Some Results on Homogeneity Results for GL_n

by

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To my friends, my family, my dear Jiahou, and my most precious Clementine

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Abstract

Let K be a nonarchimedian local field with a ring of integers R and prime ideal $\mathfrak{p}$. Let G be the group of K-points of a connected reductive algebraic group defined over K with Lie algebra $\mathfrak{g}$. In one of DeBacker's papers, he established a range of validity for the Harish-Chandra–Howe local expansion for characters of admissible irreducible representations of G under some conditions, and he established an analogous homogeneity result on the Lie algebra of G, again with some restrictions. These restrictions are, essentially, restrictions on the characteristic of the residue field k of K. While the hope of removing the characteristic restriction for G in general is not bright, in the case where $G = GL_n$ we can be more positive. Our primary goal here is to move towards a proof of a special type of case for a certain key step which plays a prominent role for the homogeneity result of GL_n without restrictions on the characteristic. Finally, in the end, we provide a full written proof of the homogeneity result for GL_3.

CHAPTER I

Introduction

1.1 Introduction

1.1.1 Basic Notation

Suppose K is a nonarchimedian local field with ring of integers R and prime ideal $\mathfrak{p} = \varpi R$, where ϖ is a uniformizer. We will also let $\mathbb{F}_q \simeq R/\mathfrak{p}$ be the residue field. Let $\bar{K}$ denote a separable closure of K and let K^{un} denote the maximal unramified extension of K in $\bar{K}$. We let v_K denote a valuation on K normalized so that $v_K(K) = \mathbb{Z}$. We fix an additive character, Λ, on K that is trivial on $\mathfrak{p}$ and nontrivial on R.

Let $\mathbb{G}$ be a connected reductive group defined over K with Lie algebra $\mathrm{Lie}(\mathbb{G})$. Let $G = \mathbb{G}(K)$ be the group of K rational points of $\mathbb{G}$, and denote the Lie Algebra of $\mathbb{G}$ by $\mathfrak{g}$. Since our main focus will be GL_n, we will often abuse notation and not distinguish between $\mathbb{G}$ and G.

The main focus of this paper is the general linear group GL_n. For a commutative ring S, we will denote by $GL_n(S)$ the set of $n \times n$ invertible matrices with entries in S, and by $M_n(S)$ the set of $n \times n$ matrices with entries in S. We will also denote by Z the center of $GL_n(K)$, and by $\mathfrak{z}$ the center of $M_n(K) = \mathfrak{gl}(K)$.

We will also use the floor function $\lfloor\cdot\rfloor$ and the ceiling function $\lceil\cdot\rceil$; when they are applied to a number x, the floor function outputs the largest integer less than or equal to x, and the ceiling function outputs the smallest integer that is greater than or equal to x.

Let Ad be the Adjoint representation of G on $\mathfrak{g}$, and ad be the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}$ inherited from Ad. For $g \in G$, $X \in \mathfrak{g}$, we will write ${}^gX$ for $\mathrm{Ad}(g)X$. For $g \in G$, $S \subseteq \mathfrak{g}$, we will set ${}^gS = \{{}^gX | X \in S\}$.

Suppose $X \in \mathfrak{g}$, we will denote the centralizer of X in G by $C_G(X)$, and the centralizer of X in $\mathfrak{g}$ by $C_{\mathfrak{g}}(X)$.

Let $X_*(\mathbb{G})$ denote the set of cocharacters or, to say, the set of 1−parameter subgroups of $\mathbb{G}$ defined over K, and let $X^*(\mathbb{G})$ denote the set of characters defined over K, so $X_*(\mathbb{G}) = \mathrm{Hom}(GL_1, \mathbb{G})^{\mathrm{Gal}(\bar{K}/K)}$ and $X^*(\mathbb{G}) = \mathrm{Hom}(\mathbb{G}, GL_1)^{\mathrm{Gal}(\bar{K}/K)}$. An element $X \in \mathfrak{g}$ is said to be *nilpotent* if there exists $\lambda \in X_*(\mathbb{G})$, such that $\lim_{t\to 0} {}^{\lambda(t)}X = 0$, and we will denote the set of nilpotent elements in $\mathfrak{g}$ by $\mathcal{N}$. Note that in the case of GL_n, $\mathcal{N}$ can be also defined as the set of all the elements in $\mathfrak{g}$ for which the Zariski closure of their $\mathbb{G}$-orbits contains zero (see [*Kempf* (1978)]).

Let dg denote a fixed Haar measure on G, and let C_c^∞ denote the space of complex-valued, locally constant, compactly supported functions on G. Suppose (π, V) is an admissible representation of G. That is, V is a complex vector space and for all compact open subgroups K of G we have that V^K, the space of K-fixed vectors in V, is finite dimensional. We let Θ_π denote the distribution character of π, that is, the map $C_c^\infty \to \mathbb{C}$ given by $\Theta_\pi(f) := \mathrm{tr}(\pi(f))$, where $\pi(f)$ is the operator on V given by

$$\pi(f)v := \int_G f(g)\pi(g)v dg$$

for $v \in V$.

When $\mathbb{G}$ is of rank r, we say an element $g \in G$ is regular semisimple provided that the coefficient of t^r in $(det(t - 1 + Ad(g)))$ is non zero, and we denote the regular semisimple elements of G by G^{reg}, similarly, we denote the set of regular semisimple elements of $\mathfrak{g}$ by $\mathfrak{g}^{reg}$.

Note that in this paper, we will make no assumption on the (prime) residual characteristic of K.

1.1.2 Apartments, Bruhat-Tits building, and related notations

Let $\mathbb{T}$ be a maximal K-split torus in $\mathbb{G}$ and recall that there is a perfect pairing $<,>: X^*(\mathbb{T}) \times X_*(\mathbb{T}) \to \mathbb{Z}$. Fix a minimal parabolic subgroup subgroup $\mathbb{P}_\varnothing$ of $\mathbb{G}$ so that $\mathbb{T} \subseteq P_\varnothing$. Let $T = \mathbb{T}(K)$ and $P_\varnothing = \mathbb{P}_\varnothing(K)$.

The choice of $\mathbb{T}$, $\mathbb{P}_\varnothing$, and $\mathbb{G}$ gives us:

- An irreducible reduced root system $\overline{\Phi} = \overline{\Phi}(\mathbb{T}, \mathbb{G}) \subset X^*(\mathbb{T}) \otimes \mathbb{R}$. The parabolic $\mathbb{P}_\varnothing$ determines a basis Δ and a set of positive roots to be denoted $\overline{\Phi}^+$.
- A family of morphisms $(U_\alpha)_{\alpha \in \Phi}$ such that for each ordering $\overline{\Phi}^+ = (\alpha_i)_{i=1,2,\dots,n}$, the natural map:
$$T \times \prod_{i=1,\dots,n} \mathbb{G}_a \xrightarrow{id \times \prod_{i=1,2,\dots n} U_{\alpha_i}} P_\varnothing$$
is an isomorphism.
- For each $\alpha \in \overline{\Phi}(\mathbb{T}, \mathbb{G})$, $n \in \mathbb{Z}$, we can define an affine root $\alpha_n : X_*(\mathbb{T}) \otimes \mathbb{R} \to \mathbb{R}$ by the affine functional given by
$$\alpha_n : v \otimes r \mapsto r < \alpha, v > +n$$

and we will denote the set of affine roots by $\Phi = \Phi(\mathbb{G}, \mathbb{T}, \nu)$.

With the above definition, we can define the standard apartment $\mathcal{A}$ as $X_*(\mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{R}$, and the hyperplane $H_\psi := \{v \in \mathcal{A} | \psi(v) = 0\}$ for $\psi \in \Phi$. These hyperplanes partition the apartment into facets, which are poly-simplicial. For any facet $F \subset \mathcal{A}$, we will denote by G_F the parahoric attached to the facet F, and its pro-unipotent radical will be denoted by G_F^+. For $x \in \mathcal{A}$, we will denote the parahoric subgroup as G_x, and it pro-unipotent radical as G_x^+. The quotient $\overline{G_{x,0}} := G_x / G_x^+$ is the group of $\mathbb{F}_q$-points of a connected reductive $\mathbb{F}_q$-group G_x.

One has the following theorem:

Theorem 1.1.1 (Goldman-Iwahori[*Goldman and Iwahori*(1963)], Iwahori-Matsumoto[*Iwahori and Matsumoto*(1965)], Bruhat-Tits[*Bruhat and Tits*(1967)])**.** *One can associate a polysimplicial complex $\mathcal{B}(G) = \mathcal{B}(\mathbb{G}, K)$ to G, called the Bruhat-Tits building of G, with the following properties:*

1. *There exists a proper continuous action of G on $\mathcal{B}(G)$ and $\forall g \in G$, the action of g on $\mathcal{B}(G)$ is a polysimplicial isomorphism.*
2. *$\mathcal{B}(G)$ is contractable and finite dimensional.*
3. *$\mathcal{B}(G)$ is locally finite(i.e. each simplex is adjacent to finitely many neighbor simplices).*

When there is no possibility for confusion, let $\mathcal{B}$ denote the Bruhat-Tits building of G, and we will refer readers to [BT1], [BT2] for further details. One of the more concrete definitions is to define $\mathcal{B}$ as the set $\mathcal{B}(\mathbb{G}, K) = G \times \mathcal{A} / \sim$, where the equivalence relation is given by $(g, x) \sim (h, y)$ if there exists $n \in N_G(T)$ such that $nx = y$ and $g^{-1}hn \in G_x$.

1.1.3 Moy-Prasad Filtration

Following [*Prasad and Moy*(1994)], [*Prasad and Moy*(1996)], for each point $x \in \mathcal{B}(G, K)$, we can associate a parahoric group G_x that is a subgroup of the stablizer of this point, we also denote the Lie algebra of G_x by $\mathfrak{g}_x = \mathfrak{g}_{x,0}$.

Moy and Prasad then introduced filtrations of these parahoric groups denoted $G_{x,r}$ and $\mathfrak{g}_{x,r}$ for $r \geq 0$. Note $G_{x,r} \trianglelefteq G_{x,0} := G_x$ and $\mathfrak{g}_{x,r} \trianglelefteq \mathfrak{g}_{x,0}$.

Moreover, since $G_{x,s} \subseteq G_{x,r}$ for $s > r$, we can define

$$G_{x,r^+} = \bigcup_{s>r} G_{x,s}, \text{ and } \mathfrak{g}_{x,r^+} = \bigcup_{s>r} \mathfrak{g}_{x,s},$$

and G_{x,r^+} is a normal subgroup of $G_{x,r}$. So it makes sense for us to consider quotient groups. Specifically, recall that for $r = 0$, we have that

$$\overline{G_{x,0}} = \overline{G_x} = G_{x,0}/G_{x,0^+}$$

is the group of $\mathbb{F}_q$ points of a a reductive $\mathbb{F}_q$-group G_x that is defined over the residue field. For $r > 0$, we have the quotient groups given by

$$G_{x,r}/G_{x,r^+} \cong \mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+} = \overline{\mathfrak{g}_{x,r}}.$$

Let (π, V) denote an irreducible admissible representation of G, and we denote the depth of the representation (π, V) by

$$\rho(\pi) := \inf\{r \in \mathbb{R}_{\geq 0} | V^{G_{x,r+}} \neq 0 \text{ for some } x \in \mathcal{B}(G)\}.$$

For $r \geq 0$, we define:

$$\mathfrak{g}_r = \bigcup_{x \in \mathcal{B}(G)} \mathfrak{g}_{x,r} \text{ and } G_r = \bigcup_{x \in \mathcal{B}(G)} G_{x,r}$$

These objects have been well studied, and they are G-domains, which means that they are G-invariant, open and closed subset of $\mathfrak{g}$ (or G, respectively). Futhermore, if we denote the set of nilpotent elements in $\mathfrak{g}$ as $\mathcal{N}$ and the set of unipotent elements in G as $\mathcal{U}$, we have

$$\mathfrak{g}_r = \bigcap_{x \in \mathcal{B}(G)} \mathfrak{g}_{x,r} + \mathcal{N} \text{ and } G_r = \bigcap_{x \in \mathcal{B}(G)} G_{x,r} \cdot \mathcal{U}.$$

Similarly, we also define

$$\mathfrak{g}_{r^+} = \bigcup_{x \in \mathcal{B}(G)} \mathfrak{g}_{x,r^+} \text{ and } G_{r^+} = \bigcup_{x \in \mathcal{B}(G)} G_{x,r^+}$$

1.2 History and Introduction

In Howe's paper [*Howe*(1973a)], he proposed two finiteness conjectures which are now what people call Howe's conjectures. One of them, which is Howe's conjecture for the Lie Algebra, states the following:

$$\dim_{\mathbb{C}} \operatorname{res}_{C_c(\mathfrak{g}/\mathcal{L})} J(\omega) < \infty.$$

Here ω is any compactly generated, invariant and closed subset of $\mathfrak{g}$, $J(\omega)$ is the space of invariant distributions on $\mathfrak{g}$ supported on ω, and $\operatorname{res}_{C_c(\mathfrak{g}/\mathcal{L})} J(\omega)$ denotes the restriction of $J(\omega)$ to the subspace of C_c^∞ consisting of locally constant functions that are translation invariant by the lattice $\mathcal{L}$ in $\mathfrak{g}$. Howe proved the conjecture for GL_n in the 1970s.

On the other hand, Harish-Chandra proved that for an admissible representation (π, V) of G, the distribution character Θ_π can be represented by a locally constant function F_π on the set of regular elements G^{reg}, which is to say,

$$\Theta_\pi(f) = \int_G f(g)F_\pi(g)dg, \ \forall f \in C_c^\infty(G^{reg}).$$

Note that throughout this paper, we will abuse notation and denote both the distribution character and the function representing it by Θ_π.

Suppose now we have $G = GL_n(K)$ or K has characteristic zero and we have a nice map e from $\mathfrak{g}$ to G in a neighborhood of zero (we can take the exponential map in the later case). Generalizing Howe's work for GL_n (in [*Howe*(1974)]), Harish-Chandra showed (Theorem 16.3 in [*Harish-Chandra et al.*(1999)*Harish-Chandra, DeBacker, and Sally*]) that Θ_π is a linear combination of Fourier transforms of nilpotent orbits in a sufficiently small neighborhood of zero in $\mathfrak{g}$, i.e. there exist constants $c_{\mathcal{O}}(\pi)$ for each nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$ respectively, such that

$$\Theta_\pi(\mathrm{e}(X)) = \sum_{\mathcal{O} \in \mathcal{O}(0)} c_{\mathcal{O}}(\pi)\hat{\mu}_{\mathcal{O}}(X).$$

This asymptotic expansion is valid for all regular semisimple element X in the Lie algebra of $\mathfrak{g}$ that are sufficiently close enough to 0. Moreover, the sum here is taken over the set of nilpotent orbits $\mathcal{O}$ in $\mathfrak{g}$, $\hat{\mu}_{\mathcal{O}}$ is the locally constant integrable function associated to the Fourier Transform of the orbital integral $\mu_{\mathcal{O}}$ and the functions $\hat{\mu}_{\mathcal{O}}$ are independent of the representation. For GL_n, the Fourier transform of $f \in C_c^\infty$ is defined by

$$\hat{f}(X) = \int_{\mathfrak{g}} f(Y)\Lambda(\mathrm{tr}(XY))\,dY$$

where dY is a fixed G-invariant measure on $\mathfrak{g}$.

This asymptotic expansion is referred to as the Harish-Chandra-Howe local expansion. Much information about the representation π can be obtained from the Harish-Chandra-Howe local character expansion, both qualitatively and quantatively. For example, when π is a discrete series representation, denoting the Steinberg Representation by St, we have the formal degree of π is given by $(-1)^r c_0(\pi)$, where $c_0(\pi) = \frac{(-1)^r \deg(\pi)}{\deg(St)}$ is the zero orbit coefficient in the local character expansion and r is the split rank of $\mathbb{G}$ over K([Harish-Chandra et al.(1999)Harish-Chandra, DeBacker, and Sally] and [Rogawski(1980)]).

At this point, a natural question arises. The expansion gives a qualitive result, and does not mention the size of the neighborhood in $\mathfrak{g}$ on which the local character expansion is valid.

Since many questions in harmonic analysis in G will need a quantitive result, it would be good to have more information about where the expansion is valid. A conjecture of Thomas Hales, Allen Moy, and Gopal Prasad (at the end of §1 in [Prasad and Moy(1994)]) states that the Harish-Chandra-Howe local character expansion should be valid on a region that depends on the depth, $\rho(\pi)$, of the representation π. For example, in the case where (π, V) is of depth zero, it means that there exists a parahoric subgroup $H \subseteq G$, such that $V^{H^+} \neq \{0\}$, where H^+ is the pro-unipotent radical of H.

Conjecture 1.2.1. *(Hales-Moy-Prasad)*

Given an irreducible admissible representation (π, V) of G, the Harish-Chandra-Howe local character expansion is valid on $G^{reg} \bigcap G_{\rho(\pi)^+}$.

Under some hypothesis on the group and the base field, J.-L. Walspurger ([Waldspurger(1993)], [Waldspurger(1995)]) proved this conjecture for integral depth representations of "classical unramified groups." Stephen DeBacker ([DeBacker(2002)]) verified the conjecture for general $\mathbb{G}$ under some hypotheses.

1.2.1 Notation and the analogous conjectures

We let R be the ring of integers of K and ϖ a uniformizer so that $\mathfrak{p} = \varpi R$ where $\mathfrak{p}$ is the prime ideal.

We realize $GL_n(K)$ as the group of $n \times n$ invertible matrices with entries in K. Moreover, we define A to be the diagonal subgroup of $GL_n(K)$.

We denote by $\mathfrak{g}$ the Lie algebra of $GL_n(K)$, which is the vector space of $n \times n$ matrices with the normal bracket operation, and we denote the nilpotent matrices in $\mathfrak{g}$ by $\mathcal{N}$. Then we have that $GL_n(K)$ acts on $\mathcal{N}$ and we let $\mathcal{O}(0)$ denote the corresponding set of nilpotent orbits.

Recall that $\mathcal{B}$ is the reduced Bruhat-Tits building of $GL_n(K)$, and let $\mathcal{A} \subseteq \mathcal{B}$ be the apartment associated to the torus A. Also, we associate to every $x \in \mathcal{B}$ and $r \in \mathbb{R}$ a lattice $\mathfrak{g}_{x,r}$, and for $r \geq 0$, we associated a compact open subgroup $G_{x,r}$ which, in the case of $GL(n,K)$, can be realized as $G_{x,0} = \mathfrak{g}_{x,0}^{\times}$ when $r = 0$ and $G_{x,r} = 1 + \mathfrak{g}_{x,r}$ for $r > 0$. Note that $\varpi \mathfrak{g}_{x,r} = \mathfrak{g}_{x,r+1}$ and $\mathfrak{g}_{x,s} \subsetneq \mathfrak{g}_{x,r}$ for $s > r$.

Conjecture 1.2.2. *Recall that*

$$\mathfrak{g}_r = \bigcup_{z \in B(G)} \mathfrak{g}_{z,r} = \bigcap_{y \in B(G)} (\mathfrak{g}_{y,r} + \mathcal{N}).$$

Consider $D_r = \sum_{z \in B(G)} C_c(\mathfrak{g}/\mathfrak{g}_{z,r})$*, we conjecture that*

$$res_{D_r} J(\mathfrak{g}_r) = res_{D_r} J(\mathcal{N}),$$

for $GL_n(K)$ *in all characteristic and for* $r = 0, 1/n, \ldots, (n-1)/n$*. Here* $J(\omega)$*, for*

$\omega = \mathcal{N}$ or $\mathfrak{g}_r$, is defined to be the invariant distributions with support in the closed G-invariant subset $\omega \subset G$.

Conjecture 1.2.3. *The stronger statement*

$$res_{D_r} \tilde{J}_{r^+} = res_{D_r} J(\mathcal{N})$$

holds. Here, we denote

$$\tilde{J}_{x,s,r^+} = \{T \in J(\mathfrak{g}) | \forall f \in C(\mathfrak{g}_{x,s}/\mathfrak{g}_{x,r^+}), \mathfrak{g}_{s^+} \cap supp(f) = \emptyset \implies T(f) = 0\},$$

and

$$\tilde{J}_{r^+} = \bigcap_{x \in \mathcal{B}} \bigcap_{s \leq r} \tilde{J}_{x,s,r^+}.$$

Note that $J(\mathcal{N}) \subseteq \tilde{J}_{r^+}$.

CHAPTER II

Moy-Prasad Filtrations

2.1 Generalized r-facets

Recall that in subsection 1.1.3, we introduced the Moy-Prasad filtration subgroups and lattices attached to a point $x \in \mathcal{B}(G)$ and $r \in \mathbb{R}_{\geq 0}$. We denoted them as

$$G_{x,r} \subseteq G_{x,0} = G_x \text{ and } \mathfrak{g}_{x,r} \subseteq \mathfrak{g}_{x,0},$$

and we also defined

$$G_{x,r^+} = \bigcup_{s>r} G_{x,s}, \text{ and } \mathfrak{g}_{x,r^+} = \bigcup_{s>r} \mathfrak{g}_{x,s}.$$

In this subsection we will partition the points in $\mathcal{B}$ into r-facets that are defined in [*DeBacker*(2004)] using the information above, and we will give some of their properties that will be used later.

Definition 2.1.1. For $x \in \mathcal{B}(G)$ and $r \in \mathbb{R}$, we define a generalized r-facet by

$$\begin{aligned} F^*(x) &:= \{y \in \mathcal{B}(G) | \mathfrak{g}_{x,r} = \mathfrak{g}_{y,r} \text{ and } \mathfrak{g}_{x,r^+} = \mathfrak{g}_{y,r^+}\} \\ &= \{y \in \mathcal{B}(G) | G_{x,|r|} = G_{y,|r|} \text{ and } G_{x,|r|^+} = G_{y,|r|^+}\}. \end{aligned}$$

and we denote

$$\mathcal{F}(r) := \{F^*(x) | x \in \mathcal{B}(G)\},$$

this is the set of generalized r-facets.

Moreover, since $\mathfrak{g}_{x,r}$, $\mathfrak{g}_{x,r^+}$, $G_{x,|r|}$, $G_{x,|r|^+}$ depend only on the facet F^* that contains x, if $x \in F^* \in \mathcal{F}(r)$ we can define

$$\mathfrak{g}_{F^*} := \mathfrak{g}_{x,r}, \ \mathfrak{g}^+_{F^*} := \mathfrak{g}_{x,r^+}$$

$$G_{F^*} := G_{x,|r|}, \ G^+_{F^*} := G_{x,|r|^+}$$

$$\mathfrak{g}_{F^*,-r} := \mathfrak{g}_{x,-r}, \ V_{F^*} := \mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+}$$

Remark 2.1.2. Following [DeBacker(2004)], the generalized r-facets satisfy the following properties:
Given $F_1^*, F_2^* \in \mathcal{F}(r)$:

1.
$$N_G(G_{F_1^*}) \cap N_G(G^+_{F_1^*}) = stab_G(F_1^*).$$
$$N_G(\mathfrak{g}_{F_1^*}) \cap N_G(\mathfrak{g}^+_{F_1^*}) = stab_G(F_1^*).$$

2. If $F_1^* \cap \overline{F_2^*} \neq \emptyset$, then
$$F_1^* \subset \overline{F_2^*}$$
.

3. If $F_1^* \subset \overline{F_2^*} \neq \emptyset$, then
$$\mathfrak{g}^+_{F_1^*} \subset \mathfrak{g}^+_{F_2^*} \subset \mathfrak{g}_{F_2^*} \subset \mathfrak{g}_{F_1^*}$$
and
$$G^+_{F_1^*} \subset G^+_{F_2^*} \subset G_{F_2^*} \subset G_{F_1^*}.$$

2.2 Chain orders and fundamental strata

2.2.1 Lattices

Let V be a K-vector space of finite dimension n. An R-lattice in V is a finitely generated R-submodule L of V such that the K-linear span KL of L is V.

In particular, we can find a K-basis $\{x_1, x_2, \ldots, x_n\}$ of V such that

$$L = \sum_{i=1}^{n} Rx_i.$$

And, moreover, an R-lattice L is a compact open subgroup of V and the set of all such lattices give a fundamental system of open neighbourhoods of 0 in V.

More generally, we can define a lattice in V to be a compact open subgroup of V, and for any lattice L, we can always find R-lattices L_1 and L_2 such that $L_1 \subseteq L \subseteq L_2$.

Note that in the case that this note is going to be dealing with, we have that $GL_n(R) \subseteq GL_n(K)$ is the unique, up to $GL_n(K)$-conjugacy, maximal compact open subgroup of G.

2.2.2 Lattice Chain

We now can define R-lattice chains. Take $V = K^n$, then we have that $G = GL_n(K) = Aut_K(V)$ and that $\mathfrak{g} = End_K(V) = \mathfrak{gl}_n(V)$.

Now define an R-lattice chain to be a non-empty set $\mathcal{L}$ consisting of R-lattices in V that is linearly ordered under inclusion and is stable under multiplication by $K^\times$.

Moreover, we can enumerate the elements of $\mathcal{L}$ such that

$$\mathcal{L} = \{L_i : i \in \mathbb{Z}\}, L_i \supsetneq L_{i+1},$$

and, furthermore, the stability condition implies that there exist an integer $e_{\mathcal{L}}$ such that $xL_i = L_{i+e_{\mathcal{L}} * v_K(x)}$ for all $i \in \mathbb{Z}$ and $x \in K^\times$.

Moreover, lattice chains are relatively easy to completely describe, as we can exhaust all the cases through direct computation.

We have the following proposition.

Proposition 2.2.1. *(See [Bushnell and Kutzko(1993)]) Let $\mathcal{L} = \{L_i\}_{i\in\mathbb{Z}}$ be an R–lattice chain in $V = K^n$. Defining $e_{\mathcal{L}}$ to be the integer such that $L_{i+e_{\mathcal{L}}} = L_i$ for any $L_i \in \mathcal{L}$, then we have that $e_{\mathcal{L}}$ can be taken from $1, 2, \ldots n$, and, moreover, we can categorize all the lattice chains up to a left action of $g \in G$ for some g, so that the equivalence classes bijectively correspond to the partitions of n.*

Example 2.2.2. In the case of GL_3, there are only three partitions up to left actions. Therefore, there are only three types of lattice chains of interest.

1. $[1, 1, 1]$.

   A lattice chain that corresponds to this partition is

$$\mathcal{L} = \{\ldots \subseteq \begin{bmatrix} \mathfrak{p} \\ \mathfrak{p} \\ \mathfrak{p} \end{bmatrix} \subseteq \begin{bmatrix} R \\ \mathfrak{p} \\ \mathfrak{p} \end{bmatrix} \subseteq \begin{bmatrix} R \\ R \\ \mathfrak{p} \end{bmatrix} \subseteq \begin{bmatrix} R \\ R \\ R \end{bmatrix} \subseteq \ldots\}$$

2. $[2, 1]$.

   A lattice chain that corresponds to this partition is

$$\mathcal{L} = \{\ldots \subseteq \begin{bmatrix} \mathfrak{p} \\ \mathfrak{p} \\ \mathfrak{p}^2 \end{bmatrix} \subseteq \begin{bmatrix} R \\ \mathfrak{p} \\ \mathfrak{p} \end{bmatrix} \subseteq \begin{bmatrix} R \\ R \\ \mathfrak{p} \end{bmatrix} \subseteq \ldots\}.$$

3. [3].

A lattice chain that corresponds to this partition is

$$\mathcal{L} = \{\ldots \subseteq \begin{bmatrix} \mathfrak{p} \\ \mathfrak{p} \\ \mathfrak{p} \end{bmatrix} \subseteq \begin{bmatrix} R \\ R \\ R \end{bmatrix} \subseteq \ldots\}.$$

Definition 2.2.3. By inclusion, we can form a partially ordered set with all chains of lattices in V, which we will denote as $\hat{\Delta}$.

Moreover, we will denote a frame of V to be a collection of $< v_i >$, the 1-dimensional K-subspaces spanning V, where $i \in \{1, 2, \ldots, n\}$. Moreover, we denote the subcomplex $\hat{\Sigma}(v_1, \ldots, v_n)$ to be the set of all chains with $(v_1, \ldots, v_n)$ as basis.

Note that it is well known that $\hat{\Delta}$ is an affine building of $GL_n(K)$, with the system of apartments given by $\hat{\Sigma}(v_1, \ldots, v_n)$ that runs through all frames of V, and one can associate the facets of the affine building with the lattices[*Garrett*(1997)][*Bruhat and Tits*(1984)].

Moreover, the stablizer group of a lattice chain $\mathcal{L}$ contains a maximal (in the stabilizer) parahoric subgroup, and we denote this parahoric by $\mathfrak{g}_{\mathcal{L}}$.

Up to conjugacy there are only three conjugacy classes of $\mathfrak{g}_{\mathcal{L}}$:

1. $\mathfrak{g}_{[3]} = \begin{bmatrix} R & R & R \\ R & R & R \\ R & R & R \end{bmatrix} \cap GL_3(K).$

2. $\mathfrak{g}_{[2,1]} = \begin{bmatrix} R & R & R \\ R & R & R \\ \mathfrak{p} & \mathfrak{p} & R \end{bmatrix} \cap GL_3(K).$

3. $\mathfrak{g}_{[1,1,1]} = \begin{bmatrix} R & R & R \\ \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & R \end{bmatrix} \cap GL_3(K).$

One can furthermore associate the barycenters of these facets of the affine building with the lattices and we can write down the Moy-Prasad filtration subgroups of these points, we denote these subgroups by $\mathfrak{g}_{\mathcal{L},r}$ and $\mathfrak{g}_{\mathcal{L},r^+}$.

	[3]	[2,1]	[1,1,1]
0	$\begin{bmatrix} R & R & R \\ R & R & R \\ R & R & R \end{bmatrix}$	$\begin{bmatrix} R & R & R \\ R & R & R \\ \mathfrak{p} & \mathfrak{p} & R \end{bmatrix}$	$\begin{bmatrix} R & R & R \\ \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & R \end{bmatrix}$
0+	$\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}$	$\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}$	$\begin{bmatrix} \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}$
1/3+			$\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix}$
1/2+		$\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}$	
2/3+			$\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}$

Table 2.1: $(\mathfrak{gl}_3)_{x,r}$, x in the standard alcove.

We exhibit the Moy-Prasad filtration lattices $\mathfrak{g}_{x,r}$ for GL_3, up to translation by $\tilde{w}$ and conjugation, in the following figure. The figure illustrate how the lattices $\mathfrak{g}_{x,r}$ vary as x and r vary for x near 0.

Since the apartment to which x belongs to can be interpreted as a copy of $\mathbb{R}^2$, we can thus present the figure as a picture in $\mathbb{R}^3$. Here the dotted hyperplanes are the graphs of $r = \alpha(x)$ for an affine root $\alpha \in \Psi$. We only present five cross sections here corresponding to $r = 0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$.

The hyperplanes here divide $\mathbb{R}^3$ into polyhedrons, and we color the polyhedrons that corresponds to different lattices with different colors.

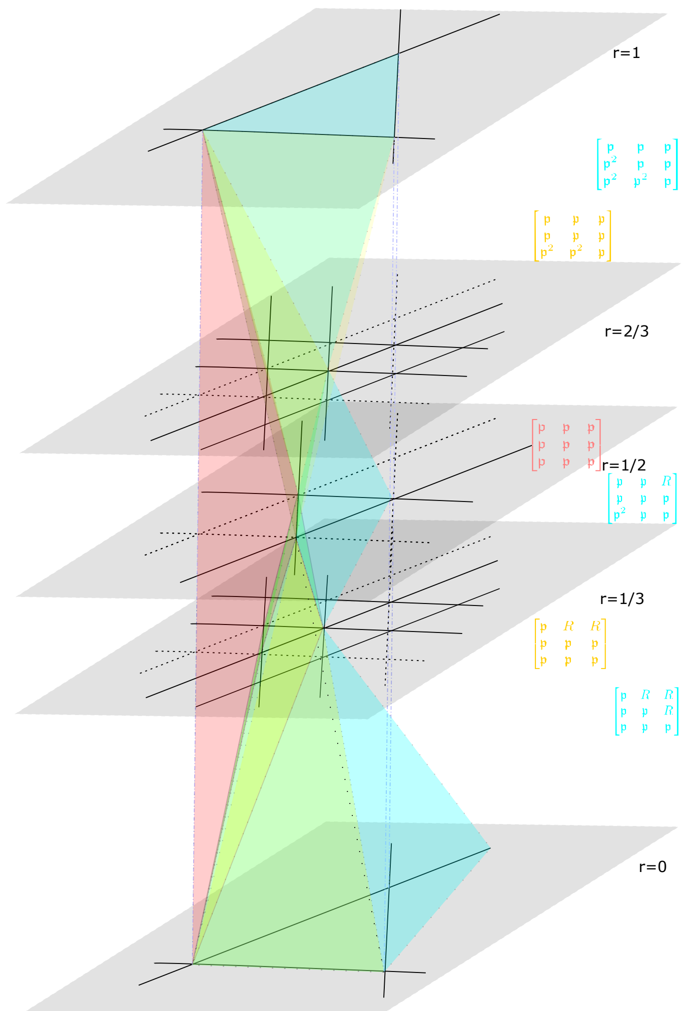

Figure 2.1: $(\mathfrak{gl}_3)_{x,r}$, x in the standard alcove.

2.2.3 Waldspurger type result

Recall that $D_{r^+} := \sum_{x\in\mathcal{B}(G)} C_c(\mathfrak{g}/\mathfrak{g}_{x,r^+})$. We define $D^r_{r^+} \subset D_{r^+}$ by $D^r_{r^+} := \sum_{x\in\mathcal{B}(G)} C_c(\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+})$.

The main result of this thesis is, either (1) under a specific conjecture or (2) in the case of GL_3:

Theorem 2.2.4. *For $GL_n(F)$, $\mathrm{res}_{D_{r^+}} \tilde{J}_{r^+} = \mathrm{res}_{D_{r^+}} J(\mathcal{N})$.*

More specifically, this result follows from the statements below:

1. $\mathrm{res}_{D_{r^+}} \tilde{J}_{r^+}$ *is completely determined by* $\mathrm{res}_{D^r_{r^+}} \tilde{J}_{r^+}$.
2. $\dim_{\mathbb{C}}(\mathrm{res}_{D_{r^+}})\tilde{J}_{r^+} \leq |\mathcal{O}(0)|$.

Proof. We assume that (1) and (2) hold to show the main statement holds.

Since (1) and (2) are true, we have

$$|\mathcal{O}(0)| = \dim_{\mathbb{C}}(\mathrm{res}\, D_{r^+} J(\mathcal{N})) \leq \dim_{\mathbb{C}} \mathrm{res}_{D_{r^+}} \tilde{J}_{r^+} \leq \dim_{\mathbb{C}} \mathrm{res}_{D^r_{r^+}} \tilde{J}_{r^+} \leq |\mathcal{O}(0)|.$$

where the first equality is true thanks to Howe [Howe(1974)]. The second equality follows from the fact that $J(\mathcal{N}) \subset \tilde{J}_{r^+}$ and the other two inequalities follow from (1) and (2) respectively.

Since $J(\mathcal{N}) \subset \tilde{J}_{r^+}$, this forces the inequalities to be equalities and we have our desired result: $\mathrm{res}_{D_{r^+}} \tilde{J}_{r^+} = \mathrm{res}_{D_{r^+}} J(\mathcal{N})$. □

CHAPTER III

Homogeneity result

3.1 Descent and recovery

3.1.1 Nice Elements

To prove Theorem 2.2.4, we first state and prove several results that we will use in the proof of the theorem.

Recall that our choice of G is GL_n and thus $\mathfrak{g} = \mathfrak{gl}_n = M_n(K)$.

Given $x \in \mathcal{B}(G)$ and $s \in \mathbb{R}$, we define $\mathfrak{g}_{x,s,s_+} = \mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+}$. Moreover, for $r \geq 0$, and $X \in \mathfrak{g}_{x,r,r_+} \cap \mathcal{N}$, there is a maximal K-split torus T and $\lambda \in X_*(T)$ such that $x \in \mathcal{A}(T)$ and ${}^{\overline{\lambda}(t)}\overline{X} \to 0$, where $\overline{X}$ is the image of X in $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+}$ and $\overline{\lambda}$ is the image of λ in $X_*(T_0/T_0^+) \subseteq X_*(\overline{G_{x,0}})$. Let $\Phi(T)$ denote the affine roots of G with respect to T and the valuation v. Recall that $\overline{\Phi}(T)$ denotes the roots of G with respect to T.

We define the set of affine roots of level r with respect of x, by $\Phi(T,r,x) = \{\psi \in \Phi(T) \mid \psi(x) = r\}$. Moreover, since $G = GL_n$, we can, without lost of generality, fix the torus T to be the group of diagonal elements in $GL_n(K)$. We then have that the set of simple roots is given by

$$\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\} \text{ where } \alpha_j(\mathrm{Diag}(a_1, \ldots, a_n)) = \frac{a_j}{a_{j+1}}.$$

Let x_0 be the vertex in $\mathcal{A}$ such that $G_{x_0,0} = GL_n(R)$. With this notation, a basis for $\Phi(T)$ is given by $\{\alpha_1 + 0, \alpha_2 + 0, \ldots, \alpha_{n-1} + 0, -h + 1\}$, where $h = \sum_{i=1}^{n-1} \alpha_i$ is the highest root. We denote the elements of this basis by $\psi_1, \psi_2, \ldots, \psi_n$ where

$$\psi_i = \begin{cases} \alpha_i + 0 & 0 < i < n \\ -h + 1 & i = n \end{cases},$$

and $\psi_i(x_0) = 0$ for $0 < i < n$ and $\psi_n(x_0) = 1$.

We denote by $\mathring{C}$ the open hull of the fundamental alcove in $GL(n)$ defined by this choice of simple basis. That is, $\mathring{C}$ is, as a set, $\{x \mid \psi_i(x) > 0,\ \forall i \in \{1, 2, \ldots, n\}\}$. Note that x_0 is in the closure of $\mathring{C}$.

Definition 3.1.1. For $W \in \mathfrak{g}_{x,r}$, denote by $\overline{W}$ the image of W in $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+}$, which we identify in the natural way with a vector subspace of $\mathrm{Mn}(\mathbb{F}_q)$. We can then define $r^*(W) := \overline{rank}(W) = rank_{\mathbb{F}_q}(\overline{W})$. Let also $n^*(W)$ denote the number of non-zero entries in the matrix representation of $\overline{W}$.

Remark 3.1.2. Note that if $X \in \mathfrak{g}_{x,r} \cap \mathcal{N}$, then $0 \leq r^*(X) < n$. Now we define an element $M \in \mathfrak{g}_{x,r}$ to be *nice* provided that $r^*(X) = n^*(X)$. For a nice matrix M, there is at most one nontrivial entry in each column or row of $\overline{M}$.

Proposition 3.1.3. *Suppose $X \in \mathcal{N} \cap g_{x,r}$, where x is in $\mathring{C}$. Then X is nice. In other words, $\overline{X}$ does not have more than one non-zero entry in any row or column.*

We need some preliminary work before proving Proposition 3.1.3.

Proposition 3.1.4. *If $x \in \mathring{C}$, then $|\Phi(T, x, r)| \leq n$.*

Proof. Suppose there are $n+1$ elements in $\Phi(T,x,r)$, which we denote by ϕ_i where i runs from 1 to $n+1$. Without loss of generality, we may assume $0 \le r < 1$.

Since a basis for $\Phi(T)$ is given by $\{\psi_i\}_{i=1,2,\dots n}$ and $0 \le r < 1$, we have $\phi_i = \sum_{j=1}^n \alpha_{i,j}\psi_j$ for some $\alpha_{i,j} \in \{0,1\}$.

Moreover, since ϕ_i has $\dot{\phi_i} \in \overline{\Phi}(T)$, we have that $\exists k_{a,i}, k_{b,i} \in \mathbb{Z}$ such that $1 \le k_{a,i} \le n$ and $k_{a,i} \le k_{b,i} \le 2n$, and $\alpha_{i,\overline{k_{a,i}}} = \alpha_{i,\overline{k_{a,i}+1}} = \dots = \alpha_{i,\overline{k_{b,i}}} = 1$, and $\alpha_{i,j} = 0$ otherwise. Here $\overline{k} \in \{1,2,3,\dots,n\}$ is congruent to k mod n.

For example, consider the case $\phi_i = \psi_1 + \psi_3$ for GL_3. Then $k_{l,i} = 3$ and $k_{r,i} = 4$ in this case.

To complete the proof, it will be enough to show that if there exist $\phi_{i_1}, \phi_{i_2} \in \{\phi \in \Phi(T) \mid \phi(x) = r\}$ with $k_{a,i_1} = k_{a,i_2}$, then $\phi_{i_1} = \phi_{i_2}$.

If $\phi_{i_1} = \phi_{i_2}$, there is nothing to prove. So suppose $\phi_{i_1} \ne \phi_{i_2}$.

We know that for for some $t \in \mathbb{Z} \cap [1,n]$, $\phi_{i_1} = \sum_{j=1}^n \alpha_{i_1,j}\psi_j$ and $\phi_{i_2} = \sum_{j=1}^n \alpha_{i_2,j}\psi_j$, with $\alpha_{i_1,t} \ne \alpha_{i_2,t}$. Since we are assuming $k_{b,i_1} \ne k_{b,i_2}$, without loss of generality, we can assume $k_{b,i_1} < k_{b,i_2}$, and then

$$\tilde{\phi} = \phi_{i_2} - \phi_{i_1} = \sum_{j=1}^n \beta_j \psi_j,$$

with $\beta_j \ge 0$ for all $j \in \{1,2,\dots\}$ and

$$\tilde{\phi}(x) = (\phi_{i_2} - \phi_{i_1})(x) = \phi_{i_2}(x) - \phi_{i_1}(x) = r - r = 0.$$

Since x is in $\mathring{C}$, we have $\psi_j(x) > 0$ for all $j \in \{1,\dots,n\}$ and therefore, $\beta_j = 0$ for all $j \in \{1,\dots,n\}$. □

Corollary 3.1.5. *If $\phi_1, \phi_2, \dots, \phi_m \in \Phi(T,r,x)$ are distinct, then $m \le n$.*

We actually know more than the above. Using the same idea as above, we prove:

Proposition 3.1.6. *Recall the definition of $k_{b,i}$ in the proof of Proposition 3.1.4. If $\exists i_1, i_2 \in \mathbb{Z}$, such that for $\phi_{i_1}, \phi_{i_2} \in \{\phi \in \Phi(T) \mid \phi(x) = r\}$ we have $k_{b,i_1} = k_{b,i_2}$, then $\phi_{i_1} = \phi i_2$.*

Proof. If $\phi_{i_1} = \phi_{i_2}$, there is nothing to prove. So suppose $\phi_{i_1} \neq \phi_{i_2}$, we know that $\alpha_{i_1,t} \neq \alpha_{i_2,t}$ for some $t \in \mathbb{Z} \cap [1,n]$, and thus since $k_{b,i_1} = k_{b,i_2}$, we conclude that $k_{a,i_1} \neq k_{a,i_2}$.

Without lost of generality, we can assume $k_{a,i_1} > k_{a,i_2}$, and then

$$\tilde{\phi} = \phi_{i_2} - \phi_{i_1} = \sum_{j=1}^{n} \beta_j \psi_j,$$

with $\beta_j \geq 0$ for all $j \in \{1,2,\ldots\}$ and

$$\tilde{\phi}(x) = (\phi_{i_2} - \phi_{i_1})(x) = \phi_{i_2}(x) - \phi_{i_1}(x) = r - r = 0.$$

Since $x \in \mathring{C}$, we have $\psi_j(x) > 0$ for all $j \in \{1,\ldots,n\}$ and therefore, $\beta_j = 0$ for all $j \in \{1,\ldots,n\}$. □

Corollary 3.1.7. *Suppose x is in $\mathring{C}$. Recall from the proof of Proposition 3.1.4 that for $\phi_i = \sum_{j=1}^{n} \alpha_{i,j}\psi_j \in \Phi(T,r,x)$, there exists $k_{a,i}, k_{b,i} \in \mathbb{Z}$ such that $1 \leq k_{a,i} \leq n$ and $k_{a,i} \leq k_{b,i} \leq 2n$, and $\alpha_{i,\overline{k_{a,i}}} = \alpha_{i,\overline{k_{a,i}+1}} = \ldots = \alpha_{i,\overline{k_{b,i}}} = 1$, and $\alpha_{i,j} = 0$ otherwise. Suppose $\Phi(T,r,x) = \{\phi_1, \phi_2, \ldots, \phi_m\}$ and to each $\phi_k \in \Phi(T,r,x)$ we denote the associated root group by R_{ϕ_k}. Then for all $X \in \mathfrak{g}_{x,r,r^+}$, we can write $X = \sum_{k=1}^{m} X_{\phi_k} \mod \mathfrak{g}_{x,r^+}$, where $X_{\phi_k} \in R_{\phi_k} \cap \mathfrak{g}_{x,r}$.*

The set $\{X_{\phi_k}\}$ defines a set of pair of integers by $(i_k, j_k) := (k_{a,k}, k_{b,k} \mod n)$. Given $k, k' \in \{1,2,\ldots,n\}$, we have

- $i_k = i_{k'} \implies \phi_k = \phi_{k'}$.

- $j_k = j_{k'} \implies \phi_k = \phi_{k'}$. $\square$

Example 3.1.8. Let us consider an example of the case where $x \in \mathring{C}$ is in a sufficiently small ϵ_1 neighborhood of the origin of $\mathcal{A}(T)$. We have $\mathfrak{g}_{x,0} = \mathfrak{g}_{x,\epsilon_2} = \begin{bmatrix} \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}$ and $\mathfrak{g}_{x,\epsilon_2^+} \neq \begin{bmatrix} \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}$, for some small $\epsilon_2 > 0$.

Since $x \in \mathring{C}$, there are only three possibilities for what $\mathfrak{g}_{x,\epsilon_2^+}$ can be. $\mathfrak{g}_{x,\epsilon_2^+} = \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}$ or $\mathfrak{g}_{x,\epsilon_2^+} = \begin{bmatrix} \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}$ or $\mathfrak{g}_{x,\epsilon_2^+} = \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}$, thus $\mathfrak{g}_{x,\epsilon_2,\epsilon_2^+} = \begin{bmatrix} 0 & R\setminus\mathfrak{p} & 0 \\ 0 & 0 & R\setminus\mathfrak{p} \\ 0 & 0 & 0 \end{bmatrix} + \mathfrak{g}_{x,\epsilon_2^+}$ or $\mathfrak{g}_{x,\epsilon_2,\epsilon_2^+} = \begin{bmatrix} 0 & R\setminus\mathfrak{p} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mathfrak{g}_{x,\epsilon_2^+}$ or $\mathfrak{g}_{x,\epsilon_2,\epsilon_2^+} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & R\setminus\mathfrak{p} \\ 0 & 0 & 0 \end{bmatrix} + \mathfrak{g}_{x,\epsilon_2^+}$ respectively.

Since $X \in \mathcal{N} \cap \mathfrak{g}_{x,\epsilon_2,\epsilon_2^+}$, we see that X will have to satisfies the conclusion of Corollary 3.1.7.

We can now prove Proposition 3.1.3:

Proof. By Corollary 3.1.7 we have that for $X \in \mathcal{N} \cap \mathfrak{g}_{x,r,r^+}$, where x is in $\mathring{C}$, X is a nice matrix, and we can thus assume our X is nice. $\square$

3.1.2 Descent and Recovery

Now that we have the above results, we may proceed dealing with only the *nice* elements X.

Proposition 3.1.9. *Suppose x is in $\mathring{C}$ and X is a nice nilpotent element $X = \sum_\phi X_\phi$ in $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+}$, with each $X_\phi \notin \mathfrak{g}_{x,r^+}$ for $\phi \in \Phi(T,r,x)$ and $X_\phi \in R_\phi$, the root group corresponding to ϕ.*

There's a way to define a one parameter subgroup λ of a torus T such that:

(a) *we have that* $<\lambda, \phi>= 2$ *for the* ϕ *occurring in the sum,*

(b) *Suppose* $\psi \in \Phi(T)$*, the set of affine roots of* G *with respect to* T *and* v*. If* $\psi(x) > r$*,* $Y_\psi \in \mathfrak{g}_\psi \smallsetminus \mathfrak{g}_{\psi^+}$ *and* $<\lambda, \psi>> 0$*, then there exists* $C \in \mathfrak{g}_{x,0^+}$ *such that* $[C, X] = Y_\psi$ mod $\mathfrak{g}_{\psi^+}$.

Note that this is equivalent to the statement that if $<\dot{\psi}, \lambda>> 0$*,* $\mathfrak{g}_\psi \smallsetminus \mathfrak{g}_{\psi^+} \subset [X, \mathfrak{g}_{x,0^+}] \smallsetminus \mathfrak{g}_{x,\psi^+} \subseteq \mathfrak{g}_{x,r^+}$.

Before starting the proof, I want to define a more general version of Jordan block notation that will be used in the proof of the main proposition.

Definition 3.1.10. Given X a *nice* nilpotent matrix in $M_n(K)$, we can define a Jordan chain as follows:

Consider the space K^n, where K is the base field. For $t \in \mathbb{N}$, we define

$$\ker^t(X) = \{v \in K^n | X^t(v) = 0\}$$

and we let d_t denote the dimension of $\ker^t(X)$. We know that $\ker^a(X) \subseteq \ker^b(X)$ if $a \leq b$ and we have a chain V of vector space maps of length $L \leq n$, where L is the minimum integer k such that $X^k = 0$:

$$0 \hookrightarrow \ker^1(X) \hookrightarrow \ker^2(X) \hookrightarrow \ldots \hookrightarrow \ker^{L-1}(X) \hookrightarrow \ker^L(X) \cong K^n.$$

Note that for $A \in \ker^a(X) \smallsetminus \ker^{a-1}(X)$, we have $X(A) \in \ker^{a-1}(X) \smallsetminus \ker^{a-2}(X)$. We look at sub-filtrations $\mathcal{U}$ of V of the form $\{0\} = U_0 \subsetneq U_1 \subsetneq \ldots \subsetneq U_s$ such that for $1 \leq t \leq s$ we have $U_t \subseteq \ker^t(X)$, $U_t/U_{t-1} \simeq K$, and $X(U_{t+1}) = U_t$. We also require that each U_t be spanned by a t element subset of the standard basis of K^n, denoted by $U_t = span <e_{w_t}, \ldots, e_{w_1}>$. We call such a sub-filtration $\mathcal{U}$ a Jordan chain provided that the only sub-filtration of V that contains $\mathcal{U}$ and has these properties is the chain $\mathcal{U}$ itself. We can associate to the sub-filtration $\mathcal{U}$ the ordered datum $[\mathcal{U}] = [w_1, w_2, \dots, w_s]$, note that the w_i are positive integers.

Remark 3.1.11. There will be at most finitely many Jordan chains associated to X, and thus we can enumerate them as $\mathcal{U}^1, \mathcal{U}^2, \mathcal{U}^3, \dots, \mathcal{U}^{M(X)}$.

With this definition, we can associate to any integer $N \in \{1, 2 \dots, n\}$ a unique Jordan Chain $\mathcal{U}^i$ indexed by $i \in I = \{1, 2, \dots, M(X)\} \subset \mathbb{N}$, with length s_i, and a unique $c \in \{1, 2, \dots, s_i\}$, such that $N = w_{i,c} \in [\mathcal{U}^i]$.

Note that if X happens to be in Jordan block form, then the set of Jordan chains is giving us exactly the same information that the Jordan blocks of X carry.

Moreover, we can associate a Young tableau to a Jordan Chain of the matrix X. A Young tableau is a Young diagram of size n filled with integers in the range $\{1, 2, \dots, n\}$. We fill the i-th row of the Young Tableau with datum $[\mathcal{U}^i]$ in order, as illustrated in the following example.

Example 3.1.12. Suppose

$$X = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Notice that, in this case, there are only two Jordan chains, namely $[\mathcal{U}^1] = [1, 2, 3, 4, 5, 6]$ and $[\mathcal{U}^2] = [7]$, which correspond to the two Jordan Blocks. We also have $w_{1,1} = 1$, $w_{1,2} = 2$, $w_{1,3} = 3$, $w_{1,4} = 4$, $w_{1,5} = 5$, $w_{1,6} = 6$ and $w_{2,1} = 7$.

Recall that every Jordan block can be denoted $J_{\lambda,d}$ where d is its dimension, and λ is the corresponding eigenvalue. We can denote the Jordan block that corresponds to $\mathcal{U}^1$ by $J_{0,6} = \begin{bmatrix} 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \end{bmatrix}$ and the one that corresponds to $\mathcal{U}^2$ by $J_{0,1} = [\,0\,]$.

In this case, the corresponding Young Tableau is:

1	2	3	4	5	6
7					

Example 3.1.13. Suppose

$$X = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \varpi & 0 & 0 & 0 \end{bmatrix}.$$

Notice that there is only one Jordan chain here, namely the chain attached to the datum $[1, 3, 5, 2, 4]$.

In this case, the corresponding Young Tableau is:

1	3	5	2	4

Example 3.1.14. Suppose

$$X = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \varpi & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Notice that, there are two Jordan chains here, namely $[\mathcal{U}^1] = [1,3,5]$ and $[\mathcal{U}^2] = [6,2,4]$. Note that we also have $[\mathcal{U}^2]$ is not $[4,2,6]$ as order matters in the definition, and $\ker^1(X) = span < e_1, e_6 >$. We also have:

$$w_{1,1} = 1,\ w_{1,2} = 3,\ w_{1,3} = 5, \text{ and } w_{2,1} = 6,\ w_{2,2} = 2,\ w_{2,3} = 4.$$

In this case, the corresponding Young Tableau is:

1	3	5
6	2	4

Definition 3.1.15. Fix X that is nice. From X we have Jordan chains $[\mathcal{U}^i] = [w_{i,1}, \dots, w_{i,s_i}]$ for $i \in I = \{1, 2, \dots M(X)\}$ with $\sum_{i=1}^{k} s_k = n$. Recall that for each $N \in \{1, \dots, n\}$, we can associate a unique $i \in I$ and a unique $c \in \{1, 2, \dots, s_i\}$ such that $N = w_{i,c}$.

We use this to define $\lambda = (\lambda_1, \dots, \lambda_n) : GL_1 \to Diag_n \hookrightarrow GL_n$ by

$$\lambda_j(t) = \lambda_{w_{i,c}}(t) = t^{s_i - 2c + 1}$$

for $0 \le j \le n$.

Example 3.1.16. Suppose

$$X = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \varpi & 0 & 0 & 0 \end{bmatrix}.$$

The λ that is associated to this element is

$$\lambda : t \mapsto \begin{bmatrix} t^4 & 0 & 0 & 0 & 0 \\ 0 & t^{-2} & 0 & 0 & 0 \\ 0 & 0 & t^2 & 0 & 0 \\ 0 & 0 & 0 & t^{-4} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

There is only one Jordan chain here, namely the one with associated datum $[1,3,5,2,4]$, and thus $\lambda(t) = \mathrm{Diag}[t^4, t^{-2}, t^2, t^{-4}, 0]$, $\sum_{k=1}^{M(X)} s_k = s_1 = n = 5$.

Example 3.1.17. Suppose

$$X = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \varpi & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The λ that is associated to this element is

$$\lambda : t \mapsto \begin{bmatrix} t^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & t^{-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & t^{-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & t^2 \end{bmatrix}.$$

Note that here there are two chains, namely $[1, 3, 5]$ and $[6, 2, 4]$, and we have that $n = 6 = s_1 + s_2 = 3 + 3$.

We now prove Proposition 3.1.9.

Proof. Since X is a nice nilpotent element, we can define λ using Definition 3.1.15. By the definition of λ, we have $< \lambda, \phi >= 2$. This proves (a).

Fix $\psi \in \Phi(T)$ such that $< \lambda, \psi >> r$.

Since X is a nice nilpotent element, we can denote $X = [X_{i,j}]$, where

$$X_{i,j} = \begin{cases} u\varpi^{v(\mathfrak{g}_{x,r,i,j})} & u \in R^\times \text{ and } i = w_{\tilde{i},c}, j = w_{\tilde{i},c+1} \\ & \text{ for some } \tilde{i} \in I = \{1, 2, \dots M(X)\} \text{ and } w_{i,c} \neq w_{i,c+1}, \\ 0 & \text{else.} \end{cases}$$

Here $v(\mathfrak{g}_{x,r,i,j}) := min(\{v(X_{i,j}) \mid X \in \mathfrak{g}_{x,r}\})$.

We will now prove (b): Take $C = \Delta_{i,j} = [a_{kl}]$, where

$$a_{kl} = \begin{cases} \varpi^{\lfloor \psi(x)-r \rfloor} & i = k, j = l \\ 0 & else \end{cases}$$

and we are going to prove that, fixing $Y_\psi \in \mathfrak{g}_\psi \setminus \mathfrak{g}_{\psi^+}$, we can find C such that $[C, X] = Y_\psi$.

Without loss of generality, this is equivalent to the following claim: For any $Q \in \{1, 2, \ldots, n\}$, let $\Lambda_Q = \mathrm{ord}_t(\lambda(t)_{Q,Q}) \in \mathbb{Z}$, the order of the Q-th diagonal entry of $\lambda(t)$. Now, for any $O, P \in \{1, 2, \ldots, n\}$, if $\Lambda_O - \Lambda_P > 0$, then there exists a matrix C that is in the form $\Delta_{i,j}$ such that $[C, X]_{\ell,m} \neq 0$ if and only if $\ell = O$ and $m = P$. Moreover, if the gradient of ψ corresponds to the root space indexed by (O, P), then $[C, X] \in \mathfrak{g}_\psi \setminus \mathfrak{g}_\psi^+$.

To prove this claim, we first prove two lemmas.

Lemma 3.1.18. *If $C = \Delta_{i,j}$, then using the notation $\delta_{f,g} = \begin{cases} 1 & f = g \\ 0 & else \end{cases}$, we have that*

$$[C, X]_{O,P} = \varpi^{\lfloor \psi(x)-r \rfloor}(\delta_{i,O} X_{j,P} - X_{O,i} \delta_{P,j})$$

for $1 \leq O, P \leq n$.

Proof. Since $C = \Delta_{i,j}$ from the definition of X we have,

$$\begin{aligned} [C, X]_{O,P} &= \sum_m (C_{O,m} X_{m,P} - X_{O,m} C_{m,P}) \\ &= C_{O,j} X_{j,P} - X_{O,i} C_{i,P}. \end{aligned}$$

The result follows. □

Recall that for any $N \in \{1, 2, \ldots, n\}$ we can find appropriate i, c, so that $N = w_{i,c}$. Now we denote i by i_N and c by c_N so we can denote $\Lambda_O = s_{i_O} - 2c_O + 1$ and $\Lambda_P = s_{i_P} - 2c_P + 1$, but then since $\Lambda_O > \Lambda_P$, we have that $s_{i_O} - 2c_O + 1 > s_{i_P} - 2c_P + 1 \iff s_{i_O} - s_{i_P} > 2(c_O - c_P)$. Notice that here we have $1 \le c_O \le s_{i_O}$ and $1 \le c_P \le s_{i_P}$.

Lemma 3.1.19. *Suppose* $1 \le O, P \le n$ *and* $\Lambda_O > \Lambda_P$. *We have* $O \ne w_{i_O, s_{i_O}}$ *or* $P \ne w_{i_P,1}$.

Proof. Suppose we have $O = w_{i_O, s_{i_O}}$ for some $i_O \in I$ and $P = w_{i_P,1}$.

Recall that for each $N \in \{1, \ldots, n\}$, we can associate $i \in I$, $c \in \{1, 2, \ldots, s_i\}$, such that $N = w_{i,c}$, and $\Lambda_N = \Lambda_{w_{i,c}} = s_i - 2c + 1$.

Therefore, we have $\Lambda_O = s_{i_O} - 2s_{i_O} + 1 = 1 - s_{i_O}$ and $\Lambda_P = s_{i_P} - 2 + 1 = s_{i_P} - 1$, and so we have $1 - s_{i_O} \le 0 \le s_{i_P} - 1$ since $s_{\tilde{i}} > 0$ for all $\tilde{i} \in I$, and thus a contradiction of the assumption $\Lambda_O > \Lambda_P$. □

We complete the proof of Proposition 3.1.9:

Note that since X is nice, to show that $[C, X]_{\ell,m} \ne 0$ if and only if $\ell = O$ and $m = P$, it is enough to show that $[C, X]_{O,P} \ne 0$. As a result of Lemma 3.1.19, either $X_{j,P} \ne 0$ or $X_{O,i} \ne 0$ for some i and j. So we can take $C = \Delta_{w_{i_O, c_O+1}, P}$ in the first case and $C = \Delta_{O, w_{i_P, c_P - 1}}$ in the second case. Since $X_{O,i} \ne 0$ in the first case and $X_{j,P} \ne 0$ in the second case, we conclude that $[C, X]_{O,P} \ne 0$. The element C has been chosen so that $[C, X]_{O,P}$ lies in the proper lattice. □

Example 3.1.20. Let's go back to one of the examples above:

$$X = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \varpi & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Recall that we can define the λ that is associated to this element by

$$\lambda : t \mapsto \begin{bmatrix} t^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & t^{-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & t^{-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & t^2 \end{bmatrix}.$$

And we will have the weight spaces:

$$\begin{bmatrix} 0 & 2 & 2 & 4 & 4 & 0 \\ -2 & 0 & 0 & 2 & 2 & -2 \\ -2 & 0 & 0 & 2 & 2 & -2 \\ -4 & -2 & -2 & 0 & 0 & -4 \\ -4 & -2 & -2 & 0 & 0 & -4 \\ 0 & 2 & 2 & 4 & 4 & 0 \end{bmatrix}$$

Recall that here are two chains, namely $[1,3,5]$ and $[6,2,4]$. Note that if

$$C = \begin{bmatrix} d_1 & a & b & 0 & 0 & ew^{-1} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_4 & c & 0 \\ 0 & 0 & 0 & d & d_5 & 0 \\ f & g & h & 0 & 0 & d_6 \end{bmatrix},$$

then we have

$$[C,X] = \begin{bmatrix} 0 & e & d_1 & a & b & 0 \\ 0 & 0 & 0 & -d_4 & -c & 0 \\ 0 & 0 & 0 & -d & -d_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_6 w & f & g & h & 0 \end{bmatrix}.$$

Thus, by choosing the entries of C in the appropriate lattices in K, we can recover $\mathfrak{g}_\psi$ for all $\psi \in \Phi$ for which $\psi(x) > r$ and $\langle \dot{\psi}, \lambda \rangle > 0$.

Now that we proved Proposition 3.1.9, we may now prove a version of descent and recovery and then prove the theorem.

We begin by unraveling the definitions.

Let us consider $f \in D_{r^+}$ with $f = \sum_i f_i$ where $f_i \in C_C(\mathfrak{g}/\mathfrak{g}_{x_i,r^+})$. Using the linearity of T, without loss of generality, we can assume $f \in C_C(\mathfrak{g}/\mathfrak{g}_{x,r^+})$ for some $x \in \mathcal{B}(G)$. We can then write $f = \sum_{\overline{Z} \in \mathfrak{g}/\mathfrak{g}_{x,0^+}} c_{\overline{Z}}[Z + \mathfrak{g}_{x,0^+}]$, where we have $[Z + \mathfrak{g}_{x,0^+}]$ is the characteristic function on the corresponding cosets and all but finitely many $c_{\overline{Z}} = 0$. Since T is linear, we can assume, without loss of generality, that $f := [Z + \mathfrak{g}_{x,r^+}]$.

Now that $Z + \mathfrak{g}_{r,r^+} \subset \mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+}$, for some $s < r$. Since we have $T \in \mathrm{res}_{D_0^+} \tilde{J}_{0^+}$, we have $T(f) = 0$ if $supp(f) \cap (\mathfrak{g}_{x,s^+} + \mathcal{N}) = \varnothing$, and thus $T(f) = 0$ unless $(Z + \mathfrak{g}_{r,s^+}) \cap \mathcal{N} \neq \varnothing$.

Therefore, without loss of generality, we may assume $Z = X + Y$ with $X \in \mathcal{N} \cap (\mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+})$ and $Y \in \mathfrak{g}_{x,s^+}$.

Theorem 3.1.21. *(Descent and Recovery) Given* $\mathrm{G} = GL_n$. *Recall that we have* $x \in B(G)$, *and* x *is in the interior of the standard alcove. Suppose* $s < r$. *Suppose* $X \in \mathcal{N} \cap (\mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+})$. *In the case where* X *is nice, there exists* $\lambda \in X_*^k(S)$ *such that for all sufficiently small* $\epsilon > 0$ *we have:*

1. $X \in \mathfrak{g}_{x+\epsilon\cdot\lambda,s^+}$ *for sufficiently small* $\epsilon > 0$.
2. $\forall Y' \in \mathfrak{g}_{x,s^+}$, $X + Y' + \mathfrak{g}_{x+\epsilon\lambda,r^+} \subset {}^{G_{x,r-s}}(X + Y' + \mathfrak{g}_{x,r^+})$.

Remark 3.1.22. Notice that we know that X is nice from Proposition 3.1.3, thus we know that the result holds for the interior of the fundamental alcove C.

Proof. From the definition of the Moy-Prasad lattices, for sufficiently small $\epsilon > 0$, we have $\mathfrak{g}_{x,s^+} \subset \mathfrak{g}_{x+\epsilon\cdot\lambda,s^+}$. Choose λ as in Proposition 3.1.9. Note that $X = \sum_\phi X_\phi$ and $< \lambda, \phi >= 2$ for all such ϕ, and we have

$$\phi(x + \epsilon \cdot \lambda) = \phi(x) + \epsilon \cdot < \dot{\phi}, \lambda >= s + 2\epsilon > s.$$

Therefore, we have $X \in \mathfrak{g}_{x+\epsilon\cdot\lambda,s^+}$ for $\epsilon > 0$.

For the second statement, we know by definition that ${}^{G_{x,(r-s)}}Y' \subset Y' + \mathfrak{g}_{x,r^+}$. We now take $C \in \mathfrak{g}_{x,r-s}$, $1 + C \in G_{x,r-s}$, and compute

$$\begin{aligned} {}^{1+C}X &= (1 + C)X(1 - C + C^2 - C^3 + \dots) \\ &= 1 + [C, X] + (X, C^2) + \dots \\ &\equiv 1 + [C, X] \mod \mathfrak{g}_{x,r^+} \end{aligned}$$

Thus it suffices to show $X + \mathfrak{g}_{x+\epsilon\cdot\lambda,r^+} \subset X + [X, \mathfrak{g}_{x,r-s}] + \mathfrak{g}_{x,r^+}$.

By Proposition 3.1.9, we have that if $\phi \in \Phi(x, r)$, and $\phi(x + \epsilon \cdot \lambda) = r + 2\epsilon > r \iff <\dot{\phi}, \lambda >> 0$, then $X + \mathfrak{g}_\phi \subset X + [X, \mathfrak{g}_{x,r-s}]$. Since $\phi(x + \epsilon\lambda) > r$ if and only if $\langle \phi, \lambda \rangle > 0$, we conclude

$$X + \mathfrak{g}_{x+\epsilon\cdot\lambda,r^+} \subset X + [X, \mathfrak{g}_{x,r-s}] + \mathfrak{g}_{x,r^+}$$

as needed. □

Conjecture 3.1.23. *Theorem 3.1.19 holds not only for x such that x is in the open hull of the standard alcove $\mathring{C}$, but also for x on the boundary of C.*

If Conjecture 3.1.23 holds, then we can replace Lemma 2.3.1 in [*DeBacker*(2002)] with this conjecture, and, with the same proof developed in section 2.4 in [*DeBacker*(2002)], one can show that $\mathrm{res}_{D_{r^+}} \tilde{J}_{r^+}$ is completely determined by $\mathrm{res}_{D^r_{r^+}} \tilde{J}_{r^+}$. Thus one can complete the proof of Theorem 2.2.4, which is the desired homogeneity result.

3.2 Homogeneity Results for GL3

We will prove the complete homogeneity result in the case of GL_3.

The study of homogeneity for GL_3 was initiated by Stephen DeBacker in his Ph.D. thesis [*DeBacker*(1997)]. However, few details were provided. I will fill in the details that are not in his thesis.

3.2.1 Conventions and notation for Section 3.2

We adopt the following notation and conventions.

We will consistently use u, v to denote elements that are from $R^\times$.

Let T_m denote the subgroup of T where each diagonal entry lies in $1 + \wp^m$. For $1 \leq i \neq j \leq 3$ and $m \in \mathbb{Z}$ we let $U_{ij}(\wp^m)$ denote the subgroup of GL_3 consisting of matrices with entries from $\wp^m$ in the ij^{th} entry, ones on the diagonal, and zeros in all other entries. For example, $U_{21}(\wp^m)$ is the group consisting of matrices of the form $\left[\begin{smallmatrix} 1 & 0 & 0 \\ z & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]$ with $z \in \mathfrak{p}^m$.

We use the following computation frequently during the process, and we record it here to reduce the length of the exposition.

For $a \in R$, $b \in R$, and $m \in \mathbb{Z}_{>0}$ we have

$$\begin{aligned} \frac{1 + \varpi^m a}{1 + \varpi^m b} \mod \varpi^{m+1} &= (1 + \varpi^m a)(1 - \varpi^m b) \mod \varpi^{m+1} \\ &= 1 + (a - b)\varpi^m \mod \varpi^{m+1}. \end{aligned}$$

3.2.2 Two results

There are two propositions that we need to prove before discussing the calculation phase of our proof.

Proposition 3.2.1. *For $G = GL_n$, $res_{D_r} \tilde{J}_{r^+} = res_{D_{r+\epsilon}} \tilde{J}_{r+\epsilon^+}$ and $res_{D^r_{r^+}} \tilde{J}_{r^+} = res_{D^{r+\epsilon}_{r+\epsilon^+}} \tilde{J}_{r+\epsilon^+}$ unless $r = \frac{k}{m}$, where $k, m, n \in \mathbb{Z}$ and $0 \leq k < m < n$, where in the case of GL_3, these r values are from $\{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}\}$.*

Proof. We want to first simplify the problem, so that the only points we have to consider are the barycenters of the facets.

Proposition 3.2.2. *Without loss of generality, $res_{D^r_{r^+}} \tilde{J}_{r^+}$ can be determined by looking at functions supported at barycenters.*

Proof. Let us first recall the definition of optimal points. To our fixed alcove $C \subset \mathcal{A}(T)$, we can associate a basis Σ_C of our affine roots Φ by

$$\Sigma_C = \{\psi \in \Phi \mid \forall x \in \overline{C}, 0 \leq \psi(x) \leq 1\}.$$

Given any nonempty subset $\Xi \subset \Sigma_C$, we define the function $f_\Xi : \overline{C} \to \mathbb{R}$, by $f_\Xi(x) = \min\{\psi(x) | \psi \in \Xi\}$. Then the set of optimal points in $\overline{C}$ are the set of points

$$\{x_\Xi \in \overline{C} \mid \varnothing \neq \Xi \subset \Sigma_C \text{ and } f_\Xi \text{ attains its maximum value at } x_\Xi\ \}.$$

Recall that by Moy and Prasad [*Prasad and Moy*(1996)], the set

$$\mathcal{O} := \{x \in \overline{C} \,|\, x \text{ is the barycenter of a facet}\}$$

is the set of optimal points in the case of type A_n. Moreover, for $z \in \overline{C}$ and $r \in \mathbb{R}$ there exists $x, y \in \mathcal{O}$, such that $\mathfrak{g}_{x,r} \subset \mathfrak{g}_{z,r} \subset \mathfrak{g}_{y,r}$ and $\mathfrak{g}_{x,r^+} \supset \mathfrak{g}_{z,r^+} \supset \mathfrak{g}_{y,r^+}$, which means $\mathfrak{g}_r = \bigcup_{x \in \mathcal{O}}({}^G\mathfrak{g}_{x,r})$ and $\mathfrak{g}_{r^+} = \bigcup_{x \in \mathcal{O}}({}^G\mathfrak{g}_{x,r^+})$. It also means

$$D_{r^+} = \sum_{z \in G \cdot \mathcal{O}} C_c(\mathfrak{g}/\mathfrak{g}_{z,r^+})$$

and

$$C(\mathfrak{g}_{z,r}/\mathfrak{g}_{z,r^+}) \subset C(\mathfrak{g}_{y,r}/\mathfrak{g}_{y,r^+}).$$

Since for GL_n we have that the set $\mathcal{O}$ is in bijective correspondence with the set of optimal points, we have that $res_{D^r_{r^+}} \tilde{J}_{r^+}$ and $res_{D_{r^+}} \tilde{J}_{r^+}$ can be determined by looking at functions supported at the barycenters of the facets of $\overline{C}$. □

Since $\mathfrak{g}_{x,r} \neq \mathfrak{g}_{x,r^+}$ for x that are barycenters of facets only if $r = \frac{k}{m}$, where $k, m, n \in \mathbb{Z}$ and $0 \leq k < m < n$, Proposition 3.2.1 follows. □

3.2.3 On depth $r = 0^+$

We will begin our discussion with the case where $r = 0$.

Proposition 3.2.3.

$$res_{D_{0^+}} \tilde{J}_{0^+} = res_{D^0_{0^+}} \tilde{J}_{0^+}.$$

Proof. Fix $f \in D_{0^+}$ with $f = \sum_i f_i$ such that $f_i \in C_C(\mathfrak{g}/\mathfrak{g}_{x_i,0^+})$. Since T is linear, without loss of generality we can assume $f \in C_C(\mathfrak{g}/\mathfrak{g}_{x,0^+})$ for some $x \in \mathcal{B}$. Therefore, we can write

$$f = \sum_{\overline{Z} \in \mathfrak{g}/\mathfrak{g}_{x,0^+}} c_{\overline{Z}} [Z + \mathfrak{g}_{x,0^+}],$$

where we have $[Z + \mathfrak{g}_{x,0^+}]$ is the characteristic function on the corresponding coset and all but finitely many $c_{\overline{Z}} = 0$. Since T is linear, we can assume, without loss of generality, that $f := [Z + \mathfrak{g}_{x,0^+}]$.

Choose s so that $Z + \mathfrak{g}_{x,0^+} \subset \mathfrak{g}_{x,s} \setminus \mathfrak{g}_{x,s^+}$. Since $T \in \tilde{J}_{0^+}$, we have $T(f) = 0$ if $supp(f) \cap (\mathfrak{g}_{x,s^+} + \mathcal{N}) = \varnothing$, and thus $T(f) = 0$ unless

$$(Z + \mathfrak{g}_{x,s^+}) \cap \mathcal{N} \neq \varnothing.$$

Therefore, without loss of generality, we may assume $Z = X + Y$ with $X \in \mathcal{N} \cap (\mathfrak{g}_{x,s} \setminus$

$\mathfrak{g}_{x,s^+})$ and $Y \in \mathfrak{g}_{x,s^+}$.

Up to conjugacy, we need only those three cases where f is invariant with respect to $\mathfrak{g}_{[1,1,1],0^+} = \begin{bmatrix} \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}$, $\mathfrak{g}_{[2,1],0^+} = \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}$, or $\mathfrak{g}_{[3],0^+} = \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}$. These three cases correspond to the three conjugacy classes of barycenters of facets.

We will examine these three cases here.

1. $\mathcal{L}_x = [3]$. In this case, we are looking at the coset $X + Y + \mathfrak{g}_{x,0^+}$, where x is the corner of C corresponding to $[3]$ as on page 17, $X \in \mathcal{N} \cap (\mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+})$ with $s < 0$, and $Y \in \mathfrak{g}_{x,s^+}$. Note that we have $\mathcal{N} \cap (\mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+}) = \varnothing$ unless $s = -m$, for $m \in \mathbb{Z}_{>0}$.

   Since T is G-invariant, after conjugating by $stab_{GL_3}(x)$ we can assume that X is $\varpi^{-m}\begin{bmatrix} 0 & u & 0 \\ 0 & 0 & v \\ 0 & 0 & 0 \end{bmatrix}$ or $\varpi^{-m}\begin{bmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

   (a) Suppose $X = \varpi^{-m}\begin{bmatrix} 0 & u & 0 \\ 0 & 0 & v \\ 0 & 0 & 0 \end{bmatrix}$. We have

$$\begin{aligned} T[X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}] &= \frac{1}{q^3} \sum_{\bar{t} \in T_m/T_{m+1}} T[{}^t(X+Y) + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}] \\ &= \frac{1}{q^2} \sum_{\alpha,\beta \in R/\mathfrak{p}} T[(X + \begin{bmatrix} 0 & \alpha & 0 \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{bmatrix} + Y) + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}] \\ &= \frac{1}{q^2} T[X + Y + \begin{bmatrix} \mathfrak{p} & R & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}] \\ &= \frac{1}{q^3} \sum_{\bar{u} \in U_{12}(\mathfrak{p}^m)/U_{12}(\mathfrak{p}^{m+1})} T[{}^u X + Y + \begin{bmatrix} \mathfrak{p} & R & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}] \\ &= \frac{1}{q^3} T[X + Y + \begin{bmatrix} \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]. \end{aligned}$$

   Note that $X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix} \subset \varpi^{-m}\begin{bmatrix} \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix} = \mathfrak{g}_{[1,1,1],s^+}$. Thus, we expressed T evaluated at $[X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]$ in terms of T evaluated at $[X + Y + \begin{bmatrix} \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]$, which has support closer to the origin with respect to the $[1,1,1]$-filtration than $[X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]$ had with respect to the $[3]$-filtration.

(b) Suppose $X = \varpi^{-m}\begin{bmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. We have

$$\begin{aligned}
T[X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}] &= \frac{1}{q} \sum_{\bar{u} \in U_{21}(\wp^m)/U_{21}(\wp^{m+1})} T[{}^u X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}] \\
&= \frac{1}{q} \sum_{\alpha \in R/\mathfrak{p}} T[X + Y + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}] \\
&= \frac{1}{q} T[X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}].
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
T[X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}] &= \frac{1}{q^4} \sum_{\bar{t} \in T_m/T_{m+1}} T[{}^t X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}] \\
&= \frac{1}{q^2} \sum_{\beta \in R/\mathfrak{p}} T[X + Y + \begin{bmatrix} 0 & 0 & \beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}] \\
&= \frac{1}{q^2} T[X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}].
\end{aligned}$$

Note that $X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix} \subset \varpi^{-m}\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix} = \mathfrak{g}_{[2,1],s^+}$. Thus, we expressed T evaluated at $[X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]$ in terms of T evaluated at $[X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]$, which has support closer to the origin with respect to the $[2,1]$-filtration than $[X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]$ had with respect to the $[3]$-filtration.

2. $\mathcal{L}_x = [2,1]$. In this case, we are looking at the coset $X + Y + \mathfrak{g}_{x,0^+}$, where x is the barycenter of an edge of C and $X \in \mathcal{N} \cap (\mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+})$ with $s < 0$ and $Y \in \mathfrak{g}_{x,s^+}$. Note that we have $\mathcal{N} \cap (\mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+}) = \varnothing$ unless $s = -m + 1/2$ or $s = -m$, for $m \in \mathbb{Z}_{>0}$.

   (a) Suppose $s = -m$. Since T is G-invariant, after conjugating by $stab_{GL_3}(x)$

we can assume that X is $\varpi^{-m}\left[\begin{smallmatrix}0&u&0\\0&0&0\\0&0&0\end{smallmatrix}\right]$. We have

$$\begin{aligned}T[X+Y+\begin{bmatrix}\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\end{bmatrix}] &= \frac{1}{q^3}\sum_{\bar{t}\in T_m/T_{m+1}} T[{}^tX+Y+\begin{bmatrix}\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\end{bmatrix}]\\ &= \frac{1}{q}\sum_{\alpha\in R/\mathfrak{p}} T[X+\begin{bmatrix}0&\alpha&0\\0&0&0\\0&0&0\end{bmatrix}+\begin{bmatrix}\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\end{bmatrix}]\\ &= \frac{1}{q}T[X+Y+\begin{bmatrix}\mathfrak{p}&R&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\end{bmatrix}].\end{aligned}$$

Note that $X+Y+\left[\begin{smallmatrix}\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\end{smallmatrix}\right] \subset \varpi^{-m}\left[\begin{smallmatrix}\mathfrak{p}&R&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\end{smallmatrix}\right] = \mathfrak{g}_{[1,1,1],s^+}$. Thus, we expressed T evaluated at $[X+Y+\left[\begin{smallmatrix}\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\end{smallmatrix}\right]]$ in terms of T evaluated at $[X+Y+\left[\begin{smallmatrix}\mathfrak{p}&R&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\end{smallmatrix}\right]]$, which has support closer to the origin with respect to the $[1,1,1]$-filtration than $[X+Y+\left[\begin{smallmatrix}\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\end{smallmatrix}\right]]$ had with respect to the $[2,1]$-filtration.

(b) Suppose $s=-m+1/2$. Since T is G-invariant, after conjugating by $stab_{GL_3}(x)$ we can assume that X is $\varpi^{-m}\left[\begin{smallmatrix}0&0&u\\0&0&0\\0&0&0\end{smallmatrix}\right]$ or $\varpi^{-m}\left[\begin{smallmatrix}0&0&u\\0&0&0\\0&v\varpi&0\end{smallmatrix}\right]$.

i. Assume that X is $\varpi^{-m}\left[\begin{smallmatrix}0&0&u\\0&0&0\\0&0&0\end{smallmatrix}\right]$. We then have

$$\begin{aligned}T[X+Y+\begin{bmatrix}\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\end{bmatrix}] &= \frac{1}{q}\sum_{\bar{u}\in U_{32}(\wp^m)/U_{32}(\wp^{m+1})} T[{}^uX+Y+\begin{bmatrix}\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\end{bmatrix}]\\ &= \frac{1}{q}T[X+Y+\begin{bmatrix}\mathfrak{p}&R&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\end{bmatrix}].\end{aligned}$$

Note that $X+Y+\left[\begin{smallmatrix}\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\end{smallmatrix}\right] \subset \varpi^{-m}\left[\begin{smallmatrix}\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\\\mathfrak{p}^2&\mathfrak{p}&\mathfrak{p}\end{smallmatrix}\right] = \mathfrak{g}_{[1,1,1],s^+}$. Thus, we expressed T evaluated at $[X+Y+\left[\begin{smallmatrix}\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\end{smallmatrix}\right]]$ in terms of T evaluated at $[X+Y+\left[\begin{smallmatrix}\mathfrak{p}&R&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\end{smallmatrix}\right]]$, which has support closer to the origin with respect to the $[1,1,1]$-filtration than $[X+Y+\left[\begin{smallmatrix}\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\end{smallmatrix}\right]]$ had with respect to the $[2,1]$-filtration.

ii. Assume that X is $\varpi^{-m}\begin{bmatrix}0&0&u\\0&0&0\\0&v\varpi&0\end{bmatrix}$. We have

$$
\begin{aligned}
T[X+Y+\begin{bmatrix}\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\end{bmatrix}] =& \frac{1}{q}\sum_{\bar{u}\in U_{32}(\wp^m)/U_{32}(\wp^{m+1})} T[{}^{u}X+Y+\begin{bmatrix}\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\end{bmatrix}]\\
=& \frac{1}{q}T[X+Y+\begin{bmatrix}\mathfrak{p}&R&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\end{bmatrix}].
\end{aligned}
$$

Note that $X+Y+\begin{bmatrix}\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\end{bmatrix}\subset\varpi^{-m}\begin{bmatrix}\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\\\mathfrak{p}^2&\mathfrak{p}&\mathfrak{p}\end{bmatrix}=\mathfrak{g}_{[1,1,1],s^+}$. Thus, we expressed T evaluated at $[X+Y+\begin{bmatrix}\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\end{bmatrix}]$ in terms of T evaluated at $[X+Y+\begin{bmatrix}\mathfrak{p}&R&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\end{bmatrix}]$, which has support closer to the origin with respect to the $[1,1,1]$-filtration than $[X+Y+\begin{bmatrix}\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\end{bmatrix}]$ had with respect to the $[2,1]$-filtration.

3. $\mathcal{L}_x=[1,1,1]$. In this case we are looking at the coset $X+Y+\mathfrak{g}_{x,0^+}$, where x is the barycenter of C and $X\in\mathcal{N}\cap(\mathfrak{g}_{x,s}\smallsetminus\mathfrak{g}_{x,s^+})$ with $s<0$ and $Y\in\mathfrak{g}_{x,s^+}$. Note that $\mathcal{N}\cap(\mathfrak{g}_{x,s}\smallsetminus\mathfrak{g}_{x,s^+})=\varnothing$ unless $s=1/3-m$ or $s=2/3-m$, for $m\in\mathbb{Z}_{>0}$.

   (a) Suppose $s=1/3-m$. Since T is G-invariant, after conjugating by $stab_{GL_3}(x)$ we can assume that X is $\varpi^{-m}\begin{bmatrix}0&u&0\\0&0&0\\0&0&0\end{bmatrix}$ or $\varpi^{-m}\begin{bmatrix}0&u&0\\0&0&v\\0&0&0\end{bmatrix}$. In both cases we have

$$
T[X+Y+\begin{bmatrix}\mathfrak{p}&R&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\end{bmatrix}]=\sum_{\alpha\in\mathfrak{p}/\mathfrak{p}^2}T[X+Y+\begin{bmatrix}0&0&0\\0&0&0\\\alpha&0&0\end{bmatrix}+\begin{bmatrix}\mathfrak{p}&R&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}^2&\mathfrak{p}&\mathfrak{p}\end{bmatrix}].
$$

   Note that $\begin{bmatrix}\mathfrak{p}&R&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}^2&\mathfrak{p}&\mathfrak{p}\end{bmatrix}=g\begin{bmatrix}\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\end{bmatrix}g^{-1}$ where $g=\begin{bmatrix}0&1&0\\0&0&1\\\varpi&0&0\end{bmatrix}$. Note that $\begin{bmatrix}0&0&0\\0&0&0\\\alpha&0&0\end{bmatrix}\in\begin{bmatrix}R&R&\mathfrak{p}^{-1}\\\mathfrak{p}&R&R\\\mathfrak{p}&\mathfrak{p}&R\end{bmatrix}$ while $Y\in\begin{bmatrix}\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\\\mathfrak{p}^2&\mathfrak{p}&\mathfrak{p}\end{bmatrix}$ and $X\in\varpi^{-m}\begin{bmatrix}\mathfrak{p}&R&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}^2&\mathfrak{p}&\mathfrak{p}\end{bmatrix}$. Therefore, $X+Y+\begin{bmatrix}0&0&0\\0&0&0\\\alpha&0&0\end{bmatrix}\subset\varpi^{-m}\begin{bmatrix}\mathfrak{p}&R&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}^2&\mathfrak{p}&\mathfrak{p}\end{bmatrix}={}^{g}\mathfrak{g}_{[2,1],s^+}$. Thus, we expressed T evaluated at $[X+Y+\begin{bmatrix}\mathfrak{p}&R&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\end{bmatrix}]$ in terms of T evaluated at $[X+Y+\begin{bmatrix}0&0&0\\0&0&0\\\alpha&0&0\end{bmatrix}+\begin{bmatrix}\mathfrak{p}&R&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}^2&\mathfrak{p}&\mathfrak{p}\end{bmatrix}]$, which has support closer to the origin with respect to the $g\cdot[2,1]$-filtration than $[X+Y+\begin{bmatrix}\mathfrak{p}&R&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\end{bmatrix}]$ had with respect to the $[1,1,1]$-filtration.

   (b) Suppose $s=2/3-m$. Since T is G-invariant, after conjugating by $stab_{GL_3}(x)$

we can assume that X is $\varpi^{-m}\begin{bmatrix} 0 & 0 & 0 \\ u\varpi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ or $\varpi^{-m}\begin{bmatrix} 0 & 0 & u \\ v\varpi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. In both cases we have

$$T[X+Y+\begin{bmatrix} \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}] = \frac{1}{q^3}\sum_{\alpha,\beta,\gamma\in R/\mathfrak{p}} T[X+\begin{bmatrix} 0 & \alpha & 0 \\ 0 & 0 & 0 \\ \beta\varpi & \gamma\varpi & 0 \end{bmatrix}+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}].$$

Note that $\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix} = g\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}g^{-1}$ where $g = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varpi & 0 & 0 \end{bmatrix}$. Moreover $\begin{bmatrix} 0 & \alpha & 0 \\ 0 & 0 & 0 \\ \beta\varpi & \gamma\varpi & 0 \end{bmatrix} \in \begin{bmatrix} R & R & \mathfrak{p}^{-1} \\ R & R & \mathfrak{p}^{-1} \\ \mathfrak{p} & \mathfrak{p} & R \end{bmatrix}$, while $Y \in \varpi^{-m}\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix} \subseteq \varpi^{-m}\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}$ and $X \in \varpi^{-m}\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}$. Therefore, $X+Y+\begin{bmatrix} 0 & \alpha & 0 \\ 0 & 0 & 0 \\ \beta\varpi & \gamma\varpi & 0 \end{bmatrix} \subset \varpi^{-m}\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix} = {}^{g}\mathfrak{g}_{[3],s^+}$. Thus, we expressed T evaluated at $[X+Y+\begin{bmatrix} \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]$ in terms of T evaluated at $[X+Y+\begin{bmatrix} 0 & \alpha & 0 \\ 0 & 0 & 0 \\ \beta\varpi & \gamma\varpi & 0 \end{bmatrix}+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]$, which has support closer to the origin with respect to the $g\cdot[3]$-filtration than $[X+Y+\begin{bmatrix} \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]$ had with respect to the $[1,1,1]$-filtration. □

3.2.4 On depth $r = 1/3^+$

Proposition 3.2.4. *For GL_3, $res_{D_{1/3^+}}\tilde{J}_{1/3^+} = res_{D_{1/3^+}}J(\mathcal{N})$.*

Proof. Fix $f \in D_{1/3^+}$ with $f = \sum_i f_i$ such that $f_i \in C_C(\mathfrak{g}/\mathfrak{g}_{x_i,1/3^+})$. Since T is linear, without loss of generality we can assume $f \in C_C(\mathfrak{g}/\mathfrak{g}_{x,1/3^+})$ for some $x \in \mathcal{B}$.

Therefore, we can write

$$f = \sum_{\overline{Z}\in\mathfrak{g}/\mathfrak{g}_{x,1/3^+}} c_{\overline{Z}}[Z+\mathfrak{g}_{x,1/3^+}],$$

where $[Z+\mathfrak{g}_{x,1/3^+}]$ is the characteristic function on the corresponding cosets and all but finitely many $c_{\overline{Z}} = 0$. Since T is linear, we can assume, without loss of generality, that $f := [Z+\mathfrak{g}_{x,1/3^+}]$.

Choose s so that $Z+\mathfrak{g}_{x,1/3^+} \subset \mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+}$. Since $T \in res_{D^+_{1/3}}\tilde{J}_{1/3^+}$, $T(f) = 0$ if $supp(f) \cap$

$(\mathfrak{g}_{x,s^+} + \mathcal{N}) = \varnothing$, and thus $T(f) = 0$ unless

$$(Z + \mathfrak{g}_{x,s^+}) \cap \mathcal{N} \neq \varnothing.$$

Therefore, without loss of generality we may assume $Z = X + Y$ with $X \in \mathcal{N} \cap (\mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+})$ and $Y \in \mathfrak{g}_{\mathfrak{g}_{x,s^+}}$.

In addition to the three cases that correspond to the conjugacy classes of barycenters of facets, we need to take up a fourth case (indexed by z which is introduced in (3(a)i) on page 46 below).

We now examine the four cases.

1. $\mathcal{L}_x = [3]$. In this case, we are looking at the coset $X + Y + \mathfrak{g}_{x,1/3^+}$, where x is at a vertex of C and $X \in \mathcal{N} \cap (\mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+})$ with $s < \frac{1}{3}$ and $Y \in \mathfrak{g}_{x,s^+}$. Note that we have $\mathcal{N} \cap (\mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+}) = \varnothing$ unless $s = -m$, for $m \in \mathbb{Z}_{\geq 0}$.

   Since T is G-invariant, after conjugating by $stab_{GL_3}(x)$ we can assume that X is $\varpi^{-m}\left[\begin{smallmatrix} 0 & u & 0 \\ 0 & 0 & v \\ 0 & 0 & 0 \end{smallmatrix}\right]$ or $\varpi^{-m}\left[\begin{smallmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$.

   (a) $X = \varpi^{-m}\left[\begin{smallmatrix} 0 & u & 0 \\ 0 & 0 & v \\ 0 & 0 & 0 \end{smallmatrix}\right]$. We have

$$\begin{aligned} T[X + Y + \left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]] &= \frac{1}{q^3} \sum_{\bar{t} \in T_m / T_{m+1}} T[{}^t(X + Y) + \left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]] \\ &= \frac{1}{q^2} \sum_{\alpha, \beta \in R/\mathfrak{p}} T[(X + \left[\begin{smallmatrix} 0 & \alpha & 0 \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{smallmatrix}\right] + Y) + \left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]] \\ &= \frac{1}{q^2} T[X + Y + \left[\begin{smallmatrix} \mathfrak{p} & R & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]] \\ &= \frac{1}{q^3} \sum_{\bar{u} \in U_{12}(\mathfrak{p}^m)/U_{12}(\mathfrak{p}^{m+1})} T[{}^u X + Y + \left[\begin{smallmatrix} \mathfrak{p} & R & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]] \\ &= \frac{1}{q^3} T[X + Y + \left[\begin{smallmatrix} \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]. \end{aligned}$$

Note that $X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix} \subset \varpi^{-m} \begin{bmatrix} \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix} = \mathfrak{g}_{[1,1,1],s^+}$. Thus, we expressed T evaluated at $[X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]$ in terms of T evaluated at $[X + Y + \begin{bmatrix} \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]$, which has support closer to the origin with respect to the $[1,1,1]$-filtration than $[X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]$ had with respect to the $[3]$-filtration.

(b) $X = \varpi^{-m} \begin{bmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. We have

$$\begin{aligned} T([X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]) &= \frac{1}{q^3} \sum_{\bar{t} \in T_m / T_{m+1}} T([{}^t X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}] \\ &= \frac{1}{q} \sum_{\alpha \in R/\mathfrak{p}} T[X + Y + \begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]) \\ &= \frac{1}{q} T([X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]) \\ &= \frac{1}{q} \sum_{\beta \in \mathfrak{p}/\mathfrak{p}^2} T([X + Y + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \beta & 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]). \end{aligned}$$

Note that $X + Y + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \beta & 0 & 0 \end{bmatrix} \subset \varpi^{-m} \begin{bmatrix} \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix} = \mathfrak{g}_{[1,1,1],s^+}$. Thus, we expressed T evaluated at $[X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]$ in terms of T evaluated at $[X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]$, which has support closer to the origin with respect to the $[1,1,1]$-filtration than $[X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]$ had with respect to the $[3]$-filtration.

2. $\mathcal{L}_x = [2,1]$.

In this case, we are looking at the coset $X + Y + \mathfrak{g}_{x,1/3^+}$, where x is the barycenter of an edge of C, $X \in \mathcal{N} \cap (\mathfrak{g}_{x,s} \setminus \mathfrak{g}_{x,s^+})$ with $s < \frac{1}{3}$, and $Y \in \mathfrak{g}_{x,s^+}$.

Note that we have $\mathcal{N} \cap (\mathfrak{g}_{x,s} \setminus \mathfrak{g}_{x,s^+}) = \varnothing$ unless $s = -m + 1/2$ or $s = -m$, for $m \in \mathbb{Z}_{\geq 0}$.

(a) Suppose $s = -m$. Since T is G-invariant, thus after conjugating by $stab_{GL_3}(x)$ we can assume that X is $\varpi^{-m} \begin{bmatrix} 0 & u & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then:

$$T(X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}) = \frac{1}{q^2} \sum_{\alpha, \beta \in R/\mathfrak{p}} T([X + Y + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ \varpi\beta & 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]).$$

Note that $Y \in \varpi^{-m}\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right] \subseteq \varpi^{-m}\left[\begin{smallmatrix} \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]$ and $X \in \varpi^{-m}\left[\begin{smallmatrix} \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]$ as well. Thus $X + Y + \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ \beta\varpi & 0 & 0 \end{smallmatrix}\right] \subset \varpi^{-m}\left[\begin{smallmatrix} \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right] = \mathfrak{g}_{[1,1,1],s^+}$. Thus, we expressed T evaluated at $[X+Y+\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]$ in terms of T evaluated at $[X+Y+\left[\begin{smallmatrix} \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]$, which has support closer to the origin with respect to the $[1,1,1]$-filtration than $[X+Y+\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]$ had with respect to the $[2,1]$-filtration.

(b) Suppose $s = -m+1/2$. Since T is G-invariant, after conjugating by $stab_{GL_3}(x)$ we can assume that X is $\varpi^{-m}\left[\begin{smallmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$ or $\varpi^{-m}\left[\begin{smallmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ 0 & v\varpi & 0 \end{smallmatrix}\right]$. We have

$$T(X+Y+\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]) = \frac{1}{q^2}\sum_{\alpha,\beta\in R/\mathfrak{p}} T[X+Y+\left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ \beta\varpi & 0 & 0 \end{smallmatrix}\right]+\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]].$$

Note that $Y \in \varpi^{-m}\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{smallmatrix}\right] \subseteq \varpi^{-m}\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]$ and $X \in \varpi^{-m}\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]$ as well. Thus $X + Y + \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ \beta\varpi & 0 & 0 \end{smallmatrix}\right] \subset \varpi^{-m}\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right] = \mathfrak{g}_{[1,1,1],s^+}$. Thus, we expressed T evaluated at $[X+Y+\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]$ in terms of T evaluated at $[X+Y+\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]$, which has support closer to the origin with respect to the $[1,1,1]$-filtration than $[X+Y+\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]$ had with respect to the $[2,1]$-filtration.

3. $\mathcal{L}_x = [1,1,1]$. In this case, we are looking at the coset $X + Y + \mathfrak{g}_{x,1/3^+}$, where x is the barycenter of C, $X \in \mathcal{N} \cap (\mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+})$ with $s < \frac{1}{3}$, and $Y \in \mathfrak{g}_{x,s^+}$. Note that we have $\mathcal{N} \cap (\mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+}) = \varnothing$ unless $s = 1/3 - m$ or $s = 2/3 - m$, for $m \in \mathbb{Z}_{\geq 0}$.

(a) Suppose $s = 1/3-m$. Since T is G-invariant, after conjugating by $stab_{GL_3}(x)$ we can assume that X is $\varpi^{-m}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ u\varpi & 0 & 0 \end{smallmatrix}\right]$ or $\varpi^{-m}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & u \\ v\varpi & 0 & 0 \end{smallmatrix}\right]$.

i. $X = \varpi^{-m}\left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ u\varpi & 0 & 0 \end{smallmatrix}\right]$. We have

$$\begin{aligned} T[X+Y+\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]] &= \frac{1}{q^3}\sum_{\bar{t}\in T_m/T_{m+1}} T[{}^tX+Y+\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]] \\ &= \frac{1}{q}T[X+Y+\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]. \end{aligned}$$

There is a unique point, call it z, on the geodesic between the barycenter of C and the vertex corresponding to $[3]$ such that

$$\mathfrak{g}_{z,2/3} = \mathfrak{g}_{z,1/2^+} \subsetneq \mathfrak{g}_{z,1/2} = \mathfrak{g}_{z,1/3^+}$$

and

$$\mathfrak{g}_{z,1/2^+} = \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix}$$

while

$$\mathfrak{g}_{z,1/2} = \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}.$$

Note that $Y \in \varpi^{-m}\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix} \subseteq \varpi^{-m}\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}$, and we also have $X = \varpi^{-m}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ u\varpi & 0 & 0 \end{bmatrix} \in \varpi^{-m}\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}$. Thus $X + Y \in \varpi^{-m}\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix} = \mathfrak{g}_{z,s^+} = \mathfrak{g}_{z,1/2-m}$. Thus, we expressed T evaluated at $[X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]$ in terms of T evaluated at $[X + Y + \mathfrak{g}_{z,1/3^+}]$, which has support closer to the origin with respect to the z-filtration than $[X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]$ had with respect to the $[1,1,1]$-filtration.

ii. $X = \varpi^{-m}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & u \\ v\varpi & 0 & 0 \end{bmatrix}$. We have

$$\begin{aligned} T[X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix}] &= \frac{1}{q^3} \sum_{\bar{t} \in T_m/T_{m+1}} T[{}^t X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix}] \\ &= \frac{1}{q} T[X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]. \end{aligned}$$

Thus $X+Y \subset \varpi^{-m}\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix} = \mathfrak{g}_{[2,1],s^+}$. Thus, we expressed T evaluated at $[X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]$ in terms of T evaluated at $[X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]$, which has support closer to the origin with respect to the $[2,1]$-filtration than $[X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]$ had with respect to the $[1,1,1]$-filtration.

(b) Suppose $s = 2/3 - m$. Since T is G-invariant, after conjugating by $stab_{GL_3}(x)$ we can assume that $X = \varpi^{-m}\begin{bmatrix} 0 & 0 & 0 \\ v\varpi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ or $X = \varpi^{-m}\begin{bmatrix} 0 & 0 & 0 \\ u\varpi & 0 & 0 \\ 0 & v\varpi & 0 \end{bmatrix}$

i. $X = \varpi^{-m}\left[\begin{smallmatrix} 0 & 0 & 0 \\ v\varpi & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$. We have

$$\begin{aligned} T([X+Y+\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]) &= \frac{1}{q}\sum_{\bar{u}\in U_{32}(\wp^m)/U_{32}(\wp^{m+1})} T([{}^u X+Y+\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]) \\ &= \frac{1}{q}T([X+Y+\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]) \\ &= \frac{1}{q^2}\sum_{\alpha\in R/\mathfrak{p}} T([X+Y+\left[\begin{smallmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]+\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]). \end{aligned}$$

In this, we have that $Y \in \varpi^{-m}\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{smallmatrix}\right] \subseteq \varpi^{-m}\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]$ and $X \in \varpi^{-m}\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]$. Thus $X+Y+\left[\begin{smallmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right] \subset \varpi^{-m}\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right] = \mathfrak{g}_{[3],s^+}$. Thus, we expressed T evaluated at $[X+Y+\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]$ in terms of T evaluated at $[X+Y+\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]$, which has support closer to the origin with respect to the $[3]$-filtration than $[X+Y+\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]$ had with respect to the $[1,1,1]$-filtration.

ii. $X = \varpi^{-m}\left[\begin{smallmatrix} 0 & 0 & 0 \\ u\varpi & 0 & 0 \\ 0 & v\varpi & 0 \end{smallmatrix}\right]$. We have

$$\begin{aligned} T([X+Y+\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]) &= \frac{1}{q}\sum_{t\in\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \mathfrak{p}^{-m}/\mathfrak{p}^{-m+1} & 1 \end{smallmatrix}\right]} T([{}^t X+Y+\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]) \\ &= \frac{1}{q}T([X+Y+\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]) \\ &= \frac{1}{q^2}\sum_{\alpha\in R/\mathfrak{p}} T([X+Y+\left[\begin{smallmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]+\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]). \end{aligned}$$

In this, we have that $Y \in \varpi^{-m}\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{smallmatrix}\right] \subseteq \varpi^{-m}\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]$ and $X \in \varpi^{-m}\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]$. Thus $X+Y+\left[\begin{smallmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right] \subset \varpi^{-m}\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right] = \mathfrak{g}_{[3],s^+}$. Thus, we expressed T evaluated at $[X+Y+\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]$ in terms of T evaluated at $[X+Y+\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]$, which has support closer to the origin with respect to the $[3]$-filtration than $[X+Y+\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]$ had with respect to the $[1,1,1]$-filtration.

4. The point z. In this case, we are looking at the coset $X+Y+\mathfrak{g}_{z,1/3^+}$, where z is

as in (3(a)i) on page 46 above, $X \in \mathcal{N} \cap (\mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+})$ with $s < \frac{1}{3}$, and $Y \in \mathfrak{g}_{x,s^+}$. From (3(a)i) we know we are only interested in the case when $s = -m + 1/2$ for $m \in \mathbb{Z}_{>0}$. So, we assume we are in this situation. Since T is G-invariant, after conjugating by $stab_{GL_3}(x)$ we can assume that X is $\varpi^{-m}\begin{bmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ or $\varpi^{-m}\begin{bmatrix} 0 & u & 0 \\ 0 & 0 & 0 \\ v\varpi & 0 & 0 \end{bmatrix}$.

(a) $X = \varpi^{-m}\begin{bmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. We have

$$T(X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}) = \sum_{\alpha \in R/\mathfrak{p}} T([X + Y + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \varpi\alpha & 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix}).$$

Note that $X + Y \in X + \varpi^{-m}\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix} \subseteq \varpi^{-m}\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix} = \mathfrak{g}_{[1,1,1],s^+}$. Thus $X + Y + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha\varpi & 0 & 0 \end{bmatrix} \subset \mathfrak{g}_{[1,1,1],s^+}$. Thus, we expressed T evaluated at $[X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]$ in terms of T evaluated at $[X + Y + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha\varpi & 0 & 0 \end{bmatrix} + \mathfrak{g}_{[1,1,1],1/3^+}]$, which has support closer to the origin with respect to the $[1,1,1]$-filtration than $[X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]$ had with respect to the z-filtration.

(b) $X = \varpi^{-m}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ v\varpi & 0 & 0 \end{bmatrix}$. We have

$$T(X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}) = \sum_{\alpha \in R/\mathfrak{p}} T[X + Y + \begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}].$$

Note that $X + Y \in X + \varpi^{-m}\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix} \subseteq \varpi^{-m}\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix} = \mathfrak{g}_{[3],s^+}$. Thus $X + Y + \begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \subset \mathfrak{g}_{[3],s^+}$. Thus, we expressed T evaluated at $[X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]$ in terms of T evaluated at $[X + Y + \begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mathfrak{g}_{[3],1/3^+}]$, which has support closer to the origin with respect to the $[3]$-filtration than $[X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]$ had with respect to the z-filtration. □

3.2.5 On depth $r = 1/2^+$

Proposition 3.2.5. *For GL_3, $res_{D_{1/2^+}} \tilde{J}_{1/2^+} = res_{D_{1/2^+}} J(\mathcal{N})$.*

Proof. Fix $f \in D_{1/2^+}$ with $f = \sum_i f_i$ such that $f_i \in C_C(\mathfrak{g}/\mathfrak{g}_{x_i,1/2^+})$. Since T is linear, without loss of generality we can assume $f \in C_C(\mathfrak{g}/\mathfrak{g}_{x,1/2^+})$ for some $x \in \mathcal{B}$. Therefore, we can write

$$f = \sum_{\overline{Z} \in \mathfrak{g}/\mathfrak{g}_{x,1/2^+}} c_{\overline{Z}}[Z + \mathfrak{g}_{x,1/2^+}],$$

where we have $[Z + \mathfrak{g}_{x,1/2^+}]$ is the characteristic function of the corresponding coset and all but finitely many $c_{\overline{Z}} = 0$. Since T is linear, we can assume, without loss of generality, that $f := [Z + \mathfrak{g}_{x,1/2^+}]$.

Now, we can choose $Z + \mathfrak{g}_{x,1/2^+} \subset \mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+}$. Since $T \in \tilde{J}_{1/2^+}$, we have $T(f) = 0$ if $supp(f) \cap (\mathfrak{g}_{x,s^+} + \mathcal{N}) = \varnothing$, and thus $T(f) = 0$ unless

$$(Z + \mathfrak{g}_{x,s^+}) \cap \mathcal{N} \neq \varnothing.$$

Therefore, without loss of generality we may assume $Z = X + Y$ with $X \in \mathcal{N} \cap (\mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+})$ and $Y \in \mathfrak{g}_{\mathfrak{g}_{x,s^+}}$.

Up to conjugacy, we need to consider four cases. These include the usual three cases where f is invariant with respect to $\mathfrak{g}_{[1,1,1],1/2^+} = \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix}$, $\mathfrak{g}_{[2,1],1/2^+} = \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}$, or $\mathfrak{g}_{[3],1/2^+} = \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}$ and an additional case for the-filtration associated to z (introduced in (3(a)i) on page 46).

We now consider these three cases.

1. $\mathcal{L}_x = [3]$. In this case, we are looking at the coset $X + Y + \mathfrak{g}_{x,1/2^+}$, where x is a vertex of C, $X \in \mathcal{N} \cap (\mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+})$ with $s < \frac{1}{2}$, and $Y \in \mathfrak{g}_{x,s^+}$. Note that $\mathcal{N} \cap (\mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+}) = \varnothing$ unless $s = -m$, for $m \in \mathbb{Z}_{\geq 0}$.

   Since T is G-invariant, after conjugating by $stab_{GL_3}(x)$ we can assume that X is $\varpi^{-m}\begin{bmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ or $\varpi^{-m}\begin{bmatrix} 0 & u & 0 \\ 0 & 0 & v \\ 0 & 0 & 0 \end{bmatrix}$.

(a) $X = \varpi^{-m}\left[\begin{smallmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$. We have

$$T[X + Y + \left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]] = \frac{1}{q^2} \sum_{\alpha,\beta \in \mathfrak{p}/\mathfrak{p}^2} T[X + Y + \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha & \beta & 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{smallmatrix}\right]].$$

Thus $X + Y + \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha & \beta & 0 \end{smallmatrix}\right] \subset \varpi^{-m}\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right] = \mathfrak{g}_{[2,1],s^+}$. Thus, we expressed T evaluated at $[X + Y + \left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]$ in terms of T evaluated at $[X + Y + \left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{smallmatrix}\right]]$, which has support closer to the origin with respect to the $[2,1]$-filtration than $[X + Y + \left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]$ had with respect to the $[3]$-filtration.

(b) $X = \varpi^{-m}\left[\begin{smallmatrix} 0 & u & 0 \\ 0 & 0 & v \\ 0 & 0 & 0 \end{smallmatrix}\right]$. We have

$$\begin{aligned} T[X + Y + \left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]] &= \sum_{\alpha \in \mathfrak{p}/\mathfrak{p}^2} T[X + Y + \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha & 0 & 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]] \\ &= \frac{1}{q} \sum_{\bar{u} \in U_{12}(\wp^m)/U_{12}(\wp^{m+1})} T[{}^{u}X + Y + \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha & 0 & 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]] \\ &= \frac{1}{q} T[X + Y + \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha & 0 & 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]. \end{aligned}$$

Note that $X + Y + \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha & 0 & 0 \end{smallmatrix}\right] \subset \varpi^{-m}\left[\begin{smallmatrix} \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right] = \mathfrak{g}_{[1,1,1],s^+}$. Thus, we expressed T evaluated at $[X + Y + \left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]$ in terms of T evaluated at $[X + Y + \left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]$, which has support closer to the origin with respect to the $[1,1,1]$-filtration than $[X + Y + \left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]$ had with respect to the $[3]$-filtration.

2. $\mathcal{L}_x = [2,1]$. In this case, we are looking at the coset $X + Y + \mathfrak{g}_{x,1/2^+}$, where x is the barycenter of an edge in C, $X \in \mathcal{N} \cap (\mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+})$, and $Y \in \mathfrak{g}_{x,s^+}$. Note that we have $\mathcal{N} \cap (\mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+}) = \varnothing$ unless $s = -m + 1/2$ or $s = -m$, for $m \in \mathbb{Z}_{\geq 0}$.

   (a) Suppose $s = -m$. Since T is G-invariant, after conjugating by $stab_{GL_3}(x)$ we can assume that X is $\varpi^{-m}\left[\begin{smallmatrix} 0 & u & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$. We have

$$T[X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]$$
$$=\frac{1}{q^2}\sum_{\bar{u}\in U_{23}(\wp^m)/U_{23}(\wp^{m+1})}\sum_{\bar{v}\in U_{31}(\wp^m)/U_{31}(\wp^{m+1})} T([{}^{uv}X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]$$
$$=\frac{1}{q^2}T([X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix}].$$

Note that $Y \in \varpi^{-m}\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix} \subseteq \varpi^{-m}\begin{bmatrix} \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}$ and $X \in \varpi^{-m}\begin{bmatrix} \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}$. Thus $X+Y+\begin{bmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \subset \varpi^{-m}\begin{bmatrix} \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix} = \mathfrak{g}_{[1,1,1],s^+}$. Thus, we expressed T evaluated at $[X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]$ in terms of T evaluated at $[X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]$, which has support closer to the origin with respect to the $[1,1,1]$-filtration than $[X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]$ had with respect to the $[2,1]$-filtration.

(b) Suppose $s=-m+1/2$. Since T is G-invariant, after conjugating by $stab_{GL_3}(x)$ we can assume that X is $\varpi^{-m}\begin{bmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ or $\varpi^{-m}\begin{bmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ 0 & v\varpi & 0 \end{bmatrix}$.

i. $X=\varpi^{-m}\begin{bmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ 0 & v\varpi & 0 \end{bmatrix}$. We have

$$T([X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]) =\frac{1}{q^3}\sum_{\bar{t}\in T_m/T_{m+1}} T([{}^{t}X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}])$$
$$=\frac{1}{q}T([X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]).$$

Note that $Y \in \varpi^{-m}\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix} \subseteq \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix}$, and $X \in \varpi^{-m}\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix}$. Thus $X+Y+\begin{bmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \subset \varpi^{-m}\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix} = \mathfrak{g}_{[1,1,1],s^+}$. Thus, we expressed T evaluated at $[X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]$ in terms of T evaluated at $[X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]$, which has support closer to the origin with respect to the $[1,1,1]$-filtration than $[X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]$ had with respect to the $[2,1]$-filtration.

ii. $X=\varpi^{-m}\begin{bmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Recall the point z of C that was introduced in (3(a)i on page 46). We have

$$T([X+Y+\begin{bmatrix}\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\\\mathfrak{p}^2&\mathfrak{p}^2&\mathfrak{p}\end{bmatrix}]) = \frac{1}{q^3}\sum_{\bar{t}\in T_m/T_{m+1}} T([{}^tX+Y+\begin{bmatrix}\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\\\mathfrak{p}^2&\mathfrak{p}^2&\mathfrak{p}\end{bmatrix}])$$
$$= \frac{1}{q^2}T([X+Y+\begin{bmatrix}\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\\\mathfrak{p}^2&\mathfrak{p}^2&\mathfrak{p}\end{bmatrix}]).$$

Let $\Pi = \begin{bmatrix}0&1&0\\0&0&1\\\varpi&0&0\end{bmatrix}$. Note that $X+Y \in \varpi^{-m}\begin{bmatrix}\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\\\mathfrak{p}^2&\mathfrak{p}^2&\mathfrak{p}\end{bmatrix} = \mathfrak{g}_{\Pi z,s^+}$ and $\begin{bmatrix}\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\\\mathfrak{p}^2&\mathfrak{p}^2&\mathfrak{p}\end{bmatrix} = \mathfrak{g}_{\Pi z,1/2^+}$. Thus, we expressed T evaluated at $[X+Y+\begin{bmatrix}\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\\\mathfrak{p}^2&\mathfrak{p}^2&\mathfrak{p}\end{bmatrix}]$ in terms of T evaluated at $[X+Y+\mathfrak{g}_{\Pi z,1/2^+}]$, which has support closer to the origin with respect to the Πz-filtration than $[X+Y+\begin{bmatrix}\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\\\mathfrak{p}^2&\mathfrak{p}^2&\mathfrak{p}\end{bmatrix}]$ had with respect to the $[2,1]$-filtration.

3. $\mathcal{L}_x = [1,1,1]$. In this case, we are looking at the coset $X+Y+\mathfrak{g}_{x,1/2^+}$, where x is the barycenter of C, $X \in \mathcal{N}\cap(\mathfrak{g}_{x,s}\setminus\mathfrak{g}_{x,s^+})$, and $Y\in\mathfrak{g}_{x,s^+}$ with $s\le 1/2$. Note that we have $\mathcal{N}\cap(\mathfrak{g}_{x,s}\setminus\mathfrak{g}_{x,s^+}) = \varnothing$ unless $s = 1/3-m$ for $m\in\mathbb{Z}_{\ge 0}$ or $s = 2/3-m$, for $m\in\mathbb{Z}_{>0}$.

   (a) Suppose $s = 1/3-m$. Since T is G-invariant, after conjugating by $stab_{GL_3}(x)$ we can assume that X is $\varpi^{-m}\begin{bmatrix}0&0&0\\0&0&0\\u\varpi&0&0\end{bmatrix}$ or $\varpi^{-m}\begin{bmatrix}0&0&0\\0&0&u\\v\varpi&0&0\end{bmatrix}$. In both cases we have

$$T([X+Y+\begin{bmatrix}\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\\\mathfrak{p}^2&\mathfrak{p}&\mathfrak{p}\end{bmatrix}]) = \sum_{\alpha\in R/\mathfrak{p},\beta\in\mathfrak{p}/\mathfrak{p}^2} T([X+Y+\begin{bmatrix}0&0&\alpha\\0&0&0\\0&\beta&0\end{bmatrix}+\begin{bmatrix}\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\\\mathfrak{p}^2&\mathfrak{p}^2&\mathfrak{p}\end{bmatrix}]).$$

   Note $X+Y\subset\varpi^{-m}\begin{bmatrix}\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\end{bmatrix} = \mathfrak{g}_{[2,1],s^+}$. Thus, we expressed T evaluated at $[X+Y+\begin{bmatrix}\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\\\mathfrak{p}^2&\mathfrak{p}&\mathfrak{p}\end{bmatrix}]$ in terms of T evaluated at $[X+Y+\begin{bmatrix}\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\\\mathfrak{p}^2&\mathfrak{p}^2&\mathfrak{p}\end{bmatrix}]$, which has support closer to the origin with respect to the $[2,1]$-filtration than $[X+Y+\begin{bmatrix}\mathfrak{p}&\mathfrak{p}&R\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}\\\mathfrak{p}^2&\mathfrak{p}&\mathfrak{p}\end{bmatrix}]$ had with respect to the $[1,1,1]$-filtration.

   (b) Suppose $s = 2/3-m$. In this case, since T is G-invariant, after conjugating by $stab_{GL_3}(x)$ we can assume that X is $\varpi^{-m}\begin{bmatrix}0&0&0\\u\varpi&0&0\\0&0&0\end{bmatrix}$ or $\varpi^{-m}\begin{bmatrix}0&0&0\\v\varpi&0&0\\0&u\varpi&0\end{bmatrix}$

i. $X = \varpi^{-m}\left[\begin{smallmatrix} 0 & 0 & 0 \\ u\varpi & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$. We have

$$\begin{aligned} T([X + Y + \left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{smallmatrix}\right]]) & \\ &= \frac{1}{q^3} \sum_{\bar{t} \in T_m/T_{m+1}} T([{}^tX + Y + \left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{smallmatrix}\right]]) \\ &= \frac{1}{q} T([X + Y + \left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{smallmatrix}\right]]). \end{aligned}$$

Note that $X + Y \subset \varpi^{-m}\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{smallmatrix}\right] = \mathfrak{g}_{[2,1],s^+}$. Thus, we expressed T evaluated at $[X + Y + \left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{smallmatrix}\right]]$ in terms of T evaluated at $[X + Y + \left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{smallmatrix}\right]]$, which has support closer to the origin with respect to the $[2,1]$-filtration than $[X+Y+\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{smallmatrix}\right]]$ had with respect to the $[1,1,1]$-filtration.

ii. $X = \varpi^{-m}\left[\begin{smallmatrix} 0 & 0 & 0 \\ v\varpi & 0 & 0 \\ 0 & u\varpi & 0 \end{smallmatrix}\right]$. We have

$$\begin{aligned} T([X + Y + \left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{smallmatrix}\right]]) &= \frac{1}{q^3} \sum_{\bar{t} \in T_m/T_{m+1}} T([{}^tX + Y + \left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{smallmatrix}\right]]) \\ &= \frac{1}{q^2} T([X + Y + \left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]) \\ &= \frac{1}{q^3} \sum_{\bar{u} \in U_{21}(\wp^m)/U_{21}(\wp^{m+1})} T([{}^uX + Y + \left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]) \\ &= \frac{1}{q^3} \sum_{\alpha \in R/\mathfrak{p}} T([X + Y + \left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]). \end{aligned}$$

Note that $X+Y \subset \varpi^{-m}\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right] = \mathfrak{g}_{[3],s^+}$. Thus, we expressed T evaluated at $[X+Y+\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{smallmatrix}\right]]$ in terms of T evaluated at $[X+Y+\left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right]]$, which has support closer to the origin with respect to the $[3]$-filtration than $[X + Y + \left[\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{smallmatrix}\right]]$ had with respect to the $[1,1,1]$-filtration.

(c) The point z. From (2(b)ii) we need only handle the case where s is chosen such that

$$\mathfrak{g}_{[1,1,1],2/3^+} = \mathfrak{g}_{z,s^+} \subsetneq \mathfrak{g}_{z,s} = \mathfrak{g}_{z,1/2^+}.$$

Since T is G-invariant, after conjugating by $stab_{GL_3}(z)$ we can assume that X is $\varpi^{-m}\begin{bmatrix} 0 & 0 & 0 \\ u\varpi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ or $\varpi^{-m}\begin{bmatrix} 0 & 0 & 0 \\ v\varpi & 0 & 0 \\ 0 & u\varpi & 0 \end{bmatrix}$. In either case we have

$$\begin{aligned} T([X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]) &= \frac{1}{q} \sum_{\bar{u} \in U_{32}(\mathfrak{p}^m)/U_{32}(\mathfrak{p}^{m+1})} T({}^{u}X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]) \\ &= \frac{1}{q} T(X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]). \end{aligned}$$

Note that $X+Y \subset \varpi^{-m}\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix} = \mathfrak{g}_{[3],s^+}$. Thus, we expressed T evaluated at $[X+Y++\mathfrak{g}_{z,1/2^+}]$ in terms of T evaluated at $[X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]$, which has support closer to the origin with respect to the [3]-filtration than $[X+Y++\mathfrak{g}_{z,1/2^+}]$ had with respect to the z-filtration. □

3.2.6 On depth $r = 2/3^+$

Proposition 3.2.6. *For GL_3, $res_{D_{2/3^+}} \tilde{J}_{2/3^+} = res_{D_{2/3^+}} J(\mathcal{N})$.*

Proof. Fix $f \in D_{2/3^+}$ with $f = \sum_i f_i$ such that $f_i \in C_C(\mathfrak{g}/\mathfrak{g}_{x_i,2/3^+})$. Since T is linear, without loss of generality we can assume $f \in C_C(\mathfrak{g}/\mathfrak{g}_{x,2/3^+})$ for some $x \in \mathcal{B}$. Therefore, we can write

$$f = \sum_{\overline{Z} \in \mathfrak{g}/\mathfrak{g}_{x,2/3^+}} c_{\overline{Z}}[Z+\mathfrak{g}_{x,2/3^+}],$$

where we have $[Z+\mathfrak{g}_{x,2/3^+}]$ is the characteristic function of the corresponding coset and all but finitely many $c_{\overline{Z}} = 0$. Since T is linear, we can assume, without loss of generality, that $f := [Z+\mathfrak{g}_{x,2/3^+}]$.

Now, we can choose $Z+\mathfrak{g}_{x,2/3^+} \subset \mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+}$. Since $T \in \tilde{J}_{2/3^+}$, we have $T(f) = 0$ if $supp(f) \cap (\mathfrak{g}_{x,s^+} + \mathcal{N}) = \varnothing$, and thus $T(f) = 0$ unless

$$(Z+\mathfrak{g}_{x,s^+}) \cap \mathcal{N} \neq \varnothing.$$

Therefore, without loss of generality we may assume $Z = X + Y$ with $X \in \mathcal{N} \cap (\mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+})$ and $Y \in \mathfrak{g}_{\mathfrak{g}_{x,s^+}}$.

Up to conjugacy, we need only those three cases where f is invariant with respect to $\mathfrak{g}_{[1,1,1],2/3^+} = \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}$, $\mathfrak{g}_{[2,1],2/3^+} = \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}$, or $\mathfrak{g}_{[3],2/3^+} = \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}$. These three cases correspond to the three conjugacy classes of barycenters of facets.

We now consider these three cases.

1. $\mathcal{L}_x = [3]$. In this case, we are looking at the coset $X + Y + \mathfrak{g}_{x,2/3^+}$, where x is a vertex of C, $X \in \mathcal{N} \cap (\mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+})$ with $s < \frac{2}{3}$, and $Y \in \mathfrak{g}_{x,s^+}$. Note that $\mathcal{N} \cap (\mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+}) = \varnothing$ unless $s = -m$, for $m \in \mathbb{Z}_{\geq 0}$.

   Since T is G-invariant, after conjugating by $stab_{GL_3}(x)$ we can assume that X is $\varpi^{-m}\begin{bmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ or $\varpi^{-m}\begin{bmatrix} 0 & u & 0 \\ 0 & 0 & v \\ 0 & 0 & 0 \end{bmatrix}$.

   (a) $X = \varpi^{-m}\begin{bmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. We have

$$T[X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}] = \frac{1}{q^2} \sum_{\alpha,\beta \in \mathfrak{p}/\mathfrak{p}^2} T[X + Y + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha & \beta & 0 \end{bmatrix} + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}].$$

   Thus $X + Y + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha & \beta & 0 \end{bmatrix} \subset \varpi^{-m}\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix} = \mathfrak{g}_{[2,1],s^+}$. Thus, we expressed T evaluated at $[X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]$ in terms of T evaluated at $[X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]$, which has support closer to the origin with respect to the $[2,1]$-filtration than $[X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]$ had with respect to the $[3]$-filtration.

   (b) $X = \varpi^{-m}\begin{bmatrix} 0 & u & 0 \\ 0 & 0 & v \\ 0 & 0 & 0 \end{bmatrix}$. We have

$$T[X + Y + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}] = \sum_{\alpha,\beta,\gamma \in \mathfrak{p}/\mathfrak{p}^2} T[X + Y + \begin{bmatrix} 0 & 0 & 0 \\ \gamma & 0 & 0 \\ \alpha & \beta & 0 \end{bmatrix} + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}].$$

   Thus $X + Y + \begin{bmatrix} 0 & 0 & 0 \\ \gamma & 0 & 0 \\ \alpha & \beta & 0 \end{bmatrix} \subset \varpi^{-m}\begin{bmatrix} \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix} = \mathfrak{g}_{[1,1,1],s^+}$. Thus, we expressed T

evaluated at $[X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]$ in terms of T evaluated at $[X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]$, which has support closer to the origin with respect to the $[1,1,1]$-filtration than $[X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]$ had with respect to the $[3]$-filtration.

2. $\mathcal{L}_x = [2,1]$. In this case, we are looking at the coset $X+Y+\mathfrak{g}_{x,2/3^+}$, where x is the barycenter of an edge in C, $X \in \mathcal{N} \cap (\mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+})$, and $Y \in \mathfrak{g}_{x,s^+}$. Note that we have $\mathcal{N} \cap (\mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+}) = \varnothing$ unless $s = -m+1/2$ or $s = -m$, for $m \in \mathbb{Z}_{\geq 0}$.

    (a) Suppose $s = -m$. Since T is G-invariant, after conjugating by $stab_{GL_3}(x)$ we can assume that X is $\varpi^{-m}\begin{bmatrix} 0 & u & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. We have

    $$T[X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}] = \sum_{\alpha \in \mathfrak{p}/\mathfrak{p}^2} T([X+Y+\begin{bmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]).$$

    Note that $Y \in \varpi^{-m}\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix} \subseteq \varpi^{-m}\begin{bmatrix} \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}$, and $X \in \varpi^{-m}\begin{bmatrix} \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}$. Thus $X+Y+\begin{bmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \subset \varpi^{-m}\begin{bmatrix} \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix} = \mathfrak{g}_{[1,1,1],s^+}$. Thus, we expressed T evaluated at $[X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]$ in terms of T evaluated at $[X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]$, which has support closer to the origin with respect to the $[1,1,1]$-filtration than $[X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]$ had with respect to the $[2,1]$-filtration.

    (b) Suppose $s = -m+1/2$. Since T is G-invariant, after conjugating by $stab_{GL_3}(x)$ we can assume that X is $\varpi^{-m}\begin{bmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ or $\varpi^{-m}\begin{bmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ 0 & v\varpi & 0 \end{bmatrix}$. In both cases we have

    $$T([X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]) = \sum_{\alpha \in \mathfrak{p}/\mathfrak{p}^2} T([X+Y+\begin{bmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]).$$

    Note that $Y \in \varpi^{-m}\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix} \subseteq \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix}$, and $X \in \varpi^{-m}\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix}$. Thus $X+Y+\begin{bmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \subset \varpi^{-m}\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix} = \mathfrak{g}_{[1,1,1],s^+}$. Thus, we expressed T evaluated at

$[X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]$ in terms of T evaluated at $[X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]$, which has support closer to the origin with respect to the $[1,1,1]$-filtration than $[X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]$ had with respect to the $[2,1]$-filtration.

3. $\mathcal{L}_x = [1,1,1]$. In this case, we are looking at the coset $X+Y+\mathfrak{g}_{x,2/3^+}$, where x is the barycenter of C, $X \in \mathcal{N} \cap (\mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+})$, and $Y \in \mathfrak{g}_{x,s^+}$ with $s \leq 2/3$. Note that we have $\mathcal{N} \cap (\mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+}) = \varnothing$ unless $s = 1/3 - m$ for $m \in \mathbb{Z}_{\geq 0}$ or $s = 2/3 - m$ for $m \in \mathbb{Z}_{>0}$.

   (a) Suppose $s = 1/3 - m$. Since T is G-invariant, thus after conjugating by $stab_{GL_3}(x)$ we can assume that X is $\varpi^{-m}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ u\varpi & 0 & 0 \end{bmatrix}$ or $\varpi^{-m}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & u \\ v\varpi & 0 & 0 \end{bmatrix}$. In both cases we have

$$T([X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]) = \frac{1}{q} \sum_{\bar{u} \in U_{23}(\wp^m)/U_{23}(\wp^{m+1})} T([{}^{u}X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}])$$
$$= \frac{1}{q} T([X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]).$$

   Note $X+Y \subset \varpi^{-m}\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix} = \mathfrak{g}_{[2,1],s^+}$. Thus, we expressed T evaluated at $[X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]$ in terms of T evaluated at $[X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]$, which has support closer to the origin with respect to the $[2,1]$-filtration than $[X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]$ had with respect to the $[1,1,1]$-filtration.

   (b) Suppose $s = 2/3 - m$. In this case, since T is G-invariant, after conjugating by $stab_{GL_3}(x)$ we can assume that X is $\varpi^{-m}\begin{bmatrix} 0 & 0 & 0 \\ u\varpi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ or $\varpi^{-m}\begin{bmatrix} 0 & 0 & 0 \\ v\varpi & 0 & 0 \\ 0 & u\varpi & 0 \end{bmatrix}$

      i. $X = \varpi^{-m}\begin{bmatrix} 0 & 0 & 0 \\ u\varpi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. We have

$$T([X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}])$$
$$= \frac{1}{q^3} \sum_{\bar{t} \in T_m/T_{m+1}} T([{}^{t}X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}])$$
$$= \frac{1}{q} T([X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]).$$

Note that $X+Y \subset \varpi^{-m}\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix} = \mathfrak{g}_{[2,1],s^+}$. Thus, we expressed T evaluated at $[X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]$ in terms of T evaluated at $[X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]$, which has support closer to the origin with respect to the $[2,1]$-filtration than $[X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]$ had with respect to the $[1,1,1]$-filtration.

ii. $X = \varpi^{-m}\begin{bmatrix} 0 & 0 & 0 \\ v\varpi & 0 & 0 \\ 0 & u\varpi & 0 \end{bmatrix}$. We have

$$\begin{aligned} T([X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]) =& \frac{1}{q^3}\sum_{\bar{t}\in T_m/T_{m+1}} T([{}^{t}X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]) \\ =& \frac{1}{q^2}T([X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]) \\ =& \frac{1}{q^3}\sum_{\bar{u}\in U_{21}(\wp^m)/U_{21}(\wp^{m+1})} T([{}^{u}X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]) \\ =& \frac{1}{q^3}\sum_{\alpha\in R/\mathfrak{p}} T([X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]). \end{aligned}$$

Note that $X+Y \subset \varpi^{-m}\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix} = \mathfrak{g}_{[3],s^+}$. Thus, we expressed T evaluated at $[X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]$ in terms of T evaluated at $[X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}]$, which has support closer to the origin with respect to the $[3]$-filtration than $[X+Y+\begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}]$ had with respect to the $[1,1,1]$-filtration. □

3.2.7 Counting

To complete the proof of the homogeneity result for GL_3, we need to show that statement 2 of Theorem 2.2.4 holds for GL_3. By the definition of $\tilde{J}_{r^+}$, it is enough to show that an element $T \in \tilde{J}_{r^+}$ is determined by its values on at most three functions of the form $[X+\mathfrak{g}_{x,r^+}]$ where $X \in \mathcal{N}\cap\mathfrak{g}_{x,r}$. It is enough to show that this is true for $x \in \{[1,1,1],[2,1],[3]\}$ and $r \in \{0,1/3,1/2,2/3\}$.

1. The case when $r = 1/3$ or $r = 2/3$. Here the only point we need to consider is

$[1,1,1]$ and, as we have seen above, up to conjugacy there are two nontrivial functions of the form $[X + \mathfrak{g}_{[1,1,1],r^+}]$ and one trivial one.

2. The case when $r = 0$. Since

$$C(\mathfrak{g}_{[1,1,1],0}/\mathfrak{g}_{[1,1,1],0^+}) \subset C(\mathfrak{g}_{[3],0}/\mathfrak{g}_{[3],0^+})$$

and

$$C(\mathfrak{g}_{[2,1],0}/\mathfrak{g}_{[2,1],0^+}) \subset C(\mathfrak{g}_{[3],0}/\mathfrak{g}_{[3],0^+}),$$

it is enough to consider functions in $C(\mathfrak{g}_{[3],0}/\mathfrak{g}_{[3],0^+})$. Since $GL_3(\mathbb{F}_q)$ has exactly three nilpotent orbits in its Lie algebra, the result follows.

3. The case when $r = 1/2$. Here the only point we need to consider is $[2,1]$ and, as we have seen above, up to conjugacy there are two nontrivial functions of the form $[X + \mathfrak{g}_{[1,1,1],r^+}]$ and one trivial one.

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