M-theory on G_2 Manifolds Moduli to Phenomenology

by

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Luận án này xin được là lời cảm ơn dàng riêng cho ba mẹ,
người đã toàn tâm cống hiến cuộc đời mình để nuôi lớn
và ủng hộ con trở thành người như hôm nay.

Cũng là lời cảm ơn tới em gái,
người luôn quan tâm anh hai những năm tháng xa nhà.

Cuối cùng là lời cảm ơn tới
các giáo sư, bạn bè, đồng nghiệp, và tất cả mọi người
đã giúp đỡ tôi về cả tinh thần và trí thức.

This study is dedicated to my parents
who have wholeheartedly devoted their life
to raise and support me to become the person I am today.

To my sister who has always cared for me
during all this time away from home.

And lastly, to my advisors,
my friends, my colleagues, and all others
who have been helping me intellectually and emotionally.

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ABSTRACT

String theories are good candidates for unifying the Standard Model and general relativity. M theory is the non-perturbative limit of all string theories. Moreover, it has been shown that M theory should be compactified on G_2 manifold with ADE singularities. How this model deals with unsolved questions in physics is a good test for the validity of M theory. In this work, we will briefly cover the background in both math and physics for M theory compactified on G_2 manifold with ADE singularities. Then, we will discuss a linearized local model of this theory and use it to tackle physics questions such as quark mass hierarchy, neutrino mass, and proton decay. We see that in this model, geometrically the hierarchy of quark masses makes the solution for moduli very hard to find. This suggests a global model will be very predictive and testable through quark masses. Next, using the solution of the moduli, we compute neutrino Dirac mass terms and derive an estimation for neutrino masses. Notably, the masses of the two heavier light neutrinos are predicted to be about 0.05 eV and 0.009 eV (0.05 eV and 0.05 eV) for normal (inverted) hierarchy.

CHAPTER I
Introduction

The Standard Model and general relativity are the two pillars of fundamental physics. However, the two theories are incompatible. Therefore, finding a consistent theory of quantum gravity that unifies the two theories has been the holy grail of physics. It is hoped that by unifying the two theories, we can also explain unsolved mysteries of modern physics such as the strong CP problem, the origin of neutrino mass, the hierarchy problem, and so on.

String theories appear to be the most promising candidates for such a unified theory. They are 10-dimensional theories that naturally contain quantum gravity. Moreover, they do not suffer from ultraviolet divergences. Additionally, they contain only one parameter which is the string length. All of the string theories are the perturbative limit of a single 11-dimensional theory, called M-theory. M-theory does not have any free parameters as the only parameter of string theories, the string length, becomes part of the geometry. Thus, M-theory's geometry determines all the physics of the theory. This makes the theory attractive to many physicists. In order to be physical, M-theory must be compactified on a G_2 manifold with ADE singularities. While the G_2 manifold facilitates supersymmetry, the ADE singularities realize nonabelian gauge groups and charged fermions.

There are significant results in the physics of M-theory on these G_2 manifolds with ADE singularities. One prediction of compactified M-Theory is the existence of $\mathcal{N} = 1$ supersymmetry and its soft breaking via gluino condensation, while simultaneously stabilizing all moduli [11, 110]. M-Theory accommodates radiative electroweak symmetry breaking [10], baryogenesis [92], a solution to the strong CP problem [12], and a mechanism for inflation [91]. Lastly, this framework can include a wide variety of hidden sector dark matter candidates and predict a supergravity spectrum in a wide class of Kähler potentials, although this is not completely a general result [10, 11].

In this work, Chapters II, III, and IV review foundational materials with some of my own interpretation. Chapter II covers some basic background for the mathematics behind G_2 manifold and ADE singularities. Chapter III explains the path from string theories duality to M-theory on G_2 manifold with ADE singularities. Following that, Chapter IV discusses some key facts about M-theory on G_2 manifold with ADE singularities. Note that the heavy details of the background, although necessary and useful, can be daunting. Readers are advised to skim through and come back for more details when needed in later chapters.

Most of the original work is in Chapter V and VI. A key challenge of studying M-theory compactified on G_2 manifold with ADE singularities is that the global geometry is notoriously difficult to construct and study. Hence, we try a different approach. The problem is simplified by linearizing the local geometry of the singular G_2 manifold. In Chapter V, we propose a linearized model for G_2 manifold with resolved E_8 singularities. Using the local moduli as parameters, we compute the quark and charged lepton mass matrices. Then, we fit the experimental values to the eigenvalues of these matrices. We found a solution for these local moduli. In Chapter VI, we use this solution to compute Dirac mass terms for neutrino. After some discussions about the vacuum expectation values (VEVs) for the scalar component of right-handed neutrinos μ_i^c, we deduce that masses of the two heavier light neutrinos are about 0.05 eV and 0.009 eV (0.05 eV and 0.05 eV)) for normal (inverted) hierarchy. Figure I plots out the general road map here.

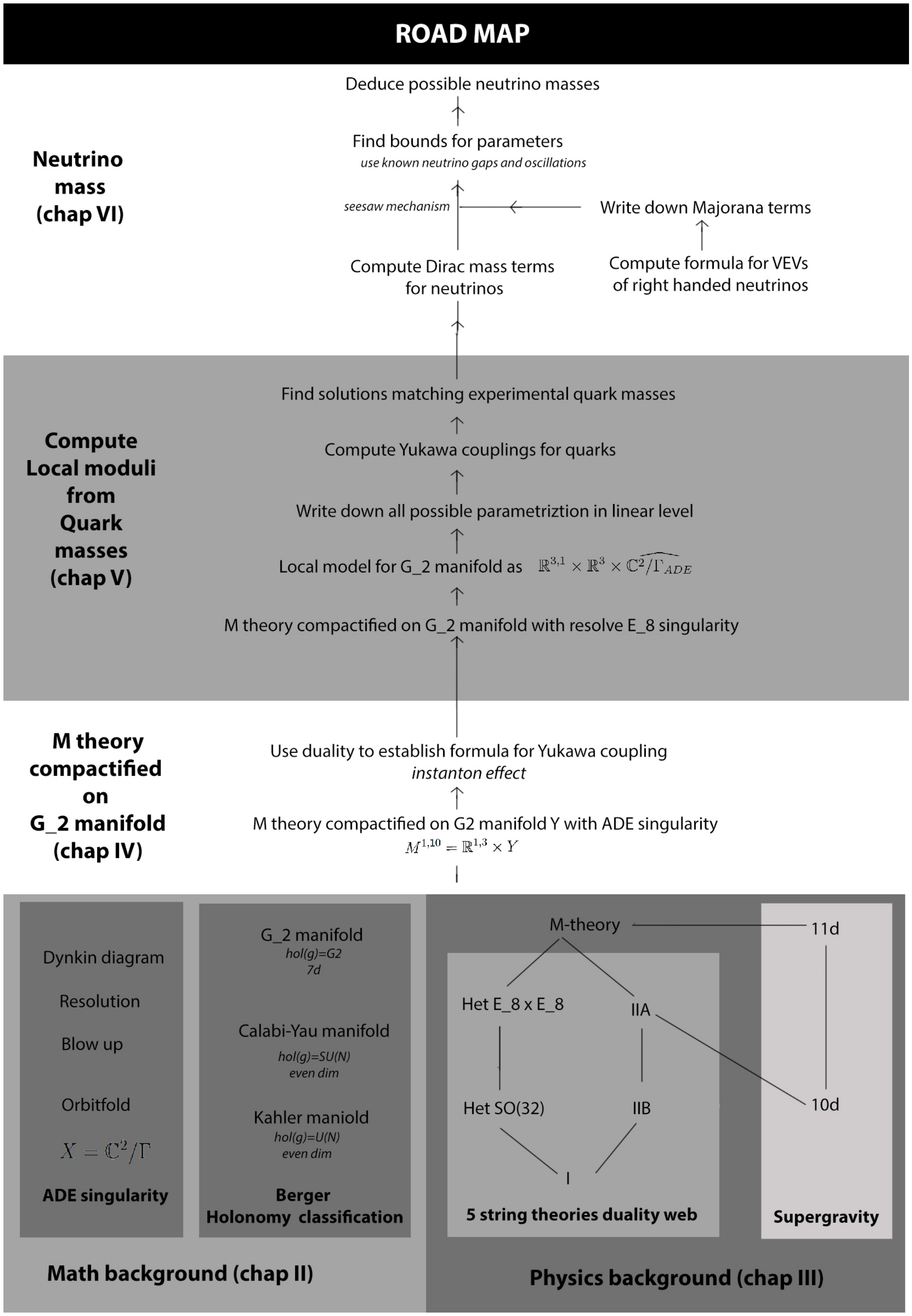


Figure I.1: Road map for this thesis

CHAPTER II
Math Background

Geometry completely determines the physics of M-theory, so it is crucial to cover some geometry background relating to M-theory. There are two major geometric ingredients: a G_2 manifold and ADE singularities. M-theory is shown to be necessarily compactified on a G_2 manifold to preserve supersymmetry in a similar manner as how 10-d string theories are compactified on a Calabi-Yau manifold. The relation between G_2 manifolds and Calabi-Yau manifolds therefore directly reflects the lift from string theories to M-theory. We will spend the majority of this chapter covering the basic geometry building up to G_2 manifolds. Readers can find more details in standard materials such as [90, 93]. The last few sections introduce ADE singularities and their resolution. This will be crucial to our discussion about the nonabelian gauge in M-theory. This construction can be found in most standard materials [63, 112].

Sec. II.1 defines the notion of holonomy. Sec. II.2 discusses the classification of holonomy called Berger classification. This makes the lift of string theory geometry to M-theory more apparent. We review the basic geometry of Calabi-Yau and G_2 manifolds in Sec. II.3 and II.4. These concepts are useful for explicit computations in differential forms and justify the construction of the M-theory. Next, we discuss ADE singularities in Sec. II.5. We try to give a concise pictorial explanation of the topic without evoking lengthy math. More explicit computation will be explored in Chapter V.

II.1: Holonomy

We consider the definition of the holonomy group on vector bundles. A similar notion can be defined for principal bundles. It can also be shown that, for principal bundles associated with a vector bundle, the two notions produce the same holonomy group.

Definition II.1. Le M be a manifold, E a vector bundle over M, and ∇^E a connection on E. Suppose $\gamma : [0,1] \to M$ is piece-wise smooth, with $\gamma(0) = \gamma(1) = x$. γ is called *loop based* at x. Define $Hol_x(\nabla^E)$ of ∇^E based at x to be

$$Hol_x(\nabla^E) = \Big\{ P_\gamma : \gamma \text{ is a loop based at x} \Big\} \subset GL(E_x) \tag{II.1.1}$$

where P_γ is the action on the vector space E_x induced by the parallel transport along γ. In this work, we only focus on connected manifolds, so the holonomy group is independent of the base point up to a conjugation. So, we will drop the subscript base point x in the notation.

II.2: Classification of Riemannian holonomy groups

Theorem II.2 (Berger). *Suppose M is a simply-connected manifold of dimension n and that g is a Riemannian metric on M that is irreducible and nonsymmetic. Then exactly one of the following seven cases holds*

(i) $Hol(g) = SO(n)$

(ii) $n = 2m$ *with* $m \geq 2$, *and* $Hol(g) = U(m)$ *in* $SO(2m)$.

(iii) $n = 2m$ *with* $m \geq 2$, *and* $Hol(g) = SU(m)$ *in* $SO(2m)$,

(iv) $n = 4m$ *with* $m \geq 2$, *and* $Hol(g) = Sp(m)$ *in* $SO(4m)$,

(v) $n = 4m$ *with* $m \geq 2$, *and* $Hol(g) = Sp(m)Sp(1)$ *in* $SO(4m)$,

(vi) $n = 7$ *and* $Hol(g) = G_2$ *in* $SO(7)$, *or*

(vii) $n = 8$ *and* $Hol(g) = Spin(7)$ *in* $SO(8)$.

The followings are some remarks about above holonomy groups. we will discuss in details about Calabi-Yau and G_2 manifold later.

- A manifold with $Hol(g) \subseteq U(m)$ are called *Kähler* manifold. This is a natural class of metrics on complex manifolds.

- A manifold with $Hol(g) \subseteq SU(m)$ are called $Calabi - Yau$ manifold. Since $SU(m) \subseteq U(m)$, it is also Kähler manifold. If g is Kähler. then the restricted holonomy group $Hol^0(g) \subseteq SU(m)$ if and only if g is Ricci-flat. Locally, Calabi-Yau metrics are the same as Ricci-flat Kähler metrics.

- A manifold with $Hol(g) \subseteq Sp(m)$ are called *hyperkähler* manifold. As $Sp(m) \subseteq SU(2m) \subseteq U(2m)$ *hyperkähler* metric is also $Calabi - Yau$.

- A metric g with $Hol(g) = Sp(m)Sp(1)$ for $m \geq 2$ is called *quaternionic hyperkähler*. It is in fact not *Kähler*. It is Einstein [1], but not Ricci-flat.

- The holonomy groups G_2 and $Spin(7)$ are called the exceptional holonomy groups.

In string/M-theory context, the physical inspiration for the holonomy type would be *the existence of invariant spinors due to supersymmetry.* The requirement for preserving n supersymmetries is the existence of n invariant spinors. Therefore, the classification by invariant spinor would be much more direct and intuitive

Theorem II.3. *Let M be an orientable, connected, simply-connected spin n-manifold for $n \geq 3$. and g an irreducible Riemannian metric on M. Define N to be the dimension of parallel spinors on M. If n is even, define $N_\pm$ to be the dimensions of the spaces of parallel spinors in $C^\infty(S_\pm)$, so that $N = N_+ + N_-$. Suppose $N \geq 1$. Then, fixing the orientation, exactly one of the following hold*

1. *$n = 4m$ for $m \geq 1$ and $Hol(g) = SU(2m)$, with $N_+ = 2$ and $N_- = 0$*
2. *$n = 4m$ for $m \geq 2$ and $Hol(g) = Sp(m)$, with $N_+ = m + 1$ and $N_- = 0$*
3. *$n = 4m + 2$ for $m \geq 1$ and $Hol(g) = SU(2m)$, with $N_+ = 1$ and $N_- = 1$*
4. *$n = 7$ and $Hol(g) = G_2$, with $N = 1$*
5. *$n = 8$ and $Hol(g) = Spin(7)$, with $N_+ = 1$ and $N_- = 0$*

So the Calabi-Yau 6-fold (case 3, $n = 6$,$m = 1$) and G_2 manifold (case 4, $n = 7$) on which string and M theories are compactified are due to the existence of 2 parallel spinors and 1 parallel spinor respectively.

[1]An Einstein manifold is a Riemannian or pseudo-Riemannian manifold whose Ricci tensor is linearly proportional to its metric tensor

II.3: Calabi-Yau manifold

Calabi-Yau manifold (CY N-fold) is

- 2N-real-dimensional or N-complex-dimensional
- Tangent vectors rotate by $SU(N)$ subgroup of $SO(2N)$ under parallel transport along closed loop
- Controlled by complex structure I and Kähler form J

II.3.1: Complex manifold

An N-dimensional *complex manifold* is a $2N$-dimensional manifold on which we can introduce local complex coordinates which can be patched together globally in a consistent way (i.e with holomorphic transition functions). More concretely, a $2N$-dimensional manifold is a complex manifold if it admits a globally defined *complex structure*, i.e a mixed tensor J_m^n satisfying $J_m^n J_n^l = -\delta_m^l$ (*almost complex structure*) which induces the vanishing of the *Niejenhuis tensor* which is defined as

$$N_{mn}^P = \delta_{[n} J_{m]}^p - J_{[m}^q J_{n]}^r \delta_r J_q^p = 0. \tag{II.3.1}$$

This is a condition required for local complex coordinates to patch together holomorphically.

It is noteworthy that if M admits an almost complex structure, it must be even-dimensional. This can be seen as follows. Suppose M is n-dimensional, and let J : TM $\rightarrow$ TM be an almost complex structure. If $J^2 = 1$ then $(\det J)^2 = (1)^n$. But if M is a real manifold, then det J is a real number – thus n must be even if M has an almost complex structure. One can show that it must be orientable as well.

Moreover, an even-dimensional manifold does not necessarily have an almost complex structure. An easy exercise in linear algebra shows that any even-dimensional vector space admits a linear complex structure. Therefore, an even-dimensional manifold always admits a $(1,1)$-rank tensor pointwise (which is just a linear transformation on each tangent space) such that $J_p^2 = 1$ at each point p. Only when this local tensor can be patched together to be defined globally does the pointwise linear complex structure yield an almost complex structure, which is then uniquely determined.

The complex struture tensor can be used to define local complex coordinates as follows.

Split the real coordinates to two sets x^j, y^j for $j = 1, \ldots, N$, so that locally $J = \begin{pmatrix} 0 & \mathbf{1}_N \\ -\mathbf{1}_N & 0 \end{pmatrix}$, and construct complex coordinates as $dz^j = dx^j + iI^j_k dy^k$ and $d\bar{z}^j = dx^j - iI^j_k dx^k$. Note that a given real differential manifold can admit many complex structures.

II.3.2: Kähler manifold

On a complex manifold, one can introduce a metric whose only non-zero components are mixed, gamely $g_{i\bar{j}}$. Using this metric to lower one index from the complex structure we obtain a 2-form with mixed indices,

$$\omega = g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}. \tag{II.3.2}$$

Equivalent defininitions would be

- $\omega(u, v) = g(Ju, v)$ for all vector fields u, v
- $\omega_{ac} = J^b_a g_{bc}$

A manifold is *Kähler* if this form is closed

$$d\omega = 0 \tag{II.3.3}$$

in which case it is called the *Kähler form*. The relevance of the Kähler condition in our setup is that the condition constrains the metric such that parallel transport does not mix holomorphic and anti-holomorphic indices, i.e, the holonomy group is at most $U(N) \simeq SU(N) \times U(1)$.

II.3.3: Calabi-Yau manifold

A further reduction to $SU(N)$ requires the absence of holonomy in the overall $U(1)$ factor and is related to the "vanishing of the first Chern class[2])". It can be shown that on a Kähler manifold, the vanishing of the first Chern class is equivalent with the vanishing of the curvature form $R_{i\bar{j}}$, i.e, the manifold is Ricci-flat. We will come back to the geometry of the Calabi-Yau manifold in Sec. II.3.6 after defining a few key concepts.

II.3.4: Hodge number

Given a topological space satisfying the Calabi-Yau conditions, we would like to count the number of free parameters in the choice of its $SU(N)$ holonomy metric. This is analog of

[2] The definition is in Eq. (II.3.15

the choice of compactification radius for a compactification on a space with the topology of a circle. This number of parameters is given by certain topological invariants, the Hodge number. Let us define this briefly in the following.

Let $TM \otimes_{\mathbb{R}} \mathbb{C}$ be the complexified tangent bundle of a complex manifold M. At each point p on M, the complex structure I satisfies $I_p^2 = -id$, so the eigenvalues of I_p are $\pm i$. Hence, we can split the tangent bundle into the bundles generated by eigenvectors of eigenvalues i and $-i$ as $TM \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}M \oplus T^{(0,1)}M$.

We can futher split the cotangent bundle $T^*M \otimes_{\mathbb{R}} \mathbb{C} = T^{*(1,0)}M \oplus T^{*(0,1)}M$. Using this, the exterior power of the cotangent bundle is

$$\Lambda^k T^*M \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{j=0}^{k} \Lambda^j T^{*(1,0)}M \otimes \Lambda^{k-j} T^{*(0,1)}M \equiv \bigoplus_{j=0}^{k} \Lambda^{j,k-j}M \tag{II.3.4}$$

where we define $\Lambda^{p,q}M$ to be $\Lambda^p T^{*(1,0)}M \otimes \Lambda^q T^{*(0,1)}M$. A section of $\Lambda^{p,q}M$ is called a (p,q)-form.

We can also split the exterior derivative d on complex $k-$forms into components $d = \partial + \bar{\partial}$ where ∂ and $\bar{\partial}$ map the vector space of (p,q)-form $C^\infty(\Lambda^{p,q}M)$ to $\Lambda^{p+1,q}M$ and $\Lambda^{p,q+1}M$ respectively. Then $d^2 = 0$ implies $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} + \bar{\partial}\partial = 0$. As $\bar{\partial}^2 = 0$, similar to de Rham cohomology $H_d^k(M)$, we may define the *Dolbeault cohomology* groups of $H_{\bar{\partial}}^{p,q}(M)$ of a complex manifold, by

$$H_{\bar{\partial}}^{p,q}(M) = \frac{Ker(\bar{\partial} : C^\infty(\Lambda^{p,q}M) \to C^\infty(\Lambda^{p,q+1}M))}{Im(\bar{\partial} : C^\infty(\Lambda^{p,q-1}M) \to C^\infty(\Lambda^{p,q}M))}. \tag{II.3.5}$$

In other words, *Dolbeault cohomology* groups contain classes of closed forms where elements of the same class are different by an exact form, with respect to $\bar{\partial}$.

In general complex manifolds, we cannot say much about this cohomology, but for compact Kähler manifolds, we have a wonderful decomposition of de Rham cohomology groups:

Theorem II.4 (Hodge decomposition)**.** *For a compact Kähler manifold M,*

$$H_d^k(M, \mathbb{C}) = \bigoplus_{j=0}^{k} H_{\bar{\partial}}^{j,k-j}(M) \tag{II.3.6}$$

This decomposition depends on the complex structure, not on any choice of a particular *Kähler* metric. Moreover, $h^{p,q} = dim H_{\bar{\partial}}^{p,q}(M)$ is called Hodge number. Then, the Betti number b^k, which is $dimH_d^k(M,\mathbb{C})$, is decomposed to

$$b^k = \sum_{j=0}^{k} h^{j,k-j} \tag{II.3.7}$$

Moreover, it can also be shown that

$$h^{p,q} = h^{q,p} = h^{m-q,m-p} = h^{m-p,m-q} \tag{II.3.8}$$

where m is the complex dimension of M. The set of Hodge numbers characterize the topology of the Kähler manifold. However, this decomposition is not a irreducible decomposition in general, so there is some loss of topological information. Yet, this is enough for us to learn a lot about the manifold. In the next chapter, we will see more concretely how the Hodge numbers relate to the parameters of the manifold.

II.3.5: Harmonic decomposition

We will try to define the Laplacian operator on Kähler manifolds. First, we will define the inner product of the complex k-form. Define a pointwise product (α,β) on M by

$$(\alpha,\beta) = \alpha_{a_1\ldots a_k}\overline{\beta_{b_1\ldots b_k}}g^{a_1b_1}\ldots g^{a_kb_k} \tag{II.3.9}$$

When M is compact, define the L^2 inner product of complex k-forms α, β by

$$\langle\alpha,\beta\rangle = \int_M (\alpha,\beta)dV_g \tag{II.3.10}$$

Then, $\langle,\rangle$ is a Hermitian inner product on the space of complex k-forms.

Next, let *the Hodge star on Kähler manifolds* be the unique map $* : \Lambda^k T^*M \otimes_{\mathbb{R}} \mathbb{C} \to \Lambda^{2m-k}T^*M \otimes_{\mathbb{R}} \mathbb{C}$ satisfying the equation $\alpha \wedge (*\beta) = (\alpha,\beta)dV_g$ for all complex k-forms α, β. Then, we can define operators d^*, ∂^*, and $\bar{\partial}^*$ by

$$d^*\alpha = -*d(*\alpha), \qquad \partial^*\alpha = -*\partial(*\alpha) \qquad \text{and} \qquad \bar{\partial}^*\alpha = -*\bar{\partial}(*\alpha). \tag{II.3.11}$$

Then d^*, ∂^*, and $\bar{\partial}^*$ take the complex k-forms to complex (k-1)-forms. Moreover, it can be shown that

$$\langle\alpha, d^*\beta\rangle = \langle d\alpha,\beta\rangle, \qquad \langle\alpha,\partial^*\beta\rangle = \langle\partial\alpha,\beta\rangle \qquad \text{and} \qquad \langle\alpha,\bar{\partial}^*\beta\rangle = \langle\bar{\partial}\alpha,\beta\rangle \tag{II.3.12}$$

Now, we can define Laplacian operators. Similar to the Riemannian case, we can make three different Laplacians on complex k-forms

$$\Delta_d = dd^* + d^*d, \qquad \Delta_\partial = \partial\partial^* \qquad \Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial. \tag{II.3.13}$$

It can be shown that

$$\Delta_\partial = \Delta_{\bar\partial} = \frac{1}{2}\Delta_d. \tag{II.3.14}$$

Then, harmonic form ω is defined to be a complex k-form that satsifies $\Delta\omega = 0$. Define $H^{p,q}(M)$ to be the vector subspace of $H^{p+q}(M, \mathbb{C})$ with harmonic (p,q)-form representatives. Now, the important result is that

Theorem II.5. *Let M be a compact Kähler manifold. Every element of $H^{p.q}(M)$ is represented by a unique harmonic (p,q)-form. Moreover, for all p, q we have $H^{p,q}(M) \simeq H^{p,q}_{\bar\partial}(M)$*

So, in fact, we can decompose the cohomology into classes of unique harmonic representatives. Thus, the Hodge numbers in fact count harmonic forms on the manifold. Each of these harmonic forms is unique in their cohomology class, so the harmonic forms. This is important in the physics context because, we will see later, the physical equations require that the massless states are zero modes of harmonic form decomposition. The light spectrum in the supergravity limit is derived from this decomposition.

II.3.6: Calabi-Yau manifold - Basic geometry

Theorem II.6. *Let (M, J, g) be a Kähler manifold. The $Hol^0(g) \subseteq SU(m)$ if and only if g is Ricci-flat.*

Calabi-Yau manifolds have slightly different definitions depending on the literature. We will consider a Calabi-Yau manifold as a compact Kähler manifold (M, J, g) with holonomy $Hol(g) \subseteq SU(m)$. The existence of a Ricci-flat metric is in fact equivalent to the vanishing first Chern class in a compact Kähler manifold. Recall the Chern classes can be defined in explicit form by the formal expansion of Tr $\exp(\frac{R}{2\pi})$, i.e

$$c_n = \frac{1}{n!(2\pi)^n}\mathrm{Tr}R^n \tag{II.3.15}$$

where R is the curvature form of the maniold

Theorem II.7. *Let (M, J) be a compact complex manifold admitting Kähler metrics with $c_1(M) = 0$. Then, there is a unique Ricci-flat Kähler metric in each Kähler class in $H^{1,1}(M)$*

on M. The Ricci-flat Kähler metrics on M form a smooth family of dimension $h^{1,1}(M)$.

The consequence of this is that *given a Kähler manifold M, the necessary and sufficient condition for being Calabi-Yau is a vanishing first Chern class.* Moreover, for each class in $H^{1,1}(M)$, there is a unique Calabi-Yau metric for the manifold. Thus, we can use this to parametrize the possible scenarios for a given topology. Concretely, each class of $H^{1,1}(M)$ is represented by a unique harmonic form, so we can parametrize the Kähler form ω by these harmonic forms

$$\omega = \sum_{a=1}^{h_{1,1}} t_a \omega_a \tag{II.3.16}$$

where t_a are real scalars, which called *Kähler moduli* of the Calabi-Yau manifold, and control the sizes of even-dimensional 2n-cycles of the internal space, whose volumes are measured by integrating the 2n-form ω^n over them (Wirtinger's formula). In particular, ω^m is the volume form of a Calabi-Yau m-fold.

Another equivalent condition for a Calabi-Yau m-fold is being a Kähler manifold admitting a nowhere vanishing (m,0) form Ω. In the case of Calabi-Yau 3-fold, the choice of a complex structure tensor is equivalent, via contraction with Ω, to the choice of a (2,1) form $I_{ij\bar{k}} = \Omega_{ijl} I^l_{\bar{k}}$. Expanding in a basis, there are $h_{2,1}$ complex parameters involved in the choice of the complex structure. These are known as *complex structure moduli* of the Calabi-Yau, and control the sizes of 3 cycles of the internal space whose volumes are measured by integrating Ω over them.

Thus, the parameters required to specify a unique metric in a given Calabi-Yau compactification space are the $h_{1,1}$ real Kähler moduli and the $h_{2,1}$ complex structure moduli.

In the string theory context, the original condition for Calabi-Yau manifolds was the existence of a covariantly constant spinor ξ. From it, we can construct the forms

$$\omega_{i\bar{j}} = -\xi^\dagger \Gamma_i \Gamma_{\bar{j}} \xi, \qquad \Omega_{ijk} = \xi^T \Gamma_i \Gamma_j \Gamma_k \xi \tag{II.3.17}$$

where Γ_i are 6d Gamma matrices.

II.3.7: Special Lagrangian

Special Lagrangian 3-cycles Π are volume minimizing 3-cycles. It is defined by

$$\omega|_\Pi = 0 \qquad Im(e^{i\phi}\Omega_3)|_\Pi = 0 \tag{II.3.18}$$

for some fixed ϕ. Then, the volume of Π is given by

(II.3.19)
$$Vol(\Pi) = \int_{\Pi} Re(e^{-i\phi}\Omega_3)$$

This submanifold is important because, in string theories, this is where the 6D-branes are wrapping and stabilized. It preserves the 4d $\mathbb{N} = 1$ supersymmetry, so it is also called a supersymmetric cycle. We will revisit this in the later review of type IIA string theory. The analog of this in a G_2 manifold is an associative 3-cycle which will be the base for the ADE singularity fibration in the M-theory model.

II.4: G_2 manifold

G_2 manifold is

- **7-dimensional**
- **tangent vectors rotate by the G_2 subgroup of $SO(7)$ under parallel transport along closed loop**
- **controlled by a 3-form ϕ and its dual $*\phi$**
- **If M is a seven-dimensional spin manifold, then M carries a non-trivial parallel spinor field if and only if the holonomy is contained in G_2.**

II.4.1: overview and definitions

The local model for a G_2 manifold is $\mathbb{R}^7$ where, if we impose $\mathbb{R}^7 = \mathbb{R}^3 \oplus \mathbb{R}^4$ with coordinate (x_1, x_2, x_3) on $\mathbb{R}^3$ and (y_0, y_1, y_2, y_3) on $\mathbb{R}^4$, we define the 3-form:

(II.4.1)
$$\varphi_{\mathbb{R}^7} = dx_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge \omega_1 + dx_2 \wedge \omega_2 + dx_3 \wedge \omega_3$$

where

(II.4.2)
$$\omega_1 = dy_0 \wedge dy_1 - dy_2 \wedge dy_3, \quad \omega_2 = dy_0 \wedge dy_2 - dy_3 \wedge dy_1, \quad \omega_3 = dy_0 \wedge dy_3 - dy_1 \wedge dy_2$$

form an orthogonal basis for the anti-self-dual 2-forms on $\mathbb{R}^4$. The stabilizer of $\varphi_{\mathbb{R}^7}$ in $GL(7, \mathbb{R})$ is isomorphic to the group G_2. Moreover, G_2 can be described as the automorphism group of the octonion algebra $\mathbb{O}$. Perhaps the most useful definition is G_2 as the subgroup of $SO(7)$ that preserves any chosen particular vector in its 8-dimensional real spinor representation. The last definition is directly used in M-theory as a G_2 manifold has one invariant spinor due to supersymmetry. However, the explicit local form in the first definition is useful in computation and visualization of the geometry in the following.

Definition II.8. A smooth 3-form φ on a 7-manifold M is a G_2-*structure* if for all $x \in M$, there exists an isomorphism $i_x : \mathbb{R}^7 \to T_x M$ so that $i_x^* \varphi = \varphi_{\mathbb{R}^7}$. This structure is sometimes called a *definite* or *positive* 3-form.

A G_2-structure φ defines a metric g_φ and orientation on M, given by volume form vol_φ, and thus a Hodge star operator $*$ on M. In fact, one has that

(II.4.3)
$$\mathrm{vol}_\varphi = \frac{1}{7}\varphi \wedge *_\varphi \varphi.$$

In $\mathbb{R}^7$, one sees explicitly that $\varphi_{\mathbb{R}^7}$ induces the flat metric on $\mathbb{R}^7$ and the standard volume form, and that the Hodge dual of $\varphi_{\mathbb{R}^7}$ is:

(II.4.4)
$$*_{\varphi_{\mathbb{R}^7}} \varphi_{\mathbb{R}^7} = dy_0 \wedge dy_1 \wedge dy_2 \wedge dy_3 - dx_2 \wedge dx_3 \wedge \omega_1 - dx_3 \wedge dx_1 \wedge \omega_2 - dx_1 \wedge dx_2 \wedge \omega_3.$$

A 3-form φ on $\mathbb{R}^7$ gives rise to a canonical symmetric bilinear form on $\mathbb{R}^7$

(II.4.5)
$$B_\varphi(X, Y) = -\frac{1}{6}(X \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi$$

with value in $\Lambda^7(\mathbb{R}^7)^*$. For a generic three-form φ, the bilinear form B_φ yields a non-degenerate pairing for some signature (p, q) with $p + q = 7$ (with respect to an oriented volume form on $\mathbb{R}^7$). In particular, there is an open set $\Lambda^3_+(\mathbb{R}^7)*$ in the space of 3-form $\Lambda^3(\mathbb{R}^7)*$ such that B_φ is a positive definite bilinear form for $\varphi \in \Lambda^3_+(\mathbb{R}^7)^*$. The $GL(7, \mathbb{R})$ acts on the 3-form φ and G_2 is its 14-dimensional stabilizer subgroup. Since G_2 leaves the positive definite pairing B_φ invariant, it is actually a subgroup of $SO(7)$.

Remark II.9. Being positive definite is crucial for defining a Riemannian metric.

Since the Lie group G_2 is the stabilizer group of the described 3-form, a 7-dimensional oriented manifold Y together with the 3-form in $\Omega^3_+(Y)$ (which is the space of smooth 3-forms and isomorphic to $\Lambda^3 T^*_p(Y) \approx \Lambda^3_+(\mathbb{R}^7)^*$ for any $p \in M$) becomes a G_2-structure manifold.

Remark II.10. $\Lambda^3_+(\mathbb{R}^7)^*$ is a convex open set in $\Lambda^3(\mathbb{R}^7)^*$. Thus, with partition of unity, we can construct a G_2 structure on any smooth paracompact 7-dimensional manifold Y.

Thus, we call φ a G_2-structure on Y. Furthermore, the positive definite pairing II.4.5 defines a Riemannian metric g_φ on Y. Namely, at any point $p \in M$ and for any basis

$\partial_1|_p, \ldots, \partial_7|_p$ at T_pY, we obtain a positive definite inner product

$$g_\varphi(X_p, Y_p) = \frac{B_\varphi(X_p, Y_p)(\partial_1|_p, \ldots, \partial_7|_p)}{\mathrm{vol}_p(\partial_1|_p, \ldots, \partial_7|_p)} \tag{II.4.6}$$

$$\mathrm{vol}_p(\partial_1|_p, \ldots, \partial_7|_p)^9 = \det[B_\varphi(\partial_i|_p, \partial_j|p)(\partial_1|_p, \ldots, \partial_7|p)] \tag{II.4.7}$$

Theorem II.11. *A G_2-structure manifold has a subgroup of G_2 as its holonomy group if and only if the 3-form φ is harmonic with respect to the constructed G_2 metric g_φ [66]*

$$d\varphi = 0, \qquad d *_{g_\varphi} \varphi = 0 \tag{II.4.8}$$

in terms of Hodge star $*_{g_\varphi}$ of the metric g.

Such a harmonic 3-form in $\Omega^3_+(Y)$ is called torsion-free. Requiring in addition a finite fundamental group $\phi_1(Y)$ ensures that Y has G_2 homology and not a proper subgroup thereof. Given a torsion free G_2-structure of a G_2-manifold, the local structure of the moduli space $\mathcal{M}$ of G_2-manifolds is known due to Joyce [89]. In particular the Betti number $b_3(Y)$ is the dimension of $\mathcal{M}$.

Definition II.12. We say that (M^7, φ) is a G_2 manifold if φ is a G_2-structure on M which is *torsion-free*, which means that

$$d\varphi = 0 \qquad d *_\varphi \varphi = 0 \tag{II.4.9}$$

The following are equivalent conditions for torsion-free (Prop 10.1.3 [90]):

- (φ, g) is torsion-free,
- $\mathrm{Hol}(g) \subset G_2$, and φ is the induced 3-form,
- $\nabla\varphi = 0$ on M, where ∇ is the Levi-Civita connection of g,
- $d\varphi = d^*\varphi = 0$ on M, and
- $d\varphi = d * \varphi = 0$ on M

II.4.2: Associative and coassociative cycles

We are interested in a distinguished class of 4-dimensional submanifolds called coassociative $4-folds$

Definition II.13. An oriented 4-dimensional submanifold N^4 of a G_2 manifold (M^7, φ) is coassociative if N is calibrated by $*\varphi$. This means that

$$*\varphi|_N = \mathrm{vol}_N \tag{II.4.10}$$

Equivalently [83], up to a choice of orientation N^4 is coassociative if and only if

$$\varphi|_N = 0 \tag{II.4.11}$$

Another important class of submanifolds that we encounter is

Definition II.14. An oriented 3-dimensional submanifold L of a G_2 maniofld (M, φ) is associative if L is calibrated by φ. This means that

$$\varphi|_L = \mathrm{vol}_L \tag{II.4.12}$$

Remark II.15. The orthogonal complement of an associative 3-plane is a coassociative 4-plane, and vice versa

Remark II.16. Associative 3-cycles of G_2 manifolds are analog of special Lagrangian 3-cycles of Calabi-Yau manifolds.

Definition II.17. Let M be a G_2 manifold. We say that M admits a coassociative fibration if there is a 3-dimensional space $\mathcal{B}$ parametrizing a family of coasociative submanifold N_b for $b \in \mathcal{B}$ of M, with the following two properties

- The family $\{N_b : b \in \mathcal{B}\}$ covers M and there is a dense open subset $\mathcal{B}^\circ$ of $\mathcal{B}$ such that N_b is smooth for all $b \in \mathcal{B}^\circ$. That is, every point $p \in M$ lies in at least one coassociative submanifold N_b, and a generic member of the family is smooth.

- On a dense open set M' of M, there is a genuine fibration of M' onto a submanifold $\mathcal{B}'$ of $\mathcal{B}$, in the sense that there is a smooth map $\pi : M' \to \mathcal{B}'$ which is a locally trivial fibration, and such that $\pi^{-1}(b) = N_b$ for each $b \in \mathcal{B}$

Remark II.18. The set $\mathcal{B} \backslash \mathcal{B}^\circ$ parametrizes singular fibres in the fibration, and the set $M \backslash M'$ consists of the points where two coassociatives in the family $\{N_b : b \in \mathcal{B}\}$ intersect.

Suppose that a G_2 manifold $(M, \varphi, g_\varphi, *_\varphi\varphi)$ can be described as a fibration by coassociative submanifolds. Then its tangent bundle TM admits a vertical subbundle V, which is the bundle of (coassociative) tangent subspaces of the coassociative fibers of M. Since coassociative subspaces come equipped with a preference of orientation, the bundle V is oriented.

A local vertical vector field is a local section of V, and hence is everywhere tangent to the coassociative fibers. In fact, for most of the known study of M-theory on G_2 manifold with ADE singularities, the G_2 manifold is assumed to have coassociative fibration. The ADE singularities live on the coassociative submanifolds N_b of the local form $\mathbb{C}^2/\Gamma$ where Γ is a $SU(2)$ finite subgroup.

Finally, it is useful to make a connection between Calabi-Yau and G_2 manifolds since M-theory is considered to be non-perturbative limit of string theories. Assume a G_2 manifold M is locally an elliptic fiberation of a 6d Calabi-Yau manifold with Kähler form ω and holomorphic 3-form Ω_3. Then, the G_2 structure ϕ on M can be identified by

$$\phi = Re(\Omega_3) + \omega \wedge dx_7 \qquad *\phi = Im(\Omega_3) \wedge dx^7 - \frac{1}{2}\omega \wedge \omega \tag{II.4.13}$$

This shows how the local spectrum of string theories is lifted to M-theory, as we can lift each harmonic mode to M-theory accordingly. Of course, this provides hints and inspiration, but never the full picture because the global G_2 structure is much more complicated.

Notice that all of the above discussion is about *smooth* G_2 manifold. In M-theory, we will later see that a singular M-theory is required. In particular, we are interested in G_2 manifolds with ADE singularities. However, the construction of G_2 manifold with ADE singularities is mathematically challenging. Only recently some works have claimed to construct G_2 manifolds with E_n singularities [113]. Thus, in the later chapters, we will only consider a local model of G_2 with ADE singularities and assume that all of the above smooth properties are still satisfied.

II.5: ADE singularity

An ADE singularity, or Du Val singularity, is a point p whose local neighborhood is isomorphic to the local neighborhood of $0 \in \mathbb{C}^2/\Gamma$ where Γ is a finite subgroup of $SL(2,\mathbb{C})$. As every finite subgroup of $SL(2,\mathbb{C})$ is conjugate to a finite subgroup of $SU(2)$, it suffices to study finite subgroups of $SU(2)$. The singularity can be smoothened out by "blowing up" the singularity. For simplicity, we consider the definition of blown-up most relevant to our work.

Definition II.19. Consider $X = \mathbb{C}^2/\Gamma$ with singularity at the origin 0. A blown-up $\tilde{X}$ of X at 0 is defined to be

$$\tilde{X} = \Big\{((x,y),(u:v)) \in X \times \mathbb{CP}^1 | xv = yu\Big\} \tag{II.5.1}$$

Recall that $\mathbb{CP}^1$ is topologically a 2-sphere. In essence, $\tilde{X}$ is exactly X except the singular point is being blown up into a sphere. $\tilde{X}$ is called a resolution of X. From here, readers can similarly define blowing-up for a general X with ADE singular point p whose local neighborhood is isomorphic to $0 \in \mathbb{C}^2/\Gamma$.

Define a projection map $\pi : \tilde{X} \to X$. Then, $\pi^{-1}(p)$ is called an exceptional divisor. If we start from II.19, the exceptional divisor is a sphere. However, oftentimes, the blowing-up may not be enough to completely smoothen the singularity, and there are still singular points on the sphere (although with a lesser degree). In this case, we can repeat the blowing-up procedure on these singular points until the manifold is completely smoothened. Then, the exceptional divisor is a collection of spheres intersecting transversely. The configuration of how these spheres intersect is exactly the Dynkin diagram of simply-laced Lie groups. These groups are classified into A, D, and E types, hence the name of this type of singularities. Figure II.1 gives a list of these diagrams. Consider one example, Figure II.2 gives a pictorial

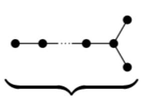
Type	diagram	finite subgroups of $SU(2)$	simple Lie group
A_n	 n nodes	$\mathbb{Z}_{n+1}$	$SU(n+1)$
D_n	 n nodes	$2\mathbb{D}_{2(n-2)}$	$SO(2n), Spin(2n)$
E_6		$2T$	E_6
E_7		$2O$	E_7
E_8		$2I$	E_8

Figure II.1: Dynkin diagrams

illustration of a singularity of type A_3. Each of the consequent $\mathbb{P}^1$ can be called a two-cycle. So, a singularity of type A_3 is one that, when completely resolved, has a configuration of A_3. Similarly, a singularity of a certain Dynkin diagram has the blown-up configuration of that diagram. We have seen that the 2-cycles $\mathbb{P}^1$ directly relate to the smoothing of singularities. We can use the volume of the 2-cycles to parametrize the resolutions. Such a method of

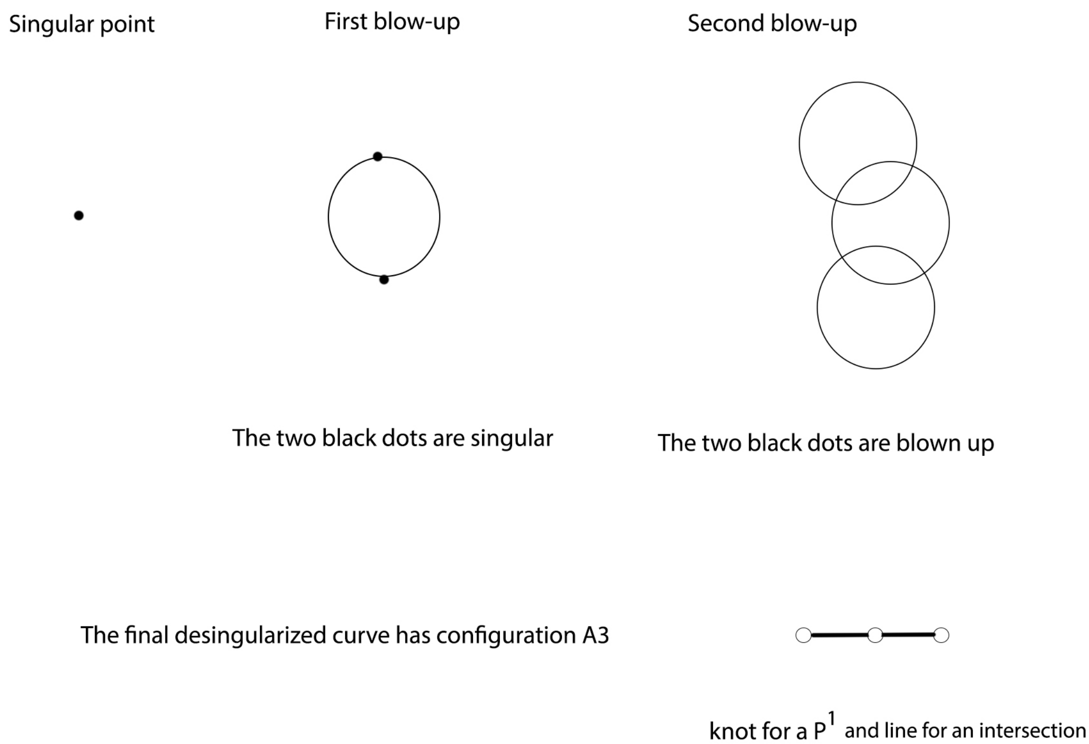


Figure II.2: An A_3 type singularity being fully resolved will have a configuration of three Riemann spheres $\mathbb{P}^1(\mathbb{C})$ which intersect according to A_3 Dynkin diagram. Note that every two spheres intersect at most at one point transversely.

smoothly parametrizing the blowing-up is called deformation.

CHAPTER III
Physics Background

The complete formulation of M-theory is still difficult to put together. Most of our inspiration and understanding about M-theory comes from the web of string dualities. In this chapter, we will briefly introduce this web as well as a dictionary for mapping the geometric objects between the theories. From this duality, we see that M-theory should be compactified on a G_2 manifold with ADE singularities. All of this discussion is strongly correlated with the mathematical tools in Chapter II. All materials here can be found in standard textbooks such as [111, 87].

III.1: Brief overview of the Standard Model

The Standard Model (SM) of particle physics is so far the most successful theory describing the three fundamental forces (the electromagnetic, weak, and strong interactions, while excluding gravity) governing our world. It will help frame our discussion to briefly describe some key features of SM and the known experimental results. All unified theories, including the M theory of our interest, must at least achieve the established success of SM to be considered a physical candidate.

One key structure of SM is gauge symmetry. In general, a gauge symmetry of a physics theory is a symmetry where all the elements in the same "gauge" orbit of the symmetry will correspond to the same physical state. In order words, the theory is unchanged under the action of its gauge symmetry. We encounter many forms of gauge symmetry in different places, but notably, in SM, the gauge symmetry is a symmetry of the fields describing the elementary particles. The gauge group of SM is

$$SU(3) \times SU(2) \times U(1). \tag{III.1.1}$$

This is a local symmetry, meaning its action on the fields is local. All the fields under the same orbit of this action describe the same particle. Therefore, the particles are characterized by the representations and charges of the gauge group.

Note that global symmetry is just a specific case of local symmetry with uniform global action. In SM physics, we use gauge group to exclusively refer to the group in III.1.1. The global symmetry in SM is the isometry group of the Minkowski spacetime, which is called the Poincaré group. The Poincaré group is a semidirect product of translations and the Lorentz group

$$\mathbb{R}^{1,3} \rtimes O(1,3) \tag{III.1.2}$$

where the Lorentz group $O(1,3)$ is the rotation group of the Minkowski spacetime. We expect the physics is invariant under Poincaré action. So, the particles are also in representations of this group. In the string/M theory description, we will see that gauge symmetry is incorporated into spacetime symmetry through singularities.

One big class of elementary particles is fermions, including quarks (up, charm, top, down, strange, bottom), leptons (electron, moun, tau, electron neutrino, moun neutrino, tau neutrino), and their antiparticles [1]. Fermionic fields are spinor fields on spacetime, so fermions split into the two irreducible spin representations of the Lorentz group, which are identified with “left-handed” and “right-handed” fermions. The fermions with different handedness usually transform in a different representation of the gauge group. Table III.1 gives a list of all SM fermions and their representations and charges under the gauge group.

Besides fermions, there is another class of particles called bosons. They are tensorial fields on spacetime. At rank 0, we have two Higgs fields H^u and H^d as scalar fields which transform trivially under the Lorentz group. The Higgses transform as $(\mathbf{1},\mathbf{2})$ of $SU(3)\times SU(2)$ with $U(1)$ charge 3 and -3 respectively. These fields are responsible for breaking electroweak symmetry ($SU(2)\times U(1)$ factor) and giving mass the fermions. At rank 1, we have a vector representation of the Lorentz group. This associates with gauge bosons whose fields are vector fields on spacetime with values in the Lie algebra of the gauge group. They act as mediators for the fundamental forces. The list stops here for SM. In string/M theories, we will see graviton bosons in the rank 2 tensor representation.

Another ingredient of SM is the framework of Lagrangians. This is an extension of Lagrangians from classical mechanics to quantum field theory. We will not devote too much time on this besides noting a few key features to recognize in a Lagrangian. First, if ϕ (or ψ) is a bosonic (or fermionic field), then a term in the quadratic form $m^2\phi^2$ (or $m\psi\bar{\psi}$) is called a mass term for the particle with mass m, This can be generalized in to n different fields in the form $m_{ij}^2\phi_i\phi_j$ (or $m_{ij}\psi_i\bar{\psi}_j$) where the masses are eigenvalues of m_{ij}, and the eigenvectors

[1] Which are in conjugate representations of the gauge group.

	SU_3	SU_2	$U(1)$
Q_1	$\mathbf{3}$	$\mathbf{2}$	1
Q_2	$\mathbf{3}$	$\mathbf{2}$	1
Q_3	$\mathbf{3}$	$\mathbf{2}$	1
u_1	$\mathbf{3}$	$\mathbf{1}$	4
u_2	$\mathbf{3}$	$\mathbf{1}$	4
u_3	$\mathbf{3}$	$\mathbf{1}$	4
d_1	$\mathbf{3}$	$\mathbf{1}$	-2
d_2	$\mathbf{3}$	$\mathbf{1}$	-2
d_3	$\mathbf{3}$	$\mathbf{1}$	-2
L_1	$\mathbf{1}$	$\mathbf{2}$	-3
L_2	$\mathbf{1}$	$\mathbf{2}$	-3
L_3	$\mathbf{1}$	$\mathbf{2}$	-3
e_1	$\mathbf{1}$	$\mathbf{1}$	6
e_2	$\mathbf{1}$	$\mathbf{1}$	6
e_3	$\mathbf{1}$	$\mathbf{1}$	6

Table III.1: SM fermions without conjugates where Q_i's are left-handed quarks, u_i's and d_i's are right-handed up-type (up, charm, top) and down-type (down, strange, bottom) quarks, L_i's are left-handed leptons, e_i's are charged right-haned leptons (electron, moun, tau). Notice that the right handed neutrinos ν_i are not here because we are not sure if they exist. If they do, they will transform in the $(\mathbf{1}, \mathbf{1})$ representation of $SU(3) \times SU(2)$ with 0 charge under $U(1)$.

are called mass eigenstates. At cubic level, we have terms called Yukawa couplings, most commonly of the form $Y_{ijk}\psi_i\bar{\psi}_j\phi_k$ where the scalar Y_{ijk} is called a Yukawa coupling strength, or, just a Yukawa coupling. Y_{ijk} determines the strength of the interaction between two fermions ψ_i and ψ_j and a scalar ϕ. Similarly, any term in the Lagrangian signifies an interaction between the particles present in that term. However, not every combination is allowed in the Lagrangian. As a Lagrangian is required to be invariant under both gauge symmetry and Lorentz symmetry, for each term in the Lagrangian, the tensor product of its fields must contain a singlet, i.e, the trivial representation for all the symmetry groups.

It is believed that supersymmetry is essential for solving many problems in physics. It acts as a symmetry between bosonic and fermionic particles. One of the original inspirations for supersymmetry was to solve the quadratically divergent contributions to the Higgs mass in the Standard Model by automatic cancellations between fermionic and bosonic Higgs interactions. If restored at the weak scale, supersymmetry eventually explains the large discrepancy between aspects of the weak force and gravity. Another motivation of supersymmetry is toward grand unification where we expect the gauge group to unify at high energy. Gauge coupling constants do not quite intersect at high energy in the Standard Model, but with supersymmetry, they do at approximately 10^{16} GeV which we call the GUT scale. Given that, we want an extension to the Standard Model that realizes supersymmetry. Such an extension with the minimum amount of particles while staying consistent with the empirical results is called the Minimal Supersymmetric Standard Model (MSSM). In the following M theory construction, we hope to reduce to MSSM in an appropriate limit in order to be consistent with phenomenology.

III.2: String theories

5 string theories with a web of dualities. See Figure III.1

- **The most natural compactifitication is on Calabi-Yau due to supersymmetry.**
- **M-theory compactified on a circle S^1 and $S^1/\mathbb{Z}_2$ reduces to Type IIA and heterotic $E_8 \times E_8$ respectively.**

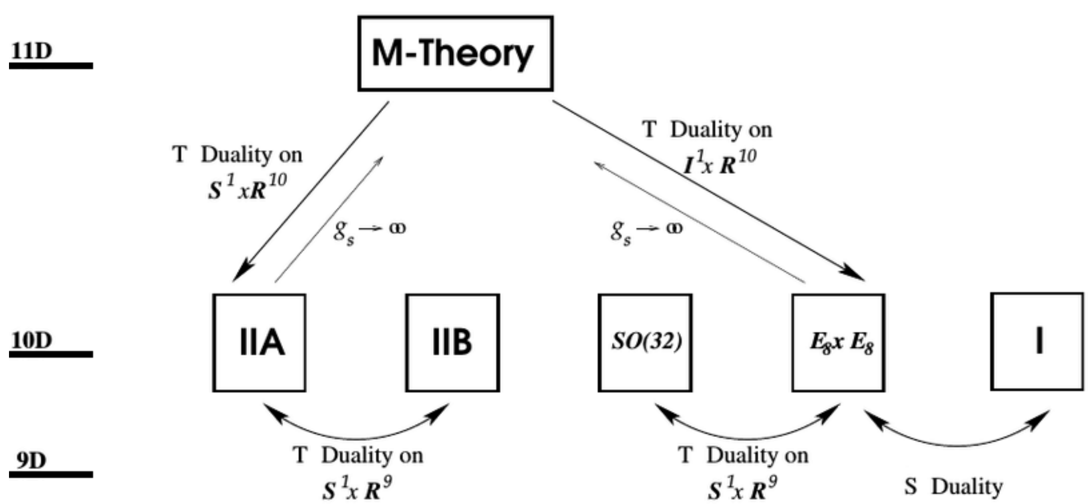


Figure III.1: String duality

There are five superstring theories that are connected by a web of dualities. All of them seem to be a limit of a non-perturbative theory called M-theory. We will briefly look over all of the string theories, as it will be useful for the later discussion of M-theory Moreover, as discussed in the previous section, it is sensible to consider string theories with supersymmetry. Such theories are called superstring theories. All discussion here will be implicitly understood as superstring.

The original string theories without supersymmetry are called bosonic string theories. As the name suggests, they contain only a bosonic spectrum. Conformal symmetry of the world sheet[2] requires bosonic string theories to be 26 dimensional. For the superstring, the local symmetries on the world sheet are enhanced from the ordinary conformal group to the $\mathcal{N}=1$ superconformal group with new (fermionic) generators. This new symmetry group reduces the appropriate dimension of the theory to 10.

There is a subtlety about supersymmetry in string theory. There technically are two types of supersymmetry: worldsheet [3] supersymmetry and spacetime supersymmetry. Worldsheet supersymmetry is the symmetry of the worldsheet Ramond-Neveu-Schwarz (RNS) action under the worldsheet supersymmetry transformations. It is a symmetry of the classical action of the superstring. It is noteworthy that *writing down a worldsheet action will automatically result in worldsheet supersymmetry*. On the other hand, we have spacetime supersymmetry. While writing down a minimal supersymmetric gravity theory, we naturally arrive at a symmetry between tensor fields (including the metrics) and spinor fields. As each well-defined supersymmetric transformation requires an invariant spinor, the spacetime supersymmetry directly relates to the holonomy of the spacetime. There is a difference between the two concepts. As string theories are perturbative theories around a fixed spacetime background, the worldsheet symmetry is in fact local supersymmetry. In contrast, if the holonomy of the global spacetime allows invariant spinors, the corresponding spacetime supersymmetry is global. In supergravity theories which are low energy limit of string theories, the GSO projection [4] matches the two types of supersymmetry precisely. This boils down to a consequence of unitarity [5].

As our world is visibly 4 spacetime dimensional. 10d string theories need to be compactified on a small 6 dimensional manifold, which is called internal space. Supersymmetry requires that the 6 dimensional manifold has one invariant spinor. From the previous chapter, this is exactly the condition for a Calabi-Yau manifold. Explicitly, all of the five superstring theories are of the form $R^{3,1} \times X$ where supersymmetry implies X to be 6d Calabi-Yau.

We perform a KK compactification for the bosonic fields and complete them to 4d $\mathcal{N}=2$ supermultiplets, which corresponds to two invariant spinors as classified in Chapter II. Depending on the content of the theory, some of these supersymmetric completion does not present in the theory, hence the theory may reduce to smaller supersymmetry. We have five possible string theories:

[2] Invariant under rescalings of the background metric (Weyl transformations)

[3] When a 1d string travels in spacetime, it swipes out a 2d surface called worldsheet

[4] Mapping out tachyons that have imaginary mass.

[5] See more in [111], volume II.

Type II theories are theories of closed oriented strings. They describe the free propagation of strings in spacetime, and the fields obey the 2d wave equation. The worldsheet theory is split into left and right moving sectors. In Type IIA, they have opposite chirality. In type IIB, they have the same chirality. There are equal contributions from left-moving and right-moving sectors, so the supersymmetry is $\mathcal{N} = 2$ as above general completion [61].

Heterotic string theories have two different algebras acting on the left and right movers. For phenomenological interest, the only option is a left-moving bosonic string sector (acted on by the usual Poincairé algebra) and a right-moving superstring sector (acted on by the supersymmetric algebra). There are only two possible gauge groups: either $E_8 \times E_8$ or $SO(32)$, hence the names. These possible gauge groups can be deduced in different ways. Notably, hexagon gravitational anomaly cancellation through the Green-Schwarz mechanism is only possible for these two gauge groups [45].

Type I theory is a theory of both open and closed unoriented strings. It is consistent only for the gauge group $SO(32)$. The right-movers and left movers are related by the open string boundary condition and transform under the same spacetime supersymmetry.

It is also noteworthy to recall that in compactification, open string endpoints are restricted to lie on subspaces. These subspaces are called D-branes. p-dimensional D-branes are called Dp-branes. Mathematically, these branes plus one temporal dimension are presented as integrating subspaces for differential forms. For example, a D2-brane evolving in time will sweep out a 3d W_3 subspace, and in fact, appears in an effective Lagrangian as $Q \int_{W_3} C_3$ where C_3 is a differential 3 form. We say C_3 electrically charges the D2-brane with charge Q.

The background in this section can be found in more detail in standard string theory textbooks [111, 87] unless stated explicitly. In the following sections, we will try to illustrate some key features of string duality. This will help understand the lift of all string theories' results to M theory. In standard literature, M theory directly duals with type IIA and $E_8 \times E_8$ heterotic string theories. The rest of the web is connected to M theory indirectly through those two string theories. Although there are several works directly dualizing M theory and the rest of the web [75, 32], in the following section we will follow the standard route. We will see that the field contents of type IIA and M theory locally are identical in low energy. The nonabelian gauge in M theory is inspired by the lift from IIA theory. Some other dualities will be briefly mentioned to give an intuition about how these are connected to M theory.

III.3: Type IIA on Calabi Yau

The bosonic fields for 10d type IIA string theory are the graviton G, the NS-NS 2-form B [6], the dilaton ϕ, and the R-R 1-form A_1 [7] and the 3-form C_3. Compactified on a Calabi-Yau manifold, their KK reductions, shown in Table III.2, are a gravity multiplet containing a graviton $G_{\mu\nu}$ and a graviphoton A_μ, $h_{1,1}$ vector multiplets containing a gauge boson $C_{i\bar{j}\mu}$ and a complex scalar $B_{i\bar{j}}$ [8], and $h_{2,1}+1$ hypermultiplets containing two complex scalars. Here i, j are internal space indices, and μ, ν are non-compact 4d spacetime indices.

IIA		Gravity	$h_{1,1}$ Vector	$h_{2,1}$ Hyper	Hyper
G	$\rightarrow$	$G_{\mu\nu}$	$h_{1,1}$ Kähler	$2h_{2,1}$ Cmplx.Str.	
B	$\rightarrow$		$B_{i\bar{j}}$		$B_{\mu\nu}$
ϕ	$\rightarrow$				ϕ
A_1 (or C_1)	$\rightarrow$	A_μ			
C_3	$\rightarrow$		$C_{i\bar{j}\mu}$	$C_{ij\bar{k}}$, $C_{\bar{i}\bar{j}k}$	C_{ijk}, $C_{\bar{i}\bar{j}\bar{k}}$

Table III.2: Type IIA field content in 4d

III.4: Type IIB on Calabi Yau

The bosonic fields for 10d type IIB string theory are the graviton G, the NSNS 2-form B, the dilaton ϕ, the RR 0-form a, 2-form C_2, and 4-form C_4 (with self-dual field strength). Compactified on a Calabi-Yau manifold, their KK reductions are given in Table III.3.

III.5: Duality between type II theories

III.5.1: T-duality

In the simplest model, 10d type II string theories can be compactified to 9d on a circle S^1 of radius R. The circle compactifications of type IIA and IIB can be shown to have the same massless spectrum. This is a manifestation of T-duality. This duality relates two theories compactified on radii R and $R' = \frac{\alpha'}{R}$. However, it is intriguing that the duality cross-maps

[6] Just a differential 2-form. It is called NS-NS because of its origin from left-moving and right-moving closed string states with fermionic fields both following anti-periodic boundary condition, called Neveu-Schwartz condition

[7] Just a differential 2-form. It is called R-R because of its origin from left-moving and right-moving closed string states with fermionic fields following periodic boundary condition, called Ramond condition

[8] Here $B_{i\bar{j}}$ represent the scalar coefficent along harmonic (1,1)-forms of the 6d internal space

IIA		Gravity	$h_{2,1}$ Vector	$h_{1,1}$ Hyper	Hyper
G	$\rightarrow$	$G_{\mu\nu}$	$2h_{2,1}$ Cmplx.Str.	$h_{1,1}$ Kähler	
B	$\rightarrow$		$B_{i\bar{j}}$		$B_{\mu\nu}$
ϕ	$\rightarrow$				ϕ
a	$\rightarrow$				a
C_2	$\rightarrow$			$C_{i\bar{j}}$	$C_{\mu\nu}$
C_3	$\rightarrow$	$C_{ijk\mu}$	$C_{ij\bar{k}\mu}$	$C_{i\bar{j}\mu\nu}$	

Table III.3: Type IIB field content in 4d

non-trivially different types of states between the the theories. For instance, the chirality is flipped for some states. We can see this effect in instanton later as well. The instanton effect for Yukawa coupling emerges as a field theory result for type IIB, but as a stringy effect for type IIA.

III.5.2: Mirror symmetry

Roughly speaking, mirror symmetry is that for every Calabi Yau manifold $\mathbf{X}_6$, there exists a mirror Calabi Yau manifold $\mathbf{Y}_6$ so that:

$$h_{1,1}(\mathrm{X}_{\mathbf{6}}) = h_{2,1}(\mathrm{Y}_{\mathbf{6}}) \qquad h_{2,1}(\mathrm{X}_{\mathbf{6}}) = h_{1,1}(\mathrm{Y}_{\mathbf{6}}) \tag{III.5.1}$$

The content of type IIA string theory on one Calabi Yau manifold is exactly the same content of type IIB string theory compactified on the mirrored manifold. One can think of mirror symmetry as a generalization of T-duality. In T-duality, type IIA compactified on a circle with radius R is equivalent to type IIB compactified on a circle with radius (proportional to) $1/R$. It implies that the limit of one theory is the other theory. Moreover, field theory description of one theory (for instance, momentum states) may have a stringy origin in the mirror (winding states).

III.6: U-duality - M-theory and type IIA duality

At the strong coupling limit $g_s \rightarrow \infty$, type IIA theory behaves similarly to a theory with a decompactified dimension. In particular, it behaves like a theory on $M_{10} \times S^1$ where the radius R of circle S^1 going to infinity. The full 11d theory is called M-theory. Notice that the radius limit is in fact a T-duality, and the strong coupling limit is an S-duality. S-duality is the duality between string theory at strong coupling and M theory at weak coupling. This combination of T-duality and S-duality is called U-duality. This setup gives the first idea about the inspiration for M-theory.

M-theory has no dimensionless coupling constant. In the duality with type IIA, both the gauge coupling g_s and string tension α' of type IIA string theory become parts of the geometry. In particular, the M-theory S^1 radius R is related to

$$g_s = (M_{11}R)^{3/2}, \qquad \alpha' = \frac{1}{M_{11}^3 R} \tag{III.6.1}$$

where M_{11} is 11d Planck scale. Moreover, the type IIA metric, dilaton, and RR 1-form are all lifted to be parts of the metric in 11d. By KK reduction on a circle, we can connect the M-theory fields with IIA:

M-theory	$\longrightarrow$	IIA	
G_{MN}	$\longrightarrow$	G_{MN}	graviton
$G_{M,10}$	$\longrightarrow$	C_1	RR 1-form
$G_{10,10}$	$\longrightarrow$	ϕ	
C_{MNP}	$\longrightarrow$	C_3	RR 3-form
C_{MN10}	$\longrightarrow$	B_2	NSNS 2-form

Table III.4: M-theory to IIA by KK reduction on S^1

And Table III.5 gives a map between other objects of the theories

IIA	$\longleftrightarrow$	M-theory on S^1
D0-branes	$\longleftrightarrow$	KK momenta of 11d supergravitons
IIA strings	$\longleftrightarrow$	M2 wrapped on S^1
D2	$\longleftrightarrow$	unwrapped M2
D4	$\longleftrightarrow$	M5 wrapped on S^1
NS5	$\longleftrightarrow$	unwrapped M5
D6	$\longleftrightarrow$	Kaluza-Klein monopole

Table III.5: IIA and M-theory on S^1 duality

III.7: Nonabelian gauge theories in string theories

First hint about nonabelian gauge in M-theory:

- **In type IIA, a stack of coincident n D-branes gives rise to a U(n) gauge group**
- **Lifted to M-theory, a A_{n-1} singularity gives rise to a SU(n) gauge group**

III.7.1: Coincident D-branes in string theories

We always have plenty of abelian $U(1)$ gauge symmetries from the vector bosonic fields. Note the fact that anytime we have a bosonic vector field, we have an abelian $U(1)$ gauge.

Realizing nonabelian gauge symmetry is more tricky. Consider a bosonic string stretching between two D-branes. The mass of the state is proportional to the distance between the two D-branes. So when the two D-branes coincide, the state becomes massless. This provides two massless bosons, corresponding to two strings connecting the two D-branes in two opposite orientations, which is a hint that gauge symmetry can be enhanced. In general, if we have a stack of n D-branes, there are n^2 possible beginning and end for a string. When the n D-branes coincide, there are then n^2 states corresponding to the same physical state. This implies a gauge symmetry. All the field components in the theory are promoted to $n \times n$ matrices, and the action is a trace of $n \times n$ matrices. This describes a $U(n)$ gauge symmetry.

III.7.2: The extension to M-theory

- **n coincident branes become n shrinking 2-spheres in M theory which make an A_{n-1} type of ADE singularities.**
- **All topologically relevant harmonic forms, corresponding to Cartan generators, must become compactly supported.**
- **Although there are n harmonic forms corresponding to n Cartan generators of $U(n)$ from n coincident branes, only $n-1$ of them can become compactly supported, and hence reduce the gauge group to $SU(n)$.**
- **This disappearing $U(1)$ means that while n D-branes have no specific orders in type IIA, the 2-spheres have a specific ordering in M theory**

n coincident branes in a string theory produce a non-abelian gauge group $U(N) = SU(N) \times U(1)$. We can examine the analog of this in the M-theory limit. Here, a string in IIA is lifted to a 2 brane called a M2 brane. A stack of n D-branes is lifted to n-2 M2 branes wrapping a n-1 Riemannian spheres [9] that intersect transversely in a configuration of the type A_n Dynkin diagram. Consequently, n coincident D-branes are lifted to shrinking those spheres to a point. Geometrically, this is an ADE singularity of A_n type. An explicit example can be seen in the construction of Taub-NUT geometry [10]. This geometry considers a spacetime of the form $M_7 \times X_4$, with a non-compact X_4 asymptotic to $\mathbb{R}^3 \times S^1$, with the $\mathbb{R}^3$ transverse to the D-branes and the M-theory compactification circle S^1 (see Figure III.2).

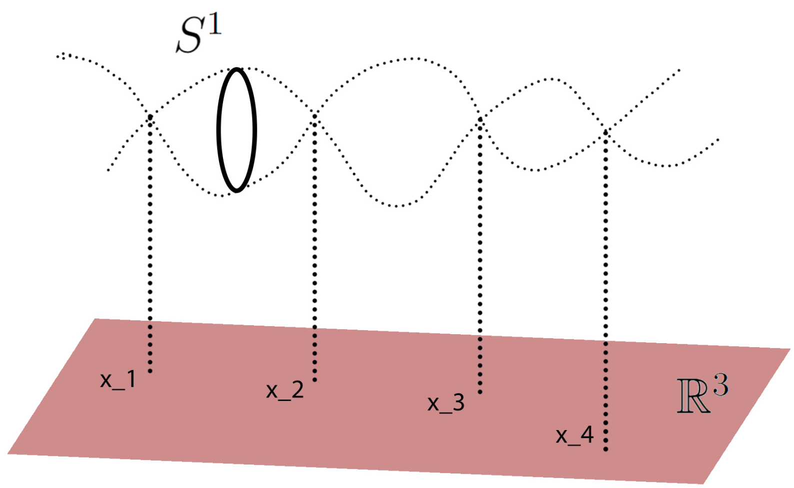


Figure III.2: Taub-NUT geometry with four D-branes transverse to four points x_a on $\mathbb{R}^3$. The S^1 fibration creates three 2-spheres in M theory.

The Taub-Nut metric is of the form

$$ds^2 = \frac{V(\vec{x})}{4} d\vec{x}^2 + \frac{V(\vec{x})^{-1}}{4}(dx^{10} + \vec{\omega}.d\vec{x})^2, \tag{III.7.1}$$

$$V(\vec{x}) = 1 + \sum_{a=1}^{N} \frac{1}{|\vec{x} - \vec{x}_a|}, \qquad \vec{\nabla} \times \vec{\omega} = \vec{\nabla} V(\vec{x}) \tag{III.7.2}$$

where $\vec{X}$ parametrize $\mathbb{R}^3$ and $\vec{\omega}$ is formally the 3d vector potential for n Dirac monopoles at n locations $\vec{x}_a$ on $\mathbb{R}^3$. This is called the n-center Taub-NUT metric, or the Kaluza-Klein

[9] Topologically a 2-sphere. This is a compplex plane with one point at infinity to compactify it. So, it is endowed with complex number algebra and division by zero

[10] See [87]

monopole. The D-branes are located transversely to $\mathbb{R}^3$ at the locations $\vec{x_a}$. The S^1 fibration degenerates into a point at the $\vec{x_a}$, so the S^1 fibration between two consecutive $\vec{x_a}$'s creates a 2-sphere which is mentioned in the previous paragraph. Intergrating over S^1 will then reduce the M_2 branes wrapping these 2-spheres to strings stretching between D-branes at the $\vec{x_a}$'s.

Interestingly, the analog in M-theory limit is $SU(N)$ with some subtlety about the disappearing $U(1)$. First, note that there exist n harmonic, normalizable two-forms, call them ω_i. These forms are localized near the centers of the Taub-NUT space, and they are the cohomological forms dual to the n non-trivial homology cycles (the spheres from earlier). Furthermore, there is one additional normalizable 2-form on the Taub-NUT geometry, which can be constructed explicitly for $n = 0$, but which does not relate to any particular topological property [78], which will be explained later.

The existence of normalizable harmonic 2-forms in a multi-center Taub-NUT geometry is a nontrivial fact. The explicit form of these forms may be found in [118, 116], who gives them

$$\omega_i = d\xi_i \tag{III.7.3}$$

$$\xi_i = V^{-1}V_i(dx^{10} + \omega.dx) - \omega_i dx \tag{III.7.4}$$

where $V_i = \frac{m}{\vec{x}-\vec{x_i}}$ and the exterior derivative in Eq. (III.7.3) is only meant to work as the derivative on the coordinates of Taub-NUT space that are not x^{10}.

ω^0 does not have a topological impact because it's not associated with the basis of 2-cycles that arises from the lines between the center of the Taub-NUT monopoles. The tricky part is the duality of cohomology and homology. Poincaré duality of these n-1 cycles gives us only n-1 compactly supported (and hence normalizable) modes. Note also that these modes are not the ω_i from Eq. (III.7.3), since although normalizable, they are not compactly supported. Now, ω^0 is an additional normalizable mode that does not appear through Poincaré duality, while the rest of the normalizable modes have compactly supported versions we can see through the homological argument.

Conveniently, this explains a missing $U(1)$ between M-theory of gauge enhancement and the type IIa D_6-brane gauge enhancement. In a more geometric perspective, we see that n D-branes have no particular order in type IIA, but in Taub-NUT geometry, there is a specific order of them at $\vec{x_a}$'s. This ordering reduces the $U(N)$ to $SU(N)$.

Finally, it is noteworthy that A_n and D_n type singularities in M theory can descend down to type IIA string theory [34, 88]. We will not go deeper into details as we get what we need for our M theory description.

III.8: Heterotic string theories

For completeness, we briefly mention the the heterotic string theories and their duality to M theory. One can use this duality to realize E_n type of ADE singularity in M theory [107] although we will not cover that here. The 10d massless fields of the heterotic theory are the 10d $\mathcal{N}=1$ gravity multiplet (whose bosonic content is the metric G_{MN}, the 2-form B_{MN}, and the dilaton ϕ) and the 10d $\mathcal{N}=1$ vector multiplets (with bosonic content given by the gauge bosons A^a_M). The difference between the $E_8 \times E_8$ and $SO(32)$ theories is just in the group theory of A^a_M under spacetime gauge symmetry. The KK reduction on a CY manifold for the gravity multiplet is

Het		Gravity	$h_{1,1}$ Chiral	$h_{2,1}$ Chiral	Chiral
G	$\longrightarrow$	$g_{\mu\nu}$	$h_{1,1}$ Kähler	$2h_{2,1}$ Cmplx. Str.	
B	$\longrightarrow$		$B_{i\bar{j}}$		$B_{\mu\nu}$
ϕ	$\longrightarrow$				ϕ

Table III.6: Heterotic string theory 4d content

The vector multiplet is decomposed differently depending on the type of compactification. This is because there are many ways to break the gauge group. We will omit the details due to limited space.

III.9: U duality - M-theory and heterotic string

Another way of compactifying M-theory to 10d, besideson an S^1, is over the quotient S^1/Z_2. This turns out to be a strong coupling limit $g_s \to \infty$ of heterotic string $E_8 \times E_8$. Consider the compactification of M-theory on a circle S^1, modded out by a Z_2 action whose generator θ behaves as

$$\theta : x^{10} \to -x^{10}, \qquad C_3 \to -C_3 \tag{III.9.1}$$

This compactification of M-theory leads to a 10d $\mathcal{N}=1$ gravity multiplet, arising from the Z_2 invariant fields of M-theory in the 11d bulk, in addition to 10d $\mathcal{N}=1$ $E_8 \times E_8$ vector multiplets, each factor propagating on each 10d boundary of spacetime. This is called Hořava-Witten theory. It has exactly the same massless spectrum as the $E_8 \times E_8$ heterotic theory. Furthermore, the effective supergravity is identical in both M-theory on S^1 and heterotic string, with S^1 radius related to heterotic coupling as $g_s = (M_{11}R)^{3/2}$.

With some subtlety, the KK reduction of M-theory to heterotic string $E_8 \times E_8$ is similar to that of type IIA.

III.10: Other dualities

There are other dualities between string theories as shown in Figure III.1. The nature of these remaining dualities are S-duality and T-duality as covered in the previous sections [84, 77]. We will omit their discussion here as it does not have a direct consequence on our later work.

III.11: Supergravity

As the name suggests, supergravity is a gravitational theory that obeys supersymmetry. There are many sensible supergravity theories in various dimensions. Notably, 10d supergravity theories and 11d supergravity are lower energy approximations of string and M theories respectively. In fact, when we consider only the massless modes of a string/M-theory, it is actually just supergravity. That is what we mean by the low energy approximation: only the massless spectrum is physical, as other heavy states do not have the required energy to exist. Every string theory has a corresponding 10d supergravity version. On the other hand, 11d supergravity is unique. For our purpose, we are mainly interested in 11d supergravity because it is the lower energy limit of M-theory. The supersymmetric condition suggests that 11d supergravity should be compactified on a G_2 manifold. This is the first clue that M-theory should be compactified on a G_2 manifold [11].

11d Supergravity theory:

- **Metric g_{MN} and an antisymmetric tensor field C_{MNP} as bosonic components.**
- **Gravitino Ψ_M, which is a Majorana spinor, as the fermionic superpartner of g_{MN} and C_{MNP}.**
- **G_2 manifolds are the most natural way to compactify the theory.**

[11] More details in [85]

The supergravity action can then be written as in [22, 55]

$$\begin{aligned} S = \frac{1}{2}\int_{M_{11}} \sqrt{-g}d^{11}x\left[R - \overline{\Psi}_M\Gamma^{MNP}D_N\Psi_P - \frac{1}{2}F_4\wedge *F^4\right] \\ -\frac{1}{192}\int_{M_{11}} \sqrt{-g}d^{11}x\overline{\Psi}_M\Gamma^{MNPQRS}\Psi_N(F_4)_{PQRS} - \frac{1}{2}\int_{M_{11}} F_4\wedge *A_4 \\ -\frac{1}{12}\int_{M_{11}} F_4\wedge F_4\wedge C_3 \end{aligned} \tag{III.11.1}$$

where we have set the eleven-dimensional Newton's constant to unity and denoted

$$(A_4)_{MNPQ} = 3\overline{\Psi}_{[M}\Gamma_{NP}\Psi_{Q]} \tag{III.11.2}$$

$$F_4 = dC_3 \tag{III.11.3}$$

We use D for the spinor covariant derivative. The indices $M, N\cdots = 0, 1, \ldots, 10$ denote curved eleven-dimensional indices.

The spinor conjugation is defined to be $\overline{\Psi}_M = \Psi^\dagger_M\Gamma^0$, We have ignored the four fermionic terms, which play no role in our analysis, and kept only bilinear terms in the gravitino field Ψ_M. The action (III.11.1) is invariant under the usual supersymmetry transformations.

Consider only the bosonic part [3],

$$S = \int \sqrt{-g}R - \frac{1}{2}F_4\wedge *F^4 - \frac{1}{6}F_4\wedge F_4\wedge C_3 \tag{III.11.4}$$

The equations of motion for C and g are of the form,

$$d * F = F\wedge F \tag{III.11.5}$$

and

$$R_{MN} - \frac{1}{2}g_{MN}R = T_{MN}(C) \tag{III.11.6}$$

where T is the energy-momentum tensor for the C-field, which is

$$T = -\frac{|F|^2}{4}g \tag{III.11.7}$$

Since the theory is supersymmetric, it is natural to look for supersymmetric vacua. In the classical theory these are just the conditions that the supersymmetry variations of the three fields vanish. In a Lorentz-invariant background the expectation value of Ψ is zero, in which case the variations of g and C vanish automatically. In order to find classically supersymmetric field configurations, we must find values of C and g for which the variation of Ψ is zero:

$$\delta_n \Psi_M \equiv \nabla_M \eta + \frac{1}{288}\left(\Gamma_M^{PQRS} F_{PQRS} - 8\Gamma^{PQR} F_{MPQR}\right)\eta = 0 \tag{III.11.8}$$

The simplest way to solve these equations is to take $F = 0$ in which case we are looking for 11-manifolds with metric g which admits a covariantly constant or parallel spinor:

$$\nabla_M \eta = 0 \tag{III.11.9}$$

We will rewrite this equation in the more symbolic form,

$$\nabla_g \eta = 0 \tag{III.11.10}$$

where by ∇_g we mean the Levi-Civita connection constructed from g. Solutions to these conditions can be classified via the holonomy group of the connection ∇_g.

This implies that $Hol(g(X))$ is G_2 or a subgroup. This is because G_2 is the maximal proper subgroup of $SO(7)$ under which the spinor representation contains a singlet (trivial representation). Specifically, a spinor of $SO(7)$ can be regarded as a fundamental of G_2 and a singlet of G_2 (which is in fact η from above):

$$\mathbf{8} \to \mathbf{7} + \mathbf{1} \tag{III.11.11}$$

Therefore if X is a compact manifold of precisely G_2-holonomy, the effective theory in four dimensions will be $\mathcal{N} = 1$ supersymmetric. We get precisely $\mathcal{N} = 1$ and no more because there is only one singlet spinor according to the above group theory.

III.11.1: Supergravity - more details

As the 11d supergravity massless spectrum is exactly that of M-theory, the formulae in this section are actually useful in M-theory discussion

Table III.7: Summary of $\mathcal{N} = 1$ supergravity multiplet. This decomposition is discussed more in Chapter IV.

Indices	Massless 4d component fields		Massless 4d
	bosonic fields	fermionic fields	$\mathcal{N} = 1$ multiplets
	metric $g_{\mu\nu}$	gravitno ψ_μ, ψ^*_μ	gravity multiplet
$I = 1, \ldots, b_2(Y)$	vectors A^I_μ	gauginos λ^I_α	vector multiplets V^I
$i = 1, \ldots, b_3(Y)$	scalars (S^i, P^i)	spinors χ^i, χ^{*i}	chiral multiplets Φ^i

To perform the Kaluza-Klein reduction let us introduce the moduli-dependent volume $V_Y(S^i)$ of the G_2-manifold Y given by

$$V_Y(S^i) = \int_Y d^7y \sqrt{\det g(S^i)_{mn}} \tag{III.11.12}$$

Furthermore, we introduce a reference G_2-manifold Y_0 with respect to some background expectation values $S^i_0 = \langle S^i \rangle$. As seen in Chapter II, this introduces a dimensionless (moduli-dependent) volume factor

$$\lambda_0(S^i) = \frac{V_Y(S^i)}{V_{Y_0}} = \frac{1}{7} \int_Y \varphi \wedge *_{g_\varphi} \varphi \tag{III.11.13}$$

in terms of the reference volume $V_{Y_0} = V_Y(S^i_0)$. Here φ is normalized by the choice of Y_0.

Then the dimensional reduction of the Einstein-Hilbert term and the C term yields the 4-dimensional action [27]

$$\begin{aligned} S^{bos}_{4d} = \frac{1}{2\kappa_4^2} \int \Big[*_4 R_S + \frac{\kappa_{IJk}}{2} (S^k F^I \wedge *_4 F^J - P^k F^I \wedge F^J) \\ -\frac{1}{2\lambda_0} \int_Y \rho^{(3)}_i \wedge *_{g_\varphi} \rho^{(3)}_j (dP^i \wedge *_4 dP^j + dS^i \wedge *_4 dS^j) \Big]. \end{aligned} \tag{III.11.14}$$

where R_S is with respect to the metric $g_{\mu\nu}$. Here we perform the Weyl rescaling of the 4-dimensional metric

$$g_{\mu\nu} \to \frac{g_{\mu\nu}}{\lambda_0(S^i)} \tag{III.11.15}$$

such that the 4-dimensional cosntants are related by

$$\kappa_4^2 = \frac{\kappa_{11}^2}{V_{Y_0}} = 8\pi G_N - 8\pi l_P^2 = \frac{8\pi}{M_P^2}. \tag{III.11.16}$$

Moreover, the coupling κ_{IJk} arise from the topoloical intersection number

$$\kappa_{IJk} = \int_Y \omega_I^{(2)} \wedge \omega_J^{(2)} \wedge \rho_k^{(3)}. \tag{III.11.17}$$

Notice that this is exactly the same as the overlapping integral we see later from type IIA and IIB theories in Sec. IV.4. We can now bring the (bosonic) action into the convential form of 4-dimensional $\mathcal{N} = 1$ supergravity [120][12]. To identify chiral multiplets – that is to identify the complex structure of the Kähler terget space – we observe that at least to the leading order the action of the membrane instantons generating non-perturbative superpotential interactions is given by [82]

$$\phi^i = -P^i + iS^i \tag{III.11.18}$$

Hence, due to holomorphy of the $\mathcal{N} = 1$ superpotential, the complex fields ϕ^i furnish complex coordinates of the Kähler target space and thus represent the complex scalar fields in the $\mathcal{N} = 1$ chiral multiplets Φ^i. This allows us to read off the Käler potential and the gauge kinetic coupling [27, 100]

$$K(\phi, \bar{\phi}) = -3 \log \Big(\frac{1}{7} \int_Y \varphi \wedge_{*g_\varphi} \varphi \Big), \tag{III.11.19}$$

$$f_{IJ}(\phi) = \frac{i}{2} \sum_k \phi^k \int_Y \omega_I^{(2)} \wedge \omega_J^{(2)} \wedge \rho_k^{(3)} = \frac{i}{2} \sum_k \kappa_{IJk} \phi^k. \tag{III.11.20}$$

The moduli space metric is then given by

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K = \frac{1}{4\lambda_0} \int_Y \rho_i^{(3)} \wedge *_{g_\phi} \rho_j^{(3)} \tag{III.11.21}$$

Note that the above result is merely from the semi-classical dimensional reduction of 11-dimensional supergravity on the G_2 manifold Y. For the resulting $\mathcal{N} = 1$ supersymmetric theory, one expects in general that the flat directions of the moduli space are lifted at the quantum level due to non-perturbative effects in M-theory [81]. We also generically expect additional non-perturbative superpotential contributions to arise from membrane instanton effects [2, 81] .

[12] More details in [29]

CHAPTER IV

M-Theory on G_2 Manifold with ADE Singularities

M-theory

- **11 d supergravity at low energy limit.**
- **Supersymmetry implies G_2 manifold.**
- **ADE singularity is required for nonabelian gauge group and light charged fermions.**
- **Nonpertubative limit of other string theories and more.**
- **Instanton effect gives the Yukawa coupling strength.**

In Chapter III, we learned that M-theory is an 11d theory in strong coupling limit of string theories. All the features of string theories are lifted to M-theory, such as a nonabelian gauge group. We also see 11d supergravity is a low energy limit of M-theory which implies M-theory should be compactified on G_2 manifold. In fact, the geometric connection between M-theory and string theories can be traced down to the local relation of G_2 manifold and Calabi-Yau manifold in Chapter II. Additionally, the lifting of coincident D-branes from string theories tells us that M-theory with A_n and D_n type singularities can produce nonabelian gauge theories. It is natural to consider a generalized case of ADE singularities in the M-theory.

Furthermore, M-theory features can be generalized to structures that have no analog in string theories. For instance, E_n singularities in M-theory have no analog in type IIA. More generally, the global geometry of G_2 manifolds can be much more than a trivial elliptical fibration of Calabi-Yau manifolds. As we see in Chapter III, all the tunable parameters such as coupling strength and string tension are part of the geometry, these non-trivial global constraints will potentially be much more predictive.

In this chapter, we will cover a few general aspects of M-theory compactified on G_2 manifold with ADE singularities. In Sec. IV.1, we look at the spectrum of M-theory on smooth

G_2 manifold through KK reduction in supergravity limit. This is indeed a continuation of the supergravity discussion in Chapter III. The gauge fields in this are all abelian, and the light fermions are uncharged. As a result, compactifying on singular G_2 manifolds is necessary.

IV.1: Kalazu-Klein reduction on G_2 manifold

Kaluza-Klein reduction of 11-dimensional $\mathcal{N} = 1$ supergravity provide the spectrum of low energy description of M theory [1]. The degrees of freedom of the massless component fields in the gravity multiplet transform in the following irreducible representations of the little group $SO(9)$: The metric g_{MN} transforms in the traceless symmetric representation $\mathbf{44}$, the 3-form $C_{[MNP]}$ in the anti-symmetric 3-tensor representation $\mathbf{84}$, and the gravitino ψ_M^α in the representation $\mathbf{128_s}$.

We now perform the Kazula-Klein reduction to 4-dimensional Minkowski space $\mathbb{M}^{1,3}$

$$g(x,y) = \nabla_{\nu\mu} dx^\nu dy^\mu + g_{mn}(y) dy^m dy^n \tag{IV.1.1}$$

where x and y are local coordinates of the $\mathbb{M}^{1,3}$ and the 7-dimensinal G_2-manifold Y.

Consider the infinitesimal fluctuation of the metric background $g \to \hat{g} + \delta\hat{g}$. Firstly, we obtain the 4-d metric fluctuation $\delta g_{\mu\nu}$, which corresponds to the gravitational degrees of the 4-d low-energy effective theory. Secondly, since the fundamental group of G_2 manifold is finite, there are no massless gravitational Kazula-Klein vectors. Finally, we determine Kazula-Klein scalar S^i, which furnish coordinates on the moduli space of G_2-metrics. Consider infinitesimal deformation under δS^i, i.e.,

$$g_{mn} dy^m dy^n \to g_{mn}(S^i) dy^m dy^n + \sum_i \delta S^i \rho^{sym}_{i,(mn)}(S^i) dy^m dy^n \tag{IV.1.2}$$

The solving Einstein's equations to linear order in the *symmetric metric fluctuations* $\rho^{sym}_{i,(mn)}$, we obtain

$$\mathrm{Ric}(g + \sum \delta S^i \rho^{sym}_i) = 0 \qquad \Longrightarrow \qquad \Delta_L \rho^{sym}_i = 0 \tag{IV.1.3}$$

in terms of the Lichnerowiz Laplacian Δ_L for the symmetric tensor fields. Recall Chapter II discussed the construction of Laplacian and the split of de Rham cohomology. Using the G_2

[1] This decomposition discussion is referenced to [79] . Also See [46, 121, 2, 1, 4, 27].

structure ϕ on Y, we construct the anti-symmetric 3-form tensors:

(IV.1.4)
$$\rho^{(3)}_{i,[mnp]} = g^{rs}\rho^{sym}_{i,[m}\phi_{np]s}.$$

Remark IV.1. $\Delta_L \rho_i^{sym} = 0$ if and only if $\Delta\rho_i^{(3)} = 0$ [76].

Thus, the massless Kaluza-Klein scalars S^i arise from the harmonic 3-forms $\rho_i^{(3)}$, which represents a basis for the vector space $H^3(Y)$ of dimension $b_3(Y)$. Then, the harmonic 3-forms $\rho_i^{(3)}$ are the first order deformations to the torsion-free G_2 structure

(IV.1.5)
$$\phi(S^i) \to \phi(S^i) + \sum_i \delta S^i \phi(S^i)$$

At a given point S^i in moduli space the harmonic 3-forms $\rho_i^{(3)}$ of Y fall into representations of the structure group G_2, and $H^3(Y)$ splits as [89]

(IV.1.6)
$$H^3(Y) = H^3_{\mathbf{1}}(Y) \oplus H^3_{\mathbf{27}}(Y), \qquad dim H^3_{\mathbf{1}}(Y) = 1 \qquad dim H^3_{\mathbf{27}}(Y) = b_3(Y) - 1$$

where the 3-form representative transform in the reppresentations **1** and **27** of G_2. The singlet corresponds to the scaling of volume of Y. The **27** corresponds to infinitesimal deformation with a constant volume at first order approximation.

In [89] Joyce shows that these infinitesimal deformations are actually unobstructed to all orders. The is to say that the vincity $U_{\phi(S^i)} \subset \mathcal{M}$ of a given torsion-free G_2-structure $\phi(S^i) \in \mathcal{M}$ — at a given point S^i, $i = 1, \ddots, b_3(Y)$, in the moduli space — is locally diffeomorphic to the de Rham cohomology $H^3(Y)$, i.e,

(IV.1.7)
$$\mathcal{P}_{\phi(S^i)} : U_{\phi(S^i)} \subset \mathcal{M} \to H^3(Y), \;\; \phi \mapsto [\phi].$$

We can locally expand the cohomology class $[\phi]$ as

(IV.1.8)
$$[\phi(S^i)] = \sum_i S^i[\rho_i^{(3)}]$$

which is a useful local description of the moduli space of Y.

Massless 4-dimensional modes arise from coefficients in the decomposition of the 11-

dimensional anti-symmetric 3-form tensor $\hat{C}$ as

$$\hat{C}(x,y) = \sum_I A^I(x) \wedge \omega_I^{(2)}(y) + \sum_I P^i(x) \wedge \rho^{(3)}(y) \tag{IV.1.9}$$

in terms of the harmonic 2-forms $\omega_I^{(2)}$ and 3-forms $\rho_i^{(3)}$ identified with non-trivial cohomology representatives of $H^2(Y)$ and $H^3(Y)$ of dimension $b_2(Y)$ and $b_3(Y)$ respectively. The 4-dimensional vectors A^I, $I = 1, \ddots, b_2(Y)$, and the 4-dimensional scalars P^i, $i = 1, \ddots, b_3(Y)$, are the only massless modes obtained.

Let us now do the dimensional reduction of the 11-dimensional gravitino $\hat{\psi}$, which geometrically is a section of $T^*M^{1,10} \otimes SM^{1,10}$, where $SM^{1,10}$ denotes a spin bundle of $M^{1,10}$.

$$\hat{\Psi}(x,y) = (\psi_\mu(x)dx^\mu + \psi_\mu^*(x)dx^\mu)\zeta(y) + (\chi(x) + \chi^*(x))\zeta_n^{(1)}(y)dy^n \tag{IV.1.10}$$

Here $(\psi_\mu, \psi_m u^*)$ and (χ, χ^*) are 4-dimensional Rarita-Schwinger and spinor fields of both chiralities in $\mathbb{M}^{1,3}$. ζ is a chapter of real spin bundle SY. Furthermore, $\zeta^{(1)}$ is a chapter of the real Rarita-Schwinger bundle $T^*Y \otimes SY$, which is locally takes the form $\theta^{(1)} \otimes \tilde{\zeta}$ in terms of the local 1-form $\theta^{(1)}$ and the spinorial chapter $\tilde{\zeta}$.

On G_2-manifold the spin bundle splits as $SY \approx T^*Y \oplus \mathbb{R}$ [89] such that the chapter ζ decompose accordingly

$$\zeta = \sum_m a_m(y)\gamma^m\eta + b(y)\eta \tag{IV.1.11}$$

Here, η is the covariantly constant Majorana spinor of the G_2-manifold and γ^m are the 7-dimensional gamma matrices. Similarly, we decompose the Rarita-Schwinger chapter $\zeta^{(1)}$ of $T^*Y \otimes SY$

$$\zeta^{(1)} = \sum_{n,m} a^{28}_{(nm)}(y)dy^n \otimes \gamma^m\eta + \sum_{n,m} a^{14}_{[nm]}(y)dy^n \otimes \gamma^m\eta + \sum_n b^7_n(y)dy^n \otimes \eta \tag{IV.1.12}$$

The superscripts in the symmetric tensor $a^{28}_{(nm)}(y)$, the anti-symmetric tensor $a^{14}_{[nm]}(y)$, and the vector $b^7_n(y)$ indicate the dimension of their respective representations with respect to the structure group G_2.

The massless 4-dimensional fermionic spectrum results from zero modes of the 7-dimensional

Dirac operator $\not{D}$ and Rarita-Schwinger operator $\not{D}^{\mathrm{RS}}$

(IV.1.13) $$\not{D}\zeta = 0 \qquad\qquad \not{D}^{\mathrm{RS}}\zeta^{(1)} = 0$$

The zero modes of these operators on G_2-manifolds are discussed in [67]. For the spinorial chapter ζ, the covariantly constant spinor η — i.e, $b(y) \equiv 1$ — is the only zero mode of the Dirac operator. In the Rarita-Schwinger chapter $\zeta^{(1)}$, the 1-form tensor $b^7(y)$ does not contribute any zero modes. All zero modes arise from the zero modes of the Lichnerowicz Laplacian and the 2-form Laplacian acting respectively on $a^{28}(y)$ and $a^{14}(y)$

(IV.1.14) $$\Delta_L a^{28}(y) = 0 \qquad\qquad \Delta a^{14}(y) = 0$$

The zero modes of the Lichnerowicz Laplacian on G_2-manifolds are again identified with hamonic 3-forms — with a singlet zero mode and $b_3(Y)$ - 1 traceless symmetric zero modes transforming in the G_2-representations **1** and **27**, respectively. Therefore, the zero modes of the Rarita-Schwinger bundle on Y are in one-to-one correspondence with non-trivial cohomology elements of both $H^3(Y)$ and $H^2(Y)$, and we arrive at the expansion of the 4-dimensional chiral fermions

(IV.1.15) $$\chi(x)\zeta^{(1)}(y) = \sum_{i=1}^{b_3(Y)} \chi^i(x)\rho^{sym}_{i,(nm)} dy^n \otimes \gamma^m\eta + \sum_{I=1}^{b_2(Y)} \lambda^I \omega^{(2)}_{I,[nm]} dy^n \otimes \gamma^m\eta$$

in terms of the bases of zero modes $\rho^{sym}_{i,(nm)}$ of the Lichnerowicz Laplacian and of the harmonic 2-forms $\omega^{(2)}_I$.

Notice that all the bosonic vector fields are Abelian. Moreover, all the fermionic fields are uncharged. This is not phenomenologically interesting. We want a theory with a nonabelian gauge and charged fermions. By duality to type IIA in Chapter III, we are inspired to put ADE singularities on G_2 manifold.

IV.2: M-theory on G_2 manifold with ADE singularities

It becomes clear that M-theory is necessarily compactified on G_2 manifold with ADE singularities. Yet, G_2 manifold with ADE singularities is a difficult construction to work with. The most common model of M-theory on G_2 manifold with ADE singularities is using the ansatz of G_2 manifold Y_7 as fibration

(IV.2.1) $$\widehat{\mathbb{C}^2/\Gamma_{ADE}} \to Y_7 \to M_3$$

where M_3 is a associative cycle defined in Chapter II, $\widehat{\mathbb{C}^2/\Gamma_{ADE}}$ is a family of resolved ADE singularity in various degree. This notation is saying that locally, $Y_7 = M_3 \times \widehat{\mathbb{C}^2/\Gamma_{ADE}}$. Then $\widehat{\mathbb{C}^2/\Gamma_{ADE}}$ is in fact coassociative. The geometry of this is discussed in Chapter II. In particular, when the ADE of $\widehat{\mathbb{C}^2/\Gamma_{ADE}}$ is blown up to a certain degree, we have a collection of the sphere called 2-cycles intersecting each other transversely with respect to the corresponding Dynkin diagram.

For each 2-cycle, we use a harmonic one-form ϕ on M_3, which can be thought of as a metric-invariant 3-vector field on M_3, to parametrize the size of the 2-cycle. Alternatively, Katz et al [37, 95] use the coefficients in the Cartan subalgebra as the parameters. Consistently, there is a one-to-one bijection between the two parametrizations given by Table V.1. Following the existing literature, we denote $\widehat{G}(f_1, f_2, f_3, \ldots, f_n)$ as the family of $\widehat{\mathbb{C}^2/\Gamma_G}$ parametrized by the coordinates f_i in Cartan subalgebra where n is the rank of G and use Table V.1 to compute the "volume" one-form ϕ when needed [2].

Note that the theory actually agrees with the distance conjecture from Vafa et al [106]. The construction of M-theory compactified on a circle is dual to type IIA string theory. Specifically, when an M2 brane wraps around one of the basis two cycles of the resolved E_8 singularity in our model, it is dual to a string wrapping around a circle in type IIA. When the moduli in our theory go to infinity, it is equivalent to the volume of the two cycles going to infinity. This is dual to the infinite radius limit of a circle in type IIA. Vafa et al [106] has already argued about the infinite tower of massless states in the type IIA side for the infinite radius limit. M-theory inherits this tower through dimensional reduction.

IV.3: Gauge group enhancement

Inherited from supergravity at the low-energy limit, the basic fields are a metric g, a 3-form potential C_3, and a gravitino spinor Ψ. We will briefly review the essential properties of the fields needed for this paper. More details are discussed in the appendix and [109, 40, 96, 80]. From Chern-Simon (CS) terms, C_3 is integrated over a manifold of the same dimension, i.e a 3 submanifold of space-time. Excluding time, this submanifold is 2d spatial. This 2d submanifold is an M_2 brane. We say C_3 electrically couples with M_2 brane. Dimensional reduction of the C_3 form on the ALE fiber produces $U(1)$ gauge fields

$$C_3 = A_i \wedge \omega^i + \ldots \tag{IV.3.1}$$

[2] More details on root system and deformation are in [94].

where A_i's are one forms (vector fields) on $\mathbb{R}^{3,1}$, and ω^i's are harmonic two forms associated with 2-cycles of ALE fibers.

The non-abelian gauge group is produced in a similar manner as n coincident D6-branes in type IIA string theory [117]. In another perspective independent of duality, the gauge symmetry at an ADE singularity comes from the symmetry of differential form under automorphism of the resolved manifold. This is in fact just isometry of the resolved manifold. Explicitly, the two forms on the resolved manifold can be expressed as an element of the lie algebra of the associated ADE group. Therefore, under an automorphic map on the resolved manifold, the form can be transformed under the action of the lie group. At singular points where some cycles shrink to a single point, the forms in the same orbit under the transformation induced from the automorphism of those cycles correspond to the same state, so the transformation is a gauge transformation. For example, a self-contained description for the gauge transformation from $SU(N)$ singularity, i.e, A_{N-1} type would be summarized in the below diagrams. The C_3 is decomposed into the basis of the 3-forms. In the local description, the basis elements contain components that are 2-forms α_i on the 2-spheres $\mathbb{CP}^1$ which resolves the singularity.

$$\begin{tikzcd}
G_2 \text{ manifold} \arrow[d, "\text{locally}"'] & C_3 \text{ field} \arrow[d, "\text{decompose}"'] \\
M_3 \times \widehat{\mathbb{C}^2/\Gamma} \arrow[r, "\text{1-forms} \wedge \text{2-forms}", "\text{basis}"'] \arrow[d] & \phi_I \wedge \alpha_J \arrow[d] \\
\widehat{\mathbb{C}^2/\Gamma} \arrow[d, "\text{embeded as } \mathbb{CP}^1"] & \alpha_J \arrow[d, "\text{lifted to}"] \\
\mathbb{C}^N \arrow[r, "\text{2-forms on 2-spheres}", "\text{centers at origins}"'] & A_{ij} dz_i \wedge d\tilde{z}_j
\end{tikzcd}$$

When embedding $\widehat{\mathbb{C}^2/\Gamma}$ into $\mathbb{C}^N$, we can explicitly write α_i in a local coordinate and see the gauge field A_{ij} transforming under the rotations of $SU(N)$. Fibering this on the M_3 base, we see the corresponding adjoint-valued form ϕ mentioned in [40]

$$\begin{tikzcd}
\text{Symmetry of } \widehat{\mathbb{C}^2/\Gamma} & A_{ij} \in \mathfrak{su}(N) \arrow[dd, "\text{Integrate the 2-cycles}"'] \\
\text{is explicitly rotations } SU(N) \arrow[d, "\text{fibering on}"] & \\
M_3 \arrow[r, "\text{Higgs bundle}"] & \phi_I \otimes A_J
\end{tikzcd}$$

where $\phi \equiv \sum_{I,J} \phi_I \otimes A_J$ is explicitly an field transform in adjoint of $SU(N)$ (through $A_{ij} \in \mathfrak{su}(N)$), thus befitting the $SU(N)$ gauge description. Similarly, we can embed D_N, E_6, E_7, and E_8 type singularities into $\mathbb{R}^{2N}$, $\mathbb{C} \otimes \mathbb{O}$ (bioctonions), $\mathbb{H} \otimes \mathbb{O}$ (quateroctonions), and $\mathbb{O} \otimes \mathbb{O}$ (octooctonions) respectively.

The moral of this is the gauge symmetry comes from the geometrical symmetry of $\widehat{\mathbb{C}^2/\Gamma}$ which can be explicitly realized by embedding into a covering space. This is an explicit connection to 7d super Yang-Mills theory on $\mathbb{R}^{3,1} \times M_3$ by Higgs bundle.(The connection has been known for a long time through duality without explicit embedding).

It has always been mentioned that M_2 branes wrapping ADE singularities will give a non-abelian gauge. In here, we can see gauge boson A_{ij} explicitly and independently from the duality description.

In a more intuitive sense, the warping of M_2 branes around non-vanishing ALE cycles creates massive vector bosons. The masses are proportional to the volume of the 2-cycles. By shrinking the 2-cycles, we are making those massive bosons massless, Moreover, the configuration of the 2-cycles (Dynkin diagram) dictates the relation of these bosons and fits them perfectly into an non-abelian gauge group. Inversely, at any point on M_3 where the volume of a 2-cycle is non-zero, the associated vector boson becomes massive and hence must be removed from the gauge group. Yet, the $U(1)$ in the Cartan subalgebra from (IV.3.1) is unaffected by this, so we still have a $U(1)$ gauge symmetry. Hence, the n-ranked gauge group is broken into an $(n-1)$-ranked subgroup and a $U(1)$ (total rank is unchanged). In general, each non-vanishing volume of a basis 2-cycle reduces the rank of the group by one and leaves a $U(1)$ behind. It is important to note that this is similar to the Higgs mechanism except that the Higgsing happens due to the geometry instead of the traditional Higgs doublets as we will discuss in the next section.

IV.3.1: Chiral Fermion

On a singularity curve for a non-abelian gauge group H, which is a resolution of a higher rank singularity of a larger gauge group G, chiral fermion solutions are localized at points where the singularity associated with H is worsened by a conical singularity [14, 19, 39, 30]. By considering the resulting extra subgroup generated by the extra shrunk two cycles, one can determine the representation of the fermions with respect to the gauge group H. We will elaborate this in V.2.1.

IV.4: Yukawa couplings from instanton effect

Instanton effect is a non-perturbative effect where semiclassical configurations provide saddle points in the euclidean path integral of the spacetime fields of the theory. Instantons are classical solutions to the equation of motion in field theory. This effect appears in string theories in interesting ways. Instanton in type IIB string theory is a field theory effect while in type IIA, it is a stringy effect from worldsheet. Heterotic string theories can realize instanton in both perspectives. There is also an analogous open worldsheet instanton for type I theory. The Yukawa coupling from the theories matches nicely through dualities. This gives us a tool to lift the effect to M-theory.

In type IIB theory, we can do the explicit computation as follows. The 10 lagrangian on D9-branes with $U(N)$ gauge group reduces at low energies (i.e low radius regime) to 10d super-Yang-Mills

$$L = -\frac{1}{4}\mathrm{Tr}\Big(F^{MN}F_{MN}\Big) + \frac{i}{2}\mathrm{Tr}\Big(\bar{\Psi}\Gamma^M D_M \Psi\Big) \tag{IV.4.1}$$

where $F = dA$. The standard KK reduction procedure in compactification to 4d is expanding Ψ and A into harmonic modes and integrate out the internal dimensions. Explicitely

$$\Psi(x^\mu, y^m) = \sum_k \chi_{(k)}(x^\mu) \otimes \psi_{(k)}(y^m) \tag{IV.4.2}$$

$$A_n(x^\mu, y^m) = \sum_k \varphi_{(k)}(x^\mu) \otimes \phi_{(k),n}(y^m) \tag{IV.4.3}$$

where x^μ and y^m are 4d and internal coordiniates, repsectively. The 4d Yukawa coupling coming from KK reduction of the coupling $A.\Psi.\Psi$ in IV.4.2. Integrating the internal part gives the Yukawa coupling

$$Y_{ijk} = \frac{g}{2} \int_{X_6} \psi_i^{\alpha\dagger} \Gamma^m \psi_j^\beta \phi_{km}^\gamma f_{\alpha\beta\gamma} \tag{IV.4.4}$$

where g is the 10d gauge coupling, α, β, and γ are $U(N)$ gauge indices and $f_{\alpha\beta\gamma}$ are $U(N)$ structure constants. ψ and ϕ are fermionic and bosonic zero modes respectively. The Yukawa couplings are hence the overlap intergrals of the three zero mode wave functions in the internal space. This is a field theory result which neatly matches with other string theories as we will see later.

In type IIA theory on orbifolds, Yukawa couplings between fields living at D6-brane interchapters arise from worldsheet instantons. These are string worldsheets wrapped on a holomorphic 2d surface with disk topology and with boundaries on the intersecting D6-branes, as in Figure IV.1. We have analogous situations for heterotic string and type I

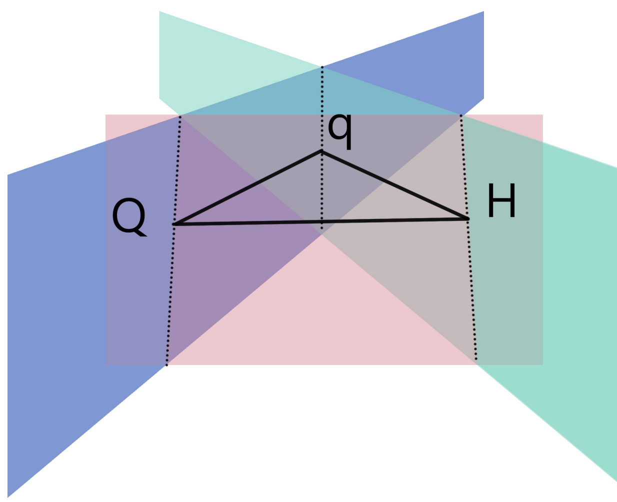


Figure IV.1: The three red, blue, and green branes intersects at the dotted lines. The worldsheet instanton is wrapped around the solid triangle whose topology is a disk. This passes through three different vertices Q, q, and H. This gives the Yukawa coupling for the term HQq in the lagrangian.

theories.

Lifting all of that to M-theory, the Yukawa coupling of three particles is determined by the volume of the 3-cycles W_3 wrapping around three singularities where the fermions are localized. Matching this with the discussion in Sec. III.11.1, we conclude the Yukawa coupling for the interaction of three fermions is

$$\frac{n_\gamma}{\Lambda} e^{Vol(W_3)} \tag{IV.4.5}$$

where n_γ is a flow direction sign which we will ignore in this work, and Λ can be visualized as the product normalizing factors of wave functions in type IIB. A careful work through the formula, one can see that there is actually a complex phase in the Yukawa coupling due to the G_2 structure form. For simplicity, we overlook this phase in this work. More details is discussed in [40]. More explicit computation and formula are presented in Chapter V.

CHAPTER V

Moduli from Quark and Charged Lepton Masses in Linearized Model

This chapter is based on a paper we published [73]. In this work, we focus on an M-theory calculation of the quark and charged lepton masses. The first step is to find an appropriate reduction from eleven to four dimensions. Suppose that spacetime is a product $\mathbb{R}^{3,1} \times Y$ where Y is a compact 7-d manifold roughly Planck scale in size. Gauge coupling unification and M-theory compactification hint at unbroken supersymmetry at the unification scale. Berger's theorem [13] requires that $\mathcal{N} = 1$ SUSY implies that the holonomy group of the manifold Y is G_2. The resultant low-energy theory can only contain $U(1)$ gauge fields. Such a compactification scheme is unrealistic since the SM contains non-Abelian gauge fields. One introduces singularities into Y to ameliorate this issue. Suppose that the local model of Y with ADE singularity is of the form $\mathbb{C}^2/\Gamma \times \mathbb{R}^3$, where Γ is a finite subgroup of $SU(2)$. Under these circumstances, a super Yang-Mills $\mathcal{N} = 1$ multiplet with gauge group $G = SU(k), SO(2k), E_6, E_7$ and E_8 respectively will be supported. These singularities can be deformed to break the symmetry of the gauge group G to a subgroup of G with equal rank.

We focus on breaking **248** of E_8 to SM particle. The matter that survives the symmetry breaking process consists of three multiplets in the **27** representation of E_6, and none in the $\overline{\mathbf{27}}$ representation of E_6 [1] [39]. This can explain why there are three and only three families. We explore the aforementioned symmetry-breaking pattern by looking to see if a realistic SM-theory can descend from a compactified M-theory construction. We calculate the Yukawa couplings under the assumption that everything originates from a deformed E_8 theory where the singularity is resolved into a lesser ranked singularity which is associated with $SU(3) \times SU(2) \times U(1) \times U(1)^4$ gauge group [2].

To explain the origin of the three families and their mass hierarchy, breaking E_8 to

[1] Acharya et al [39] explained that the net number of chiral zero modes was one. So, either **27** or $\overline{\mathbf{27}}$ was a normalizable zero mode, but not both. As a convention, we pick the normalizable zero modes to be in **27**.

[2] We separate one $U(1)$ factor out to emphasize SM gauge group.

the SM by the traditional Higgs mechanism has been unsuccessful and has shown a lack of predictability, while geometrically engineered M and F theories with E_8 points offer an alternative method of symmetry breaking. Moreover, [37] and related works suggest M-theory based on an $\hat{E}_8 - ALE$ space provide more predictability than the analogous model in F-theory[103, 119, 25]. Finally, a description of a singular G_2 manifold with Higgs bundles provides a formulation that makes the explicit computation of Yukawa couplings possible [109, 40].

We are interested in explicitly calculating the hierarchy of quark mass matrices. As that would include an explicit method for computing matter content, in gauge symmetry breaking through deformation, and their coupling constants, the results would be applicable to a wider study of other matter interactions. We also compute the mass matrix for charged leptons.

In Sec. V.2 we explicitly compute this breaking for the E_8 singularity with an explicit example of how to compute and locate the fermions on M_3. Sec. V.3 discusses the general computation for the Yukawa couplings in a local model which leads to explicit quark and charged lepton terms in Sec. V.4. After some gauge fixing for base-space M_3's parameters, numerical results are discussed in Sec. V.6. We see that the physical hierarchy is achievable with a very small set of solutions, putting a stringent constraint on the moduli of the theory. Sec. V.7 discusses the roles of both Yukawa couplings and Higgs vacuum expectation values (VEVs) in this hierarchy.

V.1: Local model

M-theory is an 11-dimensional theory that can be compactified on a compact 7d manifold X while the remaining non-compact four dimensions are the classical 4 space-time. In the supergravity limit, X is necessarily a G_2 manifold. Moreover, charged chiral particles are only possible on a singular G_2 manifold [13]. The simplest local model for such 7d manifold is given by the firing of $\widehat{\mathbb{C}^2/\Gamma_{ADE}}$ over the base M_3. Here, M_3 is an associative 3-cycle [3] in the G_2 manifold. Γ_{ADE} is a finite subgroup of $SU(2)$ acting on $\mathbb{C}^2$. $\mathbb{C}^2/\Gamma_{ADE}$ is an asymptotically locally Euclidean manifold (ALE) with ADE singularity at the origin. $\widehat{\mathbb{C}^2/\Gamma_{ADE}}$ denotes any manifold achieved from $\mathbb{C}^2/\Gamma_{ADE}$ by partially smoothing (resolving) the singularity. Locally, the manifold is of the form

$$\mathbb{R}^{3,1} \times \mathbb{R}^3 \times \widehat{\mathbb{C}^2/\Gamma_{ADE}} \tag{V.1.1}$$

[3]Equations of motion requires minimal volume, and an associative cycle is a minimal volume cycle.

	Positive Roots of E_n	Volume of Corresponding Two-Cycle
	$e_i - e_{j>i}$	$f_i - f_{j>i}$
	$-e_0 + e_i + e_j + e_k$	$f_i + f_j + f_k$
$n \geq 6$	$-2e_0 + \Sigma_{j=1}^{6} e_j$	$\Sigma_{j=1}^{6} f_j$
n=8	$-3e_0 + e_i + \Sigma_{j=1}^{8} e_j$	$f_i + \Sigma_{j=1}^{8} f_j$

Table V.1: Positive roots of E_n and the associated one-forms (sometimes called "area" in literature) controlling the sizes of 2-cycles on the ALE fiber. This is Table 1 in [37] with permission.

Note that globally, the fiber $\widehat{\mathbb{C}^2/\Gamma_{ADE}}$ varies along the base $\mathbb{R}^3$ where the singularity can be smoothed out to different degrees. More details on a recent construction of compact G_2 manifolds are in [99, 41, 42, 53].

V.2: E_8 Breaking

Our goal is to describe all the particles by resolving one single ADE singularity. E_8 is the only simple Lie group that does the job. E_8 and its breaking have been studied by several authors [103, 49, 36, 109, 62, 44, 69, 65, 108]. To understand the breaking, we first explicitly write down the simple roots of E_8 in the Dynkin diagram order (see Figure II.1) where e_i's are orthogonal vectors in $\mathbb{R}^{n,1}$. Let $\widehat{E}_8(f_1, ..., f_8)$ be the resolution of a E_8 singularity parametrized by deformation moduli f_i's which are one-forms on M_3. The simple roots are associated with the volumes of the blown-up 2-cycles by Table V.1 [37].

Each simple root, or equivalently each knot on the Dynkin diagram, will initially represent a vanishing cycle at the singularity. To break a group to a smaller group, we will "cut" a knot on their diagram so that we get the diagram of the smaller group. Each "cutting" is performed by blowing up the cycle (which was initially vanishing) associated with the knot. We recall that each cycle in the above Dynkin diagram gives rise to a boson whose mass is proportional to the volume of the cycle. Therefore, a vanishing cycle in the above Dynkin diagram will result in a massless boson. The goal is to keep the SM gauge bosons massless (zero volume cycles) while the other bosons are massive (non-zero volume cycles). We will follow the breaking path [4] of [36]. Figure V.1 summarizes the above steps. In the figure, we start with an E_8 singularity which corresponds to $\hat{E}_8(0,0,0,0,0,0,0,0)$, then turn on the volumes of the cycles associated with the crossed knots by giving non-zero values for one-form

[4] Different paths to the same subgroup will lead to the same physics. This is because if there is a diffeomorphism between X_1 and X_2 so that their hyper-Kähler structures agree, then they are isometric.

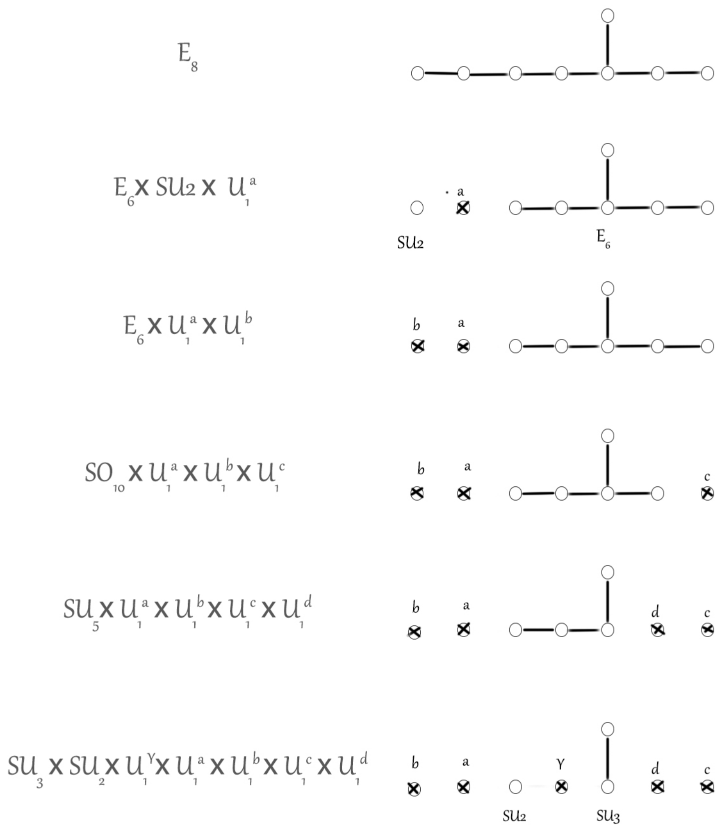


Figure V.1: Breaking of E_8 by resolving singularity

f_i's. There are five volumes needed to be turned on, so we parameterize f_i's by five non-zero one-forms a, b, c, d and Y (note that Y here is the one-form associated with hypercharge $U(1)^Y$, not the hypercharge itself). They are simply parameters that are linearly combined in a specific way so that the volumes of the cycles vanish or blow up appropriately by Table V.1. Then the final manifold is parameterized as [37]

$$\widehat{E}_8(a+b+c+d+\frac{2}{3}Y, a-b+c+d+\frac{2}{3}Y, \tag{V.2.1}$$
$$-c-d-\frac{7}{3}Y, -c-d-\frac{7}{3}Y, -c-d+\frac{8}{3}Y,$$
$$-c-d+\frac{8}{3}Y, -c+3d-\frac{4}{3}Y, 2c-2d-\frac{4}{3}Y). \tag{V.2.2}$$

We can check each step of Figure V.1 by setting all a, b, c, d, and Y in (V.2.2) to zero, then turn them on accordingly to each step, and compute the volumes using Table V.1. In the following, we can check the volumes of the cycles corresponding to the simple roots in the final step

$$\text{(V.2.3)} \qquad \begin{pmatrix} e_1 - e_2 & 2b \\ e_2 - e_3 & a - b + 2c + 2d + 3Y \\ e_3 - e_4 & 0 \\ e_4 - e_5 & -5Y \\ e_5 - e_6 & 0 \\ e_6 - e_7 & -4d + 4Y \\ e_7 - e_8 & -3c + 5d \\ -e_0 + e_6 + e_7 + e_8 & 0 \end{pmatrix}$$

This is exactly the configuration of Figure V.1. Note that one can use any different set of one-forms as long as they fulfill the desired configuration and sufficiently parameterize the independent non-vanishing cycles.

Therefore, whatever constraint we make, to avoid an unwanted shrunk cycle which will lead to an extra massless boson, we have to make non-zero volumes in the above table remain non-zero. The would mean

$$\text{(V.2.4)} \qquad b \neq 0 \qquad a - b + 2c + 2d + 3Y \neq 0$$

$$\text{(V.2.5)} \qquad Y \neq 0 \qquad Y \neq d \qquad c \neq \frac{5}{3}d.$$

Although all the shrunken simple root cycles result in massless $U(1)$ bosons, the anomaly cancellation will give most of them a mass in Sec. V.8. The only $U(1)$ remains massless is the GUT $U(1)$ (in Table V.2, it is $U(1)^d$).

V.2.1: Fermion Representations

Given a gauge group H for the theory, the corresponding cycles on the fiber are shrunk everywhere along the base manifold M_3. Those cycles correspond to the simple roots of H. A matter representation happens at the points where additional cycles associated with positive roots (see Table V.1) vanish. By letting the positive roots vanish one by one, we can find all the resulting representations. We will do a few examples showing how to calculate the representation.

First, we consider $e_2 - e_3$ cycle. Using the above table, we conclude that the associated volume is $f_2 - f_3 = a - b + 2c + 2d + 3Y$. Now, we consider the curve where this particular cycle vanishes: $a - b + 2c + 2d + 3Y = 0$. In order to know what representation emerges at this curve, we consider what kind of weight diagram is generated from $e_2 - e_3$ and the roots from the gauge group (corresponding to the globally shrunk cycles) $e_3 - e_4$ (corresponding to $SU(2)$), and $e_5 - e_6$ and $-e_0 + e_6 + e_7 + e_8$ (corresponding to $SU(3)$). In more details, we will try to find what are the positive roots we can get from $e_2 - e_3$ by adding or subtracting $e_3 - e_4$, $e_5 - e_6$, and $-e_0 + e_6 + e_7 + e_8$.

$SU(2)$ $\qquad\qquad$ $SU(3)$

$$\begin{array}{c} e_2 - e_4 \\ \uparrow{\scriptstyle e_3 - e_4} \\ e_2 - e_3 \end{array} \qquad \begin{array}{c} \text{No positive root from +/- simple roots } e_5 - e_6 \text{ or } - e_0 + e_6 + e_7 + e_8 \\ \\ e_2 - e_3 \end{array}$$

From above, we see that there are two positive roots corresponding to $SU(2)$, so the particle will behave like **2** of $SU(2)$. Only one positive root for $SU(3)$ case, so it is a singlet for $SU(3)$. Thus, this is a $(\mathbf{2}, \mathbf{1})$ of $SU(2) \times SU(3)$ (corresponding to H_2^u as in the Table V.2). Notice that the above calculation implies that $e_2 - e_4$ yields the same particle.

Next, let's try another positive root, say $-e_0+e_2+e_3+e_5$. The curve equation is $f_2+f_3+f_5 = a - b - c - d + Y = 0$. Then, we get

$SU(2)$ $\qquad\qquad$ $SU(3)$

$$\begin{array}{c} -e_0 + e_2 + e_3 + e_5 \\ \uparrow{\scriptstyle e_3 - e_4} \\ -e_0 + e_2 + e_4 + e_5 \\ \\ \\ \end{array} \qquad \begin{array}{c} -2e_0 + e_2 + e_3 + e_5 + e_6 + e_7 + e_8 \\ \uparrow{\scriptstyle -e_0 + e_6 + e_7 + e_8} \\ -e_0 + e_2 + e_3 + e_5 \\ \uparrow{\scriptstyle e_5 - e_6} \\ -e_0 + e_2 + e_3 + e_6 \end{array}$$

So by counting the positive roots, we conclude that it is **2** for $SU(2)$ and **3** or $\bar{\mathbf{3}}$ for $SU(3)$. As fundamental and anti-fundamental are just a convention, we call this order of adding $e_5 - e_6$ and $-e_0 + e_6 + e_7 + e_8$ associated with fundamental **3**. Thus this is a $(\mathbf{2}, \mathbf{3})$ of $SU(2) \times SU(3)$.

Lastly, for completeness, we will illustrate the case of $\bar{\mathbf{3}}$ with $-e_0 + e_2 + e_3 + e_4$. The curve equation is $f_2 + f_3 + f_4 = a - b - c - d - 4Y = 0$. Then, we get

$SU(2)$	$SU(3)$
No other positive root	$-2e_0 + e_2 + e_3 + e_4 + e_5 + e_7 + e_8$
	$\uparrow{\scriptstyle e_5 - e_6}$
$-e_0 + e_2 + e_3 + e_4$	$-2e_0 + e_2 + e_3 + e_4 + e_6 + e_7 + e_8$
	$\uparrow{\scriptstyle -e_0+e_6+e_7+e_8}$
	$-e_0 + e_2 + e_3 + e_4$

Notice that the order of adding $e_5 - e_6$ and $-e_0 + e_6 + e_7 + e_8$ is reversed from the previous case, so, by above convention, this is a $(\mathbf{1}, \bar{\mathbf{3}})$ of $SU(2) \times SU(3)$.
Bourjaily et al [37] have already worked out the breaking for us. The charges for relevant particles in this paper is presented in Table V.2. The location of the singularity associating with a particle is a linear combination of moduli weighted by the charges. For instance, the location of Q_1 is the curve that satisfies

$$a + b - c - d + Y = 0 \tag{V.2.6}$$

V.3: Yukawa Coupling from Volume of the Three-Cycle

In the superpotential, a cubic term ABC is allowed at tree level if the product transforms as a singlet under the gauge group. In particular, that implies the sum of charges for each of the $U(1)$'s is zero. If such a term happens, each of the particles A, B, and C will live on a different conical singularity which corresponds to different points t_A, t_B, and t_C on the base W which are solutions of equations derived from Table V.2 (similar to V.2.6). The idea of this section is that the Yukawa coupling coefficient of this term is inversely proportional to the exponential of the volume of the three-cycle wrapping around the three singularities

$$\text{Yukawa coupling} = n_{ABC} \frac{e^{-Vol(\Sigma_{ABC})}}{\Lambda_{ABC}} \tag{V.3.1}$$

where Σ_{ABC} is the three-cycle wrapping around the singularities, n_{ABC} is the sign of the term which depends subtly on the orientation of the three cycle[40] [5], Λ_{ABC} is a scale factor which is approximately the volume of G_2 manifold. We will temporarily ignore both n_{ABC} and Λ_{ABC} in our analysis in this section.

We are interested in the limit where gravity decouples. The G_2 manifold here is treated

[5] Details of how to determine n_{ABC} is in [40] and Appendix F of [68]

	SU_3	SU_2	U_1^a	U_1^b	U_1^c	U_1^d	U_1^Y
Q_1	$\mathbf{3}$	$\mathbf{2}$	1	1	−1	−1	1
Q_2	$\mathbf{3}$	$\mathbf{2}$	1	−1	−1	−1	1
Q_3	$\mathbf{3}$	$\mathbf{2}$	−2	0	−1	−1	1
u_1^c	$\bar{\mathbf{3}}$	$\mathbf{1}$	1	1	−1	−1	−4
u_2^c	$\bar{\mathbf{3}}$	$\mathbf{1}$	1	−1	−1	−1	−4
u_3^c	$\bar{\mathbf{3}}$	$\mathbf{1}$	−2	0	−1	−1	−4
d_1^c	$\bar{\mathbf{3}}$	$\mathbf{1}$	1	1	−1	3	2
d_2^c	$\bar{\mathbf{3}}$	$\mathbf{1}$	1	−1	−1	3	2
d_3^c	$\bar{\mathbf{3}}$	$\mathbf{1}$	−2	0	−1	3	2
L_1	$\mathbf{1}$	$\mathbf{2}$	1	1	−1	3	−3
L_2	$\mathbf{1}$	$\mathbf{2}$	1	−1	−1	3	−3
L_3	$\mathbf{1}$	$\mathbf{2}$	−2	0	−1	3	−3
H_1^u	$\bar{\mathbf{1}}$	$\mathbf{2}$	1	1	2	2	3
H_2^u	$\bar{\mathbf{1}}$	$\mathbf{2}$	1	−1	2	2	3
H_3^u	$\bar{\mathbf{1}}$	$\mathbf{2}$	−2	0	2	2	3
H_1^d	$\bar{\mathbf{1}}$	$\mathbf{2}$	1	1	2	−2	−3
H_2^d	$\bar{\mathbf{1}}$	$\mathbf{2}$	1	−1	2	−2	−3
H_3^d	$\bar{\mathbf{1}}$	$\mathbf{2}$	−2	0	2	−2	−3
e_1^c	$\mathbf{1}$	$\mathbf{1}$	1	1	−1	−1	6
e_2^c	$\mathbf{1}$	$\mathbf{1}$	1	−1	−1	−1	6
e_3^c	$\mathbf{1}$	$\mathbf{1}$	−2	0	−1	−1	6

Table V.2: Relevant particles from three families of E_6, for a complete listing see [37] or Appendix A.2 .

as large enough to make the calculation manageable. Then, we can focus on a local patch of M_3 which is approximately $\mathbb{R}^3$. The volume of the three-cycle in the linearization has been roughly formulated by [37]. However, a more complete analysis shows the requirement of the harmonic condition and relative rotations of the fields. By BPS equations [40], locally for each moduli ϕ ($\phi = a, b, c, d$, and Y. These are the f_i's in the previous sections), there is a harmonic function h_ϕ on M_3 base so that $\phi = dh_\phi$ [40]. For simplicity, we think of ϕ as a three vector, and $\phi = \nabla h_\phi$. The harmonic condition requires that $\Delta h_\phi = 0$. That means

$$\partial_i \phi^i = 0. \tag{V.3.2}$$

This requires that on linear level,

$$\phi = Ht + v \tag{V.3.3}$$

where H is a real traceless symmetric 3x3 matrix, v is a real three vector, t is a local real parametrization of the 3d base. Then, h_ϕ will have the form

$$\frac{1}{2} t^T H t + v^T t + c \tag{V.3.4}$$

where c is a constant term.

The location of a particle, say X, is a zero t_X of a linear combination ϕ_X of a, b, c, d, and Y with by the charges from table V.2. From the previous discussion, t_X is the critical point of a harmonic function h_{ϕ_X}. Assume the critical points are isolated. This is the same as assuming H_X is invertible. The critical point of h_{ϕ_X} or the zero point of ϕ_X is

$$t_X = -H_X^{-1} v_X. \tag{V.3.5}$$

Then, if the ABC term is allowed, i.e, $h_{\phi_A} + h_{\phi_B} + h_{\phi_C} = 0$, the volume for the three-cycle wrapping the three critical points t_A, t_B and t_C is [6]

$$\begin{aligned} Vol(\Sigma_{ABC}) &= h_{\phi_A}(t_A) + h_{\phi_B}(t_B) + h_{\phi_C}(t_C) \\ &= \frac{1}{2}(-v_A^T H_A^{-1} v_A - v_B^T H_B^{-1} v_B \\ &+ (v_A + v_B)^T (H_A + H_B)^{-1} (v_A + v_B)). \end{aligned} \tag{V.3.6}$$

Notice that the constant c in Eq. (V.3.4) plays no role here due to cancellation, so in practice, we will simply drop it. In Sec. V.4.3, explicit computation for a Yukawa coupling is shown

[6] [40] gives formulation for the general case, which has been applied to this linear case.

for a quark term.

V.3.1: Discussion of Other Features

So far, we have only considered M_3 as a flat $\mathbb{R}^3$ which obviously overlooks the very stringent global structure of a compact G_2 manifold. This structure may reduce the parametrization freedom we have in the flat local case. The singularities curves may also cut each other at some point beyond the local area due to compactness, increasing the number of possible Yukawa couplings. Additionally, the sign factors in Eq. (VI.2.2) may also change the mass matrix significantly. They are determined by the gradient flow of the h_ϕ [40, 82, 28]. It is difficult to study the gradient flow between singular points for the local model as the space is not compact. Future study of the gradient flows and hence the sign factors can reveal more of the mass matrix.

As mentioned in Sec. V.2.1, we should project out particles we do not plan to include in our theory. Projecting a specific particle includes requiring that the curves never satisfy the particle's equation derived from Table V.2. That would create more restraint on the parameters.For our local case in particular,in order to exclude a particle ϕ, it would require a vanishing determinant of the H_ϕ and v_ϕ not being in the range of H_ϕ i.e $v_\phi \notin \mathrm{R}(H_\phi)$. This constraint will make the numerical computation much more difficult, so we will not pursue it here. It is noteworthy that in general, beyond our linearization, we can locally break linearity through deformation near the unwanted singularities while keeping the rest intact. Careful study is needed on this issue.

V.4: Quark Terms

V.4.1: General Quark Terms

Recall that the quarks get mass when the Higgses receive VEVs. For example,

$$\lambda^{ij} H_k^u Q_i u_j \to \langle H_k^u \rangle \lambda^{ij} Q_i^k u_j^k. \tag{V.4.1}$$

Ellis et al [64] showed that $\tan\beta \approx 7$, from electroweak symmetry breaking, so we know both up and down VEVs in the two-Higgs-doublets model. We will discuss later how to adapt these into the six Higgs doublets in this paper. Quark terms that satisfy the vanishing sum of charges are

$$
\begin{aligned}
&Q_1u_2^cH_3^u+Q_2u_1^cH_3^u+Q_1d_2^cH_3^u+Q_2u_1^cH_3^u+\\
&Q_2u_3^cH_1^u+Q_3u_2^cH_1^u+Q_2d_3^cH_1^u+Q_3u_2^cH_1^u+\\
&Q_3u_1^cH_2^u+Q_1u_3^cH_2^u+Q_3d_1^cH_2^u+Q_1u_3^cH_2^u.
\end{aligned}
\tag{V.4.2}
$$

Note that there is no diagonal term $Q_iu_i^cH_j^u$ in this general setting. Moreover, this labeling i is completely interchangeable although manifestly picking $i = (1, 2, 3)$ corresponding to (up, charm, top) is a computationally convenient choice here. Also, some couplings between the Higgs and the quarks which could have been possible in SM are forbidden here due to the extra $U(1)$'s. Nonetheless, those terms can still be generated by the Giudice-Masiero mechanism after the breaking of supergravity [47, 7]. However, we will leave this mechanism to future study in the context of M-theory with E_8 orbifold. In the following sections, we will focus on the simplest constraints on the moduli to make the theory physical.

The relevant terms for leptons are

$$L_1e_2^cH_3^d+L_2e_3^cH_1^d+L_3e_1^cH_2^d+ \tag{V.4.3}$$

$$L_1\nu_2^cH_3^u+L_2\nu_3^cH_1^u+L_3\nu_1^cH_2^\nu. \tag{V.4.4}$$

Notice that we only have Dirac mass terms here. Majorana terms may require a quartic level, extra particles getting a VEV, or extra constraints on the moduli, so we will not discuss such terms in this Chapter. In the next chapter, we will explore the VEV options for the particles to facilitate Majorana terms.

V.4.2: Diagonal Terms and Setting $a = 0$

(V.4.2) shows that there is no diagonal term for the quark matrices. This appears to be a problem because, with the top quark mass much larger than those of up and charm quarks, the trace of the mass matrix must be non-zero. This problem is generic in our method of constructing three families from E_8 singularity. The same issue was discussed in the F-theory context in [25]. The reason for this is the conservation of charge in a and b. Hence, this directly relates to the separation of families because a and b break the adjoint of E_8 into three $\mathbf{27}'s$ in E_6. So, particles in the same family must have the same charge in a and b, making it impossible for them to form a singlet cubic term within the same family in a generic setting. One way to remedy this is to introduce a self-intersecting curve for the up-type when $Y = 0$ [25], using the fact that in grand unified theories u and Q both stay on the same curve of $\mathbf{10}$ of $SU(5)$. However, this method cannot be applied for down-type as d

does not stay on the same curve as Q. Moreover, self-intersecting requires higher-order than linearization which we will not pursue here. Alternatively, Bourjaily et al [37] also discuss the contribution of quartic terms. This will require giving large VEVs for extra particles, creating more parameters which we will not consider at this time.

In this paper, we can consider some constraints on a and b leading to possible non-zero diagonal terms. This in essence sets a relation for a and b charges. We still keep in mind the condition of non-vanishing volumes in (V.2.4) as we do not wish to unnecessarily enhance the gauge symmetry. The simplest constraint we can make is $a = 0$. Although it is intriguing to study other constraints, we will ignore them in this paper. This constraint will restrict the gauge group to $SU(3) \times SU(2) \times U(1)^Y \times U(1)^b \times U(1)^c \times U(1)^d$. In terms of geometry, this breaking of $U(1)^a$ is equivalent to restricting the basis 2-cycles in a linear relation, reducing the number of independent 2-cycles and hence the number of $U(1)$'s.

V.4.3: Quark Mass Matrices

After setting $a = 0$ together with the localization, the up-type quark mass matrix can be computed. We will show one example of the computation here for $M^u_{12} u_1 u^c_2$. It comes from the term

$$\lambda^u_{123} Q_1 u^c_2 H^u_3. \tag{V.4.5}$$

When the Higgs gets VEV at low scale, the term becomes

$$\lambda^u_{123} \langle H^u_3 \rangle u_1 u^c_2, \tag{V.4.6}$$

where $M^u_{12} = \lambda^u_{123} \langle H^u_3 \rangle$. Then, all that is left is to compute λ^u_{123}. At high scale, λ^u_{123} can be calculated from (V.3.6) and Table V.2. In the linearization language

$$H_{Q_1} = H_b - H_d + H_Y \tag{V.4.7}$$

$$v_{Q_1} = v_b - v_d + v_Y \tag{V.4.8}$$

$$H_{u_2} = -H_b - H_d - 4H_Y \tag{V.4.9}$$

$$v_{u_2} = -v_b - v_d - 4v_Y \tag{V.4.10}$$

then (V.3.6) gives

$$Vol\{\Sigma_{Q_1 u_2^c H_3^u}\} = \tag{V.4.11}$$

$$\frac{1}{2}\Big((v_b - v_d + v_Y)^T(H_b - H_d + H_Y)^{-1}(v_b - v_d + v_Y) +$$

$$(-v_b - v_d - 4v_Y)^T(-H_b - H_d - 4H_Y)^{-1}(-v_b - v_d - 4v_Y) + \tag{V.4.12}$$

$$(2v_d + 3v_Y)^T(+2H_d + 3H_Y)^{-1}(2v_d + 3v_Y)\Big)$$

Thus, $(VI.2.2)$, ignoring the overall scaling, gives

$$\lambda_{123}^u = \frac{n_{12}^u}{\lambda}\exp\Big\{$$

$$-\frac{1}{2}|(v_b - v_d + v_Y)^T(H_b - H_d + H_Y)^{-1}(v_b - v_d + v_Y) +$$

$$(-v_b - v_d - 4v_Y)^T(-H_b - H_d - 4H_Y)^{-1}(-v_b - v_d - 4v_Y) +$$

$$(2v_d + 3v_Y)^T(+2H_d + 3H_Y)^{-1}(2v_d + 3v_Y)|\Big\} \tag{V.4.13}$$

Then, we have to run these Yukawa couplings down to the SM scale to compute the mass. Note that the diagonal term $Q_3 u_3^c H_3^c$, obtained from setting $a = 0$, can be computed by the above method.

V.4.4: Six Higgs VEVs

In the six Higgs doublets model without extra $U(1)$'s, one can choose a basis for up-type and down-type Higgses so that only one pair of Higgses gets a VEV without loss of generality. Here, due to different charges for the Higgses from the extra $U(1)$'s (see Table V.2), we cannot make such a choice of basis.

We will try to translate from the two VEVs of SM Higgses to the six VEVs in our theory. By standard QFT, we can relate this by looking at the mass of W boson in the SM and identify

$$\langle H_u^{SM}\rangle^2 = \sum_i \langle H_u^i\rangle^2, \tag{V.4.14}$$

$$\langle H_d^{SM}\rangle^2 = \sum_i \langle H_d^i\rangle^2. \tag{V.4.15}$$

So, we can use spherical parametrization to write

$$\begin{aligned} \langle H^1_{u/d}\rangle &= \langle H^{SM}_{u/d}\rangle \cos\phi_{u/d}\sin\theta_{u/d}, \\ \langle H^2_{u/d}\rangle &= \langle H^{SM}_{u/d}\rangle \sin\phi_{u/d}\sin\theta_{u/d}, \\ \langle H^3_{u/d}\rangle &= \langle H^{SM}_{u/d}\rangle \cos\theta_{u/d}. \end{aligned} \tag{V.4.16}$$

Flavor changing neutral current (FCNC) is a risk in multiple Higgses models. It is dangerous as experiments put stringent bound on allowed couplings. We will discuss this in Sec. .

V.4.5: Toward Physical Coupling

Note that the Yukawa couplings in M-theory belong to the high energy scale. We will attempt to use the already existent list of high scale Yukawa coupling running from SM experimental Yukawas in Table 1 of [20] [7] and find a solution for our parameters. Babu et al [20] have studied the Yukawa coupling for minimal SO(1) unification. As three families in $SO(10)$ are the same as our three families except for the $U(1)^b$ charge [8], we assume the effect of the extra U(1)'s and extra particles (for instance, extra Higgses) from our theory in the renormalization group equations (RGEs) is similar, and the Yukawas have approximately the same magnitudes as in [20]. Admittedly, this is a major assumption that requires careful scrutiny. Unfortunately, the renormalization group for our case is a demanding work that deserves a separate study by itself, so we leave it to future study.

In order to compare with physical Yukawa couplings, we need to take into account a few modifications. First, as mentioned in [25], we need a scaling factor to normalize the wave function. For cubic Yukawa, it is roughly proportional to $V_{G_2}^{-\frac{1}{2}}$ where V_{G_2} is the volume of G_2 manifold and still a parameter in our theory (as a local model cannot determine the global volume). Thus the scaling factor for all the cubic Yukawas is a parameter in this local model.

V.4.6: Higgs VEVs

On the other hand, recall that the Higgses only get VEVs at a low scale. Therefore, precisely speaking, we can only consider the VEVs of the six Higgses after we run our M-theory Yukawa couplings down to a low scale. Unfortunately, at a high scale, we only have a set of algebraic expressions for M-theory Yukawas, making the running down to low scale complicated. Moreover, we cannot directly fit our Yukawas with the existing data of high scale running from SM Yukawas because they all assume a two Higgses model. Therefore,

[7] The GUT group is slightly different, but we assume the magnitude of the couplings are approximately the same. See also [115].

[8] Recall that U(1)$^a issettozero$

to remedy this problem, we will use a heuristic treatment assuming that the angular factors, in Eq. (V.4.16), are regarded as part of the low scale Yukawa couplings and do not change much while running to a high scale. Then, the effective VEVs at low scale is just the two VEVs from the SM, and the Yukawa couplings at the high scale used to fit with Table 1 of [20] then are

$$Y = f(\phi, \theta)\lambda \tag{V.4.17}$$

where λ is a Yukawa computed from Sec. V.4.3 and $f(\phi, \theta)$ is one of the angular functions associated with the Higgs fields from Eq. (V.4.16.) The full table of high-scale Yukawa couplings with angular factors is presented in Appendix A.1.

V.5: Yukawa matrix for gauge group $SU(3) \times SU(2) \times U(1)^Y \times U(1)^b \times U(1)^c \times U(1)^d$

First, we need to fix all extra degrees of freedom. Translation allows setting $v_d = 0$. We also have three degrees of rotation and one degree of scaling to make $v_b = (1, 0, 0)$.

Second, we will try to consider the scattering around special cases of H_b and H_d. Notice from the list in (V.2.4) that by setting all parameters to zero except b, we see that volumes of root $e_1 - e_2$ and $e_2 - e_3$ are controlled by b. They are responsible for breaking the adjoint of E_8 into three **27**'s of E_6 (see Figure V.1), hence are also responsible for separating the three SM families.

On the other hand, d controls $e_2 - e_3$, $e_6 - e_7$, and $e_7 - e_8$. The blown-up two-cycle of $e_2 - e_3$ breaks the adjoint of E_8 into two **27**'s of E_6, which transform as the fundamental and singlet of $SU(2)$ respectively, i.e, $(\mathbf{27}, \mathbf{2}) \oplus (\mathbf{27}, \mathbf{1})$. Thus d seperates one family (the top quark family) from the other two in the adjoint of E_8. The latter still has an $SU(2)$ family symmetry (which is broken when we turn b on). Additionally, $e_6 - e_7$ corresponds to breaking the **27**'s of E_6 into the presentations of $SO(10)$, separating the Higgses from quarks and leptons. Finally, $e_7 - e_8$ splits the **16**'s of $SO(10)$ into the **10** and $\bar{\mathbf{5}}$ of $SU(5)$. Thus, d also separates the up-type quarks (up, charm, top) from the down-type quarks (down, strange, bottom), i.e. an isospin breaking effect.

V.6: Numerical Evaluation

To test the compatibility of this model with the Standard Model, we perform a regression on the free parameters by a least-squares approach. Our calculations of Yukawa couplings

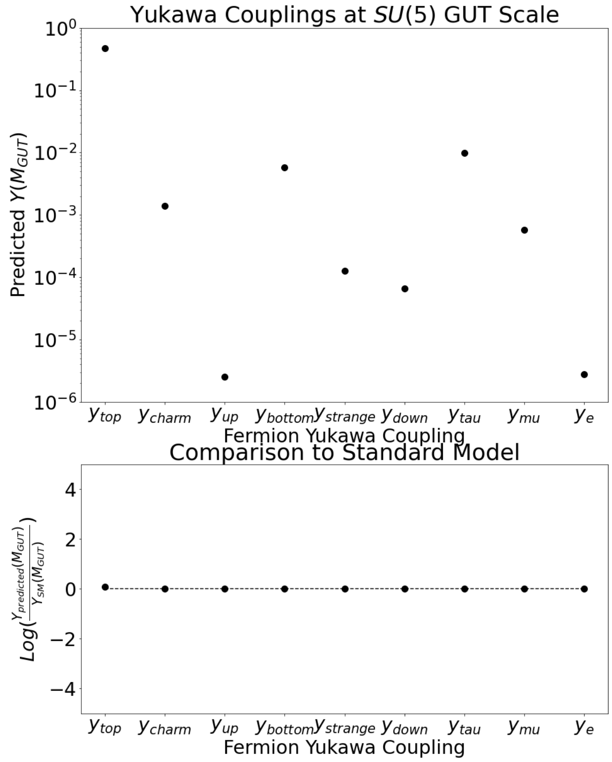


Figure V.2: One solution found numerically.

are compared to experimentally measured weak scale Yukawa couplings which have been run up to the GUT scale [9]. The theoretical uncertainty in the calculation dominates over the experimental uncertainties and we only consider theoretical uncertainty when minimizing the sum of the residuals.

Using previous arguments, we set the base parameters corresponding to $a = 0$ to zero, v_d to zero, and v_b to $(1, 0, 0)$. With three 3×3 traceless symmetric matrices H_ϕ and two $3-$vectors, we have 18 free parameters from the base space. We have four additional parameters from the Higgs VEVs, satisfying $\langle \left(H_1^2 + H_2^2 + H_3^2\right)^{1/2} \rangle = \langle H_{\mathrm{MSSM}} \rangle$. Although we have more free parameters than constraints from the data, the non-linearity in calculating the Yukawas restricts the solutions. A list of numerical solutions is in Appendix. A set of samples from the numerical evaluation is shown in Fig. V.2. We have observed some general trends among the numerical solutions. Most importantly, there exists a hierarchy of Yukawas within each family which comes from the breaking of the flavor and family

[9]See also [115].

symmetries. There is a large top quark Yukawa coupling. Finally, it appears that the hierarchy solution only happens when θ is small, an observation that is expected from the aforementioned no-neutral-current condition.

V.7: Effect of the Higgses and Yukawa couplings

We want to use this section to emphasize the necessity of both the Higgs sector and the Yukawa exponential factor (which is of stringy origin) in satisfying the hierarchy. Recall that if M is the quark mass matrix from above, then the diagonalized mass matrix Λ satisfies

$$\Lambda^2 = U^T M M^T U = V^T M^T M V \tag{V.7.1}$$

where U and V are left and right rotations to mass basis. This means the mass hierarchy is determined by the symmetrized MM^T.

First, if only one family of the Higgses get VEVs, say H_3, we will get the symmetrized up-type quark matrix of the form

$$\begin{Bmatrix} 0 & A & 0 \\ A & 0 & 0 \\ 0 & 0 & B \end{Bmatrix}.$$

Although we still have a hierarchy with one heavy and two light families, there is no hierarchy between the lighter two.

Second, if all three Higgs families get VEVs while all the Yukawa coefficients are the same (equal to 1), the theory will not have the physical hierarchy. Considering only the angular factors (dropping the common VEV factor), we have the matrix in the form

$$\begin{Bmatrix} 0 & \tilde{A} & \tilde{B} \\ \tilde{A} & 0 & \tilde{C} \\ \tilde{B} & \tilde{C} & \tilde{A} \end{Bmatrix}$$

Then, from the characteristic equation, we conclude

$$\text{(V.7.2)} \qquad \begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = \tilde{A} \\ \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = \tilde{A}^2 + \tilde{B}^2 + \tilde{C}^2 = 1 \end{cases}$$

This can be shown to imply that the quark hierarchy can never satisfy Eq. (V.7.2). Therefore, both the three families of Higgses and the stringy Yukawa suppression are needed for the hierarchy.

V.8: Anomalies

The theory may result in gauge boson triangle anomalies. Such an anomaly can be canceled by Stückelberg-Green-Schwarz mechanism and gives some bosons a mass.

V.8.1: How to Compute the Anomaly

We focus our attention on a model with gauge group $SU(3) \times SU(2) \times U(1)^n$ where the $U(1)$'s are to be examined. It can be shown that anomalies of the $U(1)$'s come from triangle loop of bosons in three configurations: $SU(3) - SU(3) - U(1)$ and $SU(2) - SU(2) - U(1)$ and $U(1) - U(1) - U(1)$. The anomaly of a triangle from three $U(1)$'s is proportional to the sum of particles that transform under the nonabelian factor weighted by the charge of $U(1)$ factors. If this sum is zero, the configuration of $U(1)$'s is anomaly-free. Otherwise, it is anomalous.

Explicitly, for

- $U^a(1) - U^b(1) - U^c(1)$ it is simply the sum, over all the particles, of the products of U(1) charges: $\sum_{i:\text{all particles}} q_i^a q_i^b q_i^c$.
- $SU(3) - SU(3) - U(1)$: Sum of U(1) charges over all triplet: $\sum_{i:\text{all triplets}} q_i$.
- $SU(2) - SU(2) - U(1)$: Sum of U(1) charges over all doublet: $\sum_{i:\text{all doublets}} q_i$.

Note that $(\mathbf{3}, \mathbf{2})$ has three $SU(2)$ doublets and two $SU(3)$ triplets.

V.8.2: Anomaly Cancelation by Stückelberg-Green-Schwarz Mechanism

[17] In string theory, an additional term is added to cancel out the anomaly. Such a term will give a mass to the anomalous boson. This is called Stückelberg-Green-Schwarz mechanism.

Table V.3: Anomaly $U(1)-U(1)-U(1)$ computation.

a-a-a	-78	b-b-b	-6	c-c-c	96
d-d-d	0	Y-Y-Y	0	a-a-b	18
a-a-c	-96	a-a-d	0	a-a-Y	0
b-b-a	26	b-b-c	-32	b-b-d	0
b-b-Y	0	c-c-a	0	c-c-b	0
c-c-d	0	c-c-Y	0	d-d-a	0
d-d-b	0	d-d-c	-288	d-d-Y	0
Y-Y- a	0	Y-Y-b	0	Y-Y-c	-288
Y-Y-d	0	a-b-c	0	a-b-d	0
a-b-Y	0	a-c-d	0	a-c-Y	0
a-d-Y	0	b-c-d	0	b-c-Y	0
b-d-Y	0	c-d-Y	288		

The anomaly-related terms in effective action is

$$\mathcal{S} = -\sum_i \int d^4x \frac{1}{4g_i^2} F_{i,\mu\nu} F_i^{\mu\nu} - \underbrace{\frac{1}{2}\int d^4x \sum_I (\partial_\mu a^I + M_i^I A_\mu^i)^2}_{\text{Stückelberg term}} \tag{V.8.1}$$

$$+ \underbrace{\frac{1}{24\pi^2} C_{ij}^I \int a^I F^i \wedge F^j}_{\text{Green-Schwarz term}} + \underbrace{\frac{1}{24\pi^2} E_{ij,k} \int A^i \wedge A^j \wedge F^k}_{\text{Chern-Simons term}} \tag{V.8.2}$$

where a^I are axions, C_{ij}^I is symmetric, E_{ijk} is antisymmetric between i and j. Then, when the anomalous variation is distributed democratically among the three vertices, the condition for canceling the anomalies is

$$t_{ijk} + E_{ijk} + E_{ikj} + M_i^I C_{jk}^I = 0 \tag{V.8.3}$$

where $t_{ijk} = \mathrm{Tr}\{t_i t_j t_k\}$. We now focus on the anomalies coming from $U(1)-U(1)-U(1)$ triangle which are computed in Table V.3. Then, the generators are commuting, so t_{ijk} is totally symmetric. Summing all equations of permutation of i, j, and k, we get

$$M_i^I C_{jk}^I + M_j^I C_{ki}^I + M_k^I C_{ij}^I = -3t_{ijk} \tag{V.8.4}$$

where we used $E_{ijk} = -E_{jik}$. We can use the value of t_{ijk} to compute possible value for M_i^I and C_{ij}^I.

Notice that simultaneous transformation

$$M_i^I \to a^I M_i^I \qquad\qquad C_{ij}^I \to \frac{1}{a^I} C_{ij}^I \tag{V.8.5}$$

for all i, j leaves the equations invariant. So, if (V.8.4) has a solution, the solution will only be unique up to the ratio of the masses. For the anomaly of $b - b - b$, the system is simply reduced to one linear equation giving $U(1)^b$ a nonzero mass, up to a scaling,

$$M_b = -3t_{bbb} = 18. \tag{V.8.6}$$

This specific number does not mean much due to scaling freedom [10]. The only significant point is $U(1)^b$ being massive. Similarly, $U(1)^c$ is also massive. Unfortunately, $U(1)^d$ remains massless. This is due to the fact that $d - d - d$ and $c - c - d$ anomalies are 0 while $c - c - c$ one is non-zero. Yet, as the Higgses are charged in $U(1)^d$ ($U(1)^b$ as well), their electroweak VEVs can give mass to the bosons. Similarly, $U(1)^Y$ is also massless.

V.9: Stabilization

One would naturally ask if the solution we found is indeed a solution that stabilizes the G_2 manifold. Our argument is that our local moduli solution can be stabilized by appropriate global moduli. Notice that in our local theory, the local moduli control the resolution of E_8 singularity but are not a complete set of moduli controlling the global G_2 manifold. Acharya et al [10] showed that M-theory is stabilized on a large class of smooth G_2 manifolds. Such smooth formulation does not precisely describe the singular manifold in our model, but we assume that our local singular region is small enough so that the contribution to overall stabilization will be a perturbation from the equations from [10]. After all, the linear formulation for the a, b, c, d, Y in our paper requires locality, otherwise linear order is not enough to describe the theory.

Moreover, we can see that heuristically, the moduli is stabilizable from Acharya et al's formulation through the remaining global moduli that do not control the singularities. Recall

[10] Study of anomaly involving $SU(2)$ and $SU(3)$ may fix this freedom.

that a, b, c, d, Y controls the volume of the two-cycles resolving the E_8 singularity. These cycles intersect transversely according to the Dynkin diagram. Two intersecting two-cycles will make a complex two torus. Thus, roughly the volume of the manifold is locally

$$V \sim \sum r_1 \times r_2 \times r_3 \times (r_a \times r_b + r_b \times r_c + r_c \times r_d + \dots) \tag{V.9.1}$$

where $r_{1,2,3}$ control the volume of the M_3 base, $r_{a,b,c,d,Y}$ control the volumes of the complex two tori. Then, each individual term is at the standard product form in eq (2) of [10]. This volume can be stabilized by exactly the same mechanism as in [10] because the equations for each r_i agree among the terms (see eq (18) and (19) of [10]). Notice that our model does not determine $r_{1,2,3}$, thus their values can be stabilized accordingly to accompany $r_{a,b,c,d,Y}$ so that the stabilization equations (eq (18) and (19)) are satisfied. This is only heuristic because there can be more moduli when we go global on G_2 manifold, and there can be the coefficients for each term (constant coefficients do not effect our argument here).

This also addresses the concern of gravity self-reaction. The effect of self-reaction on the masses can be limited by moduli stabilization. The only way for the self-reaction to affect the masses in our model is changing the location of the singularities on the internal G_2 manifold. That will result in a change in the moduli. If the moduli are at the stable point, the self-reaction will push them out of equilibrium. The stabilizing mechanism will kick in and restore the moduli to the stable point, canceling the effect of self-correction.

One can also refer additionally to the Acharya et al [6] where they argue that the moduli stabilization mechanism produces vacua within the regime of validity of supergravity. For large Yukawa's like the top, Acharya and Witten suggests that the duality with the heterotic string suggests that there is no significant back reaction [5]

V.10: Flavor changing neutral current

So far, FCNCs are considered rare events. There are several experiments that have searched for FCNCs without any evidence of them [51, 18]. Some experiments have shown evidence of FCNC [15]. These experiments put stringent bounds on the allowed FCNC [33]. For example, at $\sqrt{s} = 13$ TeV, the Yukawa couplings for flavor changing top-up and top-charm are bounded above by 0.037 and 0.071, respectively [51]. In the following, we will try to estimate the couplings which can cause FCNC. Admittedly, the following couplings are estimated at the GUT scale, so running down to a low scale is needed to be more rigorous.

For each of the up-type and down-type Higgs triad, we choose a new basis for the Higgses

$\tilde{H}_i^{u/d}$ so that the VEVs will concentrate on one Higgs $\tilde{H}_1^{u/d}$.

$$\begin{pmatrix} \tilde{H}_1^{u/d} \\ \tilde{H}_2^{u/d} \\ \tilde{H}_3^{u/d} \end{pmatrix} = \underbrace{\begin{pmatrix} \sin\theta_{1/2}\cos\phi_{1/2} & \sin\theta_{1/2}\sin\phi_{1/2} & \cos\theta_{1/2} \\ \cos\theta_{1/2}\cos\phi_{1/2} & \cos\theta_{1/2}\sin\phi_{1/2} & -\sin\theta_{1/2} \\ \cos\phi_{1/2} & -\sin\phi_{1/2} & 0 \end{pmatrix}}_{A} \begin{pmatrix} H_1^{u/d} \\ H_2^{u/d} \\ H_3^{u/d} \end{pmatrix}. \tag{V.10.1}$$

Then, we can write $H_i^{u/d}$ as linear combinations of $\tilde{H}_i^{u/d}$ (using A^T). Next, we groups the terms coupling with $\tilde{H}_i^{u/d}$ into three different groups by i. When $\tilde{H}_1^{u/d}$ gets VEVs, the first group generate mass terms for quarks and charged leptons just like in earlier sections. The second and third groups are responsible for potential FCNC. For instance, the up-type quarks terms can be separated into

$$\sum_{i,j} \tilde{Y}^u_{1ij}\tilde{H}^u_1 Q_i u^c_j + \tilde{Y}^u_{2ij}\tilde{H}^u_2 Q_i u^c_j + \tilde{Y}^u_{3ij}\tilde{H}^u_3 Q_i u^c_j \tag{V.10.2}$$

where $\tilde{Y}^u_{kij}$ have both contributions from the original Yukawa couplings and entries of A^T. The first terms generate up-type quarks mass matrix while the second and third terms, as they do not get VEVs, may create FCNC couplings in the mass eigen-basis.

Using the solution in A.4, we can compute the mass eigenstates for the fermions.

$$\Lambda_{u,d,e} = U^T_{u,d,e} M_{u,d,e} V_{u,d,e} \tag{V.10.3}$$

where $\Lambda_{u,d,e}$ are diagonal mass matrices for the three types of fermions, $M_{u,d,e}$ are the matrices of computed Yukawa couplings fro moduli, $V_{u,d,e}$ and $U_{u,d,e}$ are right and left diagonalizing orthonormal matrices whose columns are eigenvectors of $M^T_{u,d,e}M_{u,d,e}$ and $M_{u,d,e}M^T_{u,d,e}$ respectively. We use $V_{u,d,e}$ and $U_{u,d,e}$ to change the basis for all of the terms in V.10.2. Then the

FCNC couplings for $\tilde{Y}^u_{2ij}\tilde{H}^u_2 Q_i u^c_j$ and $\tilde{Y}^u_{3ij}\tilde{H}^u_3 Q_i u^c_j$ respectively are (ignore signs)

$$\tilde{Y}^u_{2ij} = \begin{pmatrix} 1.1\times10^{-1} & 6.7\times10^{-1} & 1.8\times10^{-2} \\ 6.0\times10^{-5} & 1.810^{-5} & 8.1\times10^{-3} \\ 4.2\times10^{-8} & 7.7\times10^{-8} & -1.2\times10^{-7} \end{pmatrix} \tag{V.10.4}$$

$$\tilde{Y}^u_{3ij} = \begin{pmatrix} 2.1\times10^{-1} & 6.1\times10^{-2} & 1.6\times10^{-3} \\ 3.7\times10^{-4} & 1.1\times10^{-4} & 2.9\times10^{-6} \\ 5.7\times10^{-8} & 1.6\times10^{-7} & 1.6\times10^{-8} \end{pmatrix}. \tag{V.10.5}$$

In this case the couplings for flavor changing for top-up and top-charm are about $\mathcal{O}(10^{-2})$) and $\mathcal{O}(10^{-3})$ respectively. This is consistent with [51].

Similarly, the FCNC matrices for down-type quarks are

$$\tilde{Y}^d_{2ij} = \begin{pmatrix} 3.4\times10^{-10} & 1.6\times10^{-10} & 1.3\times10^{-7} \\ 6.9\times10^{-2} & 1.5\times10^{-3} & 6.0\times10^{-8} \\ 1.3\times10^{-3} & 3.1\times10^{-3} & 1.4\times10^{-7} \end{pmatrix} \tag{V.10.6}$$

$$\tilde{Y}^d_{3ij} = \begin{pmatrix} 4.7\times10^{-10} & 7.1\times10^{-11} & 6.9\times10^{-14} \\ 9.7\times10^{-2} & 2.1\times10^{-3} & 1.5\times10^{-5} \\ 2.0\times10^{-3} & 2.3\times10^{-3} & 2.0\times10^{-7} \end{pmatrix}. \tag{V.10.7}$$

$\tilde{Y}^d_{k21} = \mathcal{O}(10^{-2})$, $\tilde{Y}^d_{k32} = \mathcal{O}(10^{-3})$, and $\tilde{Y}^u_{k31} = \mathcal{O}(10^{-3})$ are dangerous. The upper bounds for them experimentally are about $\mathcal{O}(10^{-5})$, (strange-down) $\mathcal{O}(10^{-4})$ (bottom-strange) and $\mathcal{O}(10^{-4})$ (bottom-down) for $\Delta F = 2$ processes. $\tilde{Y}^d_{k32}$ and $\tilde{Y}^u_{k31}$ are still safe for the $\mathcal{O}(10^{-3})$ bounds from $\Delta F = 1$ rare B decays [33]. As noted by [33], the diagonal terms may have interference and cause cancellations, so more careful calculations are needed. We leave this to future works.

Finally, the lepton FCNC terms

$$\tilde{Y}^e_{2ij} = \begin{pmatrix} 8.9\times10^{-6} & 7.5\times10^{-5} & 5.3\times10^{-6} \\ 6.0\times10^{-7} & 5.3\times10^{-6} & 8.5\times10^{-6} \\ 2.1\times10^{-5} & 1.8\times10^{-4} & 3.2\times10^{-6} \end{pmatrix} \tag{V.10.8}$$

$$\tilde{Y}^e_{3ij} = \begin{pmatrix} 6.0\times10^{-6} & 5.3\times10^{-5} & 3.7\times10^{-6} \\ 4.3\times10^{-7} & 3.8\times10^{-6} & 1.1\times10^{-7} \\ 1.5\times10^{-5} & 1.3\times10^{-4} & 2.3\times10^{-6} \end{pmatrix}. \tag{V.10.9}$$

These all satisfy the constraints from [33].

V.11: Conclusion

We used the geometric gauge breaking mechanism in M-theory compactified on singular G_2 manifold to help understand quark and charged lepton masses. We start with the adjoint representation of a single E_8 that contains exactly three related families of quarks and leptons. Then, we break E_8 to the Standard Model via deformations and geometric engineering, following the technique of Katz and Morrison [95]. We explicitly computed Yukawa couplings in a local model and shows their fitting with experimental results.

With this approach, we hope to understand the origin of flavors and three families and the values of quark and lepton masses. We are partially successful. We can see three families and the hierarchy of quark and lepton masses emerge. We can see the isospin breaking that makes the $SU(2)$ doublets such as top and bottom, up and down, electron and electron neutrino which all have different masses and the hierarchy of family masses. The amounts are controlled by deformation parameters that are effectively moduli. We can calculate the values of the deformation moduli that lead to the hierarchy and realistic values for the masses. Ideally, we would be able to predict the values at which the deformation moduli are stabilized, and predict the masses, but we are not yet able to do so. In principle, the moduli have to satisfy stabilization constraints, neutrino sector, global G_2 structure, and so on. So, future studies on these constraints applying to our quark and lepton context may make the theory predictive.

We are able to get some important mass values. We work with high scale Yukawa couplings. The top quark has a Yukawa coupling of order one. The up quark can be less than the down quark. More precisely, $m_{up} + m_e \lesssim m_{down}$ (ignoring an electromagnetic contribution), so that protons will be stable rather than neutrons, allowing hydrogen atoms. We can derive the conditions in the underlying theory for this inequality, or for the top Yukawa to be of order unity, but we cannot yet show they must uniquely hold. Three families and a hierarchy of masses do arise generically. The theory might not have allowed these results, so we view obtaining them in a UV complete theory as significant progress. We don't at this stage have much control over what masses are associated with the three extra $U(1)$'s, but none should be massless. Then the spectrum should contain four new Z' states. They are well motivated. In future work, it may be possible to constrain their masses. Lastly, we also leave the study of the remaining particles resulting from E_8 breaking for future study.

CHAPTER VI
Neutrino Mass

This chapter is based on a paper we published [71]. M-theory compactified on a G_2 manifold with resolved E_8 singularity is a promising candidate for a unified theory. The experimentally observed masses of quarks and charged leptons put restrictions on the moduli of the G_2 manifold. These moduli in turn uniquely determine the Dirac interactions of the neutrinos. In this chapter, we explicitly compute the Dirac terms for neutrino mass matrix using the moduli from a localized model with resolved E_8 singularities on a G_2 manifold. This is a novel approach as the Dirac terms are not assumed but derived from the structure of quarks' and charged leptons' masses. Using known mass splittings and mixing angles of neutrinos, we show the acceptable region for Majorana terms. We also analyze the theoretical region for Majorana terms induced from the expectation values of right-handed neutrinos and their conjugates through the Kolda-Martin mechanism [97, 7]. The intersection of the two regions indicates a restriction on neutrino masses. In particular, the lightest neutrino must have a small but non-zero mass. Moreover, this also puts constraints on possible Majorana contributions from Kähler potential and superpotential, which can be traced down to a restriction on the geometry. We conclude that the masses of the two heavier light neutrinos are about 0.05 eV and 0.009 eV (0.05 eV and 0.05 eV)) for normal (inverted) hierarchy. In both hierarchies, we predict the light neutrinos are mostly Dirac type. Hence neutrino-less double-beta decay will be small. This is a testable result in a near future. Some bounds on heavy neutrinos are also derived.

VI.1: Overview

The origin of the light left-handed neutrinos in the Standard Model (SM) has been a mystery. Cosmological probes have constrained the sum of the left handed neutrino masses to be $\Sigma m_\nu < 0.12$ (0.15) eV for normal (inverted) ordering [58].

Due to non-zero mixing angles, neutrino flavor eigenstates (electron, muon, and tau) are not the same as the neutrino mass eigenstates (simply labeled "1", "2", and "3"). It is not known which of these three is the heaviest. In analogy with the mass hierarchy of the charged leptons, the configuration with mass 2 being lighter than mass 3 is conventionally called the "normal hierarchy", while in the "inverted hierarchy", the opposite would hold. Several major experimental efforts are underway to help establish which is correct. Current data favors the normal hierarchy, although the confidence for this hierarchy has been decreasing over the years [58]. Neutrino mass splittings observed from neutrino oscillation are $\Delta m_{12}^2 = 7.6 \times 10^{-5}\ eV^2$, and $\Delta m_{13}^2 = 2.5 \times 10^{-3}\ eV^2$ [58]. Moreover, the oscillation angles are about $\theta_{12} = 33.44°$, $\theta_{23} = 49.0°$, and $\theta_{13} = 8.57°$ [58, 59], which can be used to explicitly compute the flavor components of mass eigenstates. In this chapter, we will assume the normal hierarchy first, and then apply a similar framework to the inverted hierarchy.

We show that viable neutrino masses can arise within the framework of M-theory with resolved E_8 singularities, which is a highly non-trivial result, given the constrained nature of M-theory constructions. From our previous work [72], we numerically compute a local solution for moduli of G_2 manifold from the experimental masses of quarks and charged leptons. As these moduli locally control the geometry structure of the manifold, they determine all other interactions in the model. Therefore, we can use them to compute the Dirac mass terms of the neutrinos. This distinguishes our approach from previous works with neutrino Dirac mass [7, 104, 48, 24, 114, 70, 56, 57, 16, 102] as we do not make an estimation, instead we compute the Dirac terms explicitly. Furthermore, these Dirac masses are insufficient, and Majorana masses are also required.

The origin of Majorana mass terms has been complicated to realize from the string theory perspective [7]. For instance, it is possible to obtain large Majorana mass terms from instanton effects [9, 35, 64], large volume compactification [52], or orbifold compactifications of the heterotic string [43]. In this work, we use the Kolda-Martin mechanism [98, 54] to generate vacuum expectation values (VEVs) for the scalar components of right-handed neutrino supermultiplets and their conjugates. The Kolda-Martin (K-M) mechanism includes effects of non-perturbative terms via the Kähler potential. A similar approach has been done by Acharya et al for an $SO(10)$ gauge group [7]. Our work expands the idea to an explicit resolved E_8 singularities model, with three generations fitting the experimental data for quarks and charged leptons, and computes neutrino Dirac terms. The computed Dirac terms put constraints on the Majorana terms through the see-saw mechanism, and the Majorana terms are generated from the VEVs of right-handed neutrinos.

Additionally, when the conjugates of the right-handed neutrinos get VEVs, we inevitably generate bilinear R-parity violating terms of the form $\epsilon_{ij} L_i H_j$. There are many works dedicated to studying these terms [23, 50, 60]. In general, due to the presence of large Majorana terms, the bilinear mixing between Higgs and leptons may spoil the Higgs physics. It is more favorable to have a small ϵ_{ij}. This puts stringent constraints on the aforementioned VEVs. In this paper, we show that there are solutions for the VEVs in which the mixing between leptons and Higgses is minimal. As a result of the constraints, with a generic un-suppressed Kähler potential coefficient, the lightest neutrino can be neither massless nor heavy.

Furthermore, the nature of the lightest neutrinos is expected to be determined in a near future. The most important process for this effort is the neutrinoless double-beta decay, in which the total lepton number is violated by two units. If the neutrinos are mostly Dirac, neutrinoless double beta decay will be small. If the light neutrinos are significantly Majorana, the experiments should be able to detect them, and therefore a good window for new physics, e.g.neutralinos (which are Majorana particles) and R-parity violating interactions[105]. In this paper, we predict that the light particles are mostly Dirac, hence the decay will be small.

This chapter is organized as follows: Sec. VI.2 will briefly cover the local model of M-theory compactified on a G_2 manifold with resolved E_8 singularities [72]. Sec. VI.3 will list all of the contributions to the neutrino mass matrix. Sec. VI.5 discusses the VEVs for the right-handed neutrinos and their conjugates through the K-M mechanism while discussing the ϵ_{ij} problem. Sec. VI.6 contains the computed Dirac matrix and sets up the framework for the neutrino mass matrix. In Sec. VI.7 we discuss the Majorana mass matrix from the experimental data and the VEVs of right-handed neutrino. In Sec. VI.8 we deduce a limit on the neutrino masses. We predict the masses of the mass eigenstate neutrinos, though we cannot yet exclude one of the normal or inverted cases. Finally, some insight about heavy neutrino masses is presented in Sec. VI.9.

VI.2: Background

VI.2.1: General setup

In our model, the resolved E_8 singularity results in matter as in Table 9 and Table 10 in [36]. In the following, the charges are listed in the same order as in Table 10 in [36] (which is in Appendix A.2 for convenience), namely in order a, b, c, d, Y. Then we have

$$\begin{aligned} E_8 \to & SU(3) \times SU(2) \times U(1)_a \times U(1)_b \\ & \times U(1)_c \times U(1)_d \times U(1)_Y . \end{aligned} \tag{VI.2.1}$$

The hypercharge Y has a factor of 6 compared to the conventional hypercharge normalization to make all the Ewe take the convention L_i to be left-handed lepton doublets, ν_i and ν_i^c to be right-handed neutrinos and its conjugates. The reason for $a = 0$ is to allow large top quark mass [72]. Note that the simple root cycles do not shrink under this condition, so there is no enhanced gauge group. This is similar to taking the diagonal $U(1)_a \times U(1)_b$. Notice that a μ term $H_i^u H_j^d$ is generally not allowed, but can be generated by the Giudice-Masiero mechanism [48].

VI.2.2: Yukawa couplings

Recall that the couplings for the interactions in the superpotential are given by the instanton effect [40, 19, 10, 86, 109]

$$Y = \frac{1}{\Lambda} e^{-V_{3 \text{ cycles}}} \tag{VI.2.2}$$

where Λ is a scaling factor proportional to the volume of the G_2 manifold [10, 40]. In our model, the local moduli are not enough to determine the volume, so we treat Λ as a parameter. $V_{3 \text{ cycles}}$ is the volume of the three cycles stretching between the three singularities where the three particles in the cubic terms are located. This volume is a function of the moduli

$$\begin{aligned} Vol(\Sigma_{ABC}) = & \frac{1}{2}(-v_A^T H_A^{-1} v_A - v_B^T H_B^{-1} v_B \\ & +(v_A + v_B)^T (H_A + H_B)^{-1} (v_A + v_B)). \end{aligned} \tag{VI.2.3}$$

Here, Σ_{ABC} is a three-cycle covering three singularities where A, B, and C localize. Moreover, each singularity's location on M_3 is determined by the critical point of

$$f = \frac{1}{2} t^T H t + v^T t + c \tag{VI.2.4}$$

where t is the local 3-d coordinate on M_3, H is a 3×3 matrix, v is a 3-vector, and c is a scalar. Using this setup, we can write down the mass matrix for quarks and charged leptons. Then, by fitting to experimental data, we can find the solutions for f_i's in the local model. We will use the fit result of b, c, d, and Y from [72]. In a full theory on a determined G_2 manifold, the moduli should uniquely determine every other quantity in the theory as they determine the geometry of the manifold. In our local model, as there is some global structure we are missing, the f_i's will determine many quantities, such as Dirac neutrino terms, but leave some other quantities, such as coefficients of Majorana terms and soft breaking terms [7], subject to tuning. Nonetheless, most of our main results will not depend on the tuning.

VI.3: Terms

VI.3.1: Neutrino-neutrino mixing terms

At tree level, the contribution from the superpotential is

$$
\begin{aligned}
W_{tree} \supset\ & y_{123}H_1^u L_2 \nu_3^c + y_{132}H_1^u L_3 \nu_2^c + y_{312}H_3^u L_1 \nu_2^c \\
& + y_{321}H_3^u L_2 \nu_1^c + y_{213}H_2^u L_1 \nu_3^c + y_{231}H_2^u L_3 \nu_1^c \\
& + y_{333}H_3^u L_3 \nu_3^c \qquad \text{(VI.3.1)}
\end{aligned}
$$

where y_{ijk} are coupling constants computed from Eq. VI.2.2. There are also contributions to the same terms from the Kähler potential with coefficients of order $\frac{1}{m_{pl}}$ which is negligible [7]. Similar to the work done by Acharya et al [7] to generate a Majorana mass term, we get contributions to the superpotential of the form

$$
W \supset \sum_{0\leq h,l,m\leq n} \sum_{i,j,k=1,2,3} \frac{C_{h,l,m}}{m_{pl}^{2n-3}} (\nu_i^c \nu_i)^h (\nu_j^c \nu_j)^l (\nu_k^c \nu_k)^m \qquad \text{(VI.3.2)}
$$

where $n = h + l + m$, and $m_{pl} = 2.4 \times 10^{18}$ GeV is the reduced Planck mass. One may notice that ν_i is not in the list of particles we listed in the local model. As ν_i has opposite charges as those of ν_i^c, its singularity will be at a critical point of the same flow function f_i of ν_i^c but with opposite Morse index [1]. Our linearization does not allow more than one critical point for each function, so we can only assume ν_i localizes somewhere outside of our local patch and depends on the global moduli. Accordingly, $C_{h,l,m}$ is determined by the same exponential formula in Eq. (VI.2.2), but we do not know the global moduli to compute the volume of the wrapping cycle. As a result, $C_{h,l,m}$ is considered a tunable parameter in our local model. As the wrapping cycle is supposed to be outside the local patch, we expect its volume to be larger relative to those in the local patch. Hence, by Eq. (VI.2.2), $C_{h,l,m}$ is expected to be smaller than the cubic couplings we encountered before.

Contributions from the Kähler potential to the same terms are expected. They can be computed from the full Kähler potential [26, 101]

$$
K = -3 \log \Big(\frac{V}{2\pi}\Big) \qquad \text{(VI.3.3)}
$$

[1] Morse index is the determinant of the second derivative matrix, i.e Hessian matrix, of f_i. It is the product of the signatures of directional concavity of f_i. In our model, the Morse index distinguishes two types of saddle points of f_i whose directional concavity signatures are (+,+,-) or (-,-,+), corresponding to morse index -1 and +1. Due to harmonic nature, f_i does not have (+,+,+) and (-,-,-).

where V is the volume of G_2 manifold. Unfortunately, the precise dependence of the volume on the global moduli in resloved E_8 orbifold is unknown. We assume it is not significant due to the generic suppression as in [7].

By solving D term and F term equations from the terms in Eq. (VI.3.2), one can find the VEVs for right-handed neutrinos and their conjugates. Assuming the leading term is quartic which we will justify later, the Majorana mass terms for conjugate right-handed neutrinos in the superpotential would have the form

$$\sum_{i,j} \frac{C_{1,1}}{m_{pl}} (\langle\tilde{\nu}_i\rangle\langle\tilde{\nu}_j\rangle) \nu_i^c \nu_j^c \tag{VI.3.4}$$

where $\langle\tilde{\nu}_i\rangle$ is the VEV of the scalar component of ν_i. We will discuss the mechanism of getting those VEVs in Sec. VI.5. In the same manner, the Dirac mass terms emerge from Eq. VI.3.1 when the Higgses get VEVs. One may concern about terms of the forms

$$\sum_{i,j} \frac{C_{1,1}}{m_{pl}} (\langle\tilde{\nu}_i\rangle\langle\tilde{\nu}_j{}^c\rangle) \nu_i^c \nu_j \qquad \text{and} \qquad \sum_{i,j} \frac{C_{1,1}}{m_{pl}} (\langle\tilde{\nu}_i{}^c\rangle\langle\tilde{\nu}_j{}^c\rangle) \nu_i \nu_j . \tag{VI.3.5}$$

But as we suppress terms in VI.3.7, the VEVs $\langle\tilde{\nu}_j{}^c\rangle$ are suppressed. Hence, the above terms are negligible.

VI.3.2: Mixing Matter with Higgs Superfields

Additionally, when the scalar components of the conjugate of right-handed neutrino superfields ν_i^c get VEVs, cubic terms of the form $Y_{ijk} H_i^u L_j \nu_k^c$ from Eq. (VI.3.1) will give rise to the mixing between L_j and H_i^u superfields. They appear in superpotential as

$$\mu_{ij} H_i^{u0} L_j \tag{VI.3.6}$$

where H_i^{u0} is Higgsino, and

$$\mu_{ij} = Y_{ijk} \langle\tilde{\nu}_k^c\rangle . \tag{VI.3.7}$$

This mixing can potentially spoil the Higgs physics, so it is generally more favorable to consider small μ_{ij} relative to Dirac mass terms in the neutrino mass matrix. This creates a stringent condition which requires $\langle\tilde{\nu}_k^c\rangle < \langle H_i^u\rangle$ while $\langle\tilde{\nu}_k\rangle$ remains large due to Eq. (VI.3.5) and the see-saw mechanism. This will be realized in Sec. VI.5.

Furthermore, the presence of R-parity violating bilinear terms (B-RPV) induces a subelectroweak scale (EWS) VEV on the scalar components of the neutrino component of the left-handed leptons fields L. In our case, below the EWS, we expect the same VEVs, generating a mixing between right-handed neutrino and Higgsinos [7]

$$\epsilon_{ij} H_i^{u0} \nu_j^c. \tag{VI.3.8}$$

Although this can create some correction to our analysis, the contribution is usually expected to be smaller than the Dirac mass terms [7].

VI.3.3: Mixing Matter with Gauginos

Finally, as in the Minimal Supersymmetric Standard Model (MSSM), the presence of VEVs will mix some fermions with gauginos through kinetic terms, namely, the Higgsinos with $\tilde{B}_1, \tilde{W}_0$ due to the Higgses VEVs [31]. Similarly, in our case, we also have ν^c-type and ν-type scalar VEVs, which will mix gauginos with matter fermions through kinetic terms. Explicitly, we have, for the $SU(2)$ states (left-handed neutrinos ν_{Li}),

$$\begin{aligned} L \supset\ & g'\langle\tilde{\nu}^c_{Li}\rangle \tilde{B}\nu_{Li} + g\langle\tilde{\nu}^c_{Li}\rangle \tilde{W}^0\nu_{Li} + g_b\langle\tilde{\nu}^c_{Li}\rangle \tilde{B}_b\nu_{Li} \\ & + g_c\langle\tilde{\nu}^c_{Li}\rangle \tilde{B}_c\nu_{Li} + g_d\langle\tilde{\nu}^c_{Li}\rangle \tilde{B}_d\nu_{Li} \end{aligned} \tag{VI.3.9}$$

where the coefficients are gauge couplings. There will be an extra (charge $\times\sqrt(2)$) coefficient for each specific particle [31]. The VEVs $\langle\tilde{\nu}^c_{Li}\rangle$ is expected to be smaller than Dirac mass terms [7]. We have similar terms for swapping $\nu^c_{Li} \leftrightarrow \nu_{Li}$. For the ν-states and ν^c-states, which are singlets under the SM gauge group, mixing takes the form

$$L \supset g_b\langle\tilde{\nu_i}^c\rangle \tilde{B}_b\nu_i + g_c\langle\tilde{\nu}^c_i\rangle \tilde{B}_c\nu_i + g_d\langle\tilde{\nu}^c_i\rangle \tilde{B}_d\nu_i. \tag{VI.3.10}$$

We have similar terms for swapping $\nu^c_i \leftrightarrow \nu_i$. In Sec. VI.3.2, we discussed that terms of the form $\tilde{B}\langle\tilde{\nu_i}^c\rangle\nu_i$ are small due to the small VEVs $\langle\tilde{\nu_i}^c\rangle$ which will be computed in Sec. VI.5. The only significant mixing is that of the forms $\langle\tilde{\nu}_i\rangle\tilde{B}\nu^c_i$. We will discuss their contribution to the mass matrix in the next section.

VI.3.4: General Mass Matrix

Combining all of the previous arguments, we can write down the general mass matrix for neutrinos. Considering the basis

$$(\tilde{B}, \tilde{W}^0, \tilde{B}_{b,c,d}, H^{u0}_{1,2,3}, L_{1,2,3}, \nu^c_{1,2,3}), \tag{VI.3.11}$$

the mass matrix will be

$$M = \left(\begin{array}{c|c} M_{\chi^0}^{8\times 8} & M_{\chi\nu}^{8\times 6} \\ \hline (M_{\chi\nu}^{8\times 6})^T & M_{\nu}^{6\times 6} \end{array}\right) \tag{VI.3.12}$$

where $M_{\chi\nu}^{8\times 6}$ is the mixing sub-matrix between gauginos, Higginos, and neutrinos. $M_{\chi^0}^{8\times 8}$ is the pure gauginos-Higginos sub-matrix, and $M_{\nu}^{6\times 6}$ is mixing between L_i and ν_i^c. Denote

$$M_{\nu}^{6\times 6} = \begin{pmatrix} 0 & D \\ D^{\intercal} & RM \end{pmatrix}$$

where D and RM are 3×3 Dirac and Majorana mass matrices. As both $M_{\chi^0}^{8\times 8}$ and $M_{\chi\nu}^{8\times 6}$ have significant entries, their contribution to the neutrino mass can be significant. In particular, as mentioned in the previous section, the only significant mixing coming from $M_{\chi\nu}^{8\times 6}$ is the mixing between gauginos and ν_i^c. However, the size of the full matrix and the unknown parameters make the analysis challenging.

VI.4: Assumption about the texture of the mass matrix

Instead, we consider several cases and handle each case separately. We need to assume that the matrices behave similarly to their simplified texture matrix. In detail, from the full mass matrix, we can extract the following texture blocks that are directly involved in neutrino masses as

$$\begin{pmatrix} A & 0 & B \\ 0 & 0 & D \\ B^T & D^{\intercal} & RM \end{pmatrix} \tag{VI.4.1}$$

where A block is the contribution from the pure gauginos terms, B block is the contribution from the gaugino and ν_i^c mixing terms. We suppress all other insignificant entries to zeros. For simplicity, we will treat the entries as scalars. Then, the eigenvalue equation is

$$-\lambda(A-\lambda)(RM-\lambda) + B^2\lambda - (A-\lambda)D^2 = 0. \tag{VI.4.2}$$

The equation is simplified to

$$-\lambda^3 + (RM + A)\lambda^2 + (B^2 - RMA + D^2) - AD^2 = 0. \tag{VI.4.3}$$

As we are interested in the masses of the lightest neutrinos which are very small, we approximate $\lambda^3 \approx 0$ and get

$$(RM + A)\lambda^2 + (B^2 - RMA + D^2) + -AD^2 = 0. \tag{VI.4.4}$$

Then, the solution is

$$\lambda = \frac{-K \pm \sqrt{K^2 + 4AD^2(RM + A)}}{2(RM + A)}. \tag{VI.4.5}$$

where $K = B^2 - RMA + D^2$

VI.4.1: Case A: $B^2 \ll RMA$

If we assume $B^2 \ll RMA$, then $K \approx -RMA$. So, we can make the approximation

$$\lambda \approx \frac{RMA \pm \sqrt{(RMA)^2 + 4AD^2(RM + A)}}{2(RM + A)}. \tag{VI.4.6}$$

As $D^2 \ll RMA$, we find

$$\lambda \approx \frac{1}{2(RM + A)}\Big(RMA \pm RMA\big(1 + \frac{2AD^2(RM + A)}{(RMA)^2}\big)\Big). \tag{VI.4.7}$$

The see-saw mechanism is apparent in this approximation. The smaller eigenvalue (ignore sign) is

$$\lambda \approx \frac{D^2}{RM}. \tag{VI.4.8}$$

This is the same as the see-saw approximation of the light eigenvalue of the pure neutrino matrix $M_\nu^{6\times 6}$.

The assumption $B^2 \ll RMA$ is in fact well motivated. The mixing terms B get contribution from $\langle \tilde{\nu}_i \rangle$ while RM gets contribution from $\langle \tilde{\nu}_i \rangle^2$. So both B^2 and RMA are propotional to $\langle \tilde{\nu}_i \rangle^2$. If gaugino mass is heavy enough, this inequality will be satisfied

Of course, the above argument oversimplifies the matrix nature of the entries. It would be interesting to study their full effects in detail in future works. For the scope of this paper, however, we will focus on the neutrino sub-matrix $M_\nu^{6\times 6}$ for computing the light neutrino masses for case A.

VI.4.2: Case B: $B^2 \gg RMA$

When the suppression factors in RM are too large, or the mass terms of the gauginos in A are not big enough, we have $B^2 \gg RMA$, and the solutions of the eigenvalue equation in VI.4.3 become

$$\text{(VI.4.9)} \qquad \lambda \approx \frac{-B^2 \pm \sqrt{B^4 + 4AD^2(RM + A)}}{2(RM + A)}$$

where we have used $K \approx B^2$. The lightest eigenvalue becomes

$$\text{(VI.4.10)} \qquad \lambda \approx \frac{D^2}{(B^2/A)}.$$

This is still the see-saw mechanism. In fact, B^2/A is proportional to the $\langle\tilde{\nu}_i\rangle^2$ just as RM in case A. The difference in case B is the involvement of gaugino mass terms and the values of gauge couplings, replacing the factor C/m_{pl} of RM in case A.

$$\text{(VI.4.11)} \qquad RM \sim \frac{C}{m_{pl}}\langle\tilde{\nu}_i\rangle^2 \qquad \leftrightarrow \qquad \frac{B^2}{A} \sim \frac{g^2}{M_g}\langle\tilde{\nu}_i\rangle^2$$

where g is a gauge coupling, and M_g is a gaugino mass. Therefore, when analyzing the masses of light neutrinos, we will try to apply the argument from case A to case B. In Sec. VI.6, VI.7, VI.8, and VI.9, we will consider the neutrino masses from case A. In Sec. VI.10, we will argue to use a similar method for case B.

VI.4.3: Case C: $B^2 \approx RMA$

This is a special case. If $B^2 - RMA \approx 0$, then the texture matrix in (VI.4.1) degenerates and implies a zero eigenvalue. Thar means the lightest neutrino mass is zero. Then, from the mass gap, we can say right the way the masses of the two heavier light neutrinos are about 0.05 eV and 0.009 eV (0.05 eV and 0.05 eV)) for normal (inverted) hierarchy.

Otherwise, if $B^2 - RMA$ has some significant non-zero values that are not covered in case A and case B, we leave the analysis to future work.

VI.5: VEVs of right-handed neutrinos and their conjugates

In order to explicitly write down the entries for $M_\nu^{6\times 6}$, in this section we will consider a semi-general method to give VEVs to right-handed neutrinos and their conjugates.

VI.5.1: Case 1: No Mixing

First, we consider a standard superpotential that gives rise to the VEVs of right-handed neutrinos and theirs conjugates without mixing of families

$$\mu\nu_i^c\nu_i + \frac{C_{n,0,0}}{m_{pl}^{2n-3}}(\nu_i^c\nu_i)^n \tag{VI.5.1}$$

where $\mu = m_{3/2}\frac{s}{m_{pl}} = \mathcal{O}(10^3)$ GeV with $m_{3/2} = \mathcal{O}(10^4)$ GeV is the mass of gravitino, $\frac{s}{m_{pl}} \equiv$ 0.1 GeV is a generic moduli VEVs contribution[7], C is dimensionless. The latter should be determined completely from the value of the moduli if we have a complete description of G_2 manifold. Unfortunately, we will use this estimated value due to our lack of knowledge for a complete G_2 structure.

D-flat directions implies

$$\sum_i q_i^j\big(|\langle\tilde{\nu}_i^c\rangle|^2 - |\langle\tilde{\nu}_i\rangle|^2\big) - \xi_j = 0 \tag{VI.5.2}$$

for $j = b, c, d, Y$ and ξ's are from Fayet–Iliopoulos terms. F-flat directions give (suppressing VEVs notation to decluster)

$$\mu\nu_i^c + \frac{nC_{n,0,0}}{m_{pl}^{2n-3}}(\nu_i^c)^n(\nu_i)^{n-1} = 0 \tag{VI.5.3}$$

$$\mu\nu_i + \frac{nC_{n,0,0}}{m_{pl}^{2n-3}}(\nu_i^c)^{n-1}(\bar{\nu}_i^c)^n = 0. \tag{VI.5.4}$$

The VEVs for $\tilde{\nu}_i^c$ can be problematic because they can create terms such as $y\langle\tilde{\nu}^c\rangle H^u L$ which may spoil Higgs physics. On the other hand, large VEVs for ν_i are needed to generate large Majorana terms for conjugate right-handed neutrinos and hence see-saw mechanism. Thus, we consider $\langle\tilde{\nu}_i^c\rangle = \epsilon_i\langle\tilde{\nu}_i\rangle$. From F-terms, this will imply

$$\langle\tilde{\nu}_i^c\rangle = \epsilon_i\langle\tilde{\nu}_i\rangle = \sqrt{\epsilon_i}\Big(-\frac{\mu m_{pl}^{2n-3}}{nC_{n,0,0}}\Big)^{\frac{1}{2(n-1)}}. \tag{VI.5.5}$$

Plugging this into the D-term, we get a restriction for Fayet–Iliopoulos coefficients.

$$\xi_b = (\epsilon_1^2 - 1)\langle\tilde{\nu}_1\rangle - (\epsilon_2^2 - 1)\langle\tilde{\nu}_2\rangle \tag{VI.5.6}$$

$$\xi_c = -\sum_{i=1}^{3}(\epsilon_i^2 - 1)\langle\tilde{\nu}_i\rangle \tag{VI.5.7}$$

$$\xi_d = -5\sum_{i=1}^{3}(\epsilon_i^2 - 1)\langle\tilde{\nu}_i\rangle \tag{VI.5.8}$$

$$\xi_Y = 0 \tag{VI.5.9}$$

This cannot give too much texture to Majorana terms without tuning $C_{n,0,0}$. From the observed data, as we will see later, a rich texture is needed. Therefore, it is inviting to consider the mixing case.

VI.5.2: Case 2: Mixing with Two Families

Consider the simplest mixing superpotential

$$\mu\nu_i\nu_i^c + \mu\nu_j\nu_j^c + \frac{C_{n-k,k,0}}{m_{pl}^{2n-3}}(\nu_i\nu_i^c)^{n-k}(\nu_j\nu_j^c)^k. \tag{VI.5.10}$$

The D-flat equations are the same as in Eq. VI.5.2. Again we consider $\langle\tilde{\nu}_i^c\rangle = \epsilon_i\langle\tilde{\nu}_i\rangle$. F-flat directions give

$$\mu\nu_i + (n-k)\frac{C_{n-k,k,0}}{m_{pl}^{2n-3}}(\nu_i)^{n-k}(\nu_j\nu_j^c)^k(\nu_i^c)^{n-k-1} = 0, \tag{VI.5.11}$$

$$\mu\nu_j + (k)\frac{C_{n-k,k,0}}{m_{pl}^{2n-3}}(\nu_i\nu_i^c)^{n-k}(\nu_j)^k(\nu_j^c)^{k-1} = 0 \tag{VI.5.12}$$

$$\text{Interchange } \nu \leftrightarrow \bar{\nu}. \tag{VI.5.13}$$

which imply

$$\langle\tilde{\nu}_i{}^c\rangle = \epsilon_i\langle\tilde{\nu}_i\rangle = \sqrt{\epsilon_i}\left[-\frac{\mu}{C_{n-k,k,0}}\frac{(n-k)^{k-1}}{k^k}m_{pl}^{2n-3}\right]^{\frac{1}{2(n-1)}}, \tag{VI.5.14}$$

$$\langle\tilde{\nu}_j^c\rangle = \epsilon_j\langle\tilde{\nu}_j\rangle = \sqrt{\epsilon_j}\left[-\frac{\mu}{C_{n-k,k,0}}\frac{k^{n-k-1}}{(n-k)^{n-k}}m_{pl}^{2n-3}\right]]^{\frac{1}{2(n-1)}}. \tag{VI.5.15}$$

A hierarchy for Majorana terms is possible here as right-handed neutrinos from different families get different VEVs.

VI.5.3: Case 3: Mixing with Three Families

We can consider the simplest mixing of three families in the superpotential

$$\mu\nu_1^c\nu_1 + \mu\nu_2^c\nu_2 + \mu\nu_3\nu_3^c + \frac{C_{h,k,l}}{m_{pl}^{2n-3}}(\nu_1\nu_1^c)^h(\nu_2\nu_2^c)^k(\nu_3\nu_3^c)^l. \tag{VI.5.16}$$

where $h+k+l=n$. The D-flat equations are the same as in Eq. VI.5.2. Again we consider $\langle\tilde{\nu}_i^c\rangle = \epsilon_i\langle\tilde{\nu}_i\rangle$. Then, F-term equations are

$$\mu + \frac{hC_{h,k,l}}{m_{pl}^{2n-3}}(\nu_1^c\nu_1)^{h-1}(\nu_2^c\nu_2)^k(\nu_3^c\nu_3)^l = 0, \tag{VI.5.17}$$

$$\text{Permute 3 pairs } (1,h),(2,k), \text{ and } (3,l), \tag{VI.5.18}$$

$$\text{permute } \nu \leftrightarrow \bar{\nu}. \tag{VI.5.19}$$

The solution is

$$\langle\tilde{\nu}_i{}^c\rangle = \epsilon_i\langle\tilde{\nu}_i\rangle = \sqrt{\epsilon_i}\left[-\frac{\mu h^{k+l+1}m_{pl}^{2n-3}}{C_{h,k,l}k^k l^l}\right]^{\frac{1}{2(n-1)}}, \tag{VI.5.20}$$

$$\text{Permute 3 pairs } (1,h),(2,k), \text{ and } (3,l). \tag{VI.5.21}$$

Note that in all of the above cases, in practice, we can drop the negative signs inside the brackets as they can be absorbed as a phase in the oscillation matrix of neutrinos. Another scenario is that one of the right-handed neutrinos completely decouples from the other two. The superpotential will then be a sum of case 1 and case 2, and the solutions are the same as case 1 and case 2.

VI.6: Mass Matrix from Neutrino Mixing (case A)

VI.6.1: Mass Matrix Setup

We investigate the matrix with only right-handed neutrinos and left-handed neutrinos. Using the moduli values computed from quarks and charged lepton mass in [72] [2], we compute Dirac

[2] Note that although we can only find one solution in [72], it is likely not unique. Study about the uniqueness of local solution is left for future study.

mass terms from the cubic Yukawa couplings at tree level

$$
\begin{aligned}
W_{tree} \supset\ & y_{123} H_1^u L_2 \nu_3^c + y_{132} H_1^u L_3 \nu_2^c + y_{312} H_3^u L_1 \nu_2^c \\
+\ & y_{321} H_3^u L_2 \nu_1^c + y_{213} H_2^u L_1 \nu_3^c + y_{231} H_2^u L_3 \nu_1^c \\
+\ & y_{333} H_3^u L_3 \nu_3^c
\end{aligned}
\tag{VI.6.1}
$$

where y_{ijk}'s are computed from the moduli. The Yukawa couplings y_{ijk} form a matrix

$$
Y = \begin{pmatrix} 0 & 6.93 \times 10^{-7} & 4.52 \times 10^{-10} \\ 7.25 \times 10^{-1} & 0 & 3.19 \times 10^{-1} \\ 2.53 \times 10^{-5} & 1.71 \times 10^{-2} & 3.22 \times 10^{-2} \end{pmatrix}.
\tag{VI.6.2}
$$

When the Higgs get VEVs, the Dirac terms are approximately

$$
D = \begin{pmatrix} 0 & 2.32 \times 10^{-5} & -3.28 \times 10^{-8} \\ 2.42 \times 10^{1} & 0 & -4.93 \times 10^{1} \\ -1.83 \times 10^{-3} & -2.64 \times 10^{0} & 1.08 \times 10^{0} \end{pmatrix} \text{GeV}.
\tag{VI.6.3}
$$

The first two diagonal entries vanish because there are no charge invariant terms for those. This comes down to the fact that when breaking from E_8, particles from the same family have the same b charge. If their charges are non-zero, they cannot couple in cubic level, which is the case for the first two families with b charge ± 1. The explanation for the size of the rest is complicated as the Yukawa is related to the moduli by exponentiated inverse matrices. However, the significant difference in sizes of the entries can be traced back to the hierarchy of the up-type quarks whose b and c charges are the same as the neutrinos.

The Majorana contribution comes form the superpotential

$$
W \supset y \nu_i^c \nu_i \nu_j^c \nu_j
\tag{VI.6.4}
$$

which was discussed in Sec. VI.3. When right-handed neutrino factors ν_i get VEVs, terms of the form in Eq. (VI.6.4) constitute the Majorana mass matrix RM. The mass matrix is in the basis of $\{L_1, L_2, L_3, \nu_1^c, \nu_2^c, \nu_3^c\}$

$$\begin{pmatrix} 0 & D \\ D^{\intercal} & RM \end{pmatrix}$$

where RM is the conjugate right-handed Majorana matrix. Notice that RM must be symmetric. RM gets large entries when right-handed neutrinos get VEVs. Before computing the VEVs for right-handed neutrinos through a variety of methods, we want to see if it is possible to get a sensible left-handed neutrino hierarchy and flavor-ratio for the mass eigenstates. According to the experimental data, orthonormal eigenvectors are approximately $V \equiv \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$ where v_i is i^{th} column of

$$V = \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13} \\ -c_{23}s_{12} - s_{13}s_{23}c_{12} & c_{23}c_{12} - s_{13}s_{23}s_{12} & c_{13}s_{23} \\ s_{23}s_{12} - s_{13}c_{23}c_{12} & -s_{23}c_{12} - s_{13}c_{23}s_{12} & c_{13}c_{23} \end{pmatrix} \tag{VI.6.5}$$

where $c_{ij} = \cos(\theta_{ij})$, $s_{ij} = \sin(\theta_{ij})$, and we omitted the possible phase for simplicity. We use the oscillation angles

$$\theta_{12} = 33.44^\circ \qquad \theta_{13} = 8.57^\circ \qquad \theta_{23} = 49.0^\circ. \tag{VI.6.6}$$

Assuming normal hierarchy, the eigenvalues are

$$\Lambda \equiv \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} = \begin{pmatrix} x & 0 & 0 \\ 0 & \sqrt{\Delta m_{21}^2 + x^2} & 0 \\ 0 & 0 & \sqrt{\Delta m_{31}^2 + x^2} \end{pmatrix} \tag{VI.6.7}$$

where x is the mass of the lightest left-handed neutrino and the mass-square differences are

$$\Delta m_{31}^2 = 2.32 \times 10^{-21}\ \text{GeV}^2 \qquad \Delta m_{21}^2 = 7.6 \times 10^{-23}\ \text{GeV}^2 \tag{VI.6.8}$$

Finally, we denote the remaining components of the left-handed neutrino eigenvectors as

$$E \equiv \begin{pmatrix} \epsilon_1 & \epsilon_2 & \epsilon_3 \end{pmatrix} \tag{VI.6.9}$$

where ϵ_i's are 3d column vectors whose entries we expect to be small but non-zero. The final eigenvector expression is

$$\underbrace{\begin{pmatrix} 0 & D \\ D^{\intercal} & RM \end{pmatrix}}_{6\times 6} \underbrace{\begin{pmatrix} V \\ E \end{pmatrix}}_{6\times 3} = \begin{pmatrix} V \\ E \end{pmatrix} \Lambda \tag{VI.6.10}$$

VI.7: Majorana Mass Matrix (case A)

VI.7.1: Majorana Mass Matrix from See-Saw Mechanism

Performing the explicit multiplication in Eq. VI.6.10, we get

$$DE = V\Lambda \implies E = D^{-1}V\Lambda, \tag{VI.7.1}$$

$$D^{\intercal}V + RME = E\Lambda \implies RMD^{-1}V\Lambda = E\Lambda - D^{\intercal}V. \tag{VI.7.2}$$

The lightest neutrino cannot be massless, otherwise $(RMD^{-1}V - E)\Lambda$ would have a vanishing third column while $D^{\intercal}$ does not. Thus, Λ is invertible. Combining the two equations we get an expression for RM

$$RM = D^{-1}V\Lambda V^{-1}D - D^{\intercal}V\Lambda^{-1}V^{-1}D. \tag{VI.7.3}$$

Notice that as Λ has very small diagonal entries, the second term is dominant

$$RM \approx -D^{\intercal}V\Lambda^{-1}V^{-1}D. \tag{VI.7.4}$$

For convenience, we absorb negative signs by a phase in V. We can investigate the small x regime by writing

$$RM_{ij} \approx (D^{\intercal}V)_{i1}\frac{1}{x}(V^{-1}D)_{1j} = \frac{(V^{-1}D)_{1i}(V^{-1}D)_{1j}}{x}. \tag{VI.7.5}$$

Thus, at small x, the Majorana terms will behave as a hyperbolic curve with respect to the lightest neutrino mass x, and the texture of RM, modulo the magnitude of x, is given by the first column of $V^{-1}D$ which is fixed.

When m_1 is close to the largest mass splitting, all m_i have the same magnitude and the approximation becomes

$$RM_{ij} \approx \sum_k (D^\intercal V)_{ik} \frac{1}{m_k} (V^{-1}D)_{kj} \tag{VI.7.6}$$

$$= \frac{\sum_k (V^{-1}D)_{ki}(V^{-1}D)_{kj}}{x} \tag{VI.7.7}$$

which is also a hyperbola with respect to x, although the texture of RM relies on all of $V^{-1}D$ here.

To build an intuition on the magnitude of RM, we try plugging in $x = \sqrt{\Delta m_{21}^2} = 10^{-11.5}$ GeV which is about the size of the second mass splitting . The diagonalized left handed neutrino mass matrix is $\mathrm{diag}(4.9 \times 10^{-5}, 8.6 \times 10^{-6}, 3.2 \times 10^{-12})$, absorbing negative signs by a phase in V, we get

$$RM = \begin{pmatrix} 6.6 \times 10^{13} & 4.6 \times 10^{12} & 1.4 \times 10^{14} \\ 4.6 \times 10^{12} & 5.8 \times 10^{11} & 9.5 \times 10^{12} \\ 1.4 \times 10^{14} & 9.5 \times 10^{12} & 2.8 \times 10^{14} \end{pmatrix} \text{GeV} \tag{VI.7.8}$$

which is a symmetric matrix as we wanted. We will see that this matrix can be constructed with appropriate right-handed neutrino VEVs. For readability, the above entries of this Majorana matrix are being rounded from the precise values needed for the hierarchy. In fact, the hierarchy and oscillation of left-handed neutrinos can only be achieved with a high level of precision in the entries of RM. We cannot round the entries up because that would destroy the final hierarchy and oscillation. This is a consequence of Eq. VI.7.2, where the entries of RM are in general much larger than those of Λ, independent of E. So for the equality in Eq.VI.7.2 to happen, entries of RM need to cancel out in RME precisely to very small non-zero numbers.

VI.7.2: Majorana mass from VEVs of ν_i^c

We will argue that contributions beyond the order of Eq. (VI.3.5) will be insignificant. In fact, the contribution from order $2N$ in the superpotential is

$$\sum_{i,j} \frac{C_{N,N,0}}{m_{pl}^{4N-3}} \langle \tilde{\nu}_i \rangle^N \langle \tilde{\nu}_j \rangle^N \langle \tilde{\nu}_i{}^c \rangle^{N-1} \langle \tilde{\nu}_j{}^c \rangle^{N-1} \nu_i^c \nu_j^c \tag{VI.7.9}$$

Plugging in the VEVs from Eq. (VI.5.20), the coefficients are of the form

$$\text{(VI.7.10)} \qquad \frac{C_{N,N,0}m_{pl}^{\frac{n-2N}{n-1}}}{\epsilon}\left[(hk)^{l+1}h^{k-h}k^{h-k}\frac{\mu^2}{C_n^2}\right]^{\frac{2N-1}{2(n-1)}}$$

where h,k,l are permuted to get other terms. Instead of separate n_i and n_j for $\langle\tilde{\nu_i}^c\rangle$ and $\langle\tilde{\nu_j}\rangle$, we can consider $n_i = n_j = n$ for some fractional n. Assume $\frac{C_{N,N,0}}{C_{N+1,N+1,0}} \approx \mathcal{O}(1)$. The h,k,l dependent part is also approximately $\mathcal{O}(1)$,and the coefficient is decreasing with respect to N if

$$\text{(VI.7.11)} \qquad \frac{\mu}{C_n m_{pl}} < 1$$

which implies

$$\text{(VI.7.12)} \qquad C_n > \frac{\mu}{m_{pl}} \approx \frac{10^3}{10^{18}} = 10^{-15}.$$

Thus, as long as the suppression coefficient is not too small, the main contribution is always at quadric order. Henceforth, we assume $C \in [10^{-15}, 1]$ which is consistent with Acharya et al [7].

VI.8: Limit for Neutrinos (Case A)

VI.8.1: Upper Bound for ϵ_i

When the right-handed neutrinos get VEVs, along with familiar Dirac terms of the form

$$\text{(VI.8.1)} \qquad y\langle H_j^u\rangle L_i\nu_k^c,$$

there are terms of the from

$$\text{(VI.8.2)} \qquad y\langle\tilde{\nu}_k^c\rangle L_i H_j^u.$$

which may potentially spoil the Higgs' physics. Therefore, it is desirable for the couplings to be smaller than those of the μ terms $\mu H_i^u H_j^d$ (generated at electroweak scale) and the Dirac terms. As our computed Dirac coupling y in Eq. (VI.6.3) is $\mathcal{O}(10^{-8})$ GeV at the smallest while the largest lepton Yukawa coupling is $\mathcal{O}(10^{-1})$. We need

$$\text{(VI.8.3)} \qquad y_{max}\langle\tilde{\nu}^c\rangle LH^u < (y\langle H^u\rangle)_{min}L\nu^c$$

where $(y\langle H^u\rangle)_{min} = \mathcal{O}(10^{-8})$ GeV, and $y_{max} = \mathcal{O}(10^{-1})$ GeV. It is sufficient to have the right-handed neutrino VEVs as

$$\langle \tilde{\nu}_i^c \rangle \lesssim 10^{-7} \text{ GeV}. \tag{VI.8.4}$$

Plugging the result from Eq. (VI.5.20) in, we get

$$\sqrt{\epsilon_i} \lesssim 10^{-7} \left[\frac{\mu h^{k+l+1} m_{pl}^{2n-3}}{C_{h,k,l} k^k l^l} \right]^{\frac{-1}{2(n-1)}} \tag{VI.8.5}$$

which implies

$$\sqrt{\epsilon_i} \lesssim 10^{-7} \left[\frac{\mu m_{pl}^{2n-3}}{C_{h,k,l}} \right]^{\frac{-1}{2(n-1)}} \tag{VI.8.6}$$

where we have again assumed the k, h, l dependent factor to be approximately $\mathcal{O}(1)$.

VI.8.2: Normal Hierarchy Analysis

Using the upper bound for ϵ we can find a lower bound for the Majorana mass term

$$\begin{aligned} RM_{ij} &= \frac{C_{1,1}}{m_{pl}} \langle \tilde{\nu}_i \rangle \langle \tilde{\nu}_j \rangle = \frac{C_{1,1} m_{pl}^{\frac{n_{ij}-2}{n_{ij}-1}} \mu^{\frac{1}{n_{ij}-1}}}{\epsilon C_{h,k,l}^{\frac{1}{n_{ij}-1}}} \\ &\geq 10^{14} \left[\frac{\mu m_{pl}^{2n_{ij}-3}}{C_{h,k,l}} \right]^{\frac{1}{(n_{ij}-1)}} \frac{C_{1,1} m_{pl}^{\frac{n_{ij}-2}{n_{ij}-1}} \mu^{\frac{1}{n_{ij}-1}}}{C_{h,k,l}^{\frac{1}{n_{ij}-1}}} \\ &= \frac{10^{14} \times C_{1,1} \times m_{pl}^{\frac{3n_{ij}-5}{n_{ij}-1}} \times \mu^{\frac{2}{n_{ij}-1}}}{C_{h,k,l}^{\frac{2}{n_{ij}-1}}} \end{aligned} \tag{VI.8.7}$$

Instead of considering separate n_i and n_j for $\langle \tilde{\nu}_i \rangle$ and $\langle \tilde{\nu}_j \rangle$, we again consider $n_i = n_j = n_{ij}$ for some fractional n_{ij}. Following the analysis of the previous section, we find

$$(D^\intercal V \Lambda^{-1} V^{-1} D)_{ij} = RM_{ij} = \frac{C_{2,1}}{m_{pl}} \langle \tilde{\nu}_i \rangle \langle \tilde{\nu}_j \rangle. \tag{VI.8.8}$$

We will analysis the upper bound for m_3 in many scenarios and deduce the rest of the neutrinos masses accordingly. For convenience, we let $m_1 = \frac{1}{k} m_3$ and $m_2 = \frac{1}{h} m_3$. Then we get

$$\text{(VI.8.9)} \qquad \frac{1}{m_3}\Big[(D^\intercal V)_{i3}(V^{-1}D)_{3j} + h(D^\intercal V)_{i2}(V^{-1}D)_{2j} \\ +k(D^\intercal V)_{i1}(V^{-1}D)_{1j}\Big] = RM_{ij}$$

which implies

$$\text{(VI.8.10)} \qquad m_3 = \frac{1}{RM_{ij}}\Big((D^\intercal V)_{i3}(V^{-1}D)_{3j} + h(D^\intercal V)_{i2}(V^{-1}D)_{2j} + k(D^\intercal V)_{i1}(V^{-1}D)_{1j}\Big)$$

$$\text{(VI.8.11)} \qquad = \frac{1}{RM_{ij}}\Big((D^\intercal V)_{i3}(D^\intercal V)_{j3} + h(D^\intercal V)_{i2}(D^\intercal V)_{j2} + k(D^\intercal V)_{i1}(D^\intercal V)_{j1}\Big)$$

Now, before we use inequality in Eq. VI.8.7 to estimate the bound, we should consider specific limiting cases and get the best bound. Explicit form of $D^T V$ will be useful

$$\text{(VI.8.12)} \qquad D^T V = \begin{pmatrix} -11.58 & 12.91 & 16.92 \\ -0.77 & 1.72 & -1.84 \\ 23.88 & -26.97 & -33.68 \end{pmatrix} \text{ GeV.}$$

First, we consider all left-handed neutrino masses are of the same order, i.e, $k = h = \mathcal{O}(1)$. For all i, j, the numerator of Eq. (VI.8.10) is at most $\mathcal{O}(10^3)$ GeV^2, and the upper bound is

$$\text{(VI.8.13)} \qquad m_3 \leq \frac{10^3}{RM_{ij}} \leq \frac{10^3 C_{h,k,l}^{\frac{2}{n-1}}}{10^{14} \times C_{1,1} \times m_{pl}^{\frac{3n-5}{n-1}} \times \mu^{\frac{2}{n-1}}} < 10^{-20} \text{ GeV}$$

for all $n \geq 2$ where we use $C \in [10^{-15}, 1]$. As the largest mass splitting is $10^{-10.5}$ GeV, it rules out the possibility of equal magnitude for left-handed neutrino masses.

A second case is when m_1 and m_2 are of the same magnitude but much smaller then m_3. Then m_3 will be approximately the mass splitting which is $10^{-10.5}$ GeV and $h \approx k \gg 1$. However, due to the smaller mass splitting about $10^{-11.5}$ GeV, we need $m_1 \approx m_2 \geq 10^{-11.5}$ which implies $h \approx k < 10$. For all i, j, we find the numerator is $\mathcal{O}(10^4)$ GeV^2 at most

$$\text{(VI.8.14)} \qquad m_3 \leq \frac{10^4}{RM_{ij}} \leq \frac{10 C_{h,k,l}^{\frac{2}{n-1}}}{10^{14} \times C_{1,1} \times m_{pl}^{\frac{3n-5}{n-1}} \times \mu^{\frac{2}{n-1}}} < 10^{-19} \text{ GeV.}$$

Thus, m_3 fails to satisfy the mass splitting constraint in this case.

Finally, when $m_1 \ll m_2, m_3$, the magnitude of each entry in RM_{ij} is determined by the magnitude of m_1. The estimate in Eq. VI.8.10 will be dominated by k and provide an upper bound larger than the mass splitting. Hence this is a viable case that agrees with experimental observation. Nonetheless, as mentioned in Sec. VI.7, m_3 cannot be massless in this model. Thus, in general, we predict the lightest neutrino to be massive but light compared the other two. This implies

$$m_3 \approx 0.05 \text{ eV} \qquad\qquad m_2 \approx 0.009 \text{ eV} \tag{VI.8.15}$$

VI.8.3: Inverted Hierarchy Analysis

We can carry out a similar analysis for the inverted hierarchy of left handed neutrino masses. Notice that the oscillations for each label i for m_i do not change. The only thing we need to modify is the diagonal mass matrix

$$\Lambda \equiv \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} = \begin{pmatrix} x & 0 & 0 \\ 0 & \sqrt{x^2 + \Delta m_{21}^2} & 0 \\ 0 & 0 & \sqrt{x^2 - \Delta m_{31}^2} \end{pmatrix} \tag{VI.8.16}$$

As m_2 is the largest, we will mimic the previous analysis as $m_1 = \frac{1}{h} m_2$ and $m_3 = \frac{1}{k} m_2$ and end up with

$$\begin{aligned} m_2 = \frac{1}{RM_{ij}} \Big(& k(D^{\intercal}V)_{i3}(V^{-1}D)_{3j} + (D^{\intercal}V)_{i2}(V^{-1}D)_{2j} \\ & + h(D^{\intercal}V)_{i1}(V^{-1}D)_{1j} \Big). \end{aligned} \tag{VI.8.17}$$

First, we consider all masses are of the same order, i.e, $k = h = O(1)$. Then, for all i, j, we arrive at the same conclusion of $m_2 < 10^{-20}$ GeV which fails to satisfy the mass splitting constraint. Unlike the normal hierarchy, the second case where $m_1 \approx m_3 \ll m_2$ is not possible with inverted hierarchy. As the large mass splitting Δm_{32}^2 requires $m_1 \approx m_3 > 10^{10.5}$ GeV, the small mass splitting $\Delta m_{12}^2 \ll \Delta m_{32}^2$ will imply $m_2 \approx m_1$. Again, we arrive at the conclusion the lightest left handed neutrino, in this case m_3, is light compared to the other two. This implies

$$m_1 \approx m_2 \approx 0.05 \text{ eV} \tag{VI.8.18}$$

The results from both hierarchies are consistent with the current knowledge of light neutrinos, for instance, the work of Gonzalo et al [74].

VI.9: Heavy neutrino mass (Case A)

We can also extract some information about heavy neutrinos by considering the eigenvector equations similar to Eq. VI.6.10

$$\begin{pmatrix} 0 & D \\ D^{\intercal} & RM \end{pmatrix} \begin{pmatrix} V' \\ E' \end{pmatrix} = \begin{pmatrix} V' \\ E' \end{pmatrix} \Lambda' \tag{VI.9.1}$$

where Λ' is the diagonal mass matrix of the heavy neutrinos. In contrast with light neutrinos, We expect V' to be small compared to E'. Similarly to light neutrino case, we can pick E to be orthonormal.This would imply

$$DE' = V'\Lambda' \tag{VI.9.2}$$

$$D^{\intercal}V' + RME' = E'\Lambda' \tag{VI.9.3}$$

As both D and V' are small compared to RM and E' respectively, we have the estimation

$$RME' \approx E'\Lambda' \tag{VI.9.4}$$

or

$$E'^{-1}RME' \approx \Lambda'. \tag{VI.9.5}$$

As E' is orthonormal, we conclude that Λ' is approximately the diagonalized matrix of RM. This means the lower bound for the heaviest eigenvalue is

$$\lambda_{max} \geq \frac{\mathrm{tr}(RM)}{3} \gtrsim 10^{14} \qquad GeV \tag{VI.9.6}$$

Using this, we can estimate the upper bound for the lightest of the heavy neutrinos.

$$\prod_{i=1,2,3} \lambda_i^{heavy} = \det RM. \tag{VI.9.7}$$

Hence,

$$\lambda_{min}^{heavy} \leq \left(\frac{\det(RM)}{\lambda_{max}}\right)^{\frac{1}{2}}. \tag{VI.9.8}$$

$\det(RM)$ is inversely proportional to the mass of the lightest neutrino, so in general $\det(RM)$ is not bounded above when the lightest neutrino becomes lighter and lighter. On the other hand, in the heaviest case, the lightest neutrino is about $10^{-11.5}$ GeV, $\det(RM)$ is about $\mathcal{O}(10^{39})$. Then, the upper bound for the lightest heavy neutrino is

$$\lambda_{min}^{heavy} \leq 10^{12.5} \qquad GeV \tag{VI.9.9}$$

VI.10: Analysis for case B

In Sec. VI.4, we see that the lightest neutrino masses of case B have similar structure to that of case A. In particular, in the simplified texture matrix, the see-saw mechanism is identical if we replace RM by B^2/A. This is the same as replacing the factors C/m_{pl} in RM mass terms by g^2/M_g. We do not have the exact values of g and M_g. g and M_g can be expected to be about $\mathcal{O}(0.1) - \mathcal{O}(1)$ [20] and $\mathcal{O}(100) - \mathcal{O}(1000)$ GeV [10]. These numbers are not rigorously computed in M theory on G_2 manifold with E_8 singularities, so they can be different in future works.

As the analysis of a full mass matrix is too complicated, we will pursue a simplified way: repeat case A by replacing RM matrix with a matrix $\overline{RM}$ whose entries are those of RM with C/m_{pl} replaced by $g^2/M_g \sim \mathcal{O}(10^{-5}) - \mathcal{O}(10^{-2})$ GeV^{-1}.

$$\overline{RM}_{ij} = \frac{g^2}{M_g}\langle\tilde{\nu}_i\rangle\langle\tilde{\nu}_j\rangle, \tag{VI.10.1}$$

and we consider simplifying the texture by

$$\begin{pmatrix} A & B & 0 \\ B^T & RM & D \\ 0 & D^T & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \overline{RM} & D \\ D^T & 0 \end{pmatrix} \tag{VI.10.2}$$

This method is motivated by a point made by Acharya et al [7]. We expect the breaking of the extra $U(1)$'s to transform a chiral superfield and a massless vector superfield into a single massive vector superfield. The degrees of freedom add up correctly and would mean that below the $U(1)$'s breaking scale we can take $\tilde{B}$'s and the linear combination of ν_i^c-states that break the $U(1)$'s to be integrated out jointly. Then, the block matrix of A, B, and RM will combine into a single Majorana matrix.

Similar to Eq. (VI.8.7), we get

$$
\begin{aligned}
\overline{RM}_{ij} &= \frac{g^2}{M_g}\frac{C_{1,1}m_{pl}^{\frac{2n_{ij}-3}{n_{ij}-1}}\mu^{\frac{1}{n_{ij}-1}}}{\epsilon C_{h,k,l}^{\frac{1}{n_{ij}-1}}} \\
&\geq 10^{14}\frac{g^2}{M_g}\left[\frac{\mu m_{pl}^{2n_{ij}-3}}{C_{h,k,l}}\right]^{\frac{1}{(n_{ij}-1)}}\frac{m_{pl}^{\frac{2n_{ij}-3}{n_{ij}-1}}\mu^{\frac{1}{n_{ij}-1}}}{C_{h,k,l}^{\frac{1}{n_{ij}-1}}} \\
&= \frac{10^{14}\frac{g^2}{M_g}\times C_{1,1}\times m_{pl}^{\frac{4n_{ij}-6}{n_{ij}-1}}\times\mu^{\frac{2}{n_{ij}-1}}}{C_{h,k,l}^{\frac{2}{n_{ij}-1}}}
\end{aligned}
\tag{VI.10.3}
$$

Then, we can consider different scenarios for the normal hierachy as like in the previous section. First, we consider all left-handed neutrino masses are of the same order, i.e, $k = h = \mathcal{O}(1)$. For all i, j the numerator is $\mathcal{O}(10^3)$ GeV^2, and the upper bound is

$$
m_3 \approx \frac{10^3}{RM_{ij}} \leq \frac{10^3 C_{h,k,l}^{\frac{2}{n-1}}}{10^{14}\times\frac{g^2}{M_g}\times C_{1,1}\times m_{pl}^{\frac{4n-6}{n-1}}\times\mu^{\frac{2}{n-1}}} < 10^{-26}\ \text{GeV}
\tag{VI.10.4}
$$

for all $n \geq 2$ where we use $C \in [10^{-15}, 1]$. As the largest mass splitting is $10^{-10.5}$ GeV, it rules out the possibility of equal magnitude for left-handed neutrino masses. We see that the bound for case B is even smaller then that of case A. This is the same for all other mass hierarchies in case B. Thus, case B has the same conclusion about the light neutrino masses as that of case A.

VI.11: Conclusion

In this paper, our primary goal is to analyze the mass matrix of neutrinos using the result from a localized model of M-theory compactified on G_2 manifold with resolved E_8 singularity [72]. We learn in this work that the neutrinos originate in the need for the full content of the representations of the resolved E_8 singularity. Similar to the work of Acharya et al [7], there are two main contributions: pure neutrino mixing, and neutralinos and Higginos mixing with neutrinos. We consider several cases to simplify to analysis in this paper.

Dirac terms of the neutrino mass matrix are explicitly computed from the moduli of the localized model on G_2 manifold. We computed the contribution on the cubic level. The texture of the neutrino masses is highly hierarchical as a result of the correlation to hierarchy from the up-type quark. From experimental data of the mixing angles and mass splittings, assuming the normal ordering, we can use the Dirac terms to compute the Majorana mass matrix as a function of the lightest neutrino mass.

The Kolda-Martin mechanism is the main theoretical tool to generate Majorana terms in this paper. In this picture, the right-handed neutrinos (and their antiparticles) get VEVs and generate Majorana masses through quadric terms. The VEVs along with the Dirac terms and experimental data oscillation angles create an upper bound for the masses of left-handed neutrinos. Considering this upper bound in both scenarios of normal and inverted hierarchies, we conclude that the last neutrino should always be light comparing the other two families regardless of the choice of hierarchy. However, the model and the computed Dirac terms generally forbid the lightest neutrino to be massless. The very light mass of the one of the neutrinos implies that the other two left-handed neutrinos have masses about 0.05 eV and 0.009 eV (0.05 eV and 0.05 eV)) for normal (inverted) hierarchy. On the other hand, we achieve some restrictions on heavy neutrinos. The bounds are not stringent enough to make a testable prediction.

For future work, we expect more predictive results when we understand better the contributions from the global structures which determine all the coefficients, including those being tunable in our local theory. Locally, it is also intriguing to explore the uniqueness of the solution. If other solutions exist, it is interesting to see the implication on physics, especially the neutrinos. As our work can be repeated for other solutions in a relatively straightforward way, it is inviting to examine a large class of solutions using bigger computational power.

CHAPTER VII
Summary and Future Directions

VII.1: Summary

M-theory is necessarily compactified on G_2 manifold with ADE singularities. By linearizing the local geometry of such manifolds, we used the quark masses to compute the local moduli. In our model, all three Higgs doublets should get a VEV. We see that the solution for the moduli is quite rare. This suggests the predictive ability of the global G_2 structure. Next, we use the computed moduli to compute Dirac terms for neutrino masses. We write down some scenarios for right-handed neutrinos to get a VEV and induce the see-saw mechanism for neutrinos. This leads us to the conclusion that all neutrinos must be massive, with an estimation that two left-handed neutrinos have masses about 0.05 eV and 0.009 eV (0.05 eV and 0.05 eV)) for the normal (inverted) hierarchy.

VII.2: Future direction

Proton decay is a problem for many unified theories. With a higher rank gauge group, many theories have Higgs color triplets and allow unstable proton. Given the lower bound of proton decay experimentally determined to be at least 1.67×10^{34} years [21], physical theories must suppress proton decay if there is any. From our moduli, we can readily compute possible proton decay. The interactions for proton decay in our model only happen in quadric terms or higher. When S_i fields (see Table A.2) get VEVs, these terms can become cubic Yukawa couplings and contribute to tree-level interactions facilitating proton decay. There have been many qualitative arguments about proton decay in M theory [37, 8], but none has incorporated explicit numerical couplings into their analysis. It is interesting to study this in future works.

CP violation is also an important question in physics. The origin of this can come from the holomorphic nature of Yukawa coupling. Notice that we omitted the complex phase in our previous discussion. This phase come from the G_2 structure 3-form [40]. Including this phase will require extra parametrization. Given how stringent G_2 manifold is, the theory is expected to have enough constraints to produce a predictive result.

The knowledge of global structure will determine all the physics, so expanding our model to a global structure is an inevitable path. Given the difficulty with the full global structure, we can take the first step with making the associative base M_3 a sphere. This is the local form of G_2 manifold from twisted connected sum. There has been some work about M-theory on this kind of manifold [79]. So, it is interesting to see our moduli computation in this setting. The first and foremost challenge is how to parametrize the singularity fibration on the 3-sphere base. We leave this to future work.

APPENDIX A
Moduli from quark mass matrices

A.1: Yukawa Tables

Here, n_{ij} takes value 1, -1, or 0 depending on the trivalent gradient flow existence and orientation whose details are in [40]. We will assume they all 1 in this local model. H and v explicitly are

$$H_\phi = \left\{\begin{matrix} u_\phi^1 & u_\phi^3 & u_\phi^4 \\ u_\phi^3 & u_\phi^2 & u_\phi^5 \\ u_\phi^4 & u_\phi^5 & -u_\phi^1 - u_\phi^2 \end{matrix}\right\}, \qquad v_\phi = \left\{\begin{matrix} v_\phi^1 \\ v_\phi^2 \\ v_\phi^3 \end{matrix}\right\}. \tag{A.1.1}$$

Table A.1: Up-type Quark terms.

Term $Q_i u_j^c H_k^u$	Coupling Y_{ijk}^u
$Q_1 u_2^c H_3^u$	$n_{12}^u \cos\theta_u \exp\Big\{$ $-\frac{1}{2}\lvert(v_b - v_c - v_d + v_Y)^T(H_b - H_c - H_d + H_Y)^{-1}(v_b - v_c - v_d + v_Y)+$ $(-v_b - v_c - v_d - 4v_Y)^T(-H_b - H_c - H_d - 4H_Y)^{-1}(-v_b - v_c - v_d - 4v_Y)+$ $(2v_d + 3v_Y)^T(2H_c + 2H_d + 3H_Y)^{-1}(2v_d + 3v_Y)\rvert\Big\}$
$Q_1 u_3^c H_2^u$	$n_{13}^u \sin\phi_u \sin\theta_u \exp\Big\{$ $-\frac{1}{2}\lvert(v_b - v_c - v_d + v_Y)^T(H_b - H_c - H_d + H_Y)^{-1}(v_b - v_c - v_d + v_Y)+$ $(-v_c - v_d - 4v_Y)^T(-H_c - H_d - 4H_Y)^{-1}(-v_c - v_d - 4v_Y)+$ $(-v_b + 2v_d + 3v_Y)^T(-H_b + 2H_c + 2H_d + 3H_Y)^{-1}(-v_b + 2v_d + 3v_Y)\rvert\Big\}$
$Q_2 u_1^c H_3^u$	$n_{21}^u \cos\theta_u \exp\Big\{$ $-\frac{1}{2}\lvert(-v_b - v_c - v_d + v_Y)^T(-H_b - H_c - H_d + H_Y)^{-1}(-v_b - v_c - v_d + v_Y)+$ $(v_b - v_c - v_d - 4v_Y)^T(H_b - H_c - H_d - 4H_Y)^{-1}(v_b - v_c - v_d - 4v_Y)+$ $(2v_d + 3v_Y)^T(2H_c + 2H_d + 3H_Y)^{-1}(2v_d + 3v_Y)\rvert\Big\}$
$Q_2 u_3^c H_1^u$	$n_{23}^u \cos\phi_u \sin\theta_u \exp\Big\{$ $-\frac{1}{2}\lvert(-v_b - v_c - v_d + v_Y)^T(-H_b - H_c - H_d + H_Y)^{-1}(-v_b - v_c - v_d + v_Y)+$ $(-v_c - v_d - 4v_Y)^T(-H_c - H_d - 4H_Y)^{-1}(-v_c - v_d - 4v_Y)+$ $(v_b + 2v_c + 2v_d + 3v_Y)^T(H_b + 2H_c + 2H_d + 3H_Y)^{-1}(v_b + 2v_c + 2v_d + 3v_Y)\rvert\Big\}$
$Q_3 u_1^c H_2^u$	$n_{31}^u \sin\phi_u \sin\theta_u \exp\Big\{$ $-\frac{1}{2}\lvert(-v_c - v_d + v_Y)^T(-H_c - H_d + H_Y)^{-1}(-v_c - v_d + v_Y)+$ $(v_b - v_c - v_d - 4v_Y)^T(H_b - H_c - H_d - 4H_Y)^{-1}(v_b - v_c - v_d - 4v_Y)+$ $(-v_b + 2v_c + 2v_d + 3v_Y)^T(-H_b + 2H_c + 2H_d + 3H_Y)^{-1}(-v_b 2v_c + 2v_d + 3v_Y)\rvert\Big\}$
$Q_3 u_2^c H_1^u$	$n_{23}^u \cos\phi_u \sin\theta_u \exp\Big\{$ $-\frac{1}{2}\lvert(-v_c - v_d + v_Y)^T(-H_c - H_d + H_Y)^{-1}(-v_c - v_d + v_Y)+$ $(-v_b - v_c - v_d - 4v_Y)^T(-H_b - H_c - H_d - 4H_Y)^{-1}(-v_b - v_c - v_d - 4v_Y)+$ $(v_b + 2v_c + 2v_d + 3v_Y)^T(H_b + 2H_c + 2H_d + 3H_Y)^{-1}(v_b + 2v_c + 2v_d + 3v_Y)\rvert\Big\}$
$Q_3 u_3^c H_3^u$	$n_{33}^u \cos\theta_u \exp\Big\{$ $-\frac{1}{2}\lvert(-v_c - v_d + v_Y)^T(-H_c - H_d + H_Y)^{-1}(-v_c - v_d + v_Y)+$ $(-v_c - v_d - 4v_Y)^T(-H_c - H_d - 4H_Y)^{-1}(-v_c - v_d - 4v_Y)+$ $(2v_c + 2v_d + 3v_Y)^T(2H_c + 2H_d + 3H_Y)^{-1}(2v_c + 2v_d + 3v_Y)\rvert\Big\}$
All else	0

Table A.2: Down-type Quark terms.

Term $Q_i d^c_j H^d_k$	Coupling Y^d_{ijk}
$Q_1 d^c_2 H^d_3$	$n^d_{12} \cos\theta_d \exp\Big\{$ $-\frac{1}{2}\vert(v_b - v_c - v_d + v_Y)^T (H_b - H_c - H_d + H_Y)^{-1}(v_b - v_c - v_d + v_Y)+$ $(-v_b - v_c + 3v_d + 2v_Y)^T(-H_b - H_c + 3H_d + 2H_Y)^{-1}(-v_b - v_c + 3v_d + 2v_Y)+$ $(2v_c - 2v_d - 3v_Y)^T(2H_c - 2H_d - 3H_Y)^{-1}(2v_c - 2v_d - 3v_Y)\vert\Big\}$
$Q_1 d^c_3 H^d_2$	$n^d_{13} \sin\phi_d \sin\theta_d \exp\Big\{$ $-\frac{1}{2}\vert(v_b - v_c - v_d + v_Y)^T (H_b - H_c - H_d + H_Y)^{-1}(v_b - v_c - v_d + v_Y)+$ $(-v_c + 3v_d + 2v_Y)^T(-H_c + 3H_d + 2H_Y)^{-1}(-v_c + 3v_d + 2v_Y)+$ $(2v_c - v_b - 2v_d - 3v_Y)^T(-H_b + 2H_c - 2H_d - 3H_Y)^{-1}(2v_c - v_b - 2v_d - 3v_Y)\vert\Big\}$
$Q_2 d^c_1 H^d_3$	$n^d_{21} \cos\theta_d \exp\Big\{$ $-\frac{1}{2}\vert(-v_b - v_c - v_d + v_Y)^T (-H_b - H_c - H_d + H_Y)^{-1}(-v_b - v_c - v_d + v_Y)+$ $(v_b - v_c + 3v_d + 2v_Y)^T(H_b - H_c + 3H_d + 2H_Y)^{-1}(v_b - v_c + 3v_d + 2v_Y)+$ $(2v_c - 2v_d - 3v_Y)^T(2H_c - 2H_d - 3H_Y)^{-1}(2v_c - 2v_d - 3v_Y)\vert\Big\}$
$Q_2 d^c_3 H^d_1$	$n^d_{23} \cos\phi_d \sin\theta_d \exp\Big\{$ $-\frac{1}{2}\vert(-v_b - v_c - v_d + v_Y)^T (-H_b - H_c - H_d + H_Y)^{-1}(-v_b - v_c - v_d + v_Y)+$ $(-v_c + 3v_d + 2v_Y)^T(-H_c + 3H_d + 2H_Y)^{-1}(-v_c + 3v_d + 2v_Y)+$ $(v_b + 2v_c - 2v_d - 3v_Y)^T(H_b + 2H_c - 2H_d - 3H_Y)^{-1}(v_b + 2v_c - 2v_d - 3v_Y)\vert\Big\}$
$Q_3 d^c_1 H^d_2$	$n^d_{31} \sin\phi_d \sin\theta_d \exp\Big\{$ $-\frac{1}{2}\vert(-v_c - v_d + v_Y)^T (-H_c - H_d + H_Y)^{-1}(-v_c - v_d + v_Y)+$ $(v_b - v_c + 3v_d + 2v_Y)^T(H_b - H_c + 3H_d + 2H_Y)^{-1}(v_b - v_c + 3v_d + 2v_Y)+$ $(-v_b + 2v_c - 2v_d - 3v_Y)^T(-H_b + 2H_c - 2H_d - 3H_Y)^{-1}(-v_b + 2v_c - 2v_d - 3v_Y)\vert\Big\}$
$Q_3 d^c_2 H^d_1$	$n^d_{32} \cos\phi_d \sin\theta_d \exp\Big\{$ $-\frac{1}{2}\vert(-v_c - v_d + v_Y)^T (-H_c - H_d + H_Y)^{-1}(-v_c - v_d + v_Y)+$ $(-v_b - v_c + 3v_d + 2v_Y)^T(-H_b - H_c + 3H_d + 2H_Y)^{-1}(-v_b - v_c + 3v_d + 2v_Y)+$ $(v_b + 2v_c - 2v_d - 3v_Y)^T(H_b + 2H_c - 2H_d - 3H_Y)^{-1}(v_b + 2v_c - 2v_d - 3v_Y)\vert\Big\}$
$Q_3 d^c_3 H^d_3$	$n^d_{33} \cos\theta_d \exp\Big\{$ $-\frac{1}{2}\vert(-v_c - v_d + v_Y)^T (-H_c - H_d + H_Y)^{-1}(-v_d + v_Y)+$ $(-v_c + 3v_d + 2v_Y)^T(-H_c + 3H_d + 2H_Y)^{-1}(-v_c + 3v_d + 2v_Y)+$ $(2v_c - 2v_d - 3v_Y)^T(2H_c - 2H_d - 3H_Y)^{-1}(2v_c - 2v_d - 3v_Y)\vert\Big\}$
All else	0

Table A.3: Electron-type terms.

Term $L_i e_j^c H_k^d$	Coupling Y_{ijk}^l
$L_1 e_2^c H_3^d$	$n_{12}^e \cos\theta_d \exp\Big\{$ $-\frac{1}{2}\lvert(v_b - v_c + 3v_d - 3v_Y)^T (H_b - H_c + 3H_d - 3H_Y)^{-1}(v_b - v_c + 3v_d - 3v_Y)+$ $(-v_b - v_c - v_d + 6v_Y)^T(-H_b - H_c - H_d + 6H_Y)^{-1}(-v_b - v_c - v_d + 6v_Y)+$ $(2v_c - 2v_d - 3v_Y)^T(2H_c - 2H_d - 3H_Y)^{-1}(2v_c - 2v_d - 3v_Y)\rvert\Big\}$
$L_1 e_3^c H_2^d$	$n_{13}^e \sin\phi_d \sin\theta_d \exp\Big\{-\frac{1}{2}\lvert(v_b - v_c + 3v_d - 3v_Y)^T (H_b - H_c + 3H_d - 3H_Y)^{-1}(v_b - v_c + 3v_d - 3v_Y)+$ $(-v_c - v_d + 6v_Y)^T(-H_c - H_d + 6H_Y)^{-1}(-v_c - v_d + 6v_Y)+$ $(-v_b + 2v_c - 2v_d - 3v_Y)^T(-H_b + 2H_c - 2H_d - 3H_Y)^{-1}(-v_b + 2v_c - 2v_d - 3v_Y)\rvert\Big\}$
$L_2 e_1^c H_3^d$	$n_{21}^e \cos\theta_d \exp\Big\{$ $-\frac{1}{2}\lvert(-v_b - v_c + 3v_d - 3v_Y)^T (-H_b - H_c + 3H_d - 3H_Y)^{-1}(-v_b - v_c + 3v_d - 3v_Y)+$ $(v_b - v_c - v_d + 6v_Y)^T(H_b - H_c - H_d + 6H_Y)^{-1}(v_b - v_c - v_d + 6v_Y)+$ $(2v_c - 2v_d - 3v_Y)^T(2H_c - 2H_d - 3H_Y)^{-1}(2v_c - 2v_d - 3v_Y)\rvert\Big\}$
$L_2 e_3^c H_1^d$	$n_{23}^e \cos\phi_d \sin\theta_d \exp\Big\{$ $-\frac{1}{2}\lvert(-v_b - v_c + 3v_d - 3v_Y)^T (-H_b - H_c + 3H_d - 3H_Y)^{-1}(-v_b - v_c + 3v_d - 3v_Y)+$ $(-v_c - v_d + 6v_Y)^T(-H_c - H_d + 6H_Y)^{-1}(-v_c - v_d + 6v_Y)+$ $(v_b + 2v_c - 2v_d - 3v_Y)^T(H_b + 2H_c - 2H_d - 3H_Y)^{-1}(v_b + 2v_c - 2v_d - 3v_Y)\rvert\Big\}$
$L_3 e_1^c H_2^d$	$n_{31}^e \sin\phi_d \sin\theta_d \exp\Big\{$ $-\frac{1}{2}\lvert(-v_c + 3v_d - 3v_Y)^T (-H_c + 3H_d - 3H_Y)^{-1}(-v_c + 3v_d - 3v_Y)+$ $(v_b - v_c - v_d + 6v_Y)^T(H_b - H_c - H_d + 6H_Y)^{-1}(v_b - v_c - v_d + 6v_Y)+$ $(-v_b + 2v_c - 2v_d - 3v_Y)^T(-H_b + 2H_c - 2H_d - 3H_Y)^{-1}(-v_b + 2v_c - 2v_d - 3v_Y)\rvert\Big\}$
$L_3 e_2^c H_1^d$	$n_{32}^e \cos\phi_d \sin\theta_d \exp\Big\{$ $-\frac{1}{2}\lvert(-v_c + 3v_d - 3v_Y)^T (-H_c + 3H_d - 3H_Y)^{-1}(-v_c + 3v_d - 3v_Y)+$ $(-v_b - v_c - v_d + 6v_Y)^T(-H_b - H_c - H_d + H_Y)^{-1}(-v_b - v_c - v_d + 6v_Y)+$ $(v_b + 2v_c - 2v_d - 3v_Y)^T(H_b + 2H_c - 2H_d - 3H_Y)^{-1}(v_b + 2v_c - 2v_d - 3v_Y)\rvert\Big\}$
$L_3 e_3^c H_3^d$	$n_{33}^e \cos\theta_d \exp\Big\{$ $-\frac{1}{2}\lvert(-v_c + 3v_d - 3v_Y)^T (-H_c + 3H_d - 3H_Y)^{-1}(-v_c + 3v_d - 3v_Y)+$ $(-v_c - v_d + 6v_Y)^T(-H_c - H_d + 6H_Y)^{-1}(-v_c - v_d + 6v_Y)+$ $(2v_c - 2v_d - 3v_Y)^T(2H_c - 2H_d - 3H_Y)^{-1}(2v_c - 2v_d - 3v_Y)\rvert\Big\}$
All else	0

The numerical result for the moduli used in Figure V.2 is in table A.4

Table A.4: Tabulated numerical values of Moduli

Parameter	Value	Parameter	Value
$H_{b\ 1,1}$	1.75219628381272	$H_{Y\ 1,1}$	0.470254977486118
$H_{b\ 1,2}$	-0.735328652705781	$H_{Y\ 1,2}$	0.701824648617083
$H_{b\ 1,3}$	-0.377020719433746	$H_{Y\ 1,3}$	-1.34973735409641
$H_{b\ 2,2}$	1.19315995302413	$H_{Y\ 2,2}$	0.641604847709697
$H_{b\ 2,3}$	0.675543994721913	$H_{Y\ 2,3}$	0.108762493856499
$H_{c\ 1,1}$	1.41562893856031	$v_{c\ 1}$	-0.250519055696569
$H_{c\ 1,2}$	1.05931181064608	$v_{c\ 2}$	-1.4656911323334
$H_{c\ 1,3}$	1.45371284442019	$v_{c\ 3}$	-0.405862359379641
$H_{c\ 2,2}$	0.509312354295455	$v_{Y\ 1}$	-0.417353540440179
$H_{c\ 2,3}$	-1.11688627765089	$v_{Y\ 2}$	0.688596965400472
$H_{d\ 1,1}$	-1.13434995566124	$v_{Y\ 3}$	-1.50894939341168
$H_{d\ 1,2}$	-0.386481794947734	ϕ_1	0.439157499240369
$H_{d\ 1,3}$	-1.34973735409641	θ_1	-1.37741142245986
$H_{d\ 2,2}$	0.336495907974189	ϕ_2	3.76691301205782
$H_{d\ 2,3}$	2.25852327647213	θ_2	6.06985994999668
Λ_0	0.907032583667471	-	-

A.2: Table 10 in [38]

	SU_3	$\times SU_2$	$\times U_1^a$	$\times U_1^b$	$\times U_1^c$	$\times U_1^d$	$\times U_1^Y$	Location
Q_1	$\mathbf{3}$	$\mathbf{2}$	1	1	-1	-1	1	$a+b-c-d+Y=0$
Q_2	$\mathbf{3}$	$\mathbf{2}$	1	-1	-1	-1	1	$a-b-c-d+Y=0$
Q_3	$\mathbf{3}$	$\mathbf{2}$	-2	0	-1	-1	1	$-2a-c-d+Y=0$
u_1^c	$\overline{\mathbf{3}}$	$\mathbf{1}$	1	1	-1	-1	-4	$a+b-c-d-4Y=0$
u_2^c	$\overline{\mathbf{3}}$	$\mathbf{1}$	1	-1	-1	-1	-4	$a-b-c-d-4Y=0$
u_3^c	$\overline{\mathbf{3}}$	$\mathbf{1}$	-2	0	-1	-1	-4	$-2a-c-d-4Y=0$
d_1^c	$\overline{\mathbf{3}}$	$\mathbf{1}$	1	1	-1	3	2	$a+b-c+3d+2Y=0$
d_2^c	$\overline{\mathbf{3}}$	$\mathbf{1}$	1	-1	-1	3	2	$a-b-c+3d+2Y=0$
d_3^c	$\overline{\mathbf{3}}$	$\mathbf{1}$	-2	0	-1	3	2	$-2a-c+3d+2Y=0$
D_1	$\mathbf{3}$	$\mathbf{1}$	-1	-1	-2	2	-2	$-a-b-2c+2d-2Y=0$
D_2	$\mathbf{3}$	$\mathbf{1}$	-1	1	-2	2	-2	$-a+b-2c+2d-2Y=0$
D_3	$\mathbf{3}$	$\mathbf{1}$	2	0	-2	2	-2	$2a-2c+2d-2Y=0$
D_1^c	$\overline{\mathbf{3}}$	$\mathbf{1}$	-1	-1	-2	-2	2	$-a-b-2c-2d+2Y=0$
D_2^c	$\overline{\mathbf{3}}$	$\mathbf{1}$	-1	1	-2	-2	2	$-a+b-2c-2d+2Y=0$
D_3^c	$\overline{\mathbf{3}}$	$\mathbf{1}$	2	0	-2	-2	2	$2a-2c-2d+2Y=0$
L_1	$\mathbf{1}$	$\mathbf{2}$	1	1	-1	3	-3	$a+b-c+3d-3Y=0$
L_2	$\mathbf{1}$	$\mathbf{2}$	1	-1	-1	3	-3	$a-b-c+3d-3Y=0$
L_3	$\mathbf{1}$	$\mathbf{2}$	-2	0	-1	3	-3	$-2a-c+3d-3Y=0$
H_1^u	$\mathbf{1}$	$\mathbf{2}$	1	1	2	2	3	$a+b+2c+2d+3Y=0$
H_2^u	$\mathbf{1}$	$\mathbf{2}$	1	-1	2	2	3	$a-b+2c+2d+3Y=0$
H_3^u	$\mathbf{1}$	$\mathbf{2}$	-2	0	2	2	3	$-2a+2c+2d+3Y=0$
H_1^d	$\mathbf{1}$	$\mathbf{2}$	1	1	2	-2	-3	$a+b+2c-2d-3Y=0$
H_2^d	$\mathbf{1}$	$\mathbf{2}$	1	-1	2	-2	-3	$a-b+2c-2d-3Y=0$
H_3^d	$\mathbf{1}$	$\mathbf{2}$	-2	0	2	-2	-3	$-2a+2c-2d-3Y=0$
e_1^c	$\mathbf{1}$	$\mathbf{1}$	1	1	-1	-1	6	$a+b-c-d+6Y=0$
e_2^c	$\mathbf{1}$	$\mathbf{1}$	1	-1	-1	-1	6	$a-b-c-d+6Y=0$
e_3^c	$\mathbf{1}$	$\mathbf{1}$	-2	0	-1	-1	6	$-2a-c-d+6Y=0$
ν_1^c	$\mathbf{1}$	$\mathbf{1}$	1	1	-1	-5	0	$a+b-c-5d=0$
ν_2^c	$\mathbf{1}$	$\mathbf{1}$	1	-1	-1	-5	0	$a-b-c-5d=0$
ν_3^c	$\mathbf{1}$	$\mathbf{1}$	-2	0	-1	-5	0	$-2a-c-5d=0$
S_1	$\mathbf{1}$	$\mathbf{1}$	-1	-1	4	0	0	$-a-b+4c=0$
S_2	$\mathbf{1}$	$\mathbf{1}$	-1	1	4	0	0	$-a+b+4c=0$
S_3	$\mathbf{1}$	$\mathbf{1}$	2	0	4	0	0	$2a+4c=0$
N_1	$\mathbf{1}$	$\mathbf{1}$	-3	1	0	0	0	$-3a+b=0$
N_2	$\mathbf{1}$	$\mathbf{1}$	0	-2	0	0	0	$-2b=0$
N_3	$\mathbf{1}$	$\mathbf{1}$	3	1	0	0	0	$3a+b=0$

Table A.5: Full content of three families from E_8

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