Free Energy and Overlaps of a Spherical Spin Glass with an External Field

by

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Dedication

To my grandfather, Lynn Wells Wildman, a proud Michigan Wolverine, class of 1946. He always believed in me and supported my dream of getting a PhD in Math. Thank you Grandpa, and Go Blue!

Acknowledgments

Earning a PhD, like so many other challenges in life, is one of those things that "takes a village." I have received tremendous support from family, friends, teachers, colleagues, and mentors throughout my educational journey, and I would like to express my gratitude to a few of them here.

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ABSTRACT

In this thesis, we analyze the free energy and the overlaps in the 2-spin spherical Sherrington Kirkpatrick (SSK) spin glass model with an external field for the purpose of understanding the transition between this model and the one without an external field. We compute the limiting values and fluctuations of the free energy as well as three types of overlaps in the setting where the strength of the external field, h, approaches zero as the dimension, N, approaches infinity. In particular, we consider overlaps with the external field, the ground state, and a replica. The methods involve a contour integral representation of the partition function along with random matrix techniques. We also provide computations for the matching between different scaling regimes and we discuss the implications of the results for susceptibility and for the geometry of the Gibbs measure.

The analysis throughout this thesis focuses on the SSK model with external field strength $h \sim N^{-\alpha}$ for $0 < \alpha < 1$. We find that the free energy exhibits a transition at $\alpha = 1/6$ in the low temperature case but at $\alpha = 1/4$ at the high temperature case. Furthermore, the overlaps do not exhibit any transition in the high temperature case, but exhibit two transitions in the low temperature case, at $\alpha = 1/6$ and $\alpha = 1/2$. These scalings are referred to as the mesoscopic and microscopic external field respectively. In the final chapter, we present a more detailed analysis of the overlaps in the microscopic setting.

CHAPTER I

Introduction

1.1 Overview of Spin Glasses and the Spherical Sherrington-Kirkpatrick Model

Since they were first developed in the 1970s, spin glass models have captured the attention of mathematicians, physicists, computer scientists, statisticians, biologists, economists, and others because of their intriguing probabilistic properties [42]. These models were initially developed by physicists to study the magnetic behavior of alloys that exhibit unusual properties at low temperatures (see, e.g. [18, 40]). The magnetic substances that are most familiar to us typically move between two magnetic phases: ferromagnetic and paramagnetic. In the ferromagnetic (low temperature) phase, the magnetic spins of the particles align with each other and remain fixed over time. In the paramagnetic (high temperature) phase, the magnetic spins are in a disordered, unaligned state and are constantly shifting over time. However, certain alloys can also exhibit a "spin glass phase" where the magnetic spins are disordered but fixed over time (sometimes called "frozen disorder"). In the alloys for which this phase exists, it occurs when the temperature is low and there is no external magnetic field (or the external field is very weak).

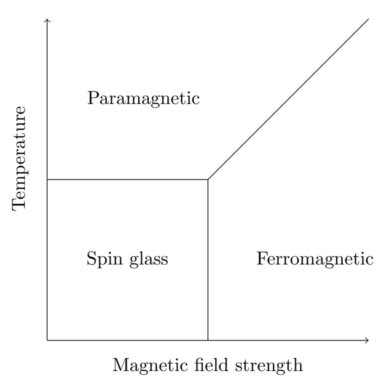


Figure I.1: This spin glass phase diagram, adapted from [10], is for the Spherical Sherrington-Kirkpatrick model with Curie-Weiss interaction. Although phase diagrams look different for each model, this one is an illustrative example.

The ferromagnetic, paramagnetic, and spin glass phases have been described as a magnetic analog of solids, liquids, and glasses respectively, and it is this analogy that gives rise to the name "spin glass" [42].

Spin glass models use probability to describe the typical magnetic behavior of particles in large systems. More specifically:

- *Disorder variables* encode the random interactions between particles.
- A *spin vector* $\boldsymbol{\sigma}$ encodes the magnetic spins of all particles at a particular instance.
- A *probability measure* $p(\boldsymbol{\sigma})$, which depends on the disorder variables, defines the spin distribution.

Because the disorder variables are random, p is a random measure. We will now look at some examples of spin glass models. We beginning with two well-known models, Edwards-Anderson and Sherrington-Kirkpatrick, before turning to the Spherical Sherrington-Kirkpatrick model, which is the focus of this thesis.

1.1.1 Important spin glass models: Edwards-Anderson and Sherrington-Kirkpatrick

Edwards-Anderson(EA) model

One of the earliest and most well-known spin glass models was developed by Sam Edwards and Philip Anderson in 1975 [18]. In the Edwards-Anderson (EA) model, we have N particles with indices $i \in \{1, 2, ..., N\}$, each of which has a spin σ_i. For simplicity, this model takes Ising spins, meaning that $\sigma_i \in \{\pm 1\}$. We denote a spin configuration by

$$\boldsymbol{\sigma} = (\sigma_1, \sigma_2, ..., \sigma_N) \in \{\pm 1\}^N. \tag{1.1.1}$$

In order to model a "typical" spin configuration, we need a way to represent the interactions between particles. For this purpose, the EA model introduces a set of i.i.d. random variables $\{J_{ij}\}$ where a positive value for J_{ij} represents a tendency of spins i and j to align while a negative value represents a tendency to anti-align. From these random variables, we get the Hamiltonian

$$\mathcal{H}_{EA}(\boldsymbol{\sigma}) = -\sum_{i \sim j} J_{ij} \sigma_i \sigma_j, \tag{1.1.2}$$

where $i \sim j$ indicates that particles i and j are adjacent (based on whatever lattice structure the model imposes). This nearest-neighbor constraint is physically realistic since the magnetic interaction between particles weakens with distance. We note that the sum $\sum_{i\sim j} J_{ij}\sigma_i\sigma_j$ is maximized by configurations $\boldsymbol{\sigma}$ in which spin pairs σ_i, σ_j generally have the same sign for positive J_{ij} and opposite signs for negative J_{ij}. This corresponds to a low energy state for the system, hence the Hamiltonian $-\sum_{i\sim j} J_{ij}\sigma_i\sigma_j$ is minimized. Finally, the EA model describes the probability distribution of spins in this system using the Gibbs measure

$$p_{EA}(\boldsymbol{\sigma}) = \frac{1}{\mathcal{Z}_N} e^{-\beta \mathcal{H}_{EA}(\boldsymbol{\sigma})} \quad \text{for } \boldsymbol{\sigma} \in \{\pm 1\}^N \tag{1.1.3}$$

where $\beta = 1/T$ denotes the inverse temperature and $\mathcal{Z}_N$ is a normalization factor (also called the partition function) defined by

$$\mathcal{Z}_N = \sum_{\boldsymbol{\sigma} \in \{\pm 1\}^N} e^{-\beta \mathcal{H}_{EA}(\boldsymbol{\sigma})}. \tag{1.1.4}$$

Since $p_{EA}(\boldsymbol{\sigma})$ is proportional to $e^{-\beta \mathcal{H}_{EA}(\boldsymbol{\sigma})}$, this probability measure reflects the fact that the system prefers low energy states (i.e. spin configurations for which $-\mathcal{H}_{EA}(\boldsymbol{\sigma})$ is large).

Sherrington-Kirkpatrick model

The Sherrington-Kirkpatrick (SK) model was developed in 1975 by David Sherrington and Scott Kirkpatrick [40]. Unlike the EA model, which is based on nearest-neighbor interactions, the SK model is an infinite-range or mean-field model, meaning that the interactions are between all pairs rather than just neighbors. While this set-up is further from the reality of physical spin systems, it is easier to analyze in certain respects and has interesting connections to other types of optimization problems.

As with the EA model, the space of spin configurations is $\{\pm 1\}^N$ but now the Hamiltonian is

$$\mathcal{H}_{SK}(\boldsymbol{\sigma}) = -\frac{1}{\sqrt{N}} \sum_{1 \le i < j \le N} J_{ij}\sigma_i\sigma_j \tag{1.1.5}$$

where $\{J_{ij}\}_{i,j=1}^N$ is a symmetric matrix whose entries are standard Gaussian random variables, independent up to the symmetry constraint. This type of matrix is known as a Gaussian Orthogonal Ensemble or GOE and its properties will be described in more detail in Chapter II (the diagonal elements are customarily taken to have twice the variance of the off-diagonal elements, although they are not included in this sum). The normalization factor $1/\sqrt{N}$ is needed to account for the fact that we are now summing over all pairs rather than just nearest neighbors, so the number of terms in the sum is $O(N^2)$ rather than $O(N)$. We will see later that this normalization is the right one to ensure the free energy is of order 1. As with the EA model, the SK Hamiltonian is associated with a

Gibbs measure

$$p_{SK}(\boldsymbol{\sigma}) = \frac{1}{\mathcal{Z}_N} e^{-\beta \mathcal{H}_{SK}(\boldsymbol{\sigma})}, \quad \mathcal{Z}_N = \sum_{\boldsymbol{\sigma} \in \{\pm 1\}^N} e^{-\beta \mathcal{H}_{SK}(\boldsymbol{\sigma})}. \tag{1.1.6}$$

The SK model is not physically realistic, since the interactions do not decay with distance. However, the model has an alternative interpretation in terms of the following optimization problem [46]: Given a set of N people, let J_{ij} represent the mutual feeling of like (positive) or dislike (negative) between individuals i and j. We would like to partition the people into two subsets such that, in general, friends are together and enemies apart (we label the subsets as ± 1). While we cannot do this perfectly, a good strategy is to choose labels $\sigma_i \in \{\pm 1\}$ that will maximize the quantity $\frac{1}{\sqrt{N}} \sum_{1 \le i < j \le N} J_{ij} \sigma_i \sigma_j$. Equivalently, we wish to find $\boldsymbol{\sigma}$ that maximizes $-\mathcal{H}_{SK}(\boldsymbol{\sigma})$. We further note that, for sufficiently large β (low temperature), the Gibbs measure concentrates near the maximizer(s) of $-\mathcal{H}_{SK}$ and the concentration becomes more pronounced as β increases. Indeed, the infinite β (or zero temperature) limit of the Gibbs measure has a point mass at the maximizer(s) of $-\mathcal{H}_{SK}$. For this reason, the optimization problem is sometimes referred to as a "zero temperature problem," while the SK model is the analogous problem "with temperature" [46].

This interpretation of the SK model as an optimization problem is illustrative of the important connections between spin glasses and optimization problems, including promising applications in areas such as machine learning (see e.g. [1]).

Lastly, we mention that the model described above is often referred to as a 2-spin SK model, which is a special case of the p-spin model [37] with Hamiltonian

$$\mathcal{H}_{SK,p}(\boldsymbol{\sigma}) = -\frac{1}{N^{(p-1)/2}} \sum_{i_1,\dots,i_p=1}^{N} g_{i_1 \dots i_p} \sigma_{i_1} \cdots \sigma_{i_p}. \tag{1.1.7}$$

This can be further generalized to mixed p-spin models, the Hamiltonians for which consist of linear combinations of $\mathcal{H}_{SK,p}$ for different p values.

1.1.2 The Spherical Sherrington-Kirkpatrick model

The Spherical Sherrington-Kirkpatrick (SSK) model, which was introduced in 1976 by Kosterlitz, Thouless, and Jones [28], is a continuous analog of the SK model. By this we mean that the spin variable $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_N)$, which was discrete in the SK model, is now located in S_{N-1}, the sphere of radius $\sqrt{N}$ in $\mathbb{R}^N$:

$$S_{N-1} = \{\boldsymbol{\sigma} \in \mathbb{R}^N : \|\boldsymbol{\sigma}\| = \sqrt{N}\}. \tag{1.1.8}$$

The Hamiltonian for the 2-spin SSK model with no external field is defined in the same way as for SK model, namely

$$\mathcal{H}_{SSK}(\boldsymbol{\sigma}) = -\frac{1}{\sqrt{N}} \sum_{1 \le i < j \le N} J_{ij} \sigma_i \sigma_j. \tag{1.1.9}$$

In this thesis we consider a variation of this model: the SSK with an external field. The Hamiltonian for this model is defined as

$$\mathcal{H}(\boldsymbol{\sigma}) = -\frac{1}{2} \sum_{i,j=1}^{N} M_{ij}\, \sigma_i\, \sigma_j - h \sum_{i=1}^{N} g_i\, \sigma_i = -\frac{1}{2} \boldsymbol{\sigma} \cdot M \boldsymbol{\sigma} - h\, \mathrm{g} \cdot \boldsymbol{\sigma} \tag{1.1.10}$$

where g is a standard Gaussian random vector and $M = \frac{1}{\sqrt{N}} J_{ij}$. More specifically, for $i \le j$, the variables M_{ij} are independent centered Gaussian random variables with variance $\frac{1}{N}$ for $i < j$ and $\frac{2}{N}$ for $i = j$. By the symmetry condition, $M_{ij} = M_{ji}$ for $i > j$, so we can sum over all pairs and divide by 2 rather than restricting to $i < j$ (some versions of this model omit the diagonal elements from the sum but we include them). The associated Gibbs measure is

$$p(\boldsymbol{\sigma}) = \frac{1}{\mathcal{Z}_N} e^{-\beta \mathcal{H}(\boldsymbol{\sigma})} \quad \text{for } \boldsymbol{\sigma} \in S_{N-1}. \tag{1.1.11}$$

The partition function $\mathcal{Z}_N$ is no longer a summation, but a surface integral defined by

$$\mathcal{Z}_N = \int_{S_{N-1}} e^{-\beta \mathcal{H}(\boldsymbol{\sigma})} \mathrm{d}\omega_N(\boldsymbol{\sigma}) \tag{1.1.12}$$

where ω_N is the normalized uniform measure on S_{N-1}. Since the disorder variables M and g are random, the Gibbs measure is a random measure, which we also call a thermal measure.

Because the SSK model has continuous spins rather than Ising spins, we lose the direct connection to the physical setting in which σ_i represents the magnetic spin of the ith particle. However, the SSK model has other benefits. The continuous nature of SSK makes certain types of analysis easier, as we will see in the subsequent chapters. It also has applications in statistics, where the free energy of the SSK model corresponds to a log-likelihood ratio used in hypothesis testing for spiked random matrices [26, 25].

1.1.3 Quantities of interest: Free energy and overlaps

The free energy per spin component is

$$\mathcal{F}_N = \mathcal{F}_N(T, h) = \frac{1}{N\beta} \log \mathcal{Z}_N. \tag{1.1.13}$$

Again, since the disorder variables M and $\mathbf{g}$ are random, the free energy $\mathcal{F}_N$ is a random variable. We are interested in the fluctuations of the free energy when $h \to 0$ as $N \to \infty$.

We also consider the behavior of the spin variables taken from the Gibbs measure. We focus on the following three particular overlaps.

- (overlap with the external field) Define

$$\mathfrak{M} = \frac{\mathbf{g} \cdot \boldsymbol{\sigma}}{N}. \tag{1.1.14}$$

- (overlap with the ground state) Let $\mathbf{u}_1$ be a unit eigenvector corresponding to the largest eigenvalue of the disorder matrix M. The vectors $\pm\mathbf{u}_1$ are the ground states in the absence of an external field, and we simply call them the ground states. Define

$$\mathfrak{G} = \frac{|\mathbf{u}_1 \cdot \boldsymbol{\sigma}|}{\sqrt{N}} \quad \text{and} \quad \mathfrak{O} = \mathfrak{G}^2 \tag{1.1.15}$$

- (overlap with a replica) Let $\boldsymbol{\sigma}^{(1)}$ and $\boldsymbol{\sigma}^{(2)}$ be two independent spin variables from the Gibbs measure for the same sample (i.e. disorder variables M_{ij} and g_i); $\boldsymbol{\sigma}^{(2)}$ is a replica of $\boldsymbol{\sigma}^{(1)}$. Define

$$\mathfrak{R} = \frac{\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}}{N}. \tag{1.1.16}$$

The factors N and $\sqrt{N}$ are included since $||\mathbf{u}_1|| = 1$, $\|\boldsymbol{\sigma}\| = \sqrt{N}$, and the expected value of $\|\mathbf{g}\|^2 = g_1^2 + \cdots + g_N^2$ is N (see below). Thus, this rescaling corresponds to normalizing the vectors.

The overlaps depend on the spin variable and also the disorder sample. Hence, there are two different expectations to consider. We consider the thermal (Gibbs) fluctuations of the overlaps for a given disorder sample. For some quantities, we also consider the sample-to-sample fluctuations of the thermal average. We denote the thermal (Gibbs) average for a given disorder sample by the bracket $\langle \cdot \rangle$. On the other hand, the sample-to-sample average of an observable O is denoted by $\bar{O}$ or $\mathbb{E}_s[O]$. For example, the thermal averages

$$\mathcal{M} = \langle \mathfrak{M} \rangle \quad \text{and} \quad \mathcal{X} = \frac{1}{h} \langle \mathfrak{M} \rangle$$

are called magnetization and susceptibility, respectively. Many of the results of this thesis are about the thermal fluctuations of overlaps for a given disorder sample. Fixing the disorder sample in this manner is also referred to as "quenched disorder."

1.2 Summary of Existing Research

The purpose of this thesis is to study the case $h \to 0$ systematically including up to the fluctuation term for the free energy and the three overlaps. Below is a survey of some of the existing research as it connects to the work in this thesis.

The free energy for the Hamiltonian (1.1.10) above when $h = 0$ converges, both in expectation and in distribution, to a deterministic value which was computed by Kosterlitz, Thouless and Jones in [28]. The Hamiltonian (1.1.10) is the 2-spin case of the more general p-spherical spin glass model which includes interactions between multiple spin coordinates. The limit of the free energy for the general spherical spin glass models which also includes the external field is given by the Crisanti-Sommers formula [15]. This formula is the spherical version of the Parisi formula [39] for the spins in hypercubes. The Parisi formula and Crisanti-Sommers formula are proved rigorously by Talagrand in [45, 44]. The result of Kosterlitz, Thouless and Jones shows that when $h = 0$, there are two phases: the spin glass phase when $T < 1$ and the paramagnetic phase when $T > 1$. On the other hand, they argued that when $h > 0$, assuming that the external field is uniform, there is no phase transition.

The subleading (in N) term of the free energy depends on the disorder and hence it describes the fluctuations of the free energy. For $h = 0$ and $T > 1$, the fluctuation term is of order N^{-1} and has the Gaussian distribution. This is proved for both the hypercube case [3, 21, 14] and the spherical case [7]. For $h = 0$ and $T < 1$, for the Hamiltonian above, the fluctuation term is of order $N^{-2/3}$ and has the GOE Tracy-Widom distribution [7]. Chen, Dey, and Panchenko performed a similar calculation for the case with Ising spins where $h > 0$ is of order 1 and g is the vector of all 1s. In this case, they find [11] that the fluctuation term is of order $N^{-1/2}$ and has the Gaussian distribution for all temperature. They claim that similar results hold for the spherical case and our results confirm this claim using a different method. We note that their result also holds for mixed p-spin with even degree terms. Chen and Sen [12] computed the ground state energy for spherical mixed p-spin models (of which SSK is a specific case) and found that the fluctuations of the ground state energy are Gaussian in the presence of an external field.

In [22], the large deviations of the free energy distribution was obtained at $T = 0$ from a non-rigorous saddle point calculation of the moments of $\mathcal{Z}_N$ in the large N limit (see also [17] for a rigorous version). From this calculation a transitional regime $h \sim N^{-1/6}$ for the fluctuations of the free energy was conjectured. A proof of the existence of this regime was obtained in [27]. In Chapter III, we obtain explicitly the fluctuations of the free energy in the regime $h \sim N^{-1/6}$ for any $T < 1$ and in the regime $h \sim N^{-1/4}$ for $T > 1$. As we show, our results match in the tail of the distribution with those of [22]. Note also the recent physics work [23] where a different spherical model of random optimization was considered, which exhibits a similar phenomenology.

The overlap with the external field has been studied in the context of magnetism and susceptibility. Kosterlitz, Thouless, and Jones [28] computed the susceptibility as h tends to zero and observed a transition at the temperature $T = 1$. Cugliandolo, Dean, and Yoshino [16] computed two different versions of this limit of the susceptibility, in the first case taking $\lim_{h\to 0}\lim_{N\to\infty}$ and in the second case taking $\lim_{N\to\infty}\lim_{h\to 0}$. In the first of these cases, they get the same result as [28] with a transition at $T = 1$, but in the second case they do not observe a transition. Furthermore, they find that the two types of limits agree for $T > 1$ but not for $T < 1$. They also extend these results to a more general class of models (beyond Gaussian) and to non-linear susceptibility. In Chapter IV, we focus on the linear susceptibility and differential susceptibility in the Gaussian case, and obtain a more detailed picture. By considering the three regimes $h > 0$ constant, $h \sim N^{-1/6}$, and $h \sim N^{-1/2}$, we see that the first limit considered by Cugliandolo et al agrees with our result for the $h \to 0$ limit of the case where $h > 0$ is a fixed constant. The second limit that they consider is analogous to our result for the $H \to 0$ limit of the $h \sim N^{-1/2}$ case where we define $H = hN^{1/2}$ (i.e. we take $N \to \infty$ for $h = HN^{-1/2}$ with fixed H and then we let $H \to 0$). However, we find in this case that the susceptibility depends on the sample and is a function of $\mathbf{g} \cdot \mathbf{u}_1$, the inner product of the external field and the ground state. This dependence was not apparent in [16], since their set-up fixes $\mathbf{g} \cdot \mathbf{u}_1 = 1$. When $\mathbf{g} \cdot \mathbf{u}_1 = 1$ we find, as they do, that there is no transition in the susceptibility between high and low temperature. However, a transition does exist for all other values of $\mathbf{g} \cdot \mathbf{u}_1$. This is discussed in Section 4.1 (specifically subsection 4.1.7) and a rigorous proof is presented in Chapter V.

The overlap with the ground state is relevant to understanding the geometry of the Gibbs measure. Subag [43] examines the geometry of the Gibbs measure for general p-spin spherical models and finds that the Gibbs measure concentrates in spherical bands around the critical points of the Hamiltonian. These bands are of the form $\text{Band}(\boldsymbol{\sigma}_0, q, q') = \{\boldsymbol{\sigma} \in S_{N-1} : q \leq R(\boldsymbol{\sigma}, \boldsymbol{\sigma}_o) \leq q'\}$ where $\boldsymbol{\sigma}_0$ is a critical point of $\mathcal{H}$ and $R(\boldsymbol{\sigma}, \boldsymbol{\sigma}_0)$ is the overlap of $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_0$. In Section 4.2, we focus specifically on the overlap with the ground state (a special case of $R(\boldsymbol{\sigma}, \boldsymbol{\sigma}_0)$ where $\boldsymbol{\sigma}_0$ is the critical point corresponding to the largest eigenvalue). In the $h = 0$ regime, as expected, we see the Gibbs measure concentrates in a band and we examine how this geometry changes for the case of positive constant h as well as the cases of $h \sim N^{-1/6}$ and $h \sim N^{-1/3}$.

The overlap with a replica has been studied extensively, both for the Ising spin models and the spherical spin models with general p-spin interaction. For $p = 2$ the non-rigorous replica method used in [28, 15, 22] obtains a replica-symmetric saddle point leading to a prediction for the overlap q as a function of h. In particular, at $h = 0$, the prediction is that $q = 1 - T$ for $T < 1$ and $q = 0$

for $T > 1$. These calculations were confirmed rigorously in [38]. Recently, Landon, Nguyen, and Sosoe extended the results further to examine the fluctuations of the overlap at high temperature [36] and low temperature [29]. They find, in particular, that the overlap has Gaussian fluctuations in the high temperature regime, whereas, in the low temperature regime, the fluctuations are of order $N^{-1/3}$ and converge to a random variable that has an explicit formula in terms of the GOE Airy point process (see subsection 2.1.4 for a description of this). In Section 4.3, we obtain similar results for $h \sim N^{-1/6}$ and $h \sim N^{-1/2}$.

1.3 Highlights of the results

1.3.1 Results for the free energy

We examine the behavior of the free energy, including its leading order and the sample-to sample fluctuation term, as $N \to \infty$ when $h = O(1)$ and when $h \to 0$. Note that throughout this thesis, we often use the notation $h = O(1)$ to mean that h is a positive constant. We find that, in each case,

$$\mathcal{F}_N(T, h) \overset{\mathcal{D}}{\simeq} F(T, h) + \text{ sample fluctuations} \tag{1.3.1}$$

where $\overset{\mathcal{D}}{\simeq}$ denotes an asymptotic expansion in distribution with respect to the disorder variables. The limiting free energy $F(T, h)$ includes all deterministic (depending only on h and T) terms whose order exceeds that of the sample fluctuations. The "sample fluctuations" refers to the largest order term that depends on the disorder sample. Our findings in each case are summarized in Table I.1. Upon computing the leading term and sample fluctuations for $\mathcal{F}_N(T, h)$ with $h = O(1)$, we made two key observations. Firstly, the free energy for $h = O(1)$ does not exhibit a transition as we see in the $h = 0$ case; this observation is consistent with the result of [11] for Ising spins. Secondly, while the limiting free energy is continuous in T and h, the sample fluctuations in the $h = O(1)$ case do not agree with those in the $h = 0$ case (neither for $T > 1$ nor for $T < 1$). This suggests the existence of transitional regimes. We found that, for $T > 1$, the transition occurs at $h \sim N^{-1/4}$ while, for $T < 1$, the transition occurs at $h \sim N^{-1/6}$. We computed the asymptotic expansion of $\mathcal{F}_N(T, h)$ in these transitional regimes.

Case	Limiting free energy $F(T,h)$	Sample fluctuations	Result
$h=0,\ T>1$	$\frac{1}{4T}$	N^{-1} Gaussian distribution	3.1.1
$h=0,\ T<1$	$1-\frac{3T}{4}+\frac{T\log T}{2}$	$N^{-\frac{2}{3}}$ TW_{GOE} distribution	3.1.2
$h=O(1)$	$\frac{\gamma_0}{2}-\frac{Ts_0(\gamma_0)}{2}-\frac{T-T\log T}{2}+\frac{h^2 s_1(\gamma_0)}{2}$	$N^{-\frac{1}{2}}$ Gaussian distribution	3.1.5
$h\sim N^{-\frac{1}{4}},\ T>1$	$\frac{1}{4T}+\frac{h^2}{2T}$	N^{-1} Gaussian distribution	3.2.2
$h\sim N^{-\frac{1}{6}},\ T<1$	$1-\frac{3T}{4}+\frac{T\log T}{2}+\frac{h^2}{2}$	$N^{-\frac{2}{3}}$ function of the GOE Airy point process and Gaussian r.v.'s	3.3.2

Table I.1: This table summarizes our findings for the leading term and fluctuations of $\mathcal{F}_N(T,h)$ in the various cases we considered. By "limiting free energy" we actually mean an asymptotic expansion of $F(T,h)$ including all terms of order greater than that of the fluctuations. The $h=0$ cases were already known [7] but are included here for completeness. In the limiting free energy for the $h=O(1)$ case, the quantity γ_0 is deterministic and depends only on T and h. The functions s_0 and s_1 are defined in Chapter II. For more details on the notation, derivation, and precise formulas for the fluctuation terms, see the corresponding result.

When comparing the fluctuations in each regime, we observe that the order of the fluctuations are largest in the $h=O(1)$ case, where they have order $N^{-1/2}$ and Gaussian distribution. This holds for all temperatures. When $T>1$ but $h=0$ or $h\to 0$, the fluctuations remain Gaussian, but their order shrinks to N^{-1}. When $T<1$ and $h=0$ or $h\to 0$, the fluctuations have order $N^{-2/3}$. In the case of $h=0$ they have GOE Tracy-Widom distribution while, in the case of $h\sim N^{-1/6}$, their distribution is a function of the GOE Airy point process and of a sequence of i.i.d. standard Gaussian random variables. See Table I.1 for the equations corresponding to each of these results.

1.3.2 Results for the overlaps

In the next three tables we state our findings for the overlap with the external field, with the ground state and with a replica. In each case the thermal average and thermal fluctuations are presented in interesting regimes of h and T. The thermal average and fluctuations in most cases depend on the disorder sample. Our findings also have implications for magnetization and susceptibility, which will be described in more detail in section 4.1.

Case	Thermal average $\langle\mathfrak{M}\rangle$	Thermal fluctuations of $\mathfrak{M}$	Result
$h = O(1)$ for all T	$hs_1(\gamma_0) + O(N^{-\frac{1}{2}})$	$N^{-\frac{1}{2}}$ Gaussian	4.1.2
(and $h = 0$, $T > 1$)			4.1.3
$h \sim N^{-\frac{1}{6}}$, $T < 1$	$h + O(N^{-\frac{1}{2}})$	$N^{-\frac{1}{2}}$ Gaussian	4.1.5
$h \sim N^{-\frac{1}{2}}$, $T < 1$	$h + \frac{\lvert n_1\rvert\sqrt{1-T}}{\sqrt{N}} \tanh\left(\frac{\lvert n_1\rvert h\sqrt{N(1-T)}}{T}\right)$	$N^{-\frac{1}{2}}$ [Gaussian + Bernoulli]	4.1.7
(and $h = 0$, $T < 1$)			4.1.3

Table I.2: This table summarizes our finding for $\mathfrak{M}$, the overlap with the external field. Here, $\gamma_0 = \gamma_0(h, T)$ in the first row is deterministic and has the same value as in Table I.1. The variable n_1 in the third row is $n_1 = \mathbf{u}_1 \cdot \mathbf{g}$. For the top two rows, the leading term in $\langle\mathfrak{M}\rangle$ and the thermal fluctuations of $\mathfrak{M}$ do not depend on the disorder sample. However, the $O(N^{-\frac{1}{2}})$ subleading terms in $\langle\mathfrak{M}\rangle$ for the top two cases and both the leading term in $\langle\mathfrak{M}\rangle$ and the thermal fluctuations of $\mathfrak{M}$ of the last row do depend on the disorder sample.

Case	Thermal average $\langle\mathfrak{G}^2\rangle$	Thermal fluctuations of $\mathfrak{G}^2$	Result
$h = O(1)$ for all T	$\frac{1}{N}\left(\frac{h^2 n_1^2}{(\gamma_0-2)^2} + \frac{T}{\gamma_0-2}\right)$	N^{-1} χ-squared (non-centered)	4.2.2
(and $h = 0$, $T > 1$)			4.2.7
$h \sim N^{-\frac{1}{6}}$, $T < 1$	$1 - T - \sum_{i=2}^{N} \frac{n_i^2 h^2 N^{1/3}}{(t+a_1-a_i)^2}$	$N^{-\frac{1}{6}}$ Gaussian	4.2.4
$h \sim N^{-\frac{1}{3}}$, $T < 1$	$1 - T + O(N^{-\frac{1}{3}})$	$N^{-\frac{1}{3}}$ r.v. that depends on disorder	4.2.6
(and $h = 0$, $T < 1$)			4.2.7

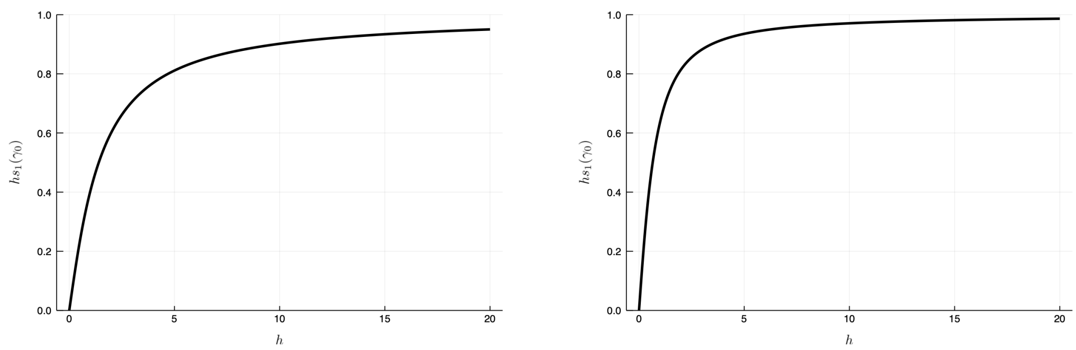
Table I.3: This table summarizes our finding for $\mathfrak{G}^2 = \mathfrak{O}$, the squared overlap with the ground state. Here $n_i = \mathbf{u}_i \cdot \mathbf{g}$ and $a_i = N^{2/3}(\lambda_i - 2)$. The quantity γ_0 in the top row is the same term from Tables I.1 and I.2. In the second row, the variable t and the total sum, which is $O(1)$, depends on the disorder sample. All leading and subleadings terms of $\langle\mathfrak{G}^2\rangle$, and the thermal fluctuations of $\mathfrak{G}^2$, except the leading term, $1 - T$, of $\langle\mathfrak{G}^2\rangle$ in the last row, depend on the disorder sample.

Case	Thermal average $\langle\mathfrak{R}\rangle$	Thermal fluctuations of $\mathfrak{R}$	Result
$h = O(1)$ for all T	$h^2 s_2(\gamma_0) + O(N^{-\frac{1}{2}})$	$N^{-\frac{1}{2}}$ Gaussian	4.3.3
(and $h = 0$, $T > 1$)			4.3.8
$h \sim N^{-\frac{1}{6}}$, $T < 1$	$1 - T + O(N^{-\frac{1}{3}})$	$N^{-\frac{1}{3}}$ r.v. that depends on disorder	4.3.5
$h \sim N^{-\frac{1}{2}}$, $T < 1$	$(1-T)\tanh^2\left(\frac{\lvert n_1\rvert h\sqrt{N(1-T)}}{T}\right)$	$O(1)$ Bernoulli	4.3.7
(and $h = 0$, $T < 1$)			4.3.8

Table I.4: This table summarizes our finding for $\mathfrak{R}$, the overlap between two independent spins. The quantity γ_0 is the same term from the preceding tables and $n_1 = \mathbf{u}_1 \cdot \mathbf{g}$. The subleading terms of $\langle\mathfrak{R}\rangle$ in the top two rows and the leading term of $\langle\mathfrak{R}\rangle$ in the third row depend on the disorder sample. The thermal fluctuations of $\mathfrak{R}$ also depend on the disorder sample for the bottom two rows.

1.3.3 Geometry of the Gibbs measure

The results for the overlaps give us information on the geometry of the spin configuration under the Gibbs measure, some of which we summarize here. Recall that the spin configuration is parameterized by the vector $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_N)$ which belongs to the $N-1$ dimensional sphere of radius $\sqrt{N}$ and we consider the limit of large N. At high temperature, $T > 1$, the spin vector $\boldsymbol{\sigma}$ is nearly orthogonal to the ground state $\pm\mathbf{u}_1$ when $h = 0$. For $h = O(1)$, the spin vector concentrates on the intersection of the sphere and the single cone around the vector g. The leading term of the cosine of the angle between the spin and the external field g depends on the temperature and the field but not on the disorder sample, and, as one can expect, is an increasing function of the field. See Figure I.2 (a). This implies that as the field becomes stronger, the cone becomes narrower. There are no transitions between $h = 0$ and $h = O(1)$.



(a) $T = 2$ (b) $T = 0.5$

Figure I.2: These are plots of the leading term of the angle between the spin and $\mathbf{g}$. The formula is given by $\mathfrak{M}^0$ in Subsection 4.1.3.2. The function depends only on T and h. (a) $T = 2$, (b) $T = 0.5$.

Now consider the low temperature regime $0 < T < 1$. When $h = 0$, the spins are concentrated on the intersection of the sphere with the double cone around the ground state $\pm\mathbf{u}_1$ such that the leading term of the cosine of the angle is $\sqrt{1-T}$. This angle was found in [28, 15, 22] and in particular, [38] showed that spins are distributed uniformly on the intersection of this double cone with the sphere. Consider increasing the external field strength h. When $h = O(1)$, the spin vector concentrates on the intersection of the sphere and the single cone around the vector $\mathbf{g}$ just like the high temperature case. See Figure I.2 (b), which is qualitatively same as Figure I.2 (a). However now between $h = 0$ and $h = O(1)$, there are two interesting transitional regimes, $h \sim N^{-1/2}$, which we call the microscopic regime, and $h \sim N^{-1/6}$, the mesoscopic regime.

In the microscopic regime, $h \sim N^{-1/2}$, at low temperature $0 < T < 1$, the results of this thesis lead us to the Conjecture 4.4.1, which implies that the double cone becomes polarized into a single cone. The spin vector prefers the cone which is closer to $\mathbf{g}$ to the other cone by the

$$e^{\frac{2h\sqrt{N}|n_1|\sqrt{1-T}}{T}} \text{ to } 1 \text{ probability ratio.}$$

The spin vector is more or less uniformly distributed on the cones. In this regime, the response of the spin to the field is the sum of (i) a linear response in the direction transverse to $\pm\mathbf{u}_1$ (i.e. along the cones) and, (ii) the response of an effective 2-level system, which may be modeled as a single one-component effective Ising spin $\frac{\boldsymbol{\sigma}}{\sqrt{N}} = \pm S\mathbf{u}_1$ of size $S = \frac{|n_1|\sqrt{1-T}}{\sqrt{N}}$ with energy scale $E = NhS = \sqrt{N}h|n_1|\sqrt{1-T}$ (leading to a mean magnetization $S\tanh(E/T)$). Note that both S and E are sample dependent, but depend only on $|n_1|$, the overlap of the ground state and the field.

For $h \sim N^{-1/6}$, all eigenvectors and eigenvalues become important. In this regime, the spins are concentrated on the intersection of the sphere and a single cone around the ground state, but the cone depends on the disorder sample. The cosine of the angle between the spin and $\mathbf{u}_1$ changes from $\sqrt{1-T}$ to a function which depends on all eigenvalues λ_i and the overlaps $n_i = \mathbf{u}_i \cdot \mathbf{g}$ of the eigenvectors and the external field. Furthermore, the spins are no longer uniformly distributed on the cone. They are pulled into the direction of g. This regime can be called "mesoscopic" as sample to sample fluctuations are strong and non trivial. Note that in the present model the magnetic response to the field, although non-trivial and sample dependent, does not exhibit jumps (so-called static avalanches or shocks) at very low temperature, as were observed and studied in other mean-field models such as the SK model; see [50, 49, 31, 32, 33].

For more details on the geometry of the Gibbs measure see Section 4.4, in particular, Table IV.1 and the summary in Subsection 4.4.3.

1.3.4 Detailed analysis of the microscopic external field

Chapter V focuses on the case where $h \sim N^{-1/2}$, also known as the microscopic regime. In particular, we present a rigorous proof of Theorem 5.2.1 (stated below), which provides the moment generating function for the overlap $\mathfrak{M}$. This is essentially a rigorous version of Result 4.1.6 whose corollary, Result 4.1.7, appears in the table above.

It is important to note that this overlap involves two types of randomness. First, we have randomness from the choice of M and g, which we refer to jointly as the "disorder sample." Second, we have randomness from the choice of spin variable. For the proofs in Chapter V, we fix an arbitrary disorder sample so that $\mathfrak{M}$ is a random variable depending on a fixed disorder sample and random spin variable. The moment generating function in Theorem 5.2.1 provides the distribution of $\mathfrak{M}$ as a function of the fixed disorder sample.

This result is valid for an arbitrary disorder sample, subject to certain constraints that hold with high probability. In particular, for any sufficiently small $\varepsilon > 0$, the event $\mathcal{E}_\varepsilon$ (defined in Section 5.1.4) provides a set of conditions on M and g that are sufficient for Theorem 5.2.1 to hold. Section 5.1.4 provides a detailed description of the event $\mathcal{E}_\varepsilon$ along with a proof that

$$\mathbb{P}(\mathcal{E}_\varepsilon) \geq 1 - N^{-\varepsilon/10} \quad \text{for all sufficently small } \varepsilon > 0 \text{ and all sufficiently large } N. \tag{1.3.2}$$

Theorem (5.2.1)**.** *Given $T < 1$ with $h = HN^{-1/2}$ for some some fixed $H \geq 0$ and $n_1 := \mathbf{u}_1 \cdot \mathbf{g}$, we have the following asymptotic formula for the moment generating function of $\mathfrak{M}$, the overlap with the external field. This formula holds on the event $\mathcal{E}_\varepsilon$ (which has probability at least $1 - N^{-\varepsilon/10}$)*

for any sufficiently small $\varepsilon > 0$ and $\xi = O(1)$.

$$\langle e^{\xi\sqrt{N}\mathfrak{M}}\rangle = e^{H\xi+\frac{T\xi^2}{2}} \frac{\cosh\left((H+T\xi)|n_1|\frac{\sqrt{1-T}}{T}\right)}{\cosh\left(H|n_1|\frac{\sqrt{1-T}}{T}\right)}\left(1+O(N^{-\frac{1}{21}+\frac{\varepsilon}{7}})\right). \tag{1.3.3}$$

Note that the leading term on the right-hand side is the product of two terms implying that it is the moment generating function of a sum of two independent random variables. The exponential term is the moment generating function of a Gaussian random variable. For the ratio of the cosh functions, we note that the moment generating function of a shifted Bernoulli random variable that takes values 1 and -1 with probabilities P and $1-P$ respectively is $Pe^t + (1-P)e^{-t}$. The ratio of cosh functions in Theorem 5.2.1 is of this form with $t = \xi|n_1|\sqrt{1-T}$ and

$$P = \frac{e^{\frac{H}{T}|n_1|\sqrt{1-T}}}{e^{\frac{H}{T}|n_1|\sqrt{1-T}} + e^{-\frac{H}{T}|n_1|\sqrt{1-T}}}. \tag{1.3.4}$$

Hence, for any large N, we can conclude that, on the event $\mathcal{E}_\varepsilon$, the scaled overlap $\sqrt{N}\mathfrak{M}$ behaves in its leading order like the independent sum of a Gaussian random variable (with mean H and variance T) and a shifted Bernoulli random variable (which takes values $|n_1|\sqrt{1-T}$ and $-|n_1|\sqrt{1-T}$ with probability P and $1-P$ respectively for the value of P stated above).

In addition to the theorem above, whose proof is published in [13], we also provide the unpublished proof of Theorem 5.3.1 for the overlap with a replica in the microscopic regime, using a similar method.

1.3.5 Application to magnetization and susceptibility

One important application of Theorem 5.2.1 is that it confirms the conjectures of [6] regarding magnetization and susceptibility. Magnetization is defined to be $\langle\mathfrak{M}\rangle$, the Gibbs average of the overlap with the external field. Susceptibility is the magnetization per unit external field strength, given by

$$\mathcal{X} = \frac{\langle\mathfrak{M}\rangle}{h}. \tag{1.3.5}$$

It follows from Theorem 5.2.1 that, on the event $\mathcal{E}_\varepsilon$, when $T < 1$ and $h = HN^{-1/2}$ for H constant, the susceptibility is

$$\mathcal{X} = 1 + \frac{|n_1|\sqrt{1-T}}{H}\tanh\left(H|n_1|\frac{\sqrt{1-T}}{T}\right) + O(N^{-\frac{1}{21}+\frac{\varepsilon}{3}}). \tag{1.3.6}$$

Of particular interest in the physics literature is the zero external field limit of the susceptibility. Cugliandolo, Dean, and Yoshino [16] discuss two ways to taking this limit, namely $\lim_{h\to 0}\lim_{N\to\infty}\mathcal{X}$ and $\lim_{N\to\infty}\lim_{h\to 0}\mathcal{X}$ (the first of these was also considered by [28]). Our results for the microscopic external field give a different way of calculating the second of these limits. In particular, since $n_1 \sim \mathcal{N}(0,1)$ for all N, we can compute

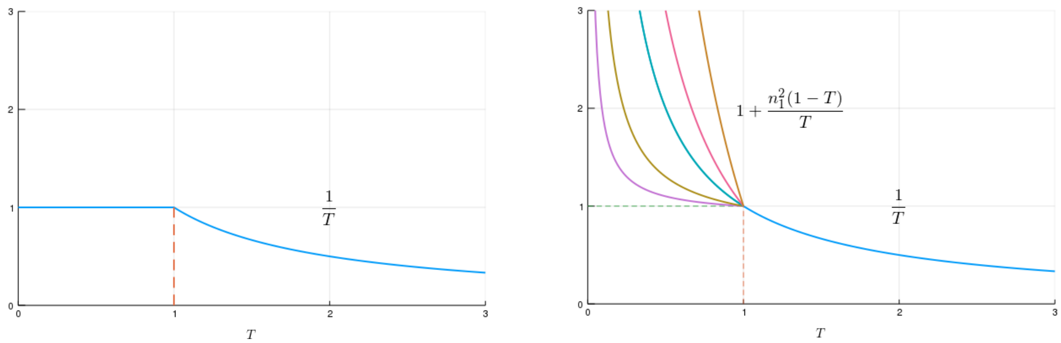
$$\lim_{H\to 0}\lim_{\substack{N\to\infty \\ h=HN^{-1/2}}} \mathcal{X} \overset{\mathcal{D}}{=} 1 + \frac{\nu^2(1-T)}{T} \qquad \text{for } T<1 \text{ and } \nu \sim \mathcal{N}(0,1) \tag{1.3.7}$$

where $\overset{\mathcal{D}}{=}$ denotes equality in distribution. This confirms the conjecture of [6]. It is also consistent with [16], which found that

$$\lim_{N\to\infty}\lim_{h\to 0}\mathcal{X} = \frac{1}{T} \qquad \text{for } T<1 \qquad \text{when } |n_1|=1 \qquad [16]. \tag{1.3.8}$$

By removing this constraint on $|n_1|$ and applying Theorem 5.2.1, we are able to show that the limiting susceptibility is not constant, as the result from [16] might suggest, but rather it is a random variable whose distribution is given explicitly in (1.3.7).

Figure I.3 (b) illustrates the dependence of the limit of $\mathcal{X}$ on the disorder variable n_1^2. This result shows that the Curie law (inverse proportionality of susceptibility and temperature) holds for the sample-to-sample average, but not for a given disorder sample. If we take a different limit, namely if we let $N\to\infty$ with $h>0$ first and then let $h\to 0$, then the limit of the susceptibility is deterministic and given by $\min\{T^{-1},1\}$. See Figure I.3 (a). This formula was previously obtained in [28], and also in [16]. See Subsections 4.1.7 and 4.1.8 for details including some further conjectures about the zero external field limit of differential susceptibility (the derivative of the magnetization with respect to external field strength). These can also be verified using Theorem 5.2.1.



(a) $\lim_{h\to 0} \lim_{\substack{N\to\infty \\ h>0}} \mathcal{X}$ (b) $\lim_{H\to 0} \lim_{\substack{N\to\infty \\ h=HN^{-1/2}}} \mathcal{X}$ for sevaral values of n_1^2

Figure I.3: Graph of the zero external field limit of the susceptibility as a function of T.

1.4 Method of analysis

Our computations are based on contour integral representations which we present in Section 2.3. The free energy and the moment generating functions of two of the overlaps can be expressed in terms of a single integral, whereas the moment generating function in the case of the overlap with a replica can be written as a double integral. The integrand for each of these integral representations contains disorder variables and hence we have random integral formulas. The single integral formula for the free energy was first observed by Kosterlitz, Thouless and Jones [28] and the authors use the method of the steepest descent to evaluate the limiting free energy. For the case of $h = 0$, this calculation was extended in [7] to find the fluctuation terms using the recent advancements in random matrix theory, in particular the rigidity results on the eigenvalues [19] and the linear statistics [24, 5, 34]. Similar ideas were also used in [8, 9, 10], including the case for the overlap with a replica in [29]. This thesis extends the integral formula approach to the case when $h = O(1)$ and $h \to 0$ in the transitional regimes. When there is an external field, the analysis becomes more involved. In this case, the dot products of the eigenvectors and the external field play an important role in the analysis.

The steepest descent analysis in chapters III and IV can be made mathematically rigorous after some efforts using probability theory and random matrix theory. However, those chapters focus on computations and interpretations assuming that various estimates in the steepest descent analysis can be obtained. In those chapters, we use the label "Result" for findings in which we do not provide rigorous proofs and the label "Theorem" for findings that we cite from prior papers that include rigorous proof. We use the label "Lemma" for short findings that we prove in full detail.

In a recent preprint [30], which was obtained independently and simultaneously with the work of chapters III and IV, Landon and Sosoe consider a similar SSK model in which the external field is a fixed vector and the disorder matrix has zero diagonal entries. Their work is mathematically rigorous and contains proofs of some of the results obtained in this thesis, namely for the free energy and some aspects of the overlaps with the external field and with a replica in Subsections 3.1.2, 3.2.1, 3.3.1, 4.1.3, 4.1.5, 4.3.1, and 4.3.2. I subsequently proved the results in Subsection 4.1.6 on the overlap with the microscopic external field rigorously in [13] and that proof is provided in Chapter V.

1.5 Organization of this thesis

Chapters I and II introduce the model, background information, and preliminary lemmas. Chapter III presents the analysis and results for free energy, while Chapter IV focuses on the overlaps. These two chapters are based on joint work with Jinho Baik, Pierre le Doussal, and Hao Wu. Chapter V presents a more detailed analysis of the overlaps for a microscopic external field ($h \sim N^{-1/2}$), which is work of the thesis writer only.

Much of the work in this thesis is already published. The contents of Chapters II, III, IV are published in [6] along with some pieces of Chapter I. The contents in the first two sections of Chapter V are published in [13], along with pieces of Chapter I. Section 5.3 contains a related, unpublished proof.

CHAPTER II

Preliminaries from Random Matrix Theory and Spin Glasses

2.1 Classical Random Matrix Results

Throughout this thesis we discuss the random matrix M (which we refer to as the interaction matrix) and the random vector g. In this section we provide definitions notations and some well known results from random matrix theory. The study of random matrices, and especially their eigenvalues began in the 1950s with Eugene Wigner's work in developing statistical models for heavy-nuclei atoms [48]. Since then, interest in random matrices has grown, not only in physics, but in many other fields including machine learning, statistics, finance, probability, and number theory to name a few. For a broad introduction to random matrix theory the reader may refer to [4], [35] . Here we focus on one simple and well-studied type of random matrix: the Gaussian Orthogonal Ensemble.

Definition 2.1.1. *A matrix from the Gaussian Orthogonal Ensemble (GOE) is a real-valued symmetric matrix*

$$M = (M)_{i,j=1}^{N} \tag{2.1.1}$$

such that, for $i \leq j$, the variables M_{ij} are independent centered Gaussian random variables with variance $\frac{1}{N}(1 + \delta_{ij})$. By the symmetric matrix condition, $M_{ij} = M_{ji}$ for $i > j$.

An alternative way of defining GOE is to omit the $1/N$ factor in the variance such that the diagonal and off-diagonal elements have variance 2 and 1 respectively. However, we choose to rescale by $1/N$ so that the eigenvalues are of order 1. We denote by

$$\lambda_1 \geq \cdots \geq \lambda_N \quad \text{and} \quad \mathbf{u}_1, \cdots, \mathbf{u}_N \tag{2.1.2}$$

the eigenvalues of M and corresponding unit eigenvectors (based on Euclidean norm).

The definition above induces a probability measure on the space of $N \times N$ symmetric matrices. An important property of GOE matrices is that this probability measure is invariant under conjugation by an orthogonal matrix (hence the name Gaussian *Orthogonal* Ensemble). This implies that the eigenvalues and eigenvectors are independent of each other. Furthermore, the eigenvectors are uniformly distributed on the unit sphere. The eigenvalues are well studied in the literature and some of their key properties are summarized below.

2.1.1 Probability notations

There are two types of randomness, one from the disorder sample M and g, and the other from the Gibbs (thermal) measure. We often need to distinguish them. We add the subscript s to denote sample probability or sample expectation such as $\mathbb{P}_s$ and $\mathbb{E}_s$. In addition, we use the following notations.

Definition 2.1.2. *When describing the limiting distributions in our results, we consider two classes of random variables, which we refer to as sample random variables and thermal random variables. To distinguish between these two classes, we denote them with the calligraphic font and the gothic font respectively. For example a standard Gaussian sample random variable and a standard Gaussian thermal variable will be denoted below by*

$$\mathcal{N} \quad \text{and} \quad \mathfrak{N} \tag{2.1.3}$$

respectively.

Definition 2.1.3. *Asymptotic notations:*

- *If $\{E_N\}_{N=1}^{\infty}$ is a sequence of events, we say that E_N holds asymptotically almost surely (or everywhere) if $\mathbb{P}_s(E_N) \to 1$ as $N \to \infty$. This probability is with respect to the choice of disorder sample.*
- *For two N-dependent random variables $A := A_N$ and $B := B_N$, the notation*

$$A = \mathcal{O}(B) \tag{2.1.4}$$

  means that, for any $\varepsilon > 0$, the inequality $A \leq BN^{\varepsilon}$ holds asymptotically almost surely.
- *The notation $\simeq$ means an asymptotic expansion up to the terms indicated on the right-hand side and the notation $A \asymp B$ indicates that A and B are of the same order as $N \to \infty$ (i.e. $c^{-1}B < A < cB$ for some constant c and sufficiently large N). When we say $A \asymp \mathcal{O}(B)$ we mean that, for any $\varepsilon > 0$, the inequality $BN^{-\varepsilon} < A < BN^{\varepsilon}$ holds asymptotically almost surely.*

- *For ease of notation, we write $h = O(1)$ to denote the case where h is a positive constant. In this case we do not mean to imply that h has any dependence on N.*

Definition 2.1.4. *Convergence notations:*

- *The convergence in distribution of a sequence of random variables X_N to a random variable X with respect to the disorder variables is denoted by $X_N \Rightarrow X$.*
- *We use the notations $\overset{\mathcal{D}}{=}$ and $\overset{\mathcal{D}}{\simeq}$ to denote an equality and an asymptotic expansion in distribution with respect to the disorder sample, respectively.*
- *We use similar notations with a different font, $\overset{\mathfrak{D}}{=}$ and $\overset{\mathfrak{D}}{\simeq}$, to denote an equality and an asymptotic expansion in distribution with respect to the Gibbs (thermal) measure, respectively.*

It is worth noting that many of our results actually hold with high probability (i.e., there exist some $D > 0, N_0 > 0$ such that, for all $N \geq N_0$, $\mathbb{P}(E_N) > 1 - N^{-D}$). While high probability is much stronger than asymptotically almost sure probability, it is much more delicate to prove and we do not discuss those proofs in Chapters III and IV. In Chapter V we provide rigorous proofs for two results from the preceding chapter, demonstrating that these results indeed hold with high probability.

2.1.2 Semicircle law

The empirical distribution of eigenvalues of M converges to the semicircle law [35]: for every continuous bounded function $f(x)$,

$$\frac{1}{N}\sum_{i=1}^{N} f(\lambda_i) \to \int f(x)\mathrm{d}\sigma_{scl}(x) \quad \text{where} \quad \mathrm{d}\sigma_{scl}(x) = \frac{\sqrt{4-x^2}}{2\pi}\mathbb{1}_{x\in[-2,2]}\mathrm{d}x \tag{2.1.5}$$

with probability 1 as $N \to \infty$.

Definition 2.1.5. *We define the following functions for later use:*

$$s_0(z) := \int \log(z-x)\mathrm{d}\sigma_{scl}(x) \quad \textit{and} \quad s_k(z) := \int \frac{\mathrm{d}\sigma_{scl}(x)}{(z-x)^k} \quad \textit{for } k = 1, 2, \cdots, \tag{2.1.6}$$

Properties: These functions can be evaluated explicitly as

$$
\begin{aligned}
s_0(z) =& \frac{1}{4} z(z - \sqrt{z^2-4}) + \log(z + \sqrt{z^2-4}) - \log 2 - \frac{1}{2}, \\
s_1(z) =& \frac{z - \sqrt{z^2-4}}{2}, \quad s_2(z) = \frac{z - \sqrt{z^2-4}}{2\sqrt{z^2-4}}, \quad s_3(z) = \frac{1}{(z^2-4)^{3/2}}, \quad s_4(z) = \frac{z}{(z^2-4)^{5/2}}
\end{aligned}
\tag{2.1.7}
$$

for z not in the real interval $[-2, 2]$. As $z \to 2$, we have

$$
s_1(z) \simeq 1 - \sqrt{z-2}, \quad s_2(z) \simeq \frac{1}{2\sqrt{z-2}} - \frac{1}{2}, \quad s_3(z) \simeq \frac{1}{8(z-2)^{3/2}}, \quad s_4(z) \simeq \frac{1}{16(z-2)^{5/2}}.
\tag{2.1.8}
$$

2.1.3 Rigidity

Definition 2.1.6. *For $i = 1, 2, \cdots, N$, let $\widehat{\lambda}_i$ be the classical location defined by the quantile conditions*

$$
\int_{\widehat{\lambda}_i}^{2} \mathrm{d}\sigma_{scl}(x) = \frac{i}{N}. \tag{2.1.9}
$$

We set $\widehat{\lambda}_0 = 2$. We also set $\widehat{a}_i = (\widehat{\lambda}_i - 2)N^{2/3}$.

Rigidity property: The rigidity result [20, 19] states that

$$
|\lambda_i - \widehat{\lambda}_i| \le (\min\{i, N+1-i\})^{-1/3} \mathcal{O}\left(N^{-2/3}\right) \tag{2.1.10}
$$

uniformly for $i = 1, 2, \cdots, N$.

The rigidity property allows us to apply the method of steepest descent to evaluate the integrals involving the eigenvalues since the eigenvalues are close enough to the classical location, and the fluctuations are small enough.

2.1.4 Edge behavior

Definition 2.1.7.

- *Define the rescaled eigenvalues*

$$
a_i := N^{2/3}(\lambda_i - 2). \tag{2.1.11}
$$

- *Define* $\{\alpha_i\}_{i=1}^{\infty}$ *to be the GOE Airy point process process to which the rescaled eigenvalues converge in distribution as* $N \to \infty$ *[47, 41]:*

$$\{a_i\} \Rightarrow \{\alpha_i\}. \tag{2.1.12}$$

Properties:

The rightmost point α_1 of the GOE Airy point process has the GOE Tracy-Widom distribution

$$a_1 \Rightarrow \alpha_1 \overset{\mathcal{D}}{=} \mathrm{TW}_{GOE}\,. \tag{2.1.13}$$

The GOE Airy point process satisfies the asymptotic property that

$$\alpha_i \simeq -\left(\frac{3\pi i}{2}\right)^{2/3} \quad \text{as } i \to \infty. \tag{2.1.14}$$

This asymptotic is due to the fact that the semicircle law is asymptotic to $\frac{\sqrt{2-x}}{\pi}\mathrm{d}x$ as $x \to 2$. The above formula and the rigidity imply that, with high probability,

$$a_i \asymp -i^{2/3} \quad \text{as } i, N \to \infty \text{ satisfying } i \leq N \tag{2.1.15}$$

2.1.5 Central limit theorem of linear statistics

For a function f which is analytic in an open neighborhood of $[-2, 2]$ in the complex plane, consider the sum of $f(\lambda_i)$. The semicircle law (2.1.5) gives its leading behavior. If we subtract the leading term, the difference

$$\sum_{i=1}^{N} f(\lambda_i) - N \int f(x)\mathrm{d}\sigma_{scl}(x) \tag{2.1.16}$$

converges to a Gaussian distribution with explicit mean and variance; see, for example, [24, 5, 34]. Note that unlike the classical central limit theorem, we do not divide by $\sqrt{N}$.

Definition 2.1.8. *Define*

$$\mathcal{L}_N(z) := \sum_{i=1}^{N} \log(z - \lambda_i) - N s_0(z). \tag{2.1.17}$$

for $z > 2$ *where* $s_0(z)$ *is given by* (2.1.7).

Properties:

The above-mentioned central limit theorem implies in this case that

$$\mathcal{L}_N(z) \Rightarrow \mathcal{N}(M(z), V(z)) \tag{2.1.18}$$

where (see Lemma A.1 in [7])

$$M(z) = \frac{1}{2}\log\left(\frac{2\sqrt{z^2-4}}{z+\sqrt{z^2-4}}\right), \qquad V(z) = 2\log\left(\frac{z+\sqrt{z^2-4}}{2\sqrt{z^2-4}}\right). \tag{2.1.19}$$

For later uses, we record that for $0 < \beta < 1$,

$$M(\beta+\beta^{-1}) = \frac{1}{2}\log(1-\beta^2), \qquad V(\beta+\beta^{-1}) = -2\log(1-\beta^2). \tag{2.1.20}$$

2.2 More Specific Preliminaries for This Research (Special sums)

In this section we collect several important results about convergence of various types of sums that we will use throughout. Before presenting those results, we recall that he external field is given by the vector

$$\mathbf{g} = (g_1, g_2, \cdots, g_N)^T, \tag{2.2.1}$$

which we assume to be a standard Gaussian vector (i.e. $\{g_i\}$ are independent standard Gaussians), and the strength of the external field is denoted by a non-negative scalar h. We also define

$$n_i = \mathbf{u}_i \cdot \mathbf{g}, \tag{2.2.2}$$

the overlap of the eigenvector and the external field. The external field and eigenvectors appear in the results and analysis of this thesis only as this combination. The variables λ_i and n_i are collectively called disorder variables. We call the joint realization of λ_i and n_i a disorder sample throughout the thesis. Note that $(n_1, \cdots, n_N)$ is a standard Gaussian vector due to the rotational invariance of the Gaussian measure. Furthermore, its entries are independent of the eigenvalues $\lambda_1, \cdots, \lambda_N$. The analysis of this thesis also applies, after some changes of formulas, to the case when the external field is a deterministic vector of length N, for example $\mathbf{g} = (1, \cdots, 1)^T$. However, we restrict to the Gaussian external field since the Gaussian assumption makes calculations simpler.

Many of the results in this section are motivated by the need to work with sums of the form

$$\frac{1}{N}\sum_{i=2}^{N}\frac{1}{(\lambda_1-\lambda_i)^k}, \qquad k = 1, 2, \cdots, \tag{2.2.3}$$

or its variations. The above quantity looks superficially close to the linear statistics (2.1.16) with $f(x) = \frac{1}{(\lambda_1 - x)^k}$ with one term removed but the function $f(x)$ is singular at $x = \lambda_1$. We note that if we replace $f(x)$ by $\frac{1}{(2-x)^k}$ and use the semicircle law, we obtain $s_k(2)$ which diverges for $k \geq 2$. Hence, the result of the previous subsection does not apply. On the other hand, for $k = 1$, $s_1(2) = 1$. This fact indicates that the above sum still converges when $k = 1$.

We present several definitions, followed by their related convergence results and some brief explanation of why these results hold. Recall the definition $a_i = (\lambda_i - 2)N^{2/3}$ and $n_i = \mathbf{u}_i \cdot \mathbf{g}$.

Definition 2.2.1. *We define the following random sums, which depend on the disorder sample:*

- *Define*
$$\Xi_N := N^{1/3}\left(\frac{1}{N}\sum_{i=2}^{N}\frac{1}{\lambda_1 - \lambda_i} - 1\right) = \sum_{i=2}^{N}\frac{1}{a_1 - a_i} - N^{1/3}. \tag{2.2.4}$$

- *Define, for $w > 0$,*
$$\mathcal{E}_N(w) := N^{1/3}\left[\frac{1}{N}\sum_{i=1}^{N}\frac{n_i^2}{wN^{-2/3} + \lambda_1 - \lambda_i} - 1\right] = \sum_{i=1}^{N}\frac{n_i^2}{w + a_1 - a_i} - N^{1/3}. \tag{2.2.5}$$

- *Define, for $z > 2$,*
$$\mathcal{S}_N(z;k) := \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{n_i^2 - 1}{(z - \widehat{\lambda}_i)^k} \qquad \textit{for } k \geq 1. \tag{2.2.6}$$

Definition 2.2.2. *We define the following limits, which depend on the GOE Airy point process $\{\alpha_i\}$:*

- *Define*
$$\Xi := \lim_{n\to\infty}\left(\sum_{i=2}^{n}\frac{1}{\alpha_1 - \alpha_i} - \frac{1}{\pi}\int_0^{\left(\frac{3\pi n}{2}\right)^{2/3}}\frac{\mathrm{d}x}{\sqrt{x}}\right). \tag{2.2.7}$$

  Landon and Sosoe showed that the limit exists almost surely [29].

- *Define $\mathcal{E}(s)$ as follows, where ν_i are i.i.d. Gaussian random variables with mean 0 and variance 1 independent of the GOE Airy point process α_i:*
$$\mathcal{E}(s) := \lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{\nu_i^2}{s + \alpha_1 - \alpha_i} - \frac{1}{\pi}\int_0^{\left(\frac{3\pi n}{2}\right)^{2/3}}\frac{\mathrm{d}x}{\sqrt{x}}\right). \tag{2.2.8}$$

  This limit exists almost surely by a similar argument as in [29] showing that Ξ exists.

Result 2.2.3. *Using the notations above, we have the following convergence results.*

- *Landon and Sosoe proved in [29] that*

$$\Xi_N \Rightarrow \Xi. \tag{2.2.9}$$

  They use this result to describe the fluctuations of the overlap with a replica when $h = 0$ *and* $T < 1$.

- *We also need another version of the result* (2.2.9) *where the constant numerators are replaced* n_i^2*:*

$$N^{1/3}\left(\frac{1}{N}\sum_{i=2}^{N}\frac{n_i^2}{\lambda_1-\lambda_i}-1\right) \Rightarrow \lim_{n\to\infty}\left(\sum_{i=2}^{n}\frac{\nu_i^2}{\alpha_1-\alpha_i}-\frac{1}{\pi}\int_0^{\left(\frac{3\pi n}{2}\right)^{2/3}}\frac{\mathrm{d}x}{\sqrt{x}}\right) \tag{2.2.10}$$

  where ν_i *are i.i.d standard Gaussians, independent of the GOE Airy point process* α_i*. This follows from* (2.2.9) *and the fact that*

$$\frac{1}{N^{2/3}}\sum_{i=2}^{N}\frac{n_i^2-1}{\lambda_1-\lambda_i} \Rightarrow \sum_{i=2}^{\infty}\frac{\nu_i^2-1}{\alpha_1-\alpha_i} \tag{2.2.11}$$

  which is a convergent series due to Kolmogorov's three series theorem and (2.1.14)*.*

- *By the same argument as for 2.2.10,*

$$\mathcal{E}_N(w) \Rightarrow \mathcal{E}(w) \quad \text{for } w > 0. \tag{2.2.12}$$

- *By the Lyapunov central limit theorem and the definition of* $\widehat{\lambda}_i$*, we have*

$$\mathcal{S}_N(z;k) \Rightarrow \mathcal{N}(0, 2s_{2k}(z)) \tag{2.2.13}$$

  as $N \to \infty$ *for* $z > 2$*. (Note that the variance of* $n_i^2 - 1$ *is* 2*.)*

Result 2.2.4. *In addition to the convergence results listed above, we also need estimates that hold for asymptotically almost every disorder sample.*

- *A consequence of* (2.2.9) *is that*

$$\frac{1}{N}\sum_{i=2}^{N}\frac{1}{\lambda_1-\lambda_i} = \frac{1}{N^{1/3}}\sum_{i=2}^{N}\frac{1}{a_1-a_i} = 1+\mathcal{O}\left(N^{-1/3}\right). \tag{2.2.14}$$

for asymptotically almost every disorder sample.

- *We have*

$$\sum_{i=2}^{N} \frac{1}{(a_1 - a_i)^k} = \mathcal{O}(1) \quad \text{and} \quad \sum_{i=2}^{N} \frac{n_i^2}{(a_1 - a_i)^k} = \mathcal{O}(1), \qquad k \geq 2, \tag{2.2.15}$$

  for asymptotically almost every disorder sample. This follows from the fact that the $a_i \asymp -i^{2/3}$ and $(a_1 - a_2)^{-1}$ is bounded of order 1 with high probability.

- *We also need the result*

$$\frac{1}{N} \sum_{i=1}^{N} \frac{n_i^2}{(z - \lambda_i)^k} = s_k(z) + \frac{\mathcal{S}_N(z; k)}{\sqrt{N}} + \mathcal{O}\left(N^{-1}\right), \qquad z > 2, \quad k > 1 \tag{2.2.16}$$

  for asymptotically almost every disorder sample. To justify (2.2.16)*, we observe that*

$$\frac{1}{N} \sum_{i=1}^{N} \frac{n_i^2}{(z - \lambda_i)^k} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{(z - \lambda_i)^k} + \frac{1}{N} \sum_{i=1}^{N} \frac{n_i^2 - 1}{(z - \lambda_i)^k}. \tag{2.2.17}$$

  We then use the central limit theorem (2.1.16) *for linear statistics for the first sum and replace λ_i by $\widehat{\lambda}_i$ in the second sum using the rigidity* (2.1.10).

2.3 Contour integral representations

The partition function is an N-fold integral over a sphere. Using the Laplace transform and Gaussian integrations, Kosterlitz, Thouless and Jones showed in [28] that this integral can be expressed as a single contour integral which involves the disorder sample. We state this result and also include its derivation in Subsection 2.3.1. By the same method, the moment generating function of each overlap can also be written as a ratio of single or double contour integrals. These results are presented in section 2.3.2.

2.3.1 Free energy

The following result holds for any disorder sample.

Lemma 2.3.1 ([28])**.** *Let M be an arbitrary N by N symmetric matrix and let* **g** *be an N dimensional vector. Let $\lambda_1 \geq \cdots \geq \lambda_N$ be the eigenvalues of the matrix M and let $\mathbf{u}_i$ be a corresponding* *unit eigenvector. Then, the partition function* $\mathcal{Z}_N$ *defined in* (1.1.13) *can be written as*

$$\mathcal{Z}_N = C_N \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} e^{\frac{N}{2}\mathcal{G}(z)} \mathrm{d}z \quad \textit{where} \quad C_N = \frac{\Gamma(N/2)}{2\pi \mathrm{i}(N\beta/2)^{N/2-1}} \tag{2.3.1}$$

and

$$\mathcal{G}(z) = \beta z - \frac{1}{N}\sum_{i=1}^{N} \log(z - \lambda_i) + \frac{h^2\beta}{N}\sum_{i=1}^{N} \frac{n_i^2}{z - \lambda_i} \quad \textit{with} \quad n_i = \mathbf{u}_i \cdot \mathbf{g}. \tag{2.3.2}$$

Here, the integration is over the vertical line $\gamma + \mathrm{i}\mathbb{R}$ *where* γ *is an arbitrary constant satisfying* $\gamma > \lambda_1$.

Proof. Let $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)$. Let $O = (\mathbf{u}_1, \cdots, \mathbf{u}_N)$ be an orthogonal matrix so that $M = O\Lambda O^T$. Let S^{N-1} be the sphere of radius 1 in $\mathbb{R}^N$ and let $\mathrm{d}\Omega_{N-1}$ be the surface area element on S^{N-1}. Then, using the change of variables $\frac{1}{\sqrt{N}} O^T \boldsymbol{\sigma} = x$,

$$\mathcal{Z}_N = \frac{1}{|S^{N-1}|} I\left(\frac{\beta N}{2}, h\sqrt{2\beta}\right) \quad \text{where} \quad I(t,s) = \int_{S^{N-1}} e^{t\sum_{i=1}^N \lambda_i x_i^2 + s\sqrt{t}\sum_{i=1}^N n_i x_i} \mathrm{d}\Omega_{N-1}(x).$$

where $n_i = (O^T\mathbf{g})_i = \mathbf{u}_i \cdot \mathbf{g}$. We take the Laplace transform of $J(t) = t^{N/2-1} I(t,s)$. Making a simple change of variables $t = r^2$ and using Gaussian integrals, the Laplace transform is equal to

$$L(z) = \int_0^\infty e^{-zt} J(t) \mathrm{d}t = 2\int_{\mathbb{R}^N} e^{-\sum_{i=1}^N (z-\lambda_i) y_i^2 + s\sum_{i=1}^N n_i y_i} \mathrm{d}^N y = 2\prod_{i=1}^N e^{\frac{s^2 n_i^2}{4(z-\lambda_i)}} \sqrt{\frac{\pi}{z - \lambda_i}}$$

for z satisfying $z > \lambda_1$. We obtain a single integral formula of the partition function by taking the inverse Laplace transform. □

Note that the sign ambiguity of $\mathbf{u}_i$ does not affect the result since the formula depends only on n_i^2.

2.3.2 Overlaps

In this section, we give the moment generating function of each of the overlaps, expressed as a ratio of contour integrals. The proofs are similar to the computations for the free energy case and we sketch them at the end of this section.

Definition 2.3.2. . *The following three functions are related to the function* $\mathcal{G}$ *and will be used to compute the three overlaps respectively. We denote by* $\eta \in \mathbb{R}$ *the parameter that will be used for the moment generating function of each overlap.*

- *For the overlap with the external field, we use the function*

$$\mathcal{G}_{\mathfrak{M}}(z) := \beta z - \frac{1}{N}\sum_{i=1}^{N}\log(z-\lambda_i) + \frac{(h+\frac{\eta}{N})^2\beta}{N}\sum_{i=1}^{N}\frac{n_i^2}{z-\lambda_i}. \tag{2.3.3}$$

Note that this is $\mathcal{G}(z)$ with h replaced by $h+\eta N^{-1}$.

- *For the (square of the) overlap with the ground state, we use the function*

$$\begin{aligned}\mathcal{G}_{\mathfrak{O}}(z) :=& \beta z - \frac{1}{N}\log\left(z-\left(\lambda_1+\frac{2\eta}{N}\right)\right) - \frac{1}{N}\sum_{i=2}^{N}\log(z-\lambda_i)\\ &+\frac{h^2\beta}{N}\frac{n_1^2}{z-(\lambda_1+\frac{2\eta}{N})} + \frac{h^2\beta}{N}\sum_{i=2}^{N}\frac{n_i^2}{z-\lambda_i}.\end{aligned} \tag{2.3.4}$$

Note that this is $\mathcal{G}(z)$ with λ_1 replaced by $\lambda_1 + \frac{\eta}{\beta N}$.

- *For the overlap with a replica, we use the function*

$$\begin{aligned}\mathcal{G}_{\mathfrak{R}}(z,w;a) :=& \beta(z+w) - \frac{1}{N}\sum_{i=1}^{N}\log\left((z-\lambda_i)(w-\lambda_i)-a^2\right)\\ &+\frac{h^2\beta}{N}\sum_{i=1}^{N}\frac{n_i^2(z+w-2\lambda_i+2a)}{(z-\lambda_i)(w-\lambda_i)-a^2}.\end{aligned} \tag{2.3.5}$$

Lemma 2.3.3. *For real parameter η, the moment generation functions of the three overlaps are as follows:*

$$\langle e^{\beta\eta\mathfrak{M}}\rangle = \frac{\int e^{\frac{N}{2}\mathcal{G}_{\mathfrak{M}}(z)}\mathrm{d}z}{\int e^{\frac{N}{2}\mathcal{G}(z)}\mathrm{d}z}, \qquad \langle e^{\beta\eta\mathfrak{O}}\rangle = \frac{\int e^{\frac{N}{2}\mathcal{G}_{\mathfrak{O}}(z)}\mathrm{d}z}{\int e^{\frac{N}{2}\mathcal{G}(z)}\mathrm{d}z}, \qquad \langle e^{\eta\mathfrak{R}}\rangle = \frac{\iint e^{\frac{N}{2}\mathcal{G}_{\mathfrak{R}}(z,w;\frac{\eta}{\beta N})}\mathrm{d}z\mathrm{d}w}{\iint e^{\frac{N}{2}\mathcal{G}_{\mathfrak{R}}(z,w;0)}\mathrm{d}z\mathrm{d}w}. \tag{2.3.6}$$

The contours are vertical lines going upward in the complex plane such that all singularities lie on the left of the contour.

Proof. First,

$$\langle e^{\beta\eta\mathfrak{M}}\rangle = \frac{1}{\mathcal{Z}_N(h)}\int_{S_{N-1}} e^{\beta\frac{\eta}{N}\mathbf{g}\cdot\boldsymbol{\sigma}} e^{\beta\left(\frac{1}{2}\boldsymbol{\sigma}\cdot M\boldsymbol{\sigma}+h\mathbf{g}\cdot\boldsymbol{\sigma}\right)}\mathrm{d}\omega_N(\boldsymbol{\sigma}) = \frac{\mathcal{Z}_N(h+\eta N^{-1})}{\mathcal{Z}_N(h)}.$$

Secondly, by definition,

$$\langle e^{\beta\eta\mathfrak{D}}\rangle = \frac{1}{\mathcal{Z}_N}\int_{S_{N-1}} e^{\beta\frac{\eta}{N}(\mathbf{u}_1\cdot\boldsymbol{\sigma})^2} e^{\beta(\boldsymbol{\sigma}\cdot M\boldsymbol{\sigma}+h\mathbf{g}\cdot\boldsymbol{\sigma})}\mathrm{d}\omega_N(\boldsymbol{\sigma}). \tag{2.3.7}$$

Since

$$\frac{1}{2}\boldsymbol{\sigma}\cdot M\boldsymbol{\sigma} + \frac{\eta}{N}(\mathbf{u}_1\cdot\boldsymbol{\sigma})^2 = \frac{1}{2}\sum_{i=1}^{N}\lambda_i(\mathbf{u}_i\cdot\boldsymbol{\sigma})^2 + \frac{\eta}{N}(\mathbf{u}_1\cdot\boldsymbol{\sigma})^2,$$

the integral in (2.3.7) is the same as that of $\mathcal{Z}_N$ with $\lambda_1 \mapsto \lambda_1 + \frac{2\eta}{N}$. Finally, using the eigenvalue-eigenvector decomposition $M = O\Lambda O^T$ and changing variables $\frac{1}{\sqrt{N}}O^T\sigma = x$ and $\frac{1}{\sqrt{N}}O^T\tau = y$, we find that

$$\langle e^{\eta\mathfrak{R}}\rangle = \frac{J(\frac{\beta N}{2}, \frac{\beta N}{2}; \frac{\eta}{N\beta}, \frac{\sqrt{\beta}h}{\sqrt{2}}, \frac{\sqrt{\beta}h}{\sqrt{2}})}{J(\frac{\beta N}{2}, \frac{\beta N}{2}; 0, \frac{\sqrt{\beta}h}{\sqrt{2}}, \frac{\sqrt{\beta}h}{\sqrt{2}})}. \tag{2.3.8}$$

where we use the notation

$$J(u,v;a,b,c) = (uv)^{\frac{N}{2}-1}\int\int e^{2a\sqrt{uv}\sum_{i=1}^{N}x_iy_i + u\sum_{i=1}^{N}\lambda_i x_i^2 + 2b\sqrt{u}\sum_{i=1}^{N}n_ix_i + v\sum_{i=1}^{N}\lambda_i y_i^2 + 2c\sqrt{v}\sum_{i=1}^{N}n_iy_i}\mathrm{d}\Omega_{N-1}^{\otimes 2}(x,y).$$

We evaluate the Laplace transform of $J(u,v,a,b,c)$. Changing variable by $u = r^2$, $v = s^2$ and $rx \mapsto x$, $sy \mapsto y$, the Laplace transform

$$Q(z,w) = \int_0^\infty\int_0^\infty e^{-zu-wv}J(u,v)\mathrm{d}u\mathrm{d}v$$

becomes a 2-dimensional Gaussian integral which evaluates to

$$Q(z,w) = 4\prod_{i=1}^{N}\frac{\pi}{\sqrt{(z-\lambda_i)(w-\lambda_i)-a^2}}e^{\frac{n_i^2((w-\lambda_i)b^2+2abc+(z-\lambda_i)c^2}{(z-\lambda_i)(w-\lambda_i)-a^2}}.$$

The inverse Laplace transform gives a double integral formula for $J(u,v)$. $\square$

CHAPTER III

Free Energy of Spherical Sherrington-Kirkpatrick Model

In this chapter we analyze the free energy of the SSK model. We begin with a summary of the known results for the limiting distribution of the fluctuations in the cases where $h = 0, T > 1$ and $h = 0, T < 1$. We then contrast this with the result for $h > 0$ for h fixed. In Section 3.2, we analyze the free energy transition between $h = 0$ and $h > 0$ in the high temperature case. This transition occurs when $h \sim N^{-1/4}$. Finally, in Section 3.3, we analyze the free energy transition between $h = 0$ and $h > 0$ in the low temperature case. This transition occurs when $h \sim N^{-1/6}$.

3.1 Fluctuations of the free energy

From the integral formula (2.3.1), using

$$C_N = \frac{\sqrt{N}\beta}{2\mathrm{i}\sqrt{\pi}(\beta e)^{N/2}}(1 + O(N^{-1})), \tag{3.1.1}$$

the free energy can be written as

$$\mathcal{F}_N = \frac{1}{2\beta}(\mathcal{G}(\gamma) - 1 - \log\beta) + \frac{1}{N\beta}\log\left(\frac{\sqrt{N}\beta}{2\mathrm{i}\sqrt{\pi}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}(z)-\mathcal{G}(\gamma))}\mathrm{d}z\right) + O(N^{-2}) \tag{3.1.2}$$

where $O(N^{-2})$ is a constant that does not depend on the disorder sample M and $\mathbf{g}$. We evaluate the integral asymptotically using the method of steepest descent. The formula for $\mathcal{G}(z)$ is given in (2.3.2) and

$$\mathcal{G}'(z) = \beta - \frac{1}{N}\sum_{i=1}^{N}\frac{1}{z-\lambda_i} - \frac{h^2\beta}{N}\sum_{i=1}^{N}\frac{n_i^2}{(z-\lambda_i)^2} \qquad \text{where} \quad n_i = \mathbf{u}_i \cdot \mathbf{g}. \tag{3.1.3}$$

For real z, $\mathcal{G}'(z)$ is an increasing function taking values from $-\infty$ to β as z moves from λ_1 to ∞. Hence, there is a unique real critical point γ satisfying

$$\mathcal{G}'(\gamma) = 0, \qquad \gamma > \lambda_1.$$

We set γ for the contour of (3.1.2) to be this critical point.

In this section, we use the formula (3.1.2) to evaluate the fluctuations of the free energy when the external field strength h is fixed. For the case $h = 0$, this computation was done in [28] for the leading deterministic term and in [7] for the subleading term. For fixed $h > 0$, the fluctuations for the SK model were computed in [11] using a method different from the one of this thesis. We first review the computation of [7] for $h = 0$ and then give a new computation for fixed $h > 0$ using the above integral formula.

The following formula will be used in one of the subsections: Since $\mathcal{G}'(\gamma) = 0$ implies that $\mathcal{G}(z) - \mathcal{G}(\gamma) = \mathcal{G}(z) - \mathcal{G}(\gamma) - \mathcal{G}'(\gamma)(z-\gamma)$, we can write

$$N(\mathcal{G}(z) - \mathcal{G}(\gamma)) = -\sum_{i=1}^{N} \left[\log(1 + \frac{z-\gamma}{\gamma-\lambda_i}) - \frac{z-\gamma}{\gamma-\lambda_i}\right] + h^2\beta \sum_{i=1}^{N} \frac{n_i^2(z-\gamma)^2}{(z-\lambda_i)(\gamma-\lambda_i)^2}. \quad (3.1.4)$$

3.1.1 No external field: $h = 0$

3.1.1.1 High temperature regime: $T > 1$

When $h = 0$, we write, using the notation (2.1.17),

$$\mathcal{G}(z) = \beta z - \frac{1}{N}\sum_{i=1}^{N} \log(z-\lambda_i) = \beta z - s_0(z) - \frac{\mathcal{L}_N(z)}{N}, \qquad s_0(z) = \int \log(z-x)\mathrm{d}\sigma_{scl}(x). \quad (3.1.5)$$

From (2.1.18), $\mathcal{L}_N(z) = \mathcal{O}(1)$ for fixed $z > 2$. Thus, $\mathcal{G}_0(z) := \beta z - s_0(z)$ is an approximation of the function $\mathcal{G}(z)$ and we first find the critical point γ_0 of $\mathcal{G}_0(z)$ satisfying $\gamma_0 > 2$, where we recall that the largest eigenvalue $\lambda_1 \to 2$. Since $\mathcal{G}_0''(z) > 0$, we find that $\min_{z\geq 2} \mathcal{G}_0'(z) = \mathcal{G}_0'(2) = \beta - 1$ from the formula (2.1.7) of $s_0'(z) = s_1(z)$. Thus, the critical point of $\mathcal{G}_0(z)$ exists only when $\frac{1}{\beta} = T > 1$. From the formula, we find that for $T > 1$, it is given by

$$\gamma_0 := \beta + \beta^{-1} = T + T^{-1}. \quad (3.1.6)$$

In this case, a simple perturbation argument (see Section 3.4) implies that $\gamma = \gamma_0 + \mathcal{O}(N^{-1})$ and

$$\mathcal{G}(\gamma) = \mathcal{G}(\gamma_0) - \frac{\mathcal{L}_N(\gamma_0)}{N} + \mathcal{O}\left(N^{-2}\right) = \frac{\beta^2}{2} + 1 + \log\beta - \frac{\mathcal{L}_N(\gamma_0)}{N} + \mathcal{O}\left(N^{-2}\right). \tag{3.1.7}$$

Even though the integral in (3.1.2) involves the disorder sample, the rigidity of the eigenvalues from Section 2.1.3 implies that, with high probability, the eigenvalues are close to the non-random classical locations (i.e. the quantiles of the semicircle law). Thus, we can still apply the method of steepest descent when the disorder sample is in an event of the high probability. Using

$$\mathcal{G}''(\gamma) \simeq \mathcal{G}_0''(\gamma_0) = s_2(\gamma_0) = \frac{\beta^2}{1-\beta^2}$$

and $\mathcal{G}^{(k)}(\gamma) = \mathcal{O}(1)$ for all $k \geq 2$, the Gaussian approximation of the integral is valid and we find that

$$\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}(z)-\mathcal{G}(\gamma))}\mathrm{d}z \simeq \frac{\mathrm{i}\sqrt{4\pi}}{\sqrt{Ns_2(\gamma_0)}} = \frac{\mathrm{i}\sqrt{4\pi(1-\beta^2)}}{\sqrt{N\beta^2}}. \tag{3.1.8}$$

Inserting everything into (3.1.2) and using the fact that $\mathcal{L}_N(\gamma_0)$ converges to a Gaussian distribution with mean and variance given by (2.1.20), we obtain the following result. This result was proved rigorously in [7].

Theorem 3.1.1 ([7]). *For $h = 0$ and $T > 1$,*

$$\mathcal{F}_N(T, 0) = \frac{1}{4T} + \frac{T}{2N}\left[\log(1-T^{-2}) - \mathcal{L}_N(\gamma_0)\right] + \mathcal{O}\left(N^{-3/2}\right) \tag{3.1.9}$$

as $N \to \infty$ with high probability, where $\gamma_0 = T + T^{-1}$ and $\mathcal{L}_N(z)$ is defined in (2.1.17)*. As a consequence,*

$$\mathcal{F}_N(T, 0) \overset{\mathcal{D}}{\simeq} \frac{1}{4T} + \frac{T}{2N}\mathcal{N}(-\alpha, 4\alpha) \qquad \alpha := -\frac{1}{2}\log(1-T^{-2}), \tag{3.1.10}$$

where $\mathcal{N}(a, b)$ is a (sample) Gaussian distribution of mean a and variance b.

3.1.1.2 Low temperature regime: $T < 1$

In contrast to the previous section, the function $\mathcal{G}_0(z) = \beta z - s_0(z)$ is no longer a good approximation of $\mathcal{G}(z)$ for $0 < T < 1$ when $h = 0$. Indeed, the function $\mathcal{G}_0(z)$ does not have a critical point satisfying $z > 2$. Hence, we need to find the critical point γ of $\mathcal{G}(z)$ directly. Since the critical point of $\mathcal{G}_0(z)$ when $T = 1$ is given by $\gamma_0 = 2$, it is reasonable to assume that when

$0 < T < 1$, γ is close to the large eigenvalue λ_1. It turns out that $\gamma = \lambda_1 + \mathcal{O}(N^{-1})$. We set $\gamma = \lambda_1 + sN^{-1}$ with $s = \mathcal{O}(1)$ and determine s. Separating out the term with $i = 1$ in the equation (3.1.3) and using (2.2.14),

$$\mathcal{G}'(\gamma) = \beta - \frac{1}{N(\gamma - \lambda_1)} - \frac{1}{N}\sum_{i=2}^{N} \frac{1}{\gamma - \lambda_i} = \beta - \frac{1}{s} - 1 + \mathcal{O}\left(N^{-1/3}\right) = 0. \tag{3.1.11}$$

Thus $s = \frac{1}{\beta-1} + \mathcal{O}\left(N^{-1/3}\right)$, which is consistent with our assumption that $s = \mathcal{O}(1)$. To evaluate

$$\mathcal{G}(\gamma) = \beta\gamma - \frac{1}{N}\sum_{i=1}^{N} \log(\gamma - \lambda_i),$$

we use (2.1.17)-(2.1.19). We need to evaluate $\sum_{i=1}^{N} \log(z - \lambda_i)$ for $z = 2 + O(N^{-2/3})$. Observe that

$$M(z) = O(\log(z-2)) \quad \text{and} \quad V(z) = O(\log(z-2)) \quad \text{as } z \to 2.$$

Hence, a formal application of (2.1.18) to this case using $s_0(z) = \frac{1}{2} + (z-2) + O((z-2)^{3/2})$ implies that for $z \to 2$ such that $|z-2| \geq N^{-d}$ for some $d > 0$,

$$\frac{1}{N}\sum_{i=1}^{N} \log(z - \lambda_i) = s_0(z) + \mathcal{O}\left(N^{-1}\right) = \frac{1}{2} + (z-2) + \mathcal{O}\left(N^{-1}\right) + O((z-2)^{3/2}). \tag{3.1.12}$$

This heuristic computation indicates that

$$\mathcal{G}(\gamma) = \beta\gamma - \frac{1}{N}\sum_{i=1}^{N} \log(\gamma - \lambda_i) = 2\beta - \frac{1}{2} + (\beta - 1)(\lambda_1 - 2) + \mathcal{O}\left(N^{-1}\right). \tag{3.1.13}$$

We now consider the integral in (3.1.2). For $k \geq 2$, we have, using the notation (2.1.11) for the scaled eigenvalues $a_i = N^{2/3}(\lambda_i - 2)$ and the estimate (2.2.15),

$$\frac{\mathcal{G}^{(k)}(\gamma)}{(-1)^k(k-1)!} = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{(\gamma - \lambda_i)^k} = \frac{N^{k-1}}{s^k} + N^{\frac{2}{3}k-1}\sum_{i=2}^{N} \frac{1}{(a_1 + sN^{-1/3} - a_i)^k} = \mathcal{O}\left(N^{k-1}\right) \tag{3.1.14}$$

with high probability. The estimate $\mathcal{G}''(\gamma) = \mathcal{O}(N)$ indicates that the main contribution to the integral comes from a neighborhood of radius N^{-1} of the critical point. However, all terms of the

Taylor series

$$N\left(\mathcal{G}(\gamma+uN^{-1})-\mathcal{G}(\gamma)\right)=\sum_{k=2}^{N}N^{1-k}\frac{\mathcal{G}^{(k)}(\gamma)}{k!}u^k$$

are of the same order $\mathcal{O}(1)$ for finite u. Hence, we cannot replace the integral with a Gaussian integral. Instead, we proceed as follows. Using the formula (3.1.4), separating out the $i=1$ term from the sum, using a Taylor approximation for the remaining sum, and using (2.2.15),

$$\begin{aligned}N\left(\mathcal{G}(\gamma+uN^{-1})-\mathcal{G}(\gamma)\right)&=-\log\left(1+\frac{u}{s}\right)+\frac{u}{s}+O\left(\sum_{i=2}^{N}\frac{u^2N^{-2/3}}{(a_1-sN^{-1/3}-a_i)^2}\right)\\&=-\log\left(1+\frac{u}{s}\right)+\frac{u}{s}+\mathcal{O}\left(N^{-2/3}\right)\end{aligned}\tag{3.1.15}$$

with high probability for finite u. From this,

$$\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}e^{\frac{N}{2}(\mathcal{G}(z)-\mathcal{G}(\gamma))}\mathrm{d}z\simeq\frac{1}{N}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{e^{\frac{u}{s}}}{1+\frac{u}{s}}\mathrm{d}u\asymp\mathcal{O}\left(N^{-1}\right).\tag{3.1.16}$$

We do not need the exact value of the integral, but only the estimate that its log is $\mathcal{O}(\log N)$.

We thus obtain the following result, which was proved rigorously in [7].

Theorem 3.1.2 ([7])**.** *For $h=0$ and $0\le T<1$,*

$$\mathcal{F}_N(T,0)=1-\frac{3T}{4}+\frac{T\log T}{2}+\frac{1-T}{2N^{2/3}}a_1+\mathcal{O}\left(N^{-1}\right)\tag{3.1.17}$$

as $N\to\infty$ with high probability. As a consequence,

$$\mathcal{F}_N(T,0)\overset{\mathcal{D}}{\simeq}1-\frac{3T}{4}+\frac{T\log T}{2}+\frac{1-T}{2N^{2/3}}\mathrm{TW}_{GOE}\,.\tag{3.1.18}$$

Remark 3.1.3. The zero temperature case $T=0$ of the theorem is the standard random matrix theory result that the largest eigenvalue of a GOE matrix converges to the Tracy-Widom distribution. We see that a formal $T\to0$ limit of the result implies this statement. Similarly, all results of this thesis, other than those that have $T>1$ restrictions, have a convergent formal limit if we take $T\to0$. Hence, even though we need a separate argument since there is no integral representation, we expect that all results are valid for the $T=0$ case as well.

3.1.2 Positive external field: $h = O(1)$

Fix $h > 0$. We use (2.1.18) and (2.2.16) to write

$$\mathcal{G}(z) = \beta z - s_0(z) + h^2\beta\left[s_1(z) + \frac{1}{\sqrt{N}}\mathcal{S}_N(z;1)\right] + \mathcal{O}\left(N^{-1}\right)$$

for $z > \lambda_1$. The random variable $\mathcal{S}_N(z;k)$ is defined in (2.2.6) and it converges in distribution to $\mathcal{N}(0, 2s_{2k}(z))$; see (2.2.13). This time, $\mathcal{G}(z)$ is approximated by the function $\mathcal{G}_0(z) = \beta z - s_0(z) + h^2\beta s_1(z)$. Its derivative $\mathcal{G}_0'(z) = \beta - s_1(z) - h^2\beta s_2(z)$ is an increasing function for $z > 2$ and $\mathcal{G}'(z) \to -\infty$ as $z \downarrow 2$ while $\mathcal{G}'(z) \to +\infty$ as $z \to +\infty$. Hence, unlike in the case of $h = 0$, there is a point $\gamma_0 > 2$ satisfying $\mathcal{G}_0'(\gamma_0) = 0$ for all $T > 0$. It satisfies the equation

$$\mathcal{G}_0'(\gamma_0) = \beta - s_1(\gamma_0) - h^2\beta s_2(\gamma_0) = 0. \tag{3.1.19}$$

A perturbation argument (see Section 3.4) implies that the critical point γ of $\mathcal{G}(z)$ has the form

$$\gamma = \gamma_0 + \gamma_1 N^{-1/2} + \mathcal{O}\left(N^{-1}\right). \tag{3.1.20}$$

We do not need a formula for γ_1 in this section, but we record it here since we use it in later sections;

$$\gamma_1 = \frac{h^2\beta\mathcal{S}_N(\gamma_0;2)}{s_2(\gamma_0) + 2h^2\beta s_3(\gamma_0)} \tag{3.1.21}$$

where we used the fact that $\frac{\mathrm{d}}{\mathrm{d}z}\mathcal{S}_N(z;1) = -\mathcal{S}_N(z;2)$. The perturbation argument also implies that

$$\mathcal{G}(\gamma) = \beta\gamma_0 - s_0(\gamma_0) + h^2\beta s_1(\gamma_0) + \frac{h^2\beta}{\sqrt{N}}\mathcal{S}_N(\gamma_0;1) + \mathcal{O}\left(N^{-1}\right). \tag{3.1.22}$$

The integral term in (3.1.2) can be evaluated using the steepest descent method as in the case of $h = 0$ and $T > 1$ since $\mathcal{G}^{(k)}(\gamma) = \mathcal{O}(1)$ for all $k \geq 2$. From the Gaussian integral approximation,

$$\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}(z)-\mathcal{G}(\gamma))}\mathrm{d}z \simeq \frac{\mathrm{i}\sqrt{4\pi}}{\sqrt{N\mathcal{G}''(\gamma)}} \asymp \mathcal{O}\left(N^{-1/2}\right). \tag{3.1.23}$$

Remark 3.1.4. We do not focus in Chapters III and IV on justifying the use of steepest descent in this context, but instead provide the computations based on this method. One can rigorously check that the steepest descent method works here, but it is also worth noting that all the contour integral computations needed in these chapters can be achieved without the use of steepest descent. In fact, the contour integrals in sections 3.1.2, 3.2.1, 3.3.1, 4.1.3, 4.1.5, 4.2.1, 4.2.2, 4.2.3, 4.3.1, and 4.3.2 require no contour deformation at all. Using the straight line contour and crude bounds on the order of the integrand, one can compute, up to leading order, the value of the integral in a neighborhood of γ and then show that the tails are of smaller order. These computations are fairly lengthy and will be omitted from this thesis. The integrals in sections 4.1.6 and 4.3.3 can be treated by a similar method, but require a slight deformation of the original contour. These proofs are included in Chapter V.

Combining the preceding information in this section, we obtain the following result.

Result 3.1.5. *For fixed $h > 0$ and $T > 0$,*

$$\mathcal{F}_N(T,h) = F(T,h) + \frac{h^2 \mathcal{S}_N(\gamma_0;1)}{2\sqrt{N}} + \mathcal{O}\left(N^{-1}\right) \tag{3.1.24}$$

as $N \to \infty$ with high probability where $\mathcal{S}_N(z;k)$ is defined in (2.2.6) *and*

$$F(T,h) := \frac{\gamma_0}{2} - \frac{Ts_0(\gamma_0)}{2} - \frac{T - T\log T}{2} + \frac{h^2 s_1(\gamma_0)}{2} \tag{3.1.25}$$

with γ_0 being the solution of the equation

$$1 - Ts_1(\gamma_0) - h^2 s_2(\gamma_0) = 0, \qquad \gamma_0 > 2. \tag{3.1.26}$$

Since $\mathcal{S}_N(\gamma_0;1)$ converges in distribution to $\mathcal{N}(0, 2s_2(\gamma_0))$ from (2.2.13), we conclude the following result.

Result 3.1.6. *For fixed $h > 0$ and $T > 0$, as $N \to \infty$,*

$$\mathcal{F}_N(T,h) \overset{\mathcal{D}}{\simeq} F(T,h) + \frac{1}{\sqrt{N}}\mathcal{N}\left(0, \frac{h^4 s_2(\gamma_0)}{2}\right). \tag{3.1.27}$$

This result shows that the order of the fluctuations of the free energy is $N^{-1/2}$ for all $T > 0$, which is different from both N^{-1} for $h = 0$, $T > 1$ and $N^{-2/3}$ for $h = 0$, $0 < T < 1$.

3.1.3 Comparison with the result of Chen, Dey, and Panchenko

Chen, Dey, and Penchenko computed the fluctuations of the free energy of the SK model with $h > 0$ in [11] when $\mathbf{g} = \mathbf{1} := (1, 1, ..., 1)^T$. We compare our result with theirs. The adaptation of the approach of [11] to the SSK model with $\mathbf{g} = \mathbf{1}$ implies that $\sqrt{N}\left(\mathcal{F}_N(T,h) - \mathbb{E}[F(T,h)]\right)$

converges in distribution as $N \to \infty$ to the centered Gaussian distribution with variance

$$\frac{h^4(1-q_0)^4}{2T^2(T^2-(1-q_0))} \tag{3.1.28}$$

where q_0 is the unique real number between 0 and 1 satisfying

$$q_0 + h^2 = \frac{T^2 q_0}{(1-q_0)^2}. \tag{3.1.29}$$

The quantity q_0 has the interpretation as the overlap of two independent spins from the Gibbs measure involving the same disorder sample, i.e. the overlap of a spin with a replica. The formula (3.1.29) was predicted using the replica saddle point method in [15] (equation (4.5)) and [22] (equation (29) with $n = 0$).

Our result (3.1.27) above is for the SSK model when $\mathbf{g}$ is a Gaussian vector, but it extends to the case $\mathbf{g} = \mathbf{1}$. The only difference is that the variance of the limiting Gaussian distribution (3.1.27) changes to

$$\frac{h^4}{2}(s_2(\gamma_0) - (s_1(\gamma_0))^2). \tag{3.1.30}$$

Using the fact that $s_2(z) = \frac{s_1(z)^2}{1-s_1(z)^2}$ for $z > 2$, it is easy to check that (3.1.28) and (3.1.30) are same with q_0 and γ_0 related by the equation

$$q_0 = 1 - Ts_1(\gamma_0). \tag{3.1.31}$$

3.1.4 Matching between $h > 0$ and $h = 0$

We have considered three different regimes: (a) $h = 0$ and $T < 1$, (b) $h = 0$ and $T > 1$, and (c) $h = O(1)$. The order of the fluctuations of the free energy in these regimes are N^{-1}, $N^{-2/3}$, and $N^{-1/2}$, respectively. In these cases, the fluctuations are governed by the disorder variables given by (a) all eigenvalues $\lambda_1, \cdots, \lambda_N$, (b) the top eigenvalue λ_1, and (c) the combinations $n_i = \mathbf{u}_i \cdot \mathbf{g}$ of the eigenvectors and the external field. These differences indicate that there should be transitional regimes as $h \to 0$. We now study the limit $h \to 0$ of the result obtained for the case $h > 0$ and determine the transitional scaling of h heuristically by matching the order of the fluctuations. We need to consider the high temperature case and the low temperature case separately.

3.1.4.1 Asymptotic property of γ_0

Throughout this thesis, we will make use of following property of the leading term γ_0 of the critical point of $\mathcal{G}(z)$ when $h = O(1)$.

Lemma 3.1.7. *Let $\gamma_0 > 2$ be the solution of the equation* (3.1.26)*, $1 - Ts_1(\gamma_0) - h^2 s_2(\gamma_0) = 0$. Then, as $h \to 0$,*

$$\gamma_0 = \begin{cases} T + T^{-1} + \dfrac{h^2}{T} + O(h^4) & \text{for } T > 1, \\ 2 + \dfrac{h^4}{4(1-T)^2} - \dfrac{h^6}{4(1-T)^4} + O(h^8) & \text{for } 0 < T < 1. \end{cases} \tag{3.1.32}$$

On the other hand, as $h \to \infty$,

$$\gamma_0 = h + \frac{T}{2} + O(h^{-1}) \qquad \text{for all } T > 0. \tag{3.1.33}$$

Proof. Consider the limit of γ_0 as $h \to 0$. For $T > 1$, the equation for γ_0 becomes $1 - Ts_1(\gamma_0) = 0$ when $h = 0$, and its solution is $T + T^{-1}$. A simple perturbation argument applied to the equation for small h implies the result. For $0 < T < 1$, we use the asymptotics

$$s_2(z) = \frac{1}{2\sqrt{z-2}} + O(1) \quad \text{and} \quad s_1(z) = 1 + O(\sqrt{z-2}) \quad \text{as } z \to 2,$$

which follow from the formulas in (2.1.7). Then, the equation for γ_0 becomes

$$1 - T - \frac{h^2}{2\sqrt{\gamma_0 - 2}} + O(h^2) + O(\sqrt{\gamma_0 - 2}) = 0 \tag{3.1.34}$$

as $h \to 0$ and $\gamma_0 \to 2$. From this equation we find the result as $h \to 0$. The limit as $h \to \infty$ follows from $s_k(z) = z^{-k} + O(z^{-k-1})$ as $z \to \infty$. □

3.1.4.2 High temperature case, $T > 1$

From (3.1.32), we find that for $T > 1$, as $h \to 0$,

$$s_0(\gamma_0) = \frac{1}{2T^2} + \log T + \frac{h^2}{T^2} + \mathcal{O}\left(h^4\right), \quad s_1(\gamma_0) = \frac{1}{T} - \frac{h^2}{T(T^2-1)} + \mathcal{O}\left(h^4\right),$$
$$s_2(\gamma_0) = \frac{1}{T^2-1} - \frac{2T^2h^2}{(T^2-1)^3} + \mathcal{O}\left(h^4\right).$$

Inserting the formulas into (3.1.25),

$$F(T,h) = \frac{1}{2T} + \frac{h^2}{2T} - \frac{h^4}{4T(T^2-1)} + O(h^6). \tag{3.1.35}$$

Therefore, we find that if we first take $N \to \infty$ with fixed $h > 0$ and then let $h \to 0$, then

$$\mathcal{F}_N(T,h) \overset{\mathcal{D}}{\simeq} \left[\frac{1}{2T} + \frac{h^2}{2T} - \frac{h^4}{4T(T^2-1)}\right] + \frac{h^2}{\sqrt{2N(T^2-1)}}\mathcal{N}(0,1) \tag{3.1.36}$$

where the terms of orders h^6 and $h^4 N^{-1/2}$ have been dropped. The fluctuations are of order $\frac{h^2}{\sqrt{N}}$. On the other hand, when $h = 0$, the fluctuations are of order N^{-1} (see (3.1.10)). These two terms are of same order when $h \sim N^{-1/4}$.

3.1.4.3 Low temperature case, $T < 1$

Using the $T < 1$ case of (3.1.32), the leading term (3.1.25) becomes

$$F(T,h) = 1 - \frac{3T}{4} + \frac{T\log T}{2} + \frac{h^2}{2} - \frac{h^4}{8(1-T)} + O(h^6) \tag{3.1.37}$$

and the variance of the Gaussian distribution in (3.1.27) becomes $\frac{h^4 s_2(\gamma_0)}{2} = \frac{h^2(T-1)}{2} + O(h^4)$. Thus, from (3.1.27), for $T < 1$, we find that if we take $N \to \infty$ first and then take $h \to 0$, then

$$\mathcal{F}_N(T,h) \overset{\mathcal{D}}{\simeq} \left[1 - \frac{3T}{4} + \frac{T\log T}{2} + \frac{h^2}{2} - \frac{h^4}{8(1-T)}\right] + \frac{1}{\sqrt{N}}\mathcal{N}\left(0, \frac{h^2(1-T)}{2}\right) \tag{3.1.38}$$

where the terms of orders h^6 and $h^3 N^{-1/2}$ have been dropped. This implies that the fluctuations of the free energy are of order $\frac{h}{\sqrt{N}}$. On the other hand, when $h = 0$, the fluctuations are of order $N^{-2/3}$ (see (3.1.18)). These two terms are of same order when $h \sim N^{-1/6}$.

3.1.4.4 Summary

In summary, a heuristic matching computation suggests that the transitional scaling is

$$\begin{aligned} h &= O(N^{-1/4}) \quad \text{for } T > 1, \\ h &= O(N^{-1/6}) \quad \text{for } T < 1. \end{aligned} \tag{3.1.39}$$

In next two sections, we compute the fluctuations of the free energy in the above transitional regimes.

3.2 Free energy for $T > 1$ and $h \sim N^{-1/4}$

3.2.1 Analysis

Assume that $T > 1$ and set

$$h = HN^{-1/4} \tag{3.2.1}$$

for fixed $H > 0$. In this case, using the notations (2.1.17) and (2.2.6),

$$\mathcal{G}(z) = \beta z - s_0(z) - \frac{\mathcal{L}_N(z)}{N} + \frac{H^2\beta}{\sqrt{N}}\left[s_1(z) + \frac{\mathcal{S}_N(z;1)}{\sqrt{N}}\right] + \mathcal{O}\left(N^{-3/2}\right) \tag{3.2.2}$$

where we recall that $\mathcal{L}_N(z)$ and $\mathcal{S}_N(z;1)$ are $\mathcal{O}(1)$ for $z > 2$. We approximate the function by $\mathcal{G}_0(z) = \beta z - s_0(z)$ and, as we discussed in sub-subsection 3.1.1.1, this function has the critical point $\gamma_0 = \beta + \beta^{-1}$ for $T > 1$. Applying a perturbation argument (see Section 3.4) and using the formulas of $s_0(z)$ and $s_1(z)$, the critical point of $\mathcal{G}(z)$ is given by

$$\gamma = \gamma_0 + \mathcal{O}\left(N^{-1/2}\right) \qquad \text{with } \gamma_0 = \beta + \beta^{-1}. \tag{3.2.3}$$

Furthermore,

$$\mathcal{G}(\gamma) = \frac{\beta^2}{2} + 1 + \log\beta + \frac{H^2\beta^2}{\sqrt{N}} + \frac{1}{N}\left[-\frac{H^4\beta^4}{2(1-\beta^2)} + H^2\beta\mathcal{S}_N(\gamma_0;1) - \mathcal{L}_N(\gamma_0)\right] + \mathcal{O}\left(N^{-3/2}\right). \tag{3.2.4}$$

Since

$$\mathcal{G}''(\gamma) = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{(\gamma-\lambda_i)^2} + \frac{2H^2\beta}{N^{3/2}}\sum_{i=1}^{N}\frac{n_i^2}{(\gamma-\lambda_i)^3} \simeq s_2(\gamma) + \frac{2H^2\beta}{N^{1/2}}s_3(\gamma) \simeq s_2(\gamma_0) \tag{3.2.5}$$

and $\mathcal{G}^{(k)}(\gamma) = O(1)$ for all $k \geq 2$, the method of steepest descent implies that

$$\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}(z)-\mathcal{G}(\gamma))}\mathrm{d}z \simeq \frac{\mathrm{i}}{N^{1/2}}\sqrt{\frac{4\pi}{s_2(\gamma_0)}} \asymp \mathcal{O}\left(N^{-1/2}\right). \tag{3.2.6}$$

Result 3.2.1. *For $h = HN^{-1/4}$ with fixed $H > 0$ and $T > 1$,*

$$\mathcal{F}_N(T,h) = \frac{1}{4T} + \frac{H^2}{2T\sqrt{N}} + \frac{T}{2N}\left[\log(1-T^{-2}) - \frac{H^4}{2T^2(T^2-1)} + \frac{H^2}{T}\mathcal{S}_N(\gamma_0;1) - \mathcal{L}_N(\gamma_0)\right] \tag{3.2.7}$$

plus $\mathcal{O}\left(N^{-3/2}\right)$*, as* $N \to \infty$ *with high probability where* $\mathcal{L}_N(z)$ *and* $\mathcal{S}_N(z;1)$ *are defined in* (2.1.17) *and* (2.2.6)*, respectively, and* $\gamma_0 = \gamma_0(h=0) = T + T^{-1}$.

The sample random variables $\mathcal{S}_N(\gamma_0;1)$ and $\mathcal{L}_N(\gamma_0)$ both converge to Gaussian distributions. Since $\mathcal{S}_N(\gamma_0;1)$ depends only on n_i's and $\mathcal{L}_N(\gamma_0)$ depends only on λ_i's, these two random variables are independent. Therefore, we obtain the following result.

Result 3.2.2. *For* $h = HN^{-1/4}$ *and* $T > 1$*, as* $N \to \infty$*,*

$$\mathcal{F}_N(T,h) \overset{\mathcal{D}}{\simeq} \left[\frac{1}{4T} + \frac{H^2}{2T\sqrt{N}}\right] + \frac{T}{2N}\mathcal{N}(-\alpha, 4\alpha), \qquad \alpha := \frac{H^4}{2T^2(T^2-1)} - \frac{1}{2}\log(1-T^{-2}). \tag{3.2.8}$$

3.2.2 Matching with $h = 0$ and $h = O(1)$ cases

If we set $H = 0$ in (3.2.7), we recover the result (3.1.10) for the case of $h = 0$. We now consider the limit $H \to \infty$. If we formally set $H = hN^{1/4}$ in (3.2.7) with h small but fixed and N large, then we have

$$\mathcal{F}_N(T,h) \simeq \frac{1}{4T} + \frac{h^2}{2T} - \frac{h^4}{4T(T^2-1)} + \frac{h^2}{2\sqrt{N}}\mathcal{S}_N(\gamma_0;1) \tag{3.2.9}$$

for asymptotically almost every disorder sample. This is the same as (3.1.24) when $h \to 0$ since $F(T,h)$ satisfies (3.1.35) as $h \to 0$. Therefore, (3.2.7) matches well with both regimes.

3.3 Free energy for $T < 1$ and $h \sim N^{-1/6}$

3.3.1 Analysis

Assume that $0 < T < 1$ and we set

$$h = HN^{-1/6} \tag{3.3.1}$$

for fixed $H > 0$. We find the critical point $\gamma > \lambda_1$. Previously we had $\gamma = \lambda_1 + \mathcal{O}\left(N^{-1}\right)$ when $h = 0$ and $\gamma = \lambda_1 + \mathcal{O}(1)$ when $h > 0$. For $h \sim N^{-1/6}$, we make the ansatz

$$\gamma = \lambda_1 + sN^{-2/3} \tag{3.3.2}$$

and find $s > 0$ assuming that $s = \mathcal{O}(1)$. From the equation $\mathcal{G}'(\gamma) = 0$, see (3.1.3), the equation of s is

$$\beta - \frac{1}{N^{1/3}}\sum_{i=1}^{N}\frac{1}{s + a_1 - a_i} - h^2\beta N^{1/3}\sum_{i=1}^{N}\frac{n_i^2}{(s+a_1-a_i)^2} = 0 \tag{3.3.3}$$

where we recall $a_i = N^{2/3}(\lambda_i - 2)$. Here, we did not change h to $HN^{-1/6}$ since we will cite this equation in several places. From (2.2.15), the second sum converges with high probability. The first sum is $1 + \mathcal{O}\left(N^{-1/3}\right)$ from (2.2.14). Thus, with $h = HN^{-1/6}$ the equation becomes, under the assumption that $s = \mathcal{O}(1)$,

$$\beta - 1 - H^2\beta \sum_{i=1}^{N} \frac{n_i^2}{(a_1 + s - a_i)^2} + \mathcal{O}\left(N^{-1/3}\right) = 0. \tag{3.3.4}$$

Let t be the unique positive solution of the equation

$$\beta - 1 - H^2\beta \sum_{i=1}^{N} \frac{n_i^2}{(t + a_1 - a_i)^2} = 0, \qquad t > 0. \tag{3.3.5}$$

Using the rigidity, we can show that $t \asymp \mathcal{O}(1)$ with high probability. From this, comparing the equations for s and t, we find that

$$s = t + \mathcal{O}\left(N^{-1/3}\right). \tag{3.3.6}$$

which is consistent with the ansatz. The last equation can also be verified by checking the inequalities

$$\mathcal{G}'(\lambda_1 + tN^{-2/3}(1 - N^{-\varepsilon})) < 0 < \mathcal{G}'(\lambda_1 + tN^{-2/3}(1 + N^{-\varepsilon}))$$

for any $0 < \epsilon < 1/3$.

We now evaluate $G(\gamma)$ which is given by

$$\mathcal{G}(\gamma) = \beta\gamma - \frac{1}{N}\sum_{i=1}^{N} \log(\gamma - \lambda_i) + \frac{H^2\beta}{N^{4/3}} \sum_{i=1}^{N} \frac{n_i^2}{\gamma - \lambda_i}. \tag{3.3.7}$$

Insert $\gamma = \lambda_1 + sN^{-2/3} = 2 + (a_1 + s)N^{2/3}$. By (3.1.12), the sum involving the log function becomes

$$\frac{1}{N}\sum_{i=1}^{N} \log(\gamma - \lambda_i) = \frac{1}{2} + N^{-2/3}(a_1 + s) + \mathcal{O}\left(N^{-1}\right).$$

The other sum in (3.3.7) is equal to

$$\frac{H^2\beta}{N^{2/3}} \sum_{i=1}^{N} \frac{n_i^2}{a_1 + s - a_i} = \frac{H^2\beta}{N^{2/3}}\left(N^{1/3} + \mathcal{E}_N(s)\right)$$

using the random variable $\mathcal{E}_N(w)$ defined by (2.2.5), which is $\mathcal{O}(1)$ outside of a set whose probability shrinks to zero. Thus,

$$\mathcal{G}(\gamma) = 2\beta - \frac{1}{2} + \frac{H^2\beta}{N^{1/3}} + \frac{1}{N^{2/3}}\left[(\beta-1)(a_1+s) + H^2\beta\mathcal{E}_N(s)\right] + \mathcal{O}\left(N^{-1}\right). \tag{3.3.8}$$

To evaluate the integral in (3.1.2), we observe that for $k \geq 2$,

$$\frac{\mathcal{G}^{(k)}(\gamma)}{(-1)^k(k-1)!} = N^{\frac{2k}{3}-1}\sum_{i=1}^{N}\frac{1}{(s+a_1-a_i)^k} + kN^{\frac{2}{3}k-\frac{2}{3}}H^2\beta\sum_{i=1}^{N}\frac{n_i^2}{(s+a_1-a_i)^{k+1}} = \mathcal{O}\left(N^{\frac{2}{3}k-\frac{2}{3}}\right).$$

For $k=2$, the leading term is

$$\mathcal{G}''(\gamma) = 2N^{2/3}H^2\beta\sum_{i=1}^{N}\frac{n_i^2}{(s+a_1-a_i)^3} + \mathcal{O}\left(N^{1/3}\right). \tag{3.3.9}$$

Since $\mathcal{G}''(\gamma) \sim N^{2/3}$, the main contribution to the integral comes from a neighborhood of radius $N^{-5/6}$ near the critical point. By the Taylor series, for $u = O(1)$,

$$N\left(\mathcal{G}(\gamma+uN^{-5/6}) - \mathcal{G}(\gamma)\right) = \sum_{k=2}^{\infty}\frac{N^{1-\frac{5}{6}k}}{k!}\mathcal{G}^{(k)}(\gamma)u^k = H^2\beta\left(\sum_{i=1}^{N}\frac{n_i^2}{(s+a_1-a_i)^3}\right)u^2 + \mathcal{O}\left(N^{-5/6}\right) \tag{3.3.10}$$

where all terms but $k=2$ are $\mathcal{O}\left(N^{-5/6}\right)$. Thus, from the Gaussian integral approximation,

$$\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}(z)-G(\gamma))}\mathrm{d}z \simeq \frac{1}{N^{5/6}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} e^{H^2\beta\left(\sum_{i=1}^{N}\frac{n_i^2}{(s+a_1-a_i)^3}\right)u^2}\mathrm{d}u \asymp \mathcal{O}\left(N^{-5/6}\right). \tag{3.3.11}$$

Combining all together in (3.1.2) and replacing s by t, we obtain the following

Result 3.3.1. *For $h = HN^{-1/6}$ and $0 < T < 1$,*

$$\mathcal{F}_N = F_0(T,h) + \frac{\widetilde{\mathcal{F}}(T,H)}{N^{2/3}} + \mathcal{O}\left(N^{-1}\right), \qquad F_0(T,h) := 1 - \frac{3T}{4} + \frac{T\log T}{2} + \frac{h^2}{2}, \tag{3.3.12}$$

as $N \to \infty$ for asymptotically almost every disorder sample. Here,

$$\widetilde{\mathcal{F}}(T,H) = \frac{1}{2}(1-T)(t+a_1) + \frac{1}{2}H^2\mathcal{E}_N(t) \tag{3.3.13}$$

where $\mathcal{E}_N(z)$ is defined in (2.2.5) *and t is the unique solution of the equation* (3.3.5)*,*

$$1 - T = H^2 \sum_{i=1}^{N} \frac{n_i^2}{(t + a_1 - a_i)^2}, \qquad t > 0. \tag{3.3.14}$$

The function $F_0(T, h)$ is equal to $F(T, h)$ of (3.1.25) if we set $\gamma_0 = 2$. The order of fluctuations is $N^{-2/3}$ as in the $h = 0$ case. But the fluctuations depend on all eigenvalues and $n_1, \cdots, n_N$. In contrast, when $h = 0$ they depend only on the largest eigenvalue. Using (2.2.12) for $\mathcal{E}_N(t)$, we obtain the next distributional result.

Result 3.3.2. *For $h = HN^{-1/6}$ and $0 < T < 1$,*

$$\mathcal{F}_N \overset{\mathcal{D}}{\simeq} F_0(T, h) + \frac{(1-T)(\varsigma + \alpha_1) + H^2 \mathcal{E}(\varsigma)}{2N^{2/3}} \tag{3.3.15}$$

as $N \to \infty$, where

$$\mathcal{E}(w) = \lim_{n \to \infty} \Big(\sum_{i=1}^{n} \frac{\nu_i^2}{w + \alpha_1 - \alpha_i} - \frac{1}{\pi} \int_0^{\left(\frac{3\pi n}{2}\right)^{2/3}} \frac{\mathrm{d}x}{\sqrt{x}} \Big) \tag{3.3.16}$$

and ς is the solution of the equation

$$1 - T = H^2 \sum_{i=1}^{\infty} \frac{\nu_i^2}{(\varsigma + \alpha_1 - \alpha_i)^2}, \qquad \varsigma > 0, \tag{3.3.17}$$

where α_i is the GOE Airy point process and ν_i are independent standard normal sample random variables.

3.3.2 Asymptotic behavior of the scaled limiting critical point t

The solution t of the equation (3.3.5),

$$1 - T - H^2 \sum_{i=1}^{N} \frac{n_i^2}{(t + a_1 - a_i)^2} = 0, \qquad t > 0, \tag{3.3.18}$$

is the scaled limiting critical point that is used in the result (3.3.12) above. We now describe the behavior of t as $H \to 0$ and $H \to \infty$. The following result is useful in the next two subsections and in two later sections.

Result 3.3.3. *The solution* t *of the equation* (3.3.18) *satisfies:*

$$t = \frac{|n_1|}{\sqrt{1-T}} H + O(H^2) \qquad \textit{as } H \to 0 \tag{3.3.19}$$

and

$$\sqrt{t} \simeq \frac{H^2}{2(1-T)} \left[1 + \frac{H^2 \mathcal{S}_N\left(2 + \frac{H^4 N^{-2/3}}{4(1-T)^2}; 2\right)}{(1-T)N^{5/6}} \right] \qquad \textit{as } H \to \infty. \tag{3.3.20}$$

The second term inside the bracket of the equation (3.3.20) *is* $\mathcal{O}\left(H^{-3}\right)$.

For the $H \to 0$ limit, we see from the equation (3.3.18) that $t \to 0$ as $H \to 0$. If we set $t = yH$, then separating the term $i = 1$, the equation becomes $1 - T = \frac{n_1^2}{y^2} + O(H^2)$. Solving it, we obtain (3.3.19).

We now consider the large-H behavior of t. We write the equation (3.3.18) as

$$\frac{1-T}{H^2} = \sum_{i=1}^{N} \frac{n_i^2}{(t + a_1 - a_i)^2} = \frac{1}{N^{4/3}} \sum_{i=1}^{N} \frac{n_i^2}{(z - \lambda_i)^2}, \qquad z = 2 + (t + a_1)N^{-2/3}. \tag{3.3.21}$$

Note that $t \to \infty$ as $H \to \infty$. We evaluate the leading term of the right-hand side of the above equation when $z \to 2$ such that $z - 2 \gg N^{-2/3}$. The equation (2.2.16) when $k = 2$ is

$$\frac{1}{N} \sum_{i=1}^{N} \frac{n_i^2}{(z - \lambda_i)^2} = s_2(z) + \frac{\mathcal{S}_N(z; 2)}{\sqrt{N}} + \mathcal{O}\left(N^{-1}\right)$$

for $z - 2 = O(1)$. We expect that this formula is still applicable to $z = 2 + (t + a_1)N^{-2/3}$ since $t \to \infty$. Since $z \to 2$, we have $s_2(z) \simeq \frac{1}{2\sqrt{z-2}}$ from (2.1.8). The equation (3.3.21) becomes

$$\frac{1-T}{H^2} \simeq \frac{1}{2N^{1/3}\sqrt{z-2}} + \frac{\mathcal{S}_N(z; 2)}{N^{5/6}}. \tag{3.3.22}$$

The sample expectation of $\mathcal{S}_N(z; 2)$ with respect to the n_i's is 0 and the variance is

$$\mathbb{E}_s[\mathcal{S}_N(z; 2)^2] = \frac{2}{N} \sum_{i=1}^{N} \frac{1}{(z - \widehat{\lambda}_i)^4} \simeq 2s_4(z) \simeq \frac{1}{8(z-2)^{5/2}}$$

from (2.1.8). Thus, we expect that $\mathcal{S}_N(z; 2) = \mathcal{O}\left((z-2)^{-5/4}\right)$ as $z \to 0$ and (3.3.22) becomes

$$\frac{1-T}{H^2} \simeq \frac{1}{2\sqrt{t}} + \frac{\mathcal{S}_N(2 + tN^{-2/3}; 2)}{N^{5/6}} \simeq \frac{1}{2\sqrt{t}} + \mathcal{O}\left(t^{-5/4}\right).$$

Solving it gives $t \simeq \frac{H^4}{4(1-T)^2}$, the leading term of (3.3.20), as $H \to \infty$. Inserting it bask to the same equation, we obtain the next term and obtain (3.3.20). The last computation also shows that the second term in the brackets of (3.3.20) is $\mathcal{O}\left(H^2 t^{-5/4}\right) = \mathcal{O}\left(H^{-3}\right)$.

3.3.3 Matching with $h = 0$

We show that a formal limit (3.3.12) as $H \to 0$ agrees with (3.1.17) which is the result for $h = 0$. The leading term satisfies

$$F_0(T, h) = 1 - \frac{3T}{4} + \frac{T \log T}{2} + O(H^2 N^{-1/3}). \tag{3.3.23}$$

For the subleading term (3.3.13), we use (3.3.19) for t and find that

$$\mathcal{E}_N(t) = \frac{n_1^2}{t} + \sum_{i=2}^{N} \frac{n_i^2}{t + a_1 - a_i} - N^{1/3} = \frac{|n_1|\sqrt{1-T}}{H} + \mathcal{O}\left(1\right) \tag{3.3.24}$$

where the $\mathcal{O}\left(1\right)$ term follows from (2.2.14). Therefore, if we set $h = HN^{-1/6}$ and take the limits $N \to \infty$ first and $H \to 0$ second, then

$$\mathcal{F}_N(T, h) = 1 - \frac{3T}{4} + \frac{T \log T}{2} + \frac{1-T}{2N^{2/3}} a_1 + O\left(H^2 N^{-1/3}\right) + \mathcal{O}\left(HN^{2/3}\right) \tag{3.3.25}$$

for asymptotically almost every disorder sample. This agrees with result (3.1.17) obtained when $h = 0$.

We remark that the two subleading terms in (3.3.25) are comparable in size when $H = O(N^{-1/3})$, or equivalently when $h = O(N^{-1/2})$. This regime is not important for the computation of the free energy, but it will become important when we discuss the overlap of the spin variable with the external field in Subsection 4.1.6.

3.3.4 Matching with $h > 0$

We show that the formal limit of (3.3.12) as $H \to \infty$ is consistent with the result (3.1.25) for $h > 0$.

3.3.4.1 Large w limit of $\mathcal{E}_N(w)$

We first consider the behavior of $\mathcal{E}_N(w)$, defined in (2.2.5), as $w \to \infty$ and then we insert $w = t$ which tends to ∞ from (3.3.20). This result is also used in other sections later.

Result 3.3.4. *As $w \to \infty$,*

$$\mathcal{E}_N(w) \simeq -\sqrt{w} + \frac{\mathcal{S}_N(W;1)}{N^{1/6}} + \mathcal{O}\left(w^{-1/2}\right), \qquad W := 2 + wN^{-2/3}. \tag{3.3.26}$$

where $\mathcal{S}_N(z;k)$ is defined in (2.2.6).

Let $\widehat{a}_i := N^{2/3}(\widehat{\lambda}_i - 2)$ be the scaled classical location of the eigenvalues. Write

$$\mathcal{E}_N(w) = \sum_{i=1}^{N} \frac{n_i^2}{w - \widehat{a}_i} - N^{1/3} + \sum_{i=1}^{N} \frac{n_i^2(a_i - \widehat{a}_i - a_1)}{(w + a_1 - a_i)(w - \widehat{a}_i)}. \tag{3.3.27}$$

Since $a_i \asymp -i^{2/3}$, we find that for any $\epsilon > 0$,

$$\sum_{i=1}^{N} \frac{1}{(w - a_i)^2} \le \frac{1}{w^{1/2-\epsilon}} \sum_{i=1}^{N} \frac{1}{(w - a_i)^{3/2+\epsilon}} = \mathcal{O}\left(w^{-1/2}\right)$$

as $w \to \infty$. Thus, considering in a similar way, the last sum in (3.3.27) is $\mathcal{O}\left(w^{-1/2}\right)$ since $a_1 = \mathcal{O}(1)$, $a_i - \widehat{a}_i = \mathcal{O}(1)$, and $w \to \infty$. Setting $W = 2 + wN^{-2/3}$, (3.3.27) can be written as

$$\mathcal{E}_N(w) = N^{1/3}\left[\frac{1}{N}\sum_{i=1}^{N} \frac{1}{W - \widehat{\lambda}_i} - 1\right] + \frac{\mathcal{S}_N(W;1)}{N^{1/6}} + \mathcal{O}\left(w^{-1/2}\right).$$

From a formal application of the semicircle law,

$$\frac{1}{N}\sum_{i=1}^{N} \frac{1}{W - \widehat{\lambda}_i} \simeq s_1(W) = 1 - \sqrt{W-2} + O(W-2) = 1 - \frac{\sqrt{w}}{N^{1/3}} + O(wN^{-2/3}).$$

Thus, we obtain (3.3.26).

The equations (3.3.20) and (3.3.26) imply the next result.

Result 3.3.5. *Let t be the solution of* (3.3.14). *Then, as $H \to \infty$,*

$$\mathcal{E}_N(t) \simeq -\frac{H^2}{2(1-T)} - \frac{H^4 \mathcal{S}_N(\Gamma_0;2)}{2(1-T)^2 N^{5/6}} + \frac{\mathcal{S}_N(\Gamma_0;1)}{N^{1/6}}, \qquad \Gamma_0 = 2 + \frac{H^4 N^{-2/3}}{4(1-T)^2}. \tag{3.3.28}$$

3.3.4.2 Large H limit

From (3.3.28), we see that the $N^{-2/3}$ term in (3.3.12) satisfies

$$\frac{\widetilde{\mathcal{F}}(T,H)}{N^{2/3}} \simeq \frac{(1-T)a_1}{2N^{2/3}} - \frac{H^4}{8(1-T)N^{2/3}} + \frac{H^2\mathcal{S}_N(\Gamma_0;1)}{2N^{5/6}} \simeq -\frac{h^4}{8(1-T)} + \frac{h^2\mathcal{S}_N(\Gamma_0;1)}{2\sqrt{N}} \quad (3.3.29)$$

writing in terms of $h = HN^{-1/6}$. Thus, we find that if we take $h = HN^{-1/6}$ and $N \to \infty$ and then take $H \to \infty$, then

$$\mathcal{F}_N \simeq \left[1 - \frac{3T}{4} + \frac{T\log T}{2} + \frac{h^2}{2} - \frac{h^4}{8(1-T)}\right] + \frac{h^2\mathcal{S}_N(\Gamma_0;1)}{2\sqrt{N}}, \qquad \Gamma_0 = 2 + \frac{H^4N^{-2/3}}{4(1-T)^2} \quad (3.3.30)$$

for asymptotically almost every disorder sample. The point Γ_0 is approximately equal to γ_0. The terms in brackets are the same as the limit of $F(T,h)$ as $h \to 0$ in (3.1.37). The $O(N^{-1/2})$ term in (3.1.37) agrees with the last term of (3.3.30) since $\gamma_0 \simeq 2 + \frac{h^4}{4(1-T)^2} = \Gamma_0$ from (3.1.32). Hence, we find that the above formula is the same as the formal $h \to 0$ limit of the result (3.1.25), which was obtained by taking $N \to \infty$ first with $h = O(1)$ fixed. Hence, the result matches with the $h = O(1)$ regime.

The last term of (3.3.30) depends on the disorder sample. We consider its sample distribution and show that the sample distributions of the $h = HN^{-1/6}$ regime and $h > 0$ regime match for $0 < T < 1$. Using (2.2.13), we replace $\mathcal{S}_N(\Gamma_0;1)$ by $\mathcal{N}(0;2s_2(\Gamma_0))$. Using $s_2(z) \simeq \frac{1}{2\sqrt{z-2}}$ as $z \to 2$, we find that

$$\frac{h^2\mathcal{S}_N(\Gamma_0;1)}{2\sqrt{N}} \overset{\mathcal{D}}{\simeq} \frac{h\sqrt{1-T}}{\sqrt{2N}}\mathcal{N}(0,1). \quad (3.3.31)$$

The right-hand side is same as the fluctuation term in (3.1.38), which shows the matching. This computation shows the matching of the $h = HN^{-1/6}$ regime and the $h > 0$ regime for $0 < T < 1$ in terms of the sample distribution as well.

3.3.5 Comparison with the large deviation result of [22]

We now compare our results with the large deviation result of [22]. To this aim we first extend their calculation from $T = 0$ to any $0 < T < 1$, which is straightforward. Denoting by $\mathbb{E}_s$ the sample expectation, we find that

$$\mathbb{E}_s[\mathcal{Z}_N^n] = \mathbb{E}_s[e^{\beta Nn\mathcal{F}_N}] \simeq e^{\beta NnF^0} e^{N2^6h^6G(\frac{\beta n}{8h^2})} \quad (3.3.32)$$

where F^0 is the same as the terms in brackets in (3.3.30), the sample-independent terms, and

$$G(x) = \frac{(1-T)^3}{3}x^3 + \frac{1-T}{4}x^2. \tag{3.3.33}$$

This formula is valid for fixed $T < 1$, n, and h to the leading order as $N \to \infty$ and in a second stage as $n, h \to 0$ so that $\frac{n}{h^2}$ is fixed. The full result for fixed n and h is in (94) and (95) of [22] and the above formula follows from it after changing $T \to 2T$, $\sigma \to 2h$, and $J_0 = 2$. Note that the term $e^{N2^6h^6G(\frac{n}{8Th^2})}$ is $O(1)$ when $h = O(N^{-1/6})$ and $n = O(h^2) = O(N^{-1/3})$. We have

$$N2^6h^6G\left(\frac{n}{8Th^2}\right) = \frac{N(1-T)^3n^3}{24T^3} + \frac{Nh^2(1-T)n^2}{4T^2}. \tag{3.3.34}$$

We compare the above formula with the one obtained using the result (3.3.12). From (3.3.12), we find that

$$\mathbb{E}_s[\mathcal{Z}_N^n] = \mathbb{E}_s[e^{\frac{Nn}{T}\mathcal{F}_N}] \simeq e^{\frac{Nn}{T}F_0(T,h)}\,\mathbb{E}_s[e^{\frac{N^{1/3}n}{T}\widetilde{\mathcal{F}}(T,H)}]. \tag{3.3.35}$$

Now we let $H \to \infty$. This term was computed in (3.3.29) in which we neglected the contribution from a_1. Including this term, using (3.3.31), and also noting that $\mathcal{S}_N(z;1)$ and a_1 are independent, we obtain

$$\mathbb{E}_s[e^{\frac{N^{1/3}n}{T}\widetilde{\mathcal{F}}(T,H)}] \simeq e^{-\frac{N^{1/3}nH^4}{8T(1-T)}}\, e^{\frac{N^{2/3}n^2H^4}{8T^2\sqrt{t}}}\; \mathbb{E}_s\left[e^{\frac{N^{1/3}n(1-T)}{2T}a_1}\right]. \tag{3.3.36}$$

We can replace $\sqrt{t} \simeq \frac{H^2}{2(1-T)}$ from (3.3.20) in the middle term. For the remaining expectation, we use the right tail of the GOE Tracy-Widom distribution $F_1(s) = \mathbb{P}(\alpha_1 < s) \sim \exp(-\frac{2}{3}s^{3/2})$ as $s \to +\infty$, and thus

$$\mathbb{E}[e^{\frac{N^{1/3}n(1-T)}{2T}a_1}] \simeq \int e^{\frac{N^{1/3}n(1-T)}{2T}a_1 - \frac{2}{3}\alpha_1^{3/2}}\mathrm{d}\alpha_1 \simeq \exp\left(\frac{1}{3}\Big(\frac{N^{1/3}n(1-T)}{2T}\Big)^3\right). \tag{3.3.37}$$

Combining the calculations together, we find that

$$\mathbb{E}[\mathcal{Z}_N^n] \simeq e^{\frac{Nn}{T}F_0(T,h)}e^{-\frac{N^{1/3}nH^4}{8T(1-T)}}\,e^{\frac{N^{2/3}n^2H^2(1-T)}{4T^2}}\,e^{\frac{Nn^3(1-T)^3}{24T^3}}. \tag{3.3.38}$$

The exponents of the last two factors, upon writing $H = hN^{1/6}$, agree with (3.3.34). Since $F^0 = F_0(T,h) - \frac{h^4}{8(1-T)}$, we find that (3.3.38) is the same as (3.3.32). This shows that the tail of the typical fluctuations obtained here matches the large deviation tails at the exponential order.

3.4 A perturbation argument

The following perturbation lemma is used to obtain (3.1.7), (3.1.22) and (3.2.4).

Lemma 3.4.1. *Let I be a closed interval of $\mathbb{R}$. Let $G(z;N)$ be a sequence of random C^4-functions for $z \in I$. Let $\epsilon = \epsilon(N) := N^{-\delta}$ for some $\delta > 0$ and assume that*

$$G(z;N) = G_0(z;N) + G_1(z;N)\epsilon + G_2(z;N)\epsilon^2 + \mathcal{O}\left(\epsilon^3\right) \tag{3.4.1}$$

and

$$G'(z;N) = G_0'(z;N) + G_1'(z;N)\epsilon + G_2'(z;N)\epsilon^2 + \mathcal{O}\left(\epsilon^3\right) \tag{3.4.2}$$

for random C^4-functions $G_k(z;N)$. Suppose that

$$G_k^{(\ell)}(z;N) = \mathcal{O}(1) \tag{3.4.3}$$

uniformly for $z \in I$ for all $k = 0, 1, 2$, $0 \le \ell \le 4$ and also assume that there is a $\gamma_0 \in I$ satisfying

$$G_0'(\gamma_0;N) = 0, \qquad |G_0''(\gamma_0;N)| \ge C > 0 \tag{3.4.4}$$

for a positive constant C. Then there is a critical point $\gamma = \gamma(N)$ of $G(z;N)$ admitting the asymptotic expansion

$$\gamma = \gamma_0 + \gamma_1\epsilon + \gamma_2\epsilon^2 + \mathcal{O}\left(\epsilon^3\right) \tag{3.4.5}$$

where

$$\gamma_1 = -\frac{G_1'(\gamma_0;N)}{G_0''(\gamma_0;N)}, \qquad \gamma_2 = -\frac{G_2'(\gamma_0;N) + G_1''(\gamma_0;N)\gamma_1 + \frac{1}{2}G_0'''(\gamma_0;N)\gamma_1^2}{G_0''(\gamma_0;N)}. \tag{3.4.6}$$

Furthermore,

$$G(\gamma;N) = G_0(\gamma_0;N) + G_1(\gamma_0;N)\epsilon + \left(\frac{1}{2}G_1'(\gamma_0;N)\gamma_1 + G_2(\gamma_0;N)\right)\epsilon^2 + \mathcal{O}\left(\epsilon^3\right). \tag{3.4.7}$$

Proof. This lemma is standard when $G(z;N)$ is deterministic. The proof for the random $G(z;N)$ does not change. For simplicity, we suppress the dependence on N in the notations; for example we write $G_0(z)$ instead of $G_0(z;N)$. In order to prove (3.4.5), it is enough to show that for any

$0 < t < \delta$, $G'(\gamma_+)G'(\gamma_-) < 0$ with $\gamma_\pm = \gamma_0 + \gamma_1\epsilon + \gamma_2\epsilon^2 \pm \epsilon^3 N^t$. From the Taylor expansion,

$$\begin{aligned} G'(\gamma_\pm) &= G_0'(\gamma_0) + (G_0''(\gamma)\gamma_1 + G_1'(\gamma_0))\epsilon \\ &\quad + \left(G_0''(\gamma_0)\gamma_2 + G_2'(\gamma_0) + G_1''(\gamma_0)\gamma_1 + \frac{1}{2}G_0'''(\gamma_0)\gamma_1^2\right)\epsilon^2 \pm G_0''(\gamma_0)\epsilon^3 N^t + \mathcal{O}\left(\epsilon^3\right). \end{aligned} \tag{3.4.8}$$

The definitions of γ_0, γ_1, and γ_2 imply that

$$G'(\gamma_\pm) = \pm G_0''(\gamma_0)\epsilon^3 N^t + \mathcal{O}\left(\epsilon^3\right) \tag{3.4.9}$$

Thus, $G'(\gamma_+)G'(\gamma_-) < 0$ for all large enough N and we obtain (3.4.5). The equation (3.4.7) follows from

$$\begin{aligned} G(\gamma) &= G_0(\gamma) + G_1(\gamma)\epsilon + G_2(\gamma)\epsilon^2 + \mathcal{O}\left(\epsilon^3\right) = G_0(\gamma_0) + (G_0'(\gamma_0)\gamma_1 + G_1(\gamma_0))\epsilon \\ &\quad + \left(G_0'(\gamma_0)\gamma_2 + \frac{1}{2}G_0''(\gamma_0)\gamma_1^2 + G_1'(\gamma_0)\gamma_1 + G_2(\gamma_0)\right)\epsilon^2 + \mathcal{O}\left(\epsilon^3\right), \end{aligned} \tag{3.4.10}$$

together with $G_0'(\gamma_0) = 0$ and (3.4.6). □

Remark 3.4.2. Here, we consider the asymptotic expansion of $G(z)$ up to the third order term. One can also consider the case where the expansion is up to the second order, then (3.4.7) is still valid up to the second order.

CHAPTER IV

Overlaps of the Spherical Sherrington-Kirkpatrick Model

This is the longest chapter of the thesis and contains our analysis of three types of overlaps: the overlap with an external field, overlap with the ground state, and the overlap with a replica. These are discussed in Sections 4.1, 4.2, and 4.3 respectively. For each type of overlap, we analyze its limiting value as well as its fluctuations in several distinct regimes, namely $h = 0$, $h > 0$ fixed, and the transitional regimes where $h \to 0$ as $N \to \infty$. None of the three overlaps exhibits any transition at high temperature. However, when $T < 1$, all three overlaps exhibit a transition at $h \sim N^{-1/6}$. The overlap with the external field and the overlap with a replica also exhibit a transition at $h \sim N^{-1/2}$. The overlap with the ground state exhibits a transition at $h \sim N^{-1/3}$. We conclude the chapter with Section 4.4, which describes the implications of our findings in terms of the geometry of the Gibbs measure.

4.1 Overlap with the external field

The overlap of a spin with the external field is

$$\mathfrak{M} = \frac{\mathbf{g} \cdot \boldsymbol{\sigma}}{N}.$$

We study the thermal fluctuation of the overlap for a given disorder sample in several regimes: $h = O(1)$, $h \sim N^{-1/6}$ and $h \sim N^{-1/2}$. We also consider the magnetization, susceptibility, and differential susceptibility,

$$\mathcal{M} = \langle \mathfrak{M} \rangle, \qquad \mathcal{X} = \frac{\mathcal{M}}{h}, \qquad \mathcal{X}_d = \frac{\mathrm{d}\mathcal{M}}{\mathrm{d}h}.$$

4.1.1 Thermal average from free energy

Before we discuss the thermal fluctuations of $\mathfrak{M}$, we first derive the thermal average of $\mathfrak{M}$, i.e. the magnetization, from the results for the free energy in two regimes, $h = O(1)$ and $h \sim N^{-1/6}$, using

$$\mathcal{M} = \langle \mathfrak{M} \rangle = \frac{\mathrm{d}\mathcal{F}_N}{\mathrm{d}h}. \tag{4.1.1}$$

Case $h = O(1)$:

For $h > 0$ and $T > 0$, the result (3.1.24) for the free energy implies that

$$\langle \mathfrak{M} \rangle = \frac{\mathrm{d}\mathcal{F}_N}{\mathrm{d}h} \simeq \frac{\mathrm{d}F(T,h)}{\mathrm{d}h} + \frac{1}{2\sqrt{N}} \frac{\mathrm{d}}{\mathrm{d}h}(h^2 \mathcal{S}_N(\gamma_0; 1)) \tag{4.1.2}$$

for asymptotically almost every disorder sample where we recall from equation (3.1.26) that $\gamma_0 > 2$ satisfies $1 - Ts_1(\gamma_0) - h^2 s_2(\gamma_0) = 0$. Using $s_0'(z) = s_1(z)$ and $s_1'(z) = -s_2(z)$,

$$\frac{\mathrm{d}F(T,h)}{\mathrm{d}h} = hs_1(\gamma_0) + \frac{1}{2}(1 - Ts_1(\gamma_0) - h^2 s_2(\gamma_0)) \frac{d\gamma_0}{\mathrm{d}h} \tag{4.1.3}$$

However, the equation for γ_0 (see equation (3.1.26)) implies that the second term is zero. On the other hand, since $\mathcal{S}_N'(z; 1) = -\mathcal{S}_N(z; 2)$,

$$\frac{\mathrm{d}}{\mathrm{d}h}(h^2 \mathcal{S}_N(\gamma_0; 1)) = 2h\mathcal{S}_N(\gamma_0; 1) - h^2 \mathcal{S}_N(\gamma_0; 2) \frac{d\gamma_0}{\mathrm{d}h}. \tag{4.1.4}$$

Using equation (3.1.26) again and $s_2'(z) = -2s_3(z)$, we find that

$$\frac{d\gamma_0}{\mathrm{d}h} = \frac{2hs_2(\gamma_0)}{Ts_2(\gamma_0) + 2h^2 s_3(\gamma_0)}. \tag{4.1.5}$$

Therefore, we conclude that, for fixed $h > 0$ and $T > 0$,

$$\langle \mathfrak{M} \rangle \simeq hs_1(\gamma_0) + \frac{1}{\sqrt{N}} \left[h\mathcal{S}_N(\gamma_0; 1) - \frac{h^3 s_2(\gamma_0) \mathcal{S}_N(\gamma_0; 2)}{Ts_2(\gamma_0) + 2h^2 s_3(\gamma_0)} \right] \tag{4.1.6}$$

for asymptotically almost every disorder sample.

Case $h \sim N^{-1/6}$ and $T < 1$:

If we use the result (3.3.12) for the free energy when $h = HN^{-1/6}$ and $0 < T < 1$, we find that

$$\langle \mathfrak{M} \rangle = N^{1/6} \frac{\mathrm{d}\mathcal{F}_N}{\mathrm{d}H} \simeq h + \frac{H\mathcal{E}_N(t)}{\sqrt{N}} + \frac{(1 - T + H^2 \mathcal{E}_N'(t))}{2\sqrt{N}} \frac{\mathrm{d}t}{\mathrm{d}H} \tag{4.1.7}$$

for asymptotically almost every disorder sample. The formula for $\mathcal{E}_N$ is given in (2.2.5) and

$$\mathcal{E}_N'(w) = -\sum_{i=1}^{N} \frac{n_i^2}{(w + a_1 - a_i)^2}. \tag{4.1.8}$$

Since t satisfies the equation (3.3.14), we see that the term $1 - T + H^2\mathcal{E}_N'(t) = 0$. Hence, for $h = HN^{-1/6}$ and $0 < T < 1$,

$$\langle \mathfrak{M} \rangle \simeq h + \frac{H\mathcal{E}_N(t)}{\sqrt{N}} \tag{4.1.9}$$

for asymptotically almost every disorder sample.

In both of these regimes, it turns out that the thermal average is indeed the leading term. However, this calculation does not give us the thermal fluctuation term. To obtain that, we use the integral representation of the overlap in the following subsections. For the overlap and magnetization, it turns out that there is another interesting regime, $h \sim N^{-1/2}$, for $0 < T < 1$. This is the regime that occurs when the two terms in (4.1.9) have the same order; it was shown in (3.3.24) that $H\mathcal{E}_N(t) \simeq \mathcal{O}(1)$ as $H \to 0$. See the following subsections for the details.

4.1.2 Setup

We obtain the thermal probability of the overlap by considering the moment generating function $\langle e^{\beta\eta\mathfrak{M}} \rangle$ with respect to the Gibbs measure (1.1.11). Here, η is the variable for the generating function and we scaled by β for convenience in subsequent formulas. It turns out that the thermal fluctuations of $\mathfrak{M}$ are of order $N^{-1/2}$ in all regimes. Hence, we scale $\eta = \xi\sqrt{N}$ and use ξ as the scaled variable for the moment generating function. From Lemma 2.3.3, we have the following formula:

$$\langle e^{\beta\xi\sqrt{N}\mathfrak{M}} \rangle = e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}) - \mathcal{G}(\gamma))} \frac{\int_{\gamma_{\mathfrak{M}} - \mathrm{i}\infty}^{\gamma_{\mathfrak{M}} + \mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{M}}(z) - \mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}))} \mathrm{d}z}{\int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}(z) - \mathcal{G}(\gamma))} \mathrm{d}z} \tag{4.1.10}$$

where

$$\mathcal{G}_{\mathfrak{M}}(z) = \beta z - \frac{1}{N}\sum_{i=1}^{N} \log(z - \lambda_i) + \frac{(h + \frac{\xi}{\sqrt{N}})^2 \beta}{N} \sum_{i=1}^{N} \frac{n_i^2}{z - \lambda_i}. \tag{4.1.11}$$

Here, we take $\gamma_{\mathfrak{M}} > \lambda_1$ to be the critical point of $\mathcal{G}_{\mathfrak{M}}(z)$ satisfying

$$\mathcal{G}_{\mathfrak{M}}'(\gamma_{\mathfrak{M}}) = 0 \tag{4.1.12}$$

and we take $\gamma > \lambda_1$ to be the critical point of $\mathcal{G}(z)$. The only difference between $\mathcal{G}_{\mathfrak{M}}$ and $\mathcal{G}$, which we studied extensively in the previous sections, is that h is changed to $h + \xi N^{-1/2}$.

We record two formulas that we use below. From the explicit formulas for $\mathcal{G}_{\mathfrak{M}}$ and $\mathcal{G}$, the equation $\mathcal{G}'_{\mathfrak{M}}(\gamma_{\mathfrak{M}}) - \mathcal{G}'(\gamma) = 0$ implies that

$$\begin{aligned}(\gamma_{\mathfrak{M}} - \gamma)\left[\frac{1}{N}\sum_{i=1}^{N}\frac{1}{(\gamma_{\mathfrak{M}}-\lambda_i)(\gamma-\lambda_i)} + \frac{h^2\beta}{N}\sum_{i=1}^{N}\frac{n_i^2(\gamma+\gamma_{\mathfrak{M}}-2\lambda_i)}{(\gamma_{\mathfrak{M}}-\lambda_i)^2(\gamma-\lambda_i)^2}\right] \\ = \left(\frac{2\xi h}{N^{3/2}} + \frac{\xi^2}{N^2}\right)\beta\sum_{i=1}^{N}\frac{n_i^2}{(\gamma_{\mathfrak{M}}-\lambda_i)^2}.\end{aligned} \tag{4.1.13}$$

The other formula that we will need is

$$\begin{aligned}N(\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}) - \mathcal{G}(\gamma)) = -\sum_{i=1}^{N}\left[\log\left(1+\frac{\gamma_{\mathfrak{M}}-\gamma}{\gamma-\lambda_i}\right) - \frac{\gamma_{\mathfrak{M}}-\gamma}{\gamma-\lambda_i}\right] + h^2\beta\sum_{i=1}^{N}\frac{n_i^2(\gamma_{\mathfrak{M}}-\gamma)^2}{(\gamma_{\mathfrak{M}}-\lambda_i)(\gamma-\lambda_i)^2} \\ + \left(\frac{2\xi h}{\sqrt{N}} + \frac{\xi^2}{N}\right)\beta\sum_{i=1}^{N}\frac{n_i^2}{\gamma_{\mathfrak{M}}-\lambda_i} =: A_1 + A_2 + A_3,\end{aligned} \tag{4.1.14}$$

which can be seen using $\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}) - \mathcal{G}(\gamma) = \mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}) - \mathcal{G}(\gamma) - \mathcal{G}'(\gamma)(\gamma_{\mathfrak{M}} - \gamma)$.

4.1.3 Positive external field: $h = O(1)$

4.1.3.1 Analysis

Fix $h > 0$. The critical point γ of $\mathcal{G}(z)$ is evaluated in subsection 3.1.2. It is shown in (3.1.20) that

$$\gamma = \gamma_0 + \gamma_1 N^{-1/2} + \mathcal{O}\left(N^{-1}\right)$$

where γ_0 and γ_1 are deterministic functions of h and T. From the formulas for $\mathcal{G}$ and $\mathcal{G}_{\mathfrak{M}}$, we see that $\mathcal{G}'_{\mathfrak{M}}(z) = \mathcal{G}'(z) + \mathcal{O}\left(N^{-1/2}\right)$ for $z > \lambda_1 + O(1)$ (cf. (2.2.16)). This implies that $\gamma_{\mathfrak{M}} - \gamma = \mathcal{O}\left(N^{-1/2}\right)$. We need to evaluate the difference precisely. From (4.1.13), we find, using the semicircle law, that

$$(\gamma_{\mathfrak{M}} - \gamma)\left(s_2(\gamma) + 2h^2\beta s_3(\gamma) + \mathcal{O}\left(N^{-1/2}\right)\right) = \frac{2\xi h\beta}{\sqrt{N}}s_2(\gamma) + \mathcal{O}\left(N^{-1}\right).$$

Thus,

$$\gamma_{\mathfrak{M}} = \gamma + \Delta N^{-1/2}, \qquad \Delta = \frac{2h\beta\xi s_2(\gamma_0)}{s_2(\gamma_0) + 2h^2\beta s_3(\gamma_0)} + \mathcal{O}\left(N^{-1/2}\right). \tag{4.1.15}$$

We now evaluate $N(\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}) - \mathcal{G}(\gamma))$ for (4.1.10) via the equation (4.1.14). Using the Taylor expansion of the logarithm function,

$$A_1 = \frac{\Delta^2}{2N} \sum_{i=1}^{N} \frac{1}{(\gamma - \lambda_i)^2} + \mathcal{O}\left(\frac{1}{N^{3/2}} \sum_{i=1}^{N} \frac{1}{(\gamma - \lambda_i)^3} \right) = \frac{\Delta^2 s_2(\gamma)}{2} + \mathcal{O}\left(N^{-1/2}\right). \tag{4.1.16}$$

Similarly,

$$A_2 = \frac{h^2 \beta \Delta^2}{N} \sum_{i=1}^{N} \frac{n_i^2}{(\gamma - \lambda_i)^3} + \mathcal{O}\left(N^{-1/2}\right) = h^2 \beta \Delta^2 s_3(\gamma) + \mathcal{O}\left(N^{-1/2}\right). \tag{4.1.17}$$

In these two equations, we replaced $\gamma_{\mathfrak{M}}$ by γ . For A_3, using (4.1.15) and the notation (2.2.6), we have

$$\begin{aligned} A_3 &= 2\xi h \beta (s_1(\gamma_{\mathfrak{M}})\sqrt{N} + \mathcal{S}_N(\gamma_{\mathfrak{M}}; 1)) + \xi^2 \beta s_1(\gamma_{\mathfrak{M}}) + \mathcal{O}\left(N^{-1/2}\right) \\ &= 2\xi h \beta s_1(\gamma)\sqrt{N} + \left[2\xi h (\mathcal{S}_N(\gamma; 1) - s_2(\gamma)\Delta) + \xi^2 s_1(\gamma)\right] \beta + \mathcal{O}\left(N^{-1/2}\right). \end{aligned} \tag{4.1.18}$$

Combining the three terms and inserting the formulas of γ and Δ,

$$\begin{aligned} N(\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}) - \mathcal{G}(\gamma)) =& 2\xi h \beta \left[\sqrt{N} s_1(\gamma_0) - s_2(\gamma_0)\gamma_1 + \mathcal{S}_N(\gamma_0; 1) \right] \\ &+ \xi^2 \left[\beta s_1(\gamma_0) - \frac{2h^2 \beta^2 s_2(\gamma_0)^2}{s_2(\gamma_0) + 2h^2 \beta s_3(\gamma_0)} \right] + \mathcal{O}\left(N^{-1/2}\right). \end{aligned} \tag{4.1.19}$$

Now we consider the integrals in (4.1.10). Since $\mathcal{G}^{(k)}(\gamma) = \mathcal{O}(1)$ for all $k \geq 2$, the method of steepest descent applies. It is also straightforward to check that

$$\mathcal{G}''_{\mathfrak{M}}(\gamma_{\mathfrak{M}}) = \mathcal{G}''(\gamma_{\mathfrak{M}}) + \mathcal{O}\left(N^{-1/2}\right) = \mathcal{G}''(\gamma) + \mathcal{O}\left(N^{-1/2}\right).$$

Hence,

$$\frac{\int_{\gamma_{\mathfrak{M}} - \mathrm{i}\infty}^{\gamma_{\mathfrak{M}} + \mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{M}}(z) - \mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}))} \mathrm{d}z}{\int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}(z) - \mathcal{G}(\gamma))} \mathrm{d}z} \simeq \sqrt{\frac{\mathcal{G}''(\gamma)}{\mathcal{G}''_{\mathfrak{M}}(\gamma_{\mathfrak{M}})}} \simeq 1.$$

Inserting the above computations into (4.1.10), moving the term involving $\sqrt{N}$ to the left, replacing $\beta\xi$ by ξ, using $\beta = 1/T$, and inserting the formula (3.1.21) for γ_1, we obtain the following.

Result 4.1.1. *For $h = O(1)$ and $T > 0$,*

$$\left\langle e^{\xi\sqrt{N}(\mathfrak{M} - h s_1(\gamma_0))} \right\rangle \simeq e^{\xi h \left[\mathcal{S}_N(\gamma_0; 1) - \frac{h^3 s_2(\gamma_0) \mathcal{S}_N(\gamma_0; 2)}{T s_2(\gamma_0) + 2h^2 s_3(\gamma_0)} \right] + \frac{\xi^2}{2} \left[T s_1(\gamma_0) - \frac{2T h^2 s_2(\gamma_0)^2}{2T s_2(\gamma_0) + h^2 s_3(\gamma_0)} \right]} \tag{4.1.20}$$

as $N \to \infty$ for asymptotically almost every disorder sample, where $\gamma_0 > 2$ is the solution of the equation (3.1.19) *and $\mathcal{S}_N(z;k)$ is defined in* (2.2.6)*.*

Since the right-hand side is an exponential of a quadratic function of ξ, we obtain the following distributional result.

Result 4.1.2. *For $h = O(1)$ and $T > 0$,*

$$\mathfrak{M} \overset{\mathfrak{D}}{\simeq} hs_1(\gamma_0) + \frac{1}{\sqrt{N}}\left[h\mathcal{S}_N(\gamma_0;1) - \frac{h^3 s_2(\gamma_0)\mathcal{S}_N(\gamma_0;2)}{Ts_2(\gamma_0) + 2h^2 s_3(\gamma_0)} + \sigma_{\mathfrak{M}}\mathfrak{N}\right] \tag{4.1.21}$$

as $N \to \infty$ for asymptotically almost every disorder sample. The thermal random variable $\mathfrak{N}$ is a standard normal random variable and the coefficient $\sigma_{\mathfrak{M}} > 0$ is given by the formula

$$\sigma_{\mathfrak{M}}^2 = Ts_1(\gamma_0) - \frac{2Th^2 s_2(\gamma_0)^2}{Ts_2(\gamma_0) + 2h^2 s_3(\gamma_0)}. \tag{4.1.22}$$

The thermal average is given by the first three terms in (4.1.21) and they agree with the formula (4.1.6) obtained from the free energy.

4.1.3.2 Discussion of the leading term

The leading term

$$\mathfrak{M}^0(h,T) := hs_1(\gamma_0(h)) \tag{4.1.23}$$

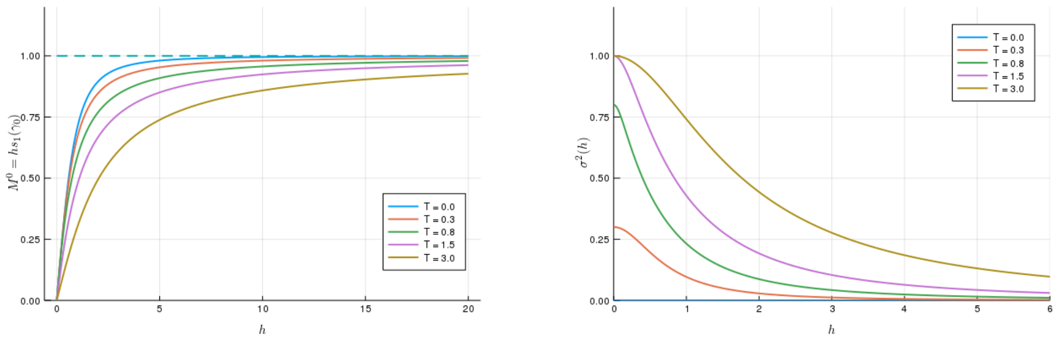
in (4.1.21) is deterministic. See Figure IV.1 (a) for a graph as a function of h. The function $\mathfrak{M}^0$ satisfies the following properties:

- For every $T > 0$, $\mathfrak{M}^0(h,T)$ is an increasing function of h.
- As $h \to \infty$,
$$\mathfrak{M}^0(h,T) = 1 - \frac{T}{2h} + O(h^{-2}) \quad \text{for all } T > 0. \tag{4.1.24}$$
- As $h \to 0$,
$$\mathfrak{M}^0(h,T) \simeq \begin{cases} \frac{h}{T} - \frac{h^3}{T(T^2-1)} & \text{for } T > 1, \\ h - \frac{h^3}{2(1-T)} & \text{for } 0 < T < 1. \end{cases} \tag{4.1.25}$$

The first property is consistent with the intuition that the overlap of the spin with the external field becomes larger as the external field becomes stronger. The proof follows from

$$\frac{\mathrm{d}}{\mathrm{d}h}\mathfrak{M}^0 = s_1(\gamma_0) - hs_2(\gamma_0)\gamma_0' = \frac{Ts_1(\gamma_0)s_2(\gamma_0) + 2h^2(s_1(\gamma_0)s_3(\gamma_0) - s_2(\gamma_0)^2)}{Ts_2(\gamma_0) + 2h^2 s_3(\gamma_0)} \tag{4.1.26}$$

and from checking that $s_1(z)s_3(z) - s_2(z)^2 > 0$ for all $z > 2$ using (2.1.7). The large-h and small-h limits follow from Lemma 3.1.7.



(a) $\mathfrak{M}^0$ (b) $\sigma^2_{\mathfrak{M}}$

Figure IV.1: Graphs of $\mathfrak{M}^0$ and $\sigma^2_{\mathfrak{M}}$ as functions of h for various values of T

4.1.3.3 Discussion of the variance

The variance of the overlap satisfies

$$\langle \mathfrak{M}^2 \rangle - \langle \mathfrak{M} \rangle^2 \simeq \frac{\sigma^2_{\mathfrak{M}}}{N}. \tag{4.1.27}$$

The term $\sigma^2_{\mathfrak{M}}(h, T) = \sigma^2_{\mathfrak{M}}$ is given in (4.1.22) and does not depend on the disorder sample. See Figure IV.1 for the graph. Here are some properties of $\sigma^2_{\mathfrak{M}}$.

- For every T, $\sigma^2_{\mathfrak{M}}(h, T)$ is a decreasing function of h.
- As $h \to \infty$,
$$\sigma^2_{\mathfrak{M}} = \frac{T}{h} + O(h^{-2}) \quad \text{for all } T > 0. \tag{4.1.28}$$
- As $h \to 0$,
$$\sigma^2_{\mathfrak{M}} \to \begin{cases} 1 & \text{for } T > 1, \\ T & \text{for } 0 < T < 1. \end{cases} \tag{4.1.29}$$

The first property follows from

$$\frac{\mathrm{d}}{\mathrm{d}h}\sigma^2_{\mathfrak{M}} = -\frac{T^2 s_2(\gamma_0)\left[(T s_2(\gamma_0)^2 - 12h^4 s_3(\gamma_0)^2 + 12h^4 s_2(\gamma_0)s_4(\gamma_0))\,\gamma_0'(h) + 4hT s_2(\gamma_0)^2\right]}{(T s_2(\gamma_0) + 2h^2 s_3(\gamma_0))^2} \tag{4.1.30}$$

by checking that $s_2(z)s_4(z) - s_3(z)^2 > 0$ for all $z > 2$. The large- and small-h limits follow from Lemma 3.1.7.

4.1.3.4 Limit as $h \to \infty$

Consider the formal limit of the result (4.1.21) as $h \to \infty$. Using (3.1.33), we have

$$h^k \mathcal{S}_N(\gamma_0; k) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{n_i^2 - 1}{(\frac{\gamma_0}{h} - \frac{\widehat{\lambda}_i}{h})^k} \simeq \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (n_i^2 - 1) \tag{4.1.31}$$

and $s_k(\gamma_0) \simeq h^{-k}$ as $h \to \infty$. Therefore, using (4.1.24) and (4.1.28), we find that if we take $N \to \infty$ with $h > 0$ and then take $h \to \infty$, we get

$$\mathfrak{M} \overset{\mathfrak{D}}{\simeq} 1 - \frac{T}{2h} + \frac{1}{\sqrt{N}} \left[\frac{\sum_{i=1}^{N} (n_i^2 - 1)}{2\sqrt{N}} + \frac{\sqrt{T}}{\sqrt{h}} \mathfrak{N} \right]. \tag{4.1.32}$$

The leading term $\mathfrak{M} \simeq 1$ is trivial since the spin is likely to be pulled to the direction of the external field if h is large.

4.1.3.5 Limit as $h \to 0$ when $T > 1$

Since $\gamma_0 \to T + T^{-1}$ as $h \to 0$ for $T > 1$ from (3.1.32), the terms $\mathcal{S}_N(\gamma_0; 1)$ and $\mathcal{S}_N(\gamma_0; 2)$ remain $O(1)$. Hence the deterministic terms in the square brackets in (4.1.21) converge to zero as $h \to 0$. We thus find, using (4.1.25) and (4.1.29), that, if we take the limit $N \to \infty$ with $h > 0$ and then take $h \to 0$, the result for $T > 1$ becomes

$$\mathfrak{M} \overset{\mathfrak{D}}{\simeq} \frac{h}{T} - \frac{h^3}{T(T^2 - 1)} + \frac{1}{\sqrt{N}} \left[\mathfrak{N} + h \mathcal{S}_N(T + \frac{1}{T}; 1) \right]. \tag{4.1.33}$$

4.1.3.6 Limit as $h \to 0$ when $T < 1$

The small-h limit (3.1.32) of γ_0 and the limit of $s_k(z)$ as $z \to 2$ obtained in (2.1.8) imply that, as $h \to 0$,

$$s_2(\gamma_0) \simeq \frac{(1-T)}{h^2} + \frac{T}{2(1-T)}, \qquad s_3(\gamma_0) \simeq \frac{(1-T)^3}{h^6}, \qquad s_4(\gamma_0) \simeq \frac{2(1-T)^5}{h^{10}} \tag{4.1.34}$$

when $0 < T < 1$. From these, we see that

$$h \mathcal{S}_N(\gamma_0; 1) - \frac{h^3 s_2(\gamma_0) \mathcal{S}_N(\gamma_0; 2)}{T s_2(\gamma_0) + 2h^2 s_3(\gamma_0)} \simeq h \mathcal{S}_N(\gamma_0; 1) - \frac{h^5 \mathcal{S}_N(\gamma_0; 2)}{2(1-T)^2}. \tag{4.1.35}$$

Thus, by (4.1.25) and (4.1.29), if take the limit $N \to \infty$ with $h > 0$ and then take $h \to 0$, then

$$\mathfrak{M} \overset{\mathcal{D}}{\simeq} h - \frac{h^3}{2(1-T)} + \frac{1}{\sqrt{N}} \left[h\mathcal{S}_N(\gamma_0; 1) - \frac{h^5 \mathcal{S}_N(\gamma_0; 2)}{2(1-T)^2} + \sqrt{T}\mathfrak{N} \right], \quad \gamma_0 \simeq 2 + \frac{h^4}{4(1-T)^2}. \tag{4.1.36}$$

Finally, we consider the terms $h\mathcal{S}_N(\gamma_0; 1)$ and $h^5\mathcal{S}_N(\gamma_0; 2)$. The sample-to-sample variance of $\mathcal{S}_N(\gamma_0; k)$ is

$$\frac{2}{N} \sum_{i=1}^{N} \frac{1}{(\gamma_0 - \widehat{\lambda}_i)^{2k}} \simeq 2s_{2k}(\gamma_0), \tag{4.1.37}$$

which is expected to hold for $\gamma_0 - 2 \gg N^{-2/3}$, i.e., $h \gg N^{-1/6}$, Thus the sample-to-sample variance is $\mathcal{O}\left(h^{-2}\right)$ for $k = 1$ and $\mathcal{O}\left(h^{-10}\right)$ for $k = 2$ from (4.1.34). Hence, we expect that $h\mathcal{S}_N(\gamma_0; 1)$ and $h^5\mathcal{S}_N(\gamma_0; 2)$ are $\mathcal{O}\left(1\right)$ for $h \gg N^{-1/6}$.

4.1.4 No external field: $h = 0$

When $h = 0$ and $T > 1$, it is well-known in spin glass theory [38, 36] that two independently chosen spins are asymptotically orthogonal, indicating that the spin variable becomes uniformly distributed on the sphere $\|\boldsymbol{\sigma}\| = \sqrt{N}$ as $N \to \infty$. For $h = 0$ the Gibbs measure is independent of g. Hence, the overlap $\mathfrak{M} = \frac{1}{N}\mathbf{g} \cdot \boldsymbol{\sigma}$ of the spin with the random Gaussian vector g is the cosine of the angle of two independent vectors which are chosen more or less uniformly at random from the sphere. Thus, we expect that $\mathfrak{M}$ is approximately $\frac{1}{\sqrt{N}}$ times a standard normal distribution. The formal limit of (4.1.33) as $h \to 0$ coincides with this result. Indeed when $T > 1$, the analysis for $h > 0$ with $h = O(1)$ extends to $h \geq 0$ and (4.1.21) holds.

When $h = 0$ and $T < 1$, it was argued in [28] that $\frac{\langle|\mathbf{u}_1 \cdot \boldsymbol{\sigma}|\rangle}{\sqrt{N}}$ converges to $\sqrt{1-T}$. (In [28], the authors claim that $\frac{\langle \mathbf{u}_1 \cdot \boldsymbol{\sigma}\rangle}{\sqrt{N}} \to \sqrt{1-T}$, but this seems to be a typographical error since $\langle \mathbf{u}_1 \cdot \boldsymbol{\sigma}\rangle = 0$ due to the symmetry of the Gibbs measure under the transformation $\boldsymbol{\sigma} \mapsto -\boldsymbol{\sigma}$.) It was also proven in [38] that the absolute value of the overlap of two independently chosen spins converges to $1 - T$. Hence, a spin variable may be written as $\frac{\boldsymbol{\sigma}}{\sqrt{N}} = \pm\sqrt{1-T}\mathbf{u}_1 + \sqrt{T}\mathbf{v}$, where the unit vector $\mathbf{v}$ is taken uniformly at random from the hyperplane perpendicular to $\mathbf{u}_1$ and the signs $\pm$ are each taken with probability $1/2$; see more discussion on such a decomposition of the spin variable in Section 4.4. Thus, using the notation $n_1 = \mathbf{u}_1 \cdot \mathbf{g}$, we expect that $\mathfrak{M} \simeq \frac{\pm n_1\sqrt{1-T} + \sqrt{T}\mathfrak{N}}{\sqrt{N}}$. Recall that $\mathbf{u}_1$ has sign ambiguity and hence n_1 is defined up to its sign. Thus, we find the following result for $h = 0$.

Result 4.1.3. *For* $h = 0$*,*

$$\mathfrak{M} \simeq \begin{cases} \frac{1}{\sqrt{N}} \mathfrak{N} & \text{for } T > 1, \\ \frac{|n_1|\sqrt{1-T}\mathfrak{B} + \sqrt{T}\mathfrak{N}}{\sqrt{N}} & \text{for } 0 < T < 1 \end{cases} \tag{4.1.38}$$

as $N \to \infty$*, for asymptotically almost every disorder sample, where* $\mathfrak{N}$ *is a standard normal random variable, and* $\mathfrak{B}$ *is independent of* $\mathfrak{N}$ *and has the distribution* $\mathbb{P}(\mathfrak{B} = 1) = \mathbb{P}(\mathfrak{B} = -1) = \frac{1}{2}$.

The right-hand side of (4.1.36) involves the thermal random variable $\mathfrak{N}$ but does not involve the other thermal random variable $\mathfrak{B}$ in (4.1.38). Hence, the formal limit of (4.1.36) as $h \to 0$ is not equal to (4.1.38) when $T < 1$. This implies that there should be a transitional regime. It turns out that there are two transitional regimes, $h \sim N^{-1/6}$ and $h \sim N^{-1/2}$. The first regime can be expected, since $\gamma_0 = 2 + O(h^4)$ as $h \to 0$, and the subleading term $O(h^4)$ is of same order as the fluctuations of the top eigenvalue λ_1 when $h \sim N^{-1/6}$. This is the same transitional regime that was observed for the free energy. The second regime $h \sim N^{-1/2}$ arises because the ratio of the integrals in (4.1.10), which was approximately equal to 1 when $h > 0$ (and when $h \sim N^{-1/6}$ as well), is no longer close to 1 when $h \sim N^{-1/2}$. This will be responsible for the appearance of $\mathfrak{B}$. We discuss these two transitional regimes in the next subsections. We will see in Subsection 4.1.6 that the result for $h = HN^{-1/2}$ actually holds even when $H = 0$, implying that (4.1.38) indeed holds.

4.1.5 Mesoscopic external field: $h \sim N^{-1/6}$ and $T < 1$

4.1.5.1 Analysis

We scale h as

$$h = HN^{-1/6}$$

for fixed $H > 0$. This scale is the same as the one considered in Subsection 3.3.1. We showed in that section that the critical point of $\mathcal{G}(z)$ is $\gamma = \lambda_1 + sN^{-2/3}$ where $s > 0$ satisfies the equation (3.3.4). To find the critical point of $\mathcal{G}_{\mathfrak{M}}(z)$, we make the ansatz that $\gamma_{\mathfrak{M}} \simeq \gamma$, Then, the equation (4.1.13) becomes

$$(\gamma_{\mathfrak{M}} - \gamma) \left[N^{1/3} \sum_{i=1}^{N} \frac{1}{(s + a_1 - a_i)^2} + H^2 \beta N^{2/3} \sum_{i=1}^{N} \frac{2n_i^2}{(s + a_1 - a_i)^3} \right]$$
$$\simeq \left(\frac{2\xi H}{N^{5/3}} + \frac{\xi^2}{N^2} \right) \beta N^{4/3} \sum_{i=1}^{N} \frac{n_i^2}{(s + a_1 - a_i)^2},$$

implying that

$$\gamma_{\mathfrak{M}} - \gamma = \mathcal{O}\left(N^{-1}\right), \tag{4.1.39}$$

which is consistent with the ansatz. We do not need to determine the $\mathcal{O}\left(N^{-1}\right)$ term in this subsection.

We now evaluate $N(\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}) - \mathcal{G}(\gamma))$ using (4.1.14). From the Taylor series of the log function,

$$A_1 \simeq \sum_{i=1}^{N} \frac{(\gamma_{\mathfrak{M}} - \gamma)^2}{(\gamma - \lambda_i)^2} = \sum_{i=1}^{N} \frac{(\gamma_{\mathfrak{M}} - \gamma)^2 N^{4/3}}{(s + a_1 - a_i)^2} = \mathcal{O}\left(N^{-2/3}\right).$$

Inserting $h = HN^{-1/6}$,

$$A_2 \simeq \frac{H^2\beta}{N^{1/3}} \left[N^2 \sum_{i=1}^{N} \frac{n_i^2}{(s + a_1 - a_i)^2}\right] (\gamma_{\mathfrak{M}} - \gamma)^2 = \mathcal{O}\left(N^{-1/3}\right).$$

The third term in (4.1.14) is

$$A_3 = \left(2\xi H + \frac{\xi^2}{N^{1/3}}\right) \beta \left[\sum_{i=1}^{N} \frac{n_i^2}{s + a_1 - a_i} + \mathcal{O}\left(N^{-1/3}\right)\right].$$

Using the random variable $\mathcal{E}_N(s)$ defined in (2.2.5), which is $\mathcal{O}(1)$, and combining all three terms,

$$N(\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}) - \mathcal{G}(\gamma)) = 2\xi H\beta N^{1/3} + 2\beta\xi H\mathcal{E}_N(s) + \beta\xi^2 + \mathcal{O}\left(N^{-1/3}\right). \tag{4.1.40}$$

Finally, consider the integrals in (4.1.10). The denominator is computed in Section 3.3.1. The numerator can be computed in the same manner. Indeed, we can check, as with the denominator, that $\mathcal{G}_{\mathfrak{M}}^{(k)}(\gamma_{\mathfrak{M}}) = \mathcal{O}\left(N^{\frac{2}{3}k - \frac{2}{3}}\right)$ for all $k \geq 2$ and

$$\mathcal{G}_{\mathfrak{M}}''(\gamma_{\mathfrak{M}}) = 2N^{2/3}H^2\beta \sum_{i=1}^{N} \frac{n_i^2}{(s + a_1 - a_i)^3} + \mathcal{O}\left(N^{1/2}\right), \tag{4.1.41}$$

which is the same as the denominator. Hence, the Gaussian integral approximations of the integrals imply that

$$\frac{\int_{\gamma_{\mathfrak{M}} - \mathrm{i}\infty}^{\gamma_{\mathfrak{M}} + \mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{M}}(z) - \mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}))} \mathrm{d}z}{\int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}(z) - \mathcal{G}(\gamma))} \mathrm{d}z} \simeq \sqrt{\frac{\mathcal{G}''(\gamma)}{\mathcal{G}_{\mathfrak{M}}''(\gamma_{\mathfrak{M}})}} \simeq 1. \tag{4.1.42}$$

Combining the above computations into (4.1.10), replacing s by t (the solution to (3.3.5)), replacing $\beta\xi$ by ξ, and using $1/\beta = T$, we obtain the following result.

Result 4.1.4. *For $h = HN^{-1/6}$ and $0 < T < 1$,*

$$\langle e^{\xi\sqrt{N}(\mathfrak{M}-h)}\rangle \simeq e^{\xi H\mathcal{E}_N(t)+\frac{T\xi^2}{2}}, \qquad \mathcal{E}_N(t) := \sum_{i=1}^{N} \frac{n_i^2}{t+a_1-a_i} - N^{1/3}, \tag{4.1.43}$$

as $N \to \infty$ for asymptotically almost every disorder sample, where $t > 0$ is the solution of the equation (3.3.14)*.*

Since the exponent of the right-hand side of (4.1.43) is a quadratic function of ξ, we obtain

Result 4.1.5. *For $h = HN^{-1/6}$ and $0 < T < 1$,*

$$\mathfrak{M} \stackrel{\mathfrak{D}}{\simeq} h + \frac{1}{\sqrt{N}}\left[H\mathcal{E}_N(t) + \sqrt{T}\mathfrak{N}\right] \tag{4.1.44}$$

as $N \to \infty$ for asymptotically almost every disorder sample, where the thermal random variable $\mathfrak{N}$ has the standard Gaussian distribution.

The thermal average is obtained from the first two terms. The average is the same as (4.1.9) that we obtained from the free energy.

4.1.5.2 Matching with $h = O(1)$

We take the formal limit $H \to \infty$ of (4.1.44). The limit of $\mathcal{E}_N(t)$ as $H \to \infty$ is obtained in (3.3.28). From this, we find that, if we take $h = HN^{-1/6}$ and let $N \to \infty$ first and then take $H \to \infty$, then

$$\mathfrak{M} \stackrel{\mathcal{D}}{\simeq} h - \frac{h^3}{2(1-T)} + \frac{1}{\sqrt{N}}\left[h\mathcal{S}_N(\gamma_0;1) - \frac{h^5\mathcal{S}_N(\gamma_0;2)}{2(1-T)^2} + \sqrt{T}\mathfrak{N}\right] \tag{4.1.45}$$

as $H \to \infty$ where $\gamma_0 \simeq 2 + \frac{h^4}{4(1-T)^2}$. This result agrees with (4.1.36), which is obtained by taking $h > 0$ fixed and letting $N \to \infty$ first and then taking $h \to 0$.

4.1.5.3 Formal limit as $H \to 0$

We take the formal limit $H \to 0$ of (4.1.44). We obtained the limit of $\mathcal{E}_N(t)$ as $H \to 0$ in (3.3.24). Hence, we find that, if we take $N \to \infty$ with $h = HN^{-1/6}$ first and then take $H \to 0$, then

$$\mathfrak{M} \stackrel{\mathcal{D}}{\simeq} h + \frac{1}{\sqrt{N}}\left[|n_1|\sqrt{1-T} + \sqrt{T}\mathfrak{N}\right]. \tag{4.1.46}$$

This formula evaluated at $H = 0$ is different from (4.1.38). In particular, the Bernoulli random variable $\mathfrak{B}(1/2)$ is missing. In the next subsection, we consider a new regime $h = O(N^{-1/2})$ in which the two terms in (4.1.46) are of the same order. We will show that this new regime interpolates between $h = O(N^{-1/6})$ and $h = 0$.

4.1.6 Microscopic external field: $h \sim N^{-1/2}$ and $T < 1$

4.1.6.1 Analysis

We set, for fixed $H > 0$,

$$h = HN^{-1/2}. \tag{4.1.47}$$

This is a new regime which did not appear in previous sections. The appearance of this scaling regime was first noticed in [22] for the zero-temperature case.

Critical points

We first compute the critical point γ of $\mathcal{G}(z)$. In previous sections, we had $\gamma = \lambda_1 + \mathcal{O}\left(N^{-2/3}\right)$ for $h \sim N^{-1/6}$ and $\gamma = \lambda_1 + \mathcal{O}\left(N^{-1}\right)$ for $h = 0$. For $h \sim N^{-1/2}$, it turns out that $\gamma = \lambda_1 + \mathcal{O}\left(N^{-1}\right)$. We make the ansatz that

$$\gamma = \lambda_1 + pN^{-1} \tag{4.1.48}$$

with $p = \mathcal{O}(1)$. Then, the critical point equation becomes

$$\beta - \frac{1}{N}\sum_{i=1}^{N} \frac{1}{\lambda_1 + pN^{-1} - \lambda_i} - \frac{H^2\beta}{N^2}\sum_{i=1}^{N} \frac{n_i^2}{(\lambda_1 + pN^{-1} - \lambda_i)^2} = 0. \tag{4.1.49}$$

Separating out $i = 1$ in both sums and using (2.2.14) and (2.2.15) for the remaining sums, the equation becomes

$$\beta - 1 - \frac{1}{p} - \frac{H^2\beta n_1^2}{p^2} + \mathcal{O}\left(N^{-1/3}\right) = 0. \tag{4.1.50}$$

The solution is

$$p = \frac{1 + \sqrt{1 + 4(\beta - 1)H^2\beta n_1^2}}{2(\beta - 1)} + \mathcal{O}\left(N^{-1/3}\right). \tag{4.1.51}$$

Hence, $p = \mathcal{O}(1)$, which is consistent with the ansatz.

Now consider the critical point of $\mathcal{G}_{\mathfrak{M}}(z)$. Due to the scale $h = HN^{-1/2}$, the function $\mathcal{G}_{\mathfrak{M}}(z)$ is the same as $\mathcal{G}(z)$ with H replaced by $H + \xi$. Thus, we find that

$$\gamma_{\mathfrak{M}} = \lambda_1 + p_{\mathfrak{M}}N^{-1} \tag{4.1.52}$$

where $p_{\mathfrak{M}} > 0$ solves the equation

$$\beta - 1 - \frac{1}{p_{\mathfrak{M}}} - \frac{(H+\xi)^2 \beta n_1^2}{p_{\mathfrak{M}}^2} + \mathcal{O}\left(N^{-1/3}\right) = 0. \tag{4.1.53}$$

Exponential terms

We evaluate $N(\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}) - \mathcal{G}(\gamma))$ using (4.1.14). For A_1, the sum with $i \geq 2$, using a Taylor approximation, is $\mathcal{O}\left(N^{-2/3}\right)$. Hence,

$$A_1 = -\log(\frac{p_{\mathfrak{M}}}{p}) + \frac{p_{\mathfrak{M}}}{p} - 1 + \mathcal{O}\left(N^{-2/3}\right).$$

The sum with $i \geq 2$ for A_2 is $\mathcal{O}\left(N^{-1}\right)$ and we obtain

$$A_2 = \frac{H^2 \beta n_1^2 (p_{\mathfrak{M}} - p)^2}{p_{\mathfrak{M}} p^2} + \mathcal{O}\left(N^{-1}\right).$$

Finally, again separating the term with $i = 1$ and using (2.2.10) for the rest of the sum,

$$A_3 = (2\xi H + \xi^2)\beta \left(\frac{n_1^2}{p_{\mathfrak{M}}} + 1\right) + \mathcal{O}\left(N^{-1/3}\right). \tag{4.1.54}$$

Therefore,

$$\begin{aligned} &N(\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}) - \mathcal{G}(\gamma)) \\ &= -\log(\frac{p_{\mathfrak{M}}}{p}) + \frac{p_{\mathfrak{M}}}{p} - 1 + \frac{H^2 \beta n_1^2 (p_{\mathfrak{M}} - p)^2}{p_{\mathfrak{M}} p^2} + (2\xi H + \xi^2)\beta \left(\frac{n_1^2}{p_{\mathfrak{M}}} + 1\right) + \mathcal{O}\left(N^{-1/3}\right). \end{aligned} \tag{4.1.55}$$

Using the equations (4.1.50) and (4.1.53) satisfied by p and $p_{\mathfrak{M}}$, the equation (4.1.55) can be written as

$$N(\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}) - \mathcal{G}(\gamma)) = -\log(\frac{p_{\mathfrak{M}}}{p}) + 2(\beta - 1)(p_{\mathfrak{M}} - p) + (2H\xi + \xi^2)\beta + \mathcal{O}\left(N^{-1/3}\right). \tag{4.1.56}$$

Integrals

We now consider the integrals in the formula (4.1.10). The ratio of the integrals in this regime turns out to give a non-trivial contribution. We first show that we cannot use a Taylor series approximation. Consider the numerator; the denominator is the same as the numerator with $\xi = 0$.

For $k \geq 2$, we use the formula for $\mathcal{G}_{\mathfrak{M}}(z)$ to get

$$\begin{aligned}\frac{\mathcal{G}_{\mathfrak{M}}^{(k)}(\gamma_{\mathfrak{M}})}{(-1)^k(k-1)!} =& \frac{1}{N}\sum_{i=1}^{N}\frac{N^{\frac{2}{3}k}}{(a_1+p_{\mathfrak{M}}N^{-1/3}-a_i)^k} + \frac{k(H+\xi)^2\beta}{N^2}\sum_{i=1}^{N}\frac{n_i^2N^{\frac{2}{3}(k+1)}}{(a_1+p_{\mathfrak{M}}N^{-1/3}-a_i)^{k+1}} \\ =& N^{\frac{2}{3}k-1}\left(\frac{N^{\frac{1}{3}k}}{p_{\mathfrak{M}}^k}+\mathcal{O}\left(1\right)\right) + k(H+\xi)^2\beta N^{\frac{2}{3}k-\frac{4}{3}}\left(\frac{N^{\frac{1}{3}(k+1)}}{p_{\mathfrak{M}}^{k+1}}+\mathcal{O}\left(1\right)\right) = \mathcal{O}\left(N^{k-1}\right).\end{aligned}$$

Since $\mathcal{G}_{\mathfrak{M}}^{(2)} = \mathcal{O}\left(N\right)$, the main contribution to the integral comes from a neighborhood of radius N^{-1} around the critical point. If we use the new variable $z = \gamma_{\mathfrak{M}} + uN^{-1}$ and the Taylor series

$$N\left(\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}+uN^{-1}) - \mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}})\right) = \sum_{k=2}^{\infty}\frac{N^{-k+1}}{k!}\mathcal{G}_{\mathfrak{M}}^{(k)}(\gamma_{\mathfrak{M}})u^k,$$

we find that all terms in the series are $\mathcal{O}\left(1\right)$ for finite u. Since all terms in the Taylor series contribute to the integral, this method will not work and we instead proceed as follows. Using $\mathcal{G}_{\mathfrak{M}}'(\gamma_{\mathfrak{M}}) = 0$, we have

$$\begin{aligned}&N(\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}+w) - \mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}})) = N(\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}+w) - \mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}) - \mathcal{G}_{\mathfrak{M}}'(\gamma_{\mathfrak{M}})w) \\ &\quad = -\sum_{i=1}^{N}\left[\log\left(1+\frac{w}{\gamma_{\mathfrak{M}}-\lambda_i}\right) - \frac{w}{\gamma_{\mathfrak{M}}-\lambda_i}\right] + \left(h+\frac{\xi}{\sqrt{N}}\right)^2\beta\sum_{i=1}^{N}\frac{n_i^2w^2}{(\gamma_{\mathfrak{M}}+w-\lambda_i)(\gamma_{\mathfrak{M}}-\lambda_i)^2}.\end{aligned}$$

Separating out $i = 1$, using a Taylor approximation of the log function, and using (2.2.15),

$$N\left(\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}+uN^{-1}) - \mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}})\right) = -\log\left(1+\frac{u}{p_{\mathfrak{M}}}\right) + \frac{u}{p_{\mathfrak{M}}} + \frac{(H+\xi)^2\beta n_1^2u^2}{(p_{\mathfrak{M}}+u)p_{\mathfrak{M}}^2} + \mathcal{O}\left(u^2N^{-2/3}\right).$$

We temporarily write the middle two terms with $x := (H+\xi)^2\beta n_1^2$ and get

$$\frac{u}{p_{\mathfrak{M}}} + \frac{xu^2}{(p_{\mathfrak{M}}+u)p_{\mathfrak{M}}^2} = u\left(\frac{1}{p_{\mathfrak{M}}}+\frac{x}{p_{\mathfrak{M}}^2}\right) - \frac{x}{p_{\mathfrak{M}}} + \frac{x}{p_{\mathfrak{M}}+x}.$$

Using (4.1.53) twice, the above formula can be written as

$$N\left(\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}+uN^{-1}) - \mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}})\right) \simeq -\log\left(1+\frac{u}{p_{\mathfrak{M}}}\right) + (\beta-1)(u-p_{\mathfrak{M}}) + 1 + \frac{(H+\xi)^2\beta n_1^2}{(p_{\mathfrak{M}}+u)}. \tag{4.1.57}$$

Thus,

$$\begin{aligned}\int_{\gamma_{\mathfrak{M}}-\mathrm{i}\infty}^{\gamma_{\mathfrak{M}}+\mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{M}}(z)-\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}))}\mathrm{d}z \simeq&\frac{1}{N}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\sqrt{\frac{p_{\mathfrak{M}}}{p_{\mathfrak{M}}+u}}e^{\frac{(\beta-1)(u-p_{\mathfrak{M}})}{2}+\frac{1}{2}+\frac{(H+\xi)^2\beta n_1^2}{2(p_{\mathfrak{M}}+u)}}\mathrm{d}u\\ =&\frac{p_{\mathfrak{M}}^{1/2}e^{-(\beta-1)p_{\mathfrak{M}}+\frac{1}{2}}}{N}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{e^{\frac{(\beta-1)(p_{\mathfrak{M}}+u)}{2}+\frac{(H+\xi)^2\beta n_1^2}{2(p_{\mathfrak{M}}+u)}}}{\sqrt{p_{\mathfrak{M}}+u}}\mathrm{d}u.\end{aligned} \tag{4.1.58}$$

The last integral is an integral formula of a modified Bessel function which can be evaluated explicitly (see e.g. [2]):

$$\int_{0_++\mathrm{i}\mathbb{R}}\frac{e^{aw+\frac{b}{w}}}{\sqrt{w}}\mathrm{d}w = 2\pi\mathrm{i}\left(\frac{b}{a}\right)^{1/4} I_{-\frac{1}{2}}(2\sqrt{ab}) = \frac{2\mathrm{i}\sqrt{\pi}}{\sqrt{a}}\cosh(2\sqrt{ab}). \tag{4.1.59}$$

Hence, we obtain

$$\int_{\gamma_{\mathfrak{M}}-\mathrm{i}\infty}^{\gamma_{\mathfrak{M}}+\mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{M}}(z)-\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}))}\mathrm{d}z \simeq \frac{2\mathrm{i}\sqrt{2\pi p_{\mathfrak{M}}}e^{-(\beta-1)p_{\mathfrak{M}}+\frac{1}{2}}}{N\sqrt{\beta-1}}\cosh\left((H+\xi)|n_1|\sqrt{\beta(\beta-1)}\right). \tag{4.1.60}$$

Note that the integral depends on ξ, unlike in the cases $h > 0$ and $h \sim N^{-1/6}$. The denominator is the case when $\xi = 0$. Thus,

$$\frac{\int e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{M}}(z)-\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}))}\mathrm{d}z}{\int e^{\frac{N}{2}(\mathcal{G}(z)-\mathcal{G}(\gamma))}\mathrm{d}z} \simeq \sqrt{\frac{p_{\mathfrak{M}}}{p}}e^{-(\beta-1)(p_{\mathfrak{M}}-p)}\frac{\cosh\left((H+\xi)|n_1|\sqrt{\beta(\beta-1)}\right)}{\cosh\left(H|n_1|\sqrt{\beta(\beta-1)}\right)}. \tag{4.1.61}$$

Combining all terms together, replacing $\beta\xi$ by ξ and using $T = 1/\beta$, we obtain the following.

Result 4.1.6. *For $h = HN^{-1/2}$ and $0 < T < 1$,*

$$\langle e^{\xi\sqrt{N}\mathfrak{M}}\rangle \simeq e^{H\xi+\frac{T\xi^2}{2}}\frac{\cosh\left((H+T\xi)|n_1|\frac{\sqrt{1-T}}{T}\right)}{\cosh\left(H|n_1|\frac{\sqrt{1-T}}{T}\right)} \tag{4.1.62}$$

as $N \to \infty$ for asymptotically almost every disorder sample.

The right-hand side is the product of two terms, implying that $\sqrt{N}\mathfrak{M}$ is a sum two independent random variables. The exponential term on right-hand side is the moment generating function of a Gaussian distribution, while the ratio of the cosh functions is the moment generating function of a shifted Bernoulli distribution. Indeed, if $\mathbb{P}(X = 1) = p$ and $\mathbb{P}(X = -1) = 1 - p$ with $p = \frac{e^a}{e^a+e^{-a}}$, then

$$\mathbb{E}[e^{\xi X}] = pe^{\xi} + (1-p)e^{-\xi} = \frac{\cosh(a+\xi)}{\cosh(a)}.$$

Hence, we deduce the following result.

Result 4.1.7. *For* $h = HN^{-1/2}$ *and* $0 < T < 1$,

$$\mathfrak{M} \overset{\mathfrak{D}}{\simeq} h + \frac{|n_1|\sqrt{1-T}\mathfrak{B}(\alpha) + \sqrt{T}\mathfrak{N}}{\sqrt{N}} \tag{4.1.63}$$

as $N \to \infty$ *for asymptotically almost every disorder sample. Here,* $\mathfrak{B}(c)$ *is a shifted Bernoulli thermal random variable with the probability mass function* $P(\mathfrak{B}(c) = 1) = c$ *and* $P(\mathfrak{B}(c) = -1) = 1 - c$ *and* α *in* (4.1.63) *is given by*

$$\alpha := \frac{e^{\frac{H|n_1|\sqrt{1-T}}{T}}}{e^{\frac{H|n_1|\sqrt{1-T}}{T}} + e^{-\frac{H|n_1|\sqrt{1-T}}{T}}}. \tag{4.1.64}$$

The thermal random variable $\mathfrak{N}$ *has the standard Gaussian distribution and it is independent of* $\mathfrak{B}(\alpha)$.

4.1.6.2 Matching with $h \sim N^{-1/6}$ and $h = 0$

As $H \to \infty$, the random variable $\mathfrak{B}(\alpha) \to 1$. The formal limit of (4.1.63) as $H \to \infty$ is

$$\mathfrak{M} \overset{\mathfrak{D}}{\simeq} h + \frac{1}{\sqrt{N}}\left[|n_1|\sqrt{1-T} + \sqrt{T}\mathfrak{N}\right], \tag{4.1.65}$$

which is the same as (4.1.46) from the $h = HN^{-1/6}$ regime. On the other hand, if we take $H \to 0$, then $\mathfrak{B}(\alpha) \overset{\mathfrak{D}}{\to} \mathfrak{B}(1/2)$. Hence, the formal limit of (4.1.63) as $H \to 0$ is the same as the $h = 0$ case (4.1.38). Therefore, the result (4.1.63) matches with both the $h \sim N^{-1/6}$ and $h = 0$ results.

4.1.7 Susceptibility

In this subsection, we discuss properties of the susceptibility, defined as the magnetization per external field strength. In the next subsection we discuss differential susceptibility

$$\mathcal{X} = \frac{\mathcal{M}}{h} = \frac{\langle \mathfrak{M} \rangle}{h} = \frac{1}{h}\frac{\mathrm{d}\mathcal{F}_N}{\mathrm{d}h}. \tag{4.1.66}$$

We denote by $\bar{\mathcal{X}}$ or $\mathbb{E}_s[\mathcal{X}]$ the sample average of $\mathcal{X}$. We denote by Var_s the sample variance. As described in Chapter II, we use the font $\overset{\mathcal{D}}{\simeq}$ to denote an asymptotic expansion in distribution with respect to the disorder sample.

4.1.7.1 Macroscopic field $h = O(1)$

From Subsection 4.1.6, for fixed $h > 0$ and $T > 0$,

$$\mathcal{X} \simeq \mathcal{X}^0 + \frac{\mathcal{X}^1}{\sqrt{N}} \tag{4.1.67}$$

for asymptotically almost every disorder sample, where

$$\mathcal{X}^0 = s_1(\gamma_0) \quad \text{and} \quad \mathcal{X}^1 = \mathcal{S}_N(\gamma_0; 1) - \frac{h^2 s_2(\gamma_0)\mathcal{S}_N(\gamma_0; 2)}{T s_2(\gamma_0) + 2h^2 s_3(\gamma_0)} \tag{4.1.68}$$

and γ_0 is the solution of the equation (3.1.19) and $\mathcal{S}_N(z; k)$ is defined in (2.2.13).

The leading term $\mathcal{X}^0$ is deterministic and satisfies:

- $\mathcal{X}^0$ is a decreasing function of h,
- As $h \to \infty$,
$$\mathcal{X}^0(h, T) = \frac{1}{h} - \frac{T}{2h^2} + O(h^{-3}) \quad \text{for all } T > 0 \tag{4.1.69}$$
- As $h \to 0$,
$$\mathcal{X}^0(h, T) \simeq \begin{cases} \frac{1}{T} - \frac{h^2}{T(T^2-1)} & \text{for } T > 1 \\ 1 - \frac{h^2}{2(1-T)} & \text{for } 0 < T < 1. \end{cases} \tag{4.1.70}$$

See Figure IV.2a for the graph of $\mathcal{X}^0$ as a function of h.

The subleading term $\mathcal{X}^1$ depends on the disorder sample. We consider its sample-to-sample fluctuations. From (2.2.13), $\mathcal{S}_N(\gamma_0; 1)$ and $\mathcal{S}_N(\gamma_0; 2)$ converge to the centered bivariate Gaussian distribution with

$$\mathrm{Var}_s[\mathcal{S}_N(\gamma_0; 1)] \to 2s_2(\gamma_0), \qquad \mathrm{Var}_s[\mathcal{S}_N(\gamma_0; 2)] \to 2s_4(\gamma_0), \tag{4.1.71}$$

and

$$\mathrm{Cov}_s(\mathcal{S}_N(\gamma_0; 1), \mathcal{S}_N(\gamma_0; 2)) = \mathbb{E}_s\left[\frac{1}{N}\sum_{i=1}^{N} \frac{(n_i^2 - 1)^2}{(\gamma_0 - \widehat{\lambda}_i)^3}\right] \to 2s_3(\gamma_0). \tag{4.1.72}$$

as $N \to \infty$. Hence, as $N \to \infty$,

$$\mathcal{X}^1 \overset{\mathcal{D}}{\simeq} \mathcal{N}(0, \sigma_s^2) \tag{4.1.73}$$

where the sample variance is

$$\sigma_s^2 = \frac{2s_2(\gamma_0)^2 \left(T^2 s_2(\gamma_0) + 2Th^2 s_3(\gamma_0) + h^4 s_4(\gamma_0)\right)}{\left(Ts_2(\gamma_0) + 2h^2 s_3(\gamma_0)\right)^2}. \tag{4.1.74}$$

See Figure IV.2b for the graph of σ_s^2. The graph shows that σ_s^2 is a monotonically decreasing function of h. It is easy to check that:

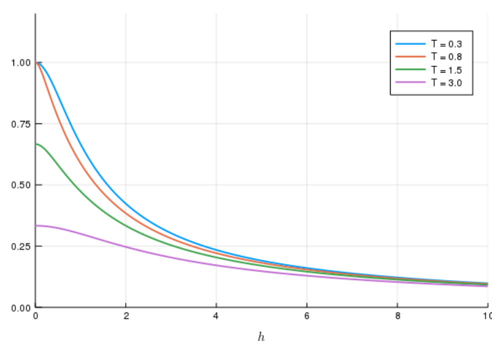
- As $h \to \infty$,

$$\sigma_s^2 \simeq \frac{1}{2h^2} \qquad \text{for all } T > 0. \tag{4.1.75}$$

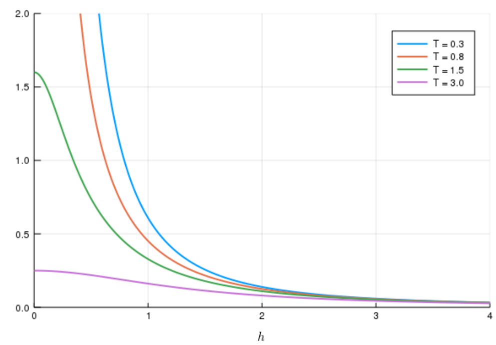
- As $h \to 0$,

$$\sigma_s^2 \simeq \begin{cases} \dfrac{2}{T^2 - 1} & \text{for } T > 1, \\ \dfrac{1 - T}{h^2} & \text{for } T < 1. \end{cases} \tag{4.1.76}$$

The above formula suggests that there is an interesting transition as T approaches the critical temperature $T = 1$ in the case where $h \to 0$. The behavior near the point $(T, h) = (1, 0)$ is worth studying, but we leave this subject for the future.



(a) Graph of $\mathcal{X}^0(h)$



(b) Graph of $\sigma_s^2(h)$

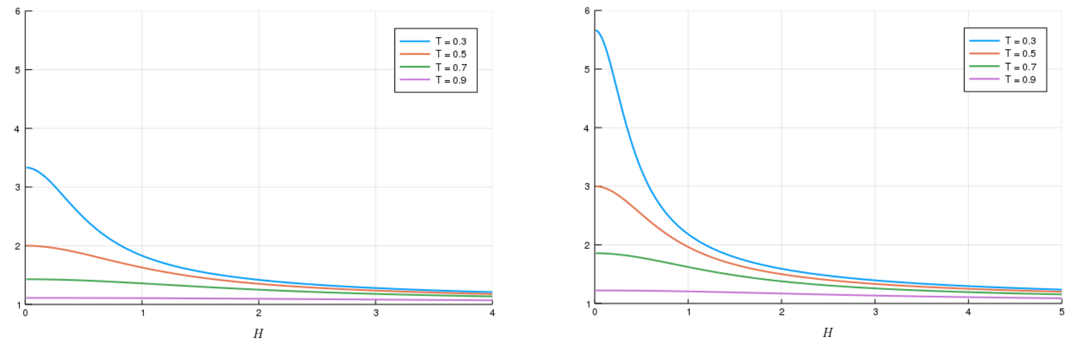
Figure IV.2: Graphs of $\mathcal{X}^0(h)$ and $\sigma_s^2(h)$ as functions of h for various values of T

4.1.7.2 Mesoscopic external field: $h \sim N^{-1/6}$ and $T < 1$

From Subsection 4.1.6, for $h = HN^{-1/6}$ with fixed $H > 0$ and $0 < T < 1$,

$$\mathcal{X} \simeq 1 + \frac{\mathcal{E}_N(t)}{N^{1/3}} \tag{4.1.77}$$

for asymptotically almost every disorder sample, where $\mathcal{E}_N(t)$ is given in (4.1.43).



(a) $\mathcal{X}_{\text{micro}}$ as a function of H when $|n_1| = 1$ (b) $\mathcal{X}_{\text{micro}}$ as a function of H when $|n_1| = \sqrt{2}$

Figure IV.3: Graphs of $\mathcal{X}_{\text{micro}}(H, T)$ as functions of H for various values of $0 < T < 1$.

The behavior of $\mathcal{E}_N(t)$ as $H \to \infty$ and $H \to 0$ is discussed in Subsubsections 4.1.5.2 and 4.1.5.3. The sample-to-sample fluctuation of $\mathcal{E}_N(t)$ is shown in Subsection 3.3.1 and we see that $\mathcal{E}_N(t) \overset{\mathcal{D}}{\simeq} \mathcal{E}(\varsigma)$ where

$$\mathcal{E}(\varsigma) = \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{\nu_i^2}{\varsigma + \alpha_1 - \alpha_i} - \frac{1}{\pi} \int_0^{\left(\frac{3\pi n}{2}\right)^{2/3}} \frac{\mathrm{d}x}{\sqrt{x}} \right) \tag{4.1.78}$$

and $\varsigma > 0$ solves $1 - T = H^2 \sum_{i=1}^{\infty} \frac{\nu_i^2}{(\varsigma+\alpha_1-\alpha_i)^2}$. Here, α_i is the GOE Airy point process and ν_i are i.i.d standard normal random variables independent of α_i.

4.1.7.3 Microscopic external field: $h \sim N^{-1/2}$ and $T < 1$

The thermal average of (4.1.63) implies that for $h = HN^{-1/2}$ with fixed $H > 0$ and $0 < T < 1$,

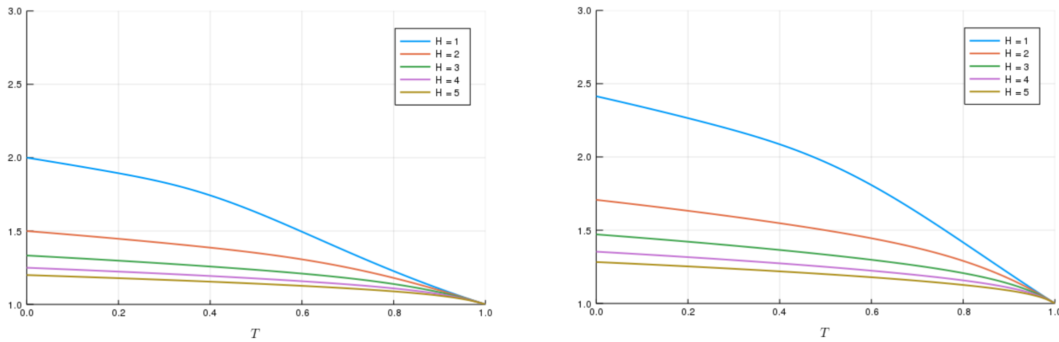
$$\mathcal{X} \simeq 1 + \frac{|n_1|\sqrt{1-T}}{H} \tanh\left(\frac{H|n_1|\sqrt{1-T}}{T} \right) =: \mathcal{X}_{\text{micro}} \tag{4.1.79}$$

for asymptotically almost every disorder sample. The function $\mathcal{X}_{\text{micro}}$ is a decreasing function in both H and T (see Figures IV.3 and IV.4). From the formula for $\mathcal{X}_{\text{micro}}$, we conclude that

$$\mathcal{X}_{\text{micro}} \simeq 1 + \frac{|n_1|\sqrt{1-T}}{H} \qquad \text{as } H \to \infty \tag{4.1.80}$$

and

$$\mathcal{X}_{\text{micro}} \simeq 1 + \frac{n_1^2(1-T)}{T} - \frac{H^2 n_1^4 (1-T)^2}{3T^3} \qquad \text{as } H \to 0. \tag{4.1.81}$$



(a) $\mathcal{X}_{\text{micro}}$ as a function of T when $|n_1| = 1$ (b) $\mathcal{X}_{\text{micro}}$ as a function of T when $|n_1| = \sqrt{2}$

Figure IV.4: Graphs of $\mathcal{X}_{\text{micro}}(H, T)$ as functions of T for various values of H.

4.1.7.4 The zero-external-field limit of the susceptibility

We consider two different limits of the susceptibility depending on how $h \to 0$ and $N \to \infty$ are taken. The first limit is obtained from (4.1.70):

$$\lim_{h\to 0} \lim_{\substack{N\to\infty \\ h>0}} \mathcal{X} = \begin{cases} \frac{1}{T} & \text{for } T > 1 \\ 1 & \text{for } T < 1. \end{cases} \tag{4.1.82}$$

See Figure I.3 (a). This result (4.1.82) was previously obtained in [28], and also in [16]. The limit does not depend on the disorder sample.

The second limit is obtained from (4.1.81) for $0 < T < 1$:

$$\lim_{H\to 0} \lim_{\substack{N\to\infty \\ h=HN^{-1/2}}} \mathcal{X} = 1 + \frac{n_1^2(1-T)}{T} \qquad \text{for } 0 < T < 1. \tag{4.1.83}$$

See Figure I.3 (b). This limit depends on the disorder sample, but only on one variable, n_1^2. Observe that this limit blows up at $T = 0$ while the limit (4.1.82) is finite at $T = 0$. The sample-to-sample average of (4.1.83) satisfies

$$\lim_{H\to 0} \lim_{\substack{N\to\infty \\ h=HN^{-1/2}}} \bar{\mathcal{X}} = \frac{1}{T} \qquad \text{for } 0 < T < 1. \tag{4.1.84}$$

4.1.8 Differential susceptibility

We also consider the differential susceptibility given by

$$\mathcal{X}_d = \frac{\mathrm{d}}{\mathrm{d}h}\langle \mathfrak{M} \rangle = \frac{\mathrm{d}^2 \mathcal{F}_N}{\mathrm{d}h^2} = \frac{N}{T}\left(\langle \mathfrak{M}^2 \rangle - \langle \mathfrak{M} \rangle^2\right). \tag{4.1.85}$$

The results (4.1.21), (4.1.44), and (4.1.63) imply the following formulas. All formulas hold for asymptotically almost every disorder sample.

(a) For fixed $h > 0$ and $T > 0$,

$$\mathcal{X}_d \simeq s_1(\gamma_0) - \frac{2h^2 s_2(\gamma_0)^2}{T s_2(\gamma_0) + 2h^2 s_3(\gamma_0)} =: \mathcal{X}_d^0. \tag{4.1.86}$$

(b) For $h = HN^{-1/6}$ with fixed $H > 0$ and $0 < T < 1$,

$$\mathcal{X}_d \simeq 1. \tag{4.1.87}$$

(c) For $h = HN^{-1/2}$ with fixed $H > 0$ and $0 < T < 1$,

$$\mathcal{X}_d \simeq 1 + \frac{n_1^2(1-T)}{T\cosh^2\left(\frac{H|n_1|\sqrt{1-T}}{T}\right)} =: \mathcal{X}_{d,\text{micro}}. \tag{4.1.88}$$

The limits for the macroscopic and mescopic regimes do not depend on the disorder samples, but the limit for the microscopic regime depends on the disorder variable n_1^2. The macroscopic limit satisfies the following property as $h \to 0$:

$$\mathcal{X}_d^0 \simeq \begin{cases} \dfrac{1}{T} - \dfrac{3h^2}{T(T^2-1)} + O(h^4) & T > 1, \\ 1 - \dfrac{3h^2}{2(1-T)^2} + O(h^4) & 0 < T < 1. \end{cases} \tag{4.1.89}$$

On the other hand the microscopic limit satisfies, for $0 < T < 1$,

$$\mathcal{X}_{d,\text{micro}} \simeq \begin{cases} 1 + O(e^{-\frac{2H|n_1|\sqrt{1-T}}{T}}) & \text{as } H \to \infty. \\ 1 + \dfrac{n_1^2(1-T)}{T} - \dfrac{H^2 n_1^4 (1-T)^2}{T^3} & \text{as } H \to 0. \end{cases} \tag{4.1.90}$$

The zero-external-field limit is the same as the susceptibility of the last section even though the subleading terms differ by a factor of 3. In both cases the limit is

$$\lim_{H\to 0} \mathcal{X}_{d,\text{micro}} = \lim_{H\to 0} \mathcal{X}_{\text{micro}} = 1 + \frac{n_1^2(1-T)}{T} \tag{4.1.91}$$

and this value depends on the disorder variable n_1^2. Note that the sample-to-sample average of n_1^2 is 1. This result shows that both susceptibilities satisfy Curie's law in the sample-to-sample average sense, but not in the quenched disorder sense. In other words, the sample mean of the susceptibility is inversely proportional to temperature, but not the susceptibility for an arbitrary fixed disorder sample.

We note that if we take $T \to 0$ with $H > 0$ fixed in (4.1.79) and (4.1.88), then

$$\mathcal{X}_{\text{micro}} \simeq 1 + \frac{|n_1|}{H} \quad \text{and} \quad \mathcal{X}_{d,\text{micro}} \simeq 1 \qquad \text{at } T = 0. \tag{4.1.92}$$

This shows that $\mathcal{X}_{d,\text{micro}}(T=0)$ does not diverge as $H \to 0$ but $\mathcal{X}_{\text{micro}}(T=0)$ does.

4.2 Overlap with the ground state

Recall that $\pm\mathbf{u}_1$ denote the unit eigenvectors corresponding to the largest eigenvalue of M. The overlap of the spin with the ground state and the squared overlap are defined as

$$\mathfrak{G} = \frac{|\mathbf{u}_1 \cdot \boldsymbol{\sigma}|}{\sqrt{N}}, \qquad \mathfrak{O} = \mathfrak{G}^2 = \frac{1}{N}(\mathbf{u}_1 \cdot \boldsymbol{\sigma})^2, \tag{4.2.1}$$

respectively. The overlap $\mathfrak{G} = 1$ when $T = h = 0$ since the Hamiltonian is maximized when $\boldsymbol{\sigma}$ is parallel to $\pm\mathbf{u}_1$. The overlap measures how close the spin is to the ground state. Since it is more convenient to analyze, we consider $\mathfrak{O}$ in this section.

As with the overlap with the external field, there are no transitions when $T > 1$ as $h \to 0$. However, when $T < 1$, there are two interesting transitional regimes given by $h \sim N^{-1/6}$ and $h \sim N^{-1/3}$. The second regime did not appear for the overlap with the external field. On the other hand, the regime $h \sim N^{-1/2}$, which we studied for the free energy and the overlap with the external field, does not reveal any new features of $\mathfrak{O}$. Instead, $\mathfrak{O}$ has the same properties for $h \sim N^{-1/2}$ as it does for $h = 0$.

The moment generating function of $\mathfrak{O}$ has the integral formula given in Lemma 2.3.3,

$$\langle e^{\beta\eta\mathfrak{O}} \rangle = e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{O}}(\gamma_{\mathfrak{O}}) - \mathcal{G}(\gamma))} \frac{\int_{\gamma_{\mathfrak{O}}-\mathrm{i}\infty}^{\gamma_{\mathfrak{O}}+\mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{O}}(z) - \mathcal{G}_{\mathfrak{O}}(\gamma_{\mathfrak{O}}))} \mathrm{d}z}{\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}(z) - \mathcal{G}(\gamma))} \mathrm{d}z} \tag{4.2.2}$$

where

$$\mathcal{G}_{\mathfrak{D}}(z) = \beta z - \frac{1}{N}\log(z-\lambda_1-b) - \frac{1}{N}\sum_{i=2}^{N}\log(z-\lambda_i) + \frac{h^2\beta n_1^2}{N(z-\lambda_1-b)} + \frac{h^2\beta}{N}\sum_{i=2}^{N}\frac{n_i^2}{z-\lambda_i}. \tag{4.2.3}$$

We take take $\gamma_{\mathfrak{D}}$ and γ to be the critical points of $\mathcal{G}_{\mathfrak{D}}$ and $\mathcal{G}$ respectively, and we use the notation

$$b := \frac{2\eta}{N}. \tag{4.2.4}$$

The difference between $\mathcal{G}_{\mathfrak{D}}$ and $\mathcal{G}$ is that, in the case of $\mathcal{G}_{\mathfrak{D}}$, λ_1 is changed to $\lambda_1 + b$.

The following two formulas will be used in the analysis below. First, we have

$$N(\mathcal{G}_{\mathfrak{D}}(\gamma_{\mathfrak{D}}) - \mathcal{G}(\gamma)) = N(\mathcal{G}_{\mathfrak{D}}(\gamma_{\mathfrak{D}}) - \mathcal{G}(\gamma) - \mathcal{G}'(\gamma)(\gamma_{\mathfrak{D}}-\gamma)) = D_1 + D_2 + D_3 + D_4 \tag{4.2.5}$$

where

$$\begin{aligned}
D_1 &:= -\log\left(1 + \frac{\gamma_{\mathfrak{D}}-\gamma-b}{\gamma-\lambda_1}\right) + \frac{\gamma_{\mathfrak{D}}-\gamma}{\gamma-\lambda_1}, \\
D_2 &:= -\sum_{i=2}^{N}\left[\log\left(1+\frac{\gamma_{\mathfrak{D}}-\gamma}{\gamma-\lambda_i}\right) - \frac{\gamma_{\mathfrak{D}}-\gamma}{\gamma-\lambda_i}\right], \\
D_3 &:= h^2\beta n_1^2\left[\frac{1}{\gamma_{\mathfrak{D}}-\lambda_1-b} - \frac{1}{\gamma-\lambda_1} + \frac{\gamma_{\mathfrak{D}}-\gamma}{(\gamma-\lambda_1)^2}\right], \\
D_4 &:= h^2\beta(\gamma_{\mathfrak{D}}-\gamma)^2\sum_{i=2}^{N}\frac{n_i^2}{(\gamma_{\mathfrak{D}}-\lambda_i)(\gamma-\lambda_i)^2}.
\end{aligned} \tag{4.2.6}$$

Second, we can show from the equation $\mathcal{G}_{\mathfrak{D}}'(\gamma_{\mathfrak{D}}) - \mathcal{G}'(\gamma) = 0$ that

$$\begin{aligned}
&(\gamma_{\mathfrak{D}}-\gamma)\Bigg[\frac{1}{N(\gamma_{\mathfrak{D}}-\lambda_1-b)(\gamma-\lambda_1)} + \frac{1}{N}\sum_{i=2}^{N}\frac{1}{(\gamma_{\mathfrak{D}}-\lambda_i)(\gamma-\lambda_i)} \\
&+ \frac{h^2\beta n_1^2}{N}\frac{\gamma+\gamma_{\mathfrak{D}}-2\lambda_1-b}{(\gamma_{\mathfrak{D}}-\lambda_1-b)^2(\gamma-\lambda_1)^2} + \frac{h^2\beta}{N}\sum_{i=2}^{N}\frac{n_i^2(\gamma+\gamma_{\mathfrak{D}}-2\lambda_i)}{(\gamma_{\mathfrak{D}}-\lambda_i)^2(\gamma-\lambda_i)^2}\Bigg] \\
&= b\Bigg[\frac{1}{N(\gamma_{\mathfrak{D}}-\lambda_1-b)(\gamma-\lambda_1)} + \frac{h^2\beta n_1^2}{N}\frac{\gamma+\gamma_{\mathfrak{D}}-2\lambda_1-b}{(\gamma_{\mathfrak{D}}-\lambda_1-b)^2(\gamma-\lambda_1)^2}\Bigg].
\end{aligned} \tag{4.2.7}$$

4.2.1 Macroscopic external field: $h = O(1)$

4.2.1.1 Analysis

Fix $h > 0$. The fluctuations of $\mathfrak{O}$ turn out to be of order N^{-1}. Thus we set

$$\eta = \xi N \quad \text{so that} \quad b = 2\xi. \tag{4.2.8}$$

The critical point of $\mathcal{G}(z)$ is obtained in Subsection 3.1.2 and is given by $\gamma = \gamma_0 + \mathcal{O}\left(N^{-1/2}\right)$ where γ_0 solves the equation (3.1.19). We do not need an explicit formula for the term $\mathcal{O}\left(N^{-1/2}\right)$ in this section. Since $\mathcal{G}'_{\mathfrak{O}}(z) = \mathcal{G}'(z) + \mathcal{O}\left(N^{-1}\right)$ for $z > 2$, a perturbation argument implies that the critical point of $\mathcal{G}_{\mathfrak{O}}(z)$ is given by

$$\gamma_{\mathfrak{O}} = \gamma + \mathcal{O}\left(N^{-1}\right). \tag{4.2.9}$$

We use (4.2.5) to compute $N(\mathcal{G}_{\mathfrak{O}}(\gamma_{\mathfrak{O}}) - \mathcal{G}(\gamma))$. From the semi-circle law, we have $D_2 = \mathcal{O}\left((\gamma_{\mathfrak{O}} - \gamma)^2 N\right) = \mathcal{O}\left(N^{-2}\right)$ and $D_4 = \mathcal{O}\left(N^{-1}\right)$. On the other hand, D_1 and D_3 are easy to compute and we find that

$$N(\mathcal{G}_{\mathfrak{O}}(\gamma_{\mathfrak{O}}) - \mathcal{G}(\gamma)) = -\log\left(1 - \frac{2\xi}{\gamma_0 - 2}\right) + \frac{2h^2\beta n_1^2\xi}{(\gamma_0-2)^2(1-\frac{2\xi}{\gamma_0-2})} + \mathcal{O}\left(N^{-1/2}\right). \tag{4.2.10}$$

Since $\mathcal{G}_{\mathfrak{O}}^{(k)}(\gamma_{\mathfrak{O}}) = \mathcal{O}(1)$ for all $k \geq 2$, the ratio of the integrals (4.2.2) can be evaluated using the method of steepest descent. For $k = 2$,

$$\mathcal{G}''_{\mathfrak{O}}(\gamma_{\mathfrak{O}}) \simeq s_2(\gamma_0) + h^2\beta s_3(\gamma_0),$$

which does not depend on ξ. Since $\mathcal{G}(\gamma)$ is the special case of $\mathcal{G}_{\mathfrak{O}}(\gamma)$ when $\xi = 0$, we conclude that

$$\frac{\int_{\gamma_{\mathfrak{O}}-i\infty}^{\gamma_{\mathfrak{O}}+i\infty} e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{O}}(z)-\mathcal{G}_{\mathfrak{O}}(\gamma_{\mathfrak{O}}))}\mathrm{d}z}{\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\frac{N}{2}(\mathcal{G}(z)-\mathcal{G}(\gamma))}\mathrm{d}z} \simeq \sqrt{\frac{\mathcal{G}''(\gamma)}{\mathcal{G}''_{\mathfrak{O}}(\gamma_{\mathfrak{O}})}} \simeq 1.$$

Inserting these results into (4.2.2), replacing ξ with $(\gamma_0 - 2)\xi$, and using $\beta = 1/T$, we obtain the following.

Result 4.2.1. *For $h = O(1)$ and $T > 0$,*

$$\langle e^{\frac{\gamma_0-2}{T}\xi N\mathfrak{O}}\rangle \simeq (1-2\xi)^{-1/2}\, e^{\frac{h^2 n_1^2\xi}{T(\gamma_0-2)(1-2\xi)}} \tag{4.2.11}$$

as $N \to \infty$ for asymptotically almost every disorder, where $\gamma_0 > 2$ is the solution of the equation (3.1.19).

Note that if X is a non-central Gaussian random variable $\mu + \mathfrak{N}$, i.e. if X^2 is a non-centered chi-squared distribution with 1 degree of freedom, then

$$E[e^{\xi X^2}] = (1-2\xi)^{-1/2} e^{\frac{\mu^2 \xi}{1-2\xi}}. \tag{4.2.12}$$

Therefore, we obtain the next result from the one above.

Result 4.2.2. *For $h = O(1)$ and $T > 0$,*

$$\mathfrak{O} \stackrel{\mathfrak{D}}{\simeq} \frac{\mathfrak{O}^0}{N} \quad \textit{where} \quad \mathfrak{O}^0 = \frac{T}{\gamma_0 - 2} \left| \frac{h|n_1|}{\sqrt{T(\gamma_0 - 2)}} + \mathfrak{N} \right|^2 \tag{4.2.13}$$

as $N \to \infty$ for asymptotically almost every disorder, where the thermal random variable $\mathfrak{N}$ has the standard Gaussian distribution.

4.2.1.2 Limits as $h \to \infty$ and $h \to 0$

Consider the formal limit of (4.2.13) as $h \to \infty$. From (3.1.33), we find that if we take $h > 0$ and let $N \to \infty$ first and then $h \to \infty$, we get

$$\mathfrak{O} \stackrel{\mathcal{D}}{\simeq} \frac{1}{N} \left[n_1^2 + \frac{2|n_1|\sqrt{T}}{\sqrt{h}} \mathfrak{N} \right] \tag{4.2.14}$$

for all $T > 0$. On the other hand, the equation (3.1.32) implies that if we take $h > 0$ and let $N \to \infty$ first and then $h \to 0$, we obtain

$$\mathfrak{O} \stackrel{\mathfrak{D}}{\simeq} \frac{T^2}{N(T-1)^2} \left[\mathfrak{N}^2 + \frac{2h|n_1|}{T-1} \mathfrak{N} \right] \qquad \text{for } T > 1 \tag{4.2.15}$$

and

$$\mathfrak{O} \stackrel{\mathfrak{D}}{\simeq} \frac{16}{N} \left[\frac{(1-T)^4 n_1^2}{h^6} + \frac{\sqrt{T}(1-T)^3 |n_1|}{h^5} \mathfrak{N} \right] \qquad \text{for } 0 < T < 1. \tag{4.2.16}$$

For $0 < T < 1$, the above result indicates that the overlap is of order 1 when $h \sim N^{-1/6}$. We study this regime in the next subsection.

4.2.2 Mesoscopic external field: $h \sim N^{-1/6}$ and $T < 1$

4.2.2.1 Analysis

We set

$$h = HN^{-1/6} \tag{4.2.17}$$

for fixed $H > 0$. If we insert $h = HN^{-1/6}$ in to the formula, the equation (4.2.16) indicates that the fluctuations are of order $N^{-1/6}$. Thus, we set

$$\eta = \xi N^{1/6} \quad \text{so that} \quad b = 2\xi N^{-5/6} \tag{4.2.18}$$

in (4.2.2) and (4.2.4).

The critical point γ of $\mathcal{G}(z)$ is obtained in Subsection 3.3.1 and it is given by $\gamma = \lambda_1 + sN^{-2/3}$ where $s > 0$ is the solution of the equation (3.3.4). We now consider the critical point of $\mathcal{G}_{\mathfrak{D}}(z)$. From the formula, we see that $\mathcal{G}'_{\mathfrak{D}}(z)$ is an increasing function of z for $z > \lambda_1 + b$. Using $b > 0$ and the explicit formula of the functions, we can easily check that $\mathcal{G}'_{\mathfrak{D}}(\gamma) < \mathcal{G}'(\gamma) = 0$ and $\mathcal{G}'_{\mathfrak{D}}(\gamma + b) > \mathcal{G}'(\gamma) = 0$. Hence, we find that $\gamma < \gamma_{\mathfrak{D}} < \gamma + b$, and thus, $\gamma_{\mathfrak{D}} - \gamma = \mathcal{O}(N^{-5/6})$. We now set

$$\gamma_{\mathfrak{D}} = \gamma + \Delta N^{-5/6} \tag{4.2.19}$$

and determine Δ using (4.2.7). The right-hand side of the equation (4.2.7) is equal to

$$\frac{2\xi}{N^{5/6}} \left[\frac{N^{1/3}}{s^2} + \frac{2H^2 \beta n_1^2 N^{2/3}}{s^3} \right] = \frac{4\xi H^2 \beta n_1^2}{N^{1/6} s^3} \left(1 + \mathcal{O}\left(N^{-1/3}\right)\right).$$

For the left-hand side of the equation, the first two terms are of smaller order than the last two terms. Using $\gamma_{\mathfrak{D}} = \gamma + \mathcal{O}\left(N^{-5/6}\right)$ and $b = \mathcal{O}\left(N^{-5/6}\right)$ for the other two sums, the left-hand side is equal to

$$\frac{\Delta}{N^{5/6}} \left[\frac{2H^2 \beta n_1^2 N^{2/3}}{s^3} + 2H^2 \beta N^{2/3} \sum_{i=2}^{N} \frac{n_i^2}{(s + a_1 - a_i)^3} + \mathcal{O}\left(N^{1/3}\right) \right].$$

Therefore,

$$\Delta = \frac{2\xi n_1^2 s^{-3}}{\sum_{i=1}^{N} n_i^2 (s + a_1 - a_i)^{-3}} + \mathcal{O}\left(N^{-1/6}\right). \tag{4.2.20}$$

We now evaluate $N(\mathcal{G}_{\mathfrak{D}}(\gamma_{\mathfrak{D}}) - \mathcal{G}(\gamma))$ using (4.2.5). It is easy to check that $D_1 = \mathcal{O}\left(N^{-1/6}\right)$

and $D_2 = \mathcal{O}\left(N^{-1/3}\right)$. Evaluating the first two leading terms,

$$D_3 = H^2\beta n_1^2\left[\frac{2\xi}{s^2}N^{1/6} + \frac{(\Delta-2\xi)^2}{s^3} + \mathcal{O}\left(N^{-1/6}\right)\right].$$

Finally,

$$D_4 = H^2\beta\Delta^2\sum_{i=2}^{N}\frac{n_i^2}{(s+a_1-a_i)^3} + \mathcal{O}\left(N^{-1/6}\right).$$

Putting these together and also using the explicit formula of Δ, we obtain

$$N(\mathcal{G}_{\mathfrak{D}}(\gamma_{\mathfrak{D}}) - \mathcal{G}(\gamma)) = \frac{2H^2\beta n_1^2}{s^2}\xi N^{1/6} + \frac{4H^2\beta n_1^2\left[\sum_{i=2}^{N} n_i^2(s+a_1-a_i)^{-3}\right]}{s^3\left[\sum_{i=1}^{N} n_i^2(s+a_1-a_i)^{-3}\right]}\xi^2 + \mathcal{O}\left(N^{-1/6}\right). \tag{4.2.21}$$

It remains to consider the integrals in (4.2.2). The scale $h = HN^{-1/6}$ is the same as the one in Subsection 3.3.1. Since $\gamma_{\mathfrak{D}} = \lambda_1 + sN^{-2/3} + \Delta N^{-5/6} = \lambda_1 + sN^{-2/3} + \mathcal{O}\left(N^{-5/6}\right)$ and $b = O(N^{-5/6})$, the calculation from Subsection 3.3.1 applies with only small changes. We find from the explicit formulas that $\mathcal{G}_{\mathfrak{D}}^{(k)}(\gamma_{\mathfrak{D}}) = \mathcal{O}\left(N^{\frac{2}{3}k-\frac{2}{3}}\right)$ for all $k \geq 2$ and

$$\mathcal{G}_{\mathfrak{D}}''(\gamma_{\mathfrak{D}}) = H^2\beta t^2\sum_{i=1}^{N}\frac{n_i^2}{(s+a_1-a_i)^3} + \mathcal{O}\left(N^{-1/6}\right).$$

Thus, as in Subsection 3.3.1, the main contribution to the integral comes from a neighborhood of radius $N^{-5/6}$ around the critical point, and the numerator can be evaluated using a Gaussian integral. Since the leading term of $\mathcal{G}_{\mathfrak{D}}''(\gamma_{\mathfrak{D}})$ does not depend on ξ and the denominator is the case of the numerator with $\xi = 0$, we find that

$$\frac{\int_{\gamma_{\mathfrak{D}}-\mathrm{i}\infty}^{\gamma_{\mathfrak{D}}+\mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{D}}(z)-\mathcal{G}_{\mathfrak{D}}(\gamma_{\mathfrak{D}}))}\mathrm{d}z}{\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}(z)-\mathcal{G}(\gamma))}\mathrm{d}z} \simeq 1. \tag{4.2.22}$$

From the above computations, we obtain an asymptotic formula for $\langle e^{\beta\xi N^{1/6}\mathfrak{D}}\rangle$. Moving a term of order $N^{1/6}$ to the left, changing $\beta\xi$ to ξ, replacing β by $1/T$, and replacing s by t, which solves the equation (3.3.14), we arrive at the following result.

Result 4.2.3. *For* $h = HN^{-1/6}$ *and* $0 < T < 1$,

$$\langle e^{\xi N^{1/6}(\mathfrak{O} - \frac{H^2 n_1^2}{t^2})} \rangle \simeq \exp\left(\frac{2H^2 T n_1^2 \left[\sum_{i=2}^N n_i^2 (t + a_1 - a_i)^{-3} \right]}{t^3 \left[\sum_{i=1}^N n_i^2 (t + a_1 - a_i)^{-3} \right]} \xi^2 \right) \tag{4.2.23}$$

as $N \to \infty$ *for asymptotically almost every disorder sample, where* $t > 0$ *is the solution of the equation* (3.3.14)*.*

The right-hand side depends strongly on the disorder sample, as the formula involves all of the a_i and n_i. The above result implies the following.

Result 4.2.4. *For* $h = HN^{-1/6}$ *and* $0 < T < 1$,

$$\mathfrak{O} \overset{\mathfrak{D}}{\simeq} \frac{H^2 n_1^2}{t^2} + \frac{\sigma_{\mathfrak{O}} \mathfrak{N}}{N^{1/6}} = \left[1 - T - H^2 \sum_{i=2}^N \frac{n_i^2}{(t + a_1 - a_i)^2} \right] + \frac{\sigma_{\mathfrak{O}} \mathfrak{N}}{N^{1/6}} \tag{4.2.24}$$

as $N \to \infty$ *for asymptotically almost every disorder sample, where the thermal random variable* $\mathfrak{N}$ *has the standard normal distribution and* $\sigma_{\mathfrak{O}} > 0$ *satisfies*

$$\sigma_{\mathfrak{O}}^2 = \frac{4H^2 T n_1^2 \left[\sum_{i=2}^N n_i^2 (t + a_1 - a_i)^{-3} \right]}{t^3 \left[\sum_{i=1}^N n_i^2 (t + a_1 - a_i)^{-3} \right]}. \tag{4.2.25}$$

The equality of the leading terms in the two formulas of (4.2.24) *follows from the equation* (3.3.14) *that* t *satisfies.*

4.2.2.2 Matching with $h = O(1)$

We consider the $H \to \infty$ limit. From (3.3.20), we have $t \simeq \frac{H^4}{4(1-T)^2}$. Hence, the term(4.2.24) satisfies

$$\sigma_{\mathfrak{O}}^2 \simeq \frac{4H^2 T n_1^2}{t^3} \simeq \frac{4^4 T n_1^2 (1-T)^6}{H^{10}}.$$

Therefore, the first formula of (4.2.24) implies that if we take $h = HN^{-1/6}$ and let $N \to \infty$ first and then $H \to \infty$, we get

$$\mathfrak{O} \overset{\mathfrak{D}}{\simeq} \frac{16}{N} \left[\frac{(1-T)^4 n_1^2}{h^6} + \frac{\sqrt{T}(1-T)^3 |n_1|}{h^5} \mathfrak{N} \right]. \tag{4.2.26}$$

This formula matches the formal limit given in (4.2.16). Thus this regime matches with the $h = O(1)$ regime.

4.2.2.3 Formal limit as $H \to 0$

Using (3.3.19) for t, the denominator of (4.2.25) becomes $n_1^2 + \mathcal{O}(H^3)$ as $H \to 0$. Thus, if we take $h = HN^{-1/6}$ and let $N \to \infty$ first and then take $H \to 0$, we get

$$\mathfrak{O} \overset{\mathfrak{D}}{\simeq} 1 - T - H^2 \sum_{i=2}^{N} \frac{n_i^2}{(a_1 - a_i)^2} + \frac{2H\sqrt{T}}{N^{1/6}} \left[\sum_{i=2}^{N} \frac{n_i^2}{(a_1 - a_i)^3} \right]^{1/2} \mathfrak{N}. \tag{4.2.27}$$

The last two terms of (4.2.27) are of orders $H^2 = h^2 N^{1/3}$ and $HN^{-1/6} = h$, respectively. These two terms have the same order if $h \sim N^{-1/3}$. We study this regime in the next subsection. Note that, in this regime, the two terms are of order $N^{-1/3}$.

4.2.3 Microscopic external field: $h \sim N^{-1/3}$ and $T < 1$

4.2.3.1 Analysis

Set

$$h = HN^{-1/3} \tag{4.2.28}$$

for fixed $H > 0$. In the last part of the previous sub-subsection, a formal calculation indicated that the order of fluctuation in this regime is $N^{-1/3}$. We set

$$\eta = \xi N^{1/3} \quad \text{so that} \quad b = 2\xi N^{-2/3}. \tag{4.2.29}$$

The regime $h \sim N^{-1/3}$ did not appear in previous sections. Hence, we first find the critical point γ of $\mathcal{G}(z)$. Previously we saw that $\gamma = \lambda_1 + \mathcal{O}\left(N^{-2/3}\right)$ when $h \sim N^{-1/6}$ and $\gamma = \lambda_1 + \mathcal{O}\left(N^{-1}\right)$ when $h \sim N^{-1/2}$. We expect that, in this regime, γ is between the above two cases, so we set $\gamma = \lambda_1 + w$ for some w and we assume $N^{-1} \ll w \ll N^{-2/3}$. The equation for the critical point is, using (2.2.14),

$$\mathcal{G}'(\gamma) = \beta - \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\gamma - \lambda_i} - \frac{H^2 \beta}{N^{5/3}} \sum_{i=1}^{N} \frac{n_i^2}{(\gamma - \lambda_i)^2} = \beta - \frac{1}{Nw} - 1 + \mathcal{O}\left(N^{-1/3}\right) - \frac{H^2 \beta n_1^2}{N^{5/3} w^2} = 0. \tag{4.2.30}$$

Under the assumption for w, we see that $\frac{1}{Nw} \ll \frac{1}{N^{5/3} w^2}$, and hence $w = \mathcal{O}\left(N^{-5/6}\right)$. Explicitly solving the equation $\beta - 1 - \frac{H^2\beta n_1^2}{N^{5/3}w^2} = 0$, we find that

$$\gamma = \lambda_1 + rN^{-5/6} \quad \text{where} \quad r = \sqrt{\frac{H^2\beta n_1^2}{\beta - 1}} + \mathcal{O}\left(N^{-1/6}\right). \tag{4.2.31}$$

For later use, we record that, upon inserting $\gamma = \lambda_1 + rN^{-5/6}$ into the equation (4.2.30), r satisfies the following more detailed equation, using the notation Ξ_N defined in (2.2.9):

$$\beta - \frac{1}{rN^{1/6}} - 1 - \frac{\Xi_N}{N^{1/3}} + \mathcal{O}\left(N^{-1/2}\right) - \frac{H^2\beta n_1^2}{r^2} - \frac{H^2\beta}{N^{1/3}} \sum_{i=2}^{N} \frac{n_i^2}{(a_1 - a_i)^2} = 0. \tag{4.2.32}$$

The critical point $\gamma_{\mathfrak{D}}$ of $\mathcal{G}_{\mathfrak{D}}(z)$ is easy to obtain since $b = \frac{2\xi}{N^{2/3}}$ has the same order as the fluctuations of the eigenvalues λ_i. The critical point equation is the same as in the case of $\mathcal{G}(z)$ except that λ_1 is changed to $\lambda_1 + b$. Thus we have

$$\gamma_{\mathfrak{D}} = \lambda_1 + b + r_{\mathfrak{D}} N^{-5/6} \quad \text{where} \quad r_{\mathfrak{D}} = r + \mathcal{O}\left(N^{-1/6}\right). \tag{4.2.33}$$

For our computation, it turns out that we need an improved estimate for $r_{\mathfrak{D}} - r$. The equation $\mathcal{G}'_{\mathfrak{D}}(\gamma_{\mathfrak{D}}) = 0$ is, in terms of $r_{\mathfrak{D}}$,

$$\beta - \frac{1}{r_{\mathfrak{D}} N^{1/6}} - 1 - \frac{H^2\beta n_1^2}{r_{\mathfrak{D}}^2} + \mathcal{O}\left(N^{-1/3}\right) = 0.$$

This equation is the same as the equation (4.2.32) up to order $N^{-1/6}$. Therefore, we obtain an improved estimate $r_{\mathfrak{D}} = r + \mathcal{O}\left(N^{-1/3}\right)$. As a consequence,

$$\gamma_{\mathfrak{D}} - \gamma = b + \mathcal{O}\left(N^{-7/6}\right) = 2\xi N^{-2/3} + \mathcal{O}\left(N^{-7/6}\right). \tag{4.2.34}$$

We now evaluate $N(\mathcal{G}_{\mathfrak{D}}(\gamma_{\mathfrak{D}}) - \mathcal{G}(\gamma))$ using (4.2.5). We have

$$\begin{aligned} D_1 &= \frac{2\xi N^{1/6}}{r} + \mathcal{O}\left(N^{-1/3}\right), \\ D_2 &= -\sum_{i=2}^{N}\left[\log\left(1+\frac{2\xi}{a_1-a_i}\right) - \frac{2\xi}{a_1-a_i}\right] + \mathcal{O}\left(N^{-1/6}\right), \\ D_3 &= \frac{2\xi H^2\beta n_1^2}{r^2}N^{1/3} + \mathcal{O}\left(N^{-1/6}\right), \\ D_4 &= 4\xi^2 H^2\beta\sum_{i=2}^{N}\frac{n_i^2}{(a_1+2\xi-a_i)(a_1-a_i)^2} + \mathcal{O}\left(N^{-1/6}\right). \end{aligned}$$

Note that r appears only in D_1 and D_3. Using the equation (4.2.32), the sum $D_1 + D_3$ can be expressed without using r:

$$D_1 + D_3 = 2\xi N^{1/3}\left[\beta - 1 - \frac{\Xi_N}{N^{1/3}} - \frac{H^2\beta}{N^{1/3}}\sum_{i=2}^{N}\frac{n_i^2}{(a_1-a_i)^2}\right] + \mathcal{O}\left(N^{-1/6}\right). \tag{4.2.35}$$

On the other hand, using the notation Ξ_N in (2.2.9) again, we can write

$$D_2 = -\left[\sum_{i=2}^{N}\log\left(1+\frac{2\xi}{a_1-a_i}\right) - 2\xi N^{1/3}\right] + 2\xi\Xi_N + \mathcal{O}\left(N^{-1/6}\right). \tag{4.2.36}$$

Adding D_1, D_2, D_3, and D_4, and combining two sums that are multiplied by $H^2\beta$, we find that

$$\begin{aligned} N(\mathcal{G}_{\mathfrak{D}}(\gamma_{\mathfrak{D}}) - \mathcal{G}(\gamma)) = 2\xi(\beta-1)N^{1/3} &+ \left[2\xi N^{1/3} - \sum_{i=2}^{N}\log\left(1+\frac{2\xi}{a_1-a_i}\right)\right] \\ &- 2\xi H^2\beta\sum_{i=2}^{N}\frac{n_i^2}{(a_1+2\xi-a_i)(a_1-a_i)} + \mathcal{O}\left(N^{-1/6}\right) \end{aligned} \tag{4.2.37}$$

We note that the term in brackets is $\mathcal{O}(1)$ due to (2.2.9).

Finally, we consider the integrals in (4.2.2), beginning with the numerator. Using $\gamma_{\mathfrak{D}} = \lambda_1 + b + rN^{-5/6} + \mathcal{O}\left(N^{7/6}\right)$ and the explicit formula for $\mathcal{G}_{\mathfrak{D}}(z)$, we find that

$$\mathcal{G}_{\mathfrak{D}}^{(k)}(\gamma_{\mathfrak{D}}) = \mathcal{O}\left(N^{\frac{5}{6}k-\frac{5}{6}}\right)$$

for $k \geq 2$. Since $\mathcal{G}_{\mathfrak{D}}''(\gamma_{\mathfrak{D}}) = \mathcal{O}\left(N^{\frac{5}{6}}\right)$, the main contribution to the integral comes from a neighborhood of radius $N^{-\frac{11}{12}}$ about the critical point. For $k = 2$, we find explicitly that

$$\mathcal{G}''_{\mathfrak{D}}(\gamma_{\mathfrak{D}}) = \frac{2H^2\beta n_1^2}{r^3}N^{-5/6} + \mathcal{O}\left(N^{-1}\right).$$

Hence,

$$N(\mathcal{G}_{\mathfrak{D}}(\gamma_{\mathfrak{D}} + wN^{-\frac{11}{12}}) - \mathcal{G}_{\mathfrak{D}}(\gamma_{\mathfrak{D}})) = \sum_{k=2}^{\infty} \frac{N^{1-\frac{11}{12}k}\mathcal{G}_{\mathfrak{D}}^{(k)}(\gamma_{\mathfrak{D}})w^k}{k!} = \frac{H^2\beta n_1^2}{r^3}w^2 + \mathcal{O}\left(N^{-\frac{1}{12}}\right) \quad (4.2.38)$$

for finite w, and the integral can be evaluated as a Gaussian integral. Since the leading term of (4.2.38) does not depend on ξ, we find that the ratio of the integrals in (4.2.2) is asymptotically equal to 1.

Combining the computations above, we obtain an asymptotic formula for $\left\langle e^{\beta\xi N^{1/3}\mathfrak{D}}\right\rangle$. Moving a term and using $\beta = 1/T$, we arrive at the following result.

Result 4.2.5. *For $h = HN^{-1/3}$ and $0 < T < 1$,*

$$\left\langle e^{\frac{\xi}{T}N^{1/3}(\mathfrak{D}-(1-T))}\right\rangle \simeq e^{\xi N^{1/3}} \prod_{i=2}^{N} \frac{\exp\left(-\frac{\xi H^2 n_i^2}{T(a_1+2\xi-a_i)(a_1-a_i)}\right)}{\sqrt{1+\frac{2\xi}{a_1-a_i}}} \quad (4.2.39)$$

as $N \to \infty$ for asymptotically almost every disorder sample.

We remark that the right-hand side is $\mathcal{O}(1)$ since

$$\xi N^{1/3} - \frac{1}{2}\sum_{i=2}^{N} \log\left(1 + \frac{2\xi}{a_1 - a_i}\right) = \mathcal{O}(1).$$

The formula (4.2.39) is a product of the moment generating functions of non-centered chi-squared distributions (see (4.2.12)). Hence, we obtain the following.

Result 4.2.6. *For $h = HN^{-1/3}$ and $0 < T < 1$,*

$$\mathfrak{D} \overset{\mathfrak{D}}{\simeq} 1 - T + \frac{T}{N^{1/3}}\mathfrak{W}_N, \qquad \mathfrak{W}_N = N^{1/3} - \sum_{i=2}^{N} \frac{\left|\frac{H|n_i|}{\sqrt{T(a_1-a_i)}} + \mathfrak{n}_i\right|^2}{a_1 - a_i} \quad (4.2.40)$$

as $N \to \infty$ for asymptotically almost every disorder sample, where the thermal random variables $\mathfrak{n}_i$ are independent standard normal random variables.

Here, we emphasize that n_i are sample random variables (given by the dot product of each eigenvector of M with the external field) while $\mathfrak{n}_i$ are thermal random variables. Note that $\mathfrak{W}_N = \mathcal{O}(1)$ since $N^{1/3} - \sum_{i=2}^{N} \frac{\mathfrak{n}_i^2}{a_1 - a_i} = \mathcal{O}(1)$.

4.2.3.2 Matching with the mesocopic field, $h \sim N^{-1/6}$

We take the formal limit $H \to \infty$ of (4.2.40) and compare with (4.2.27). Then, using $N^{1/3} - \sum_{i=2}^{N} \frac{1}{a_1 - a_i} = \mathcal{O}(1)$ from (2.2.14),

$$\mathfrak{W}_N = -\frac{H^2}{T} \sum_{i=2}^{n} \frac{n_i^2}{(a_1 - a_i)^2} - \frac{2H}{\sqrt{T}} \sum_{i=2}^{N} \frac{|n_i|\mathfrak{n}_i}{(a_1 - a_i)^{3/2}} + \mathcal{O}(1). \tag{4.2.41}$$

The second sum is a sum of independent (thermal) Gaussian random variables, and hence it has a Gaussian distribution. Therefore, if take $h = HN^{-1/3}$ and let $N \to \infty$ first and then take $H \to \infty$, we get

$$\mathfrak{O} \overset{\mathfrak{D}}{\simeq} 1 - T - \frac{H^2}{N^{1/3}} \sum_{i=2}^{N} \frac{n_i^2}{(a_1 - a_i)^2} + \frac{2H\sqrt{T}}{N^{1/3}} \left[\sum_{i=2}^{N} \frac{n_i^2}{(a_1 - a_i)^3} \right]^{1/2} \mathfrak{N}. \tag{4.2.42}$$

In order to compare this with the result (4.2.27), we use the notation $h = H_{\text{micro}} N^{-1/3} = H_{\text{meso}} N^{-1/6}$. The equations (4.2.42) and (4.2.27) are same once we set $H = H_{\text{micro}}$ and $H = H_{\text{meso}}$, respectively.

4.2.4 No external field: $h = 0$

For $0 < T < 1$, the calculations of the previous subsection for $h = HN^{-1/3}$ go through; we obtain the result by setting $H = 0$ in (4.2.40). For $T > 1$, the computations in Subsection 4.2.1 for $h = O(1)$ also apply to $h = 0$; see (4.2.15).

Result 4.2.7. *For $h = 0$,*

$$\mathfrak{O} \overset{\mathfrak{D}}{\simeq} \begin{cases} \dfrac{T^2}{N(T-1)^2} \mathfrak{N}^2 & \text{for } T > 1, \\ 1 - T + \dfrac{T}{N^{1/3}} \left(N^{1/3} - \displaystyle\sum_{i=2}^{N} \frac{\mathfrak{n}_i^2}{a_1 - a_i} \right) & \text{for } 0 < T < 1. \end{cases} \tag{4.2.43}$$

where the thermal random variable $\mathfrak{N}$ has the standard normal distribution, and $\mathfrak{n}_i$ are independent standard normal thermal random variables.

4.2.5 The thermal average

We use the notation

$$\Omega = \langle \mathfrak{O} \rangle = \langle \mathfrak{G}^2 \rangle \tag{4.2.44}$$

to denote the thermal average of the squared overlap of a spin with the ground state. Previous subsections imply the following results.

(i) For $h \geq 0$ and $T > 1$, or for $h = O(1)$ with $h > 0$ and $0 < T < 1$,

$$\Omega \simeq \frac{\Omega^0}{N}, \qquad \Omega^0 := \frac{T}{\gamma_0 - 2}\left[\frac{h^2 n_1^2}{T(\gamma_0 - 2)} + 1\right]. \tag{4.2.45}$$

From the asymptotic formulas (3.1.33) and (3.1.32) of γ_0,

$$\Omega^0 \simeq n_1^2 + \frac{T - (T-4)n_1^2}{h} \qquad \text{as } h \to \infty \text{ for all } T > 0 \tag{4.2.46}$$

and

$$\Omega^0 \simeq \begin{cases} \dfrac{T^2}{(T-1)^2} + \dfrac{h^2(n_1^2 - 1)T^2}{(T-1)^4} & \text{as } h \to 0 \text{ for } T > 1 \\ \dfrac{16 n_1^2 (1-T)^4}{h^6} + \dfrac{4T(1-T)^2 + 32 n_1^2 (1-T)^4}{h^4} & \text{as } h \to 0 \text{ for } 0 < T < 1. \end{cases} \tag{4.2.47}$$

See Figure IV.5 for graphs of Ω^0.

(ii) For $h = HN^{-1/6}$ with $0 < T < 1$,

$$\Omega \simeq \frac{H^2 n_1^2}{t^2} = 1 - T - H^2 \sum_{i=2}^{N} \frac{n_i^2}{(t + a_1 - a_i)^2}. \tag{4.2.48}$$

(iii) For $h = HN^{-1/3}$ with $T < 1$ (including the case when $H = 0$),

$$\Omega \simeq 1 - T + \frac{1}{N^{1/3}}\left[T\left(N^{1/3} - \sum_{i=2}^{N} \frac{1}{a_1 - a_i}\right) - H^2 \sum_{i=2}^{N} \frac{n_i^2}{(a_1 - a_i)^2}\right]. \tag{4.2.49}$$

If we collect only the order 1 terms, then as $N \to \infty$ with $T < 1$,

$$\Omega \to \begin{cases} 0 & \text{for } h > 0 \\ 1 - T - H^2 \sum_{i=2}^{N} \frac{n_i^2}{(t+a_1-a_i)^2} & \text{for } h = HN^{-1/6} \\ 1 - T & \text{for } h = HN^{-1/3} \text{ (including } H = 0). \end{cases} \tag{4.2.50}$$

The sample-to-sample standard deviation of the thermal average of squared overlap satisfies for $0 < T < 1$,

$$\sqrt{\overline{\Omega^2} - (\overline{\Omega})^2} = \begin{cases} O(N^{-1}) & \text{for } h = O(1) \\ O(1) & \text{for } h \sim N^{-1/6} \\ O(N^{-1/3}) & \text{for } h \sim N^{-1/3} \text{ (including } h = 0). \end{cases} \tag{4.2.51}$$

The order is largest when $h \sim N^{-1/6}$.

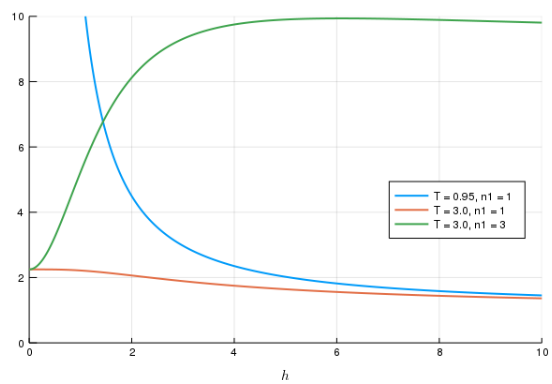


Figure IV.5: Graphs of Ω^0 for $h = O(1)$ as functions of h for different combinations of T and n_1.

4.2.6 Order of thermal fluctuations

For $0 < T < 1$, the standard deviation of the thermal fluctuations satisfies

$$\sqrt{\langle \mathfrak{O}^2 \rangle - \langle \mathfrak{O} \rangle^2} = \begin{cases} \mathcal{O}\left(N^{-1}\right) & \text{for } h = O(1) \\ \mathcal{O}\left(N^{-1/6}\right) & \text{for } h \sim N^{-1/6} \\ \mathcal{O}\left(N^{-1/3}\right) & \text{for } h \sim N^{-1/3} \text{ (including } h = 0). \end{cases} \tag{4.2.52}$$

for asymptotically almost every disorder sample. The thermal fluctuations are largest when $h \sim N^{-1/6}$.

4.3 Overlap with a replica

Let

$$\mathfrak{R} = \mathfrak{R}^{1,2} = \frac{\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}}{N} \tag{4.3.1}$$

be the overlap of a spin $\boldsymbol{\sigma}^{(1)}$ and its replica $\boldsymbol{\sigma}^{(2)}$, chosen independently from S_{N-1} using the Gibbs measure with the same disorder sample. From Lemma 2.3.3, we have

$$\langle e^{\eta\mathfrak{R}} \rangle = e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}};a)-2\mathcal{G}(\gamma))} \frac{\iint e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{R}}(z,w;a)-\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}};a))}\mathrm{d}z\mathrm{d}w}{\left(\int e^{\frac{N}{2}(\mathcal{G}(z)-\mathcal{G}(\gamma))}\mathrm{d}z\right)^2} \tag{4.3.2}$$

where

$$\mathcal{G}_{\mathfrak{R}}(z,w;a) = \beta(z+w) - \frac{1}{N}\sum_{i=1}^{N} \log\left((z-\lambda_i)(w-\lambda_i)-a^2\right) + \frac{h^2\beta}{N}\sum_{i=1}^{N} \frac{n_i^2(z+w-2\lambda_i+2a)}{(z-\lambda_i)(w-\lambda_i)-a^2} \tag{4.3.3}$$

and we set

$$a = \frac{\eta}{\beta N}. \tag{4.3.4}$$

We take γ to be the critical point of $\mathcal{G}(z)$ and we chose $\gamma_{\mathfrak{R}} > \lambda_1 + |a|$ such that $(\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}})$ is a critical point of $\mathcal{G}_{\mathfrak{R}}(z,w;a)$. We calculate $\gamma_{\mathfrak{R}}$ below.

The partial derivative of $\mathcal{G}_{\mathfrak{R}}$ with respect to z is

$$\frac{\partial \mathcal{G}_{\mathfrak{R}}}{\partial z} = \beta - \frac{1}{N}\sum_{i=1}^{N} \frac{w-\lambda_i}{(z-\lambda_i)(w-\lambda_i)-a^2} - \frac{h^2\beta}{N}\sum_{i=1}^{N} \frac{n_i^2(w-\lambda_i+a)^2}{((z-\lambda_i)(w-\lambda_i)-a^2)^2} \tag{4.3.5}$$

and $\frac{\partial \mathcal{G}_{\mathfrak{R}}}{\partial w}$ is similar. Since $\frac{\partial \mathcal{G}_{\mathfrak{R}}}{\partial z}$ is an increasing function for real z (and similarly with $\frac{\partial \mathcal{G}_{\mathfrak{R}}}{\partial w}$), there exists a critical point of the form $(z,w) = (\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}})$ where $\gamma_{\mathfrak{R}}$ solves the equation

$$\beta - \frac{1}{N}\sum_{i=1}^{N} \frac{\gamma_{\mathfrak{R}}-\lambda_i}{(\gamma_{\mathfrak{R}}-\lambda_i-a)(\gamma_{\mathfrak{R}}-\lambda_i+a)} - \frac{h^2\beta}{N}\sum_{i=1}^{N} \frac{n_i^2}{(\gamma_{\mathfrak{R}}-\lambda_i-a)^2} = 0, \quad \gamma_{\mathfrak{R}} > \lambda_1 + |a|. \tag{4.3.6}$$

There may be other critical points, but $(\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}})$ is the one that we use for our steepest descent analysis. For simplicity, we refer to this critical point as $\gamma_{\mathfrak{R}}$ rather than $(\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}})$. For $a = 0$, $\mathcal{G}_{\mathfrak{R}}(z,w;0) = \mathcal{G}(z) + \mathcal{G}(w)$, and in this case, the critical point is $(z,w) = (\gamma, \gamma)$.

We use the following two formulas in this section. The first formula is

$$N\left(\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}};a)-2\mathcal{G}(\gamma)\right)=N(\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}};a)-2\mathcal{G}(\gamma)-2\mathcal{G}'(\gamma)(\gamma_{\mathfrak{R}}-\gamma))=B_1+B_2 \tag{4.3.7}$$

where

$$B_1:=-\sum_{i=1}^{N}\left[\log\left(1+\frac{2(\gamma_{\mathfrak{R}}-\gamma)}{\gamma-\lambda_i}+\frac{(\gamma_{\mathfrak{R}}-\gamma)^2-a^2}{(\gamma-\lambda_i)^2}\right)-\frac{2(\gamma_{\mathfrak{R}}-\gamma)}{\gamma-\lambda_i}\right]$$

and

$$B_2:=2h^2\beta\sum_{i=1}^{N}n_i^2\left[\frac{1}{\gamma_{\mathfrak{R}}-\lambda_i-a}-\frac{1}{\gamma-\lambda_i}+\frac{\gamma_{\mathfrak{R}}-\gamma}{(\gamma-\lambda_i)^2}\right].$$

The second formula is

$$\begin{aligned}(\gamma_{\mathfrak{R}}-\gamma-a)&\left[\sum_{i=1}^{N}\frac{\gamma_{\mathfrak{R}}-\lambda_i}{(\gamma_{\mathfrak{R}}-\lambda_i-a)(\gamma_{\mathfrak{R}}-\lambda_i+a)(\gamma-\lambda_i)}+h^2\beta\sum_{i=1}^{N}\frac{n_i^2(\gamma+\gamma_{\mathfrak{R}}-2\lambda_i-a)}{(\gamma_{\mathfrak{R}}-\lambda_i-a)^2(\gamma-\lambda_i)^2}\right]\\ &=-a\sum_{i=1}^{N}\frac{1}{(\gamma_{\mathfrak{R}}-\lambda_i+a)(\gamma-\lambda_i)},\end{aligned} \tag{4.3.8}$$

which follows from subtracting the critical point equations for $\gamma_{\mathfrak{R}}$ and γ.

We also make use of the following lemma.

Lemma 4.3.1. *The point $\gamma_{\mathfrak{R}}$ satisfies $\gamma<\gamma_{\mathfrak{R}}<\gamma+a$.*

Proof. Let

$$g(z)=\beta-\frac{1}{N}\sum_{i=1}^{N}\frac{z-\lambda_i}{(z-\lambda_i-a)(z-\lambda_i+a)}-\frac{h^2\beta}{N}\sum_{i=1}^{N}\frac{n_i^2}{(z-\lambda_i-a)^2}.$$

Since $g(\gamma_{\mathfrak{R}})=0$, it is enough to show that $g(\gamma)<0$ and $g(\gamma+a)>0$. Using $a>0$, we see that

$$g(\gamma)<\beta-\frac{1}{N}\sum_{i=1}^{N}\frac{1}{\gamma-\lambda_i}-\frac{h^2\beta}{N}\sum_{i=1}^{N}\frac{n_i^2}{(\gamma-\lambda_i)^2}=\mathcal{G}'(\gamma)=0.$$

On the other hand,

$$\begin{aligned} g(\gamma+a) =& \beta - \frac{1}{N}\sum_{i=1}^{N} \frac{\gamma-\lambda_i+a}{(\gamma-\lambda_i)(\gamma-\lambda_i+2a)} - \frac{h^2\beta}{N}\sum_{i=1}^{N}\frac{n_i^2}{(\gamma-\lambda_i)^2} \\ >& \beta - \frac{1}{N}\sum_{i=1}^{N}\frac{1}{\gamma-\lambda_i} - \frac{h^2\beta}{N}\sum_{i=1}^{N}\frac{n_i^2}{(\gamma-\lambda_i)^2} = \mathcal{G}'(\gamma) = 0. \end{aligned}$$

□

4.3.1 Macroscopic external field: $h = O(1)$

4.3.1.1 Analysis

Fix $h > 0$. It turns out that the fluctuations are of order $N^{-1/2}$. Hence, we set

$$\eta = \beta\xi\sqrt{N} \quad \text{so that} \quad a = \xi N^{-1/2}. \tag{4.3.9}$$

The critical point of $\mathcal{G}(z)$ is given in (3.1.20) by $\gamma = \gamma_0 + \gamma_1 N^{-1/2} + \mathcal{O}\left(N^{-1}\right)$. Consider the critical point $\gamma_{\mathfrak{R}}$. By Lemma 4.3.1, $\gamma_{\mathfrak{R}} = \gamma + O(N^{-1/2})$. We now use the equation (4.3.8). Using the semi-circle law approximation, we find that

$$\gamma_{\mathfrak{R}} - \gamma - a = -\frac{a\left(s_2(\gamma_0) + O(N^{-\frac{1}{2}})\right)}{s_2(\gamma_0) + 2h^2\beta s_3(\gamma_0) + O(N^{-\frac{1}{2}})}. \tag{4.3.10}$$

Thus,

$$\gamma_{\mathfrak{R}} = \gamma + \frac{\xi A}{\sqrt{N}} + O(N^{-1}) \quad \text{where} \quad A = \frac{2h^2\beta s_3(\gamma_0)}{s_2(\gamma_0) + 2h^2\beta s_3(\gamma_0)}. \tag{4.3.11}$$

We evaluate $N(\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}}; a) - 2\mathcal{G}(\gamma))$ using (4.3.7). From a Taylor approximation,

$$B_1 = \sum_{i=1}^{N} \frac{(\gamma_{\mathfrak{R}}-\gamma)^2 + a^2}{(\gamma-\lambda_i)^2} + \mathcal{O}\left(N^{-1/2}\right) = \left((\gamma_{\mathfrak{R}}-\gamma)^2 + a^2\right) N s_2(\gamma) + \mathcal{O}\left(N^{-1/2}\right). \tag{4.3.12}$$

On the other hand, using the geometric series for $\frac{1}{\gamma_{\mathfrak{R}}-\lambda_i-a} = \frac{1}{(\gamma-\lambda_i)+(\gamma_{\mathfrak{R}}-\gamma-a)}$ and using (2.2.16),

$$\begin{aligned} B_2 &= \sum_{i=1}^{N} n_i^2 \left[\frac{a}{(\gamma-\lambda_i)^2} + \frac{(\gamma_{\mathfrak{R}}-\gamma-a)^2}{(\gamma-\lambda_i)^3} + O\left(\frac{(\gamma_{\mathfrak{R}}-\gamma-a)^3}{(\gamma-\lambda_i)^4}\right)\right] \\ &= a\left(s_2(\gamma) + N^{-1/2}\mathcal{S}_N(\gamma;2)\right) + (\gamma_{\mathfrak{R}}-\gamma-a)^2 s_3(\gamma) + \mathcal{O}\left(N^{-1/2}\right) \end{aligned} \tag{4.3.13}$$

where $\mathcal{S}_N(z;k)$ is defined in (2.2.6). The leading term is $as_2(\gamma)$ which is $O(N^{1/2})$ and the rest is $\mathcal{O}(1)$. Inserting $\gamma = \gamma_0 + \gamma_1 N^{-1/2} + \mathcal{O}(N^{-1})$ and using $s_2'(z) = -2s_3(z)$, we find that

$$\begin{aligned}&N(\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}}; a) - 2\mathcal{G}(\gamma)) = \xi^2(1+A^2)s_2(\gamma_0) \\ &\quad + 2h^2\beta\left(\xi^2(A-1)^2 s_3(\gamma_0) + \xi\mathcal{S}_N(\gamma_0;2) + \xi\sqrt{N}s_2(\gamma_0) - 2\xi s_3(\gamma_0)\gamma_1\right) + \mathcal{O}\left(N^{-\frac{1}{2}}\right).\end{aligned} \tag{4.3.14}$$

We now consider the integrals in (4.3.2). Since all partial derivatives of $\mathcal{G}_{\mathfrak{R}}(z,w)$ evaluated at the critical point $(z,w) = (\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}})$ are $\mathcal{O}(1)$, the two dimensional method of steepest descent applies. Since the second derivatives evaluated at the critical point do not depend on ξ, we find that the ratio of the integrals in (4.3.2) is asymptotically equal to 1.

Combining the computations above, we find that

$$\begin{aligned}\log\langle e^{\beta\xi\sqrt{N}\mathfrak{R}}\rangle \simeq& \frac{1}{2}\xi^2(1+A^2)s_2(\gamma_0) \\ &+ h^2\beta\left(\xi^2(A-1)^2 s_3(\gamma_0) + \xi\mathcal{S}_N(\gamma_0;2) + \xi\sqrt{N}s_2(\gamma_0) - 2\xi s_3(\gamma_0)\gamma_1\right)\end{aligned} \tag{4.3.15}$$

where A is given by (4.3.11). Using the formula (3.1.21) of γ_1, we obtain

$$\mathcal{S}_N(\gamma_0;2) - 2s_3(\gamma_0)\gamma_1 = \frac{Ts_2(\gamma_0)}{Ts_2(\gamma_0) + 2h^2 s_3(\gamma_0)}\mathcal{S}_N(\gamma_0;2). \tag{4.3.16}$$

Hence, we conclude the following.

Result 4.3.2. *For $h > 0$ and $T > 0$,*

$$\log\langle e^{\xi\sqrt{N}(\mathfrak{R} - h^2 s_2(\gamma_0))}\rangle \simeq \frac{h^2 T s_2(\gamma_0)\mathcal{S}_N(\gamma_0;2)}{Ts_2(\gamma_0) + 2h^2 s_3(\gamma_0)}\xi + \frac{T^2 s_2(\gamma_0)(Ts_2(\gamma_0) + 4h^2 s_3(\gamma_0))}{2(Ts_2(\gamma_0) + 2h^2 s_3(\gamma_0))}\xi^2 \tag{4.3.17}$$

as $N \to \infty$ for asymptotically almost every disorder sample, where $\gamma_0 > 2$ is the solution of the equation $1 - Ts_1(\gamma_0) - h^2 s_2(\gamma_0) = 0$, and $\mathcal{S}_N(z;k)$ is defined in (2.2.6).

As a consequence, we obtain the following.

Result 4.3.3. *For $h > 0$ and $T > 0$,*

$$\mathfrak{R} \overset{\mathfrak{D}}{\simeq} h^2 s_2(\gamma_0) + \frac{1}{\sqrt{N}}\left[\frac{h^2 T s_2(\gamma_0)\mathcal{S}_N(\gamma_0;2)}{Ts_2(\gamma_0) + 2h^2 s_3(\gamma_0)} + \sigma_{\mathfrak{R}}\mathfrak{N}\right] \tag{4.3.18}$$

as $N \to \infty$ for asymptotically almost every disorder sample, where the thermal random variable $\mathfrak{N}$ has the standard normal distribution and $\sigma_{\mathfrak{R}} > 0$ satisfies

$$\sigma_{\mathfrak{R}}^2 = \frac{T^2 s_2(\gamma_0)(Ts_2(\gamma_0) + 4h^2 s_3(\gamma_0))}{Ts_2(\gamma_0) + 2h^2 s_3(\gamma_0)}. \tag{4.3.19}$$

4.3.1.2 Discussion of the leading term

The leading term

$$\mathfrak{R}^0 = \mathfrak{R}^0(T, h) = h^2 s_2(\gamma_0) = 1 - Ts_1(\gamma_0) \tag{4.3.20}$$

in (4.3.18) depends on neither the choice of spin configuration nor the disorder sample. See Figure IV.6a for the graph of $\mathfrak{R}^0$ as a function of h.

The value (4.3.20) for $\mathfrak{R}^0$ reproduces the prediction q_0 for the overlap obtained in [15, 22] from the replica saddle methods which predicts that q_0 is determined by (3.1.29), The equivalence is checked using that $s_2(z) = s_1(z)^2/(1 - s_1(z)^2)$ and $q_0 = 1 - Ts_1(\gamma_0)$.

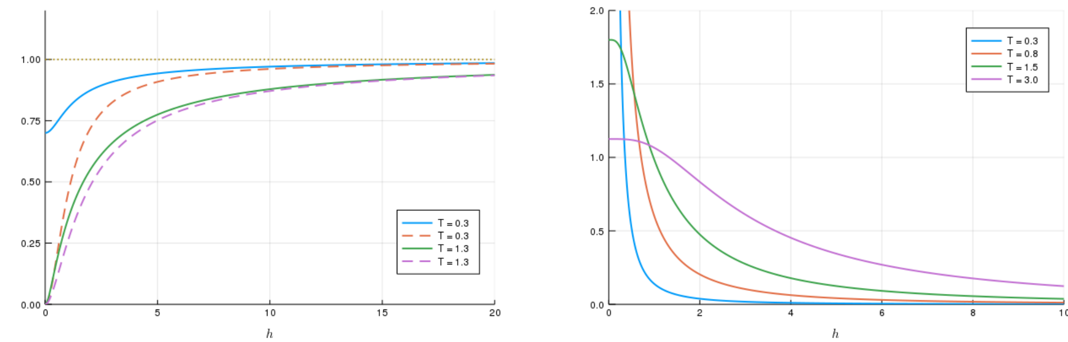
It is easy to check the following properties using a computation similar to the one in Subsection 4.1.3:

- For every $T > 0$, $\mathfrak{R}^0$ is an increasing function of $h > 0$.
- As $h \to \infty$,
$$\mathfrak{R}^0 = 1 - \frac{T}{h} + O(h^{-2}) \qquad \text{for all } T > 0. \tag{4.3.21}$$
- As $h \to 0$,
$$\mathfrak{R}^0 = \begin{cases} \dfrac{h^2}{T^2 - 1} - \dfrac{2T^2 h^4}{(T^2 - 1)^2} + O(h^6) & \text{for } T > 1, \\ 1 - T + \dfrac{Th^2}{2(1 - T)} + O(h^4) & \text{for } 0 < T < 1. \end{cases} \tag{4.3.22}$$

4.3.1.3 Discussion of the thermal variance

The thermal variance of $\mathfrak{R}$ satisfies

$$\langle \mathfrak{R}^2 \rangle - \langle \mathfrak{R} \rangle^2 \simeq \frac{\sigma_{\mathfrak{R}}^2}{N} \tag{4.3.23}$$



(a) Graph of $\mathfrak{R}^0$ (solid line) and $(\mathfrak{M}^0)^2$ (dashed line) as a function of h for $T = 0.3$ and $T = 1.3$

(b) Graph of $\sigma_{\mathfrak{R}}^2$ as a function of h.

Figure IV.6: Graphs of $\mathfrak{R}^0$ and $\sigma_{\mathfrak{R}}^2$.

for $\sigma_{\mathfrak{R}}^2$ given in (4.3.19) and it does not depend on the disorder sample. See Figure IV.6b for the graph. It is a decreasing function of h, and satisfies

$$\sigma_{\mathfrak{R}}^2 = \frac{2T^2}{h^2} - \frac{5T^3}{2h^3} + O(h^{-4}) \quad \text{as } h \to \infty \text{ for all } T > 0 \tag{4.3.24}$$

and

$$\sigma_{\mathfrak{R}}^2(h, T) = \begin{cases} \dfrac{T^2}{T^2 - 1} + O(h^4) & \text{as } h \to 0 \text{ for } T > 1, \\ \dfrac{2T^2(1-T)}{h^2} + O(1) & \text{as } h \to 0 \text{ for } 0 < T < 1. \end{cases} \tag{4.3.25}$$

4.3.1.4 Limit as $h \to \infty$

As $h \to \infty$, using (3.1.33) and $s_k(z) = z^{-k} + O(z^{-k-2})$ as $z \to \infty$, we find that

$$\frac{h^2 T s_2(\gamma_0) \mathcal{S}_N(\gamma_0; 2)}{T s_2(\gamma_0) + 2h^2 s_3(\gamma_0)} \simeq \frac{T \sum_{i=1}^N (n_i^2 - 1)}{2h\sqrt{N}}. \tag{4.3.26}$$

Thus, we see that, for every $T > 0$, if we take $N \to \infty$ with $h > 0$ and then take $h \to \infty$,

$$\mathfrak{R} \overset{\mathcal{D}}{\simeq} 1 - \frac{T}{h} + \frac{T}{h\sqrt{N}} \left[\frac{\sum_{i=1}^N (n_i^2 - 1)}{2\sqrt{N}} + \sqrt{2}\mathfrak{N} \right]. \tag{4.3.27}$$

4.3.1.5 Limit as $h \to 0$ when $T > 1$

Using (3.1.32), if we take $N \to \infty$ with $h > 0$ and then take $h \to 0$, we see that, for $T > 1$,

$$\mathfrak{R} \overset{\mathfrak{D}}{\simeq} \frac{h^2}{T^2-1} - \frac{2T^2h^4}{(T^2-1)^2} + \frac{1}{\sqrt{N}}\left[\frac{T}{\sqrt{T^2-1}}\mathfrak{N} + h^2\mathcal{S}_N(T+\frac{1}{T};2)\right]. \tag{4.3.28}$$

4.3.1.6 Limit as $h \to 0$ when $T < 1$

Similarly, from (3.1.32), if we take $N \to \infty$ with $h > 0$ and then take $h \to 0$, we see that, for $0 < T < 1$,

$$\mathfrak{R} \overset{\mathfrak{D}}{\simeq} (1-T) + \frac{Th^2}{2(1-T)} + \frac{T}{h\sqrt{N}}\left[\frac{h^5\mathcal{S}_N(\gamma_0;2)}{2(1-T)^2} + \sqrt{2(1-T)}\mathfrak{N}\right]. \tag{4.3.29}$$

From the discussions around the equation (4.1.37), we expect that $h^5\mathcal{S}_N(\gamma_0;2) = \mathcal{O}(1)$ as $h \to 0$ if $h \gg N^{-1/6}$. This indicates that there may be a transition when $h \sim N^{-1/6}$. We study this regime in the next subsection. On the other hand, the thermal fluctuation term becomes of order 1 if $h^{-1}N^{-1/2} = O(1)$. This indicates a new regime $h \sim N^{-1/2}$, which we study in a later section.

4.3.2 Mescoscopic external field: $h \sim N^{-1/6}$ and $T < 1$

4.3.2.1 Analysis

Set

$$h = HN^{-1/6} \tag{4.3.30}$$

for fixed $H > 0$. It turns out that the order of the fluctuations of $\mathfrak{R}$ is $N^{-1/3}$. Hence, we set

$$\eta = \beta\xi N^{1/3} \quad \text{so that} \quad a = \xi N^{-2/3}. \tag{4.3.31}$$

The critical point of $\mathcal{G}(z)$ is given by $\gamma = \lambda_1 + sN^{-2/3}$ where $s > 0$ solves the equation (3.3.3). Inserting $h = HN^{-1/6}$, the equation takes the form

$$\beta - \frac{1}{N^{1/3}}\sum_{i=1}^{N}\frac{1}{s+a_1-a_i} - H^2\beta\sum_{i=1}^{N}\frac{n_i^2}{(s+a_1-a_i)^2} = 0. \tag{4.3.32}$$

The solution satisfies $s = t + \mathcal{O}\left(N^{-1/3}\right)$ where t solves the equation (3.3.14).

For the critical point of $\mathcal{G}_{\mathfrak{R}}$, Lemma 4.3.1) shows that $\gamma < \gamma_{\mathfrak{R}} < \gamma + a$. Hence, $\gamma_{\mathfrak{R}} - \gamma - a = O(N^{-2/3})$. However, we can get a sharper bound on this difference. The right-hand side of (4.3.8) is $\mathcal{O}\left(aN^{4/3}\right)$ and the bracket term of the left-hand side of the same equation is $\mathcal{O}\left(N^{5/3}\right)$, with the leading contribution coming from the second sum. Hence, we find that

$$\gamma_{\mathfrak{R}} = \gamma + a - \epsilon, \qquad \epsilon = \mathcal{O}\left(N^{-1}\right). \tag{4.3.33}$$

We now evaluate (4.3.7). The first sum B_1 is

$$-\sum_{i=1}^{N}\left[\log\left(1+\frac{2(a-\epsilon)}{\gamma-\lambda_i}-\frac{(2a-\epsilon)\epsilon}{(\gamma-\lambda_i)^2}\right)-\frac{2(a-\epsilon)}{\gamma-\lambda_i}\right] \simeq -\sum_{i=1}^{N}\left[\log\left(1+\frac{2\xi}{s+a_1-a_i}\right)-\frac{2\xi}{s+a_1-a_i}\right]$$

and this sum is $\mathcal{O}(1)$. For the second sum, we get

$$B_2 = 2\xi N^{1/3}H^2\beta\sum_{i=1}^{N}\frac{n_i^2}{(s+a_1-a_i)^2}+\mathcal{O}\left(N^{-1/3}\right).$$

Therefore, $N(\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}};a)-2\mathcal{G}(\gamma))$ is equal to

$$-\sum_{i=1}^{N}\left[\log\left(1+\frac{2\xi}{s+a_1-a_i}\right)-\frac{2\xi}{s+a_1-a_i}\right]+2\xi N^{1/3}H^2\beta\sum_{i=1}^{N}\frac{n_i^2}{(s+a_1-a_i)^2}+\mathcal{O}(N^{-1/3}). \tag{4.3.34}$$

Using the equation (4.3.32) for s, we can write

$$N(\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}};a)-2\mathcal{G}(\gamma)) = 2\xi\beta N^{1/3}-\sum_{i=1}^{N}\log\left(1+\frac{2\xi}{s+a_1-a_i}\right)+\mathcal{O}(N^{-1/3}). \tag{4.3.35}$$

Finally, we compute the integrals in (4.3.2). A calculation similar to the one from Subsection 3.3.1 shows that the kth partial derivatives of $\mathcal{G}_{\mathfrak{R}}$ evaluated at $(z,w)=(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}})$ are $\mathcal{O}\left(N^{\frac{2}{3}k-\frac{2}{3}}\right)$. Since the second derivatives are $\mathcal{O}\left(N^{\frac{2}{3}}\right)$, the main contribution to the integral comes from a neighborhood of radius $N^{-5/6}$ around the critical point. Moreover, from explicit computations, we find that

$$\frac{\partial^2\mathcal{G}_{\mathfrak{R}}}{\partial z^2}(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}}) = \frac{\partial^2\mathcal{G}_{\mathfrak{R}}}{\partial w^2}(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}}) \simeq xN^{2/3}, \qquad \frac{\partial^2\mathcal{G}_{\mathfrak{R}}}{\partial z\partial w}(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}}) \simeq yN^{2/3}$$

where

$$x = 2H^2\beta \sum_{i=1}^{N} \frac{n_i^2(s+a_1-a_i+\xi)}{(s+a_1-a_i)^3(s+a_1-a_i+2\xi)}, \qquad y = 2H^2\beta \sum_{i=1}^{N} \frac{n_i^2\xi}{(s+a_1-a_i)^3(s+a_1-a_i+2\xi)}.$$

Using the method of steepest descent with the change of variables $z = \gamma_{\mathfrak{R}} + uN^{-5/6}$ and $w = \gamma_{\mathfrak{R}} + vN^{-5/6}$, the integral becomes

$$\int_{\gamma_{\mathfrak{R}}+\mathrm{i}\mathbb{R}} \int_{\gamma_{\mathfrak{R}}+\mathrm{i}\mathbb{R}} e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{R}}(z,w;a)-\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}};a))}\mathrm{d}z\mathrm{d}w \simeq \frac{1}{N^{5/3}} \int_{\mathrm{i}\mathbb{R}} \int_{\mathrm{i}\mathbb{R}} e^{\frac{1}{4}(xu^2+xv^2+2yuv)}\mathrm{d}u\mathrm{d}v. \qquad (4.3.36)$$

Evaluating the Gaussian integral, inserting the formulas of x and y, and noting that the denominator is the same as the numerator when $\xi = 0$, the ratio of the integrals becomes

$$\frac{\int\int e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{R}}(z,w;a)-\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}};a))}\mathrm{d}z\mathrm{d}w}{\left(\int e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{R}}(z)-\mathcal{G}(\gamma))}\mathrm{d}z\right)^2} \simeq \sqrt{\frac{\sum_{i=1}^{N} \frac{n_i^2}{(s+a_1-a_i)^3}}{\sum_{i=1}^{N} \frac{n_i^2}{(s+a_1-a_i)^2(s+a_1-a_i+2\xi)}}}. \qquad (4.3.37)$$

Combining the above calculations and replacing s by t, we obtain the following result after moving a term of order $N^{1/3}$.

Result 4.3.4. *For $h = HN^{-1/6}$ and $0 < T < 1$,*

$$\left\langle e^{\frac{1}{T}\xi N^{1/3}(\mathfrak{R}-(1-T))}\right\rangle \simeq e^{\xi N^{1/3}-\frac{1}{2}\sum_{i=1}^{N}\log\left(1+\frac{2\xi}{t+a_1-a_i}\right)} \sqrt{\frac{\sum_{i=1}^{N} \frac{n_i^2}{(t+a_1-a_i)^3}}{\sum_{i=1}^{N} \frac{n_i^2}{(t+a_1-a_i)^2(t+a_1-a_i+2\xi)}}} \qquad (4.3.38)$$

as $N \to \infty$ for asymptotically almost every disorder sample, where $t > 0$ is the solution of the equation (3.3.14)*.*

The term in the exponent on the right-hand side is $\mathcal{O}(1)$.

Result 4.3.5. *For $h = HN^{-1/6}$ and $0 < T < 1$,*

$$\mathfrak{R} \overset{\mathfrak{D}}{\simeq} 1 - T + \frac{T}{N^{1/3}}\Upsilon_N(t) \qquad (4.3.39)$$

as $N \to \infty$ for asymptotically almost every disorder sample, where $t > 0$ is the solution of the equation (3.3.14) *and $\Upsilon_N(t)$ is a random variable defined by the generating function given by the right-hand side of* (4.3.38)*.*

4.3.2.2 Matching with $h = O(1)$

We take the formal limit of the result (4.3.39) as $H \to \infty$. From (3.3.20), $t \to \infty$. The big square root term of the generating function on the right-hand side of (4.3.38) is approximately 1. On the other hand,

$$\xi N^{1/3} - \frac{1}{2}\sum_{i=1}^{N} \log\left(1 + \frac{2\xi}{t + a_1 - a_i}\right) \simeq \xi\left(N^{1/3} - \sum_{i=1}^{N} \frac{1}{t + a_1 - a_i}\right) + \xi^2 \sum_{i=1}^{N} \frac{1}{(t + a_1 - a_i)^2}$$

Setting $x = \lambda_1 + tN^{-2/3}$, we have, using a formal application of the semi-circle law,

$$N^{1/3} - \sum_{i=1}^{N} \frac{1}{t + a_1 - a_i} = N^{1/3}\left(1 - \frac{1}{N}\sum_{i=1}^{N} \frac{1}{x - \lambda_i}\right) \simeq N^{1/3}\left(1 - s_1(x)\right).$$

Using (2.1.8), the above equation becomes

$$N^{1/3} - \sum_{i=1}^{N} \frac{1}{t + a_1 - a_i} \simeq N^{1/3}\sqrt{x - 2} \simeq \sqrt{t}.$$

For the other term,

$$\sum_{i=1}^{N} \frac{1}{(t + a_1 - a_i)^2} = \frac{1}{N^{4/3}}\sum_{i=1}^{N} \frac{1}{(x - \lambda_i)^2} \simeq \frac{1}{N^{1/3}} s_2(x) \simeq \frac{1}{N^{1/3} 2\sqrt{x - 2}} \simeq \frac{1}{2\sqrt{t}}.$$

Hence, the generating function on the right-hand side of (4.3.38) is approximately $e^{\sqrt{t}\xi + \frac{\xi^2}{2\sqrt{t}}}$. Therefore,

$$\Upsilon_N(t) \overset{\mathfrak{D}}{\simeq} \sqrt{t} + t^{-1/4}\mathfrak{N}$$

for a thermal standard normal random variable $\mathfrak{N}$. Inserting the large H formula (3.3.20) for t and replacing $H = hN^{1/6}$, we find that if we take $h = HN^{-1/6}$ and let $N \to \infty$ first and then take $H \to \infty$, we get

$$\mathfrak{R} \overset{\mathfrak{D}}{\simeq} 1 - T + \frac{Th^2}{2(1 - T)} + \frac{T}{hN^{1/2}}\left[\frac{h^5 \mathcal{S}_N(\gamma_0; 2)}{2(1 - T)^2} + \sqrt{2(1 - T)}\mathfrak{N}\right]. \qquad (4.3.40)$$

This is the same as (4.3.29) which is obtained by first taking $N \to \infty$ with $h > 0$ fixed and then taking $h \to 0$. Therefore, the result matches with the $h = O(1)$ case.

4.3.2.3 Limit as $H \to 0$

From (3.3.19), $t = O(H) \to 0$ as $H \to 0$. The generating function on the right-hand side of (4.3.38) converges to

$$e^{\xi N^{1/3} - \frac{1}{2}\sum_{i=2}^{N} \log\left(1+\frac{2\xi}{t+a_1-a_i}\right)}$$

where the term $i = 1$ cancels out with the limit of the big square root term. Using the moment generating function (4.2.12) for the chi-squared distribution, we find that if we take $h = HN^{-1/6}$ and $N \to \infty$ and then take $H \to 0$, then

$$\mathfrak{R} \stackrel{\mathfrak{D}}{\simeq} 1 - T + \frac{T}{N^{1/3}} \left(N^{1/3} - \sum_{i=2}^{N} \frac{\mathfrak{n}_i^2}{a_1 - a_i} \right). \tag{4.3.41}$$

for independent thermal standard Gaussian random variables $\mathfrak{n}_i$.

4.3.3 Microscopic external field: $h \sim HN^{-1/2}$ and $T < 1$

4.3.3.1 Analysis

Set

$$h = HN^{-1/2} \tag{4.3.42}$$

for fixed $H > 0$. It turns out that the fluctuations are of order $\mathcal{O}(1)$. In other words, the leading term of $\mathfrak{R}$ converges to a random variable. We set

$$\eta = \beta\xi \quad \text{so that} \quad a = \xi N^{-1}. \tag{4.3.43}$$

The critical point of $\mathcal{G}(z)$ is $\gamma = \lambda_1 + pN^{-1}$ from (4.1.48). Consider the critical point of $\mathcal{G}_{\mathfrak{R}}$. Lemma 4.3.1 implies that $\gamma_{\mathfrak{R}} = \lambda_1 + \mathcal{O}(N^{-1})$. We set

$$\gamma_{\mathfrak{R}} = \lambda_1 + q_{\mathfrak{R}} N^{-1}, \qquad q_{\mathfrak{R}} > |\xi|, \tag{4.3.44}$$

for some $q_{\mathfrak{R}}$. Separating $i = 1$ in the equation (4.3.6), we find that $q_{\mathfrak{R}}$ is the solution of the equation

$$\beta - 1 - \frac{q_{\mathfrak{R}}}{q_{\mathfrak{R}}^2 - \xi^2} - \frac{H^2\beta n_1^2}{(q_{\mathfrak{R}} - \xi)^2} + \mathcal{O}\left(N^{-1/3}\right) = 0. \tag{4.3.45}$$

When $\beta = T^{-1} > 1$, the equation $\beta - 1 - \frac{x}{x^2-\xi^2} - \frac{H^2\beta n_1^2}{(x-\xi)^2} = 0$ has a unique solution for x and $q_{\mathfrak{R}}$ is approximated by this solution with error $\mathcal{O}\left(N^{-1/3}\right)$.

Using (4.3.7) and separating out the $i = 1$ term, we find that $N(\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}}; a) - 2\mathcal{G}(\gamma))$ is equal to

$$-\log\left(\frac{q_{\mathfrak{R}}^2 - \xi^2}{p^2}\right) + \frac{2(q_{\mathfrak{R}} - p)}{p} + 2H^2\beta n_1^2\left[\frac{1}{q_{\mathfrak{R}} - \xi} - \frac{1}{p} + \frac{q_{\mathfrak{R}} - p}{p^2}\right] + \mathcal{O}\left(N^{-1/3}\right). \quad (4.3.46)$$

Using the fact that p satisfies equation (4.1.50), this can be written as

$$\begin{aligned} &N(\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}}; a) - 2\mathcal{G}(\gamma)) \\ &= -\log\left(\frac{q_{\mathfrak{R}}^2 - \xi^2}{p^2}\right) + 2(\beta - 1)(q_{\mathfrak{R}} - p) + 2H^2\beta n_1^2\left[\frac{1}{q_{\mathfrak{R}} - \xi} - \frac{1}{p}\right] + \mathcal{O}\left(N^{-1/3}\right). \end{aligned} \quad (4.3.47)$$

We now consider the integrals in (4.3.2). As in Subsection 4.1.6 of the overlap with the external field when $h \sim N^{-1/2}$, the main contribution to the integral comes from a neighborhood of radius N^{-1} around the critical point in both variables. Changing variables to $z = \gamma_{\mathfrak{R}} + uN^{-1}$ and $w = \gamma_{\mathfrak{R}} + vN^{-1}$, we find that all terms of the Taylor series are of the same order, so we see, as in Subsection 4.1.6, that the integral is not approximated by a Gaussian integral. Therefore, we proceed by writing

$$\begin{aligned} &N(\mathcal{G}_{\mathfrak{R}}(z, w; a) - \mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}}; a)) \\ &= N(\mathcal{G}_{\mathfrak{R}}(z, w; a) - \mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}}; a) - (\mathcal{G}_{\mathfrak{R}})_z(\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}}; a)(z - \gamma_{\mathfrak{R}}) - (\mathcal{G}_{\mathfrak{R}})_w(\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}}; a)(w - \gamma_{\mathfrak{R}})) \\ &= -\sum_{i=1}^{N}\left[\log\left(\frac{(z - \lambda_i)(w - \lambda_i) - a^2}{(\gamma_{\mathfrak{R}} - \lambda_i)^2 - a^2}\right) - \frac{(\gamma_{\mathfrak{R}} - \lambda_i)(z + w - 2\gamma_{\mathfrak{R}})}{(\gamma_{\mathfrak{R}} - \lambda_i)^2 - a^2}\right] \\ &\quad + h^2\beta\sum_{i=1}^{N} n_i^2\left[\frac{z + w - 2\lambda_i + 2a}{(z - \lambda_i)(w - \lambda_i) - a^2} - \frac{2}{\gamma_{\mathfrak{R}} - \lambda_i - a} + \frac{z + w - 2\gamma_{\mathfrak{R}}}{(\gamma_{\mathfrak{R}} - \lambda_i - a)^2}\right]. \end{aligned}$$

Inserting the change of variables and separating $i = 1$ out,

$$\begin{aligned} N(\mathcal{G}_{\mathfrak{R}}(z, w; a) - \mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}}; a)) \simeq &-\log\left(\frac{(u + q_{\mathfrak{R}})(v + q_{\mathfrak{R}}) - \xi^2}{q_{\mathfrak{R}}^2 - \xi^2}\right) + \frac{q_{\mathfrak{R}}(u + v)}{q_{\mathfrak{R}}^2 - \xi^2} \\ &+ H^2\beta n_1^2\left[\frac{u + v + 2q_{\mathfrak{R}} + 2\xi}{(u + q_{\mathfrak{R}})(v + q_{\mathfrak{R}}) - \xi^2} - \frac{2}{q_{\mathfrak{R}} - \xi} + \frac{u + v}{(q_{\mathfrak{R}} - \xi)^2}\right] \end{aligned}$$

for finite u and v. Using the equation (4.3.45), this can be written as

$$\begin{aligned} &N(\mathcal{G}_{\mathfrak{R}}(z, w; a) - \mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}}; a)) \\ &\simeq -\log\left(\frac{(u + q_{\mathfrak{R}})(v + q_{\mathfrak{R}}) - \xi^2}{q_{\mathfrak{R}}^2 - \xi^2}\right) + (\beta - 1)(u + v) + H^2\beta n_1^2\left[\frac{u + v + 2q_{\mathfrak{R}} + 2\xi}{(u + q_{\mathfrak{R}})(v + q_{\mathfrak{R}}) - \xi^2} - \frac{2}{q_{\mathfrak{R}} - \xi}\right]. \end{aligned}$$

Thus, the numerator integral in (4.3.2) is asymptotically equal to

$$\frac{\sqrt{q_{\mathfrak{R}}^2-\xi^2}}{N^2}\int\int \frac{e^{\frac{1}{2}(\beta-1)(u+v)+\frac{H^2\beta n_1^2}{2}\left[\frac{u+v+2q_{\mathfrak{R}}+2\xi}{(u+q_{\mathfrak{R}})(v+q_{\mathfrak{R}})-\xi^2}-\frac{2}{q_{\mathfrak{R}}-\xi}\right]}}{\sqrt{(u+q_{\mathfrak{R}})(v+q_{\mathfrak{R}})-\xi^2}}\mathrm{d}u\mathrm{d}v$$

where the contours are from $-\mathrm{i}\infty$ to $\mathrm{i}\infty$ such that all singularities lie on the left of the contours. The denominator integral is the same with $\xi = 0$.

Combining the above calculations into (4.3.2) and making simple translations for the integral, we find that

$$\langle e^{\beta\xi\mathfrak{R}}\rangle \simeq \frac{\int\int \frac{1}{\sqrt{uv-\xi^2}}e^{\frac{1}{2}(\beta-1)(u+v)+\frac{H^2\beta n_1^2(u+v+2\xi)}{2(uv-\xi^2)}}\mathrm{d}u\mathrm{d}v}{\left(\int \frac{1}{\sqrt{u}}e^{\frac{1}{2}(\beta-1)u+\frac{H^2\beta n_1^2}{2u}}\mathrm{d}u\right)^2} \tag{4.3.48}$$

where the contours are upward vertical lines such that the points ξ (in the numerator) and 0 (in the denominator) lie on the left of the contours. We now evaluate the integrals using (recall (4.1.59))

$$\int \frac{e^{au+\frac{b}{u}}}{\sqrt{u}}\mathrm{d}u = \frac{2\mathrm{i}\sqrt{\pi}}{\sqrt{a}}\cosh(2\sqrt{ab}). \tag{4.3.49}$$

Consider the double integral in the numerator. For each v, we change the variable u to z by setting $uv-\xi^2 = z$. We can define the branch cut appropriately such that the contour for z does not cross the branch cut. The numerator becomes

$$\iint \frac{1}{v\sqrt{z}}e^{\frac{\beta-1}{2}(\frac{z+\xi^2}{v}+v)+\frac{H^2\beta n_1^2}{2z}(\frac{z+\xi^2}{v}+v+2\xi)}\mathrm{d}z\mathrm{d}v.$$

The z-integral can be evaluated using (4.3.49). Writing the resulting $\cosh$ term as the sum of two exponentials, we can evaluate the w-integral again using (4.3.49). The above double integral becomes

$$-\frac{2\pi}{\beta-1}\Bigg[e^{\sqrt{(\beta-1)\beta}H|n_1|}\cosh\left(\sqrt{(\beta-1)\beta}H|n_1|+(\beta-1)\xi\right) + e^{-\sqrt{(\beta-1)\beta}H|n_1|}\cosh\left(\sqrt{(\beta-1)\beta}H|n_1|-(\beta-1)\xi\right)\Bigg].$$

Writing $\cosh$ as the sum of two exponentials again, the expression above becomes a linear combination of $e^{(\beta-1)\xi}$ and $e^{-(\beta-1)\xi}$. The denominator in (4.3.48) is the same as the numerator when $\xi = 0$. Thus, using $\beta = 1/T$ and re-scaling ξ, we obtain the following

Result 4.3.6. *For $h = HN^{-1/2}$ and $0 < T < 1$,*

$$\langle e^{\xi \frac{\mathfrak{R}}{1-T}} \rangle \simeq \frac{\cosh\left(\frac{2\sqrt{1-T}H|n_1|}{T}\right) e^{\xi} + e^{-\xi}}{\cosh\left(\frac{2\sqrt{1-T}H|n_1|}{T}\right) + 1} \tag{4.3.50}$$

as $N \to \infty$ for asymptotically almost every disorder sample.

Recognizing that the right-hand side is the moment generating function of a shifted Bernoulli random variable, we obtain the following result.

Result 4.3.7. *For $h = HN^{-1/2}$ and $0 < T < 1$,*

$$\frac{\mathfrak{R}}{1-T} \overset{\mathfrak{D}}{\simeq} \mathfrak{B}(\theta), \qquad \theta := \frac{\cosh\left(\frac{2\sqrt{1-T}H|n_1|}{T}\right)}{\cosh\left(\frac{2\sqrt{1-T}H|n_1|}{T}\right) + 1} \tag{4.3.51}$$

as $N \to \infty$ for asymptotically almost every disorder sample, where the thermal random variable $\mathfrak{B}(c)$ is the (shifted) Bernoulli distribution taking values 1 and -1 with probability c and $1 - c$, respectively.

4.3.3.2 Limits as $H \to \infty$

If we formally take the limit as $H \to \infty$ of the result (4.3.51), then

$$\mathfrak{R} \overset{\mathfrak{D}}{\simeq} 1 - T. \tag{4.3.52}$$

This is the same as the leading term of (4.3.41) which is obtained by taking $h = HN^{-1/6}$ and letting $N \to \infty$ first and then taking $H \to 0$.

4.3.4 No external field: $h = 0$

For $0 < T < 1$, the analysis in Subsection 4.3.3 for $h = HN^{-1/2}$ extends to $H = 0$ case as well. For $T > 1$, the analysis in Subsection (4.3.1) applies to all $h \geq 0$. We note that, for $h = 0$ and $T > 1$, $\gamma_0 = T + T^{-1}$ and $s_2(\gamma_0) = \frac{1}{T^2-1}$. We have the following result.

Result 4.3.8. *For $h = 0$,*

$$\mathfrak{R} \overset{\mathfrak{D}}{\simeq} \begin{cases} \frac{T}{\sqrt{N(T^2-1)}} \mathfrak{N} & \text{for } T > 1, \\ (1-T)\,\mathfrak{B}(1/2) & \text{for } 0 < T < 1. \end{cases} \tag{4.3.53}$$

4.4 Geometry of the spin configuration

The results on three types of overlaps tell us how the spin variables are distributed on the sphere. We discuss the geometry of the spin configuration vector $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_N)$ from the Gibbs measure in this section. Recall that $\mathbf{u}_1$ is a unit vector which is parallel to the eigenvector corresponding to the largest eigenvalue of the disorder matrix. In this section, we choose $\mathbf{u}_1$, among two opposite directions, as the one satisfying $\mathbf{u}_1 \cdot \mathbf{g} \geq 0$. Recall the notation $n_1 = \mathbf{g} \cdot \mathbf{u}_1$ and that the external field $\mathbf{g}$ is a standard Gaussian vector. Note that $n_1 = |n_1|$ because of the choice of $\mathbf{u}_1$. The normalized spin vector can be decomposed as

$$\hat{\boldsymbol{\sigma}} := \frac{\boldsymbol{\sigma}}{\sqrt{N}} = a\mathbf{u}_1 + b\frac{\mathbf{g} - n_1\mathbf{u}_1}{\|\mathbf{g} - n_1\mathbf{u}_1\|} + \mathbf{v}, \qquad \mathbf{v} \cdot \mathbf{u}_1 = \mathbf{v} \cdot \mathbf{g} = 0, \tag{4.4.1}$$

where a and b are components of the normalized spin vector in the $\mathbf{u}_1$ and $\mathbf{g}_1 - n_1\mathbf{u}_1$ directions, respectively. The vector $\mathbf{v}$ is perpendicular to both $\mathbf{u}_1$ and $\mathbf{g}$, and it satisfies

$$\|\mathbf{v}\|^2 = 1 - a^2 - b^2. \tag{4.4.2}$$

Note that $\|\mathbf{g} - n_1\mathbf{u}_1\|^2 = \|\mathbf{g}\|^2 - n_1^2 \simeq N + \mathcal{O}\left(N^{1/2}\right)$ and $n_1 = \mathcal{O}(1)$. Thus, if we ignore subleading terms from each component, the above decomposition becomes

$$\hat{\boldsymbol{\sigma}} \simeq a\mathbf{u}_1 + b\frac{\mathbf{g}}{\sqrt{N}} + \mathbf{v} = a\mathbf{u}_1 + b\hat{\mathbf{g}} + \mathbf{v}, \qquad \hat{\mathbf{g}} := \frac{\mathbf{g}}{\sqrt{N}}. \tag{4.4.3}$$

The components a and b are related to the overlaps by the formulas

$$\mathfrak{O} = (\hat{\boldsymbol{\sigma}} \cdot \mathbf{u}_1)^2 = a^2, \qquad \mathfrak{M} = \hat{\boldsymbol{\sigma}} \cdot \hat{\mathbf{g}} = \frac{an_1}{\sqrt{N}} + b\frac{\|\mathbf{g} - n_1\mathbf{u}_1\|}{\sqrt{N}} \simeq \frac{an_1}{\sqrt{N}} + b \tag{4.4.4}$$

up to $\mathcal{O}(N^{-1})$ terms. Furthermore, $\mathbf{v}$ satisfies the equation

$$\mathfrak{R} = \hat{\boldsymbol{\sigma}}^{(1)} \cdot \hat{\boldsymbol{\sigma}}^{(2)} = a_1a_2 + b_1b_2 + \mathbf{v}^{(1)} \cdot \mathbf{v}^{(2)}. \tag{4.4.5}$$

4.4.1 The signed overlap with a replica for the microscopic field, $h \sim N^{-1/2}$ and $T < 1$

Consider the decomposition for $h = HN^{-1/2}$ and $0 < T < 1$. The overlap with the ground state is given in Result 4.2.6 for $h \sim N^{-1/3}$ and Result 4.2.7 for $h = 0$. Since the leading terms of the both results are same, given by $1 - T$, the leading term holds also for $h \sim N^{-1/2}$. Thus, we find that $a^2 \simeq 1 - T$ in this regime, and hence $|a| \simeq \sqrt{1 - T}$. On the other hand, Result 4.1.7 on $\mathfrak{M}$

implies that

$$\frac{an_1}{\sqrt{N}} + b \overset{\mathfrak{D}}{\simeq} h + \frac{|n_1|\sqrt{1-T}\mathfrak{B}(\alpha)}{\sqrt{N}} + \frac{\sqrt{T}\mathfrak{N}}{\sqrt{N}}. \tag{4.4.6}$$

Noting $h \sim N^{-1/2}$, we find that $b = \mathcal{O}\left(N^{-1/2}\right)$. From the formulas of a and b, we also find that $\|\mathbf{v}\|^2 = 1 - a^2 - b^2 \simeq T$. Finally, Result 4.3.7 implies that

$$a_1 a_2 + b_1 b_2 + \mathbf{v}^{(1)} \cdot \mathbf{v}^{(2)} \overset{\mathfrak{D}}{\simeq} (1-T)\mathfrak{B}(\theta). \tag{4.4.7}$$

Here, θ is given in (4.3.51) and α in (4.4.6) is given by (4.1.64). They satisfy the relation $\theta = \alpha^2 + (1-\alpha)^2$. Now, we make the following ansatz on a. For $h = 0$ and $0 < T < 1$, the spin configurations are equally likely to be on either of the double cones around $\mathbf{u}_1$ with the cosine of the angle given by $\sqrt{1-T}$. This means that $a \overset{\mathfrak{D}}{\simeq} \sqrt{1-T}\,\mathfrak{B}(1/2)$ for $h = 0$ and $0 < T < 1$. For $h \sim N^{-1/2}$, we make the ansatz that

$$a = \hat{\boldsymbol{\sigma}} \cdot \mathbf{u}_1 \overset{\mathfrak{D}}{\simeq} \sqrt{1-T}\,\mathfrak{B}(\varphi) \tag{4.4.8}$$

for some φ which we determine now. Note that if X_1 and X_2 are independent (thermal) random variables distributed as $\mathfrak{B}(\varphi)$, then their product $X_1 X_2$ is $\mathfrak{B}(\varphi^2 + (1-\varphi)^2)$-distributed. Thus, the equations (4.4.6) and (4.4.7) become

$$\frac{|n_1|\sqrt{1-T}\,\mathfrak{B}(\varphi)}{\sqrt{N}} + b \overset{\mathfrak{D}}{\simeq} h + \frac{|n_1|\sqrt{1-T}\mathfrak{B}(\alpha)}{\sqrt{N}} + \frac{\sqrt{T}\mathfrak{N}}{\sqrt{N}}$$

and

$$(1-T)\,\mathfrak{B}(\varphi^2 + (1-\varphi)^2) + \mathcal{O}\left(N^{-1}\right) + \mathbf{v}^{(1)} \cdot \mathbf{v}^{(2)} \overset{\mathfrak{D}}{\simeq} (1-T)\mathfrak{B}(\theta).$$

Since $\theta = \alpha^2 + (1-\alpha)^2$, it is reasonable to assume that the solutions are $\varphi = \alpha$, and

$$a \overset{\mathfrak{D}}{\simeq} \sqrt{1-T}\,\mathfrak{B}(\alpha), \qquad b \overset{\mathfrak{D}}{\simeq} h + \frac{\sqrt{T}\mathfrak{N}}{\sqrt{N}}.$$

This calculation leads us to the following conjecture on the signed overlap of the spin variable with a replica.

Conjecture 4.4.1. *For a given disorder sample, let $\mathbf{u}_1$ be the unit vector corresponding to the ground state such that $\mathbf{u}_1 \cdot \mathbf{g} \geq 0$. Then, for $h = HN^{-1/2}$ and $0 < T < 1$, the signed overlap with* *the ground state satisfies*

$$\frac{\boldsymbol{\sigma}\cdot\mathbf{u}_1}{\sqrt{N}} \overset{\mathfrak{D}}{\simeq} \sqrt{1-T}\,\mathfrak{B}(\alpha), \qquad \alpha = \frac{e^{\frac{H|n_1|\sqrt{1-T}}{T}}}{e^{\frac{H|n_1|\sqrt{1-T}}{T}} + e^{-\frac{H|n_1|\sqrt{1-T}}{T}}}, \tag{4.4.9}$$

as $N \to \infty$ *for asymptotically almost every disorder sample.*

The above conjecture implies that for $h = HN^{-1/2}$ the spin configuration vector concentrates on the intersection of the sphere and the double cone around $\mathbf{u}_1$ where the cosine of the angle is $\sqrt{1-T}$, just like the $h = 0$ case. However, while for $H = 0$ the spin vector is equally likely to be on either of the cones, for $H > 0$ the spin prefers the cone that is closer to g than the other cone. As $H \to \infty$, the polarization parameter $\alpha \to 1$ and hence for $h \gg N^{-1/2}$, the spin vector is concentrated on one of the cones.

4.4.2 Spin decompositions in various regimes

The results of the overlaps give us information about the decomposition of the spin for other regimes of h as well. From the first equation of (4.4.4), we find a^2, and hence $|a|$. The discussion of the previous subsection implies that for $h \gg N^{-1/2}$, the spin vector concentrates on one of the cones. Thus, we expect that $a = |a|$ for such h. Using this formula of a, we then obtain b from the second equation of (4.4.4), from which we also find $\|\mathbf{v}\|^2 = 1 - a^2 - b^2$. Finally, the equation (4.4.5) implies $\mathbf{v}^{(1)}\cdot\mathbf{v}^{(2)}$, and hence, the overlap $\hat{\mathbf{v}}^{(1)}\cdot\hat{\mathbf{v}}^{(2)}$ of the unit transversal vector $\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$ with its replica. We summarize the findings in Table IV.1. The result for the last row follows from the last subsection.

The result for the regime $h \sim N^{-1/6}$ (fourth row) follows from Results 4.2.4, 4.1.5, and 4.3.5. The term $\mathcal{A} = \mathcal{A}(T, hN^{1/6})$ is given by the leading term in Result 4.2.4,

$$\mathcal{A} = \sqrt{1 - T - h^2N^{1/3}\sum_{i=2}^{N}\frac{n_i^2}{(t + a_1 - a_i)^2}} = \frac{hN^{1/6}|n_1|}{t}, \tag{4.4.10}$$

where $t > 0$ is the number that makes the two formulas of $\mathcal{A}$ equal. For every disorder sample, $\mathcal{A}$ is a decreasing function of $H = hN^{1/6}$, changing from $\sqrt{1-T}$ for $H = 0$ to 0 as $H \to \infty$.

The result for the regime $h = O(1)$ (second row) follows from Result 4.2.2, 4.1.2, and 4.3.3. The variable $\gamma_0 = \gamma_0(T, h) > 2$ is the solution of the equation (3.1.26). It satisfies $\gamma_0 \simeq h + \frac{T}{2}$ as $h \to \infty$ and $\gamma_0 \simeq 2 + \frac{h^4}{4(1-T)^2}$ as $h \to 0$: See Lemma 3.1.7. The function $s_1(z)$ is the Stieltjes transform of the semicircle law. It satisfies $s_1(z) = z^{-1} + O(z^{-3})$ as $z \to \infty$ and $s_1(z) \simeq 1 - \sqrt{z-2}$ as

Case	$a = \hat{\boldsymbol{\sigma}} \cdot \mathbf{u}_1$	$b \simeq \hat{\boldsymbol{\sigma}} \cdot \hat{\mathbf{g}}$	$\|\mathbf{v}\|$	$\hat{\mathbf{v}}^{(1)} \cdot \hat{\mathbf{v}}^{(2)}$
$h \to \infty$	0	1	0	0
$h = O(1)$	$\frac{\sqrt{\mathfrak{D}^0}}{\sqrt{N}}$	$hs_1(\gamma_0)$	$\sqrt{1 - h^2 s_1(\gamma_0)^2}$	$\frac{h^2 s_1(\gamma_0)^4}{(1-s_1(\gamma_0)^2)(1-h^2 s_1(\gamma_0)^2)}$
$h \to 0$, $hN^{\frac{1}{6}} \to \infty$	$\frac{4(1-T)^2 \|n_1\|}{h^3\sqrt{N}}$	h	1	$1 - T$
$h \sim N^{-\frac{1}{6}}$	$\mathcal{A}(T, hN^{1/6})$	h	$\sqrt{1 - \mathcal{A}^2}$	$\frac{1-T-\mathcal{A}^2}{1-\mathcal{A}^2}$
$hN^{\frac{1}{6}} \to 0$, $hN^{\frac{1}{2}} \to \infty$	$\sqrt{1-T}$	h	$\sqrt{T}$	$o(1)$
$h \sim N^{-\frac{1}{2}}$ (and $h = 0$)	$\sqrt{1-T}\,\mathfrak{B}(\alpha)$	$h + \frac{\sqrt{T}\mathfrak{N}}{\sqrt{N}}$	$\sqrt{T}$	$o(1)$

Table IV.1: This table summarized the findings of the decomposition of the spin variable $\hat{\boldsymbol{\sigma}} \simeq a\mathbf{u}_1 + b\hat{\mathbf{g}} + \mathbf{v}$ in different regimes for $0 < T < 1$. We indicate the leading order terms, except that we have $o(1)$ at two places. The $o(1)$ term in the fifth row is complicated to state and the $o(1)$ term in the last row is not determined from our analysis. The unit transversal vector is $\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$.

$z \to 2$: see (2.1.8). See Sub-subsection 4.1.3.2 for properties of $\mathfrak{M}^0 = hs_1(\gamma_0)$. The term $\frac{\sqrt{\mathfrak{D}^0}}{\sqrt{N}}$ is from Result 4.2.2 and is given by

$$\frac{\sqrt{\mathfrak{D}^0}}{\sqrt{N}} = \frac{1}{\sqrt{N}} \left| \frac{h|n_1|}{\gamma_0 - 2} + \frac{\sqrt{T}\mathfrak{N}}{\sqrt{\gamma_0 - 2}} \right|. \tag{4.4.11}$$

For the last column, Result 4.3.3 and the formula $b \simeq hs_1(\gamma_0)$ imply that $\mathbf{v}^{(1)} \cdot \mathbf{v}^{(2)} \simeq h^2 s_2(\gamma_0) - h^2 s_1(\gamma_0)^2$. We use the identity $s_2(z) = s_1(z)^2/(1 - s_1(z)^2)$ for $z > 0$ to simplify the formula.

The third row follows either from the fourth row or from the second row. Starting from the fourth row, we use (4.2.26), which shows that

$$\mathcal{A}^2 \simeq \frac{16(1-T)^4 n_1^2}{h^6 N} \tag{4.4.12}$$

as $hN^{1/6} \to \infty$. We can also see this formula from (4.4.10) because $t \simeq \frac{h^4 N^{2/3}}{4(1-T)^2}$ (see (3.3.20)). Note that $\mathcal{A} = o(1)$ in this regime. On the other hand, if we start from the second row, we use (4.2.16) to find the same formula for a. Other columns can be found from $s_1(\gamma_0) \simeq 1 - \frac{h^2}{2(1-T)}$ as $h \to 0$. Note that the two components a and b are comparable in size for $h \sim N^{-1/8}$.

The quantity a in the fifth row follows either from the fourth row or from the last row. The formula (4.2.27) shows that $\mathcal{A}^2 \simeq 1 - T$ as $hN^{1/6} \to 0$. We also see this formula from (4.4.10) by dropping the $o(1)$ term. If we start from the last row, the polarization parameter α satisfies $\alpha \to 1$ as $hN^{1/2} \to \infty$, and hence $a \simeq \sqrt{1-T}$, giving the same formula for a. The other columns follow from this result. One can show using Result 4.2.6 and (4.2.41) that the subleading term in a (not shown in Table IV.1) is comparable to the leading term of b, which is h, when $h \sim N^{-1/3}$.

4.4.3 Summary

Three quantities contain thermal random variables: a for the regimes $h = O(1)$ and $h \sim N^{-1/2}$, and b for the regime $h \sim N^{-1/2}$. Among those, a for the regime $h \sim N^{-1/2}$ is $\mathcal{O}(1)$ but the other two quantities are of smaller order $\mathcal{O}\left(N^{-1/2}\right)$.

The table shows that $a = O(1)$ for $h \leq O(N^{-1/6})$ and $b = O(1)$ for $h \geq O(1)$. As h increases, the $\mathbf{u}_1$ component of a typical spin vector decreases while the $\hat{\mathbf{g}}$ component increases. The above result shows that the crossover occurs in the regime $N^{-1/6} \ll h \ll O(1)$ in which both components are $o(1)$.

The last column of the table is the overlap of the unit transversal vector $\hat{\mathbf{v}}$ with its replica. This overlap is $o(1)$ for $h \ll N^{-1/6}$. If the error were $O(N^{-1/2})$, it would give a strong indication that the thermal distribution of $\hat{\mathbf{v}}$ is uniform on the transverse space (i.e. the set of unit vectors that are perpendicular to $\mathbf{u}_1$ and $\mathbf{g}$). The above result does not show the error, but we expect that the distribution on the transverse space is close to being uniform. On the other hand, for $h \geq O(N^{-1/6})$, the overlap of the unit transversal vector is non-zero and $\mathcal{O}(1)$. This implies that $\hat{\mathbf{v}}$ is not uniformly distributed on the transverse space.

Overall, for $0 < T < 1$, as we increase the external field, we expect the following geometry of the spin vector that is randomly chosen using the Gibbs (thermal) measure for a quenched disorder, i.e. for asymptotically almost every disorder sample.

- For $h \ll N^{-1/6}$, the spin vector is on a double cone around $\mathbf{u}_1$ (possibly preferring one cone to the other), and the thermal distribution on the transverse space is close to being uniform.
- For $h \sim N^{-1/6}$, the spin vector is polarized to a single cone around $\mathbf{u}_1$, but the cone itself depends non-trivially on the disorder sample. The thermal distribution on the transverse space is not uniform and depends on the disorder sample.
- For $N^{-1/6} \ll h \ll O(1)$, the spin vector entirely lies on the transverse space with only $o(1)$ components on the ground state and external field directions. Although the thermal distribution is not uniform, it does not depend on the disorder sample.

- For $h = O(1)$, the spin vector is on a cone around $\mathbf{g}$ and the thermal distribution on the transverse space is not uniform. The cone and the distribution on the transverse space do not depend on the disorder sample.
- For $h \to \infty$, the spin vector is parallel to $\mathbf{g}$.

The results of this thesis do not describe the distribution of $\hat{\mathbf{v}}$ on the transverse space in detail. This can be achieved by studying the overlaps $\sigma \cdot \mathbf{u}_i$ with other eigenvectors. This analysis can be done using the method of this thesis and we leave this work as a future project.

The items in the table can be expressed via a single formula across all regimes by using the following decomposition of the spin configuration vector:

$$\hat{\boldsymbol{\sigma}} \overset{\mathfrak{D}}{\simeq} \mathcal{A}\mathfrak{B}(\alpha)\mathbf{u}_1 + hs_1(\gamma_0)\hat{\mathbf{g}} + \sqrt{1 - \mathcal{A}^2 - h^2 s_1(\gamma_0)^2}\hat{\mathbf{v}} + \mathcal{O}\left(N^{-1/2}\right) \tag{4.4.13}$$

where $\hat{\mathbf{v}}$ is a unit vector in the transverse space, i.e. $\hat{\mathbf{v}} \cdot \mathbf{u}_1 = \hat{\mathbf{v}} \cdot \hat{\mathbf{g}} = 0$ and $\|\hat{\mathbf{v}}\| = 1$. All items in the middle three columns of the table other than two items, a for the regime $h = O(1)$ and b for the regime $h \sim N^{-1/2}$, are of order greater than $\mathcal{O}\left(N^{-1/2}\right)$. Hence, the above formula is meaningful for all items except those two.

CHAPTER V

Detailed Analysis of the Overlaps for a Microscopic External Field

The purpose of this chapter is to rigorously prove our results for the the overlap with the external field and overlap with a replica in the microscopic field regime ($h \sim N^{-1/2}$). The microscopic regime is the most delicate one to analyze and has important implications for magnetic susceptibility as well as the geometry of the Gibbs measure. This chapter focuses on the proofs of Theorems 5.2.1 and 5.3.1. These are similar to Results 4.1.6 and 4.3.6 in the previous chapter, but we provide a more precise statement of each result as well as a rigorous proof. In particular, we specify bounds for the order of the subleading term and the probability with which the result holds.

Section 5.1 provides preliminary lemmas, which include more precise versions of some of the material in Chapter II. The proof of Theorem 5.2.1 can be found in Section 5.2. Sections 5.1 and 5.2 are also published in [13]. In Section 5.3 we prove Theorem 5.3.1. A comparable result was obtained via a different method in [30] (see Theorem 2.14).

5.1 Preliminary lemmas

5.1.1 Eigenvalue spacing

Recall from Chapter II that we define the rescaled eigenvalues

$$a_i := N^{2/3}(\lambda_i - 2) \tag{5.1.1}$$

and, as $N \to \infty$, the rescaled eigenvalues converge in distribution to the GOE Airy point process [47, 41]. We denote this as $\{\alpha_i\}_{i=1}^{\infty}$ satisfying

$$\{a_i\} \Rightarrow \{\alpha_i\}. \tag{5.1.2}$$

Heuristically, we expect that, for $1 \ll i \ll N$,

$$a_i \approx \alpha_i \approx -\left(\frac{3\pi i}{2}\right)^{2/3} \tag{5.1.3}$$

since the semicircle law is asymptotic to $\frac{\sqrt{2-x}}{\pi}\mathrm{d}x$ as $x \to 2$. The above approximation and the rigidity property suggest that,

$$a_i \asymp -i^{2/3} \quad \text{as } i, N \to \infty \text{ satisfying } i \ll N. \tag{5.1.4}$$

For proofs throughout this chapter, we need a more rigorous version of the approximation above, which we obtain in the following lemma.

Lemma 5.1.1. *(adapted from [29]) There exist some integer K and some $c > 0$, which do not depend on N, such that, for all $k > K$, we have*

$$\mathbb{P}\left(\bigcap_{N^{2/5} \geq j \geq k} \left\{a_1 - a_j \geq c j^{2/3}\right\}\right) \geq 1 - \frac{2}{k^{1/2}}. \tag{5.1.5}$$

Proof. In line (6.33) of [29], Landon and Sosoe obtain the result that there exists some K_1 (not depending on N) such that, for all $k > K_1$,

$$\mathbb{P}\left(\bigcap_{N^{2/5} \geq j \geq k} \left\{N^{2/3}(\lambda_j - 2) \leq -\left(\frac{3\pi j}{2}\right)^{2/3} + \frac{1}{10} j^{2/3}\right\}\right) \geq 1 - \frac{1}{k^{1/2}}. \tag{5.1.6}$$

(Note that the original statement of this inequality in the arxiv version of [29] contains a typo, but the result above is what follows from the preceding lines of [29] and we confirmed this with the authors.) Next, we observe that there exists some K' such that, for all $k > K'$, we have

$$\mathbb{P}\left(N^{2/3}(2 - \lambda_1) \leq \frac{1}{10} k^{2/3}\right) \geq 1 - \frac{1}{k^{1/2}} \tag{5.1.7}$$

for N sufficiently large. This comes from the fact that the GOE Tracy-Widom distribution has sub-exponential tails. Neither K_1 nor K' depends on N, so we take K to be the maximum of these two values and, combining (5.1.6) and (5.1.7), we conclude the desired result. $\square$

5.1.2 Special sums

The preliminaries in Chapter II also included convergence results for sums of the form

$$\frac{1}{N}\sum_{i=2}^{N}\frac{1}{(\lambda_1-\lambda_i)^m}. \tag{5.1.8}$$

In particular, recall Landon and Sosoe proved [29] that

$$\Xi_N := N^{1/3}\left(\frac{1}{N}\sum_{i=2}^{N}\frac{1}{\lambda_1-\lambda_i}-1\right)\Rightarrow \Xi \tag{5.1.9}$$

for a random variable Ξ as $N\to\infty$. The limiting random variable Ξ can be expressed in terms of the GOE Airy kernel point process as

$$\Xi = \lim_{n\to\infty}\left(\sum_{i=2}^{n}\frac{1}{\alpha_1-\alpha_i}-\frac{1}{\pi}\int_0^{\left(\frac{3\pi n}{2}\right)^{2/3}}\frac{\mathrm{d}x}{\sqrt{x}}\right) \tag{5.1.10}$$

where the limit exists almost surely.

We also claimed in Chapter II another version of the result (5.1.9) where the constant numerators are replaced n_i^2:

$$N^{1/3}\left(\frac{1}{N}\sum_{i=2}^{N}\frac{n_i^2}{\lambda_1-\lambda_i}-1\right)\Rightarrow \lim_{n\to\infty}\left(\sum_{i=2}^{n}\frac{\nu_i^2}{\alpha_1-\alpha_i}-\frac{1}{\pi}\int_0^{\left(\frac{3\pi n}{2}\right)^{2/3}}\frac{\mathrm{d}x}{\sqrt{x}}\right) \tag{5.1.11}$$

where ν_i are i.i.d standard Gaussians, independent of the GOE Airy point process α_i. This follows from (5.1.9) and the fact that

$$\frac{1}{N^{2/3}}\sum_{i=2}^{N}\frac{n_i^2-1}{\lambda_1-\lambda_i}\Rightarrow\sum_{i=2}^{\infty}\frac{\nu_i^2-1}{\alpha_1-\alpha_i} \tag{5.1.12}$$

which is a convergent series due to Kolmogorov's three series theorem and Lemma 5.1.1. Note that Lemma 5.1.1 enables to verify (5.1.11) rigorously by including details that we omitted in Chapter II.

The last task related to the special sums is to provide a rigorous version of Result 2.2.4 that includes more specific bounds on the order of the sub-leading terms and the probability with which each statement holds (in Result 2.2.4 we simply said "for asymptotically almost every disorder sample”). The precise version is provided in the following two lemmas.

Lemma 5.1.2. *For any $\delta > 0$,*

$$\frac{1}{N}\sum_{i=2}^{N}\frac{1}{\lambda_1-\lambda_i} = 1+O(N^{-\frac{1}{3}+\delta}) \quad \textit{and} \quad \frac{1}{N}\sum_{i=2}^{N}\frac{n_i^2}{\lambda_1-\lambda_i} = 1+O(N^{-\frac{1}{3}+\delta}) \tag{5.1.13}$$

with probability at least $1-N^{-\delta/2}$. (This lemma is adapted from a similar result in [30]).

Proof. Define an event

$$E_\delta := \left\{\lambda_1-\lambda_2 \geq N^{-\frac{2}{3}(1+\delta)}\right\} \cap \left\{\bigcap_{i=1}^{N}\left\{|\lambda_i-\widehat{\lambda}_i| \leq N^{-\frac{2}{3}+\delta}\left(\min\{i, N+1-i\}\right)^{-1/3}\right\}\right\}. \tag{5.1.14}$$

The first equation in (5.1.13) holds on this event, which we can see by writing

$$\begin{aligned}\frac{1}{N}\sum_{i=2}^{N}\frac{1}{\lambda_1-\lambda_i} &= \frac{1}{N}\sum_{i=2}^{\lfloor N^{\delta/3}\rfloor}\frac{1}{\lambda_1-\lambda_i} + \frac{1}{N}\sum_{i=\lfloor N^{\delta/3}\rfloor+1}^{N}\frac{1}{\lambda_1-\lambda_i} \\ &= O(N^{-\frac{1}{3}+\delta}) + \left(1+O(N^{-\frac{1}{3}+\delta})\right)\end{aligned} \tag{5.1.15}$$

where, for the first sum, we use $\lambda_1-\lambda_i \geq N^{-\frac{2}{3}(1+\delta)}$ and, for the second sum, we use eigenvalue rigidity and the semicircle law. The second equation in (5.1.13) also holds on E_δ using the same reasoning along with the fact that the sum in (5.1.12) is convergent. It remains only to show that

$$\mathbb{P}(E_\delta) \geq 1-N^{-\delta/2}. \tag{5.1.16}$$

From Lemma 3.4 from [29], we have

$$\mathbb{P}(\lambda_1-\lambda_2 \geq N^{-\frac{2}{3}(1+\delta)}) \geq 1-N^{-\frac{2}{3}\delta+\delta'} \tag{5.1.17}$$

for any $\delta' > 0$. This, along with (2.1.10), implies the lemma. □

We also consider a similar class of sums with a larger exponent in the denominator and get the following lemma.

Lemma 5.1.3. *For any $\delta > 0$*

$$\sum_{i=2}^{N}\frac{1}{(a_1-a_i)^m} = O(N^\delta) \quad \textit{and} \quad \sum_{i=2}^{N}\frac{n_i^2}{(a_1-a_i)^m} = O(N^\delta), \qquad m \geq 2, \tag{5.1.18}$$

with probability at least $1 - N^{-\frac{\delta}{3m}}$.

Proof. To prove the first of these inequalities we consider the event

$$\mathcal{F}_{\delta,m} = \left\{a_1 - a_2 > N^{-\frac{\delta}{2m}}\right\} \cap \left\{\bigcap_{i=1}^{N} \left\{|\lambda_i - \widehat{\lambda}_i| \le N^{-\frac{2}{3}+\delta} \left(\min\{i, N+1-i\}\right)^{-1/3}\right\}\right\} \tag{5.1.19}$$

The event $\left\{a_1 - a_2 > N^{-\frac{\delta}{2m}}\right\}$ occurs with probability at least $1 - N^{-\frac{\delta}{2m}+\delta'}$ for any $\delta' > 0$ (see [29] Lemma 3.4). Using this fact along with the eigenvalue rigidity result (2.1.10), we can conclude that the event $\mathcal{F}_{\delta,m}$ occurs with probability at least $1 - N^{-\frac{\delta}{3m}}$. Now we show that the first inequality in Lemma 5.1.3 holds on the event $\mathcal{F}_{\delta,m}$. In particular, on that event, we have

$$\begin{aligned}
\sum_{i=2}^{N} \frac{1}{(a_1 - a_i)^m} &= \sum_{i=2}^{\lfloor N^{\delta/2} \rfloor} \frac{1}{(a_1 - a_i)^m} + \sum_{i=\lfloor N^{\delta/2} \rfloor+1}^{N} \frac{1}{(a_1 - a_i)^m} \\
&< N^{\delta/2} \cdot \frac{1}{(N^{-\frac{\delta}{2m}})^m} + 2 \sum_{i=\lfloor N^{\delta/2} \rfloor+1}^{N} \frac{1}{(-a_i)^m} \\
&\le N^{\delta} + 2 \sum_{i=\lfloor N^{\delta/2} \rfloor+1}^{N} \frac{1}{(-\hat{a}_i)^m} \left(1 + \frac{|a_i^m - \hat{a}_i^m|}{(-a_i)^m}\right) \\
&< N^{\delta} + 4 \sum_{i=\lfloor N^{\delta/2} \rfloor+1}^{N} \frac{1}{(-\hat{a}_i)^m}
\end{aligned} \tag{5.1.20}$$

The summation in the last line is well approximated by the integral

$$N^{-\frac{2m}{3}+1} \int_{-2}^{\hat{\lambda}_{\lfloor N^{\delta/2} \rfloor+1}} \frac{1}{(2-x)^m} \mathrm{d}\sigma_{scl}(x) < 4N^{-\frac{2m}{3}+1} \int_{-2}^{\hat{\lambda}_{\lfloor N^{\delta/2} \rfloor+1}} \frac{1}{(2-x)^{m-\frac{1}{2}}} \mathrm{d}x \tag{5.1.21}$$

Using the approximation $2 - \hat{\lambda}_{\lfloor N^{\delta/2} \rfloor+1} \approx cN^{-\frac{2}{3}+\frac{\delta}{3}}$ from (5.1.3), we see that the right hand side of the inequality above is of order $N^{-m\delta/3}$. Thus $\sum_{i=2}^{N} \frac{1}{(a_1-a_i)^m} = O(N^{\delta})$ on the event $\mathcal{F}_{\delta,m}$. Because n_i are standard Gaussians, the sum $\sum_{i=2}^{N} \frac{n_i^2}{(a_1-a_i)^m}$ has the same order with comparable probability. □

5.1.3 Chi-squared distributions

One quantity that we make use of throughout this chapter is $n_1 = \mathbf{u}_1^T \mathbf{g}$. We note that n_1 has a standard normal distribution which means that n_1^2 has a chi-squared distribution with one degree of freedom. We prove many results that hold on the event where n_1^2 is roughly of order 1. More specifically, we have the following lemma

Lemma 5.1.4. *For any sufficiently small $\delta > 0$,*

$$\mathbb{P}\left(N^{-\delta} < n_1^2 < \delta \log N\right) \geq 1 - N^{-\delta/2} \tag{5.1.22}$$

The proof of this lemma is straightforward from the probability density function for chi-squared random variables. We note for the purpose of future results that n_1 is independent of the eigenvalues of M.

5.1.4 Defining the event on which our result holds

For any $\varepsilon > 0$, we define an event $\mathcal{E}_\varepsilon$ as follows:

$$\begin{aligned}\mathcal{E}_\varepsilon := &\left\{N^{-\varepsilon} < n_1^2 < \varepsilon \log N\right\} \cap \\ &\left\{\frac{1}{N}\sum_{i=2}^{N} \frac{1}{\lambda_1 - \lambda_i} = 1 + O(N^{-\frac{1}{3}+\varepsilon}) \text{ and } \frac{1}{N}\sum_{i=2}^{N} \frac{n_i^2}{\lambda_1 - \lambda_i} = 1 + O(N^{-\frac{1}{3}+\varepsilon})\right\} \\ &\cap \left\{\sum_{i=2}^{N} \frac{1}{(a_1 - a_i)^m} \leq N^\varepsilon \text{ and } \sum_{i=2}^{N} \frac{n_i^2}{(a_1 - a_i)^m} \leq N^\varepsilon \text{ for } m = 2, 3\right\}\end{aligned} \tag{5.1.23}$$

Lemma 5.1.5. *For $\varepsilon > 0$ sufficiently small and N sufficiently large,*

$$\mathbb{P}(\mathcal{E}_\varepsilon) \geq 1 - N^{-\varepsilon/10} \tag{5.1.24}$$

Proof. The event $\mathcal{E}_\varepsilon$ as defined above is the intersection of three events, each with probability close to 1. For sufficiently large N, we know from Lemma 5.1.4 that the first event in the intersection has probability at least $1 - N^{-\varepsilon/2}$ and, from Lemma 5.1.2, the second event in the intersection has probability at least $1 - N^{-\varepsilon/2}$. The third event in the intersection is actually composed of two events, the one for $m = 2$ and the one for $m = 3$. By Lemma 5.1.3, these hold with probability $1 - N^{-\varepsilon/6}$ and $1 - N^{-\varepsilon/9}$ respectively. Putting these together, we see that, even if the complements of all of these events are disjoint, we have $\mathbb{P}(\mathcal{E}_\varepsilon) \geq 1 - N^{-\varepsilon/10}$ for any sufficiently small $\varepsilon > 0$ and sufficiently large N. □

Throughout the rest of this chapter, we will prove various results assuming that we are on the event $\mathcal{E}_\varepsilon$. We can then conclude that those results hold with probability at least $1 - N^{-\varepsilon/10}$.

5.2 Overlap with the microscopic external field (detailed proof)

5.2.1 Introduction

In this section, we present a rigorous proof of Theorem 5.2.1, which provides the moment generating function for the overlap $\mathfrak{M}$. This is essentially a rigorous version of Result 4.1.6/4.1.7.

Theorem 5.2.1. *Given $T < 1$ with $h = HN^{-1/2}$ for some some fixed $H \geq 0$ and $n_1 := \mathbf{u}_1 \cdot \mathbf{g}$, we have the following asymptotic formula for the moment generating function of $\mathfrak{M}$, the overlap with the external field. This formula holds on the event $\mathcal{E}_\varepsilon$ (which has probability at least $1 - N^{-\varepsilon/10}$) for any sufficiently small $\varepsilon > 0$ and $\xi = O(1)$.*

$$\langle e^{\xi\sqrt{N}\mathfrak{M}} \rangle = e^{H\xi + \frac{T\xi^2}{2}} \frac{\cosh\left((H + T\xi)|n_1|\frac{\sqrt{1-T}}{T}\right)}{\cosh\left(H|n_1|\frac{\sqrt{1-T}}{T}\right)} \left(1 + O(N^{-\frac{1}{21}+\frac{\varepsilon}{7}})\right). \tag{5.2.1}$$

Note that the leading term on the right-hand side is the product of two terms implying that it is the moment generating function of a sum of two independent random variables. The exponential term is the moment generating function of a Gaussian random variable. For the ratio of the cosh functions, we note that the moment generating function of a shifted Bernoulli random variable that takes values 1 and -1 with probabilities P and $1 - P$ respectively is $Pe^t + (1 - P)e^{-t}$. The ratio of cosh functions in Theorem 5.2.1 is of this form with $t = \xi|n_1|\sqrt{1-T}$ and

$$P = \frac{e^{\frac{H}{T}|n_1|\sqrt{1-T}}}{e^{\frac{H}{T}|n_1|\sqrt{1-T}} + e^{-\frac{H}{T}|n_1|\sqrt{1-T}}}. \tag{5.2.2}$$

Hence, for any large N, we can conclude that, on the event $\mathcal{E}_\varepsilon$, the scaled overlap $\sqrt{N}\mathfrak{M}$ behaves in its leading order like the independent sum of a Gaussian random variable (with mean H and variance T) and a shifted Bernoulli random variable (which takes values $|n_1|\sqrt{1-T}$ and $-|n_1|\sqrt{1-T}$ with probability P and $1 - P$ respectively for the value of P stated above).

We can use Theorem 5.2.1 to obtain various information about the overlaps, including formulas for all moments of $\mathfrak{M}$. Of particular interest are the first moment (Gibbs expectation) and the variance with respect to the Gibbs measure. Since $\mathfrak{M}$ is of order $N^{-1/2}$ in the case of a microscopic external field, we examine the scaled overlap $\mathfrak{M}\sqrt{N}$. For the expectation, we get

$$\langle \mathfrak{M}\sqrt{N} \rangle = H + |n_1|\sqrt{1-T}\tanh\left(H|n_1|\frac{\sqrt{1-T}}{T}\right) + O\left(N^{-\frac{1}{21}+\frac{\varepsilon}{3}}\right) \tag{5.2.3}$$

and for the variance with respect to the Gibbs measure, we get

$$\mathrm{Var}(\mathfrak{M}\sqrt{N}) = T + n_1^2(1-T)\left(1 - \tanh^2\left(H|n_1^2|\frac{\sqrt{1-T}}{T}\right)\right) + O\left(N^{-\frac{1}{21}+\frac{\varepsilon}{3}}\right), \tag{5.2.4}$$

where both of these formulas hold on the event $\mathcal{E}_\varepsilon$.

In the proof of Theorem 5.2.1, we make use of Lemma 2.3.3, which can be restated as follows:

$$\langle e^{\beta\xi\sqrt{N}\mathfrak{M}}\rangle = e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}})-\mathcal{G}(\gamma))}\frac{\int_{\gamma_{\mathfrak{M}}-\mathrm{i}\infty}^{\gamma_{\mathfrak{M}}+\mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{M}}(z)-\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}))}\mathrm{d}z}{\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}(z)-\mathcal{G}(\gamma))}\mathrm{d}z} \tag{5.2.5}$$

where

$$\mathcal{G}_{\mathfrak{M}}(z) = \beta z - \frac{1}{N}\sum_{i=1}^{N}\log(z-\lambda_i) + \frac{(h+\frac{\xi}{\sqrt{N}})^2\beta}{N}\sum_{i=1}^{N}\frac{n_i^2}{z-\lambda_i} \tag{5.2.6}$$

and we use γ and $\gamma_{\mathfrak{M}}$ to denote critical points of $\mathcal{G}(z)$ and $\mathcal{G}_{\mathfrak{M}}(z)$ respectively, which satisfy $\gamma > \lambda_1$ and $\gamma_{\mathfrak{M}} > \lambda_1$. In the next two lemmas, we show that these critical points are unique and we compute upper and lower bounds for them. After accomplishing this, we turn to the more delicate task of computing the integrals in the formula for the generating function of $\mathfrak{M}$. This is more difficult for $h \sim N^{-1/2}$ than in the other scaling regimes because the critical point is very close to a branch point. Since a straightforward application of Taylor approximation and steepest descent analysis does not work in this case, we directly compute the integral in a neighborhood of the critical point and then show that the tails of the integral are of smaller order.

5.2.2 Critical point analysis

We begin by computing the critical point γ of $\mathcal{G}(z)$. In Section 4.1.6, we used the ansatz that $\gamma = \lambda_1 + pN^{-1}$ with $N^{-\delta} < p < N^{\delta}$ for any $\delta > 0$ and sufficiently large N on some event whose probability tends to 1 as $N \to \infty$. Here we take a more rigorous approach. Without making any assumption about the order of p, we set

$$\gamma = \lambda_1 + pN^{-1} \tag{5.2.7}$$

and then prove that the order of p indeed satisfies the ansatz from Section 4.1.6 (in fact we prove something more precise). In particular, we can define p via the formula for $\mathcal{G}'(z)$ and prove the following lemma.

Lemma 5.2.2. *There exists a unique $p > 0$ satisfying the equation*

$$\mathcal{G}'(\lambda_1 + pN^{-1}) = \beta - \frac{1}{N}\sum_{i=1}^{N} \frac{1}{\lambda_1 + pN^{-1} - \lambda_i} - \frac{H^2\beta}{N^2}\sum_{i=1}^{N} \frac{n_i^2}{(\lambda_1 + pN^{-1} - \lambda_i)^2} = 0 \quad (5.2.8)$$

and, for any sufficiently small $\varepsilon > 0$ and sufficiently large N, we have $T < p < \varepsilon \log N$ on the event $\mathcal{E}_\varepsilon$, which occurs with probability at least $1 - N^{-\varepsilon/10}$.

Proof. The existence and uniqueness of p can be seen from the fact that, for $x \in (\lambda_1, \infty)$, the function $\mathcal{G}'(x)$ is monotonically increasing with $\mathcal{G}'(x) \to \beta$ as $x \to \infty$ and $\mathcal{G}'(x) \to -\infty$ as $x \to \lambda_1$. Having established that (5.2.8) has a unique solution $p > 0$, we turn to the task of bounding p. On the event $\mathcal{E}_\varepsilon$, the last sum in equation (5.2.8) is $O(N^{-\frac{2}{3}+\varepsilon})$ for any sufficiently small $\varepsilon > 0$. From this, we get

$$\beta - \frac{1}{N}\sum_{i=2}^{N} \frac{1}{\lambda_1 - \lambda_i} - \frac{1}{p} - \frac{H^2\beta n_1^2}{p^2} + O(N^{-\frac{2}{3}+\varepsilon}) \ < \ 0 \ < \ \beta - \frac{1}{p} - \frac{H^2\beta n_1^2}{p^2} \quad (5.2.9)$$

on $\mathcal{E}_\varepsilon$. Further applying the definition of $\mathcal{E}_\varepsilon$ to the sum on the left hand side and rearranging terms, we get

$$\beta - 1 + O(N^{-\frac{1}{3}+\varepsilon}) \ < \ \frac{1}{p} + \frac{H^2\beta n_1^2}{p^2} \ < \ \beta. \quad (5.2.10)$$

Hence, on $\mathcal{E}_\varepsilon$, the expression $\frac{1}{p} + \frac{H^2\beta n_1^2}{p^2}$ is bounded above and below by order 1 quantities. The upper bound ensures that $p > \frac{1}{\beta} = T$ (note this is not a sharp bound). The lower bound on $\frac{1}{p} + \frac{H^2\beta n_1^2}{p^2}$ ensures that $p = O(\varepsilon \log N)$ provided that $|n_i| = O(\varepsilon \log N)$. Since $|n_i| < (\varepsilon \log N)^{1/2}$ for sufficiently large N on $\mathcal{E}_\varepsilon$, we can definitely ensure that $|n_i| < C\varepsilon \log N$ for any constant C and sufficiently large N. □

Having proved the lemma, we apply the bounds on the order of p to equation (5.2.8) and conclude that p satisfies

$$\beta - 1 - \frac{1}{p} - \frac{H^2\beta n_1^2}{p^2} + O(N^{-\frac{1}{3}+\varepsilon}) = 0 \quad (5.2.11)$$

with probability $1 - N^{-\varepsilon/10}$. We note that, when $h = HN^{-1/2}$, the equation for $\mathcal{G}_{\mathfrak{M}}$ is same as the one for $\mathcal{G}$ with H replaced by $H + \xi$. Thus $\gamma_{\mathfrak{M}} = \lambda_1 + p_{\mathfrak{M}}N^{-1}$ where $p_{\mathfrak{M}} > 0$ solves the equation

$$\beta - 1 - \frac{1}{p_{\mathfrak{M}}} - \frac{(H+\xi)^2\beta n_1^2}{p_{\mathfrak{M}}^2} + O(N^{-\frac{1}{3}+\varepsilon}) = 0, \quad (5.2.12)$$

and the lemma below follows by the same reasoning as in the lemma above.

Lemma 5.2.3. *There exists a unique $p_{\mathfrak{M}} > 0$ satisfying the equation*

$$\mathcal{G}'_{\mathfrak{M}}(\lambda_1 + p_{\mathfrak{M}} N^{-1}) = \beta - \frac{1}{N}\sum_{i=1}^{N} \frac{1}{\lambda_1 + p_{\mathfrak{M}} N^{-1} - \lambda_i} - \frac{(H+\xi)^2\beta}{N^2}\sum_{i=1}^{N} \frac{n_i^2}{(\lambda_1 + p_{\mathfrak{M}} N^{-1} - \lambda_i)^2} = 0 \tag{5.2.13}$$

and, for any sufficiently small $\varepsilon > 0$ and sufficiently large N, we have $T < p_{\mathfrak{M}} < \varepsilon \log N$ on the event $\mathcal{E}_\varepsilon$, which occurs with probability at least $1 - N^{-\varepsilon/10}$.

5.2.3 Contour integral computation

We now consider the ratio of the integrals in the formula (5.2.5). For the integral in the numerator, we have the following lemma.

Lemma 5.2.4. *For fixed $H > 0$ with $h = HN^{-1/2}$ and $T < 1$*

$$\int e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{M}}(z) - \mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}))} \mathrm{d}z$$
$$= \frac{2\mathrm{i}\sqrt{2\pi p_{\mathfrak{M}}}e^{-(\beta-1)p_{\mathfrak{M}} + \frac{1}{2}}}{N\sqrt{\beta - 1}} \cosh\left((H+\xi)|n_1|\sqrt{\beta(\beta-1)}\right)\left(1 + O(N^{-\frac{1}{21}+\frac{\varepsilon}{7}})\right) \tag{5.2.14}$$

on the event $\mathcal{E}_\varepsilon$, which occurs with probability at least $1 - N^{-\varepsilon/10}$ for any sufficiently small $\varepsilon > 0$.

Proof. To compute this integral, we need a formula for $N(\mathcal{G}_{\mathfrak{M}}(z) - \mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}))$ in terms of u where $z = \gamma_{\mathfrak{M}} + uN^{-1}$. We will begin by focusing on the central portion of the integral and then we will handle the tails separately. When we are on the event $\mathcal{E}_\varepsilon$ and $|u| = o(N^{\frac{1}{3}-\varepsilon})$, we get the following computation:

$$\begin{aligned}
&N(\mathcal{G}_{\mathfrak{M}}(z) - \mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}})) = N(\mathcal{G}_{\mathfrak{M}}(z) - \mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}) - \mathcal{G}'_{\mathfrak{M}}(\gamma_{\mathfrak{M}})uN^{-1}) \\
&= -\sum_{i=1}^{N}\left[\log\left(1 + \frac{uN^{-1}}{\gamma_{\mathfrak{M}} - \lambda_i}\right) - \frac{uN^{-1}}{\gamma_{\mathfrak{M}} - \lambda_i}\right] + \frac{(H+\xi)^2\beta}{N}\sum_{i=1}^{N}\frac{n_i^2 u^2 N^{-2}}{(\gamma_{\mathfrak{M}} + uN^{-1} - \lambda_i)(\gamma_{\mathfrak{M}} - \lambda_i)^2} \\
&= -\log\left(1 + \frac{u}{p_{\mathfrak{M}}}\right) + \frac{u}{p_{\mathfrak{M}}} + O\left(\sum_{j=2}^{N}\frac{|u|^2 N^{-2}}{(\gamma_{\mathfrak{M}} - \lambda_j)^2}\right) \\
&\qquad + \frac{(H+\xi)^2\beta n_1^2 u^2}{(p_{\mathfrak{M}} + u)p_{\mathfrak{M}}^2} + O\left(\sum_{j=2}^{N}\frac{|u|^2 N^{-3}}{(\gamma_{\mathfrak{M}} - \lambda_j)^3}\right).
\end{aligned} \tag{5.2.15}$$

Using properties of the event $\mathcal{E}_\varepsilon$, the quantity in the last line above can be simplified as follows:

$$
\begin{aligned}
&= -\log\left(1+\frac{u}{p_{\mathfrak{M}}}\right) + \frac{u}{p_{\mathfrak{M}}} + \frac{(H+\xi)^2\beta n_1^2 u^2}{(p_{\mathfrak{M}}+u)p_{\mathfrak{M}}^2} + O(|u|^2 N^{-\frac{2}{3}+\varepsilon}) \\
&= -\log\left(1+\frac{u}{p_{\mathfrak{M}}}\right) + (\beta - 1 + O(N^{-\frac{1}{3}+\varepsilon}))(u - p_{\mathfrak{M}}) + 1 + \frac{(H+\xi)^2\beta n_1^2}{(p_{\mathfrak{M}}+u)} + O(|u|^2 N^{-\frac{2}{3}+\varepsilon}) \\
&= -\log\left(1+\frac{u}{p_{\mathfrak{M}}}\right) + (\beta-1)(u-p_{\mathfrak{M}}) + 1 + \frac{(H+\xi)^2\beta n_1^2}{(p_{\mathfrak{M}}+u)} + O\left((|u|+1)N^{-\frac{1}{3}+\varepsilon}\right).
\end{aligned} \tag{5.2.16}
$$

Now, set $g(u) = \frac{1}{2}\left(-\log\left(1+\frac{u}{p_{\mathfrak{M}}}\right) + (\beta-1)(u-p_{\mathfrak{M}}) + 1 + \frac{(H+\xi)^2\beta n_1^2}{(p_{\mathfrak{M}}+u)}\right)$ and let $0 < \delta < \frac{1}{6} - \frac{\varepsilon}{2}$. Then we see that, on the event $\mathcal{E}_\varepsilon$,

$$
\begin{aligned}
&\int_{\gamma_{\mathfrak{M}}-\mathrm{i}\infty}^{\gamma_{\mathfrak{M}}+\mathrm{i}\infty} \exp\left[\frac{N}{2}(\mathcal{G}_{\mathfrak{M}}(z) - \mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}))\right] \mathrm{d}z \\
&\qquad = \frac{1}{N}\left(\int_{-\mathrm{i}N^\delta}^{\mathrm{i}N^\delta} \exp\left(g(u) + O\left((|u|+1)N^{-\frac{1}{3}+\varepsilon}\right)\right)\mathrm{d}u + \text{ integrals of tails}\right).
\end{aligned} \tag{5.2.17}
$$

For the purposes of computing this, it helps to deform the contour by shifting it leftward so that, instead of the vertical contour from $\gamma_{\mathfrak{M}} - \mathrm{i}\infty$ to $\gamma_{\mathfrak{M}} + \mathrm{i}\infty$, we consider a contour from $\lambda_1 - \mathrm{i}\infty$ to $\lambda_1 + \mathrm{i}\infty$ which is a straight vertical line except near λ_1 where it passes to the right of λ_1. The integral on this contour will be the same as on the original contour and we get

$$
\begin{aligned}
&\int_{\lambda_1-\mathrm{i}\infty}^{\lambda_1+\mathrm{i}\infty} \exp\left[\frac{N}{2}(\mathcal{G}_{\mathfrak{M}}(z) - \mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}))\right] \mathrm{d}z \\
&\qquad = \frac{1}{N}\left(\int_{-\mathrm{i}N^\delta}^{\mathrm{i}N^\delta} \exp\left(g(u-p_{\mathfrak{M}}) + O\left((|u|+1)N^{-1/3}\right)\right)\mathrm{d}u + \text{ integrals of tails}\right) \\
&\qquad = \frac{1}{N}\left(\int_{-\mathrm{i}N^\delta}^{\mathrm{i}N^\delta} \exp(g(u-p_{\mathfrak{M}}))\left(1 + O\left((|u|+1)N^{-\frac{1}{3}+\varepsilon}\right)\right)\mathrm{d}u + \text{ integrals of tails}\right).
\end{aligned} \tag{5.2.18}
$$

Note that we have implemented a leftward shift of $\gamma_{\mathfrak{M}} - \lambda_1$ with respect to z, which corresponds to a leftward shift of $p_{\mathfrak{M}}$ with respect u. We make this shift in the integrand rather than in the contour bounds, so the contour with respect to u is a vertical line along the imaginary axis, but passing to the right of the origin. Next, we compute the integral on the portion of the contour from $-\mathrm{i}N^\delta$ to $\mathrm{i}N^\delta$. Call this portion of the contour C. We define C more specifically (in terms of u) to be composed of three pieces:

- C_1 is the straight line from $-\mathrm{i}N^\delta$ to $-\mathrm{i}p_{\mathfrak{M}}$.
- C_2 is the semicircle given by $p_{\mathfrak{M}}e^{\mathrm{i}\theta}$ with $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.
- C_3 is the straight line from $\mathrm{i}p_{\mathfrak{M}}$ to $\mathrm{i}N^\delta$.

We show that $\exp(g(u - p_{\mathfrak{M}}))$ is bounded on C_1, C_2, C_3 by bounding the real part of $g(u - p_{\mathfrak{M}})$. On C_1 and C_3, we have

$$\mathrm{Re}(g(u - p_{\mathfrak{M}})) = -\frac{1}{2}\log\left(\frac{|u|}{p_{\mathfrak{M}}}\right) - 2p_{\mathfrak{M}}(\beta - 1) + 1 < 1. \tag{5.2.19}$$

On C_2, we have

$$\begin{aligned}\mathrm{Re}(g(p_{\mathfrak{M}}e^{\mathrm{i}\theta} - p_{\mathfrak{M}})) =& -\frac{1}{2}\log(|e^{\mathrm{i}\theta}|) + (\beta - 1)\cdot \mathrm{Re}(p_{\mathfrak{M}}e^{\mathrm{i}\theta} - 2p_{\mathfrak{M}}) + 1 + \mathrm{Re}\left(\frac{(H+\xi)^2\beta n_1^2}{p_{\mathfrak{M}}e^{\mathrm{i}\theta}}\right)\\ &< 1 + \frac{(H+\xi)^2\beta n_1^2}{p_{\mathfrak{M}}}.\end{aligned} \tag{5.2.20}$$

Since the real part of $g(u - p_{\mathfrak{M}})$ is bounded, the magnitude of $\exp(g(u - p_{\mathfrak{M}}))$ is also bounded by some constant (call it c) so we have

$$\begin{aligned}&\int_{C_1} \exp(g(u - p_{\mathfrak{M}}))\left(1 + O\left((|u|+1)N^{-\frac{1}{3}+\varepsilon}\right)\right)\mathrm{d}u\\ &= \int_{-\mathrm{i}N^\delta}^{-\mathrm{i}p_{\mathfrak{M}}} \exp(g(u - p_{\mathfrak{M}}))\left(1 + O\left((|u|+1)N^{-\frac{1}{3}+\varepsilon}\right)\right)\mathrm{d}u\\ &= \int_{-\mathrm{i}N^\delta}^{-\mathrm{i}p_{\mathfrak{M}}} \exp(g(u - p_{\mathfrak{M}}))\mathrm{d}u + O(c\cdot 2N^{2\delta-\frac{1}{3}+\varepsilon})\\ &= \int_{C_1} \exp(g(u - p_{\mathfrak{M}}))\mathrm{d}u + O(N^{2\delta-\frac{1}{3}+\varepsilon}).\end{aligned} \tag{5.2.21}$$

Similarly, we have

$$\int_{C_3} \exp(g(u - p_{\mathfrak{M}}))\left(1 + O\left((|u|+1)N^{-\frac{1}{3}+\varepsilon}\right)\right)\mathrm{d}u = \int_{C_3} \exp(g(u - p_{\mathfrak{M}}))\mathrm{d}u + O(N^{2\delta-\frac{1}{3}+\varepsilon}). \tag{5.2.22}$$

Finally, for C_2, we get

$$\begin{aligned}\int_{C_2} \exp(g(u-p_{\mathfrak{M}}))\left(1+O\left((|u|+1)N^{-\frac{1}{3}+\varepsilon}\right)\right)\mathrm{d}u \\ =\int_{-\pi/2}^{\pi/2} \exp(g(p_{\mathfrak{M}}e^{\mathrm{i}\theta}-p_{\mathfrak{M}}))\left(1+O\left(N^{-\frac{1}{3}+\varepsilon}\right)\right)p_{\mathfrak{M}}\mathrm{i}e^{\mathrm{i}\theta}\mathrm{d}\theta \\ =\int_{-\pi/2}^{\pi/2} \exp(g(p_{\mathfrak{M}}e^{\mathrm{i}\theta}-p_{\mathfrak{M}}))p_{\mathfrak{M}}\mathrm{i}e^{\mathrm{i}\theta}\mathrm{d}\theta+O\left(p_{\mathfrak{M}}c\pi N^{-\frac{1}{3}+\varepsilon}\right) \\ =\int_{C_2} \exp(g(u-p_{\mathfrak{M}}))\mathrm{d}u+O(N^{-\frac{1}{3}+\varepsilon}).\end{aligned} \tag{5.2.23}$$

Thus, we conclude that, on the event $\mathcal{E}_\varepsilon$, for any $0<\delta<\frac{1}{6}-\frac{\varepsilon}{2}$,

$$\begin{aligned}\int_{-\mathrm{i}N^\delta}^{\mathrm{i}N^\delta} \exp(g(u-p_{\mathfrak{M}}))\left(1+O\left((|u|+1)N^{-\frac{1}{3}+\varepsilon}\right)\right)\mathrm{d}u \\ =\int_{-\mathrm{i}N^\delta}^{\mathrm{i}N^\delta} \exp(g(u-p_{\mathfrak{M}}))\mathrm{d}u+O(N^{2\delta-\frac{1}{3}+\varepsilon}).\end{aligned} \tag{5.2.24}$$

We use lemma 5.2.5 to show that the integral of the tails has order $O(N^{-\delta/3})$. This has the same order as $O(N^{2\delta-\frac{1}{3}+\varepsilon})$ when $\delta=\frac{1}{7}(1-3\varepsilon)$, which is positive for any $0<\varepsilon<\frac{1}{3}$. Since we are free to choose any $0<\delta<\frac{1}{6}-\frac{\varepsilon}{2}$, we set $\delta=\frac{1}{7}(1-3\varepsilon)$ and get

$$\begin{aligned}\int e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{M}}(z)-\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}))}\mathrm{d}z=\frac{1}{N}\left(\int_{-\mathrm{i}N^\delta}^{\mathrm{i}N^\delta} \exp(g(u-p_{\mathfrak{M}}))\mathrm{d}u+O\left(N^{-\frac{1}{21}+\frac{\varepsilon}{7}}\right)\right) \\ =\frac{1}{N}\left(\int_{-\mathrm{i}N^\delta}^{\mathrm{i}N^\delta}\sqrt{\frac{p_{\mathfrak{M}}}{u}}e^{\frac{(\beta-1)(u-2p_{\mathfrak{M}})}{2}+\frac{1}{2}+\frac{(H+\xi)^2\beta n_1^2}{2u}}\mathrm{d}u+O\left(N^{-\frac{1}{21}+\frac{\varepsilon}{7}}\right)\right) \\ =\frac{p_{\mathfrak{M}}^{1/2}e^{-(\beta-1)p_{\mathfrak{M}}+\frac{1}{2}}}{N}\left(\int_{-\mathrm{i}N^\delta}^{\mathrm{i}N^\delta}\frac{1}{\sqrt{u}}e^{\frac{(\beta-1)u}{2}+\frac{(H+\xi)^2\beta n_1^2}{2u}}\mathrm{d}u+O\left(N^{-\frac{1}{21}+\frac{\varepsilon}{7}}\right)\right).\end{aligned} \tag{5.2.25}$$

The integral $\int_{0_++\mathrm{i}\mathbb{R}}\frac{1}{\sqrt{u}}\exp\left(\frac{(\beta-1)u}{2}+\frac{(H+\xi)^2\beta n_1^2}{2u}\right)\mathrm{d}u$ can be evaluated using the contour integral formula for the modified Bessel function (see e.g. [2]):

$$\int_{0_++\mathrm{i}\mathbb{R}}\frac{1}{\sqrt{w}}e^{aw+\frac{b}{w}}\mathrm{d}w=2\pi\mathrm{i}\left(\frac{b}{a}\right)^{1/4}I_{-\frac{1}{2}}(2\sqrt{ab})=\frac{2\mathrm{i}\sqrt{\pi}}{\sqrt{a}}\cosh(2\sqrt{ab}). \tag{5.2.26}$$

Since this integral converges, the integral in the last line of equation (5.2.25) must converge to the same value. Furthermore, the tails of the integral in (5.2.26) beyond order N^δ only contribute

$O(N^{-\delta/2})$ to the value of the integral. This can be seen by applying integration by parts to the integral $\mathrm{i}\int_{N^\delta}^{\infty}\frac{1}{\sqrt{\mathrm{i}y}}e^{\mathrm{i}(ay-\frac{b}{y})}\mathrm{d}y$ where the factor $\frac{1}{\sqrt{y}}e^{-\frac{ib}{y}}$ is differentiated and the factor e^{iay} is integrated. This gives us $O(N^{-\delta/2})$, which is less than $O(N^{-\frac{1}{21}+\frac{\varepsilon}{7}})$ since $\delta=\frac{1}{7}(1-3\varepsilon)$. Hence, we conclude that, on the event $\mathcal{E}_\varepsilon$,

$$\int e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{M}}(z)-\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}))}\mathrm{d}z$$
$$=\frac{2\mathrm{i}\sqrt{2\pi p_{\mathfrak{M}}}e^{-(\beta-1)p_{\mathfrak{M}}+\frac{1}{2}}}{N\sqrt{\beta-1}}\cosh\Big((H+\xi)|n_1|\sqrt{\beta(\beta-1)}\Big)\Big(1+O(N^{-\frac{1}{21}+\frac{\varepsilon}{7}})\Big). \quad (5.2.27)$$

□

We now return to the task of computing the moment generating function of $\mathfrak{M}$ using equation (5.2.5). The integral in the denominator of that formula can be viewed as a special case of the numerator in which $\xi=0$ and $p_{\mathfrak{M}}$ is replaced with p. Therefore, on the event $\mathcal{E}_\varepsilon$,

$$\frac{\int e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{M}}(z)-\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}))}\mathrm{d}z}{\int e^{\frac{N}{2}(\mathcal{G}(z)-\mathcal{G}(\gamma))}\mathrm{d}z}$$
$$=\sqrt{\frac{p_{\mathfrak{M}}}{p}}e^{-(\beta-1)(p_{\mathfrak{M}}-p)}\frac{\cosh\Big((H+\xi)|n_1|\sqrt{\beta(\beta-1)}\Big)}{\cosh\Big(H|n_1|\sqrt{\beta(\beta-1)}\Big)}\Big(1+O(N^{-\frac{1}{21}+\frac{\varepsilon}{7}})\Big). \quad (5.2.28)$$

To compute the moment generating function of $\mathfrak{M}$ from the formula (5.2.5), it remains only to evaluate the factor $e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}})-\mathcal{G}(\gamma))}$, which is also computed in Section 4.1.6. Although we require more precision for the order of the sub-leading term than we did in the preceding chapter, this can easily be achieved by repeating our previous steps using the assumptions that hold on the event $\mathcal{E}_\varepsilon$ and carefully tracking the order of each term. This yields the result

$$N(\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}})-\mathcal{G}(\gamma))=-\log(\frac{p_{\mathfrak{M}}}{p})+2(\beta-1)(p_{\mathfrak{M}}-p)+(2H\xi+\xi^2)\beta+O(N^{-\frac{1}{3}+\varepsilon}). \quad (5.2.29)$$

Thus we can conclude that

$$\langle e^{\beta\xi\sqrt{N}\mathfrak{M}}\rangle=e^{\frac{(2H\xi+\xi^2)\beta}{2}}\frac{\cosh\Big((H+\xi)|n_1|\sqrt{\beta(\beta-1)}\Big)}{\cosh\Big(H|n_1|\sqrt{\beta(\beta-1)}\Big)}\Big(1+O(N^{-\frac{1}{21}+\frac{\varepsilon}{7}})\Big). \quad (5.2.30)$$

Replacing $\beta\xi$ by ξ and using $T=1/\beta$, we obtain the result stated in Theorem 5.2.1.

5.2.4 Integral Approximation Lemma

The proof of Theorem 5.2.1 in the preceding subsections required us to compute a contour integral. In that computation, we relied on the fact that the dominant contribution to the integral comes from a neighborhood of the critical point. In this section, we prove that fact by providing an upper bound for the value of the integral outside of a neighborhood of the critical point and showing that the upper bound shrinks to zero as $N \to \infty$. This is the most technical part of the contour integral computations.

Lemma 5.2.5 (Tail approximation for overlap with external field when $h = H^{-1/2}$ and $T < 1$)**.** *For any $\delta > 0$,*

$$\int_{\mathrm{i}N^\delta}^{\mathrm{i}\infty} \exp\left[\frac{N}{2}(\mathcal{G}_{\mathfrak{M}}(\lambda_1 + uN^{-1}) - \mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}))\right] \mathrm{d}u = O(N^{-\delta/3}) \quad \textit{as } N \to \infty \tag{5.2.31}$$

on the event $\mathcal{E}_\varepsilon$.

Proof. First, observe that we are using a vertical contour with real part equal to λ_1 as opposed to the original contour, which had real part equal to γ. This is due to a contour deformation that we did when computing the integral on the central portion of the contour. To show that the integrals of the tails tend to zero, we deform the contour yet again. Instead of the vertical line contour given by $\lambda_1 + i(N^\delta + t)N^{-1}$ with $t \in [0, \infty)$, we consider the contour C_4 given by $\lambda_1 - f(t)N^{-1} + i(N^\delta + t)N^{-1}$ where $f(t) = (t+1)^\Delta - 1$ for some $0 < \Delta < \frac{1}{3}$. We briefly comment on this choice of contour: In order to show decay of the tails of the integral, we want the real part of $z(t)$ to approach $-\infty$ as $t \to \pm\infty$. However, in order to control the integrand, we want the real part of $|z(t) - \lambda_1|$ to be smaller than $\lambda_1 - \lambda_2$ when $|t| < N$. We choose $f(t) = (t+1)^\Delta - 1$ because we also need $f(0) = 0$ and $f'(t)$ bounded.

To bound $\int_{\mathrm{i}N^\delta}^{\mathrm{i}\infty} \exp[\frac{N}{2}(\mathcal{G}_{\mathfrak{M}}(\lambda_1 + uN^{-1}) - \mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}))]\mathrm{d}u$ we observe that

$$\begin{aligned}
&\left|\int_{\mathrm{i}N^\delta}^{\mathrm{i}\infty} \exp\left[\frac{N}{2}(\mathcal{G}_{\mathfrak{M}}(\lambda_1 + uN^{-1}) - \mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}))\right] \mathrm{d}u\right| \\
&= \int_0^\infty \left|(f'(t) + i) \exp\left[\frac{N}{2}(\mathcal{G}_{\mathfrak{M}}(\lambda_1 + (-f(t) + i(N^\delta + t))N^{-1}) - \mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}))\right]\right| \mathrm{d}t \\
&\le \int_0^\infty \left| -\frac{\Delta}{(t+1)^{1-\Delta}} + i\right| \cdot \left|\exp\left[\frac{N}{2}(\mathcal{G}_{\mathfrak{M}}(\lambda_1 + (-f(t) + i(N^\delta + t))N^{-1}) - \mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}))\right]\right| \mathrm{d}t \\
&\le \int_0^\infty \sqrt{2}\left|\exp\left[\frac{N}{2}(\mathcal{G}_{\mathfrak{M}}(\lambda_1 + (-f(t) + i(N^\delta + t))N^{-1}) - \mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}))\right]\right| \mathrm{d}t
\end{aligned} \tag{5.2.32}$$

Thus, it suffices to show $\int_0^\infty |\exp\left[\frac{N}{2}(\mathcal{G}_{\mathfrak{M}}(\lambda_1 + (-f(t) + i(N^\delta + t))N^{-1}) - \mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}))\right]|\mathrm{d}u = O(N^{-\delta/3})$. We use the notation $\mathcal{G}_{\mathfrak{M}}(z) = A(z) + B(z)$ where

$$A(z) := \beta z - \frac{1}{N}\sum_{j=1}^{N}\log(z - \lambda_j), \qquad B(z) := \frac{(H+\xi)^2\beta}{N^2}\sum_{j=1}^{N}\frac{n_j^2}{z-\lambda_j}. \tag{5.2.33}$$

We begin by noting that

$$\begin{aligned}&\int_0^\infty \left|\exp\left[\frac{N}{2}\left(\mathcal{G}_{\mathfrak{M}}(z(t)) - \mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}})\right)\right]\right|\mathrm{d}t \\ &\leq \int_0^\infty \left|\exp\left[\frac{N}{2}\left(A(z(t)) - A(\gamma_{\mathfrak{M}})\right)\right]\right| \cdot \left|\exp\left[\frac{N}{2}\left(B(z(t)) - B(\gamma_{\mathfrak{M}})\right)\right]\right|\mathrm{d}t\end{aligned} \tag{5.2.34}$$

Therefore, in order to show that the integral on the tail has order $O(N^{-\delta/3})$, it is enough two prove the following two things:

- $\int_0^\infty \left|\exp\left(\frac{N}{2}(A(\lambda_1 - f(t)N^{-1} + i(N^\delta + t)N^{-1}) - A(\gamma_{\mathfrak{M}}))\right)\right|\mathrm{d}t = O(N^{-\delta/3})$ and
- $\left|\exp\left(\frac{N}{2}(B(\lambda_1 - f(t)N^{-1} + i(N^\delta + t)N^{-1}) - B(\gamma_{\mathfrak{M}}))\right)\right|$ is bounded for $t > 0$.

The integral in the first bullet point can be rewritten as

$$\begin{aligned}&\int_0^\infty \left|\exp\left(\frac{N}{2}(A(\lambda_1 - f(t)N^{-1} + i(N^\delta + t)N^{-1}) - A(\gamma_{\mathfrak{M}}))\right)\right|\mathrm{d}t \\ &\qquad = \int_0^\infty \exp\left(\frac{N\beta}{2}(\lambda_1 - f(t)N^{-1} - \gamma_{\mathfrak{M}})\right) \\ &\qquad\qquad \cdot \left|\exp\left[-\frac{1}{2}\sum_{j=1}^{N}\log\left(\frac{\lambda_1 - f(t)N^{-1} + i(N^\delta + t)N^{-1} - \lambda_j}{\gamma_{\mathfrak{M}} - \lambda_j}\right)\right]\right|\mathrm{d}t\end{aligned} \tag{5.2.35}$$

and this can be further simplified as

$$\begin{aligned}&= \int_0^\infty \exp\left(-\frac{\beta(p_{\mathfrak{M}} + f(t))}{2}\right) \\ &\quad \cdot \left|\exp\left[-\frac{1}{2}\sum_{j=1}^{N}\log\left(\frac{\lambda_1 - f(t)N^{-1} + i(N^\delta + t)N^{-1} - \lambda_j}{\gamma_{\mathfrak{M}} - \lambda_j}\right)\right]\right|\mathrm{d}t \\ &= \int_0^\infty \exp\left(-\frac{\beta(p_{\mathfrak{M}} + f(t))}{2}\right)\cdot\exp\left[-\frac{1}{2}\sum_{j=1}^{N}\log\left|\frac{\lambda_1 - f(t)N^{-1} + i(N^\delta + t)N^{-1} - \lambda_j}{\gamma_{\mathfrak{M}} - \lambda_j}\right|\right]\mathrm{d}t\end{aligned} \tag{5.2.36}$$

We begin by showing that this integral restricted to the interval $[2(\gamma_{\mathfrak{M}}-\lambda_N)N,\infty)$ is of order $O(e^{-N/2})$. If we integrate over just the first factor in the expression above, we get

$$\begin{aligned}\int_{2(\gamma_{\mathfrak{M}}-\lambda_N)N}^{\infty}&\exp\left(-\frac{\beta(p_{\mathfrak{M}}+f(t))}{2}\right)\mathrm{d}t\\&=\exp\left(-\frac{\beta(p_{\mathfrak{M}}-1)}{2}\right)\int_{2(\gamma_{\mathfrak{M}}-\lambda_N)N}^{\infty}\exp\left(-\frac{(t+1)^{\Delta}}{2}\right)\mathrm{d}t\\&=O(\exp(-N^{\Delta}))\end{aligned}\tag{5.2.37}$$

In the last equality above, we used the fact (see Lemma 5.2.3) that $p_{\mathfrak{M}}>T$ on the event $\mathcal{E}_\varepsilon$. Since this integral converges, it suffices to show that
$\exp\left[-\frac{1}{2}\sum_{j=1}^{N}\log\left|\frac{\lambda_1-f(t)N^{-1}+i(N^{\delta}+t)N^{-1}-\lambda_j}{\gamma_{\mathfrak{M}}-\lambda_j}\right|\right]$ is of order $O(e^{-N/2})$ for $t\geq 2(\gamma_{\mathfrak{M}}-\lambda_N)N$.

$$\begin{aligned}&\exp\left[-\frac{1}{2}\sum_{j=1}^{N}\log\left|\frac{\lambda_1-f(t)N^{-1}+i(N^{\delta}+t)N^{-1}-\lambda_j}{\gamma_{\mathfrak{M}}-\lambda_j}\right|\right]\\&\leq\exp\left[-\frac{1}{2}\sum_{j=1}^{N}\log\left|\frac{(N^{\delta}+t)N^{-1}}{\gamma_{\mathfrak{M}}-\lambda_j}\right|\right]\leq\exp\left[-\frac{1}{2}\sum_{j=1}^{N}\log\left|\frac{2(\gamma_{\mathfrak{M}}-\lambda_N)}{\gamma_{\mathfrak{M}}-\lambda_j}\right|\right]\\&\leq\exp\left[-\frac{N}{2}\log(2)\right]<e^{-N/2}\end{aligned}\tag{5.2.38}$$

Next, we show that the integral is of order $O(N^{-\delta/3})$ on the interval $[0,2(\gamma_{\mathfrak{M}}-\lambda_N)N]$. For t in this interval we have

$$\begin{aligned}&\exp\left[-\frac{1}{2}\sum_{j=1}^{N}\log\left|\frac{\lambda_1-f(t)N^{-1}+i(N^{\delta}+t)N^{-1}-\lambda_j}{\gamma_{\mathfrak{M}}-\lambda_j}\right|\right]\\&=\exp\left[-\frac{1}{2}\sum_{j=1}^{N}\log\left|\frac{a_1-a_j-f(t)N^{-1/3}+i(N^{\delta}+t)N^{-1/3}}{p_{\mathfrak{M}}N^{-1/3}+a_1-a_j}\right|\right].\end{aligned}\tag{5.2.39}$$

To obtain an upper bound for this quantity, we begin with the $j=1$ term and observe that

$$\exp\left[-\frac{1}{2}\log\left|\frac{-f(t)+i(N^{\delta}+t)}{p_{\mathfrak{M}}}\right|\right]\leq\exp\left[-\frac{1}{2}\log\left|\frac{N^{\delta}}{p_{\mathfrak{M}}}\right|\right]\leq\left(\frac{p_{\mathfrak{M}}}{N^{\delta}}\right)^{1/2}.\tag{5.2.40}$$

For the summation of the $j \geq 2$ terms, we get

$$\begin{aligned}
&\exp\left[-\frac{1}{2}\sum_{j=2}^{N}\log\left|\frac{a_1 - a_j - f(t)N^{-1/3} + i(N^\delta + t)N^{-1/3}}{p_{\mathfrak{M}}N^{-1/3} + a_1 - a_j}\right|\right] \\
&\leq \exp\left[-\frac{1}{4}\sum_{j=2}^{N}\log\left|\frac{(a_1 - a_j - f(t)N^{-1/3})^2 + (N^\delta + t)^2 N^{-2/3}}{(p_{\mathfrak{M}}N^{-1/3} + a_1 - a_j)^2}\right|\right] \\
&\leq \exp\left[-\frac{1}{4}\sum_{j=2}^{N}\log\left|1 + \frac{-2(f(t) + p_{\mathfrak{M}})(a_1 - a_j)N^{-1/3} - p_{\mathfrak{M}}^2 N^{-2/3} + (N^\delta + t)^2 N^{-2/3}}{(p_{\mathfrak{M}}N^{-1/3} + a_1 - a_j)^2}\right|\right].
\end{aligned} \tag{5.2.41}$$

Because $-p_{\mathfrak{M}}^2 N^{-2/3} + (N^\delta + t)^2 N^{-2/3} > 0$, the last line of this inequality is bounded above by

$$\begin{aligned}
&\exp\left[-\frac{1}{4}\sum_{j=2}^{N}\log\left|1 - \frac{2(f(t) + p_{\mathfrak{M}})(a_1 - a_j)N^{-1/3}}{(p_{\mathfrak{M}}N^{-1/3} + a_1 - a_j)^2}\right|\right] \\
&\leq \exp\left[-\frac{1}{4}\sum_{j=2}^{N}\log\left|1 - \frac{2(f(t) + p_{\mathfrak{M}})N^{-1/3}}{a_1 - a_j}\right|\right] \\
&\leq \exp\left[\frac{1}{4}\sum_{j=2}^{N}\left(\frac{2(f(t) + p_{\mathfrak{M}})N^{-1/3}}{a_1 - a_j} + \left(\frac{2(f(t) + p_{\mathfrak{M}})N^{-1/3}}{a_1 - a_j}\right)^2\right)\right] \\
&= \exp\left[\frac{f(t) + p_{\mathfrak{M}}}{2}\sum_{j=2}^{N}\frac{N^{-1/3}}{a_1 - a_j} + ((f(t) + p_{\mathfrak{M}})N^{-1/3})^2\sum_{j=2}^{N}\frac{1}{(a_1 - a_j)^2}\right].
\end{aligned} \tag{5.2.42}$$

Next, using the properties of the event $\mathcal{E}_\varepsilon$, we see that the last line above has upper bound

$$\begin{aligned}
&\exp\left[\frac{f(t) + p_{\mathfrak{M}}}{2}\left((1 + O(N^{-\frac{1}{3}+\varepsilon})) + O(N^{-\frac{2}{3}+\Delta+\varepsilon})\right)\right] \\
&\qquad = \exp\left[\frac{f(t) + p_{\mathfrak{M}}}{2}\left(1 + O(N^{-\frac{1}{3}+\varepsilon})\right)\right].
\end{aligned} \tag{5.2.43}$$

Combining this with the upper bound from the $j = 1$ term in (5.2.40), we conclude that

$$\begin{aligned}
&\exp\left[-\frac{1}{2}\sum_{j=1}^{N}\log\left|\frac{\lambda_1 - f(t)N^{-1} + i(N^\delta + t)N^{-1} - \lambda_j}{\gamma_{\mathfrak{M}} - \lambda_j}\right|\right] \\
&\qquad \leq \left(\frac{p_{\mathfrak{M}}}{N^\delta}\right)^{1/2}\exp\left[\frac{f(t) + p_{\mathfrak{M}}}{2}\left(1 + O(N^{-\frac{1}{3}+\varepsilon})\right)\right].
\end{aligned} \tag{5.2.44}$$

Finally, plugging this back into the original integral, we get

$$\begin{aligned}
&\int_0^{2(\gamma_{\mathfrak{M}}-\lambda_N)N} \left|\exp\left(\frac{N}{2}(A(\lambda_1 - f(t)N^{-1} + i(N^\delta + t)N^{-1}) - A(\gamma_{\mathfrak{M}}))\right)\right| \mathrm{d}t \\
&\leq \int_0^{2(\gamma_{\mathfrak{M}}-\lambda_N)N} \exp\left(-\frac{\beta(p_{\mathfrak{M}} + f(t))}{2}\right) \cdot \left(\frac{p_{\mathfrak{M}}}{N^\delta}\right)^{1/2} \exp\left[\frac{f(t) + p_{\mathfrak{M}}}{2}(1 + O(N^{-\frac{1}{3}+\varepsilon}))\right] \mathrm{d}t \\
&= \left(\frac{p_{\mathfrak{M}}}{N^\delta}\right)^{1/2} \int_0^{2(\gamma_{\mathfrak{M}}-\lambda_N)N} \exp\left[-\frac{\beta - 1 + O(N^{-\frac{1}{3}+\varepsilon})}{2} \cdot (f(t) + p_{\mathfrak{M}})\right] \mathrm{d}t
\end{aligned} \tag{5.2.45}$$

Since $\beta > 1$, there exists some $C'' > 0$ such that the integral is bounded above by

$$\left(\frac{p_{\mathfrak{M}}}{N^\delta}\right)^{1/2} \int_0^{2(\gamma_{\mathfrak{M}}-\lambda_N)N} \exp\left[-C''((t+1)^\Delta - 1)\right] \mathrm{d}t$$

$$= O\left(\left(\frac{p_{\mathfrak{M}}}{N^\delta}\right)^{1/2}\right) = O\left(\left(\frac{\varepsilon \log N}{N^\delta}\right)^{1/2}\right) = O(N^{-\delta/3}) \tag{5.2.46}$$

Lastly, it remains to show that $\left|\exp\left(\frac{N}{2}(B(\lambda_1 - f(t)N^{-1} + i(N^\delta + t)N^{-1}) - B(\gamma_{\mathfrak{M}}))\right)\right|$ is bounded and it suffices to show that $\mathrm{Re}\left(\frac{N}{2}(B(\lambda_1 - f(t)N^{-1} + i(N^\delta + t)N^{-1}) - B(\gamma_{\mathfrak{M}}))\right)$ is bounded above.

$$\begin{aligned}
&\mathrm{Re}\left[\frac{N}{2}(B(\lambda_1 - f(t)N^{-1} + i(N^\delta + t)N^{-1}) - B(\gamma_{\mathfrak{M}})]\right) \\
&= \mathrm{Re}\left[\frac{N}{2} \cdot \frac{(H+\xi)^2\beta}{N^2} \sum_{j=1}^N \left(\frac{n_j^2}{\lambda_1 - f(t)N^{-1} + i(N^\delta + t)N^{-1} - \lambda_j} - \frac{n_j^2}{\gamma_{\mathfrak{M}} - \lambda_j}\right)\right]
\end{aligned} \tag{5.2.47}$$

We observe that the real part of the $j = 1$ term in the summation is negative and, furthermore, $\sum_{j=2}^N (-\frac{n_j^2}{\gamma_{\mathfrak{M}} - \lambda_j})$ is negative. Removing these terms, we see that the quantity above has upper bound

$$\begin{aligned}
&\mathrm{Re}\left[\frac{(H+\xi)^2\beta}{2N} \sum_{j=2}^N \frac{n_j^2}{\lambda_1 - f(t)N^{-1} + i(N^\delta + t)N^{-1} - \lambda_j}\right] \\
&= \frac{(H+\xi)^2\beta}{2N} \sum_{j=2}^N \frac{n_j^2(\lambda_1 - f(t)N^{-1} - \lambda_j)}{(\lambda_1 - f(t)N^{-1} - \lambda_j)^2 + (N^\delta + t)^2 N^{-2}}.
\end{aligned} \tag{5.2.48}$$

Now consider two cases. For $t < N$, the expression in the last line is bounded above by

$$\frac{(H+\xi)^2\beta}{2N}\sum_{j=2}^{N}\frac{n_j^2}{\lambda_1 - f(t)N^{-1} - \lambda_j} = \frac{(H+\xi)^2\beta}{2}\sum_{j=2}^{N}\frac{n_j^2 N^{-1/3}}{a_1 - a_j - f(t)N^{-1/3}} \tag{5.2.49}$$

This will be $O(1)$ because $\sum_{j=2}^{N}\frac{n_j^2 N^{-1/3}}{a_1 - a_j} = 1 + O(N^{-\frac{1}{3}+\varepsilon})$ on the event $\mathcal{E}_\varepsilon$ and, for sufficiently small ε, we have $f(t)N^{-1/3} < \frac{1}{2}(a_1 - a_2)$ since $f(t)N^{-1/3} = O(N^{\Delta - \frac{1}{3}})$ where $\Delta < \frac{1}{3}$ and $a_1 - a_2 > N^{-\varepsilon/3}$ on $\mathcal{E}_\varepsilon$.

In the case where $t \geq N$, we instead use the upper bound

$$\frac{(H+\xi)^2\beta}{2N}\sum_{j=2}^{N}\frac{n_j^2(\lambda_1 - \lambda_j)}{(N^\delta + t)^2 N^{-2}} \leq \frac{(H+\xi)^2\beta}{2N}\sum_{j=2}^{N} 4n_j^2 \tag{5.2.50}$$

Since n_j^2 are i.i.d. chi-squared random variables, the right-hand side is $O(1)$ with overwhelming probability. □

5.3 Applying this method to the overlap of two replicas (unpublished result)

Using a method similar to the proofs in Section 5.2, we can prove Theorem 5.3.1 below for $\mathfrak{R}$, the overlap of two replicas (this is a rigorous re-formulation of Result 4.3.6). The generating function for the overlap with a replica involves a double integral rather than a single integral, but we can use the same contour as in Section 5.2 for both integrals and then transform to polar coordinates in order to prove the desired decay properties outside a neighborhood of the critical point. While our method works to prove this theorem, the details were omitted from the published paper [13] because the theorem also follows from Theorem 2.14 of [30], as we explain below.

Theorem 5.3.1. *Given $T < 1$ and $h = HN^{-1/2}$ for some some fixed $H \geq 0$, we have the following asymptotic formula for the moment generating function of $\mathfrak{R}$, the overlap with a replica. This formula holds on the event $\mathcal{E}_\varepsilon$ (which has probability at least $1 - N^{-\varepsilon/10}$) for any sufficiently small $\varepsilon > 0$ and $\xi = O(1)$.*

$$\langle e^{\xi\frac{\mathfrak{R}}{1-T}}\rangle = \frac{\cosh\left(\frac{2\sqrt{1-T}H|n_1|}{T}\right)e^{\xi} + e^{-\xi}}{\cosh\left(\frac{2\sqrt{1-T}H|n_1|}{T}\right) + 1} + O(N^{-\frac{1}{21}+\frac{\varepsilon}{7}}) \tag{5.3.1}$$

Note that the leading order term on the right hand side is the moment generating function of a shifted Bernoulli random variable that takes values 1 and -1 with probability P and $1-P$

respectively, where

$$P = \frac{\cosh\left(\frac{2\sqrt{1-T}H|n_1|}{T}\right)}{\cosh\left(\frac{2\sqrt{1-T}H|n_1|}{T}\right)+1}. \tag{5.3.2}$$

Thus, for large N, we can conclude that, on the event $\mathcal{E}_\varepsilon$, the overlap $\mathfrak{R}$ behaves in its leading order like a shifted Bernoulli random variable. This conclusion also follows from Theorem 2.14 of [30], which states that, for sufficiently $\varepsilon > 0$, there exists $\varepsilon_1 > 0$ such that with probability at least $1 - N^{-\varepsilon_1}$ and all $t > 0$,

$$\begin{aligned}&\left\langle \mathbf{1}_{\{|N^{-1}\sigma^{(1)}\cdot\sigma^{(2)}\mp(1-\beta^{-1})|\le t\}}\right\rangle \\ &\qquad = \frac{1}{2} \pm \frac{1}{2}\tanh^2\left(\sqrt{v_1^2\theta(\beta-1)}\right) + N^\varepsilon O\left(t + N^{-2/3+\varepsilon}t^{-2} + N^{-1/3}\right)\end{aligned} \tag{5.3.3}$$

where their parameter θ is equivalent to $H^2\beta$ in our notation and their v_1 is analogous to our n_1, but for a deterministic rather than a random choice of g. While their theorem is formulated and proved in a different manner than Theorem 5.3.1, their result implies ours. The unpublished proof of Theorem 5.3.1 is provided below:

Proof of Theorem 5.3.1

To prove this theorem, we begin by observing that, at the scaling $h \sim N^{-1/2}$, the fluctuations of $\mathfrak{R}$ are of order 1 (i.e. the leading term converges to a random variable) so we set

$$\eta = \beta\xi \quad \text{and thus} \quad a = \frac{\eta}{\beta N} = \frac{\xi}{N}. \tag{5.3.4}$$

We make use of Lemma 2.3.3, which can be restated as follows:

$$\langle e^{\beta\xi\mathfrak{R}}\rangle = e^{\frac{N}{2}(\mathcal{G}_\mathfrak{R}(\gamma_\mathfrak{R},\gamma_\mathfrak{R};a)-2\mathcal{G}(\gamma))}\frac{\iint e^{\frac{N}{2}(\mathcal{G}_\mathfrak{R}(z,w;a)-\mathcal{G}_\mathfrak{R}(\gamma_\mathfrak{R},\gamma_\mathfrak{R};a))}\mathrm{d}z\mathrm{d}w}{\left(\int e^{\frac{N}{2}(\mathcal{G}(z)-\mathcal{G}(\gamma))}\mathrm{d}z\right)^2} \tag{5.3.5}$$

where

$$\mathcal{G}_\mathfrak{R}(z,w;a) = \beta(z+w) - \frac{1}{N}\sum_{i=1}^N \log\left((z-\lambda_i)(w-\lambda_i)-a^2\right) + \frac{H^2\beta}{N^2}\sum_{i=1}^N \frac{n_i^2(z+w-2\lambda_i+2a)}{(z-\lambda_i)(w-\lambda_i)-a^2} \tag{5.3.6}$$

and $\gamma_\mathfrak{R} > \lambda_1$ is chosen such that $(\gamma_\mathfrak{R},\gamma_\mathfrak{R})$ is a critical point of $\mathcal{G}_\mathfrak{R}$. In Section 4.3, we rigorously proved that $\mathcal{G}_\mathfrak{R}$ has a unique critical point of this form. We provide further analysis of the critical point in the next subsection and then turn to the task of computing the integrals.

5.3.1 Critical point analysis

From Lemma 5.2.2, we know that the critical point γ of $\mathcal{G}$ is

$$\gamma = \lambda_1 + \frac{p}{N} \tag{5.3.7}$$

where $T < p < \varepsilon \log N$ on $\mathcal{E}_\varepsilon$ for sufficiently large N, and p satisfies the equation

$$\beta - 1 - \frac{1}{p} - \frac{H^2\beta n_1^2}{p^2} + O(N^{-\frac{1}{3}+\varepsilon}) = 0. \tag{5.3.8}$$

In Lemma 4.3.1, we rigorously proved that the function $\mathcal{G}_{\mathfrak{R}}(z, w; a)$ has a unique critical point of the form $(z, w) = (\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}})$ and that, for any scaling of h, it satisfies

$$\gamma < \gamma_{\mathfrak{R}} < \gamma + a. \tag{5.3.9}$$

Applying this lemma to the case of $h \sim N^{-1/2}$, we can verify that $\gamma_{\mathfrak{R}}$ has the form

$$\gamma_{\mathfrak{R}} = \lambda_1 + \frac{q_{\mathfrak{R}}}{N} \tag{5.3.10}$$

with $q_{\mathfrak{R}} > \xi$ and $T < q_{\mathfrak{R}} < \varepsilon \log N + \xi$ on $\mathcal{E}_\varepsilon$ for sufficiently large N. This follows from the lemma by noting that

$$pN^{-1} = \gamma - \lambda_1 < \gamma_{\mathfrak{R}} - \lambda_1 < \gamma + a - \lambda_1 = pN^{-1} + \xi N^{-1} \tag{5.3.11}$$

Thus, we have $p < q_{\mathfrak{R}} < p + \xi$ where ξ is of order 1 and $T < p < \varepsilon \log N$ on $\mathcal{E}_\varepsilon$.

Then the critical point equation becomes, separating the $i = 1$ term out,

$$\beta - 1 - \frac{q_{\mathfrak{R}}}{q_{\mathfrak{R}}^2 - \xi^2} - \frac{H^2\beta n_1^2}{(q_{\mathfrak{R}} - \xi)^2} + O(N^{-\frac{1}{3}+\varepsilon}) = 0. \tag{5.3.12}$$

5.3.2 Contour integral computation

Having verified the location of $\gamma_{\mathfrak{R}}$, we proceed to the computation of the contour integrals. The first quantity we need is $N(\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}}; a) - 2\mathcal{G}(\gamma))$. The intermediate steps of this computation are provided in Chapter II and we do not repeat them here. The result of the calculation is that, on the event $\mathcal{E}_\varepsilon$,

$$\begin{aligned}&N(\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}};a)-2\mathcal{G}(\gamma))\\&=-\log\left(\frac{q_{\mathfrak{R}}^2-\xi^2}{p^2}\right)+2(\beta-1)(q_{\mathfrak{R}}-p)+2H^2\beta n_1^2\left[\frac{1}{q_{\mathfrak{R}}-\xi}-\frac{1}{p}\right]+O(N^{-\frac{1}{3}+\varepsilon}).\end{aligned} \tag{5.3.13}$$

We also need the quantity $N(\mathcal{G}_{\mathfrak{R}}(z,w;a)-\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}};a))$ which we calculate to be

$$\begin{aligned}&N(\mathcal{G}_{\mathfrak{R}}(z,w;a)-\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}};a))\\&=N(\mathcal{G}_{\mathfrak{R}}(z,w;a)-\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}};a)-(\mathcal{G}_{\mathfrak{R}})_z(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}};a)(z-\gamma_{\mathfrak{R}})-(\mathcal{G}_{\mathfrak{R}})_w(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}};a)(w-\gamma_{\mathfrak{R}}))\\&=-\sum_{i=1}^N\left[\log\left(\frac{(z-\lambda_i)(w-\lambda_i)-a^2}{(\gamma_{\mathfrak{R}}-\lambda_i)^2-a^2}\right)-\frac{(\gamma_{\mathfrak{R}}-\lambda_i)(z+w-2\gamma_{\mathfrak{R}})}{(\gamma_{\mathfrak{R}}-\lambda_i)^2-a^2}\right]\\&\quad+\frac{H^2\beta}{N}\sum_{i=1}^N n_i^2\left[\frac{z+w-2\lambda_i+2a}{(z-\lambda_i)(w-\lambda_i)-a^2}-\frac{2}{\gamma_{\mathfrak{R}}-\lambda_i-a}+\frac{z+w-2\gamma_{\mathfrak{R}}}{(\gamma_{\mathfrak{R}}-\lambda_i-a)^2}\right].\end{aligned}$$

For the ratio of the integrals, we set $z=\gamma_{\mathfrak{R}}+\frac{u}{N}$ and $w=\gamma_{\mathfrak{R}}+\frac{v}{N}$. The $i=1$ term gives the main contribution and we obtain

$$\begin{aligned}&N(\mathcal{G}_{\mathfrak{R}}(z,w;a)-\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}};a))\\&=-\log\left(\frac{(u+q_{\mathfrak{R}})(v+q_{\mathfrak{R}})-\xi^2}{q_{\mathfrak{R}}^2-\xi^2}\right)+\frac{q_{\mathfrak{R}}(u+v)}{q_{\mathfrak{R}}^2-\xi^2}\\&\quad+H^2\beta n_1^2\left[\frac{u+v+2q_{\mathfrak{R}}+2\xi}{(u+q_{\mathfrak{R}})(v+q_{\mathfrak{R}})-\xi^2}-\frac{2}{q_{\mathfrak{R}}-\xi}+\frac{u+v}{(q_{\mathfrak{R}}-\xi)^2}\right]+O\left(\frac{|u|+|v|}{N^{\frac{1}{3}-\varepsilon}}\right).\end{aligned} \tag{5.3.14}$$

Note that this equation differs from the one provided in the preceding chapter in the sense that it includes the dependence on u and v in the subleading term. This will be used to bound certain integrals later in the proof. Using equation (5.3.12), the above can be written as

$$\begin{aligned}&N(\mathcal{G}_{\mathfrak{R}}(z,w;a)-\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}};a))\\&=-\log\left(\frac{(u+q_{\mathfrak{R}})(v+q_{\mathfrak{R}})-\xi^2}{q_{\mathfrak{R}}^2-\xi^2}\right)+(\beta-1)(u+v)\\&\quad+H^2\beta n_1^2\left[\frac{u+v+2q_{\mathfrak{R}}+2\xi}{(u+q_{\mathfrak{R}})(v+q_{\mathfrak{R}})-\xi^2}-\frac{2}{q_{\mathfrak{R}}-\xi}\right]+O\left(\frac{|u|+|v|+1}{N^{\frac{1}{3}-\varepsilon}}\right).\end{aligned} \tag{5.3.15}$$

We now compute the integral of the numerator. We will take C_1 and C_2 be finite subsections of the contours that we get when we make the change of variables $z=\gamma_{\mathfrak{R}}+\frac{u}{N}$ and $w=\gamma_{\mathfrak{R}}+\frac{v}{N}$ (possibly after some allowable deformation of the contour). We require that $|u|,|v|=O(N^\delta)$ on C_1

and C_2 respectively where δ is some positive constant to be determined later. Thus, we write the integral in the numerator as

$$\begin{aligned}
&\int_{\gamma_{\mathfrak{R}}+i\mathbb{R}}\int_{\gamma_{\mathfrak{R}}+i\mathbb{R}} e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{R}}(z,w;a)-\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}};a))}\mathrm{d}z\mathrm{d}w \\
&= \frac{\sqrt{q_{\mathfrak{R}}^2-\xi^2}}{N^2}\Bigg(\int_{C_1}\int_{C_2}\frac{e^{\frac{1}{2}(\beta-1)(u+v)+\frac{H^2\beta n_1^2}{2}\left[\frac{u+v+2q_{\mathfrak{R}}+2\xi}{(u+q_{\mathfrak{R}})(v+q_{\mathfrak{R}})-\xi^2}-\frac{2}{q_{\mathfrak{R}}-\xi}\right]+O\left(\frac{|u|+|v|+1}{N^{1/3-\varepsilon}}\right)}}{\sqrt{(u+q_{\mathfrak{R}})(v+q_{\mathfrak{R}})-\xi^2}}\mathrm{d}u\mathrm{d}v \qquad (5.3.16)\\
&\qquad\qquad + \text{integral outside } C_1\times C_2\Bigg).
\end{aligned}$$

If we let m denote the maximal magnitude of the integrand on these contours, we can simplify the integral as follows (where c_1, c_2 denote the lengths of C_1 and C_2 respectively):

$$\begin{aligned}
&\int_{C_1}\int_{C_2}\frac{e^{\frac{1}{2}(\beta-1)(u+v)+\frac{H^2\beta n_1^2}{2}\left[\frac{u+v+2q_{\mathfrak{R}}+2\xi}{(u+q_{\mathfrak{R}})(v+q_{\mathfrak{R}})-\xi^2}-\frac{2}{q_{\mathfrak{R}}-\xi}\right]+O\left(\frac{|u|+|v|+1}{N^{1/3-\varepsilon}}\right)}}{\sqrt{(u+q_{\mathfrak{R}})(v+q_{\mathfrak{R}})-\xi^2}}\mathrm{d}u\mathrm{d}v \\
&= \int_{C_1}\int_{C_2}\frac{e^{\frac{1}{2}(\beta-1)(u+v)+\frac{H^2\beta n_1^2}{2}\left[\frac{u+v+2q_{\mathfrak{R}}+2\xi}{(u+q_{\mathfrak{R}})(v+q_{\mathfrak{R}})-\xi^2}-\frac{2}{q_{\mathfrak{R}}-\xi}\right]}}{\sqrt{(u+q_{\mathfrak{R}})(v+q_{\mathfrak{R}})-\xi^2}}\left(1+O(N^{\delta-\frac{1}{3}+\varepsilon})\right)\mathrm{d}u\mathrm{d}v \qquad (5.3.17)\\
&= \left(1+O(N^{\delta-\frac{1}{3}+\varepsilon})c_1c_2m\right)\int_{C_1}\int_{C_2}\frac{e^{\frac{1}{2}(\beta-1)(u+v)+\frac{H^2\beta n_1^2}{2}\left[\frac{u+v+2q_{\mathfrak{R}}+2\xi}{(u+q_{\mathfrak{R}})(v+q_{\mathfrak{R}})-\xi^2}-\frac{2}{q_{\mathfrak{R}}-\xi}\right]}}{\sqrt{(u+q_{\mathfrak{R}})(v+q_{\mathfrak{R}})-\xi^2}}\mathrm{d}u\mathrm{d}v.
\end{aligned}$$

Thus, we need to show that we can pick contours C_1, C_2 such that c_1, c_2 are $O(N^\delta)$ and m is bounded. For this purpose, it suffices to take the full contours to be straight lines parallel to the imaginary axis with real part equal to $\xi - q_{\mathfrak{R}}$, except within a one unit radius of the point $\xi - q_{\mathfrak{R}}$, where the contour takes a semicircular detour around the point and define both C_1 and C_2 to be the portion of this contour where the variable of integration has magnitude less than N^δ. For these contours, the magnitude of the integrand is bounded above by $\exp((\beta-1)(1+\xi)+H^2\beta n_1^2)$ while c_1 and c_2 are still $O(N^\delta)$. Thus, the quantity $O(N^{\delta-\frac{1}{3}+\varepsilon})c_1c_2m$ becomes $O(N^{3\delta-\frac{1}{3}+\varepsilon})$.

We then show in section 5.3.3 that the integral outside the central region converges to zero at an exponential rate and the integral (5.3.16) becomes

$$\frac{\sqrt{q_{\mathfrak{R}}^2-\xi^2}}{N^2}(1+O(N^{3\delta-\frac{1}{3}+\varepsilon}))\int_{C_1}\int_{C_2}\frac{e^{\frac{1}{2}(\beta-1)(u+v)+\frac{H^2\beta n_1^2}{2}\left[\frac{u+v+2q_{\mathfrak{R}}+2\xi}{(u+q_{\mathfrak{R}})(v+q_{\mathfrak{R}})-\xi^2}-\frac{2}{q_{\mathfrak{R}}-\xi}\right]}}{\sqrt{(u+q_{\mathfrak{R}})(v+q_{\mathfrak{R}})-\xi^2}}\mathrm{d}u\mathrm{d}v \qquad (5.3.18)$$

Now we plug this integral into the formula (5.3.5) along with the computation from equation

(5.3.13). By simple translations of the variables we get the following (where C is now the vertical contour with real part ξ that passes to the right of ξ and is truncated at magnitude N^δ).

$$\langle e^{\beta\xi\mathfrak{R}}\rangle = \frac{\int_C\int_C \frac{1}{\sqrt{uv-\xi^2}} e^{\frac{1}{2}(\beta-1)(u+v)+\frac{H^2\beta n_1^2(u+v+2\xi)}{2(uv-\xi^2)}}\mathrm{d}u\mathrm{d}v\left(1+O(N^{3\delta-\frac{1}{3}+\varepsilon})\right)}{\left(\int_C \frac{1}{\sqrt{u}} e^{\frac{1}{2}(\beta-1)u+\frac{H^2\beta n_1^2}{2u}}\mathrm{d}u\right)^2\left(1+O(N^{3\delta-\frac{1}{3}+\varepsilon})\right)}. \tag{5.3.19}$$

We now evaluate the integrals in (5.3.19) using (recall (5.2.26))

$$\int \frac{e^{au+\frac{b}{u}}}{\sqrt{u}}\mathrm{d}u = \frac{2\mathrm{i}\sqrt{\pi}}{\sqrt{a}}\cosh(2\sqrt{ab}). \tag{5.3.20}$$

While our integrals are defined on a finite contour, they must converge to the same value as the integral on the full contour as $N\to\infty$. As we observed in the single-integral case, the tails of the integral in (5.3.20) beyond $|u| = N^\delta$ only contribute $O(N^{-\delta/2})$ to the value of the intergral. We find that the terms $O(N^{3\delta-\frac{1}{3}+\varepsilon})$ and $O(N^{-\delta/2})$ are of equal order when $\delta = \frac{2}{21}(1-3\varepsilon)$ and both terms become $O(N^{-\frac{1}{21}+\frac{\varepsilon}{7}})$.

Now consider the double integral in the numerator. For each v, we change the variable u to z by setting $uv-\xi^2 = z$. We can define the branch cut appropriately such that the contour for z does not cross the branch cut, although we note that the branch cut will change depending on the value of v. The double integral becomes

$$\iint \frac{1}{v\sqrt{z}} e^{\frac{\beta-1}{2}(\frac{z+\xi^2}{v}+v)+\frac{H^2\beta n_1^2}{2z}(\frac{z+\xi^2}{v}+v+2\xi)}\mathrm{d}z\mathrm{d}v. \tag{5.3.21}$$

The z-integral can be evaluated using (5.3.20) and the double integral becomes

$$\frac{2\mathrm{i}\sqrt{2\pi}}{\sqrt{\beta-1}}\int \frac{1}{\sqrt{v}} e^{\frac{\beta-1}{2}(\frac{\xi^2}{v}+v)+\frac{H^2\beta n_1^2}{2v}}\cosh\left(\frac{\sqrt{(\beta-1)\beta}H|n_1|(v+\xi)}{v}\right)\mathrm{d}v. \tag{5.3.22}$$

Writing $\cosh$ as the sum of two exponentials, the v-integral becomes the sum of two integrals which we can evaluate again using (5.3.20). We find that the double integral is equal to

$$\begin{aligned}-\frac{2\pi}{\beta-1}\Big[&e^{\sqrt{(\beta-1)\beta}H|n_1|}\cosh\left(\sqrt{(\beta-1)\beta}H|n_1|+(\beta-1)\xi\right)\\ &+e^{-\sqrt{(\beta-1)\beta}H|n_1|}\cosh\left(\sqrt{(\beta-1)\beta}H|n_1|-(\beta-1)\xi\right)\Big]\end{aligned} \tag{5.3.23}$$

Writing $\cosh$ as the sum of two exponentials again, the above becomes a linear combination of $e^{(\beta-1)\xi}$ and $e^{-(\beta-1)\xi}$. The denominator in (5.3.19) is same as the numerator when $\xi = 0$. We find that

$$\langle e^{\beta\xi\mathfrak{R}}\rangle = \frac{\cosh\left(2\sqrt{(\beta-1)\beta}H|n_1|\right)e^{(\beta-1)\xi} + e^{-(\beta-1)\xi}}{\cosh\left(2\sqrt{(\beta-1)\beta}H|n_1|\right)+1}\left(1+O(N^{-\frac{1}{21}+\frac{\varepsilon}{7}})\right) \tag{5.3.24}$$

Using $\beta = 1/T$ and re-scaling ξ, we find that

$$\langle e^{\xi\frac{\mathfrak{R}}{1-T}}\rangle = \frac{\cosh\left(\frac{2\sqrt{1-T}H|n_1|}{T}\right)e^{\xi} + e^{-\xi}}{\cosh\left(\frac{2\sqrt{1-T}H|n_1|}{T}\right)+1}\left(1+O(N^{-\frac{1}{21}+\frac{\varepsilon}{7}})\right) \tag{5.3.25}$$

Thus, we can conclude the result given in theorem 5.3.1.

5.3.3 Approximating the integral outside the central region

The goal of this section is to show that, for the double contour integral that we use to compute overlap of two independent spins, the integral outside of the central region shrinks to zero (here "central region" is where both variables of integration have order less than N^{δ}). We can use a similar approach to what was done for the single-integral examples. We break the equation $G(z, w; a)$ into two parts

$$\begin{aligned} A(z,w;a) &= \beta(z+w) - \frac{1}{N}\sum_{i=1}^{N}\log((z-\lambda_i)(w-\lambda_i)-a^2) \\ B(z,w;a) &= \frac{h^2\beta}{N}\sum_{i=1}^{N}\frac{n_i^2(z+w-2\lambda_i+2a)}{(z-\lambda_i)(w-\lambda_i)-a^2} \end{aligned} \tag{5.3.26}$$

Our goal will be to show that the integral of

$$\begin{aligned} \exp\Bigg[\frac{N}{2}\Big(&A(\gamma_{\mathfrak{R}}+\mathrm{i}tN^{-\delta'}, \gamma_{\mathfrak{R}}+\mathrm{i}rN^{-1/2}; a) - A(\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}}; a) \\ &+ B(\gamma_{\mathfrak{R}}+\mathrm{i}tN^{-\delta'}, \gamma_{\mathfrak{R}}+\mathrm{i}rN^{-\delta'}; a) - B(\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}}; a)\Big)\Bigg] \end{aligned} \tag{5.3.27}$$

tends to zero exponentially (i.e. it has order $O(\exp(-N^{\delta'}))$ for some $\delta' > 0$) if we integrate on the region outside a circle of radius N^{δ} centered at the origin.

The general approach will be to show that the integral of the "A part" goes to zero exponentially and that the "B part" is either bounded or has growth that is at most polynomial in N. To show that the "A part" goes to zero exponentially, we will have to consider two subregions:

1. The region in which one of the variables of integration has magnitude greater than a multiple of N and the other can be anything.

2. The region in which both variables of integration have order less than N while the distance of their coordinates (as a point in $\mathbb{R}^2$) from the origin is at least $2N^\delta$.

For this integration we take the contours to be the curves parametrized by $u(t) = x(t) + iy(t)$ and $v(r) = x(r) + iy(r)$ where

$$x(t) = \begin{cases} \cos(2\pi t) & t \in [-1, 1] \\ 0 & t \in [-N^\delta, -1] \cup [1, N^\delta] \\ 1 - (|t| - N^\delta + 1)^\Delta & t \in (-\infty, -N^\delta) \cup (N^\delta, \infty) \end{cases} \tag{5.3.28}$$

$$y(t) = \begin{cases} \sin(2\pi t) & t \in [-1, 1] \\ t & t \in (-\infty, -1) \cup (1, \infty) \end{cases} \tag{5.3.29}$$

With this parametrization, the integral becomes

$$\begin{aligned} &\left| \int\int \exp\left[\frac{N}{2} (\mathcal{G}_{\mathfrak{R}}(\lambda_1 + a + uN^{-1}, \lambda_1 + a + vN^{-1}; a) - \mathcal{G}_{\mathfrak{R}}(\lambda_1 + a, \lambda_1 + a; a)) \right] \mathrm{d}u \; \mathrm{d}v \right| \\ &\leq \int\int |(x'(t) + iy'(t))(x'(r) + iy'(r))| \\ &\quad \cdot \left| \exp\left[\frac{N}{2} \left(\mathcal{G}_{\mathfrak{R}}\left(\lambda_1 + a + \tfrac{x(t)+iy(t)}{N}, \lambda_1 + a + \tfrac{x(r)+iy(r)}{N}; a \right) - \mathcal{G}_{\mathfrak{R}}(\lambda_1 + a, \lambda_1 + a; a) \right) \right] \right| \mathrm{d}r \; \mathrm{d}t \end{aligned} \tag{5.3.30}$$

Note that $x'(t)$ and $y'(t)$ are bounded.

Showing the A part tends to zero exponentially when one variable is greater than a multiple of N

Without loss of generality, let $t > 3N(\gamma_{\mathfrak{R}} - \lambda_N - a)$ and $r \in \mathbb{R}$. (In the next two sections, we will consider $N^\delta < t < 3N(\gamma_{\mathfrak{R}} - \lambda_N - a)$ for various ranges of r. The cases where t is negative or where the roles of t and r are reversed can be treated similarly). The A part of the integral on the desired region becomes

$$\int_{-\infty}^{\infty}\int_{3N(\gamma_{\mathfrak{R}}-\lambda_N-a)}^{\infty}\exp\left(\frac{N\beta}{2}(2(\lambda_1+a-\gamma_{\mathfrak{R}})+(x(t)+x(r))N^{-1})\right)$$
$$\cdot\left|\exp\left[-\frac{1}{2}\sum_{i=1}^{N}\log\left(\frac{(\lambda_1+a+x(t)N^{-1}-\lambda_i+iy(t)N^{-1})(\lambda_1+a+x(r)N^{-1}-\lambda_i+iy(r)N^{-1})-a^2}{(\gamma_{\mathfrak{R}}-\lambda_i)^2-a^2}\right)\right]\right|\mathrm{d}t\ \mathrm{d}r \quad (5.3.31)$$

We will show that, on the region of integration, the second factor of this integrand is bounded while the integral of the first factor tends to zero exponentially on the event $\mathcal{E}_\varepsilon$. We will begin by considering the integral of the first factor and for these calculations that we recall that $x(t)$ and $x(r)$ are bounded above by 1 and are strictly negative for $|r|,|t|>N^\delta$. Using this we have

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{3N(\gamma_{\mathfrak{R}}-\lambda_N-a)}^{\infty}\exp\left(\frac{N\beta}{2}(2(\lambda_1+a-\gamma_{\mathfrak{R}})+(x(t)+x(r))N^{-1})\right)\mathrm{d}t\ \mathrm{d}r\\
&\quad=\int_{-\infty}^{\infty}\int_{3N(\gamma_{\mathfrak{R}}-\lambda_N-a)}^{\infty}\exp\left(-\beta\left(q_{\mathfrak{R}}-\xi-\frac{1}{2}(x(t)+x(r))\right)\right)\mathrm{d}t\ \mathrm{d}r\\
&\quad=\exp(-\beta(q_{\mathfrak{R}}-\xi))\int_{-\infty}^{\infty}\exp\left(\frac{\beta}{2}x(r)\right)\mathrm{d}r\cdot\int_{3N(\gamma_{\mathfrak{R}}-\lambda_N-a)}^{\infty}\exp\left(\frac{\beta}{2}x(t)\right)\mathrm{d}t\\
&\quad=O(1)\cdot O(N^\delta)\cdot O\left(\exp\left(-\frac{\beta}{2}(3N(\gamma_{\mathfrak{R}}-\lambda_N-a)-N^\delta)\right)\right)
\end{aligned} \quad (5.3.32)$$

where the we get $\exp(-\beta(q_{\mathfrak{R}}-\xi))=O(1)$ in the last line from the fact that $q_{\mathfrak{R}}>T$ on $\mathcal{E}_\varepsilon$. The quantity in the last line will be less than $\exp(-\beta N)$ for sufficiently large N, so it remains to show that, for $t>3N(\gamma_{\mathfrak{R}}-\lambda_N-a)$, the quantity $\left|\exp\left[-\frac{1}{2}\sum_{i=1}^{N}\log\left(\frac{(\lambda_1+a+x(t)N^{-1}-\lambda_i+iy(t)N^{-1})(\lambda_1+a+x(r)N^{-1}-\lambda_i+iy(r)N^{-1})-a^2}{(\gamma_{\mathfrak{R}}-\lambda_i)^2-a^2}\right)\right]\right|$ is bounded. This quantity can be rewritten as

$$\left|\exp\left[-\frac{1}{2}\sum_{i=1}^{N}\log\left(\frac{(\lambda_1-\lambda_i+a)^2-a^2+(x(t)+x(r))(\lambda_1-\lambda_i+a)N^{-1}-rtN^{-2}+i(r+t)(\lambda_1-\lambda_i+a)N^{-1}}{(\gamma_{\mathfrak{R}}-\lambda_i)^2-a^2}\right)\right]\right| \quad (5.3.33)$$

If $r > -N(\gamma_{\mathfrak{R}} - \lambda_N - a)$ then this quantity is bounded above by

$$\begin{aligned} &\exp\left[-\frac{1}{2}\sum_{i=1}^{N}\log\left(\frac{(t+r)(\lambda_1-\lambda_i+a)N^{-1}}{(\gamma_{\mathfrak{R}}-\lambda_i)^2-a^2}\right)\right] \\ &\quad\leq \exp\left[-\frac{1}{2}\sum_{i=1}^{N}\log\left(\frac{2(\gamma_{\mathfrak{R}}-\lambda_N-a)(\lambda_1-\lambda_i+a)}{(\gamma_{\mathfrak{R}}-\lambda_i-a)(\gamma_{\mathfrak{R}}-\lambda_i+a)}\right)\right] \\ &\quad= \exp\left[-\frac{1}{2}\sum_{i=1}^{N}\log\left(2\cdot\frac{q_{\mathfrak{R}}-\xi+a_1-a_N}{q_{\mathfrak{R}}-\xi+a_1-a_i}\cdot\frac{\xi+a_1-a_i}{q_{\mathfrak{R}}+\xi+a_1-a_i}\right)\right] < 1. \end{aligned} \tag{5.3.34}$$

The inequality in the last line above (which is definitely not optimal) follows from the fact that, when $a_1 - a_i > q_{\mathfrak{R}}$, the first fraction is greater than 1 and the second fraction is greater than $\frac{1}{2}$, but, when $a_1 - a_i < q_{\mathfrak{R}}$, the first fraction is greater than $N^{2/3}(\varepsilon \log N)^{-1}$ and the second is greater than $\xi(\varepsilon \log N)^{-1}$ on the event $\mathcal{E}_\varepsilon$. Thus, we are taking the log of a number greater than 1 for every index i.

Thus, we are done with the case in which $r > -N(\gamma_{\mathfrak{R}} - \lambda_N - a)$. We now turn to the case of $r < -N(\gamma_{\mathfrak{R}} - \lambda_N - a)$. In this case, the quantity in line (5.3.33) is bounded above by

$$\begin{aligned} &\exp\left[-\frac{1}{2}\sum_{i=1}^{N}\log\left|\frac{(\lambda_1-\lambda_i+a)^2-a^2-rtN^{-2}+(x(t)+x(r))N^{-1}(\lambda_1-\lambda_i+a)}{(\gamma_{\mathfrak{R}}-\lambda_i)^2-a^2}\right|\right] \\ &\quad\leq \exp\left[-\frac{1}{2}\sum_{i=1}^{N}\log\left|\frac{|rt|N^{-2}-(t^{\Delta}+|r|^{\Delta})N^{-1}(\lambda_1-\lambda_i+a)}{(\gamma_{\mathfrak{R}}-\lambda_i)^2-a^2}\right|\right]. \end{aligned} \tag{5.3.35}$$

Given that $|t|$ and $|r|$ both have order at least N, the first term in the numerator above has strictly greater order than the second term. Thus, the quantity above has upper bound

$$\begin{aligned} &\exp\left[-\frac{1}{2}\sum_{i=1}^{N}\log\left|\frac{\frac{1}{2}|rt|N^{-2}}{(\gamma_{\mathfrak{R}}-\lambda_i)^2-a^2}\right|\right] \leq \exp\left[-\frac{1}{2}\sum_{i=1}^{N}\log\left|\frac{\frac{3}{2}(\gamma_{\mathfrak{R}}-\lambda_N-a)^2}{(\gamma_{\mathfrak{R}}-\lambda_i)^2-a^2}\right|\right] \\ &\leq \exp\left[-\frac{1}{2}\sum_{i=1}^{N}\log\left(\frac{3}{2}\left(1-\frac{2a}{\gamma_{\mathfrak{R}}-\lambda_N+a}\right)\right)\right] = \exp\left[-\frac{1}{2}\sum_{i=1}^{N}\log\left(\frac{3}{2}\left(1-O\left(N^{-1}\right)\right)\right)\right] \\ &\leq \exp\left[-\frac{N}{2}\log\left(\frac{4}{3}\right)\right]. \end{aligned} \tag{5.3.36}$$

Thus we have shown that the integral of the A part is less than $\exp(-\beta N)$ on the region where $t > 3N(\gamma_{\mathfrak{R}} - \lambda_N - a)$ and $r \in \mathbb{R}$.

Showing the A part tends to zero exponentially when both variables have order less than N and the radius is at least $2N^{\delta}$

Similarly to the previous case, the integral on this region can be expressed as follows (we begin with an integral with variables of integration t and r but will later transform to polar coordinates):

$$\int\int \exp\left(\frac{N\beta}{2}(2(\lambda_1+a-\gamma_{\mathfrak{R}})+(x(t)+x(r))N^{-1})\right) \cdot \left|\exp\left[-\frac{1}{2}\sum_{i=1}^{N}\log\left(\tfrac{(\lambda_1+a+x(t)N^{-1}-\lambda_i+iy(t)N^{-1})(\lambda_1+a+x(r)N^{-1}-\lambda_i+iy(r)N^{-1})-a^2}{(\gamma_{\mathfrak{R}}-\lambda_i)^2-a^2}\right)\right]\right| \mathrm{d}t\ \mathrm{d}r. \quad (5.3.37)$$

The first factor of the integrand simplifies to $\exp(-\beta(q_{\mathfrak{R}}-\xi-\frac{1}{2}(x(t)+x(r))))$. We now focus on computing an upper bound for the term in the absolute value, which can be simplified as

$$\begin{aligned}&\left|\exp\left[-\frac{1}{2}\sum_{i=1}^{N}\log\left(\tfrac{(\lambda_1-\lambda_i+a)^2-a^2+(x(t)+x(r))(\lambda_1-\lambda_i+a)N^{-1}-rtN^{-2}+i(r+t)(\lambda_1-\lambda_i+a)N^{-1}}{(\gamma_{\mathfrak{R}}-\lambda_i)^2-a^2}\right)\right]\right|\\ &=\left|\exp\left[-\frac{1}{2}\sum_{i=1}^{N}\log\left(\tfrac{(a_1-a_i+\xi N^{-\frac{1}{3}})^2+(x(t)+x(r))(a_1-a_i+\xi N^{-\frac{1}{3}})N^{-\frac{1}{3}}-rtN^{-\frac{2}{3}}+i(r+t)(a_1-a_i+\xi N^{-\frac{1}{3}})N^{-\frac{1}{3}}}{(q_{\mathfrak{R}}N^{-\frac{1}{3}}+a_1-a_i)^2-\xi^2N^{-\frac{2}{3}}}\right)\right]\right|\end{aligned} \quad (5.3.38)$$

At this point, it helps to consider the contribution from the $i=1$ term of the summation separately from the contribution of the terms $2\leq i\leq N$. The contribution from the $i=1$ term can be bounded as follows:

$$\begin{aligned}&\left|\exp\left[-\frac{1}{2}\log\left(\frac{\xi^2N^{-\frac{2}{3}}+(x(t)+x(r))\xi N^{-\frac{2}{3}}-rtN^{-\frac{2}{3}}+i(r+t)\xi N^{-\frac{2}{3}}}{q_{\mathfrak{R}}^2N^{-\frac{2}{3}}-\xi^2N^{-\frac{2}{3}}}\right)\right]\right|\\ &=\exp\left[-\frac{1}{2}\log\left|\frac{\xi^2+(x(t)+x(r))\xi-rt+i(r+t)\xi}{q_{\mathfrak{R}}^2-\xi^2}\right|\right]\end{aligned} \quad (5.3.39)$$

To bound this quantity we assume without loss of generality that $|t|\geq|r|$ and consider two cases:

- If $|r|<1$ then $|t|>2N^{\delta}-1$ and the quantity in (5.3.39) is bounded above by

$$\exp\left[-\frac{1}{2}\log\left|\frac{r+t}{q_{\mathfrak{R}}^2-\xi^2}\right|\right]\leq\exp\left[-\frac{1}{2}\log\left|\frac{N^{\delta}}{q_{\mathfrak{R}}^2-\xi^2}\right|\right]=\sqrt{(q_{\mathfrak{R}}^2-\xi^2)}N^{-\frac{\delta}{2}} \quad (5.3.40)$$

- In the case where $|r|\geq 1$, we recall that $|t|\geq|r|$ so $|t|\geq\sqrt{2}N^{\delta}$ in this region. Using this fact, we conclude that the quantity in (5.3.39) is bounded above by

$$\exp\left[-\frac{1}{2}\log\left|\frac{\xi^2+(x(t)+x(r))\xi-rt}{q_{\mathfrak{R}}^2-\xi^2}\right|\right] \leq \exp\left[-\frac{1}{2}\log\left|\frac{rt}{2(q_{\mathfrak{R}}^2-\xi^2)}\right|\right]$$
$$\leq \exp\left[-\frac{1}{2}\log\left|\frac{N^\delta}{2(q_{\mathfrak{R}}^2-\xi^2)}\right|\right] \leq \sqrt{2(q_{\mathfrak{R}}^2-\xi^2)}N^{-\frac{\delta}{2}} \quad (5.3.41)$$

In either case, we conclude that the contribution from the $i = 1$ term is at most $\sqrt{(q_{\mathfrak{R}}^2-\xi^2)}N^{-\frac{\delta}{2}}$.

Next, we need to bound the contribution from the $2 \leq i \leq N$ terms in the last line of (5.3.38). We use the notation $X_i = a_1 - a_i + \xi N^{-\frac{1}{3}}$ and $Y = (x(t)+x(r))N^{-\frac{1}{3}}$. Note that both X_i and Y depend on N and Y also depends on r, t. Furthermore, Y is negative for all r, t in this region. Using this notation, we see that, on the event $\mathcal{E}_\varepsilon$, the contribution from the $2 \leq i \leq N$ terms in the last line of (5.3.38) becomes

$$\left|\exp\left[-\frac{1}{2}\sum_{i=2}^{N}\log\left(\frac{X_i^2+X_iY-rtN^{-\frac{2}{3}}+iX_i(r+t)N^{-\frac{1}{3}}}{(X_i+(q_{\mathfrak{R}}-\xi)N^{-\frac{1}{3}})^2-\xi^2N^{-\frac{2}{3}}}\right)\right]\right|$$
$$=\exp\left[-\frac{1}{4}\sum_{i=2}^{N}\log\left(\frac{X_i^4+(X_iY-rtN^{-\frac{2}{3}})^2+2X_i^3Y-2X_i^2rtN^{-\frac{2}{3}}+X_i^2(r+t)^2N^{-\frac{2}{3}}}{\left((X_i+(q_{\mathfrak{R}}-\xi)N^{-\frac{1}{3}})^2-\xi^2N^{-\frac{2}{3}}\right)^2}\right)\right]. \quad (5.3.42)$$

Combining the last two terms in the numerator and removing a negative term from the denominator, we see that the last line above has upper bound

$$\exp\left[-\frac{1}{4}\sum_{i=2}^{N}\log\left(\frac{X_i^4+(X_iY-rtN^{-\frac{2}{3}})^2+2X_i^3Y+X_i^2(r^2+t^2)N^{-\frac{2}{3}}}{(X_i+(q_{\mathfrak{R}}-\xi)N^{-\frac{1}{3}})^4}\right)\right]$$
$$\leq \exp\left[-\frac{1}{4}\sum_{i=2}^{N}\log\left(\frac{X_i^4+2X_i^3Y}{(X_i+(q_{\mathfrak{R}}-\xi)N^{-\frac{1}{3}})^4}\right)\right]$$
$$= \exp\left[-\frac{1}{4}\sum_{i=2}^{N}\log\left(1+\frac{(2Y-4(q_{\mathfrak{R}}-\xi)N^{-\frac{1}{3}})X_i^3+O(N^{-\frac{2}{3}}X_i^2(q_{\mathfrak{R}}-\xi)^2)}{(X_i+(q_{\mathfrak{R}}-\xi)N^{-\frac{1}{3}})^4}\right)\right] \quad (5.3.43)$$
$$\leq \exp\left[-\frac{1}{4}\sum_{i=2}^{N}\log\left(1-\frac{2[C(q_{\mathfrak{R}}-\xi)-x(t)-x(r)]N^{-\frac{1}{3}}}{a_1-a_i}\right)\right]$$

where the last line holds for some constant C sufficiently large. Because t and r have order at most N, we can choose $\Delta < \frac{1}{3}$ and apply the Taylor approximation for $\log$ to conclude that this quantity is bounded above by

$$\exp\left[\frac{1}{4}\sum_{i=2}^{N}\left(\frac{2[C(q_{\mathfrak{R}}-\xi)-x(t)-x(r)]N^{-\frac{1}{3}}}{a_1-a_i}+\left(\frac{2[C(q_{\mathfrak{R}}-\xi)-x(t)-x(r)]N^{-\frac{1}{3}}}{a_1-a_i}\right)^2\right)\right]$$
$$=\exp\left[\frac{C(q_{\mathfrak{R}}-\xi)-x(t)-x(r)}{2}\sum_{i=2}^{N}\frac{N^{-\frac{1}{3}}}{a_1-a_i}+\left(C(q_{\mathfrak{R}}-\xi)-x(t)-x(r)\right)^2N^{-\frac{2}{3}}\sum_{i=2}^{N}\frac{1}{(a_1-a_i)^2}\right]$$
$$\leq\exp\left[\frac{C(q_{\mathfrak{R}}-\xi)-x(t)-x(r)}{2}(1+O(N^{-\frac{1}{3}+\varepsilon}))+O(N^{2\Delta-\frac{2}{3}+\varepsilon})\right]. \tag{5.3.44}$$

If we further require $\Delta<\frac{1}{3}-\frac{\varepsilon}{2}$, then we can replace $O(N^{2\Delta-\frac{2}{3}+\varepsilon})$ with $O(1)$. Finally, we plug this upper bound back into (5.3.38) along with the contribution from the $i=1$ term of the summation and we find that the integral is bounded above by

$$\begin{aligned}
&\int\int\exp\left(-\beta\frac{2(q_{\mathfrak{R}}-\xi)-x(t)-x(r)}{2}\right)\cdot\frac{\sqrt{2(q_{\mathfrak{R}}^2-\xi^2)}}{N^{\frac{\delta}{2}}}\\
&\qquad\cdot\exp\left[\frac{C(q_{\mathfrak{R}}-\xi)-x(t)-x(r)}{2}(1+O(N^{-\frac{1}{3}+\varepsilon}))+O(1)\right]\mathrm{d}t\;\mathrm{d}r\\
&=\exp\left(\left(\frac{C}{2}-\beta\right)(q_{\mathfrak{R}}-\xi)(1+o(1))\right)\frac{\sqrt{2(q_{\mathfrak{R}}^2-\xi^2)}}{N^{\frac{\delta}{2}}}\\
&\qquad\cdot\int\int\exp\left(-\frac{\beta-1+O(N^{-\frac{1}{3}+\varepsilon})}{2}(-x(t)-x(r))\right)\mathrm{d}t\;\mathrm{d}r\\
&=O(N^{1-\frac{\varepsilon}{2}}\log N)\int\int\exp\left(-\frac{\beta-1+O(N^{-\frac{1}{3}+\varepsilon})}{2}(-x(t)-x(r))\right)\mathrm{d}t\;\mathrm{d}r.
\end{aligned}\tag{5.3.45}$$

Since the term in front of the integral has sublinear growth, it now suffices to show that the integral has exponentially shrinking order. To achieve this, we will transform the integral to polar coordinates where $t=R\cos\theta$ and $r=R\sin\theta$. Observe that, because of how $x(t)$ and $x(r)$ are defined, the quantity $-x(t)-x(r)$ is always positive in this region. Furthermore, the integral that we get on this region (between the circle of radius $2N^{\delta}$ and the square of side length $6N(\gamma_{\mathfrak{R}}-\lambda_N+a)$) is bounded above by the integral on the whole region outside the circle of radius $2N^{\delta}$. If we now consider the integral on this infinite region, we observe that the value of the integral is the same on each octant (i.e. the integral for $\theta\in(0,\frac{\pi}{4})$ and $R\in(2N^{\delta},\infty)$ is the same as if we took $\theta\in\left(\frac{k\pi}{4},\frac{(k+1)\pi}{4}\right)$ for any integer k). Thus, it suffices to consider the integral over $0<\theta<\frac{\pi}{4}$ and multiply the result by 8. In particular, it suffices to show that the following integral tends to zero exponentially.

$$\int_{2N^\delta}^{\infty} \int_{0}^{\frac{\pi}{4}} \exp(-C(-x(R\cos\theta) - x(R\sin\theta)))\mathrm{d}\theta \ \mathrm{d}R \tag{5.3.46}$$

On this region of integration, we have $R\cos\theta > \sqrt{2}N^\delta$ and thus

$$-x(R\cos\theta) = -1 + (R\cos\theta - N^\delta + 1)^\Delta \tag{5.3.47}$$

However, the picture for $-x(R\sin\theta)$ is a bit more complicated. We get

$$-x(R\sin\theta) = \begin{cases} -1 + (R\sin\theta - N^\delta + 1)^\Delta & \theta < \arcsin\left(\frac{N^\delta}{R}\right) \\ \text{in } [-1, 0] & \theta > \arcsin\left(\frac{N^\delta}{R}\right) \end{cases} \tag{5.3.48}$$

Therefore, we can split the integral into two parts, one with $\theta \in [0, \arcsin(\frac{N^\delta}{R})]$ and the second with $\theta \in [\arcsin(\frac{N^\delta}{R}), \frac{\pi}{4}]$. We begin by computing the second of these integrals.

$$\begin{aligned} &\int_{2N^\delta}^{\infty} \int_{\arcsin(\frac{N^\delta}{R})}^{\frac{\pi}{4}} R\exp(-C(-x(R\cos\theta) - x(R\sin\theta)))\mathrm{d}\theta \ \mathrm{d}R \\ &\leq \int_{2N^\delta}^{\infty} \int_{\arcsin(\frac{N^\delta}{R})}^{\frac{\pi}{4}} R\exp\left(-C(-2 + (R\cos\theta - N^\delta + 1)^\Delta + (R\sin\theta - N^\delta + 1)^\Delta)\right)\mathrm{d}\theta \ \mathrm{d}R \end{aligned} \tag{5.3.49}$$

Using the inequality $||z||_1 \geq ||z||_{2/\Delta}$, the quantity above has upper bound

$$\begin{aligned} &e^{2C} \int_{2N^\delta}^{\infty} \int_{\arcsin(\frac{N^\delta}{R})}^{\frac{\pi}{4}} R\exp\left(-C\left[(R\cos\theta - N^\delta + 1)^2 + (R\sin\theta - N^\delta + 1)^2\right]^{\frac{\Delta}{2}}\right)\mathrm{d}\theta \ \mathrm{d}R \\ &\leq e^{2C} \int_{2N^\delta}^{\infty} \int_{\arcsin(\frac{N^\delta}{R})}^{\frac{\pi}{4}} R\exp\left(-C\left[R^2 - 2\sqrt{2}(N^\delta - 1)R + 2(N^\delta - 1)^2\right]^{\frac{\Delta}{2}}\right)\mathrm{d}\theta \ \mathrm{d}R \\ &= e^{2C} \int_{2N^\delta}^{\infty} \int_{\arcsin(\frac{N^\delta}{R})}^{\frac{\pi}{4}} R\exp\left(-C\left[R - \sqrt{2}(N^\delta - 1)\right]^{\Delta}\right)\mathrm{d}\theta \ \mathrm{d}R. \end{aligned} \tag{5.3.50}$$

Since the integrand no longer depends on θ, this has upper bound

$$\begin{aligned}
&\frac{\pi}{4}e^{2C}\int_{2N^\delta}^{\infty} R\exp\left(-C\left[R-\sqrt{2}N^\delta\right]^\Delta\right)\mathrm{d}R \\
&=\frac{\pi}{4}e^{2C}\left[\frac{R}{-C(R-\sqrt{2}N^\delta)^{1+\Delta}}\exp\left(-C\left[R-\sqrt{2}N^\delta\right]^\Delta\right)\right]_{2N^\delta}^{\infty} \\
&=\frac{\pi}{4}e^{2C}\left[\frac{2N^\delta}{C(2N^\delta-\sqrt{2}N^\delta)^{1+\Delta}}\exp\left(-C\left[2N^\delta-\sqrt{2}N^\delta\right]^\Delta\right)-0\right] \\
&=C'N^{-\delta\Delta}\exp\left(-C''N^{\delta\Delta}\right)
\end{aligned} \tag{5.3.51}$$

where C' and C'' are constants. Thus, we conclude that

$$\int_{2N^\delta}^{\infty}\int_{\arcsin(\frac{N^\delta}{R})}^{\frac{\pi}{4}} R\exp(-C(-x(R\cos\theta)-x(R\sin\theta)))\mathrm{d}\theta\,\mathrm{d}R \le C'N^{-\delta\Delta}\exp\left(-C''N^{\delta\Delta}\right). \tag{5.3.52}$$

This integral tends to zero exponentially and it remains to show that the other integral does as well.

$$\begin{aligned}
&\int_{2N^\delta}^{\infty}\int_{0}^{\arcsin(\frac{N^\delta}{R})} R\exp(-C(-x(R\cos\theta)-x(R\sin\theta)))\mathrm{d}\theta\,\mathrm{d}R \\
&\le \int_{2N^\delta}^{\infty}\int_{0}^{\arcsin(\frac{N^\delta}{R})} R\exp(-C(-2+(R\cos\theta-N^\delta+1)^\Delta))\mathrm{d}\theta\,\mathrm{d}R \\
&\le e^{2C}\int_{2N^\delta}^{\infty}\int_{0}^{\arcsin(\frac{N^\delta}{R})} R\exp\left(-C\left(R\sqrt{1-\left(\frac{N^\delta}{R}\right)^2}-N^\delta+1\right)^\Delta\right)\mathrm{d}\theta\,\mathrm{d}R
\end{aligned} \tag{5.3.53}$$

Since the integrand no longer depends on θ, it can be simplified as

$$\begin{aligned}
&e^{2C}\int_{2N^\delta}^{\infty} R\arcsin\left(\frac{N^\delta}{R}\right)\exp\left(-C\left(\sqrt{R^2-N^{2\delta}}-N^\delta+1\right)^\Delta\right)\mathrm{d}R \\
&\le e^{2C}\int_{2N^\delta}^{\infty} R\cdot\frac{\pi}{6}\exp\left(-C\left(\sqrt{R^2-N^{2\delta}}-N^\delta\right)^\Delta\right)\mathrm{d}R.
\end{aligned} \tag{5.3.54}$$

At this point we make a change of variable $T=\sqrt{R^2-N^{2\delta}}$ and note that $dT=\frac{R}{T}\mathrm{d}R$. Then the last line of the inequality above becomes

$$\frac{\pi}{6}e^{2C}\int_{\sqrt{3}N^\delta}^{\infty} T\exp\left(-C(T-N^\delta)^\Delta\right)dT \tag{5.3.55}$$

The computation of this integral follows as in (5.3.51) and we conclude that, for some constants C'

and C'' (not necessarily the same values as before) this integral is bounded above by

$$C' N^{-\delta\Delta} \exp\left(-C'' N^{\delta\Delta}\right) \tag{5.3.56}$$

Again, we have exponential decay, so we conclude that the A part decays exponentially on the region between the circle of radius $2N^\delta$ and the square of side length $6N(\gamma_{\mathfrak{R}} - \lambda_N + a)$.

Showing the B part has at most polynomial growth

In each of the cases above, the integral of the A part shrinks exponentially with N. Therefore, it is not necessary to show that the B part is bounded, but it suffices to show that it grows as a polynomial of N.

$$\begin{aligned}
&\left|\exp\left(\frac{N}{2}\left(B(\lambda_1 + a + u(t)N^{-1}, \lambda_1 + a + v(r)N^{-1}; a) - B(\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}}; a)\right)\right)\right| \\
&\leq \exp\left(\frac{N}{2}\Re(B(\lambda_1 + a + u(t)N^{-1}, \lambda_1 + a + v(r)N^{-1}; a))\right) \\
&= \exp\left(\frac{H^2\beta}{2N}\sum_{i=1}^{N}\Re\tfrac{n_i^2(2\lambda_1+2a(x(t)+x(r))N^{-1}-2\lambda_i+2a+i(y(r)+y(t))N^{-1})}{(\lambda_1+a-\lambda_i+x(t)N^{-1}+iy(t)N^{-1})(\lambda_1+a-\lambda_i+x(r)N^{-1}+iy(r)N^{-1})-a^2}\right) \\
&= \exp\left(\frac{H^2\beta}{2N}\sum_{i=1}^{N}\frac{\text{numerator}}{\text{denominator}}\right)
\end{aligned} \tag{5.3.57}$$

where the numerator and denominator are given as follows:

$$\begin{aligned}
&\text{numerator} = \\
&(2(\lambda_1-\lambda_i+2a)+(x(t)+x(r))N^{-1})\left[(\lambda_1-\lambda_i+a+x(t)N^{-1})(\lambda_1-\lambda_i+a+x(r)N^{-1})-y(t)y(r)N^{-2}-a^2\right] \\
&\qquad -(y(t)+y(r))\left[(y(t)+y(r))(\lambda_1-\lambda_i+a)N^{-1}+(x(t)y(r)+x(r)y(t))N^{-2}\right]N^{-2}
\end{aligned} \tag{5.3.58}$$

$$\begin{aligned}
\text{denominator} = &\left[(\lambda_1 - \lambda_i + a + x(t)N^{-1})(\lambda_1 - \lambda_i + a + x(r)N^{-1}) - y(t)y(r)N^{-2} - a^2\right]^2 \\
&+ \left[y(t)(\lambda_1 - \lambda_i + a + x(r)N^{-1})N^{-1} + y(r)(\lambda_1 - \lambda_i + a + x(t)N^{-1})N^{-1}\right]^2
\end{aligned} \tag{5.3.59}$$

It is straightforward to show that when $i = 1$ this fraction has order 1. Now show that the fraction is of polynomial order (uniformly for $2 \leq i \leq N$). We assume without loss of generality that $|t| \geq N^\delta$. I will also assume for now that $|r| \geq 1$ (it should be relatively easy to separately check for $|r| < 1$). Given these assumptions, we have $y(t) = t$ and $y(r) = r$.

The second line of the expression for the numerator will negative, so we have

$$\begin{aligned}
&\text{numerator} \\
&\leq (2(\lambda_1 - \lambda_i + 2a) - |x(t) + x(r)|N^{-1}) \\
&\qquad \cdot \left[(\lambda_1 - \lambda_i + a)^2 + x(t)x(r)N^{-2} - (\lambda_1 - \lambda_i)|x(t) + x(r)|N^{-1} - trN^{-2} - a^2\right] \\
&\leq 2(\lambda_1 - \lambda_i + 2a)^3 + (x(t) + x(r))^2(\lambda_1 - \lambda_i + a)N^{-1} \\
&\qquad + |tr|N^{-2}[(\lambda_1 - \lambda_i + 2a) + |x(t) + x(r)|N^{-1}] \\
&= O(1) + O(|t|^{2\Delta}N^{-1}) + O(|tr|N^{-2}) + O(|tr| \cdot |t|^{\Delta}N^{-3})
\end{aligned} \tag{5.3.60}$$

For the denominator, we have:

- If $|rt| \gg N^{\frac{2}{3}}$ then denominator = $\Omega(r^2t^2N^{-4})$
- If $|rt| \ll N^{\frac{2}{3}}$ then denominator = $\Omega((\lambda_1 - \lambda_i + a)^4)$
- If $|rt| = \Theta(N^{\frac{2}{3}})$ then denominator = $\Omega(r^2t^2(\lambda_1 - \lambda_i)^2N^{-2})$

Putting this information together, we see that, when $|rt|$ is sufficiently large (e.g. $|rt| \geq N$) then the fraction has order

$$O\left(\frac{|t|^{2\Delta}N^3 + |tr|N^2 + |tr| \cdot |t|^{\Delta}N}{r^2t^2}\right)$$

This is of polynomial order since $r, t > 1$ and have larger order in the denominator. On the other hand, for smaller values of $|rt|$ (e.g. $|rt| < N$), the numerator has order 1 and the denominator is bounded below by a negative power of N. Thus, the growth is again of polynomial order.

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