{CFT}_2} = -\frac{1}{2}\log\frac{\det \mathcal{A}}{(2\pi)^{n+2}} = -2\log N + \mathcal{O}(1)\,. \tag{4.2.16}$$

4.3 Discussion

In this chapter, we have explored logarithmic corrections to asymptotically AdS_5 supersymmetric extremal, rotating, electrically charged black holes and black strings. For each case we examined the microstate counting in the context of $\mathcal{N} = 4$ SYM whereby it reduces to a combinatorial contribution from the space of solutions. We also approached the logarithmic corrections to the entropy by considering the microstate counting in the near-horizon geometry and its dual CFT_2, where the logarithmic corrections arise as subleading contributions in the Cardy formula for the degeneracy of states. We found that the results from both approaches precisely match for both AdS_5 black holes and rotating black strings. It is instructive to write our result as $(1-d)\log N$ to note that the logarithmic correction has two contributions, one that has a completely combinatorial origin and is rather universal, namely, $\log N$, while the other contribution from the change of ensemble depending on the number of independent chemical potentials of the theory, $-d\log N$. Since we have 3 independent chemical potentials, we obtain $-2\log N$ as a correction to the microscopic entropy.

Our agreement in using the Cardy formula to its logarithmic precision should come more as a surprise than as a foregone conclusion. There is precedent where the Cardy formula leads to the wrong answer for logarithmic corrections [21]. Although the subtleties in applying the Cardy formula beyond its intrinsic regime are numerous, we expect that our positive results indicate the existence of resolutions which take into account particular properties of the spectrum [141, 142].

It would be interesting to derive the logarithmic corrections directly from the macroscopic one-loop contribution in type IIB supergravity. It is also natural to extend our near-horizon analysis to asymptotically AdS black holes in other dimensions. This route is certain to encounter obstructions in the form of zero modes, as is the case for asymptotically AdS_4 and AdS_6 black holes. Indeed, it has been shown that the one-loop supergravity contribution to the logarithmic corrections for asymptotically AdS_4 black holes [97] is different from the one obtained in the near-horizon approach [95, 96]. Our work indicates that given the absence of obstructions (zero modes) in odd-dimensional AdS spacetimes the counting can be performed at the near-horizon level, paving the way for a quantum entropy formula à la Sen [143]. It will also be interesting to explore the implications of our near-horizon results within supergravity localization along the lines of [48, 144].

Chapter 5

Four dimensional AdS Spacetimes

5.1 Summary of results

In this chapter, we take a practical, bottom-up approach to the question of logarithmic contributions in four dimensions. Our main result is the computation of logarithmic correction in $\mathcal{N} = 2$ minimal gauged supergravity. We also present results for minimally coupled fields as well as for the Einstein-Maxwell theory with a negative cosmological constant.

The black hole we are interested is the AdS-Kerr-Newman geometry [75, 145, 146]. In the extremal case, we will also consider the near horizon geometry which includes a warped circle fibration over AdS_2. Our results also give the logarithmic correction to the free energy of thermal AdS_4. They can also be applied to the hyperbolic black hole [147] from which we obtain the logarithmic corrections to the corresponding entanglement entropy.

The microcanonical entropy of the black hole is given by

$$S = \frac{A}{4G} + C \log \lambda + \ldots \ , \qquad (\lambda \to +\infty) \tag{5.1.1}$$

where A is the area of the horizon and the subleading logarithmic term is the explicit quantum correction we seek. We are interested in the coefficient of the $\log \lambda$ in the "isometric" scaling regime where all length scales (in Planck units) are multiplied by λ and we take $\lambda \to +\infty$.

The logarithmic correction receives two types of contributions

$$C = C_{\text{local}} + C_{\text{global}} \ . \tag{5.1.2}$$

The global contribution C_{global} is an integer that captures the contribution from the zero modes and from the change of ensemble from canonical to microcanonical. The more interesting local contribution, C_{local}, receives contributions from the non-zero modes and can

Multiplet	$a_{\rm E}$	c	b_1	b_2
Free scalar	$\frac{1}{360}$	$\frac{1}{120}$	$\frac{1}{288}(\Delta(\Delta-3)-2)^2$	0
Free fermion	$-\frac{11}{360}$	$\frac{1}{20}$	$\frac{1}{72}\left(\Delta-\frac{3}{2}\right)^2\left(\left(\Delta-\frac{3}{2}\right)^2-2\right)$	0
Free vector	$\frac{31}{180}$	$\frac{1}{10}$	0	0
Free gravitino	$-\frac{229}{720}$	$-\frac{77}{120}$	$-\frac{1}{9}$	0
Einstein-Maxwell	$\frac{53}{45}$	$\frac{137}{60}$	$-\frac{13}{36}$	0
$\mathcal{N}=2$ gravitini	$-\frac{589}{360}$	$-\frac{137}{60}$	0	$\frac{13}{18}$
$\mathcal{N}=2$ gravity multiplet	$-\frac{11}{24}$	0	$-\frac{13}{36}$	$\frac{13}{18}$

Table 5.1: Results for the Seeley-DeWitt coefficient a_4 responsible for the logarithmic corrections. The results for a_0 and a_2 are given in Table B.1 in Appendix B.4.

be computed using the heat kernel expansion. It is given by an integral over the Euclidean spacetime

$$C_{\rm local} \equiv \int d^dx\sqrt{g}\, a_4(x)\ , \tag{5.1.3}$$

where the so-called fourth Seeley-DeWitt coefficient is a sum of four-derivative terms

$$a_4(x) = -a_{\rm E}E_4 + c\, W^2 + b_1 R^2 + b_2 R F_{\mu\nu}F^{\mu\nu}, \tag{5.1.4}$$

evaluated on the background. The backgrounds we consider are solutions of Einstein-Maxwell theory with a negative cosmological constant. Using the equations of motion, a general four-derivative expression such as $a_4(x)$ can always be decomposed in the above basis. The expression of Euler, E_4, and the Weyl tensor squared, W^2, are given in (5.3.2). The heat kernel expansion provides a way to compute these coefficients from any two-derivative action using the formula (5.2.23). The results are summarized for the theories studied in this chapter in Table 5.1.

Our final result for the Seeley-DeWitt coefficient of minimal $\mathcal{N} = 2$ gauged supergravity takes the form

$$(4\pi)^2 a_4(x) = \frac{11}{24}E_4 - \frac{13}{36}R^2 + \frac{13}{18}RF_{\mu\nu}F^{\mu\nu} \ . \tag{5.1.5}$$

Evaluating this expression on the BPS Kerr-Newman black hole, we obtain

$$C_{\text{local}} = \frac{11}{6} - \frac{26}{3}\frac{a(\ell^2 - 4\ell - a^2)}{(\ell - a)(a^2 + 6a\ell + \ell^2)}, \tag{5.1.6}$$

where $a = J/M$ is the rotation parameter and ℓ is the AdS_4 radius. The integer corrections, C_{global}, are summarized in Table 5.2.

We observe that the logarithmic correction for a BPS black hole in gauged supergravity has a richer structure than in flat space: *the logarithmic correction is non-topological, i.e., its coefficient is not a pure number but depends on black hole parameters.* Our result is, to our knowledge, the first computation of the Seeley-DeWitt coefficient $a_4(x)$ in gauged supergravity. We find that the non-topological contribution comes from the additional four-derivative terms R^2 and $RF_{\mu\nu}F^{\mu\nu}$. In the flat space limit, these terms both vanish and the logarithmic correction becomes topological and gives $C_{\text{local}} = \frac{11}{6}$. This was shown in [148,149] and is a non-trivial consequence of supersymmetry.

We suspect that the non-topological piece can be interpreted as a contribution from the AdS boundary. It is possible to interpret the logarithmic correction as the Atiyah-Singer index of an appropriate supercharge [150]. We surmise that the non-topological term should correspond to the η-invariant which is a correction due to the presence of a boundary.

We note that according to microscopic computations [95,99,100,104,151], we expect the full logarithmic entropy correction to be topological. Such expectation has been confirmed in various 11d supergravity computations [97,100,104,151]. There is, however, no contradiction because the 4d minimal gauged supergravity is by itself not the low-energy effective theory of a UV complete theory as matter multiplets, arising from Kaluza-Klein reduction, need to be included. Nonetheless, our result shows that supersymmetry is not enough to guarantee a topological logarithmic correction. This observation suggests that the topological nature of the logarithmic correction could be used to indicate which low-energy theories admit a UV completion[1].

[1] The possibility of using the topological nature of logarithmic correction for such questions was emphasized to us by Alejandra Castro and was discussed in [152].

5.2 Logarithmic corrections in AdS_4

In this section, we review logarithmic corrections to black hole entropy and the heat kernel method for their computation [21–23]. This method has been chiefly applied to asymptotically flat black holes. We also explain how to apply it to asymptotically AdS black holes.

5.2.1 Euclidean quantum gravity

We consider theories of Einstein gravity in D dimensions coupled to matter fields. We restrict to theories with a scaling property so that purely bosonic terms have two derivatives, terms with two fermions have one derivative and terms with four fermions have no derivative. This covers a wide range of theories, such as Einstein gravity with minimally coupled scalars, fermions and gauge fields, but also a variety of supergravity theories at a generic point in the moduli space. We also allow for the presence of a cosmological constant.

We now consider a black hole solution in this theory. To define the quantum entropy of the black hole, we use the fact that this black hole appears as a saddle-point of the Euclidean path integral

$$Z(\beta, \mu_\alpha) = \int D\Psi \, e^{-S_E(\Psi)} \ , \tag{5.2.1}$$

where $\mathcal{S}_E$ is the Euclidean action and the integration is done while fixing the temperature β, thermodynamically conjugate to the mass, M, and appropriate chemical potentials μ^α associated to the U(1) charges q_α.

Upon studying the black hole solutions, we probe the Euclidean spacetimes via a continuation to imaginary time and analytically continue the action. For the case of the Kerr solutions, these quasi-Euclidean metrics are complex and do indeed give appropriate thermodynamics, see for example [153]. Our computations focus on the small fluctuations around the complex saddle points and we do not expect the subtleties of analytic continuation to affect these quantum corrections. Therefore, for the sake of this chapter, we consider these quasi-Euclidean solutions (which we call Euclidean) as is, and leave the subtleties of spacetimes with complex metrics for future study. We simply comment that complex solutions in Euclidean gravity is an evolving subject and we refer the reader to a few examples in the literature [154, 155] as well as more recent discussions on this matter [156, 157].

The black hole entropy is given by the Legendre transform

$$S = \log Z + \beta M + \sum_\alpha \mu^\alpha q_\alpha \ . \tag{5.2.2}$$

At leading order, the classical approximation $\log Z = -S_E^{\text{classical}}$ is the Euclidean on-shell action. It is a classic result of Gibbons and Hawking [153] that the transform leads to the Bekenstein-Hawking entropy formula

$$S = \frac{\text{Area}(q_\alpha)}{4G} + \dots \,. \tag{5.2.3}$$

At one-loop order around the saddle-point, we obtain

$$Z(\beta, \mu_\alpha) \sim \frac{1}{\sqrt{\det \mathcal{Q}}} e^{-S_E^{\text{classical}}}, \tag{5.2.4}$$

where $\mathcal{Q} = \frac{\delta^2 S_E}{\delta \Psi^2}$ is the quadratic operator for the fluctuating fields on the background. This expression is divergent and needs to be regulated. The one-loop correction to the black hole entropy is

$$\delta S = -\frac{1}{2} \log \det \mathcal{Q}. \tag{5.2.5}$$

5.2.2 Scaling regime

The result for the logarithmic correction is highly sensitive to the precise scaling regime we consider. To isolate the logarithmic correction, we consider a reference configuration with fixed length scales $\ell_i^{(0)}$. In the example of AdS-Schwarzschild, these length scales can be taken to be the AdS_4 radius ℓ and the horizon size r_+. We then consider a rescaled configuration where all length scales are multiplied by the <u>same</u> factor $\lambda \gg 1$: $\ell_i = \lambda \ell_i^{(0)}$. We are then interested in the coefficient of $\log \lambda$ in the one-loop correction to the entropy of the rescaled configuration.

This scaling regime is "isometric" because it only magnifies the geometry without deforming it. As a result, the eigenvalues of $\mathcal{Q}$ are given by

$$\kappa_n = \lambda^2 \kappa_n^{(0)}, \tag{5.2.6}$$

where $\kappa_n^{(0)}$ are the eigenvalues of the reference configuration. As explained in the next section, this relation is important to ensure that the logarithmic correction depends only on the small s expansion of the heat kernel.

For more general scaling regimes, there will not be any simple relation between the eigenvalues of the scaled versus reference configuration, because the geometry gets deformed. In this case, the logarithmic correction cannot be computed by the heat kernel expansion and would require knowledge of the heat kernel at general values of s. For a background with k independent length scales $\ell_1, \dots, \ell_k$ (in Planck units), the general logarithmic correction would take the form

$$S = \frac{A}{4G} + \sum_{i=1}^{k} C_i \log \ell_i + \dots \ , \tag{5.2.7}$$

with an independent coefficient C_i for each independent length scale ℓ_i. In these terms, the heat kernel expansion can only give us the sum

$$C = \sum_{i=1}^{k} C_i \ , \tag{5.2.8}$$

without being able to access the individual C_i. Indeed, C is the coefficient of $\log\lambda$ if we write $\ell_i = \lambda \ell_i^{(0)}$ with $\ell_i^{(0)}$ fixed.

Let us now contrast this regime with the flat space regime of [21–23]. In flat space, we do not rescale the mass m of massive fields. As a consequence, it can be shown that massive fields do not contribute to the logarithmic correction of flat space black holes. In AdS, the prescription is to fix the conformal dimension, or equivalently the combination $m\ell$, so we get non-trivial logarithmic corrections for massive fields as a function of the conformal dimension. Clearly we can see that in the flat space limit $\ell \to +\infty$, only fields with $m = 0$ can contribute and in that limit, we actually reproduce the scaling regime of [21–23]. We indeed see that we reproduce known results for flat space black holes by taking the flat space limit of our results.

It can also be shown that higher loops do not contribute to the logarithmic correction as they are suppressed by positive powers of λ [22]. Summarizing, the logarithmic correction to the entropy arises only at one-loop from the two-derivative Lagrangian and can be unambiguously computed in the low-energy effective theory.

For extremal black holes, we need to be more careful. In particular, the thermal circle is infinite which naively makes the Euclidean on-shell action divergent. To obtain a well-defined $\beta \to +\infty$ limit, we remove a divergence that can be viewed as an infinite shift in the ground state energy. This can be made precise using the quantum entropy function [143] in which the quantum entropy is defined using the $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence in the near horizon geometry. This procedure was used, for example, in [19–21].

5.2.3 Heat kernel expansion

We will now describe the main technical tool which makes possible the exact computation of the logarithmic correction for a variety of black holes: the heat kernel expansion [158–160].

The one-loop correction to the partition function decomposes as a contribution Z_{nz} from the non-zero modes and a contribution Z_{zm} from the zero modes of the corresponding kinetic operators, so that we have

$$Z_{\text{1-loop}}(\beta,\mu_\alpha) = Z_{\text{nz}} Z_{\text{zm}}\, e^{-\mathcal{S}_E^{\text{class.}}} \ . \tag{5.2.9}$$

The one-loop corrected Bekenstein-Hawking entropy, defined in the microcanonical ensemble, takes the form

$$S = \frac{A}{4G} + (C_{\text{local}} + C_{\text{global}}) \log\lambda + \ldots \ . \tag{5.2.10}$$

Here C_{local} is the local contribution computed using the heat kernel. The global term C_{global} is an integer correction due to the zero modes and the change of ensemble from canonical to microcanonical. We now explain how to compute the local contribution. It originates from the non-zero modes

$$\log Z_{\text{nz}} = -\frac{1}{2}{\sum_n}' \log\kappa_n \ , \tag{5.2.11}$$

where κ_n are the eigenvalues of the quadratic operator $\mathcal{Q}$ and the primed sum runs only over the non-zero eigenvalues $\kappa_n \neq 0$. This can be computed by introducing the heat kernel

$$K(x,s) = \sum_n e^{-\kappa_n s} f_n^\ell(x) f_n^{\ell'}(x) G_{\ell\ell'} \ , \tag{5.2.12}$$

where $\{f_n^\ell\}$ are the ortho-normalized eigenfunctions of $\mathcal{Q}$ with eigenvalues $\{\kappa_n\}$ and $G_{\ell\ell'}$ is the metric on field space. In particular, we have

$$\int_{\mathcal{M}} d^D x\sqrt{g}\, K(x,s) = \sum_n e^{-s\kappa_n} = {\sum_n}' e^{-s\kappa_n} + N_{\text{zm}} \ , \tag{5.2.13}$$

where N_{zm} is the number of zero modes. We will make use of the relation

$$\log\kappa - \log\kappa^{(0)} = -\lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{ds}{s}\left(e^{-s\kappa} - e^{-s\kappa^{(0)}}\right) \ . \tag{5.2.14}$$

In our scaling regime, the eigenvalues are rescaled according to (5.2.6). This allows us to show that we have

$$\log Z_{\text{nz}} - \log Z_{\text{nz}}^{(0)} = \frac{1}{2}\int_\epsilon^{\epsilon\lambda} \frac{ds}{s}\left(\int_{\mathcal{M}} d^D x\sqrt{g}\, K(x,s) - N_{\text{zm}}\right) \ . \tag{5.2.15}$$

The above expression makes it clear that only the range of very small s contributes due to a cancellation between Z_{nz} and $Z_{\text{nz}}^{(0)}$. We can then use the heat kernel expansion which states the existence of a small s expansion of the form

$$K(x,s) = \sum_{n\geqslant 0} s^{n-D/2} a_{2n}(x) \tag{5.2.16}$$

where D is the dimension of spacetime. The coefficients $a_{2n}(x)$ are known as Seeley-DeWitt coefficients. For smooth manifolds, $a_{2n}(x)$ is a sum of $2n$-derivative terms constructed from the fields appearing in the action [158].

We are mainly interested in $D = 4$ for which we have

$$K(x,s) = s^{-2}a_0(x) + s^{-1}a_2(x) + s^0 a_4(x) + \mathcal{O}(s) \ . \tag{5.2.17}$$

We want to compute the $\log\lambda$ contribution in $\log Z_{\text{nz}}$. The integral (5.2.15) makes it clear that this comes from the a_4 coefficient and we have

$$\log Z_{\text{nz}} = C_{\text{local}} \log\lambda + \dots \ , \tag{5.2.18}$$

where we have defined

$$C_{\text{local}} \equiv \int d^4x\sqrt{g}\, a_4(x) \ . \tag{5.2.19}$$

We refer to this as the local contribution as it is given by an integral over spacetime. In general spacetime dimension D, $a_4(x)$ should be replaced by $a_D(x)$ in the above formula. Note that this vanishes when D is odd so there is no local contribution in odd-dimensional spacetimes.

The power of the heat kernel expansion lies in the fact that there is a general expression for $a_4(x)$ summarized in [158]. This allows to compute C_{local} without computing the eigenvalues of $\mathcal{Q}$.

The other Seeley-DeWitt coefficients $a_0(x)$ and $a_2(x)$ capture one-loop corrections to the cosmological constant, Newton's constant and the other couplings in the Lagrangians. This is discussed, for completeness, in Appendix B.4.

Bosonic fluctuations. We write the operator of quadratic fluctuations for bosons as

$$\mathcal{Q}^n_m = (\Box)I^n_m + 2(\omega^\mu D_\mu)^n_m + P^n_m \ , \tag{5.2.20}$$

where the Latin indices m, n refer to the different fields and D_μ is the spacetime covariant derivative. We define $\mathcal{D}_\mu = D_\mu + \omega_\mu$ to complete the square so that

$$\mathcal{Q}_m^n = (\mathcal{D}^\mu \mathcal{D}_\mu) I_m^n + E_m^n, \tag{5.2.21}$$

with

$$E \equiv P - \omega^\mu \omega_\mu - (D^\mu \omega_\mu) \ . \tag{5.2.22}$$

The Seeley-DeWitt coefficient $a_4(x)$ is then given explicitly by the formula

$$\begin{aligned}(4\pi)^2 a_4(x) &= \mathrm{Tr}\Big[\frac{1}{2}E^2 + \frac{1}{6}RE + \frac{1}{12}\Omega_{\mu\nu}\Omega^{\mu\nu} \\ &\qquad + \frac{1}{360} I(5R^2 + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu})\Big] \ ,\end{aligned} \tag{5.2.23}$$

where $\Omega_{\mu\nu} = [D_\mu + \omega_\mu, D_\nu + \omega_\nu]$ is the curvature associated to the connection $\mathcal{D}_\mu$.

Fermionic fluctuations. For fermionic fields, the quadratic Lagrangian takes the form $\mathcal{L} = \bar{\psi}\mathcal{D}\psi$ where $\mathcal{D} = \not{D} + L$ is a Dirac-type operator and ψ denotes all the fermions of the theory. The prescription is then to use the fact that

$$\log\det\mathcal{D} = \frac{1}{2}\log\det\mathcal{D}^\dagger\mathcal{D}, \tag{5.2.24}$$

so that we can apply the heat kernel method to $\mathcal{Q} = \mathcal{D}^\dagger\mathcal{D}$. We have, more explicitly,

$$\omega^\mu = \frac{1}{2}\big(\gamma^\mu L - L^\dagger\gamma^\mu\big), \qquad P = \mathcal{R} + \big(\not{D}L\big) - L^\dagger L, \tag{5.2.25}$$

where $\mathcal{R} = -\frac{1}{4}R$ for spin $\frac{1}{2}$ and $\mathcal{R} = -\frac{1}{4}g_{\mu\nu} + \frac{1}{2}\gamma^{\rho\sigma}R_{\mu\nu\rho\sigma}$ for spin $\frac{3}{2}$.

5.2.4 Global contribution

The global contribution consists of an integer correction which is the sum of two contributions

$$C_{\text{global}} = C_{\text{ens}} + C_{\text{zm}}. \tag{5.2.26}$$

The first term corresponds to the correction due to changing from the grand canonical to the microcanonical ensemble [22].

The zero modes are associated to asymptotic symmetries: gauge transformations with parameters that do not vanish at infinity and are thus, not normalizable. In the path integral, we can treat them by making a change of variable to the parameters of the asymptotic symmetry group. For a field Ψ, the Jacobian of this change of variable introduces a factor

$$\lambda^{\beta_\Psi} \ , \tag{5.2.27}$$

which contributes a logarithmic correction $\beta_\Psi \log L$ to the entropy. As a result, the total contribution from the zero modes is

$$C_{\rm zm} = \sum_\Psi (\beta_\Psi - 1) n^0_\Psi, \tag{5.2.28}$$

where we are summing over all fields Ψ (including ghosts) and we denote by n^0_Ψ the number of zero modes for Ψ. There is a -1 because we include here the $-N_{\rm zm}$ which was in the non-zero mode contribution (5.2.15) (and not included in $C_{\rm local}$). The value of β_Ψ can be computed by normalizing correctly the path integral measure. We refer to [21] for a more detailed discussion. As an illustration, we report below the values of β_Ψ for the gauge field, the Rarita-Schwinger field and the graviton in D spacetime dimensions

$$\beta_A = \frac{D}{2} - 1, \qquad \beta_\psi = D - 1, \qquad \beta_g = \frac{D}{2} \ . \tag{5.2.29}$$

5.3 Black hole backgrounds

In this section, we present the background geometries for which we compute the logarithmic corrections. They are solutions of Einstein-Maxwell theory with a negative cosmological constant.

We give the integrated four-derivative terms as a precursor to the computations of the logarithmic corrections and describe the extremal limit and the near horizon geometry. At this level, we already observe that the local contribution $C_{\rm local}$ for extremal black holes is the same in the full geometry and in the near horizon geometry so that the only difference is due to the zero mode contribution.

In the following subsections, we review the metrics of AdS-Schwarzschild, thermal AdS_4 and the Reissner-Nordström AdS_4 black hole as simple examples before we consider the general Kerr-Newman AdS_4 black hole solution with particular emphasis on its BPS limit. We compute the curvature invariants in both the full solution and the near horizon before giving the general result for the logarithmic corrections to the entropy. The results are written in terms of the theory-dependent coefficients $a_{\rm E}, c, b_1$ and b_2. The computation of these coefficients for the theories of interest will be the subject of subsequent sections.

5.3.1 General structure

The local contribution to the logarithmic correction is given by the Seeley-DeWitt coefficient $a_4(x)$ using (5.2.19). For solutions of Einstein-Maxwell-AdS theory, a general four-derivative term can be decomposed as

$$(4\pi)^2 a_4(x) = -a_{\rm E}\, E_4 + c\, W^2 + b_1 R^2 + b_2 R F_{\mu\nu} F^{\mu\nu} \ , \tag{5.3.1}$$

after using the equations of motion for the background fields. Here we write the curvature invariants in terms of the Euler density and the Weyl tensor squared given explicitly as

$$E_4 \equiv R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \ , \tag{5.3.2}$$

$$W^2 \equiv R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 2 R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2 \ . \tag{5.3.3}$$

Note that the equations of motion implies that $R = 4\Lambda = -12/\ell^2$. The difference with the previous flat space computations lies in the last two terms in (5.3.1), which vanish if $R = 0$. These terms are responsible for making the logarithmic correction non-topological.

To regularize the integral over spacetime, we use the same prescription as in holographic renormalization, which gives an unambiguous finite answer. A consistency check on this procedure is that for the Euler term, the regularized integral gives

$$\chi = \frac{1}{32\pi^2} \int d^4x \sqrt{g}\, E_4, \qquad \text{(regularized)} \tag{5.3.4}$$

where χ is the Euler characteristic of spacetime. This is possible because our regularization procedure produces the same boundary as the one appearing in the Gauss-Bonnet-Chern theorem, as we explain in Appendix B.5. Thus, we see that the logarithmic correction is topological if and only if $a_4(x)$ contains only the Euler term, that is, $c = b_1 = b_2 = 0$.

5.3.2 AdS-Schwarzschild black hole

The Euclidean AdS-Schwarzschild black hole is described by the line element

$$ds^2 = f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \qquad f(r) = 1 + \frac{r^2}{\ell^2} - \frac{2m}{r} \ , \tag{5.3.5}$$

where m is the mass of the black hole and ℓ is the radius of AdS_4. Here-forth, Euclidean time is identified with a period proportional to the inverse Hawking temperature β,

$$t \sim t + \beta, \qquad \beta = \frac{4\pi r_+}{1 + \frac{3r_+^2}{\ell^2}} \ , \tag{5.3.6}$$

where r_+ is the position of the horizon given by the largest real root of $f(r_+) = 0$. The curvature invariants in (5.3.1) for this solution are

$$E_4 = \frac{24}{\ell^4} + \frac{48m^2}{r^6} \ , \qquad W^2 = \frac{48m^2}{r^6} \ , \qquad R^2 = \frac{144}{\ell^4} \ , \qquad RF_{\mu\nu}F^{\mu\nu} = 0 \ . \tag{5.3.7}$$

The integrated curvature invariant are divergent due to the infinite volume. To regularize these divergences, we utilize the same prescription as holographic renormalization [161,162]. Such choice of renormalization is natural given that the logarithmic contributions are corrections to the on-shell action and it allows us to obtain finite and unambiguous results in all cases. A more systematic understanding of this prescription would require a quantum version of holographic renormalization.

The prescription is to impose a cutoff at large $r = r_c$. At the boundary, we add a counter term written in terms of intrinsic data

$$a_4^{\text{CT}} = \int_{\partial M} d^3y\sqrt{h}\,(c_1 + c_2\mathcal{R}) \ , \tag{5.3.8}$$

where $\mathcal{R}$ is the Ricci curvature of the boundary ∂M. The coefficients c_1, c_2 are determined by the requirement that $a_4 + a_4^{\text{CT}}$ remains finite as we take $r_c \to +\infty$. The regularized integrated invariants take the form

$$\begin{aligned}
&\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\,E_4 = 4 \ , && \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\,W^2 = \frac{4(\ell^2 + r_+^2)^2}{\ell^2(\ell^2 + 3r_+^2)} \ , \\
&\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\,R^2 = \frac{24r_+^2(\ell^2 - r_+^2)}{\ell^2(\ell^2 + 3r_+^2)} \ , && \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\,RF_{\mu\nu}F^{\mu\nu} = 0 \ .
\end{aligned} \tag{5.3.9}$$

As expected from the Gauss-Bonnet-Chern theorem, the Euler characteristic is

$$\chi = \frac{1}{32\pi^2}\int d^4x\sqrt{g}\,E_4 = 2. \tag{5.3.10}$$

In fact, we verify that with the holographic renormalization procedure, the integral of the Euler density is always the Euler characteristic of the spacetime, for all the backgrounds considered in this chapter. This suggests that the holographic counterterm reproduces exactly the boundary term comparable to that of the Gauss-Bonnet-Chern theorem. This is evidence that our renormalization procedure is correct and we refer to Appendix B.5 for details.

The final result for $C_{\rm local}$ for AdS-Schwarzschild takes the form

$$C_{\rm local} = \frac{4}{\ell^2(\ell^2+3r_+^2)}\left((c-a_{\rm E})\ell^4 + (2c-3a_{\rm E}+6b_1)\ell^2 r_+^2 + (c-6b_1)r_+^4\right) \ . \tag{5.3.11}$$

Thermal AdS$_4$

We are mainly interested in logarithmic corrections to black hole entropy. However, the dominant saddle-point in the canonical ensemble is not always a black hole in AdS. For temperatures below the Hawking-Page transition [163], it is a thermal AdS. Our computation gives the logarithmic corrections to the free energy of AdS$_4$. The metric of the AdS spacetime with only radiation, *thermal AdS*, is given by

$$ds^2 = \left(1+\frac{r^2}{\ell^2}\right)dt^2 + \frac{dr^2}{\left(1+\frac{r^2}{\ell^2}\right)} + r^2 d\Omega^2 \ . \tag{5.3.12}$$

The curvature invariants for the thermal AdS background read

$$E_4 = \frac{24}{\ell^4}, \qquad W^2 = 0, \qquad R^2 = \frac{144}{\ell^4} \ , \qquad F_{\mu\nu}F^{\mu\nu} = 0 \ . \tag{5.3.13}$$

Using the same regularization procedure as above, the integrated invariants all vanish

$$\begin{aligned} &\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\,E_4 = 0 \ , && \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\,W^2 = 0 \ , \\ &\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\,R^2 = 0 \ , && \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\,RF_{\mu\nu}F^{\mu\nu} = 0 \ . \end{aligned} \tag{5.3.14}$$

This shows that on thermal AdS$_4$, we have $a_4(x) = 0$ so that the local contribution vanishes

$$C_{\rm local} = 0 \ , \tag{5.3.15}$$

and the logarithmic correction comes only from the zero mode contribution. Thus we may use $\mathrm{C}_{\rm local}$ as an order parameter indicating the Hawking page transition. In the case of Einstein-Maxwell theory, we must include a fixed gauge potential Φ as thermal AdS [75, 164, 165]. Since it is a pure gauge, it does not affect the logarithmic term of the entropy.

5.3.3 Reissner-Nordström black hole

We now turn to the AdS-Reissner-Nordström black hole and its extremal limit. It is important to note that this black hole is not a BPS solution of minimal gauged supergravity. A non-zero rotation is necessary to solve the BPS equations as we discuss in the next section.

Non-extremal black hole

The Euclidean Reissner-Nordström black hole in AdS is described by

$$ds^2 = f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \,, \qquad A = \frac{iq_e}{r}dt - q_m \cos\theta \, d\phi \,. \tag{5.3.16}$$

with

$$f(r) = 1 + \frac{r^2}{\ell^2} - \frac{2m}{r} + \frac{q_e^2 + q_m^2}{r^2} \,, \tag{5.3.17}$$

where m, q_e and q_m characterize the mass, the electric charge and the magnetic charge of the black hole, respectively. The horizon r_+ is the largest root of $f(r) = 0$ and the Hawking temperature is

$$T_H = \beta^{-1} = \frac{f'(r_+)}{4\pi} = \frac{1}{4\pi r_+}\left(1 + \frac{3r_+^2}{\ell^2} - \frac{(q_e^2 + q_m^2)}{r_+^2}\right) \,. \tag{5.3.18}$$

The curvature invariants are computed to be

$$\begin{aligned} R^2 &= \frac{144}{\ell^4} \,, & E_4 &= \frac{24}{\ell^4} + \frac{8\left(6m^2r^2 - 12m(q_e^2+q_m^2)r + 5(q_e^2+q_m^2)^2\right)}{r^8} \,, \\ F_{\mu\nu}F^{\mu\nu} &= -\frac{2(q_e^2 - q_m^2)}{r^4} \,, & W^2 &= \frac{48\left(mr - (q_e^2+q_m^2)\right)^2}{r^8} \,. \end{aligned} \tag{5.3.19}$$

The integrated invariants can be computed using the same renormalization procedure as described above for the AdS-Schwarzschild case. The results are

$$\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\, E_4 = 4 \,, \tag{5.3.20}$$

$$\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\, W^2 = \frac{2}{5}\left(2 - \frac{14r_+^2}{\ell^2} + \frac{16\pi r_+}{\beta} + \frac{(\ell^4 + \ell^2 r_+^2 + 4r_+^4)}{\pi\ell^4 r_+}\beta\right) , \tag{5.3.21}$$

$$\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\, R^2 = \frac{12r_+(r_+^2 + \ell^2)}{\pi\ell^4}\beta - \frac{24r_+^2}{\ell^2} \,, \tag{5.3.22}$$

$$\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\, RF_{\mu\nu}F^{\mu\nu} = \frac{6(3r_+^4 + \ell^2 r_+^2 - 2\ell^2 q_m^2)}{\pi\ell^4 r_+}\beta - \frac{24r_+^2}{\ell^2} \,. \tag{5.3.23}$$

The final result for the Reissner-Nordström black hole takes the following form

$$
\begin{aligned}
C_{\text{local}} \;&=\; \frac{2}{5}\Bigg(2(c-5a_{\text{E}}) - \frac{2r_+^2}{\ell^2}(7c+30(b_1+b_2)) + \frac{16\pi r_+}{\beta}c \\
&\quad + \frac{\beta}{\pi\ell^4 r_+}\Big(c\ell^4 + \big(c+30b_1+15b_2\big)\ell^2 r_+^2 + \big(4c+30b_1+45b_2\big)r_+^4 - 30b_2\ell^2 q_m^2\Big)\Bigg) \;.
\end{aligned}
\tag{5.3.24}
$$

The appearance of q_m indicates that if the final result has a non-vanishing b_2, the logarithmic correction does not preserves the electromagnetic duality. As we will see in section 5.6, if we consider $\mathcal{N}=2$ supergravity, we do have a non-trivial b_2.

Extremal limit

The result for the extremal black hole is obtained by taking the $T \to 0$ or $\beta \to +\infty$ limit. This limit is naively divergent and we will describe how to implement it in this context. The prescription is as follows. First, the outer horizon is a function of β, and must be substituted as an explicit expression in terms of β. We then take the $\beta \to \infty$ limit while keeping the charges fixed and subsequently impose the extremal values of the charges. The low-temperature expansion yields

$$
r_+ = r_0 + \frac{2\pi\ell_2^2}{\beta} + O(\beta^{-2}), \tag{5.3.25}
$$

where r_0 is the position of the extremal horizon and ℓ_2 is the AdS$_2$ radius and can be expressed as

$$
r_0^2 = \frac{1}{6}\ell(\sqrt{\ell^2+12q^2}-\ell)\;, \qquad \ell_2^2 = \frac{r_0^2}{1+\frac{6r_0^2}{\ell^2}} = \frac{\ell^2}{6}\left(1-\frac{\ell}{\sqrt{\ell^2+12q^2}}\right)\;. \tag{5.3.26}
$$

In the $\beta \to +\infty$, we generally have

$$
\int d^4x\sqrt{g}\,a_4(x) = C_1\beta + C_0 + O(\beta^{-1})\;. \tag{5.3.27}
$$

The first term, linear in β, is divergent. As this expression is a correction to the effective action, we can interpret this term as a shift of the ground state energy due to one-loop fluctuations. As a result, we ignore this term and define the limit $\beta \to +\infty$ to be the constant term C_0. The resulting four-derivative terms are

$$\lim_{\beta\to+\infty}\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\,E_4 = 4\,, \tag{5.3.28}$$

$$\lim_{\beta\to+\infty}\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\,W^2 = -\frac{2(r_0^2-\ell_2^2)^2}{3r_0^2\ell_2^2}\,, \tag{5.3.29}$$

$$\lim_{\beta\to+\infty}\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\,R^2 = -\frac{2(r_0^2-\ell_2^2)^2}{r_0^2\ell_2^2}\,, \tag{5.3.30}$$

$$\lim_{\beta\to+\infty}\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\,RF_{\mu\nu}F^{\mu\nu} = -\frac{(r_0^2-\ell_2^2)(r_0^4+r_0^2\ell_2^2-4q_m^2\ell_2^2)}{r_0^4\ell_2^2}\,. \tag{5.3.31}$$

This leads to the final result

$$\lim_{\beta\to+\infty}C_{\text{local}} = -4\,a_{\text{E}} - \frac{r_0^2-\ell_2^2}{r_0^2\ell_2^2}\left(\left(\frac{2}{3}c+2b_1\right)\left(r_0^2-\ell_2^2\right)+b_2\left(r_0^2+\ell_2^2-\frac{4\ell_2^2q_m^2}{r_0^2}\right)\right)\,. \tag{5.3.32}$$

Note that in the flat space limit, we have $r_0=\ell_2$ and the logarithmic correction is manifestly topological, but such cancellation does not occur for AdS black holes.

Near horizon geometry

As we would like to investigate where the quantum degrees of freedom live for asymptotically AdS spacetimes, we compare the basis of curvature invariants of the full solution to that of the near horizon geometry. Let us first consider the extremal black hole. The near horizon geometry can be obtained using the change of coordinates

$$r\to r_0+\epsilon\,\tilde r,\qquad t\to \ell_2^2\,\frac{\tilde t}{\epsilon} \tag{5.3.33}$$

and taking the limit $\epsilon\to 0$. The result is the $\text{AdS}_2\times S^2$ geometry

$$ds^2=\ell_2^2\left(\tilde r^2 d\tilde t^2+\frac{d\tilde r^2}{\tilde r^2}\right)+r_0^2\,d\Omega_2^2,\qquad A=-\frac{i\ell_2^2 q_e}{r_0^2}\tilde r d\tilde t+q_m\cos\theta\,d\phi\,, \tag{5.3.34}$$

where ℓ_2 and r_0 are defined in (5.3.26). For the gauge field, a pure gauge term needs to be added to obtain a smooth $\epsilon\to 0$ limit. We can express everything in terms of the two scales ℓ_2 and r_0. The AdS_4 radius and the extremal charges are given by

$$\frac{6}{\ell^2}=\frac{1}{\ell_2^2}-\frac{1}{r_0^2}\,,\qquad q_e^2+q_m^2=\frac{r_0^2(r_0^2+\ell_2^2)}{2\ell_2^2}\,. \tag{5.3.35}$$

In particular we see that we must have $r_0>\ell_2$. Note that in flat space we obtain $r_0=\ell_2$.

The infinite volume of AdS_2 is regularized by removing the divergence through a redefinition of the ground state energy in the dual CFT_1 [143, 166]. This leads to a regularized volume of unit AdS_2 which is -2π. The integrated invariants can then be computed and we find

$$C_{\text{local}} = -4a_{\text{E}} - \frac{r_0^2 - \ell_2^2}{r_0^2 \ell_2^2} \left(\left(\frac{2}{3}c + 2b_1 \right) \left(r_0^2 - \ell_2^2 \right) + b_2 \left(r_0^2 + \ell_2^2 - \frac{4\ell_2^2 q_m^2}{r_0^2} \right) \right) . \tag{5.3.36}$$

This expression matches the result (5.3.32) obtained by taking the $\beta \to +\infty$ limit of the non-extremal C_{local}. Hence, the computation of C_{local} for an extremal black hole can be done either in the full geometry or in the near horizon region. The difference in logarithmic correction between the full geometry and the near horizon geometry come exclusively from zero modes.

5.3.4 Kerr-Newman black hole

We now turn to the AdS-Kerr-Newman black hole [75, 146]. This solution is particularly interesting because it has a regular BPS limit unlike the Reissner-Nordström black hole [167–169].

Non-extremal black hole

As given in [75], the line element takes the form ,

$$ds^2 = -\frac{\Delta_r}{\rho^2} \left(dt - \frac{a \sin^2\theta}{\Xi} d\phi \right)^2 + \frac{\rho^2 dr^2}{\Delta_r} + \frac{\rho^2 d\theta^2}{\Delta_\theta} + \frac{\Delta_\theta \sin^2\theta}{\rho^2} \left(a\, dt - \frac{r^2 + a^2}{\Xi} d\phi \right)^2 , \tag{5.3.37}$$

where we have defined

$$\Delta_r = \left(r^2 + a^2 \right) \left(1 + \frac{r^2}{\ell^2} \right) - 2mr + q_e^2 + q_m^2, \qquad \Delta_\theta = 1 - \frac{a^2}{\ell^2} \cos^2\theta , \tag{5.3.38}$$

with $\rho^2 = r^2 + a^2 \cos^2\theta$ and $\Xi = 1 - \frac{a^2}{\ell^2}$. The gauge field is given by

$$A = -\frac{q_e r}{\rho^2} \left(dt - \frac{a \sin^2\theta}{\Xi} d\phi \right) - \frac{q_m \cos\theta}{\rho^2} \left(a dt - \frac{r^2 + a^2}{\Xi} d\phi \right) , \tag{5.3.39}$$

The parameters satisfy $a^2 < \ell^2$ and we take $a \geqslant 0$ without loss of generality.[2] The physical mass M, angular momentum J, electric charge Q_e and magnetic charge Q_m are given by

$$M = \frac{m}{\Xi^2}, \qquad J = \frac{am}{\Xi^2}, \qquad Q_e = \frac{q_e}{\Xi}, \qquad Q_m = \frac{q_m}{\Xi}, \tag{5.3.40}$$

[2] The general result is obtained by replacing $a \to |a|$ everywhere.

and the inverse temperature is

$$\beta = \frac{4\pi\left(r_+^2 + a^2\right)}{r_+\left(1 + \frac{a^2}{\ell^2} + 3\frac{r_+^2}{\ell^2} - \frac{a^2+q_e^2+q_m^2}{r_+^2}\right)}. \tag{5.3.41}$$

For the non-extremal black hole, the general form is

$$C_{\text{local}} = -4\,a_{\text{E}} + (6A_1 + cW_1)\beta + (24A_2 + cW_2) + \frac{c\,W_3}{\beta}\ , \tag{5.3.42}$$

where the logarithmic corrections depends on five independent parameters $\{r_+, \beta, \ell, a, q_m\}$. The Euler term simply gives a pure number in agreement with the formula

$$\chi = \frac{1}{32\pi^2}\int d^4x\sqrt{g}E_4 = 2\ . \tag{5.3.43}$$

The expressions A_i and W_i are independent of β and take the form

$$\begin{aligned} A_1 &= \frac{(2b_1+b_2)(a^2+\ell^2)r_+^3 + (2b_1+3b_2)r_+^5 + ((2b_1-b_2)a^2 - 2b_2q_m^2)r_+\ell^2}{\pi\ell^2(\ell^2-a^2)(a^2+r_+^2)}\ ,\\ A_2 &= -\frac{b_1a^2 + (b_1+b_2)r_+^2}{\ell^2-a^2}\ , \end{aligned} \tag{5.3.44}$$

and we have isolated the contribution W_i from the Weyl squared term, explicitly given as

$$\begin{aligned} W_1 &= \frac{1}{16\pi a^5 r_+^4\ell^2(\ell^2-a^2)(a^2+r_+^2)}\big[3ar_+(a^8(\ell^2-r_+^2)^2 + r_+^8(\ell^2+3r_+^2)^2)\\ &\quad -4a^3r_+^3(r_+^4(\ell^4-9r_+^4) + a^4(\ell^4+12\ell^2r_+^2+3r_+^4) + 2a^5r_+^5(\ell^4-14\ell^2r_+^2+5r_+^4)\\ &\quad -3(a^2+r_+^2)(a^2(\ell^2-r_+^2) - r_+^2(\ell^2+3r_+^2))^2(r_+^4-a^4)\arctan(a/r_+)\big]\ ,\\ W_2 &= \frac{a^2+r_+^2}{2a^5r_+^3(\ell^2-a^2)}\big[4a^3\ell^2r_+^3 + 3ar_+(a^4(\ell^2-r_+^2) - r_+^4(\ell^2+3r_+^2))\\ &\quad -3(a^2(\ell^2-r_+^2) - r_+^2(\ell^2+3r_+^2))(r_+^4-a^4)\arctan(a/r_+)\big]\ ,\\ W_3 &= \frac{\pi\ell^2(a^2+r_+^2)}{a^5r_+^2(\ell^2-a^2)}\big[ar_+(3a^4+2a^2r_+^2+3r_+^4) - 3(r_+^2+a^2)(r_+^4-a^4)\arctan(a/r_+)\big]\ . \end{aligned} \tag{5.3.45}$$

We have checked that we reproduce the Reissner-Nordström results of section 5.3.3 in the limit $a = 0$.

Extremal limit

As was done in the Reissner-Nordström black hole, the extremal limit can be found by taking the limit $T \to 0$ or $\beta \to +\infty$ while keeping the charges fixed. To do this appropriately, we use that for small temperatures

$$r_+ = r_0 + \frac{2\pi\ell_2^2}{\beta} + O(\beta^{-2}) \ , \tag{5.3.46}$$

and we take the $\beta \to +\infty$ limit while keeping r_0, ℓ, a, q_m fixed. The procedure yields a finite piece in β as well as a piece linear in β, which can be removed by a renormalization of the ground state energy. The final result can be written in terms of the four independent parameters $\{r_0, \ell, a, q_m\}$. It takes the form

$$\begin{aligned} C_{\rm local} \ &= \ -4a_{\rm E} + \frac{1}{2ar_0^5(\ell^2-a^2)(a^2+r_0^2)(a^2+\ell^2+6r_0^2)}\Big[-3a^7 r_0 (16b_1 r_0^4 + c(\ell^2-r_0^2)^2). \\ &\quad +a^5 r_0^3 (c\ell^4 + 2(11c-12b_2)\ell^2 r_0^2 - 3(13c-8b_2+80b_1)r_0^4) \\ &\quad +a^3 r_0^5 (15c\ell^4 + 2(25c+24b_2)\ell^2 r_0^2 - (49c+336b_1-48b_2)r_0^4 - 48b_2\ell^2 q_m^2) \\ &\quad +3ar_0^7 (c\ell^4 + 2(3c-4b_2)\ell^2 r_0^2 - (7c+48b_1+24b_2)r_0^4 + 16b_2\ell^2 q_m^2) \\ &\quad -3c(a^2+r_0^2)(a^2(r_0^2-\ell^2) + r_0^2(\ell^2+3r_0^2))^2 \arctan(a/r_0) \Big] \ . \end{aligned} \tag{5.3.47}$$

We can also compare with the computation performed in the near horizon geometry obtained via

$$r \to r_0 + \epsilon\,\tilde r, \qquad t \to \ell_2^2\,\frac{\tilde t}{\epsilon}, \qquad \phi \to \phi - \frac{ia\ell_2^2(\ell^2-a^2)}{\ell^2(a^2+r_0^2)}\,\frac{t}{\epsilon} \ , \tag{5.3.48}$$

while taking $\epsilon \to 0$. This leads to

$$\begin{aligned} d\tilde s^2 \ &= \ \frac{\ell_2^2(r_0^2+a^2\cos^2\theta)}{a^2+r_0^2}\left(\tilde r^2 d\tilde t^2 + \frac{d\tilde r^2}{\tilde r^2}\right) + \frac{\ell^2(r_0^2+a^2\cos^2\theta)}{\ell^2-a^2\cos^2\theta}d\theta^2 \\ &\quad + \frac{\ell^2(a^2+r_0^2)^2(\ell^2-a^2\cos^2\theta)\sin^2\theta}{(\ell^2-a^2)^2(r_0^2+a^2\cos^2\theta)}\left(d\phi - \frac{2\ell_2^2 a r_0(\ell^2-a^2)}{\ell^2(a^2+r_0^2)^2}\,ir dt\right)^2 \ , \end{aligned} \tag{5.3.49}$$

where the AdS_2 radius is

$$\ell_2 = \ell\sqrt{\frac{a^2+r_0^2}{a^2+\ell^2+6r_0^2}} \ . \tag{5.3.50}$$

The near horizon geometry is a warped version of AdS_2 with a circle fiber, similar to the near horizon of extreme Kerr (NHEK), which we recover in the appropriate limit. The near horizon gauge field takes the form

$$\begin{aligned} \widetilde A \ &= \ \frac{1}{r_0^2+a^2\cos^2\theta}\Big[-\frac{i\ell_2^2}{a^2+r_0^2}(q_e(r_0^2-a^2\cos^2\theta) + 2q_m\, a r_0\cos\theta)\tilde r d\tilde t \\ &\qquad + \frac{\ell^2}{\ell^2-a^2}(q_e\, a r_0 \sin^2\theta + q_m(a^2+r_0^2)\cos\theta) d\tilde\phi \Big] \ . \end{aligned} \tag{5.3.51}$$

We can perform more general near horizon limits by taking at the same time a near-extremal limit. Instead of setting $q_e = q_e^*$, we can consider a deformation $q_e = q_e^* + \delta q_e \epsilon^2$ parametrized by the same ϵ as in (5.3.48). Moreover, keeping subleading corrections in β^{-1} would yield corrections to the entropy in the near-extremal regime. The non-zero energy associated to this large diffeomorphism can be understood in terms of the Schwarzian action of Jackiw-Teitelboim gravity [170].

We are now in a position to compute the logarithmic corrections in the near horizon geometry and we find that the result is equal to (5.3.47) obtained by taking the extremal limit appropriately, *i.e.*, fixing the charges while taking $\beta \to +\infty$. Thus, the local contribution is the same in the near horizon region and the full geometry.

BPS limit

The BPS limit can be obtained by imposing the additional conditions to the extremal black hole

$$r_0 = \sqrt{a\ell}, \qquad q_m = 0 \ . \tag{5.3.52}$$

The resulting black hole preserves half of the supersymmetries [168]. Its charges are given by

$$M = \frac{\sqrt{a\ell}}{\left(1-\frac{a}{\ell}\right)^2}, \qquad Q_e = \frac{\sqrt{a\ell}}{1-\frac{a}{\ell}}, \qquad Q_m = 0, \qquad J = \frac{a\sqrt{a\ell}}{\left(1-\frac{a}{\ell}\right)^2} \tag{5.3.53}$$

and it satisfies a BPS bound:

$$M = Q_e + \frac{J}{\ell} \ . \tag{5.3.54}$$

The BPS result can be written in terms of the two independent parameters ℓ and a

$$\begin{aligned} C_{\rm local} \;\; = \;\; -4a_{\rm E} + \frac{3\ell_2^2}{2a\ell^2(\ell^2-a^2)}\Bigg[&(9c-8b_2)a\ell^2 - (9c+48b_1-8b_2)a^2\ell - (c+16b_1)a^3 \\ &+c\ell^3 - \frac{c(a+\ell)^4}{\sqrt{al}}\arctan(\sqrt{a/\ell})\Bigg], \end{aligned} \tag{5.3.55}$$

where the AdS$_2$ radius given in (5.3.50) is

$$\ell_2 = \ell\sqrt{\frac{a(a+\ell)}{a^2+6a\ell+\ell^2}} \qquad \text{(BPS case)} \ . \tag{5.3.56}$$

It is clear from this formula that there is no non-rotating BPS solution as the limit $a \to 0$ is singular.

5.3.5 AdS-Rindler geometry

Our computation of the logarithmic correction can also be applied to the so-called hyperbolic black hole of [147], i.e., the AdS_4-Rindler geometry. The entropy of this black hole is the entanglement entropy

$$S_{\mathrm{EE}} = -\mathrm{Tr}\, \rho_A \log \rho_A, \tag{5.3.57}$$

associated to a ball-shaped boundary subregion A. Here ρ_A is the reduce density matrix defined by tracing over the complement $\bar{A}$

$$\rho_A = \mathrm{Tr}_{\bar{A}} |0\rangle\langle 0|, \tag{5.3.58}$$

where $|0\rangle$ is the global vacuum. Here the only length scale is the AdS_4 radius ℓ so we are considering the regime of large ℓ and computing

$$S_{\mathrm{EE}} = \frac{\mathrm{Area}}{4G} + (C_{\mathrm{local}} + C_{\mathrm{zm}}) \log \ell + \ldots . \tag{5.3.59}$$

The geometry of the hyperbolic black hole is given by

$$ds^2 = \left(\frac{\rho^2}{\ell^2} - 1 \right) dt^2 + \frac{d\rho^2}{\frac{\rho^2}{\ell^2} - 1} + \rho^2 ds^2_{H_2} \ , \qquad ds^2_{H_2} = du^2 + \sinh^2 u \, d\phi^2, \tag{5.3.60}$$

where $\rho \geqslant \ell$, $u \geqslant 0$ and $t \sim t + \beta$. The inverse temperature is given by

$$\beta = 2\pi \ell \ . \tag{5.3.61}$$

We regularize the integral over spacetime using holographic renormalization. In this case, there is also a divergence coming from the volume of H_2 and we take a regulator such that $\mathrm{vol}(H_2) = -2\pi$. The integrated four-derivative invariants are given by

$$\begin{aligned} \frac{1}{(4\pi)^2} \int d^4x \sqrt{g}\, E_4 &= 2 \ , & \frac{1}{(4\pi)^2} \int d^4x \sqrt{g}\, W^2 &= 0 \ , \\ \frac{1}{(4\pi)^2} \int d^4x \sqrt{g}\, R^2 &= 12 \ , & \frac{1}{(4\pi)^2} \int d^4x \sqrt{g}\, R F_{\mu\nu} F^{\mu\nu} &= 0 \ . \end{aligned} \tag{5.3.62}$$

This implies that we have

$$C_{\mathrm{local}} = -2a_{\mathrm{E}} + 12 b_1 \ . \tag{5.3.63}$$

Note that the Gauss-Bonnet-Chern theorem gives

$$\chi = \frac{1}{32\pi^2} \int d^4x \sqrt{g}\, E_4 = 1 \ , \tag{5.3.64}$$

as expected since M is topologically $D_2 \times H_2$ where D_2 is a disk and we have $\chi(M) = \chi(D_2)\chi(H_2) = 1$ since $\chi(D_2) = \chi(H_2) = 1$. This is a non-trivial consistency check for our regularization procedure.

5.3.6 Global contribution

We now compute the global contribution (5.2.26) which comes from the zero modes and the change of ensemble. The results are summarized in Table 5.2. In the full geometry, the contribution from the bosonic zero modes in the full asymptotically AdS_4 geometry vanishes [22]. Indeed, the fact that AdS_4 admits a 2-form zero mode follows from the general result of Camporesi and Higuchi who established that AdS_{2M} admits a M-form zero mode [171]. This 2-form zero mode is central in generating the logarithmic correction in asymptotically AdS_4 backgrounds embedded in eleven-dimensional supergravity [97, 172]. However, in the four-dimensional theories we consider in this manuscript, there is no contribution from such a 2-form zero mode.

Hence we have

$$C_{\text{zm}} = 0 \qquad \text{(full geometry)} . \tag{5.3.65}$$

In the near horizon geometry, additional zero modes come from the AdS_2 factor. The metric contributes -3 zero modes. In the near horizon geometry of BPS black holes, we also have 8 fermionic zero modes. The zero mode contribution for extremal black holes in the near horizon geometry is then given by

$$C_{\text{zm}} = -3 + 8\,\delta_{\text{BPS}} \qquad \text{(near horizon geometry)}, \tag{5.3.66}$$

where $\delta_{\text{BPS}} = 1$ in the BPS case and 0 otherwise. It is interesting to observe that this contribution can be interpreted in the context of nearly AdS_2 holography [170]. The asymptotic symmetry group of AdS_2 is $\text{Diff}(S^1)/\text{SL}(2,\mathbb{R})$. Upon a choice of configuration, the number of broken symmetries is $n_0 = +\infty - 3$, the infinite piece being absorbed in a renormalization of the energy. So the -3 zero modes come from the unbroken $\text{SL}(2,\mathbb{R})$ symmetry of AdS_2. A similar argument for BPS black holes explains the 8 fermionic zero modes as arising from the eight fermionic generators of the $\text{PSU}(1,1|2)$ near horizon symmetry. These patterns of symmetry breaking can be studied using Jackiw-Teitelboim gravity [170, 173–179].

We also include in C_{global} the correction that comes from the change of ensemble from canonical to microcanonical. Following [22], the change of ensemble gives a contribution

$$C_{\text{ens}} = -K , \tag{5.3.67}$$

Background spacetime	C_{zm}	C_{ens}	C_{global}
Schwarzschild	0	−3	−3
Reissner-Nordström	0	−3	−3
Kerr	0	−1	−1
Kerr-Newman	0	−1	−1
BPS Kerr-Newman	0	−1	−1
Reissner-Nordström near horizon	−3	−3	−6
Kerr-Newman near horizon	−3	−1	−4
BPS Kerr-Newman near horizon	5	−1	4
Thermal AdS_4	0	−3	−3
AdS_4-Rindler	0	−3	−3

Table 5.2: Global contribution to the logarithmic correction.

where K is the number of rotational symmetries of the black hole.

5.4 Minimally coupled matter

To obtain the logarithmic corrections, we need to compute the coefficients $a_{\text{E}}, c, b_1, b_2$ that appear in the general expression (5.3.1). Our ultimate aim is to evaluate logarithmic corrections in theories that can arise as consistent low-energy truncations from string and M-theory. However, in the next sections, we compute these logarithmic corrections in Einstein-Maxwell theory with a negative cosmological constant and in minimal $\mathcal{N} = 2$ gauged supergravity. As a warm-up, we also present the logarithmic corrections to AdS black holes due to minimally coupled fields, as was done for flat space black holes in [22].

5.4.1 Minimal theories

In this subsection, we compute C_{local} for minimal scalars, fermions, vectors and gravitini.

Free scalar. We consider a scalar field of mass m described by the action

$$S = -\frac{1}{2}\int d^4x\sqrt{g}\left((\partial\phi)^2 + m^2\phi^2\right). \tag{5.4.1}$$

The result for a scalar field is obtained by setting

$$P = E = m^2, \qquad \Omega = 0\,, \tag{5.4.2}$$

in equation (5.2.23). As explained in section 5.2.2, we consider a regime where every length scales with a factor λ. So here m scales as λ^{-1} and what is fixed is the conformal dimension

$$\Delta = \frac{1}{2}\left(3 + \sqrt{9 + 4m^2\ell^2}\right) \ . \tag{5.4.3}$$

This is to be contrasted with flat space where massive fields do not contribute to the logarithmic correction as explained in [21–23].

The heat kernel takes the form

$$(4\pi)^2 a_4(x) = -\frac{1}{360}E_4 + \frac{1}{120}W^2 + \frac{1}{288}(\Delta(\Delta-3)-2)^2 R^2 \ . \tag{5.4.4}$$

The explicit result for C_{local} can be obtained using (5.2.19) and (5.3.1). We report the result for the extremal black hole

$$C_{\text{local}} = -\frac{1}{90} - \frac{r_0^2}{20\ell^4}(24 + 5(\Delta+1)\Delta(\Delta-3)(\Delta-4)) \ . \tag{5.4.5}$$

Free fermion. We consider a free Dirac fermion with Euclidean action

$$S = \int d^{d+1}x\sqrt{g}\,\bar{\psi}\,(\gamma^\mu\nabla_\mu - m)\,\psi \tag{5.4.6}$$

This is dual to an operator with scaling dimension [180]

$$\Delta = \frac{3}{2} + m\ell. \tag{5.4.7}$$

The result is

$$(4\pi)^2 a_4(x) = \frac{11}{360}E_4 + \frac{1}{20}W^2 + \frac{1}{72}\left(\Delta - \tfrac{3}{2}\right)^2\left(\left(\Delta - \tfrac{3}{2}\right)^2 - 2\right)R^2. \tag{5.4.8}$$

Free vector. We now consider a free Maxwell field a^μ with the Lagrangian

$$\mathcal{L} = -\frac{1}{4}f_{\mu\nu}f^{\mu\nu}, \tag{5.4.9}$$

where $f_{\mu\nu} = \nabla_\mu a_\nu - \nabla_\nu a_\mu$. We add the gauge fixing term

$$\mathcal{L}_{\text{g.f.}} = \frac{1}{2}(\nabla_\mu a^\mu)^2, \tag{5.4.10}$$

so that the total Lagrangian becomes

$$\mathcal{L} + \mathcal{L}_{\text{g.f.}} = a^\mu \Box a_\mu - a^\mu R_{\mu\nu} a^\nu \ . \tag{5.4.11}$$

The gauge-fixing induces two massless scalar fields with fermionic statistics. We obtain the result

$$(4\pi)^2 a_4(x) = -\frac{31}{180} E_4 + \frac{1}{10} W^2 \ . \tag{5.4.12}$$

Free Rarita-Schwinger field. We consider here a Majorana spin-$\frac{3}{2}$ field described by the Lagrangian

$$\mathcal{L}_{3/2} = -\bar{\psi}_\mu \gamma^{\mu\rho\nu} \nabla_\rho \psi_\nu . \tag{5.4.13}$$

We use the gauge-fixing condition $\gamma^\mu \psi_\mu = 0$. This is implemented with the gauge-fixing term

$$\mathcal{L}_{\text{g.f}} = -\frac{1}{2} (\bar{\psi}_\mu \gamma^\mu) \gamma^\rho \nabla_\rho (\gamma^\nu \psi_\nu) \ , \tag{5.4.14}$$

so that the total Lagrangian is

$$\mathcal{L}_{3/2} + \mathcal{L}_{\text{g.f}} = \bar{\chi}_\mu \gamma^\nu D_\nu \chi^\mu , \tag{5.4.15}$$

after using the field redefinition $\psi_\mu = \chi_\mu - \frac{1}{2}\gamma_\mu \gamma^\nu \chi_\nu$. The gauge-fixing leads to three Majorana ghosts which are free massless fermions. We refer the reader to section B.3.3 for details on the gauge-fixing procedure. Hence, we find

$$(4\pi)^2 a_4(x) = \frac{229}{720} E_4 - \frac{77}{120} W^2 - \frac{1}{9} R^2 . \tag{5.4.16}$$

5.4.2 Logarithmic corrections

The results for minimally coupled scalars are summarized in Table 5.1.

Massless fields

For massless fields, we can present the result as

$$a_{\text{E}} = \frac{1}{720}(2n_S + 22n_F + 124n_V - 229n_\psi) \ , \tag{5.4.17}$$

$$c = \frac{1}{120}(n_S + 6n_F + 12n_V - 77n_\psi) \ , \tag{5.4.18}$$

$$b_1 = \frac{1}{72}(n_S - 8n_\psi) \ . \tag{5.4.19}$$

where n_S, n_F, n_V, n_ψ the number of scalars, spin-$\frac{1}{2}$ Majorana fermion, vector and gravitini. The result for AdS-Schwarzschild takes the form

$$\begin{aligned} C_{\text{local}} &= \frac{1}{180\ell^2(\ell^2+3r_+^2)}\left[\ell^4(4n_S+14n_F-52n_V-233n_\psi)\right. \\ &\left. +3r_+^2\ell^2(22n_S+2n_F-76n_V-239n_\psi)+18r_+^4(-3n_S+2n_F+4n_V+n_\psi)\right]. \end{aligned} \tag{5.4.20}$$

It is easily seen that in the flat space limit, we have

$$\lim_{\ell\to+\infty} C_{\text{local}} = \frac{1}{180}(4n_S+14n_F-52n_V-233n_\psi). \tag{5.4.21}$$

which reproduces the results of [22].

Corrections to entanglement entropy

Our result can also be applied to compute logarithmic correction to entanglement entropy. We consider a ball-shaped region A in the boundary. The entanglement entropy of A is given by the area of the hyperbolic black hole discussed in section 5.3.5. The logarithmic corrections to entanglement entropy are given by

$$S_{\text{EE}} = \frac{\text{Area}}{4G} + C\,\log\beta + \dots. \tag{5.4.22}$$

The contribution of a minimal scalar field of conformal dimension Δ gives

$$C = \frac{29}{180} + \frac{1}{24}(\Delta+1)\Delta(\Delta-3)(\Delta-4)\ . \tag{5.4.23}$$

We have here $C = C_{\text{local}}$ since there is no zero mode for the scalar field. Quantum corrections to entanglement entropy can also be interpreted in terms of bulk entanglement entropy [181]. It would be interesting to see if we can understand (5.4.23) as the logarithmic piece of the bulk entanglement entropy of a scalar field in the Rindler wedge.

5.5 Einstein-Maxwell-AdS theory

We now consider Einstein-Maxwell theory with a negative cosmological constant. This is the minimal theory that contains the AdS-Kerr-Newman black hole and is the bosonic part of minimal $\mathcal{N}=2$ gauged supergravity studied in section 5.6. The action is given by

$$S = \int d^4x\sqrt{g}\,(R-2\Lambda-F_{\mu\nu}F^{\mu\nu})\,, \tag{5.5.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength with A_μ the gauge potential. Note that we find it convenient to use a convention $4\pi G = 1$.

The computation is easily performed using the algorithm described in Appendix B.1. We have also performed an independent computation by hand, as detailed in Appendix B.2.

5.5.1 Bosonic fluctuations

We consider variations of the metric and gauge field

$$\delta g_{\mu\nu} = \sqrt{2} h_{\mu\nu} \ , \qquad \delta A_\mu = \frac{1}{2} a_\mu \ , \tag{5.5.2}$$

where $h_{\mu\nu}$ and a_α are the graviton and graviphoton, respectively. We impose a particular gauge to the theory by adding a suitable gauge-fixing Lagrangian

$$S = -\int d^4x \sqrt{\det g} \left\{ \left(D^\mu h_{\mu\rho} - \frac{1}{2} D_\rho h \right) \left(D^\nu h^\rho_\nu - \frac{1}{2} D^\rho h \right) + \frac{1}{2} \left(D^\mu a_\mu \right) \left(D^\nu a_\nu \right) \right\} , \tag{5.5.3}$$

and the corresponding ghost action to the action (5.5.1). We then expand the action up to quadratic order. The linear order variation yields the equation of motion for the background fields

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = 2 F_{\mu\rho} F_\nu{}^\rho - \frac{1}{2} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \ , \tag{5.5.4}$$

$$D^\mu F_{\mu\nu} = 0 \ . \tag{5.5.5}$$

Note that the equations of motion implies that $R = 4\Lambda = -12/\ell^2$. It is also worth mentioning the Bianchi identity for the gravitational field and gauge field

$$D_{[\mu} F_{\nu\rho]} = 0 \ , \tag{5.5.6}$$

$$R_{\mu[\nu\rho\sigma]} = 0 \ , \tag{5.5.7}$$

as they serve as handy tools for simplifying our calculations. The quadratic action can be put in the canonical form (5.2.21). The details can be found in appendix B.2.1 where we present the explicit form of the quadratic fluctuations. This allows us to extract the matrices

I, E and Ω:

$$\phi_m I^{mn}\phi_n = h_{\mu\nu}\left(\frac{1}{2}g^{\mu\alpha}g^{\nu\beta} + \frac{1}{2}g^{\mu\beta}g^{\nu\alpha} - \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\right)h_{\alpha\beta} + a_\alpha g^{\alpha\beta}a_\beta, \tag{5.5.8}$$

$$\begin{aligned}\phi_m E^{mn}\phi_n =& h_{\mu\nu}\left(R^{\mu\alpha\nu\beta} + R^{\mu\beta\nu\alpha} - g^{\mu\nu}R^{\alpha\beta} - g^{\alpha\beta}R^{\mu\nu} + \Lambda g^{\mu\nu}g^{\alpha\beta}\right)h_{\alpha\beta} \\ &+ a_\alpha\left(\frac{3}{2}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu} - \Lambda g^{\alpha\beta}\right)a_\beta + \frac{\sqrt{2}}{2}h_{\mu\nu}\left(D^\mu F^{\alpha\nu} + D^\nu F^{\alpha\mu}\right)a_\alpha \\ &+ \frac{\sqrt{2}}{2}a_\alpha\left(D^\mu F^{\alpha\nu} + D^\nu F^{\alpha\mu}\right)h_{\mu\nu}\ ,\end{aligned} \tag{5.5.9}$$

$$\begin{aligned}\phi_m\left(\Omega^{\rho\sigma}\right)^{mn}\phi_n =& h_{\mu\nu}\left\{\frac{1}{2}\left(g^{\nu\beta}R^{\mu\alpha\rho\sigma} + g^{\nu\alpha}R^{\mu\beta\rho\sigma} + g^{\mu\beta}R^{\nu\alpha\rho\sigma} + g^{\mu\alpha}R^{\nu\beta\rho\sigma}\right)\right. \\ &\left.+ \left[\omega^\rho, \omega^\sigma\right]^{\mu\nu\alpha\beta}\right\}h_{\alpha\beta} + a_\alpha\left\{R^{\alpha\beta\rho\sigma} + \left[\omega^\rho, \omega^\sigma\right]^{\alpha\beta}\right\}a_\beta \\ &+ h_{\mu\nu}\left(D^{[\rho}\omega^{\sigma]}\right)^{\mu\nu\alpha}a_\alpha + a_\alpha\left(D^{[\rho}\omega^{\sigma]}\right)^{\alpha\mu\nu}h_{\mu\nu}\ ,\end{aligned} \tag{5.5.10}$$

where ω^ρ is the spin-connection given by

$$\begin{aligned}\phi_m\left(\omega^\rho\right)^{mn}\phi_n =& \frac{\sqrt{2}}{2}h_{\mu\nu}\Big(g^{\alpha\mu}F^{\rho\nu} + g^{\alpha\nu}F^{\rho\mu} - g^{\mu\rho}F^{\alpha\nu} - g^{\nu\rho}F^{\alpha\mu} - g^{\mu\nu}F^{\rho\alpha}\Big)a_\alpha \\ &- \frac{\sqrt{2}}{2}a_\alpha\Big(g^{\alpha\mu}F^{\rho\nu} + g^{\alpha\nu}F^{\rho\mu} - g^{\mu\rho}F^{\alpha\nu} - g^{\nu\rho}F^{\alpha\mu} - g^{\mu\nu}F^{\rho\alpha}\Big)h_{\mu\nu}\ .\end{aligned} \tag{5.5.11}$$

We then find the trace of (5.5.8)-(5.5.10). The computation is tedious, but it may also be illuminating for some readers. We present the intermediate steps in appendix B.2.2. The final contribution to the heat kernel coefficient is

$$(4\pi)^2 a_4^{\mathrm{EM}}(x) = -\frac{277}{180}E_4 + \frac{38}{15}W^2 + \frac{7}{18}R^2\ . \tag{5.5.12}$$

The reader familiar with the literature might notice that we have not treated the *trace mode* in the graviton. Traditionally, as in the literature [148, 153, 182], one decomposes the fields appearing in the Lagrangian into the irreducible fields $\phi(A, B)$ which transform according to the irreducible (A, B) representation of SO(4). For example, in [182], the authors considered the decomposition of fluctuation of geometry $h_{\mu\nu}$ into a $(1,1)$ symmetric traceless tensor, a scalar characterizing the trace part transforming in $(0,0)$ and the corresponding vector ghost field in $(\frac{1}{2}, \frac{1}{2})$. Here, we choose the operator I^{mn} as the effective metric, which is equivalent to making this decomposition.

Ghost contribution

The addition of the gauge-fixing Lagrangian (5.5.3) gives an action for the ghosts

$$\mathcal{S}_{\text{ghost},b} = \frac{1}{2}\int d^4x\sqrt{g}\Big\{2b_\mu\big(g^{\mu\nu}\Box + R^{\mu\nu}\big)c_\nu + 2b\Box c - 4bF^{\rho\nu}D_\rho c_\nu\Big\}, \tag{5.5.13}$$

where b_μ and c_μ are vector fields and b and c are scalar fields. From these expression, we can extract the matrices E and Ω as

$$\begin{aligned}\phi_n E^n_m \phi^m =& b_\mu\left(R^\mu{}_\nu\right)b^\nu + c_\mu\left(R^\mu{}_\nu\right)c^\nu,\\ \phi_n\left(\Omega_{\alpha\beta}\right)^n_m\phi^m =& b_\mu\left(R^\mu{}_{\nu\alpha\beta}\right)b^\nu + c_\mu\left(R^\mu{}_{\nu\alpha\beta}\right)c^\nu - \frac{1}{2}\left(b_\mu - ic_\mu\right)\left(D^\mu F_{\alpha\beta}\right)(b+ic)\\ &+\frac{1}{2}(b+ic)\left(D_\nu F_{\alpha\beta}\right)\left(b^\nu - ic^\nu\right),\end{aligned} \tag{5.5.14}$$

The result for the Seeley-DeWitt coefficient is

$$a_4^{\text{ghosts}}(x) = \frac{13}{36}E_4 - \frac{1}{4}W^2 - \frac{3}{4}R^2\ , \tag{5.5.15}$$

where we have already included here the minus sign due to the opposite statistics.

5.5.2 Logarithmic correction

Adding the above results, the heat kernel for Einstein-Maxwell theory takes the form,

$$(4\pi)^2 a_4^{\text{B}}(x) = -\frac{53}{45}E_4 + \frac{137}{60}W^2 - \frac{13}{36}R^2\ . \tag{5.5.16}$$

We can read off the coefficients from (5.3.1) to be

$$a_{\text{E}} = \frac{53}{45},\qquad c = \frac{137}{60},\qquad b_1 = -\frac{13}{36}\ ,\qquad b_2 = 0. \tag{5.5.17}$$

We note that in the flat limit $\ell \to +\infty$, the coefficients a and c match the known flat space computations in [148,183,184] while the coefficients b_1 and b_2 are unique to AdS. We can also note that the result does not explicitly depend on $F^{\mu\nu}$ as $b_2 = 0$. This implies that the final result is invariant under electric-magnetic duality. This property has also been observed in the asymptotically flat case in [183,184]. Another sanity check is to consider the truncation of the terms involving $F_{\mu\nu}$ in the fluctuations. Then (5.5.16) reduces to the neutral limit which was first obtained in [182]; we show this in detail in Appendix B.2.4.

We can evaluate this result for the BPS Kerr-Newman solution described in section 5.3.4.

The result is

$$C_{\text{local}} = -\frac{212}{45} + \frac{\ell_2^2}{120a\ell^2(\ell^2-a^2)}\Bigg(629a^3 - 579a^2\ell + 3699a\ell^2 \qquad (5.5.18)$$
$$+411\ell^3 - 411\frac{(a+\ell)^4}{\sqrt{a\ell}}\arctan(\sqrt{a/\ell})\Bigg) .$$

5.6 Minimal $\mathcal{N}=2$ gauged supergravity

We now consider the simplest supersymmetric theory with a consistent truncation to Einstein-Maxwell theory with a negative cosmological constant. This is minimal $\mathcal{N}=2$ gauged supergravity [167, 185–187]. In this section, we compute the logarithmic corrections in this theory. We find that in contrast to flat space, the logarithmic correction for BPS black holes is not topological. The results of this section were obtained using a Mathematica algorithm described in Appendix B.1 which we have made publicly available [188].

Ultimately, we would like to compute the logarithmic correction for AdS black holes where a microscopic counting is available. Although the techniques of this chapter are applicable in those cases, the computations are more involved due to additional matter multiplets.

5.6.1 Fermionic fluctuations

The bosonic Lagrangian of minimal $\mathcal{N}=2$ supergravity is the same as (5.5.1). Hence, the result of the previous section can be applied and gives (5.5.16). In this section, we will compute the contribution from the fermions. In the conventions of [187], the fermionic Lagrangian takes the form

$$\mathcal{L}_f = \frac{1}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho + \frac{i}{4}F^{\mu\nu}\bar{\psi}_\rho\gamma_\mu\gamma^{\rho\sigma}\gamma_\nu\psi_\sigma - \frac{1}{2\ell}\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu \ , \qquad (5.6.1)$$

where the gravitino ψ_μ is a Dirac spin-$\frac{3}{2}$ field with charge one, in units of the AdS_4 length, under the U(1) gauge symmetry. The action of the covariant derivative is

$$D_\mu\psi_\nu = \nabla_\mu\psi_\nu - \frac{i}{\ell}A_\mu\psi_\nu \ . \qquad (5.6.2)$$

We now put the fermionic Lagrangian in a form suitable for the heat kernel computation. Firstly, we fix the gauge by adding the following gauge-fixing Lagrangian

$$\mathcal{L}_{\text{g.f.}} = -\frac{1}{4}(\bar{\psi}_\mu\gamma^\mu)\gamma^\nu D_\nu(\gamma^\rho\psi_\rho) \ . \qquad (5.6.3)$$

This choice is convenient because after we perform the field redefinition

$$\psi_\mu = \sqrt{2}\left(\chi_\mu - \frac{1}{2}\gamma_\mu\gamma^\nu\chi_\nu\right) \ , \tag{5.6.4}$$

we obtain a simpler kinetic term. The resulting Lagrangian takes the form

$$\mathcal{L}_f \ = \ g^{\mu\nu}\bar{\chi}_\mu\gamma^\rho D_\rho\chi_\nu + \frac{i}{2}F^{\mu\nu}\bar{\chi}_\rho\gamma_\mu\gamma^{\rho\sigma}\gamma_\nu\chi_\sigma - \frac{1}{\ell}\bar{\chi}_\mu\gamma^{\mu\nu}\chi_\nu \ . \tag{5.6.5}$$

More details on this computation are given in Appendix B.3. We then write the Dirac spinor as

$$\chi^\mu = \chi_1^\mu + i\chi_2^\mu \ , \tag{5.6.6}$$

where χ_1 and χ_2 are Majorana spin-$\frac{3}{2}$ spinors [3]. We use the label $A = 1, 2$ for the two Majorana spinors. The covariant derivative acting on χ_A^μ takes the form

$$D_\mu\chi_A^\nu = \left(\delta_{AB}\nabla^\mu + \frac{1}{\ell}\,\varepsilon_{AB}A_\mu\right)\chi_B^\nu \ , \tag{5.6.7}$$

where ε_{AB} is the antisymmetric symbol with $\varepsilon_{12} = 1$. This is necessary if we want to preserve the reality condition. It is useful to use the Majorana flip identities (B.3.17) reviewed in Appendix B.3. The computation detailed there leads to the Lagrangian in terms of Majorana spinors

$$\mathcal{L}_f \ = \ \delta_{AB}g_{\mu\nu}\bar{\chi}_A^\mu\gamma^\rho D_\rho\chi_B^\nu - \frac{1}{2}\varepsilon_{AB}F^{\mu\nu}\bar{\chi}_A^\rho\gamma^\mu\gamma_{\rho\sigma}\gamma^\nu\chi_B^\sigma - \frac{1}{\ell}\delta_{AB}\bar{\chi}_A^\mu\gamma_{\mu\nu}\chi_B^\nu \ . \tag{5.6.8}$$

Finally, we reinterpret this Lagrangian as being a Euclidean Lagrangian in which χ_μ^A are Euclidean spinors satisfying $\bar{\chi}_\mu^A = (\chi_\mu^A)^\dagger$. This Lagrangian can then be used in the algorithm to obtain the result for the heat kernel. Note that we can question the validity of the Wick rotation here because Majorana spinors actually do not exist in four Euclidean dimensions. This can be addressed by using symplectic Majorana spinors. We find, however, that this procedure actually gives the same result as the naive Wick rotation.

Symplectic Majoranas

The Lagrangian (5.6.8) is written in terms of Majorana spinors in $(1, 3)$ signature. We would like to Wick rotate this Lagrangian to $(0, 4)$ signature. As mentioned above, this appears problematic because Majorana spinors do not exist in $(0, 4)$ signature. A better approach is

[3]For definiteness, we can use here the "really real" representation of the Clifford algebra in which the Majorana condition is just the reality condition [189].

to use symplectic Majorana spinors which exist in both $(1,3)$ and $(0,4)$ signature [190].

It is shown in [191] that one can map Majorana spinors χ^μ_A to symplectic Majorana spinors λ^μ_A using

$$\begin{aligned} \chi^\mu_1 &= \frac{1}{2}(\lambda^\mu_1 + \gamma^5\lambda^\mu_2) \ , \\ \chi^\mu_2 &= \frac{i}{2}(\lambda^\mu_1 - \gamma^5\lambda^\mu_2) \ . \end{aligned} \tag{5.6.9}$$

This allows us to write the Lagrangian in terms of λ^μ_A. We find that the two flavors actually decouple as

$$\mathcal{L}_f = \mathcal{L}_1 + \mathcal{L}_2, \tag{5.6.10}$$

with

$$\mathcal{L}_1 = g_{\mu\nu}\bar{\lambda}^\mu_1\gamma^\rho(\nabla_\rho + i\ell^{-1}A_\rho)\lambda^\nu_1 - \frac{i}{2}F^{\mu\nu}\bar{\lambda}^\rho_1\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda^\sigma_1 - \frac{1}{\ell}\bar{\lambda}^\mu_1\gamma_{\mu\nu}\lambda^\nu_1 \ , \tag{5.6.11}$$

$$\mathcal{L}_2 = g_{\mu\nu}\bar{\lambda}^\mu_2\gamma^\rho(\nabla_\rho - i\ell^{-1}A_\rho)\lambda^\nu_2 - \frac{i}{2}F^{\mu\nu}\bar{\lambda}^\rho_2\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda^\sigma_2 + \frac{1}{\ell}\bar{\lambda}^\mu_2\gamma_{\mu\nu}\lambda^\nu_2 \ . \tag{5.6.12}$$

The Wick rotation is done by reinterpreting λ^μ_A as symplectic Majorana spinors in $(0,4)$ signature with $\bar{\lambda}^\mu_A = (\lambda^\mu_A)^\dagger$. This can then be used in the algorithm, described in Appendix (B.1), to compute the heat kernel[4]. We obtain the gravitini contribution

$$(4\pi)^2 a_4^{\text{F,gravitini}}(x) = \frac{139}{90}E_4 - \frac{32}{15}W^2 - \frac{2}{9}R^2 + \frac{8}{9}RF_{\mu\nu}F^{\mu\nu} \ . \tag{5.6.13}$$

Note that the result is ultimately the same as what we obtain naively by using directly the Majorana Lagrangian (5.6.8) in the algorithm.

Ghost contribution

The gauge-fixing of the gravitini leads to three pairs of ghosts. The gauge condition $\gamma^\mu\psi^A_\mu = 0$ leads to a bc ghost Lagrangian given as

$$\mathcal{L}_{bc} = \delta_{AB}\bar{b}_A\gamma_\mu\delta_c\psi^\mu_B, \tag{5.6.14}$$

where $\delta_c\psi^\mu$ is the supersymmetry transformation with parameter c. This gives

$$\mathcal{L}_{bc} = \delta_{AB}\bar{b}_A\left(\gamma^\mu D_\mu + \frac{2}{\ell}\right)c_B \ . \tag{5.6.15}$$

[4] The contribution to the heat kernel of $\mathcal{L}_1$ and $\mathcal{L}_2$ are equal because the two Lagrangian differs by $\ell \to -\ell$ and the four-derivative terms are invariant under that change.

We can get a diagonal kinetic term by a suitable redefinition. This leads to two pairs of ghosts which are charged spin-$\frac{1}{2}$ fermions with mass $m = \frac{2}{\ell}$. In addition, implementing the gauge-fixing term in the path integral leads to an additional pair of massless charged ghosts, giving us a ghost for ghost phenomena [192, 193]. Details are given in Appendix B.3.

The total heat kernel of the fermionic ghosts is

$$(4\pi)^2 a_4^{\mathrm{F,ghosts}}(x) = \frac{11}{120}E_4 - \frac{3}{20}W^2 + \frac{2}{9}R^2 - \frac{1}{6}RF_{\mu\nu}F^{\mu\nu} \ , \tag{5.6.16}$$

where we have already included the minus sign due to the opposite statistics of ghosts.

5.6.2 Logarithmic correction

The total fermionic contribution is

$$(4\pi)^2 a_4^{\mathrm{F}}(x) = \frac{589}{360}E_4 - \frac{137}{60}W^2 + \frac{13}{18}RF_{\mu\nu}F^{\mu\nu} \ . \tag{5.6.17}$$

Adding this to (5.5.16) from the bosonic computation, we obtain for minimal $\mathcal{N} = 2$ supergravity

$$(4\pi)^2 a_4(x) = \frac{11}{24}E_4 - \frac{13}{36}R^2 + \frac{13}{18}RF_{\mu\nu}F^{\mu\nu} \ . \tag{5.6.18}$$

We find that the full result is not only given by the Euler term as other four-derivative invariants are present. This indicates that the logarithmic correction is non-universal. We note that the W^2 term, which would give another non-universal contribution, does cancel between bosons and fermions. This is expected from the flat space result [148], which we recover in the flat limit.

Evaluation

We can evaluate the heat kernel coefficient on the backgrounds summarized in section 5.3. For the non-extremal Kerr-Newman black hole, we get

$$C_{\mathrm{local}} = \frac{11}{6} + \frac{26\left[r_+\beta(r_+^4 - \ell^2(a^2+q_m^2)) - \pi\ell^2(r_+^4 - a^4)\right]}{3\pi\ell^2(\ell^2-a^2)(a^2+r_+^2)} \ . \tag{5.6.19}$$

We note that the fermionic contribution breaks electromagnetic duality as it generates a non-zero b_2. This is reflected by the dependence in the magnetic charge q_m in the above expression.

The result for extremal Kerr-Newman takes the form

$$C_{\rm local} = \frac{11}{6} + \frac{26\ell_2^2\left[(a^2(a+\ell)+r_0^2(3a-\ell))(a^2(a-\ell)+r_0^2(3a+\ell))+2\ell^2 q_m^2(r_0^2-a^2)\right]}{3\ell^2(\ell^2-a^2)(a^2+r_0^2)^2}, \tag{5.6.20}$$

where here the AdS_2 radius is $\ell_2 = \ell\sqrt{\frac{a^2+r_0^2}{a^2+\ell^2+6r_0^2}}$. As explained in section 5.3.4, this is obtained by either taking the extremal limit of (5.6.19) or by doing the computation in the near horizon geometry.

We are particularly interested in evaluating the logarithmic corrections on BPS black holes. Rotation is necessary to have a regular BPS background in minimal gauged supergravity as the extremal AdS-Reissner-Nordström is singular in the BPS limit [75, 168, 194]. We obtain the BPS result by imposing the BPS constraints $r_0 = \sqrt{a\ell}$ and $q_m = 0$ on (5.6.20). The contribution is

$$C_{\rm local} = \frac{11}{6} - \frac{26}{3}\frac{a(\ell^2-4\ell-a^2)}{(\ell-a)(a^2+6a\ell+\ell^2)}, \tag{5.6.21}$$

where the first term comes from the topological Euler term and the second term comes R^2 and RF^2 and constitute the non-topological piece. We discuss the significance of this non-topological term in the next subsection.

5.6.3 Implications

We shall now comment on the non-topological nature of the logarithmic correction. For the flat space Kerr-Newman black hole, the heat kernel $a_4(x)$ is the sum of only two terms: the Euler term and the Weyl squared term. Although $W^2 = 0$ for extremal non-rotating black holes in flat space, it is non-zero for extremal rotating black holes. It was shown in [148, 195] that supersymmetry ensures that the coefficient c multiplying W^2 actually cancels. This shows that supersymmetry makes the logarithmic correction topological in ungauged supergravity.

In AdS_4, the W^2 term never vanishes in the near horizon geometry (even without rotation) and there are two additional terms. We also obtain that supersymmetry ensures $c = 0$ due to cancellations between bosons and fermions. This could be expected from the flat space results of [148] which we reproduce in the flat limit $\ell \to +\infty$. Hence, even with a negative cosmological constant, supersymmetry makes the logarithmic correction less complicated as it removes the non-topological term W^2. This term is a complicated function of the black hole parameters. For the BPS Kerr-Newman AdS black hole, it takes the form

$$\frac{1}{(4\pi)^2}\int d^2x\sqrt{g}\,W^2 = \frac{3\ell_2^2}{2a\ell^2(\ell^2-a^2)}\left((\ell-a)(\ell^2+10a\ell+a^2) - \frac{(a+\ell)^4}{\sqrt{a\ell}}\arctan(\sqrt{a/\ell})\right). \tag{5.6.22}$$

The two other four-derivative, R^2 and RF^2, terms do not cancel so supersymmetry does not imply that the logarithmic corrections are topological. However, we see that the BPS result (5.6.21) is still a simpler function of a and ℓ as it has $c = 0$. It is a rational function rather than a transcendental one.

It is natural to expect topological logarithmic corrections in the UV given the known examples of microscopic counting of black hole entropy [95, 99, 100, 104, 151, 196–198]. This is also automatic if the 4d theory comes from an odd-dimensional theory by Kaluza-Klein reduction because $C_{\text{local}} = 0$ in odd dimensions. Hence, the logarithmic correction can be a useful probe of whether a low-energy effective theory can have a UV completion. The idea is that from a bottom-up perspective, we should prefer low-energy theories which have topological logarithmic correction. This is only possible if $c = b_1 = b_2 = 0$ which is a rather strong constraint on the low-energy Lagrangian, analogous to anomaly cancellation.

5.7 Discussion

In this chapter, we have computed the logarithmic corrections to the entropy of black holes in minimal gauged supergravity using the four dimensional heat kernel expansion. The inclusion of a negative cosmological constant leads to new features compared to the case of asymptotically flat black holes. In the especially interesting case of BPS black holes, the logarithmic corrections present a richer structure and can be non-topological.

The original explicit logarithmic corrections performed for asymptotically $\text{AdS}_4 \times S^7$ black holes based on Sen's entropy function formalism, using the near horizon geometry, did not agree with the field theory computations [95, 96]. It was only in [97] that agreement was found by considering the full geometry. The results of this manuscript clarify that the difference between the two approaches comes from the contributions of the zero modes which are indeed different in the two geometries. Namely, we have shown that for extremal black holes the *local* contribution to the logarithmic correction, C_{local}, is the same when computed either from the full AdS_4 asymptotic region or for the near horizon geometry. This result elucidates the question of where the degrees of freedom responsible for the quantum entropy live.

For the BPS Kerr-Newman black hole in minimally gauged supergravity, we have found that the logarithmic correction, given in (5.6.21), is non-topological. To obtain this result, we have used holographic renormalization to regularize the divergent volume integrals. This appears to be the right prescription as, for example, it gives the correct counterterm to obtain the Euler characteristic when integrating the Euler density, see Appendix B.5 for more details.

The non-topological nature of the logarithmic corrections suggest that they might contain more information than the flat space counterpart, providing a wider "infrared window into the microstates". Moreover, this non-topological nature is interesting because for all the available examples of microscopic counting and for BPS black holes in flat space [148], the logarithmic correction is always topological, *i.e.*, the coefficient of the logarithm is a pure number. In minimal gauged supergravity, we find that it is instead a rather non-trivial function of the black hole parameters.

It is illuminating to compare this result to recent investigations using supergravity localization [98,102,150]. In [150], the general structure of the logarithmic correction of 4d $\mathcal{N} = 2$ gauged supergravity on BPS backgrounds was studied using index theory. It was shown that the universal piece coming from the Euler term arises from the application of the Atiyah-Singer theorem to an appropriate supercharge. We surmise that the non-universal piece that we obtained should be interpreted as the contribution from the η invariant, not considered in [150], which is a non-topological correction due to the presence of a boundary [199]. Supergravity localization has the potential of ultimately providing the full quantum entropy of the black holes and it would be fruitful to test it against one-loop supergravity results such as ours.

Our work also clarifies the role of supersymmetry. One could think that supersymmetry guarantees that the logarithmic corrections are topological. This is suggested by the index theory interpretation [150] and by results in flat spacetime [148, 149]. In this work, we have seen that supersymmetry is not enough to make the other terms cancel which shows the logarithmic corrections can be non-topological for BPS black holes. Nevertheless, supersymmetry does play a role in making C_{local} simpler as it cancels the coefficient, c, of the Weyl squared term (5.6.22). This simplifies the logarithmic correction for the BPS black hole as its dependence on a and ℓ becomes rational instead of transcendental.

We might hope to use the topological nature of logarithmic corrections as a criterion for a low-energy theory to admit a UV completion. In the available examples of microscopic countings, the logarithmic correction is indeed topological [95, 99, 100, 104, 151, 196–198]. Such a criteria would greatly constrain effective supergravity theories as it gives rather stringent conditions similar to anomaly cancellation. Note that in odd dimensions, the logarithmic correction is automatically topological because the local contribution is trivially zero. So if the four-dimensional theory comes from the dimensional reduction of an odd-dimensional theory, such as 11d supergravity, the logarithmic correction computed in 4d has to be topological. For ten dimensional theories, there is a local contribution in 10d and, as a result, the topological criterion should be much more constraining.

We have obtained the logarithmic correction for the simplest gauged supergravity in four dimensions. Our goal is to grow this direction towards more interesting theories and to relate our results to other approaches such as the computations performed in eleven-dimensional supergravity [97,100,104,151]. It should be possible to perform the same computation in the gauged $U(1)^4$ supergravity which comes from eleven dimensional supergravity on $AdS_4 \times S^7$. Similarly, the logarithmic correction to the entropy of black holes in $AdS_4 \times SE_7$ has been computed both in field theory and supergravity for a large class of Sasaki-Einstein seven-dimensional manifolds [104]. In both these cases, the topological nature follows from the fact that the parent theory is odd-dimensional. It would be interesting to see explicitly how this is realized from a four-dimensional perspective.

More challenging would be the cases where the AdS_4 black holes are embedded in ten-dimensional theories such as massive IIA supergravity. A matching of the Bekenstein-Hawking entropy at leading order was presented in [200–202]. The available sub-leading, microscopic analysis confirms the topological nature of the logarithmic term [99]. However, the supergravity computations need to be in agreement with the nontrivial nature of C_{local}. We hope to address some of these issues in the future.

Chapter 6

Concluding Remarks

In this thesis, we have focused on exploring quantum gravity in the context of holography and its relation to AdS black holes. While this is a very broad subject, we have focused on investigating the entropy of black holes via gravity and field theoretic computations. The aim of part I was to explore the AdS/CFT and Kerr/CFT correspondence in a certain parameter space to study the matching between the macroscopic entropy via the Bekenstein-Hawking formula and the microscopic entropy counting via CFT_{d-1} and CFT_2. For example, we have also been able to extend these holographic approaches to the entropy to the context of near-extremality, with the hopes that we can eventually probe the full non-extremal regime. This is an outstanding problem since most field theoretic computations heavily rely on supersymmetry.

In part II, we shifted our attention to the entropy at subleading order by considering two distinct methods: the Kerr/CFT correspondence and the heat kernel. Although the two dimensional effective CFT_2 is not completely understand from the Kerr/CFT perspective, we can still connect and take advantage of the correspondence. Expanding beyond the saddle point, we can investigate the logarithmic term of the entropy via the Cardy formula to obtain a correction that matches precisely with field theory predictions. This supports the use of Kerr/CFT and prompts us to investigate further the strength of the conjecture. Moreover, the success of implementing the near-horizon geometry to extract the entropy, as for example in the Kerr/CFT correspondence, leaves room for speculation as to what extent the black hole horizon provides a window into the full understanding of quantum gravity. By calculating the quantum corrections to the entropy, we have found that the near-horizon accounts does contain the quantum degrees of freedom accounting for the logarithmic correction, up to zero modes and changes in the ensemble. From these two methods to extract the logarithmic corrections, it would be interesting to understand how zero modes, the heat kernel and the Kerr/CFT correspondence are related in diverse dimensions.

Appendix A

Verifying Black Hole Equations of Motion

A.1 Verifying the equations of motion for the near-horizon

Gravitational theories are nonlinear and, therefore, a truncated sector of a solution need not be a solution itself. For that reason, we explicitly verify that, in each instance, the near-horizon limit satisfies the equations of motion. This fact alone should inspire trust in the consistency of the resulting geometry and the potential existence and closure of a dual field theory sector. Returning to the analogy with the BMN paradigm, this is equivalent to checking the equations of motion for the plane wave background [203].

A.1.1 AdS_5

We verify the equations of motion for the near-horizon geometry for AdS_5. The Lagrangian describing the solution in [24] is

$$\mathcal{L} = \left(R + 12g^2\right) * 1 - \frac{1}{2} * F \wedge F + \frac{1}{3\sqrt{3}} F \wedge F \wedge A, \tag{A.1.1}$$

and the equations of motion are

$$R_{ab} - \frac{1}{2} F_{ac} F_b{}^c + \frac{1}{3} g_{ab} \left(\frac{1}{4} F^2 + 12g^2\right) = 0, \qquad d \star F - \frac{1}{\sqrt{3}} F \wedge F = 0. \tag{A.1.2}$$

In order to facilitate the computation, we turn to a veilbein description for the near-horizon geometry,

$$\begin{aligned}
e_1 &= \sqrt{\frac{a}{10ag^2+2g}}\tilde{r}d\tau, \\
e_2 &= \sqrt{\frac{a}{10ag^2+2g}}\frac{d\tilde{r}}{\tilde{r}}, \\
e_3 &= \sqrt{\frac{2a}{g-ag^2}}d\theta, \\
e_4 &= p_1\left(p_2\left((-3ag\cos 2\theta+ag-4)d\tilde{\psi}-6ag\sin^2\theta d\tilde{\phi}\right)+3a\left(a^2g^2+ag-2\right)\tilde{r}d\tau\right), \\
e_5 &= p_3\left(3a(1-ag)\tilde{r}d\tau+2p_2d\tilde{\phi}\right),
\end{aligned} \tag{A.1.3}$$

where

$$\begin{aligned}
p_1 &= -\frac{\cos\theta}{(1-ag)(5ag+1)\sqrt{2(ag+2)(3ag\cos(2\theta)-ag+4)}}, \\
p_2 &= \sqrt{\frac{a(ag+2)}{g}}(5ag+1), \\
p_3 &= \frac{\sin\theta}{(5ag+1)\sqrt{(1-ag)(3ag\cos(2\theta)-ag+4)}}.
\end{aligned} \tag{A.1.4}$$

Note that this coframe describes the near-horizon, which is computed using [37]. After applying the near-horizon geometry and gauge fixing, the gauge potential is

$$\begin{aligned}
A_{(1),\mathrm{near}} = &-\frac{\sqrt{6}(1-ag)}{\sqrt{ag+2}\sqrt{5ag+1}}e_1-\frac{\sqrt{6ag}\cos\theta}{\sqrt{3ag\cos 2\theta-ag+4}}e_4 \\
&-\frac{2\sqrt{3ag(1-ag)}\sin\theta}{(\sqrt{ag+2}\sqrt{3ag\cos 2\theta-ag+4}}e_5.
\end{aligned} \tag{A.1.5}$$

Note that the exterior derivative and the near-horizon geometry limit commute to give an equivalent expression for the gauge field,

$$\begin{aligned}
F_{(2),\mathrm{near}} = dA_{(1),\mathrm{near}} = &\frac{\sqrt{3ag(ag+2)}}{a}e_1\wedge e_2+2g\sin\theta\sqrt{\frac{3(1-ag)}{3ag\cos 2\theta-ag+4}}e_3\wedge e_4 \\
&+g\cos\theta\sqrt{\frac{6(ag+2)}{(3ag\cos 2\theta-ag+4)}}e_3\wedge e_5.
\end{aligned} \tag{A.1.6}$$

Then,

$$F_{(2),\text{near}} \wedge F_{(2),\text{near}} = 6g^{3/2}\frac{\sqrt{ag+2}}{\sqrt{a(3ag\cos 2\theta - ag + 4)}}\left(2\sqrt{1-ag}\sin\theta e_1 \wedge e_2 \wedge e_3 \wedge e_4 \right. \\ \left. -\sqrt{2ag+4}\cos\theta e_1 \wedge e_2 \wedge e_3 \wedge e_5\right). \tag{A.1.7}$$

The other term gives

$$\begin{aligned} d \star F_{(2),\text{near}} =& d\left[\sqrt{6}g\cos\theta\sqrt{\frac{ag+2}{3ag\cos 2\theta - ag + 4}}e_1 \wedge e_2 \wedge e_4 - \frac{\sqrt{3ag(ag+2)}}{a}e_3 \wedge e_4 \wedge e_5 \right. \\ & \left. +2\sqrt{3}g\sin\theta\sqrt{\frac{1-ag}{3ag\cos 2\theta - ag + 4}}e_1 \wedge e_2 \wedge e_5\right] \\ =& \frac{2\sqrt{3(ag+2)}g^{3/2}}{\sqrt{a(3ag\cos 2\theta - ag + 4)}}\left(2\sqrt{1-ag}\sin\theta e_1 \wedge e_2 \wedge e_3 \wedge e_4 \right. \\ & \left. -\sqrt{2ag+4}\cos\theta e_1 \wedge e_2 \wedge e_3 \wedge e_5\right). \end{aligned} \tag{A.1.8}$$

Comparing equations (A.1.7) and (A.1.8), we can see that the equation of motion for the gauge potential is satisfied. For the Einstein equations, the geometric data we need are the nonzero components of the Ricci tensor

$$\begin{aligned} R_{00,\text{near}} &= -R_{11,\text{near}} = \frac{g(11ag+4)}{2a}, \\ R_{22,\text{near}} &= -\frac{g(5ag-2)}{2a}, \\ R_{33,\text{near}} &= -\frac{g\left(9a^2g^2\cos 2\theta - a^2g^2 + 14ag - 4\right)}{a(3ag\cos 2\theta - ag + 4)}, \\ R_{34,\text{near}} &= -\frac{3\sqrt{2}g^2\sqrt{2-ag-a^2g^2}\sin\theta\cos\theta}{3ag\cos 2\theta - ag + 4}, \\ R_{44,\text{near}} &= -\frac{g\left(21a^2g^2\cos 2\theta - 11a^2g^2 - 12ag\cos 2\theta + 28ag - 8\right)}{2a(3ag\cos 2\theta - ag + 4)}. \end{aligned} \tag{A.1.9}$$

We also need the explicit nonzero contractions of the gauge field $F_{ac,\text{near}} F_{b\ \ ,\text{near}}^{\ c} \equiv \mathcal{F}_{ab,\text{near}}$

$$\begin{aligned}
&\mathcal{F}_{00,\text{near}} = \frac{3g(ag+2)}{a}, \qquad \mathcal{F}_{11,\text{near}} = -\mathcal{F}_{00,\text{near}}, \qquad \mathcal{F}_{22,\text{near}} = 3g^2, \\
&\mathcal{F}_{33,\text{near}} = \frac{12g^2(1-ag)\sin^2\theta}{3ag\cos 2\theta - ag + 4}, \qquad \mathcal{F}_{34,\text{near}} = -\frac{3g^2 \sin 2\theta \sqrt{2(1-ag)(2+ag)}}{3ag\cos 2\theta - ag + 4}, \\
&\mathcal{F}_{44,\text{near}} = \frac{6g^2(ag+2)\cos^2\theta}{3ag\cos 2\theta - ag + 4}, \qquad F_{ab,\text{near}} F^{ab,\text{near}} = -\frac{12g}{a}.
\end{aligned} \tag{A.1.10}$$

The equations of motion are then verified once we impose these expressions.

A.1.2 AdS_4

The 4d $\mathcal{N} = 4$ gauged supergravity can be obtained by the truncation of the 11d supergravity [41]

$$\begin{aligned}
\mathcal{L}_4 =& R * 1 - \frac{1}{2} * \mathrm{d}\varphi \wedge \mathrm{d}\varphi - \frac{1}{2} e^{2\varphi} * \mathrm{d}\chi \wedge \mathrm{d}\chi - \frac{1}{2} e^{-\varphi} * F_{(2)2} \wedge F_{(2)2} - \frac{1}{2}\chi F_{(2)2} \wedge F_{(2)2} \\
& - \frac{1}{2\left(1+\chi^2 e^{2\varphi}\right)} \left(e^{\varphi} * F_{(2)1} \wedge F_{(2)1} - e^{2\varphi}\chi F_{(2)1} \wedge F_{(2)1} \right) \\
& - g^2 \left(4 + 2\cosh\varphi + e^{\varphi}\chi^2 \right) * 1,
\end{aligned} \tag{A.1.11}$$

where φ and χ are the dilaton and axion. The subscript in parenthesis denotes the degree of the form. The solution has two pairwise equal charges and therefore two gauge potential

$A_{(1)1}$ and $A_{(1)2}$. The equations of motion are

$$\begin{aligned}
0 =& \mathrm{d}\left(\frac{1}{1+\chi^2 e^{2\varphi}}\left(e^{\varphi}\star F_{(2)1} - e^{2\varphi}\chi F_{(2)1}\right)\right),\\
0 =& \mathrm{d}\left(e^{-\varphi}\star F_{(2)2} + \chi F_{(2)2}\right),\\
0 =& -\mathrm{d}\star\mathrm{d}\varphi - e^{2\varphi}\star\mathrm{d}\chi\wedge\mathrm{d}\chi + \frac{1}{2}e^{-\varphi}\star\mathrm{d}A_{(1)2}\wedge\mathrm{d}A_{(1)2} - g^2(2\mathrm{sinh}\varphi + e^{\varphi}\chi^2)\star 1\\
&+\frac{e^{\varphi}\left(e^{2\varphi}\chi^2-1\right)}{2\left(e^{2\varphi}\chi^2+1\right)^2}\star\mathrm{d}A_{(1)1}\wedge\mathrm{d}A_{(1)1} + \frac{e^{2\varphi}\chi}{\left(e^{2\varphi}\chi^2+1\right)^2}\mathrm{d}A_{(1)1}\wedge\mathrm{d}A_{(1)1},\\
0 =& -\mathrm{d}(e^{2\varphi}\star d\chi) - \frac{1}{2}\mathrm{d}A_{(1)2}\wedge\mathrm{d}A_{(1)2} + \frac{e^{3\varphi}\chi}{\left(e^{2\varphi}\chi^2+1\right)^2}\star\mathrm{d}A_{(1)1}\wedge\mathrm{d}A_{(1)1} - 2g^2 e^{\varphi}\chi\star 1\\
&+\frac{e^{2\varphi}-e^{4\varphi}\chi^2}{2\left(e^{2\varphi}\chi^2+1\right)^2}\mathrm{d}A_{(1)1}\wedge\mathrm{d}A_{(1)1},\\
0 =& R_{ab} - \frac{1}{2}g_{ab}R - \frac{1}{2}\left(\nabla_a\varphi\nabla_b\varphi - \frac{1}{2}\nabla^c\varphi\nabla_c\varphi g_{ab}\right) - \frac{1}{2}e^{-\varphi}\left(F_{ac,2}F^c_{b,2} - \frac{1}{4}F_{cd,2}F_2^{cd}g_{ab}\right)\\
&-\frac{1}{2}e^{2\varphi}\left(\nabla_a\chi\nabla_b\chi - \frac{1}{2}\nabla^c\chi\nabla_c\chi g_{ab}\right) - \frac{e^{\varphi}}{2(1+\chi^2 e^{2\varphi})}\left(F_{ac,1}F^c_{b,1} - \frac{1}{4}F_{cd,1}F_1^{cd}g_{ab}\right)\\
&+\frac{1}{2}g^2(4+2\mathrm{cosh}\varphi + e^{\varphi}\chi^2)g_{ab}.
\end{aligned} \tag{A.1.12}$$

The convenient veilbein for this black hole solution is

$$\begin{aligned}
e_0 &= G_1\sqrt{\cos 2\theta + x^2y^2}\,\tilde{r}d\tau,\\
e_1 &= G_1\sqrt{\cos 2\theta + x^2y^2}\,\frac{d\tilde{r}}{\tilde{r}},\\
e_2 &= \sqrt{2}\sqrt{\frac{\cos 2\theta + x^2y^2}{h^2\left(-2\cos 2\theta + x^4y^4 - 2x^2y^2 - 1\right)}}d\theta,\\
e_3 &= G_4\sin\theta\sqrt{\frac{-2\cos 2\theta + x^4y^4 - 2x^2y^2 - 1}{\left(x^2y^2-1\right)^2\left(\cos 2\theta + x^2y^2\right)}}(G_5\tilde{r}d\tau + d\tilde{\phi}),
\end{aligned} \tag{A.1.13}$$

where G_1, G_4, G_5 are constants

$$
\begin{aligned}
G_1 &= \frac{xy\sqrt{2\left(x^4-1\right)\left(y^4-1\right)}}{g}\Big(x^{10}\left[y^{10}+y^6\right)+x^8\left(6y^8-8y^4\right)+x^6y^2\left(y^8-10y^4+5\right)\\
&\quad -2x^4\left(4y^8-7y^4+1\right)+x^2y^2\left(5y^4-3\right)-2y^4\Big]^{-1/2},\\
G_4 &= \frac{\sqrt{2}\left(x^2y^2-1\right)}{g\left|x^2y^2-3\right|},\\
G_5 &= G_1^2\frac{g^2\left(x^2+y^2\right)\left(x^2y^2-3\right)\left(x^2y^2-1\right)^{3/2}}{\sqrt{2}xy\sqrt{\left(x^4-1\right)\left(y^4-1\right)}},
\end{aligned}
\tag{A.1.14}
$$

that depend on the parameters δ_1, δ_2 of the solution, which we have redefined as

$$
\delta_1 \equiv \ln x, \qquad \delta_2 \equiv \ln y. \tag{A.1.15}
$$

The scalar fields in the near-horizon are

$$
\begin{aligned}
\chi_{\text{near}} &= -\frac{x\cos(\theta)\left(x^2-y^2\right)\left(x^2y^2+1\right)\sqrt{2\left(y^4-1\right)\left(x^2y^2-1\right)}}{y\left(x^2\left(y^4-1\right)\cos(2\theta)+x^6y^4-x^4y^2-x^2+y^2\right)\sqrt{x^4-1}},\\
\left(e^{\varphi}\right)_{\text{near}} &= \frac{x^2\left(y^4-1\right)\cos(2\theta)+x^6y^4-x^4y^2-x^2+y^2}{x^2\left(y^4-1\right)\left(\cos(2\theta)+x^2y^2\right)},
\end{aligned}
\tag{A.1.16}
$$

and for the gauge potentials after adding a pure gauge term for convenience,

$$
\begin{aligned}
A_{(1)1,\text{near}} =& \frac{M_1}{\cos(2\theta)+x^2y^2}\left[\frac{M_2\left(-\left(x^4-1\right)y^2\cos(2\theta)+x^4y^2\left(y^4-2\right)-x^2\left(y^4-1\right)+y^2\right)}{G_1\sqrt{\cos(2\theta)+x^2y^2}}e_1\right.\\
&\left.+M_4\sin^2\theta\left(\frac{\csc(\theta)\left(x^2y^2-1\right)\sqrt{\cos(2\theta)+x^2y^2}}{G_4\sqrt{-2\cos(2\theta)+x^4y^4-2x^2y^2-1}}e_4-\frac{G_5}{G_1\sqrt{\cos(2\theta)+x^2y^2}}e_1\right)\right],\\
A_{(1)2,\text{near}} =& \frac{M_1}{\cos(2\theta)+x^2y^2}\left[\frac{M_2\left(-x^2\left(y^4-1\right)\cos(2\theta)+x^6y^4-x^4y^2+x^2\left(1-2y^4\right)+y^2\right)}{G_1\sqrt{\cos(2\theta)+x^2y^2}}e_1\right.\\
&\left.+M_4\sin^2\theta\left(\frac{\csc(\theta)\left(x^2y^2-1\right)\sqrt{\cos(2\theta)+x^2y^2}}{G_4\sqrt{-2\cos(2\theta)+x^4y^4-2x^2y^2-1}}e_4-\frac{G_5}{G_1\sqrt{\cos(2\theta)+x^2y^2}}\right)e_1\right],
\end{aligned}
\tag{A.1.17}
$$

where

$$
\begin{aligned}
M_1 &= \frac{x^2y^2-1}{\sqrt{2}}, \\
M_2 &= -G_5 \frac{2\left(x^2y^2+1\right)}{g\left(x^2+y^2\right)\left(x^2y^2-3\right)\left(x^2y^2-1\right)^2}, \\
M_4 &= \frac{4}{3g-gx^2y^2}.
\end{aligned}
\tag{A.1.18}
$$

A.1.3 AdS_7

In this section, we are interested in charged, rotating AdS_7 black hole solutions as studied in [50, 51]. The Lagrangian is

$$
\begin{aligned}
\mathcal{L}_7 =& R \star 1 - \frac{1}{2}\sum_{i=1}^{2} \star \mathrm{d}\varphi_i \wedge \mathrm{d}\varphi_i - \frac{1}{2}\sum_{I=1}^{2} X_I^{-2} \star F_{(2)}^I \wedge F_{(2)}^I - \frac{1}{2} X_1^2 X_2^2 \star F_{(4)} \wedge F_{(4)} \\
&+ 2g^2\left(8X_1X_2 + 4X_1^{-1}X_2^{-2} + 4X_1^{-2}X_2^{-1} - X_1^{-4}X_2^{-4}\right) \star 1 \\
&+ gF_{(4)} \wedge A_{(3)} + F_{(2)}^1 \wedge F_{(2)}^2 \wedge A_{(3)},
\end{aligned}
\tag{A.1.19}
$$

where

$$
X_1 = \mathrm{e}^{-\varphi_1/\sqrt{10}-\varphi_2/\sqrt{2}}, \quad X_2 = \mathrm{e}^{-\varphi_1/\sqrt{10}+\varphi_2/\sqrt{2}}, \quad F_{(2)}^I = \mathrm{d}A_{(1)}^I, \quad F_{(4)} = \mathrm{d}A_{(3)}, \tag{A.1.20}
$$

where we have fixed a typographical error corresponding to a minus sign in one of the terms in the Lagrangian. The bosonic fields include two scalars φ_1 and φ_2, the graviton, a 3-form potential $A_{(3)}$, and two U(1) gauge potentials $A_{(1)}^I, I = 1, 2$. We study two different solutions to this Lagrangian. The first solution is more general with two charges set equal but different angular momenta. The equations of motion corresponding to the scalars and gauge fields are

$$\begin{aligned}
\Box\varphi_1 =& \frac{8}{\sqrt{10}}g^2\left(4X_1X_2-3X_1^{-1}X_2^{-2}-3X_1^{-2}X_2^{-1}+2X_1^{-4}X_2^{-4}\right)+\frac{1}{2\sqrt{10}}\sum_{I=1}^{2}X_I^{-2}F^{Iab}F^I_{ab}\\
&-\frac{1}{12\sqrt{10}}X_1^2X_2^2F^{abcd}F_{abcd},\\
\Box\varphi_2 =& \frac{1}{2\sqrt{2}}\left(X_1^{-2}F^{1ab}F^1_{ab}-X_2^{-2}F^{2ab}F^2_{ab}\right)+4\sqrt{2}g^2\left(X_1^{-1}X_2^{-2}-X_1^{-2}X_2^{-1}\right),\\
0 =& \,\mathrm{d}\left(X_1^{-2}\star F^1_{(2)}\right)-F^2_{(2)}\wedge F_{(4)},\\
0 =& \,\mathrm{d}\left(X_2^{-2}\star F^2_{(2)}\right)-F^1_{(2)}\wedge F_{(4)},\\
0 =& \,\mathrm{d}\left(X_1^2X_2^2\star F_{(4)}\right)-2gF_{(4)}-F^1_{(2)}\wedge F^2_{(2)},
\end{aligned} \tag{A.1.21}$$

and for the graviton, we have

$$\begin{aligned}
0 =& R_{ab}-\frac{1}{2}R\,g_{ab}-g^2\left(8X_1X_2+4X_1^{-1}X_2^{-2}+4X_1^{-2}X_2^{-1}-X_1^{-4}X_2^{-4}\right)g_{ab}\\
&-\sum_{i=1}^{2}\left(\frac{1}{2}\nabla_a\varphi_i\nabla_b\varphi_i-\frac{1}{4}\nabla^c\varphi_i\nabla_c\varphi_i g_{ab}\right)-\sum_{I=1}^{2}X_I^{-2}\left(\frac{1}{2}F_a{}^{Ic}F^I_{bc}-\frac{1}{8}F^{Icd}F^I_{cd}\,g_{ab}\right)\\
&-X_1^2X_2^2\left(\frac{1}{12}F_a{}^{cde}F_{bcde}-\frac{1}{96}F^{cdef}F_{cdef}\,g_{ab}\right).
\end{aligned} \tag{A.1.22}$$

We can truncate this solution as constructed in [50], where the two charges and angular momenta are set equal. This truncation can be done by letting $X=X_1=X_2=e^{-\varphi/\sqrt{10}}$, $\varphi_2=0$ and $A_{(1)}=A^1_{(1)}=A^2_{(1)}$ and the Lagrangian of interest becomes

$$\begin{aligned}
\mathcal{L}_7 =& R\star 1-\frac{1}{2}\star\mathrm{d}\varphi_1\wedge\mathrm{d}\varphi_1-X^{-2}\star F_{(2)}\wedge F_{(2)}-\frac{1}{2}X^4\star F_{(4)}\wedge F_{(4)}\\
&+2g^2\left(8X^2+8X^{-3}-X^{-8}\right)\star 1+F_{(2)}\wedge F_{(2)}\wedge A_{(3)}+gF_{(4)}\wedge A_{(3)},
\end{aligned} \tag{A.1.23}$$

and the equations of motion are

$$\begin{aligned}
0 =& \,\mathrm{d}\star\mathrm{d}\varphi-\frac{2X^{-2}}{\sqrt{10}}\star F_{(2)}\wedge F_{(2)}+\frac{2X^4}{\sqrt{10}}\star F_{(4)}\wedge F_{(4)}-\frac{16g^2}{\sqrt{10}}\left(2X^2-3X^{-3}+X^{-8}\right)\star 1,\\
0 =& \,d\left(X^{-2}\star F_{(2)}\right)-F_{(2)}\wedge F_{(4)},\\
0 =& \,\mathrm{d}\left(X_1^4\star F_{(4)}\right)-2gF_{(4)}-F^1_{(2)}\wedge F^2_{(2)},
\end{aligned} \tag{A.1.24}$$

and for the graviton

$$\begin{aligned}0 =& R_{ab} - \frac{1}{2}R\, g_{ab} - \frac{1}{2}\left(\nabla_a\varphi\nabla_b\varphi - \frac{1}{2}\nabla^c\varphi\nabla_c\varphi g_{ab}\right) - X^{-2}\left(F_a{}^cF_{bc} - \frac{1}{4}F^{cd}F_{cd}\, g_{ab}\right) \\ & - g^2\left(8X^2 + 8X^{-3} - X^{-8}\right) - \frac{1}{12}X^4\left(F_a{}^{cde}F_{bcde} - \frac{1}{8}F^{cdef}F_{cdef}\, g_{ab}\right).\end{aligned} \tag{A.1.25}$$

The fields corresponding to the solution in [50] are

$$\begin{aligned} X &= H^{-1/5}, \\ A_{(1)} &= \frac{2m\sinh(\delta)\cosh(\delta)}{\rho^4\Xi H}(dt - a\sigma) + \frac{\alpha_{70}\, dt}{\Xi_-}, \\ A_{(3)} &= \frac{\left(am\sinh^2(\delta)\right)\sigma\wedge d\sigma}{\rho^2\Xi\Xi_-} + \alpha_{71}\, dt\wedge d\theta\wedge d\psi + \alpha_{72}\, dt\wedge d\xi\wedge d\phi + \alpha_{73}\, dt\wedge d\xi\wedge d\psi,\end{aligned} \tag{A.1.26}$$

where we have added pure gauge terms to both potentials $A_{(1)}$ and $A_{(3)}$ for convenience. More precisely, after taking the near-horizon geometry, we have

$$\alpha_{70} = -1, \qquad \alpha_{71} = -\beta\sin\theta\sin^2\xi, \qquad \alpha_{72} = \beta\sin2\xi, \qquad \alpha_{73} = \beta\cos\theta\sin2\xi, \tag{A.1.27}$$

where

$$\beta = -\frac{4\left(e^{2\delta} - 1\right)}{\left(-13e^{2\delta} - 9e^{4\delta} + 9e^{6\delta} + 5\right)g^2}. \tag{A.1.28}$$

A convenient veilbein for the near-horizon is

$$\begin{aligned} e_1 &= p_1\tilde{r}d\tau, \\ e_2 &= p_1\frac{d\tilde{r}}{\tilde{r}}, \\ e_3 &= p_2\left(p_3\tilde{r}d\tau + p_4\left(\sin^2\xi(d\phi + \cos\theta d\psi) + 2d\tilde{\chi}\right)\right), \\ e_4 &= p_5 d\xi, \\ e_5 &= p_5\sin\xi d\theta, \\ e_6 &= p_5\sin\theta\sin\xi d\psi, \\ e_7 &= p_5\sin\xi\cos\xi(d\phi + \cos\theta d\psi),\end{aligned} \tag{A.1.29}$$

where

$$\begin{aligned}
p_1 &= \frac{1}{g}\frac{2^{3/5}3^{1/5}(e^{2\delta}+1)^{1/5}}{\sqrt{6e^{2\delta}+27e^{4\delta}+43}},\\
p_2 &= \frac{1}{2^{2/5}3^{3/10}g}\frac{1}{(1-3e^{2\delta})(5-3e^{2\delta})}\frac{1}{\sqrt{(e^{2\delta}+1)^{3/5}(9e^{2\delta}-7)}},\\
p_3 &= -\frac{16\left(3e^{2\delta}-5\right)^{3/2}\left(2e^{2\delta}+3e^{4\delta}-1\right)}{6e^{2\delta}+27e^{4\delta}+43}\sqrt{\frac{3(9e^{2\delta}-7)}{(-2e^{2\delta}+3e^{4\delta}-5)}},\\
p_4 &= -30e^{2\delta}+27e^{4\delta}+7,\\
p_5 &= \frac{1}{g}\frac{2^{8/5}}{3^{3/10}}\frac{(e^{2\delta}+1)^{1/5}}{\sqrt{(-2e^{2\delta}+3e^{4\delta}-5)}}.
\end{aligned} \tag{A.1.30}$$

In the near-horizon limit, the fields in the veilbein basis become

$$\begin{aligned}
X_{\text{near}} &= \frac{2^{2/5}}{3^{1/5}(e^{2\delta}+1)^{1/5}},\\
A_{(1),\text{near}} &= \frac{2^{2/5}3^{3/10}\left(e^{2\delta}+1\right)^{4/5}\left(\left(15-9e^{2\delta}\right)e_1+\sqrt{6e^{2\delta}+27e^{4\delta}+43}e_3\right)}{\sqrt{44e^{2\delta}+210e^{4\delta}+108e^{6\delta}+243e^{8\delta}-301}},\\
A_{(3),\text{near}} &= \frac{(54e^{2\delta}+27e^{4\delta}-101)(e_1\wedge e_5\wedge e_6-e_1\wedge e_4\wedge e_7)}{2^{9/5}3^{1/10}(e^{2\delta}+1)^{1/10}\sqrt{9e^{2\delta}-7}\sqrt{6e^{2\delta}+27e^{4\delta}+43}}\\
&\quad-\frac{\left(99e^{2\delta}-117e^{4\delta}+81e^{6\delta}-215\right)(e_3\wedge e_4\wedge e_7-e_3\wedge e_5\wedge e_6)}{2^{4/5}3^{1/10}(e^{2\delta}+1)^{1/10}\sqrt{9e^{2\delta}-7}\,(6e^{2\delta}+27e^{4\delta}+43)}.
\end{aligned} \tag{A.1.31}$$

A.1.4 AdS_6

The field content consists of the graviton, a 2-form $A_{(2)}$, the scalar φ and one U(1) gauge potential $A_{(1)}$ after truncation, as shown in [55]. After appropriate rescaling and gauge transformations, the 6d Lagrangian is given by

$$\begin{aligned}
\mathcal{L}_6 =& R\star 1-\frac{1}{2}\star \mathrm{d}\varphi\wedge\mathrm{d}\varphi - X^{-2}\left(\star F_{(2)}\wedge F_{(2)}+g^2\star A_{(2)}\wedge A_{(2)}\right)-\frac{1}{2}X^4\star F_{(3)}\wedge F_{(3)}\\
&+g^2\left(9X^2+12X^{-2}-X^{-6}\right)\star 1-F_{(2)}\wedge F_{(2)}\wedge A_{(2)}-\frac{g^2}{3}A_{(2)}\wedge A_{(2)}\wedge A_{(2)},
\end{aligned} \tag{A.1.32}$$

where

$$X=e^{-\varphi/\sqrt{8}}. \tag{A.1.33}$$

The equations of motion are

$$
\begin{aligned}
G_{ab} =& \frac{1}{2}\nabla_a\varphi\nabla_b\varphi - \frac{1}{4}\nabla^c\varphi\nabla_c\varphi g_{ab} + X^{-2}\left(F_a^c F_{bc} - \frac{1}{4}F^{cd}F_{cd}g_{ab}\right) \\
&+ X^{-2}g^2\left(A_a^c A_{bc} - \frac{1}{4}A^{cd}A_{cd}g_{ab}\right) + X^4\left(\frac{1}{4}F_a^{cd}F_{bcd} - \frac{1}{24}F^{cde}F_{cde}g_{ab}\right) \\
&+ \frac{g^2}{2}\left(9X^2 + 12X^{-2} - X^{-6}\right)g_{ab},
\end{aligned}
\tag{A.1.34}
$$

$$
\begin{aligned}
\Box\varphi &= \frac{1}{\sqrt{8}}X^{-2}\left(F^{ab}F_{ab} + g^2A^{ab}A_{ab}\right) - \frac{1}{3\sqrt{8}}X^4F^{abc}F_{abc} + \frac{3}{\sqrt{2}}g^2\left(3X^2 - 4X^{-2} + X^{-6}\right), \\
\mathrm{d}\left(X^{-2}\star F_{(2)}\right) &= -F_{(2)}\wedge F_{(3)}, \\
\mathrm{d}\left(X^4\star F_{(3)}\right) &= -F_{(2)}\wedge F_{(2)} - g^2A_{(2)}\wedge A_{(2)} - 2g^2X^{-2}\star F_{(2)}.
\end{aligned}
\tag{A.1.35}
$$

We omit the veilbein and several other details for this black hole as the expressions are quite long. The scalar and the $U(1)$ gauge field in the near-horizon limit take the form

$$
\begin{aligned}
\chi^4_{\text{near}} &= \frac{g\left(a\left(b+gy^2\right)+y^2(bg+1)\right)\left(a\left(b+gz^2\right)+z^2(bg+1)\right)}{(ag+bg+1)\left(a^2b(bg+1)+a\left(b^2+bg\left(y^2+z^2\right)+g^2y^2z^2\right)+gy^2z^2(bg+1)\right)}, \\
A_{(1),\text{near}} &= W_1\left(W_2\tilde{r}d\tilde{t} + W_3d\tilde{\phi}_1 + W_4d\tilde{\phi}_2\right),
\end{aligned}
\tag{A.1.36}
$$

where

$$
\begin{aligned}
W_1 =& \frac{\sqrt{ab}}{\sqrt{ag+bg+1}\left(a^2b(bg+1)+a\left(b^2+bg\left(y^2+z^2\right)+g^2y^2z^2\right)+gy^2z^2(bg+1)\right)}, \\
W_2 =& \Big[\left(b^2-a^2\right)\Xi_a\Xi_b(ag+bg+1)\left(a^2\left(3b^2+bg\left(y^2+z^2\right)-g^2y^2z^2\right)\right. \\
&\left. +a\left(b^2g\left(y^2+z^2\right)+b\left(-2g^2y^2z^2+y^2+z^2\right)-2gy^2z^2\right)-y^2z^2(bg+1)^2\right] \\
&\Big[a^4g^2\left(b^2g^2+6bg+1\right)+2a^3g\left(b^3g^3+7b^2g^2+7bg+1\right) \\
&+a^2\left(b^4g^4+14b^3g^3+30b^2g^2+14bg+1\right)+2ab\left(3b^3g^3+7b^2g^2+7bg+3\right) \\
&+b^2(bg+1)^2\Big]^{-1}, \\
W_3 =& \frac{b\left(a^2-y^2\right)\left(a^2-z^2\right)\left(b^2g^2-1\right)}{\sqrt{\frac{ab}{ag+bg+1}}}, \\
W_4 =& -\frac{a\left(a^2g^2-1\right)\left(b^2-y^2\right)\left(b^2-z^2\right)}{\sqrt{\frac{ab}{ag+bg+1}}}.
\end{aligned}
\tag{A.1.37}
$$

Note that we have added a pure gauge term to the 1-form $\alpha_6 dt$, where $\alpha_6 = -1$.

Appendix B

Subleading Corrections via the Heat Kernel: Supplemental Computations

B.1 Mathematica algorithm

We describe the Mathematica algorithm written with xAct [204] and xPert [205] to compute the Seeley-DeWitt coefficients $a_4(x)$ presented in this paper. An executable code reproducing the results of this paper is available at [188]. The purpose of the algorithm is to compute $a_4(x)$ via the expression

$$(4\pi)^2 a_4(x) = \mathrm{Tr}\left[\frac{1}{2}E^2 + \frac{1}{6}RE + \frac{1}{12}\Omega_{\mu\nu}\Omega^{\mu\nu} + \frac{1}{360}(5R^2 + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu})\right] , \tag{B.1.1}$$

where E and Ω are determined by the two-derivative action as defined in section (5.2.3). This computation is straightforward but tedious to do by hand, especially for fermions. The algorithm uses xTensor and our own implementation of Euclidean spinors (as xSpinor can only treat Lorentzian spinors). The resulting expression is then reduced using various spinorial and geometrical identities, as well as the equations of motion.

For bosons, the algorithm uses xPert [205] to expand any Lagrangian to quadratic order. It then extracts the matrices P and ω which allows us to compute E and Ω, evaluates and simplifies $a_4(x)$. For fermions, the input is the matrix L which defines the quadratic Lagrangian as $\mathcal{L} = \bar{\psi}(\not{D} + L)\psi$ where ψ refers to all the fermionic fields of the theory. The heat kernel method is then applied to the operator $\mathcal{Q} = \mathcal{D}^\dagger\mathcal{D}$ using the formula (5.2.25) to obtain P and ω. The algorithmic approach is useful because we can automatize simplification using gamma matrix identities.

The algorithm was used in this paper to verify the bosonic result, also computed by hand, and to obtain the fermionic result, which appeared too tedious to compute by hand. It has also been used to obtain the results for minimal couplings, which can also be easily obtained by hand.

Let us mention various checks that have been performed on this algorithm. It gives the correct logarithmic contribution for various results in the literature such as minimally coupled fields [158] and ungauged $\mathcal{N} \geqslant 2$ supergravity [148]. The same algorithm was used in [206] to compute the logarithmic correction in the non-BPS branch of ungauged $\mathcal{N} \geqslant 2$ supergravity. The results of [206] were subsequently checked by a completely independent approach [148] computing directly $a_4(x)$ from eigenvalues, with agreement in all cases. Given that the Lagrangians involved in these computations were fairly complicated, this gives us confidence that the algorithm performs correctly.

B.2 Bosonic computation

For the interested reader, we present a self-contained computation of the heat kernel coefficient $a_4(x)$ for the Einstein-Maxwell-AdS theory.

B.2.1 Quadratic fluctuations in Einstein-Maxwell AdS theory

The action is given by

$$S = \int d^4x \sqrt{g} \left(R - 2\Lambda - F_{\mu\nu} F^{\mu\nu} \right), \tag{B.2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength with A_μ the gauge potential. Note that we find it convenient to use the convention $4\pi G = 1$. We consider variations of the metric and gauge field

$$\delta g_{\mu\nu} = \sqrt{2} h_{\mu\nu}, \qquad \delta A_\mu = \frac{1}{2} a_\mu \ , \tag{B.2.2}$$

where $h_{\mu\nu}$ and a_α are the graviton and graviphoton respectively. We impose a particular gauge to the theory by adding a suitable gauge-fixing Lagrangian

$$S = -\int d^4x \sqrt{\det g} \left\{ \left(D^\mu h_{\mu\rho} - \frac{1}{2} D_\rho h \right) \left(D^\nu h^\rho_\nu - \frac{1}{2} D^\rho h \right) + \frac{1}{2} \left(D^\mu a_\mu \right) \left(D^\nu a_\nu \right) \right\}, \tag{B.2.3}$$

and the corresponding ghost action to the action (B.2.1). We then expand the action up to quadratic order. The linear order variation yields the equation of motion for the background fields

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+g_{\mu\nu}\Lambda =2F_{\mu\rho}F_{\nu}{}^{\rho}-\frac{1}{2}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\ , \tag{B.2.4}$$

$$D^{\mu}F_{\mu\nu} =0\ . \tag{B.2.5}$$

Note that the equations of motion implies that $R=4\Lambda=-12/\ell^2$. It is also worth mentioning the Bianchi identity for the gravitational and gauge fields

$$D_{[\mu}F_{\nu\rho]} =0\ , \tag{B.2.6}$$

$$R_{\mu[\nu\rho\sigma]} =0\ . \tag{B.2.7}$$

Writing the quadratic action in the standard form (5.2.20), we find

$$\begin{aligned}
\phi_m\mathcal{Q}^{mn}\phi_n =&G^{\mu\nu\alpha\beta}h_{\mu\nu}\Box h_{\alpha\beta}+g^{\alpha\beta}a_\alpha\Box a_\beta-a_\alpha R^{\alpha\beta}a_\beta\\
&+h_{\mu\nu}\Big\{R^{\mu\alpha\nu\beta}+R^{\mu\beta\nu\alpha}-\frac{1}{2}\left(g^{\mu\alpha}R^{\nu\beta}+g^{\mu\beta}R^{\nu\alpha}\right.\\
&\left.+g^{\nu\alpha}R^{\mu\beta}+g^{\nu\beta}R^{\mu\alpha}\right)-2\left(F^{\mu\alpha}F^{\nu\beta}+F^{\mu\beta}F^{\nu\alpha}\right)\\
&+\frac{1}{2}\left(g^{\mu\nu}g^{\alpha\beta}-g^{\mu\alpha}g^{\nu\beta}-g^{\mu\beta}g^{\nu\alpha}\right)\left(F_{\theta\varphi}F^{\theta\varphi}\right)+2G^{\alpha\beta\mu\nu}\Lambda\Big\}h_{\alpha\beta}\\
&-h_{\mu\nu}\left\{\frac{1}{4}\left(D_\rho K^\rho\right)^{\mu\nu\alpha}-\frac{1}{2}\left(K^\rho\right)^{\mu\nu\alpha}D_\rho\right\}a_\alpha\\
&-a_\alpha\left\{\frac{1}{4}\left(D_\rho K^\rho\right)^{\mu\nu\alpha}+\frac{1}{2}\left(K^\rho\right)^{\mu\nu\alpha}D_\rho\right\}h_{\mu\nu},
\end{aligned} \tag{B.2.8}$$

where

$$\left(K^\rho\right)^{\mu\nu\alpha}=2\sqrt{2}\Big(g^{\alpha\mu}F^{\rho\nu}+g^{\alpha\nu}F^{\rho\mu}-g^{\mu\rho}F^{\alpha\nu}-g^{\nu\rho}F^{\alpha\mu}-g^{\mu\nu}F^{\rho\alpha}\Big). \tag{B.2.9}$$

Note that we have used the symmetry properties of the graviton to write the term proportional to Λ using the DeWitt metric

$$G^{\mu\nu\alpha\beta}=\frac{1}{2}\left(g^{\mu\alpha}g^{\nu\beta}+g^{\mu\beta}g^{\nu\alpha}-g^{\mu\nu}g^{\alpha\beta}\right), \tag{B.2.10}$$

as $h_{\mu\nu}(2g^{\mu\alpha}g^{\nu\beta}\Lambda-g^{\mu\nu}g^{\alpha\beta}\Lambda)h_{\alpha\beta}=h_{\mu\nu}(2G^{\alpha\beta\mu\nu}\Lambda)h_{\alpha\beta}$. We note that the results pertaining to the cosmological constant terms agree with what we expect from [182]. With (B.2.8), we can explicitly read out the matrices I^{mn}, ω^ρ and P^{mn} in (5.2.20):

$$\phi_m I^{mn}\phi_n = h_{\mu\nu}G^{\mu\nu\alpha\beta}h_{\alpha\beta} + a_\alpha g^{\alpha\beta}a_\beta \ . \tag{B.2.11}$$

$$\phi_m (\omega^\rho)^{mn}\phi_n = \frac{1}{4}h_{\mu\nu}(K^\rho)^{\mu\nu\alpha}a_\alpha - \frac{1}{4}a_\alpha (K^\rho)^{\mu\nu\alpha}h_{\mu\nu} \ . \tag{B.2.12}$$

$$\begin{aligned}\phi_m P^{mn}\phi_n = & h_{\mu\nu}\Big\{R^{\mu\alpha\nu\beta} + R^{\mu\beta\nu\alpha} - \frac{1}{2}\left(g^{\mu\alpha}R^{\nu\beta} + g^{\mu\beta}R^{\nu\alpha}\right.\\ &\left. + g^{\nu\alpha}R^{\mu\beta} + g^{\nu\beta}R^{\mu\alpha}\right) - 2\left(F^{\mu\alpha}F^{\nu\beta} + F^{\mu\beta}F^{\nu\alpha}\right)\\ &+\frac{1}{2}\left(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha}\right)\left(F_{\theta\varphi}F^{\theta\varphi}\right)\\ &+2g^{\mu\alpha}g^{\nu\beta}\Lambda - g^{\mu\nu}g^{\alpha\beta}\Lambda\Big\}h_{\alpha\beta} \ ,\\ &- a_\alpha R^{\alpha\beta}a_\beta + \frac{\sqrt{2}}{2}h_{\mu\nu}\left\{D^\mu F^{\alpha\nu} + D^\nu F^{\alpha\mu}\right\}a_\alpha\\ &+ \frac{\sqrt{2}}{2}a_\alpha\left\{D^\mu F^{\alpha\nu} + D^\nu F^{\alpha\mu}\right\}h_{\mu\nu} \ .\end{aligned} \tag{B.2.13}$$

B.2.2 Trace computation

Here, we present various trace computations. The field strength Ω is given by

$$\phi_m \Omega^{mn}_{\mu\nu}\phi_n = \phi_m[D_\mu + \omega_\mu, D_\nu + \omega_\nu]\phi_n = \phi_m\Big\{[D_\mu, D_\nu]^{mn} + D_{[\mu}\omega_{\nu]}{}^{mn} + [\omega_\mu, \omega_\nu]^{mn}\Big\}\phi_n \ . \tag{B.2.14}$$

To compute the matrix E and Ω using (5.2.22) and (B.2.14), we need $[D_\mu . D_\nu]$, $(D^\rho\omega_\rho)^{mn}$ and $(\omega^\rho)^{mp}(\omega_\rho)_p{}^n$:

$$\begin{aligned}\phi_m [D^\rho, D^\sigma]^{mn}\phi_n = & h_{\mu\nu}[D^\rho, D^\sigma]h^{\mu\nu} + a_\alpha [D^\rho, D^\sigma]a^\alpha\\ = & h_{\mu\nu}\left\{g^{\nu\beta}R^{\mu\alpha\rho\sigma} + g^{\mu\beta}R^{\nu\alpha\rho\sigma}\right\}h_{\alpha\beta} + a_\alpha R^{\alpha\beta\rho\sigma}a_\beta\\ = & \frac{1}{2}h_{\mu\nu}\left\{g^{\nu\beta}R^{\mu\alpha\rho\sigma} + g^{\nu\alpha}R^{\mu\beta\rho\sigma} + g^{\mu\beta}R^{\nu\alpha\rho\sigma} + g^{\mu\alpha}R^{\nu\beta\rho\sigma}\right\}h_{\alpha\beta}\\ & + a_\alpha R^{\alpha\beta\rho\sigma}a_\beta,\end{aligned} \tag{B.2.15}$$

$$\phi_m (D^\rho\omega_\rho)^{mn}\phi_n = -\frac{\sqrt{2}}{2}h_{\mu\nu}\left(D^\nu F^{\alpha\mu} + D^\mu F^{\alpha\nu}\right)a_\alpha + \frac{\sqrt{2}}{2}a_\alpha\left(D^\nu F^{\alpha\mu} + D^\mu F^{\alpha\nu}\right)h_{\mu\nu}, \tag{B.2.16}$$

$$\begin{aligned}\phi_m (\omega^\rho)^{mp}(\omega_\rho)_p{}^n\phi_n = & \frac{1}{16}h_{\mu\nu}(K^\rho)^{\mu\nu\alpha}(-K_\rho)^{\delta\gamma\beta}g_{\alpha\beta}h_{\delta\gamma}\\ & + \frac{1}{16}a_\alpha (K^\rho)^{\mu\nu\alpha}(-K_\rho)^{\delta\gamma\ \beta}G_{\mu\nu\delta\gamma}a_\beta.\end{aligned} \tag{B.2.17}$$

For (B.2.17), using the definition of $(K^\rho)^{\mu\nu\alpha}$ (B.2.9), we find

$$\begin{aligned}\phi_m\left(\omega^\rho\right)^{mp}\left(\omega_\rho\right)_p{}^n\phi_n =&h_{\mu\nu}\left(-2F^{\mu\delta}F^{\nu\gamma}-2F^{\mu\gamma}F^{\nu\delta}+2F^{\gamma\rho}F^{\delta}{}_{\rho}g^{\mu\nu}-F^{\nu\rho}F^{\delta}{}_{\rho}g^{\mu\gamma}\right.\\&-F^{\nu\rho}F^{\gamma}{}_{\rho}g^{\mu\delta}-F^{\mu\rho}F^{\delta}{}_{\rho}g^{\nu\gamma}-F^{\mu\rho}F^{\gamma}{}_{\rho}g^{\nu\delta}+2F^{\mu\rho}F^{\nu}{}_{\rho}g^{\gamma\delta}\\&\left.-\tfrac{1}{2}F_{\rho\sigma}F^{\rho\sigma}g^{\mu\nu}g^{\gamma\delta}\right)h_{\gamma\delta}+a_\alpha\left(-2F^{\alpha\gamma}F^{\beta}{}_{\gamma}-F_{\theta\phi}F^{\theta\phi}g^{\alpha\beta}\right)a_\beta.\end{aligned} \tag{B.2.18}$$

Note that (B.2.18) is the same expression as in the asymptotically flat case [183] and changes only when we plug in the equations of motion (B.2.4),

$$\begin{aligned}\phi_m\left(\omega^\rho\right)^{mp}\left(\omega_\rho\right)_p{}^n\phi_n =&h_{\mu\nu}\left(-2F^{\mu\alpha}F^{\nu\beta}-2F^{\mu\beta}F^{\nu\alpha}-\tfrac{1}{2}F_{ad}F^{ad}g^{\mu\alpha}g^{\nu\beta}-\tfrac{1}{2}F_{ad}F^{ad}g^{\mu\beta}g^{\nu\alpha}\right.\\&+\tfrac{1}{2}F_{ad}F^{ad}g^{\mu\nu}g^{\beta\alpha}+g^{\beta\alpha}R^{\mu\nu}-\tfrac{1}{2}g^{\nu\alpha}R^{\mu\beta}-\tfrac{1}{2}g^{\nu\beta}R^{\mu\alpha}-\tfrac{1}{2}g^{\mu\alpha}R^{\nu\beta}\\&\left.-\tfrac{1}{2}g^{\mu\beta}R^{\nu\alpha}+g^{\mu\nu}R^{\beta\alpha}+\Lambda g^{\mu\alpha}g^{\nu\beta}+\Lambda g^{\mu\beta}g^{\nu\alpha}-2\Lambda g^{\mu\nu}g^{\beta\alpha}\right)h_{\alpha\beta}\\&+\frac{1}{2}a_\alpha\left(-\tfrac{3}{2}F_{\theta\phi}F^{\theta\phi}g^{\alpha\beta}-R^{\alpha\beta}+\Lambda g^{\alpha\beta}\right)a_\beta.\end{aligned} \tag{B.2.19}$$

Extracting the information from the quadratic action, we find

$$\begin{aligned}\phi_m E^{mn}\phi_n =&h_{\mu\nu}\left(R^{\mu\alpha\nu\beta}+R^{\mu\beta\nu\alpha}-g^{\mu\nu}R^{\alpha\beta}-g^{\alpha\beta}R^{\mu\nu}+\Lambda g^{\mu\nu}g^{\alpha\beta}\right)h_{\alpha\beta}\\&+a_\alpha\left(\frac{3}{2}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}-\Lambda g^{\alpha\beta}\right)a_\beta+\frac{\sqrt{2}}{2}h_{\mu\nu}\left(D^\mu F^{\alpha\nu}+D^\nu F^{\alpha\mu}\right)a_\alpha\\&+\frac{\sqrt{2}}{2}a_\alpha\left(D^\mu F^{\alpha\nu}+D^\nu F^{\alpha\mu}\right)h_{\mu\nu},\end{aligned} \tag{B.2.20}$$

$$\begin{aligned}\phi_m\left(\Omega^{\rho\sigma}\right)^{mn}\phi_n =&h_{\mu\nu}\left\{\frac{1}{2}\left(g^{\nu\beta}R^{\mu\alpha\rho\sigma}+g^{\nu\alpha}R^{\mu\beta\rho\sigma}+g^{\mu\beta}R^{\nu\alpha\rho\sigma}+g^{\mu\alpha}R^{\nu\beta\rho\sigma}\right)\right.\\&\left.+\left[\omega^\rho,\omega^\sigma\right]^{\mu\nu\alpha\beta}\right\}h_{\alpha\beta}+a_\alpha\left\{R^{\alpha\beta\rho\sigma}+\left[\omega^\rho,\omega^\sigma\right]^{\alpha\beta}\right\}a_\beta\\&+h_{\mu\nu}\left(D^{[\rho}\omega^{\sigma]}\right)^{\mu\nu\alpha}a_\alpha+a_\alpha\left(D^{[\rho}\omega^{\sigma]}\right)^{\alpha\mu\nu}h_{\mu\nu}.\end{aligned} \tag{B.2.21}$$

We explicitly compute the traces involving the endomorphism E

$$\begin{aligned}\mathrm{Tr}(RE) =& \mathrm{Tr}R\left[R^{\mu\alpha\nu\beta}+R^{\mu\beta\nu\alpha}-g^{\mu\nu}R^{\alpha\beta}-g^{\alpha\beta}R^{\mu\nu}+\Lambda g^{\mu\nu}g^{\alpha\beta}\right]\\&+\mathrm{Tr}\left(\frac{3}{2}g^{\alpha\beta}RF_{\mu\nu}F^{\mu\nu}-R\Lambda g^{\alpha\beta}\right)\\=&R\left(R^{\mu\alpha\nu\beta}+R^{\mu\beta\nu\alpha}-g^{\mu\nu}R^{\alpha\beta}-g^{\alpha\beta}R^{\mu\nu}+g^{\mu\nu}g^{\alpha\beta}\Lambda\right)G_{\mu\nu\alpha\beta}\\&+R\left(\frac{3}{2}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}-\Lambda g^{\alpha\beta}\right)g_{\alpha\beta},\end{aligned} \tag{B.2.22}$$

which after expanding, we find

$$\begin{aligned}\mathrm{Tr}(RE) &= R\left(-2R+(D-2)R+\left(D-\frac{D^2}{2}\right)\Lambda\right)+R\left(\frac{3D}{2}F_{\mu\nu}F^{\mu\nu}-d\Lambda\right)\\ &= -32\Lambda^2+6RF_{\mu\nu}F^{\mu\nu},\end{aligned} \tag{B.2.23}$$

where $D=4$ is the dimension of the space and we have imposed $R=4\Lambda$. Next, we consider

$$\begin{aligned}\mathrm{Tr}(E^2) =&\mathrm{Tr}\left(R^{\mu\alpha\nu\beta}+R^{\mu\beta\nu\alpha}-g^{\mu\nu}R^{\alpha\beta}-g^{\alpha\beta}R^{\mu\nu}+g^{\mu\nu}g^{\alpha\beta}\Lambda\right)\\ &\times\left(R^{\rho\tau\sigma\delta}+R^{\rho\delta\sigma\tau}-g^{\rho\sigma}R^{\tau\delta}-g^{\tau\delta}R^{\rho\sigma}+g^{\rho\sigma}g^{\tau\delta}\Lambda\right)\\ &+\mathrm{Tr}\left(\left(\frac{3}{2}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}-\Lambda g^{\alpha\beta}\right)\left(\frac{3}{2}g^{\tau\delta}F_{\theta\phi}F^{\theta\phi}-\Lambda g^{\tau\delta}\right)\right)\\ &+\frac{1}{2}\mathrm{Tr}\left(\left(D^\mu F^{\alpha\nu}+D^\nu F^{\alpha\mu}\right)\left(D^\rho F^{\beta\sigma}+D^\sigma F^{\beta\rho}\right)\right)\\ &+\frac{1}{2}\mathrm{Tr}\left(\left(D^\mu F^{\alpha\nu}+D^\nu F^{\alpha\mu}\right)\left(D^\rho F^{\beta\sigma}+D^\sigma F^{\beta\rho}\right)\right).\end{aligned} \tag{B.2.24}$$

For the first term of (B.2.24), we have

$$\begin{aligned}&\left(R^{\mu\alpha\nu\beta}+R^{\mu\beta\nu\alpha}-g^{\mu\nu}R^{\alpha\beta}-g^{\alpha\beta}R^{\mu\nu}+g^{\mu\nu}g^{\alpha\beta}\Lambda\right)\\ &\times\left(R^{\rho\tau\sigma\delta}+R^{\rho\delta\sigma\tau}-g^{\rho\sigma}R^{\tau\delta}-g^{\tau\delta}R^{\rho\sigma}+g^{\rho\sigma}g^{\tau\delta}\Lambda\right)G_{\mu\nu\rho\sigma}G_{\alpha\beta\tau\delta}\\ &\quad=16\Lambda^2-4R_{ab}R^{ab}+3R_{abcd}R^{abcd}.\end{aligned} \tag{B.2.25}$$

The second term of (B.2.24) gives

$$\mathrm{Tr}\left(\frac{3}{2}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}-\Lambda g^{\alpha\beta}\right)\left(\frac{3}{2}g^{\tau\delta}F_{\theta\phi}F^{\theta\phi}-\Lambda g^{\tau\delta}\right)=4\Lambda^2-12\Lambda F_{\alpha\beta}F^{\alpha\beta}+9\left(F_{\theta\phi}F^{\theta\phi}\right)^2, \tag{B.2.26}$$

where $g^{\alpha\beta}g^{\tau\delta}g_{\alpha\tau}g_{\beta\delta}=D=4$. Now, for the remaining terms in the trace in (B.2.24), we need the following identities

$$\begin{aligned}(D_\rho F_{\mu\nu})(D^\rho F^{\mu\nu}) &= -R^{\mu\nu}R_{\mu\nu}+\frac{1}{2}R^2-\Lambda R-\frac{1}{2}RF_{\rho\sigma}F^{\rho\sigma}+R_{\mu\nu\rho\alpha}F^{\mu\nu}F^{\rho\alpha}\ ,\\ (D_\mu F_\rho{}^\nu)(D_\nu F^{\rho\mu}) &= \frac{1}{2}\left(2R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}-R_{\mu\nu}R^{\mu\nu}+\frac{1}{2}R^2-\Lambda R-\frac{1}{2}RF_{\rho\sigma}F^{\rho\sigma}\right)\ .\end{aligned} \tag{B.2.27}$$

These identities can be found by using the Bianchi identity (B.2.6) and (B.2.7) followed by an integration of parts, dropping the boundary terms along the way, and imposing the commutator relations (B.2.15) of the covariant derivatives acting on the gauge field. Note that (B.2.27) are on-shell since we have explicitly imposed the Maxwell equations and Einstein equations. Then,

$$\begin{aligned} X \equiv& \frac{1}{2}\mathrm{Tr}\left(\left(D^\mu F^{\alpha\nu} + D^\nu F^{\alpha\mu}\right)\left(D^\rho F^{\beta\sigma} + D^\sigma F^{\beta\rho}\right)\right) \\ &+ \frac{1}{2}\mathrm{Tr}\left(\left(D^\mu F^{\alpha\nu} + D^\nu F^{\alpha\mu}\right)\left(D^\rho F^{\beta\sigma} + D^\sigma F^{\beta\rho}\right)\right) \\ =& \left(D^\mu F^{\alpha\nu} + D^\nu F^{\alpha\mu}\right)\left(D^\rho F^{\beta\sigma} + D^\sigma F^{\beta\rho}\right) g_{\alpha\beta} G_{\mu\nu\rho\sigma} \ . \end{aligned} \tag{B.2.28}$$

Imposing the Bianchi identities simplifies our expression to

$$X = 2(D_\rho F_{\alpha\sigma})(D^\rho F^{\alpha\sigma}) + 2(D_\rho F_{\alpha\sigma})(D^\sigma F^{\alpha\rho}) \ . \tag{B.2.29}$$

Finally, using (B.2.27), we find

$$\begin{aligned} X =& 2\left(-R^{\mu\nu}R_{\mu\nu} + \frac{1}{2}R^2 - \Lambda R - \frac{1}{2}RF_{\rho\sigma}F^{\rho\sigma} + R_{\mu\nu\rho\alpha}F^{\mu\nu}F^{\rho\alpha}\right) \\ &+ \left(R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} - R_{\mu\nu}R^{\mu\nu} + \frac{1}{2}R^2 - \Lambda R - \frac{1}{2}RF_{\rho\sigma}F^{\rho\sigma} + R_{\mu\nu\rho\alpha}F^{\mu\nu}F^{\rho\alpha}\right) \\ =& 3\left(-R^{\mu\nu}R_{\mu\nu} + \frac{1}{2}R^2 - \Lambda R - \frac{1}{2}RF_{\rho\sigma}F^{\rho\sigma} + R_{\mu\nu\rho\alpha}F^{\mu\nu}F^{\rho\alpha}\right) \ . \end{aligned} \tag{B.2.30}$$

Putting all the contributions together, the trace of the square of E is therefore

$$\begin{aligned} \mathrm{Tr}E^2 =& \left(16\Lambda^2 - 4R_{ab}R^{ab} + 3R_{abcd}R^{abcd}\right) + \left(4\Lambda^2 - 12\Lambda F_{\alpha\beta}F^{\alpha\beta} + 9\left(F_{\theta\phi}F^{\theta\phi}\right)^2\right) \\ &+ 3\left(-R^{\mu\nu}R_{\mu\nu} + \frac{1}{2}R^2 - \Lambda R - \frac{1}{2}RF_{\rho\sigma}F^{\rho\sigma} + R_{\mu\nu\rho\alpha}F^{\mu\nu}F^{\rho\alpha}\right) \\ =& 32\Lambda^2 - 7R_{\mu\nu}R^{\mu\nu} + 3R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 18\Lambda F_{\rho\sigma}F^{\rho\sigma} + 3R_{\mu\nu\rho\alpha}F^{\mu\nu}F^{\rho\alpha} + 9\left(F_{\theta\phi}F^{\theta\phi}\right)^2 \ . \end{aligned} \tag{B.2.31}$$

The necessary traces are summarised below:

$$\mathrm{Tr}(I) = 14 \ , \tag{B.2.32}$$

$$\mathrm{Tr}(RE) = -32\Lambda^2 + 6RF_{\mu\nu}F^{\mu\nu} \ , \tag{B.2.33}$$

$$\begin{aligned} \mathrm{Tr}(E^2) =& 32\Lambda^2 - 7R_{\mu\nu}R^{\mu\nu} + 3R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 18\Lambda F_{\rho\sigma}F^{\rho\sigma} \\ &+ 3R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} + 9\left(F_{\theta\phi}F^{\theta\phi}\right)^2 \ , \end{aligned} \tag{B.2.34}$$

$$\begin{aligned} \mathrm{Tr}(\Omega^2) =& -224\Lambda^2 + 60\Lambda F_{\mu\nu}F^{\mu\nu} - 54F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + 56R_{\mu\nu}R^{\mu\nu} \\ &- 18F^{\mu\nu}F^{\rho\sigma}R_{\mu\nu\rho\sigma} - 7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \ . \end{aligned} \tag{B.2.35}$$

Substituting the traces (B.2.32)-(B.2.35) into (5.2.23), we obtain the fourth heat kernel coefficient of Einstein-Maxwell AdS theory without ghosts

$$(4\pi)^2 a_4^{\text{EM}}(x) = -\frac{880}{180}\Lambda^2 + \frac{196}{180}R_{\mu\nu}R^{\mu\nu} + \frac{179}{180}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \ . \tag{B.2.36}$$

Ghost contribution

The addition of the gauge-fixing Lagrangian (B.2.3) introduces an action for the ghosts, given by

$$\mathcal{S}_{\text{ghost},b} = \frac{1}{2}\int d^4x\sqrt{g}\Big\{2b_\mu\big(g^{\mu\nu}\Box + R^{\mu\nu}\big)c_\nu + 2b\Box c - 4bF^{\rho\nu}D_\rho c_\nu\Big\} \ , \tag{B.2.37}$$

where b_μ and c_μ are vector fields and b and c are scalar fields. From these expressions, we can extract the matrices E and Ω as

$$\begin{aligned} \phi_n E^n_m \phi^m =& b_\mu\left(R^\mu{}_\nu\right)b^\nu + c_\mu\left(R^\mu{}_\nu\right)c^\nu, \\ \phi_n\left(\Omega_{\alpha\beta}\right)^n_m\phi^m =& b_\mu\left(R^\mu{}_{\nu\alpha\beta}\right)b^\nu + c_\mu\left(R^\mu{}_{\nu\alpha\beta}\right)c^\nu - \frac{1}{2}\left(b_\mu - ic_\mu\right)\left(D^\mu F_{\alpha\beta}\right)(b+ic) \\ &+ \frac{1}{2}(b+ic)\left(D_\nu F_{\alpha\beta}\right)\left(b^\nu - ic^\nu\right) \ , \end{aligned} \tag{B.2.38}$$

Note that in the case of the ghost fields, we are raising and lowering the indices with $g^{\alpha\beta}$ and $\mathbf{1}$. The result for the heat kernel is

$$a_4^{\text{ghosts,EM}}(x) = \frac{13}{36}E_4 - \frac{1}{4}W^2 - \frac{3}{4}R^2 \ . \tag{B.2.39}$$

where we have already included here the negative sign due to the opposite statistics.

B.2.3 Logarithmic correction

Adding the above results, the heat kernel for Einstein-Maxwell-AdS theory takes the form,

$$(4\pi)^2 a_4^{\text{B}}(x) = -\frac{53}{45}E_4 + \frac{137}{60}W^2 - \frac{13}{36}R^2 \ . \tag{B.2.40}$$

We can read off the coefficients from (5.3.1) to be

$$a_{\text{E}} = \frac{53}{45}, \qquad c = \frac{137}{60}, \qquad b_1 = -\frac{13}{36} \ , \qquad b_2 = 0 \ . \tag{B.2.41}$$

B.2.4 Neutral limit

As previously mentioned, if we properly truncate the fluctuations and the resulting curvature invariants, we recover the result obtained in [182] for the theory of pure gravity with a negative cosmological constant. Let us show this explicitly as a sanity check of our results. In this limit, we must truncate the fluctuation of a_α in (B.2.20)

$$\phi_m E^{mn} \phi_n = h_{\mu\nu} \left(R^{\mu\alpha\nu\beta} + R^{\mu\beta\nu\alpha} - g^{\mu\nu} R^{\alpha\beta} - g^{\alpha\beta} R^{\mu\nu} + \Lambda g^{\mu\nu} g^{\alpha\beta} \right) h_{\alpha\beta} \ . \tag{B.2.42}$$

This yields the following traces

$$\begin{aligned} &\mathrm{Tr}(RE) = -16\Lambda^2 \ , \\ &\mathrm{Tr}(E^2) = 16\Lambda^2 - 4R_{\mu\nu}R^{\mu\nu} + 3R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \ , \\ &\mathrm{Tr}(I) = 10 \ . \end{aligned} \tag{B.2.43}$$

Note that the trace of I is 10 instead of 14 because we no longer have the fluctuation a_μ, this is the kind of intermediate result that makes a naive truncation of the final answer yield the wrong result. Moreover, the field strength Ω is simply the commutator of ∇. Its trace is well-known and takes the value $\mathrm{Tr}(\Omega^2) = -6R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. Combining these results, we have

$$\begin{aligned} 180(4\pi)^2 a_4^{\mathrm{bulk},\Lambda} =& \frac{1}{2}\Big[60(-16\Lambda^2) + 180(16\Lambda^2 - 4R_{\mu\nu}R^{\mu\nu} + 3R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}) + 30(-6)R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} \\ & + 10(5R^2 + 2R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} - 2R_{\mu\nu}R^{\mu\nu} \Big] \\ =& -120\Lambda^2 + 190R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} \ . \end{aligned} \tag{B.2.44}$$

In the second line, we used the field equation $R_{\mu\nu} = g_{\mu\nu}\Lambda$. For the ghost contribution, we do not have the ghost of the graviphoton, *i.e.*, the scalar ghosts b and c. Therefore, the matrix E in the neutral limit remains the same and Ω reduces to

$$\phi_n \left(\Omega_{\alpha\beta}\right)^n_m \phi^m = b_\mu \left(R^\mu_{\nu\alpha\beta} \right) b^\nu + c_\mu \left(R^\mu_{\nu\alpha\beta} \right) c^\nu \ . \tag{B.2.45}$$

The only change in the trace is the I operator

$$\mathrm{Tr}(I) = \mathrm{Tr}(g^{\mu\nu}) + \mathrm{Tr}(g^{\mu\nu}) = 8 \ , \tag{B.2.46}$$

which yields the ghost contribution in the neutral limit

$$
\begin{aligned}
-(4\pi)^2 a_4^{\text{ghost,B}}(x) &= \frac{1}{3}R^2 + R_{\mu\nu}R^{\mu\nu} - \frac{1}{6}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + (\frac{1}{9}R^2 + \frac{2}{45}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{2}{45}R_{\mu\nu}R^{\mu\nu}) \\
&= \frac{4}{9}R^2 + \frac{43}{45}R_{\mu\nu}R^{\mu\nu} - \frac{43}{45}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \\
&= \frac{11}{90}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{164}{15}\Lambda^2 \ ,
\end{aligned}
\tag{B.2.47}
$$

where we use the field equation $R_{\mu\nu} = g_{\mu\nu}\Lambda$ once more. In the neutral limit, the heat kernel coefficient is therefore

$$
180(4\pi)^2 a_4^{\Lambda}(x) = 212 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2088\Lambda^2 \ , \tag{B.2.48}
$$

which agrees with the result in [182].

B.3 Fermionic computation

In this appendix, we present the details of the computation for the gravitini of minimal gauged supergravity described in section 5.6.

B.3.1 Majorana Lagrangian

The fermionic Lagrangian is given as [185]

$$
\mathcal{L} = \frac{1}{2}\bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho + \frac{i}{4}F^{\mu\nu}\bar{\psi}_\rho \gamma_\mu \gamma^{\rho\sigma}\gamma_\nu \psi_\sigma - \frac{1}{2\ell}\bar{\psi}_\mu \gamma^{\mu\nu}\psi_\nu \ , \tag{B.3.1}
$$

where ψ_μ is a complex spinor with spin $\frac{3}{2}$. It can be written as

$$
\psi_\mu = \psi_\mu^1 + i\psi_\mu^2 \ , \tag{B.3.2}
$$

where ψ_μ^A are Majorana spinors. For definiteness, we use the really real representation of the Clifford algebra given by explicitly as [189]

$$
\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} . \tag{B.3.3}
$$

In this case, the Majorana condition reduces to

$$\psi^* = \psi, \tag{B.3.4}$$

which is just the reality condition. We choose the gauge $\gamma^\mu \psi_\mu = 0$. This can be implemented by the gauge-fixing term

$$\mathcal{L}_{\text{g.f.}} = -\frac{1}{4}(\bar{\psi}_\mu \gamma^\mu)\gamma^\nu D_\nu (\gamma^\rho \psi_\rho) \ . \tag{B.3.5}$$

More details about the Faddeev-Popov procedure are given in the following subsection when we consider the ghost contribution. We then perform the field redefinition

$$\psi_\mu = \sqrt{2}\left(\chi_\mu - \frac{1}{2}\gamma_\mu \gamma^\nu \chi_\nu\right) \ , \tag{B.3.6}$$

which leads to the Lagrangian

$$\mathcal{L}_{\text{Fermi}} + \mathcal{L}_{\text{g.f.}} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\bar{\psi}F\psi} + \mathcal{L}_{\bar{\psi}\psi} \ , \tag{B.3.7}$$

where

$$\begin{aligned} \mathcal{L}_{\text{kin}} &= \frac{1}{2} g^{\mu\nu} \bar{\psi}_\mu \gamma^\rho \nabla_\rho \psi_\nu \ , \\ \mathcal{L}_{\bar{\psi}F\psi} &= \frac{i}{4} F^{\mu\nu} \bar{\psi}_\rho \gamma_\mu \gamma^{\rho\sigma} \gamma_\nu \psi_\sigma \ , \\ \mathcal{L}_{\bar{\psi}\psi} &= -\frac{1}{2\ell} \bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu \ . \end{aligned} \tag{B.3.8}$$

For the kinetic term, we can use the fact that (see [19, 148] for details)

$$\bar{\chi}_\mu \gamma^\nu D_\nu \chi^\mu = \frac{1}{2}\bar{\psi}_\mu \left(\gamma^{\mu\nu\rho} D_\nu - \frac{1}{2}\gamma^\mu \gamma^\nu \gamma^\rho D_\nu\right) \psi_\rho \ . \tag{B.3.9}$$

Note that gauge invariance guarantees that this identity holds also for the gauge connection. For the mass term, we have

$$
\begin{aligned}
\mathcal{L}_{\bar{\psi}\psi} &= -\frac{1}{2\ell}\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu \\
&= -\frac{1}{\ell}\left(\bar{\chi}_\mu - \frac{1}{2}\bar{\chi}_\rho\gamma^\rho\gamma_\mu\right)\gamma^{\mu\nu}\left(\chi_\nu - \frac{1}{2}\gamma_\nu\gamma^\sigma\chi_\sigma\right) \\
&= -\frac{1}{\ell}\left(\bar{\chi}_\mu\gamma^{\mu\nu}\chi_\nu - \frac{1}{2}\bar{\chi}_\rho\gamma^\rho\gamma_\mu\gamma^{\mu\nu}\chi_\nu - \frac{1}{2}\bar{\chi}_\mu\gamma^{\mu\nu}\gamma_\nu\gamma^\sigma\chi_\sigma + \frac{1}{4}\bar{\chi}_\rho\gamma^\rho\gamma_\mu\gamma^{\mu\nu}\gamma_\nu\gamma^\sigma\chi_\sigma\right) \\
&= -\frac{1}{\ell}\left(\bar{\chi}_\mu\gamma^{\mu\nu}\chi_\nu - 3\bar{\chi}_\mu\gamma^\mu\gamma^\nu\chi_\nu + 3\bar{\chi}_\rho\gamma^\rho\gamma^\sigma\chi_\sigma\right) \\
&= -\frac{1}{\ell}\bar{\chi}_\mu\gamma^{\mu\nu}\chi_\nu \ ,
\end{aligned}
\tag{B.3.10}
$$

where we have used $\gamma^{\mu\nu} = \gamma^\mu\gamma^\nu - g^{\mu\nu}$ so that $\gamma_\mu\gamma^{\mu\nu} = 3\gamma^\nu$ and $\gamma_\mu\gamma^{\mu\nu}\gamma_\nu = 12$. Finally, we have

$$
\begin{aligned}
\mathcal{L}_{\bar{\psi}F\psi} &= \frac{i}{4}F^{\mu\nu}\bar{\psi}_\rho\gamma_\mu\gamma^{\rho\sigma}\gamma_\nu\psi_\sigma \\
&= \frac{i}{2}F^{\mu\nu}\left(\bar{\chi}_\rho - \frac{1}{2}\bar{\chi}_\alpha\gamma^\alpha\gamma_\rho\right)\gamma_\mu\gamma^{\rho\sigma}\gamma_\nu\left(\chi_\sigma - \frac{1}{2}\gamma_\sigma\gamma^\beta\chi_\beta\right) \\
&= \frac{i}{2}F^{\mu\nu}\left(\bar{\chi}_\rho\gamma_\mu\gamma^{\rho\sigma}\gamma_\nu\chi_\sigma - \frac{1}{2}\bar{\chi}_\alpha\gamma^\alpha\gamma_\rho\gamma_\mu\gamma^{\rho\sigma}\gamma_\nu\chi_\sigma - \frac{1}{2}\bar{\chi}_\rho\gamma_\mu\gamma^{\rho\sigma}\gamma_\nu\gamma_\sigma\gamma^\beta\chi_\beta\right. \\
&\qquad \left. + \frac{1}{4}\bar{\chi}_\alpha\gamma^\alpha\gamma_\rho\gamma_\mu\gamma^{\rho\sigma}\gamma_\nu\gamma_\sigma\gamma^\beta\chi_\beta\right) \ .
\end{aligned}
\tag{B.3.11}
$$

To simplify, we use the following gamma matrix identities

$$
\begin{aligned}
\gamma^\mu\gamma_\nu\gamma_\mu &= -2\gamma_\nu, \\
\gamma_\rho\gamma_\mu\gamma^{\rho\sigma} &= \gamma_\rho\gamma_\mu(\gamma^\rho\gamma^\sigma - g^{\rho\sigma}) = -\gamma_\mu\gamma^\sigma - 2\delta^\sigma_\mu \ , \\
\gamma_\rho\gamma_\mu\gamma^{\rho\sigma}\gamma_\nu\gamma_\sigma &= (-\gamma_\mu\gamma^\sigma - 2\delta^\sigma_\mu)\gamma_\nu\gamma_\sigma = 4\gamma_{\mu\nu} \ .
\end{aligned}
\tag{B.3.12}
$$

For the second term in (B.3.11), we have

$$
\begin{aligned}
-\frac{1}{2}\bar{\chi}_\alpha\gamma^\alpha\gamma_\rho\gamma_\mu\gamma^{\rho\sigma}\gamma_\nu\chi_\sigma &= \frac{1}{2}\bar{\chi}_\alpha\gamma^\alpha(\gamma_\mu\gamma^\sigma + 2\delta^\sigma_\mu)\gamma_\nu\chi_\sigma \\
&= \frac{1}{2}\bar{\chi}_\alpha\gamma^\alpha\gamma_\mu\gamma^\sigma\gamma_\nu\chi_\sigma + \bar{\chi}_\alpha\gamma^\alpha\gamma_\nu\chi_\mu \\
&= \frac{1}{2}\bar{\chi}_\alpha\gamma^\alpha\gamma_\mu(2\delta^\sigma_\nu - \gamma_\nu\gamma^\sigma)\chi_\sigma + \bar{\chi}_\alpha\gamma^\alpha\gamma_\nu\chi_\mu \\
&= \bar{\chi}_\alpha\gamma^\alpha\gamma_\mu\chi_\nu - \frac{1}{2}\bar{\chi}_\alpha\gamma^\alpha\gamma_\mu\gamma_\nu\gamma^\sigma\chi_\sigma + \bar{\chi}_\alpha\gamma^\alpha\gamma_\nu\chi_\mu \ .
\end{aligned}
\tag{B.3.13}
$$

After contracting with $F^{\mu\nu}$, the first and last terms in (B.3.11) cancel due to antisymmetry. Finally, symmetry arguments show that the third term in (B.3.11) gives the same simplification as (B.3.13). At the end, we obtain

$$\begin{aligned} \mathcal{L}_{\bar{\psi}F\psi} &= \frac{i}{2}F^{\mu\nu}\left(\bar{\chi}_\rho\gamma_\mu\gamma^{\rho\sigma}\gamma_\nu\chi_\sigma - \bar{\chi}_\alpha\gamma^\alpha\gamma_\mu\gamma_\nu\gamma^\sigma\chi_\sigma + \bar{\chi}_\alpha\gamma^\alpha\gamma_{\mu\nu}\gamma^\beta\chi_\beta\right) \\ &= \frac{i}{2}F^{\mu\nu}\bar{\chi}_\rho\gamma_\mu\gamma^{\rho\sigma}\gamma_\nu\chi_\sigma \ . \end{aligned} \tag{B.3.14}$$

We thus obtain the Lagrangian

$$\mathcal{L}_f = g^{\mu\nu}\bar{\chi}_\mu\gamma^\rho D_\rho\chi_\nu + \frac{i}{2}F^{\mu\nu}\bar{\chi}_\rho\gamma_\mu\gamma^{\rho\sigma}\gamma_\nu\chi_\sigma - \frac{1}{\ell}\bar{\chi}_\mu\gamma^{\mu\nu}\chi_\nu. \tag{B.3.15}$$

We introduce the complex spinor χ_μ as

$$\chi_\mu = \chi^1_\mu + i\chi^2_\mu \ , \tag{B.3.16}$$

in terms of Majorana spinors. We use the label $A = 1, 2$ for the two spinors and make use of Majorana flip identities [189]

$$\bar{\lambda}\gamma_{\mu_1\ldots\mu_r}\chi = t_r\bar{\chi}\gamma_{\mu_1\ldots\mu_r}\lambda \ , \tag{B.3.17}$$

where

$$t_r = \begin{cases} -1 & r = 1, 2 \bmod 4 \\ 1 & r = 3, 4 \bmod 4 \end{cases} . \tag{B.3.18}$$

The sign t_r reflects the symmetry of the gamma matrices under charge conjugation. Another useful identity is [189]

$$\bar{\lambda}\gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_p}\chi = (-1)^p\bar{\chi}\gamma^{\mu_p}\ldots\gamma^{\mu_2}\gamma^{\mu_1}\lambda. \tag{B.3.19}$$

Each term in the Lagrangian can be simplified using Majorana flips and is either proportional to the identity matrix δ_{AB} or the antisymmetric matrix ε_{AB} (wtih $\varepsilon^{12} = 1$). In the kinetic term, the cross-terms cancel

$$\begin{aligned} \bar{\chi}^1_\mu\gamma^\rho D_\rho\chi^2_\nu &= -D_\rho\bar{\chi}^2_\nu\gamma^\rho\chi^1_\mu \\ &= \bar{\chi}^2_\nu\gamma^\rho D_\rho\chi^1_\mu \end{aligned} \tag{B.3.20}$$

where we used a Majorana flip and integration by parts. Hence we have

$$\mathcal{L}_{\text{kin}} = \delta_{AB} g^{\mu\nu} \bar{\chi}^A_\mu \gamma^\rho D_\rho \chi^B_\nu \ . \tag{B.3.21}$$

We then have

$$\begin{aligned} F^{\mu\nu} \bar{\psi}^1_\rho \gamma_\mu \gamma^{\rho\sigma} \gamma_\nu \psi^1_\sigma &= F^{\mu\nu} (\gamma_\mu \psi^1_\rho)^\dagger \gamma^{\rho\sigma} (\gamma_\nu \psi^1_\sigma) \\ &= -F^{\mu\nu} (\gamma_\nu \psi^1_\sigma)^\dagger \gamma^{\rho\sigma} (\gamma_\mu \psi^1_\rho) \\ &= -F^{\mu\nu} \bar{\psi}^1_\rho \gamma_\mu \gamma^{\rho\sigma} \gamma_\nu \psi^1_\sigma \\ &= 0 \end{aligned} \tag{B.3.22}$$

where we used a Majorana flip in the second line. This shows that

$$\mathcal{L}_{\bar{\psi} F \psi} = -\frac{1}{2} \varepsilon_{AB} F^{\mu\nu} \bar{\chi}^A_\rho \gamma_\mu \gamma^{\rho\sigma} \gamma_\nu \chi^B_\sigma \ . \tag{B.3.23}$$

Finally, we have

$$\begin{aligned} \bar{\chi}^1_\mu \gamma^{\mu\nu} \chi^2_\nu &= -\bar{\chi}^2_\nu \gamma^{\mu\nu} \chi^1_\mu \\ &= \bar{\chi}^2_\mu \gamma^{\mu\nu} \chi^1_\nu \end{aligned} \tag{B.3.24}$$

where we used a Majorana flip and antisymmetry of $\gamma^{\mu\nu}$. This shows that mass term is

$$\mathcal{L}_{\bar{\psi}\psi} = -\frac{1}{\ell} \delta_{AB} \bar{\chi}^A_\mu \gamma^{\mu\nu} \chi^B_\nu \ . \tag{B.3.25}$$

The final Lagrangian is then

$$\mathcal{L}_f + \mathcal{L}_{\text{g.f.}} = \delta_{AB} g^{\mu\nu} \bar{\chi}^A_\mu \gamma^\rho D_\rho \chi^B_\nu - \frac{1}{2} \varepsilon_{AB} F^{\mu\nu} \bar{\chi}^A_\rho \gamma_\mu \gamma^{\rho\sigma} \gamma_\nu \chi^B_\sigma - \frac{1}{\ell} \delta_{AB} \bar{\chi}^A_\mu \gamma^{\mu\nu} \chi^B_\nu \tag{B.3.26}$$

This Lagrangian could now be interpreted as a Euclidean Lagrangian by performing the Wick rotation and using $\bar{\chi}_A = \chi_A^\dagger$. This can then be used in the algorithm to compute the logarithmic corrections.

B.3.2 Symplectic Lagrangian

We are ultimately interested in the fermionic Lagrangian in $(0,4)$ signature. It is known that Majorana spinors do not exist in $(0,4)$ signature [191,207]. Instead, we should use symplectic Majorana spinors. Thus, we first convert our Lagrangian from Majorana to symplectic Majorana spinors in $(1,3)$ signature, where both Majorana and symplectic Majorana spinors exist. We then perform the Wick rotation to obtain the Lagrangian in $(0,4)$ signature.

Symplectic Majorana spinors

The symmetries of the gamma matrices are captured by matrices A, B and C. The matrix A expresses the Hermitian conjugate of a gamma matrix as

$$(\gamma^\mu)^\dagger = (-1)^t A\gamma^\mu A^{-1} \tag{B.3.27}$$

and we can take $A = -\gamma^0$. The charge conjugation matrix gives the transpose as

$$(\gamma^\mu)^t = t_0 t_1 C\gamma^\mu C^{-1}, \tag{B.3.28}$$

and satisfies $C^t = -t_0 C$. Here t_0, t_1 can take the values ± 1. The matrix B captures the complex conjugate

$$(\gamma^\mu)^* = -t_0 t_1 B\gamma^\mu B^{-1}\ . \tag{B.3.29}$$

and can be obtained as

$$B = (CA^{-1})^t\ . \tag{B.3.30}$$

We also define

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \tag{B.3.31}$$

given in the representation (B.3.3). Note that we have

$$(\gamma^5)^2 = 1, \qquad (\gamma^5)^\dagger = \gamma^5, \qquad (\gamma^5)^t = (\gamma^5)^* = -\gamma^5\ . \tag{B.3.32}$$

There are two possible choice of charge conjugation matrix which we will denote C_+ and C_-. The C_+ matrix has $t_0 = 1, t_1 = -1$. It gives a matrix $B_+ = (C_+ A^{-1})^t = \mathbf{1}$. The Majorana condition is then written as

$$\psi^* = B_+\psi = \psi\ . \tag{B.3.33}$$

To define symplectic Majoranas, we need to use another charge conjugation matrix $C_- = C_+\gamma^5$ which has $t_0 = t_1 = 1$. It gives the matrix

$$B_- = (C_- A^{-1})^t = \gamma^5\ . \tag{B.3.34}$$

The symplectic Marojana condition can be written as

$$(\lambda^\mu_A)^* = B_-\varepsilon_{AB}\lambda^\mu_B\ . \tag{B.3.35}$$

The mapping between Majoranas and symplectic Majoranas in $(1,3)$ signature is given in [191] and takes the form

$$\lambda_1^\mu = \chi_1^\mu - i\chi_\mu^2 \ , \tag{B.3.36}$$
$$\lambda_2^\mu = \gamma^5(\chi_1^\mu + i\chi_2^\mu) \ , \tag{B.3.37}$$

where we have used that $B_- = \gamma^5$. This gives

$$\chi_1^\mu = \frac{1}{2}(\lambda_1^\mu + \gamma^5\lambda_2^\mu) \ , \tag{B.3.38}$$
$$\chi_2^\mu = \frac{i}{2}(\lambda_1^\mu - \gamma^5\lambda_2^\mu) \ . \tag{B.3.39}$$

It is also useful to note that the Dirac conjugated defined as $\bar{\chi}_A^\mu = (\chi_A^\mu)i\gamma^0$ gives

$$\bar{\chi}_1^\mu = \frac{1}{2}(\bar{\lambda}_1^\mu - \bar{\lambda}_2^\mu\gamma^5) \ , \tag{B.3.40}$$
$$\bar{\chi}_2^\mu = -\frac{i}{2}(\bar{\lambda}_1^\mu + \bar{\lambda}_2^\mu\gamma^5) \ . \tag{B.3.41}$$

We will write the Majorana Lagrangian (B.3.26) as

$$\mathcal{L}_f + \mathcal{L}_{\text{g.f.}} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\chi F\chi} + \mathcal{L}_{\chi\chi} \tag{B.3.42}$$

where

$$\begin{aligned} \mathcal{L}_{\text{kin}} &= \delta_{AB} g_{\mu\nu}\bar{\chi}_A^\mu\gamma^\rho D_\rho\chi_B^\nu \ , \\ \mathcal{L}_{\chi F\chi} &= -\frac{1}{2}\varepsilon_{AB}F^{\mu\nu}\bar{\chi}_A^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\chi_B^\sigma \ , \\ \mathcal{L}_{\chi\chi} &= -\frac{1}{\ell}\delta_{AB}\bar{\chi}_A^\mu\gamma_{\mu\nu}\chi_B^\nu \ . \end{aligned} \tag{B.3.43}$$

We will now convert these terms one by one.

Kinetic term

For the kinetic term, we compute

$$\begin{aligned} g_{\mu\nu}\bar{\chi}_1^\mu\gamma^\rho\nabla_\rho\chi_1^\nu &= \frac{1}{4}g_{\mu\nu}(\bar{\lambda}_1^\mu - \bar{\lambda}_2^\mu\gamma^5)\gamma^\rho\nabla_\rho(\lambda_1^\mu + \gamma^5\lambda_2^\mu) \\ &= \frac{1}{4}g_{\mu\nu}(\bar{\lambda}_1^\mu\gamma^\rho\nabla_\rho\lambda_1^\mu - \bar{\lambda}_2^\mu\gamma^5\gamma^\rho\nabla_\rho\gamma^5\lambda_2^\mu + \bar{\lambda}_1^\mu\gamma^\rho\nabla_\rho\gamma^5\lambda_2^\mu - \bar{\lambda}_2^\mu\gamma^5\gamma^\rho\nabla_\rho\lambda_1^\mu) \\ &= \frac{1}{4}g_{\mu\nu}(\bar{\lambda}_1^\mu\gamma^\rho\nabla_\rho\lambda_1^\mu + \bar{\lambda}_2^\mu\gamma^\rho\nabla_\rho\lambda_2^\mu + \bar{\lambda}_1^\mu\gamma^\rho\nabla_\rho\gamma^5\lambda_2^\mu - \bar{\lambda}_2^\mu\gamma^5\gamma^\rho\nabla_\rho\lambda_1^\mu). \end{aligned} \tag{B.3.44}$$

The other contribution is

$$\begin{aligned} g_{\mu\nu}\bar{\chi}_2^\mu\gamma^\rho\nabla_\rho\chi_2^\nu &= \frac{1}{4}g_{\mu\nu}(\bar{\lambda}_1^\mu+\bar{\lambda}_2^\mu\gamma^5)\gamma^\rho\nabla_\rho(\lambda_1^\mu-\gamma^5\lambda_2^\mu) \\ &= \frac{1}{4}g_{\mu\nu}(\bar{\lambda}_1^\mu\gamma^\rho\nabla_\rho\lambda_1^\mu-\bar{\lambda}_2^\mu\gamma^5\gamma^\rho\nabla_\rho\gamma^5\lambda_2^\mu-\bar{\lambda}_1^\mu\gamma^\rho\nabla_\rho\gamma^5\lambda_2^\mu+\bar{\lambda}_2^\mu\gamma^5\gamma^\rho\nabla_\rho\lambda_1^\mu) \\ &= \frac{1}{4}g_{\mu\nu}(\bar{\lambda}_1^\mu\gamma^\rho\nabla_\rho\lambda_1^\mu+\bar{\lambda}_2^\mu\gamma^\rho\nabla_\rho\lambda_2^\mu-\bar{\lambda}_1^\mu\gamma^\rho\nabla_\rho\gamma^5\lambda_2^\mu+\bar{\lambda}_2^\mu\gamma^5\gamma^\rho\nabla_\rho\lambda_1^\mu). \end{aligned} \tag{B.3.45}$$

Therefore, we get

$$\mathcal{L}_{\text{kin}} = \frac{1}{2}g_{\mu\nu}(\bar{\lambda}_1^\mu\gamma^\rho\nabla_\rho\lambda_1^\mu+\bar{\lambda}_2^\mu\gamma^\rho\nabla_\rho\lambda_2^\mu). \tag{B.3.46}$$

We now compute the contribution from the gauge connection. The Majorana spinors $(\chi_1^\mu\ \chi_2^\mu)$ form a doublet under the SO(2) $\cong$ U(1) gauge symmetry. We have

$$D_\mu\chi_A^\nu = \left(\delta_{AB}\nabla_\mu+\frac{1}{\ell}\varepsilon_{AB}A_\mu\right)\chi_B^\nu\ , \tag{B.3.47}$$

or more explicitly

$$D_\mu\chi_1^\nu = \nabla_\mu\chi_1^\nu+\frac{1}{\ell}A_\mu\chi_2^\nu\ , \tag{B.3.48}$$

$$D_\mu\chi_2^\nu = \nabla_\mu\chi_2^\nu-\frac{1}{\ell}A_\mu\chi_1^\nu\ . \tag{B.3.49}$$

Using (B.3.36), we see that

$$\begin{aligned} D_\mu\lambda_1^\nu &= D_\mu(\chi_1^\nu-i\chi_2^\nu) \\ &= \nabla_\mu\chi_1^\nu+\frac{1}{\ell}A_\mu\chi_2^\nu-i\left(\nabla_\mu\chi_2^\nu-\frac{1}{\ell}A_\mu\chi_1^\nu\right) \\ &= \nabla_\mu\lambda_1^\nu+\frac{i}{\ell}A_\mu\lambda_1^\nu\ , \end{aligned} \tag{B.3.50}$$

and

$$\begin{aligned} D_\mu\lambda_2^\nu &= \gamma^5D_\mu(\chi_1^\nu+i\chi_2^\nu) \\ &= \gamma^5\left(\nabla_\mu\chi_1^\nu+\frac{1}{\ell}A_\mu\chi_2^\nu\right)+i\gamma^5\left(\nabla_\mu\chi_2^\nu-\frac{1}{\ell}A_\mu\chi_1^\nu\right) \\ &= \nabla_\mu\lambda_2^\nu-\frac{i}{\ell}A_\mu\lambda_2^\nu\ . \end{aligned} \tag{B.3.51}$$

As a result, we see that λ_1^μ and λ_2^μ are singlet under the U(1) gauge symmetry and have opposite charges. By gauge invariance, the kinetic term including the gauge connection is then

$$\mathcal{L}_{\rm kin} = \frac{1}{2} g_{\mu\nu} (\bar{\lambda}_1^{\mu} \gamma^{\rho} D_{\rho} \lambda_1^{\mu} + \bar{\lambda}_2^{\mu} \gamma^{\rho} D_{\rho} \lambda_2^{\mu}) \ . \tag{B.3.52}$$

Mass term

We have

$$\mathcal{L}_{\chi\chi} = -\frac{1}{\ell} \delta_{AB} \bar{\chi}_A^{\mu} \gamma_{\mu\nu} \chi_B^{\nu}. \tag{B.3.53}$$

We compute

$$\begin{aligned} \bar{\chi}_1^{\mu} \gamma_{\mu\nu} \chi_1^{\nu} &= \frac{1}{4} (\bar{\lambda}_1^{\mu} - \bar{\lambda}_2^{\mu} \gamma^5) \gamma_{\mu\nu} (\lambda_1^{\nu} + \gamma^5 \lambda_2^{\nu}) \\ &= \frac{1}{4} \left(\bar{\lambda}_1^{\mu} \gamma_{\mu\nu} \lambda_1^{\nu} - \bar{\lambda}_2^{\mu} \gamma^5 \gamma_{\mu\nu} \gamma^5 \lambda_2^{\nu} + \bar{\lambda}_1^{\mu} \gamma_{\mu\nu} \gamma^5 \lambda_2^{\nu} - \bar{\lambda}_2^{\mu} \gamma^5 \gamma_{\mu\nu} \lambda_1^{\nu} \right) \\ &= \frac{1}{4} \left(\bar{\lambda}_1^{\mu} \gamma_{\mu\nu} \lambda_1^{\nu} - \bar{\lambda}_2^{\mu} \gamma_{\mu\nu} \lambda_2^{\nu} + \bar{\lambda}_1^{\mu} \gamma_{\mu\nu} \gamma^5 \lambda_2^{\nu} - \bar{\lambda}_2^{\mu} \gamma^5 \gamma_{\mu\nu} \lambda_1^{\nu} \right) , \end{aligned} \tag{B.3.54}$$

as well as

$$\begin{aligned} \bar{\chi}_2^{\mu} \gamma_{\mu\nu} \chi_2^{\nu} &= \frac{1}{4} (\bar{\lambda}_1^{\mu} + \bar{\lambda}_2^{\mu} \gamma^5) \gamma_{\mu\nu} (\lambda_1^{\nu} - \gamma^5 \lambda_2^{\nu}) \\ &= \frac{1}{4} \left(\bar{\lambda}_1^{\mu} \gamma_{\mu\nu} \lambda_1^{\nu} - \bar{\lambda}_2^{\mu} \gamma_{\mu\nu} \lambda_2^{\nu} - \bar{\lambda}_1^{\mu} \gamma_{\mu\nu} \gamma^5 \lambda_2^{\nu} + \bar{\lambda}_2^{\mu} \gamma^5 \gamma_{\mu\nu} \lambda_1^{\nu} \right) . \end{aligned} \tag{B.3.55}$$

We see that the cross terms cancel upon addition of the two contributions and we end up with

$$\mathcal{L}_{\chi\chi} = -\frac{1}{2\ell} \left(\bar{\lambda}_1^{\mu} \gamma_{\mu\nu} \lambda_1^{\nu} - \bar{\lambda}_2^{\mu} \gamma_{\mu\nu} \lambda_2^{\nu} \right) . \tag{B.3.56}$$

Gauge interaction term

We compute

$$\begin{aligned} & F^{\mu\nu} \bar{\chi}_1^{\rho} \gamma_{\mu} \gamma_{\rho\sigma} \gamma_{\nu} \chi_2^{\sigma} \\ &= \frac{i}{4} F^{\mu\nu} (\bar{\lambda}_1^{\rho} - \bar{\lambda}_2^{\rho} \gamma^5) \gamma_{\mu} \gamma_{\rho\sigma} \gamma_{\nu} (\lambda_1^{\sigma} - \gamma^5 \lambda_2^{\sigma}) \\ &= \frac{i}{4} F^{\mu\nu} \left(\bar{\lambda}_1^{\rho} \gamma_{\mu} \gamma_{\rho\sigma} \gamma_{\nu} \lambda_1^{\sigma} + \bar{\lambda}_2^{\rho} \gamma^5 \gamma_{\mu} \gamma_{\rho\sigma} \gamma_{\nu} \gamma^5 \lambda_2^{\sigma} - \bar{\lambda}_1^{\rho} \gamma_{\mu} \gamma_{\rho\sigma} \gamma_{\nu} \gamma^5 \lambda_2^{\sigma} - \bar{\lambda}_2^{\rho} \gamma^5 \gamma_{\mu} \gamma_{\rho\sigma} \gamma_{\nu} \lambda_1^{\sigma} \right) \\ &= \frac{i}{4} F^{\mu\nu} \left(\bar{\lambda}_1^{\rho} \gamma_{\mu} \gamma_{\rho\sigma} \gamma_{\nu} \lambda_1^{\sigma} + \bar{\lambda}_2^{\rho} \gamma_{\mu} \gamma_{\rho\sigma} \gamma_{\nu} \lambda_2^{\sigma} - \bar{\lambda}_1^{\rho} \gamma_{\mu} \gamma_{\rho\sigma} \gamma_{\nu} \gamma^5 \lambda_2^{\sigma} - \bar{\lambda}_2^{\rho} \gamma^5 \gamma_{\mu} \gamma_{\rho\sigma} \gamma_{\nu} \lambda_1^{\sigma} \right) , \end{aligned} \tag{B.3.57}$$

and

$$
\begin{aligned}
&F^{\mu\nu}\bar{\chi}_2^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\chi_1^\sigma \\
&= -\frac{i}{4}F^{\mu\nu}(\bar{\lambda}_1^\rho + \bar{\lambda}_2^\rho\gamma^5)\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu(\lambda_1^\sigma + \gamma^5\lambda_2^\sigma) \\
&= -\frac{i}{4}F^{\mu\nu}\left(\bar{\lambda}_1^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda_1^\sigma + \bar{\lambda}_2^\rho\gamma^5\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\gamma^5\lambda_2^\sigma + \bar{\lambda}_1^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\gamma^5\lambda_2^\sigma + \bar{\lambda}_2^\rho\gamma^5\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda_1^\sigma\right) \\
&= -\frac{i}{4}F^{\mu\nu}\left(\bar{\lambda}_1^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda_1^\sigma + \bar{\lambda}_2^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda_2^\sigma + \bar{\lambda}_1^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\gamma^5\lambda_2^\sigma + \bar{\lambda}_2^\rho\gamma^5\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda_1^\sigma\right).
\end{aligned}
\tag{B.3.58}
$$

Finally, we get

$$
\mathcal{L}_{\chi F\chi} = -\frac{1}{2}\varepsilon_{AB}F^{\mu\nu}\bar{\chi}_A^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\chi_B^\sigma = -\frac{i}{4}F^{\mu\nu}\left(\bar{\lambda}_1^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda_1^\sigma + \bar{\lambda}_2^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda_2^\sigma\right). \tag{B.3.59}
$$

Final Lagrangian

The final Lagrangian, written in terms of symplectic Majorana spinors, takes the form

$$
\mathcal{L}_f = \frac{1}{2}\delta_{AB}g_{\mu\nu}\bar{\lambda}_A^\mu\gamma^\rho D_\rho\lambda_A^\nu - \frac{i}{4}F^{\mu\nu}\left(\bar{\lambda}_1^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda_1^\sigma + \bar{\lambda}_2^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda_2^\sigma\right) - \frac{1}{2\ell}\left(\bar{\lambda}_1^\mu\gamma_{\mu\nu}\lambda_1^\nu - \bar{\lambda}_2^\mu\gamma_{\mu\nu}\lambda_2^\nu\right). \tag{B.3.60}
$$

We rescale $\lambda_A \to \sqrt{2}\lambda_A$ and write explicitly the gauge covariant derivative. At the end, the two symplectic Majorana spinors decouple and we can write

$$
\mathcal{L}_f = \mathcal{L}_1 + \mathcal{L}_2, \tag{B.3.61}
$$

where

$$
\mathcal{L}_1 = g_{\mu\nu}\bar{\lambda}_1^\mu\gamma^\rho(\nabla_\rho + i\ell^{-1}A_\rho)\lambda_1^\nu - \frac{i}{2}F^{\mu\nu}\bar{\lambda}_1^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda_1^\sigma - \frac{1}{\ell}\bar{\lambda}_1^\mu\gamma_{\mu\nu}\lambda_1^\nu\,, \tag{B.3.62}
$$

$$
\mathcal{L}_2 = g_{\mu\nu}\bar{\lambda}_2^\mu\gamma^\rho(\nabla_\rho - i\ell^{-1}A_\rho)\lambda_2^\nu - \frac{i}{2}F^{\mu\nu}\bar{\lambda}_2^\rho\gamma_\mu\gamma_{\rho\sigma}\gamma_\nu\lambda_2^\sigma + \frac{1}{\ell}\bar{\lambda}_2^\mu\gamma_{\mu\nu}\lambda_2^\nu\,. \tag{B.3.63}
$$

We can now reinterpret this Lagrangian to be in $(0,4)$ signature. To perform the Wick rotation, we define Euclidean gamma matrices

$$
\hat{\gamma}^1, \hat{\gamma}^2, \hat{\gamma}^3, \hat{\gamma}^4, \tag{B.3.64}
$$

where $\hat{\gamma}^i = \gamma^i$ for $i = 1,2,3$ and $\hat{\gamma}^4 = -i\gamma^0$. They satisfy

$$
(\hat{\gamma}^\mu)^\dagger = \hat{\gamma}^\mu\,. \tag{B.3.65}
$$

We also take the Hermitian conjugate to be

$$\bar{\lambda}^\mu_A = (\lambda^\mu_A)^\dagger \ . \tag{B.3.66}$$

Note that it is clear that both flavors give the same contribution because $\mathcal{L}_1$ and $\mathcal{L}_2$ are equal up to an exchange of $\ell \leftrightarrow -\ell$ and the four-derivative terms only involve ℓ^2. As a result, it is enough to do the computation for $\mathcal{L}_1$ and multiply the final heat kernel by two.

B.3.3 Ghosts

In this section, we discuss the contributions from ghosts.

Faddeev-Popov procedure

Given the crucial role of a proper treatment of the ghosts, we include here details about the Faddeev-Popov procedure. This was first explained in [192] (see also [19]). The fermionic path integral is schematically of the form

$$Z = \int D\bar{\psi}_\mu D\psi_\nu \, e^{-S[\bar{\psi}_\mu, \psi_\nu]}. \tag{B.3.67}$$

The Faddeev-Popov procedure corresponds to inserting in the path integral

$$1 = \int D\epsilon D\bar{\epsilon} \, \delta(\xi - \gamma^\mu \psi_\mu^{(\epsilon)}) \delta(\bar{\xi} - \gamma^\mu \bar{\psi}_\mu^{(\epsilon)}) \Delta_{\mathrm{FP}}^{-1}, \tag{B.3.68}$$

where the Faddeev-Popov determinant is

$$\Delta_{\mathrm{FP}} = \det \left(\frac{\delta(\gamma^\mu \psi_\mu^{(\epsilon)})}{\delta\epsilon} \right) \det \left(\frac{\delta(\gamma^\mu \bar{\psi}_\mu^{(\bar{\epsilon})})}{\delta\bar{\epsilon}} \right), \tag{B.3.69}$$

and $\psi_\mu^{(\epsilon)} = \psi_\mu + \mathcal{D}_\mu \epsilon$ is the infinitesimal transform of ψ^μ under a supersymmetry transformation. Here, ξ is an arbitrary spinor. We then insert

$$1 = \frac{1}{\det \not{D}} \int D\xi D\bar{\xi} \, \exp\left(-\bar{\xi} \not{D} \xi\right) \ . \tag{B.3.70}$$

As a result, we have

$$
\begin{aligned}
Z &= \frac{\Delta_{\rm FP}^{-1}}{\det \not{D}} \int D\bar{\psi}_\mu D\psi_\nu D\xi D\bar{\xi} D\epsilon D\bar{\epsilon}\, \exp\left(-\bar{\xi}\not{D}\xi\right) \delta(\xi - \gamma^\mu \psi_\mu^{(\epsilon)}) \delta(\bar{\xi} - \gamma^\mu \bar{\psi}_\mu^{(\epsilon)})\, e^{-S[\bar{\psi}_\mu, \psi_\nu]} \\
&= \frac{\Delta_{\rm FP}^{-1}}{\det \not{D}} \int D\bar{\psi}_\mu D\psi_\nu D\epsilon D\bar{\epsilon}\, \exp\left(-\bar{\psi}_\nu \gamma^\nu \not{D} \gamma^\mu \psi_\mu\right)\, e^{-S[\bar{\psi}_\mu, \psi_\nu]}, \qquad \text{(B.3.71)}
\end{aligned}
$$

where we have performed the integral over $\xi, \bar{\xi}$ and performed the field redefinition $\psi_\mu \to \psi_\mu - \mathcal{D}_\mu \epsilon$. By supersymmetry, the action is invariant under this redefinition. We see that the correct gauge-fixing term appears. Now, we can rewrite the prefactor in terms of b, c ghosts and an additional d ghost

$$
\begin{aligned}
\Delta_{\rm FP}^{-1} &= \int DbDc\, \exp\left(-b\gamma^\mu \mathcal{D}_\mu c\right), \qquad \text{(B.3.72)} \\
\frac{1}{\det \not{D}} &= \int DdD\bar{d} \exp\left(-\bar{d}\gamma^\mu D_\mu d\right),
\end{aligned}
$$

where $b, c, d, \bar{d}$ are spin $\frac{1}{2}$ ghosts with bosonic statistics.

Ghost Lagrangian

The ghost Lagrangian

$$
\mathcal{L}_{\rm ghost} = \bar{b}_A \left(\gamma^\mu D_\mu + \frac{2}{\ell}\right) c_A + \bar{c}_A \left(\gamma^\mu D_\mu + \frac{2}{\ell}\right) b_A + \bar{e}_A \gamma^\mu D_\mu e_A\ , \qquad \text{(B.3.73)}
$$

where b_A, c_A, e_A are Majorana spinors. We map them to symplectic Majoranas using

$$
\begin{aligned}
b_1 &= \frac{1}{2}(\beta_1 + \gamma^5 \beta_2), \qquad b_2 = \frac{i}{2}(\beta_1 - \gamma^5 \beta_2)\ , \qquad \text{(B.3.74)} \\
c_1 &= \frac{1}{2}(\eta_1 + \gamma^5 \eta_2), \qquad c_2 = \frac{i}{2}(\eta_1 - \gamma^5 \eta_2)\ , \qquad \text{(B.3.75)} \\
e_1 &= \frac{1}{2}(\epsilon_1 + \gamma^5 \epsilon_2), \qquad e_2 = \frac{i}{2}(\epsilon_1 - \gamma^5 \epsilon_2)\ . \qquad \text{(B.3.76)}
\end{aligned}
$$

We have

$$\begin{aligned}\bar{b}_1\gamma^\mu D_\mu c_1 &= \frac{1}{4}(\bar{\beta}_1 - \bar{\beta}_2\gamma^5)\gamma^\mu D_\mu(\eta_1 + \gamma^5\eta_2) \\ &= \frac{1}{4}\left(\bar{\beta}_1\gamma^\mu D_\mu\eta_1 - \bar{\beta}_2\gamma^5\gamma^\mu D_\mu\gamma^5\eta_2 + \bar{\beta}_1\gamma^\mu D_\mu\gamma^5\eta_2 - \bar{\beta}_2\gamma^5\gamma^\mu D_\mu\eta_1\right) \\ &= \frac{1}{4}\left(\bar{\beta}_1\gamma^\mu D_\mu\eta_1 + \bar{\beta}_2\gamma^\mu D_\mu\eta_2 + \bar{\beta}_1\gamma^\mu D_\mu\gamma^5\eta_2 - \bar{\beta}_2\gamma^5\gamma^\mu D_\mu\eta_1\right),\end{aligned} \tag{B.3.77}$$

and

$$\begin{aligned}\bar{b}_2\gamma^\mu D_\mu c_2 &= \frac{1}{4}(\bar{\beta}_1 + \bar{\beta}_2\gamma^5)\gamma^\mu D_\mu(\eta_1 - \gamma^5\eta_2) \\ &= \frac{1}{4}\left(\bar{\beta}_1\gamma^\mu D_\mu\eta_1 + \bar{\beta}_2\gamma^\mu D_\mu\eta_2 - \bar{\beta}_1\gamma^\mu D_\mu\gamma^5\eta_2 + \bar{\beta}_2\gamma^5\gamma^\mu D_\mu\eta_1\right).\end{aligned} \tag{B.3.78}$$

We see that the cross terms cancel upon summation so that

$$\bar{b}_A\gamma^\mu D_\mu c_A = \frac{1}{2}(\bar{\beta}_1\gamma^\mu D_\mu\eta_1 + \bar{\beta}_2\gamma^\mu D_\mu\eta_2)\ . \tag{B.3.79}$$

Similarly,

$$\bar{c}_A\gamma^\mu D_\mu b_A = \frac{1}{2}(\bar{\eta}_1\gamma^\mu D_\mu\beta_1 + \bar{\eta}_2\gamma^\mu D_\mu\beta_2)\ , \tag{B.3.80}$$

and

$$\bar{e}_A\gamma^\mu D_\mu e_A = \frac{1}{2}(\bar{\epsilon}_1\gamma^\mu D_\mu\epsilon_1 + \bar{\epsilon}_2\gamma^\mu D_\mu\epsilon_2)\ . \tag{B.3.81}$$

The mass term gives

$$\begin{aligned}\bar{b}_1 c_1 &= \frac{1}{4}(\bar{\beta}_1 - \bar{\beta}_2\gamma^5)(\eta_1 + \gamma^5\eta_2) \\ &= \frac{1}{4}(\bar{\beta}_1\eta_1 - \bar{\beta}_2\eta_2 + \bar{\beta}_1\gamma^5\eta_2 - \bar{\beta}_2\gamma^5\eta_1),\end{aligned} \tag{B.3.82}$$

and

$$\begin{aligned}\bar{b}_2 c_2 &= \frac{1}{4}(\bar{\beta}_1 + \bar{\beta}_2\gamma^5)(\eta_1 - \gamma^5\eta_2) \\ &= \frac{1}{4}(\bar{\beta}_1\eta_1 - \bar{\beta}_2\eta_2 - \bar{\beta}_1\gamma^5\eta_2 + \bar{\beta}_2\gamma^5\eta_1),\end{aligned} \tag{B.3.83}$$

so that

$$\bar{b}_A c_A = \frac{1}{2}(\bar{\beta}_1\eta_1 - \bar{\beta}_2\eta_2), \tag{B.3.84}$$

and

$$\bar{c}_A b_A = \frac{1}{2}(\bar{\eta}_1\beta_1 - \bar{\eta}_2\beta_2). \tag{B.3.85}$$

We now rescale all ghosts by a factor $\sqrt{2}$. Finally, we obtain

$$\begin{aligned}\mathcal{L}_{\text{ghosts}} &= \bar{\beta}_1\left(\gamma^\mu D_\mu + \frac{2}{\ell}\right)\eta_1 + \bar{\eta}_1\left(\gamma^\mu D_\mu + \frac{2}{\ell}\right)\beta_1 \\ &\quad +\bar{\beta}_2\left(\gamma^\mu D_\mu - \frac{2}{\ell}\right)\eta_2 + \bar{\eta}_2\left(\gamma^\mu D_\mu - \frac{2}{\ell}\right)\beta_2 \\ &\quad +\bar{\epsilon}_1\gamma^\mu D_\mu\epsilon_1 + \bar{\epsilon}_2\gamma^\mu D_\mu\epsilon_2\end{aligned} \tag{B.3.86}$$

We now Wick rotate and interpret the Dirac conjugates as Hermitian conjugates in Euclidean signature

$$\bar{\beta}_1 = \eta_1^\dagger, \qquad \bar{\beta}_2 = \eta_2^\dagger, \qquad \bar{\epsilon}_1 = \epsilon_1^\dagger\ . \tag{B.3.87}$$

The above choice of Hermitian conjugate makes the kinetic term diagonal and suitable for the heat kernel computation. We can also use the more natural choice $\bar{\beta}_1 = \beta_1^\dagger, \bar{\beta}_2 = \beta_2^\dagger$ and make the kinetic diagonal by a simple field redefinition.

B.3.4 Result

The heat kernel can now be computed using the algorithm described in section B.1. The gravitini contribution is computed using the Lagrangian (B.3.61). The result is

$$(4\pi)^2 a_4(x) = \frac{139}{90}E_4 - \frac{32}{15}W^2 - \frac{2}{9}R^2 + \frac{8}{9}RF_{\mu\nu}F^{\mu\nu}\ . \tag{B.3.88}$$

In total we have one massless pair of ghosts and two massive pairs. A massless pair of ghosts gives the contribution

$$(4\pi)^2 a_4(x) = \frac{11}{360}E_4 - \frac{1}{20}W^2 - \frac{1}{18}RF_{\mu\nu}F^{\mu\nu}, \tag{B.3.89}$$

and each of the massive pair gives

$$(4\pi)^2 a_4(x) = \frac{11}{360}E_4 - \frac{1}{20}W^2 + \frac{1}{9}R^2 - \frac{1}{18}RF_{\mu\nu}F^{\mu\nu}. \tag{B.3.90}$$

In all these formulas, we have already included the minus sign due to the opposite statistics of ghosts. Finally, the total fermionic contribution gives

$$(4\pi)^2 a_4(x) = \frac{589}{360}E_4 - \frac{137}{60}W^2 + \frac{13}{18}RF_{\mu\nu}F^{\mu\nu}\ , \tag{B.3.91}$$

as reported in (5.6.17).

A different gauge-fixing term

Another possible gauge-fixing term is to take

$$\begin{aligned} \mathcal{L}_{\text{g.f.}} &= -\frac{1}{4}(\bar{\psi}_\mu\gamma^\mu)(\gamma^\nu D_\nu - m)(\gamma^\rho\psi_\rho) \\ &= -\frac{1}{4}\delta_{AB}(\bar{\psi}^A_\mu\gamma^\mu)(\gamma^\nu D_\nu - m)(\gamma^\rho\psi^B_\rho)\ . \end{aligned} \tag{B.3.92}$$

This is natural because with $m = \frac{2}{\ell}$, the three pairs of ghosts become identical. Of course the final result should not depend on this choice. This adds the term

$$\begin{aligned} \mathcal{L}_{\text{new}} &= \frac{1}{4}m\bar{\psi}_\mu\gamma^\mu\gamma^\nu\psi_\nu \\ &= \frac{1}{2}m\left(\bar{\chi}_\mu - \frac{1}{2}\bar{\chi}_\alpha\gamma^\alpha\gamma_\mu\right)\gamma^\mu\gamma^\nu\left(\chi_\nu - \frac{1}{2}\gamma_\nu\gamma^\beta\chi_\beta\right) \\ &= \frac{1}{2}m\,\delta_{AB}\bar{\chi}^A_\mu\gamma^\mu\gamma^\nu\chi^B_\nu . \end{aligned} \tag{B.3.93}$$

Using that $g_{\mu\nu} = \gamma^\mu\gamma^\nu - \gamma^{\mu\nu}$, we can simplify the Majorana Lagrangian with the choice $m = \frac{2}{\ell}$ so that it takes the form

$$\mathcal{L}_f = \delta_{AB}g^{\mu\nu}\bar{\chi}^A_\mu\gamma^\rho D_\rho\chi^B_\nu - \frac{1}{2}\varepsilon_{AB}F^{\mu\nu}\bar{\chi}^A_\rho\gamma_\mu\gamma^{\rho\sigma}\gamma_\nu\chi^B_\sigma + \frac{1}{\ell}\delta_{AB}\bar{\chi}^A_\mu g^{\mu\nu}\chi^B_\nu\ . \tag{B.3.94}$$

After converting to symplectic Majoranas and performing the computation, this gives the gravitini contribution

$$(4\pi)^2 a_4(x) = \frac{139}{90}E_4 - \frac{32}{15}W^2 - \frac{1}{3}R^2 + \frac{8}{9}RF_{\mu\nu}F^{\mu\nu}\ . \tag{B.3.95}$$

The ghost contribution is also modified. Indeed, the Lagrangian of the e-ghost is determined by the gauge-fixing Lagrangian and hence acquires the same mass of the b, c ghosts. So we end up with three identical pairs of charged ghosts for a total ghost contribution of

$$(4\pi)^2 a_4(x) = -\frac{11}{120}E_4 - \frac{3}{20}W^2 + \frac{1}{3}R^2 - \frac{1}{6}RF_{\mu\nu}F^{\mu\nu}\ . \tag{B.3.96}$$

The total contribution is then

$$(4\pi)^2 a_4(x) = \frac{589}{360}E_4 - \frac{137}{60}W^2 + \frac{13}{18}RF_{\mu\nu}F^{\mu\nu}\ , \tag{B.3.97}$$

which, as expected, is the same as (B.3.91).

B.4 Renormalization of the couplings

Our focus in this paper has been on the Seeley-DeWitt coefficient a_4 which is responsible for the local contribution to the logarithmic corrections. The other Seeley-DeWitt coefficients a_0 and a_2 capture the one-loop renormalization of the couplings. Indeed, the effective Euclidean action takes the form

$$S = S_{\text{classical}} + S_{\text{1-loop}} + \ldots, \tag{B.4.1}$$

where the one-loop correction is

$$S_{\text{1-loop}} = -\frac{1}{2}\int_{\epsilon}^{+\infty}\frac{ds}{s}\int d^4x\sqrt{g}\,K(x,s), \qquad K(x,s) = s^{-2}a_0(x) + s^{-1}a_2(x) + \ldots, \tag{B.4.2}$$

which gives

$$S_{\text{1-loop}} = \int d^4x\sqrt{g}\left(-\frac{1}{4\epsilon^2}a_0(x) - \frac{1}{2\epsilon}a_2(x) + \ldots\right). \tag{B.4.3}$$

The coefficient a_0 is a constant while a_2 is a general two-derivative term

$$(4\pi)^2 a_2(x) = d_1 R + d_2 F_{\mu\nu}F^{\mu\nu}\ . \tag{B.4.4}$$

Thus, we get

$$S_{\text{1-loop}} = \frac{1}{16\pi^2}\int d^4x\sqrt{g}\left(-\frac{1}{4\epsilon^2}a_0 - \frac{1}{2\epsilon}(d_1 R + d_2 F_{\mu\nu}F^{\mu\nu}) + \ldots\right). \tag{B.4.5}$$

From this expression, we can see that a_0, d_1 and d_2 are respectively renormalizations of the cosmological constant, Newton's constant and the electric charge. Here ϵ represents a UV cutoff. We can compute a_0 and a_2 using the formulas [158]

$$(4\pi)^2 a_0 = \mathrm{Tr}\,1\ , \tag{B.4.6}$$

$$(4\pi)^2 a_2 = \mathrm{Tr}\left(E + \frac{1}{6}R\right)\ . \tag{B.4.7}$$

This allows us to compute the coefficients a_0, d_1, d_2 for the theories considered in this paper. The results are summarized in Table B.1 below.

The renormalization of the cosmological constant, a_0, depends only on the number of fields and is as in flat space. However, our computations for the renormalization of Newton's constant, d_1, generalize previous flat space discussions in, for example, [208, 209]. To compare with those papers we note that our ϵ above has dimensions of $[L]^2$. More precisely, considering a massless scalar in AdS, $\Delta = 3$, leads to the same contribution as the one presented in [209]: $d_1 = 1/6$. Similarly, the massless Dirac fermion ($\Delta = 3/2$) leads to $d_1 = 1/3$ and the free vector to $d_1 = -2/3$ which coincide with [209].

Multiplet	a_0	d_1	d_2
Free scalar	1	$\frac{1}{12}(2 - \Delta(\Delta - 3))$	0
Free Dirac fermion	-4	$-\frac{1}{12}(5 + 4\Delta(\Delta - 3))$	0
Free vector	2	$-\frac{2}{3}$	0
Free gravitino	-2	$-\frac{1}{2}$	0
Einstein-Maxwell	4	$-\frac{10}{3}$	6
$\mathcal{N} = 2$ gravitini	-4	7	-8
$\mathcal{N} = 2$ gravity multiplet	0	$\frac{11}{3}$	-2

Table B.1: Seeley-DeWitt coefficients a_0 and a_2 for the theories studied in this paper

B.5 Holographic renormalization and the Gauss-Bonnet-Chern theorem

Since the local contribution is given by an integral over the Euclidean spacetime, the result for the logarithmic correction is sensitive to the choice of regularization procedure. In this work, we have used holographic renormalization to regulate these integrals. This is natural because the logarithmic correction can be viewed as a term in the effective bulk action. We have found that this prescription always gives a finite and unambiguous result.

For the Euler density, a natural counterterm is provided by the Gauss-Bonnet-Chern theorem

$$\frac{1}{32\pi^2}\int_{\mathcal{M}} d^d x\sqrt{g}\, E_4 + \frac{1}{32\pi^2}B = \chi \ , \tag{B.5.1}$$

where B is is the boundary term found by Chern in [210] and χ is the Euler characteristic of spacetime, which is an integer. Our regularization prescription gives precisely the same counterterm. Indeed, we find that

$$\lim_{r_c\to+\infty}\left[\frac{1}{32\pi^2}\int_{\mathcal{M}} d^d x\sqrt{g}\, E_4 + \int_{\partial\mathcal{M}} d^3y\sqrt{h}\,(c_1 + c_2\mathcal{R})\right] = \chi \ . \tag{B.5.2}$$

where c_1 and c_2 are chosen to cancel the r_c^3 and r_c divergences. This works for all the geometries considered in this paper. Note that a naive regularization procedure where we simply remove the divergent term would not give this result. In fact, it would lead to a non-topological result, depending on the black hole parameters. The holographic counterterm gives a finite contribution, necessary to obtain a topological result which is the same as the one appearing in the Gauss-Bonnet-Chern theorem. This gives us confidence that our regularization procedure is physically sensible. In this appendix, we show explicitly the matching of the two counterterms for the AdS-Schwarzschild black hole.

B.5.1 Chern's boundary term

To illustrate the above points, we consider the application of the Gauss-Bonnet-Chern theorem [210] (see section 8 of [211] for a review and [212] for a simple AdS application) to the AdS-Schwarzschild solution. The theorem takes the form

$$\chi = \frac{1}{32\pi^2}\int d^4x\sqrt{g}\,E_4 + \frac{1}{32\pi^2}B, \tag{B.5.3}$$

where B is the boundary term [210]

$$B \equiv -2\int \epsilon_{abcd}\theta^a_b\mathcal{R}^c_d + \frac{4}{3}\int \epsilon_{abcd}\theta^a_b\theta^c_e\theta^e_d \ . \tag{B.5.4}$$

where θ_{ab} is the second fundamental form, and $\mathcal{R}$ is the Riemann curvature tensor at the boundary. For AdS-Schwarzschild, the bulk contribution is

$$\begin{aligned} \frac{1}{32\pi^2}\int d^4x\sqrt{g}\,E_4 &= \int_{r_+}^{r_c} r^2 dr\int d\Omega^2\int_0^\beta d\tau\ \left(\frac{24}{\ell^4}+\frac{48m^2}{r^6}\right) \\ &= \frac{\beta}{\pi}\left[2m^2\left(\frac{1}{r_+^3}-\frac{1}{r_c^3}\right)+\frac{1}{\ell^4}\left(r_c^3-r_+^3\right)\right] \ , \end{aligned} \tag{B.5.5}$$

where r_+ is the horizon radius and we have introduced a cutoff $r = r_c$ that should be taken to infinity at the end. For the boundary term, we first compute the fundamental form

$$\theta_{01} = \frac{m\ell^2 + r_c^3}{r_c^2\ell^2}\,d\tau \ , \tag{B.5.6}$$

$$\theta_{12} = -\sqrt{\frac{r_c^3 + r_c\ell^2 - 2m\ell^2}{r_c\ell^2}}\,d\theta \ , \tag{B.5.7}$$

$$\theta_{12} = -\sqrt{\frac{r_c^3 + r_c\ell^2 - 2m\ell^2}{r_c\ell^2}}\,\sin\theta\, d\phi \ . \tag{B.5.8}$$

Note that the vielbeins are in general non-trivial linear combination of dx^μ, but the fundamental forms here are simple monomials of dx^μ. We can read out the coordinate component ${\mathcal{R}^a}_{b\mu\nu}$ on the slice $r = r_c$ using that

$$\mathcal{R}^a_b = {\mathcal{R}^a}_{bcd} e^c \wedge e^d = \mathcal{R}^a_{b\mu\nu} dx^\mu \wedge dx^\nu \ . \tag{B.5.9}$$

This gives

$$\begin{aligned} {\mathcal{R}^0}_{2\tau\theta} &= -\frac{m\ell^2 + r_c^3}{\ell^2 r_c^2}\sqrt{\frac{r_c^3 + r_c\ell^2 - 2m\ell^2}{r_c\ell^2}} \ , && \text{(B.5.10)} \\ {\mathcal{R}^0}_{3\tau\phi} &= -\frac{m\ell^2 + r_c^3}{\ell^2 r_c^2}\sqrt{\frac{r_c^3 + r_c\ell^2 - 2m\ell^2}{r_c\ell^2}} \sin\theta \ , && \text{(B.5.11)} \\ {\mathcal{R}^0}_{3\theta\phi} &= \frac{2m\ell^2 - r_c^3}{r_c\ell^2} \ . && \text{(B.5.12)} \end{aligned}$$

The boundary contribution is then

$$\begin{aligned} B &= 8(4\pi\beta)\frac{(m\ell^2 + r_c^3)(2m\ell^2 - r_c^3)}{\ell^4 r_c^3} && \text{(B.5.13)} \\ &= 4\pi\beta\left[-16m^2\frac{1}{r_c^3} + 8m\frac{1}{\ell^2} - \frac{8}{\ell^4}r_c^3\right] \ . \end{aligned}$$

The total contribution of the Euler characteristic is

$$4\pi\beta\left[16m^2\frac{1}{r_+^3} + 8m\frac{1}{\ell^2} - \frac{8}{\ell^4}r_+^3\right] \ . \tag{B.5.14}$$

Using the fact that the periodicity of τ is the inverse Hawking temperature (5.3.6) and that $f(r_+) = 0$ gives $2m = r_+\left(1 + \frac{r_+^2}{\ell^2}\right)$, we obtain

$$\chi = \frac{\beta}{\pi}\left[2m^2\frac{1}{r_+^3} + m\frac{1}{\ell^2} - \frac{1}{\ell^3}r_+^3\right] = 2 \ , \tag{B.5.15}$$

which is the correct result for the Euler characteristic of a black hole.

B.5.2 Holographic renormalization

We now show that our regularization procedure, using the prescription of holographic renormalization, gives the same boundary term. This is already clear from the fact that the regularized Euler integral gives the correct χ but here we directly compare the boundary terms. The boundary geometry at $r = r_c$ is

$$ds^2 = \left(1 + \frac{r_c^2}{\ell^2} - \frac{2m}{r_c}\right) dt^2 + r_c^2 d\theta^2 + r_c^2 \sin^2\theta \, d\phi^2, \tag{B.5.16}$$

and the Ricci scalar on the boundary is $\mathcal{R} = \frac{2}{r_c^2}$. The holographic counterterms are given by

$$\begin{aligned}
a^{\text{CT}} &= \int d^3y\sqrt{h}(c_1 + c_2\mathcal{R}) && \text{(B.5.17)}\\
&= \int_0^\beta dt \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \sqrt{1 + \frac{r_c^2}{\ell^2} - \frac{2m}{r_c}} r_c^2 \left(c_1 + c_2\mathcal{R}\right) \\
&= 4\pi\beta \left(c_1 r_c^2 + 2c_2\right) \sqrt{\frac{r_c^2}{\ell^2} - \frac{2m}{r_c} + 1} && \text{(B.5.18)}\\
&= 4\pi\beta \left[\frac{c_1}{\ell} r_c^3 + \frac{(c_1\ell^2 + 4c_2)}{2\ell} r_c - c_1 m\ell\right] + \mathcal{O}(r_c^{-1}). && \text{(B.5.19)}
\end{aligned}$$

In order to remove the divergence in (B.5.5), we demand

$$c_1 = -\frac{8}{\ell^3} \,, \quad c_2 = \frac{2}{\ell}. \tag{B.5.20}$$

Plugging back in (B.5.17), we see that the counterterm is exactly equal to Chern's boundary term. Note that this is non-trivial because the renormalization introduces a finite correction. It would be interesting to have a more geometrical understanding of this identification.

B.6 Vanishing of boundary terms

The proper application of the heat kernel expansion in AdS following [158] requires the addition of boundary terms. They come from the fact that the computation needs to be done on a regularized geometry defined by a cutoff $r < r_c$. In this appendix, we show that these boundary terms vanish and thus can be ignored. One source of boundary terms is the fact that $a_4(x)$ actually contains total derivatives, highlighted below as

$$a_4(x) = \cdots + \frac{1}{30}(5\Box E + \Box R) \ . \tag{B.6.1}$$

This gives a contribution

$$C_{\text{local}} = \cdots + \frac{1}{(4\pi)^2}\frac{1}{30}\int d^3y\sqrt{h}\, n^\mu \nabla_\mu (5 \text{Tr}\, E + R\, \text{Tr}\, 1) \ . \tag{B.6.2}$$

Using the fact that $\mathrm{Tr}\, E$ is a linear combination of two-derivative terms, it can be generally written as

$$\mathrm{Tr}\, E = \alpha_1 R + \alpha_2 F_{\mu\nu} F^{\mu\nu}, \tag{B.6.3}$$

for some coefficient α_1 and α_2. We can then compute the contribution (B.6.2) on our background. It is of order $O(r_c^{-1})$ and hence vanishes in the limit $r_c \to +\infty$.

Another contribution comes from the formula of $a_4(x)$ on a manifold with boundaries, which include additional boundary terms. This can be written as an additional boundary contribution to C

$$C_{\text{bdy}} = \int d^3y \sqrt{h}\, a_4^{\partial}(y), \tag{B.6.4}$$

where

$$a_4^{\partial} = B_1\, \mathrm{Tr}\, 1 + B_2\, \mathrm{Tr}\, E, \tag{B.6.5}$$

and B_1 and B_2 are geometric invariants of the boundary depending on both intrinsic and extrinsic data

$$\begin{aligned} B_1 = \frac{1}{360} \Big[& 24 K_{:bb} + 20 R K + 4 R_{anan} K - 12 R_{anbn} K_{ab} + 4 R_{abcb} K_{ac} + 480 S^2 K + 480 S^3 \\ & + \frac{1}{21} \left[(280 \Pi_+ + 40 \Pi_-) K^3 + (168 \Pi_+ - 264 \Pi_-) K_{ab} K_{ab} K_{cc} \right] + 120 S_{:aa} \\ & + \frac{1}{21} (224 \Pi_+ + 320 \Pi_-) K_{ab} K_{bc} K_{ac} + 120 S R + 144 S K^2 + 48 S K_{ab} K_{ab} \Big] \ , \\ B_2 = \frac{1}{3} & (K + 6S) \ , \end{aligned} \tag{B.6.6}$$

where K_{ab} is the extrinsic curvature of the boundary. Here $\Pi_{\pm}$ and S capture the choice of boundary conditions for the fields at infinity. For normalizable boundary conditions in AdS_4, it can be checked that they are constants.

We can evaluate this term on our background using the general expression (B.6.3) for $\mathrm{Tr}\, E$, which diverges and hence, we use the same regularization prescription as in holographic renormalization. Once the dust settles, we find that this contribution vanishes. Hence, no boundary term of this type gives a contribution to the logarithmic correction.

B.7 Black hole curvature invariants

In the main text, we emphasize the role of universality considering by the Euler characteristic and the square of the Weyl tensor. It is also common to express the curvature invariants in terms of $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$, $R_{\mu\nu} R^{\mu\nu}$ and R^2. In this appendix, we explicitly write out the curvature invariants for the Kerr-Newman-AdS black hole studied in section 5.3.4 and show how they are related to their flat space counterparts.

The curvature invariants are

$$\begin{aligned}
R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} &= \frac{24}{\ell^4} + \frac{8}{4(r^2+a^2\cos^2\theta)^6}\Big[-24m^2\left(a^2\cos^2\theta+r^2\right)^3 + 192r^4\left((q_e^2+q_m^2)-2mr\right)^2 \\
&\quad -192r^2\left((q_e^2+q_m^2)-3mr\right)\left((q_e^2+q_m^2)-2mr\right)\left(a^2\cos^2\theta+r^2\right) \\
&\quad +4\left((q_e^2+q_m^2)-6mr\right)\left(7(q_e^2+q_m^2)-18mr\right)\left(a^2\cos^2\theta+r^2\right)^2\Big] \\
&= \frac{24}{\ell^4} + \tilde{R}_{\mu\nu\alpha\beta}\tilde{R}^{\mu\nu\alpha\beta}\ , \\
R_{\mu\nu}R^{\mu\nu} &= \frac{36}{\ell^4} + \frac{4(q_e^2+q_m^2)^2}{\left(r^2+a^2\cos^2\theta\right)^4} \\
&= \frac{36}{\ell^4} + \tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu}\ , \\
R^2 &= \frac{144}{\ell^4}\ , \\
F_{\mu\nu}F^{\mu\nu} &= -\frac{2\big(q_e^2-q_m^2\big)\left(r^4-6a^2r^2\cos^2\theta+a^4\cos^4\theta\right)+16q_eq_m ra\,\cos\theta\big(r^2-a^2\cos^2\theta\big)}{\left(r^2+a^2\cos^2\theta\right)^4}\ .
\end{aligned} \tag{B.7.1}$$

Each of the invariants are a sum of two terms. The first term is proportional to ℓ^{-4} and the second term which has no ℓ dependence agrees with the analogous invariants $\tilde{R}_{\mu\nu\alpha\beta}\tilde{R}^{\mu\nu\alpha\beta}$ and $\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu}$ of asymptotically flat Kerr-Newman black holes. Thus, taking $\ell\to\infty$, we smoothly recover the same expressions for the invariants in [172, 183]. The invariants for Reissner-Nordström and Schwarzschild can be obtained by specializing the parameters. We can find the expressions for E_4 using (5.3.2):

$$\begin{aligned}
E_4 = \frac{24}{\ell^4} + \frac{8}{\left(r^2+a^2\cos^2\theta\right)^6}\Big(&6m^2\left(r^6-15a^2r^4\cos^2\theta+15a^4r^2\cos^4\theta-a^6\cos^6\theta\right) \\
&-12mr(q_e^2+q_m^2)\left(r^4-10a^2r^2\cos^2\theta+5a^4\cos^4\theta\right) \\
&+(q_e^2+q_m^2)^2\left(5r^4-38a^2r^2\cos^2\theta+5a^4\cos^4\theta\right)\Big)
\end{aligned} \tag{B.7.2}$$

Moreover, upon integration, the ℓ^{-4} term diverges because of the divergent AdS_4 volume, and we tame this divergence by holographic renormalisation. On the other hand, the second term is finite and does not require holographic renormalization. The integrated curvature invariants after proper renormalization take the form of

$$
\begin{aligned}
&\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\,R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\\
&\quad=-\frac{3\left(a^2-r_+^2\right)\left(a^2+r_+^2\right)^2\left(a^2\left(r_+^2-\ell^2\right)+r_+^2\left(\ell^2+3r_+^2\right)\right)\arctan\left(\frac{a}{r_+}\right)}{a^5\ell^2\Xi r_+^3}\\
&\qquad+\left(a^2\left(\ell^2-r_+^2\right)-r_+^2\left(\ell^2+3r_+^2\right)\right)^2\\
&\qquad\times\frac{\beta\left(3a^5r_++2a^3r_+^3+3\left(a^2-r_+^2\right)\left(a^2+r_+^2\right)^2\arctan\left(\frac{a}{r_+}\right)+3ar_+^5\right)}{8\pi a^5\ell^4\Xi r_+^4\left(a^2+r_+^2\right)}\\
&\qquad+\frac{2\pi\left(a^2+r_+^2\right)\left(3a^5r_++2a^3r_+^3+3\left(a^2-r_+^2\right)\left(a^2+r_+^2\right)^2\arctan\left(\frac{a}{r_+}\right)+3ar_+^5\right)}{a^5\beta\Xi r_+^2}\\
&\qquad+\frac{ar_+\left(a^6\left(3\ell^2-7r_+^2\right)+a^4\left(3\ell^2r_+^2-11r_+^4\right)+a^2r_+^4\left(\ell^2-9r_+^2\right)-3r_+^6\left(\ell^2+3r_+^2\right)\right)}{a^5\ell^2\Xi r_+^3}\;,\\
&\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\,R_{\mu\nu}R^{\mu\nu}\\
&\quad=\frac{96\beta r_+^4\left(\beta r_+\left(a^2+\ell^2+2r_+^2\right)-2\pi\ell^2\left(a^2+r_+^2\right)\right)-96\beta^2r_+^5\left(a^2+r_+^2\right)}{32\pi\beta\ell^4\Xi r_+^4}\\
&\qquad+\left(a^2\left(\beta r_+^2-\ell^2(\beta+4\pi r_+)\right)+\ell^2r_+^2(\beta-4\pi r_+)+3\beta r_+^4\right)^2\\
&\qquad\times\frac{\left(3a^5r_++2a^3r_+^3+3(a-r_+)(a+r_+)\left(a^2+r_+^2\right)^2\arctan\left(\frac{a}{r_+}\right)+3ar_+^5\right)}{32\pi\beta\ell^4\Xi r_+^4\left(a^5\left(a^2+r_+^2\right)\right)}\;,\\
&\qquad\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\,R^2=\frac{12\left(\beta r_+\left(\ell^2+r_+^2\right)-2\pi\ell^2\left(a^2+r_+^2\right)\right)}{\pi\ell^4\Xi}\;,\\
&\qquad\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}F_{\mu\nu}F^{\mu\nu}=2r_+^2-\frac{\beta r_+\left(3r_+^4+(a^2+\ell^2)r_+^2-a^2\ell^2-2\ell^2q_m^2\right)}{2\pi\ell^2\left(a^2+r_+^2\right)}\;.
\end{aligned}
\tag{B.7.3}
$$

Appendix C

C.1 Special Functions

Here we summarize the definitions of special functions used in the paper. The Dedekind eta function is defined as

$$\eta(q) = q^{\frac{1}{24}} \prod_{k=1}^{\infty} \left(1 - q^k\right) , \quad \mathrm{Im}(\tau) > 0 \, , \tag{C.1.1}$$

with $q = e^{2\pi i \tau}$. The Pochhammer symbol is defined as

$$(z; q)_\infty = \prod_{k=0}^{\infty} (1 - z q^k) \, . \tag{C.1.2}$$

The elliptic theta functions which are relevant to us have the following product form:

$$\theta_0(u;\tau) = \prod_{k=0}^{\infty} (1 - e^{2\pi i (u + k\tau)})(1 - e^{2\pi i(-u + (k+1)\tau)}) \, , \tag{C.1.3a}$$

$$\begin{aligned} \theta_1(u;\tau) &= -i e^{\frac{\pi i \tau}{4}} (e^{\pi i \tau} - e^{-\pi i \tau}) \prod_{k=1}^{\infty} (1 - e^{2\pi i k \tau})(1 - e^{2\pi i (k\tau + u)})(1 - e^{2\pi i (k\tau - u)}) \\ &= i e^{\frac{\pi i \tau}{4}} e^{-\pi i u} \theta_0(u;\tau) \prod_{k=1}^{\infty} (1 - e^{2\pi i k \tau}) \, . \end{aligned} \tag{C.1.3b}$$

The elliptic gamma function and the "tilde" elliptic gamma function are defined as

$$\Gamma(z; p, q) = \prod_{j,k=0}^{\infty} \frac{1 - p^{j+1} q^{k+1} z^{-1}}{1 - p^j q^k z} \, , \tag{C.1.4a}$$

$$\widetilde{\Gamma}(u; \sigma, \tau) = \prod_{j,k=0}^{\infty} \frac{1 - e^{2\pi i [(j+1)\sigma + (k+1)\tau - u]}}{1 - e^{2\pi i [j\sigma + k\tau + u]}} \, . \tag{C.1.4b}$$

C.1.1 Asymptotic Behavior

For a small $|\tau|$ with fixed $0 < \arg\tau < \pi$, the Pochhammer symbol can be approximated as

$$\log(q;q)_\infty = -\frac{\pi i}{12}(\tau + \frac{1}{\tau}) - \frac{1}{2}\log(-i\tau) + \mathcal{O}(e^{\frac{2\pi \sin(\arg\tau)}{|\tau|}})\,. \tag{C.1.5}$$

To study asymptotic behaviors of elliptic functions, it is useful to introduce the function $\{u\}_\tau$, as

$$\{u\}_\tau \equiv u - \lfloor \mathrm{Re}(u) - \cot(\arg\tau)\mathrm{Im}(u)\rfloor \quad (u \in \mathbb{C})\,, \tag{C.1.6}$$

which satisfies

$$\{u\}_\tau = \{\tilde{u}\}_\tau + \check{u}\tau\,, \qquad \{-u\}_\tau = \begin{cases} 1 - \{u\}_\tau & (\tilde{u} \notin \mathbb{Z})\,, \\ -\{u\}_\tau & (\tilde{u} \in \mathbb{Z})\,, \end{cases} \tag{C.1.7}$$

where we have defined $\tilde{u}, \check{u} \in \mathbb{R}$ as

$$u = \tilde{u} + \check{u}\tau\,. \tag{C.1.8}$$

The elliptic theta function $\theta_0(u;\tau)$ can be approximated for a small $|\tau|$ with fixed $0 < \arg\tau < \pi$ as

$$\begin{aligned}\log\theta_0(u;\tau) = &\frac{\pi i}{\tau}\{u\}_\tau(1-\{u\}_\tau) + \pi i\{u\}_\tau - \frac{\pi i}{6\tau}(1+3\tau+\tau^2) \\ &+ \log(1-e^{-\frac{2\pi i}{\tau}(1-\{u\}_\tau)})\left(1-e^{-\frac{2\pi i}{\tau}\{u\}_\tau}\right) + \mathcal{O}(e^{\frac{2\pi\sin(\arg\tau)}{|\tau|}})\,.\end{aligned} \tag{C.1.9}$$

The elliptic theta function $\theta_1(u;\tau)$ is approximated for a small $|\tau|$ with fixed $0 < \arg\tau < \pi$ as

$$\begin{aligned}\log\theta_1(u;\tau) = &\frac{\pi i}{\tau}\{u\}_\tau(1-\{u\}_\tau) - \frac{\pi i}{4\tau}(1+\tau) + \pi i\lfloor \mathrm{Re}(u) - \cot(\arg\tau)\mathrm{Im}(u)\rfloor + \frac{1}{2}\log\tau \\ &+ \log(1-e^{-\frac{2\pi i}{\tau}(1-\{u\}_\tau)})\left(1-e^{-\frac{2\pi i}{\tau}\{u\}_\tau}\right) + \mathcal{O}(e^{\frac{2\pi\sin(\arg\tau)}{|\tau|}})\,.\end{aligned} \tag{C.1.10}$$

For a small $|\tau|$ with fixed $0 < \arg\tau < \pi$, the elliptic gamma function can be approximated as

$$\log\widetilde{\Gamma}(u;\tau) = 2\pi i\, Q(\{u\}_\tau;\tau) + \mathcal{O}(|\tau|^{-1}e^{\frac{2\pi\sin(\arg\tau)}{|\tau|}\min(\{\tilde{u}\},1-\{\tilde{u}\})})\,, \tag{C.1.11}$$

provided $\tilde{u} \not\to \mathbb{Z}$ (see [27] for example), and the function $Q(\cdot\,;\cdot)$ is defined as:

$$Q(u;\tau) \equiv -\frac{B_3(u)}{6\tau^2} + \frac{B_2(u)}{2\tau} - \frac{5}{12}B_1(u) + \frac{\tau}{12}\,, \tag{C.1.12}$$

with $B_n(u)$ being the n-th Bernoulli polynomial.

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