Learning to Forget
Advanced Online Recursive Identification for Estimation and Adaptive Control

by

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Thy knowledge is become wonderful to me: it is high, and I cannot reach to it.

– Psalm 139:6.

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To Augusta Catherine Maria, who came to us on March 26th, 2021, and whose smiles and giggles saw me through it all. And to Ashley again, first friend, then partner, now wife, who never left my side.

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ABSTRACT

In adaptive control and online parameter estimation, recursive identification algorithms, such as Recursive Least Squares (RLS) or gradient methods, are often used to learn system dynamics online. An established method of coping with changing system parameters is the use of forgetting in the recursive identifier, but standard constant-rate forgetting can cause divergence of the filter covariance in RLS and has an unacceptably low re-convergence rate when the system is subject to abrupt instead of gradual changes. In this work, we present novel theory and algorithms for mitigating these two drawbacks by the introduction of modified RLS algorithms with variable-rate forgetting, matrix forgetting, and *a priori* bounded covariance. Next, we exhaustively investigate the role of persistent excitation in the global asymptotic stability (GAS) of RLS without forgetting, and finally we present results guaranteeing the GAS of sequential gradient descent using a class of cost functions that include the non-strictly convex least-squares costs corresponding to limiting case of forgetting all past data. Our results provide a robust forgetting-based solution to the loss of persistency and abrupt change of parameters problems for RLS, as well as a template for directly generalizing existing persistency conditions to achieve necessary and sufficient regressor properties for the GAS of RLS. Finally, the work opens new directions in stability analyses for gradient methods, such as the introduction of time-generalized fixed point theorems that can be used in the case where Lyapunov's direct method cannot be applied.

CHAPTER 1

Introduction

The year is 2054 and an F43, the newest and most advanced fighter in the world, is hit by an air-to-air missile while dog-fighting in the Pacific. In a tenth of a second, flames engulf the fighter's left wing and a small stub of contorted steel near the fuselage is all that remains. The aircraft begins to roll unstably about the nosecone. Knowing that he is over unfriendly territory, the pilot chooses not to eject. The mental and physical strains of this risky option have been well known to generations of pilots. Instead, he waits; less than a second later the fighter has restabilized and begins a flight back to base. In several minutes, the battered F43 lands safely on its home carrier, with repair crews rushing to meet it. The life of a serviceman and hundreds of thousands of dollars in equipment have been saved–but how?

Unlike the fighters of today, the F43 has an adaptive autopilot that can redesign itself quickly in response to damage inflicted upon the aircraft. The heart of this autopilot is an advanced online system identification algorithm, which infers flight parameters in real-time from online data streams and can detect and respond to parameter changes in as little as 30 samples of data–or 50 milliseconds at 600 Hz. Although it might seem far-fetched that such a system could exist, progress in control and estimation theory has made this kind of technology feasible. This thesis will describe fundamental algorithms that can enable technologies like the autopilot of the hypothetical F43 in applications ranging from aerospace, automotive systems, and robotics to chemical plants and power and electrical

systems.

1.1 Recursive Online System Identification

The main type of algorithms we will discuss will be system identification algorithms, and specifically recursive online system identification algorithms. To understand exactly what this means, it is useful to separate the term into its constituent parts:

i) *System Identification:* a system is any assembly of interacting material parts. Examples are mechanical systems, such as aircraft, automobiles, and spacecraft; chemical systems, such as controlled chemical plants and reactors like distillation columns; electrical systems, such as power stations, the electric grid, or radar installations; and biological systems such as individual cells, organs, or the entire human body. Systems usually admit the flow of data through the system in some form. The standard paradigm of systems theory distinguishes several kinds of data streams:

Inputs are data streams that are supplied from outside the system and affect the behavior of the system. Inputs can be *controlled*–in which case they are specified by the user and flow into special parts of the system called *actuators*–or *uncontrolled* (also called *exogenous*) in which case they cannot be manipulated by the user and affect the system through paths that are often complex and unknown. Uncontrolled inputs are often called *disturbances* because they disturb what would otherwise be the ordinary, predictable, or desired behavior of the system.

Outputs are data streams that are measured by one or more *sensors* and are available to the user, usually corrupted with noise.

System states are a special set of data streams that taken collectively uniquely specify the internal configuration of the system at each instant. If the entire set of states were known at a single instant, and all future inputs were known, then an analyst could, in principle, determine the states and outputs at every point in the future.

There are other approaches to system theory, such as the *behavior approach* developed by Willems and his collaborators [1–4], but the input, output, and state approach just described is standard and accounts for the majority of research and all major applications in the field.

When a system is considered for engineering purposes, a *model* connecting the behavior of the inputs to the outputs and possibly unmeasured system characteristics is often useful or necessary. *System identification* is the scientific study of how models can be obtained from input and output data from a system. The kind of models used in system identification and how they are determined from data will be discussed at length in a later chapter. For now, we mention that we will be interested primarily in discrete, dynamic models. A dynamic model is a model that predicts the response of a dynamic system from known inputs and initial conditions. The data and predicted response can be given continuously in time or supplied only at finitely-separated instants. We will consider only the latter case, in which the dynamics are said to be *discrete*. This is to better fit the analysis to the operation of the most common hardware, which all relies on digital electronics and sampled-data. The reader should be aware, however, that continuous-time versions exist of all the fundamental algorithms we will present.

$ii)$ *online*: system identification can be performed entirely in the laboratory by means of controlled (in the scientific sense) experiments. Typically, this approach operates by applying a series of non-destructive inputs to the system, or a prototype of the system, recording the corresponding system outputs, and batch processing the experimental data into a set of model parameters. In contrast to this approach, *online identification* attempts to identify the system in real-time while it is operating. Because of this more ambitious objective, online identification lacks most of the luxuries available to the offline, laboratory-based approach. Instead of being able to choose experiments at will, we must work with the data that is produced during system operation. Notably,

when the system is operating within a closed-loop control architecture, the inputs will tend to minimize the range of system dynamics they excite. Although this is highly desirable from a control perspective, this behavior will result in data that is not rich enough in information to identify a full model of the system. Identification and control can therefore be seen to have competing interests in online operation. Thus enters the notion of *persistency*, also called *persistency of excitation*, which is a property that can be used to decide if the input and output data is rich enough to fully identify a model of the system. The simultaneous satisfaction of the goals of control and identification during closed-loop online operation is sometimes referred to as the problem of *dual control* (cf. [5–9]), the general solution of which remains open at the time of writing.

$iii)$ *recursive*: not only must the identification algorithm operate online, but it must be at least in principle capable of computing model estimates efficiently and with little computation power. Hardware constraints are often a limiting factor, since online algorithms usually run on embedded platforms such as those used in flight control applications. A standard way of creating efficient algorithms is to assume a recursive form for the estimate, given in general by

$$\theta_{k+1} = \theta_k + U_k(y_k, u_k, \theta_k), \tag{1.1}$$

where θ_k is the current model estimate, given as a parameter vector in a finite-dimensional real vector space, θ_{k+1} is the updated estimate, and U_k is a step-dependent update function that incorporates the current output y_k, input u_k, as well as possibly internal state variables. An algorithm that can be written in this form can be bounded in complexity by the update function. In several important cases, this complexity is $\mathcal{O}(p^2)$, where p is the number of system outputs, which is frequently far less than the number of model parameters, enabling the computation of an efficient estimate.

Recursive online identification thus forms a subfield of general system identification, which in turn forms a subfield of estimation theory and systems theory.

What sort of algorithms do we consider in recursive online identification? A Great variety of options exist depending on the choice of model type and how the engineer decides to measure the quality of the estimate. The simplest option, which we will later find has much to recommend it from the Bayesian viewpoint, is to begin with a well-established offline algorithm and modify it for recursive online implementation. For the important class of linear input-output models (also called ARMA models), the elementary least-squares algorithm of Gauss [10] and Legendre [11] is difficult to beat–though many have tried. Another recommending factor is that a recursive version of least-squares, which is straightforwardly called Recursive Least-Squares (RLS), exists that produces exactly the same estimates as the offline batch version. This is not the case with many offline algorithms.

1.2 Parameter Changes

Thus far we have presented nothing new. The idea of using RLS for system identification dates back to the 1970s and 80s [12, 13] and RLS itself dates back to the 1950s [14]. If RLS works well enough, what is there to do? Recalling the F43 scenario, we see that what we want is not simply an algorithm that can identify a system, but an algorithm that can adapt to changes in a system; the aerodynamic coefficients of the hypothetical F43 are doubtlessly much different *after* the missile hits it than before, and we want an algorithm that is robust to these kind of changes.

Parameters can change either gradually or abruptly. Gradual changes happen slowly over time. These kind of changes usually result from component wear or gradual environmental drift, and the task of the identifier in this situation is to track this trend. In contrast, abrupt changes happen over a very short period, and are usually due to the sudden failure of a component or to damage being inflicted upon the system. When the missile hit our

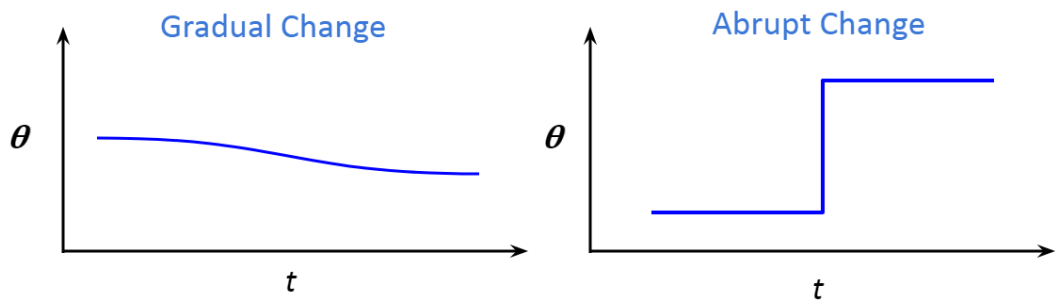

Figure 1.1: Intuitive view of gradual and abrupt changes

F43, it caused an abrupt change in the system parameters. The task of the identifier in this case is to detect the change and reconverge to the new parameters as quickly as possible.

Except for the simplest cases of very slowly changing parameters, standard recursive identifiers track parameter changes too slowly, since the estimate is continuously biased by older data. This motivated the introduction of forgetting factors from the 1980s to the early 2000s (see, for instance, [15, 16], [12, pp. 52-53]). A forgetting factor was initially seen as a number λ in the interval $(0, 1]$ that is applied geometrically at each data point. Hence, at step k, the measurement y_k has weight 1, the measurement y_{k-1} has weight λ, y_{k-2} has weight λ^2 and so forth. There are several standard algorithms in which the forgetting factor can be incorporated in a way that the recursive form of the algorithm is preserved. The most important example of this RLS, in which the forgetting factor is incorporated as a multiplicative factor in the covariance update.

When a forgetting factor is included, RLS is more responsive to gradual parameter changes and more quick to respond to abrupt parameter changes. However, the re-convergence rate after abrupt changes is still slower than desired, and might take as many as several hundred to several thousand steps depending on the magnitude of the forgetting factor. The rate of re-convergence is quicker for smaller forgetting factors, but small forgetting factors degrade the quality of the estimate, since less data is effectively being used in the production of the estimate. Small forgetting factors also lead issues of stability and numerical robustness, and hence in most applications large forgetting factors, ranging from $0.999 - 0.999999$, are used. Another major drawback of using forgetting in this way

emerges when we consider the effect that a gradual or abrupt loss of persistency will have on the estimator.

1.3 Persistency

As mentioned before, persistency is a property of the data being used in an estimator. In system identification, the data under consideration consist of input and output measurements of the system, $y_0, \ldots, y_k$ and $u_0, \ldots, u_k$. The identification procedures we will consider organize these measurements into a vector

$$\phi_k = \begin{bmatrix} -y_{k-1} & \ldots & -y_{k-n} & u_k & \ldots & u_{k-m} \end{bmatrix}, \tag{1.2}$$

where n and m are related to the selected model order. The vector ϕ_k is then called the *regressor* and the sequence $\phi_0, \phi_1, \ldots$ is called the *regressor sequence*. Persistency can be defined as a property of the regressor sequence. Specifically, if there exists an $N > 0$ such that any $N+1$ consecutive regressors, when formed into the positive-semidefinite matrices $\phi_k^{\mathrm{T}}\phi_k$ and added have minimum eigenvalue greater than a positive constant α. That is, for all $j \geq 0$,

$$\sum_{i=0}^{N} \phi_{i+j}^{\mathrm{T}}\phi_{i+j} \geq \alpha I_n. \tag{1.3}$$

This formulation of persistency can be shown to guarantee a wide range of convergence and consistency results for RLS and exponential convergence results for RLS with a constant forgetting factor λ. The question remains, however: what happens when persistency is *lost*?

Loss of persistency occurs during the operation of most adaptive control systems, since as the control objective is met, the outputs and inputs tend to reach a steady state of strictly "lower dimension" than the parameter space of the system in the sense that too few system

modes are excited to uniquely identify all of the system parameters. Hence, supposing that there are no changes in the parameters, the only data that contributes information about the steady state non-identifiable parameters is that taken during transient operation. When a forgetting factor is used, however, this data is weighted successively less and less as the system continues in steady state, and asymptotically is excluded from the estimate. This causes a divergence in the estimates that can lead to *bursting* in adaptive controllers [17,18].

We can also consider the effect of the loss of persistency on a recursive online identification algorithm without necessarily assuming that the estimates will be used in an adaptive controller. In this case we simply assume that the regressor loses persistency, without considering the cause of this loss. When this occurs for RLS with a constant forgetting factor λ, one finds that the covariance matrix for RLS begins to diverge exponentially, which ultimately causes the estimate itself to diverge, since random noise and even small numerical roundoff errors will eventually be amplified enough to disturb even an estimate that has converged very closely to the true parameters. This divergence is caused entirely by the fact that there is both forgetting and loss of persistency simultaneously–were even one of these two circumstances different, no divergence would occur. Since the loss of persistency cannot be easily mitigated in practice, we naturally come to the question of how an algorithm can be designed that has the advantages of forgetting, but does not exhibit the instabilities caused by a constant forgetting factor when the regressor is not persistent.

1.4 Two Questions to Begin the Work

As seen above, consideration of extant online identification algorithms, and more specifically RLS, has led us to the following two questions, which can be taken as the beginning of our work. Namely,

i) How can a RLS or another existing recursive identification algorithm be modified to handle both gradual and abrupt parameter changes well? (Currently RLS with a

constant forgetting factor has some success with gradual changes.)

ii) Does there exist a way to include forgetting in RLS that does not suffer from instabilities when persistency is lost?

From these two questions, we will move on successively to others. However, this is the natural starting place of the work we will present herein.

1.5 Outline

Let us outline the remainder of the thesis. In the next chapter, we will discuss the fundamental notions of system identification that will be assumed throughout the five papers that constitute the main body of the work. In order these papers are

i) "A Modified Recursive Least Squares Algorithm with Forgetting and Bounded Covariance" published in the 2019 American Control Conference as [19].

ii) "Convergence and Consistency of Recursive Least Squares with Variable-Rate Forgetting" published in *Automatica* as [20].

iii) "Recursive Least Squares with Matrix Forgetting" published in the 2020 American Control Conference as [21]

iv) "Necessary and Sufficient Regressor Conditions for the Global Asymptotic Stability of Recursive Least Squares" published in *Systems & Control Letters* as [22].

v) "Sequential Gradient-Descent Optimization of Data-Dependent, Rank-Deficient Cost Functions with a Unique Common Global Minimizer", which is to appear in the 2021 Conference on Decision and Control and which is under revision for submission to the *IEEE Transactions on Automatic Control*.

Each of these papers is accompanied by a preface that attempts to put the work in the larger context of the research program of the thesis–something that is not always possible in individual publications. They begin with our second main question: how do we modify RLS to be robust to the loss of persistency? The first papers evaluates one possibility–bounding the covariance *a priori*. Although this is an interesting idea, originally presented by Goodwin et al. in [23], and although progress on Goodwin's idea proved to be possible, the course of the research shifted when we discovered variable-rate forgetting factors, especially those that increase or decrease monotonically as a function of the measurement residual. The development of these ideas is presented in the second and third papers, both for scalar and matrix-valued forgetting factors. Variable-rate forgetting provides, in its various versions, answers to both questions, although, as shown in paper *iii*) and [24], guaranteeing total robustness to the loss of persistency might be as expensive as computing an SVD of the covariance at each timestep.

With answers that are at least workable to both of our beginning questions, we turned to a further investigations of features that we found while pursuing our work up until that point. One major question was the role played by the "miraculous" (*cf.* [25]) condition of persistency in convergence. If the loss of persistency can be as catastrophic as it is with constant forgetting, yet only be a sufficent condition for convergence, then surely the study of weaker-than-persistency conditions is valuable to understand the behavior of RLS. Indeed this is the case, and in paper *iv*) we present a comprehensive theory of *necessary and sufficient* conditions for the convergence of ordinary RLS.

Finally, we examine the case in which forgetting is taken to an extreme and only the most recent data point is used to update the estimate. In this case, there is in general no analytic minimizer and thus the RLS update cannot be used. We forge ahead using gradient-descent, and hence set forth the problem of finding sufficient conditions under which gradient descent operating sequentially on each cost, will converge to the true system parameters.

CHAPTER 2

Background on System Identification

In this chapter we will formulate the system identification problem rigorously and then derive and discuss the RLS algorithm. Before adding mathematics to the conceptual notion of "finding a system model", we will try to clarify exactly what we want.

There are several possible questions that can motivate the identification of a system. One example is: how can the outputs of a system be correctly predicted if we know that a specific sequence of inputs will be applied? We will call this question the *prediction* question. Note that in this case, we are ambivalent to the particular way that the output prediction is made. We do not have a specific model set in mind and the choice of one model set over another may be the result of nothing more than convention, familiarity, or, perhaps more humorously put, an attempt to maximize the likelihood of a high citation rate[1]. Also note that we are not attempting to determine a state vector, but only outputs. In many cases, there is no obvious choice of state vector or the canonical choices have so many dimensions that we may prefer to work without them.

There are multiple concrete interpretations of predicting "outputs" given an "input". First, we may consider the apparently simplest case of predicting the output y_k that will be observed simultaneously with the application of a specific input u_k. Experience with dynamical systems shows that knowledge only of u_k is insufficient to form good predictions

[1]This is only humorous in part however. Everyone who conducts research has come across cliques, fads, and informal 'schools', and it is not difficult to find research in which the models are apparently chosen for no other reason than to appeal to one of these.

of y_k, and thus this approach is usually extended as follows: given the previous outputs $y_0, \ldots, y_{k-1}$ and the previous inputs $u_0, \ldots, u_{k-1}$, predict the y_k that will be observed for a specific u_k. We might call this "output predictability of the first kind". We may of course extend this to the prediction of a finite number of future outputs $y_k, \ldots, y_{k+m}$ that are to be predicted for a specific input sequence $u_k, \ldots, u_{k+m}$, under the condition that the previous inputs and outputs are known, which we might call "output predictability of the second kind", and finally, letting $m \to \infty$, we might consider the problem of predicting the remainder of the whole sequence $y_k, y_{k+1}, \ldots$ *ad inf.* for a specific input sequence $u_k, u_{k+1}, \ldots$ *ad inf.*, or, by setting $k = 0$, the problem of determining the entire output sequence from a specific input sequence, either of which we might call "output predictability of the third kind." It follows immediately by induction that output predictability of the first kind implies that of the second and third kind, and by setting $m = 0$, it follows that the second kind implies the first, but the third kind need not imply the first or second. That being said, we will henceforth only consider output predictability of the first kind, thus obtaining the second and third kinds as implications.

Alternatively, we may not be ambivalent to the way that the output predictions are made. In this case, the fundamental causes of an observed sequence of outputs may be known through, for instance, the application of basic scientific laws and principles, and only a finite number of (usually constant) parameters are to be determined. Hence we may ask: how can we determine the best values of a finite set of parameters $\theta_1, \ldots, \theta_n$, so that the output predicted by the scientific model is as close as possible to the observed output. We will call this question the *fitting* question. Note that there is no need, even with perfect, noiseless measurements, for the predicted and observed outputs to match, since the scientific model may not include all of the mechanisms that noticeably affect the output of the system.

These two goals are different in their essential object. There first requires only some mathematical device–any one–that succeeds in making good predictions, while the second

uses prediction as a way of measuring the accuracy of a hypothesis (albeit from a quite specialized class). The pursuit of the first goal leads naturally to the development of generic model structures, like transfer functions of a specified order [12, 26, 27], linear state space models, or nonlinear models like Volterra series [28–30] or neural nets [31–33]. The pursuit of the second goal leads to detailed procedures for estimating physically meaningful quantities in first-principles models. Clearly we must think carefully and decide on which one of these goals better fits out intentions when we set out to perform system identification. In what follows, we will consider only the goal of prediction, since the chief purpose of our identified models will be for the synthesis of controllers.

2.1 Model Classes

Model types can be broadly classified along the lines of: linear or nonlinear, state-space or input-output (stateless), finite or infinite dimension, and so forth. For simplicity we restrict our attention to linear models, since this class is most frequently used in applications. Ultimately, we will also only use input-output models, but before doing so, we will briefly review state space models, since these kinds of models are becoming increasingly represented in the literature [34].

2.1.1 State Models

With the advent of "modern" control in the 1960s, state space models of linear systems became increasingly popular. Among many theoretical innovations, these models allowed the determination of a system's instantaneous output y_k using only the knowledge of the state vector x_k. The state vector at the next step is uniquely determined by the current state vector and corresponding input value u_k. Altogether, these relationships are expressed by

the now ubiquitous equations

$$x_{k+1} = Ax_k + Bu_k, \tag{2.1}$$

$$y_k = Cx_k + Du_k. \tag{2.2}$$

For practical systems whose frequency response tends asymptotically to zero, we have $D = 0$.

In general, the state vector is not directly measured, and must be determined by the use of an observer, such as those given in the classic works of Kalman [35] and Luenberger [36]. The use of these observers, however, requires knowledge of the state space matrices A, B, C, D, which are typically themselves sought in the identification process. Therefore, established identification procedures that produce state space models must generally operate without explicit knowledge of the state. This assumption leads to realization-type algorithms, such as the Eigenvector Realization Algorithm [37], or modal identification [38, 39], that return state representations in a generic basis, which cannot be easily specified or constrained by the user. It has also been noted that these algorithms are not robust to measurement noise. For instance, the authors in [34] carefully observe that a more realistic model for identification is given by

$$x_{k+1} = Ax_k + Bu_k + w_k, \tag{2.3}$$

$$y_k = Cx_k + Du_k + \nu_k. \tag{2.4}$$

where $(w_k)_{k\geq 0}$ and $(\nu_k)_{k\geq 0}$ are zero-mean Gauss-Markov processes as found in the standard formulation of the Kalman filter [35].

At most $n^2 + np + nm + mp$ parameters are needed to specify a state space model. However, frequently far fewer are required. In this case, one way of writing the state space

model is

$$x_{k+1} = A(\theta)x_k + B(\theta)u_k, \tag{2.5}$$

$$y_k = C(\theta)x_k + D(\theta)u_k. \tag{2.6}$$

The use of such parameterized state space models is usually more appropriate to the goal of fitting rather than prediction, since the specific parameterization is frequently the result of the application of scientific principles. It is also the case that in these kind of models the notion of identifiability [40] can become a driving factor, since it may not be possible to uniquely determine the entire parameter vector simply from the available measurements. Although state space models are an interesting and increasingly present class of models, the computational burden, arbitrariness of the state basis, and lack of robustness to noise make them poorly suited to the output prediction problem for the purpose of adaptive control, and hence most adaptive control algorithms (see, e.g. [12, 13]) use input-output model types, although it is possible that special state realizations may be performed during the controller synthesis stage [41–43].

2.1.2 Input-Output Algorithms

The most basic input-output algorithm in discrete time is given by the so-called AR (auto-regressive) model

$$y_k + a_1 y_{k-1} + \cdots + a_n y_{k-n} = b_0 u_k + b_1 u_{k-1} + \cdots + n_m u_{k-m}, \quad k \geq 0, \tag{2.7}$$

with initial conditions $y_{-n}, \ldots y_{-1}$. In the SISO case the output and input sequences, along with the coefficients are scalar-valued, while in the MIMO case, the output and input sequences will be vector-valued while the coefficients will be matrix-valued. In order to account for disturbances, model uncertainty, and noise, an exogenous term is frequently

added in the form of

$$y_k + \sum_{i=1}^{n} a_k y_{k-i} = \sum_{i=0}^{n} b_i u_{k-i} + d_0 e_0 + \cdots + d_p e_p, \quad k \geq 0, \tag{2.8}$$

where $(e_p)_{k\geq 0}$ is a stochastic process that is almost always zero-mean and Gauss-Markov. This modification of the AR model is frequently called the ARMAX model, in which the "MA" stands for "moving avergage", and the "X" stands for "eXogenous". Different choices of filter coefficients $d_0, \ldots d_p$ can lead the stochastic process to mimic different disturbance behavior and noise spectra. Exogenous input filters are discussed further in [26]. The inclusion of a nontrivial filter for the exogenous input requires an estimate of the noise or disturbance spectrum, which is not straightforward to compute online, and in many cases cannot be simply calibrated.

In what follows we will only consider ARMAX models with the exogenous input given by zero-mean Gaussian white noise ν_k and $d_i = \delta_{0,i}$, which is typically referred to as the ARX model, since the moving average is removed. In this case the exogenous inputs acts as a noise term that affects the outputs y_k recursively. For this model, the SISO case can be written as

$$y_k = \phi_k \theta + \nu_k, \tag{2.9}$$

where ϕ_k is the regressor given in (1.2). We will call this form the *linear regressor form*, since it expresses the relationship between y_k and θ in the form of a linear regression model. For the MIMO case, a similar regression form can be computed by the use of the identity $Ax = (I_n \otimes x^{\mathrm{T}})\mathrm{vec}(A)$. Note that in this case, the regressor will be an $p \times q$ matrix, where q is the number of identified parameters.

An important distinction arises between the inclusion of noise in recursion relation (2.9)

and the inclusion of noise by the process

$$\begin{aligned} y_k &= \phi_k \theta, \quad k \geq 0, \\ \tilde{y}_k &= y_k + \nu_k, \end{aligned} \tag{2.10}$$

which can be found, for instance, as a special case of the models presented in Chapter 4 of [44]. In the model structure (2.10), the noise is included only in the measured output $\tilde{y}_k$, but is not assumed to act in the recursion, and hence the regressor in this model is noiseless. This is in contradistinction to (2.9) in which each y_k is assumed to be noisy and hence the regressor is also noisy. This renders the ARX model closer to an Error-In-Variables model in statistics (see, for example, [45, Ch. 8, pp. 300-330], [46, 47]), which is considerably less tractable than the case of a noiseless regressor, and whose complete analysis, even for the simplest case of a static regressor (such as are frequently assumed in statistics), is open at the time of writing.

Hence for the purposes of analysis, we will assume that the regressor is noiseless in conformity with the model type (2.10). However, since the actual regressor, which includes measured outputs, must be constructed for the implementation of the algorithms we will consider, and since only noisy output data is available in practice, the implementation of the algorithms we discuss will conform more closely to (2.9). Although we have observed that this inconsistent use of the two models makes little difference in practical experiments and numerical analysis, nonetheless, the treatment of the noisy regressor case would form a more perfect theoretical basis than the current assumption of a noiseless regressor, and we see this as future work.

2.2 Recursive Least Squares

We will now discuss the tradition recursive least squares algorithm, found, for instance, in [13,14,48,49]. Throughout this section, we will assume the following sizes of the various

parameters: $\theta \in \mathbb{R}^n$, and $y_k \in \mathbb{R}^p$ and $\phi_k \in \mathbb{R}^{p\times n}$ for all $k \geq 0$. To find the parameter vector that best predicts the outputs y_k, for the linear regression form (2.9), we minimize the cost

$$J_k(\theta) = \frac{1}{2}\sum_{i=0}^{k} \|y_i - \phi_i\theta\|^2 + R(\theta, \theta_0), \tag{2.11}$$

where $R(\theta, \theta_0)$ is a regularization term that is user-defined. The most popular choice of regularization is

$$R(\theta, \theta_0) = \frac{1}{2}(\theta - \theta_0)^{\mathrm{T}} P_0^{-1}(\theta - \theta_0), \tag{2.12}$$

where P_0 is a positive definite matrix, which has both algorithmic and statistical reasons to recommend it, as we will discuss below. With this choice of regularization, the least-squares cost (2.11) is a strictly convex quadratic form in θ and hence has a unique global minimizer, given by the *normal equations*

$$\theta_{k+1} = \left(P_0^{-1} + \sum_{i=0}^{k} \phi_i^{\mathrm{T}}\phi_i\right)^{-1} \left(\sum_{i=0}^{k} \phi_i^{\mathrm{T}} y_i + P_0^{-1}\theta_0\right). \tag{2.13}$$

This form of the equations provides a full solution to the optimization problem, but is inefficient for online implementation due to the recalculation of the $n \times n$ covariance matrix at each step. Hence, we seek a way to recursively update the matrix $P_0^{-1} + \sum_{i=0}^{k} \phi_i^{\mathrm{T}}\phi_i$. To do so, set $P_k^{-1} \triangleq P_0^{-1} + \sum_{i=0}^{k} \phi_i^{\mathrm{T}}\phi_i$. We obtain the recursion equation for the inverse

$$P_{k+1}^{-1} = P_k^{-1} + \phi_k^{\mathrm{T}}\phi_k, \tag{2.14}$$

and by the matrix inversion lemma [50], it follows that

$$P_{k+1} = P_k - P_k\phi_k^{\mathrm{T}}(I_n + \phi_k P_k \phi_k^{\mathrm{T}})^{-1}\phi_k P_k. \tag{2.15}$$

A straightforward application of the matrix inversion lemma to (2.13) and some calculation yields the recursive form for the estimate,

$$\theta_{k+1} = \theta_k + P_{k+1}\phi_k^{\mathrm{T}}(y_k - \phi_k\theta_k), \tag{2.16}$$

and thus we have derived the ordinary RLS algorithm

$$\theta_{k+1} = \theta_k + P_{k+1}\phi_k^{\mathrm{T}}(y_k - \phi_k\theta_k), \tag{2.17}$$

$$P_{k+1} = P_k - P_k\phi_k^{\mathrm{T}}(I_n + \phi_k P_k \phi_k^{\mathrm{T}})^{-1}\phi_k P_k,\ k \geq 0, \tag{2.18}$$

where the initial values $\theta_0 \in \mathbb{R}^n$ and $P_0 \in \mathbf{P}^n$ are specified by the regularization term.

2.2.1 Convergence and Consistency

The convergence and consistency of RLS is a main topic throughout several of the papers presented in the sequel. In this section we will quickly introduce two basic results that can be used for comparison with the more complex results that follow.

Definition 1. *A sequence of estimates $(\theta_k)_{k\geq 0} \subset \mathbb{R}^p$ of a parameter $\theta \in \mathbb{R}^p$ is convergent to θ if $\lim_{k\to\infty} \theta_k = \theta$.*

Note that convergence is not simply the convergence of the sequence of estimates in the sense of mathematical analysis, but more specifically the convergence of the sequence of estimates to the 'true' parameter vector θ. Hence both divergent (in the analytic sense) and biased (in the statistical sense) estimators are not convergent in this sense.

Theorem 1. *If the regressor sequence is persistent and $\nu_k = 0$ for all k, then the sequence of RLS estimates is convergent to the true parameters θ.*

Proof. Let $\tilde{\theta}_k \stackrel{\triangle}{=} \theta_k - \theta$ and note that the assumption that $\nu_k = 0$ implies that $y_k = \phi_k\theta$.

From (2.17) and the recursion equation $P_{k+1}^{-1} = P_k^{-1} + \phi_k^{\mathrm{T}}\phi_k$, it follows that

$$\tilde{\theta}_{k+1} = P_{k+1}P_k^{-1}\tilde{\theta}_k = P_{k+1}P_0^{-1}\theta_0.$$

Since $(\phi_k)_{k\geq 0}$ is persistent, it follows that there exists and $\alpha > 0$ such that $\sum_{k=0}^{\infty}\phi_k^{\mathrm{T}}\phi_k \geq \sum_{k=0}^{\infty}\alpha = \infty$ and thus $P_k \to 0$ as $k \to \infty$. Hence $\tilde{\theta}_k \to 0$ as $k \to \infty$. □

The next property is an essentially stochastic property in which θ_k is naturally viewed as a random variable, and θ_0 and P_0 as the mean and covariance of a prior distribution. We will discuss the Bayesian formulation of the RLS estimator in further detail in the next section.

Definition 2. *Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and, for all $k \geq 0$, let θ_k be a measurable function $\Omega \to \mathbb{R}^n$. Then $(\theta_k)_{k\geq 0}$ is consistent with respect to θ if, for all $\varepsilon > 0$,*

$$\lim_{k\to\infty} \mathbb{P}(\{\omega \in \Omega\colon \|\theta_k(\omega) - \theta\| < \varepsilon\}) = 1. \tag{2.19}$$

If $\theta_k \sim \mathcal{N}(\theta, P_k)$ for all $k \geq 0$, it is straightforward to show that $(\theta_k)_{k\geq 0}$ is consistent if and only if $P_k \to 0$ as $k \to \infty$. The following result is an immediate corollary to Theorem 6 in chapter 4

Theorem 2. *If the regressor sequence is persistent and $(\nu_k)_{k\geq 0}$ is a zero-mean Gaussian white noise process, then the sequence of RLS estimates $(\theta_k)_{k\geq 0}$ is consistent with respect to the true parameters θ.*

2.2.2 A Bayesian Perspective on RLS

In this section we will show that the RLS cost function with the regularization given by (2.12) can be interpreted as a Bayesian maximum a priori (MAP) estimate when θ is considered as a random variable. We will also assume that, for all $k \geq 0$, $\nu_k \sim \mathcal{N}(0, 1)$. We assume that θ has the prior density $\mathcal{N}(\theta_0, P_0^{-1})$ and that the data $y_0, \ldots, y_k$ is available,

which is connected to θ by the linear regression form (2.9). We seek $f_{\theta|y_0,\ldots,y_k}$, the posterior distribution of θ given the these data. From Bayes theorem and the law of total probability, it follows that

$$f_{\theta|y_0,\ldots,y_k} = \frac{f_{y_0,\ldots,y_k|\theta} f_\theta}{\int f_{y_0,\ldots,y_k|\theta} f_\theta \, d\theta}, \tag{2.20}$$

Since $y_k - \phi_k\theta = \nu_k$, it follows that $y_k - \phi_k\theta \sim \mathcal{N}(0,1)$ and hence

$$f_{y_k|\theta}(y_k,\theta) = \frac{1}{N} e^{-\frac{1}{2}\|y_k - \phi_k\theta\|^2}, \tag{2.21}$$

where N is a normalization factor. Furthermore, since the ν_k are independent, it follows that $f_{y_0,\ldots,y_k|\theta} = f_{y_0|\theta} \cdots f_{y_k|\theta}$. Thus, it follows that the *a posteriori* distribution is given by

$$\begin{aligned} &f_{\theta|y_0,\ldots,y_k}(\theta, y_0, \ldots, y_k) \\ &= \frac{1}{N'(y_0,\ldots,y_k)} \exp\left(-\frac{1}{2}\sum_{i=0}^{k} \|y_i - \phi_i\theta\|^2 - \frac{1}{2}(\theta-\theta_0)^{\mathrm{T}} P_0^{-1}(\theta-\theta_0) \right), \end{aligned} \tag{2.22}$$

where $N'(y_0,\ldots,y_k)$ is a function only of the specific data values. Hence, to maximize this distribution, it follows that the argument of the exponential function should be minimized, which is exactly the minimization of (2.11)–(2.12).

CHAPTER 3

A Modified Recursive Least Squares Algorithm with Forgetting and Bounded Covariance

3.1 Preface

The following paper was published in the 2019 American Control Conference as [19]. The objective of the research was to generalize the Exponential Forgetting and Resetting Algorithm (EFRA) of Goodwin et al. [23], which achieved *a priori* bounds on the filter covariance via an *ad hoc* modification of the usual RLS update. In order to produce these covariance bounds, Goodwin et al. introduced four new parameters, γ, α, β, and δ, the first two of which acted as generalizations of a constant-rate forgetting factor λ, while the last two were proportionality constants used to balance covariance growth by the adding the positive-definite term βI into the update, while simultaneously subtracting the term positive-definite term δP_k^2. The feasible set of these parameters is not simple, as can be seen by the constraints given in Theorem 3.1.

The ability to enforce *a priori* covariance bounds is appealing from the perspective of preventing instabilities due to the covariance divergence during losses of persistency, which is experienced by RLS with constant-rate forgetting, since the ability to guarantee bounded covariance stops these instabilities at their source. Unfortunately, EFRA was not derived from an RLS-like cost function, so it has no apparent connection to MAP estimation in the Bayesian framework. It thus represents one of a class of algorithms that

are created as modifications of well-known methods that proceed from sound statistical and philosophical principles, but themselves are justified only by technical properties (such as bounded covariance) that are far removed from the basic considerations of the problem. Furthermore, we were able to show that EFRA does not match the RLS update anywhere in the closure of the feasible set of its parameters. That is, RLS is never a special or limiting case of EFRA. A second, practical drawback is the complexity of the formulas for the covariance bounds; it was very difficult to start out with two covariance bounds in mind and actually find EFRA parameters that achieved those covariance bounds. In many cases, reasonable covariance bounds are simply not in the feasible set.

With these results, we set down the task of creating a bounded-covariance modification of RLS that did not have these limitations. The result, poetically called "Modified Recursive Least Squares" (MRLS), is presented in the latter half of the paper, along with a derivation showing the limit in which it approaches RLS and an heuristic algorithm for selecting parameters corresponding to given covariance bounds. While no convergence or reachability analysis of this algorithm is given, our experience is that it works well.

The paper was well-received at ACC 2019, but soon after completing it, I discovered variable-rate forgetting and how to design variable forgetting factors that are both robust to the loss of persistency and much more responsive to abrupt changes in the system parameters. As a result, MRLS lost most of its importance for applications and we shifted the focus of our work. The MRLS paper is included here for completeness in tracing our thoughts in approaches to solving the problems of abrupt parameter changes and sudden loss of persistency, as well as a bridge to connect ideas like EFRA to those like RLS-VRF.

3.2 Introduction

The recursive least squares (RLS) algorithm is one of the key fundamental tools of identification, signal processing, estimation, and control [51, Section 2.2], [52, Chapter

13], [53, 54], [55, Chapter 12]. RLS provides a recursive technique for minimizing the least-squares cost function $J(x) = (Ax - b)^{\mathrm{T}}(Ax - b)$, where each row of A provides an additional data point that can be used to update the previous estimate of x. A unique minimizer of A exists if and only if $A^{\mathrm{T}}A$ is positive definite; an equivalent condition is that A is left invertible. Since A may not be left invertible for a limited amount of data, the RLS cost function includes a regularization term.

The simplest approach to deriving RLS is to define a recursion for the quadratic term in x appearing in $J(x)$ in terms of a covariance matrix. The matrix inversion can then be used to arrive at the final update equations. The term "covariance matrix" arises from the relationship between RLS and the Kalman filter [56,57]. In particular, by defining the state update $x_{k+1} = x_k$, which models the assumption that x is constant, $Ax = b$ can be viewed as the measurement equation $b = Ax$, where A is the "C" matrix for use in the Kalman filter. With these dynamics and measurement equation, the Riccati difference equation with initial condition given by the regularization term yields the RLS covariance update equation. Note that the state update $x_{k+1} = x_k$ does not include disturbance noise, and thus the Riccati covariance update lacks a constant driving term. Consequently, the solution of the Riccati difference equation is monotonically decreasing.

A useful variation of RLS is obtained by modifying the cost function to include a forgetting factor $\lambda \in (0, 1)$. By using λ, older data are discounted relative to more recent data. Consequently, RLS can respond more quickly to changes in x. In terms of the Kalman filter, the forgetting factor corresponds to the state equation $x_{k+1} = \frac{1}{\sqrt{\lambda}}x_k$, which, since $\lambda < 1$, are unstable. This instability allows the covariance to increase, which explains the ability of RLS to respond more quickly to changes in x.

An unfortunate side effect of the forgetting factor occurs when the data are not persistently exciting, which is reflected by the situation where $(\frac{1}{\sqrt{\lambda}}I, A)$ is not observable. In this case, sensor noise causes the covariance to diverge, leading to divergence of the estimate of x [58].

In order to overcome covariance divergence, a modified covariance equation is given in [23]. This modified covariance equation includes terms that bound the covariance. Most importantly, the modified covariance equation allows the covariance to increase, thus providing the ability to adapt to changes in x. Consequently, the *exponential forgetting and resetting algorithm* (EFRA) of [23] provides forgetting while preventing divergence. Unfortunately, EFRA does not include RLS as a special or limiting case.

The contribution of the present paper is to derive a modified RLS (MRLS) variation of [23] that includes RLS without forgetting as a limiting case and can closely approximate RLS with forgetting. Like EFRA, MRLS provides forgetting action while bounding the covariance. An additional benefit of MRLS relative to EFRA is greater simplicity in setting the upper and lower covariance bounds.

The contents of the paper are as follows. Section II reviews recursive least squares. Section III introduces EFRA and shows that it does not have an RLS limit. Section IV gives the modified RLS algorithm and its proof. Section V shows that MRLS can approximate RLS. Section VI gives some results which help simplify MRLS coefficient selection. Finally, Sections VII-IX give examples that illustrate the performance of MRLS, RLS, and EFRA in different scenarios.

3.3 Review of Recursive Least Squares

In this section, we briefly review of recursive least squares (RLS) with forgetting factor λ.

Theorem 3.1. *For all $k \geq 1$, let $\phi(k) \in \mathbb{R}^{p\times n}$ and $y(k) \in \mathbb{R}^p$. Furthermore, let $\theta_0 \in \mathbb{R}^n$, let $P_0 \in \mathbb{R}^{n\times n}$ be positive definite, let $\lambda \in (0, 1]$, and, for all $k \geq 0$, denote the minimizer of the function*

$$J_k(\theta) \triangleq \sum_{i=1}^{k} \lambda^{k-i}(y(i) - \phi(i)\theta)^{\mathrm{T}}(y(i) - \phi(i)\theta) + \lambda^k(\theta - \theta_0)^{\mathrm{T}}P_0^{-1}(\theta - \theta_0) \tag{3.1}$$

by

$$\theta_k \stackrel{\triangle}{=} \operatorname{argmin}_{\theta \in \mathbb{R}^n} J_k(\theta). \tag{3.2}$$

Then, for all $k \geq 1$, θ_k *is given by*

$$\theta_k = \theta_{k-1} + P_{k-1}\phi(k)^{\mathrm{T}}(\lambda I + \phi(k)P_{k-1}\phi(k)^{\mathrm{T}})^{-1}(y(k) - \phi(k)\theta_{k-1}), \tag{3.3}$$

$$P_k = \frac{1}{\lambda}P_{k-1} - \frac{1}{\lambda}P_{k-1}\phi(k)^{\mathrm{T}}\left[\lambda I + \phi(k)P_{k-1}\phi(k)^{\mathrm{T}}\right]^{-1}\phi(k)P_{k-1}. \tag{3.4}$$

It can be seen from (3.4) that the current covariance matrix is given by the sum of a positive-definite matrix and a negative-semidefinite matrix. In the case where $\lambda = 1$, the positive-definite matrix is the previous covariance matrix, and thus the sequence of covariance matrices is nonincreasing with respect to the positive-semidefinite matrix partial ordering. In the case where $\lambda < 1$, the sequence of covariance matrices is not necessarily nonincreasing. By allowing eigenvalues of the covariance matrix to increase, the effect of the forgetting factor is to discount prior information and facilitate future learning.

3.4 Exponential Forgetting and Resetting Algorithm

Although the use of the forgetting factor allows eigenvalues of the covariance to increase and thus facilitate learning, an undesirable side effect is that, in the absence of persistent excitation and in the presence of noise, the covariance may diverge (see Section 3.10). An extension of RLS given by the exponential forgetting and resetting algorithm (EFRA) [23] overcomes this problem.

Theorem 3.1. *Let $\alpha \in (0,1)$, $\gamma \in (0,\alpha)$, $\beta > 0$, and $\delta > 0$, and assume that*

$$(\alpha-\gamma)^2 + 4\beta\delta < (1-\alpha)^2. \tag{3.5}$$

Furthermore, define

$$\sigma \triangleq \frac{\alpha-\gamma}{2\delta}\left(\sqrt{1+\frac{4\beta\delta}{(\alpha-\gamma)^2}}-1\right), \tag{3.6}$$

$$\nu \triangleq \frac{\gamma}{2\delta}\left(1+\sqrt{1+\frac{4\beta\delta}{\gamma^2}}\right), \tag{3.7}$$

let $P_0 \in \mathbb{R}^{n\cdot n}$ be positive semidefinite, and assume that

$$\sigma I \leq P_0 \leq \nu I. \tag{3.8}$$

Furthermore, let $\theta_0 \in \mathbb{R}^n$, for all $k \geq 0$, let $y(k) \in \mathbb{R}$ and $\phi(k) \in \mathbb{R}^{1\times n}$, and consider the update equations

$$\theta_{k+1} = \theta_k + \frac{\alpha}{1+\phi(k)P_k\phi(k)^{\mathrm{T}}}P_k\phi(k)^{\mathrm{T}}(y(k)-\phi(k)\theta_k) \tag{3.9}$$

$$P_{k+1} = (1+\gamma)P_k - \frac{\alpha}{1+\phi(k)P_k\phi(k)^{\mathrm{T}}}P_k\phi(k)\phi(k)^{\mathrm{T}}P_k + \beta I - \delta P_k^2. \tag{3.10}$$

Then the following statements hold:

i) For all $k \geq 0$, $\sigma I \leq P_k \leq \nu I$.

ii) For all $k \geq 0$, $\beta I + \gamma P_k - \delta P_k^2 \geq 0$.

iii) If there exists $k_0 \geq 0$ such that, for all $k \geq k_0$, $\phi(k) = 0$, then $\lim_{k\to\infty} P_k = \nu I$.

Note that (3.10) includes the terms βI and δP_k^2, which do not appear in (3.4). These terms enforce the bounds given by $i)$. Note, in addition, that γ plays the role of $1/\lambda - 1$ in

RLS.

Comparing (3.9), (3.10) with (3.3), (3.4), it can be seen that these results coincide in the case where $\beta = \delta = 0$ and $1/\lambda = 1 + \gamma = \alpha$. Since $0 < \alpha < 1$, it follows that $-1 < \gamma < 0$, which contradicts the assumption that $0 < \gamma < \alpha$. Therefore, there is no choice of parameters α, γ, β, and δ for which RLS is a special or limiting case of EFRA. Finally, for each choice of α and γ, it can be shown that the ratio ν/σ cannot be set arbitrarily. For example, let $\alpha = 1/2$ and $\gamma = 1/4$. From (3.5), it follows that $\beta\delta < 0.046875$. Furthermore, (3.6) and (3.7) imply that

$$\frac{\nu}{\sigma} = \frac{\frac{1}{4} + \sqrt{\frac{1}{16} + 4\beta\delta}}{\sqrt{\frac{1}{16} + 4\beta\delta} - \frac{1}{4}}. \tag{3.11}$$

Thus, for all $\beta > 0$ and $\delta < 0.046875/\beta$, it follows that $\nu/\sigma > 3$.

3.5 Modified RLS

Inspired by EFRA, we now derive modified RLS (MRLS), which, like EFRA, has bounded covariance, but, unlike EFRA, can approximate RLS. Furthermore, attaining an arbitrary choice of covariance bounds is simpler for MRLS than for EFRA.

Theorem 3.1. *Let $\beta \in (0, \infty)$, $\delta \in (0, \infty)$, and $\gamma \in [1, \frac{3}{2})$, assume that*

$$\gamma + 2\beta\delta < \tfrac{3}{2}, \tag{3.12}$$

and define

$$\bar{\alpha} \triangleq \frac{2[\sqrt{(\gamma-1)^2 + 4\beta\delta}(2 - \gamma - \sqrt{(\gamma-1)^2 + 4\beta\delta}) + \gamma - 1]}{1 - (2 - \gamma - \sqrt{(\gamma-1)^2 + 4\beta\delta})^2}. \tag{3.13}$$

Then, $\bar{\alpha} \in (0,1)$. Next, let $\alpha \in (0,\bar{\alpha})$, and define

$$\sigma(\alpha) \triangleq \frac{\gamma - 1 - \alpha + \sqrt{(\gamma - 1 - \alpha)^2 + 4\beta\delta}}{2\delta}. \tag{3.14}$$

Then,

$$0 < \sigma(\alpha) < (1-\alpha)\sigma(0) < \sigma(0). \tag{3.15}$$

Furthermore, let $\theta_0 \in \mathbb{R}^n$ and $P_0 \in \mathbb{R}^{n\times n}$, and assume that P_0 is positive definite and satisfies

$$\sigma(\alpha) I \leq P_0 \leq \sigma(0) I. \tag{3.16}$$

For all $k \geq 0$, let $\varepsilon_k \in (0,\infty)$, $\eta_k \in (0,\infty)$, $y(k) \in \mathbb{R}^p$, and $\phi(k) \in \mathbb{R}^{p\times n}$, and consider the update equations

$$\theta_{k+1} = \theta_k + \eta_k P_k \phi(k)^{\mathrm{T}} [\varepsilon_k I + \phi(k) P_k \phi(k)^{\mathrm{T}}]^{-1} (y(k) - \phi(k)\theta_k), \tag{3.17}$$

$$P_{k+1} = \gamma P_k - \alpha P_k \phi(k)^{\mathrm{T}} [\varepsilon_k I + \phi(k) P_k \phi(k)^{\mathrm{T}}]^{-1} \phi(k) P_k + \beta I - \delta P_k^2. \tag{3.18}$$

Then, for all $k \geq 1$,

$$\sigma(\alpha) I \leq P_k \leq \sigma(0) I. \tag{3.19}$$

Proof. To prove that $\bar{\alpha} \in (0,1)$, note that (3.12) implies that

$$\begin{aligned}(1-\gamma)^2 + 4\beta\delta &= 1 - 2\gamma + \gamma^2 + 4\beta\delta \\ &< 1 - 2\gamma + \gamma^2 + 3 - 2\gamma \\ &= (2-\gamma)^2.\end{aligned}$$

Hence, the numerator of (3.13) is positive. Furthermore, since $1 \leq \gamma < \frac{3}{2}$, it follows that

$$\begin{aligned} 1 &\leq \sqrt{1 + 2(\gamma - 1)(2 - \gamma)} = \sqrt{(2 - \gamma)^2 + (1 - \gamma)^2} \\ &< 2 - \gamma + \sqrt{(1 - \gamma)^2 + 4\beta\delta}. \end{aligned}$$

Since $2 - \gamma - \sqrt{(1 - \gamma)^2 + 4\beta\delta} > 0$ and $2 - \gamma + \sqrt{(1 - \gamma)^2 + 4\beta\delta} > 1$, it follows that

$$\begin{aligned} &2 - \gamma - \sqrt{(1 - \gamma)^2 + 4\beta\delta} \\ &< (2 - \gamma - \sqrt{(1 - \gamma)^2 + 4\beta\delta})(2 - \gamma + \sqrt{(1 - \gamma)^2 + 4\beta\delta}) \\ &= (2 - \gamma)^2 - (1 - \gamma)^2 - 4\beta\delta \\ &= 1 - (2\gamma - 2 + 4\beta\delta) \\ &< 1. \end{aligned}$$

Hence, the denominator of (3.13) is positive and therefore, $\bar{\alpha} > 0$. Furthermore,

$$\begin{aligned} &2[\sqrt{(1 - \gamma)^2 + 4\beta\delta}(2 - \gamma - \sqrt{(1 - \gamma)^2 + 4\beta\delta}) + \gamma - 1] \\ &= -(2 - \gamma - \sqrt{(1 - \gamma)^2 + 4\beta\delta})^2 \\ &+ (2 - \gamma)^2 - (1 - \gamma)^2 - 4\beta\delta + \gamma - 1 \\ &= 1 - 2(\gamma - 1) + (\gamma - 1)^2 \\ &- (\gamma - 1)^2 - 4\beta\delta + \gamma - 1 - (2 - \gamma - \sqrt{(1 - \gamma)^2 + 4\beta\delta})^2 \\ &= 1 - (\gamma - 1) - 4\beta\delta - (2 - \gamma - \sqrt{(1 - \gamma)^2 + 4\beta\delta})^2 \\ &< 1 - (2 - \gamma - \sqrt{(1 - \gamma)^2 + 4\beta\delta})^2. \end{aligned}$$

Hence, $\bar{\alpha} < 1$. Therefore, $\bar{\alpha} \in (0, 1)$.

To show that $\sigma(\alpha) > 0$, note that since $\beta > 0$ and $\delta > 0$ it follows that

$$|\gamma - \alpha - 1| < \sqrt{(\gamma - \alpha - 1)^2 + 4\beta\delta}, \tag{3.20}$$

and thus $\gamma - \alpha - 1 + \sqrt{(\gamma - \alpha - 1)^2 + 4\beta\delta} > 0$. Since $\alpha > 0$ it follows that $\gamma - \alpha - 1 + \sqrt{(\gamma - \alpha - 1)^2 + 4\beta\delta} < \gamma - 1 + \sqrt{(\gamma - 1)^2 + 4\beta\delta}$, hence $\sigma(\alpha) < \sigma(0)$.

To show that $\sigma(\alpha) < (1 - \alpha)\sigma(0)$, define the positive numbers

$$b \stackrel{\triangle}{=} \gamma - 1, \quad c \stackrel{\triangle}{=} 4\beta\delta, \tag{3.21}$$

$$f \stackrel{\triangle}{=} \sqrt{b^2 + c} = \sqrt{(\gamma - 1)^2 + 4\beta\delta}, \tag{3.22}$$

$$g \stackrel{\triangle}{=} 1 - b = 2 - \gamma, \tag{3.23}$$

and note that

$$\bar{\alpha} = \frac{2[f(g - f) + b]}{1 - (g - f)^2}. \tag{3.24}$$

Furthermore, note that, since $\alpha < \bar{\alpha}$ and $g - f < 1$, it follows that $0 < 2[f(g - f) + b] - \alpha[1 - (g - f)^2]$. Hence,

$$\begin{aligned}
0 &< 2\alpha[f(g - f) + b] - \alpha^2[1 - (g - f)^2] \\
&= \alpha^2 g^2 - 2\alpha f^2 + \alpha^2 f^2 + 2\alpha g f - 2\alpha^2 g f - \alpha^2 + 2b\alpha \\
&= \alpha^2 g^2 - 2\alpha f^2 + \alpha^2 f^2 + 2\alpha g f - 2\alpha^2 g f \\
&\quad - \alpha^2 + 2b\alpha + f^2 - f^2 \\
&= \alpha^2 g^2 + (1 - \alpha)^2 f^2 + 2\alpha(1 - \alpha) g f - \alpha^2 + 2b\alpha - f^2 \\
&= [\alpha g + (1 - \alpha) f]^2 - [\alpha^2 - 2b\alpha + f^2] \\
&= [\alpha g + (1 - \alpha) f]^2 - [(b - \alpha)^2 + c] \\
&= [\alpha g + (1 - \alpha) f - \sqrt{(b - \alpha)^2 + c}] \\
&\quad \cdot [\alpha g + (1 - \alpha) f + \sqrt{(b - \alpha)^2 + c}].
\end{aligned}$$

Since $\alpha < 1$, it follows that $\alpha g + (1-\alpha)f + \sqrt{(b-\alpha)^2 + c} > 0$. Therefore,

$$\begin{aligned}
0 &< \alpha g + (1-\alpha)f - \sqrt{(b-\alpha)^2 + c} \\
&= \alpha(1-b) + (1-\alpha)\sqrt{b^2+c} - \sqrt{(b-\alpha)^2+c} + b - b \\
&= (1-\alpha)b + (1-\alpha)\sqrt{b^2+c} - [b - \alpha + \sqrt{(b-\alpha)^2+c}] \\
&= (1-\alpha)[b + \sqrt{b^2+c}] - [b - \alpha + \sqrt{(b-\alpha)^2+c}],
\end{aligned}$$

Hence,

$$\begin{aligned}
\sigma(\alpha) &= \frac{b - \alpha + \sqrt{(b-\alpha)^2+c}}{2\delta} \\
&< (1-\alpha)\frac{b + \sqrt{b^2+c}}{2\delta} = (1-\alpha)\sigma(0).
\end{aligned}$$

Next, let $k \geq 0$, suppose that P_k is positive semidefinite, and define

$$M_k = \begin{bmatrix} P_k & P_k\phi(k)^{\mathrm{T}} \\ \phi(k)P_k & \phi(k)P_k\phi(k)^{\mathrm{T}} \end{bmatrix}. \tag{3.25}$$

Since P_k is positive semidefinite, M_k can be written as

$$M_k = \begin{bmatrix} P_k^{1/2} \\ \phi(k)P_k^{1/2} \end{bmatrix} \begin{bmatrix} P_k^{1/2} & P_k^{1/2}\phi(k)^{\mathrm{T}} \end{bmatrix}, \tag{3.26}$$

and thus M_k is positive semidefinite. Since $\varepsilon_k > 0$, it follows that

$$N_k = M_k + \begin{bmatrix} 0 & 0 \\ 0 & \varepsilon_k I \end{bmatrix} \tag{3.27}$$

is also positive semidefinite. Therefore, since $\phi(k)P_k\phi(k)^{\mathrm{T}} + \varepsilon_k I$ is positive definite, it

follows that the Schur complement of N_k is positive semidefinite, and thus

$$P_k\phi(k)^{\mathrm{T}}[\varepsilon_k + \phi(k)P_k\phi(k)^{\mathrm{T}}]^{-1}\phi(k)P_k \leq P_k. \tag{3.28}$$

Next, we show that, for all $k \geq 1$, $P_k \leq \sigma(0)I$. Let $k \geq 1$. By (3.16), $P_0 \leq \sigma(0)I$. Hence, suppose that $P_{k-1} \leq \sigma(0)I$. Now, define

$$a \triangleq 1 - \sqrt{(1-\gamma)^2 + 4\beta\delta}, \tag{3.29}$$

and note that

$$P_k - \sigma(0)I = a(P_{k-1} - \sigma(0)I) - \delta(P_{k-1} - \sigma(0)I)^2 - \alpha G_{k-1}. \tag{3.30}$$

From (3.12) it follows that

$$(\gamma-1)^2 + 4\beta\delta = \gamma^2 - 4\gamma + 1 + 2\gamma + 4\beta\delta < (\gamma-2)^2 < 1,$$

and thus $\sqrt{(\gamma-1)^2 + 4\beta\delta} < 1$, which implies that $a > 0$. Since a is positive, it follows from $P_{k-1} \leq \sigma(0)I$ and (3.30) that $P_k - \sigma(0)I \leq 0$. Hence, $P_k \leq \sigma(0)I$.

Next, we show that, for all $k \geq 1$, $P_k \geq \sigma(\alpha)I$. Let $k \geq 1$. By assumption, $P_0 \geq \sigma(\alpha)I$. Hence, suppose that $P_{k-1} \geq \sigma(\alpha)I$. Now, define

$$G_{k-1} \triangleq P_{k-1}\phi(k-1)^{\mathrm{T}}[\varepsilon_{k-1}I + \phi(k-1)P_{k-1}\phi(k-1)^{\mathrm{T}}]^{-1}\phi(k-1)P_{k-1}, \tag{3.31}$$

$$f(\mu) \triangleq (\gamma-\alpha)\mu + \beta - \delta\mu^2. \tag{3.32}$$

Since P_{k-1} is positive definite, it follows that $P_{k-1} \geq G_{k-1}$, and thus

$$P_k = \gamma P_{k-1} - \alpha G_{k-1} + \beta I - \delta P_{k-1}^2, \tag{3.33}$$

$$\geq (\gamma - \alpha) P_{k-1} + \beta I - \delta P_{k-1}^2. \tag{3.34}$$

Since $\sigma(\alpha) I \leq P_{k-1} \leq \sigma(0) I$, it follows that $\operatorname{spec}(P_{k-1}) \subset [\sigma(\alpha), \sigma(0)]$, and thus

$$\min_{[\sigma(\alpha),\sigma(0)]} f(\mu) \leq \min_{\operatorname{spec}(P_{k-1})} f(\mu) \leq \lambda_{\min}(P_k), \tag{3.35}$$

where the minimum in (3.35) exists because f is continuous and $[\sigma(\alpha), \sigma(0)]$ is compact. Since f is concave, its unique stationary point is its maximizer, and thus the minimizer of f over $[\sigma(\alpha), \sigma(0)]$ is either $\sigma(\alpha)$ or $\sigma(0)$. Since $f(\sigma(0)) = \sigma(0)$, $f(\sigma(\alpha)) = (1-\alpha)\sigma(\alpha)$, and $\sigma(\alpha) < (1-\alpha)\sigma(0)$, it follows that $f(\sigma(0)) < f(\sigma(\alpha))$ and thus $\sigma(\alpha)$ is the unique minimizer of f over $[\sigma(\alpha), \sigma(0)]$. Therefore, $\lambda_{\min}(P_k) \geq \sigma(\alpha)$, and hence $P_k \geq \sigma(\alpha) I$. $\square$

3.6 Approximation of RLS by MRLS

In this section, we show that, in the case where $\lambda = 1$, MRLS approximates RLS as a limiting case. Although the same statement cannot be made in the case where $\lambda < 1$, we show that the MRLS estimates approximate the RLS estimates in the case where $\lambda \approx 1$.

First, we show that RLS is a limiting case of MRLS in the case where $\lambda = 1$. To see this, note that, for all $\gamma \in (1, \frac{3}{2})$,

$$\lim_{\beta \downarrow 0} \bar{\alpha} = \frac{\gamma^2 - 3\gamma + 2}{\gamma^2 - 3\gamma + 2} = 1. \tag{3.36}$$

Hence, by L'Hôpital's rule,

$$\lim_{\gamma \downarrow 1} \lim_{\beta \downarrow 0} \bar{\alpha} = \lim_{\gamma \downarrow 1} \frac{\gamma^2 - 3\gamma + 2}{\gamma^2 - 3\gamma + 2} = 1. \tag{3.37}$$

Hence, letting $\beta \to 0, \gamma \to 1, \delta \to 0, \alpha \to 1, \varepsilon_k = 1, \eta_k = 1$, and setting $\lambda = \lim_{\gamma \downarrow 1} 1/\gamma = 1$, it follows that (3.17), (3.18) approximate (3.3), (3.4).

Next, let $\gamma \in (1, \frac{3}{2})$. Letting $\beta \to 0$, $\delta \to 0$, $\alpha \to 1$, $\varepsilon_k = 1$, $\eta_k = 1/\gamma$, and setting $\lambda = \lim_{\gamma \downarrow 1} 1/\gamma$, it follows that (3.17), (3.18) are given by

$$\theta_{k+1} = \theta_k + P_k \phi(k)^{\mathrm{T}} [\lambda I + \phi(k) P_k \phi(k)^{\mathrm{T}}]^{-1} (y(k) - \phi(k)\theta_k), \tag{3.38}$$

$$P_{k+1} = \frac{1}{\lambda} P_k - P_k \phi(k)^{\mathrm{T}} [\lambda I + \phi(k) P_k \phi(k)^{\mathrm{T}}]^{-1} \phi(k) P_k. \tag{3.39}$$

Comparing (3.39) and (3.4), we see the only difference is in the second term. For $\lambda \approx 1$, this difference is small, and the examples in Section VIII show that (3.38), (3.39) numerically approximate (3.3), (3.4).

3.7 Coefficient Selection for MRLS

In this section we give a heuristic procedure for choosing MRLS parameters that yield an approximate forgetting factor and specified covariance bounds. The procedure is easy to apply but is not guaranteed to be successful in every case.

Let λ, x, and y be the desired forgetting factor, upper covariance bound, and lower covariance bound, respectively. Next, consider the following steps:

$i)$ Let $\gamma = \frac{1}{\lambda}$.

$ii)$ Choose $\alpha \in (0, 1)$ such that

$$\frac{x}{y} < \frac{\gamma - 1}{\gamma - \alpha - 1}. \tag{3.40}$$

$iii)$ Compute

$$\delta \triangleq \frac{(\gamma-1)(x-y)+\alpha y}{x^2-y^2}, \tag{3.41}$$

$$\beta \triangleq \frac{xy[\alpha x-(\gamma-1)(x-y)]}{x^2-y^2}. \tag{3.42}$$

$iv)$ Compute $\bar{\alpha}$ using (3.13).

$v)$ If $\alpha < \bar{\alpha}$, then the coefficients γ, α, δ, β yield an MRLS filter with $\sigma(0) = x$ and $\sigma(\alpha) = y$, where $\sigma(\alpha)$ is defined by (3.14). If $\alpha > \bar{\alpha}$, then decrease α and repeat steps $ii) - v)$.

$vi)$ Choose ε_k and η_k.

To obtain an estimator that mimics RLS but has bounded covariance, choose $\varepsilon_k = \lambda$ and $\eta_k = 1$. If β is small then $\bar{\alpha} \approx 1$. In this case, let $\alpha \approx 1$ to most effectively approximate RLS.

The following examples illustrate the behavior of MRLS and its relationship to EFRA and RLS.

3.8 Example 1: MRLS Approximation of RLS

Consider the system

$$G(\mathbf{q}) = \frac{\mathbf{q}-\frac{1}{12}}{\mathbf{q}^2-\frac{1}{12}\mathbf{q}-\frac{1}{12}}. \tag{3.43}$$

with input

$$u(k) = 1 + \sin\frac{2\pi k}{10} + \sin\frac{2\pi k}{20} + \sin\frac{2\pi k}{100}. \tag{3.44}$$

Suppose that the measurement of the output is corrupted by noise with standard deviation $\sigma = 0.01$. Let $\lambda = 0.999$, $P_0 = 100$, and $\theta_0 = 0$. Consider the MRLS parameters

$$\begin{aligned} &\alpha = 0.99998, \;\; \gamma = 1.001, \;\; \delta = 10^{-7}, \\ &\beta = 0.001, \;\; \eta = 1, \;\; \varepsilon = 0.999. \end{aligned} \tag{3.45}$$

The covariance bounds corresponding to these parameters are $\sigma(\alpha) = 0.001$ and $\sigma(0) = 1.011 \times 10^4$, which closely approximate RLS. Figure 3.1 shows the similarity between the responses for RLS and MRLS.

Now consider the MRLS parameters

$$\begin{aligned} &\alpha = 0.999998, \;\; \gamma = 1.001, \;\; \delta = 1, \\ &\beta = 0.001, \;\; \eta = 1, \;\; \varepsilon = 0.999. \end{aligned} \tag{3.46}$$

The covariance bounds corresponding to these parameters are $\sigma(\alpha) = 0.001$ and $\sigma(0) = 0.0321$. With these parameters, Figure 3.1 shows the difference between the responses for RLS and MRLS.

3.9 Example 2: Sudden Change of Parameters

Consider the system described by the time-dependent transfer function

$$G_k(\mathbf{q}) = \begin{cases} \frac{\mathbf{q}+0.2}{\mathbf{q}^2-0.6\mathbf{q}+0.08}, & k \leq 50000, \\ \frac{2\mathbf{q}+0.5}{\mathbf{q}^2+0.4\mathbf{q}-0.05}, & k > 50000, \end{cases} \tag{3.47}$$

which has a sudden change of parameters at $k = 50,000$. The driving signal is given by

$$u(k) = 1 + \sin\frac{2\pi k}{10} + \sin\frac{2\pi k}{20} + \sin\frac{2\pi k}{100}. \tag{3.48}$$

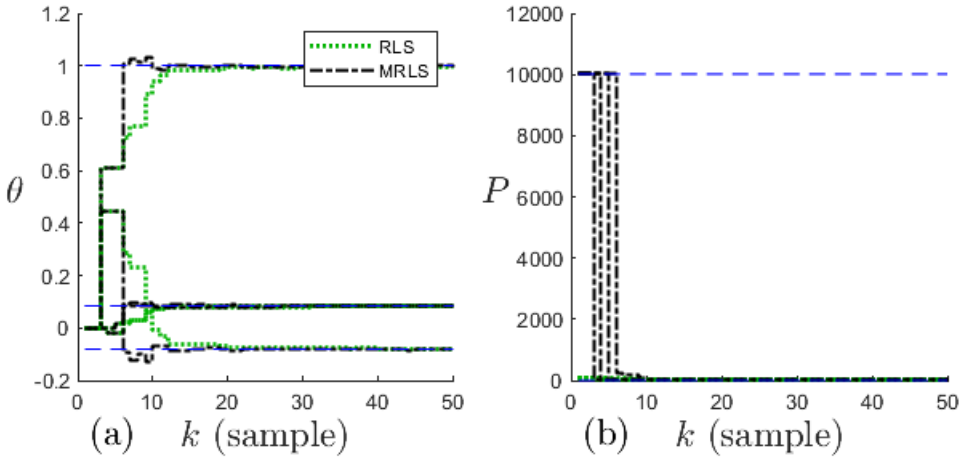

Figure 3.1: Example 1: MRLS with coefficients that closely approximate RLS. (a) shows the parameter estimates and their true values; (b) shows the eigenvalues of the covariance matrix with the MRLS covariance bounds.

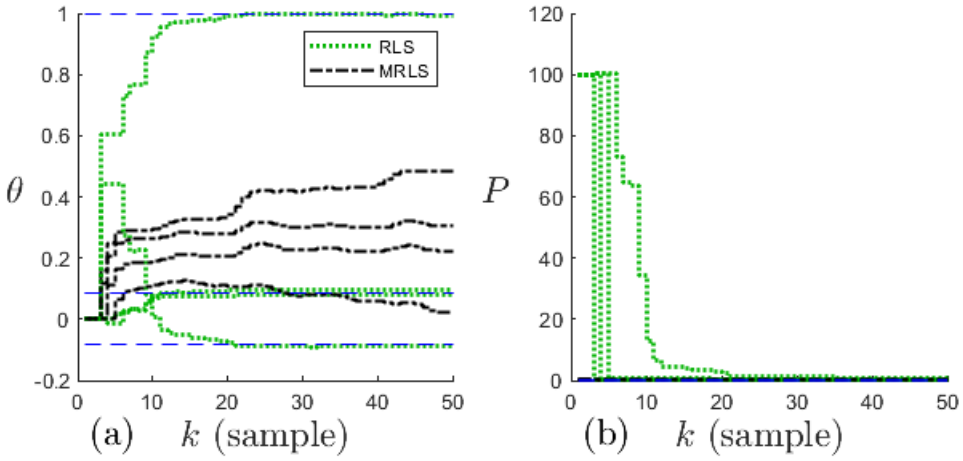

Figure 3.2: Example 1: MRLS with coefficients that differ significantly from RLS. (a) shows the parameter estimates and their true values; (b) shows the eigenvalues of the covariance matrix with the MRLS covariance bounds.

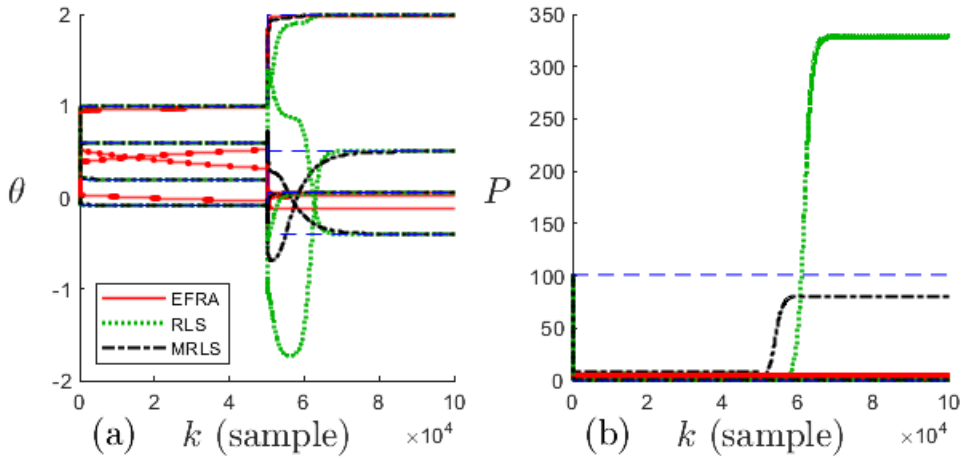

Figure 3.3: Example 2: Sudden change of parameters. (a) shows the parameter estimates and their true values; (b) shows the eigenvalues of the covariance matrix with the EFRA and MRLS covariance bounds.

Suppose that the output measurement is noise free. Let $\lambda = 0.999$, $P_0 = 100$, and $\theta_0 = 0$. Consider the EFRA parameters

$$\alpha = 0.375, \quad \gamma = 0.001, \quad \delta = 0.05, \quad \beta = 1.2525, \tag{3.49}$$

and the MRLS parameters

$$\begin{aligned} &\alpha = 0.991, \quad \gamma = 1.001, \quad \delta = 0.00001, \\ &\beta = 0.001, \quad \eta = 1, \quad \varepsilon = 0.999. \end{aligned} \tag{3.50}$$

Figure 3.3 shows the parameter estimates and covariance eigenvalues for RLS, EFRA, and MRLS, along with the EFRA and MRLS covariance bounds. The choice of constraint on the covariance prevents convergence of EFRA to the true value both before and after the parameter change. In contrast, RLS and MRLS quickly converge to the true parameter values, and then reconverge to the new values after approximately $20,000$ steps.

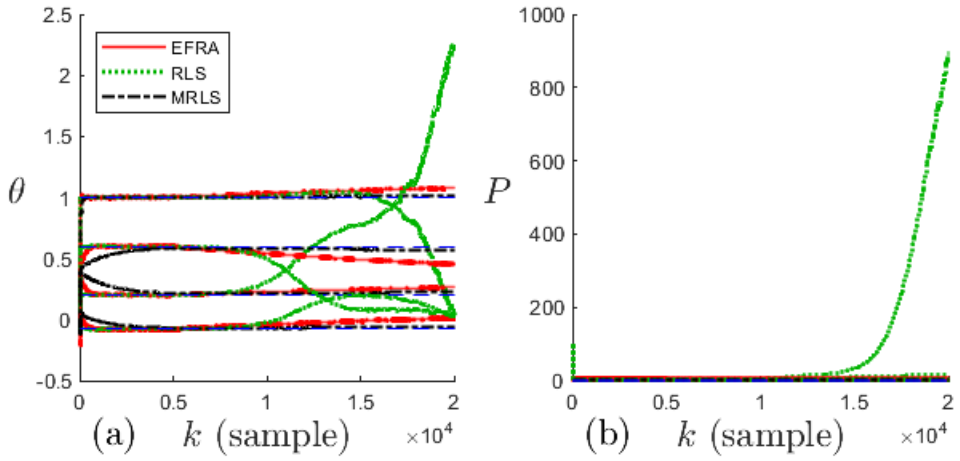

Figure 3.4: Example 3: Loss of persistency at $k = 10,000$. (a) shows the parameter estimates with the true values shown by blue dashed lines; (b) shows the eigenvalues of the covariance matrix with EFRA covariance bounds (red dashed) and MRLS covariance bounds (blue dashed).

3.10 Example 3: Sudden Loss of Persistency with Sensor Noise

Consider the system

$$G(\mathbf{q}) = \frac{\mathbf{q} + 0.2}{\mathbf{q}^2 - 0.6\mathbf{q} + 0.08} \tag{3.51}$$

with the driving signal

$$u(k) = \begin{cases} \nu(k) & k \leq 5000, \\ \sin \frac{k}{10} & k > 5000, \end{cases} \tag{3.52}$$

where, for all $k \geq 1$, $\nu(k) \sim \mathcal{N}(0, 1)$. Note that u suddenly loses persistency at $k = 5000$. Suppose that the measurement of the output is corrupted by noise with standard deviation

$\sigma = 0.01$. Let $\lambda = 0.999$, $P_0 = 100$, and $\theta_0 = 0$. Consider the EFRA parameters

$$\alpha = 0.375, \;\; \gamma = 0.001, \;\; \delta = 0.05, \;\; \beta = 1.2525, \tag{3.53}$$

and the MRLS parameters

$$\alpha = 0.991, \;\; \gamma = 1.001, \;\; \delta = 1, \tag{3.54}$$

$$\beta = 0.001, \;\; \eta = 1, \;\; \varepsilon = 0.999. \tag{3.55}$$

Figure 3.4 shows the parameter estimates and covariance eigenvalues for RLS, EFRA, and MRLS, along with the EFRA and MRLS covariance bounds. When the driving signal loses persistency at $k = 5000$, the RLS covariance diverges, and the RLS parameter estimates diverge from the true parameter values. Since EFRA and MRLS both have bounded covariance, neither can diverge. The MRLS covariance is bounded to be close to zero, causing a lag in the initial estimate convergence, but also maintaing the estimates close to the converged values, even when persistency is lost.

3.11 Conclusions

In this paper we derived a modified recursive least squares (MRLS) algorithm with forgetting and bounded covariance. Unlike EFRA [23], it is possible to select the parameters of MRLS such that MRLS approximates RLS as a limiting case in the absence of forgetting, and approximately in the case of forgetting. In addition, the simpler constraints of MRLS enable a straightforward process for choosing the MRLS parameters that yield specified covariance bounds.

Examples were given to compare the performance of RLS, MRLS, and EFRA. These examples showed that, by suitably choosing the covariance bounds, it is possible to use MRLS effectively in various situations, including sudden changes in parameters and lack

of persistency.

The derivation of MRLS raises questions for future research. The highest priority is to investigate whether or not there is a bounded covariance algorithm that exactly yields RLS with forgetting as a limiting case. Another question is whether or not analytical bounds can be found for the error between RLS with forgetting and the closest MRLS approximation to RLS. The numerical examples in Section VII suggest that these bounds are tight.

CHAPTER 4

Covergence and Consistency Recursive Least Squares with Variable-Rate Forgetting

4.1 Preface

The bounded-covariance algorithms EFRA and MRLS generalized the number of parameters in the covariance update, including those corresponding to the constant forgetting factor, but these parameters themselves remained constant during operation.

Another approach advocated in the literature, such as [16, 59], is to instead make the forgetting factor time-varying. A brief section on variable forgetting factors appeared in the 1987 text of [26]. However, time-varying forgetting, which we will refer to as *variable-rate forgetting* (VRF), was not largely adopted in system identification applications, even after these works appeared. One major reason for this lack of adoption was the complexity that a *variable* forgetting factor introduces into the covariance update. Now we deal not only with a nonlinear vector-matrix difference equation, but with a time-varying nonlinear equation. The lack of theoretical guarantees–indeed, any theory at all–made VRF a frightening prospect for applications. Although several variable forgetting formulas existed in the literature, they were not connected by any unifying analysis, and the questions of which one to choose and how to use it were not considered.

The following paper, published in Automatica as [60], presented the first thorough analysis of convergence and convergence-rate for VRF, as well as the first detailed analysis of

consistency. Surprisingly, under the assumption of persistency of excitation, convergence occurs *regardless of the particular value of the forgetting factor*![1]. The rate of convergence increases as the amount of forgetting increases, which is consistent with earlier results, such as those found in [24], that for persistent regressors, constant-rate forgetting is exponentially stable while ordinary RLS is only asymptotically stable (linear convergence rates being strictly possible).

Consistency is more difficult. In general, it can be shown that the asymptotic covariance is bounded by functions of the cumulative weight due to variable-rate forgetting, but no widely general sufficient condition can be given at present. Results for more restricted cases are proven, however, such as the case where the infinite product of forgetting factors converges. Convergence is not necessary, however, since the forgetting law $\beta_k = 1 + 1/k$, whose infinite product diverges, is also consistent.

Finally, and most important for practical applications, was the discovery of a new kind of forgetting formula. This is a formula that is "error-actuated" in the sense that when some measure of the residual is large, the forgetting factor increases–since such a misfit is taken to indicate parameters that poorly predict model performance and are thus likely to be spurious or obsolete. Although the simpleest form is actuated using the instantaneous error, forms of the forgetting formula using averaging and statistical weighting, such as those in [61–63] or [41–43], can be used in situations where noise is present in the calculation of the residual.

This work presents the basis of our solution to the abrupt change of parameters problem, which is developed further in "Recursive least squares with Matrix Forgetting", and provided the basis for the work in "Necessary and Sufficient Regressor Conditions for the Global Asymptotic Stability of Recursive Least Squares" and "Sequential Gradient-Descent Optimization of Data-Dependent, Rank-Deficient Cost Functions with a Unique Common Global Minimizer". The latter two, while not directly involving VRF emerge

[1] so long as the cumulative weight is non-decreasing, which is equivalent to the assumption that the forgetting factor is actually *forgetting*.

from the questions of 1) given the importance and ubiquity of persistency, is it necessary for RLS convergence? and 2) given the importance forgetting, is it still possible to identify the system parameters when only the most recent measurement is used in the update?

4.2 Introduction

Recursive least squares (RLS) is one of the foundational algorithms of systems and control theory, especially for signal processing, identification, and adaptive control [49,64]. An early exposition of RLS is given in [65].

Standard RLS employs a constant forgetting factor λ, which enhances the importance of recent data over older data. Although λ can be set by the user, the performance of RLS is often extremely sensitive to the chosen value. Consequently, choosing a suitable value of λ is typically a trial and error process.

To remedy this problem, various techniques have been proposed to automatically vary the forgetting factor in response to the fit error. In particular, [16] reports a method for sequentially updating the forgetting factor to conserve the amount information used in the estimate, and [59] reports an update-based algorithm that uses noise statistics to control the forgetting factor. [66] gives a gradient-based algorithm for computing a forgetting factor that locally minimizes the mean-square error of the estimate, and [67] derives a Newton-type gradient-descent algorithm that combines sequential estimation with minimization of the mean-squared error. Finally, [68] gives a formula based on exponentiation of the squared residual.

The present paper approaches the problem of varying the forgetting factor by deriving a generalization of RLS that includes time-dependent cost scaling and regularization. This formulation involves a growing-window cost function, and thus is distinct from the formulation of [69], which uses a sliding-window cost function. The growing-window cost function is advantageous since it directly generalizes traditional RLS and has the ability to

weigh recent data more heavily than older data.

The first contribution of the paper is given by Theorem 1, which introduces RLS with variable-rate forgetting (VRF), a novel extension of RLS in which the role of the constant forgetting factor λ in RLS is replaced by a variable forgetting factor β_k. By setting $\beta_k = \frac{1}{\lambda}$ for all k, VRF specializes to RLS with constant-rate-forgetting (CRF). The variable-rate-forgetting extensions of RLS given in [16, 59, 66–68] are special cases of Theorem 3 with specific choices of β_k. In addition, Theorem 1 refines the variable-rate weighting used in [64, pp. 17, 18]. In particular, we factor α_k in [64, Eq. (2.12)] as $\beta_k \cdots \beta_0$, where $1/\beta_k$ serves as the instantaneous forgetting factor at step k. This formulation allows the user to specify β_k at each step based on the current residual or knowledge of system changes. The second and third contributions of this paper are given by Theorems 4, 6, and Corollary 4.1, which prove conditions on β_k ensuring convergence under the assumption of persistency (Theorem 4) and consistency under the assumption of persistency and that the regressor and sensor noise are uncorrelated (Theorem 6, Corollary 4.1). Specific examples of β_k for consistent and non-consistent algorithms are given in Corollary 4.2. The fourth contribution is two choices of β_k that may be useful in practice. In Section 4.7, we demonstrate these choices on an abruptly changing system with and without measurement noise and compare the performance of VRF and CRF for the given example.

The notation used throughout this paper is as follows. The symbols $\mathbb{S}^n$, $\mathbb{N}^n$, and $\mathbb{P}^n$ denote the sets of real $n \times n$ symmetric, positive-semidefinite, and positive-definite matrices, respectively. For all $A \in \mathbb{S}^n$, $\lambda_i(A)$ denotes the ith largest eigenvalue of A, $\lambda_{\max}(A) \triangleq \lambda_1(A)$, and $\lambda_{\min}(A) \triangleq \lambda_n(A)$. $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x \in \mathbb{R}$. If X is a set, then the notation $(x_k)_{k\geq 0} \subset X$ indicates that $(x_k)_{k\geq 0}$ is a sequence in X. If $(a_k)_{k\geq 0}, (b_k)_{k\geq 0} \subset \mathbb{R}$, then the notation $a_k \sim \mathcal{O}(b_k)$ indicates that there exists $M > 0$ and $K \geq 0$ such that, for all $k \geq K$, $a_k \leq M b_k$. Finally, for all $k \geq 0$ and $N \geq 0$, we define $\xi(k, N) \triangleq \lfloor \frac{k}{N+1} \rfloor$.

4.3 Problem Formulation

Let $\lambda \in (0, 1]$, $\theta_0 \in \mathbb{R}^n$, and $P_0 \in \mathbb{P}^n$. Furthermore, for all $k \geq 0$, let $\phi_k \in \mathbb{R}^{p\times n}$, $y_k \in \mathbb{R}^p$, $e_k \triangleq y_k - \phi_k\theta$ and define $J_k\colon \mathbb{R}^n \to [0, \infty)$ by

$$J_k(\theta) \triangleq \sum_{i=0}^{k} \lambda^{k-i}\|e_k\|^2 + \lambda^{k+1}(\theta - \theta_0)^{\mathrm{T}} P_0^{-1}(\theta - \theta_0). \tag{4.1}$$

Equation (4.1) is the cost function for CRF, the minimization of which produces the least squares estimate of θ given $y_0, \ldots, y_k$. Since J_k is quadratic and strictly convex, it follows that its unique global minimizer, $\theta_{k+1} \triangleq \operatorname{argmin}_{\theta\in\mathbb{R}^n} J_k(\theta)$, is the only local minimizer. The following proposition gives the traditional RLS update equations for computing θ_{k+1} [**?**, 49, 64].

Proposition 1. *Under the notation and assumptions of the preceding paragraph, for all $k \geq 0$, define $J_k\colon \mathbb{R}^n \to [0, \infty)$ by* (4.1)*. Then*

$$\theta_{k+1} = \theta_k + P_{k+1}\phi_k^{\mathrm{T}}(y_k - \phi_k\theta_k), \tag{4.2}$$

where

$$P_{k+1} = \frac{1}{\lambda}P_k - \frac{1}{\lambda}P_k\phi_k^{\mathrm{T}}(\lambda I_p + \phi_k P_k \phi_k^{\mathrm{T}})^{-1}\phi_k P_k. \tag{4.3}$$

In this paper, we introduce a generalization of (4.1) in which the forgetting factor is variable, prove a result analogous to Proposition 1 for the generalization, and analyze convergence and consistency for the family of algorithms thus obtained. To generalize (4.1), for all $k \geq 0$, let $\beta_k > 0$, define

$$\rho_k \triangleq \prod_{i=0}^{k} \beta_i, \quad \rho_{-1} \triangleq 1, \tag{4.4}$$

and define the cost function $J_k \colon \mathbb{R}^n \to [0, \infty)$ by

$$J_k(\theta) \triangleq \sum_{i=0}^{k} \frac{\rho_i}{\rho_k} \|e_k\|^2 + \frac{1}{\rho_k} (\theta - \theta_0)^{\mathrm{T}} P_0^{-1} (\theta - \theta_0). \tag{4.5}$$

Since (4.5) is quadratic and strictly convex, like (4.1), its unique global minimizer is the only local minimizer. Theorem 3 provides recursive update equations for this minimizer.

4.4 RLS with Variable-Rate Forgetting

Note that (4.5) can be written as

$$J_k(\theta) = \theta^{\mathrm{T}} A_k \theta - 2 b_k^{\mathrm{T}} \theta + c_k, \tag{4.6}$$

where

$$A_k \triangleq \sum_{i=0}^{k} \frac{\rho_i}{\rho_k} \phi_i^{\mathrm{T}} \phi_i + \frac{1}{\rho_k} P_0^{-1}, \tag{4.7}$$

$$b_k \triangleq \sum_{i=0}^{k} \frac{\rho_i}{\rho_k} \phi_i^{\mathrm{T}} y_i + \frac{1}{\rho_k} P_0^{-1} \theta_0, \tag{4.8}$$

$$c_k \triangleq \sum_{i=0}^{k} \frac{\rho_i}{\rho_k} y_i^{\mathrm{T}} y_i + \frac{1}{\rho_k} \theta_0^{\mathrm{T}} P_0^{-1} \theta_0. \tag{4.9}$$

Since A_k is positive definite, we define the positive-definite matrix

$$P_k \triangleq A_{k-1}^{-1}, \tag{4.10}$$

where $A_{-1} \triangleq P_0^{-1}$.

The following result, *RLS with variable-rate forgetting (VRF)*, generalizes Proposition 1 to the minimizer of (4.5).

Theorem 3. *Let $\theta_0 \in \mathbb{R}^n$, $P_0 \in \mathbb{P}^n$, and, for all $k \geq 0$, let $\phi_k \in \mathbb{R}^{p \times n}$, $y_k \in \mathbb{R}^p$, and*

$\beta_k \in (0, \infty)$. *Then the minimizer* θ_{k+1} *of* (4.5) *is given by*

$$\theta_{k+1} = \theta_k + P_{k+1}\phi_k^{\mathrm{T}}(y_k - \phi_k\theta_k), \tag{4.11}$$

and

$$P_{k+1} = L_k - L_k\phi_k^{\mathrm{T}}(I_p + \phi_k L_k \phi_k^{\mathrm{T}})^{-1}\phi_k L_k, \tag{4.12}$$

$$L_k \stackrel{\triangle}{=} \beta_k P_k. \tag{4.13}$$

The proof of Theorem 3 requires the following lemma.

Lemma 1. *Let* $P_0 \in \mathbb{P}^n$ *and, for all* $k \geq 0$*, let* $\beta_k > 0$*, define* ρ_k *by* (4.4)*, and define* P_k *by* (4.10)*. Then, for all* $k \geq 0$*,*

$$P_{k+1}^{-1} = \frac{1}{\beta_k}P_k^{-1} + \phi_k^{\mathrm{T}}\phi_k \tag{4.14}$$

$$= \frac{1}{\rho_k}\left(P_0^{-1} + \sum_{i=0}^{k}\rho_i\phi_i^{\mathrm{T}}\phi_i\right). \tag{4.15}$$

Proof. Let $k \geq 0$. It follows from (4.7) that $A_k = \frac{1}{\beta_k}A_{k-1} + \phi_k^{\mathrm{T}}\phi_k$, which, using (4.10), implies (4.14). Furthermore, (4.14) implies $P_1^{-1} = \frac{1}{\rho_0}(P_0^{-1} + \rho_0\phi_0^{\mathrm{T}}\phi_0)$, which confirms (4.15) for $k = 0$. Next, let $k > 0$ and suppose for induction that (4.15) holds for $k-1$. From (4.14) it follows that $P_{k+1}^{-1} = \frac{1}{\beta_k}P_k^{-1} + \phi_k^{\mathrm{T}}\phi_k = \frac{1}{\rho_k}\left(P_0^{-1} + \sum_{i=0}^{k-1}\rho_i\phi_i^{\mathrm{T}}\phi_i\right) + \frac{\rho_k}{\rho_k}\phi_k^{\mathrm{T}}\phi_k = \frac{1}{\rho_k}\left(P_0^{-1} + \sum_{i=0}^{k}\rho_i\phi_i^{\mathrm{T}}\phi_i\right)$. □

Proof of Theorem 3. Let $k \geq 0$. To prove (4.12), note that it follows from (4.13), (4.14), and the matrix inversion lemma that

$$P_{k+1} = \left(\frac{1}{\beta_k}P_k^{-1} + \phi_k^{\mathrm{T}}\phi_k\right)^{-1} = L_k - L_k\phi_k^{\mathrm{T}}\left(I_p + \phi_k L_k\phi_k^{\mathrm{T}}\right)^{-1}\phi_k L_k.$$

To prove (4.11), note that (4.8), (4.10), and (4.14) imply that

$$\begin{aligned}
\theta_{k+1} &= P_{k+1}\left(\sum_{i=0}^{k}\frac{\rho_i}{\rho_k}\phi_i^{\mathrm{T}}y_i + \frac{1}{\rho_k}P_0^{-1}\theta_0\right)\\
&= P_{k+1}\left(\phi_k^{\mathrm{T}}y_k + \frac{\rho_{k-1}}{\rho_k}\left[\sum_{i=0}^{k-1}\frac{\rho_i}{\rho_{k-1}}\phi_i^{\mathrm{T}}y_i + \frac{1}{\rho_{k-1}}P_0^{-1}\theta_0\right]\right)\\
&= P_{k+1}\left(\phi_k^{\mathrm{T}}\phi_k + \frac{1}{\beta_k}P_k^{-1}\right)\theta_k + P_{k+1}\phi_k^{\mathrm{T}}(y_k - \phi_k\theta_k)\\
&= \theta_k + P_{k+1}\phi_k^{\mathrm{T}}(y_k - \phi_k\theta_k), \qquad \square
\end{aligned}$$

For all $k \geq 0$, let $\beta_k = \frac{1}{\lambda}$. Then (4.5) specializes to (4.1), and (4.11)–(4.13) specialize to (4.3) and (4.2). Theorem 3 thus includes Proposition 1 as a special case.

4.5 Convergence of VRF

4.5.1 Asymptotic Convergence

Definition 3. *A sequence* $(S_k)_{k\geq 0} \subset \mathbb{N}^n$ *is* persistent *if there exist* $N \geq 1$ *and* $\alpha > 0$ *such that, for all* $j \geq 0$*,*

$$\alpha I_n \leq \sum_{i=0}^{N} S_{i+j}. \tag{4.16}$$

The numbers α *and* N *are, respectively, the* lower bound *and* persistency window *of* $(S_k)_{k\geq 0}$*. The sequence* $(\phi_k)_{k\geq 0} \subset \mathbb{R}^{n\times m}$ *is* persistent *if* $(\phi_k^{\mathrm{T}}\phi_k)_{k\geq 0}$ *is persistent.*

Theorem 4. *Let* $(\phi_k)_{k\geq 0} \subset \mathbb{R}^{n\times m}$*, be persistent, let* $\theta \in \mathbb{R}^n$*, and, for all* $k \geq 0$*, let* $y_k = \phi_k\theta$*. Furthermore, let* $a > 1$ *and, for all* $k \geq 0$*, let* $\beta_k \geq 1$*. Finally, let* $\theta_0 \in \mathbb{R}^n$*, let* $P_0 \in \mathbb{P}^n$*, and, for all* $k \geq 0$*, define* θ_{k+1} *by* (4.11)–(4.13)*. Then* $\lim_{k\to\infty}\theta_k = \theta$*.*

Proof. Let $k \geq 0$ and define $\tilde{\theta}_k \triangleq \theta_k - \theta$. Using (4.11) and (4.14) it follows that $\tilde{\theta}_{k+1} =$

$(I_n - P_{k+1}\phi_k^{\mathrm{T}}\phi_k)\tilde{\theta}_k = \frac{1}{\beta_k}P_{k+1}P_k^{-1}\tilde{\theta}_k$, thus $\tilde{\theta}_k = \frac{1}{\rho_{k-1}}P_kP_0^{-1}\tilde{\theta}_0$. From (4.15), it follows that

$$\begin{aligned}
\overline{\lim}_{k\to\infty}\|\tilde{\theta}_k\|^2 &\le \overline{\lim}_{k\to\infty}\frac{\lambda_{\max}(P_k^2)}{\rho_{k-1}^2}\|P_0^{-1}\theta_0\|^2\\
&\le \overline{\lim}_{k\to\infty}\frac{\|P_0^{-1}\theta_0\|^2}{\lambda_{\max}^2\left(P_0^{-1}+\sum_{i=0}^{k-1}\rho_i\phi_i^{\mathrm{T}}\phi_i\right)}\\
&\le \overline{\lim}_{k\to\infty}\frac{\|P_0^{-1}\theta_0\|^2}{[\lambda_{\max}(P_0^{-1})+\xi(k,N+1)\alpha]^2} = 0. \qquad \square
\end{aligned}$$

4.5.2 Convergence Rate

Definition 4. *Let $(S_i)_{i\ge0}\subset\mathbb{N}^n$ be persistent with lower bound α and window N. Then the* upper bound $\beta\in(0,\infty)\cup\{\infty\}$ *of $(S_i)_{i\ge0}$ is*

$$\beta \triangleq \sup_{j\ge0}\lambda_{\max}\left(\sum_{i=0}^{N}S_{i+j}\right). \tag{4.17}$$

Lemma 2. *Let $(S_i)_{i\ge0}\subset\mathbb{N}^n$ be persistent with window N, lower bound α, and upper bound β, and let $(a_i)_{i\ge0}$ be a nondecreasing sequence of nonnegative numbers. Then, for all $k\ge0$,*

$$\alpha\ell_{\xi(k,N)-1}I_n \le \sum_{i=0}^{k}a_iS_i \le \beta r_{\xi(k,N)}I_n, \tag{4.18}$$

where $\ell_j \triangleq \sum_{i=0}^{j}a_{i(N+1)}$ and $r_j \triangleq \sum_{i=0}^{j}a_{i(N+1)+N}$.

Proof. In the case where $\beta=\infty$, the upper bound of (4.18) is immediate. Hence, assume $\beta<\infty$. Let $k\ge0$. Since $(a_i)_{i\ge0}$ is nondecreasing, for all $j\ge0$ and $i\in\{0,\dots,N\}$, $a_{i+j}\le a_{N+j}$ and $a_j\le a_{i+j}$. From (4.16) and (4.17) it follows that $\alpha a_jI_n \le a_j\sum_{i=0}^{N}S_{i+j}\le\sum_{i=0}^{N}a_{i+j}S_{i+j}$, thus

$$\alpha\ell_{\xi(k,N)-1}I_n \le \sum_{q=0}^{\xi(k,N)-1}\sum_{i=0}^{N}a_{i+q(N+1)}S_{i+q(N+1)} \le \sum_{i=0}^{k}a_iS_i.$$

Similarly, $\sum_{i=0}^{N} a_{i+j} S_{i+j} \leq a_{N+j} \sum_{i=0}^{N} S_{i+j} \leq a_{N+j} \beta I_n$, and thus

$$\sum_{i=0}^{k} a_i S_i \leq \sum_{q=0}^{\xi(k,N)-1} a_{q(N+1)+N} \beta I_n + a_k \beta I_n \leq \beta r_{\xi(k,N)} I_n. \qquad \square$$

Theorem 5. *Under the assumptions and notation of Theorem 4,*

$$\|\tilde{\theta}_k\| \sim \mathcal{O}\left(1 / \sum_{i=0}^{\xi(k,N)-1} \rho_{i(N+1)}\right). \tag{4.19}$$

Proof. Let $M = \|P_0^{-1}\tilde{\theta}_0\|/\alpha$. From Lemma 2, it follows that, for all $k \geq 0$, $\lambda_{\min}\left(P_0^{-1}\right) + \alpha \sum_{i=0}^{\xi(k,N)-1} \rho_{i(N+1)} \leq \lambda_{\min}\left(P_0^{-1} + \sum_{i=0}^{k} \rho_i \phi_k^{\mathrm{T}} \phi_k\right)$, and therefore, for all $k \geq 0$,

$$\begin{aligned}
\|\tilde{\theta}_k\| &\leq \|P_k\| \|P_0^{-1}\tilde{\theta}_0\| \leq \frac{\|P_0^{-1}\tilde{\theta}_0\|}{\lambda_{\min}\|P_k^{-1}\|} \\
&= \frac{\|P_0^{-1}\tilde{\theta}_0\|}{\lambda_{\min}\left(P_0^{-1}\right)} \left(1 + \frac{\alpha}{\lambda_{\min}\left(P_0^{-1}\right)} \sum_{i=0}^{\xi(k,N)-1} \rho_{i(N+1)}\right)^{-1} \\
&\leq M \left(\sum_{i=0}^{\xi(k,N)-1} \rho_{i(N+1)}\right)^{-1}. \qquad \square
\end{aligned}$$

The following corollary shows that Theorem 5 can be used to prove convergence rates for RLS without forgetting as well as with CRF.

Corollary 4.1. *Under the assumptions and notation of Theorem 4, assume that there exists* $\gamma \in [1, \infty)$ *such that, for all* $k \geq 0$, $\beta_k = \gamma$. *Then*

$$\|\tilde{\theta}_k\| \sim \begin{cases} \mathcal{O}(1/\xi(k,N)), & \gamma = 1, \\ \mathcal{O}(\gamma^{-(N+1)\xi(k,N)}), & \gamma > 1, \end{cases} \tag{4.20}$$

Proof. In the case where $\gamma = 1$, (4.20) is immediate from Theorem 5. Hence, suppose that $\gamma > 1$. From Theorem 5, it follows that there exist $M_0 > 0$ and $K_0 \geq 0$ such that, for all $k \geq K$, $\|\tilde{\theta}_k\| \leq M_0 / \sum_{i=0}^{\xi(k,N)-1} \rho_{i(N+1)}$. Let $M = M_0 \gamma^N$ and $K = \max(K_0, N+1)$.

Therefore, since, for all $k \geq 0$, $\rho_{k(N+1)} = \gamma^{k(N+1)+1}$, it follows that, for all $k \geq K$,

$$\begin{aligned}\|\tilde{\theta}_k\| &\leq \frac{M_0}{\sum_{i=0}^{\xi(k,N)-1} \gamma^{i(N+1)+1}} = \frac{M_0}{\gamma} \frac{\gamma^{N+1}-1}{\gamma^{(N+1)\xi(k,N)}-1} \\ &= \frac{M_0}{\gamma} \frac{\gamma^{N+1}-1}{\gamma^{(N+1)\xi(k,N)}-1} \frac{\gamma^{(N+1)\xi(k,N)}}{\gamma^{(N+1)\xi(k,N)}} \leq \frac{M}{\gamma^{(N+1)\xi(k,N)}}. \end{aligned} \qquad \square$$

Since generally, for all $k \geq 0$, $\beta_k \geq 1$, it follows that $1/\sum_{i=0}^{\xi(k,N)-1} \rho_{i(N+1)} \leq 1/\xi(k,N)$. Thus, this analysis suggests that, in the case where $\beta_k > 1$ for an infinite set of indices, the asymptotic convergence rate of VRF is faster than the asymptotic convergence rate of RLS without forgetting.

4.6 Consistency of VRF

A sequence $(X_k)_{k\geq 0}$ of vector-valued random variables on Ω is a *consistent estimator* of $\theta \in \mathbb{R}^n$ if, for all $\varepsilon > 0$,

$$\lim_{k\to\infty} \mathbb{P}(\{\omega \in \Omega\colon \|X_k(\omega) - \theta\| < \varepsilon\}) = 1. \tag{4.21}$$

When θ is understood, for brevity, we call such sequences *consistent*. In this section we give conditions on β_k which are necessary and sufficient for the consistency of VRF when the measurements of $\phi_k\theta$ are corrupted by noise.

Theorem 6. *Let $(\phi_k)_{k\geq 0}$ be a persistently exciting sequence with window N, lower bound α, and upper bound $\beta < \infty$. Let $\theta \in \mathbb{R}^n$, $P_0 \in \mathbb{P}^n$, and $\theta_0 \sim \mathcal{N}(\theta, P_0)$. Let $(\nu_k)_{k\geq 0}$ be an $\mathbb{R}^p$-valued stationary Gaussian white-noise process with variance V and uncorrelated with θ_0, and define $y_k = \phi_k\theta + \nu_k$. Furthermore, for all $k \geq 0$, let $\beta_k \geq 1$, and define θ_{k+1}*

by (4.11)–(4.13)*. Then, for all* $k \geq 0$, θ_k *is a Gaussian random variable with mean* $\bar{\theta}$*. Then*

$$\frac{\alpha\lambda_{\min}(V)}{\beta^2}\underline{\lim}_{k\to\infty}\frac{q_{\mathrm{l},\xi(k,N)}}{s^2_{\mathrm{u},\xi(k,N)}} \leq \underline{\lim}_{k\to\infty}\lambda_{\min}(\mathrm{var}(\theta_k)) \tag{4.22}$$

$$\leq \overline{\lim}_{k\to\infty}\lambda_{\max}(\mathrm{var}(\theta_k)) \leq \frac{\beta\lambda_{\max}(V)}{\alpha^2}\overline{\lim}_{k\to\infty}\frac{q_{\mathrm{u},\xi(k,N)}}{s^2_{\mathrm{l},\xi(k,N)}}, \tag{4.23}$$

where, for all $j \geq 0$, $s_{\mathrm{l},j} \triangleq \sum_{i=0}^{j-1}\rho_{i(N+1)}$, $\quad s_{\mathrm{u},j} \triangleq \sum_{i=0}^{j}\rho_{i(N+1)+N}$, $q_{\mathrm{l},j} \triangleq \sum_{i=0}^{j-1}\rho^2_{i(N+1)}$, *and* $q_{\mathrm{u},j} \triangleq \sum_{i=0}^{j}\rho^2_{i(N+1)+N}$.

Proof. With base case $\theta_0 \sim \mathcal{N}(\theta, P_0)$, suppose for induction that $\theta_k \sim \mathcal{N}(\theta, \mathrm{var}(\theta_k))$. Define $\tilde{\theta}_k \triangleq \theta_k - \theta$. From (4.11), it follows that $\tilde{\theta}_{k+1} = \beta_k^{-1}P_{k+1}P_k^{-1}\tilde{\theta}_k + P_{k+1}\phi_k^{\mathrm{T}}\nu_k$. Since $\theta_k \sim \mathcal{N}(\theta, \mathrm{var}(\theta_k))$, it follows from Lemma 3 that $\tilde{\theta}_k \sim \mathcal{N}(0, \mathrm{var}(\theta_k))$. Next, define $z_k \triangleq P_k^{-1}\tilde{\theta}_k$. Since $\tilde{\theta}_k \sim \mathcal{N}(0, \mathrm{var}(\theta_k))$, it follows from Lemma 3 that $z_k \sim \mathcal{N}(0, P_k^{-1}\mathrm{var}(\theta_k)P_k^{-1})$. Since ν_k is uncorrelated with $\nu_0, \ldots, \nu_{k-1}, \theta_0$, it follows that ν_k and z_k are also uncorrelated. Furthermore, $z_{k+1} = \beta_k^{-1}z_k + \phi_k^{\mathrm{T}}\nu_k$, and thus $[z_k\ \nu_k]^{\mathrm{T}} \sim \mathcal{N}(0_{2\times 1}, \mathrm{diag}(\mathrm{var}(z_k), V))$. Therefore, Lemma 3 implies that $z_{k+1} \sim \mathcal{N}(0, \mathrm{var}(z_{k+1}))$ and $\mathrm{var}(z_{k+1}) = \beta_k^{-2}\mathrm{var}(z_k) + \phi_k^{\mathrm{T}}V\phi_k$. Since $\theta_{k+1} = P_{k+1}z_{k+1} + \theta$, it follows from Lemma 3 that $\theta_{k+1} \sim \mathcal{N}(\theta, P_{k+1}\mathrm{var}(z_{k+1})P_{k+1})$. Thus, for all $k \geq 0$, θ_k is a Gaussian random variable with mean θ. Since $\mathrm{var}(z_0) = P_0^{-1}P_0P_0^{-1} = P_0^{-1}$, it follows that $\mathrm{var}(z_{k+1}) = \rho_k^{-2}\left(P_0^{-1} + \sum_{i=0}^{k}\rho_i^2\phi_i^{\mathrm{T}}V\phi_i\right)$. For convenience, define $M_k \triangleq \sum_{i=0}^{k}\rho_i\phi_i^{\mathrm{T}}\phi_i$, $M_{\nu,k} \triangleq \sum_{i=0}^{k}\rho_i^2\phi_i^{\mathrm{T}}V\phi_i$, $H_{0,k} \triangleq (P_0^{-1} + M_k)^{-1}P_0^{-1}(P_0^{-1} + M_k)^{-1}$, $H_{\nu,k} \triangleq (P_0^{-1} + M_k)^{-1}M_{\nu,k}(P_0^{-1} + M_k)^{-1}$. For all $k \geq 0$, it follows from Lemma 2 that

$$\alpha s_{\mathrm{l},\xi(k,N)}I_n \leq M_k \leq \beta s_{\mathrm{u},\xi(k,N)}I_n, \tag{4.24}$$

$$\alpha\lambda_{\min}(V)q_{\mathrm{l},\xi(k,N)}I_n \leq M_{\nu,k} \leq \beta\lambda_{\max}(V)q_{\mathrm{u},\xi(k,N)}I_n. \tag{4.25}$$

Since $\beta_k \geq 1$, it follows that $q_{\ell,\xi(k,N)} \to \infty$ as $k \to \infty$, and thus $\lambda_{\max}(M_k) \to \infty$ as

$k \to \infty$. From this result and Lemma 6 it follows that

$$\overline{\lim}_{k\to\infty}\lambda_{\max}(H_{0,k}) \leq \overline{\lim}_{k\to\infty}\lambda_{\max}(P_0^{-1})/\lambda_{\max}(M_k)^2 = 0.$$

Hence, $\overline{\lim}_{k\to\infty}\lambda_{\max}(H_{0,k}) = 0$. Noting that $\mathrm{var}(\theta_k) = H_{0,k} + H_{\nu,k}$, it follows from Lemmas 5 and 6, (4.24), and (4.25) that

$$\begin{aligned}
&\overline{\lim}_{k\to\infty}\ \lambda_{\max}(\mathrm{var}(\theta_k)) \leq \overline{\lim}_{k\to\infty}\lambda_{\max}(H_{0,k}) + \overline{\lim}_{k\to\infty}\lambda_{\max}(H_{\nu,k}) \\
&= \overline{\lim}_{k\to\infty}\lambda_{\max}(H_{\nu,k}) \leq \overline{\lim}_{k\to\infty}\frac{\lambda_{\max}(M_{\nu,k})}{\lambda_{\max}(P_0^{-1}+M_k)^2} \\
&\leq \overline{\lim}_{k\to\infty}\frac{\lambda_{\max}(M_{\nu,k})}{\lambda_{\max}(M_k)^2} \leq \frac{\beta\lambda_{\max}(V)}{\alpha^2}\overline{\lim}_{k\to\infty}\frac{q_{\mathrm{u},\xi(k,N)}}{s^2_{\mathrm{l},\xi(k,N)}},
\end{aligned}$$

Since $H_{0,k} \in \mathbb{P}^n$ and $\underline{\lim}_{k\to\infty}\lambda_{\min}(H_{0,k}) \leq \overline{\lim}_{k\to\infty}\lambda_{\max}(H_{0,k}) = 0$, it follows that $\underline{\lim}_{k\to\infty}\lambda_{\min}(H_{0,k}) = 0$. Thus, from Lemmas 5, 6, and 7, [70, Fact 10.4.13], (4.24), and (4.25), it follows that

$$\begin{aligned}
&\frac{\alpha\lambda_{\min}(V)}{\beta^2}\underline{\lim}_{k\to\infty}\frac{q_{\mathrm{l},\xi(k,N)}}{s^2_{\mathrm{u},\xi(k,N)}} \leq \underline{\lim}_{k\to\infty}\frac{\lambda_{\min}(M_{\nu,k})}{\lambda_{\min}(M_k)^2} \\
&= \underline{\lim}_{k\to\infty}\frac{\lambda_{\min}(M_{\nu,k})}{[\lambda_{\max}(P_0^{-1}) + \lambda_{\min}(M_k)]^2} \leq \underline{\lim}_{k\to\infty}\frac{\lambda_{\max}(M_{\nu,k})}{\lambda_{\min}(P_0^{-1}+M_k)^2} \\
&\leq \underline{\lim}_{k\to\infty}[\lambda_{\min}(H_{0,k}) + \lambda_{\min}(H_{\nu,k})] = \underline{\lim}_{k\to\infty}\lambda_{\min}(\mathrm{var}(\theta_k)).
\end{aligned}$$

□

Corollary 4.1. *Under the notation and assumptions of Theorem 6, consider the following statements:*

i) $\overline{\lim}_{k\to\infty} q_{\mathrm{u},\xi(k,N)}/s^2_{\mathrm{l},\xi(k,N)} = 0.$

ii) $(\theta_k)_{k\geq 0}$ *is consistent.*

iii) $\underline{\lim}_{k\to\infty} q_{\mathrm{l},\xi(k,N)}/s^2_{\mathrm{u},\xi(k,N)} = 0.$

Then $i) \implies ii) \implies iii)$.

Proof. To prove $i) \implies ii)$, let $\overline{\lim}_{k\to\infty} q_{\mathrm{u},\xi(k,N)}/s^2_{\mathrm{l},\xi(k,N)} = 0$. Then $\lim_{k\to\infty} \lambda_{\max}(\mathrm{var}(\theta_k)) = 0$. Thus, from Lemma 4, it follows that $(\theta_k)_{k\geq 0}$ is consistent. To prove $ii) \implies iii)$, suppose that $(\theta_k)_{k\geq 0}$ is consistent. Then, from Lemma 4, it follows that $\underline{\lim}_{k\to\infty} \lambda_{\min}(\mathrm{var}(\theta_k)) = 0$, and therefore $\underline{\lim}_{k\to\infty} q_{\mathrm{l},\xi(k,N)}/s^2_{\mathrm{u},\xi(k,N)} = 0$. □

Corollary 4.2. *Under the notation and assumptions of Theorem 6, the following statements hold: i) assume that $\prod_{k\geq 0}\beta_k$ is finite. Then $(\theta_k)_{k\geq 0}$ is consistent; ii) let $\beta_0 = 1$ and for all $k > 0$, let $\beta_k = 1 + \frac{1}{k}$. Then $(\theta_k)_{k\geq 0}$ is consistent; iii) let $\gamma \in [1, \infty)$, and, for all $k \geq 0$, let $\beta_k = \gamma$. Then $(\theta_k)_{k\geq 0}$ is consistent if and only if $\gamma = 1$.*

Proof. To prove $i)$, suppose that $\prod_{k\geq 0}\beta_k = \rho$ and let $\varepsilon > 0$. Thus there exists $K > 0$ such that, for all $i \geq K$, $\rho - \varepsilon < \rho_i < \rho + \varepsilon$. Let $k_\varepsilon > 0$ be the smallest integer such that $\xi(k_\varepsilon, N)(N+1) \geq K$, and define $B_\varepsilon \stackrel{\triangle}{=} \sum_{i=0}^{\xi(k_\varepsilon,N)} \rho^2_{i(N+1)+N}$ and $C_\varepsilon \stackrel{\triangle}{=} \sum_{i=0}^{\xi(k_\varepsilon,N)} \rho_{i(N+1)}$. Then, for all $k > k_\varepsilon$,

$$\frac{q_{\mathrm{u},\xi(k,N)}}{s^2_{\mathrm{l},\xi(k,N)}} \leq \frac{B_\varepsilon + (\rho+\varepsilon)^2(\xi(k,N) - \xi(k_\varepsilon,N) - 1)}{\left(C_\varepsilon + (\rho-\varepsilon)(\xi(k,N) - \xi(k_\varepsilon,N) - 1)\right)^2}. \tag{4.26}$$

Since the limit superior of the left-hand side of (4.26) is zero, it follows that $(\theta_k)_{k\geq 0}$ is consistent. To prove $ii)$, for all $k \geq 0$, let $\beta_k = 1 + 1/k$. Then, for all $i \geq 0$, $\rho_i = i + 1$, and thus $q_{\mathrm{u},\xi(k,N)}$ and $s^2_{\mathrm{l},\xi(k,N)}$ are polynomials of degree three and four, respectively. Hence the limit superior is zero, and therefore $(\theta_k)_{k\geq 0}$ is consistent. To prove $iii)$, suppose that $\gamma = 1$. Then $\overline{\lim}_{k\to\infty} q_{\mathrm{u},\xi(k,N)}/s^2_{\mathrm{l},\xi(k,N)} = \overline{\lim}_{k\to\infty} \xi(k,N)^{-1} = 0$. Hence $(\theta_k)_{k\geq 0}$ is consistent. Conversely, suppose $\gamma > 1$. Then, for all $i \geq 0$, $\rho_i = \gamma^{i+1}$, and thus

$$\begin{aligned}\underline{\lim}_{k\to\infty} \frac{q_{\mathrm{l},\xi(k,N)}}{s^2_{\mathrm{u},\xi(k,N)}} &= \frac{1}{\gamma^{2N}} \frac{(1-\gamma^{(N+1)})^2}{1-\gamma^{2(N+1)}} \underline{\lim}_{k\to\infty} \frac{1-\gamma^{2(N+1)(\xi(k,N)+1)}}{(1-\gamma^{(N+1)(\xi(k,N)+1)})^2} \\ &= \frac{1}{\gamma^{2N}} \frac{\gamma^{(N+1)}-1}{\gamma^{(N+1)}+1},\end{aligned}$$

which is positive because $\gamma > 1$. Therefore, $(\theta_k)_{k\geq 0}$ is not consistent. □

Corollary 4.2 shows that if $\prod_{k\geq 0}\beta_k$ converges, then VRF is consistent, but also that the

converse is false, since $\prod_{k>0} 1 + \frac{1}{k} = \infty$. Furthermore, CRF is consistent if and only if $\lambda = 1$, and thus RLS with a constant, nontrivial forgetting factor is not consistent.

4.7 Example: Abruptly Changing Parameters

Consider a mass-spring-damper system with $m = 5$ kg, $k = 1$ N/m, and $b = 1$ N·sec/m sampled at 1 sample/sec, and suppose that at 100 samples the parameters of the system abruptly change to $k = 10$ N/m and $b = 0.01$ N·sec/m. This process is modeled by the time-varying discrete-time transfer function

$$G_k(\mathbf{q}) = \begin{cases} \dfrac{0.4606\mathbf{q} + 0.4307}{\mathbf{q}^2 - 1.64\mathbf{q} + 0.8187}, & k < 100, \\ \dfrac{0.4218\mathbf{q} + 0.4215}{\mathbf{q}^2 - 0.3116\mathbf{q} + 0.998}, & k \geq 100, \end{cases} \tag{4.27}$$

where $\mathbf{q}$ is the forward shift operator. For all $k \geq 0$, let $u_k \sim \mathcal{N}(0, 1)$, and define

$$\beta_k \triangleq 1 + \eta \, \mathrm{sat}_\gamma(\|y_k - \phi_k \theta_k\|), \tag{4.28}$$

where $\eta, \gamma > 0$, and sat_γ is the unit-slope saturation function with saturation level γ. Figure 4.1 shows the performance of VRF with $\gamma = \eta = 1$ and CRF with $\lambda = 0.99$. VRF converges to the initial system parameters and reconverges to the modified parameters in about 10 samples, illustrating Theorem 4. In contrast, while CRF converges to the initial parameters, reconvergence to the modified parameters is still not achieved at 200 samples. Next, consider the same system with the output corrupted by additive noise $\nu_k \sim \mathcal{N}(0, 0.05)$, and define

$$\beta_k \triangleq \begin{cases} 1 + \eta \, \mathrm{sat}_\gamma(E_\tau), & E_\tau > 1, \\ 1, & E_\tau \leq 1, \end{cases} \tag{4.29}$$

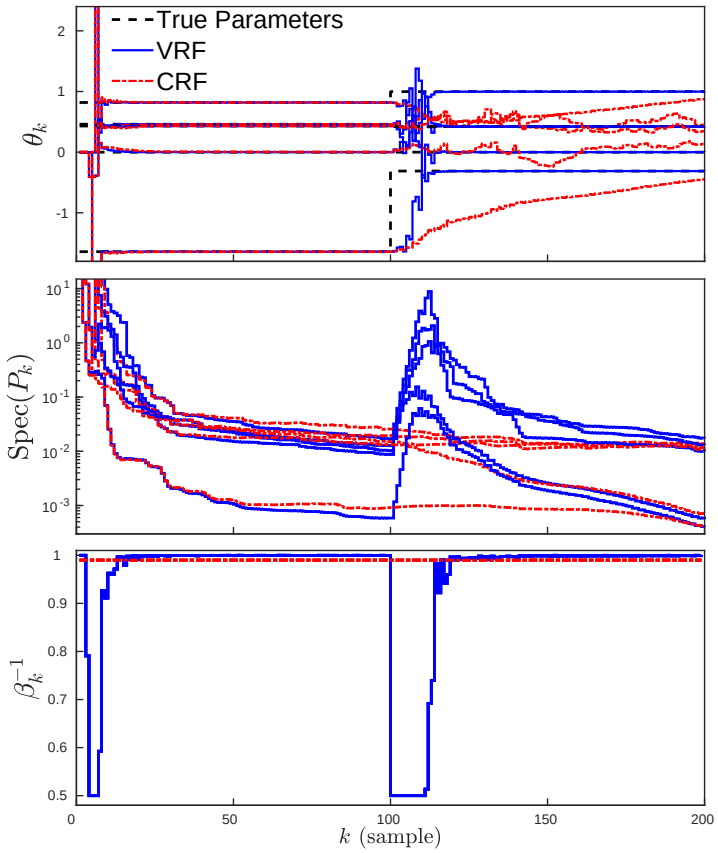

Figure 4.1: The parameter estimate θ_k given by VRF with β_k defined by (4.28) reconverges after an abrupt change in the system as guaranteed by Theorem 4. In contrast, The parameter estimate given by CRF with $\lambda = 0.99$ requires many samples to reconverge.

where $\tau \in \mathbb{N}$ and $E_\tau \triangleq \left(\frac{1}{\tau}\sum_{i=k-\tau}^{k} \|y_i - \phi_i\theta_i\|^2\right)^{1/2}$. Figure 4.2 shows the performance of VRF with $\eta = 1$, $\gamma = 5$, and $\tau = 10$, and CRF with $\lambda = 0.99$. VRF converges to the initial parameters and then reconverges to the new parameters in roughly 30 samples. As in the previous case, CRF converges to the initial parameters, but at 200 samples has still not reconverged to the modified parameters. ◇

4.8 Lemmas

Lemma 3. *Let $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. Let $X \sim \mathcal{N}(\mu, P)$ and define $Y \triangleq AX + b$. Then $Y \sim \mathcal{N}(A\mu + b, APA^{\mathrm{T}})$.*

Lemma 4. *Let (Ω, Σ, P) be a probability space, let $\theta \in \mathbb{R}^n$, and let $(X_k \colon \Omega \to \mathbb{R}^n)_{k\geq 0}$ be*

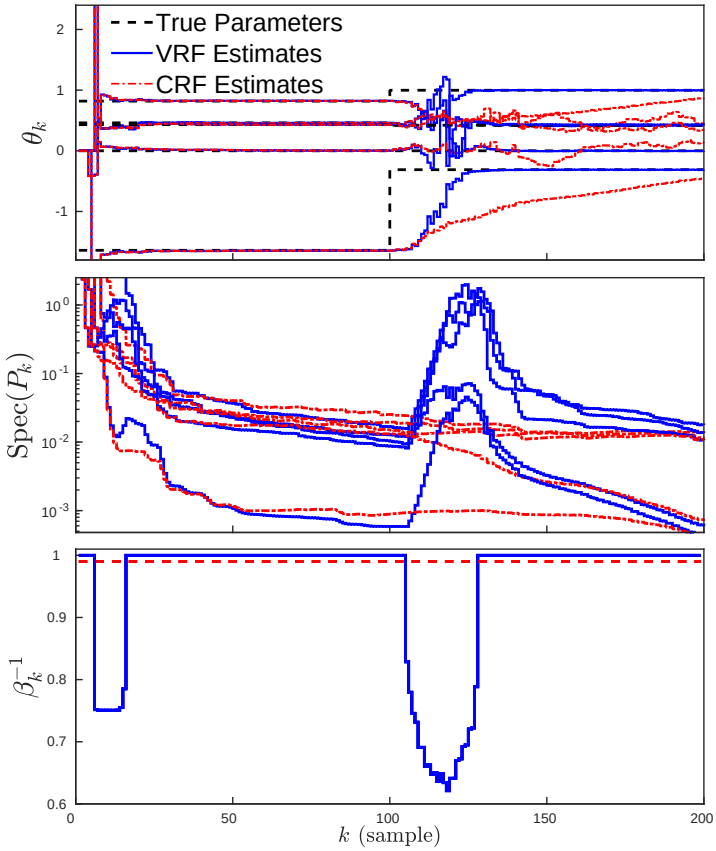

Figure 4.2: The parameter estimate θ_k given by VRF with β_k defined by (4.29) reconverges after an abrupt change in the system with noisy measurements. In contrast, the parameter estimate given by CRF with $\lambda = 0.99$ requires many samples to reconverge.

a sequence of random variables such that, for all $k \geq 0$, $X_k \sim \mathcal{N}(\theta, \Sigma_k)$. *Then* $(X_k)_{k\geq 0}$ *is a consistent estimator for* θ *if and only if* $\lim_{k\to\infty} \Sigma_k = 0$.

Lemma 5. *Let* $(A_k)_{k\geq 0}, (B_k)_{k\geq 0} \subset (\mathbb{N}^n)$. *Then*

$$\overline{\lim}_{k\to\infty} \lambda_{\max}(A_k + B_k) \leq \overline{\lim}_{k\to\infty} \lambda_{\max}(A_k) + \overline{\lim}_{k\to\infty} \lambda_{\max}(B_k),$$

$$\underline{\lim}_{k\to\infty} \lambda_{\min}(A_k + B_k) \geq \underline{\lim}_{k\to\infty} \lambda_{\min}(A_k) + \underline{\lim}_{k\to\infty} \lambda_{\min}(B_k).$$

Lemma 6. *Let* $A \in \mathbb{N}^n$ *and* $B \in \mathbb{P}^n$. *Then, for all* $i = 1, \ldots, n$,

$$\frac{\lambda_{\min}(A)}{\lambda_i(B)^2} \leq \lambda_i(B^{-1}AB^{-1}) \leq \frac{\lambda_{\max}(A)}{\lambda_i(B)^2}. \tag{4.30}$$

Now assume that $A \in \mathbb{P}^n$. Then there exist $0 < b_1 \leq b_2$ and $0 < a_1 \leq a_2$ such that

$$a_1 I_n \leq A \leq a_2 I_n, \tag{4.31}$$

$$b_1 I_n \leq B \leq b_2 I_n. \tag{4.32}$$

Furthermore, for all a_1, a_2, b_1, b_2 satisfying (4.31)*,* (4.32)*,*

$$\frac{a_1}{b_2^2} I_n \leq B^{-1} A B^{-1} \leq \frac{a_2}{b_1^2} I_n. \tag{4.33}$$

Lemma 7. *Let $a \in [0, \infty)$, let $(b_k)_{k\geq 0}, (c_k)_{k\geq 0} \subset [0, \infty)$, and assume that $\lim_{k\to\infty} b_k = \infty$. Then, for all $p \geq 0$,*

$$\underline{\lim}_{k\to\infty} \frac{c_k}{(a + b_k)^p} = \underline{\lim}_{k\to\infty} \frac{c_k}{b_k^p}. \tag{4.34}$$

CHAPTER 5

A Note On Artificial Uncertainty

We have seen in the foregoing paper, first published in 2020 and containing investigations that were substantially completed by mid 2019, that varying the forgetting factor in RLS is one method for achieving adaptability of the parameter estimates during abrupt changes while maintaining good estimate quality during periods of little or no change. Moreover, unlike CRF, the VRF forgetting factor can be designed to converge to unity during periods in which persistency is lost (although that is not shown in the paper), and thus is much less susceptible to instability when regressor is not persistent. The following paper will show how, by combining VRF with the Variable-direction forgetting concept given in [24], a matrix-valued forgetting factor algorithm can be derived in which the forgetting factor is set to unity exactly along the covariance eigenspace in which the information is stationary at any given instant.

Astute readers will recognize that general the idea of variable-rate forgetting is not original to our work, and indeed can be found in [16] or [26, pp. 305-309]. (*Cf.* especially equation 11.6 of page 305 of [26] with our definition of β_k and ρ_k above.) Our contribution therefore, is not the mere introduction of a variable forgetting factor, but, from a theoretical perspective, an extended analysis of convergence and consistency (hence the title) that far surpasses anything previously attempted[1], and from a practical perspective, the introduction

[1]The complexity of our consistency result, in particular, which I consider at most a partial result, shows why the often muddled and confused treatment of stochastic convergence in the system theory literature was incapable of achieving any results at all. Hence the problem was universally ignored.

of the error-actuated formula

$$\beta_k = 1 + \eta \|y_k - \phi_k \theta\|^\alpha. \tag{5.1}$$

The idea that β_k should be chosen as a direct function of the residual is surprisingly absent in the sporadic variable-rate forgetting literature. Moreover, the methods that earlier investigators have proposed, such as the "constant information" method of [16] have considerable and unnecessary drawbacks–consider, for instance, the implications on consistency for a choice of forgetting factor that maintains *constant* information in the Fisherian sense, and we find yet another reason why the question of consistency, along with any stochastic analysis whatsoever, is ignored in [16]. The idea of error-actuation, or residual-actuation, appears to be both novel and highly effective, at least for a certain class of problems. For this reason, VRF, using modified versions of (5.1), has been successfully applied in two major new approaches to adaptive control: Data-Driven Retrospective Cost Adaptive Control [61–63], and Preditive Cost Adaptive Control [41–43].

In this note, I will present some new, unpublished results that give an interpretation of variable-rate forgetting in terms of measurement statistics. We will see that the inclusion of any forgetting whatsoever can be regarded as an artificial increase in the uncertainty associated with past measurements at the current step. Incredibly, this *artificial uncertainty* will actually change at each step, and thus the estimator will see a fixed data point as having variable, non-decreasing measure of uncertainty associated with it as time and the number of measurements increase.

5.1 Artificial Uncertainty

We have seen that the RLS algorithm behaves very differently under the influence of a forgetting factor compared to the elementatry form of the algorithm. Let us investigate this behavior from a Bayesian perspective.

Consider the measurement model

$$y_k = \phi_k \theta + \nu_k, \quad k \geq 0, \tag{5.2}$$

where $(\nu_k)_{k\geq 0}$ is zero-mean standard Gaussian white noise. We assume that θ has an *a priori* distribution given by $f(\theta) = Ne^{-S(\theta)}$, where S grows quickly enough as $\|\theta\| \to \pm\infty$ so that

$$\frac{1}{N} \stackrel{\triangle}{=} \int e^{-S(\theta)}\, d\theta < \infty. \tag{5.3}$$

We will now obtain the maximum *a posteriori* estimate of θ at step k under the assumption that we have measurements of $y_0, \ldots, y_k$. From Bayes theorem, it follows that

$$f_{\theta|y_0,\ldots,y_k} = \frac{f_{y_0,\ldots,y_k|\theta} f}{f_{y_0,\ldots,y_k}} \tag{5.4}$$

$$= \frac{f_{y_0,\ldots,y_k|\theta} f}{\int f_{y_0,\ldots,y_k|\theta}(y_0,\ldots,y_k;\theta) f(\theta)\, d\theta}. \tag{5.5}$$

Put $D_k(y_0,\ldots,y_k) \stackrel{\triangle}{=} \int f_{y_0,\ldots,y_k|\theta}(y_0,\ldots,y_k;\theta) f(\theta)\, d\theta$. After we obtain an expression for $f_{y_0,\ldots,y_k|\theta}$ below, it will be easy to show that D_k is well-defined for all k. Since $y_j = \phi_j\theta + \nu_j$ for all $j = 0,\ldots,k$, it follows that

$$f_{y_0,\ldots,y_k|\theta,\nu_0,\ldots,\nu_k}(y_0,\ldots,y_k,\nu_0,\ldots\nu_k,\theta) = \prod_{j=0}^{k} \delta(y_j - \phi_j\theta - \nu_j), \tag{5.6}$$

and thus, since the ν_j are independent, the law of total probability implies that

$$f_{y_0,\ldots,y_k|\theta}(y_0,\ldots,y_k;\theta) = \int \cdots \int d\nu_0 \ldots d\nu_k \prod_{j=0}^{k} \delta(y_j - \phi_j\theta - \nu_j) f_{\nu_j}(\nu_j) \tag{5.7}$$

$$= \prod_{j=0}^{k} \int d\nu_j\, \delta(y_j - \phi_j\theta - \nu_j) M_j \exp\left[-\frac{\|v_j\|^2}{2\sigma_{k,j}^2}\right]. \tag{5.8}$$

Note that we are allowing the variance of ν_j to depend on the total number of measurements, k. The justification for this decision will become apparent shortly. From the Fourier transform relation

$$\delta(x) = \int dz\, e^{iz^{\mathrm{T}}x}, \tag{5.9}$$

which is understood in the distributional sense of Schwartz [71], we have

$$f_{y_0,\ldots,y_k|\theta}(y_0,\ldots,y_k;\theta) = \prod_{j=0}^{k} M_j \int d\nu_j \int dz_j\, e^{iz_j^{\mathrm{T}}(y_j-\phi_j\theta-\nu_j)} \exp\left[-\frac{\|v_j\|^2}{2\sigma_{k,j}^2}\right] \tag{5.10}$$

$$= \prod_{j=0}^{k} \int M_j \int d\nu_j\, dz_j\, \exp\left[iz_j^{\mathrm{T}}(y_j-\phi_j\theta-\nu_j) - \frac{\|v_j\|^2}{2\sigma_{k,j}^2}\right]. \tag{5.11}$$

Each integrand $I_j(z_j,\nu_j) = \exp\left[iz_j^{\mathrm{T}}(y_j-\phi_j\theta-\nu_j) - \frac{\|v_j\|^2}{2\sigma_{k,j}^2}\right]$ is continuous as a function of z_j or ν_j alone, and hence we may permute the order of integration, which yields

$$f_{y_0,\ldots,y_k|\theta}(y_0,\ldots,y_k;\theta)$$

$$= \prod_{j=0}^{k} M_j \int dz_j e^{iz_j^{\mathrm{T}}(y_j-\phi_j\theta)} \int d\nu_j\, \exp\left[-iz_j^{\mathrm{T}}\nu_j - \frac{\|v_j\|^2}{2\sigma_{k,j}^2}\right]. \tag{5.12}$$

Using the formula

$$\int e^{b^{\mathrm{T}}x-\frac{1}{2}x^{\mathrm{T}}Ax}\, dx = \frac{\pi^{n/2}}{\sqrt{\det A}} e^{\frac{1}{2}b^{\mathrm{T}}A^{-1}b}, \tag{5.13}$$

The second integral with respect to ν_j is found to be $C_j e^{-\frac{1}{2}\sigma_{k,j}^2\|z_j\|^2}$, where C_j is a normalization factor. Thus we have

$$f_{y_0,\ldots,y_k|\theta}(y_0,\ldots,y_k;\theta) = \prod_{j=0}^{k} C_j' \int dz_j e^{iz_j^{\mathrm{T}}(y_j-\phi_j\theta)-\frac{1}{2}\sigma_{k,j}^2\|z_j\|^2} \tag{5.14}$$

$$= \prod_{j=0}^{k} C_j' e^{-\frac{1}{2\sigma_{k,j}^2}\|y_j-\phi_j\theta\|^2} = K_k e^{-\frac{1}{2}\sum_{j=0}^{k}\sigma_{k,j}^{-2}\|y_j-\phi_j\theta\|^2}, \tag{5.15}$$

where C_j' and K_k are constants, and therefore the posterior distribution is given by

$$f_{\theta|y_0,\ldots,y_k}(y_0,\ldots,y_k;\theta) = K_k' \frac{e^{-\frac{1}{2}\sum_{j=0}^{k}\sigma_{k,j}^{-2}\|y_j-\phi_j\theta\|^2-\frac{1}{2}S(\theta)}}{D_k(y_0,\ldots,y_k)}, \tag{5.16}$$

where K_k' is a constant, and hence the maximum *a priori* estimate for θ is given by

$$\text{argmax}_\theta f_{\theta|y_0,\ldots,y_k}(y_0,\ldots,y_k;\theta) = \text{argmin}_\theta \sum_{j=0}^{k}\sigma_{k,j}^{-2}\|y_j-\phi_j\theta\|^2 + \frac{1}{2}S(\theta). \tag{5.17}$$

Setting $S(\theta)$ to be the slightly modified Gaussian prior $\frac{1}{\sigma_{k,0}^2}(\theta-\theta_0)^{\mathrm{T}}P_0^{-1}(\theta-\theta_0)$, we recover the familiar problem of minimizing the least squares cost

$$J_k(\theta) = \frac{1}{2}\sum_{j=0}^{k}\frac{1}{\sigma_{k,j}^2}\|y_j-\phi_j\theta\|^2 + \frac{1}{2\sigma_{k,0}^2}(\theta-\theta_0)^{\mathrm{T}}P_0^{-1}(\theta-\theta_0). \tag{5.18}$$

By reference to the VRF cost, we infer that

$$\sigma_{k,j} = \sqrt{\frac{\rho_k}{\rho_i}} = \sqrt{\beta_k\ldots\beta_{i+1}}. \tag{5.19}$$

That is, VRF treats the measurement y_j with noise ν_j as having the covariance

$$\text{Cov}(y_j) = \phi_j P_0 \phi_j^{\mathrm{T}} + \beta_k\ldots\beta_{i+1} \approx \beta_k\ldots\beta_{i+1}. \tag{5.20}$$

The forgetting factor is therefore responsible for increasing the formal covariance of the measurement y_i. Since the forgetting factor is determined by the user, this amounts to a user-defined covariance, or artificial uncertainty, that is placed upon the measurement to induce the algorithm to react in a desired way–that is, to be responsive to parameter changes. What is more fascinating is that the covariance assigned to a fixed measurement is revised at each new step. Thus, the data taken at step 6 may be associated with more uncertainty at step 15 than it is at step 12. It is the same data–it does not change–but our

degree of trust in it changes, and this is the key observation.

Setting $\beta_k = \frac{1}{\lambda}$ for all k to obtain CRF, we get $\sigma_{k,j} = 1/\sqrt{\lambda^{k-j}}$. Thus at step k, CRF presumes the following level of uncertainty for the trailing measurements:

$$\begin{array}{cccccc} k & & 1 & & k-3 & \frac{1}{\lambda^{3/2}} & \dots \\ k-1 & & \frac{1}{\sqrt{\lambda}} & & k-4 & \frac{1}{\lambda^2} & \dots \\ k-2 & & \frac{1}{\lambda} & & k-5 & \frac{1}{\lambda^{5/2}} & \dots \end{array}$$

Thus, CRF sees each fixed data point as having successively more uncertainty as the estimation process progresses. As $k \to \infty$, the covariance associated with each fixed measurement will become infinite, and the weight of the measurement in the estimate will become negligible. This, conceptually, is why CRF can track slowly varying parameters well, but also why any constant forgetting factor will result in an estimator that is not consistent.[2]

It would be possible to write at length about the applications of artificial uncertainty, especially in connecting RLS with Kalman filtering and regarding its interpretation as a nonlinear control mechanism for the filter covariance. However, time and space have limited me to this brief note, which hopefully has helped to enlighten the reader and provide an intuitive view of the causes of some of the results we have observed so far.

[2]Clearly the rate at which the weight of a fixed data point becomes negligible will affect consistency. For the exponential rate of CRF, non-consistency is assured (as proven in the foregoing paper) regardless of the specific value of the forgetting factor.

CHAPTER 6

Recursive Least Squares with Matrix Forgetting

6.1 Preface

This paper extends the work of [20] to the case where the forgetting factor is matrix-valued. This is inspired by [24], in which *variable-direction forgetting* (VDF) is presented. When the regressor is not persistent, there exist one or more eigenvectors of the covariance matrix with eigenvalue zero. In these directions, new data is not entering into the estimate, and thus forgetting in these directions will cause the covariance matrix to diverge exponentially. VDF stops this from happening by setting the forgetting factor to unity for all of the axes in which no new measurement information is available. In order to do this, VDF requires a matrix-valued forgetting factor, which is included as an *ad hoc* modification of the ordinary RLS update with forgetting.

The introduction of a matrix-valued forgetting factor raises several questions. First, and most obvious, is how to combine VRF and VDF. As it happens, this is a simple task, since one need only replace the constant forgetting factor applied along information-containing eigenaxes with a variable forgetting factor, such as that given in the previous paper, [20]. Such a modification is one way to ensure that the resulting algorithm is robust both to parameter changes and the loss of persistency. Of course, this need not be the only application of a matrix-valued forgetting factor, and thus a modification allowing for general matrix-valued forgetting factors is presented.

One question that was not answered in [24] was whether or not VDF, and by extension a general matrix-valued forgetting RLS algorithm could be derived from a least-squares cost in the same way as VRF or ordinary RLS. The major theoretical result in this paper answers this question in the affirmative–matrix forgetting does indeed minimize a least-sqaures cost. The detailed structure of this cost, however, is very different from those corresponding to scalar forgetting factors. Instead of including the forgetting in between the residuals (note that the size would be wrong in general), it must be included in the regularization term. This regularization now depends on the step k, and must be calculated recursively, since the current value of the regularization will depend essentially on previous values in a complicated way.

Perhaps the most philosophically disturbing feature of this cost is the fact that it allows from the minimum to become negative! The physical interpretation of a negative minimum value for a least-squares cost is unclear to say the least. What can be said is that the negative minimum is the true global minimum, since the cost is strictly convex, and the minimizer is truly the matrix-valued RLS estimate. One interesting implication of the specialization to VRF is the fact that the same iteration can arise as the minimizer to multiple, very different least-squares costs. The significance of this implication can be seen by the following considerations: suppose we are using VRF, which we interpret to be the minimizer of the cost given in [20]. This will have a matrix intepretation in which the forgetting factor is given by $\beta_k I$. Now assume that we modify this to $\beta_k I + \eta_k$, where η_k is a matrix that is not just a scalar multiple of the identity. The resulting algorithm minimimizes an entirely different cost, regardless of how small η_k is! In practice, η_k could be so small that its effect is negligible and not measurable in practice. Yet, technically speaking, the cost minimized by the algorithm would be the matrix-forgetting cost, and *not* the VRF cost.

6.2 Introduction

Recursive least squares (RLS) is an algorithm that is widely used in signal processing, identification, and adaptive control [**?**, 49, 55, 64]. Standard RLS employs a forgetting factor λ that enhances the importance of recent data over older data, but unfortunately, the performance of RLS is often extremely sensitive to the choice of λ. To overcome this problem, various extensions of RLS have been developed to include variable-rate forgetting [16, 59, 66–68] or covariance resetting [72].

An additional weakness of RLS is that the use of a fixed- or variable-rate forgetting factor may cause the covariance to diverge when the input signal is not persistently exciting [73–77]. A variety of techniques have been developed for overcoming the divergence due to lack of persistency [23, 73, 78]. One approach to this problem is to restrict forgetting to the subspace in which the data provide new information about the parameters, [79–85]. Consequently, the direction of the forgetting is varied based on the information content of the measurements.

The main contribution of this paper is a modified cost function whose minimization yields a matrix forgetting RLS algorithm that can be specialized into a combined variable-rate and variable-direction RLS algorithm (RLS-VRDF). The resulting extension of RLS thus seeks to overcome both changing parameters and loss of persistency. In addition, since RLS-VRDF is obtained by minimizing a cost function, this modification of RLS has a known optimal interpretation, in contrast with extensions of RLS obtained by direct modification of the RLS update equations [23, 78]. Since this paper gives a general matrix forgetting RLS algorithm and a cost function which it minimizes, and then numerically investigates RLS-VRDF, while [86] only includes RLS-VRF and [87] only includes RLS-VDF, the contribution of this paper goes beyond [86, 87].

Matrix forgetting algorithms are also given in [21, 88]. However, [21] assumes an ARMA model of the system and develops matrix forgetting that only applies to the Instrumental Variable Method. In contrast, this paper makes no assumptions about the system

and directly generalizes standard RLS. In [88], a matrix forgetting algorithm is derived by modifying the standard RLS cost function, but results in a covariance matrix that is not generally symmetric. The algorithm also assumes that there is a single output and restricts the forgetting matrix to be both constant and symmetric. In this paper, the covariance matrix is guaranteed to be symmetric, but the forgetting matrix need not be either symmetric or constant–allowing for a wide range of choices, such as RLS-VRDF. Furthermore, there is no assumption on the number of outputs.

The paper is organized as follows. In Section 6.3, we introduce preliminary results on least squares optimization, including a recursive update algorithm for a general least squares cost which does not use the matrix inversion lemma (Lemma 8), and show that standard RLS can be obtained as a special case of this cost. In Section 6.4, we specialize Proposition 3 in Section 6.3 to the case of matrix forgetting. Then, in Section 6.5, we specialize Theorem 7 in Section 6.4 further to the cases of RLS-VRF, RLS-VDF, and RLS-VRDF. Finally, in Section 6.6, we show the performance of the different algorithms on a system identification example with both abruptly changing parameters and abrupt loss of persistency.

6.3 Preliminary Results on Least Squares Optimization

The following result on least squares optimization is an immediate consequence of Lemma 9 in the appendix.

Proposition 2. *For all $k \geq 0$, let $y_k \in \mathbb{R}^p$, $\phi_k \in \mathbb{R}^{p\times n}$, and $\alpha_k \in \mathbb{R}^n$, let $Q_k \in \mathbb{R}^{p\times p}$ be positive semidefinite, let $R_k \in \mathbb{R}^{n\times n}$ be symmetric, define $J_k \colon \mathbb{R}^n \to [0,\infty)$ by*

$$J_k(\theta) \triangleq \sum_{i=0}^{k} (y_i - \phi_i\theta)^{\mathrm{T}} Q_i (y_i - \phi_i\theta) + (\theta - \alpha_k)^{\mathrm{T}} R_k (\theta - \alpha_k), \tag{6.1}$$

and assume that

$$A_k \triangleq \sum_{i=0}^{k} \phi_i^{\mathrm{T}} Q_i \phi_i + R_k \tag{6.2}$$

is positive definite. Then, for all $k \geq 0$, J_k is quadratic and strictly convex, and thus has a unique global minimizer, which is also the only local minimizer. For all $k \geq 0$, define

$$\theta_{k+1} \triangleq \operatorname{argmin}_{\theta \in \mathbb{R}^n} J_k(\theta). \tag{6.3}$$

Then, for all $k \geq 0$,

$$\theta_{k+1} = -A_k^{-1} b_k, \tag{6.4}$$

and the minimum value of J_k is given by

$$J_k(\theta_{k+1}) = c_k - b_k^{\mathrm{T}} A_k^{-1} b_k, \tag{6.5}$$

where

$$b_k \triangleq -\sum_{i=0}^{k} \phi_i^{\mathrm{T}} Q_i y_i - R_k \alpha_k, \tag{6.6}$$

$$c_k \triangleq \sum_{i=0}^{k} y_i^{\mathrm{T}} Q_i y_i + \alpha_k^{\mathrm{T}} R_k \alpha_k. \tag{6.7}$$

Note that in the case where R_k is positive definite (positive semidefinite) it follows that (6.5) is positive (nonegative). If however R_k is not positive semidefinite, (6.5) may be negative.

The next result is a recursive variation of Proposition 2.

Proposition 3. *Under the notation and assumptions of Proposition 2, let $\theta_0 \in \mathbb{R}^n$ and*

define $R_{-1} \in \mathbb{R}^{n\times n}$ *and* $\alpha_{-1} \in \mathbb{R}^n$ *such that* $R_{-1}(\alpha_{-1} - \theta_0) = 0$. *Then, for all* $k \geq 0$,

$$A_k = A_{k-1} + \phi_k^{\mathrm{T}} Q_k \phi_k + R_k - R_{k-1}, \tag{6.8}$$

$$\theta_{k+1} = \theta_k + A_k^{-1}\phi_k^{\mathrm{T}} Q_k(y_k - \phi_k\theta_k) + A_k^{-1}[R_k(\alpha_k - \theta_k) - R_{k-1}(\alpha_{k-1} - \theta_k)]. \tag{6.9}$$

Proof. Since $A_{-1} = R_{-1}$, it follows that

$$\begin{aligned} A_0 &= \phi_0^{\mathrm{T}} Q_0 \phi_0 + R_0 \\ &= R_{-1} + \phi_0^{\mathrm{T}} Q_0 \phi_0 + R_0 - R_{-1} \\ &= A_{-1} + \phi_0^{\mathrm{T}} Q_0 \phi_0 + R_0 - R_{-1}, \end{aligned}$$

which confirms (6.8) for $k = 0$. Since A_0 is positive definite, Lemma 9 implies that

$$\begin{aligned} \theta_1 &= -A_0^{-1} b_0 \\ &= A_0^{-1}(\phi_0^{\mathrm{T}} Q_0 y_0 + R_0\alpha_0) \\ &= A_0^{-1}(\phi_0^{\mathrm{T}} Q_0 y_0 + \phi_0^{\mathrm{T}} Q_0 \phi_0\theta_0 - \phi_0^{\mathrm{T}} Q_0 \phi_0\theta_0 + R_0\alpha_0) \\ &= A_0^{-1}(\phi_0^{\mathrm{T}} Q_0 \phi_0\theta_0 + R_0\theta_0) + A_0^{-1}\phi_0^{\mathrm{T}} Q_0(y_0 - \phi_0\theta_0) + A_0^{-1}R_0(\alpha_0 - \theta_0) \\ &= \theta_0 + A_0^{-1}\phi_0^{\mathrm{T}} Q_0(y_0 - \phi_0\theta_0) + A_0^{-1}R_0(\alpha_0 - \theta_0) - A_0^{-1}R_{-1}(\alpha_{-1} - \theta_0), \end{aligned}$$

which confirms (6.9) for $k = 0$.

Now let $k \geq 1$. From (6.2) it follows that

$$\begin{aligned} A_k &= \sum_{i=0}^{k} \phi_i^{\mathrm{T}} Q_i \phi_i + R_k \\ &= \sum_{i=0}^{k-1} \phi_i^{\mathrm{T}} Q_i \phi_i + \phi_k^{\mathrm{T}} Q_k \phi_k + R_k \\ &= A_{k-1} - R_{k-1} + \phi_k^{\mathrm{T}} Q_k \phi_k + R_k, \end{aligned}$$

which confirms (6.8). Furthermore, b_k can be written recursively as

$$b_k = b_{k-1} - \phi_k^{\mathrm{T}} Q_k y_k + R_{k-1}\alpha_{k-1} - R_k \alpha_k.$$

It thus follows from Lemma 9 that

$$\begin{aligned}
\theta_{k+1} &= -A_k^{-1} b_k \\
&= -A_k^{-1}(b_{k-1} - \phi_k^{\mathrm{T}} Q_k y_k + R_{k-1}\alpha_{k-1} - R_k\alpha_k) \\
&= -A_k^{-1}(b_{k-1} - \phi_k^{\mathrm{T}} Q_k y_k + \phi_k^{\mathrm{T}} Q_k \phi_k \theta_k - \phi_k^{\mathrm{T}} Q_k \phi_k \theta_k) - A_k^{-1}(R_{k-1}\alpha_{k-1} - R_k\alpha_k) \\
&= -A_k^{-1}(b_{k-1} - \phi_k^{\mathrm{T}} Q_k \phi_k \theta_k) + A_k^{-1}\phi_k^{\mathrm{T}} Q_k (y_k - \phi_k\theta_k) + A_k^{-1}(R_k\alpha_k - R_{k-1}\alpha_{k-1}) \\
&= A_k^{-1}(A_{k-1}\theta_k + \phi_k^{\mathrm{T}} Q_k \phi_k \theta_k) + A_k^{-1}\phi_k^{\mathrm{T}} Q_k (y_k - \phi_k\theta_k) + A_k^{-1}(R_k\alpha_k - R_{k-1}\alpha_{k-1}) \\
&= A_k^{-1}(A_{k-1} + \phi_k^{\mathrm{T}} Q_k \phi_k + R_k - R_{k-1})\theta_k + A_k^{-1}\phi_k^{\mathrm{T}} Q_k (y_k - \phi_k\theta_k) \\
&\quad + A_k^{-1}(R_k\alpha_k - R_{k-1}\alpha_{k-1}) - A_k^{-1}(R_k - R_{k-1})\theta_k \\
&= \theta_k + A_k^{-1}\phi_k^{\mathrm{T}} Q_k (y_k - \phi_k\theta_k) + A_k^{-1}[R_k(\alpha_k - \theta_k) - R_{k-1}(\alpha_{k-1} - \theta_k)],
\end{aligned}$$

which confirms (6.9). □

The following corollary of Proposition 3 is the classical least squares result, which we provide for comparison with the matrix forgetting result in Section IV.

Corollary 1. *Let $\lambda \in (0, 1]$ and $\theta_0 \in \mathbb{R}^n$, and let $P_0 \in \mathbb{R}^{n\times n}$ be positive definite. For all $k \geq 0$, let $y_k \in \mathbb{R}^p$ and $\phi_k \in \mathbb{R}^{p\times n}$, and define $J_k \colon \mathbb{R}^n \to [0, \infty)$ by*

$$\bar{J}_k(\theta) \triangleq \sum_{i=0}^{k} \lambda^{k-i}(y_i - \phi_i\theta)^{\mathrm{T}}(y_i - \phi_i\theta) + \lambda^k(\theta - \theta_0)^{\mathrm{T}} P_0^{-1}(\theta - \theta_0). \tag{6.10}$$

Then, for all $k \geq 0$, J_k is quadratic and strictly convex, and thus has a unique global minimizer, which is also the only local minimizer. For all $k \geq 0$, define

$$\theta_{k+1} \triangleq \operatorname{argmin}_{\theta\in\mathbb{R}^n} \bar{J}_k(\theta). \tag{6.11}$$

Then, for all $k \geq 0$,

$$\theta_{k+1} = \theta_k + P_{k+1}\phi_k^{\mathrm{T}}(y_k - \phi_k\theta_k) \tag{6.12}$$

where

$$P_{k+1} = \frac{1}{\lambda}P_k - \frac{1}{\lambda}P_k\phi_k^{\mathrm{T}}(\lambda I_p + \phi_k P_k \phi_k^{\mathrm{T}})^{-1}\phi_k P_k. \tag{6.13}$$

6.4 RLS with Matrix Forgetting

Note that (6.9) requires computation of the $n \times n$ inverse A_k^{-1}. In Corollary 1, the matrix inversion lemma was used to replace the $n \times n$ inverse with a $p \times p$ inverse. As can be seen in the proof of Corollary 1, this reduction in complexity was because R_k is equal to a constant, namely P_0^{-1}, for all $k \geq 0$, and thus the term $R_k - R_{k-1}$ vanishes. An alternative approach is to choose a variable R_k that avoids the need for an $n \times n$ inverse. The following result uses a specific choice of R_k and α_k to obtain a version of RLS with matrix forgetting, thereby providing an explicit quadratic cost function which is minimized by matrix forgetting RLS. Hereafter, R_k^+ denotes the Moore-Penrose pseudoinverse of R_k.

Theorem 7. *Let* $\theta_0 \in \mathbb{R}^n$*, and let* $P_0 \in \mathbb{R}^{n\times n}$ *be positive definite. Furthermore, for all* $k \geq 0$*, let* $\phi_k \in \mathbb{R}^{p\times n}$*, let* $Q_k \in \mathbb{R}^{p\times p}$ *be positive definite, let* $B_k \in \mathbb{R}^{n\times n}$ *be nonsingular,*

and define $J_k \colon \mathbb{R}^n \to [0, \infty)$ *by* (6.1) *with*

$$R_k \triangleq \begin{cases} P_0^{-1}, & k = -1, \\ \sum_{i=0}^{k} B_i^{-\mathrm{T}} A_{i-1} B_i^{-1} - \sum_{i=1}^{k} A_{i-1}, & k \geq 0 \end{cases} \tag{6.14}$$

and

$$\alpha_k \triangleq \begin{cases} \theta_0, & k = -1, \\ R_k^{+} S_k, & k \geq 0, \end{cases} \tag{6.15}$$

where $A_{-1} \triangleq P_0^{-1}$,

$$\begin{aligned} S_k \triangleq & \sum_{i=0}^{k} \Pi_{k-1} \cdots \Pi_i B_i^{-\mathrm{T}} A_{i-1} B_i^{-1} \theta_i \\ & - \sum_{i=1}^{k} \Pi_{k-1} \cdots \Pi_i A_{i-1} \theta_i, \end{aligned} \tag{6.16}$$

and, for all $i \geq 0$, $\Pi_i \triangleq R_i R_i^{+}$. *Then, for all* $k \geq 0$, A_k *defined by* (6.2) *is positive definite. Furthermore, for all* $k \geq 0$, *let* $y_k \in \mathbb{R}^p$, *define* $P_{k+1} \triangleq A_k^{-1}$, *and define* θ_{k+1} *by* (6.3). *Then, for all* $k \geq 0$, θ_{k+1} *is given by*

$$P_{k+1} = L_k - L_k \phi_k^{\mathrm{T}} (Q_k^{-1} + \phi_k L_k \phi_k^{\mathrm{T}})^{-1} \phi_k L_k, \tag{6.17}$$

$$\theta_{k+1} = \theta_k + P_{k+1} \phi_k^{\mathrm{T}} Q_k (y_k - \phi_k \theta_k) + P_{k+1} (\Pi_k - I_n) \gamma_k, \tag{6.18}$$

where

$$L_k \triangleq B_k P_k B_k^{\mathrm{T}}, \tag{6.19}$$

$$\gamma_k \triangleq R_{k-1} \alpha_{k-1} + (B_k^{-\mathrm{T}} A_{k-1} B_k^{-1} - A_{k-1}) \theta_k. \tag{6.20}$$

Proof. Note that, for all $k \geq 0$, R_k satisfies

$$R_k = R_{k-1} + B_k^{-\mathrm{T}} A_{k-1} B_k^{-1} - A_{k-1}, \tag{6.21}$$

and α_k satisfies

$$\alpha_k = R_k^+ R_{k-1} \alpha_{k-1} + R_k^+ (B_k^{-\mathrm{T}} A_{k-1} B_k^{-1} - A_{k-1}) \theta_k. \tag{6.22}$$

Furthermore, note that A_{-1} is positive definite. Hence, suppose for induction that A_{k-1} is positive definite. Since $\alpha_{-1} = \theta_0$, it follows that $R_{-1}(\alpha_{-1} - \theta_0) = 0$ and therefore, from Lemma 1, it follows that, for all $k \geq 0$,

$$\begin{aligned} A_k &= A_{k-1} + \phi_k^{\mathrm{T}} Q_k \phi_k + R_k - R_{k-1}, \\ &= A_{k-1} + \phi_k^{\mathrm{T}} Q_k \phi_k + B_k^{-T} A_{k-1} B_k^{-1} - A_{k-1} \\ &= \phi_k^{\mathrm{T}} Q_k \phi_k + B_k^{-T} A_{k-1} B_k^{-1}. \end{aligned}$$

Since B_k is nonsingular and A_{k-1} is positive definite, it follows that $B_k^{-T} A_{k-1} B_k^{-1}$ is positive definite. Thus A_k is positive definite.

Next, define $L_k \triangleq B_k P_k B_k^{\mathrm{T}}$. From Lemma 8 with $A = B_k^{-T} A_{k-1} B_k^{-1}$, $U = \phi_k^{\mathrm{T}}$, $C = Q_k$, and $V = \phi_k$, it follows that, for all $k \geq 0$,

$$\begin{aligned} P_{k+1} &= (B_k^{-T} A_{k-1} B_k^{-1} + \phi_k^{\mathrm{T}} Q_k \phi_k)^{-1} \\ &= L_k - L_k \phi_k^{\mathrm{T}} (Q_k^{-1} + \phi_k L_k \phi_k^{\mathrm{T}})^{-1} \phi_k L_k, \end{aligned}$$

which confirms (6.17). Furthermore, from Lemma 1, it follows that, for all $k \geq 0$,

$$\begin{aligned}
&\theta_{k+1} - \theta_k - P_{k+1}\phi_k^{\mathrm{T}} Q_k(y_k - \phi_k\theta_k) \\
&= P_{k+1}R_k(\alpha_k - \theta_k) - P_{k+1}R_{k-1}(\alpha_{k-1} - \theta_k) \\
&= P_{k+1}[R_k(R_k^+ R_{k-1}\alpha_{k-1} + R_k^+[B_k^{-\mathrm{T}} A_{k-1} B_k^{-1} - A_{k-1}]\theta_k - \theta_k) - R_{k-1}(\alpha_{k-1} - \theta_k)] \\
&= P_{k+1}[(\Pi_k R_{k-1}\alpha_{k-1} - R_{k-1}\alpha_{k-1}) + \Pi_k(B_k^{-\mathrm{T}} A_{k-1} B_k^{-1} - A_{k-1})\theta_k - (R_k - R_{k-1})\theta_k] \\
&= P_{k+1}[(\Pi_k - I_n)R_{k-1}\alpha_{k-1} + (\Pi_k - I_n)(B_k^{-\mathrm{T}} A_{k-1} B_k^{-1} - A_{k-1})\theta_k] \\
&= P_{k+1}(\Pi_k - I_n)\gamma_k,
\end{aligned}$$

which confirms (6.18). □

If R_k is nonsingular, then $\Pi_k = I_n$. Assuming that this is the case for all $k \geq 0$, Theorem 7 specializes to the following result.

Corollary 2. *Under the notation and assumptions of Theorem 7, let $k \geq 0$ and assume that R_k is nonsingular. Then θ_{k+1} is given by*

$$P_{k+1} = L_k - L_k\phi_k^{\mathrm{T}}(Q_k^{-1} + \phi_k L_k \phi_k^{\mathrm{T}})^{-1}\phi_k L_k, \tag{6.23}$$

$$\theta_{k+1} = \theta_k + P_{k+1}\phi_k^{\mathrm{T}} Q_k(y_k - \phi_k\theta_k), \tag{6.24}$$

where L_k is defined by (6.19)*.*

Note that, in Corollary 2, B_k can be viewed as a matrix forgetting factor. To see this, let $\lambda \in (0, 1]$ and let $B_k = \frac{1}{\sqrt{\lambda}} I_n$. Then (6.23)-(6.24) specialize to the traditional RLS equations with forgetting factor λ. Hereafter, we refer to this specialization as RLS with constant-rate forgetting (RLS-CRF).

6.5 Specializations

Note that the nonsingular matrix B_k in (6.19), (6.23), (6.24) can be chosen arbitrarily. In particular, the following specializations of Corollary 2 choose B_k in order to achieve variable-rate and variable-direction forgetting.

Variable-Rate Forgetting (VRF). For all $k \geq 0$, let $\beta_k \in (0, \infty)$, $B_k = \beta_k I_n$, and $Q_k = I_p$. Then (6.19), (6.17), and (6.18) are given by

$$L_k = \beta_k P_k, \tag{6.25}$$

$$P_{k+1} = L_k - L_k \phi_k^{\mathrm{T}} (I_p + \phi_k L_k \phi_k^{\mathrm{T}})^{-1} \phi_k L_k, \tag{6.26}$$

$$\theta_{k+1} = \theta_k + P_{k+1} \phi_k^{\mathrm{T}} (y_k - \phi_k \theta_k). \tag{6.27}$$

Equations (6.25)–(6.27) give RLS with variable-rate forgetting (RLS-VRF) [86].

Variable-Direction Forgetting (VDF). Compute the singular value decomposition $P_k = U_k \Sigma_k U_k^{\mathrm{T}}$, where $U_k \in \mathbb{R}^{n \times n}$ is orthonormal, and define

$$\psi_k \triangleq \phi_k U_k. \tag{6.28}$$

Next, let $\varepsilon > 0$ be larger than the noise-to-signal ratio or, if no noise is present, larger than the machine zero. Finally, let $\lambda \in (0, 1)$, and define

$$\overline{\Lambda}_k(i,i) \triangleq \begin{cases} \sqrt{\lambda}, & \|\psi_{k,i}\| > \varepsilon, \\ 1, & \text{otherwise}, \end{cases} \tag{6.29}$$

where $\psi_{k,i}$ is the ith column of ψ_k. Finally, define

$$B_k \triangleq U_k \overline{\Lambda}_k^{-1} U_k^{\mathrm{T}}, \tag{6.30}$$

Then, with B_k given by (6.30), equations (6.19), (6.23), (6.24) give RLS with variable-

direction forgetting (RLS-VDF) [87].

Variable-Rate and -Direction Forgetting (VRDF). For all $k \geq 0$ and $i = 1, \ldots, p$, let $\beta_{i,k} \in (0, \infty)$, define

$$\overline{D}_k(i,i) \triangleq \begin{cases} \sqrt{\beta_{i,k}}, & \|\psi_{k,i}\| > \varepsilon, \\ 1, & \text{otherwise}, \end{cases} \tag{6.31}$$

with U_k, ψ_k, and $\psi_{k,i}$ defined as in (6.28), and let

$$B_k = U_k \overline{D}_k U_k^{\mathrm{T}}. \tag{6.32}$$

Then, with B_k given by (6.32), equations (6.19), (6.23), (6.24) give variable-rate and -direction forgetting (RLS-VRDF), which combines RLS-VRF and RLS-VDF.

6.6 Example: abrupt loss and recovery of persistency with abruptly changing parameters.

Consider a mass-spring-damper system with $m = 5$ kg, $k = 1$ N/m, and $b = 1$ N·s/m sampled at 1 sample/s, and suppose that at 200 samples the parameters of the system abruptly change to $k = 10$ N/m and $b = 0.01$ N·s/m and then at 1200 samples the parameters of the system abruptly change again to $k = 0.1$ N/m and $b = 10$ N·s/m. This system is modeled by the time-varying discrete-time transfer function

$$G_k(\mathbf{q}) = \begin{cases} \dfrac{0.4606\mathbf{q} + 0.4307}{\mathbf{q}^2 - 1.64\mathbf{q} + 0.8187}, & k < 200, \\ \dfrac{0.4218\mathbf{q} + 0.4215}{\mathbf{q}^2 - 0.3116\mathbf{q} + 0.998}, & 200 \leq k \leq 1200, \\ \dfrac{0.2834\mathbf{q} + 0.1482}{\mathbf{q}^2 - 1.127\mathbf{q} + 0.1353}, & k > 1200, \end{cases} \tag{6.33}$$

where $\mathbf{q}$ is the forward shift operator. Let the input to (6.33) be given by

$$u_k = \begin{cases} \tilde{u}_k, & k < 100 \text{ or } k > 1000, \\ \sin(0.01k), & 100 \leq k \leq 1000, \end{cases} \tag{6.34}$$

where, for all $k \geq 0$,

$$\tilde{u}_k \triangleq \sin(0.01k) + \sin(0.1k) + \sin(k) + \sin(10k). \tag{6.35}$$

Note that, for all $100 < k < 1000$, (6.34) is not persistently exciting. Furthermore, suppose that the output is corrupted with additive Gaussian white noise with standard deviation $\sigma = 0.025$. Finally, define

$$\beta_k \triangleq \begin{cases} 1 + \eta \, \mathrm{sat}_\gamma(E_\tau), & E_\tau > 1, \\ 1, & E_\tau \leq 1, \end{cases} \tag{6.36}$$

where η and γ are positive numbers, sat_γ is the unit-slope saturation function with saturation level γ, τ is a positive integer, and

$$E_\tau \triangleq \left(\frac{1}{\tau} \sum_{i=k-\tau}^{k} \| y_i - \phi_i \theta_i \|^2 \right)^{1/2}. \tag{6.37}$$

The forgetting factor produced by (6.36) increases when output measurements differ significantly from the predicted output, which occurs when the parameters change. To reduce sensitivity to noise, the output error is first fed through a moving average given by (6.37) and forgetting is only activated when the average output error exceeds unity.

Figure 6.1 shows the performance of RLS-CRF and RLS-VDF, both with $\lambda = 0.99$, and RLS-VRDF with β_k given by (6.36) with $\gamma = \eta = 1$. Note that, after both parameter changes, RLS-VRDF reconverges to the modified parameters more quickly than RLS-CRF and RLS-VDF. During the loss of persistency between $k = 100$ and $k = 1000$ samples, the

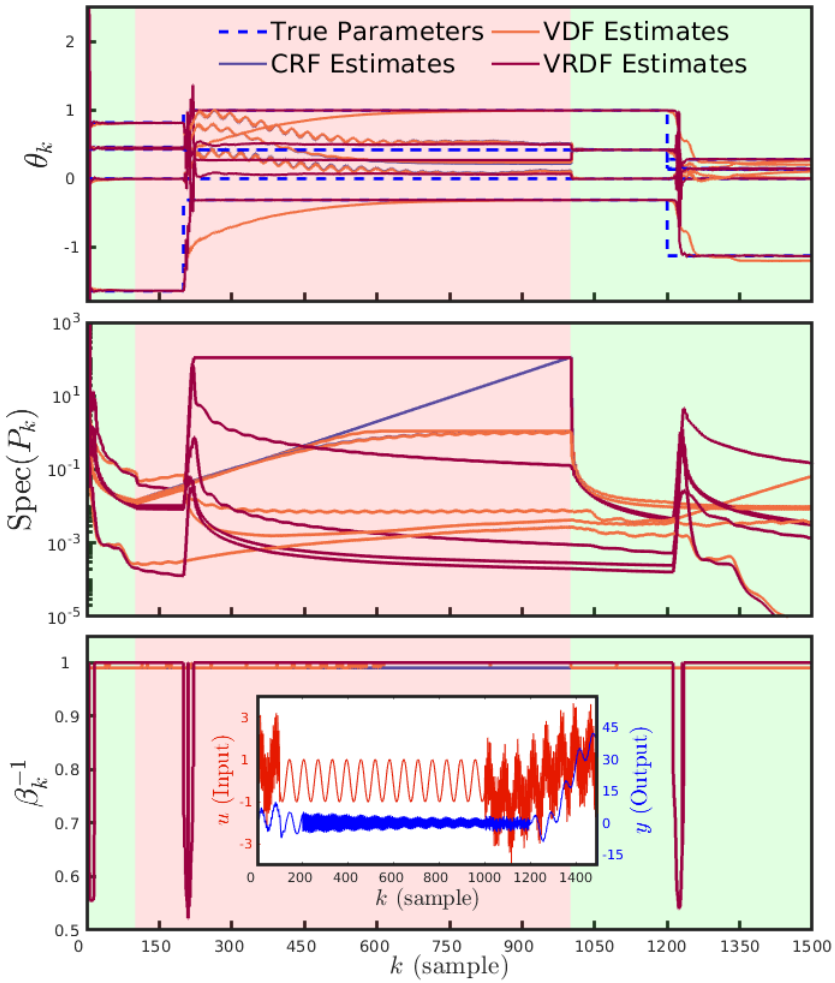

Figure 6.1: Parameter estimates θ_k, spectrum of P_k, and spectrum of B_k given by RLS-CRF and RLS-VDF with $\lambda = 0.99$ and by RLS-VRDF with β_k defined by (6.36). Intervals where the input is persistently exciting are shaded in light green, while the interval where persistency is lost is shaded in light red. The input and output signals are shown in the middle of the bottom figure for reference.

covariance of RLS-CRF begins to diverge while the covariances of RLS-VDF and RLS-VRDF remain bounded. ◇

Conclusions and Future Research

Future research will focus on techniques for reducing the computational complexity of the singular value decomposition of P_k. A starting point for this objective is the recursive

SVD presented in [89].

Appendix

Lemma 8. *(matrix inversion lemma) Let $A \in \mathbb{R}^{n\times n}$, $U \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{p\times p}$, and $V \in \mathbb{R}^{p\times n}$, and assume that A, C, and $A + UCV$ are nonsingular. Then*

$$(A+UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}. \tag{6.38}$$

Lemma 9. *Let $A \in \mathbb{R}^{n\times n}$, assume that A is positive definite, let $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$, and define $f : \mathbb{R}^n \to \mathbb{R}$ by*

$$f(x) \stackrel{\triangle}{=} x^{\mathrm{T}}Ax + 2b^{\mathrm{T}}x + c. \tag{6.39}$$

Then the unique minimizer of f is

$$x_{\mathrm{opt}} = -A^{-1}b, \tag{6.40}$$

and the minimum value of f is

$$f(x_{\mathrm{opt}}) = c - b^{\mathrm{T}}A^{-1}b. \tag{6.41}$$

CHAPTER 7

Necessary and Sufficient Regressor Conditions for the Global Asymptotic Stability of Recursive Least Squares

7.1 Preface

In the past two papers we have given results on the extension of ordinary RLS by generalizing the forgetting factor to vary based on step or direction. In this paper, we take a step back and resolve a basic theoretical issue in RLS itself–the existence of necessary and sufficient regressor conditions for the convergence of RLS to the true system parameters. Although persistency has been known to be a *sufficient* conditions for RLS convergence since the 1980s (see, for example, the sections on RLS in [12, 90]), examples of RLS convergence in which the regressor is not persistent were given in [24] and [91]. Hence persistency is not a *sufficient* condition for RLS convergence.

This prompts the question of whether or not a regressor condition can be found that is both necessary and sufficient for RLS convergence. This paper answers the question in the affirmative. In fact, there is an easily determined necessary and sufficient condition, but this condition has several drawbacks–such as having no apparent connection with the sufficient condition of persistency–that lead us to search out more effective necessary and sufficient conditions. Ultimately our search leads us to an understanding of the basic role

played by *bounded interval partitions* in RLS.

Since the paper describes the results and their development in detail, I will not further outline it here. Instead, I would like to say a few words on why this problem is important and how it connects with the foregoing work. Ultimately, we desired a complete theory connecting regressor behavior to estimator convergence, both for RLS and similar estimators, such as gradient descent (the subject of the following paper). This is desired for several reasons. First, it allows us to design effective online identification systems in which a model is desired for its own sake. Persistency is often one of the most difficult conditions to enforce in practice, and thus we would like to ensure that we are seeking the weakest possible conditions necessary to guarantee convergence–that is, that we are aware of necessary and sufficient conditions for estimate convergence. Second, when we are dealing with indirect adaptive control algorithms, the requirement of persistency is often too restrictive, since, as we mentioned in the introduction, adaptive systems have a tendency to lose persistency as the control objective is more closely met.

The logical place to begin the development of this theory is with the ordinary RLS algorithm, which is among the simplest and most widely used identification algorithms in existence. Hence the conditions of weak persistency of excitation and the existence of minimum eigenvalue Oresme partitions should be seen as a conceptual basis from which to develop further theory to understand the operation of these algorithms in the presence of forgetting and closed-loop operation.

7.2 Introduction

Many problems in system identification, signal processing, and control depend on least squares algorithms to estimate unknown parameters [26]. In real-time identification and adaptive control, an estimate of the unknown parameters is often required at each time step, and recursive estimation algorithms, such as recursive least squares (RLS) [48, 55],

are commonly used to meet this requirement. When using RLS, a persistently exciting (PE) regressor implies GAS, hence convergence of the RLS parameter estimates to the true parameter values in the absence of noise. However, PE is sufficient but not necessary for the GAS of RLS, and examples of non-PE regressors for which RLS is GAS have been reported in [24] and [91]. Since PE is ubiquitous in stability analyses for recursive estimators, the existence of non-PE regressors that ensure GAS prompts the question of whether or not PE can be directly generalized into a necessary and sufficient condition for the GAS of RLS.

There has been substantial research on weakening the assumption of PE in convergence and stability proofs for parameter estimation and adaptive control. Work on adaptive control methods that do not require PE, such as [92], has been done since the 1980s, and [93] studied boundedness of the RLS error without explicitly assuming PE in 1990. More recently, [94, 95] have developed concurrent learning algorithms that require excitation over only a finite time interval, a strictly weaker condition than PE. For gradient-based parameter estimators, PE is shown to be necessary and sufficient to guarantee exponential stability for gradient descent in [13, pp. 71-73] and [96–99], but [100] and [101] have shown that PE is not necessary for GAS. Recently, [102] has shown that the gradient-type algorithms proposed in 1967 and 1977 by [103] and [99], respectively, can converge asymptotically under conditions strictly weaker than PE, and a procedure for generating gradient-type estimators that converge without assuming PE is given in [104], and subsequently expanded in [105]. The extant literature thus concerns both the analysis of strictly-weaker-than-PE stability for traditional gradient-based estimators as well as the synthesis of estimators with asymptotically stable error dynamics under strictly-weaker-than-PE conditions.

In contrast to most of this work, analysis of strictly-weaker-than-PE conditions for RLS stability is sparse. A straightforward proof shows that the condition (cf. [101], [104])

$$\lim_{n\to\infty} \lambda_{\min}\left[\sum_{k=0}^{n} \phi_k^{\mathrm{T}}\phi_k\right] = \infty \tag{7.1}$$

is necessary and sufficient for the GAS of RLS, and it is shown in [93] that the component of the RLS error contained in the subspace

$$E \triangleq \left\{ x \in \mathbb{R}^n \colon \sum_{k=0}^{\infty} \|\phi_k x\|^2 = \infty \right\}, \tag{7.2}$$

whose defining condition is related to (7.1) in the case where $E = \mathbb{R}^n$, is driven asymptotically to zero. Since (7.1) is already a necessary and sufficient condition for the GAS of RLS convergence, it is natural to ask why further conditions should be sought.

There are two major reasons. First, the search for necessary and sufficient conditions is motivated by the existence of non-PE regressors for which RLS is GAS, and thus, a satisfactory condition should enable these regressors to be precisely characterized. The truism that these are exactly the regressors that do not satisfy the PE condition but satisfy (7.1) is not helpful, since it does not articulate the fundamental structural characteristics that cause these regressors to have the properties in question. A precise characterization that captures these fundamental structural features can be obtained by directly generalizing PE. In this context, a "direct generalization" is understood as a condition obtained by directly weakening or transforming the PE hypotheses, rather than by supplementing the PE hypotheses or their negation with additional conditions such as (7.1). A direct generalization that is also necessary and sufficient would by default maintain a clear and simple connection to PE, and thus reveal explicitly the gap between PE and non-PE sequences for which RLS is GAS. Creating a similarly precise characterization using only (7.1) would be difficult, and no such result has appeared before in the literature.

Second, PE is an intrinsically local condition; that is, a condition formulated in terms of finite sums only, without global analytic objects such as limits or infinite series. In contrast, (7.1) is global, involving a limit related to an infinite series of matrices. As well as the possibility that this could be difficult to use in applications, it also destroys the locality of PE. Thus, instead of possessing only a global necessary and sufficient condition, it is

reasonable to seek a necessary and sufficient condition that preserves locality.

The main contribution of this paper is two novel necessary and sufficient conditions for the GAS of RLS. The first, Weak persistent excitation (WPE), is a direct generalization of PE that relaxes the uniform summation window length and uniform lower bound conditions of PE carefully enough to maintain sufficiency while gaining necessity. Using WPE, we are able to state precisely how PE's required uniformity in both of these parameters allows for non-PE regressors that still satisfy (7.1). The second, the existence of a minimum eigenvalue Oresme partition (MEOP), is a matrix generalization of a standard comparison method for nonnegative series divergence proofs. Unlike condition (7.1) or WPE, the existence of a MEOP is a local condition, which requires only finite sums to verify, and thus both preserves the locality of the PE definition and provides an alternative route to proving GAS for RLS in cases for which the limits occurring in (7.1) and WPE are difficult to compute. While some implications among these three conditions are straightforward to prove, showing the equivalence of (7.1), WPE, and the existence of a MEOP is nontrival. Thus, the main result of this paper is the following:

Condition (7.1) *holds iff the regressor is WPE iff there exists a MEOP of the regressor sequence.*

That is, any one of these conditions is equivalent to the GAS of RLS.

The outline of this paper is as follows. After fixing notation and definitions, Section 7.4 briefly reviews RLS and defines PE in Section 7.4.2. Section 7.5 motivates the definition for WPE in depth by providing a step-by-step search for the 'right' generalization of PE, then introduces WPE and shows WPE is equivalent to a slightly rephrased version of condition (7.1) in Theorem 11, which is a nontrivial result for the nonscalar case. Finally, Section 7.6 defines the MEOP property for sequences of positive-semidefinite matrices and shows in Theorem 12 that such a sequence is WPE if and only if it has a MEOP. This theorem is

followed by two corollaries and a proposition, which allow us to characterize, in consistent terms, exactly the property that must be satisfied for a regressor sequence to not be PE but still ensure the GAS of RLS.

7.3 Preliminaries

We define $\mathbb{N} \triangleq \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 \triangleq \{0, 1, 2, 3, \dots\}$. The sets of $n \times n$ real symmetric, real symmetric positive-semidefinite, and real symmetric positive-definite matrices are denoted by $\mathbf{S}^n$, $\mathbf{N}^n$, and $\mathbf{P}^n$, respectively. For $S \in \mathbf{S}^n$, $\lambda_{\min}(S)$ denotes the smallest eigenvalue of S, $S > 0$ indicates that S is positive-definite, and $S \geq 0$ indicates that S is positive-semidefinite. I_n denotes the $n \times n$ identity matrix. $\mathcal{N}(A)$ denotes the nullspace of A. $(a_k) = (a_k)_{k \in \mathbb{N}_0} \subset X$ denotes the sequence $(a_0, a_1, \dots)$ whose components are elements of the set X. The sum over the empty set is defined to be zero. The kth harmonic sum is denoted by

$$H_k \triangleq \begin{cases} \sum_{p=1}^{k} 1/p, & k > 0, \\ 0, & k = 0. \end{cases} \tag{7.3}$$

For all $x \in \mathbb{R}$, we define $\lfloor x \rfloor \triangleq \max\{n \in \mathbb{N}_0 \colon n \leq x\}$.

Let $\mathcal{D} \subset \mathbb{R}^n$. Then, $x_e \in \mathcal{D}$ is an *equilibrium* of the sequence $(f_k \colon \mathcal{D} \to \mathcal{D})$ if, for all $k \in \mathbb{N}_0$, $f_k(x_e) = x_e$. The following are standard definitions of stability for an equilibrium of a discrete-time system.

Definition 5. *Let $\mathcal{D} \subset \mathbb{R}^n$, for all $k \in \mathbb{N}_0$, let $f_k \colon \mathcal{D} \to \mathcal{D}$, let x_e be an equilibrium of (f_k), and consider the initial value problem*

$$x_{k+1} = f_k(x_k), \tag{7.4}$$

$$x_{k_0} \in \mathcal{D}. \tag{7.5}$$

Then the equilibrium x_e of (7.4)-(7.5) *is*

$i)$ Lyapunov stable *if, for all $k_0 \in \mathbb{N}_0$ and $\varepsilon > 0$, there exists $\delta(k_0, \varepsilon) > 0$ such that $\|x_{k_0} - x_e\| < \delta$ implies that, for all $k \geq k_0$, $\|x_k - x_e\| < \varepsilon$.*

$ii)$ uniformly Lyapunov stable *if, for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that, for all $k_0 \in \mathbb{N}_0$, $\|x_{k_0} - x_e\| < \delta$ implies that, for all $k \geq k_0$, $\|x_k - x_e\| < \varepsilon$.*

$iii)$ globally attractive *if, for all $k \in \mathbb{N}_0$ and $x_{k_0} \in \mathcal{D}$, $\lim_{k\to\infty} x_k = x_e$.*

$iv)$ globally asymptotically stable *if it is Lyapunov stable and globally attractive.*

Definition 6. *Let $a > 0$. A function $f\colon [0, a) \to [0, \infty)$ is class $\mathcal{K}$ if it is increasing and $f(0) = 0$.*

Theorem 8. *Assume that the system* (7.4)-(7.5) *has an equilibrium $x_e \in \mathcal{D}$, and let $\mathcal{N} \subset \mathcal{D}$ be a neighborhood of x_e. If there exists a continuous function $V\colon \mathbb{N}_0 \times \mathcal{N} \to [0, \infty)$ and class $\mathcal{K}$ functions g and h such that, for all $k \in \mathbb{N}_0$ and $x \in \mathcal{N}$,*

$i)$ $V_k(x_e) = 0$,

$ii)$ $g(\|x\|) \leq V(k, x) \leq h(\|x\|)$,

$iii)$ $\Delta V(k, x) \stackrel{\triangle}{=} V(k + 1, f_k(x)) - V(k, x) \leq 0$,

then x_e is uniformly Lyapunov stable.

7.4 Recursive Least Squares with PE Regressors

7.4.1 Recursive Least Squares

The RLS algorithm was introduced in 1950 by [14], and treatments of the fundamental RLS theory can be found in many introductory textbooks, such as [13, 26, 27, 49]. In this section we briefly review the facts about RLS necessary to understand the remainder of

the paper. Further information on RLS using similar notation and conventions as we have adopted here can be found in [24, 48, 106].

Let $\theta, \theta_0 \in \mathbb{R}^n$, let $P_0 \in \mathbf{P}^n$, and, for all $k \in \mathbb{N}_0$, let $\phi_k \in \mathbb{R}^{p\times n}$ and define $y_k \triangleq \phi_k\theta$ and

$$J_k(\tilde{\theta}) \triangleq \sum_{i=0}^{k}(y_i - \phi_i\tilde{\theta})^{\mathrm{T}}(y_i - \phi_i\tilde{\theta}) + (\tilde{\theta} - \theta_0)^{\mathrm{T}}P_0^{-1}(\tilde{\theta} - \theta_0). \tag{7.6}$$

For all $k \in \mathbb{N}_0$, J_k is strictly convex and hence has a unique global minimizer, which we denote by $\theta_{k+1} \triangleq \min_{\tilde{\theta}\in\mathbb{R}^n} J_k(\tilde{\theta})$. The RLS algorithm is given by the update equations

$$\theta_{k+1} = \theta_k + P_{k+1}\phi_k^{\mathrm{T}}(y_k - \phi_k\theta_k), \tag{7.7}$$

$$P_{k+1} = P_k - P_k\phi_k^{\mathrm{T}}\left(I_p + \phi_k P_k \phi_k^{\mathrm{T}}\right)^{-1}\phi_k P_k, \tag{7.8}$$

which enable the recursive calculation of θ_{k+1} in terms of the previous minimizer θ_k, and current values of the output y_k, regressor ϕ_k, and covariance P_{k+1}. Defining the RLS error $\tilde{\theta}_k \triangleq \theta_k - \theta$ and $S_k \triangleq \phi_k^{\mathrm{T}}\phi_k$, the update equations

$$\tilde{\theta}_{k+1} = P_{k+1}P_k^{-1}\tilde{\theta}_k, \tag{7.9}$$

$$P_{k+1}^{-1} = P_k^{-1} + S_k, \tag{7.10}$$

follow from (7.7) and (7.8). Equations (7.9) and (7.10) can be solved to yield

$$\tilde{\theta}_k = P_k P_0^{-1}\tilde{\theta}_0, \tag{7.11}$$

$$P_k^{-1} = P_0^{-1} + \sum_{i=0}^{k-1} S_i. \tag{7.12}$$

Proving a stability property of the equilibrium θ of (7.7)-(7.8) is thus equivalent to proving the same property of the zero equilibrium for (7.9)-(7.10). Furthermore, since ϕ_k enters into (7.9) and (7.10) only in the form of S_k, it follows that every property of (ϕ_k) affecting

the RLS error is equivalent to a property of (S_k), and thus we shall henceforth consider only properties of (S_k).

It is possible that (7.9)-(7.10) has equilibria other than the zero equilibrium. In particular, if there exists $p \in \mathcal{N}(S_k) \setminus \{0\}$ for all $k \in \mathbb{N}_0$, then p is a nonzero equilibrium of (7.9)-(7.10). Thus we restrict our attention to the case where $\cap_{k\in\mathbb{N}_0}\mathcal{N}(S_k) = \varnothing$, which corresponds to the case where RLS converges to the true parameters regardless of the initial conditions. The following result shows that (7.9)-(7.10) is uniformly Lyapunov stable regardless of the properties of the regressor. Thus, necessary and sufficient conditions on (S_k) for the GAS of (7.9)-(7.10) are equivalent to necessary and sufficient conditions on (S_k) for global attractivity of the zero equilibrium. Similar results are found in [24, Theorem 6].

Proposition 4. *Let $(S_k) \subset \mathbf{N}^n$ and $P_0 \in \mathbf{P}^n$. Then the zero equilibrium of* (7.9)-(7.10) *is uniformly Lyapunov stable.*

Proof. Define the Lyapunov candidate function

$$V(k,\tilde{\theta}) \triangleq \tilde{\theta}^{\mathrm{T}} P_k^{-1}(I_n + P_k^{-2})^{-1} P_k^{-1}\tilde{\theta}. \tag{7.13}$$

For all $k \in \mathbb{N}_0$, $V(k,\tilde{\theta})$ is a continuous function of $\tilde{\theta}$ that satisfies $V(k,0) = 0$. Moreover, noting that $(I + P_k^{-2})^{-1} \in \mathbf{P}^n$ satisfies $I - P_k^{-1}(I + P_k^{-2})^{-1}P_k^{-1} = (I + P_k^{-2})^{-1}$, it follows that $V(k,\tilde{\theta}) \leq \|\tilde{\theta}\|^2$. Furthermore, since $S_k \geq 0$, it follows that P_k^{-1} is nondecreasing, and thus

$$\begin{aligned}
\tilde{\theta}^{\mathrm{T}} P_k^{-1}(I_n + P_k^{-2})^{-1} P_k^{-1}\tilde{\theta} &\geq \lambda_{\min}((I_n + P_k^{-2})^{-1})\|P_k^{-1}\tilde{\theta}\|^2 \\
&= \|I_n + P_k^{-2}\|^{-1}\|P_k^{-1}\tilde{\theta}\|^2 \\
&\geq (1 + \|P_k^{-1}\|^2)^{-1}\|P_k^{-1}\|^2\|\tilde{\theta}\|^2 \\
&\geq (1 + \|P_0^{-1}\|^2)^{-1}\|P_0^{-1}\|^2\|\tilde{\theta}\|^2,
\end{aligned}$$

where the final step follows from the fact that $x/(1+x)$ is a nondecreasing function of x.

Thus, for all $k \in \mathbb{N}_0$ and $\tilde{\theta} \in \mathbb{R}^n$, $V(k, \tilde{\theta})$ is bounded above and below by class $\mathcal{K}$ functions. Finally, since

$$\begin{aligned}\Delta V(k, \tilde{\theta}) &= V(k+1, P_{k+1}P_k^{-1}\tilde{\theta}) - V(k, \tilde{\theta}) \\ &= \tilde{\theta}^{\mathrm{T}} P_k^{-1}[(I + P_{k+1}^{-2})^{-1} - (I + P_k^{-2})^{-1}]P_k^{-1}\tilde{\theta}, \end{aligned} \tag{7.14}$$

and since $P_{k+1}^{-1} \geq P_k^{-1}$ implies that $(I + P_{k+1}^{-2})^{-1} \leq (I + P_k^{-2})^{-1}$, it follows that $V(k, \tilde{\theta})$ satisfies Theorem 5, and thus (7.9)-(7.10) is uniformly Lyapunov stable. □

Although the RLS error is always uniformly Lyapunov stable, global attractivity, and hence GAS, depends strongly on properties of the regressor. In the next section, we investigate PE as a property of (S_k) that guarantees global attractivity of the zero equilibrium of (7.9)-(7.10).

7.4.2 PE and RLS Stability

Definition 7. $(S_k) \subset \mathbf{N}^n$ *is* persistently exciting (PE) *if there exist* $\alpha > 0$ *and* $N \in \mathbb{N}_0$ *such that, for all* $j \in \mathbb{N}_0$, $\sum_{i=0}^{N} S_{i+j} \geq \alpha I_n$.

If (S_k) is bounded, then $\sum_{i=0}^{N} S_{i+j}$ has an upper bound in addition to the lower bound given in Definition 7, and many definitions of PE, such as [49, p. 64], include an upper bound to reflect this assumption. However, since the upper bound is not relevant to GAS, we have omitted it to simplify Definition 7.

Theorem 9. *Let* (S_k) *be PE, and, for all* $k \in \mathbb{N}_0$, *define* P_k *by* (7.10). *Then i)* $\lim_{k\to\infty} P_k = 0$, *and ii)* (7.9)-(7.10) *is GAS.*

Proof. A proof of *i*) is given in [24, Proposition 3] and the uniform Lyapnunov stability of (7.9)-(7.10) follows from Proposition 4. Hence, to prove attractivity and thus *ii*), let $k_0 \in \mathbb{N}_0$ and $\theta_{k_0} \in \mathbb{R}^n$. From (7.9) it follows that $\tilde{\theta}_k = P_k P_{k_0}^{-1}\theta_{k_0}$ and thus $\lim_{k\to\infty} \tilde{\theta}_k = 0$ implies $\lim_{k\to\infty} P_k = 0$. □

The converse of Theorem 9 is false. To demonstrate this with an example, we first prove the following proposition.

Proposition 5. *Let* $(S_k) \subset \mathbf{N}^n$ *be PE. Then, for all* $k \in \mathbb{N}_0$, $\sum_{i=0}^{k} S_i \geq \left\lfloor \frac{k+1}{N+1} \right\rfloor \alpha I_n$.

Proof. Note that $\left\lfloor \frac{k+1}{N+1} \right\rfloor \alpha I_n \leq \sum_{j=0}^{\left(\lfloor \frac{k+1}{N+1} \rfloor - 1\right)} \sum_{i=0}^{N} S_{i+j(N+1)} \leq \sum_{i=0}^{(N+1)\lfloor \frac{k+1}{N+1} \rfloor - 1} S_i \leq \sum_{i=0}^{k} S_i$. □

Example 1. Let $S_k = \frac{1}{k+1}$. From (7.12), it follows that

$$\lim_{k\to\infty} P_k = \lim_{k\to\infty} \left(P_0^{-1} + H_k\right)^{-1} = 0,$$

and thus (7.9)-(7.10) is GAS. Noting that, for all $k \in \mathbb{N}_0$, $\sum_{i=0}^{k} S_i = H_{k+1} < \log(k+1)+1$, suppose for contradiction that (S_k) is PE with lower bound $\alpha > 0$ and window parameter $N \in \mathbb{N}_0$. Then $\frac{\alpha k}{2N} < \log(k+1) + 1$, which is false for all sufficiently large $k \in \mathbb{N}_0$, contradicting Proposition 5. ◇

7.5 From PE to WPE

Having shown that PE is sufficient but not necessary for GAS, we now present WPE, a direct generalization of PE that is both necessary and sufficient for GAS. Since the full relationship between PE and WPE as given by Definition 10 might not be immediately apparent, we justify our claim that WPE is a direct generalization of PE by beginning with PE and providing a step-by-step process in which WPE is deduced.

First, we prove that (7.1) is necessary and sufficient for the GAS of RLS. Although we are ultimately seeking an alternative to (7.1), it nonetheless convenient for checking the necessity and sufficiency of proposed generalizations of PE.

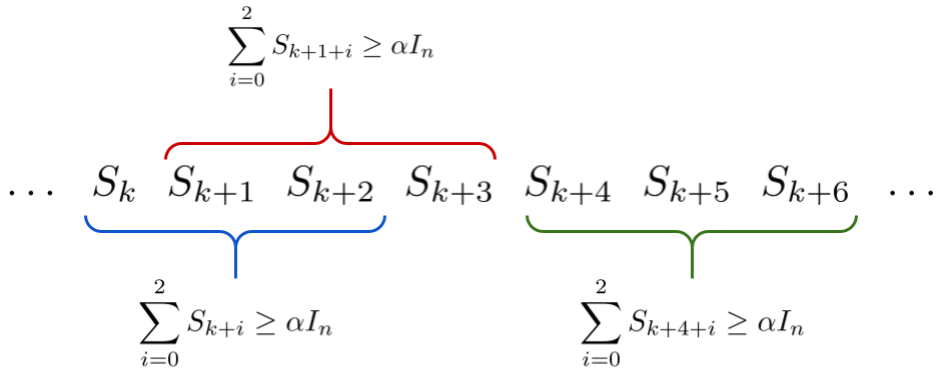

Figure 7.1: A PE sequence with $N = 2$ (window length of 3) showing several summation windows. PE is characterized by strict uniformity in the window length parameter N and the lower bound α, both of which must have values that are independent of the position of the window in the sequence.

Theorem 10. *The zero equilibrium of* (7.9)-(7.10) *is GAS if and only if*

$$\lim_{n\to\infty} \lambda_{\min}\left(\sum_{k=0}^{n} S_k\right) = \infty. \tag{7.15}$$

Proof. Sufficiency is immediate. To prove the converse direction, note that, for all $n \in \mathbb{N}_0$, $\lambda_{\min}(P_n^{-1}) \leq \lambda_{\max}(P_0^{-1}) + \lambda_{\min}\left(\sum_{k=0}^{n} S_k\right)$. From (7.9), it follows that $\lim_{k\to\infty} \|\tilde{\theta}_k\| = \lim_{k\to\infty} \|P_k P_0^{-1} \tilde{\theta}_0\| = 0$, hence $\lim_{k\to\infty} \|P_k\| = \lim_{k\to\infty} \lambda_{\min}^{-1}(P_k^{-1}) = 0$, and therefore, $\lim_{k\to\infty} \lambda_{\min}\left(\sum_{i=0}^{k} S_i\right) \geq \lim_{k\to\infty} \lambda_{\min}(P_k^{-1}) - \lambda_{\max}(P_0^{-1}) = \infty$. □

To generalize the hypotheses of PE, we first note that PE is a condition concerning finite windows of sequential indices. In this context, a *window* is a finite tuple[1] of consecutive indices. For PE, each index k is associated with a unique window $W_k = (k, k+1, \ldots, k+N)$, as illustrated in Figure 7.1. Henceforth, we shall informally refer to this kind of figure as a *window diagram*. As shown in Figure 7.1, PE requires that N, which determines the window length, and the lower bound α be uniform within the sequence (W_k) of windows. From this perspective, a natural generalization of PE would be to relax the requirement of

[1]We will not distinguish between tuples with elements in $\mathbb{N}_0$ and subsets of $\mathbb{N}_0$, since the ordering of the former is identical to the usual ordering of $\mathbb{N}_0$ and since we do not consider tuples containing multiple copies of a single element.

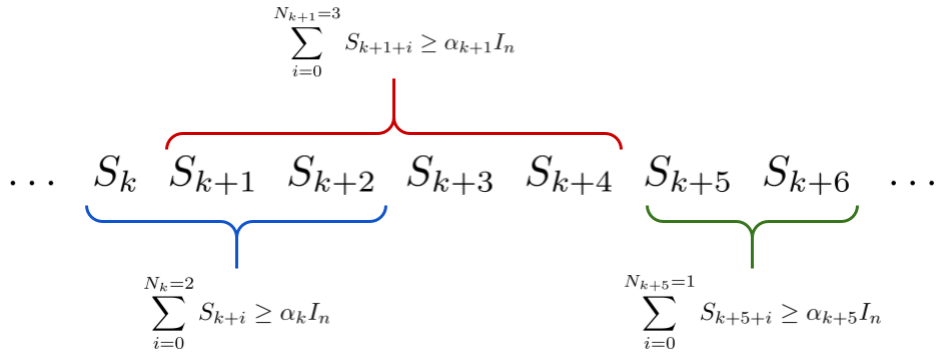

Figure 7.2: A GPE-1/GPE-2 sequence showing several summation windows. In contrast to PE, the window length and lower bound are allowed to vary based on the position of the window in the sequence. The GPE-1 and GPE-2 conditions completely relax the uniformity in N and α demanded by PE, but lose sufficiency for guaranteeing GAS by doing so.

uniformity for N and α. Naively doing so, we obtain the following condition.

Definition 8. *The sequence* $(S_k) \subset \mathbf{N}^n$ *is* GPE-1 *if, for all* $k \in \mathbb{N}_0$*, there exist* $N_k \geq 0$ *and* $\alpha_k > 0$ *such that* $\sum_{i=0}^{N_k} S_{k+i} \geq \alpha_k I_n$.

Figure 7.2 shows the window diagram for GPE-1, illustrating the nonuniformity in N and α. Unfortunately, GPE-1 relaxes PE too much. Although necessary, GPE-1 is not sufficient for (7.15), as can be seen from the sequence $(1/k^2)$, which satisfies GPE-1 with $N_k = 0$ and $\alpha_k = 1/k^2$, but not (7.15). This example suggests that GPE-1 is insufficient because (α_k) is not required to diverge, and thus cannot ultimately force the divergence of $\lim_{n\to\infty} \lambda_{\min} (\sum_{k=0}^{n} S_k)$. Hence, our first refinement of GPE-1 is to require that $\sum_{k=0}^{\infty} \alpha_k = \infty$.

Definition 9. *The sequence* $(S_k) \subset \mathbf{N}^n$ *is* GPE-2 *if, for all* $k \in \mathbb{N}_0$*, there exist* $N_k \geq 0$ *and* $\alpha_k > 0$ *such that* $\sum_{i=0}^{N_k} S_{k+i} \geq \alpha_k I_n$ *and* $\sum_{k=0}^{\infty} \alpha_k = \infty$.

The window diagram for GPE-2 can be pictured in the same way as for GPE-1 (Figure 7.2), since the only difference is that GPE-2 requires $\sum_{k=0}^{\infty} \alpha_k = \infty$. GPE-2 is also necessary but not sufficient for (7.15), as can be seen from the sequence $(s_k) = (1, 1, 1/2^2, 1/2^2, 1/3^2, 1/3^2, \ldots, 1/k^2, 1/k^2, \ldots)$, which satisfies GPE-2 with $N_k = k + 1$

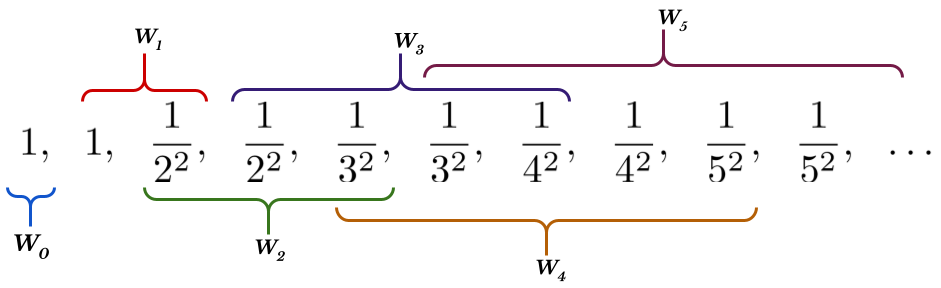

Figure 7.3: The first five summation windows defined by $N_k = k + 1$ for the GPE-2 sequence $1, 1, 1/2^2, 1/2^2, 1/3^2, 1/3^2, \ldots$. Note the overcounting-like effect due to the intersecting windows.

and $\alpha_k = \sum_{i=0}^{N_k} s_{i+k}$. In particular, the requirement that $\sum_{k=0}^{\infty} \alpha_k = \infty$ is satisfied because

$$\begin{aligned}\sum_{k=0}^{\infty} \alpha_k &= 1 + \left(1 + \frac{1}{2^2}\right) + \left(\frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{3^2}\right) + \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{4^2}\right) + \cdots \\ &> 1 + \frac{2}{2^2} + \frac{3}{3^2} + \frac{4}{4^2} + \cdots = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty.\end{aligned}$$

However, since $\sum_{k=0}^{\infty} s_k = 2\sum_{k=0}^{\infty} 1/k^2 < \infty$, it follows that (7.15) is not satisfied.

To understand how this can occur, consider Figure 7.3, which shows (s_k) with the first five windows of defined by (N_k) overlaid. Note that adjacent windows have large intersections, which exaggerates the effect of individual terms of (s_k) on $\sum_{k=0}^{\infty} \alpha_k$ in a way similar to overcounting. When there are large intersections between adjecent windows, it is possible that $\sum_{k=0}^{\infty}(s_k + s_{k+1} \cdots + s_{k+N_k})$ diverges even though $\sum_{k=0}^{\infty} s_k$ converges, which suggests that intersections between distinct windows must be disallowed. To formulate a non-intersection requirement, we replace hypotheses formulated in terms of N_k with hypotheses formulated in terms of windows. PE, GPE-1, and GPE-2 use sequences of windows (W_k) for which every window W_k begins at k (that is, for all k, $\min W_k = k$), but such a window sequence will have intersections unless each window has a length of 1. Hence, we dispense with the property that $\min W_k = k$ for all $k \in \mathbb{N}_0$, and restrict GPE-2 to window sequences whose consecutive elements are strictly adjacent in the sense that, for

all $k \in \mathbb{N}_0$, $\min W_{k+1} = \max W_k + 1$. Collecting the results of this analysis, we find that, for all $k, k' \in \mathbb{N}_0$, the following three conditions must hold:

i) W_k is a *bounded interval* of $\mathbb{N}_0$; that is, a tuple $(n, n+1, \ldots, n+m) \subset \mathbb{N}_0$ such that $n, m \in \mathbb{N}_0$.

ii) $k \neq k'$ implies that $W_k \cap W_{k'} = \varnothing$.

iii) $\cup_{k \in \mathbb{N}_0} W_k = \mathbb{N}_0$.

The last two conditions state that (W_k) is a *partition* of $\mathbb{N}_0$ in the set-theoretic sense. Thus, we define a *bounded interval partition* (BIP) of $\mathbb{N}_0$ to be a sequence (W_k) of subsets of $\mathbb{N}_0$ satisfying the three conditions given above. That is, a BIP is a partition of $\mathbb{N}_0$ whose elements are bounded intervals and whose order is inherited from the order of their minimal elements. The next definition, WPE, modifies GPE-2 to include a non-intersection requirement phrased in terms of BIPs.

Definition 10. *The sequence* (S_k) *is* weakly persistently exciting (WPE) *if there exists a BIP* (W_k) *and a sequence* $(\alpha_k) \subset [0, \infty)$ *such that* $\sum_{k=0}^{\infty} \alpha_k = \infty$ *and, for all* $k \in \mathbb{N}_0$, $\sum_{i \in W_k} S_i \geq \alpha_k I_n$.

Figure 7.4 shows a window diagram for WPE. Three observations are immediate. First, unlike GPE-1 or 2, WPE is sufficient for (7.1), as is seen by noting that

$$\lambda_{\min} \left[\sum_{k=0}^{n} S_k \right] \geq \sum_{k=0}^{K_n} \lambda_{\min} \left[\sum_{j \in W_k} S_j \right] \geq \sum_{k=0}^{K_n} \alpha_k,$$

where $K_n \triangleq \max\{\ell \in \mathbb{N}_0 \colon \max W_\ell \leq n\}$. Since $K_n \to \infty$ as $n \to \infty$, it follows that $\lim_{n \to \infty} \lambda_{\min} \left(\sum_{k=0}^{n} S_k \right) \geq \sum_{k=0}^{\infty} \alpha_k = \infty$. Second, every PE sequence is WPE, as can be seen by setting $\alpha_k = \alpha$ for all $k \in \mathbb{N}_0$ and taking the BIP $W_k = (kN + k + 1, \ldots, (k+1)N + k + 2)$

$$W_0 = (0, \ldots, N+1),\ W_1 = (N+2, \ldots, 2N+3),\ W_2 = (2N+3, \ldots, 3N+4),\ \ldots$$

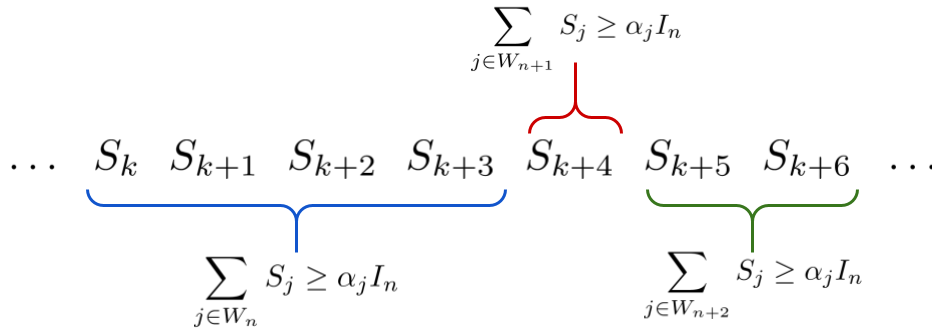

Figure 7.4: A WPE sequence showing several summation windows. Like GPE-1 and 2, but unlike PE, the window length and lower bound values are allowed to vary based on their position in the sequence. However, unlike GPE-1 and 2, WPE does not allow intersections between the windows. This diagram assumes that the nth window W_n begins at the kth element of the sequence.

Third, WPE is strictly weaker than PE. For example, the sequence

$$1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, \ldots,$$

in which the nth 1 is followed by n 0's, is WPE but not PE. These observations show that WPE improves PE as a sufficient condition for GAS, but do not show that WPE is necessary. The following result proves that it is.

Theorem 11. *The sequence* $(S_k) \subset \mathbf{N}^n$ *satisfies condition* (7.15) *if and only if it is WPE.*

Proof. Sufficiency is straightforward and an outline of the proof has been given in the previous paragraph. To prove necessity, note that (7.15) is equivalent to the existence of a nondecreasing, unbounded sequence $(h_k) \subset [0, \infty)$ such that, for all $n \in \mathbb{N}_0$, $\sum_{k=0}^{n} S_k \geq h_n I_n$. Let $\varepsilon > 0$. Since (h_k) is unbounded, it follows that there exists $m_0 > 0$ such that $h_{m_0} \geq \varepsilon$. Defining $\tilde{U}_0 = \{0, \ldots, m_0\}$, it follows that $\sum_{j\in\tilde{U}_0} S_j \geq h_{m_0} I_n \geq \varepsilon I_n$. Next, let $k \in \mathbb{N}_0$ and suppose for strong induction that there exist pairwise disjoint intervals $\tilde{U}_0, \ldots, \tilde{U}_k$ such that for all $0 \leq \ell \leq k$, $\sum_{j\in\tilde{U}_\ell} S_j \geq \varepsilon$. Since (h_k) is unbounded and nondecreasing, it follows that there exists $M > 0$ such that, for all $m \geq M$, $h_m \geq \lambda_{\max}\left(\sum_{j\in\cup_{\ell=0}^{k}\tilde{U}_\ell} S_j\right) + \varepsilon$. Hence, let $m_{k+1} = \max(M, \max_{0\leq\ell\leq k}(\max U_\ell) + 1)$ and de-

fine $\tilde{U}_{k+1} = \{\max_{0\le\ell\le k}(\max \tilde{U}_\ell) + 1, \ldots, m_{k+1}\}$. Since m_{k+1} is the maximum of a finite, bounded subset of $\mathbb{N}_0$, it follows that $m_{k+1} < \infty$, and thus $\tilde{U}_{k+1}$ is a bounded interval. Furthermore, since, for all $0 \le \ell \le k$, $\tilde{U}_{k+1} \cap \tilde{U}_\ell = \varnothing$, it follows that

$$\begin{aligned}
\varepsilon I_n &\le \left[\lambda_{\max}\left(\sum_{j\in\cup_{\ell=0}^{k}\tilde{U}_\ell} S_j\right) + \varepsilon\right] I_n - \sum_{j\in\cup_{\ell=0}^{k}\tilde{U}_\ell} S_j \\
&\le h_{m_{k+1}} I_n - \sum_{j\in\cup_{\ell=0}^{k}\tilde{U}_\ell} S_j \le \sum_{j=0}^{m_{k+1}} S_j - \sum_{j\in\cup_{\ell=0}^{k}\tilde{U}_\ell} S_j \\
&= \sum_{j\in\cup_{\ell=0}^{k+1}\tilde{U}_\ell} S_j - \sum_{j\in\cup_{\ell=0}^{k}\tilde{U}_\ell} S_j = \sum_{j\in\tilde{U}_{k+1}} S_j. \qquad (7.16)
\end{aligned}$$

Thus $(\tilde{U}_k)$ is a collection of disjoint finite intervals such that, for all $k \in \mathbb{N}_0$, $\sum_{j\in\tilde{U}_k} S_j \ge \varepsilon I_n$. Finally, define $U_k = \tilde{U}_k \cup \{\ell\colon \max \tilde{U}_k + 1 \le \ell \le \min \tilde{U}_{k+1} - 1\}$. Since $(\tilde{U}_k)$ is a collection of disjoint finite intervals, it follows that (U_k) is a collection of disjoint finite intervals. Furthermore, since, for all $p \in \mathbb{N}_0$, there exists $k \in \mathbb{N}_0$ such that either $p \in \tilde{U}_k$ or $p \in \{\ell\colon \max \tilde{U}_k + 1 \le \ell \le \min \tilde{U}_{k+1} - 1\}$, it follows that there exists $k \in \mathbb{N}_0$ such that $p \in U_k$, and thus (U_k) is a BIP of $\mathbb{N}_0$. Finally, since, for all $k \in \mathbb{N}_0$, $\sum_{j\in U_k} S_j \ge \sum_{j\in\tilde{U}_k} S_j \ge \varepsilon I_n$, setting $\alpha_k = \varepsilon$ for all $k \in \mathbb{N}_0$, it follows that (S_k) is WPE. □

The logical statements of PE, GPE-1, GPE-2, and WPE are summarized in Table 7.1 along with their status as necessary and/or sufficient conditions for GAS.

Condition	Logical Statement	Necessity/Sufficiency
PE	$\exists N \ge 0\, \exists\, \alpha > 0\, \forall k \ge 0 \left[\sum_{i=0}^{N} S_{k+i} \ge \alpha I_n\right]$	S but not N
GPE-1	$\forall k \ge 0\, \exists N_k \ge 0\, \exists \alpha_k > 0 \left[\sum_{i=0}^{N_k} S_{k+i} \ge \alpha_k I_n\right]$	N but not S
GPE-2	$\forall k \ge 0\, \exists N_k \ge 0\, \exists \alpha_k > 0 \left[\sum_{i=0}^{N_k} S_{k+i} \ge \alpha_k I_n \wedge \sum_{k=0}^{\infty} \alpha_k = \infty\right]$	N but not S
WPE	$\exists$ BIP$(W_k)\, \forall k \ge 0\, \exists \alpha_k > 0 \left[\sum_{i\in W_k} S_i \ge \alpha_k I_n \wedge \sum_{k=0}^{\infty} \alpha_k = \infty\right]$	Both N and S

Table 7.1: Summary of PE and its non-uniform generalizations. Only WPE is both necessary and sufficient for the GAS of RLS.

Note that by summing over k on each side of the inequality $\lambda_{\min}\left[\sum_{j\in W_k} S_j\right] \le \alpha_k I_n$ under the assumption that $\sum_{k=0}^{\infty} \alpha_k = \infty$, we obtain the conditions that, for all $k \in \mathbb{N}_0$,

$\lambda_{\min}\left[\sum_{j\in W_k} S_j\right] > 0$ and $\sum_{k=0}^{\infty} \lambda_{\min}\left[\sum_{j\in W_k} S_j\right] = \infty$, which are equivalent to the predicate of WPE given in Table 7.1 and shows that WPE can be stated without reference to (α_k). However, we have chosen to retain (α_k) in Definition 10 to more explicitly show its connection with PE. Finally, we note that the term 'weakly persistently exciting' is used in [107, p. 73] to refer to a regressor sequence with an asymptotically convergent and positive-definite mean. This usage is unrelated to the concept of WPE described in this paper.

7.6 A Local Necessary and Sufficient Condition Using Minimum Eigenvalue Oresme Partitions

In this section, we formulate an equivalent to WPE that is local in the sense that it is formulated without limits or infinite sums. Consider the following divergence proof of the harmonic series:

$$\begin{aligned} 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots &= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots \\ &\geq 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\cdots \\ &= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots = \infty. \end{aligned}$$

This proof, originally due to N. Oresme (c. 1320–1382), implicitly uses the BIP

$$W_1 = (1),\ W_2 = (2),\ W_3 = (3,4),\ W_4 = (5,6,7,8),\ \ldots$$

to define $g_k \stackrel{\triangle}{=} \sum_{j\in W_k} 1/j$. Since $\inf_{k\in\mathbb{N}_0} g_k = 1/2 > 0$, it follows that $\sum_{k=1}^{\infty} g_k = \infty$ and hence $\sum_{k=1}^{\infty} 1/k = \infty$. It is straightforward to see how this strategy can be generalized to all nonnegative sequences by systematizing the formation of (g_k).

Definition 11. *The sequence* $(g_k) \subset [0, \infty)$ *is an* Oresme partition (OP) *of the sequence* $(a_k) \subset [0, \infty)$ *if there exists a BIP* (W_k) *of* $\mathbb{N}_0$ *such that* $g_k = \sum_{j \in W_k} a_j$ *and* $\inf_{k \in \mathbb{N}_0} g_k > 0$.

We now state the crux of Oresme's proof as a proposition.

Proposition 6. *If* $(a_k) \subset [0, \infty)$ *has an OP, then* $\sum_{k=0}^{\infty} a_k = \infty$.

An immediate question is whether or not the converse holds: does the divergence of a nonnegative series imply the existence of an OP? We will show that this is true, and, moreover, that the following definition, which generalizes Definition 11 to positive-semidefinite matrices, yields a condition equivalent to WPE.

Definition 12. *The sequence* $(g_k) \subset [0, \infty)$ *is a* minimum eigenvalue Oresme partition (MEOP) *of the sequence* $(S_k) \subset \mathbf{N}^n$ *if there exists a BIP* (W_k) *of* $\mathbb{N}_0$ *such that* $g_k = \lambda_{\min}\left(\sum_{j \in W_k} S_j\right)$ *and* $\inf_{k \in \mathbb{N}_0} g_k > 0$.

The existence of a MEOP is a local property, since it can be formulated using only hypotheses involving finite sums. Furthermore, since Theorem 12 will show that having a MEOP is equivalent to being WPE, it follows that having a MEOP is a local necessary and sufficient condition for the GAS of RLS. To prove Theorem 12, we need the following lemma.

Lemma 10. *Let* $(a_k) \subset [0, \infty)$ *satisfy* $\sum_{k=0}^{\infty} a_k = \infty$. *Then, for all* $p \in \mathbb{N}_0$, *there exists* $n > p$ *such that* $\sum_{j=p}^{n} a_j \geq a_0$.

Proof. Since the result is immediate for $a_0 = 0$, let $a_0 > 0$ and suppose for contrapositive that there exists $l \geq 0$ such that, for all $n > l$, $\sum_{j=l}^{n} a_j < a_0$. Hence, for all $n > l$, $\sum_{j=0}^{n} a_j = \sum_{j=0}^{l-1} a_j + \sum_{j=l}^{n} a_j \leq \sum_{j=0}^{l-1} a_j + a_0$, and thus $\sum_{k=0}^{\infty} a_k$ is finite. □

Theorem 12. *The sequence* $(S_k) \subset \mathbf{N}^n$ *is WPE if and only if it has a MEOP.*

Proof. Let (g_k) be a MEOP of (S_k). Then sufficiency follows immediately from setting $\alpha_k = g_k$ for all $k \in \mathbb{N}_0$. To prove necessity, suppose that (S_k) is WPE and let (U_k) be a BIP such that, for all $k \in \mathbb{N}_0$, $\lambda_{\min}\left(\sum_{j \in U_k} S_j\right) > 0$ and $\sum_{k=0}^{\infty} \lambda_{\min}\left(\sum_{j \in U_k} S_j\right) = \infty$.

For all $k \in \mathbb{N}_0$, set $\lambda_k = \lambda_{\min}\left(\sum_{j\in U_k} S_j\right)$. Next, define $W_0 = \{0\}$, and suppose for strong induction that there exist pairwise disjoint bounded intervals $W_0, W_1, \ldots, W_k$ such that, for all $0 \leq j \leq k$, $\sum_{i\in W_j} \lambda_i \geq \lambda_0$. Since $\sum_{k=0}^{\infty} \lambda_k = \infty$, Lemma 10 implies that there exists $n_{k+1} > \max(W_k) + 1$ such that $\sum_{j=\max(W_k)+1}^{n_{k+1}} \lambda_j \geq \lambda_0$. Defining $W_{k+1} = \{\max(W_k) + 1, \ldots, n_{k+1}\}$, it follows that $W_0, \ldots, W_{k+1}$ are pairwise disjoint bounded intervals such that, for all $0 \leq j + 1 \leq k$, $\sum_{i\in W_j} \lambda_i \geq \lambda_0$, and thus the principle of strong induction implies that there exists a BIP (W_k) with these properties. Next, define the BIP (V_k), for all $k \geq 0$, by $V_k = \bigcup_{j\in W_k} U_j$. Since $\lambda_0 > 0$ by construction, and since $\lambda_{\min}\left(\sum_{i\in V_k} S_i\right) = \lambda_{\min}\left(\sum_{j\in W_k}\sum_{i\in U_j} S_i\right) \geq \sum_{j\in W_k} \lambda_{\min}\left(\sum_{i\in U_k} S_i\right) \geq \lambda_0$, it follows that $\inf_{k\in\mathbb{N}_0} \lambda_{\min}\left(\sum_{i\in V_k} S_i\right) = \lambda_0 > 0$, and hence (g_k) defined for all $k \in \mathbb{N}_0$ by $g_k \stackrel{\triangle}{=} \lambda_{\min}\left(\sum_{i\in V_k} S_i\right)$ is a MEOP for (S_k). □

Since WPE is equivalent to (7.15), setting $n = 1$ in Theorem 12 yields the result promised after Proposition 6.

Corollary 3. *Let $(a_k) \subset [0, \infty)$. Then $\sum_{k=0}^{\infty} a_k = \infty$ if and only if there exists an OP of (a_k).*

Moreover, Theorem 12 also implies the following fact, which is implicit in the proof of Theorem 11.

Corollary 4. *The sequence (S_k) is WPE with lower bound sequence (α_k) if and only if it is WPE with a constant lower bound sequence $\alpha_k = \alpha > 0$.*

This result implies that WPE could be equivalently stated without altering the uniform lower bound of requirement of PE. However, we have chosen to retain the variable lower bound to emphasize the flexibility of the WPE hypotheses, the full range of which can be helpful in practice. For example, the sequence

$$1, 0, \frac{1}{2}, 0, 0, \frac{1}{4}, 0, 0, 0, \ldots$$

is more easily verified to be WPE using Definition 10 and the BIP $W_1 = (1, 2),\ W_2 = (3, 4, 5),\ W_3 = (5, 6, 7, 8),\ \ldots$ than it would be from a definition that required a uniform lower bound. Note, however, the powerful converse of Corollary 4. Although such a BIP might be difficult to find, the fact that the sequence is WPE implies that a BIP does exist for which the uniform lower bound property holds.

Finally, PE can be stated in terms of BIPs and MEOPs.

Proposition 7. *Let $(S_k) \subset \mathbf{N}^n$. Then (S_k) is PE if and only if it has a MEOP whose BIP (U_k) consists of constant-length intervals. That is, For all $k, k' \in \mathbb{N}_0$, $|U_k| = |U_{k'}|$.*

Proof. Sufficiency follows immediately by using the BIP $U_0 = (0, \ldots, N)$, $U_1 = (N + 1, \ldots, 2N + 1), \ldots$. To prove necessity, suppose that (g_k) is a MEOP of (S_k) with BIP (U_k) of constant-length intervals and set $N = 2|U_0|$ and $\alpha = \inf_{k\in\mathbb{N}_0} g_k > 0$. Since, for all $k \in \mathbb{N}_0$, $|U_k| = N/2$, it follows that, for all $j \in \mathbb{N}_0$, there exists $k(j) \in \mathbb{N}_0$ such that $U_{k(j)} \subset (j, j + 1, \ldots, j + N)$, and thus $\sum_{i=0}^{N} S_{i+j} \geq \sum_{i\in U_{k(j)}} S_i \geq \alpha$. □

This proposition allows us to articulate, in consistent terms, the essential structural property of non-PE regressors for which RLS is GAS. Specifically, defining a *uniform BIP* to be a BIP for which each interval has the same length and a *variable BIP* as a BIP for which there exist at least two intervals with different lengths, we have the following result.

Corollary 5. *The non-PE sequences that guarantee the GAS of RLS are exactly the sequences that have a MEOP with a variable BIP, but no MEOP with a uniform BIP.*

CHAPTER 8

Sequential Gradient-Descent Optimization of Data-Dependent, Rank-Deficient Cost Functions with a Unique Common Global Minimizer

8.1 Preface

Finally, our last paper shifts the focus from RLS to a related algorithm, sequential gradient descent. Although not explicitly stated in the paper, the motivation for this work also lies in the study of forgetting. Suppose we start with VRF and look at the case in which the forgetting factor is error-actuated and an abrupt change has just taken place. In this circumstance, the forgetting factor will be very large during the first few steps after the abrupt change takes place, and the data used in the estimate will be approximately the same as if we were to use only the most recent few data points. The most extreme case of this would be using only the single most recent measurement. That is, minimizing

$$J_k(\theta) = \frac{1}{2}\|y_k - \phi_k\theta\|^2. \tag{8.1}$$

Although it might be difficult to imagine instances in which this *instantaneous least-squares cost* is preferable to cumulative RLS (possibly with forgetting), it nonetheless appears to be the case that in certain applications of DDRCAC, use of the instantaneous

cost is an effective way of increasing the numerical stability of the controller while still maintaining good estimate quality.

When ϕ_k is not full-rank, (8.1) does not have an analytic formula for its minimizer. In fact, this cost is in general convex, but not *strictly* convex, and thus admits an uncountably infinite number of minimizers, consisting of all the points in some subspace of the parameter space. Thus the ordinary RLS update and its variants cannot be used to minimize (8.1). Instead, we use the gradient-descent update

$$\theta_{k+1} = \theta_k - \mu_k \nabla J_k(\theta), \tag{8.2}$$

in which the minimizer is approximated but not computed exactly. Note that in this case the cost depends on the step as well as the estimate, which makes this algorithm, *sequential gradient descent*, essentially different from the elementary *fixed-cost gradient descent* that is found in most optimization textbooks. The question that now concerns us is another one of convergence of the estimates to the true system parameters, which, like that of the previous paper, can be phrased in terms of global asymptotic stability of the dynamical system defined by the recursive formula for the estimator. We note that while it is possible to consider *subiterations* of (8.2); that is, multiple gradient-descent iterations using the same cost and executed in the time between successive measurement updates; these subiterations will only affect the convergence-rate, not asymptotic convergence or global asymptotic stability, and hence we may take, without loss of generality, the form given in (8.2).

The goal is to provide conditions for the stepsize μ_k so that sequential gradient descent is GAS for a "reasonable" set of costs. In this case, our set of costs will be general quadratic cost functions, which includes all least-squares costs that appear in system identification. The conditions on the stepsize are harder to state simply. Starting with the concept of a bounded interval partition developed in the preceding paper, we are able to deduce a condition called *weak ultimate positivity* which requires the stricter concept of a *uniformly*

bounded interval partition. If the sequence defined by the stepsize multiplied by the Hessian of the cost at each step is weakly ultimately positive, and the stepsize is bounded by the reciprocal of the maximum eigenvalue of the Hessian, then sequential gradient-descent is GAS.

One final thing worth noting about this work is that theorems asserting results about *stability* are proven using results from *fixed-point theory* and not the more familiar Lyapunov theory. To see why Lyapunov's second method cannot be used with non-strictly convex costs, at least in its standard form, consider a Lyapunov candidate function $V : \mathbb{R}^n \times \mathbb{N}_0 \to \mathbb{R}$. Since J_k is not strictly convex, there exist multiple minima, and hence multiple stationary points of J_k. Thus, the gradient map

$$f_k(\theta) = \theta - \mu_k \nabla J_k(\theta) \tag{8.3}$$

has multiple distinct fixed points. Exactly one of these is the equilibrium, in the Lyapunov sense, of the sequence (f_k), and thus there exists at least one non-equilibrium point θ^* that is also a fixed point of f_k. The Lyapunov difference evaluated at this point is

$$\Delta V_k(\theta^*) \stackrel{\triangle}{=} V_k \circ f_k(\theta^*) - V_k(\theta^*) = 0, \tag{8.4}$$

Since θ^* is a fixed point of f_k, and hence the main condition of Lyapunov's second method is spoiled. We will see that under certain circumstances, a modification of the Banach fixed-point theorem can succeed where Lyapunov's method fails. At the time of writing it is not clear whether or not these modifications might facilitate more stability proofs that are not possible using Lyapunov's second method, but the existence of another, completely different approach to the problem is worthy of note.

8.2 Introduction and Problem Statement

8.2.1 Background on Online Parameter Estimation

System identification typically requires online estimation of parameters from a linear regression model. Since data is obtained sequentially during online operation, the task of online identification leads to the problem of optimizing a sequence of costs that are updated at each step by the most recent data. For cumulative least squares costs, Recursive Least Squares (RLS) [**?**, 14, 49, 64, 65, 108–110] is a well-established method, which includes sophisticated forgetting schemes [16,59,66–68,106,111] as well as techniques for avoiding divergence when the regressor lacks persistency of excitation [24,92,94,95,112]. However, RLS has the drawback of requiring the propagation of a covariance matrix and is restricted to a cumulative quadratic cost.

Gradient methods [113–115], [13, pp. 58–61] neither require covariance propagation nor assume a cumulative cost, and applications of gradient algorithms such as the stochastic gradient [116, 117], multi-innovation [118–123], and conjugate gradient [124–127] methods to system identification, adaptive control, and adaptive filtering have been studied extensively. Although stability conditions for particular gradient algorithms, such as the instantaneous and instantaneous normalized projected gradient methods [13, pp. 71-73], [96–99] are known, the stability of gradient-based identification methods for general quadratic costs is not well-studied. This can be compared to fixed-cost optimization, where the convergence criteria of gradient methods for large classes of costs are well-established [128, pp. 466-475], [129, pp. 28-35].

A third approach to sequential optimization is Online Convex Optimization (OCO) [130–132], which has emerged as a subfield of machine learning. The OCO literature has studied algorithms closely related to the gradient methods used in identification (cf. section 8.2.5) and has treated problems similar to those of interest in system identification, such as finding a common global minimizer [133–135] or tracking a time-varying global mini-

mizer [136–138] for sets of strongly convex costs. However, as shown in section 8.2.5, the OCO objective of regret minimization may not be effective for system identification since the regret can be minimized without guaranteeing attractivity to the true parameters. This is especially true for non-strictly convex costs, which may have multiple global minimizers.

In this paper, we address the global asymptotic stability of gradient descent for the purpose of online system identification. That is, gradient descent with the objective of determining true system parameters using only sequentially available input/output data (cf. [115], [13, pp. 58–61]). We refer to this approach as *sequential-cost gradient descent* (SGD) to distinguish it from other problems that also use gradient descent (e.g., fixed-cost optimization). We restrict our attention to quadratic costs, which are the most relevant for identification, including possibly rank-deficient costs, which are convex but not strictly convex, and hence we allow for the existence of multiple global minimizers at each step.

Section 8.2.2 fixes notation and terms, Section 8.2.3 further describes the motivation of the main problem, Section 8.2.4 defines the main problem formally as **P1**, and section 8.2.5 discusses the relationship of the present work to OCO. In Section 8.3, we present three fixed-point results that are subsequently used to prove global asymptotic stability conditions for SGD in Section 8.4. Finally, these conditions are specialized to least squares costs in Section 8.5 and illustrated with examples in Section 8.6.

8.2.2 Notation and Terminology

We define $\mathbb{N} \triangleq \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 \triangleq \{0\} \cup \mathbb{N}$. The symbols $\mathbf{S}^n$, $\mathbf{N}^n$, and $\mathbf{P}^n$ denote the sets of real $n \times n$ symmetric, positive-semidefinite, and positive-definite matrices, respectively. For $A \in \mathbf{S}^n$, $\lambda_i(A)$ denotes the ith largest eigenvalue of A, $\lambda_{\max}(A) \triangleq \lambda_1(A)$, and $\lambda_{\min}(A) \triangleq \lambda_n(A)$. For all $A \in \mathbb{R}^{n \times n}$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ denote the range and null space of A, respectively, and A^+ denotes the generalized inverse of A. The notation $|U|$ denotes the cardinality of the set U. The notation $(x_k)_{k \in \mathbb{N}_0} \subset U$ indicates that the components $x_0, x_1, \dots$ of the sequence $(x_k)_{k \in \mathbb{N}_0}$ are elements of U. For convenience, we write (x_k)

for $(x_k)_{k\in\mathbb{N}_0}$. The empty product and empty sum are defined to be 1 and 0, respectively. Given the n-tuple $\mathcal{J} = (j_1, \ldots, j_n)$ of indices and $r \times r$ matrices $A_{j_1}, \ldots, A_{j_n}$, we define $\prod_{j\in\mathcal{J}} A_j \triangleq A_{j_n} A_{j_{n-1}} \cdots A_{j_1}$.

8.2.3 Identification Using Costs with Multiple Minimizers

Let $\mathcal{D} \subset \mathbb{R}^n$ and let $\mathcal{J}$ be a set of differentiable functionals $J\colon \mathcal{D} \to \mathbb{R}$. For all $J \in \mathcal{J}$, let M_J denote the set of global minimizers of J, and denote $M_{\mathcal{J}} \triangleq \cap_{J\in\mathcal{J}} M_J$. It is frequently useful in system identification to consider sets $\mathcal{J}$ such that, for all $J \in \mathcal{J}$, $|M_J| > 1$ but $|M_{\mathcal{J}}| = 1$. For example, the set of instantaneous least squares costs $J_k(x) = \frac{1}{2}\|y_k - \phi_k x\|^2$, $k \geq 0$, where $(y_k) \subset \mathbb{R}^m$ and $(\phi_k) \subset \mathbb{R}^{m\times n}$ is a sequence of regressor matrices such that $\operatorname{rank}(\phi_k) < \min(m, n)$, satisfies this property in the case where $y_k = \phi_k x^*$ and $\cap_{k\geq 1}\mathcal{N}(\phi_k) = \{0\}$. The objective is to identify the single element $x^* \in M_{\mathcal{J}}$, which corresponds to the true system parameters, using only knowledge of the individual costs in $\mathcal{J}$. In particular, since each set M_J contains elements other than x^*, perhaps infinitely many, an algorithm for determining x^* must be capable of distinguishing between points that are only minimizers of a proper subset of the costs in $\mathcal{J}$ and the universal minimizer of every cost in $\mathcal{J}$. Section 8.2.4 proposes a simple gradient descent strategy for pursuing this objective.

8.2.4 sequential-cost gradient descent

Based on the discussion in the preceding paragraph, we make the following assumption:

A1. *There exists a set* $\mathcal{J} \triangleq \{J\colon \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}\}$ *of differentiable functionals such that, for all* $J \in \mathcal{J}$, $|M_J| \geq 1$, *and* $|M_{\mathcal{J}}| = 1$.

Thus, we allow for the possibility that, for all $J \in \mathcal{J}$, $|M_J| > 1$, but need not assume this *a priori*.

A2. *$\mathcal{J}$ contains a sequence (J_k) such that $\cap_k M_{J_k} = M_{\mathcal{J}}$.*

We refer to a sequence satisfying **A2** as an *exhaustive sequence* in $\mathcal{J}$.

Since we are interested in online operation, where each cost J_k is not available until step k, we also make an assumption restricting the availability of costs.

A3. *At each step k, the only available cost is J_k.*

Although it is possible to use the previous costs $J_0, \ldots, J_{k-1}$ in addition to J_k at step k, we shall show that it is possible to identify x^* using only the current cost J_k, which is significantly more computationally efficient than holding multiple costs in memory.

Since the most important in system identification applications are quadratic functions, for simplicity in this initial research, we make the following final assumption:

A4. *For all $k \geq 0$, J_k is a quadratic function. That is, there exist $A_k \in \mathbf{N}^n \backslash \{0\}$, $b_k \in \mathcal{R}(A_k)$, and $c_k \in \mathbb{R}$, such that $J_k(x) \stackrel{\triangle}{=} \frac{1}{2} x^{\mathrm{T}} A_k x + b_k^{\mathrm{T}} x + c_k$.*

Note that by restricting attention to quadratic functions, we also assume that $\mathcal{D} = \mathbb{R}^n$. The assumption that $b_k \in \mathcal{R}(A_k)$ is necessary to ensure the existence of finite-norm minimizers, while the assumption that A_k is not necessarily full rank implies that J_k is convex, but not necessarily strictly convex, and hence that there may exist multiple global minimizers. Note that a quadratic cost sequence satisfies **A1-2** if and only if $|\cap_{k \geq 0} [-A_k^+ b_k + \mathcal{N}(A_k)]| = 1$.

Let $(\mu_k) \subset [0, \infty)$. Then the *gradient iteration* of (J_k) is defined as

$$x_{k+1} = x_k - \mu_k \nabla J_k(x_k), \tag{8.5}$$

$$x_0 \in \mathbb{R}^n, \tag{8.6}$$

where μ_k is the *step size* at step k. We refer to the use of the gradient iteration to determine x^* as sequential-cost gradient descent. Note that x^* is an equilibrium point of the gradient iteration (8.5)–(8.6). The main problem that we address in this paper may be stated as follows:

P1. *Under assumptions **A1–A4**, determine sufficient conditions such that x^* is a GAS equilibrium of* (8.5)–(8.6).

Guaranteeing GAS will prove *a fortiori* that the SGD estimates converge to x^* regardless of the initialization.

8.2.5 Relationship with Online Convex Opimization

The SGD algorithm is equivalent to Online Gradient Descent [132, pp. 9-11], [130, pp. 130-134], [131, pp. 179-183], [133] in OCO, possibly with the addition of a projection step, and thus we might initially consider using results from OCO to help answer **P1**. Unfortunately, since OCO is based on regret minimization, it cannot guarantee GAS when $\mathcal{J}$ has costs with multiple global minimizers.

To see this, assume that **A1–A4** hold and recall that the regret of an OCO algorithm is defined [132, pp. 1-2], [130, p. 112], [131, pp. 159-161] by

$$R_T \triangleq \sum_{k=0}^{T} J_k(x_k) - \min_{x \in \mathbb{R}^n} \sum_{k=0}^{T} J_k(x), \tag{8.7}$$

where x_k is the estimated minimum of J_k at step k. In the OCO framework, the goal is to provide guarantees on the asymptotic growth of R_T, and the main figure of merit is how well R_T can be bounded (possibly asymptotically by a function of T). The ideal performance is $R_T = 0$ for all $T > 0$, but even the best OCO algorithms guarantee only sublinear growth of R_T, since the OCO framework allows J_k to be chosen adversarially

[132, p. 6].

The task given in **P1** is to prove GAS of x^*, and hence convergence of the estimate sequence to the true parameters. Hypothetically, if GAS could be guaranteed by bounding R_T, then the methods of OCO might be used to answer **P1**. Unfortunately, as the following example shows, if $\mathcal{J}$ has costs with multiple global minimizers, then even achieving the ideal OCO performance of $R_T = 0$ for all $T > 0$ is insufficient to guarantee GAS, or even attractivity of x^*.

Example 1. Let $A \in \mathbf{N}^n \backslash \{0\}$, $\|A\| \leq 1$, and $\operatorname{rank}(A) < n$, define $J_1(x) = x^T A x$ and $J_2 = x^T(I_n - A)x$, and let $\mathcal{J} = \{J_1, J_2\}$ and $(J_k) = (J_1, J_2, J_1, J_2, \dots)$. Then $\mathcal{J}$ satisfies **A1** and (J_k) satisfies **A2**. Since $\mathcal{N}(A),\ \mathcal{N}(I_n - A) \neq \{0\}$, there exist $y_1 \in \mathcal{N}(A) \setminus \{0\}$ and $y_2 \in \mathcal{N}(I_n - A) \setminus \{0\}$, and thus, for all $\alpha \in \mathbb{R}$, the sequence

$$x_k \triangleq \begin{cases} \alpha^k y_1, & k = 0, 2, 4, \dots \\ \alpha^k y_2, & k = 1, 3, 5, \dots \end{cases} \tag{8.8}$$

is well-defined and has $R_T = 0$ for all $T > 0$. For all $\alpha > 1$, however, it follows that $\lim_{k\to\infty} \|x_k - x^*\| = \infty$. ◇

In Example 1, since both J_1 and J_2 have an infinite number of global minimizers, there are sequences with identically zero regret that diverge infinitely far from the true parameters. Hence, guarantees on R_T, such as those provided by OCO, do not guarantee attractivity, and thus cannot satisfactorily answer **P1**. In the remainder of the paper, we instead pursue a solution strategy based directly on the definition of stability.

8.3 Fixed-Point Theory

This section reviews three fixed-point results that are essential for the main results of the paper. Although straightforward, to the authors' knowledge, they have not appeared

before in the system identification literature.

Let (M, d) be a metric space, and let $f\colon M \to M$. Then, f is *nonexpansive* if there exists a *nonexpansion coefficient* $q \in (0, 1]$ such that, for all $x, y \in M$, $d(f(x), f(y)) \leq qd(x, y)$. The point $p \in M$ is a *fixed point* of f if $f(p) = p$.

Definition 13. *Let (M, d) be a metric space, and let (f_k) be a sequence of functions on M. Then $p \in M$ is a fixed point of (f_k) if, for all $k \in \mathbb{N}_0$, $f_k(p) = p$. The set of all fixed points of (f_k) is denoted by Fix$[(f_k)]$.*

Proposition 8. *For all $k \in \mathbb{N}_0$, let $f_k\colon M \to M$ be nonexpansive with nonexpansion coefficient q_k. Let $x_0 \in M$, define $x_{k+1} = f_k(x_k)$, and assume that p is a fixed point of (f_k). Then $\lim_{k\to\infty} d(x_k, p) \leq (\prod_{k=0}^{\infty} q_k)\, d(x_0, p)$, and, furthermore, if $\prod_{k=0}^{\infty} q_k = 0$, then p is the only fixed point of (f_k) and $\lim_{k\to\infty} x_k = p$.*

Proof. Since, for all $k \in \mathbb{N}_0$, f_k is nonexpansive, it follows that $d(x_{k+1}, p) = d(f_k(x_k), f_k(p)) \leq q_k d(x_k, p)$, hence, $d(x_k, p) \leq \prod_{\ell=0}^{k-1} q_\ell\, d(x_0, p)$, and thus $\lim_{k\to\infty} d(x_k, p) \leq (\prod_{k=0}^{\infty} q_k)\ d(x_0, p)$. Setting $\prod_{k=0}^{\infty} q_k = 0$, it follows that $\lim_{k\to\infty} d(x_k, p) = 0$. Suppose that $p' \in$ Fix$[(f_k)]$. Then $d(p', p) \leq \lim_{k\to\infty} d(p', x_k) + \lim_{k\to\infty} d(x_k, p) = 0$. □

Proposition 9. *Let (f_k) be a sequence of nonexpansive functions on (M, d) with fixed point p. Then p is a Lyapunov stable equilibrium of the system*

$$x_{k+1} = f_k(x_k), \tag{8.9}$$

$$x_0 \in M. \tag{8.10}$$

Proof. Let $\varepsilon > 0$ and $x_0 \in B_\varepsilon(p)$. Since (f_k) is nonexpansive, it follows that, for all $k \geq 0$, $d(x_k, p) = d(f_{k-1}(x_{k-1}), f_k(p)) \leq d(x_{k-1}, p) \leq d(x_0, p) < \varepsilon$. □

A *bounded interval* of $\mathbb{N}_0$ is a set $\{n, n+1, \ldots, n+m\} \subset \mathbb{N}_0$, where $n, m \in \mathbb{N}_0$. A *bounded interval partition* of $\mathbb{N}_0$ is a partition of $\mathbb{N}_0$ whose elements are bounded intervals, and a *uniformly bounded interval partition* P of $\mathbb{N}_0$ is a bounded interval partition of $\mathbb{N}_0$ such that $\sup_{U \in P} |U| < \infty$.

Proposition 10. *Let (f_k) be a sequence of nonexpansive functions on (M, d) with fixed point p, let (U_k) be a bounded interval partition of $\mathbb{N}_0$, and, for all $k \in \mathbb{N}_0$, define*

$$F_k \stackrel{\triangle}{=} f_{\max U_k} \circ \cdots \circ f_{\min U_k}. \tag{8.11}$$

Then the following statements hold:

i) p is a fixed point of (F_k).

ii) (F_k) is nonexpansive.

iii) For all $x \in M$, $\lim_{k\to\infty} F_k \circ \cdots \circ F_0(x) = p$ if and only if $\lim_{k\to\infty} f_k \circ \cdots \circ f_0(x) = p$.

Proof. i) and ii) are immediate from (8.11). To prove iii), let $z \in M$, and define $(y_k), (x_k) \subset M$ by $x_0 = z$, $y_0 = z$, and, for all $k \in \mathbb{N}_0$, by $x_{k+1} = f_k(x_k)$ and $z_{k+1} = F_k(z_k)$. Suppose that $\lim_{k\to\infty} x_k = p$. Since (z_k) is a subsequence of (x_k), it follows that $\lim_{k\to\infty} z_k = p$. Conversely, suppose that $\lim_{k\to\infty} z_k = p$, and let $\varepsilon > 0$. Then there exists $k \in \mathbb{N}_0$ such that, for all $k \geq K$, $d(z_k, p) < \varepsilon$. Since (z_k) is a subsequence of (x_k), it follows that there exists $M \geq 0$ such that $z_K = x_M$. Let $m > M$. Then, since f_k is

nonexpansive, it follows that

$$\begin{aligned} d(x_m, p) &= d(f_m^{n_m} \circ \cdots \circ f_M^{n_M}(x_M), f_m^{n_m} \circ \cdots \circ f_M^{n_M}(p)) \\ &\leq d(x_M, p) = d(z_K, p) < \varepsilon, \end{aligned}$$

and thus $\lim_{k\to\infty} x_k = p$. □

8.4 Global Asymptotic Stability

In this section, we state and prove sufficient conditions for GAS of SGD in Theorem 13, answering **P1**. This is our main result, the proof of which requires the following three lemmas.

Lemma 11. *Let (U_k) be a bounded interval partition of $\mathbb{N}_0$ and, for all $k \geq 0$, let $A_k \in \mathbf{N}^n$, with $\|A_k\| \leq 1$ and $\lim_{k\to\infty} \left\|\prod_{j\in U_k} A_j\right\| = 1$. Then $\lim_{k\to\infty} \|A_k\| = 1$.*

Proof. Let $\varepsilon > 0$. Since $\lim_{k\to\infty} \left\|\prod_{j\in U_k} A_j\right\| = 1$ there exists $K \geq 0$ such that, for all $k \geq K$ and $j \in U_k$, $1 - \varepsilon < \left\|\prod_{j\in U_k} A_j\right\| \leq \prod_{j\in U_k} \|A_j\| \leq \|A_j\| \leq 1$. Thus, for all $j \geq \min U_K$, $1 - \varepsilon \leq \|A_j\| \leq 1$.

Lemma 12. *Let (U_k) be a uniformly bounded interval partition of $\mathbb{N}_0$ and let $(a_k) \subset \mathbb{R}$ be a sequence such that $\lim_{k\to\infty} a_k = 0$. Then $\lim_{k\to\infty} \sum_{j\in U_k} a_j = 0$.*

Proof. Let $\varepsilon > 0$ and $\sup_{k\in\mathbb{N}_0} |U_k| = M$. Since $\lim_{k\to\infty} a_k = 0$, it follows that there exists $K \in \mathbb{N}_0$ such that, for all $k \geq K$, $|a_k| < \varepsilon/M$. Since (U_k) is an interval partition, it follows that there exists $K_1 \geq 0$ such that, for all $k \geq K_1$, $\min U_k > K$. Let $k \geq K_1$. Since, for all

$j \in U_k$, $j \geq K$, it follows that $\left|\sum_{j\in U_k} a_j\right| \leq \sum_{j\in U_k} |a_j| < \frac{\varepsilon}{M}|U_k| \leq \frac{\varepsilon}{M}\sup_{k\in\mathbb{N}_0}|U_k| = \varepsilon$. $\square$

To see that the assumption of uniform boundedness is essential, let $U_0 = \{0\}$, $U_1 = \{1\}$, $U_3 = \{2, 3\}$, $U_4 = \{4, 5, 6\}$, $U_5 = \{7, 8, 9, 10\}$, ... Then (U_k) is a bounded, but not uniformly bounded interval partition. For all $k \in \mathbb{N}_0$, let $\{a_j\}_{j\in U_k} = \{\frac{1}{k+1}, \ldots, \frac{1}{2k}\}$. Then $\sum_{j\in U_k} a_j = \ln(2) + \varepsilon_{2k} - \varepsilon_k$, where $\lim_{k\to\infty} \varepsilon_k = 0$. Thus $\lim_{k\to\infty} \sum_{j\in J_k} a_j = \ln(2)$ even though $\lim_{k\to\infty} a_k = 0$.

Definition 14. *Let $(S_k) \subset \mathbf{N}^n$. Then (S_k) is ultimately positive if $\liminf_{k\to\infty} \lambda_{\min}(S_k) > 0$ and weakly ultimately positive if there exists a uniformly bounded interval partition (U_k) of $\mathbb{N}_0$ such that $\left(\sum_{j\in U_k} S_j\right)$ is ultimately positive. The sequence $(\phi_k) \subset \mathbb{R}^{n\times m}$ is ultimately positive, or weakly ultimately positive if $(\phi_k^{\mathrm{T}}\phi_k)$ is ultimately positive, or weakly ultimately positive, respectively.*

Lemma 13. *For all $k \in \mathbb{N}_0$, let $A_k \in \mathbf{N}^n\backslash\{0\}$,*

$$\mu_k \in [0, \lambda_{\max}^{-1}(A_k)], \tag{8.12}$$

and define $q_k \triangleq \left\|\prod_{j\in U_k}(I_n - \mu_j A_j)\right\|$. Furthermore, assume that $(\mu_k A_k)$ is weakly ultimately positive. Then $\limsup_{k\to\infty} q_k < 1$.

Proof. From (8.12), it follows that $\limsup_{k\to\infty} q_k \leq 1$. Hence, suppose for contradiction that $\limsup_{k\to\infty} q_k = 1$. From Lemma 11, it follows that $\limsup_{k\to\infty} \|I_n - \mu_k A_k\| = 1$, and thus $\limsup_{k\to\infty}(1 - \|I_n - \mu_k A_k\|) = 0$. For all $k \in \mathbb{N}_0$, let x_k be the unit eigenvector of A_k corresponding to $\lambda_{\max}(A_k)$ and let $\xi \in B_1(0)$. Since $\mu_k A_k x_k = (1 - \|I - \mu_k A_k\|)x_k$,

Lemma 12 implies that

$$\begin{aligned}
&\liminf_{k\to\infty}\left[\lambda_{\min}\left(\sum_{j\in U_k}\mu_j A_j\right)\right] \leq \liminf_{k\to\infty}\left\|\left[\sum_{j\in J_k}\mu_j A_j\right]\xi\right\| \\
&\leq \liminf_{k\to\infty}\left\|\sum_{j\in J_k}\mu_j A_j x_j\right\| \leq \liminf_{k\to\infty}\sum_{j\in J_k}\|\mu_j A_j x_j\| \\
&= \liminf_{k\to\infty}\sum_{j\in J_k}(1-\|I_n-\mu_j A_j\|) \leq \limsup_{k\to\infty}\sum_{j\in J_k}(1-\|I_n-\mu_j A_j\|) = 0,
\end{aligned}$$

which contradicts the assumption that $(\mu_k A_k)$ is weakly ultimately positive. □

Theorem 13. *Under the notation and assumptions **A1–A4**, let* $(\mu_k) \subset [0,\infty)$ *satisfy* (8.12) *and assume that* $(\mu_k A_k)$ *is weakly ultimately positive. Then* $x^* \in M_{\mathcal{J}}$ *is the globally asymptotic stable equilibrium of* (8.5)–(8.6).

Proof. The point x^* is an equilibrium of (8.5)–(8.6) by definition. Consider the sequence $(f_k\colon \mathbb{R}^n \to \mathbb{R}^n)$ defined for all $k \geq 0$ by $f_k(x) \stackrel{\triangle}{=} x - \mu_k \nabla J_k(x)$, where $(J_k) \subset \mathcal{J}$ is exhaustive. Since (8.12) implies that $\|I-\mu_k A_k\| = 1-\mu_k\lambda_{\min}(A_k) \in [0,1]$, it follows that, for all $x, y \in \mathbb{R}^n$, $\|f_k(x)-f_k(y)\| \leq \|I-\mu_k A_k\|\|x-y\| \leq \|x-y\|$, and thus (f_k) is nonexpansive. From Proposition 9, it follows that x^* is Lyapunov stable.

To prove attractivity, let (U_k) be a uniformly bounded interval partition for which $(\mu_k A_k)$ is weakly ultimately positive, for all $k \in \mathbb{N}_0$, define F_k by (8.11) with f_k given as in the previous paragraph, and define $q_k \stackrel{\triangle}{=} \left\|\prod_{j\in U_k}(I_n-\mu_j A_j)\right\|$. Using (8.11), it follows that, for all $k \in \mathbb{N}_0$, $\|F_k(x)-F_k(y)\| = \left\|\left[\prod_{j\in J_k}(I_n-\mu_j A_j)\right](x-y)\right\| \leq q_k\|x-y\|$. Finally, $\|I_n-\mu_k A_k\| \leq 1$ implies that $q_k \in [0,1]$. Next, since $(\mu_k A_k)$ is weakly ultimately positive, it follows that Lemma 13 holds, and thus (13) implies that $1-\limsup_{k\to\infty} q_k \in$

$(0,1)$. Hence, let $\alpha \in (0, 1 - \limsup_{k\to\infty} q_k)$, and, for all $k \in \mathbb{N}_0$, define

$$\tilde{q}_k \triangleq \begin{cases} q_k, & q_k \neq 0, \\ \alpha, & q_k = 0. \end{cases}$$

Since, for all $k \in \mathbb{N}_0$, $\tilde{q}_k \in (0,1]$ and, for all $x, y \in \mathbb{R}^n$, $\|F_k(x) - F_k(y)\| \leq \tilde{q}_k \|x - y\|$, it follows that $\tilde{q}_k$ is a nonexpansion coefficient for f_k. Since $\tilde{q}_k \in (0,1]$, it follows that $\prod_{k=0}^{\infty} \tilde{q}_k$ is either zero or positive. Suppose for contradiction that $\prod_{k=0}^{\infty} \tilde{q}_k$ is positive. Then $\lim_{k\to\infty} \tilde{q}_k = 1$, which implies that q_k is ultimately positive, and hence $\lim_{k\to\infty} q_k = \lim_{k\to\infty} \tilde{q}_k = 1$, contradicting Lemma 13. Thus $\prod_{k=0}^{\infty} \tilde{q}_k = 0$.

From **A1**, it follows that x^* is the unique fixed point of (f_k). From Proposition 10, $i)$ and 8 it follows that x^* is the unique fixed point of (F_k), and, for all $x \in \mathbb{R}^n$, $\lim_{k\to\infty} F_k \circ \cdots \circ F_0(x) = x^*$. Thus, it follows from Proposition 10, $iii)$ that $\lim_{k\to\infty} x_k = x^*$. Since the initialization is arbitrary, it follows that x^* is GAS. □

Note that Theorem 13 combined with the uniqueness of the universal fixed point in Proposition 8 implies that if (μ_k) satisfies (8.12), $(\mu_k A_k)$ is weakly ultimately positive, and $\cap_{k\geq 0}[-A_k^+ b_k + \mathcal{N}(A_k)] \neq \varnothing$, then, moreover, $|\cap_{k\geq 0}[-A_k^+ b_k + \mathcal{N}(A_k)]| = 1$ and $x^* \in \cap_{k\geq 0}[-A_k^+ b_k + \mathcal{N}(A_k)]$ is the GAS equilibrium of (8.5)–(8.6).

8.5 Least Squares Costs

Least squares costs form a significant subset of the quadratic costs commonly used in practice. For this special case, the results in the previous section can be specialized.

Definition 15. *The sequence (J_k) of quadratic cost functions is a least squares sequence if, for all $k \in \mathbb{N}_0$, there exist $\phi_k \in \mathbb{R}^{p\times n}\backslash\{0\}$, $y_k \in \mathbb{R}^p$, $\ell_k \in \{0, \ldots, k\}$ and, for all*

$1 \leq i \leq \ell_k$, $W_{k,i} \in \mathbf{P}^p$ *such that*

$$J_k(x) = \frac{1}{2} \sum_{i=0}^{\ell_k} (y_{k-i} - \phi_{k-i}x)^{\mathrm{T}} W_{k,i} (y_{k-i} - \phi_{k-i}x). \tag{8.13}$$

Note that in the case where $\sum_{i=0}^{\ell_k} \phi_{k-i}^{\mathrm{T}} W_{k,i} \phi_{k-i}$ is rank-deficient, J_k has an infinite number of global minimizers.

Theorem 14. *Let* $(\phi_k) \subset \mathbb{R}^{p \times n}$ *be weakly ultimately positive, and, for all* $k \in \mathbb{N}_0$*, define* $y_k = \phi_k x^*$*, where* $x^* \in \mathbb{R}^n$*, let* $0 \leq \ell_k \leq k$*, and assume that, for all* $1 \leq i \leq \ell_k$*,* $W_{k,i} \in \mathbf{P}^n$ *and* $W_{k,i} \geq \xi I_n$*, where* $\xi > 0$*. Finally, let* $(\mu_k) \subset [0, \infty)$ *be an ultimately positive sequence such that, for all* $k \in \mathbb{N}_0$*,*

$$\mu_k \leq \frac{1}{\lambda_{\max} \left(\sum_{i=0}^{\ell_k} \phi_{k-i}^{\mathrm{T}} W_{k,i} \phi_{k-i} \right)}, \tag{8.14}$$

Then x^* *is the globally asymptotically stable equilibrium of* (8.5)–(8.6) *with* J_k *given, for all* $k \geq 0$*, by* (8.13)*.*

Proof. Note that (8.13) can be written as $J_k(\tilde{x}) \triangleq \frac{1}{2} \tilde{x}^{\mathrm{T}} A_k \tilde{x}$, where $A_k = \sum_{i=0}^{\ell_k} \phi_{k-i}^{\mathrm{T}} W_{k,i} \phi_{k-i}$ and $\tilde{x} = x^* - x$. Then μ_k satisfies (8.12) by construction. To show that $(\mu_k A_k)$ is weakly ultimately positive, let (U_k) be a uniformly bounded interval partition with respect to which (ϕ_k) is weakly ultimately positive, and for all $k \in \mathbb{N}_0$, define $\mu_k^- = \min_{j \in U_k} \mu_j$. Since (μ_k) is ultimately positive, (μ_k^-) is ultimately positive, and thus

$$\begin{aligned} \liminf_{k \to \infty} \lambda_{\min} \left(\sum_{j \in U_k} \mu_j A_j \right) &\geq \xi \liminf_{k \to \infty} \mu_k^- \lambda_{\min} \left(\sum_{j \in U_k} \sum_{i=0}^{\ell_j} \phi_{j-i}^{\mathrm{T}} \phi_{j-i} \right) \\ &\geq \xi \liminf_{k \to \infty} \mu_k^- \lambda_{\min} \left(\sum_{j \in U_k} \phi_j^{\mathrm{T}} \phi_j \right) > 0. \end{aligned}$$ □

8.6 Examples

The following examples illustrate Theorems 13 and 14.

Example 2. Considering the cost set and sequence of Example 1, for all $k \in \mathbb{N}_0$, let $\mu_k = \min(1, \lambda_{\max}^{-1}(A))$, and let $(U_k) = (\{0,1\}, \{2,3\}, \{4,5\}, \dots)$. Then Theorem 14 implies that $x^* = 0$ is the GAS equilibrium of (8.5)–(8.6). $\diamond$

Example 3. Let $\theta = [1\ 2]^{\mathrm{T}}$, and, for all $k \in \mathbb{N}_0$, define

$$\phi_k = \begin{cases} \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{k} \end{bmatrix}, & k = 0, 2, 4, \dots, \\ \begin{bmatrix} \frac{1}{\sqrt{k}} & 0 \\ 0 & 0 \end{bmatrix}, & k = 1, 3, 5, \dots, \end{cases} \tag{8.15}$$

$y_k = \phi_k \theta$, and let $J_k(x) = \frac{1}{2}\|y_k - \phi_k x\|^2$. Let $\mu_k = k^2$ and consider the uniformly bounded interval partition given in Example 2. Then Theorem 14 implies that θ is the GAS equilibrium of (8.5)–(8.6). Note that θ is GAS even though μ_k is unbounded. $\diamond$

Example 4. Let $\theta \in \mathbb{R}^n$, let $N_w \geq 0$, let $(\phi_k) \subset \mathbb{R}^{p \times n}$ be weakly ultimately positive, and assume that $\beta \stackrel{\triangle}{=} \sup_{j \geq 0} \lambda_{\max}\left(\phi_j^{\mathrm{T}} \phi_j\right) < \infty$. For all $k \in \mathbb{N}_0$, let $y_k = \phi_k \theta$, and define $J_k(\hat{\theta}) = \frac{1}{2} \sum_{i=0}^{N_w} \|y_{k-i} - \phi_{k-i}\hat{\theta}\|^2$. Finally, let (μ_k) be an ultimately positive sequence such that $\mu_k \in [0, \frac{1}{N_w \beta}]$. Then (μ_k) satisfies (8.14), since $\lambda_{\max}\left(\sum_{i=0}^{N_w} \phi_{k-i}^{\mathrm{T}} \phi_{k-i}\right) \leq N_w \beta$, and Theorem 14 implies that θ is the GAS equilibrium of (8.5), (8.6). $\diamond$

8.7 Conclusion

Sufficient conditions were given under which the sequential-cost gradient descent is GAS with an equilibrium corresponding to the true system parameters. GAS was obtained regardless of whether or not the individual costs have a unique minimizer (that is, are

strictly convex). Since quadratic costs are the most common type in system identification, we restricted attention to this important class of costs. Specialized conditions were given for least squares costs, including rank-deficient least squares costs with an infinite number of global minimizers. Future work will consider extensions to sequences of nonquadratic convex cost functions, the effect of noise, and extension to costs without a common global minimizer.

CHAPTER 9

Conclusion

In the foregoing chapters, we have discussed online system identification using both the RLS and sequential gradient descent. We were mainly interested in solving online system identification problems in the case where the system parameters could change either gradually or abruptly. We restricted our attention to systems whose output behavior can be described by single or multivariable ARX-type models.

First we considered RLS with constant-rate forgetting as a proposed solution to the changing parameters problem. We showed how RLS with constant-rate forgetting can be useful for tracking slowly varying parameters but also that the covariance diverges when the regressor loses persistency. We first investigated a solution to the issue of covariance divergence by modifying RLS to include *a priori* covariance bounds. However, this modification had the drawback that it was only an *ad hoc* modification of RLS, and practically speaking it did not produce a good solution to the abrupt change of parameters problem.

Next, we investigated variable-rate forgetting, and presented novel analyses of convergence and consistency as well as the novel idea of an error-actuated variable forgetting factor. The combination of these contributions allowed us to produce a solution to the abrupt change of parameters problem while at the same time minimizing the risk of divergence due to non-persistent regressors. A further refinement allowed us to incorporate VRF with variable-direction forgetting, as presented in [24], to produce variable-rate-and-direction forgetting (VRDF), which we provided a least-squares cost for in the general form

of matrix forgetting. Although potentially computationally expensive due to an SVD of the covariance matrix, VRDF guarantees estimator robustness when persistency is lost, rather than simply mitigating it.

After presenting our forgetting-based solution to the changing parameters problem, we further investigated the regressor conditions that are necessary and sufficient for the global asymptotic convergence of RLS without forgetting. Since persistency of excitation is both known to be a logically too strict condition (being sufficient but not necessary for RLS convergence to the true parameters) and also difficult to guarantee in practice, an understanding of the weakest possible conditions that still guarantee parameter convergence was useful. We found that WPE and the existence of a MEOP were two equivalent necessary and sufficient conditions that can both be understood in a "local" sense in time and are direct generalizations of PE.

Finally, we investigated the case of extreme "forgetting" in which only the most recent measurement is used in updating the estimate. Since the associated least squares cost has in general an uncountably infinite number of minimizers, the RLS update cannot be applied and therefore we had to use gradient descent instead. This led to an investigation of the global asymptotic stability of online gradient descent (in which the cost changes at each step) for general quadratic cost functions. We provided conditions for the gradient stepsize that gauranteed the gradient iteration was globally asymptotically stable with respect to the time-varying equilibrium defined by the true parameters of the system both for general quadratic costs and the special case of least-squares costs.

9.1 Future Work

The analyses of VRF and VRDF can be extended to achieve more complete consistency results, especially in the case where the forgetting factor is error-actuated. Extensions to filtering theory especially might be available through the further development of the artifi-

cial uncertainty concept, which can also provide a starting point for more comprehensive results on consistency. VRDF could be improved by increasing the practicality of computing the SVD. This might be possible if an algorithm could be formulated in which the SVD could be recursively updated.

The investigations on necessary and sufficient regressor conditions for RLS parameter convergence can be extended to include constant and variable-rate forgetting. Furthermore, necessary and sufficient conditions for consistency should also be investigated for RLS both with and without forgetting.

Finally, we are currently extending the results of the gradient descent analysis to the case of general cost functions, possibly with spatially bounded Hessians, which would remove the current restriction to general quadratic costs.

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