Propagation and Reduction of Coherent States in Bargmann Spaces

by

Jenia Rousseva

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Applied and Interdisciplinary Mathematics)
in the University of Michigan
2022

Doctoral Committee:

Professor Christine Aidala, Co-Chair
Professor Alejandro Uribe-Ahumada, Co-Chair
Professor Daniel Burns Jr.
Professor John Schotland, Yale University

Jenia Rousseva

jenia@umich.edu

ORCID iD: 0000-0002-7900-7840

Dedication

To my parents

Acknowledgments

First and foremost, I would like to thank my main advisor Alejandro Uribe-Ahumada for all of your guidance and encouragement as I dove into a field that was new to me. I will never forget the kindness and patience your have shown me over these past years.

I would also like to thank my other committee members: Christine Aidala, Daniel Burns, and John Schotland for reading my thesis and for your thought-provoking questions.

I am grateful to the University of Michigan Department of Mathematics not only for the opportunities to challenge myself intellectually, but also for the warm community it has provided me. Thank you to my peers who have always made me feel like we are “in this together” throughout all the trials and tribulations of graduate school. A special thank you goes to the burger group: Will and Jade Clark, Anthony Della Pella, Jon Guzman, Marshall Scott, and Jerry Waters for your friendship and many wonderful memories.

Last, but not least, thank you to my parents for their unwavering support throughout my life.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF APPENDICES

LIST OF SYMBOLS

ABSTRACT

Coherent states are special types of wavefunctions that minimize a generalized uncertainty principle for a suitable pair of operators. Equivalently, they are eigenstates of an appropriate annihilation operator. Their applications are extensive throughout physics including in quantum optics, nuclear physics, quantum field theory, path integral formulations, and quantum information through the study of entanglement and quantum measurement.

This thesis explores two main topics. First, we consider the Schrödinger evolution of a Gaussian coherent state under a non-Hermitian Hamiltonian. We develop a symbol calculus and use it to construct an approximate solution to the time-dependent Schrödinger equation. We find the evolution equations of the center and the Gaussian matrix of the coherent state, which form a system. This result generalizes the previously-known case where the classical Hamiltonian is quadratic.

In the second part of the thesis, we apply a quantum version of dimensional reduction to construct Gaussian coherent states in the Bargmann space of complex projective space. The semiclassical properties of these reduced states are controlled by a suitable notion of symbol. Making use of these properties, we provide norm estimates and a propagation result for Hermitian Hamiltonians. As a special case of these reduced states, we define and examine spin-squeezed states that live naturally in the Bargmann space of the Riemann sphere.

CHAPTER I

General Introduction

The first discovery of "coherent states" (although they were not called by this name at the time) was in 1926 by Erwin Schrödinger who aimed to find solutions of the quantum harmonic oscillator that most closely resembles the oscillating behavior of the classical harmonic oscillator; or, in other words, the states that minimize the Heisenberg uncertainty principle [Sch26]. As a result, coherent states are often referred to as minimum uncertainty states.

The term "coherent states" was introduced in the context of quantum optics by Roy Glauber in 1963 [Gla63]. Glauber introduced these states as superpositions of Fock states of the quantized electromagnetic field, that up to a complex factor, are not modified by the action of photon annihilation operators. His work gave rise to the definition of coherent states as eigenstates of an annihilation operator. Today, we refer to Glauber's states as *standard* or *canonical* coherent states. They describe a reservoir with an undetermined number of photons, which is in a sense "close" to the classical description where the concept of a photon is nonexistent [Gaz09].

Coherent states are ubiquitous in quantum mechanics and we shall not attempt to give an exhaustive list of their applications. They appear in nuclear, atomic, and condensed matter physics, quantum field theory, quantization and de-quantization problems, path integral formulations and, more recently, quantum information in the analysis of entanglement or quantum measurement. Some good references on the theory and applications of coherent states are [KsS85, Per86, ZFG90, Gaz09].

1.1 Coherent states in $L^2(\mathbb{R}^d)$

Most commonly, coherent states are defined in the Hilbert space $L^2(\mathbb{R}^d)$. We proceed to describe how they arise in this setting. However, this will not be the preferred Hilbert space in this thesis. In the following section, we will introduce the Bargmann space of $\mathbb{C}^d$ and present an argument for studying coherent states in this space rather than in $L^2(\mathbb{R}^d)$. At the end of this chapter, we will present a definition of Gaussian coherent states in the Bargmann space of $\mathbb{C}^d$.

For the rest of this section, we will quote freely from Combescure and Robert [CR12].

Let $\widehat{X} = (\widehat{X}_1, \ldots, \widehat{X}_d)$ where $\widehat{X}_j$ is the multiplication operator by the coordinate x_j for $j = 1, \ldots, d$. Similarly, define $\widehat{P} = (\widehat{P}_1, \ldots, \widehat{P}_d)$ be the momentum operator in $L^2(\mathbb{R}^d)$ where $\widehat{P}_j = -i\hbar\frac{\partial}{\partial x_j}$ for $j = 1, \ldots, d$. The operators $\widehat{X}$ and $\widehat{P}$ are self-adjoint and satisfy the Heisenberg commutation relation

$$\left[\widehat{X}_j, \widehat{P}_k\right] = \delta_{j,k} i\hbar \, \mathrm{I} \, .$$

where $\delta_{j,k}$ is the Kronecker delta and I is the identity operator. Furthermore, we define the *standard annihilation and creation operators*, respectively, as

$$a_j = \frac{1}{\sqrt{2\hbar}} \left(\widehat{X}_j + i\widehat{P}_j\right), \qquad a_j^* = \frac{1}{\sqrt{2\hbar}} \left(\widehat{X}_j - i\widehat{P}_j\right)$$

for $j = 1, \ldots, d$. These operators satisfy the *canonical commutation relations (CCRs)*

$$[a_j, a_k^*] = \delta_{j,k} \, \mathrm{I} \, . \tag{1.1}$$

The $d-$dimensional *quantum harmonic oscillator* of frequency 1 is

$$\widehat{H}_{os} = \tfrac{1}{2} \left(\widehat{X}^2 + \widehat{P}^2\right)$$

or equivalently, in terms of annihilation and creation operators

$$\widehat{H}_{os} = \hbar \sum_{j=1}^{d} \left(a_j^* a_j + \frac{d}{2} \right) .$$

The ground state, or the lowest energy state, of $\widehat{H}_{os}$ is a Gaussian function in $L^2(\mathbb{R}^d)$ given by

$$\varphi_0(x) := (\pi\hbar)^{-d/4} e^{-\hbar^{-1} x x^T/2}.$$

This is the simplest example of a *coherent state* that we will see. One may verify that φ_0 is an eigenstate of the annihilation operator a with eigenvalue 0. According to Proposition 12.10 of [Hal13], the fact that φ_0 is an eigenstate of a implies that φ_0 minimizes Heisenberg's uncertainty principle. This means we get equality in the product of the uncertainty (standard deviation) in the measurement of each of the position and momentum operators in the state φ_0:

$$\left(\Delta_{\varphi_0} \widehat{X}_j \right) \left(\Delta_{\varphi_0} \widehat{P}_j \right) = \frac{\hbar}{2}, \quad j = 1, \ldots, d\,. \tag{1.2}$$

More precisely, the uncertainty in the measurement of each of the operators in the state φ_0 is the same: $\Delta_{\varphi_0} \widehat{X}_j = \Delta_{\varphi_0} \widehat{P}_j = \sqrt{\hbar/2}$.

The *Weyl-Heisenberg translation operator* is a unitary operator represented by $\widehat{T}_Z$ whose action on a state $\varphi \in L^2(\mathbb{R}^d)$ is expressed as

$$\left(\widehat{T}_Z \varphi \right)(x) = e^{-i\hbar^{-1}\, q\, p^T/2}\, e^{i\hbar^{-1}\, x\, p^T}\ \varphi(x - q).$$

Physically speaking, $\widehat{T}_Z$ translates the state φ by $Z = (q, p)$ in phase space $\mathbb{R}^{2d}$. Hence, we can apply the translation operator to the state φ_0 to shift its center from $(0, 0)$ to $Z = (q, p)$. This results in

$$\varphi_Z(x) = \left(\widehat{T}_Z \varphi_0 \right)(x) = (\pi\hbar)^{-d/4}\, e^{-i\hbar^{-1}\, q\, p^T/2}\, e^{i\hbar^{-1}\, x\, p^T}\, e^{-\hbar^{-1}(x-q)(x-q)^T/2}.$$

The states φ_Z are the *standard coherent states* introduced by Schrödinger in 1926. Note that these states are localized in a neighborhood of size $\sqrt{\hbar}$ around the point Z in all of the position and momentum coordinates.

A natural question arises: is φ_Z also a minimum uncertainty state? The answer is yes; if we define $\alpha = \frac{1}{\sqrt{2}}(q + ip)$, one may check that $a\varphi_Z = \alpha\varphi_Z$, so φ_Z an eigenstate of the annihilation operator. Hence, by Proposition 12.10 in [Hal13], φ_Z satisfies (1.2).

A more general class of Gaussian coherent states in $L^2(\mathbb{R}^d)$ are the so-called *squeezed coherent states*. A squeezed state centered at the point $(0,0) \in \mathbb{R}^{2d}$ is defined as

$$\varphi_0^\Gamma(x) = c_\Gamma\, e^{i\,\hbar^{-1}\, x\Gamma x^T/2} \tag{1.3}$$

where Γ is an $d \times d$ complex symmetric matrix and $\Im\Gamma > 0$. This condition ensures that the state is in $L^2(\mathbb{R}^d)$ and also controls that "squeezing" behavior of the state. The factor

$$c_\Gamma = (\pi\hbar)^{-d/4}\, \det(\Im\Gamma)^{1/4}$$

is chosen so that the L^2-norm of the state is equal to one. Observe that $\Gamma = iI_d$, where I_d is the $d-$dimensional identity matrix, gives exactly φ_0, so one may think of squeezed states as generalizations of the ground state of the harmonic oscillator and the standard coherent states.

Once again, we may construct squeezed states with center $Z = (q,p)$ in phase space by applying the translation operator:

$$\varphi_Z^\Gamma(x) = \left(\widehat{T}_Z\varphi_0^\Gamma\right)(x) = c_\Gamma\, e^{i\hbar^{-1}\left[-\frac{1}{2}qp^T + xp^T + \frac{1}{2}(x-q)\Gamma(x-q)^T\right]}. \tag{1.4}$$

The squeezed coherent states are also minimum uncertainty states, but for a different suitable pair of operators $\widehat{\mathcal{X}}_j$ and $\widehat{\mathcal{P}}_j$, respectively, that are linear combinations of the operators $\widehat{X}_j$ and $\widehat{P}_j$. In that sense, $\widehat{\mathcal{X}}_j$ and $\widehat{\mathcal{P}}_j$ satisfy (1.2) for each j.

1.2 Bargmann Space and the Bargmann Transform

While most of the existing literature on coherent states considers them as elements in $L^2(\mathbb{R}^d)$, they can also be defined as objects in the Bargmann space[1] of $\mathbb{C}^d$ as we shall see in §1.3. In fact, we will be working in Bargmann spaces throughout the rest of this thesis. One advantage to working in the Bargmann space of $\mathbb{C}^d$ is that it is a Hilbert space of functions on phase space $\mathbb{R}^{2d} \cong \mathbb{C}^d$, the natural setting of Hamiltonian mechanics, rather than on configuration space $\mathbb{R}^d$. Another reason we favor the Bargmann space of $\mathbb{C}^d$ over $L^2(\mathbb{R}^d)$ is because the SU(2) states, *or spin-squeezed states* that we construct in Chapter VII "live" more naturally in the Bargmann space of the Riemann sphere. In the remaining part of this section, we present some of the history and important properties of the Bargmann space of $\mathbb{C}^d$.

The Bargmann space of $\mathbb{C}^d$ and its properties were first outlined by Valentine Bargmann in 1961 [Bar61]. We will quote some results from his paper, but avoid many of the technical details.

According to Bargmann, in 1928, Fock wanted to find operators that satisfy the canonical commutation relations for the annihilation and creation operators given in (1.1), but for a space of holomorphic functions on $\mathbb{C}^d$, i.e., functions that are independent of the coordinates $\overline{z}_j$ for $j = 1, \ldots, d$. He found that the operators $\hbar \frac{\partial}{\partial z_j}$ and z_j, where the latter represents multiplication by the coordinate z_j, satisfy (1.1). However, this alone does not give a representation of the CCRs because they require the existence of a Hilbert space *and* annihilation and creation operators $a_j^{\mathcal{B}}$ and $a_j^{*\mathcal{B}}$ that are adjoints of each other and satisfy $[a_j^{\mathcal{B}}, a_k^{*\mathcal{B}}] = \delta_{j,k} \hbar \, \mathrm{I}$ [Hal00]. Hence, Bargmann aimed to find an inner product on the space of holomorphic functions on $\mathbb{C}^d$ such that $\hbar \frac{\partial}{\partial z_j}$ and z_j are adjoints of one another. It turns out that these operators are precisely the desired annihilation and creation operators, respectively,

[1] Sometimes this is referred to as the Segal-Bargmann space as in [Hal00], and other authors such as [CR12] refer to it as the Bargmann-Fock space. We will stick with the shorter name of "Bargmann space."

which we label as:

$$a_j^{\mathcal{B}} = \hbar\frac{\partial}{\partial z_j}, \qquad a_j^{*\mathcal{B}} = \text{multiplication by } z_j, \qquad j = 1, \ldots, d\,. \tag{1.5}$$

One can also define "position" and "momentum" operators in this space as

$$A_j^{\mathcal{B}} = \frac{1}{\sqrt{2}}\left(a_j^{*\mathcal{B}} + a_j{}^{\mathcal{B}}\right), \qquad B_j^{\mathcal{B}} = \frac{i}{\sqrt{2}}\left(a_j^{*\mathcal{B}} - a_j{}^{\mathcal{B}}\right), \qquad j = 1, \ldots, d\,.$$

After some calculations to find the appropriate weight function ρ, which an interested reader may read about in [Bar61], Bargmann defines the inner product on this space as

$$\langle f, g\rangle := \int \overline{f(z)}\, g(z)\, \rho(z)\, dL(z)$$

where $dL(z)$ denotes the $2d-$dimensional Lebesgue measure on $\mathbb{C}^d$ and

$$\rho(z) = (\pi\hbar)^{-d} e^{-\hbar^{-1}|z|^2}.$$

Using this information, we can define the Bargmann space of $\mathbb{C}^d$ as given in [Hal00].

Definition I.1. The *Bargmann space*, denoted by $\mathcal{B}(\mathbb{C}^d)$, is the space of holomorphic functions $f : \mathbb{C}^d \to \mathbb{C}$ which satisfy the square-integrability condition:

$$\|f\|^2_{\mathcal{B}(\mathbb{C}^d)} = (\pi\hbar)^{-d} \int_{\mathbb{C}^d} |f(z)|^2\, e^{-\hbar^{-1}|z|^2}\, dL(z) < \infty,$$

provided that $dL(z)$ is the $2d-$dimensional Lebesgue measure on $\mathbb{C}^d$.

Bargmann also found a unitary map $V : L^2(\mathbb{R}^d) \to \mathcal{B}(\mathbb{C}^d)$ that intertwines the annihilation and creation operators in $L^2(\mathbb{R}^d)$ with those in (1.5). This is given by

$$(Vf)(z) = \int_{\mathbb{R}^d} \mathcal{A}(z, x)\, f(x)\, dx \tag{1.6}$$

where the *kernel* $\mathcal{A}(z,x)$ is

$$\mathcal{A}(z,x) = (\pi\hbar)^{-d/4}\, e^{-\hbar^{-1}(zz^T - 2\sqrt{2}\,xz^T + xx^T)/2} \qquad x \in \mathbb{R}^d, z \in \mathbb{C}^d\,.$$

We also give the map introduced by Bargmann a formal name:

Definition I.2. The unitary map $V : L^2(\mathbb{R}^d) \to \mathcal{B}(\mathbb{C}^d)$ defined in (1.6) is the *Bargmann transform.*

One can further prove that V is an isometry. Hence, we can think of the Bargmann space of $\mathbb{C}^d$ as simply a different, but unitarily equivalent representation of the CCRs.

Next, we will state a few more interesting properties about the Bargmann space. We will quote results from [Hal00].

Lemma I.3. *1. For all $z \in \mathbb{C}^d$, $\exists$ a constant C_z such that*

$$|f(z)|^2 \leq C_z \, \|f\|^2_{L^2(\mathbb{C}^d,\rho)}, \qquad \forall f \in \mathcal{B}(\mathbb{C}^d).$$

2. $\mathcal{B}(\mathbb{C}^d)$ is a closed subspace of $L^2(\mathbb{C}^d,\rho)$, and is therefore a Hilbert space.

A detailed proof of this lemma can be found in [Hal00]. The first point tells us that pointwise evaluation is continuous, meaning that for each $z \in \mathbb{C}^d$, the map $\mathcal{B}(\mathbb{C}^d) \ni f \mapsto f(z)$ is a continuous linear functional on $\mathcal{B}(\mathbb{C}^d)$. This property for holomorphic function spaces does not exist for non-holomorphic L^2-spaces.

Remark 1. Since we may express a holomorphic function as a power series around the origin, Bargmann proved that the set of orthonormal basis vectors in Bargmann space is given by $\{z^n/\sqrt{n!}\}_{n=0}^{\infty}$ [Bar61]. Therefore, we may decompose holomorphic functions in terms of homogeneous polynomials. This will be useful in Chapter V.

A remarkable property of Bargmann space is that it has a *reproducing kernel* given by

$$K(z,w) = e^{\hbar^{-1} z\overline{w}^T}, \qquad z, w \in \mathbb{C}^d. \tag{1.7}$$

Integrating a function in the Bargmann space of $\mathbb{C}^d$ against the kernel gives back (reproduces) the function itself as shown in property (2) of the following theorem.

Theorem I.4. *The reproducing kernel in* (1.7) *has the following properties:*

(1) $K(z,w)$ *is holomorphic in* z *and anti-holomorphic in* w*, and satisfies*

$$K(w,z) = \overline{K(z,w)}.$$

(2) For each fixed $z \in \mathbb{C}^d$*,* $K(z,w)$ *is square-integrable* $d\rho(w)$*. For all* $f \in \mathcal{B}(\mathbb{C}^d)$*,*

$$f(z) = \int_{\mathbb{C}^d} K(z,w) f(w)\, \rho(w)\, dL(w).$$

(3) If $f \in L^2(\mathbb{C}^d, \rho)$*, let* Πf *denote the orthogonal projection of* f *onto the closed subspace* $\mathcal{B}(\mathbb{C}^d)$*. Then,*

$$\Pi f(z) = \int_{\mathbb{C}^d} K(z,w) f(w) \rho(w)\, dL(w).$$

(4) For all $z, u \in \mathbb{C}^d$*,*

$$K(z,u) = \int_{\mathbb{C}^d} K(z,w)\, K(w,u)\, \rho(w)\, dL(w).$$

(5) For all $z \in \mathbb{C}^d$*,*

$$|f(z)|^2 \leq K(z,z)\, \|f\|^2 = e^{\hbar^{-1}|z|^2} \|f\|^2$$

and the constant $e^{\hbar^{-1}|z|^2}$ *is optimal in the sense that for each* $z \in \mathbb{C}^d$ *there exists a non-zero* $f(z) \in \mathcal{B}(\mathbb{C}^d)$ *for which equality holds.*

(6) Given any $z \in \mathbb{C}^d$*, if* $\psi_z(\cdot) \in \mathcal{B}(\mathbb{C}^d)$ *satisfies*

$$f(z) = \int_{\mathbb{C}^d} \overline{\psi_z(w)}\, f(w)\, \rho(w)\, dL(w).$$

for all $f \in \mathcal{B}(\mathbb{C}^d)$*, then* $\overline{\psi_z(w)} = K(z,w)$.

The requirement that f be a holomorphic function is also important in the definition of Bargmann space. If f were any square-integrable function, it could be localized into an arbitrarily small neighborhood of phase space, which would violate the uncertainty principle. However, since $f \in \mathcal{B}(\mathbb{C}^d)$ is holomorphic and it satisfies the pointwise bound in Property (5) of Theorem I.4, there is a limit to how concentrated f can be in a neighborhood of any point in phase space.

Finally, we would like to define coherent states in Bargmann space. From the point of view of functional analysis, these are the unique elements $\psi_z \in \mathcal{B}(\mathbb{C}^d)$ such that

$$f(z) = \langle \psi_z, f \rangle_{\mathcal{B}(\mathbb{C}^d)}, \qquad \forall f \in \mathcal{B}(\mathbb{C}^d). \tag{1.8}$$

An alternative way to define the standard coherent states in $L^2(\mathbb{R}^d)$ as given in [Hal00] is:

Theorem I.5. *The* $\varphi_z \in L^2(\mathbb{R}^d)$ *are the unique states that satisfy*

$$(Vf)(z) = \langle \varphi_z, f \rangle_{L^2(\mathbb{R}^d)}$$

where $V : L^2(\mathbb{R}^d) \to \mathcal{B}(\mathbb{C}^d)$ *is the Bargmann transform.*

Since the Bargmann transform is unitary,

$$(Vf)(z) = \langle \varphi_z, f \rangle_{L^2(\mathbb{R}^d)} = \langle V\varphi_z, Vf \rangle_{\mathcal{B}(\mathbb{C}^d)}.$$

By comparing the above with (1.8), we have that $V\varphi_z = \psi_z$, which means the *standard coherent states in Bargmann space* can be obtained by applying the Bargmann transform to the standard coherent states in $L^2(\mathbb{R}^d)$. Similarly, we may construct more general Gaussian states in Bargmann space by applying the Bargmann transform to (1.3), as we shall soon observe.

Moreover, recalling Property (6) in Theorem I.4 we see that the *standard coherent state in Bargmann space* is the reproducing kernel, i.e., $\psi_z(w) = \overline{K(z,w)} = K(w,z) = e^{\hbar^{-1}\overline{z}w^T}$. Using the property in (1.8), we arrive at the interesting fact that the reproducing kernel is just the inner product of two standard coherent states:

$$K(z,w) = \langle \psi_z, \psi_w \rangle.$$

1.3 Gaussian Coherent States in Bargmann Space

For the rest of this thesis, we will use a slightly modified version of Definition I.1 for the Bargmann space.[2] We will include the weight $e^{-\hbar^{-1}|z|^2/2}$ in the definition of the space and drop the constant factor $(\pi\hbar)^{-d}$.

More precisely, $\mathcal{B}(\mathbb{C}^d)$ is now the space of functions $\psi : \mathbb{C}^d \to \mathbb{C}$, $\psi(z) = f(z)\, e^{-\hbar^{-1}|z|^2/2}$, where f is a holomorphic function and the following condition is satisfied:

$$\|\psi\|^2_{\mathcal{B}(\mathbb{C}^d)} = \int_{\mathbb{C}^d} |f(z)|^2\, e^{-\hbar^{-1}|z|^2}\, dL(z) < \infty.$$

Remark 2. In the subsequent chapters, we will often take derivatives with respect to z of $\psi \in \mathcal{B}(\mathbb{C}^d)$. We will only differentiate $f(z)$ and not the weight $e^{-\hbar^{-1}|z|^2/2}$ because the resulting ψ would not be in $\mathcal{B}(\mathbb{C}^d)$. To avoid being cumbersome, we will not write this out explicitly every time, but implicitly the calculation we are performing is

$$e^{-\hbar^{-1}|z|^2/2}\, \frac{\partial^n}{\partial z_j^n} \left(e^{\hbar^{-1}|z|^2/2}\, \psi(z) \right), \qquad j = 1, \ldots, d\,.$$

Before we introduce the Gaussian coherent states in $\mathcal{B}(\mathbb{C}^d)$, we must define the generalized

[2] Definition I.1 is more commonly used including in Bargmann's paper [Bar61] and the work of B. Hall [Hal00].

unit disk

$$\mathcal{D}_d = \{d \times d \text{ complex symmetric matrices } A \text{ such that } A^*A < I_d\}$$

where I_d is the $d \times d$ identity matrix.

We also have an analogous version of the translation operator in $\mathcal{B}(\mathbb{C}^d)$.

Definition I.6. Let $\psi = f(z)\, e^{-\hbar^{-1}|z|^2/2} \in \mathcal{B}(\mathbb{C}^d)$ be centered at $z = 0$ in phase space.[3] The translation operator shifts the center of ψ to $z = w$ in the following manner:

$$\left(\widehat{T}_w f\right)(z) = e^{-\hbar^{-1}|w|^2/2}\, e^{\hbar^{-1} z\overline{w}^T} f(z - w).$$

Definition I.7. The *Gaussian coherent states in Bargmann space* are of the form: $\forall A \in \mathcal{D}_d$ and $w \in \mathbb{C}^d$, let $Q_A(z) = zAz^T$ (where $z \in \mathbb{C}^d$ is considered a row vector). The associated state is

$$\psi_{A,w}(z) := e^{\hbar^{-1} Q_A(z-w)/2}\, e^{\hbar^{-1} z\overline{w}^T}\, e^{-\hbar^{-1}|w|^2/2}\, e^{-\hbar^{-1}|z|^2/2}\,. \tag{1.9}$$

$\psi_{A,w}$ is the quantum translation of $\psi_{A,0}$ by w, which is called the *center* of $\psi_{A,w}$.

Remark 3. If $A = 0$, we will refer to ψ_w as a *standard coherent state* in Bargmann space. Whenever, $A \neq 0$, we can think of $\psi_{A,w}$ as a *squeezed coherent state* in Bargmann space.

A natural question is how are these states related to the states φ_Z^Γ in (1.4)? We can obtain $\psi_{A,w}$ by taking the Bargmann transform of φ_Z^Γ (up to some constants possibly). It turns out that (see Proposition 36 in [CR12]) the matrix $A \in \mathcal{D}_d$ in (1.9) is the Cayley transform of the matrix Γ in (1.4):

$$A = -(\Gamma - iI)^{-1}(\Gamma + iI)\,.$$

[3] We will typically identity $\mathbb{R}^{2d} \cong \mathbb{C}^d$ using $z_j = \frac{1}{\sqrt{2}}(q_j - ip_j)$ where $q_j, p_j \in \mathbb{R}$ for $j = 1, \ldots, d$.

1.4 Structure of the Thesis

There are two main parts in this thesis: propagation of coherent states via the Schrödinger equation (Chapters II–IV) and dimensional reduction of coherent states (Chapters V–VII). First, we study the propagation of coherent states under the dynamics of the time-dependent Schrödinger equations for both Hermitian and non-Hermitian quantum Hamiltonians. Although these topics have been studied extensively in the literature for coherent states defined in $L^2(\mathbb{R}^d)$, we will present a new approach for constructing approximate semiclassical solutions using a symbol calculus. These techniques are developed in Chapter II. In Chapter III, we restrict our attention to evolution with Hermitian Hamiltonians. We construct solutions to arbitrary order in $\hbar$ to Schrödinger's equation for a more general class of coherent states. Chapter IV deals with the case of non-Hermitian Hamiltonians. Due to the fact that the geometry of the dynamics is more complicated than for Hermitian Hamiltonians, we only consider states that are initially Gaussian in the non-Hermitian case.

In Chapter V, we describe the process of dimensionally reducing the Gaussian coherent states (1.9) to construct Gaussian states on complex projective spaces. These states have semiclassical properties governed by a symbol calculus. Chapter VI is where we consolidate the ideas of reduction and propagation and we demonstrate how the reduced Gaussian states evolve under Hermitian Hamiltonians. Finally, in Chapter VII we present the special case of the reduction of squeezed states in $\mathcal{B}(\mathbb{C}^2)$ which correspond to SU(2), or spin-squeezed, states. This chapter may be of particular interest to physicists.

CHAPTER II

A Symbol Calculus

In Chapters III and IV we will be concerned with constructing time-dependent wavefunctions ψ, which live in special subspaces of $C^\infty(\mathbb{C}^d \times \mathbb{R})$ associated to a smooth curve, such that ψ is a solution to Schrödinger's equation to arbitrary order in $\hbar$. In Chapter III, we will consider the case where the quantum Hamiltonian in Schrödinger's equation is Hermitian. Then, in Chapter IV, we allow for the quantum Hamiltonian to be non-Hermitian. This chapter is devoted to defining our special subspaces I_γ^m and to developing the necessary *symbol calculus* needed in later chapters.

2.1 The spaces I_γ^m

In this section, we construct the spaces of wavefunctions associated to a smooth γ curve which we will denote by I_γ^m. First, we present the Schwartz space which appears in our definition of I_γ^m.

Definition II.1. The *Schwartz space* is the topological vector space

$$\mathcal{S}(\mathbb{R}^{2d}) := \left\{ f \in C^\infty(\mathbb{R}^{2d}) \, ; \forall \alpha, \beta \in \mathbb{N}^d, \ \|f\|_{\alpha,\beta} < \infty \right\}$$

where $C^\infty(\mathbb{R}^{2d})$ is the function space of smooth functions from $\mathbb{R}^{2d}$ to $\mathbb{C}$ and the semi-norm, which defines the topology of the space, is

$$\|f\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^{2d}} |x^\beta D^\alpha f(x)|.$$

Definition II.2. Let $\gamma : t \to w(t)$ be a smooth curve on $\mathbb{C}^d$. The space I_γ^m is the space of all functions of the general form

$$\psi(z,t,\hbar) = \hbar^m\, e^{i\hbar^{-1} f(t)}\, e^{i\hbar^{-1}\omega(z,w(t))}\, \varphi\left(\frac{z-w(t)}{\sqrt{\hbar}}, t, \hbar\right)\, e^{-\hbar^{-1}|z-w(t)|^2/2}, \tag{2.1}$$

where ω is the standard symplectic form $\omega = i dz \wedge d\overline{z}$, so $\omega(z,w(t)) = \Im(z\overline{w}(t)^T)$ and the following conditions hold:

1. f is real-valued,
2. $\forall t$, $\varphi\, e^{-\hbar^{-1}|z-w(t)|^2/2} \in \mathcal{S}(\mathbb{R}^{2d})$,
3. φ is holomorphic in z, i.e., $\frac{\partial \varphi}{\partial \overline{z}_j} = 0$ for $j = 1, \ldots, d$,
4. φ has an asymptotic expansion of the form $\varphi(\zeta,t,\hbar) \sim \sum_{j=0}^{\infty} \hbar^{j/2} \varphi_j(\zeta,t)$ where $\forall t, j$, $\varphi_j(\zeta,t)\, e^{-|\zeta|^2/2} \in \mathcal{S}(\mathbb{R}^{2d})$ with estimates locally uniform in t. More precisely, assume that $\forall j, N \in \mathbb{N}$, $\forall \alpha, \beta \in \mathbb{N}^d$ and $\forall K \subset \mathbb{R}$ compact, $\exists C$ such that $\forall (\zeta,t) \in \mathbb{C}^d \times K$,

$$\left| \partial_\zeta^\alpha \partial_t^\beta\, \varphi_j(\zeta,t)\, e^{-|\zeta|^2/2} \right| \le C\,(1+|\zeta|)^{-N}. \tag{2.2}$$

   Furthermore, $\forall \alpha, \beta \in \mathbb{N}^d$, $\forall M, N \in \mathbb{N}$ and $\forall K \subset \mathbb{R}$, $\exists C$ such that $\forall (\zeta,t) \in \mathbb{C}^d \times K$,

$$\left| \partial_\zeta^\alpha \partial_t^\beta\, e^{-|\zeta|^2/2} \left(\varphi(\zeta,t,\hbar) - \sum_{j=0}^{N} \hbar^{j/2} \varphi_j(\zeta,t) \right) \right| \le C\,(1+|\zeta|)^{-M}\, \hbar^{(N+1)/2}. \tag{2.3}$$

Moreover, ψ is of *order* m in I_γ^m.

Remarks 4. A few comments about the I_γ^m spaces:

1. The I_γ^m are vector spaces and they are subspaces of $C^\infty(\mathbb{C}^d \times \mathbb{R})$.
2. The functions $\psi \in I_\gamma^m$ are regarded as functions of both $z \in \mathbb{C}^d$ and $t \in \mathbb{R}$ simultaneously.
3. $I_\gamma^m \subset I_\gamma^{m+1/2}$ in the case that the leading order term in φ, $\varphi_0 = 0$.

4. Observe that (3) and (4) in Definition II.2 imply that $\forall t$, $\psi \in \mathcal{B}(\mathbb{C}^d)$.

Definition II.3. The elements $\psi_1, \psi_2 \in I_\gamma^m$ are *equal modulo* $I_\gamma^{m+1/2}$ if $\psi_1 - \psi_2 \in I_\gamma^{m+1/2}$.

2.2 Symbols

Definition II.4. The *principal symbol* of $\psi \in I_\gamma^m$ as in (2.1) is

$$\sigma_\psi^m(\zeta, t) := \varphi_0(\zeta, t)\, e^{-|\zeta|^2/2}\,. \tag{2.4}$$

Note that these are $\hbar-$independent Schwartz functions.

Example II.5. Choosing $\varphi_0(\zeta, t) = p(\zeta, t)\, e^{Q_A(\zeta)/2}$ where p is a polynomial in $\zeta \in \mathbb{C}^d$ and $A \in \mathcal{D}_d$ is an example of such a symbol.

Definition II.6. The *space of symbols* is

$$S := \left\{ \sigma_\psi^m(\zeta, t) = \varphi_0(\zeta, t)\, e^{-|\zeta|^2/2} \in \mathcal{S}(\mathbb{R}^{2d})\,;\ \forall t, \varphi_0 \text{ is holomorphic in } \zeta \right\}.$$

We make the following observations about symbols:

1. As Schwartz functions, $\sigma_{\hbar^p \psi}^m = \sigma_\psi^m$ for any power p. This fact will be useful later on.

2. One has a short exact sequence

$$0 \hookrightarrow I_\gamma^{m+1/2} \hookrightarrow I_\gamma^m \xrightarrow{\boldsymbol{\sigma}^m} S \longrightarrow 0, \qquad \forall m$$

   where $\boldsymbol{\sigma}^m$ is the symbol map. The symbol map induces a linear bijection

$$I_\gamma^m / I_\gamma^{m+1/2} \cong S, \qquad \forall m.$$

3. If $\psi \in I_\gamma^m$ is such that $\boldsymbol{\sigma}^m(\psi) = 0 \in S$, then by the short exact sequence, $\psi \in I_\gamma^{m+1/2}$. Thus, $\boldsymbol{\sigma}^{m+1/2}(\psi) \in S$ is well-defined.

The following norm estimate will be useful.

Lemma II.7. *Given $\psi \in I_\gamma^m$, $\forall t \in \mathbb{R}$,*

$$\lim_{\hbar \to 0} \hbar^{-(m+d/2)} \|\psi(\cdot, t, \hbar)\|_{\mathcal{B}(\mathbb{C}^d)} = \|\sigma_\psi^m(\cdot, t)\|_{L^2(\mathbb{C}^d)}$$

provided $\sigma_\psi^m(\zeta, t) \neq 0$.

Proof. We have

$$\begin{aligned}
\|\psi(\cdot, t, \hbar)\|^2_{\mathcal{B}(\mathbb{C}^d)} &= \int_{\mathbb{C}^d} \left| \hbar^m \, e^{i\hbar^{-1} f(t)} \, e^{i\hbar^{-1}\omega(z, w(t))} \varphi\left(\frac{z - w(t)}{\sqrt{\hbar}}, t, \hbar\right) \, e^{-\hbar^{-1}|z-w(t)|^2/2} \right|^2 dL(z) \\
&= \hbar^{2m} \int_{\mathbb{C}^d} \left| \varphi\left(\frac{z - w(t)}{\sqrt{\hbar}}, t, \hbar\right) \, e^{-\hbar^{-1}|z-w(t)|^2/2} \right|^2 dL(z).
\end{aligned}$$

Let $\zeta = (z - w(t))/\sqrt{\hbar}$, so $dL(z) = (\sqrt{\hbar})^{2d} \, dL(\zeta)$ which leads to

$$\|\psi(\cdot, t, \hbar)\|^2_{\mathcal{B}(\mathbb{C}^d)} = \hbar^{2m+d} \int_{\mathbb{C}^d} |\varphi\left(\zeta, t, \hbar\right)|^2 \, e^{-|\zeta|^2} dL(\zeta).$$

We need to bound the above expression from both above and below. By (2.3),

$$\left| \left(\varphi\left(\zeta, t, \hbar\right) - \varphi_0\left(\zeta, t\right)\right) \, e^{-|\zeta|^2/2} \right| \leq C(1 + |\zeta|)^{-M} \, \hbar^{1/2}. \tag{2.5}$$

Using the reverse triangle inequality,

$$\left| \left(\varphi\left(\zeta, t, \hbar\right) - \varphi_0\left(\zeta, t\right)\right) \, e^{-|\zeta|^2/2} \right| \geq \left| |\varphi(\zeta, t, \hbar)| \, e^{-|\zeta|^2/2} - |\varphi_0(\zeta, t)| \, e^{-|\zeta|^2/2} \right|,$$

so

$$|\varphi(\zeta, t, \hbar)| \, e^{-|\zeta|^2/2} - |\varphi_0(\zeta, t)| \, e^{-|\zeta|^2/2} \leq C(1 + |\zeta|)^{-M} \, \hbar^{1/2}.$$

It is clear that

$$|\varphi(\zeta, t, \hbar)| \, e^{-|\zeta|^2/2} \leq C(1 + |\zeta|)^{-M} \, \hbar^{1/2}. \tag{2.6}$$

Inequality (2.5) is equivalent to

$$\left|(\varphi_0\,(\zeta,t,) - \varphi\,(\zeta,t,\hbar))\; e^{-|\zeta|^2/2}\right| \leq C(1+|\zeta|)^{-M}\,\hbar^{1/2}$$

and by applying the reverse triangle inequality again, we conclude that

$$|\varphi_0(\zeta,t)|\, e^{-|\zeta|^2/2} - |\varphi(\zeta,t,\hbar)|\, e^{-|\zeta|^2/2} \leq C(1+|\zeta|)^{-M}\,\hbar^{1/2}$$

which implies

$$|\varphi_0(\zeta,t)|\, e^{-|\zeta|^2/2} - C(1+|\zeta|)^{-M}\,\hbar^{1/2} \leq |\varphi(\zeta,t,\hbar)|\, e^{-|\zeta|^2/2}. \tag{2.7}$$

Combining (2.6) and (2.7) we obtain

$$\begin{aligned} |\varphi_0(\zeta,t)|\, e^{-|\zeta|^2/2} - C(1+|\zeta|)^{-M}\,\hbar^{1/2} &\leq |\varphi(\zeta,t,\hbar)|\, e^{-|\zeta|^2/2} \\ &\leq |\varphi_0(\zeta,t)|\, e^{-|\zeta|^2/2} + C(1+|\zeta|)^{-M}\,\hbar^{1/2}. \end{aligned} \tag{2.8}$$

Squaring the left side of (2.8) we get

$$\begin{aligned} &\left(|\varphi_0(\zeta,t)|\, e^{-|\zeta|^2/2} - C(1+|\zeta|)^{-M}\,\hbar^{1/2}\right)^2 \\ &\qquad = |\varphi_0(\zeta,t)|^2\, e^{-|\zeta|^2} - 2\,C\,\hbar^{1/2}(1+|\zeta|)^{-M}|\varphi_0(\zeta,t)|\, e^{-|\zeta|^2/2} + C^2\,\hbar\,(|1+|\zeta|)^{-2M}. \end{aligned}$$

Then, integrating over $\mathbb{C}^d$, we have

$$\mathrm{I} = \int_{\mathbb{C}^d} |\varphi_0(\zeta,t)|^2\, e^{-|\zeta|^2}\, dL(\zeta) = \|\sigma_\psi^m(\cdot,t)\|^2_{L^2(\mathbb{C}^d)},$$

and

$$\mathrm{II} = 2\,C\,\hbar^{1/2}\int_{\mathbb{C}^d}(1+|\zeta|)^{-M}|\varphi_0(\zeta,t)|\, e^{-|\zeta|^2/2} dL(\zeta) \leq K\,\hbar^{1/2}\int_{\mathbb{C}^d}(1+|\zeta|)^{-(M+N)} dL(\zeta)$$

for some other constant K since $|\varphi_0(\zeta,t)|\, e^{-|\zeta|^2/2} \leq C'(1+|\zeta|)^{-N}$. Provided $M+N$ is sufficiently large enough, the integral in II is bounded, so $\mathrm{II} = O(\hbar^{1/2})$. By similar reasoning,

$$\mathrm{III} = C^2\, \hbar \int_{\mathbb{C}^d} (1+|\zeta|)^{-2M} = O(\hbar)$$

for M large enough, e.g., $M > d$. The same analysis can be applied to the right side of (2.8). Then, in the limit as $\hbar \to 0$, $\hbar^{-(2m+d)} \|\psi(\cdot,t,\hbar)\|^2_{\mathcal{B}(\mathbb{C}^d)}$ approaches $\|\sigma^m_\psi(\cdot,t)\|^2_{L^2(\mathbb{C}^d)}$ from both above and below. $\square$

2.3 Action of operators

This section is dedicated to studying the action of operators on elements in the I^m_γ spaces. In general, our operators will be the Weyl quantization on $z \in \mathbb{C}^d$ of functions that satisfy the following conditions:

Assumption II.8. *Let $F : \mathbb{C}^d \times \mathbb{R} \to \mathbb{C}$ be a smooth function in (z,t) and let F and all its partial derivatives have at most polynomial growth at infinity.*

In Chapters III and IV, $F(z,t)$ will represent a classical Hamiltonian and its Weyl quantization on z and will be a quantum operator that is either Hermitian (in the case of a real-valued F), or non-Hermitian (if F is complex-valued).

Theorem II.9. *Assume $F(z,t)$ satisfies II.8. For each $t \in \mathbb{R}$, let $\widehat{F}$ denote the Weyl quantization on z of F. Then, for any $\psi \in I^m_\gamma$, $\widehat{F}(\psi) \in I^m_\gamma$.*

Remark 5. We will show later on that the principal symbol of $\widehat{F}\psi$ as an element in I^m_γ is $\sigma^m_{\widehat{F}\psi}(\zeta,t) = F(w(t),t)\, \sigma^m_\psi(\zeta,t)$.

Notation 1. In what follows it will be useful for us to rewrite the $\psi \in I^m_\gamma$ in (2.1) as

$$\psi(z,t,\hbar) = \hbar^m e^{i\hbar^{-1}\Phi(z,t)}\, e^{-\hbar^{-1}|z|^2/2}\, \varphi\left(\frac{z-w(t)}{\sqrt{\hbar}}, t, \hbar\right), \tag{2.9}$$

where $\Phi(z,t)$ is given by

$$\Phi(z,t) := f(t) - iz\overline{w}(t)^T + \frac{i}{2}|w(t)|^2.$$

Our main goal in this section is to prove the above theorem, but we will first prove a few preliminary propositions.

Proposition II.10. *For any $\psi \in I^m_\gamma$, $(z_j - w_j(t))\,\psi \in I^{m+1/2}_\gamma$ and $\sigma^{m+1/2}_{(z_j-w_j(t))\,\psi}(\zeta,t) = \zeta_j\,\sigma^m_\psi(\zeta,t)$ for $j = 1,\ldots,d$.*

Proof. For any $\psi \in I^m_\gamma$,

$$\begin{aligned}(z_j - w_j(t))\,\psi &= \hbar^m\, e^{i\hbar^{-1}\Phi(z,t)}\, e^{-\hbar^{-1}|z|^2/2}\sqrt{\hbar}\left[\left(\frac{z_j - w_j(t)}{\sqrt{\hbar}}\right)\varphi\left(\frac{z-w(t)}{\sqrt{\hbar}},t,\hbar\right)\right]\\ &= \hbar^{m+1/2}\, e^{i\hbar^{-1}\Phi(z,t)}\, e^{-\hbar^{-1}|z|^2/2}\,\widetilde{\varphi}\left(\frac{z-w(t)}{\sqrt{\hbar}},t,\hbar\right) \in I^{m+1/2}_\gamma\end{aligned}$$

where $\widetilde{\varphi} := \hbar^{-1/2}(z_j - w_j(t))\,\varphi$.

For the principal symbol, we have that

$$\sigma^{m+1/2}_{(z_j-w_j(t))\,\psi} = \sigma^{m+1/2}_{\hbar^{1/2}\left(\frac{z_j-w_j(t)}{\sqrt{\hbar}}\right)\psi} = \sigma^{m+1/2}_{\left(\frac{z_j-w_j(t)}{\sqrt{\hbar}}\right)\psi}$$

since $\sigma^m_{\hbar^{1/2}\psi} = \sigma^m_\psi$ for any m. Defining $\zeta_j := (z_j - w_j(t))/\sqrt{\hbar}$ one obtains the result. ☐

Remark 6. The previous proposition tells us that multiplying an element of I^m_γ by a factor of $(z-w(t))^\ell$ is in the same space as multiplying the element by $\hbar^{\ell/2}$ for any $\ell \in \mathbb{N}$.

Proposition II.11. *For any $\psi \in I^m_\gamma$, $\left(\hbar\frac{\partial}{\partial z_j} - \overline{w}_j(t)\right)\psi \in I^{m+1/2}_\gamma$ and*

$$\sigma^{m+1/2}_{(\hbar\frac{\partial}{\partial z_j} - \overline{w}_j(t))\psi}(\zeta,t) = \frac{\partial\sigma^m_\psi}{\partial\zeta_j}(\zeta,t)$$

for $j = 1,\ldots,d$.

Proof. We first calculate[1]

$$\begin{aligned}
\hbar\frac{\partial\psi}{\partial z_j} &= \hbar^{m+1}\left[\frac{\partial}{\partial z_j}\left(e^{i\hbar^{-1}\Phi(z,t)}\right)\varphi + \hbar^{-1/2}e^{i\hbar^{-1}\Phi(z,t)}\frac{\partial\varphi}{\partial z_j}\right] e^{-\hbar^{-1}|z|^2/2} \\
&= \hbar^{m+1}\left[i\hbar^{-1}\frac{\partial\Phi}{\partial z_j}\, e^{i\hbar^{-1}\Phi(z,t)}\varphi + \hbar^{-1/2}e^{i\hbar^{-1}\Phi(z,t)}\frac{\partial\varphi}{\partial z_j}\right] e^{-\hbar^{-1}|z|^2/2} \\
&= i\frac{\partial\Phi}{\partial z_j}\,\psi + \hbar^{m+1/2}\, e^{i\hbar^{-1}\Phi(z,t)}\,\frac{\partial\varphi}{\partial z_j}\, e^{-\hbar^{-1}|z|^2/2}.
\end{aligned}$$

Using the fact that $\frac{\partial\Phi}{\partial z_j}(z,t) = -iw_j(t)$, the above simplifies to

$$\left(\hbar\frac{\partial}{\partial z_j} - \overline{w}_j(t)\right)\psi = \hbar^{m+1/2}\,\frac{\partial\varphi}{\partial z_j}\left(\frac{z-w(t)}{\sqrt{\hbar}}, t, \hbar\right) e^{i\hbar^{-1}\Phi(z,t)}\, e^{-\hbar^{-1}|z|^2/2},$$

which is in $I_\gamma^{m+1/2}$. To find the principal symbol, we can rewrite the previous result as

$$\left(\hbar\frac{\partial}{\partial z_j} - \overline{w}_i(t)\right)\psi = \hbar^{m+1/2}\, e^{i\hbar^{-1}f(t)}\, e^{i\hbar^{-1}\omega(z,w(t))}\,\frac{\partial\varphi}{\partial z_j}\left(\frac{z-w(t)}{\sqrt{\hbar}}, t, \hbar\right) e^{-\hbar^{-1}|z-w(t)|^2/2}.$$

Then, substituting $\zeta_j := (z_j - w_j(t))/\sqrt{\hbar}$ and recalling that $\sigma^m_{\hbar^{1/2}\psi} = \sigma^m_\psi$,

$$\sigma^{m+1/2}_{\left(\hbar\frac{\partial}{\partial z_j} - \overline{w}_j(t)\right)\psi}(\zeta,t) = \frac{\partial\varphi_0}{\partial\zeta_j}(\zeta,t)\, e^{-|\zeta|^2/2} = \frac{\partial\sigma^m_\psi}{\partial\zeta_j}(\zeta,t). \qquad \square$$

Now we prove Theorem II.9.

Proof. Without loss of generality, let $m = 0$ and because $\widehat{F}$ only acts on the z variable we can also let $w(t) = 0$. For now we will assume that the φ in $\psi \in I_\gamma^0$ has a single $\hbar-$independent term denoted by $\varphi(z/\sqrt{\hbar}, t)$ satisfying (2.2) rather than an asymptotic expansion. Hence,

$$\psi(z,t) = e^{i\hbar^{-1}f(t)}\,\varphi\left(\frac{z}{\sqrt{\hbar}}, t\right)\, e^{-\hbar^{-1}|z|^2/2} \in I_\gamma^0.$$

[1]Notice that we don't differentiate $e^{-\hbar^{-1}|z|^2/2}$ with respect to z.

For each $t \in \mathbb{R}$, consider a Taylor expansion of $F(z,t)$ about $z = 0$:

$$F(z, \overline{z}, t) = P_N(z, \overline{z}, t) + \sum_{|a|+|b|=N} z^a \, \overline{z}^b \, R_{a,b}\,(z, \overline{z}, t) \tag{2.10}$$

where P_N is a polynomial of degree N in $(z, \overline{z})$ and $R_{a,b}(z, \overline{z}, t)$ is a function of the derivatives of F, so it is also a smooth function with polynomials bounds on its derivatives. We need to show that

$$\widehat{F}(\psi) = e^{i\hbar^{-1} f(t)} \, \mu\left(\frac{z}{\sqrt{\hbar}}, t, \hbar\right) e^{-\hbar^{-1}|z|^2/2} \tag{2.11}$$

is in I^0_γ. In other words, $\exists\, \mu_j(\zeta, t)$ where $\zeta = z/\sqrt{\hbar}$ which satisfy (2.2) and

$$e^{-|\zeta|^2/2}\left(\mu(\zeta, t, \hbar) - \sum_{j=0}^{N} \mu_j(\zeta, t)\right)$$

is bounded in the manner of (2.3). Weyl quantizing (2.10) on z and applying it to ψ we have

$$\widehat{F}(\psi) = \widehat{P}_N(\psi) + \sum_{|a|+|b|=N} \widehat{z^a \, \overline{z}^b \, R_{a,b}}(\psi)\,.$$

Now

$$\begin{aligned}
\widehat{P}_N(\psi)(z,t) &= e^{i\hbar^{-1} f(t)} \, e^{-\hbar^{-1}|z|^2/2} \sum_{|a|+|b|=N} \hbar^{|b|} \, c_{a,b} \, z^a \, \partial_z^b \, \varphi\left(\frac{z}{\sqrt{\hbar}}, t\right) \\
&= e^{i\hbar^{-1} f(t)} \, e^{-\hbar^{-1}|z|^2/2} \sum_{j=0}^{N} \hbar^{j/2} \, \mu_j\left(\frac{z}{\sqrt{\hbar}}, t\right)
\end{aligned}$$

because we can apply Propositions II.10 and II.11 repeatedly. Additionally, the μ_j's satisfy (2.2) because they represent monomials multiplied by derivatives in z of φ and φ itself satisfies

(2.2). Rearranging the expression, we obtain

$$e^{-\hbar^{-1}|z|^2/2}\sum_{j=0}^{N}\hbar^{j/2}\,\mu_j\left(\frac{z}{\sqrt{\hbar}},t\right)=e^{-i\hbar^{-1}f(t)}\,\widehat{P}_N(\psi)(z,t)=\widehat{P}_N\left(\varphi\left(\frac{z}{\sqrt{\hbar}},t\right)e^{-\hbar^{-1}|z|^2/2}\right).$$

Similarly, we can rearrange (2.11):

$$e^{-\hbar^{-1}|z|^2/2}\,\mu\left(\frac{z}{\sqrt{\hbar}},t,\hbar\right)=\widehat{F}\left(\varphi\left(\frac{z}{\sqrt{\hbar}},t\right)e^{-\hbar^{-1}|z|^2/2}\right).$$

Returning to the expression we wish to bound:

$$\begin{aligned}e^{-|\zeta|^2/2}\left(\mu(\zeta,t,\hbar)-\sum_{j=0}^{N}\hbar^{j/2}\,\mu_j(\zeta,t)\right)&=\left(\widehat{F}-\widehat{P}_N\right)\left(\varphi\left(\frac{z}{\sqrt{\hbar}},t\right)e^{-\hbar^{-1}|z|^2/2}\right)\Big|_{z=\sqrt{\hbar}\zeta}\\&=\sum_{|a|+|b|=N}\widehat{z^a\,\overline{z}^b\,R_{a,b}}\left(\varphi\left(\frac{z}{\sqrt{\hbar}},t\right)e^{-\hbar^{-1}|z|^2/2}\right)\Big|_{z=\sqrt{\hbar}\zeta}.\end{aligned}$$

Next, we use the formula for Weyl quantization in Bargmann space from [Her97] on the remainder term. It is sufficient to consider a single term in the expansion:

$$\begin{aligned}I_1(\zeta,t,\hbar)&=\widehat{z^a\,\overline{z}^b\,R_{a,b}}\left(\varphi\left(\frac{z}{\sqrt{\hbar}},t\right)e^{-\hbar^{-1}|z|^2/2}\right)\Big|_{z=\sqrt{\hbar}\zeta}\\&=\frac{e^{-|\zeta|^2/2}}{(\pi\hbar)^d\,2^{|a|}}\int_{\mathbb{C}^d}(u+\sqrt{\hbar}\zeta)^a\,\overline{u}^bR_{a,b}\left(\frac{u+\sqrt{\hbar}\zeta}{2},\overline{u},t\right)\varphi\left(\frac{u}{\sqrt{\hbar}},t\right)e^{\hbar^{-1/2}\zeta\overline{u}^T}e^{-\hbar^{-1}|u|^2}du\,d\overline{u}.\end{aligned}$$

We perform a change of variables $(u,\overline{u})=\sqrt{\hbar}(v,\overline{v})$, so that

$$\begin{aligned}I_1(\zeta,t,\hbar)&=\frac{\hbar^d\,e^{-|\zeta|^2/2}}{(\pi\hbar)^d\,2^{|a|}}\int_{\mathbb{C}^d}\hbar^{|a+b|/2}(v+\zeta)^a\,\overline{v}^b\,R_{a,b}\left(\frac{\sqrt{\hbar}(v+\zeta)}{2},\sqrt{\hbar}\,\overline{v},t\right)\varphi\left(v,t\right)e^{\zeta\overline{v}^T}\,e^{-|v|^2}\,dv\,d\overline{v}\\&=\frac{\hbar^{N/2}\,e^{-|\zeta|^2/2}}{\pi^d\,2^{|a|}}\int_{\mathbb{C}^d}(v+\zeta)^a\,\overline{v}^b\,R_{a,b}\left(\frac{\sqrt{\hbar}(v+\zeta)}{2},\sqrt{\hbar}\,\overline{v},t\right)\varphi\left(v,t\right)\,e^{\zeta\overline{v}^T}\,e^{-|v|^2}\,dv\,d\overline{v}.\end{aligned}$$

Let us rewrite the $e^{\zeta\overline{v}^T}$ factor slightly. Note that $\zeta\overline{v}^T=\zeta\cdot v+i\omega(\zeta,v)=\zeta\cdot v+iv\cdot J\zeta$ where $\cdot$ represents the real scalar product and J is the standard complex structure. Then, completing the square, we have $e^{-|v|^2/2}\, e^{\zeta\cdot v} = e^{-\frac{1}{2}\left(|v-\zeta|^2-|\zeta|^2\right)} = e^{-|v-\zeta|^2/2}\, e^{|\zeta|^2/2}$, so we can rewrite the integrand as

$$I_1(\zeta,t,\hbar) = \frac{\hbar^{N/2}}{\pi^d\, 2^{|a|}} \int_{\mathbb{C}^d} e^{iv\cdot J\zeta}\, (v+\zeta)^a\, \overline{v}^b\, R_{a,b}\left(\frac{\sqrt{\hbar}(v+\zeta)}{2}, \sqrt{\hbar}\,\overline{v}, t\right)\, e^{-|v-\zeta|^2/2} \varphi\,(v,t)\; e^{-|v|^2/2} dv\, d\overline{v}.$$

For each $t\in\mathbb{R}$, $I_1(\zeta,t,\hbar)$ resembles a Fourier transform evaluated at $J\zeta$ in the v variable, but it is not a true Fourier transform because ζ appears in some of the factors in the integrand. We proceed to show that I_1 is Schwartz.

(1) First, we prove that for any $\alpha\in\mathbb{N}^d$, $\zeta^\alpha\, I_1(\zeta,t,\hbar)$ is bounded. For brevity, let

$$G(v,\overline{v},\zeta,t,\hbar) := (v+\zeta)^a\, \overline{v}^b\, R_{a,b}\left(\frac{\sqrt{\hbar}(v+\zeta)}{2}, \sqrt{\hbar}\,\overline{v}, t\right)\, e^{-|v-\zeta|^2/2} \varphi\,(v,t)\; e^{-|v|^2/2}$$

and consider

$$\begin{aligned}\zeta^\alpha\, I_1(\zeta,t,\hbar) &= \frac{\hbar^{N/2}}{\pi^d\, 2^{|a|}} \int_{\mathbb{C}^d} \left(\zeta^\alpha e^{iv\cdot J\zeta}\right) G(v,\overline{v},\zeta,t,\hbar)\, dv\, d\overline{v} \\ &= \frac{-i^{-|\alpha|}\, \hbar^{N/2}}{\pi^d\, 2^{|a|}} \int_{\mathbb{C}^d} e^{iv\cdot J\zeta} \left[(-J\partial_v)^\alpha\right]^T G(v,\overline{v},\zeta,t,\hbar)\, dv\, d\overline{v}\end{aligned}$$

where we have integrated by parts. We can calculate $\left[(-J\partial_v)^\alpha\right]^T G(v,\overline{v},\zeta,t,\hbar)$ using Leibniz's rule, but of course this would be cumbersome and unnecessary for our purposes. We need to justify that the integral does in fact converge. The function $R_{a,b}$ and all its derivatives are bounded by polynomials in ζ, so any derivatives or products by monomials of $R_{a,b}$ are also bounded. For each $t\in\mathbb{R}$, $\varphi(v,t)\, e^{-|v|^2/2} \in \mathcal{S}(\mathbb{R}^{2d})$, so its derivatives are also Schwartz. Lastly, $\partial_v^\alpha\left(e^{-|v-\zeta|^2/2}\right)$ is bounded in ζ. Therefore, the result of $\left[(-J\partial_v)^\alpha\right]^T G(v,\overline{v},\zeta,t,\hbar)$ is bounded by polynomials in ζ. Hence, $\zeta^\alpha\, I_1(\zeta,t,\hbar)$ converges and is bounded by powers of ζ.

(2) We also need that for any $\alpha \in \mathbb{N}^d$, $\partial_\zeta^\alpha I(\zeta, t, \hbar)$ is bounded.

$$
\begin{aligned}
\partial_\zeta^\alpha I_1(\zeta, t, \hbar) &= \frac{\hbar^{N/2}}{\pi^d\, 2^{|a|}} \int_{\mathbb{C}^d} \partial_\zeta^\alpha \, \left(e^{iv \cdot J\zeta}\, G(v, \overline{v}, \zeta, t, \hbar)\right) dv\, d\overline{v} \\
&= \frac{\hbar^{N/2}}{\pi^d\, 2^{|a|}} \int_{\mathbb{C}^d} \partial_\zeta^\alpha (e^{iv \cdot J\zeta})\, G(v, \overline{v}, \zeta, t, \hbar) + e^{iv \cdot J\zeta}\, \partial_\zeta^\alpha G(v, \overline{v}, \zeta, t, \hbar)\, dv\, d\overline{v} \\
&= \frac{\hbar^{N/2}}{\pi^d\, 2^{|a|}} \int_{\mathbb{C}^d} e^{iv \cdot J\zeta} \left((-iJv)^\alpha + \partial_\zeta^\alpha G(v, \overline{v}, \zeta, t, \hbar)\right)\, dv\, d\overline{v}
\end{aligned}
$$

where we have used $v \cdot J\zeta = -\zeta \cdot Jv$. Once again, the calculation of $\partial_\zeta^\alpha G$ is an application of Leibniz's rule. Omitting the details, we will get a finite sum in powers of v with respect to ζ, so the integral converges and is bounded by powers of ζ.

Overall, $I_1 = O(\hbar^{N/2})$.

It remains to prove $\widehat{F}(\psi) \in I_\gamma^m$ for a φ that has an asymptotic expansion $\varphi \sim \sum_{j=0}^{\infty} \hbar^{j/2}\, \varphi(\zeta, t)$. In this case,

$$
\widehat{F}(\psi) = e^{i\hbar^{-1} f(t)}\, \nu \left(\frac{z}{\sqrt{\hbar}}, t, \hbar\right)\, e^{-\hbar^{-1}|z|^2/2}
$$

which can be rearranged as

$$
e^{-\hbar^{-1}|z|^2/2}\, \nu \left(\frac{z}{\sqrt{\hbar}}, t, \hbar\right) = \widehat{F}\left(\varphi\left(\frac{z}{\sqrt{\hbar}}, t, \hbar\right) e^{-\hbar^{-1}|z|^2/2}\right).
$$

We define

$$
\psi_N(z, t, \hbar) := e^{i\hbar^{-1} f(t)} e^{-\hbar^{-1}|z|^2/2} \sum_{j=0}^{N} \hbar^{j/2}\, \varphi_j \left(\frac{z}{\sqrt{\hbar}}, t\right).
$$

In a similar manner,

$$
\widehat{F}(\psi_N) = e^{i\hbar^{-1} f(t)} \sum_{j=0}^{N} \hbar^{j/2}\, \nu_j \left(\frac{z}{\sqrt{\hbar}}, t, \hbar\right)\, e^{-\hbar^{-1}|z|^2/2}
$$

can be rearranged as

$$e^{-\hbar^{-1}|z|^2/2}\sum_{j=0}^{N}\hbar^{j/2}\,\nu_j\left(\frac{z}{\sqrt{\hbar}},t,\hbar\right)=\widehat{F}\left(\sum_{j=0}^{N}\hbar^{j/2}\varphi_j\left(\frac{z}{\sqrt{\hbar}},t\right)e^{-\hbar^{-1}|z|^2/2}\right).$$

Then, it is sufficient to prove that the following satisfies (2.3):

$$\begin{aligned}
I_2(\zeta,t,\hbar)&=e^{-|\zeta|^2/2}\left(\nu(\zeta,t,\hbar)-\sum_{j=0}^{N}\hbar^{j/2}\,\nu_j\left(\zeta,t,\hbar\right)\right)\\
&=\widehat{F}\left[e^{-\hbar^{-1}|z|^2/2}\left(\varphi\left(\frac{z}{\sqrt{\hbar}},t,\hbar\right)-\sum_{j=0}^{N}\hbar^{j/2}\,\varphi_j\left(\frac{z}{\sqrt{\hbar}},t\right)\right)\right]\Bigg|_{z=\sqrt{\hbar}\zeta}\\
&=\frac{e^{-|\zeta|^2/2}}{(\pi\hbar)^d}\int_{\mathbb{C}^d}F\left(\frac{u+\sqrt{\hbar}\zeta}{2},\overline{u},t\right)R_N\left(\frac{u}{\sqrt{\hbar}},t,\hbar\right)e^{\hbar^{-1/2}\zeta\overline{u}^T}\,e^{-\hbar^{-1}|u|^2}\,du\,d\overline{u}
\end{aligned}$$

where we have let $R_N:=\varphi-\sum_{j=0}^{N}\hbar^{j/2}\,\varphi_j$. Making the same change of variables as before: $(u,\overline{u})=\sqrt{\hbar}\,(v,\overline{v})$, we have

$$I_2(\zeta,t,\hbar)=\frac{e^{-|\zeta|^2/2}}{\pi^d}\int_{\mathbb{C}^d}F\left(\frac{\sqrt{\hbar}(v+\zeta)}{2},\sqrt{\hbar}\overline{v},t\right)R_N(v,t,\hbar)\,e^{\zeta\overline{v}^T}\,e^{-|v|^2}\,dv\,d\overline{v},$$

which we can rewrite as

$$I_2(\zeta,t,\hbar)=\frac{1}{\pi^d}\int_{\mathbb{C}^d}e^{iv\cdot J\zeta}\,F\left(\frac{\sqrt{\hbar}(v+\zeta)}{2},\sqrt{\hbar}\overline{v},t\right)R_N(v,t,\hbar)\,e^{-|v-\zeta|^2/2}\,e^{-|v|^2/2}\,dv\,d\overline{v}.$$

Observe that this integral is analogous to I_1. Since by our assumption F and all its derivatives are bounded by polynomials, so we may apply the same analysis as before, to prove that I_2 is Schwartz. The powers of $\hbar$ result from the fact that $\left|R_N(v,t,\hbar)\,e^{-|v|^2/2}\right|$ satisfies (2.3) and $I_2=O(\hbar^{(N+1)/2})$. □

Using Propositions II.10 and II.11, we can prove the following:

Proposition II.12. *Assume $F(z,t)$ satisfies II.8. Furthermore, assume that for each $t\in\mathbb{R}$, $F(\cdot,t):\mathbb{C}^d\to\mathbb{C}^d$ vanishes to order ℓ on the curve γ. Let $\widehat{F}$ denote the Weyl quantization on*

z of F. Then,

$$\widehat{F} : I_\gamma^m \to I_\gamma^{m+\ell/2}.$$

Proof. Assume that ℓ is such that if $F(\cdot,t)$ vanishes to order j on γ for each t, then $\widehat{F} : I_\gamma^m \to I_\gamma^{m+j/2}$, for all $j \leq \ell$. Inductively, we can assume that $F(\cdot,t)$ vanishes to order $\ell+1$ on γ for each t.

By Taylor-Hadamard's lemma, we can express $F(\cdot,t)$ as

$$F(z,\overline{z},t) = (z - w(t))\, F_1(z,\overline{z},t) + (\overline{z} - \overline{w}(t))\, F_2(z,\overline{z},t)$$

where $F_1(\cdot,t)$ and $F_2(\cdot,t)$ both vanish to order ℓ on γ.

By Weyl quantization, we have that $\widehat{(z-w(t))} \circ \widehat{F_1} = \widehat{(z-w(t))F_1} - i\hbar\{\widehat{z-w(t), F_1}\}$ and $\widehat{(\overline{z}-\overline{w}(t))} \circ \widehat{F_2} = \widehat{(\overline{z}-\overline{w}(t))F_2} - i\hbar\{\widehat{\overline{z}-\overline{w}(t), F_2}\}$ because z is linear in both position and momentum. Let $\psi \in I_\gamma^m$ so that

$$\widehat{(z-w(t))F_1}(\psi) = \widehat{(z-w(t)}\left(\widehat{F_1}(\psi)\right) + i\hbar\{\widehat{z-w(t), F_1}\}(\psi).$$

Now $\widehat{F_1}(\psi) \in I_\gamma^{m+\ell/2}$, so $\widehat{(z-w(t))}\left(\widehat{F_1}(\psi)\right) \in I_\gamma^{m+(\ell+1)/2}$ by Proposition II.10.

Also, $\{\widehat{z-w(t), F_1}\}(\psi) \in I_\gamma^{m+(\ell-1)/2}$ because $\{z-w(t), F_1\}$ vanishes to order $\ell-1$ since we just have derivatives of F_1. Hence, $\hbar \cdot \{\widehat{z-w(t), F_1}\}(\psi) \in I_\gamma^{m+(\ell+1)/2}$, so we have that $\widehat{(z-w(t))F_1}(\psi) \in I_\gamma^{m+(\ell+1)/2}$.

We can also check that

$$\widehat{(\overline{z}-\overline{w}(t))F_2}(\psi) = \widehat{(\overline{z}-\overline{w}(t))}\left(\widehat{F_2}(\psi)\right) + i\hbar\{\widehat{\overline{z}-\overline{w}(t), F_2}\}(\psi) \in I_\gamma^{m+(\ell+1)/2}$$

by similar reasoning. Therefore, we have shown that $\widehat{F} : I_\gamma^m \to I_\gamma^{m+(\ell+1)/2}$. $\square$

2.4 Calculating Symbols

In Chapters III and IV we will consider the Schrödinger evolution of wavefunctions $\psi \in I_\gamma^m$ under Hermitian and non-Hermitian Hamiltonians, respectively. We will look at the principal symbols in order to obtain transport equations for the parameters in ψ. We now establish some rules for computing symbols.

First, consider the cases where $\ell = 0, 1$, and 2 in Proposition II.12 and as let us compute the principal symbols of $\widehat{F}\psi$.

Corollary II.13. *Assume $F(z,t)$ satisfies II.8. For each $t \in \mathbb{R}$, let $\widehat{F}$ denote the Weyl quantization on z of F. Then, for any $\psi \in I_\gamma^m$,*

1. *The principal symbol of $\widehat{F}\psi$ as an element in I_γ^m is*

$$\sigma^m_{\widehat{F}\psi}(\zeta,t) = F(w(t),t)\,\sigma^m_\psi(\zeta,t). \tag{2.12}$$

2. *If F vanishes to first order on γ, then the principal symbol of $\widehat{F}\psi$ as an element in $I_\gamma^{m+1/2}$ is*

$$\sigma^{m+1/2}_{\widehat{F}\psi}(\zeta,t) = \left(\nabla_z F(w(t),t)\zeta^T + \nabla_{\bar{z}} F(w(t),t)\nabla_\zeta^T\right)\sigma^m_\psi(\zeta,t). \tag{2.13}$$

3. *If F vanishes to second order on γ, then the principal symbol of $\widehat{F}\psi$ as an element in I_γ^{m+1} is*

$$\sigma^{m+1}_{\widehat{F}\psi}(\zeta,t) = \widehat{\mathcal{Q}}(\sigma^m_\psi)(\zeta,t). \tag{2.14}$$

Here $\widehat{\mathcal{Q}}$ is the Weyl quantization in ζ of the Hessian of $F(\zeta,t)$ with $\hbar = 1$:

$$\widehat{\mathcal{Q}} := \tfrac{1}{2}\zeta R_t \zeta^T + \zeta S_t \nabla_\zeta^T + \tfrac{1}{2}\mathit{Tr}(S_t) + \tfrac{1}{2}\nabla_\zeta Q_t \nabla_\zeta^T,$$

and

$$R_t := F_{zz}(w(t),t) \qquad S_t := F_{z\overline{z}}(w(t),t) \qquad Q_t := F_{\overline{z}\overline{z}}(w(t),t).$$

where $F_{zz} = \left(\frac{\partial^2 F}{\partial z_j \partial z_\ell}\right)$, *etc.*[2]

Proof. For each $t \in \mathbb{R}$, a second order Taylor expansion of F about $z = w(t)$ gives

$$F_2(z,t) = F^{(0)}(z,t) + F^{(1)}(z,t) + F^{(2)}(z,t)$$

where

$$F^{(0)}(z,t) = F(w(t),t)$$
$$F^{(1)}(z,t) = \nabla_z F(w(t),t)(z-w(t))^T + \nabla_{\overline{z}} F(w(t),t)(\overline{z}-\overline{w}(t))^T$$
$$F^{(2)}(z,t) = \tfrac{1}{2}(z-w(t))\,R_t\,(\overline{z}-\overline{w}(t))^T + (z-w(t))\,S_t\,(\overline{z}-\overline{w}(t))^T + \tfrac{1}{2}(\overline{z}-\overline{w}(t))\,Q_t\,(\overline{z}-\overline{w}(t))^T.$$

Then, the Weyl quantization of F on z and applied to ψ is

$$\widehat{F}_2(\psi)(z,t,\hbar) = \widehat{F}^{(0)}\psi + \widehat{F}^{(1)}\psi + \widehat{F}^{(2)}\psi$$

with

$$\begin{aligned}
\widehat{F}^{(0)}\psi &= F(w(t),t)\,\psi \\
\widehat{F}^{(1)}\psi &= \nabla_z F(w(t),t)(z-w(t))^T\psi + \nabla_{\overline{z}} F(w(t),t)\,(\hbar\nabla_z - \overline{w}(t))^T\,\psi \\
\widehat{F}^{(2)}\psi &= \tfrac{1}{2}(z-w(t))R_t(z-w(t))^T\psi + (z-w(t))S_t\,(\hbar\nabla_z - \overline{w}(t))^T\,\psi + \tfrac{1}{2}\hbar\,\mathrm{Tr}(S_t)\psi \\
&\quad + \tfrac{1}{2}\,(\hbar\nabla_z - \overline{w}(t))\,Q_t\,(\hbar\nabla_z - \overline{w}(t))^T\,\psi.
\end{aligned}$$

Using the calculation from Proposition II.11 and the fact that $\nabla_z \Phi(z,t) = -iw(t)$ we

[2] R_t and Q_t are symmetric and there is no condition on S_t.

calculate[3]

$$
\begin{aligned}
&\left(\hbar\nabla_z - \overline{w}(t)\right) Q_t \left(\hbar\nabla_z - \overline{w}(t)\right)^T \psi \\
&\qquad = \sqrt{\hbar}\left(\hbar\nabla_z - \overline{w}(t)\right) Q_t\, \nabla_z^T \varphi\, e^{i\hbar^{-1}\Phi(z,t)}\, e^{-\hbar^{-1}|z|^2/2} \\
&\qquad = \left[\hbar^{3/2}\nabla_z \left(Q_t\, \nabla_z^T \varphi\, e^{i\hbar^{-1}\Phi(z,t)}\right) - \sqrt{\hbar}\,\overline{w}(t)\, Q_t\, \nabla_z^T \varphi\, e^{i\hbar^{-1}\Phi(z,t)}\right] e^{-\hbar^{-1}|z|^2/2} \\
&\qquad = \left[\hbar^{3/2}\left(\hbar^{-1/2}\nabla_z Q_t \nabla_z^T \varphi + i\hbar^{-1}\nabla_z \Phi\, Q_t \nabla_z^T \varphi\right) - \sqrt{\hbar}\,\overline{w}(t)\, Q_t\, \nabla_z^T \varphi\right] e^{i\hbar^{-1}\Phi(z,t)}\, e^{-\hbar^{-1}|z|^2/2} \\
&\qquad = \left[\hbar\nabla_z Q_t \nabla_z^T \varphi + i\sqrt{\hbar}\,(-i\overline{w}(t)) Q_t\, \nabla_z^T \varphi - \sqrt{\hbar}\,\overline{w}(t)\, Q_t\, \nabla_z^T \varphi\right] e^{i\hbar^{-1}\Phi(z,t)}\, e^{-\hbar^{-1}|z|^2/2} \\
&\qquad = \hbar\, \nabla_z Q_t \nabla_z^T \varphi\, e^{i\hbar^{-1}\Phi(z,t)} e^{-\hbar^{-1}|z|^2/2}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\widehat{F}^{(1)}\psi &= \nabla_z F(w(t),t)(z-w(t))^T\, \psi + \sqrt{\hbar}\, \nabla_{\overline{z}} F(w(t),t) \nabla_z^T \varphi\, e^{i\hbar^{-1}\Phi(z,t)}\, e^{-\hbar^{-1}|z|^2/2} \\
\widehat{F}^{(2)}\psi &= \frac{1}{2}(z-w(t))\, R_t\, (z-w(t))^T \psi + \sqrt{\hbar}(z-w(t))\, S_t\, \nabla_z^T \varphi\, e^{i\hbar^{-1}\Phi(z,t)} e^{-\hbar^{-1}|z|^2/2} \\
&\qquad + \frac{\hbar}{2}\mathrm{Tr}(S_t)\psi + \frac{\hbar}{2}\nabla_z Q_t \nabla_z^T \varphi\, e^{i\hbar^{-1}\Phi(z,t)} e^{-\hbar^{-1}|z|^2/2}.
\end{aligned}
$$

which we can rewrite as

$$
\begin{aligned}
\widehat{F}^{(1)}\psi = \hbar^m \left[\nabla_z F(w(t),t)(z-w(t))^T + \sqrt{\hbar}\, \nabla_{\overline{z}} F(w(t),t) \nabla_z^T\right] \varphi\left(\frac{z-w(t)}{\sqrt{\hbar}}, t, \hbar\right) \\
\times\, e^{i\hbar^{-1}f(t)}\, e^{i\hbar^{-1}\omega(z,w(t))}\, e^{-\hbar^{-1}|z-w(t)|^2/2}
\end{aligned}
$$

and

$$
\begin{aligned}
\widehat{F}^{(2)}\psi = \hbar^m \left[\tfrac{1}{2}(z-w(t))\, R_t\, (z-w(t))^T + \sqrt{\hbar}(z-w(t))\, S_t \nabla_z^T + \tfrac{1}{2}\,\hbar \mathrm{Tr}(S_t)\psi + \tfrac{1}{2}\,\hbar \nabla_z Q_t \nabla_z^T\right] \\
\times\, \varphi\left(\frac{z-w(t)}{\sqrt{\hbar}}, t, \hbar\right) e^{i\hbar^{-1}f(t)}\, e^{i\hbar^{-1}\omega(z,w(t))}\, e^{-\hbar^{-1}|z-w(t)|^2/2}.
\end{aligned}
$$

[3] Note: $\nabla_z Q_t \nabla_z \varphi = \sum_{j,\ell=1}^d Q_{j,\ell}(t)\, \frac{\partial^2 \varphi}{\partial z_j \partial z_\ell}$

Making the change of variables $\zeta = (z - w(t))/\sqrt{\hbar}$, we find that

$$\sigma^m_{\widehat{F}\psi}(\zeta, t) = \sigma^m_{\widehat{F^{(0)}}\psi}(\zeta, t) = F(w(t), t)\, \sigma^m_\psi(\zeta, t).$$

Now if $F(\cdot, t)$ vanishes to first order on γ, then $\widehat{F}\psi$ is an element of $I^{m+1/2}_\gamma$ and its principal symbol is given by

$$\sigma^{m+1/2}_{\widehat{F}\psi}(\zeta, t) = \sigma^{m+1/2}_{\widehat{F^{(1)}}\psi}(\zeta, t) = \left(\nabla_z F(w(t), t)\zeta^T + \nabla_{\overline{z}} F(w(t), t)\nabla^T_\zeta\right) \sigma^m_\psi(\zeta, t).$$

Lastly, if $F(\cdot, t)$ vanishes to second order on γ, then $\widehat{F}\psi$ is an element of I^{m+1}_γ and its symbol is

$$\sigma^{m+1}_{\widehat{F}\psi}(\zeta, t) = \sigma^{m+1}_{\widehat{F^{(2)}}\psi}(\zeta, t) = \widehat{\mathcal{Q}}(\sigma^m_\psi)(\zeta, t). \qquad \square$$

Recall that Schrödinger's equation is $i\hbar\frac{\partial\psi}{\partial t} = \widehat{F}\psi$. We have computed symbols for the right side of the equation, so let us also calculate symbols for the left side.

Lemma II.14. *For any $\psi \in I^m_\gamma$, for each $t \in \mathbb{R}$, $\hbar\frac{\partial\psi}{\partial t} \in I^m_\gamma$.*

1. *The principal symbol of $i\hbar\frac{\partial\psi}{\partial t}$ as an element in I^m_γ is*

$$\sigma^m_{i\hbar\frac{\partial\psi}{\partial t}}(\zeta, t) = -\left(\dot{f}(t) + \Im(\dot{\overline{w}}(t)w(t)^T)\right) \sigma^m_\psi(\zeta, t)\,. \tag{2.15}$$

2. *If $\sigma^m_{i\hbar\frac{\partial\psi}{\partial t}}(\zeta, t) = 0$, then the principal symbol of $i\hbar\frac{\partial\psi}{\partial t}$ as an element in $I^{m+1/2}_\gamma$ is*

$$\sigma^{m+1/2}_{i\hbar\frac{\partial\psi}{\partial t}}(\zeta, t) = i\left(\dot{\overline{w}}(t)\zeta^T - \dot{w}(t)\nabla^T_\zeta\right) \sigma^m_\psi(\zeta, t). \tag{2.16}$$

3. *If $\sigma^m_{i\hbar\frac{\partial\psi}{\partial t}}(\zeta, t) = \sigma^{m+1/2}_{i\hbar\frac{\partial\psi}{\partial t}}(\zeta, t) = 0$, then the principal symbol of $i\hbar\frac{\partial\psi}{\partial t}$ as an element in I^{m+1}_γ is*

$$\sigma^{m+1}_{i\hbar\frac{\partial\psi}{\partial t}}(\zeta, t) = i\frac{\partial\sigma^m_\psi}{\partial t}(\zeta, t). \tag{2.17}$$

Proof. First, we calculate that

$$
\begin{aligned}
i\hbar\frac{\partial\psi}{\partial t} &= \left(-\hbar^m\frac{\partial\Phi}{\partial t}(z,t)\,\varphi - i\hbar^{m+1/2}\dot{w}(t)\nabla_z\varphi + i\hbar^{m+1}\frac{\partial\varphi}{\partial t}\right)e^{i\hbar^{-1}\Phi(z,t)}\,e^{-\hbar^{-1}|z|^2/2}\\
&= h^m\left(\frac{\partial\Phi}{\partial t}(z,t)\,\varphi - i\hbar^{1/2}\dot{w}(t)\nabla_z\varphi + i\hbar\frac{\partial\varphi}{\partial t}\right)e^{i\hbar^{-1}f(t)}\,e^{i\hbar^{-1}\omega(z,w(t))}\,e^{-\hbar^{-1}|z-w(t)|^2/2}\\
&= h^m\left(\left(-\dot{f}(t) - \Im(\dot{w}(t)\overline{w}(t)^T) + i(z-w(t))\dot{\overline{w}}(t)^T\right)\varphi - i\hbar^{1/2}\dot{w}(t)\nabla_z\varphi + i\hbar\frac{\partial\varphi}{\partial t}\right)\\
&\qquad\times e^{i\hbar^{-1}f(t)}\,e^{i\hbar^{-1}\omega(z,w(t))}\,e^{-\hbar^{-1}|z-w(t)|^2/2} \qquad (2.18)
\end{aligned}
$$

where we have calculated and substituted

$$
\begin{aligned}
\frac{\partial\Phi}{\partial t}(z,t) &= \dot{f}(t) - iz\dot{\overline{w}}(t)^T + \frac{i}{2}\left(w(t)\dot{\overline{w}}(t)^T + \dot{w}(t)\overline{w}(t)^T\right)\\
&= \dot{f}(t) - iz\dot{\overline{w}}(t)^T + \frac{i}{2}w(t)\dot{\overline{w}}(t)^T + \frac{i}{2}w(t)\dot{\overline{w}}(t)^T - \frac{i}{2}w(t)\dot{\overline{w}}(t)^T + \frac{i}{2}\dot{w}(t)\overline{w}(t)^T\\
&= \dot{f}(t) - iz\dot{\overline{w}}(t) + iw(t)\dot{\overline{w}}(t)^T + \frac{i}{2}\left(\dot{w}(t)\overline{w}(t)^T - w(t)\dot{\overline{w}}(t)^T\right)\\
&= \dot{f}(t) + \Im(\dot{\overline{w}}(t)w(t)^T) - i(z-w(t))\dot{\overline{w}}(t)^T\,.
\end{aligned}
$$

Next, consider the terms by their order in $\hbar$. Recall from Proposition II.10 that multiplying by a factor of $(z-w(t))$ is of the same order as multiplying by $\sqrt{\hbar}$.

$$
\begin{aligned}
O(\hbar^m):&\quad \hbar^m e^{i\hbar^{-1}f(t)}\,e^{i\hbar^{-1}\omega(z,w(t))}\left(-\dot{f}(t) - \Im(\dot{\overline{w}}(t)w(t)^T\right)\varphi\left(\frac{z-w(t)}{\sqrt{\hbar}},t,\hbar\right)e^{-\hbar^{-1}|z-w(t)|^2/2}\\
O(\hbar^{m+1/2}):&\quad \hbar^m\,e^{i\hbar^{-1}f(t)}\,e^{i\hbar^{-1}\omega(z,w(t))}\left(i(z-w(t))\dot{\overline{w}}(t)^T\varphi\left(\frac{z-w(t)}{\sqrt{\hbar}},t,\hbar\right)\right.\\
&\qquad\qquad\left. -i\hbar^{1/2}\dot{w}(t)\nabla_z^T\varphi\left(\frac{z-w(t)}{\sqrt{\hbar}},t,\hbar\right)\right)e^{-\hbar^{-1}|z-w(t)|^2/2}\\
O(\hbar^{m+1}):&\quad \hbar^{m+1}\,e^{i\hbar^{-1}f(t)}\,e^{i\hbar^{-1}\omega(z,w(t))}\,i\frac{\partial\varphi}{\partial t}\left(\frac{z-w(t)}{\sqrt{\hbar}},t,\hbar\right)e^{-\hbar^{-1}|z-w(t)|^2/2}
\end{aligned}
$$

Now make the substitution $\zeta = (z-w(t))/\sqrt{\hbar}$ in each of the expressions above. The symbol of $i\hbar\frac{\partial\psi}{\partial t}$ as an element in I_γ^m, is

$$\sigma^m_{i\hbar\frac{\partial\psi}{\partial t}}(\zeta,t) = -\left(\dot{f}(t) + \Im(\dot{\overline{w}}(t)w(t)^T)\right)\sigma^m_\psi(\zeta,t).$$

If $\sigma^m_{i\hbar\frac{\partial\psi}{\partial t}}(\zeta,t)=0$, then the principal symbol of $i\hbar\frac{\partial\psi}{\partial t}$ as an element in $I_\gamma^{m+1/2}$ is given by

$$\sigma^{m+1/2}_{i\hbar\frac{\partial\psi}{\partial t}}(\zeta,t) = i\left(\dot{\overline{w}}(t)\zeta^T - \dot{w}(t)\nabla_\zeta^T\right)\sigma^m_\psi(\zeta,t).$$

Finally, if $\sigma^m_{i\hbar\frac{\partial\psi}{\partial t}}(\zeta,t) = \sigma^{m+1/2}_{i\hbar\frac{\partial\psi}{\partial t}}(\zeta,t)=0$, then the principal symbol of $i\hbar\frac{\partial\psi}{\partial t}$ as an element in I_γ^{m+1} is given by

$$\sigma^{m+1}_{i\hbar\frac{\partial\psi}{\partial t}}(\zeta,t) = i\frac{\partial\sigma^m_\psi}{\partial t}(\zeta,t).$$ $\square$

We finish this section with a result that applies to propagation under both Hermitian and non-Hermitian Hamiltonians.

Definition II.15. Define the *time-dependent Schrödinger operator* $\widetilde{\Box}$ which acts on functions of (z,t) as

$$\widetilde{\Box} := i\hbar\frac{\partial}{\partial t} - \widehat{F}$$

where $\widehat{F}$ is the Weyl quantization on z of $F(z,t)$ with the assumptions in II.8.

Theorem II.16. *For $\psi \in I_\gamma^m$, $\widetilde{\Box}\psi \in I_\gamma^m$ and its principal symbol is*

$$\sigma^m_{\widetilde{\Box}\psi}(\zeta,t) = -\left(\dot{f}(t) + \Im(\dot{\overline{w}}(t)w(t)^T) + F(w(t),t)\right)\sigma^m_\psi(\zeta,t)\,. \tag{2.19}$$

Proof. By linearity,

$$\sigma^m_{\widetilde{\Box}\psi}(\zeta,t) = \sigma^m_{i\hbar\frac{\partial\psi}{\partial t}}(\zeta,t) - \sigma^m_{\widehat{F}\psi}(\zeta,t).$$

The result follows after substituting equations (2.15) and (2.12). $\square$

The equation for $f(t)$ can easily be obtained from the previous theorem:

Corollary II.17. *If $\dot{f}(t) = -F(w(t),t) - \Im\left(\dot{\overline{w}}(t)w(t)^T\right)$ for each $t\in\mathbb{R}$, then $\widetilde{\Box}\psi \in I_\gamma^{m+1/2}$.*

Proof. Setting $\sigma^m_{\widetilde{\Box}\psi}(\zeta, t) = 0$ in (2.19) gives the equation for $\dot{f}(t)$ and implies that $\widetilde{\Box}\psi \in I_\gamma^{m+1/2}$. $\Box$

Remark 7. From this corollary, we see that since f is real-valued by assumption, in order for $\widetilde{\Box}\psi$ to be an element in $I_\gamma^{m+1/2}$, F must be a real-valued function. In Chapter IV, we will see a modification of $\widetilde{\Box}$ to accommodate the case where F is allowed to be complex-valued.

At this point one may wonder if we can obtain a condition(s) for which $\widetilde{\Box}\psi \in I_\gamma^{m+\ell/2}$, for $\ell = 1, 2, \ldots$ for some $\psi \in I_\gamma^m$. The answer is yes, but the cases where $F(z,t)$ is strictly real-valued and where it is allowed to be complex-valued must be treated separately due the different behavior of the trajectory of $w(t)$.

CHAPTER III

Propagation of Coherent States under Hermitian Hamiltonians

In this chapter, we impose the condition that the classical Hamiltonian be a real-valued function. In turn, its quantum counterpart will be a Hermitian Hamiltonian. This is a standard axiom of quantum mechanics because the Hermiticity guarantees that the energy spectrum is real and the time evolution under the Schrödinger equation is unitary, and hence norm-preserving.

The Schrödinger evolution under Hermitian quantum Hamiltonians of Gaussian coherent states in $L^2(\mathbb{R}^d)$ as defined in (1.4) has been studied quite extensively in the literature. Restricting the classical Hamiltonian to be at most a quadratic function in $(x, p) \in \mathbb{R}^d$, leads to an exact solution, and the resulting coherent state is another Gaussian state whose center moves along the Hamilton trajectory and the parameter $\Gamma(t)$ that appears in (1.4) evolves according to a matrix Riccati equation. Relaxing the assumption that the classical Hamiltonian is at most quadratic presents a naturally more difficult problem. The general case can be approximated semiclassically as a non-trivial perturbation of the quadratic case. One can prove that up to leading order in $\hbar$, the evolved state resembles the solution in the quadratic case. A thorough summary of these results is given in [CR12].

In what follows, we employ a novel method using the symbol calculus developed in the previous chapter to show that analogous results for propagation by Hermitian Hamiltonians hold for elements defined in the Bargmann space of $\mathbb{C}^d$, both for a more general class of coherent states, and for the Gaussian states we defined in (1.9).

3.1 Main Results

Throughout this chapter, assume that $F(z,t)$ has the same smoothness conditions as in II.8 but we require that it is strictly real-valued. Let $\widehat{F}$ be the Weyl quantization on z of F. Let $U(t)$ be the quantum evolution operator as a fundamental solution of Schrödinger's equation

$$i\hbar\frac{\partial}{\partial t}U(t) = \widehat{F}U(t), \qquad U(0) = \mathbb{I}.$$

For any given initial coherent state, ψ_0, $U(t)\,\psi_0$ is a solution to Schrödinger's equation. Furthermore, $U(t)$ has the following properties:

1. $U(t,s) = U(s,t)$ where $U(t,s) = U(t)\,U^{-1}(s)$,

2. $U(t,s)\,U(s,t') = U(t,t')$.

Theorem III.1. *Let $w \in \mathbb{C}^d$ and let γ be the Hamilton trajectory of $F(z,t)$ starting at w. Assume that $f(t)$ satisfies the condition in Corollary II.17 and that the initial coherent state is of the form*

$$\psi_w(z) = e^{i\hbar^{-1}\omega(z,w)}\,\varphi\left(\frac{z-w}{\sqrt{\hbar}},\hbar\right)\,e^{-\hbar^{-1}|z-w|^2/2}.$$

Then, $\forall N \in \mathbb{N}$, $\exists \psi_N \in I^0_\gamma$ such that

$$i\hbar\frac{\partial \psi_N}{\partial t} - \widehat{F}(\psi_N) \in I^{(N+3)/2}_\gamma, \qquad \psi_N|_{t=0} = \psi_w$$

and $\forall T \in (0,\infty)$, $\exists C_{T,N} < \infty$ such that for every $t \in [-T,T]$,

$$\|U(t)\psi_w - \psi_N(t)\|_{\mathcal{B}(\mathbb{C}^d)} \leq \hbar^{(N+1+d)/2}\,|t|\,C_{T,N}.$$

Remark 8. Although the initial state ψ_w is not in I^0_γ because it is time-independent, it is still natural to define its symbol as

$$\sigma_{\psi_w}(\zeta) = \varphi_0(\zeta)\,e^{-|\zeta|^2/2}$$

using the usual substitution $\zeta = (z - w(t))/\sqrt{\hbar}$ where φ_0 is the leading term in the asymptotic expansion of φ. Let us consider how this symbol evolves.

We know from the existing theory[1] [CR12] that if F is at most a quadratic function in the z variable, the solution to the Schrödinger problem

$$i\hbar \frac{\partial \psi_N}{\partial t} - \widehat{F}(\psi_N) = 0, \qquad \psi_N|_{t=0} = \psi_w \tag{3.1}$$

is exact, and the propagator $U(t)$ is a metaplectic operator that is associated with the Jacobian of the linear flow $\phi_t : \mathbb{C}^d \to \mathbb{C}^d$ at w where $\{\phi_t\}$ is the Hamilton flow of F:

$$\psi_N = \mathrm{Mp}(\mathrm{Jac}(\phi_t)_w)\,(\psi_w).$$

where Mp is the metaplectic representation[2] in the Bargmann space of $\mathbb{C}^d$. At the level of symbols, we have

$$\sigma_{\psi_N}(\zeta, t) = \mathrm{Mp}(d(\phi_t)_w)\,(\sigma_{\psi_w})\,(\zeta). \tag{3.2}$$

Finally, we know from the literature [CR12] that the solution to (3.1) in the case where F is a general real-valued function is approximated by the quadratic case, so the principal symbol is still given by (3.2).

3.2 Intermediate Results

Making use of the results in Chapter II we may prove:

Proposition III.2. *If $f(t)$ satisfies (II.17) and if $\gamma : t \to w(t)$ is a Hamilton trajectory of F, then for any $\psi \in I^m_\gamma$, we have $\widetilde{\Box}\psi \in I^{m+1}_\gamma$. Furthermore, the principal symbol of $\widetilde{\Box}\psi$ as*

[1] Although the theory is outlined for coherent states in $L^2(\mathbb{R}^d)$, via the Bargmann transform, it holds in our setting.

[2] More on the metaplectic representation is provided in §6.3.1.

an element of I_γ^{m+1} is given by

$$\sigma_{\widetilde{\Box}\psi}^{m+1}(\zeta,t) = \left(i\frac{\partial \varphi_0}{\partial t}(\zeta,t) - \widehat{\mathcal{Q}}^H(\varphi_0)(\zeta,t)\right) e^{-|\zeta|^2/2}. \tag{3.3}$$

Here $\widehat{\mathcal{Q}}^H$ is the Weyl quantization in ζ of the Hessian of $F(\zeta,t)$ with $\hbar = 1$:

$$\widehat{\mathcal{Q}}^H := \tfrac{1}{2}\zeta R_t \zeta^T + \zeta S_t \nabla_\zeta^T + \tfrac{1}{2} Tr(S_t) + \tfrac{1}{2}\nabla_\zeta \overline{R}_t \nabla_\zeta^T,$$

where $R_t^T = R_t$ and $\overline{S}_t^T = S_t$.

Proof. Recall that Corollary II.17 ensures $\widetilde{\Box}\psi \in I_\gamma^{m+1/2}$. To prove that $\widetilde{\Box}\psi \in I_\gamma^{m+1}$, we must show that $\sigma_{\widetilde{\Box}\psi}^{m+1/2}(\zeta,t) = 0$. Using equations (2.13) and (2.16), we calculate

$$\begin{aligned}
\sigma_{\widetilde{\Box}\psi}^{m+1/2}(\zeta,t) &= \sigma_{i\hbar\frac{\partial\psi}{\partial t}}^{m+1/2}(\zeta,t) - \sigma_{\widehat{F}\psi}^{m+1/2}(\zeta,t) \\
&= i\left(\dot{\overline{w}}(t)\zeta^T - \dot{w}(t)\nabla_\zeta^T\right)\sigma_\psi^m(\zeta,t) - \left(\nabla_z F(w(t),t)\zeta^T + \nabla_{\overline{z}} F(w(t),t)\nabla_\zeta^T\right)\sigma_\psi^m(\zeta,t) \\
&= \left(i\dot{\overline{w}}(t) - \nabla_z F(w(t),t)\right)\zeta^T\sigma_\psi^m(\zeta,t) - \left(i\dot{w}(t) + \nabla_{\overline{z}} F(w(t),t)\right)\nabla_\zeta^T\sigma_\psi^m(\zeta,t).
\end{aligned}$$

Since γ is a Hamiltonian trajectory, we have Hamilton's equations:

$$\dot{w}(t) = i\nabla_{\overline{z}} F(w(t),t) \quad \text{and} \quad \dot{\overline{w}}(t) = -i\nabla_z F(w(t),t)$$

which allow us to simplify the above expression to:

$$\begin{aligned}
\sigma_{\widetilde{\Box}\psi}^{m+1/2}(\zeta,t) &= \left(\nabla_z F(w(t),t) - \nabla_z F(w(t),t)\right)\zeta^T\sigma_\psi^m(\zeta,t) \\
&\qquad - \left(-\nabla_{\overline{z}} F(w(t),t) + \nabla_{\overline{z}} F(w(t),t)\right)\nabla_\zeta^T\sigma_\psi^m(\zeta,t) = 0.
\end{aligned}$$

Therefore, $\widetilde{\Box}\psi \in I_\gamma^{m+1}$. Lastly, to find the principal symbol of $\widetilde{\Box}\psi$ as an element in I_γ^{m+1}, we make use of equations (2.14) and (2.17):

$$\sigma^{m+1}_{\widetilde{\Box}\psi}(\zeta,t) = \sigma^{m+1}_{i\hbar\frac{\partial\psi}{\partial t}}(\zeta,t) - \sigma^{m+1}_{\widehat{F}\psi}(\zeta,t) = i\frac{\partial\sigma^m_\psi}{\partial t}(\zeta,t) - \widehat{\mathcal{Q}}^H(\sigma^m_\psi)(\zeta,t)\,.$$

Note that since $\widehat{F}$ is Hermitian we must require that $Q_t := \overline{R}_t$ and $\overline{S_t}^T = S_t$ in the quadratic operator $\widehat{\mathcal{Q}}$ in (2.14), so we have re-labeled this new operator as $\widehat{\mathcal{Q}}^H$. Using the fact that $\sigma^m_\psi(\zeta,t) = \varphi_0(\zeta,t)\, e^{-|\zeta|^2/2}$, we obtain the desired equation. $\square$

Proposition III.3. $\widetilde{\Box}\psi \in I^{m+3/2}_\gamma$ *provided* $\sigma^m_\psi(\zeta,t)$ *satisfies the transport equation*

$$i\frac{\partial\sigma^m_\psi}{\partial t}(\zeta,t) = \widehat{\mathcal{Q}}^H(\sigma^m_\psi)(\zeta,t), \qquad \sigma^m_\psi(\zeta,t)|_{t=0} = \varphi_0(\zeta,0)\, e^{-|\zeta|^2/2}. \tag{3.4}$$

Proof. We have $\widetilde{\Box}\psi \in I^{m+3/2}_\gamma$ provided that the principal symbol of $\widetilde{\Box}\psi$ as an element in I^{m+1}_γ is zero. Setting (3.3) equal to zero leads to (3.4). $\square$

Lemma III.4. *For each* $t \in \mathbb{R}$, (3.4) *has a unique solution* $\sigma^m_\psi(\zeta,t) \in \mathcal{S}(\mathbb{R}^{2d})$.

Proof. Equation (3.4) is Schrödinger's equation with $\hbar = 1$. Since $\widehat{\mathcal{Q}}^H$ is a quadratic operator, the propagator $U(t) = e^{-it\widehat{\mathcal{Q}}^H}$ is a metaplectic operator that is well-defined and unitary according to Corollary 11 in Chapter 3 of [CR12]. Therefore, the solution is $\sigma^m_\psi(\zeta,t) = U(t)\,\sigma^m_\psi(\zeta,0)$. This solution is in $\mathcal{S}(\mathbb{R}^{2d})$ because metaplectic operators map Schwartz functions to Schwartz functions. See the proof of this fact in Appendix A. $\square$

Assuming that for each $t \in \mathbb{R}$, $\widetilde{\Box}\psi \in I^{m+3/2}_\gamma$, we would like to show that $\widetilde{\Box}\psi$ can be made even smaller on the order of $\hbar$. To accomplish this, we systematically add correction terms in the following manner:

Proposition III.5. *Let* $\psi \in I^m_\gamma$ *be as in* (2.1). *Assume that* $\dot{f}(t)$ *satisfies* (II.17), $\gamma : t \to w(t)$ *is a Hamilton trajectory of* $F(z,t)$, *and equation* (3.4) *also holds. Then, for any* $N \in \mathbb{N}$, $\exists \rho_1, \ldots, \rho_N$ *with* $\rho_j \in I^m_\gamma$ *such that if we define*

$$\psi_N = \psi + \hbar^{1/2}\rho_1 + \hbar\rho_2 + h^{3/2}\rho_3 + \cdots + h^{N/2}\rho_N, \tag{3.5}$$

then

$$\widetilde{\Box}\psi_N \in I_\gamma^{m+(N+3)/2}, \qquad \psi_N\big|_{t=0} = \psi\big|_{t=0}.$$

Proof. We proceed by induction on N. Assume that $\widetilde{\Box}\psi_{N-1} \in I_\gamma^{m+N/2+1}$. Let

$$\psi_N = \psi_{N-1} + \hbar^{N/2}\rho_N$$

for $\rho_N \in I_\gamma^m$. We would like to choose ρ_N so that $\widetilde{\Box}\psi_N \in I_\gamma^{m+(N+3)/2}$. By Proposition III.2, $\widetilde{\Box}\rho_N \in I_\gamma^{m+1}$, so $\hbar^{N/2}\,\widetilde{\Box}\rho_N \in I_\gamma^{m+N/2+1}$. This gives $\widetilde{\Box}\psi_N \in I_\gamma^{m+N/2+1}$.

Let $\beta_{N-1}(\zeta,t) := \sigma^{m+N/2+1}_{\widetilde{\Box}\psi_{N-1}}(\zeta,t)$ be the principal symbol of $\widetilde{\Box}\psi_{N-1}$ as an element in $I_\gamma^{m+N/2+1}$. We will show below that $\exists\rho_N$ such that

$$\sigma^{m+1}_{\widetilde{\Box}\rho_N}(\zeta,t) = -\beta_{N-1}(\zeta,t), \qquad \sigma_{\widetilde{\Box}\rho_N}(\zeta,t)|_{t=0} = 0\,. \tag{3.6}$$

This choice of ρ_N gives us that $\sigma^{m+N/2+1}_{\widetilde{\Box}\psi_N}(\zeta,t) = 0$ which implies $\widetilde{\Box}\psi_N \in I_\gamma^{m+(N+3)/2}$. □

Lemma III.6. *For each $t \in \mathbb{R}$, the IVP (3.6) has a unique solution that is a Schwartz function in $\zeta \in \mathbb{C}^d$.*

Proof. The initial-value problem (3.6) can be rewritten slightly. Recall that

$$\sigma^{m+1}_{\widetilde{\Box}\rho_N}(\zeta,t) = i\frac{\partial\sigma^m_{\rho_N}}{\partial t}(\zeta,t) - \widehat{\mathcal{Q}}^H(\sigma^m_{\rho_N})(\zeta,t),$$

so rearranging the terms gives

$$i\frac{\partial\sigma^m_{\rho_N}}{\partial t}(\zeta,t) = \widehat{\mathcal{Q}}^H(\sigma^m_{\rho_N})(\zeta,t) - \beta_{N-1}(\zeta,t), \qquad i\frac{\partial\sigma^m_{\rho_N}}{\partial t}(\zeta,0) - \widehat{\mathcal{Q}}^H(\sigma^m_{\rho_N})(\zeta,0) = 0. \tag{3.7}$$

Therefore, the transport equation in (3.7) is just the non-homogeneous version of the equation in (3.4) for $t \in [-T,T]$ with $T \in (0,\infty)$. Thus, we can apply Duhamel's principle to solve for

$\sigma^m_{\rho_N}(\zeta, t)$:

$$\sigma^m_{\rho_N}(\zeta, t) = U(t)\, \sigma^m_{\rho_N}(\zeta, 0) + i \int_0^t U(t-s)\, \beta_{N-1}(\zeta, s)\, ds$$

where $U(t) = e^{-it\widehat{\mathcal{Q}}^H}$. $\square$

3.3 Proof of Theorem III.1

Proof. The remainder term is

$$R_N(z, t, \hbar) := i\hbar \frac{\partial}{\partial t} \psi_N(z, t, \hbar) - \widehat{F} \psi_N(z, t, \hbar) = \widetilde{\Box} \psi_N(z, t, \hbar)$$

where ψ_N is as in (3.5). Using the fact that $\widetilde{\Box}\psi_N \in I^{m+(N+3)/2}_{\gamma}$ from Proposition III.5 and the norm estimate from Lemma II.7, we conclude that

$$\sup_{t \in [-T,T]} \left\| \widetilde{\Box} \psi_N(\cdot, t, \hbar) \right\|_{\mathcal{B}(\mathbb{C}^d)} \leq \hbar^{m+(N+3+d)/2}\, C_{T,N}.$$

Then, by Duhamel's principle,

$$U(t)\psi_w - \psi_N(t) = \frac{i}{\hbar} \int_0^t U(t, s) R_N(s)\, ds.$$

Taking the norm in Bargmann space and using the fact that $U(t, s)$ is a unitary operator,

$$\begin{aligned}
\left\| U(t)\psi_w - \psi_N(t) \right\|_{\mathcal{B}(\mathbb{C}^d)} &\leq \hbar^{-1} \int_0^t \left\| U(t, s) R_N(s) \right\|_{\mathcal{B}(\mathbb{C}^d)} ds \\
&= \hbar^{-1} \int_0^t \left\| R_N(s) \right\|_{\mathcal{B}(\mathbb{C}^d)} ds \\
&\leq \hbar^{-1} \int_0^t \sup_{s \in [0,t]} \left\| R_N(s) \right\|_{\mathcal{B}(\mathbb{C}^d)} ds \\
&\leq \hbar^{-1} \hbar^{m+(N+3+d)/2} \int_0^t C_{T,N}\, ds \\
&= \hbar^{m+(N+1+d)/2}\, |t|\, C_{T,N}.
\end{aligned}$$

$\square$

3.4 Special Case: Propagation of Gaussian States

The problem:

$$\widetilde{\Box}\psi_{A(t),w(t)} = 0, \qquad \psi_{A(t),w(t)}\big|_{t=0} = \psi_{A,w}$$

where $\psi_{A,w} \in \mathcal{B}(\mathbb{C}^d)$ is as in (1.9) was solved in [RU21] using methods from [CR12]. It turns out that we can obtain the same equation for $A(t)$ as in [RU21] using our symbol calculus. Let the ansatz be

$$\psi_{A(t),w(t)}(z,t,\hbar) = e^{i\hbar^{-1}f(t)}\, e^{i\hbar^{-1}\omega(z,w(t))}\, e^{i\chi(t)}\, e^{\hbar^{-1}Q_{A(t)}(z-w(t))/2}\, e^{-\hbar^{-1}|z-w(t)|^2/2}. \tag{3.8}$$

where $\chi(t)$ solves $\dot{\chi}(t) = -\frac{1}{2}\mathrm{Tr}(\overline{R}_t A_t + S_t)$ with initial condition $\chi(0) = 0$. This requirement is necessary to cancel out trace terms that would otherwise appear in the symbols. We may also write (3.8) in form of (2.9) with only one term in the expansion of φ:

$$\varphi_0\left(\frac{z-w(t)}{\sqrt{\hbar}}, t, \hbar\right) = e^{i\chi(t)}\, e^{\hbar^{-1}Q_{A(t)}(z-w(t))/2}.$$

Remark 9. The symbol of (3.8) as an element of I^0_γ is $\sigma^0_{\psi_{A(t),w(t)}}(\zeta,t) = e^{i\chi(t)}\, e^{Q_{A(t)}(\zeta)/2}\, e^{-|\zeta|^2/2}$.

The following is analogous to Corollary A.5 in [RU21], but the proof follows the method of 3.2.

Theorem III.7. *Assume that $\dot{f}(t)$ satisfies (II.17) and $\gamma : t \to w(t)$ is a Hamilton trajectory of $F(z,t)$. Then, for each t, $\widetilde{\Box}\psi_{A(t),w(t)} \in I^{3/2}_\gamma$ provided $A(t)$ satisfies the matrix Riccati equation*

$$i\dot{A}_t = R_t + (S_t A_t + A_t S_t^T) + A_t \overline{R}_t A_t, \qquad A(0) = A\,.$$

Proof. Since $\dot{f}(t)$ satisfies (II.17) and Hamilton's equations hold, $\sigma^0_{\widetilde{\Box}\psi_{A(t),w(t)}}(\zeta,t) = \sigma^{1/2}_{\widetilde{\Box}\psi_{A(t),w(t)}}(\zeta,t) = 0$, so $\widetilde{\Box}\psi_{A(t),w(t)} \in I^1_\gamma$. Setting $\sigma^1_{\widetilde{\Box}\psi_{A(t),w(t)}}(\zeta,t) = 0$ and using (3.4) gives the equation for $\dot{A}(t)$. ∎

CHAPTER IV

Propagation of Coherent States under Non-Hermitian Hamiltonians

We now relax the assumption that the classical Hamiltonian be real-valued and allow it to take on complex values. Hence, the quantum Hamiltonian is a non-Hermitian operator. Non-Hermitian operators can have complex eigenvalues and the Schrödinger time evolution is no longer guaranteed to be unitary. This makes the analysis of non-Hermitian systems intrinsically much more challenging, as we shall see in this chapter.

The appearance of non-Hermitian Hamiltonians in the study of various phenomena is not uncommon. From a purely mathematical perspective, the study of the pseudo-spectra of non-Hermitian operators has received much attention in recent years, for example, see [TE20]. In quantum mechanics, states that correspond to "resonance" peaks are associated with the eigenvectors of a non-Hermitian Hamiltonian which decay in time [ROM10, Moi11]. In the context of nuclear physics, the optical model, which is used for describing the elastic and inelastic scattering in nucleon-nucleus interactions, contains a complex-valued potential whose imaginary part describes the absorption processes that occur via compound nucleus formation and subsequent decay [ROM10]. Furthermore, in open quantum systems, non-Hermitian Hamiltonians have been used to study quantum dissipation, i.e., energy loss with the environment [ROM10, WQM$^+$21].

More recently, a special class of non-Hermitian Hamiltonians, which remain invariant under parity inversion (reflection of the spatial coordinates) ($\mathcal{P}$) and time reversal ($\mathcal{T}$) have been considered. Typically, referred to as $\mathcal{PT}$–symmetric Hamiltonians, these operators, pioneered by [BB98] in 1998, possess real spectra and are norm-preserving. However, in 2002, Mostafazadeh [Mos02] argued that $\mathcal{PT}$ symmetry is not a necessary and sufficient condition for non-Hermitian systems to have real eigenvalues. He proposed that $\mathcal{PT}$–symmetric Hamiltonians were a special case of *pseudo-Hermitian Hamiltonians* whose energy spectrum is either real ($\mathcal{PT}$ symmetry is exact/unbroken) or appears in conjugate complex number pairs ($\mathcal{PT}$ symmetry is broken). He defined these pseudo-Hermitian operators, here denoted by $\widehat{F}$, as having the property: $\widehat{F}^\dagger = \widehat{\eta}\,\widehat{F}\,\widehat{\eta}^{-1}$ where $\widehat{\eta}$ is a Hermitian linear automorphism on some inner product space. Note that taking $\widehat{\eta}$ to be the identity operator reduces to the Hermitian case. As an example, these pseudo-Hermitian operators appear in quantum cosmology, particularly in the Wheeler-DeWitt equation for a Freedman-Robertson-Walker (FRW) model coupled to a massive real scalar field which can be re-formulated as the Schrödinger equation in the two-component representation [Mos02, Mos98]. Since the work of Mostafazadeh, there has been a greater interest in time-dependent pseudo-Hermitian Hamiltonians using Dyson maps (see [LdPM20], [dMFF06], and [Fri22]).

Now that we have presented a brief history on non-Hermitian Hamiltonians, let us return our attention to how coherent states propagate under the Schrödinger evolution with a non-Hermitian quantum Hamiltonian. In our analysis, we place minimal assumptions on the Hamiltonian and do not require any form of symmetry or pseudo-Hermiticity. This problem has been analyzed in the case of quadratic non-Hermitian quantum Hamiltonians for the states $\varphi_Z^\Gamma \in L^2(\mathbb{R}^d)$ that we described in §1.1 by [LST18, GS12]. Graefe and Shubert [GS12] work with single coherent states, whereas Lasser et al. [LST18] and Troppman [Tro17] propagate Hagedorn wave-packets and also consider the evolution of excited states. In these works, the authors show that the solution is exact, consistent with the Hermitian case. Arnaiz [Arn21], extends the work of these authors to construct small-time approximate solutions to Schrödinger's equation of Hagedorn wave-packets with a more general non-Hermitian Hamiltonian. The author assumes certain geometric control conditions on the Hamiltonian that ensure the energies decay exponentially.

We will use the work of these authors to guide our own analysis of the propagation of elements in our spaces I_γ^m under general non-Hermitian Hamiltonians. We will only develop the theory for Gaussian states rather than for more general elements in I_γ^m due to the fact that the underlying geometry is much more complicated than in case of propagation with Hermitian Hamiltonians. Our approach will be different from that of the authors whose work we mentioned because we will employ the symbol calculus developed in Chapter II and we will avoid the use of excited states.

In particular, given the assumption on F in II.8, the system we would like to solve is

$$\widetilde{\Box}\widetilde{\psi} = \left(i\hbar\frac{\partial}{\partial t} - \widehat{F}\right)\widetilde{\psi} = 0, \qquad \widetilde{\psi}|_{t=0} = \psi_{A,w}\,. \tag{4.1}$$

where $w(0) = w$ and $A(0) = A$ and $\psi_{A,w}$ is given in (1.9).

The main goals of this chapter are:

1. Find the transport equations for $w(t)$ and $A(t)$ in (4.1).

2. In the case where F is *not* at most a quadratic in z, construct an approximate solution that solves (4.1) to arbitrary order in $\hbar$.

4.1 Main Results

Theorem IV.1. *Let $F = H + i\Gamma$ where H and Γ are real-valued. Assume the smoothness conditions in II.8 hold for F and let $\widehat{F}$ be its Weyl quantization on z.*

(1) Given $w(0) = x(0) + iy(0) \in \mathbb{C}^d$ and $A(0) \in \mathcal{D}_d$, there exists $T \in (0,\infty)$ and a curve $\gamma : t \to w(t)$ and $A(t) \in \mathcal{D}_d$ such that for $t \in [-T, T]$ the system

$$(\dot{x}(t), \dot{y}(t)) = \Xi_H(w(t), t) + J_{A(t)}\left(\Xi_\Gamma(w(t), t)\right) \tag{4.2}$$

$$i\dot{A}_t = R_t + 2S_tA_t + A_tQ_tA_t \tag{4.3}$$

is satisfied.

Here $\Xi_H := \frac{1}{2}(-\nabla_y H, \nabla_x H)$ *and* $\Xi_\Gamma := \frac{1}{2}(-\nabla_y \Gamma, \nabla_x \Gamma)$ *denote the real Hamilton vector fields of* H *and* Γ*, respectively, and* $R_t = F_{zz}(w(t), t), S_t = F_{z\overline{z}}(w(t), t)$ *and* $Q_t = F_{\overline{zz}}(w(t), t)$*. Also, for each* t*,* $J_{A(t)}$ *is a linear complex structure on* $\mathbb{R}^{2d}$ *associated to* $A(t)$*. More precisely, the matrix form of* $J_{A(t)}$ *is given in* (C.1)*.*

Moreover, if $\widetilde{\psi}$ *is*

$$\widetilde{\psi}(z, t, \hbar) = e^{\hbar^{-1}\int_0^t \Gamma(w(s),s)\,ds}\, e^{i\hbar^{-1} f(t)}\, e^{i\hbar^{-1}\omega(z, w(t))}\, e^{i\chi(t)}\, e^{\hbar^{-1} Q_{A(t)}(z-w(t))/2}\, e^{-\hbar^{-1}|z-w(t)|^2/2}$$

where $f(t)$ *satisfies* $\dot{f}(t) = -H(w(t), t) - \Im(\dot{\overline{w}}(t) w(t)^T)$ *and* $\chi(t)$ *solves* $\dot{\chi}(t) = -\frac{1}{2}\,Tr(Q_t A_t + S_t)$*, then*

$$i\hbar \frac{\partial \widetilde{\psi}}{\partial t} = \widehat{F}(\widetilde{\psi}) + e^{\hbar^{-1}\int_0^t \Gamma(w(s),s)\,ds}\, \eta$$

where $\eta \in I_\gamma^{3/2}$*.*

(2) If $\Xi_\Gamma(w(t), t) \neq 0$ *for all* $t \in [-T, T]$*, then* $\forall N \in \mathbb{N}$*,* $\exists \psi_N \in I_\gamma^0$ *such that*

$$\widetilde{\psi} = e^{\hbar^{-1}\int_0^t \Gamma(w(s),s)\,ds} \psi_N$$

satisfies

$$i\hbar \frac{\partial \widetilde{\psi}}{\partial t} = \widehat{F}(\widetilde{\psi}) + e^{\hbar^{-1}\int_0^t \Gamma(w(s),s)\,ds}\, \eta_N$$

where $\eta_N \in I_\gamma^{(N+3)/2}$*.*

Remarks 10. Some observations on the previous statements:

1. Observe that the remainder in part (2) in small compared to $\widetilde{\psi}$: $\forall t \in [-T, T]$, using Lemma II.7, we have

$$\frac{\|e^{\hbar^{-1}\int_0^t \Gamma(w(s),s)\,ds}\, \eta_N\|_{\mathcal{B}(\mathbb{C}^d)}}{\|\widetilde{\psi}\|_{\mathcal{B}(\mathbb{C}^d)}} = \frac{\|\eta_N\|_{\mathcal{B}(\mathbb{C}^d)}}{\|\psi_N\|_{\mathcal{B}(\mathbb{C}^d)}} \leq C_T\, \hbar^{(N+3)/2}$$

for some constant C_T.

2. For each $t \in [-T, T]$, $w(t)$ is the center of a generalized state ψ.

3. In the case where $\Gamma = 0$, the system given by equations (4.2) and (4.3) decouples. The problem simplifies to the Hermitian case and trajectory for $w(t)$ in (4.2) follows Hamilton's equations.

4. A remarkable fact is that when $\Gamma(z, t) \neq 0$ for all $t \in [-T, T]$, the trajectory of the center of the coherent state depends on the initial squeezing parameter $A(0)$.

5. The condition on the gradient of Γ in part (2) is necessary in order to solve transport equations that arise in the construction of ψ_N.

6. Equation (4.2) is consistent with the equations found in [GS12] and [BBLU22].

7. Without referencing $J_{A(t)}$, equation (4.2) can be written as

$$i(\dot{\bar{w}}(t)^T - A_t \dot{w}(t)^T) = \nabla_z^T F(w(t), t) + A_t \nabla_{\bar{z}}^T F(w(t), t)\,.$$

There is some very interesting geometry behind equation (4.2). In the next section, we will consider an example with a simple non-Hermitian Hamiltonian that can be solved exactly as a motivation for this geometry.

4.2 A Simple Case

It has been known for a while in the field of mathematical physics that the center of a quantum state leaves classical phase space when we propagate the state by the Schrödinger equation with a non-Hermitian Hamiltonian. To elaborate, the solutions for the center of the coherent state formally continue to follow Hamilton's equations, but the classical center is not in $\mathbb{C}^d$. Therefore, in order to study the evolution of coherent states under non-Hermitian quantum Hamiltonians further, on the classical side we need to "complexify" phase space.

On the quantum side, this requires that the Gaussian states have a center that is a point in $\mathbb{C}^d \times \mathbb{C}^d$ rather than in $\mathbb{C}^d$. As an illustration of this phenomenon, we present the following example.

Example IV.2. Let $F(z,t) = \frac{1}{2}z^2$ for $z \in \mathbb{C}$. The quantum operator is $\widehat{F} = \frac{1}{2}\hat{z}^2$, where $\hat{z}$ is multiplication by z. Let the ansatz be

$$\psi_{A(t),(W(t),Z(t))}(z,t,\hbar) = e^{i\hbar^{-1}f(t)}\, e^{\hbar^{-1}A(t)(z-W(t))^2/2}\, e^{\hbar^{-1}zZ(t)}\, e^{-\hbar^{-1}W(t)Z(t)/2}\, e^{-\hbar^{-1}|z|^2/2}$$

where the center of this state is $(W(t),Z(t)) \in \mathbb{C}^d \times \mathbb{C}^d$ and $Z(t) \neq \overline{W}(t)$ except at $t=0$. Our goal is to solve

$$\widetilde{\Box}\psi_{A(t),(W(t),Z(t))} = 0, \qquad \psi_{A(t),(W(t),Z(t))}\Big|_{t=0} = e^{\hbar^{-1}z\overline{W}(0)}\, e^{-\hbar^{-1}|W(0)|^2/2}\, e^{-\hbar^{-1}|z|^2/2}\,. \tag{4.4}$$

In this case, Hamilton's equations are

$$\begin{aligned} \dot{W}(t) &= i\frac{\partial F}{\partial \overline{z}}(Z(t)) = 0 \\ \dot{\overline{Z}}(t) &= -i\frac{\partial F}{\partial z}(W(t)) = -iW(t) \end{aligned}$$

and solving them we have

$$\begin{aligned} W(t) &= W(0) \\ Z(t) &= -itW(0) + Z(0) = -itW(0) + \overline{W}(0)\,. \end{aligned}$$

From these equations, we can clearly see that for $t \neq 0$, $Z(t)$ is *not* the complex conjugate of $W(t)$ which means that the center of the coherent state leaves classical phase space. Next, we calculate that

$$i\hbar\frac{\partial}{\partial t}\psi_{A(t),(W(t),Z(t))} = \Big[-\dot{f}(t) + \frac{i}{2}\Big(\dot{A}(t)(z-W(t))^2 - 2A(t)(z-W(t))\,\dot{W}(t) + 2z\dot{Z}(t) \\ -\dot{W}(t)Z(t) - W(t)\dot{Z}(t)\Big)\Big]\,\psi_{A(t),(W(t),Z(t))}$$

and after substituting Hamilton's equations we are left with

$$i\hbar\frac{\partial}{\partial t}\psi_{A(t),(W(t),Z(t))} = \left[-\dot{f}(t) + \frac{i}{2}\dot{A}(t)(z-W(t))^2 + zW(t) - \frac{1}{2}W(t)^2\right]\psi_{A(t),(W(t),Z(t))}\,.$$

Therefore,

$$\begin{aligned}\widetilde{\Box}\psi_{A(t),(W(t),Z(t))} &= i\hbar\frac{\partial}{\partial t}\psi_{A(t),(W(t),Z(t))} - \tfrac{1}{2}z^2\psi_{A(t),(W(t),Z(t))} \\ &= \left[-\dot{f}(t) + \frac{i}{2}\dot{A}(t)(z-W(t))^2 + zW(t) - \frac{1}{2}W(t)^2 - \frac{1}{2}z^2\right]\psi_{A(t),(W(t),Z(t))} \\ &= \left[-\dot{f}(t) + \tfrac{1}{2}\left(i\dot{A}(t)-1\right)(z-W(t))^2\right]\psi_{A(t),(W(t),Z(t))}\,.\end{aligned}$$

Setting $\widetilde{\Box}\psi_{A(t),(W(t),Z(t))} = 0$ and matching powers of $(z - W(t))$ gives two initial value problems:

$$\begin{aligned}&(1)\quad \dot{f}(t) = 0, \qquad f(0) = 0 \\ &(2)\quad i\dot{A}(t) = 1, \qquad A(0) = 0\end{aligned}$$

which evaluate to $f(t) = 0$ and $A(t) = -it$. Plugging in the equations for $f(t)$, $W(t)$, $Z(t)$, and $A(t)$, the solution to (4.4) is

$$\psi_{A(t),(W(t),Z(t))}(z,t,\hbar) = e^{-i\hbar^{-1}t\,(z-W(0))^2/2}\,e^{\hbar^{-1}z\left(\overline{W}(0)-itW(0)\right)}\,e^{-\hbar^{-1}W(0)\left(\overline{W}(0)-itW(0)\right)/2}\,e^{-\hbar^{-1}|z|^2/2}$$

which simplifies to

$$\psi_{A(t),(W(t),Z(t))}(z,t,\hbar) = e^{-i\hbar^{-1}t\,z^2/2}\, e^{\hbar^{-1}z\overline{W}(0)}\, e^{-\hbar^{-1}|W(0)|^2/2}\, e^{-\hbar^{-1}|z|^2/2}. \tag{4.5}$$

On the other hand, Theorem IV.1 says that the solution to (4.4) is a standard Gaussian state, i.e.,

$$\psi_{A(t),w(t)}(z,t,\hbar) = e^{-i\hbar^{-1}t(z-w(t))^2/2}\, e^{\hbar^{-1}z\overline{w}(t)}\, e^{-\hbar^{-1}w(t)\overline{w}(t)/2}\, e^{-\hbar^{-1}|z|^2/2}$$

with a center in classical phase space that we now compute. Let $z = x + iy$, so

$$F(x,y) = \tfrac{1}{2}z^2 = \tfrac{1}{2}(x^2 - y^2 + ixy).$$

Then, set $w(t) = x(t) + iy(t)$, so that the real Hamiltonian fields are

$$\Xi_H(w(t),t) = \tfrac{1}{2}(y(t), x(t)), \qquad \Xi_\Gamma(w(t),t) = \tfrac{1}{2}(-x(t), y(t)).$$

Using the matrix expression for $J_{A(t)}$ in equation (C.1), we find

$$J_{-it} = \begin{pmatrix} -t & 1 \\ 1 & -t \end{pmatrix}^{-1} \begin{pmatrix} -1 & t \\ -t & 1 \end{pmatrix} = \frac{1}{t^2-1} \begin{pmatrix} 2t & -(t^2+1) \\ t^2+1 & -2t \end{pmatrix}.$$

Hence,

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} y(t) \\ x(t) \end{pmatrix} + \frac{1}{t^2-1}\begin{pmatrix} 2t & -(t^2+1) \\ t^2+1 & -2t \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} -x(t) \\ y(t) \end{pmatrix} = \frac{1}{1-t^2}\begin{pmatrix} tx + y \\ x + ty \end{pmatrix}.$$

Solving the system gives

$$x(t) = \frac{x(0) + y(0)\,t}{1-t^2}, \qquad y(t) = \frac{x(0)\,t + y(0)}{1-t^2}.$$

Therefore, as a point in $\mathbb{C}^d$, the center is

$$w(t) = \frac{1}{1-t^2}\left(w(0) + it\overline{w}(0)\right) .$$

Observe that the trajectory of $w(t)$ tends to infinity for $t = \pm 1$, so this solution exists for $|t| < 1$ which is consistent with the condition that $|A(t)| < 1$. For $|t| > 1$, the solution still exists, but it is not square-integrable, i.e., $\|\psi_{A(t),w(t)}\|_{\mathcal{B}(\mathbb{C}^d)}$ tends to infinity.

Plugging in the solution for $w(t)$ into $\psi_{A(t),w(t)}$ and simplifying results in

$$\psi_{A(t),w(t)}(z,t,\hbar) = e^{-i\hbar^{-1}tz^2/2}\, e^{\hbar^{-1}z\overline{w}(0)}\, e^{-\hbar^{-1}\left(|w(0)|^2 + it\overline{w}(0)^2\right)/(2(1-t^2))}\, e^{-\hbar^{-1}|z|^2/2} . \tag{4.6}$$

It is a nontrivial computation to show that up to a $z-$independent constant, the solution in (4.6) is equal to (4.5):

$$\psi_{A(t),(W(t),Z(t))} = \alpha_\hbar\, \psi_{A(t),w(t)}$$

where

$$\alpha_\hbar := e^{\hbar^{-1}\left(t^2|w(0)|^2 + it\overline{w}(0)^2\right)/(2(1-t^2))} .$$

In general, we will need a way to project the complex center back onto classical phase space. In the following section, we use the work of [GS12] and [LST18] as motivation for how to set up the underlying geometry.

4.3 Complexification of Classical Phase Space

To "complexify" $\mathbb{C}^d$ we embed it in $\mathbb{C}^d \times \mathbb{C}^d$:

$$\begin{aligned} \mathbb{C}^d &\hookrightarrow \mathbb{C}^d \times \mathbb{C}^d \\ w &\longmapsto (w, \overline{w}). \end{aligned}$$

Generally, we will call the new variables $(W,Z) \in \mathbb{C}^d \times \mathbb{C}^d$ where Z is not necessarily equal to $\overline{W}$. We give a special name to the set of points where this is true.

Definition IV.3. The *real locus* is the set

$$\mathfrak{R} = \{(W, Z) \in \mathbb{C}^d \times \mathbb{C}^d \mid Z = \overline{W}\}.$$

Recall that a Gaussian state in $\mathcal{B}(\mathbb{C}^d)$ centered at $w \in \mathbb{C}^d$ is given by

$$\psi_{A,w}(z) = e^{\hbar^{-1} Q_A(z-w)/2}\, e^{\hbar^{-1} z\overline{w}^T}\, e^{-\hbar^{-1}|w|^2/2} e^{-\hbar^{-1}|z|^2/2}$$

for $A \in \mathcal{D}_d$ and $z \in \mathbb{C}^d$. The *phase* of the above state may be re-written as

$$\tfrac{1}{2}\left(Q_A(z-w) - |z|^2 - |w|^2\right) + z\overline{w}^T = \tfrac{1}{2}\left(Q_A(z) - |z|^2\right) + z(\overline{w}^T - Aw^T) + \tfrac{1}{2}\left(Q_A(w) - w\overline{w}^T\right).$$

Writing the phase in this manner allows us to make the following observation:

Lemma IV.4. *Let $w \in \mathbb{C}^d$. Then, if $W, Z \in \mathbb{C}^d$ are such that*

$$Z^T - AW^T = \overline{w}^T - Aw^T \tag{4.7}$$

then $\forall \hbar$,

$$\psi_{A,w}(z) = C_\hbar\, \psi_{A,(W,Z)}(z) \tag{4.8}$$

where

$$\psi_{A,(W,Z)}(z) = e^{\hbar^{-1} Q_A(z-W)/2}\, e^{\hbar^{-1} zZ^T}\, e^{-\hbar^{-1} WZ^T/2}\, e^{-\hbar^{-1}|z|^2/2}, \tag{4.9}$$

and

$$C_\hbar = e^{\hbar^{-1}\left(Q_A(w) - Q_A(W) - w\overline{w}^T + WZ^T\right)/2}.$$

From (4.8) we see that up to a constant, the state $\psi_{A,(W,Z)}$ with a center in complex phase space is physically the same as the state $\psi_{A,w}$ whose center is in real phase space! This generalizes our observations in Example IV.2.

As a consequence of Lemma IV.4, we introduce the following definition:

Definition IV.5. Given $A \in \mathcal{D}_d$ and $w \in \mathbb{C}^d$, define the set

$$\Lambda_{A,w} := \{(W, Z) \mid Z^T - AW^T = \overline{w}^T - Aw^T\} \tag{4.10}$$

to be the *space of complex centers* $(W, Z) \in \mathbb{C}^d \times \mathbb{C}^d$ *of states* (4.9) *which are physically equivalent to* $\psi_{A,w}$.

Remark 11. For all $w \in \mathbb{C}^d$, $\psi_{A,w} = \psi_{A,(W=w,Z=\overline{w})}$ is the state on the real locus.

4.3.1 Lagrangian subspaces

In this section we study of the geometry of $\Lambda_{A,w}$. We can immediately show that each $\Lambda_{A,w}$ intersects the real locus at a single point.

Lemma IV.6. $\Lambda_{A,w} \cap \mathfrak{R} = \{(w, \overline{w});\ w \in \mathbb{C}^d\}$.

Proof. We choose $(W, Z) \in \Lambda_{A,w} \cap (W, \overline{W})$. Then, $Z^T - AW^T = \overline{w}^T - Aw^T$ and $Z = \overline{W}$ together imply that $\overline{W}^T - AW^T = \overline{w}^T - Aw^T$ which means $\overline{W}^T - \overline{w}^T = A(W^T - w^T)$ and $W^T - w^T = \bar{A}(\overline{W}^T - \overline{w}^T)$. Hence,

$$W^T - w^T = \bar{A}A(W^T - w^T) = A^*A(W^T - w^T),$$

since $\bar{A} = A^*$ because $A = A^T$. However, one is not an eigenvalue of A^*A because $A \in \mathcal{D}_d$, so $W = w$. $\square$

For fixed $A \in \mathcal{D}_d$ and $w \in \mathbb{C}^d$, $\Lambda_{A,w}$ has a special structure that will be important for our purposes in that it is a *positive Lagrangian subspace*. Before we can prove this, we must state some relevant definitions.

First of all, in our embedding of $\mathbb{C}^d$ into $\mathbb{C}^d \times \mathbb{C}^d$, complex conjugation takes on a different definition.

Definition IV.7. The *Bargmann conjugation* is the complex conjugation on $\mathbb{C}^d \times \mathbb{C}^d$ defined as

$$\overline{(W,Z)}^{\mathcal{B}} = (\overline{Z}, \overline{W}).$$

We will distinguish this conjugation from the "standard conjugation": $\overline{(W,Z)} = (\overline{W}, \overline{Z})$. The standard conjugation holds true on the real locus $\mathfrak{R}$.

Let

$$\Omega = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}$$

where I_d is the $d \times d$ identity matrix and note that $\Omega^2 = -I_{2d}$.

Using the *Bargmann conjugation* and Ω, we may define what it means for a subspace to be *(positive) Lagrangian* under our complexification.

Definition IV.8. A linear subspace of $\Lambda \subset \mathbb{C}^d \times \mathbb{C}^d$ is called *Lagrangian* if $\Omega|_\Lambda = 0$ and $\dim \Lambda = d$. If in addition, the quadratic form

$$h^{\mathcal{B}}_{\Lambda}(z, z') = \frac{i}{2} z\, \Omega(\overline{z'}^{\mathcal{B}})^T$$

is positive on Λ, i.e., $h^{\mathcal{B}}_{\Lambda}(z, z) > 0$ for all $z \in \Lambda$, then Λ is a *positive Lagrangian.*

Lemma IV.9. *The subspace* $\Lambda_{A,0} := \{(W, WA)\,; W \in \mathbb{C}^d\}$ *is a positive Lagrangian subspace.*

Proof. First, $\dim \Lambda_{A,0} = d$, so we need to show that $\Omega|_{\Lambda_{A,0}} = 0$.

Let $z = (W, WA) \in \Lambda_{A,0}$ and $z' = (W', W'A) \in \Lambda_{A,0}$. Then,

$$\begin{aligned} \tfrac{1}{2}\, z\Omega(z')^T &= -\tfrac{1}{2}(W, WA) \begin{pmatrix} -(W'A)^T \\ (W')^T \end{pmatrix} \\ &= -\tfrac{1}{2}\left(-W(W'A)^T + WA(W')^T\right) = \tfrac{1}{2} W(A - A^T)(W')^T = 0 \end{aligned}$$

because $A = A^T$. Next, we need to show $h^{\mathcal{B}}_{\Lambda_{A,0}}(z,z) > 0, \forall z \in \Lambda_{A,0}$.

$$h^{\mathcal{B}}_{\Lambda_{A,0}}(z,z) = -\tfrac{1}{2}(W, WA)\begin{pmatrix} -\overline{W}^T \\ \\ (\overline{W}\,\overline{A})^T \end{pmatrix}$$
$$= -\tfrac{1}{2}\left(-W\overline{W}^T + WA(\overline{W}\,\overline{A})^T\right) = \tfrac{1}{2}W\left(I_d - AA^*\right)\overline{W}^T > 0$$

because $AA^* < I_d$. Therefore, $\Lambda_{A,0}$ is a positive Lagrangian subspace. □

The subspace $\Lambda_{A,0}$ is precisely the *graph* of A. All of the other subspaces $\Lambda_{A,w}$, as defined in equation (4.10), are translations of $\Lambda_{A,0}$, so by the previous lemma they are also positive Lagrangian subspaces.

For a fixed $A \in \mathcal{D}_d$, as we vary w on $\mathbb{C}^d$, the positive Lagrangian subspaces $\Lambda_{A,w}$ partition complex phase space $\mathbb{C}^d \times \mathbb{C}^d$. We will refer to this partition of complex phase space by Lagrangian subspaces as a *Lagrangian foliation*. The blue lines in Figure 4.1 represent a foliation for fixed $A \in \mathcal{D}_d$. A single blue line in Figure 4.1 represents a *leaf*; that is, a $\Lambda_{A,w}$ for fixed A and w. In this way, we may think of a coherent state $\psi_{A,(W,Z)}$ as an object that is centered at a point on the leaf. Of course, varying $A \in \mathcal{D}_d$ gives rise to a different foliation.

4.3.2 Projection

Since complex phase space can be partitioned by Lagrangian subspaces, every point in $\mathbb{C}^d \times \mathbb{C}^d$ is exactly in one of the Lagrangians of the foliation for a given $A \in \mathcal{D}_d$. Also, by Lemma IV.6, each Lagrangian subspace intersects the real locus at exactly one point. These two facts allow us to define a projection from a point in $\mathbb{C}^d \times \mathbb{C}^d$ onto the real locus.

Definition IV.10. Let $(W,Z) \in \mathbb{C}^d \times \mathbb{C}^d$ and fix $A \in \mathcal{D}_d$. Then, the projection map $\Pi_A : \mathbb{C}^d \times \mathbb{C}^d \to \mathfrak{R}$ is given by

$$\Pi_A(W,Z) = w \in \mathbb{C}^d \qquad s.t. \quad Z^T - AW^T = \overline{w}^T - Aw^T. \tag{4.11}$$

Remark 12. We could, in theory, have defined the projection onto $\overline{w} \in \mathfrak{R}$, but we find that projecting onto the w coordinate is suitable for our purposes.

Figure 4.1 provides a visualization of the projection.

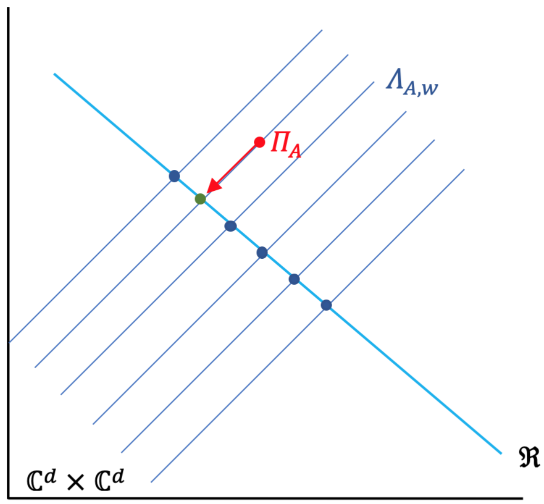

Figure 4.1: The red dot represents the center $(W, Z) \in \mathbb{C}^d \times \mathbb{C}^d$ of a coherent state and the green dot is the projection of the red dot onto the real locus. Hence, the green dot represents the "real center" of a coherent state. By Lemma IV.4 the two dots correspond to centers of states that are physically equivalent.

Lemma IV.11. *The projection* Π_A *may also be expressed as*

$$\Pi_A(W, Z) = \left[\overline{Z} + (Z - \overline{W} - WA)\bar{A}\right](I - A\bar{A})^{-1} = w \in \mathbb{C}^d.$$

Proof. From (4.11) we have that

1. $\overline{w} - wA = Z - WA \implies \overline{w} = Z - WA + wA$
2. $w - \overline{w}\bar{A} = \overline{Z} - \overline{W}\bar{A}$.

Substituting (1) into (2), we have

$$\begin{aligned}
\overline{z} - \overline{W}\bar{A} &= w - (Z - WA + wA)\bar{A} \\
\overline{z} - \overline{W}\bar{A} &= w(I_d - A\bar{A}) - ZA + WA\bar{A} \\
w &= \left[\overline{Z} + (Z - \overline{W} - WA)\bar{A}\right](I_d - A\bar{A})^{-1}.
\end{aligned}$$

□

Remark 13. Π_A has the following properties:

1. If $A = 0$, then (4.11) is satisfied if $w = \overline{Z}$ which implies

   $$\Pi_{A=0}(W, Z) = \overline{Z} = w \in \mathbb{C}^d$$

   and the projection is holomorphic (commutes with $-i$).

2. $\Lambda_{A,0} = \ker \Pi_A$

3. Π_A is the identity on the real locus:

   $$\Pi_A(W, \overline{W}) = [W + (\overline{W} - \overline{W} - WA)\bar{A}](I - A\bar{A})^{-1} = W(I - A\bar{A})(I - A\bar{A})^{-1} = W.$$

4. Π_A is $\mathbb{R}-$linear.

4.3.3 Complex structure

According to [BBLU22], there exists a complex structure that is related to the projection.

Proposition IV.12. *For $A \in \mathcal{D}_d$, there exists an $\mathbb{R}-$linear map $J_A \colon \mathfrak{R} \to \mathfrak{R}$ such that*

$$\forall v \in \mathbb{C}^d \times \mathbb{C}^d, \qquad \Pi_A(iv) = J_A\left(\Pi_A(v)\right). \tag{4.12}$$

Moreover, $J_A^2 = -I$.

Proof. Define: $\forall m \in \mathfrak{R}$, $J_A(m) := \Pi_A(im)$. This definition is imposed on us by the desired (4.12). We now check that J_A has the desired properties:

1. $\mathbb{R}-$linearity: If $m_1, m_2 \in \mathfrak{R}$ and $r \in \mathbb{R}$, then

$$J_A(m_1 + rm_2) = \Pi_A(im_1 + irm_2) = \Pi_A(im_1) + r\Pi_A(im_2) = J_A(m_1) + rJ_A(m2)$$

   where we have used the $\mathbb{R}-$linearity of Π_A in the next-to-last step.

2. Let $v \in \mathbb{C}^d \times \mathbb{C}^d$, and define $m = \Pi_A(v)$. Since $\Pi_A(m) = m = \Pi_A(v)$, $v - m \in \ker(\Pi_A)$. Now using that $\ker(\Pi_A)$ is a $\mathbb{C}-$linear subspace $iv - im \in \ker(\Pi_A)$ which implies

$$\Pi_A(iv) = \Pi_A(im) = J_A(m) = J_A\left(\Pi_A(v)\right).$$

   This proves (4.12).

3. Next, we show that $J_A^2 = -I$. Let $m \in \mathfrak{R}$. Observe, that

$$\Pi_A(i^2 m) = \Pi_A\left(i(im)\right) = J_A(\Pi_A(im)) = J_A^2(m)$$

   but also,

$$\Pi_A(i^2 m) = \Pi_A(-m) = -\Pi_A(m) = -m$$

   which implies $J_A^2(m) = -m$. □

From the definition of J_A in (4.12) we have that

$$\Pi_A(iZ, i\overline{Z}) = J_A(Z), \qquad \forall Z \in \mathbb{C}^d$$

and from the definition of Π_A in (4.11), we can conclude that

$$i\overline{Z} - iZA = \overline{J_A(Z)} - J_A(Z)\,A, \qquad \forall Z \in \mathbb{C}^d. \tag{4.13}$$

The above identity will be important in the equations of motion.

Remarks 14. A couple of comments about J_A:

1. If $A = 0$, J_0 is the multiplication by $-i$ operator on $\mathbb{C}^d$.

2. J_A cannot have i as an eigenvalue.

   Proof. If $\exists\, Z \in \mathbb{C}^d$ such that $J_A(Z) = iZ$, then by (4.13):

$$-i\overline{Z} - iZA = -i\bar{Z} - iZA \implies -i\overline{Z} = -iZ,$$

   which is only true if $Z = 0$. □

   This fact will be useful at the end of this chapter in solving an important ODE.

4.4 A Modified Equation

We will now study the dynamics of the Gaussian state $\psi_{A,w}$ as given in (1.9) under the Schrödinger equation. Let the evolved state be $\widetilde{\psi}$, and recall that the system we want to solve is

$$\widetilde{\Box}\widetilde{\psi} = \left(i\hbar\frac{\partial}{\partial t} - \widehat{F}\right)\widetilde{\psi} = 0, \qquad \widetilde{\psi}|_{t=0} = \psi_{A,w}$$

with $w(0) = w$ and $A(0) = A$.

From the existing literature on the case where $F(z,t)$ is assumed to be at most quadratic in z [GS12, LST18], we know that the solution $\widetilde{\psi}$ will not be in our spaces I^m_γ because the Bargmann norm of $\widetilde{\psi}(\cdot,t,\hbar)$ for each $t \in \mathbb{R}$ does not have a fixed order in $\hbar$. However, an inspection of the exact solution to $\widetilde{\Box}\psi = 0$ in the case where $F(z,t)$ is at most a quadratic function in z, shows that the solution is of the form

$$\widetilde{\psi} = e^{-\hbar^{-1}g(t)}\psi, \qquad \psi \in I^m_\gamma$$

where $g(t)$ is a real-valued function that satisfies $\dot{g}(t) = -\Gamma(w(t), t)$ and $w(t)$ is the solution to (4.2). Next, we find the equation that ψ solves. Observe that

$$i\hbar\frac{\partial\psi}{\partial t} = i\dot{g}(t)\, e^{\hbar^{-1}g(t)}\widetilde{\psi} + e^{\hbar^{-1}g(t)}\left(i\hbar\frac{\partial\widetilde{\psi}}{\partial t}\right).$$

Since by assumption $\widetilde{\Box}\widetilde{\psi} = 0$,

$$\begin{aligned}
\widetilde{\Box}\psi = i\hbar\frac{\partial\psi}{\partial t} - \widehat{F}\psi &= i\dot{g}(t)\, e^{\hbar^{-1}g(t)}\widetilde{\psi} + e^{\hbar^{-1}g(t)}\left(i\hbar\frac{\partial\widetilde{\psi}}{\partial t}\right) - \widehat{F}\psi \\
&= e^{\hbar^{-1}g(t)}\left[i\dot{g}(t)\,\widetilde{\psi} + i\hbar\frac{\partial\widetilde{\psi}}{\partial t} - \widehat{F}\widetilde{\psi}\right] \\
&= e^{\hbar^{-1}g(t)}\left[i\dot{g}(t)\,\widetilde{\psi} + \widetilde{\Box}\widetilde{\psi}\right] = e^{\hbar^{-1}g(t)} i\dot{g}(t)\,\widetilde{\psi} = i\dot{g}(t)\psi.
\end{aligned}$$

Hence, the equation for ψ is $(\widetilde{\Box} - i\dot{g}(t))\psi = 0$.

For brevity of notation, let us define a new operator:

Definition IV.13. Let $\Box := \widetilde{\Box} + i\Gamma(w(t), t)$.

Finally, the IVP we will construct a solution for is

$$\Box\psi = 0, \qquad \psi\Big|_{t=0} = \psi_{A,w}.$$

4.5 Symbol Calculations Revisited

Throughout the rest of this chapter, we will be working with a subset of elements in I_γ^m where the function φ is a polynomial times a Gaussian function, rather than a general Schwartz function, namely,

$$\begin{aligned}
\rho(z, t, \hbar) &= q\left(\frac{z - w(t)}{\sqrt{\hbar}}, t\right)\psi(z, t, \hbar) \\
&= \hbar^m q\left(\frac{z - w(t)}{\sqrt{\hbar}}, t\right) e^{i\hbar^{-1}\Phi(z,t)}\, e^{i\chi(t)}\, e^{\hbar^{-1}Q_{A(t)}(z - w(t))/2}\, e^{-\hbar^{-1}|z|^2/2}
\end{aligned} \tag{4.14}$$

where q is a polynomial.

Remark 15. The principal symbol of ρ as an element in I_γ^m is

$$\sigma_\rho^m(\zeta,t) = q(\zeta,t)\,\sigma_\psi^m(\zeta,t) = q(\zeta,t)\,e^{i\chi(t)}\,e^{Q_{A(t)}(\zeta)/2}\,e^{-|\zeta|^2/2}.$$

We now revisit some of the symbol calculus that we developed in Chapter II. Since we are now working with a class of functions of the form (4.14), we can obtain more explicit equations for the dynamics of $w(t)$ and $A(t)$.

Lemma IV.14. *For $\rho \in I_\gamma^m$ of the form in (4.14), for each $t \in \mathbb{R}$,*

1. *The principal symbol of $i\hbar\frac{\partial\rho}{\partial t}$ as an element in I_γ^m is*

$$\sigma^m_{i\hbar\frac{\partial\rho}{\partial t}}(\zeta,t) = -\left(\dot{f}(t) - \Im(\dot{\overline{w}}(t)w(t)^T)\right)\,\sigma_\rho^m(\zeta,t).$$

2. *If $\sigma^m_{i\hbar\frac{\partial\rho}{\partial t}}(\zeta,t) = 0$, then the principal symbol of $i\hbar\frac{\partial\rho}{\partial t}$ as an element in $I_\gamma^{m+1/2}$ is*

$$\sigma^{m+1/2}_{i\hbar\frac{\partial\rho}{\partial t}}(\zeta,t) = i\left(\dot{\overline{w}}(t) - \dot{w}(t)A(t)\right)\,\zeta^T\sigma_\rho^m(\zeta,t) - i\dot{w}(t)\nabla_\zeta^T q(\zeta,t)\,\sigma_\psi^m(\zeta,t).$$

3. *If $\sigma^m_{i\hbar\frac{\partial\rho}{\partial t}}(\zeta,t) = \sigma^{m+1/2}_{i\hbar\frac{\partial\rho}{\partial t}}(\zeta,t) = 0$, then the principal symbol of $i\hbar\frac{\partial\psi}{\partial t}$ as an element in I_γ^{m+1} is*

$$\sigma^{m+1}_{i\hbar\frac{\partial\rho}{\partial t}}(\zeta,t) = \tfrac{1}{2}\left(i\zeta\dot{A}_t\zeta^T + Tr(Q_tA_t + S_t)\right)\sigma_\rho^m(\zeta,t) + i\frac{\partial q}{\partial t}(\zeta,t)\,\sigma_\psi^m(\zeta,t).$$

Proof. First, we calculate

$$i\hbar\frac{\partial\rho}{\partial t} = i\hbar q\frac{\partial\psi}{\partial t} + i\hbar\left(-\hbar^{-1/2}\dot{w}(t)\nabla_z^T q + \frac{\partial q}{\partial t}\right)\psi = -i\sqrt{\hbar}\,\dot{w}(t)\nabla_z^T q\,\psi + i\hbar\frac{\partial q}{\partial t}\psi + i\hbar\frac{\partial\psi}{\partial t}q,$$

but we already know $i\hbar\frac{\partial\psi}{\partial t}$ from (2.18), so

$$\begin{aligned}i\hbar\frac{\partial\rho}{\partial t} &= -i\sqrt{\hbar}\dot{w}(t)\nabla_z^T q\,\psi + i\hbar\frac{\partial q}{\partial t}\psi - \dot{f}(t)\rho - \Im(\dot{\overline{w}}(t)w(t)^T)\rho \\ &\quad + i(z-w(t))\left(\dot{\overline{w}}(t)^T - A(t)\dot{w}(t)^T\right)\rho + \frac{i}{2}(z-w(t))\dot{A}_t(z-w(t))^T\rho + \frac{\hbar}{2}\mathrm{Tr}(Q_tA_t+S_t)\rho.\end{aligned}$$

Now we group the terms by their order in $\hbar$:

$$\begin{aligned}O(\hbar^m):&\quad -\left(\dot{f}(t) + \Im(\dot{\overline{w}}(t)w(t)^T)\right)\rho(z,t,\hbar) \\ O(\hbar^{m+1/2}):&\quad i(z-w(t))\left(\dot{\overline{w}}(t)^T - A(t)\dot{w}(t)^T\right)\rho(z,t,\hbar) - i\sqrt{\hbar}\,\dot{w}(t)\nabla_z^T q\left(\frac{z-w(t)}{\sqrt{\hbar}},t\right)\psi(z,t,\hbar) \\ O(\hbar^{m+1}):&\quad \tfrac{1}{2}\left(i(z-w(t))\dot{A}_t(z-w(t))^T + \hbar\mathrm{Tr}(Q_tA_t+S_t)\right)\rho(z,t,\hbar) \\ &\qquad\qquad + i\hbar\frac{\partial q}{\partial t}\left(\frac{z-w(t)}{\sqrt{\hbar}},t\right)\psi(z,t,\hbar)\end{aligned}$$

Note that factors of $\hbar$ are part of the ρ and the ψ.

To find the symbols of $i\hbar\frac{\partial\rho}{\partial t}$ as an element of I_γ^m, $I_\gamma^{m+1/2}$ and I_γ^{m+1}, we consider the $O(\hbar^m)$, $O(\hbar^{m+1/2})$ and $O(\hbar^{m+1})$ terms respectively, and make the substitution $\zeta = (z-w(t))/\sqrt{\hbar}$. Remember that $\sigma^m_{\hbar^p\rho} = \sigma^m_\rho$ for any power p and any m. Then, the symbol of $i\hbar\frac{\partial\rho}{\partial t} \in I_\gamma^m$, is

$$\sigma^m_{i\hbar\frac{\partial\rho}{\partial t}}(\zeta,t) = -\left(\dot{f}(t) + \Im(\dot{\overline{w}}(t)w(t)^T)\right)\sigma^m_\rho(\zeta,t).$$

If $\sigma^m_{i\hbar\frac{\partial\rho}{\partial t}}(\zeta,t) = 0$, then the principal symbol of $i\hbar\frac{\partial\rho}{\partial t}$ is an element in $I_\gamma^{m+1/2}$ given by

$$\sigma^{m+1/2}_{i\hbar\frac{\partial\rho}{\partial t}}(\zeta,t) = i\left(\dot{\overline{w}}(t) - \dot{w}(t)A(t)\right)\zeta^T\sigma^m_\rho(\zeta,t) - i\dot{w}(t)\nabla_\zeta^T q(\zeta,t)\,\sigma^m_\psi(\zeta,t).$$

Finally, if $\sigma^m_{i\hbar\frac{\partial\rho}{\partial t}}(\zeta,t) = \sigma^{m+1/2}_{i\hbar\frac{\partial\rho}{\partial t}}(\zeta,t) = 0$, then the principal symbol is an element in I_γ^{m+1} given by

$$\sigma^{m+1}_{i\hbar\frac{\partial\rho}{\partial t}}(\zeta,t) = \tfrac{1}{2}\left(i\zeta\dot{A}_t\zeta^T + \mathrm{Tr}(Q_tA_t+S_t)\right)\sigma^m_\rho(\zeta,t) + i\frac{\partial q}{\partial t}(\zeta,t)\,\sigma^m_\psi(\zeta,t). \qquad \square$$

Lemma IV.15. *Assume $F(z,t)$ satisfies II.8. For each $t \in \mathbb{R}$, let $\widehat{F}$ be the Weyl quantization on z of F. Then, for $\rho \in I_\gamma^m$ of the form in (4.14),*

1. *The principal symbol of $\widehat{F}\rho$ as an element in I_γ^m is*

$$\sigma^m_{\widehat{F}\rho}(\zeta,t) = (F(w(t),t)\,\sigma^m_\rho(\zeta,t) = (H(w(t),t) + i\Gamma(w(t),t))\ \sigma^m_\rho(\zeta,t)$$

2. *If $\widehat{F}$ vanishes to first order on γ, then the principal symbol of $\widehat{F}\rho$ as an element of $I_\gamma^{m+1/2}$ is*

$$\sigma^{m+1/2}_{\widehat{F}\rho}(\zeta,t) = (\nabla_z F(w(t),t) + \nabla_{\overline{z}} F(w(t),t) A_t)\ \zeta^T \sigma^m_\rho(\zeta,t) \\ + \nabla_{\overline{z}} F(w(t),t) \nabla^T_\zeta (q(\zeta,t)) \sigma^m_\psi(\zeta,t)$$

3. *If $\widehat{F}\rho$ vanishes to second order on γ, then the principal symbol of $\widehat{F}\rho$ as an element of I_γ^{m+1} is*

$$\sigma^{m+1}_{\widehat{F}\rho}(\zeta,t) = \tfrac{1}{2}\left[\zeta(R_t + 2S_tA_t + A_tQ_tA_t)\zeta^T + Tr(Q_tA_t + S_t)\right]\sigma^m_\rho(\zeta,t) \\ + \left(\zeta\, S_t\, \nabla^T_\zeta + \tfrac{1}{2}\nabla_\zeta\, Q_t \nabla^T_\zeta\right)(q(\zeta,t))\,\sigma^m_\psi(\zeta,t).$$

Proof. Consider a second order expansion of $F(z,t)$ about $z = w(t)$. The Weyl quantization of F on z and applied to ρ is $\widehat{F}_2(\rho)(z,t,\hbar) = \widehat{F}^{(0)}\rho + \widehat{F}^{(1)}\rho + \widehat{F}^{(2)}\rho$ with

$$\begin{aligned}
\widehat{F}^{(0)}\rho &= F(w(t),t)\rho = H(w(t),t)\rho + i\Gamma(w(t),t)\rho \\
\widehat{F}^{(1)}\rho &= \nabla_z F(w(t),t)(z - w(t))^T\rho + \nabla_{\overline{z}} F(w(t),t)\left(\hbar\nabla_z - \overline{w}(t)\right)^T\rho \\
\widehat{F}^{(2)}\rho &= \tfrac{1}{2}(z - w(t))R_t(z - w(t))^T\rho + (z - w(t))S_t\left(\hbar\nabla_z - \overline{w}(t)\right)^T\rho + \tfrac{1}{2}\hbar\,\mathrm{Tr}(S_t)\rho \\
&\quad + \tfrac{1}{2}\left(\hbar\nabla_z - \overline{w}(t)\right)Q_t\left(\hbar\nabla_z - \overline{w}(t)\right)^T\rho.
\end{aligned}$$

where R_t, S_t, and Q_t are the previously defined Hessian matrices. Now we compute some derivatives. First,

$$\hbar\nabla_z\rho = \hbar\nabla_z\left[q\left(\frac{z-w(t)}{\sqrt{\hbar}},t\right)\psi\right] = \hbar\left(\hbar^{-1/2}\psi\,\nabla_z q + q\nabla_z\psi\right)$$
$$= (z-w(t))A_t\,\rho + \overline{w}(t)\rho + \sqrt{\hbar}\nabla_z q\,\psi\,. \tag{4.15}$$

which implies

$$\left(\hbar\nabla_z - \overline{w}(t)\right)\rho = (z-w(t))A_t\,\rho + \sqrt{\hbar}\nabla_z q\;\psi$$

Then,

$$\begin{aligned}
\left(\hbar\nabla_z - \overline{w}(t)\right)Q_t\left(\hbar\nabla_z - \overline{w}(t)\right)^T\rho &= \left(\hbar\nabla_z - \overline{w}(t)\right)\Big(Q_tA_t(z-w(t))^T\rho + \sqrt{\hbar}\,Q_t\nabla_z^Tq\cdot\psi\Big)\\
&= \left(\hbar\nabla_z - \overline{w}(t)\right)Q_tA_t(z-w(t))^T\rho + \sqrt{\hbar}\left(\hbar\nabla_z - \overline{w}(t)\right)Q_t\nabla_z^Tq\cdot\psi\\
&= \underbrace{\hbar\nabla_z\left(Q_tA_t(z-w(t))^T\rho\right)}_{\mathrm{I}} - \overline{w}(t)Q_tA_t(z-w(t))^T\rho\\
&\quad + \underbrace{\hbar^{3/2}\nabla_z\left(Q_t\nabla_z^Tq\cdot\psi\right)}_{\mathrm{II}} - \sqrt{\hbar}\overline{w}(t)Q_t\nabla_z^Tq\cdot\psi
\end{aligned}$$

Using (4.15):

$$\begin{aligned}
\mathrm{I} &= \hbar\nabla_z\left(Q_tA_tz^T\rho\right) - \hbar\nabla_z(Q_tA_tw(t)^T\rho)\\
&= \hbar\nabla_z\rho\,Q_tA_tz^T + \hbar\nabla_z(Q_tA_tz^T)\rho - \hbar\nabla_z\rho\,Q_tA_tw(t)^T\\
&= (z-w(t))A_tQ_tA_tz^T\rho + \overline{w}(t)Q_tA_tz^T\rho + \hbar\nabla_zq\,Q_tA_tz^T\psi - \hbar\mathrm{Tr}(Q_tA_t)\rho\\
&\quad - (z-w(t))A_tQ_tA_tw(t)^T\rho - \overline{w}(t)Q_tA_tw(t)^T\rho - \hbar\nabla_zq\,Q_tA_tw(t)^T\psi\\
&= (z-w(t))A_tQ_tA_t(z-w(t))^T\rho + (z-w(t))A_tQ_t\overline{w}(t)^T\rho\\
&\qquad - \sqrt{\hbar}(z-w(t))A_tQ_t\nabla_z^Tq\cdot\psi + \hbar\mathrm{Tr}(Q_tA_t)
\end{aligned}$$

$$
\begin{aligned}
\mathrm{II} &= \hbar^{3/2}\nabla_z\psi\, Q_t\nabla_z^T q + \hbar^{3/2}\hbar^{-1/2}\nabla_z Q_t\nabla_z^T q\cdot\psi \\
&= \sqrt{\hbar}(z-w(t))A_tQ_t\nabla_z^T q\cdot\psi + \sqrt{\hbar}\overline{w}(t)Q_t\nabla_z^T q\cdot\psi + \hbar\nabla_z Q_t\nabla_z^T q\cdot\psi.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
(\hbar\nabla_z - \overline{w}(t))\, Q_t\, (\hbar\nabla_z - \overline{w}(t))^T \rho &= (z-w(t))A_t\, Q_t\, A_t(z-w(t))^T\rho \\
&\quad + \hbar\,\mathrm{Tr}(Q_tA_t)\,\rho + \hbar\,\nabla_z Q_t\nabla_z^T q\cdot\psi.
\end{aligned}
$$

Putting everything together we have,

$$
\widehat{F}^{(1)}\rho = (z-w(t))\left[\nabla_z^T F(w(t),t) + A_t\nabla_{\overline{z}}^T F(w(t),t)\right]\rho + \sqrt{\hbar}\nabla_z q\nabla_{\overline{z}}^T F(w(t),t)\,\psi
$$

$$
\begin{aligned}
\widehat{F}^{(2)}\rho &= \tfrac{1}{2}(z-w(t))(R_t + 2S_tA_t + A_tQ_tA_t)(z-w(t))^T\rho + \sqrt{\hbar}(z-w(t))S_t\nabla_z^T q\cdot\psi \\
&\quad + \tfrac{1}{2}\,\hbar\,\nabla_z Q_t\nabla_z^T q\cdot\psi + \tfrac{1}{2}\,\hbar\,\mathrm{Tr}(S_t + Q_tA_t)\rho.
\end{aligned}
$$

From (2.12), we already know that the principal symbol of $\widehat{F}\rho$ is $\sigma^m_{\widehat{F}\rho}(\zeta,t) = F(w(t),t)\sigma^m_{\widehat{F}\rho}(\zeta,t)$.

If $F(\cdot,t)$ vanishes to first order on γ, the principal symbol of $\widehat{F}\psi$ is an element of $I_\gamma^{m+1/2}$ given by

$$
\begin{aligned}
\sigma^{m+1/2}_{\widehat{F}\psi}(\zeta,t) = \sigma^{m+1/2}_{\widehat{F}^{(1)}\psi}(\zeta,t) &= \left(\nabla_z F(w(t),t) + \nabla_{\overline{z}}F(w(t),t)A_t\right)\,\zeta^T\sigma^m_\rho(\zeta,t) \\
&\quad + \nabla_{\overline{z}}F(w(t),t)\,\nabla_\zeta^T(q(\zeta,t))\,\sigma^m_\psi(\zeta,t)
\end{aligned}
$$

Lastly, if $F(\cdot,t)$ vanishes to second order on γ, the principal symbol of $\widehat{F}\psi$ is an element of I_γ^{m+1} given by

$$
\begin{aligned}
\sigma^{m+1}_{\widehat{F}\psi}(\zeta,t) = \sigma^{m+1}_{\widehat{F}^{(2)}\psi}(\zeta,t) &= \tfrac{1}{2}\left[\zeta(R_t + 2S_tA_t + A_tQ_tA_t)\zeta^T + \mathrm{Tr}(Q_tA_t + S_t)\right]\sigma^m_\rho(\zeta,t) \\
&\quad + \left(\zeta\, S_t\,\nabla_\zeta^T + \tfrac{1}{2}\nabla_\zeta\, Q_t\nabla_\zeta^T\right)(q(\zeta,t))\,\sigma^m_\psi(\zeta,t). \quad \square
\end{aligned}
$$

4.6 Propagation under the Modified Equation

Using the results from the previous section, we now look at how the operator $\Box$ acts on the elements $\rho \in I_\gamma^m$ given in (4.14). In the language of symbol calculus:

Proposition IV.16. *For $\rho \in I_\gamma^m$ as in (4.14), if $f(t)$ satisfies*

$$\dot{f}(t) = -H(w(t),t) - \Im(\dot{\overline{w}}(t)w(t)^T) \tag{4.16}$$

where $H(w(t),t) = \Re(F(w(t),t))$, then $\Box\rho \in I_\gamma^{m+1/2}$.

Proof. We must show that $\sigma^m_{\Box\rho}(\zeta,t) = 0$. By linearity,

$$\begin{aligned}
\sigma^m_{\Box\rho}(\zeta,t) &= \sigma^m_{\widetilde{\Box}\rho}(\zeta,t) + \sigma^m_{i\Gamma(w(t),t)\,\rho}(\zeta,t) \\
&= \sigma^m_{i\hbar\frac{\partial\rho}{\partial t}}(\zeta,t) - \sigma^m_{\widehat{F}\rho}(\zeta,t) + \sigma^m_{i\Gamma(w(t),t)\,\rho}(\zeta,t) \\
&= -\left(\dot{f}(t) + \Im(\dot{\overline{w}}(t)w(t)^T) + F(w(t),t)\right)\sigma^m_\rho(\zeta,t) + i\Gamma(w(t),t)\,\sigma^m_\rho(\zeta,t) \\
&= \left(-\dot{f}(t) - \Im(\dot{\overline{w}}(t)w(t)^T) - H(w(t),t) - i\Gamma(w(t),t) + i\Gamma(w(t),t)\right)\sigma^m_\rho(\zeta,t) \\
&= \left(-\dot{f}(t) - \Im(\dot{\overline{w}}(t)w(t)^T) - H(w(t),t)\right)\sigma^m_\rho(\zeta,t)
\end{aligned}$$

after applying the results from Lemmas IV.14 and IV.15. From the last expression, we can see that setting $\dot{f}(t) = -H(w(t),t) - \Im(\dot{\overline{w}}(t)w(t)^T)$ gives $\sigma^m_{\Box\rho}(\zeta,t) = 0$. $\Box$

4.6.1 Propagation of Gaussian States

For now assume that the polynomial $q(\zeta,t) = 1$ in (4.14), so $\rho = \psi$.

Remark 16. The principal symbol of such a ψ is

$$\sigma^m_\psi(\zeta,t) = e^{i\chi(t)}\, e^{Q_{A(t)}(\zeta)/2}\, e^{-|\zeta|^2/2}. \tag{4.17}$$

Proposition IV.17. *Let ψ be as in (4.14) and suppose that $f(t)$ satisfies (4.16). If the* *system*

$$i(\dot{\overline{w}}(t)^T - A_t\dot{w}(t)^T) = \nabla_z^T F(w(t),t) + A_t \nabla_{\overline{z}}^T F(w(t),t) \tag{4.18}$$

$$i\dot{A}_t = R_t + 2S_tA_t + A_tQ_tA_t \tag{4.19}$$

is also satisfied, then $\Box\psi \in I_\gamma^{m+3/2}$.

Proof. Since $f(t)$ satisfies (4.16), then by Proposition IV.16, we know that $\sigma_{\Box\psi}^m(\zeta,t) = 0$, so $\Box\psi \in I_\gamma^{m+1/2}$. Using Lemmas IV.14 and IV.15, we have

$$\begin{aligned}
\sigma_{\Box\psi}^{m+1/2}(\zeta,t) &= \sigma_{i\hbar\frac{\partial\psi}{\partial t}}^{m+1/2}(\zeta,t) - \sigma_{\widehat{F}\psi}^{m+1/2}(\zeta,t) \\
&= i\left(\dot{\overline{w}}(t) - \dot{w}(t)A(t)\right)\zeta^T\sigma_\psi^m(\zeta,t) - \left(\nabla_z F(w(t),t) + \nabla_{\overline{z}}F(w(t),t)A_t\right)\zeta^T\sigma_\psi^m(\zeta,t) \\
&= \left(\left(\dot{\overline{w}}(t) - \dot{w}(t)A(t)\right)\zeta^T - \left(\nabla_z F(w(t),t) + \nabla_{\overline{z}}F(w(t),t)A_t\right)\right)\zeta^T\sigma_\psi^m(\zeta,t)
\end{aligned}$$

since $\nabla_\zeta^T q(\zeta,t) = 0$ and $\frac{\partial q}{\partial t}(\zeta,t) = 0$. Setting $\sigma_{\Box\psi}^{m+1/2}(\zeta,t) = 0$ gives equation (4.18). Then, consider

$$\begin{aligned}
\sigma_{\Box\psi}^{m+1}(\zeta,t) &= \sigma_{i\hbar\frac{\partial\psi}{\partial t}}^{m+1}(\zeta,t) - \sigma_{\widehat{F}\psi}^{m+1}(\zeta,t) \\
&= \tfrac{1}{2}\left(i\zeta\dot{A}_t\zeta^T + \mathrm{Tr}(Q_tA_t + S_t)\right)\sigma_\psi^m(\zeta,t) \\
&\quad - \tfrac{1}{2}\left[\zeta(R_t + 2S_tA_t + A_tQ_tA_t)\zeta^T + \mathrm{Tr}(Q_tA_t + S_t)\right]\sigma_\psi^m(\zeta,t) \\
&= \tfrac{1}{2}\zeta\left(i\dot{A}_t - (R_t + 2S_tA_t + A_tQ_tA_t)\right)\zeta^T\sigma_\psi^m(\zeta,t).
\end{aligned}$$

If we set $\sigma_{\Box\psi}^{m+1}(\zeta,t) = 0$ we obtain (4.19) and hence $\Box\psi \in I_\gamma^{m+3/2}$. $\Box$

Remark 17. The system of equations is coupled because the matrices R_t, S_t, and Q_t depend on $w(t)$. If F is at most quadratic in z, then R_t, S_t, and Q_t will only depend on t and not $w(t)$.

4.6.2 Equations of Motion of the Center

The equation in (4.18) is consistent with the equation of motion of the real center of ψ in the following sense:

Proposition IV.18. *Let $w(t) = x(t) + iy(t)$ for $x, y \in \mathbb{R}^d$. Then,*

$$(\dot{x}(t), \dot{y}(t)) = \Xi_H(w(t), t) + J_{A(t)}\left(\Xi_\Gamma(w(t), t)\right)$$

implies

$$i(\dot{\overline{w}}(t) - \dot{w}(t)\, A_t) = \nabla_z F(w(t), t) + \nabla_{\overline{z}} F(w(t), t)\, A_t.$$

Proof. If $z = x + iy$, then $\nabla_z = \frac{1}{2}\left(\nabla_x - i\nabla_y\right)$, so $\Xi_H = \frac{1}{2}(-\nabla_y H, \nabla_x H)$ for any real-valued Hamiltonian H. Therefore, in terms of complex notation, we can write

$$\dot{w}(t) = i\nabla_{\overline{z}} H(w(t), t) + J_{A(t)}\left(i\nabla_{\overline{z}}\Gamma(w(t), t)\right). \tag{4.20}$$

By (4.13),

$$\begin{aligned} i(\dot{\overline{w}}(t) - \dot{w}(t)\, A_t) &= i\Big[-i\nabla_z H(w(t), t) + \overline{J_{A(t)}(i\nabla_{\overline{z}}\Gamma(w(t), t))} \\ &\qquad - i\nabla_{\overline{z}} H(w(t), t)\, A_t - J_{A(t)}(i\nabla_{\overline{z}}\Gamma(w(t), t))\, A_t\Big] \\ &= \nabla_z H(w(t), t) + \nabla_{\overline{z}} H(w(t), t)\, A_t \\ &\qquad + i\left[\overline{J_{A(t)}(i\nabla_{\overline{z}}\Gamma(w(t), t))} - J_{A(t)}(i\nabla_{\overline{z}}\Gamma(w(t), t))\, A_t\right]. \end{aligned}$$

Again, by (4.13), if $Z = i\nabla_{\overline{z}}\Gamma(w(t), t)$, then

$$i\nabla_z\Gamma(w(t), t) + i\nabla_{\overline{z}}\Gamma(w(t), t)\, A_t = i\left[\overline{J_{A(t)}(i\nabla_{\overline{z}}\Gamma(w(t), t))} - J_{A(t)}(i\nabla_{\overline{z}}\Gamma(w(t), t))\, A_t\right],$$

which leads to

$$\begin{aligned} i(\dot{\overline{w}}(t) - \dot{w}(t)\,A_t) &= \nabla_z H(w(t),t) + \nabla_{\overline{z}} H(w(t),t)\,A_t + i\nabla_z \Gamma(w(t),t) + i\nabla_{\overline{z}}\Gamma(w(t),t)\,A_t \\ &= \nabla_z F(w(t),t) + \nabla_{\overline{z}} F(w(t),t)\,A_t\,. \end{aligned} \qquad \square$$

For the rest of this chapter, let ψ be given by

$$\psi(z,t,\hbar) = e^{i\hbar^{-1}f(t)}\,e^{i\hbar^{-1}\omega(z,w(t))}\,e^{i\chi(t)}\,e^{\hbar^{-1}Q_{A(t)}(z-w(t))/2}\,e^{-\hbar^{-1}|z-w(t)|^2/2}$$

where $f(t)$ satisfies $\dot{f}(t) = -H(w(t),t) - \Im(\dot{\overline{w}}(t)w(t)^T)$ and $w(t) = x(t) + iy(t)$ and $A(t)$ satisfy the the following system

$$\begin{aligned} (\dot{x}(t),\dot{y}(t)) &= \Xi_H(w(t),t) + J_{A(t)}\,(\Xi_\Gamma(w(t),t)) \\ i\dot{A}_t &= R_t + 2S_tA_t + A_tQ_tA_t. \end{aligned} \tag{4.21}$$

Lemma IV.19. *A solution to the system* (4.21) *exists.*

Proof. Consider $u = (w(t), A(t))$ and reformulate the system as a Cauchy problem: Then, since F is analytic, by the Cauchy-Kowalevski Theorem (1.25 in [Fol95]) there exists a unique analytic solution u to the problem. $\square$

4.6.3 Adding Corrections

We will eventually show that we can construct a $\psi_N \in I^m_\gamma$ such that $\Box\psi_N$ is arbitrarily small in $\hbar$. We will employ the same technique of adding correction terms as in solving the Schrödinger equation for a Hermitian $\widehat{F}$.

However, there are two modifications we must make in the case of a non-Hermitian $\widehat{F}$:

1. The ansatz for the correction terms needs to be more general; namely, let each correction term $\rho_j \in I^m_\gamma$, $j = 1,\ldots,N$ be of the form (4.14).

2. An important distinction from the Hermitian case is that in this setting the first

correction term ρ_1 is preceded by an extra one-half power of $\hbar$:

$$\psi_N = \psi + \hbar\,\rho_1 + \hbar^{3/2}\rho_2 + \cdots + \hbar^{(N+1)/2}\rho_N, \qquad \psi, \rho_j \in I_\gamma^m.$$

This is due to that fact that in the non-Hermitian case, Hamilton's equations do not hold, so we will only be able to prove that $\Box\rho_1 \in I_\gamma^{m+1/2}$ rather than $\Box\rho_1 \in I_\gamma^{m+1}$.

Lemma IV.20. *The principal symbol of $\Box\rho$ as an element of $I_\gamma^{m+1/2}$ is*

$$\sigma_{\Box\rho}^{m+1/2}(\zeta, t) = \mathcal{L}(q(\zeta, t))\,\sigma_\psi^m(\zeta, t)$$

where

$$\mathcal{L} := \left[J_{A(t)}\left(\nabla_{\overline{z}}\Gamma(w(t), t)\right) - i\nabla_{\overline{z}}\Gamma(w(t), t)\right]\nabla_\zeta^T$$

and $\sigma_\psi^m(\zeta, t)$ is given in (4.17).

Proof. By linearity,

$$\begin{aligned}
\sigma_{\Box\rho}^{m+1/2}(\zeta, t) &= \sigma_{\widetilde{\Box}\rho}^{m+1/2}(\zeta, t) = \sigma_{i\hbar\frac{\partial\rho}{\partial t}}^{m+1/2}(\zeta, t) - \sigma_{\widehat{F}\rho}^{m+1/2}(\zeta, t) \\
&= i\left(\dot{\overline{w}}(t) - \dot{w}(t)A(t)\right)\,\zeta^T\sigma_\rho^m(\zeta, t) - i\dot{w}(t)\nabla_\zeta^T q(\zeta, t)\,\sigma_\psi^m(\zeta, t) \\
&\quad - \left(\nabla_\zeta F(w(t), t) + \nabla_{\overline{\zeta}}F(w(t), t)A_t\right)\,\zeta^T\sigma_\rho^m(\zeta, t) - \nabla_{\overline{\zeta}}F(w(t), t)\nabla_\zeta^T(q(\zeta, t))\sigma_\psi^m(\zeta, t) \\
&= \left[i\left(\dot{\overline{w}}(t) - \dot{w}(t)A(t)\right) - \left(\nabla_\zeta F(w(t), t) + \nabla_{\overline{\zeta}}F(w(t), t)A_t\right)\right]\zeta^T\,\sigma_\rho^m(\zeta, t) \\
&\quad + \left(-i\dot{w}(t) - \nabla_{\overline{\zeta}}F(w(t), t)\right)\nabla_\zeta^T(q(\zeta, t))\,\sigma_\psi^m(\zeta, t)\,.
\end{aligned}$$

where we have applied Lemmas IV.14 and IV.15. After substituting equation (4.18), the above reduces to

$$\sigma_{\Box\rho}^{m+1/2}(\zeta, t) = \left(-i\dot{w}(t) - \nabla_{\overline{z}}F(w(t), t)\right)\nabla_\zeta^T(q(\zeta, t))\,\sigma_\psi^m(\zeta, t)\,.$$

Now substituting (4.20) for $\dot{w}(t)$:

$$\begin{aligned}(-i\dot{w}(t) - \nabla_{\overline{z}}F(w(t),t)) &= -i\left(i\nabla_{\overline{z}}H(w(t),t) + J_{A(t)}\left(i\nabla_{\overline{z}}\Gamma(w(t),t)\right)\right) - \nabla_{\overline{z}}F(w(t),t)\\ &= \nabla_{\overline{z}}H(w(t),t) + J_{A(t)}\left(\nabla_{\overline{z}}\Gamma(w(t),t)\right) - \nabla_{\overline{z}}H(w(t),t) - i\nabla_{\overline{z}}\Gamma(w(t),t)\\ &= J_{A(t)}\left(\nabla_{\overline{z}}\Gamma(w(t),t)\right) - i\nabla_{\overline{z}}\Gamma(w(t),t)\,.\end{aligned}$$

Thus,

$$\sigma_{\Box\rho}^{m+1/2}(\zeta,t) = \left[J_{A(t)}\left(\nabla_{\overline{z}}\Gamma(w(t),t)\right) - i\nabla_{\overline{z}}\Gamma(w(t),t)\right]\nabla_{\zeta}^{T}(q(\zeta,t))\,\sigma_{\psi}^{m}(\zeta,t)\ . \qquad \square$$

Remarks 18. Some remarks on the operator $\mathcal{L}$:

1. $\mathcal{L}$ is an operator that acts on $q(\zeta,t)$ and $\sigma_{\psi}^{m}(\zeta,t)$ can be thought of as the "weight" in some other space.

2. Recall that if $A = 0$, then $J_0 = -i$, so $\mathcal{L} = -2i\,\nabla_{\overline{z}}\Gamma(w(t),t)\,\nabla_{\zeta}^{T}$.

3. Also, recall from Remark 14 that i is not an eigenvalue of $J_{A(t)}$, therefore

$$J_{A(t)}\left(\nabla_{\overline{z}}\Gamma(w(t),t)\right) - i\nabla_{\overline{z}}\Gamma(w(t),t) \neq 0$$

   for any $t \in \mathbb{R}$ as long as $\nabla_{\overline{z}}\Gamma(w(t),t) \neq 0$.

Proposition IV.21. *Let $\psi \in I_{\gamma}^{m}$ be given by*

$$\psi(z,t,\hbar) = \hbar^{m}\,e^{i\hbar^{-1}\Phi(z,t)}\,e^{i\chi(t)}\,e^{\hbar^{-1}Q_{A(t)}(z-w(t))/2}\,e^{-\hbar^{-1}|z|^2/2}.$$

Assume that $\dot{f}(t)$ satisfies (4.16), and $\dot{w}(t)$ and $\dot{A}(t)$ satisfy the coupled system (4.21). Therefore, for all $N \in \mathbb{N}$, $\exists\,\rho_1,\ldots,\rho_N \in I_{\gamma}^{m}$ of the form

$$\rho_j(z,t,\hbar) = q_j\left(\frac{z-w(t)}{\sqrt{\hbar}},t\right)\,\psi(z,t,\hbar)$$

where the q_j are polynomials. If we define ψ_N in the following manner:

$$\psi_N = \psi + \hbar\rho_1 + \hbar^{3/2}\rho_2 + \cdots + \hbar^{(N+1)/2}\rho_N, \tag{4.22}$$

then,

$$\Box\psi_N \in I_\gamma^{m+(N+3)/2}, \qquad \psi_N|_{t=0} = \psi|_{t=0} = \psi_{A,w}$$

where $\psi_{A,w} \in \mathcal{B}(\mathbb{C}^d)$ as given in (1.9).

Proof. The proof is analogous to the proof of Proposition III.5. The main difference is in the type of equation that the correction terms ρ_j must satisfy. Nevertheless, we re-construct the proof to show where this difference arises.

By induction on N, assume that $\Box\psi_{N-1} \in I_\gamma^{m+N/2+1}$. Let $\psi_N = \psi_{N-1} + \hbar^{(N+1)/2}\rho_N$ for ρ_N of the form (4.22). By Lemma IV.20, $\Box\rho_N \in I^{m+1/2}$ which implies $\hbar^{(N+1)/2}\,\rho_N \in I_\gamma^{m+N/2+1}$. Therefore, $\Box\psi_N \in I_\gamma^{m+N/2+1}$.

Let $\beta_{N-1}(\zeta,t) := \sigma_{\Box\psi_{N-1}}^{m+N/2+1}(\zeta,t)$ be the principal symbol of $\Box\psi_{N-1}$ as an element in $I_\gamma^{m+N/2+1}$. In the next lemma, we show that $\exists\rho_N$ such that $\sigma_{\Box\rho_N}^{m+1/2}(\zeta,t) = -\beta_{N-1}(\zeta,t)$ for each t, i.e.,

$$\mathcal{L}(q_N(\zeta,t))\,\sigma_\psi^m(\zeta,t) = -\beta_{N-1}(\zeta,t), \qquad \sigma_{\Box\rho_N}^{m+1/2}\big|_{t=0} = 0\,. \tag{4.23}$$

This ensures that $\sigma_{\Box\psi_N}^{m+N/2+1}(\zeta,t) = 0$, and therefore, $\Box\psi \in I_\gamma^{m+(N+3)/2}$. $\Box$

Lemma IV.22. *Assume that $\nabla_{\bar z}\Gamma(w(t),t) \neq 0$[1] for all $t \in [-T,T]$. Then for each $t \in [-T,T]$, the system (4.23) has a solution for $q_N(\zeta,t)$.*

Proof. First, we need the following result:

Claim: As a symbol, $\beta_{N-1}(\zeta,t) = \sigma_{\Box\psi_{N-1}}^{m+N/2+1}(\zeta,t)$ is of the form $\beta_{N-1}(\zeta,t) = p(\zeta,t)\,\sigma_\psi^m(\zeta,t)$ where $p(\zeta,t)$ is a polynomial and $\sigma_\psi^m(\zeta,t)$ is given by (4.17).

[1]This is equivalent to the condition $\Xi_\Gamma(w(t),t) \neq 0$.

Proof of the Claim: By linearity,

$$\begin{aligned}\sigma^{m+N/2+1}_{\Box\psi_{N-1}}(\zeta,t) &= \sigma^{m+N/2+1}_{i\hbar\frac{\partial}{\partial t}\psi_{N-1}}(\zeta,t) - \sigma^{m+N/2+1}_{\widehat{F}(\psi_{N-1})}(\zeta,t) + \sigma^{m+N/2+1}_{i\Gamma(w(t),t)\psi_{N-1}}(\zeta,t) \\ &= \sigma^{m+N/2+1}_{i\hbar\frac{\partial}{\partial t}\psi_{N-1}}(\zeta,t) - \sigma^{m+N/2+1}_{\widehat{F}(\psi_{N-1})}(\zeta,t) + i\Gamma(w(t),t)\,\sigma^{m+N/2+1}_{\psi_{N-1}}(\zeta,t).\end{aligned}$$

Now let us consider each term separately. Rewriting ψ_{N-1} in (4.22):

$$\psi_{N-1}(z,t,\hbar) = \left(1 + \sum_{j=1}^{N-1} \hbar^{(j+1)/2}\, q_j\left(\frac{z-w(t)}{\sqrt{\hbar}}, t\right)\right)\psi(z,t,\hbar)\,. \tag{4.24}$$

There are no $O(\hbar^{m+N/2+1})$ terms in ψ_{N-1} by construction, so considering ψ_{N-1} as an element in $I^{m+N/2+1}_{\gamma}$, its symbol $\sigma^{m+N/2+1}_{\psi_{N-1}}(\zeta,t)$ must be zero.

Computing $\sigma^{m+N/2+1}_{i\hbar\frac{\partial}{\partial t}\psi_{N-1}}(\zeta,t)$ involves differentiating (4.24) with respect to t but the result will be another polynomial times ψ since ψ is Gaussian. Then, considering the terms that are $O(\hbar^{m+N/2+1})$ and changing variables to $\zeta = (z-w(t))/\sqrt{\hbar}$, we get that $\sigma^{m+N/2+1}_{i\hbar\frac{\partial}{\partial t}\psi_{N-1}}(\zeta,t)$ is some polynomial multiplied by $\sigma^{m}_{\psi}(\zeta,t)$.

Lastly, to compute $\sigma^{m+N/2+1}_{\widehat{F}(\psi_{N-1})}(\zeta,t)$, take the $(N+2)-$order Taylor expansion of $F(z,t)$ about $z = w(t)$. Weyl quantize this expansion on z and apply it to ψ_{N-1}. The resulting expression will involve a sum of partial derivatives in z of ψ_{N-1} multiplied by powers of $(z-w(t))$ all multiplying ψ itself. Hence, when we consider the terms that are $O(\hbar^{m+N/2+1})$ and change variables to $\zeta = (z-w(t))/\sqrt{\hbar}$, we get that $\sigma^{m+N/2+1}_{\widehat{F}(\psi_{N-1})}(\zeta,t)$ is also a polynomial times $\sigma^{m}_{\psi}(\zeta,t)$.

Combining everything we get $\psi_{N-1}(z,t,\hbar) = p(\zeta,t)\,\sigma^{m}_{\psi}(\zeta,t)$ for some polynomial p. $\square$

Applying the claim, (4.23) simplifies to

$$\mathcal{L}(q_N(\zeta,t)) = p(\zeta,t). \tag{4.25}$$

By our assumption, $\nabla_{\overline{z}}\Gamma(w(t),t) \neq 0$ for all $t \in [-T,T]$ and by Remark 18, $\mathcal{L}$ is never zero for any t. Hence, by a suitable linear change of coordinates in (4.25), $\mathcal{L} = \frac{\partial}{\partial \zeta_1}$. Integrating the right side of (4.25) with respect to ζ_1 gives a solution for $q_N(\zeta, t)$. $\square$

4.7 The Quadratic Case

Assume that $F(z,t) : \mathbb{C}^d \times \mathbb{R} \to \mathbb{C}$ is at most a quadratic function in z and let $\widehat{F}$ be its Weyl quantization on the z coordinate. In this case, the solution to

$$\Box\psi = 0, \qquad \psi\big|_{t=0} = \psi_{A,w}$$

is exact and the solutions for $w(t) = x(t) + iy(t)$ and $A(t)$ are given by the system in (4.21). Given that F is at most quadratic, the Hessian matrices R_t, S_t, and Q_t are independent of $w(t)$. Hence, we may solve equation for $A(t)$ first and use that solution to obtain $w(t)$.

Let us explain the evolution of a Gaussian state under at most a quadratic non-Hermitian Hamiltonian in the context of the geometry of Lagrangian manifolds. Figure 4.2 provides a visualization of these mechanics.

(1) The initial state $\psi_{A,w}$ has with center w on the real locus $\mathcal{R}$. The center of this state is represented by the green dot as the intersection of the leaf $\Lambda_{A,w}$ (blue line) and the real locus $\mathfrak{R}$. Then, evolve $\psi_{A,w}$ by Schrödinger's equation using a non-Hermitian quantum Hamiltonian for some time $t > 0$. As the state evolves, the parameter A varies with t, so we get a different foliation of Lagrangian subspaces $\Lambda_{A(t),w(t)}$ represented by the yellow lines. Meanwhile, the center of the state (green dot) moves away from the real locus onto one of the leaves of the foliation $\Lambda_{A(t),w(t)}$.

(2) Project the center of the evolved state (which corresponds to the green dot on the yellow line) back onto the real locus. The purple dot represents the center of the final state. Note by Lemma IV.4, these two states are physically equivalent. The equation that we found for $w(t)$ in terms of the complex structure $J_{A(t)}$ in the system 4.21 corresponds to the purple dot.

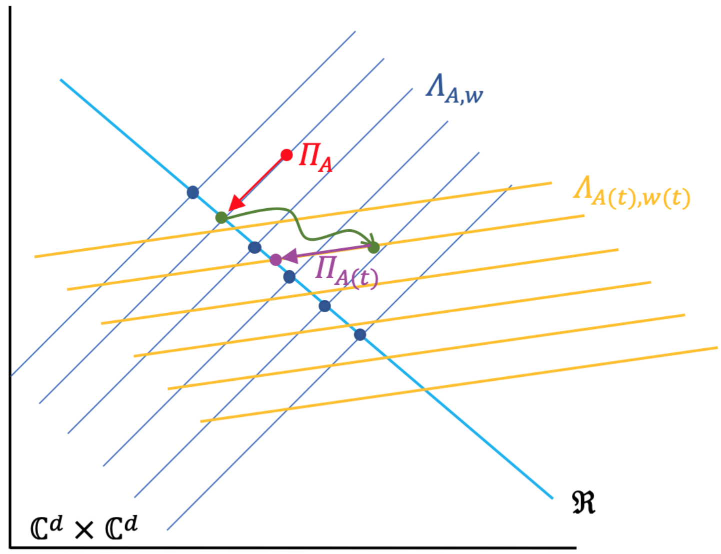

Figure 4.2: Dynamics of the evolution a Gaussian state by a non-Hermitian Hamiltonian

We are using the term "lines" broadly here. Under the evolution of a non-Hermitian *quadratic* Hamiltonian, the leaves of the foliation $\Lambda_{A(t),w(t)}$ will be "lines," but for general non-Hermitian Hamiltonians, these will be more complicated submanifolds. This is described in [BBLU22].

CHAPTER V

Reduction of Coherent States

We now turn our attention to the second part of this thesis. In this chapter, we will describe the procedure for a quantum version of dimensional reduction which allows us to reduce the states $\psi_{A,w} \in \mathcal{B}(\mathbb{C}^d)$ defined in (1.9) to construct a family of Gaussian coherent states on complex projective spaces. Of special interest, are the states in the Bargmann space of $\mathbb{CP}^1$, that is, squeezed SU(2) coherent states which we will discuss in Chapter VII. This reduction procedure holds more generally for Kähler manifolds (see [RU21] for more information), but we will explain it in the context of the coherent states that we are interested in. In particular, we will study the semiclassical properties of the reduction of $\psi_{A,w}$.

5.1 Dimensional Reduction of Gaussian Coherent States

We proceed to describe the notion of quantum reduction.

Notation 2. For the rest of this thesis, unless otherwise noted, we will set $k = \hbar^{-1}$.

Generally, the Bargmann space of $\mathbb{C}^d$ may be split up as

$$\mathcal{B}(\mathbb{C}^d) = \bigoplus_{\ell=0}^{\infty} \mathcal{W}_\ell^{(k)}$$

where

$$\mathcal{W}_\ell^{(k)} = \{f(z)e^{-k|z|^2/2}\,;\ f \text{ is a homogeneous polynomial of degree } \ell\}.$$

and the $\mathcal{W}_\ell^{(k)}$ are mutually orthogonal.[1] Equivalently, the $\mathcal{W}_\ell^{(k)}$ are the eigenspaces with

[1]See Remark 1 for why this is justified.

eigenvalue $\hbar\ell$ of the number operator in Bargmann space, namely,

$$\widehat{\mathcal{N}} = \hbar \sum_{j=1}^{d} z_j \frac{\partial}{\partial z_j}. \tag{5.1}$$

The $\mathcal{W}_\ell^{(k)}$ are invariant under the angular momentum operators in Bargmann space given by

$$\widehat{L}_1 = \frac{1}{2}\left(z_1 \frac{\partial}{\partial z_2} + z_2 \frac{\partial}{\partial z_1}\right),\ \widehat{L}_2 = -\frac{i}{2}\left(z_1 \frac{\partial}{\partial z_2} - z_2 \frac{\partial}{\partial z_1}\right),\ \widehat{L}_3 = \frac{1}{2}\left(z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2}\right). \tag{5.2}$$

These operators also satisfy the commutation relations $[\widehat{L}_1, \widehat{L}_2] = i\widehat{L}_3$ and cyclic permutations. Since the spectrum of $\widehat{L}_3$ ranges from (half)-integers $-\ell/2$ to $\ell/2$, one may think of $\mathcal{W}_\ell^{(k)}$ as the Hilbert space of spin $\ell/2$.

Example V.1. In the case where $d = 2$, if $\ell = 1$, then

$$\mathcal{W}_1^{(k)} = \operatorname{span}\{z_1\, e^{-k|z|^2/2},\ z_2\, e^{-k|z|^2/2}\}$$

and we can think of one of the elements corresponding to spin $\frac{1}{2}$ and the other to spin $-\frac{1}{2}$. For $\ell = 2$, the elements in $\mathcal{W}_2^{(k)}$ are quadratic polynomials corresponding to spin $-1, 0$, and 1:

$$\mathcal{W}_2^{(k)} = \operatorname{span}\{z_1^2\, e^{-k|z|^2/2}, z_1\, z_2\, e^{-k|z|^2/2},\ z_2^2\, e^{-k|z|^2/2}\}.$$

Observe that the total dimension of $\mathcal{W}_\ell^{(k)}$ is $\ell + 1$.

Remark 19. This is an example of geometric quantization which is a way of associating Hilbert spaces with symplectic manifolds and when applied to $\mathbb{CP}^{d-1}$ gives the spaces $\mathcal{W}_\ell^{(k)}$ [Woo97].

Let $P_k : \mathcal{B}(\mathbb{C}^d) \to \mathcal{W}_k^{(k)}$ be the orthogonal projection. Note that we are choosing $\ell = k$ which corresponds to reducing at energy $|z|^2 = 1$.[2] Quantum mechanically, $|z|^2 = 1$ becomes

[2] $|z|^2$ is the classical harmonic oscillator in the Bargmann space of $\mathbb{C}^d$.

$\hbar\ell = 1$ because $\hbar\ell$ is an eigenvalue of the the number operator given in (5.1). This gives $\ell = k$ with our definition $\hbar = 1/k$.

The "reduced" space is

$$\mathcal{B}(\mathbb{CP}^{d-1}) := \left\{\text{restrictions to } S^{2d-1} \text{ of homogeneous polynomials of degree } k\right\},$$

with the Hilbert space structure of $L^2(S^{2d-1})$.

The procedure for reducing $\mathcal{B}(\mathbb{C}^d)$ to $\mathcal{B}(\mathbb{CP}^{d-1})$ is presented in the following definition.

Definition V.2. The *quantum reduction operator* $\mathcal{R}_k$ is the composition

$$\mathcal{R}_k := R_k \circ P_k : \mathcal{B}(\mathbb{C}^d) \to \mathcal{B}(\mathbb{CP}^{d-1})$$

where $R_k : \mathcal{W}_k^{(k)} \to \mathcal{B}(\mathbb{CP}^{d-1})$ is the restriction to S^{2d-1} operator.

We have an integral expression for the (normalized) reduction operator:

$$\forall \psi \in \mathcal{B}(\mathbb{C}^d),\ \forall z \in S^{2d-1} \qquad \mathcal{R}_k(\psi)(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt}\,\psi(e^{it}z)\,dt.$$

One may check that $\mathcal{R}_k(\psi)$ is a homogeneous polynomial of degree k in z, and therefore, it is an element in $\mathcal{B}(\mathbb{CP}^{d-1})$.

5.2 Main Results

In order to state our main results more generally, we introduce a coordinate-free version of the generalized unit disk $\mathcal{D}_d$ that we defined in §1.3.

Definition V.3. Let $\mathcal{H}$ be a complex vector space with a Hermitian inner product, and let $\mathcal{G} : \mathcal{H} \to \mathbb{R}$ be the standard Gaussian, $\mathcal{G}(v) = e^{-\|v\|^2/2}$. Let us then define

$$\mathcal{D}(\mathcal{H}) = \left\{\text{quadratic forms } Q : \mathcal{H} \to \mathbb{C} \text{ such that } e^{Q/2}\,\mathcal{G} \in L^2(\mathcal{H})\right\}.$$

One can then show that $Q \in \mathcal{D}(\mathbb{C}^d)$ iff the symmetric matrix A associated with Q in the usual sense is in $\mathcal{D}_d$.

We now define our main objects of study for the rest of this thesis.

Definition V.4. The *reduced squeezed states* are given by

$$\Psi_{A,w} := \mathcal{R}_k(\psi_{A,w}), \quad w \in S^{2d-1}. \tag{5.3}$$

Remark 20. Note when $A = 0$ this corresponds to the standard or "non-squeezed" coherent states. By making use of Stirling's formula[3] we can compute

$$\forall z \in S^{2N-1} \quad \Psi_{0,w}(z) = \frac{e^{-k}}{2\pi} \int_0^{2\pi} e^{-ikt}\, e^{ke^{it} z\overline{w}^T}\, dt = \frac{e^{-k}\, k^k}{k!} (z\overline{w}^T)^k \sim \frac{1}{\sqrt{2\pi k}} (z\overline{w}^T)^k.$$

Our main results are summarized in the following theorem.

Theorem V.5. *Let $A \in \mathcal{D}_d$ and $w \in \mathbb{C}^d$ be such that $|w| = 1$. Then $\Psi_{A,w} = \mathcal{R}_k(\psi_{A,w})$ has the following properties:*

(1) Its micro-support (or semiclassical wave-front set) as $k \to \infty$ consists of the S^1 orbit of w, that is, $\{e^{it}w\ ;\ t \in [0, 2\pi]\}$. On $\mathbb{CP}^{d-1}$, it is a single point

$$\varpi := \pi(w) \in \mathbb{CP}^{d-1}$$

where π is the (general) Hopf fibration, i.e., the projection

$$\pi : S^{2d-1} \to \mathbb{CP}^{d-1}.$$

(2) If $\eta \in \mathcal{H}_w := (\mathbb{C}w)^\perp$ (the Hermitian orthogonal space to the complex line spanned by

[3]Stirling's formula is an approximation for factorials given by: $k! \sim \sqrt{2\pi k}\left(\frac{k}{e}\right)^k$.

w), one has

$$\begin{aligned}\sigma_A(\eta) &:= \lim_{k\to\infty} \sqrt{k}\,\Psi_{A,w}\left(w + \eta\,\sqrt{k}\right)\\ &= \frac{1}{2\pi}\,e^{-|\eta|^2/2}\int_{-\infty}^{\infty} e^{Q_A(isw+\eta)/2}\,e^{-s^2/2}\,ds.\end{aligned} \tag{5.4}$$

and moreover,

$$\sigma_A(\eta) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{Q_A(w)+1}}e^{Q_{\rho_w(A)}(\eta)/2}\,e^{-|\eta|^2/2} \tag{5.5}$$

for some $Q_{\rho_w(A)} \in \mathcal{D}(\mathcal{H}_w)$.

(3) For all A, $B \in \mathcal{D}_d$ one has

$$\langle \Psi_{A,w}, \Psi_{B,w}\rangle_{\mathcal{B}(\mathbb{CP}^{d-1})} = \frac{2\pi}{k^d}\int_{\mathcal{H}_w} \sigma_A(\eta)\,\overline{\sigma_B}(\eta)\,dL(\eta) + O(k^{-d-1}) \tag{5.6}$$

where dL stands for Lebesgue measure.

Remarks 21. Some comments on the previous theorem:

1. Since $A \in \mathcal{D}_d$ and $|w| = 1$, $\Re(Q_A(w)+1) > 0$. The branch of the square root in (5.5) is the natural analytic extension to the right half of the complex plane.

2. The space $\mathcal{H}_w$ is in fact a subspace of $T_w S^{2d-1}$; it is the horizontal subspace at w of the natural connection on the Hopf fibration $\pi : S^{2d-1} \to \mathbb{CP}^{d-1}$. The differential $d\pi_w$ induces an isometry $\mathcal{H}_w \cong T_{\varpi}\mathbb{CP}^{d-1}$, where the latter space is given the Fubini-Study metric. We will tacitly use this identification in what follows.

3. In case $w = (1, \vec{0})$[4] (the general case can be reduced to this by the action of a unitary matrix), one has that $\rho_w(A)$ is the lower $(d-1)\times(d-1)$ principal minor of

$$A - \frac{Aw^T wA}{wAw^T + 1}.$$

[4]The notation $\vec{0}$ signifies $d-1$ zeros.

The derivation of this is provided in Lemma V.23.

4. Theorem V.15 gives the asymptotic behavior of (5.3) *at* ϖ.

From (5.6), we obtain the norm of a reduced state as an asymptotic expansion:

$$\|\Psi_{A,w}\|^2_{\mathcal{B}(\mathbb{CP}^{d-1})} = \frac{2\pi}{k^d}\int_{\mathcal{H}_w} |\sigma_A(\eta)|^2 \, dL(\eta) + O(k^{-d-1}), \tag{5.7}$$

and as a corollary we obtain:

Corollary V.6. *If $A, B \in \mathcal{D}_d$ are such that $\sigma_A = \sigma_B$, then*

$$\|\Psi_{A,w} - \Psi_{B,w}\|^2_{\mathcal{B}(\mathbb{CP}^{d-1})} = O(k^{-d-1}).$$

Proof. Expanding the norm and substituting (5.6) and (5.7) we get

$$\begin{aligned}
\|\Psi_{A,w} - \Psi_{B,w}\|^2_{\mathcal{B}(\mathbb{CP}^{d-1})} &= \|\Psi_{A,w}\|^2_{\mathcal{B}(\mathbb{CP}^{d-1})} + \|\Psi_{B,w}\|^2_{\mathcal{B}(\mathbb{CP}^{d-1})} - 2\Re\langle \Psi_{A,w}, \Psi_{B,w}\rangle_{\mathcal{B}(\mathbb{CP}^{d-1})} \\
&= \frac{2\pi}{k^d}\int_{\mathcal{H}_w} \left(|\sigma_A(\eta)|^2 + |\sigma_A(\eta)|^2 - 2\Re\left(\sigma_A(\eta)\,\overline{\sigma_B}(\eta)\right)\right) dL(\eta) + O(k^{-d-1}) \\
&= \frac{2\pi}{k^d}\int_{\mathcal{H}_w} |\sigma_A(\eta) - \sigma_B(\eta)|^2 \, dL(\eta) + O(k^{-d-1}) \\
&= O(k^{-d-1}).
\end{aligned}$$

□

5.2.1 Symbols of the Reduced States

The concept of (principal) symbol played a significant role in Chapters II - IV in studying the Schrödinger evolution by both Hermitian and non-Hermitian Hamiltonians of coherent states. In fact, symbols are also a central part of the analysis on the reduction of coherent states.

In Chapter II, we defined the principal symbol of an element in I^m_γ as a Schwartz function in ζ that also depends on $t \in \mathbb{R}$. In this chapter, the states $\psi_{A,w} \in \mathcal{B}(\mathbb{C}^d)$ have a t-independent symbol which is analogous to Definition II.4 and is given by

$$\sigma_{\psi_{A,w}}(\zeta) = e^{Q_A(\zeta)/2}\, e^{-|\zeta|^2/2} \tag{5.8}$$

where we have rescaled the variable: $\zeta = (z-w)/\sqrt{\hbar}$.

The reduced states $\Psi_{A,w}$ also have symbols. Observe that the function σ_A characterizes $\Psi_{A,w}$ to leading order in k. It is a $k-$independent function which is also Schwartz and so we can think of it as the (principal) symbol of $\Psi_{A,w}$. Formally,

Definition V.7. The function $\sigma_A : \mathcal{H}_w \to \mathbb{C}$ given by the expressions (5.4) and (5.5) will be considered as a function of the Bargmann space of the tangent space $T_\varpi\mathbb{CP}^{d-1}$ (with $\hbar = 1$), and will be called the *symbol* of (5.3).

Remark 22. The symbol of a standard spin coherent state is simply $\sigma_{A=0}(\eta) = \frac{1}{\sqrt{2\pi}}e^{-|\eta|^2/2}$. As we'll see below, we can obtain any Gaussian as the symbol of the reduction of a $\psi_{A,w}$ for a suitable $A \in \mathcal{D}_d$.

Remark 23. It is very convenient to extend by linearity the definition of symbols of reduced states at the same center $w \in S^{2d-1}$. We will also agree that multiplying $\Psi_{A,w}$ by a power of k results in a function having the same symbol as $\Psi_{A,w}$. This is in line with our observations in §2.2.

So what is the geometric meaning of the symbol? Intuitively, the symbol captures the asymptotic behavior of the coherent state in a neighborhood of size $O(1/\sqrt{k})$ of its center. As a mathematical object, the symbol is a Schwartz function on the tangent space at the center of the state. Roughly speaking, it arises by performing the rescaling $z = w + \frac{\eta}{\sqrt{k}}$ in suitable coordinates, where w is the center of the state, and taking the leading term as $k \to \infty$. That is, generally speaking, what is happening when we go from the behavior described by Theorem V.15 and part (2) of Theorems V.5. The result is a function of η. An example is of course (5.4), where it is crucial that η is in the horizontal subspace $\mathcal{H}_w$. More details on the geometry of the symbol in the context of Kähler quantization are given in §3 of [RU21].

5.2.2 On the Reduction of Excited States

Lasser et al. [LST18] and Arnaiz [Arn21] discuss the construction of excited coherent states in $L(\mathbb{R}^d)$ via the repeated application of appropriate creation operators to ground states of the form (1.4). The excited states are of the form a polynomial times the ground state. By the Bargmann transform, we expect that a similar notion exists for states in defined the Bargmann space of $\mathbb{C}^d$. Associated to $A \in \mathcal{D}_d$, there is a sequence of annihilation and creation operators. Considering $\psi_{A,w} \in \mathcal{B}(\mathbb{C}^d)$ (1.9) as ground states, we can repeatedly apply these particular creation operators to obtain excited states which are again of the form a polynomial times $\psi_{A,w}$. A natural question to ask is can we reduce excited states? The answer, perhaps not surprisingly, is yes!

Let $\widetilde{p}(z)$ be a k-independent polynomial and define

$$\Phi_{A,w} := \mathcal{R}_k\left(p_w \cdot \psi_{A,w}\right)$$

with $p_w(z) = \widetilde{p}(\sqrt{k}(z-w))$.

As an extension of part (2) of Theorem V.5, we have

$$\begin{aligned} \varsigma_A(\eta) &:= \lim_{k\to\infty} \sqrt{k}\,\Phi_{A,w}\left(w + \eta\,\sqrt{k}\right) \\ &= \frac{1}{2\pi}\, e^{-|\eta|^2/2} \int_{-\infty}^{\infty} \widetilde{p}(isw+\eta)\; e^{Q_A(isw+\eta)/2}\, e^{-s^2/2}\, ds \end{aligned} \tag{5.9}$$

for $\eta \in \mathcal{H}_w$. The function ς_A is the symbol of the reduced excited state $\Phi_{A,w}$ and has the same properties as σ_A.

In §5.5 we prove (5.9). In the case where $\widetilde{p}(z) = 1$, the result simplifies to part (2) of Theorem V.5. In the remaining parts of this thesis, however, we will be concerned with the reduction of the ground states $\psi_{A,w}$, namely, $\Psi_{A,w}$.

5.3 An Algebraic Formula

One can compute an exact algebraic expression for the reduced states (5.3), which will be useful for the numerical computations that we present in Chapter VII.

Proposition V.8. *For all $z, w \in S^{2d-1}$ one has:*

$$\Psi_{A,w}(z) = e^{-k}\, e^{kQ_A(w)/2} \sum_{\ell \geq k/2}^{k} \frac{k^\ell}{(k-\ell)!(2\ell-k)!} \left(\frac{1}{2} Q_A(z)\right)^{k-\ell} \left(z(\overline{w}^T - Aw^T)\right)^{2\ell-k}. \quad (5.10)$$

Proof. Since

$$Q_A(z-w) = Q_A(z) - 2zAw^T + Q_A(w),$$

we can re-write

$$\psi_{A,w}(z) = e^{-k}\, e^{kQ_A(w)/2}\, e^{kQ_A(z)/2}\, e^{kz(\overline{w}^T - Aw^T)}$$

Therefore

$$\psi_{A,w}(e^{it}z) = e^{-k}\, e^{kQ_A(w)/2} \sum_{\ell=0}^{\infty} \frac{k^\ell}{\ell!} \left(e^{2it}Q_A(z)/2 + e^{it}z(\overline{w}^T - Aw^T)\right)^\ell.$$

Now apply the binomial theorem to the ℓ-th term of the series:

$$\left(e^{2it}Q_A(z)/2 + e^{it}z(\overline{w}^T - 2Aw^T)\right)^\ell = \sum_{j=0}^{\ell} \binom{\ell}{j} e^{it(j+\ell)} \left(Q_A(z)/2\right)^j \left(z(\overline{w}^T - Aw^T)\right)^{\ell-j}.$$

When we multiply by e^{-ikt} and integrate over $t \in [0, 2\pi]$ only the terms with $j + \ell = k$ survive. For each ℓ, there exists exactly one such term precisely when $0 \leq k - \ell \leq \ell$. This gives the range $k/2 \leq \ell \leq k$, and the expression (5.10) follows. □

Remark 24. Note that the case $A = 0$ (standard coherent states in the Bargmann space of $\mathbb{C}^d$), up to a multiplicative constant, the reduced state is indeed just the standard SU(d) state $(z\overline{w})^k$.

Remark 25. The previous expression is exact but is "redundant to leading order" because the mapping

$$\mathcal{D}_d \ni A \mapsto \rho_w(A) \in \mathcal{D}_{d-1}$$

is not injective, and the symbol controls the reduced state to leading order.

In Chapter VII we will make explicit choices of A when $d = 2$ to avoid this redundancy. After some normalizations we will propose a non-redundant expression for SU(2) squeezed coherent states (see Definition VII.4 and (7.3)).

5.4 Estimates

5.4.1 Estimates on Gaussian States

We begin by establishing some fundamental estimates on Gaussian states $\psi_{A,w}$.

Lemma V.9. *Let $A \in \mathcal{D}_d$. Then*

$$\exists \kappa \in [0,1) \ \forall z \in \mathbb{C}^d \qquad |Q_A(z)| \leq \kappa |z|^2. \tag{5.11}$$

Proof. Let $A \in \mathcal{D}_d$. By the Autonne-Takagi factorization, there exists a unitary matrix U and a diagonal matrix D such that $A = UDU^T$, and D is diagonal with entries $\kappa_j(A) \geq 0$, $j = 1, \ldots, d$, the square roots of the eigenvalues of A^*A. Let $z \in \mathbb{C}^d$ and $\gamma = zU$. Then

$$|Q_A(z)| = |Q_D(\gamma)| = \left| \sum_{j=1}^{d} \gamma_j^2 \kappa_j \right| \leq \kappa |\gamma|^2 = \kappa |z|^2,$$

where $\kappa = \max_j \kappa_j$. The assumption that $A \in \mathcal{D}_d$ is equivalent to $\kappa < 1$. □

In particular $\Re\left(Q_A(z)\right) \leq \kappa |z|^2$. On the other hand,

$$\psi_{A,w}(z) = e^{kQ_A(z-w)/2}\, e^{-k|z-w|^2/2}\, e^{ik\Im(z\overline{w}^T)},$$

where ω is the symplectic usual form

$$\omega(z,w) = \Im(z\overline{w}^T).$$

Therefore, the Husimi function of $\psi_{A,w}$ is equal to

$$|\psi_{A,w}|^2(z) = e^{k[\Re(Q_A(z-w))-|z-w|^2]} \leq e^{-k[(1-\kappa)|z-w|^2]}. \tag{5.12}$$

Since $\kappa < 1$, the phase in (5.12) is non-positive and is zero precisely at $z = w$. Away from w the Husimi function is exponentially decreasing. From this it follows that the semi-classical microsupport of $\psi_{A,w}$ is $\{w\}$.

The proof of the previous lemma can easily be modified to show the equivalence of the two definitions of $\mathcal{D}_d$ and $\mathcal{D}(\mathbb{C}^d)$.

As another observation, we note that coherent states with different center don't "overlap" in the semiclassical limit.

Lemma V.10. *Given A, $B \in \mathcal{D}_d$ and $v, w \in \mathbb{C}^d$, then*

$$v \neq w \quad \Rightarrow \quad \langle \psi_{A,w}, \psi_{B,v} \rangle = O(k^{-\infty}).$$

Proof. Let us write $\langle \psi_{A,w}, \psi_{B,v} \rangle = \int_{\mathbb{C}^d} e^{\varphi(z,\overline{z})}\, dL(z)$ where

$$\varphi = Q_A(z-w)/2 + \overline{Q_B(z-v)}/2 + z\overline{w}^T + \overline{z}v^T - |v|^2/2 - |w|^2/2 - |z|^2.$$

Let us look for critical points of the phase. Note that

$$\frac{\partial \varphi}{\partial z} = (z-w)A + \overline{w} - \overline{z}, \quad \text{and} \tag{5.13}$$

$$\frac{\partial \varphi}{\partial \overline{z}} = (z-v)B + \overline{v} - \overline{z}. \tag{5.14}$$

Claim: If $A \in \mathcal{D}_d$, the mapping $\mathbb{C}^d \ni z \mapsto zA - \overline{z} \in \mathbb{C}^d$ is bijective.

Proof of the claim. Since the map is $\mathbb{R}$-linear, it is enough to prove that its kernel is zero. Note that

$$zA = \overline{z} \quad \Rightarrow \quad \overline{z}\overline{A} = z \quad \Rightarrow \quad zA\overline{A} = z.$$

Since A is symmetric this means that $zAA^* = z$. Since $A \in \mathcal{D}_d$, 1 is not an eigenvalue of AA^*, and therefore $z = 0$.

Since (5.13) being equal to zero is equivalent to $zA - \overline{z} = wA - \overline{w}$, we see that $\frac{\partial \varphi}{\partial z} = 0$ iff $z = w$. Similarly, $\frac{\partial \varphi}{\partial \overline{z}} = 0$ iff $z = v$. So if $v \neq w$ the phase φ does not have any critical points. □

5.4.1.1 Covariance

We find that the Gaussian states in the Bargmann space of $\mathbb{C}^d$ have the following useful covariance property. The group $\mathrm{U}(d)$ acts on $\mathbb{C}^d$ on the right (since we are working with row vectors), which induces an action (representation) on $\mathcal{B}(\mathbb{C}^d)$ given by

$$\forall g \in \mathrm{U}(d), \quad \psi \in \mathcal{B}(\mathbb{C}^d) \qquad (g \cdot \psi)(z) := \psi(zg).$$

The following is straightforward, and is very useful:

Lemma V.11. *One has*

$$g \cdot \psi_{A,w} = \psi_{gAg^T, wg^{-1}}. \tag{5.15}$$

Remark 26. The covariance property (5.15) justifies simplifying to the case where $w = (1, \vec{0})$ in certain contexts later on. Even if our state $\psi_{A,w}$ is not initially centered at $w = (1, \vec{0})$, we can always rotate the center to that point via a unitary rotation. The rotation will in effect change the squeezing parameter A. It would have greatly simplified our analysis if we could find a rotation which both moves the center to $w = (1, \vec{0})$ *and* diagonalizes A, but in general that is not possible!

5.4.2 Pointwise Estimates of the Reduced States

Let $A \in \mathcal{D}_d$ and $w \in S^{2d-1}$. We now obtain a point-wise estimate of $\Psi_{A,w}$.

From the definition (after a short calculation),

$$\forall z \in S^{2d-1} \qquad \Psi_{A,w}(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{k\varphi(z,t)}\,dt \tag{5.16}$$

where the phase is

$$\varphi(z,t) := e^{it}z\overline{w}^T + \frac{1}{2}(e^{it}z - w)A(e^{it}z - w)^T - it - \frac{1}{2}(|z|^2 + |w|^2). \tag{5.17}$$

Lemma V.12. *The phase satisfies $\Re(\varphi) \leq 0$. Moreover, its critical points (with respect to t) satisfying $\Re(\varphi) = 0$ are precisely the solutions of $e^{it}z = w$.*

Proof. We already know from (5.11) that $\Re(\varphi) = 0$ iff $e^{it}z = w$. On the other hand, the critical points of the phase are solutions of

$$e^{it}z\overline{w}^T + (e^{it}z - w)Az^T = 1.$$

This is indeed satisfied if $e^{it}z = w$. $\square$

As a corollary of the previous Lemma,

Corollary V.13. *If $z \neq e^{-it}w$ for some $t \in \mathbb{R}$, then $\Psi_{A,w}(z) = O(k^{-\infty})$.*

This means that $\Psi_{A,w}$ concentrates on the circle $\{e^{-it}w\ ;\ t \in [0, 2\pi]\}$. The previous conclusions naturally lead to the (general) *Hopf fibration* given by the projection

$$\pi : S^{2d-1} \to \mathbb{CP}^{d-1}$$

because for fixed w, every z that is in the same Hopf fiber (circle) is also a critical point of (5.17).

Definition V.14. We will denote *"center" of the reduced state* $\Psi_{A,w}$ by

$$\varpi := \pi(w) = \{e^{-it}w\ ;\ t \in [0, 2\pi]\}.$$

The previous results tell us that $\Psi_{A,w}$ and all its derivatives are rapidly decreasing away from ϖ. To evaluate $\Psi_{A,w}$ asymptotically at ϖ, let us apply the method of stationary phase (Theorem 7.7.5 in [Hör90]) to (5.16). Thus, assume that $e^{it_0}z = w$ for some t_0. The second derivative of the phase at $t = t_0$ is equal to $i(1 + wAw^T)$. This implies:

Theorem V.15. *With the previous notation,*

$$\Psi_{A,w}(e^{-it_0}w) = \frac{1}{\sqrt{2\pi k}} \frac{e^{-ikt_0}}{\sqrt{wAw^T + 1}} + O(k^{-3/2}) \qquad \textit{as } k \to \infty.$$

Remark 27. From the previous theorem we can observe that if $w \mapsto e^{i\theta}w$

$$\Psi_{A,e^{i\theta}w}(z) = \Psi_{e^{2i\theta}A,w}(e^{-i\theta}z) = e^{-ik\theta}\,\Psi_{e^{2i\theta}A,w}(z)$$

for some $\theta \in [0, 2\pi)$ where the last step follows from the fact that $\Psi_{A,w}$ is homogeneous in z of degree k. Hence, rotating the center of the state before reducing results in an additional phase factor multiplying the reduced state.

The result of Theorem V.15 is not very illustrative of the behavior of $\psi_{A,w}$ for z near the critical points because it gives us this "all-or-nothing" behavior. Let's see why by considering a simple example in dimension $d = 1$.

Example V.16. Let $w = 0 \in \mathbb{C}$, so $\psi_{A,0} = e^{kAz^2/2}\, e^{-k|z|^2/2}$. Then, consider

$$|\psi_{A,0}(z)|^2 = \left|e^{kA^2/2}\right| \cdot e^{-k|z|^2}.$$

Fixing z, as $k \to \infty$,

$$|\psi_{A,0}(z)|^2 \sim \begin{cases} 0 & \text{if } z \neq 0 \text{ since } |A| < 1 \\ 1 & \text{if } z = 0. \end{cases}$$

However, if we rescale the coordinate as $z = \xi/\sqrt{k}$, then as $k \to \infty$, $z \to 0$, so

$$|\psi_{A,0}(\xi)|^2 = \left|e^{A\xi^2/2}\right|^2 \cdot e^{-k|z|^2}.$$

Hence, by "zooming in" at the center of the state, we get a Gaussian state with no k dependence.

This example motivates the need to do *local scaling asymptotics* where we magnify the geometry at the center of the state using the change of coordinates $z = w + \eta/\sqrt{k}$. This serves as the motivation for part (2) of Theorem V.5.

5.5 Proof of Part (2) of Theorem V.5

We will now prove (5.9) for the reasons discussed in §5.2.2. Recall that the result simplifies to part (2) of Theorem V.5 when $\widetilde{p}(z) = 1$.

Proof. Fix $w \in S^{2d-1}$ and $\eta \in \mathcal{H}_w$, and introduce the notation

$$\Upsilon_A(\eta, k) := \Phi_{A,w}\left[w + \frac{\eta}{\sqrt{k}}\right]. \tag{5.18}$$

We need to show that

$$\sqrt{k}\,\Upsilon_A(\eta, k) = \frac{1}{2\pi} e^{-|\eta|^2/2} \int_{-\infty}^{\infty} \widetilde{p}(isw + \eta)\, e^{Q_A(isw+\eta)/2}\, e^{-s^2/2}\, ds + O(1/\sqrt{k}). \tag{5.19}$$

Note that

$$\Upsilon_A(\eta, k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \widetilde{p}\left(\sqrt{k}\left(e^{it}(w + \eta/\sqrt{k}) - w\right)\right)\ \psi_{A,w}\left(e^{it}(w + \eta/\sqrt{k})\right) e^{-ikt} dt.$$

For each k we split the domain of integration into three parts,

$$\Upsilon_A(\eta, k) = \frac{1}{2\pi}\int_{-\pi}^{-a_k} + \frac{1}{2\pi}\int_{-a_k}^{a_k} + \frac{1}{2\pi}\int_{a_k}^{\pi} =: I_1 + I_2 + I_3,$$

respectively, where (a_k) is a sequence of positive numbers tending to zero that we will specify later. In particular, we will choose this sequence so that I_1 and I_3 are negligible with respect to I_2.

First let us estimate I_3. Recall that $|\psi_{A,w}(z)| \leq e^{-Ck|z-w|^2}$ with $C = (1-\kappa)/2 \in (0, 1/2]$, where κ is the largest eigenvalue of A^*A (see (5.12)). Therefore,

$$\left|\psi_{A,w}\left(e^{it}(w + \eta/\sqrt{k})\right) e^{-ikt}\right| \leq e^{-Ck|e^{it}(w+\eta/\sqrt{k})-w|^2} = e^{-Ck(|e^{it}-1|^2|w|^2+|\eta|^2/k)} \leq e^{-Ck|e^{it}-1|^2}$$

since $e^{-C|\eta|^2} \leq 1$.

Also, since p is a polynomial, $\widetilde{p}(z) = \sum_{|m|=0}^{\alpha} c_m z^m$ for $z \in \mathbb{C}^d$ with $m \in \mathbb{N}_0^d$ and

$$\left|\widetilde{p}\left(\sqrt{k}\left(e^{it}(w + \eta/\sqrt{k}) - w\right)\right)\right| \leq \sum_{|m|=0}^{\alpha} k^{|m|/2}\, |c_m| \left(\left|e^{it} - 1\right|^2 + \frac{|\eta|^2}{k}\right)^{|m|/2}.$$

Note that we have used that $\eta \cdot \overline{w} = 0$ and $|w|^2 = 1$ in the previous two calculations.

Hence,

$$\begin{aligned}
|I_3| &\leq \frac{1}{2\pi}\int_{a_k}^{\pi} \sum_{|m|=0}^{\alpha} k^{|m|/2}\, |c_m| \left(\left|e^{it} - 1\right|^2 + \frac{|\eta|^2}{k}\right)^{|m|/2} e^{-Ck|e^{it}-1|^2} dt \\
&= \frac{1}{2\pi}\sum_{|m|=0}^{\alpha} k^{|m|/2}\, |c_m| \int_{a_k}^{\pi} \left(\left|e^{it} - 1\right|^2 + \frac{|\eta|^2}{k}\right)^{|m|/2} e^{-Ck|e^{it}-1|^2} dt \\
&\leq \frac{1}{2\pi}\sum_{|m|=0}^{\alpha} k^{|m|/2}\, |c_m| \left(4 + \frac{|\eta|^2}{k}\right)^{|m|/2} \int_{a_k}^{\pi} e^{-Ck|e^{it}-1|^2} dt \\
&\leq D(\eta, k) \max_{t \in [a_k, \pi]} e^{-Ck|e^{it}-1|^2} = D(\eta, k)\, e^{-Ck|e^{ia_k}-1|^2}.
\end{aligned}$$

where

$$D(\eta, k) := \frac{1}{2\pi} \sum_{|m|=0}^{\alpha} k^{|m|/2} \, |c_m| \, (4 + |\eta|^2/k)^{|m|/2} \leq C_1(\eta) \, k^{\alpha/2}$$

for some $C_1(\eta)$.

Since $|e^{it} - 1|^2 = t^2 + t^4 R(t)$ for some function $R(t)$ bounded in a neighborhood of zero, we conclude that for each $\eta \in \mathcal{H}_w$

$$|I_3| \leq C_1(\eta) \, k^{\alpha/2} \, e^{-Cka_k^2(1+a_k^2 R(a_k))}$$

and similarly for I_1. We now pick

$$a_k = \left(\frac{\log(k^{\alpha/2+1})}{Ck} \right)^{1/2}$$

with C the above constant. Therefore,

$$I_3 = O\left(1/k\right) \quad \text{and similarly for } I_1. \tag{5.20}$$

We now turn to $I_2 = \frac{1}{2\pi} \int_{-a_k}^{a_k} \widetilde{p}\left(\sqrt{k}\left(e^{it}(w + \eta/\sqrt{k}) - w\right)\right) \; \psi_{A,w}\left(e^{it}(w + \eta/\sqrt{k})\right) e^{-ikt} dt$. After some algebra, one finds that this integral has the following form:

$$I_2 = \frac{1}{2\pi} \int_{-a_k}^{a_k} \widetilde{p}\left(\sqrt{k}\left(e^{it}(w + \eta/\sqrt{k}) - w\right)\right) \; e^{k\phi(t) + \sqrt{k}\psi(t) + \varrho(t)} dt$$

where

$$\begin{aligned}
\phi(t) &= \frac{1}{2} Q_A(w)(1 - 2e^{it} + e^{2it}) + e^{it} - it - 1 \\
\psi(t) &= (e^{2it} - e^{it}) \eta A w^T \\
\varrho(t) &= \frac{1}{2}(e^{2it} Q_A(\eta) - |\eta|^2).
\end{aligned}$$

The only critical point of the phase is at $t = 0$. After a Taylor expansion at zero of the phase, one obtains:

Lemma V.17. *The integral I_2 is of the form*

$$I_2 = \frac{e^{[Q_A(\eta)-|\eta|^2]/2}}{2\pi} \int_{-a_k}^{a_k} \widetilde{p}\left(\sqrt{k}\left(e^{it}(w+\eta/\sqrt{k})-w\right)\right)\, e^{f_k(t)+g_k(t)}\,dt \tag{5.21}$$

with

$$f_k(t) = -k(Q_A(w)+1)t^2/2 + it\sqrt{k}\,\eta\, Aw^T$$

and

$$g_k(t) = it^3 kG(t) + t^2\sqrt{k}H(t) + itF(t),$$

where F, G, H are smooth k-independent functions (in particular bounded in a neighborhood of zero).

Proof. We would like to expand e^{it} and e^{2it} in powers ot t. We make use of the following formulae obtained using the Hadamard Lemma:

$$\begin{aligned}
e^x &= 1 + xf(x), & f(x) &:= \int_0^1 e^{ux}dx \\
e^x &= 1 + x + x^2 g(x), & g(x) &:= \int_0^1\int_0^1 ue^{uvx}dvdu \\
e^x &= 1 + x + \frac{1}{2}x^2 + x^3 h(x), & h(x) &:= \int_0^1\int_0^1\int_0^1 u^2ve^{uvwx}dwdvdu.
\end{aligned}$$

Then, after rearranging the terms in the exponent of the integrand of (5.21), we get

$$\begin{aligned}
k\phi(t) + \sqrt{k}\psi(t) + \varrho(t) &= -\frac{k}{2}(Q_A(w)+1)t^2 + \sqrt{k}\eta Aw^T it + \frac{1}{2}Q_A(\eta) - \frac{1}{2}|\eta|^2 \\
&= -\frac{k}{2}(Q_A(w)+1)t^2 + \sqrt{k}\eta Aw^T it + \frac{1}{2}Q_A(\eta) - \frac{1}{2}|\eta|^2 \\
&\quad + kit^3H(t) + \sqrt{k}t^2G(t) + itF(t)\,.
\end{aligned}$$

We have defined

$$F(t) := Q_A(\eta)f(2it)$$
$$G(t) := \eta A w^T(g(it) - 4g(2it))$$
$$H(t) = Q_A(w)(h(it) - 4h(2it)) - h(it).$$

where $Q_A(w)$, $\eta A w^T$, and $Q_A(\eta)$ are complex constants for fixed η and w. $\square$

We now make the change of variables $t = s/\sqrt{k}$ in (5.21), to obtain

$$I_2 = \frac{e^{[Q_A(\eta)-|\eta|^2]/2}}{2\pi\sqrt{k}} \int_{-\infty}^{\infty} \widetilde{p}\left(\sqrt{k}(e^{is/\sqrt{k}} - 1)w + e^{is/\sqrt{k}}\eta\right) e^{-(Q_A(w)+1)s^2/2+is\,\eta\,Aw^T}\, e^{g_k(s/\sqrt{k})} \times \chi\left(\frac{s}{\sqrt{k}\,a_k}\right) ds$$

where χ is the characteristic function of $[-1,1]$. We claim that

$$e^{g_k(s/\sqrt{k})}\,\chi\left(\frac{s}{\sqrt{k}\,a_k}\right) \text{ is uniformly bounded and converges to one } \forall s \in \mathbb{R}. \tag{5.22}$$

To see this, observe first that the support of $\chi\left(\frac{s}{\sqrt{k}\,a_k}\right)$ is equal to the set of s such that

$$|s| \le C^{-1/2}\log(k)^{1/2}, \tag{5.23}$$

which inequality implies that $\frac{|s|^j}{\sqrt{k}} \le \frac{\log(k)^{j/2}}{\sqrt{k}}$ for $j = 0, 1\ldots$, since $C < 1$. Then, since

$$g_k(s/\sqrt{k}) = \left[is^3G(s/\sqrt{k}) + s^2H(s/\sqrt{k}) + isF(s/\sqrt{k})\right]\frac{1}{\sqrt{k}}, \tag{5.24}$$

for all s in the support of $\chi\left(\frac{s}{\sqrt{k}\,a_k}\right)$ $g_k(s/\sqrt{k})$ is uniformly bounded by a constant times $\frac{\log(k)^{3/2}}{\sqrt{k}}$, which tends to zero.

By (5.22) and the Lebesgue dominated convergence theorem, $\sqrt{k}I_2$ converges to the right-hand side of (5.19). It remains to estimate the rate of convergence. Let us define

$\mathcal{E}(s,k) := e^{g_k(s/\sqrt{k})} - 1$ for s satisfying (5.23) and zero otherwise, so that

$$e^{g_k(s/\sqrt{k})}\chi\left(\frac{s}{\sqrt{k}\,a_k}\right) = \chi\left(\frac{s}{\sqrt{k}\,a_k}\right)[1+\mathcal{E}(s,k)]\,.$$

Applying Taylor's theorem to $|\mathcal{E}|^2$ near $s=0$, for each k, one gets

$$|\mathcal{E}(s,k)|^2 = \frac{2s}{\sqrt{k}}\Re\left[g_k'(b/\sqrt{k})\left(e^{\overline{g_k}(b/\sqrt{k})}-1\right)\right] \leq \frac{2|s|}{\sqrt{k}}\left|g_k'(b/\sqrt{k})\left(e^{\overline{g_k}(b/\sqrt{k})}-1\right)\right|$$

for $|s| < C^{-1}\log(k)^{1/2}$ and where $b=b(s)$ is between zero and s, and therefore $|b(s)| \leq C^{-1}\log(k)^{1/2}$. From this and (5.24) it follows that

$$\left|g_k'(b/\sqrt{k})\right| \leq \frac{C_1}{\sqrt{k}} \quad \text{and} \quad \left|e^{\overline{g_k}(b/\sqrt{k})}-1\right| \leq C_2$$

for some constants $C_j>0$, for each s satisfying (5.23). Therefore $\exists C_3>0$ such that

$$|\mathcal{E}(s,k)|\,\chi\left(\frac{s}{\sqrt{k}\,a_k}\right) \leq \frac{C_3}{\sqrt{k}} \tag{5.25}$$

for all $s\in\mathbb{R}$ and for all $k\in\mathbb{N}$.

Substituting back into I_2, we get that $I_2 = J_1 + J_2$ where

$$J_1 := \frac{e^{[Q_A(\eta)-|\eta|^2]/2}}{2\pi\sqrt{k}}\int_{-\infty}^{\infty}\widetilde{p}\left(\sqrt{k}(e^{is/\sqrt{k}}-1)w + e^{is/\sqrt{k}}\eta\right)e^{-(Q_A(w)+1)s^2/2+is\,\eta\,Aw^T}\,\chi\left(\frac{s}{\sqrt{k}\,a_k}\right)\,ds$$

and

$$J_2 := \frac{e^{[Q_A(\eta)-|\eta|^2]/2}}{2\pi\sqrt{k}}\int_{-\infty}^{\infty}\widetilde{p}\left(\sqrt{k}(e^{is/\sqrt{k}}-1)w + e^{is/\sqrt{k}}\eta\right)e^{-(Q_A(w)+1)s^2/2+is\,\eta\,Aw^T}\,\mathcal{E}(s,k)$$
$$\times\,\chi\left(\frac{s}{\sqrt{k}\,a_k}\right)\,ds.$$

We now use the classic estimate $\frac{1}{\sqrt{\pi}}\int_{-x}^{x} e^{-s^2}\, ds = 1 - \frac{e^{-x^2}}{x\sqrt{\pi}} + O(\frac{e^{-x^2}}{x^2})$ to conclude that

$$J_1 = \frac{e^{[Q_A(\eta)-|\eta|^2]/2}}{2\pi\sqrt{k}} \int_{-\infty}^{\infty} \widetilde{p}\left(\sqrt{k}(e^{is/\sqrt{k}}-1)w + e^{is/\sqrt{k}}\eta\right) e^{-(Q_A(w)+1)s^2/2 + is\,\eta\, Aw^T}\, ds \\ + O(1/k^{1/C}\, \log(k)^{1/2})$$

and, using (5.25), that $|J_2| \leq \frac{D}{k}$ where D is a constant that depends on η. Given that $C < 1$ we can conclude that

$$I_2 = \frac{1}{2\pi\sqrt{k}} e^{-|\eta|^2/2} \int_{-\infty}^{\infty} \widetilde{p}(isw+\eta)\, e^{Q_A(isw+\eta)/2}\, e^{-s^2/2}\, ds + O(1/k).$$

In view of (5.20)

$$\Upsilon_A(\eta,k) = \frac{1}{2\pi\sqrt{k}} e^{-|\eta|^2/2} \int_{-\infty}^{\infty} \widetilde{p}(isw+\eta)\, e^{Q_A(isw+\eta)/2}\, e^{-s^2/2}\, ds + O(1/k),$$

and the proof is complete. □

5.6 Inner Product estimates

In this section we prove part (3) of Theorem V.5, namely, we compute the inner product estimate in (5.6):

Let $A, B \in \mathcal{D}_d, w \in S^{2d-1}$ and $\eta \in \mathcal{H}_w$, then

$$\langle \Psi_{A,w}, \Psi_{B,w} \rangle_{\mathcal{B}(\mathbb{CP}^{d-1})} = \frac{2\pi}{k^d} \int_{\mathcal{H}_w} \sigma_A(\eta)\, \overline{\sigma_B}(\eta)\, dL(\eta) + O(k^{-d-1}). \tag{5.6}$$

Proof. By equivariance, without loss of generality we can take $w = (1, \vec{0})$. (See Remark 26.) We introduce a standard parametrization of a dense open set $\mathcal{U} \in \mathbb{CP}^{d-1}$, containing the point $\varpi = \pi(w)$, namely, the set $\mathcal{U}$ which is the complement to the hyperplane $\{z_1 = 0\}$.

One identifies $\mathcal{U} \cong \mathbb{C}^{d-1}_\zeta$ by the coordinates

$$\zeta_j = \frac{z_{j+1}}{z_1}, \quad j = 1, \ldots, d-1.$$

Define next a section of $\pi : S^{2d-1} \to \mathbb{CP}^{d-1}$ over $\mathcal{U}$ by

$$S_\varpi : \mathbb{C}^{d-1} \to S^{2d-1}, \quad S_\varpi(\zeta) = \frac{1}{\sqrt{1+|\zeta|^2}}\,(1,\zeta).$$

Note that ϖ corresponds to the origin $\zeta = 0$, and $S_\varpi(0) = w$.

The left-hand side of (5.6) is an integral over S^{2d-1} of a function that is S^1 invariant. Therefore, we can compute it (up to a factor of 2π) by pulling it back by the section S_ϖ and integrating with respect to the appropriate measure on $\mathbb{C}^{d-1}$. A calculation shows that

$$\langle \Psi_{A,w}, \Psi_{B,w}\rangle_{\mathcal{B}(\mathbb{CP}^{d-1})} = 2\pi \int_{\mathbb{C}^{d-1}} \Psi_{A,w}(S_\varpi(\zeta))\,\overline{\Psi_{B,w}}(S_\varpi(\zeta))\frac{dL(\zeta)}{(1+|\zeta|^2)^d} = \mathrm{I} + \mathrm{II}$$

where $\mathrm{I} = \int_{|\zeta|\le 1} \Psi_{A,w}(S_\varpi(\zeta))\,\overline{\Psi_{B,w}}(S_\varpi(\zeta))\frac{dL(\zeta)}{(1+|\zeta|^2)^d}$ and II is the integral of the same integrand over $|\zeta| > 1$.

We will show that II is rapidly decreasing. We first find a bound for $|\Psi_{A,w}(S_\varpi(\zeta))|$. To begin with,

$$\begin{aligned}
|\Psi_{A,w}(S_\varpi(\zeta))| &\le \frac{1}{2\pi}\int_0^{2\pi} \left|\psi_{A,w}(e^{it}S_\varpi(\zeta))\right|\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left|e^{kQ_A(e^{it}S_\varpi(\zeta)-w)/2}e^{-k|e^{it}S_\varpi(\zeta)-w|^2/2}e^{ik\omega(e^{it}S_\varpi(\zeta),w)/2}\right|\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{k\Re[Q_A(e^{it}S_\varpi(\zeta)-w)/2]}e^{-k|e^{it}S_\varpi(\zeta)-w|^2/2}\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{-k\widetilde{Q}_A(e^{it}S_\varpi(\zeta)-w)/2}\,dt
\end{aligned}$$

where $\widetilde{Q}_A(z) := -\Re(Q_A(z)) + |z|^2$ is a real positive definite quadratic form. Denote by $c_A > 0$

the smallest eigenvalue of Q_A. Then $\forall z$, $\widetilde{Q}_A(z) \geq c_A|z|^2$. Hence

$$\begin{aligned}
|\Psi_{A,w}(S_\varpi(\zeta))| &\leq \frac{1}{2\pi}\int_0^{2\pi} e^{-kc_A|e^{it}S_\varpi(\zeta)-w|^2/2}\,dt = \frac{1}{2\pi}\int_0^{2\pi} e^{-kc_A|S_\varpi(\zeta)-e^{-it}w|^2/2}\,dt \\
&\leq \max_{t\in[0,2\pi]} e^{-kc_A|S_\varpi(\zeta)-e^{-it}w|^2/2} = e^{-kc_A \min_{t\in[0,2\pi]}|S_\varpi(\zeta)-e^{-it}w|^2/2} = e^{-kc_A(1-\rho(\zeta))}.
\end{aligned}$$

where $\rho(\zeta) = (1+|\zeta|^2)^{-1/2}$. This last step results from the fact that

$$|S_\varpi(\zeta) - e^{-it}w|^2 = |(\rho - e^{-it}, \rho\zeta)|^2 = |\rho - e^{-it}|^2 + \rho^2|\zeta|^2$$

which is minimized at $t = 0$, and $|\rho-1|^2+\rho^2|\zeta|^2 = \rho^2(1+|\zeta|^2)+1-2\rho = 1+1-2\rho = 2(1-\rho)$.

All in all, we have $|\Psi_{A,w}(S_\varpi(\zeta))| \leq e^{-kc_A(1-\rho(\zeta))}$ and by similar analysis, we obtain $|\Psi_{B,w}(S_\varpi(\zeta))| \leq e^{-kc_B(1-\rho(\zeta))}$ for some $c_B > 0$. Therefore,

$$|\mathrm{II}| \leq 2\pi \int_{|\zeta|>1} |\Psi_{A,w}(S_\varpi(\zeta))|\;|\Psi_{B,w}(S_\varpi(\zeta))|\frac{dL(\zeta)}{(1+|\zeta|^2)^d} \leq 2\pi\int_{|\zeta|>1} e^{-k(c(1-\rho(\zeta))}\frac{dL(\zeta)}{(1+|\zeta|^2)^d}$$

where $c := c_A+c_B$. If we change to polar coordinates, then $r = |\zeta|$ and $1-\rho(\zeta) = 1-(1+r^2)^{-1/2}$, so

$$|\mathrm{II}| \leq 2\pi\cdot(2\pi)^{d-1}\int_{r=1}^{\infty} e^{-kc\left(1-\frac{1}{\sqrt{1+r^2}}\right)}\frac{r^{2d-3}\,dr}{(1+r^2)^d} \leq Ce^{-kc\left(1-\frac{1}{\sqrt{2}}\right)}$$

where $C > 0$, and thus II tends to zero rapidly as $k \to \infty$.

Now let's consider the integral I. We change variables to $\zeta = \eta/\sqrt{k}$, so that $|\eta| \leq \sqrt{k}$ provided $|\zeta| < 1$. Thus,

$$\begin{aligned}
|\mathrm{I}| &\leq \frac{2\pi}{k^{d-1}}\int_{\mathbb{C}^{d-1}}\left|\Psi_{A,w}(S_\varpi(\eta/\sqrt{k}))\right|\left|\Psi_{B,w}(S_\varpi(\eta/\sqrt{k}))\right|\chi(|\eta|/\sqrt{k})\frac{dL(\eta)}{(1+|\eta|^2/k)^d} \\
&= \frac{2\pi}{k^{d-1}}\int_{\mathbb{C}^{d-1}}|\Upsilon_A(\eta,k)|\,|\Upsilon_B(n,k)|\,\chi(|\eta|/\sqrt{k})\frac{dL(\eta)}{(1+|\eta|^2/k)^d}
\end{aligned}$$

where χ is a cutoff function. We define

$$f_k(\eta) := |\Upsilon_A(\eta, k)|\,|\Upsilon_B(n, k)|\,\frac{\chi(|\eta|/\sqrt{k})}{(1+|\eta|^2/k)^d}.$$

Now $f_k(\eta) > 0$ is a sequence in $L^1(\mathbb{C}^{d-1}, dL)$ and $\exists\, c, C > 0$ such that $f_k(\eta)$ is dominated by $Ce^{-c|\eta|^2}$, $\forall k, \eta$ such that $|\eta| \leq \sqrt{k}$. Moreover, $f_k(\eta)$ converges to $|\Upsilon_A(\eta, k)||\Upsilon_B(\eta, k)|$ pointwise as $k \to \infty$, so by the Dominated Convergence Theorem and by part 2 of Theorem 1.3,

$$\langle \Psi_{A,w}, \Psi_{B,w}\rangle_{\mathcal{B}^{(k)}_{\mathbb{CP}^{d-1}}} = \frac{2\pi}{k^d}\int_{\mathbb{C}^{d-1}} \sigma_A(\eta)\,\overline{\sigma_B}(\eta)\,dL(\eta) + O(k^{-d-1}).$$

(The additional factor of $1/k$ comes from the definition $\lim\limits_{k\to\infty} \Upsilon_A(\eta, k) = \sigma_A(\eta)/\sqrt{k}$, and similarly for $\Upsilon_B(\eta, k)$.) □

5.7 Reduction of Symbols

In this section we will prove that the "symbol of the reduction is the reduction of the symbol". We will define a "new" reduction operator that acts directly on the symbol of the state $\psi_{A,w} \in \mathcal{B}(\mathbb{C}^d)$ and maps it to the symbol of the reduced state $\Psi_{A,w}$ (Lemma V.22). The reduction operator will be useful to prove the propagation theorem of Chapter VI. First, we set the framework. Throughout this section note that $\hbar = k = 1$, and we work entirely in the category of symplectic vector spaces.

5.7.1 More General Bargmann Spaces

Let (E, ω, J) be a Kähler vector space. We take the sign convention that the associated positive definite metric is $g(u, v) = \omega(u, Jv)$. A nice reference for the material in this section is [Dau80]. We will quote freely from that article.

Definition V.18. Let

$$\mathcal{B}(E) = \left\{\psi : E \to \mathbb{C}\;;\; \psi(v) = f(v)\,e^{-\|v\|^2/2} \text{ where } \bar\partial f = 0 \text{ and } \psi \in L^2(E)\right\}$$

be the *Bargmann space of* E. Here $\overline{\partial}$ is the d-bar operator associated with J and $\|v\|^2 = g(v, v)$.

Remark 28. In our applications, E is the tangent space at a point in a Kähler manifold. In the case of the squeezed states $\psi_{A,w} \in \mathcal{B}(\mathbb{C}^d)$, $E = T_w S^{2d-1}$ and the symbol of a squeezed state, $\sigma_{\psi_{A,w}}$ as given in (5.8), is an element of $\mathcal{B}(T_w S^{2d-1})$.

The Heisenberg group of E is unitarily represented in $\mathcal{B}(E)$, as follows. If $a \in E$, define the operator $\rho(a) : \mathcal{B} \to \mathcal{B}$ by $\rho(a)(\psi)(v) = e^{i\omega(a,v)}\psi(v - a)$. Then $\rho(a) \circ \rho(b) = e^{i\omega(a,b)}\rho(a + b)$, so these operators form part of the Heisenberg representation of the Heisenberg group of E. Recall that $\psi \in \mathcal{B}$ is said to be a smooth vector if and only if for all $\phi \in \mathcal{B}$ the function $a \mapsto \langle \rho(a)(\psi), \phi\rangle$ is smooth (this is the analogue of Schwartz functions in Bargmann space).

Definition V.19. We will denote by

$$\mathcal{B}^\infty(E) \subset \mathcal{B}(E)$$

the *subspace of smooth vectors* for this representation.

5.7.2 Reduction

If $S \subset E$ is a subspace, we denote by S° and $S^\perp$ its *symplectic annihilator* and *orthogonal complement*, respectively. Note that

$$J(S^\circ) = S^\perp. \tag{5.26}$$

From now on we fix a co-isotropic subspace $\mathcal{C} \subset E$ (this means that $\mathcal{C}^\circ \subset \mathcal{C}$). Let us define

$$\mathcal{H} := \mathcal{C} \cap J(\mathcal{C}), \text{ the maximal complex subspace of } \mathcal{C}.$$

Note that automatically $\mathcal{H}$ is a Kähler (in particular, symplectic) subspace of E.

Lemma V.20. *One has:*

$$\mathcal{C} \cap (\mathcal{C}^\circ)^\perp = \mathcal{H}, \tag{5.27}$$

and therefore the projection $\pi : \mathcal{C} \to \mathcal{C}/\mathcal{C}^\circ =: F$ identifies the reduction, F, of $\mathcal{C}$ with the maximal complex subspace of $\mathcal{C}$. Under this identification the symplectic structures of $\mathcal{H}$ and F agree.

Proof. By (5.26), $J(\mathcal{C}) = (\mathcal{C}^\circ)^\perp$ and (5.27) follows, which implies that π restricted to $\mathcal{H}$ is a bijection. The rest follows from the usual characterization of the symplectic structure of a reduction. □

By the previous discussion, the reduction $F = \mathcal{C}/\mathcal{C}^\circ$ of $\mathcal{C}$ inherits the structure of a Kähler vector space. Let $\mathcal{B}(F)$ denote its Bargmann space, and $\mathcal{B}^\infty_F \subset \mathcal{B}_\mathcal{F}$ the subspace of smooth vectors. Our objective is to introduce a natural "reduction" operator

$$\mathcal{R} : \mathcal{B}^\infty(E) \to \mathcal{B}^\infty(F) \tag{5.28}$$

associated with $\mathcal{C}$. Here "natural" is with respect to symplectic linear transformations. There is an obvious map, namely, restriction to $\mathcal{H}$ followed by the identification $\mathcal{H} \cong F$, but this is not the right one for our purposes.

Definition V.21. We define $R : \mathcal{B}^\infty(E) \to \mathcal{B}^\infty(F)$ to be the operator of restriction to $\mathcal{C}$ followed by integration over $\mathcal{C}^\circ$, with respect to the measure induced by the Euclidean inner product.

The point of this definition is that it describes the abstract way to construct the symbol of a reduced state from the symbol of a Gaussian state in Bargmann space:

Lemma V.22. *In the context of Theorem V.5, one has:*

$$\sigma_A(\eta) = \frac{1}{2\pi} R(\sigma_{\psi_{A,w}})(\eta),$$

where $E = \mathbb{C}^d$, $\mathcal{C} = T_w S^{2d-1}$ *and* $\sigma_{\psi_{A,w}}(z) = e^{Q_A(z)/2 - |z|^2/2}$ *is the symbol of* $\psi_{A,w}$.

Proof. Simply note that iw is a unitary basis of $\mathcal{C}^\circ$ and $\eta \in \mathcal{H}_w$. Therefore

$$e^{-|\eta|^2/2} \int_{-\infty}^{\infty} e^{Q_A(isw+\eta)/2}\, e^{-s^2/2}\, ds$$

is exactly the definition of $R(\sigma_{\psi_{A,w}})(\eta)$. $\square$

Remark 29. The space $F = T_\varpi \mathbb{CP}^{d-1}$, so the symbol of the reduced state $\Psi_{A,w}$, σ_A, is a Schwartz function in the Bargmann space of $(T_\varpi \mathbb{CP}^{d-1})$.

We now explicitly compute σ_A in a special case:

Lemma V.23. *If* $w = (1, \vec{0})$, *then (see* (5.5)*)*

$$\sigma_A(\eta) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{Q_A(w)+1}}\, e^{Q_{\rho_w(A)}(\eta)/2}\, e^{-|\eta|^2/2}$$

where $\rho_w(A) \in \mathcal{D}_{d-1}$ *is the lower* $(d-1)\times(d-1)$ *principal minor of*

$$A - \frac{Aw^T wA}{wAw^T + 1}.$$

Proof. Since

$$Q_A(isw+\eta) = i^2 s^2 wAw^T + 2is\eta Aw^T + \eta A \eta^T = -s^2 Q_A(w) + 2is\eta Aw^T + Q_A(\eta)$$

equation (5.4) may be re-written as

$$\begin{aligned}
\sigma_A(\eta) &= \frac{1}{2\pi} e^{-|\eta|^2/2}\, e^{Q_A(\eta)/2} \int_{-\infty}^{\infty} e^{-s^2(Q_A(w)+1)/2}\, e^{si\eta Aw^T}\, ds \\
&= \frac{1}{2\pi} e^{-|\eta|^2/2}\, e^{Q_A(\eta)/2} \int_{-\infty}^{\infty} e^{-b^2 s^2/2}\, e^{cs}\, ds
\end{aligned}$$

where $b^2 := Q_A(w) + 1$ with b in the right side of the complex plane and $c := i\eta Aw^T$. Now

$\Re(b^2) = \Re(Q_A(w)+1) > 0$ since $A \in \mathcal{D}_d$ and $|w| = 1$, so then we can evaluate the integral,

$$\begin{aligned}\sigma_A(\eta) &= \frac{1}{2\pi} e^{-|\eta|^2/2}\, e^{Q_A(\eta)/2}\, \frac{\sqrt{2\pi}}{b} e^{(c/b)^2/2} \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{Q_A(w)+1}} e^{Q_A(\eta)/2}\, e^{-(\eta A w^T)^2/(2(Q_A(w)+1))}\, e^{-|\eta|^2/2}.\end{aligned}$$

Since $\eta \in \mathcal{H}_{(1,\vec{0})}$, our choice of w forces the first coordinate of η to be zero, so we take $\eta = (0, \eta_1, \ldots, \eta_{d-1})$ and write

$$\frac{1}{2}\left[Q_A(\eta) - \frac{(\eta A w^T)^2}{Q_A(w)+1}\right] = \frac{1}{2}\eta\left[A - \frac{Aw^T w A}{wAw^T+1}\right]\eta^T.$$

Therefore the matrix $\rho_w(A)$ is the lower $(d-1)\times(d-1)$ principal minor of the matrix

$$A - \frac{Aw^T w A}{wAw^T+1}. \qquad \square$$

Remark 30. Recall from Remark 25 that the mapping $\mathcal{D}_d \ni A \mapsto\in \rho_w(A) \in \mathcal{D}_{d-1}$ is not injective, so multiple matrices in $\mathcal{D}_d$ can given rise to the principal minor above.

Corollary V.24. *The symbol of the $\Psi_{A,w}$ for any $w \in S^{2d-1}$ is given by equation (5.5):*

$$\sigma_A(\eta) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{Q_A(w)+1}}\, e^{Q_{\rho_w(A)}(\eta)/2}\, e^{-|\eta|^2/2}$$

for a suitable $Q_{\rho_w(A)} \in \mathcal{D}(\mathcal{H}_w)$.

Proof. By equivariance of the construction under the action of $\mathrm{U}(d)$, we can assume without loss of generality that $w = (1, \vec{0})$. But that case was settled in Lemma V.23. $\square$

CHAPTER VI

Propagation of Reduced States under Hermitian Hamiltonians

Given that the first part of this thesis was devoted to studying the Schrödinger evolution of coherent states, it is natural to inquire about the dynamics of the reduced states $\Psi_{A,w}$ of the previous chapter. Recall from §3.4 that under the evolution of a general Hermitian Hamiltonian, a state that is initially Gaussian remains approximately a Gaussian state. The center of $\psi_{A(t),w(t)}$ follows the solutions to Hamilton's equations and the squeezing parameter $A(t)$ is governed by a Riccati equation. We will show that the same phenomenon occurs for $\Psi_{A,w}$, i.e., in the semiclassical limit $k \to \infty$, the reduced state remains a reduced state. Additionally, we consider the dynamics of its symbol σ_A.

6.1 Preliminaries

We begin with classical dynamics. Let $h : \mathbb{CP}^{d-1} \to \mathbb{R}$ be a smooth function that denotes the classical Hamiltonian on complex projective space. Furthermore, we must require that h is the projection $\pi : S^{2d-1} \to \mathbb{CP}^{d-1}$ of a smooth time-independent Hamiltonian $H : \mathbb{R}^{2d} \to \mathbb{R}$ that commutes with the circle action. Formally speaking, we will refer to H as the *canonical lift* of h as in:

Definition VI.1. The *canonical lift* of h is

$$H : \mathbb{C}^d \setminus \{0\} \to \mathbb{R}, \quad H(z) := |z|^2 h\left(\pi\left[\frac{z}{|z|}\right]\right), \tag{6.1}$$

where, recall, $\pi : S^{2d-1} \to \mathbb{CP}^{d-1}$ is the (general) Hopf fibration.

Clearly, H is positive-homogeneous of degree two and S^1 invariant, in the following sense:

$$\forall \lambda \in \mathbb{C}^*,\ z \in \mathbb{C}^d \setminus \{0\} \qquad H(\lambda z) = |\lambda|^2 H(z).$$

Conversely, any $H : \mathbb{C}^d \setminus \{0\} \to \mathbb{R}$ with this property is related to a smooth function h on $\mathbb{CP}^{d-1}$ by (6.1).

Now if H is the canonical lift of h, then by construction the Weyl quantization in z of H, denoted by $\widehat{H}$, commutes with the number operator in Bargmann space (5.1), so we can define the operator

$$\widehat{h} : \mathcal{B}(\mathbb{CP}^{d-1}) \to \mathcal{B}(\mathbb{CP}^{d-1})$$

by restricting $\widehat{H}$ to $\mathcal{B}(\mathbb{CP}^{d-1})$ to be the *quantum Hamiltonian on* $\mathbb{CP}^{d-1}$.

Recall that we described the propagation of a Gaussian coherent state in $\mathcal{B}(\mathbb{C}^d)$ by a Hermitian Hamiltonian in Theorem III.7. For convenience, we recall the results here.

Let $H : \mathbb{R}^{2d} \to \mathbb{R}$ be a smooth Hamiltonian which agrees with the canonical lift of a smooth $h : \mathbb{CP}^{d-1} \to \mathbb{R}$ outside a small neighborhood of the origin, $\widehat{H} : \mathcal{B}(\mathbb{C}^d) \to \mathcal{B}(\mathbb{C}^d)$ be its quantization in Bargmann space and $U(t)$ be the fundamental solution of the Schrödinger equation $i\hbar\frac{\partial}{\partial t}U = \widehat{H}U$. Also, let $w \in \mathbb{C}^d$, $t \mapsto w(t)$ be the trajectory of H through w. For each $t \in \mathbb{R}$, let

$$S_t := H_{\overline{z}z}(w(t)) \quad \text{and} \quad R_t := H_{zz}(w(t))$$

with $H_{\overline{z}z} = \left(\frac{\partial^2 H}{\partial \overline{z}_j z_l}\right)$ etc. Then, for $t \in [-T, T]$ with $T \in (0, \infty)$:

$$U(t)(\psi_{A,w}) = e^{ikf(t)}\, e^{i\chi(t)}\, \psi_{A(t),w(t)} \left(1 + O(k^{-1/2})\right) \tag{6.2}$$

where $f(t)$, $A(t)$ and $\chi(t)$ solve

$$\begin{aligned}
\dot{f}(t) &= -H(w) - \Im(\dot{\overline{w}}(t)w(t)^T) \\
i\dot{A}_t &= R_t + (S_tA_t + A_tS_t^T) + A_t\overline{R}_tA_t \\
\dot{\chi}(t) &= -\tfrac{1}{2}\mathrm{Tr}(\overline{R}_tA_t + S_t)
\end{aligned} \tag{6.3}$$

with $A(0) = 0$ and $\chi(0) = 0$.

From (3.2), the symbol of $\psi_{A,w}$ moves according to:

$$e^{i\chi(t)}\,\sigma_{\psi_{A(t),w(t)}} = \mathrm{Mp}(d(\phi_t)_w)\left(\sigma_{\psi_{A,w}}\right)$$

where Mp is the metaplectic representation of $\mathcal{B}(\mathbb{C}^d)$ and $\phi_t : \mathbb{C}^d \to \mathbb{C}^d$ is the Hamilton flow of H. We make the assumption that we can identify the tangent spaces $T_w\mathbb{C}^d \cong T_{w(t)}\mathbb{C}^d$ using translations. However, no such identification exists among tangent spaces of $\mathbb{CP}^{d-1}$, which complicates the description of the symbol of a propagated reduced state, although, under certain conditions, we may obtain an analogous result.

6.2 Main Results

We will denote the quantum propagator on the Bargmann space of the projective space as

$$V(t) = e^{-ikt\widehat{h}} : \mathcal{B}(\mathbb{CP}^{d-1}) \to \mathcal{B}(\mathbb{CP}^{d-1}).$$

This is the fundamental solution of Schrödinger's equation: $i\hbar\frac{\partial}{\partial t}V(t) = \widehat{h}V(t)$.

The following theorem states, that up to leading order in k, the propagation of a reduced squeezed state remains a squeezed state.

Theorem VI.2. *Let $t \in [-T, T]$ with $T \in (0, \infty)$. The evolution $V(t)(\Psi_{A,w})$ of a reduced*

Gaussian state is of the form

$$V(t)(\Psi_{A,w}) = e^{ikf(t)}\, e^{i\chi(t)}\, \Psi_{A(t),w(t)} \left(1 + O(k^{-1/2})\right) \tag{6.4}$$

where $f(t)$, $A(t)$, and $\chi(t)$ are as in (6.3) with H the canonical lift of h.

Next we address the problem of computing the symbol of $V(t)(\Psi_{A,w})$ for each t. Recall that this symbol is an element of the $\hbar = 1$ Bargmann space of $T_{\pi(w(t))}\mathbb{CP}^{d-1}$. We can certainly combine (6.4) with Corollary V.24 to obtain the symbol of $V(t)(\Psi_{A,w})$, which we denote by $\sigma_A(t)$. However, in general this symbol lives in a different space than the symbol σ_A of $\Psi_{A,w}$. It is true that, since the entire construction of reduction is covariant with respect to the $\mathrm{U}(d)$ action which is transitive on the projective space, for a given t, we can apply an element of $\mathrm{U}(d)$ and rotate $w(t)$ back to the initial w. However, this element in $\mathrm{U}(d)$ is not unique.

To avoid this problem, we examine the special case when the point $\varpi \in \mathbb{CP}^{d-1}$ is fixed:

$$\varpi = \pi(w) \quad \text{is a critical point of} \;\; h, \text{and } h(\varpi) = 0. \tag{6.5}$$

As we will see in Lemma VI.9, these assumptions in particular imply that w is a critical point of $H : \mathbb{C}^d \to \mathbb{R}$, so $w(t) = w$. Therefore, the symbol stays on the same tangent space when we propagate the state. The flow is governed by the Hessian of h at ϖ.

By our construction, we have an analogous result as (5.5) for the symbol of the reduced states.

Theorem VI.3. *Under the assumption (6.5), for each $t \in \mathbb{R}$, the symbol of $V(t)(\Psi_{A,w})$ is equal to $Mp(\varphi_t)(\sigma_A)$, where σ_A is the symbol of $\Psi_{A,w}$, $\varphi_t : T_\varpi\mathbb{CP}^{d-1} \to T_\varpi\mathbb{CP}^{d-1}$ is the flow of the Hessian of h at ϖ, and Mp is the metaplectic representation in the Bargmann space of $T_\varpi\mathbb{CP}^{d-1}$.*

6.3 Intermediate Results

6.3.1 The Metaplectic Representation and Reduction

In this section we will prove that "reduction commutes with propagation" on the level of symbols. This result is needed to prove how the symbol of a reduced state propagates.

Once again, let (E, ω, J) be a Kähler vector space. Denote by $P_E : L^2(E) \to \mathcal{B}(E)$ the orthogonal projector (it turns out that $\mathcal{B}(E)$ is closed in $L^2(E)$). If $\Phi : E \to E$ is a symplectic transformation, then one can form the unitary operator $U_\Phi : L^2(E) \to L^2(E)$ which is simply

$$U_\Phi(\psi) = \psi \circ \Phi^{-1}.$$

One of the main results of [Dau80] is the following:

Theorem VI.4. *([Dau80] §6) Let $Sp(E)$ denote the group of symplectic transformations of E. The assignment*

$$Sp(E) \ni \Phi \mapsto \mathcal{W}(\Phi) := \eta_{J,\Phi}\, P_E \circ U_\Phi : \mathcal{B}(E) \to \mathcal{B}(E)$$

where

$$\eta_{J,\Phi} = 2^{-N} \left(\det\left[(I - iJ) + \Phi(1 + iJ)\right]\right)^{1/2} \tag{6.6}$$

is the metaplectic representation.

Our goal here is to prove that the metaplectic representation is natural with respect to symplectic quotients, in the following sense.

Assumption VI.5. *Let $\mathcal{C} \subset E$ be a co-isotropic subspace, as above, and let $\Phi : E \to E$ a linear symplectic isomorphism satisfying $\Phi(\mathcal{C}) = \mathcal{C}$. From this it follows that Φ maps $\mathcal{C}^\circ$ onto itself.*

Assumption VI.6. *The restriction of* Φ *to* $\mathcal{H}^\circ = \mathcal{C}^\circ + J(\mathcal{C}^\circ)$ *is the identity:* $\Phi|_{\mathcal{H}^\circ} : \mathcal{H}^\circ \to \mathcal{H}^\circ$*. Denote by* $F = \mathcal{C}/\mathcal{C}^\circ$ *the reduction of* $\mathcal{C}$*, and by* $\phi : F \to F$ *the reduction of* Φ*:*

$$\forall v \in \mathcal{C} \qquad \phi([v]) = [\Phi(v)],$$

where $[v] \in F$ *denotes the projection of* v*.* ϕ *itself is a symplectomorphism.*

Proposition VI.7. *Under the previous assumptions VI.5 and VI.6, the following diagram commutes,*

$$\begin{array}{ccc} \mathcal{B}(E) & \xrightarrow{\mathcal{W}(\Phi)} & \mathcal{B}(E) \\ \downarrow & & \downarrow \\ \mathcal{B}(F) & \xrightarrow{\mathcal{W}(\phi)} & \mathcal{B}(F) \end{array}$$

where the vertical arrows are the reduction operator R*.*

The proof of the previous statement is given in Proposition 4.7 of [RU21].

6.3.2 Classical Dynamics

The next lemma follows from Definition VI.1.

Lemma VI.8. *The trajectories of the Hamilton flow of the canonical lift of* h *on* S^{2d-1} *project onto trajectories of the Hamilton flow of* h*.*

The next lemma is needed to prove Theorem VI.3.

Lemma VI.9. *Consider* $h \in C^\infty(\mathbb{CP}^{d-1})$ *and* $\varpi \in \mathbb{CP}^{d-1}$ *a critical point. Let* $w \in \pi^{-1}(\varpi)$ *and* $\mathcal{H} \subset T_w S^{2d-1}$ *be the horizontal space at* w*, which we identify with* $T_\varpi \mathbb{CP}^{d-1}$*.*

If, in addition, $h(\varpi) = 0$ *then* w *is a critical point of the canonical lift,* H*, of* h*, and with respect to the decomposition* $T_w\mathbb{C}^d = \mathcal{H} \oplus \mathcal{H}^\circ$ *the Hessian of* H *at* w *has the block form*

$$\begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}$$

where $*$ is the Hessian of h at ϖ.

The proof of the previous result is given in Lemma 5.2 in [RU21].

Remark 31. If ϖ is a critical point of h but $h(\varpi)$ is not necessarily zero, then we can apply the previous lemma to $\tilde{h} = h - h(\varpi)$. Clearly the canonical lifts of these functions are related by $\widetilde{H} = H - h(\varpi)|z|^2$, and since $\{H, |z|^2\} = 0$ the Hamilton flow of H restricted to the unit sphere agrees with that of $\widetilde{H}$ up to a the action of $e^{ith(\varpi)} \in S^1$.

6.3.3 Quantum Propagation

A first observation is:

Proposition VI.10. *One has:*

$$\hat{h}(\Psi_{A,w}) = h((\pi(w))\Psi_{A,w} \left(1 + O(k^{-1/2})\right).$$

Proof. The analogous result for the action of $\widehat{H}$ on Gaussian coherent states in $L^2(\mathbb{R}^d)$ is well-known. Since $[\widehat{H}, \mathcal{R}] = 0$ with $\mathcal{R}$ being the reduction operator in (5.28), the result follows immediately. □

Proof of Theorem VI.2:

Proof. Let $U(t) = \exp\left[-ikt\widehat{H}\right]$. Simply notice that $[U, \mathcal{R}] = 0$, as $\widehat{H}$ and the number operator commute and $\mathcal{R}$ is a normalized spectral projector of the latter, and then apply the result on propagation of squeezed states in $\mathcal{B}(\mathbb{C}^d)$ (6.2). □

Proof of Theorem VI.3:

Proof. We will apply Proposition VI.7, with $E = T_w\mathbb{C}^d$, $\mathcal{C} = T_w S^{2d-1}$ and $\Phi : E \to E$ equal to the differential at w of the time t map of the Hamilton flow of H, $\Phi = d(\phi_t)_w$. Let us identify the various relevant subspaces of E. One has

$$\mathcal{C}^\circ = \{siw \; ; \; s \in \mathbb{R}\} \quad \text{and} \quad J(\mathcal{C}^\circ) = \{sw \; ; \; s \in \mathbb{R}\},$$

and $\mathcal{H}$ is the horizontal subspace $\mathcal{H} = (\mathbb{C}w)^{\perp}$, where the orthogonal is with respect to the standard Hermitian form on $\mathbb{C}^d$. The reduction $\mathcal{C}/\mathcal{C}^{\circ}$ is naturally identified with W and with $T_{\varpi}\mathbb{CP}^{d-1}$. Finally, observe that $\mathcal{H}^{\circ} = \mathbb{C}w$.

We need to verify that the hypotheses (1) and (2) of Proposition VI.7 are satisfied. This follows by Lemma VI.9, because Φ is the time t map of the Hamilton flow of the Hessian of H at w. Therefore Proposition VI.7 applies to the present situation, which concludes the proof in view of Theorem VI.2. $\square$

CHAPTER VII

Application: Spin-Squeezed States

In this chapter, we examine the reduction of squeezed states in Bargmann space of complex dimension $d = 2$. The reduced states correspond to SU(2), or *spin-squeezed* coherent states naturally in the Bargmann space of $\mathbb{CP}^1$ (which is also known as the Riemann sphere or the Bloch sphere in physics). We will provide formulas for the spin-squeezed states in the standard orthonormal basis of $\mathcal{B}(\mathbb{CP}^1)$ and look at an example of propagating one of these states by a simple Hermitian quantum Hamiltonian.

Definition VII.1. A standard orthonormal basis of $\mathcal{B}(\mathbb{CP}^1)$ is

$$|n\rangle = \frac{1}{\pi}\sqrt{\frac{k+1}{2}}\sqrt{\binom{k}{n}}\, z_1^n\, z_2^{k-n}, \quad 0 \le n \le k. \tag{7.1}$$

This is a basis of eigenvectors of the operator $\widehat{L}_3$ in (5.2) corresponding to

$$\sigma_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

the eigenvalue associated with $|n\rangle$ being $n - \frac{k}{2}$.

Recall that (5.10) gave us an exact algebraic expression for the reduced states $\Psi_{A,w}$. We now present an expression that agrees asymptotically with (5.10) and only involves a single parameter $\mu \in \mathcal{D}_1$. By equivariance of the construction under the action of SU(2), it suffices to write the approximation in the case $w = (1, 0)$.

Proposition VII.2. *Let $\mu \in \mathbb{C}$, $|\mu| < 1$ and $[\mu] := \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix}$. Then*

$$\Psi_{[\mu],(1,0)} = \pi k^k e^{-k} \sqrt{\frac{2}{(k+1)!}} \sum_{0 \le \ell \le k/2} \left(\frac{1}{2k}\right)^{\ell} \frac{1}{\sqrt{(k-2\ell)!}} \sqrt{\binom{2\ell}{\ell}} \mu^{\ell} \, |k - 2\ell\rangle. \tag{7.2}$$

Furthermore, for any $A = \begin{pmatrix} a & c \\ c & b \end{pmatrix} \in \mathcal{D}_2$, if we let

$$\mu = \rho_{(1,0)}(A) = b - \frac{c^2}{1+a},$$

then $|\mu| < 1$ and one has

$$\Psi_{A,w}(z) = \Psi_{[\mu],(1,0)}(1 + O(1/\sqrt{k})),$$

where the error estimate is in norm.

Proof. (7.2) is a straightforward calculation, starting with (5.10). If $\mu = \rho_{(1,0)}(A)$, then $|0, \mu\rangle$ and $\Psi_{A,w}$ have the same symbol, and the proposition follows from Corollary V.6. □

With the goal of obtaining the simplest expression for spin squeezed states (asymptotically), we now normalize (7.2) and simplify it.

Lemma VII.3. *The wave function*

$$|o, \mu\rangle := \sum_{0 \le \ell \le k/2} \left(\frac{1}{2k}\right)^{\ell} \frac{(2\ell)!}{\ell!} \sqrt{\binom{k}{2\ell}} \mu^{\ell} \, |k - 2\ell\rangle \tag{7.3}$$

agrees to leading order with $\frac{k}{\sqrt{\pi}} \Psi_{[\mu],(1,0)}$, and its norm satisfies

$$\langle o, \mu || o, \mu\rangle = (1 - |\mu|^2)^{-1/2} + O(1/k). \tag{7.4}$$

(Here $o = \pi(1,0) \in \mathbb{CP}^1$.)

Proof. Identifying $\mathcal{H}_{(1,0)}$ with the z_2 complex plane one finds that

$$\sigma_{[\mu]}(z_2) = \frac{1}{\sqrt{2\pi}} e^{\mu z_2^2/2}\, e^{-|z_2|^2/2}$$

and therefore, by (5.7), after some calculations we obtain

$$\|\Psi_{[\mu],(1,0)}\| \sim \frac{\sqrt{\pi}}{k}(1-|\mu|^2)^{-1/4}. \tag{7.5}$$

If we multiply $\Psi_{[\mu],(1,0)}$ by $k/\sqrt{\pi}$, by (7.5) the result has a norm squared that asymptotically is given by (7.4). We then apply Stirling's formula to $k/\sqrt{\pi}\,\Psi_{[\mu],(1,0)}$ and simplify to obtain (7.3). □

For future reference, note that the symbol of $|o,\mu\rangle$ (see Remark 23) is

$$\frac{1}{\sqrt{2}\,\pi}\, e^{\mu z_2^2/2}\, e^{-|z_2|^2/2}.$$

We have plotted the magnitudes of the components of the ℓ^2-normalized $|o,\mu\rangle$ for $\mu = 3/4$ and $k = 30$ in Figure 7.1.

We now let SU(2) act on the kets $|0,\mu\rangle$:

Definition VII.4. Let $S_k : \mathrm{SU}(2) \to \mathcal{U}\left(\mathcal{B}(\mathbb{CP}^1)\right)$ be the natural representation of SU(2) in $\mathcal{B}(\mathbb{CP}^1)$. For any $p \in \mathbb{CP}^1$, let $g \in \mathrm{SU}(2)$ be such that $p = g \cdot o$. If $\mu \in \mathcal{D}_1$, let

$$|p,\mu\rangle = S_k(g)(|o,\mu\rangle). \tag{7.6}$$

We call any such state *a squeezed SU(2) Gaussian state with center p and parameter μ.*

We note that the notation (7.6) is ambiguous, since g is not unique for a given p, but the ambiguity is a unitary factor (the squeezed coherent states are properly labeled by points on S^3).

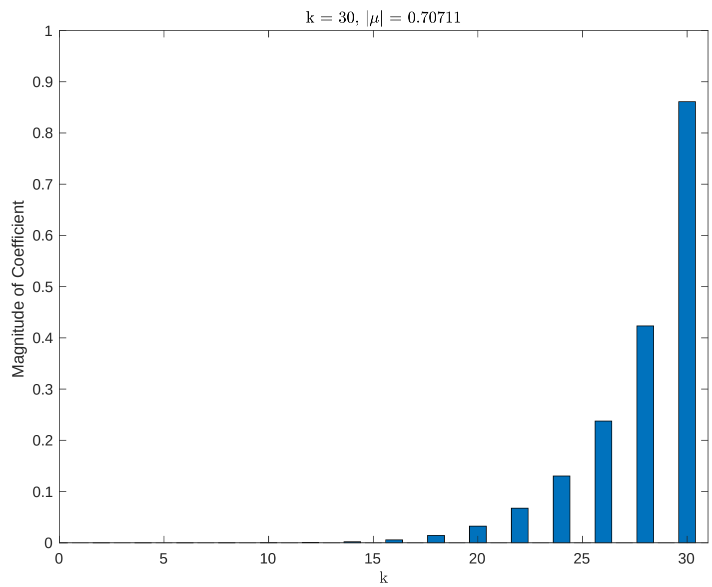

Figure 7.1: Plot of the components of the ℓ^2-normalized kets (7.3) for $k = 30$ and $\mu = \frac{1}{2}(1-i)$. Observe that the magnitudes decay as k decreases. The decay is faster the closer $|\mu|$ is to zero.

It is worthwhile to give another description of $|o, \mu\rangle$, based on another common realization of the Hilbert space of $\mathbb{CP}^1$. Using a trivialization of the Hopf fibration $S^3 \to \mathbb{CP}^1$, one can identify the Bargmann space of $\mathbb{CP}^1$ with the space

$$\mathcal{B}(\mathbb{CP}^1) \cong \left\{ \Psi(\zeta) = \frac{f(\zeta)}{(1+|\zeta|^2)^{k/2}} \mid \overline{\partial} f = 0 \right\} \cap L^2(\mathbb{C}, dm) \text{ where } dm = \frac{2\pi}{i} \frac{d\zeta\, \bar{\partial}\zeta}{(1+|\zeta|^2)^2}. \quad (7.7)$$

One can check that, in the above, f must be a polynomial of degree at most k in the complex variable ζ. The identification is simply by pulling back elements in $\mathcal{B}(\mathbb{CP}^1)$ by the section $S_\varpi : \mathbb{C} \to S^3$ given by

$$S_\varpi(\zeta) = \frac{1}{\sqrt{1+|\zeta|^2}}(1, \zeta).$$

The pull-back of the basis kets is

$$S_\varpi^* |n\rangle = \frac{1}{\pi} \frac{1}{(1+|\zeta|^2)^{k/2}} \sqrt{\frac{k+1}{2}} \sqrt{\binom{k}{n}}\, \zeta^{k-n} \quad (7.8)$$

and this allows us to find the pull-back of the squeezed states

$$S_{\varpi}^{*}|o,\mu\rangle = \frac{1}{\pi}\frac{k!}{(1+|\zeta|^2)^{k/2}}\sqrt{\frac{k+1}{2}}\sum_{0\le\ell\le k/2}\left(\frac{1}{2k}\right)^{\ell}\frac{1}{\ell!(k-2\ell)!}\,\mu^{\ell}\,\zeta^{2\ell}. \tag{7.9}$$

Figure 7.2 shows the Husimi function $|S_{\varpi}^{*}|o,\mu\rangle|^2$ of the ket $|o,\mu\rangle$ and its level sets as a function of ζ, for a choice of μ and k. Notice that the Husimi function is always nonnegative, so it can be thought of as the phase space probability density function for the state. Yet another motivation for using the Bargmann space is that it is a natural home for the Husimi function.

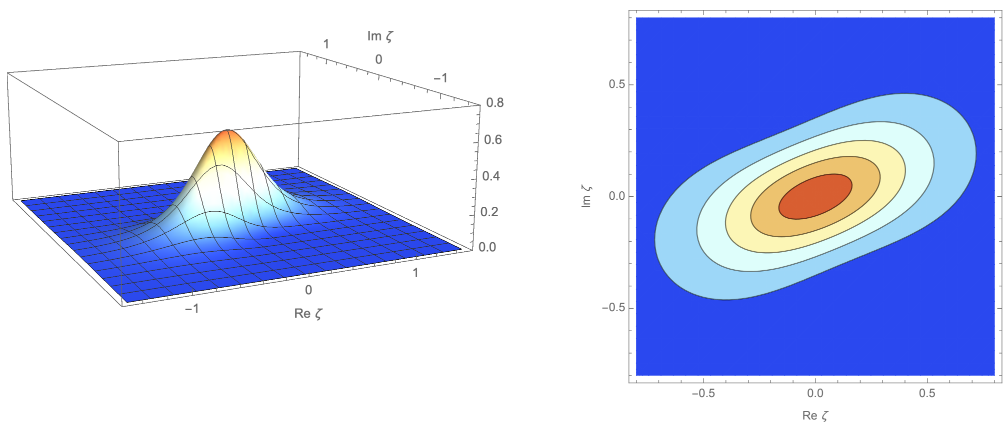

Figure 7.2: Plot of $|S_{\varpi}^{*}|o,\mu\rangle|^2$ and its levels sets for $k=10$ and $\mu=\frac{1}{2}(1-i)$.

7.1 Example: Propagation with a Hermitian Hamiltonian

Let the classical angular momentum operators in Bargmann space $L_j:\mathbb{C}^2\to\mathbb{R}$ be given by

$$\begin{aligned}
L_1 &= \Re(z_1\overline{z}_2) = \frac{1}{2}(q_1q_2+p_1p_2),\\
L_2 &= \Im(z_1\overline{z}_2) = \frac{1}{2}(q_1p_2-p_1q_2),\\
L_3 &= \frac{1}{2}\left(|z_1|^2-|z_2|^2\right) = \frac{1}{4}(q_1^2+p_1^2-q_2^2-p_2^2)
\end{aligned}$$

where we have let $z_j = \frac{1}{\sqrt{2}}(q_j - ip_j)$. Then $\{L_1, L_2\} = L_3$ and cyclic permutations. Observe that these are the classical analogues of the operators in (5.2):

$$\widehat{L}_1 = \frac{1}{2}\left(z_1\frac{\partial}{\partial z_2} + z_2\frac{\partial}{\partial z_1}\right), \; \widehat{L}_2 = -\frac{i}{2}\left(z_1\frac{\partial}{\partial z_2} - z_2\frac{\partial}{\partial z_1}\right), \; \widehat{L}_3 = \frac{1}{2}\left(z_1\frac{\partial}{\partial z_1} - z_2\frac{\partial}{\partial z_2}\right).$$

These functions are the components of the moment map of the SU(2) action on $\mathbb{C}^2$ with respect to the standard Pauli matrices, and they all commute with the circle action. Therefore they descend to smooth functions

$$\ell_j : \mathbb{CP}^1 \to \mathbb{R}$$

which are the components of the SU(2) Hamiltonian action on the complex projective line. Since the L_j are quadratic, they are the canonical lift of the ℓ_j.

Using the coordinate $\zeta = z_2/z_1$ and writing $\zeta = x + iy$, the ℓ_j's are defined as

$$\ell_1 := \Re\frac{\zeta}{1+|\zeta|^2}, \qquad \ell_2 := \Im\frac{\zeta}{1+|\zeta|^2}, \qquad \ell_3 := \frac{1}{2}\frac{|\zeta|^2-1}{|\zeta|^2+1}.$$

Each ℓ_j has two critical points. Since the SU(2) action is an isometry, the Hessian of ℓ_j at any fixed point ϖ generates a unitary transformation of $T_\varpi\mathbb{CP}^1$, which is simply a rotation. Under the quantum propagation of $\hat{\ell}_j$ a squeezed state at ϖ simply rotates, and so does its symbol.

Let

$$h = a^2\ell_1^2 - b^2\ell_2^2, \quad a, b \geq 0$$

be a classical Hamiltonian on $\mathbb{CP}^1$. The point $\varpi = \pi(1,0)$ is a critical point of h, and $h(\varpi) = 0$. To apply Theorem VI.3 we need to identify the Hessian of h at ϖ.

Using the approximation $\frac{1}{1+|\zeta|^2} \sim 1 - |\zeta|^2$, one readily checks that the Taylor expansion of h at the origin begins with

$$h(\zeta) \sim \frac{(a^2+b^2)}{4}(\zeta^2 + \overline{\zeta}^2) + \frac{a^2-b^2}{2}\zeta\overline{\zeta}.$$

Let us now choose $a = b = 1/\sqrt{2}$ so that $h(\zeta) \sim \frac{1}{4}(\zeta^2 + \overline{\zeta}^2)$. If $\mathfrak{z}$ is a complex coordinate on $T_\varpi\mathbb{CP}^1$, the symbol $\sigma(\mathfrak{z}, t) = f(\mathfrak{z}, t)e^{-|\mathfrak{z}|^2/2}$ of a propagated squeezed state centered at the origin solves the Schrödinger equation[1]

$$i\frac{\partial f(\mathfrak{z}, t)}{\partial t} = \frac{1}{4}\left(\mathfrak{z}^2 + \frac{d^2}{d\mathfrak{z}^2}\right) f(\mathfrak{z}, t).$$

We choose the time-evolved ansatz to be $f(\mathfrak{z}, t) = \nu(t)\, e^{\mu(t)\mathfrak{z}^2/2}$. We can now apply 6.2 and (6.3) (which now gives an exact solution) with $R_t = 1/4$ and $S_t = 0$, and conclude that ν and μ satisfy

$$\dot{\mu} = \frac{1}{2i}(1 + \mu^2) \quad \text{and} \quad \dot{\nu} = -\frac{i}{4}\mu\,\nu.$$

Let us impose the initial conditions $\mu(0) = 0$ and $\nu(0) = 1/(\pi\sqrt{2})$, which correspond to the symbol of the standard SU(2) coherent state at the origin. We find that the solutions to these ODEs are

$$\mu(t) = -i\tanh(t/2), \qquad \nu(t) = \frac{1}{\pi\sqrt{2}}\frac{1}{\sqrt{\cosh(t/2)}},$$

and therefore

$$\sigma(\mathfrak{z}, t) = \frac{1}{\pi\sqrt{2}}\frac{1}{\sqrt{\cosh(t/2)}}e^{-i\tanh(t/2)\mathfrak{z}^2/2}\, e^{-|\mathfrak{z}|^2/2}.$$

In terms of the standard squeezed states (7.3) with $\mu(0) = 0$ in this case, we can conclude that the evolved state is given by

$$e^{-ikt\hat{h}}|o, 0\rangle = \nu(t)|o, \mu(t)\rangle\left(1 + O(1/\sqrt{k})\right) \tag{7.10}$$

where the functions $\nu(t)$ and $\mu(t)$ and the Hamiltonian $\hat{h}$ are as above.

Figure 7.3 compares numerically the left-hand-side and right-hand-side of (7.10) for $k = 30$ and $t = 2$. In order to compute the left-hand-side of (7.10), we have written the quantum

[1] Note that $\hbar = 1$ here.

Hamiltonian $\hat{h}$ as

$$\hat{h} = a^2 \hat{L}_1 - b^2 \hat{L}_2$$

where, as matrices in the basis of $\mathcal{B}(\mathbb{CP}^1)$ (7.1), $\hat{L}_1$ and $\hat{L}_2$ are given by

$$\hat{L}_1|n\rangle = \frac{1}{2k}\left[\sqrt{n(k-n+1)}\,|n-1\rangle + \sqrt{(k-n)(n+1)}\,|n+1\rangle\right]$$
$$\hat{L}_2|n\rangle = \frac{i}{2k}\left[\sqrt{n(k-n+1)}\,|n-1\rangle - \sqrt{(k-n)(n+1)}\,|n+1\rangle\right]$$

for $n = 0, \ldots, k$. Notice that these matrices only have nonzero entries along the sub-diagonal and the super-diagonal. These matrices can be found using the operators in Lemma 3.2 and Lemma 3.4 in [BGPU03].

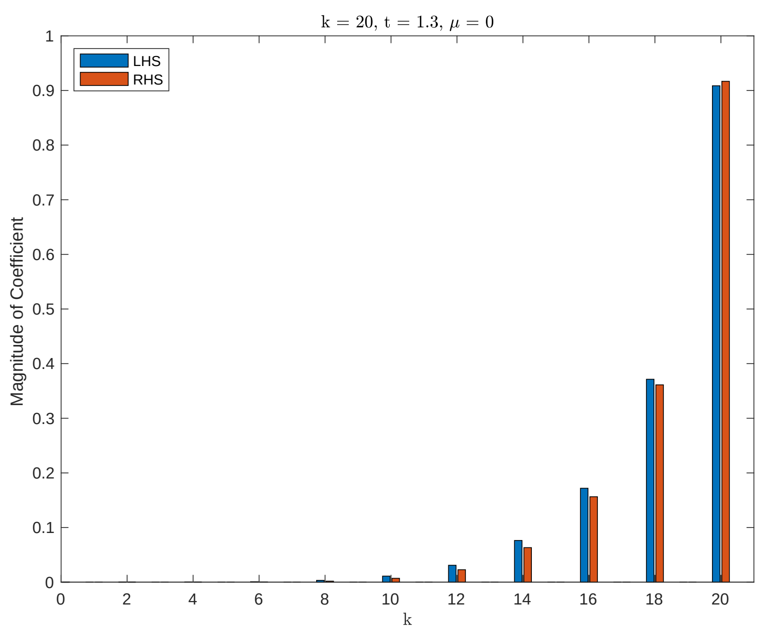

Figure 7.3: Plot of the magnitudes of the components of the normalized vectors on both sides of (7.10) for $k = 20$ and $t = 1.3$. The difference in the ℓ^2-norm is $|\mathrm{LHS}-\mathrm{RHS}| \approx 2.58 \times 10^{-2}$.

APPENDIX A

Metaplectic Operators acting on Schwartz Functions

We prove that metaplectic operators map Schwartz functions to Schwartz functions which is used in the proof of Lemma III.4.

Let $\mathbf{H}_{2d}$ be the Heisenberg group of $\mathbb{R}^{2d}$ and let $U_{\mathcal{B}(\mathbb{C}^d)}$ denote unitary operators in $\mathcal{B}(\mathbb{C}^d)$. Denote the *Heisenberg representation* by $\rho : \mathbf{H}_{2d} \to U_{\mathcal{B}(\mathbb{C}^d)}$.

Definition A.1. An element $\psi \in \mathcal{B}(\mathbb{C}^d)$ is a *smooth vector* if and only if $\forall \phi \in \mathcal{B}(\mathbb{C}^d)$ the map $\mu : \mathbf{H}_{2d} \to \mathbb{C}$ given by

$$\mu(g) = \langle \rho(g)(\psi), \phi \rangle$$

is smooth.

The next lemma follows from equation (12) in [How80].

Lemma A.2. *The space of smooth vectors in $\mathcal{B}(\mathbb{C}^d)$ is $\mathcal{B}(\mathbb{C}^d) \cap \mathcal{S}(\mathbb{R}^{2d})$.*

The following result from Chapter 4 in [Fol95] states that the metaplectic operators intertwine the Heisenberg representation.

Lemma A.3. *Let W be a metaplectic operator associated with a group isomorphism $G : \boldsymbol{H}_{2d} \to \boldsymbol{H}_{2d}$. Then,*

$$\rho\left(G(g)\right) = W \circ \rho(g) \circ W^{-1}, \qquad g \in \boldsymbol{H}_{2d}.$$

Proposition A.4. *Let W be a metaplectic operator associated with the group isomorphism $G : \boldsymbol{H}_{2d} \to \boldsymbol{H}_{2d}$. Then, $W : \mathcal{S}(\mathbb{R}^{2d}) \to \mathcal{S}(\mathbb{R}^{2d})$.*

Proof. Let $\psi \in \mathcal{B}(\mathbb{C}^d) \cap \mathcal{S}(\mathbb{R}^{2d})$ and choose $\phi \in \mathcal{B}(\mathbb{C}^d)$. By Lemma A.2, ψ is a smooth vector in Bargmann space. Hence, using the definition of smooth vector, consider the function $\mu(g) = \langle \rho(g)(W(\psi)), \phi \rangle$ for $g \in \mathbf{H}_{2d}$. We need to show that $\mu(g)$ is smooth. Observe that

$$\mu(g) = \langle \rho(g)(W(\psi)),\, \phi \rangle = \langle W^{-1}\rho(g)(W(\psi)),\, W^{-1}\phi \rangle = \langle \rho(G^{-1}(g))(\psi), W^{-1}\phi \rangle$$

where we have used the fact that W^{-1} is unitary in the first step and Lemma A.3 in the second step. Let $\mu = \nu \circ G^{-1}$ where $\nu(f) := \langle \rho(f)\,\psi, W^{-1}\,\phi \rangle$ for each $f \in \mathbf{H}_{2d}$. Now ν is a smooth because ψ is a smooth vector which implies that μ is also smooth. $\square$

APPENDIX B

Borel Summation

The following proposition follows from Proposition III.5. It shows that we can find a wavefunction that solves the Schrödinger problem to all orders in $\hbar$.

Proposition B.1. *Given an initial condition,*

$$\psi\big|_{t=0} = \hbar^m \, e^{i\hbar^{-1}\Phi(z,0)} \, e^{-\hbar^{-1}|z|^2/2} \, \varphi\left(\frac{z-w(0)}{\hbar}, 0, \hbar\right),$$

for each t, $\exists \psi(z,t) \in I^m_\gamma$ s.t. for each N, T, $\exists C_{N,T}$ s.t.

$$\sup_{z\in\mathbb{C}^d} |\widetilde{\square}\psi| \leq C_{N,T}\, \hbar^{m+(N+1)/2}, \quad \forall |t| \leq T.$$

Remark 32. Note that $\psi\big|_{t=0}$ is not actually in I^m_γ because it has no t dependence.

Proof. Let $\rho_0 := \psi$ in our expansion in (3.5). Without loss of generality, let $m = 0$. Then,

$$\psi_N = \sum_{j=0}^{N} \hbar^{j/2}\rho_j, \qquad \rho_j \in I^0_\gamma.$$

Choose a C^∞ function χ such that $0 \leq \chi \leq 1$, $\chi \equiv 1$ on $[0, 1]$ and $\chi \equiv 0$ on $[2, \infty)$. We define

$$\psi := \sum_{j=0}^{\infty} \hbar^{j/2}\rho_j \, \chi(\lambda_j \hbar) \tag{B.1}$$

where the sequence $\lambda_j \to \infty$ remains to be chosen. Since $\lambda_j \to \infty$, there are for each

$\hbar > 0$ at most finitely many nonzero terms in the sum (B.1). We have

$$\psi - \psi_N = \psi - \sum_{j=0}^{N} \hbar^{j/2}\rho_j = \sum_{j=N+1}^{\infty} \hbar^{j/2}\chi(\lambda_j\hbar)\,\rho_j + \sum_{j=0}^{N} \hbar^{j/2}\left(\chi(\lambda_j\hbar) - 1\right)\rho_j.$$

For each j, $|\widetilde{\Box}\rho_j| \leq \hbar\, C_j$ for some constant C_j by Proposition (III.2) and using the fact that ρ_j is Schwartz, so

$$\left|\chi(\lambda_j\hbar)\,\widetilde{\Box}\rho_j\right| \leq \hbar\, C_j\chi(\lambda_j\hbar).$$

If we choose the λ_j so that $\lambda_j \geq 2^{j+1}C_j$, then

$$\hbar\, C_j\chi(\lambda_j\hbar) = \hbar\, C_j\,\frac{\lambda_j\hbar}{\lambda_j\hbar}\chi(\lambda_j\hbar) = \frac{C_j}{\lambda_j}\,(\lambda_j\hbar)\,\chi(\lambda_j\hbar) \leq 2\,\frac{C_j}{\lambda_j} \leq 2\cdot 2^{j-1} = 2^{-j} \tag{B.2}$$

because $(\lambda_j\hbar)\,\chi(\lambda_j\hbar) \leq 2$. We will also assume that $\lambda_j \leq \lambda_{j+1}$.

Since $\widetilde{\Box}$ is a linear operator and χ is a piece-wise constant function,

$$\widetilde{\Box}(\psi - \psi_N) = \widetilde{\Box}\psi - \widetilde{\Box}\psi_N = \sum_{j=N+1}^{\infty} \hbar^{j/2}\chi(\lambda_j\hbar)\,\widetilde{\Box}\rho_j + \sum_{j=0}^{N} \hbar^{j/2}\left(\chi(\lambda_j\hbar) - 1\right)\widetilde{\Box}\rho_j.$$

Then,

$$\begin{aligned}
\left|\widetilde{\Box}(\psi - \psi_N)\right| &\leq \sum_{j=N+1}^{\infty} h^{j/2}\left|\chi(\lambda_j\hbar)\,\widetilde{\Box}\rho_j\right| + \sum_{j=0}^{N} \hbar^{j/2}\left|\left(\chi(\lambda_j\hbar) - 1\right)\widetilde{\Box}\rho_j\right| \\
&= \sum_{j=N+1}^{\infty} h^{j/2}\left|\chi(\lambda_j\hbar)\,\widetilde{\Box}\rho_j\right| + \sum_{j=0}^{N} \hbar^{j/2}\left|\left(1 - (\chi(\lambda_j\hbar)\right)\widetilde{\Box}\rho_j\right| \\
&= \sum_{j=N+1}^{\infty} h^{j/2}\chi(\lambda_j\hbar)\left|\widetilde{\Box}\rho_j\right| + \sum_{j=0}^{N} \hbar^{j/2}(1 - (\chi(\lambda_j\hbar))\left|\widetilde{\Box}\rho_j\right| \\
&=: \mathrm{I} + \mathrm{II}
\end{aligned}$$

Using the estimate in (B.2),

$$\mathrm{I} \leq \sum_{j=N+1}^{\infty} \hbar^{j/2+1} C_j \, \chi(\lambda_j \hbar) \leq \sum_{j=N+1}^{\infty} \frac{\hbar^{j/2}}{2^j} \leq \hbar^{(N+1)/2}.$$

Furthermore,

$$\mathrm{II} \leq \sum_{j=0}^{N} \hbar^{j/2+1} C_j (1 - \chi(\lambda_j \hbar)).$$

Since $\chi \equiv 1$ on $[0,1]$, $\mathrm{II} = 0$ if $0 < \hbar \leq \lambda_N^{-1}$ because $\lambda_N \geq \lambda_j$, $\forall j$, so $\lambda_N^{-1} \leq \lambda_j^{-1}$.

Otherwise, if $\lambda_N^{-1} \leq \hbar \leq 1$, then $1 \leq \lambda_N \hbar$, so

$$\mathrm{II} \leq \sum_{j=0}^{N} \hbar \, C_j \leq \sum_{j=0}^{N} \hbar \, (\lambda_N \hbar)^{(N-1)/2} C_j = \hbar \, \hbar^{(N-1)/2} \, \lambda_N^{(N-1)/2} \sum_{j=0}^{N} C_j = \widetilde{C}_N \, \hbar^{(N+1)/2}$$

where $\widetilde{C}_N := \lambda_N^{(N-1)/2} \sum_{j=0}^{N} C_j$.[1]

Therefore, for any N, $|\widetilde{\Box}(\psi - \psi_N)| \leq C_N \hbar^{(N+1)/2}$. $\square$

[1]The choice of $(\lambda_N \hbar)^{(N-1)/2}$ is not unique; this sum can be made arbitrarily small. The restriction on the bound comes from the sum in I.

APPENDIX C

The Complex Structure $J_{A(t)}$ as a Matrix

In this appendix, we derive a matrix version of the complex structure $J_{A(t)}$ that is found in equation (4.2) which may be more useful for computations. Assume that we are working with column vectors.

We begin with (4.18), which we reproduce below for convenience:

$$i(\dot{\overline{w}}(t) - A_t\, \dot{w}(t)) = \nabla_z F(w(t), t) + A_t \nabla_{\overline{z}} F(w(t), t).$$

Recall that $w(t) = x(t) + iy(t)$ and $\nabla_z = \frac{1}{2}(\nabla_x - i\nabla_y)$. Let $A_t = B_t + iD_t$ where B_t and D_t real-symmetric matrices which denote the real and imaginary parts of A_t, respectively. Then, on the left-hand side we have

$$\begin{aligned}
i(\dot{\overline{w}}(t) - A_t\, \dot{w}(t)) &= i\left[\dot{x}(t) - i\dot{y}(t) - (B_t + iD_t)(\dot{x}(t) + i\dot{y}(t))\right] \\
&= i\left[\dot{x}(t) - i\dot{y}(t) - (B_t\, \dot{x}(t) - D_t\, \dot{y}(t)) - i(D_t\, \dot{x}(t) + B_t\, \dot{y}(t))\right] \\
&= i\left[(I_d - B_t)\, \dot{x}(t) + D_t\, \dot{y}(t) - i\left[D_t\, \dot{x}(t) + (I_d + B_t)\, \dot{y}(t)\right]\right] \\
&= D_t\, \dot{x}(t) + (I_d + B_t)\, \dot{y}(t) + i\left[(I_d - B_t)\, \dot{x}(t) + D_t\, \dot{y}(t)\right].
\end{aligned}$$

Furthermore,

$$
\begin{aligned}
\nabla_z F(w(t),t) + A_t\,\nabla_{\overline{z}} F(w(t),t) &= \tfrac{1}{2}(\nabla_x F - i\nabla_y F) + \tfrac{1}{2}(B_t + iD_t)(\nabla_x F + i\nabla_y F) \\
&= \tfrac{1}{2}\left[\nabla_x(H+i\Gamma) - i\nabla_y(H+i\Gamma)\right] + \tfrac{1}{2}(B_t+iD_t)\nabla_x(H+i\Gamma) + \tfrac{1}{2}\,i(B_t+iD_t)\nabla_y(H+i\Gamma) \\
&= \tfrac{1}{2}\left(\nabla_x H + i\nabla_x\Gamma - i\nabla_y H + \nabla_y\Gamma + B_t\,\nabla_x H + iD_t\,\nabla H_x\right) \\
&\quad + \tfrac{1}{2}\left(iB_t\,\nabla_x\Gamma - D_t\,\nabla_x\Gamma + iB_t\,\nabla_y H - D_t\,\nabla_y H - B_t\,\nabla_y\Gamma - iD_t\,\nabla_y\Gamma\right) \\
&= \tfrac{1}{2}\left(\nabla_x H + \nabla_y\Gamma + B_t\,\nabla_x H - D_t\,\nabla_x\Gamma - D_t\,\nabla_y H - B_t\,\nabla_y\Gamma\right) \\
&\quad + \tfrac{1}{2}\,i\left(\nabla_x\Gamma - \nabla_y H + D_t\,\nabla_x H + B_t\,\nabla_x\Gamma + B_t\,\nabla_y H - D_t\,\nabla_y\Gamma\right) \\
&= \tfrac{1}{2}\left[(I_d+B_t)\nabla_x H - D_t\,\nabla_y H - D_t\,\nabla_x\Gamma + (I_d-B_t)\nabla_y\Gamma\right] \\
&\quad + \tfrac{1}{2}\,i\left[D_t\,\nabla_x H - (I_d-B_t)\nabla_y H + (I_d+B_t)\nabla_x\Gamma - D_t\,\nabla_y\Gamma\right].
\end{aligned}
$$

Finally, in matrix form $i(\dot{\overline{w}}(t) - A_t\dot{w}(t)) = \nabla_z F(w(t),t) + A_t\nabla_{\overline{z}}F(w(t),t)$ is

$$
\begin{pmatrix} D_t\,\dot{x}(t) + (I_d+B_t)\,\dot{y}(t) \\ (I_d-B_t)\,\dot{x}(t) + D_t\,\dot{y}(t) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (I_d+B_t)\nabla_x H - D_t\,\nabla_y H \\ D_t\,\nabla_x H - (I_d-B_t)\nabla_y H \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -D_t\,\nabla_x\Gamma + (I_d-B_t)\nabla_y\Gamma \\ (I_d+B_t)\nabla_x\Gamma - D_t\,\nabla_y\Gamma \end{pmatrix}
$$

$$
\begin{aligned}
\begin{pmatrix} D_t & I_d+B_t \\ I_d-B_t & D_t \end{pmatrix}\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} &= \begin{pmatrix} D_t & I_d+B_t \\ I_d-B_t & D_t \end{pmatrix}\cdot\frac{1}{2}\begin{pmatrix} -\nabla_y H(w(t),t) \\ \nabla_x H(w(t),t) \end{pmatrix} \\
&\quad + \begin{pmatrix} -(I_d-B_t) & -D_t \\ D_t & I_d+B_t \end{pmatrix}\cdot\frac{1}{2}\begin{pmatrix} -\nabla_y\Gamma(w(t),t) \\ \nabla_x\Gamma(w(t),t) \end{pmatrix}
\end{aligned}
$$

$$
\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \Xi_H(w(t),t) + \begin{pmatrix} D_t & I_d+B_t \\ I_d-B_t & D_t \end{pmatrix}^{-1}\begin{pmatrix} -(I_d-B_t) & -D_t \\ D_t & I_d+B_t \end{pmatrix}\Xi_\Gamma(w(t),t)\,.
$$

Therefore, as a $2d \times 2d$ matrix,

$$J_{A(t)} = \begin{pmatrix} \Im(A_t) & I_d + \Re(A_t) \\ I_d - \Re(A_t) & \Im(A_t) \end{pmatrix}^{-1} \begin{pmatrix} -(I_d - \Re(A_t)) & -\Im(A_t) \\ \Im(A_t) & I_d + \Re(A_t) \end{pmatrix}. \tag{C.1}$$

BIBLIOGRAPHY

[Arn21] Víctor Arnaiz. Construction of quasimodes for non-selfadjoint operators via propagation of Hagedorn wave-packets. 2021.

[Bar61] Valentine Bargmann. On a Hilbert space of analytic functions and an associated integral transform. *Communications on Pure and Applied Mathematics*, 14:187–214, 1961.

[BB98] Carl M. Bender and Stefan Boettcher. Real spectra in non-Hermitian hamiltonians having PT symmetry. *Physical Review Letters*, 80:5243–5246, 1998.

[BBLU22] Reebu Bhattacharyya, Daniel Burns, Ernesto Lupercio, and Alejandro Uribe. The exponential map of the complexification of the group of analytic Hamiltonian diffeomorphisms. *Pure and Applied Mathematics Quaterly*, 18:33–70, 2022.

[BGPU03] A. Bloch, F. Golse, T. Paul, and A. Uribe. Dispersionless Toda and Toeplitz operators. *Duke Mathematical Journal*, 17(1):157–196, March 2003.

[CR12] Monique Combescure and Didier Robert. *Coherent States and Applications in Mathematical Physics*. Springer, 2012.

[Dau80] Ingrid Daubechies. Coherent states and projective representation of the linear canonical transformations. *Journal of Mathematical Physics*, 21(6):1377–1389, June 1980.

[dMFF06] C. Figueira de Morisson Faria and A. Fring. Time evolution of non-Hermitian Hamiltonian systems. *Journal of Physics A: Mathematical and General*, 39:9269–9289, 2006.

[Fol95] Gerald B. Folland. *Introduction to Partial Differential Equations*. Princeton University Press, 1995.

[Fri22] Andreas Fring. An introduction to PT-symmetric quantum mechanics – time-dependent systems, 2022.

[Gaz09] Jean-Pierre Gazeau. *Coherent States in Quantum Physics*. WILEY-VCH Verlag GmbH & Co., 2009.

[Gla63] Roy J. Glauber. Coherent and incoherent states of the radiation field. *Physical Review*, 131:2766–2788, 1963.

[GS12] Eva-Maria Graefe and Roman Shubert. Complexified coherent states and quantum evolution with non-Hermitian Hamiltonians. *Journal of Physics A: Mathematical and Theoretical*, 45, 2012.

[Hal00] Brian C. Hall. Holomorphic methods in mathematical physics. *Contemporary Mathematics*, 260:1–59, 2000.

[Hal13] Brian C. Hall. *Quantum Theory for Mathematicians*. Springer, 2013.

[Her97] Jan Herczyński. Semiclassical estimates for Weyl quantisation. *Journal of Functional Analysis*, 147:1–25, 1997.

[Hör90] Lars Hörmander. *The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis*. Springer, 1990.

[How80] Roger Howe. On the role of the Heisenberg group in harmonic analysis. *Bulletin of the American Mathematical Society*, 3:821–843, 1980.

[KsS85] John R. Klauder and Bo sture Skagerstam. *Coherent States: Applications In Physics And Mathematical Physics*. World Scientific Publishing Co., 1985.

[LdPM20] F. S. Luiz, M. A. de Ponte, and M. H. Y. Moussa. Unitarity of the time-evolution and observability of non-Hermitian Hamiltonians for time-dependent Dyson maps. *Physica Scripta*, 95, 2020.

[LST18] Caroline Lasser, Roman Shubert, and Stephanie Troppmann. Non-Hermitian propagation of Hagedorn wavepackets. *Journal of Mathematical Physics*, 59, 2018.

[Moi11] Nimrod Moiseyev. *Non-Hermitian Quantum Mechanics*. Cambridge University Press, 2011.

[Mos98] Ali Mostafazadeh. Two-component formulation of the Wheeler-Dewitt equation. *Journal of Mathematical Physics*, 39:205–214, 1998.

[Mos02] Ali Mostafazadeh. Pseudo-Hermiticity versus PT symmetry: The necessary condition for the reality of the spectrum of a non-Hermitian Hamiltonian. *Journal of Mathematical Physics*, 43:205–214, 2002.

[Per86] Askold Perelomov. *Generalized Coherent States and Their Applications*. Springer Berlin Heidelberg, 1986.

[ROM10] Erasmo Recami, Vladislav S. Olkhovsky, and Sergei P. Maydanyuk. On nonselfadjoint operators for observables in quantum mechanics and quantum field theory. *International Journal of Modern Physics*, 25:1785–1818, 2010.

[RU21] Jenia Rousseva and Alejandro Uribe. Reduction and coherent states. *Letters in Mathematical Physics*, 2021.

[Sch26] Erwin Schrödinger. Der stetige Übergang von der mikro-zur makro-mechanik. *Die Naturawissenschaften*, 28:664–666, 1926. English translation: "The Continuous Transition from Micro-to Macro-Mechanics", in *Collected Papers on Wave Mechanics*, pp. 41-44.

[TE20] Lloyd N. Trefethen and Mark Embree. *Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators*. Princeton University Press, 2020.

[Tro17] Stephanie Troppman. Non-Hermitian Schrödinger dynamics with Hagedorn's wave-packets, 2017.

[Woo97] N.M.J. Woodhouse. *Geometric Quanitzation*. Oxford Mathematical Monographs, 1997.

[WQM+21] Hong Wang, Yue Qin, Jingxu Ma, Heng Shen, and Xiaojun Jia. Application of non-Hermitian Hamiltonian model in open quantum optical systems. *Chinese Physics B*, 30, 2021.

[ZFG90] Wei-Min Zhang, Da Hsuan Feng, and Robert Gilmore. Coherent states: Theory and some applications. *Review of Modern Physics*, 62:867–927, 1990.