On the Coefficients of some Nonabelian Equivariant Cohomology Theories

by

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ABSTRACT

In this thesis, we give a complete calculation of the coefficients of ordinary equivariant cohomology with constant coefficients, graded by the real representation ring of a finite group, where the group is the dihedral group of order $2p$ for an odd prime p, and when the group is the quaternion group. Another independent topic will be equivariant complex cobordism. We calculate the coefficient ring of homotopical equivariant complex cobordism for the symmetric group on three elements. We also study the relation between the coefficient ring of equivariant complex cobordism and the universal Lazard ring of equivariant formal group laws for finite abelian groups, and prove a result generalizing classical Quillen's Theorem.

CHAPTER I

Introduction

1.1 Overview

Algebraic topology uses algebraic invariants to study topological spaces. By passing from topological spaces to spectra, we can study stable invariants. With an action by a compact Lie group G, we can study equivariant invariants as well. Equivariant stable homotopy theory has proven to be very useful in recent years, with applications in solving the 50-year-old Kervaire invariant one problem [HHR16]; in computing algebraic K-theory and topological Hochschild homology [BMS19], [NS18]; and also in chromatic homotopy theory, derived algebraic geometry, and representation theory.

In equivariant stable homotopy theory, the role of abelian groups is replaced by Mackey functors. Given a G-Mackey functor $\underline{M}$, we can construct the Eilenberg-Mac Lane spectrum $H\underline{M}$ with properties analogous to its non-equivariant counterparts [LMM81]. We can suspend an equivariant spectrum by S^V for $V \in RO(G)$, the real representation ring of G, and define $RO(G)$-graded homology and cohomology theories [HHR16, LMSM86].

Calculation of the $RO(G)$-graded cohomology of a point for a non-trivial finite group G has been a fundamental question for equivariant stable homotopy theory. The recent development of the slice spectral sequence [HHR16], requires such understanding as ingredients. One of the purposes of the thesis is to present some calculations for nonabelian compact Lie groups.

Another focus of this thesis will be on the homotopical equivariant complex cobordism spectrum MU_G. It is a complex stable theory, so the homotopy groups are $\mathbb{Z}$-graded. We will give a calculation of the coefficient ring π_*MU_G for G is the symmetric group on three elements.

In [Qui69], Quillen proved that the complex cobordism ring MU_* is isomorphic to the Lazard ring, the universal ring for 1-dimensional commutative associative formal group laws, and that the formal group law associated to the complex orientation of MU is the universal one. Let G be an abelian compact Lie group. An $RO(G)$-graded multiplicative equivariant generalized cohomology theory is called complex-oriented if it has a Thom isomorphism with respect to every G-equivariant complex bundle. Cole, Greenlees and Kriz [CGK00] defined the notion of equivariant formal group laws for abelian compact Lie groups. Suppose that E_G is a complex-oriented cohomology theory, then it gives rise to an G-equivariant formal group law. There exists a universal G-equivariant formal group law defined over the G-equivariant Lazard ring L_G, which leads to a map

$$L_G \to (MU_G)_*.$$

Greenlees [Gre01] showed that the map is surjective with Euler-torsion and infinitely Euler-divisible kernel for any finite abelian group, and conjectured that it is an isomorphism for all abelian compact Lie groups. The conjecture was recently proved for $G = \mathbb{Z}/2$ by Hanke and Wiemeler [HW18] based on explicit calculations of Strickland [Str01], and then fully proved by Hausmann [Hau22] using global homotopy theory.

We shall give an alternative method for proving the equivariant analogue of Quillen's result when the group G is a finite cyclic group.

1.2 Organization of this thesis

We compute the $RO(G)$-graded coefficients of ordinary equivariant cohomology with constant coefficients for G is the dihedral group of order $2p$ for a prime $p > 2$ and G is the quaternion group Q_8 in Chapter 2. The main theorems are Theorem 10 and Theorem 19.

We compute the coefficient ring for the equivariant homotopical complex cobordism MU_G for $G = \Sigma_3$, the symmetric group on three elements, in Chapter 3. The main result is given in Theorem 27. We will also investigate the theory of equivariant formal groups laws, and give a proof of the equivariant Quillen Theorem for finite cyclic groups. This result is Theorem 29.

In the appendix we will present fundamentals of equivariant stable homotopy theory, which are the foundations of this thesis.

CHAPTER II

The $RO(G)$-graded Stable Homotopy Groups of Equivariant Eilenberg-Mac Lane Cohomology

In this chapter, we study the $RO(G)$-graded coefficients of equivariant Eilenberg-Mac Lane cohomology. Given a G-Mackey functor $\underline{M}$, we can construct the Eilenberg-Mac Lane spectrum $H\underline{M}$ with properties analogous to its non-equivariant counterparts [LMM81]. We can suspend an equivariant spectrum by S^V for $V \in RO(G)$, the real representation ring of G, and define $RO(G)$-graded homology and cohomology theories [HHR16, LMSM86].

We will first review the theory of Mackey functors, which will play the role of abelian groups as "coefficients" in the world of equivariant stable homotopy theory. Next we will go over some fundamental techniques of such calculations and review the computations for $G = \mathbb{Z}/p$. Finally, we will fully compute the $RO(G)$-graded coefficients for $H\underline{\mathbb{Z}}$ in the cases when $G = D_{2p}$ and $G = Q_8$.

No results for a non-abelian group is known until Kriz and the author's recent work [KL20, Lu22]. In this thesis, we will recall the computations for G a dihedral group of order $2p$, and will compute the $RO(G)$-graded coefficients when $G = Q_8$. There are multiple ways to compute, and the main tools we use ares the G-equivariant cellular structures on representation spheres and the method of isotropy separation. We will comment on how to induct structures for complex groups from known structures of known cases.

2.1 Structure of Mackey functors

In the ordinary cohomology, the coefficient is taken to be an abelian group. Mackey functors are abelian group-valued presheaves on the Burnside category, and they will be the coefficients for $RO(G)$-graded cohomology. There are several equivalent definitions of Mackey functors, we will choose to work with one particularly handy definition for finite groups.

Let G be a finite group. Let GSet be the category of finite left G-sets and equivariant maps between them. A Mackey functor $\underline{M}$ consists of two functors M_* and M^*, where M_* is a covariant functor from the category of G-sets to Ab, and M^* is contravariant $G\text{Set} \to \text{Ab}$. They need to satisfy the following conditions:

(1). They agree on objects, so we could write $\underline{M}(S)$ for a G-set S.

(2). $\underline{M}(A \coprod B) \cong \underline{M}(A) \oplus \underline{M}(B)$.

(3). Given a pullback

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\beta"] & B \arrow[d, "\gamma"] \\ D \arrow[r, "\delta"] & C \end{tikzcd}$$

we have $\underline{M}_*(\beta)\underline{M}^*(\alpha) = \underline{M}^*(\delta)\underline{M}_*(\gamma)$.

It suffices to define a Mackey functor on orbits. With natural transformations as morphisms, the category of G-Mackey functors is an abelian category. In fact, it is also a symmetric monoidal category with box products between Mackey functors.

Suppose that H, K are subgroups of G, and $f : G/H \to G/K$ is a G-map. Then we call $M^*(f)$ a restriction map and $M_*(f)$ a transfer map.

We could use the orbit category of G to describe a G-Mackey functor visually, which is called a Lewis diagram. We will put $\underline{M}(G/G)$ on the top and $\underline{M}(G/e)$ on the bottom. Restriction maps are going downwards and transfer maps are going upwards. For example, a quick look at the subgroups of the quaternion group Q_8 gives the following framework of a Lewis diagram of a Q_8-Mackey functor $\underline{M}$:

$$\begin{tikzcd}
& \underline{M}(Q_8/Q_8) \arrow[dl, shift left] \arrow[d, shift left] \arrow[dr, shift left] & \\
\underline{M}(Q_8/\langle i \rangle) \arrow[ur, shift left] \arrow[dr, shift left] & \underline{M}(Q_8/\langle j \rangle) \arrow[u, shift left] \arrow[d, shift left] & \underline{M}(Q_8/\langle k \rangle) \arrow[ul, shift left] \arrow[dl, shift left] \\
& \underline{M}(Q_8/\langle -1 \rangle) \arrow[ul, shift left] \arrow[u, shift left] \arrow[ur, shift left] \arrow[d, shift left] & \\
& \underline{M}(Q_8/e) \arrow[u, shift left] &
\end{tikzcd}$$

Example. Constant Mackey functor $\underline{\mathbb{Z}}$. The restriction maps are identities, and the transfer maps are indices of subgroup inclusion.

$$\begin{tikzcd}
& & \mathbb{Z} \arrow[dll, bend left, "2"] \arrow[d, bend left, "2"] \arrow[drr, bend left, "2"] & & \\
\mathbb{Z} \arrow[urr, bend left, "1"] \arrow[drr, bend left, "2"] & & \mathbb{Z} \arrow[u, bend left, "1"] \arrow[d, bend left, "2"] & & \mathbb{Z} \arrow[ull, bend left, "1"] \arrow[dll, bend left, "2"] \\
& & \mathbb{Z} \arrow[ull, bend left, "1"] \arrow[u, bend left, "1"] \arrow[urr, bend left, "1"] \arrow[d, bend left, "2"] & & \\
& & \mathbb{Z} \arrow[u, bend left, "1"] & &
\end{tikzcd}$$

Example. Fixed point Mackey functor. Given a left $\mathbb{Z}[G]$ module M, the fixed point Mackey functor $\underline{M}$ is defined by $\underline{M}(G/H) = M^H$, restriction given by inclusion of fixed point, and transfer given by summing over cosets. For example, the fixed point Mackey functor $\underline{\mathbb{Z}[Q_8/\langle -1 \rangle]}$ is given by

$$\begin{tikzcd}
& & \mathbb{Z} \arrow[dll, bend left, "{[1,1]}"] \arrow[d, bend left, "{[1,1]}"] \arrow[drr, bend left, "{[1,1]}"] & & \\
\mathbb{Z} \oplus \mathbb{Z} \arrow[urr, bend left, "{(2,2)}"] \arrow[drr, bend left, "A_i"] & & \mathbb{Z} \oplus \mathbb{Z} \arrow[u, bend left, "{(2,2)}"] \arrow[d, bend left, "A_j"] & & \mathbb{Z} \oplus \mathbb{Z} \arrow[ull, bend left, "{(2,2)}"] \arrow[dll, bend left, "A_k"] \\
& & \mathbb{Z}[Q_8/\langle -1 \rangle] \arrow[ull, bend left, "B_i"] \arrow[u, bend left, "B_j"] \arrow[urr, bend left, "B_k"] \arrow[d, bend left, "1"] & & \\
& & \mathbb{Z}[Q_8/\langle -1 \rangle] \arrow[u, bend left, "2"] & &
\end{tikzcd}$$

Here round brackets stand for row vectors while square brackets stand for column vectors. The matrices are given by choosing the order $\{1, i, j, k\}$ on the basis, and they are

$$A_i = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, A_j = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, A_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$B_i = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, B_j = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, B_k = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Similar calculations give the explicit structure for other fixed point Mackey functors of other orbits of Q_8.

In general, if X is a finite G-CW complex, the quotient X^n/X^{n-1} is a wedge of spheres

$$X^n/X^{n-1} = X_{n+} \wedge S^n$$

where X_n is a discrete left G-set. We have a chain complex of Mackey functors $\underline{C}_*(X;M)$ given by

$$\underline{C}_n(X;\underline{M}) = \underline{\pi}_0(H\underline{M} \wedge X_{n+}).$$

It is also true that for any finite left G-set S,

$$\underline{C}_n(X;\underline{M})(S) = \underline{M}(S \times X_n).$$

The category of G-Mackey functors has a symmetric monoidal product called the box product denoted by $\square$. Given any finite group G we can define the box product of G-Mackey functors in terms of a double coend or a left Kan extension. The unit for the box product is the Burnside Mackey functor, and thus the category of G-Mackey functors forms a symmetric monoidal category.

2.2 The case $G = \mathbb{Z}/2$

In this section we will go over the computation of $\pi_\star H\underline{\mathbb{Z}}$ in the unpublished work of Stong. We will give two parallel methods of calculations. One by using cellular method, one by using isotropy separation.

We denote the nontrivial element of $\mathbb{Z}/2$ by σ and denote the nontrivial irreducible representation by α.

Cellular approach:

We may proceed by finding a cell structure on $S^{n\alpha}$. There is a cofiber sequence

$$\mathbb{Z}/2_+ \to S^0 \to S^\alpha.$$

Fix $n < 0$. By dimension axiom, $H_*^{\mathbb{Z}/2}(S^0;\underline{\mathbb{Z}}) = \mathbb{Z}$. By induction, $S^{(n+1)\alpha}$ is obtained from $S^{n\alpha}$ by adding a cell. We need only to determine the map

$$H_n^{\mathbb{Z}/2}(S^n \wedge \mathbb{Z}/2_+;\underline{\mathbb{Z}}) \to H_n^{\mathbb{Z}/2}(S^{n\alpha};\underline{\mathbb{Z}}).$$

For $n = 0$ it is the transfer map in the Mackey functor $\underline{\mathbb{Z}}$, hence multiplication by 2. For $n > 0$, note that

$$H_n^{\mathbb{Z}/2}(S^{n\alpha};\underline{\mathbb{Z}}) \to H_n^{\mathbb{Z}/2}(S^{n\alpha}/S^{(n-1)\alpha};\underline{\mathbb{Z}})$$

is an isomorphism. The composite

$$S^n \wedge \mathbb{Z}/2_+ \to S^{n\alpha} \to S^{n\alpha}/S^{(n-1)\alpha} \cong S^n \wedge \mathbb{Z}/2_+$$

is the suspension of the $n = 0$ case, hence it is also multiplication by 2. The calculations for $n \geq 0$ is similar.

Tate diagram approach:

Consider the Tate diagram for $H\underline{\mathbb{Z}}$:

$$\begin{tikzcd} H\underline{\mathbb{Z}}_h \arrow[r] \arrow[d] & H\underline{\mathbb{Z}} \arrow[r] \arrow[d] & \Phi^{\mathbb{Z}/2} H\underline{\mathbb{Z}} \arrow[d] \\ H\underline{\mathbb{Z}}_h \arrow[r] & H\underline{\mathbb{Z}}^h \arrow[r] & \widehat{H\underline{\mathbb{Z}}} \end{tikzcd}$$

Lemma 1. *The Borel cohomology is*

$$H\underline{\mathbb{Z}}^h_\star = \mathbb{Z}[a][u^{\pm 1}]/(2a)$$

where $|a| = -\alpha$, $|u| = 2 - 2\alpha$.

Proof. We may choose a filtration of $E\mathbb{Z}/2_+$ as

$$S(\alpha)_+ \subset S(2\alpha)_+ \subset ... \subset S(\infty\alpha)_+ = E\mathbb{Z}/2_+$$

There is one free $\mathbb{Z}/2$-cell in each degree and this gives the standard 2-periodic resolution

$$...\mathbb{Z}G \xrightarrow{1+\sigma} \mathbb{Z}G \xrightarrow{1-\sigma} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

Then the Borel cohomology spectral sequence collapses at the E_2 page and gives the desired result. □

The Tate cohomology is obtained by inverting a, and we have the following calculation for the geometric fixed points [tD70]. It could also be read off from the fact that both are a-periodic.

Lemma 2. *The geometric fixed points are given by*

$$\Phi^{\mathbb{Z}/2} H\underline{\mathbb{Z}}_\star = \mathbb{Z}/2[a^{\pm 1}, u].$$

With understanding of the building blocks, the calculation of $H\underline{\mathbb{Z}}$ could be now inferred.

Both methods yield the following result of the $RO(\mathbb{Z}/2)$-graded coefficients for ordinary equivariant cohomology $H\underline{\mathbb{Z}}$. This result would also be useful for the calculations in the case for

$G = D_{2p}$.

Denote, for $\ell \geq 0$,

$$B_\ell = \widetilde{H}_*^{D_{2p}}(S^{\ell\alpha}, \underline{\mathbb{Z}}) = \widetilde{H}_*^{\mathbb{Z}/2}(S^{\ell\alpha}, \underline{\mathbb{Z}}), \tag{II.1}$$

$$B^\ell = \widetilde{H}^*_{D_{2p}}(S^{\ell\alpha}, \underline{\mathbb{Z}}) = \widetilde{H}^*_{\mathbb{Z}/2}(S^{\ell\alpha}, \underline{\mathbb{Z}}). \tag{II.2}$$

Proposition 3. *Let n denote the grading. We have*

$$B_{\ell,n} = \begin{cases} \mathbb{Z} & n = \ell \textit{ even} \\ \mathbb{Z}/2 & 0 \leq n < \ell \textit{ even} \\ 0 & \textit{else,} \end{cases}$$

$$B^{\ell,n} = \begin{cases} \mathbb{Z} & n = \ell \textit{ even} \\ \mathbb{Z}/2 & 3 \leq n \leq \ell \textit{ odd} \\ 0 & \textit{else.} \end{cases}$$

□

2.3 The case $G = D_{2p}$

We present $G = D_{2p}$ as

$$\{\zeta, \tau \,|\, \zeta^p = 1, \tau^2 = 1, \zeta\tau = \tau\zeta^{-1}\}.$$

The group G has two one-dimensional representations: the trivial representation denoted by ϵ and the sign representation denoted by α. The group G also admits $(p-1)/2$ two-dimensional representations, denoted by γ_i's, given by

$$\gamma_i : \zeta \mapsto \begin{bmatrix} \cos(\frac{2\pi i}{p}) & -\sin(\frac{2\pi i}{p}) \\ \sin(\frac{2\pi i}{p}) & \cos(\frac{2\pi i}{p}) \end{bmatrix}, \tau \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, 1 \leq i \leq \frac{p-1}{2}.$$

Let $S(m\gamma_i)$ be the unit sphere of the representation $m\gamma_i$. We will subdivide the standard $\mathbb{Z}/p$-equivariant cells of $S(m\gamma_i)$ to obtain a D_{2p}-equivariant CW stucture on $S(m\gamma_i)$.

To do this, first identify the non-equivariant underlying spaces of $S(m\gamma_i)$ with the unit disk in $\mathbb{C}^m$. Then $\zeta \in G$ simply acts by coordinate-wise ζ_p^i multiplication where $\zeta_p = e^{2\pi i/p}$.

First observe that the free $\mathbb{Z}/p$-equivariant CW-structure on $S(m\gamma_i)$ has equivariant cells freely generated by the following non-equivariant cells for $1 \leq k \leq m$:

$$\{(z_1, ..., z_k, 0, ..., 0) \in S(m\gamma_i) \mid z_k \in [0, 1]\}, \tag{II.3}$$

$$\{(z_1,...,z_k,0,...,0)\in S(m\gamma_i)\mid z_k\in[0,1]\cdot e^{\lambda i},\ (p-1)\pi/p\le\lambda\le(p+1)\pi/p\}. \tag{II.4}$$

Both (II.3) and (II.4) are stable under the action of τ. However, they are not D_{2p}-cells since τ acts non-trivially on them. However, noticing that they can be identified with unit disks of the representations

$$(k-1)\alpha+(k-1)\epsilon,\ k\alpha+(k-1)\epsilon, \tag{II.5}$$

respectively, we could combine with the $\mathbb{Z}/2$-equivariant structures described in Section 2.2 to obtain D_{2p}-equivariant cells. To be precise, we consider the following cells for $S(m\gamma_i)$:

Type A.

$$a_{k,\ell},\ 0\le\ell\le k-1, 1\le k\le m,$$

generated by

$$\{(z_1,...,z_k,0,...,0)\in S(m\gamma_i)\,|\,\mathrm{Im}(z_\ell)\ge 0, z_{\ell+1},...,z_{k-1}\in[-1,1], z_k\in[0,1]\}.$$

The cell $a_{k,\ell}$ has dimension $k+\ell-1$ and has isotropy $\mathbb{Z}/2$ for $\ell=0$ and $\{e\}$ for $\ell>0$.

Type B.

$$b_{k,\ell},\ 0\le\ell\le k-1, 1\le k\le m,$$

generated by

$$\{(z_1,...,z_k,0,...,0)\in S(m\gamma_i)\,|\,\mathrm{Im}(z_\ell)\ge 0, z_{\ell+1},...,z_{k-1}\in[-1,1], z_k\in[-1,0]\}.$$

Since it is symmetric to $a_{k,\ell}$, the cell $b_{k,\ell}$ has dimension $k+\ell-1$ and has isotropy $\mathbb{Z}/2$ for $\ell=0$ and $\{e\}$ for $\ell>0$.

Type C.

$$c_k,\ 1\le k\le m,$$

generated by

$$\{(z_1,...,z_k,0,...,0)\in S(m\gamma_i)\,|\,z_k\in[0,1]\cdot e^{i\lambda}, 0\le\lambda\le\pi/p\}.$$

The cell c_k has dimension $2k-1$ and has isotropy $\{e\}$.

The spaces given by the generators are homeomorphic to (closed) disks. The attaching maps from the boundary of n-cells are equivariant and are mapping to lower dimensional cells. Finally the open cells form a partition of $S(m\gamma_i)$, so it is a regular G-CW complex. The topology is quotient topology and agrees with the induced topology on $S(m\gamma_i)$. In other words, these cells give a D_{2p}-equivariant CW decomposition for each $S(m\gamma_i)$, only with different D_{2p}-actions for different $S(m\gamma_i)$'s.

Based on the equivariant CW-structure, we are ready to write down the differentials. As being unit sphere in the representations, the CW structure is regular, i.e., the attaching maps are embeddings, hence the incidence coefficients are either $+1$ or -1. We orient all cells as submanifolds (with boundaries) of the complex vector space $\mathbb{C}^m$. The induced orientation of the boundary of a cell is chosen by the following rule: the induced orientation followed by the outward normal direction together make up the standard orientation of $\mathbb{C}^m$. For example, the induced orientation of $S^1 \subset \mathbb{C}$ is going clockwise, hence the incidence number between $a_{2,1}$ and c_1 is -1.

Lemma 4. *Given $1 \leq i \leq (p-1)/2$, let $1 \leq j \leq p-1$ be the multiplicative inverse of i. Let $\zeta_i = \zeta^j$. With respect to the CW-structure and orientations described above, the D_{2p}-equivariant cell chain complex of $S(m\gamma_i)$ in the sense of Bredon [Bre67] has differential*

$$da_{1,0} = 0$$
$$db_{1,0} = 0$$
$$dc_1 = \zeta_i^{\frac{p+1}{2}} b_{1,0} - a_{1,0}$$
$$da_{2,1} = -a_{2,0} - (1 + \zeta_i + ... + \zeta_i^{\frac{p-1}{2}})c_1 + (\zeta_i + ... + \zeta_i^{\frac{p-1}{2}})\tau c_1$$
$$db_{2,1} = -b_{2,0} - (1 + \zeta_i + ... + \zeta_i^{\frac{p-1}{2}})c_1 + (\zeta_i + ... + \zeta_i^{\frac{p-1}{2}})\tau c_1$$
$$da_{k,0} = a_{k-1,0} - b_{k-1,0} \qquad k > 1$$
$$db_{k,0} = a_{k-1,0} - b_{k-1,0} \qquad k > 1$$
$$da_{k,1} = a_{k-1,1} - b_{k-1,1} + (-1)^{k-1}a_{k,0} \qquad k > 2$$
$$db_{k,1} = a_{k-1,1} - b_{k-1,1} + (-1)^{k-1}b_{k,0} \qquad k > 2$$

For $k > 3$, $1 < \ell < k-1$,

$$da_{k,\ell} = a_{k-1,\ell} - b_{k-1,\ell} + (-1)^{k-\ell}a_{k,\ell-1} + (-1)^{k-1}\tau a_{k,\ell-1}$$
$$db_{k,\ell} = a_{k-1,\ell} - b_{k-1,\ell} + (-1)^{k-\ell}b_{k,\ell-1} + (-1)^{k-1}\tau b_{k,\ell-1}$$

For $k > 2$, by abbreviating the action of $\sum_{j=1}^{(p-1)/2} \zeta_i^j$ to σ,

$$da_{k,k-1} = -a_{k,k-2} + (-1)^{k-1}\tau a_{k,k-2} - (1+\sigma)c_{k-1} + (-1)^{k-2}\sigma\tau c_{k-1}$$
$$db_{k,k-1} = -b_{k,k-2} + (-1)^{k-1}\tau b_{k,k-2} - (1+\sigma)c_{k-1} + (-1)^{k-2}\sigma\tau c_{k-1}$$

Finally, for $k > 1$,

$$dc_k = -a_{k,k-1} + (-1)^k\tau a_{k,k-1} + \zeta_i^{\frac{p+1}{2}} b_{k,k-1} + (-1)^{k-1}\zeta_i^{\frac{p+1}{2}}\tau b_{k,k-1}.$$

Proof. We present here a computation for the differential of $a_{k,k-1}$ for $k > 2$. By equivariance, it suffices to work on the generator, which is given by

$$\{(z_1, ..., z_k, 0, ..., 0) \in S(m\gamma_i) \,|\, \mathrm{Im}(z_{k-1}) \geq 0, z_k \in [0,1]\}.$$

Note that z_k is uniquely determined by the values of $z_1, ..., z_{k-1}$, and the dimension of the cell is $2k-2$. Hence we only need to consider cells of dimension $2k-3$ to which $a_{k,k-1}$ attaches. They are precisely those cells with z_{k-1} coordinates lying on the boundary of $a_{k,k-1}$, namely,

$$a_{k,k-2}, \tau a_{k,k-2},\ c_{k-1}, \zeta_i c_{k-1}, ..., \zeta_i^{(p-1)/2} c_{k-1}, \zeta_i \tau c_{k-1}, ..., \zeta_i^{(p-1)/2} \tau c_{k-1}.$$

Here cells in the orbit of c_{k-1} are those with $\mathrm{Im}(z_{k-1}) \geq 0$.

It remains to determine the incidence numbers between $a_{k,k-1}$ and these cells. By the rule set above, we could use the basis

$$(e_1, ie_1, e_2, ie_2, ..., e_{k-1}, ie_{k-1}) \tag{II.6}$$

to determine the orientation of $a_{k,k-1}$, and the orientation of $\tau a_{k,k-2}$ could be described by

$$(e_1, -ie_1, e_2, -ie_2, ..., e_{k-2}, -ie_{k-2}, e_{k-1}). \tag{II.7}$$

On a point of $\tau a_{k,k-2}$ that $a_{k,k-1}$ attaches, the induced orientation is given by

$$(e_1, ie_1, ..., e_{k-2}, ie_{k-2}, -e_{k-1}) \tag{II.8}$$

since juxtaposing with outward normal direction $-ie_{k-1}$ gives the same orientation as (II.6). It is straightforward to compare orientations (II.7) and (II.8) and this gives the sign

$$da_{k,k-1} = ... + (-1)^{k-1} \tau a_{k,k-2} + ...$$

in the formula. All the other computations follow by direct inspection in a similar way. □

Since $S^{m\gamma_i}$ is the unreduced suspension of $S(m\gamma_i)$, the D_{2p}-equivariant CW structure of $S^{m\gamma_i}$ is easily derived.

We will next prove (Proposition 7 below) that the choice of two-dimensional representation γ_i doesn't matter in the computation of ordinary equivariant cohomology. Hence the cohomology could be indexed by $k\epsilon + \ell\alpha + m\gamma$.

Let $\underline{A}$ denote the Burnside ring Green functor. We will compute the D_{2p}-Mackey functor-valued chain complex $\underline{C}_*(S^{\gamma_i}; \underline{M})$ for constant coefficient $\underline{\mathbb{Z}}$ and Burnside coefficient $\underline{A}$, we start with describing some D_{2p}-Mackey functors. Despite the fact that the group D_{2p} is non-abelian, its conjugacy relations among subgroups are simple and we can depict a D_{2p}-Mackey functor $\underline{M}$ by a Lewis diagram of the following form:

$$
\begin{tikzcd}
 & \underline{M}(D_{2p}/e) \arrow[dl, bend right=10] \arrow[dr, bend left=10] & \\
\underline{M}(D_{2p}/\langle\tau\rangle) \arrow[ur, bend left=10] \arrow[dr, bend left=10] & & \underline{M}(D_{2p}/\langle\zeta\rangle) \arrow[ul, bend right=10] \arrow[dl, bend left=10] \\
 & \underline{M}(D_{2p}/D_{2p}) \arrow[ul, bend left=10] \arrow[ur, bend right=10] &
\end{tikzcd}
$$

Example 5. *Constant Mackey functor $\underline{\mathbb{Z}}$.*

$$
\begin{tikzcd}
 & \mathbb{Z} \arrow[dl, "2"] \arrow[dr, bend left, "p"] & \\
\mathbb{Z} \arrow[ur, bend left, "1"] \arrow[dr, "p"] & & \mathbb{Z} \arrow[ul, "1"] \arrow[dl, bend left, "2"] \\
 & \mathbb{Z} \arrow[ul, bend left, "1"] \arrow[ur, "1"] &
\end{tikzcd}
$$

Example 6. *Given a $\mathbb{Z}[G]$-module M, we have fixed-point Mackey functor $\underline{M}$ defined by $\underline{M}(G/H) = M^H$, restriction given by inclusion, and transfer given by summing over cosets. For example the fixed point Mackey functor $\underline{\mathbb{Z}[D_{2p}/\langle\tau\rangle]}$ is given by*

$$
\begin{tikzcd}
 & \mathbb{Z}[D_{2p}/\langle\tau\rangle] \arrow[dl, "B"] \arrow[dr, bend left, "{(1,1,\ldots,1)}"] & \\
\frac{p-1}{2}\mathbb{Z} \oplus \mathbb{Z} \arrow[ur, bend left, "A"] \arrow[dr, "{(2,\ldots,2,1)}"] & & \mathbb{Z} \arrow[ul, "{[1,1,\ldots,1]}"] \arrow[dl, bend left, "2"] \\
 & \mathbb{Z} \arrow[ul, bend left, "{[1,1,\ldots,1]}"] \arrow[ur, "1"] &
\end{tikzcd}
$$

Here round brackets stand for row vectors while square brackets stand for column vectors, and

$$
A = \begin{bmatrix} 0 & I_1 \\ I_{\frac{p-1}{2}} & 0 \\ J_{\frac{p-1}{2}} & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & I_{\frac{p-1}{2}} & J_{\frac{p-1}{2}} \\ 2 & 0 & 0 \end{bmatrix}.
$$

where I_n is the $n \times n$ identity matrix and J_n is the $n \times n$ minor diagonal identity matrix.

Similarly, the fixed point Mackey functor $\underline{\mathbb{Z}[D_{2p}/e]}$ is given by the following diagram.

$$\begin{tikzcd}
 & \mathbb{Z}[D_{2p}/e] \arrow[dl, bend left, "{(I_p,I_p)}"] \arrow[dr, bend left, "D"] & \\
p\mathbb{Z} \arrow[ur, bend left, "{[I_p,I_p]}"] \arrow[dr, bend left, "{(1,1,\dots,1)}"] & & \mathbb{Z}[D_{2p}/\langle\zeta\rangle] \arrow[ul, bend left, "C"] \arrow[dl, bend left, "{(1,1)}"] \\
 & \mathbb{Z} \arrow[ul, bend left, "{[1,1,\dots,1]}"] \arrow[ur, bend left, "{[1,1]}"] &
\end{tikzcd}$$

where the matrices are represented by

$$C = ([1, ..., 1, 0, ..., 0], [0, ..., 0, 1, ..., 1]),$$

$$D = [(1, ..., 1, 0, ..., 0), (0, ..., 0, 1, ..., 1)].$$

The matrices above are derived by arranging the order of cells carefully: The basis of $\mathbb{Z}[D_{2p}/\langle\tau\rangle]$ can be identified with cells generated by $a_{1,0}$. Recalling that ζ_i acts by $2\pi/p$-rotation, we put a geometric counterclockwise order on the cells

$$a_{1,0}, \zeta_i a_{1,0}, ..., \zeta_i^{p-1} a_{1,0}.$$

We also put an order on the generators of $\mathbb{Z}[D_{2p}/\langle\tau\rangle]^{\langle\tau\rangle}$ by

$$\zeta_i a_{1,0} + \zeta_i^{p-1} a_{1,0}, ..., \zeta_i^{\frac{p-1}{2}} a_{1,0} + \zeta_i^{\frac{p+1}{2}} a_{1,0}, a_{1,0},$$

and this is why the upper left pair of arrows in the diagram for $\underline{\mathbb{Z}[D_{2p}/\langle\tau\rangle]}$ has the given matrix representation.

The basis of $\mathbb{Z}[D_{2p}/e]$ can be identified with cells generated by c_1. We arrange them in the following order:

$$c_1, \zeta_i c_1, ..., \zeta_i^{p-1} c_1, \tau c_1, \tau\zeta_i c_1, ..., \tau\zeta_i^{p-1} c_1.$$

The fixed point submodules are endowed with the induced order of basis.

Now fix $\underline{M} = \underline{\mathbb{Z}}$. In this case, by the double coset formula, the associated chain complex of Mackey functors can be calculated as fixed point Mackey functos. Hence using the examples above, the Mackey functor-valued D_{2p}-equivariant chain complexes for S^{γ_i} is the following:

$$\underline{\mathbb{Z}} \longleftarrow \underline{\mathbb{Z}[D_{2p}/\langle\tau\rangle]} \oplus \underline{\mathbb{Z}[D_{2p}/\langle\tau\rangle]} \longleftarrow \underline{\mathbb{Z}[D_{2p}/e]}.$$

The differentials are derived from Lemma 4. Since the differentials are D_{2p}-equivariant, we immediately see that all chain complexes for the different S^{γ_i}'s are isomorphic.

However, the isomorphism is not induced by any D_{2p}-equivariant map between the representation spheres. To prove Proposition 7 we instead want to construct a functor $\mathscr{H} : Ch_{\geq 0}(Mack) \to D\mathscr{S}p_G$ such that

(1). $\mathscr{H}\underline{M} = H\underline{M}$.

(2). $\mathscr{H}\underline{C}_*(X;\underline{M}) \simeq X \wedge H\underline{M}$.

Construction:

Let $\mathscr{H}$ be the composition of the following functors

$$Ch_{\geq 0}(Mack) \xrightarrow{K} sMack \xrightarrow{H} sD\mathscr{S}p_G \xrightarrow{|\cdot|} D\mathscr{S}p_G$$

where K is the functor in Dold-Puppe correspondence which is an equivalence of first two categories; H is the Eilenberg-Mac Lane functor and $|\cdot|$ is geometric realization functor. The Eilenberg-Mac Lane construction is functorial; a recent account of this is in [BO15].

As an example we compute the case when $X = G/H_+$. Then $\underline{C}_*(X;\underline{M})$ is concentrated on degree 0. All the functors are computable, and we have

$$\mathscr{H}\underline{C}_*(X;\underline{M}) = H\underline{M}_{G/H} \simeq H\underline{M} \wedge G/H_+.$$

The last equivalence can be verified by computing the homotopy groups of $H\underline{M} \wedge G/H_+$, and using the uniqueness of Eilenberg-Mac Lane spectra.

In fact, one could make it into an natural isomorphism. By the projection formula

$$G \ltimes_H H\underline{M} \cong G/H_+ \wedge H\underline{M}$$

and adjunction, it arises from the natural map of H-spectrum $H\underline{M} \to H\underline{M}_{G/H}$ induced by inclusion at the coset eH:

$$\underline{M} \hookrightarrow \underline{M}_{G/H}.$$

For any finite G-CW complex X, we realize it as a simplicial G-set and the functor $\mathscr{H}$ is constructed as above. Then Proposition 7 follows directly.

Hence we have proved the following

Proposition 7. *Let $\underline{M}$ be a D_{2p}-Mackey functor. The D_{2p}-stable homotopy type of the $H\underline{A}$-module spectrum $H\underline{M} \wedge S^{\gamma_i}$ does not depend on the choice of i.*

□

To proceed, we will use the isotropy separation sequence

$$S(m\gamma)_+ \to S^0 \to S^{m\gamma}.$$

Recall that there is a cellular filtration on $S(m\gamma)$ by the $\mathbb{Z}/p$-equivariant cells generated by (II.3), (II.4) of dimension less or equal to s. For $k \geq 1$, the filtration degree $2k-1$ part is generated by cells $b_{k,\ell}, c_k$ and the degree $2k-2$ part is generated by cells $a_{k,\ell}$. Using the differentials computed above, the corresponding homological spectral sequence has the following E^1-term:

$$E^1_{2k-1,*} = B_{k-1}[k-1], \text{ for } 1 \leq k \leq m$$

$$E^1_{2k,*} = B_k[k-1], \text{ for } 1 \leq k \leq m.$$

The nontrivial differential d^1 is also determined by Lemma 4, which is an isomorphism except for $E^1_{4j,0} \to E^1_{4j-1,0} : \mathbb{Z} \xrightarrow{p} \mathbb{Z}$. On the two vertical edges $s = 0, 2m$, the terms also survive and the spectral sequence collapses to the E^2 page. In the case of cohomology, one just needs to turn subscripts into superscripts, mirror the computations by reversing arrows and use restriction maps of Mackey functors. Thus, we have proved the following

For $m > 0$, we have

$$H_*^{D_{2p}}(S(m\gamma), \underline{\mathbb{Z}}) = \mathbb{Z} \oplus {}_0A_m \oplus B_m[m-1],$$

$$H^*_{D_{2p}}(S(m\gamma), \underline{\mathbb{Z}}) = \mathbb{Z} \oplus {}^0A^m \oplus B^m[m-1].$$

□

Now we need to suspend with representation spheres $S^{\ell\alpha}$. It is worth noting that our D_{2p}-CW structure is designed in a way such that the subsequent quotients of $S^{\ell\alpha}$ suspension have a nice form, given by

$$F_{2k+1}/F_{2k} \cong D_{2p} \ltimes_{\mathbb{Z}/2} S^{k+(k+1)\alpha}, \quad F_{2k+2}/F_{2k+1} \cong D_{2p} \ltimes_{\mathbb{Z}/2} S^{(k+1)+(k+1)\alpha}.$$

The corresponding spectral sequence has differential d^1 given by $S^{\ell\alpha}$-suspension of the connecting map $F_{2k+2}/F_{2k+1} \to \Sigma F_{2k+1}/F_{2k}$ of the triad

$$(F_{2k+2}, F_{2k+1}, F_{2k})$$

which stably does not depend on ℓ.

To determine this map, note that it is a stable D_{2p}-equivariant map

$$D_{2p}/(\mathbb{Z}/2)_+ \to D_{2p}/(\mathbb{Z}/2)_+$$

By adjunction, it is equivalent to a $\mathbb{Z}/2$-equivariant stable map

$$S^0 \to D_{2p}/(\mathbb{Z}/2)_+$$

which is classified by an element in

$$A(\mathbb{Z}/2) \oplus \mathbb{Z}^{\oplus(p-1)/2} \tag{II.9}$$

by the Wirthmüller isomorphism (see Appendix A.2). Now we see that that $\mathbb{Z}/2$-equivariantly, the orbit $D_{2p}/(\mathbb{Z}/2)$ is the wedge sum of one fixed point G/G_+ and $(p-1)/2$ free orbits G/e_+. Now the connecting map could be read off from the attaching maps from $a_{k+1,k}$ to c_k, namely from

$$da_{k,k-1} = -a_{k,k-2} + (-1)^{k-1}\tau a_{k,k-2} - (1+\sigma)c_{k-1} + (-1)^{k-2}\sigma\tau c_{k-1}.$$

This shows that the connecting map does not depend on k, and is in (II.9) represented by the element

$$(1, 1, \ldots, 1).$$

In the case of constant Mackey functor $\underline{\mathbb{Z}}$, it corresponds to multiplication by p.

Therefore the spectral sequence of $\Sigma^{\ell+\ell\alpha}S(m\gamma)_+$, whose E^1 page is a shift of the conjunction of both cohomology and homology E^1 page for $S(m\gamma)$, and it also collapses at the E^2-page.

Define ${}_sA_t$ and ${}^sA^t$ to be

$$({}_sA_t)_n = \begin{cases} \mathbb{Z}/p & \text{when } 2s < n < 2t-1, n \equiv 3 \mod 4, \\ 0 & \text{else,} \end{cases} \tag{II.10}$$

$$({}^sA^t)^n = \begin{cases} \mathbb{Z}/p & \text{when } 2s < n < 2t-1, n \equiv 0 \mod 4, \\ 0 & \text{else.} \end{cases} \tag{II.11}$$

We obtain the following result:

Proposition 8. *For $m > 0$, we have*

$$H_*^{D_{2p}}(\Sigma^{\ell\alpha}S(m\gamma)_+, \underline{\mathbb{Z}}) = B_\ell \oplus B_{\ell+m}[m-1] \oplus {}_\ell A_{\ell+m}[-\ell],$$

$$H^*_{D_{2p}}(\Sigma^{\ell\alpha}S(m\gamma)_+, \underline{\mathbb{Z}}) = B^\ell \oplus B^{\ell+m}[m-1] \oplus {}^\ell A^{\ell+m}[-\ell].$$

□

Example 9. As an example, we illustrate how to compute

$$H_*^{D_{2p}}(\Sigma^{-4\alpha} S(5\gamma)_+, \underline{\mathbb{Z}}).$$

First we compute the D_{2p}-equivariant homology and cohomology of $S(5\gamma)$. The following is the E^1 page of the homological spectral sequence for $H_*^{D_{2p}}(S(5\gamma), \underline{\mathbb{Z}})$.

$$\begin{array}{r|cccccccccc}
0 & \mathbb{Z} & 0 & 0 & \mathbb{Z} \xleftarrow{p} & \mathbb{Z} & 0 & 0 & \mathbb{Z} \xleftarrow{p} & \mathbb{Z} & 0 \\
-1 & & \mathbb{Z}/2 \leftarrow & \mathbb{Z}/2 & & & \mathbb{Z}/2 \leftarrow & \mathbb{Z}/2 & & & \mathbb{Z}/2 \\
-2 & & & & \mathbb{Z}/2 \leftarrow & \mathbb{Z}/2 & & & \mathbb{Z}/2 \leftarrow & \mathbb{Z}/2 & \\
-3 & & & & & & \mathbb{Z}/2 \leftarrow & \mathbb{Z}/2 & & & \mathbb{Z}/2 \\
-4 & & & & & & & & \mathbb{Z}/2 \leftarrow & \mathbb{Z}/2 & \\
-5 & & & & & & & & & & \mathbb{Z}/2 \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9
\end{array}$$

E^1 page for $H_*^{D_{2p}}(S(5\gamma), \underline{\mathbb{Z}})$

The differential d^1 is a multiplication by p when there is a $\mathbb{Z}$ in the target (which is supported by c_k, k even). The exception is filtration degree $2m - 1 = 9$, where there is no differential with that target, and filtration degree 0, where there is no differential with that source. There is no room for higher differentials for dimensional reasons. Hence the spectral sequence collapses to the E^2 page. The two vertical edges and the $t = 0$ line give the three summands in Proposition 2.3.

The following is the E_1 page of the cohomological spectral sequence.

0	$\mathbb{Z}$	0	0	$\mathbb{Z} \xrightarrow{p}$	$\mathbb{Z}$	$\mathbb{Z}/2 \to$	$\mathbb{Z}/2$	$\mathbb{Z} \xrightarrow{p}$	$\mathbb{Z}$	$\mathbb{Z}/2$
-1								$\mathbb{Z}/2 \to$	$\mathbb{Z}/2$	
-2										$\mathbb{Z}/2$
	0	1	2	3	4	5	6	7	8	9

E_1 page for $H^*_{D_{2p}}(S(5\gamma), \underline{\mathbb{Z}})$

Now let us suspend by $-4-4\alpha$. Since the filtration on $S(5\gamma)$ is given by

$$S^0, S^\alpha, S^{1+\alpha}, ..., S^{4+4\alpha}, S^{4+5\alpha},$$

the filtered quotients are given by

$$S^{-4-4\alpha}, S^{-4-3\alpha}, S^{-3-3\alpha}, ..., S^{-1}, S^0, S^\alpha.$$

The following is the E_1 page, which is a shift of a juxtaposition of the dual of a truncation (at filtration degree 7) of the cohomological E_1 page and a truncation (at filtration degree 1) of the homological E^1 page.

1	$\mathbb{Z}/2$									
0	$\mathbb{Z}$	$\mathbb{Z}/2 \leftarrow$	$\mathbb{Z}/2$	$\mathbb{Z} \xleftarrow{p}$	$\mathbb{Z}$	0	0	$\mathbb{Z} \xleftarrow{p}$	$\mathbb{Z}$	0
-1										$\mathbb{Z}/2$
	0	1	2	3	4	5	6	7	8	9

E^1 page for $\tilde{H}^{D_{2p}}_*(\Sigma^{-4-4\alpha} S(5\gamma)_+, \underline{\mathbb{Z}})$

1	$\mathbb{Z}/2$									
0	$\mathbb{Z}$	0	0	$\mathbb{Z}/p$	0	0	0	$\mathbb{Z}/p$	0	0
-1										$\mathbb{Z}/2$
	0	1	2	3	4	5	6	7	8	9

E^2 page for $\tilde{H}^{D_{2p}}_*(\Sigma^{-4-4\alpha} S(5\gamma)_+, \underline{\mathbb{Z}})$

The full calculational result now follows by the long exact sequence from the cofiber sequence.

Theorem 10. *For $m > 0$, we have*

$$H_*^{D_{2p}}(S^{m\gamma+\ell\alpha}, \underline{\mathbb{Z}}) = {}_{\ell-1}A_{\ell+m}[-\ell+1] \oplus B_{\ell+m}[m], \tag{II.12}$$

$$H^*_{D_{2p}}(S^{m\gamma+\ell\alpha}, \underline{\mathbb{Z}}) = {}^{\ell}A^{\ell+m}[-\ell+1] \oplus B^{\ell+m}[m]. \tag{II.13}$$

Proof. We use the cofiber sequence

$$\Sigma^{\ell\alpha} S(m\gamma)_+ \to S^{\ell\alpha} \to S^{\ell\alpha+m\gamma}$$

to finish our computation. Looking at the long exact sequence in homology, the morphism $B_\ell \to B_\ell$ is the transfer map p, which is an isomorphism except in the top dimension when ℓ is even, and this gives an extra $\mathbb{Z}/p$. Besides, all the other components are shifted up by 1. Hence we have proved that

$$H_*^{D_{2p}}(S^{\ell\alpha+m\gamma}, \underline{\mathbb{Z}}) = B_{\ell+m}[m] \oplus {}_{\ell-1}A_{\ell+m}[-\ell+1].$$

In cohomology the restriction maps always give isomorphisms, hence

$$H^*_{D_{2p}}(S^{\ell\alpha+m\gamma}, \underline{\mathbb{Z}}) = B^{\ell+m}[m] \oplus {}^{\ell}A^{\ell+m}[-\ell+1].$$

□

2.4 The case $G = Q_8$

We will write the generators of the quaternion group $G = Q_8$ with i, j satisfying relations

$$\{i, j \mid i^4, i^2 j^{-2}, ijij^{-1}\}.$$

The quaternion group G has four one-dimensional real representations, given by scalar action of generators i and j:

$$i \mapsto \pm 1, \quad j \mapsto \pm 1.$$

We will denote the trivial representation by 1 and the other three representations by α, β, γ, whose kernels are respectively $\langle i \rangle, \langle j \rangle, \langle ij \rangle$.

The group G also has a four-dimensional irreducible real representation, where G acts by left multiplication, and we will denote this representation by ρ. This representation is of quaternionic type.

Hence the representations $1, \alpha, \beta, \gamma, \rho$ form an additive basis for $RO(Q_8)$, and the grading could be denoted as $* + k\alpha + \ell\beta + m\gamma + n\rho$, where $*$ represents the $\mathbb{Z}$-grading.

In fact, we could do more on the reduction of the $RO(Q_8)$-grading: The representations α, β, γ are symmetric up to automoprhisms of Q_8, hence without loss of generality we may assume that k, ℓ have the same sign. Furthermore, by universal coefficient theorem and Spanier-Whitehead duality, if we flip all the signs in the grading, i.e., changing $*+k\alpha+\ell\beta+m\gamma+n\rho$ to $-*-k\alpha-\ell\beta-m\gamma-n\rho$, there is an isomorphism

$$H\underline{\mathbb{Z}}^{Q_8}_{*+k\alpha+\ell\beta+m\gamma+n\rho} \cong H\underline{\mathbb{Z}}^{-*-k\alpha-\ell\beta-m\gamma-n\rho}_{Q_8}$$

so we may further restrict to $k \geq \ell \geq 0$. Therefore, we reduce to the calculations of $\mathbb{Z}$-graded homology and cohomology of

$$\Sigma^{k\alpha+\ell\beta+m\gamma+n\rho} H\underline{\mathbb{Z}}$$

for $k \geq \ell \geq 0$.

The representation ρ is free, and its matrix representation is given by even permutations of signs. For example, the generator i acts by the following matrix:

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

For $n > 0$, let $S(n\rho)$ be the unit sphere in the representation $n\rho$. Observe that $S(\infty\rho)$ is a model for the universal classifying space EQ_8. The freeness, together with symmetry, gives us a guide for how to subdivide the representation spheres to obtain an equivariant CW structure.

By identifying the non-equivariant underlying space of $S(n\rho)$ as a subspace of $\mathbb{R}^{4n}$, we will index the Euclidean coordinates as

$$(x_1, y_1, z_1, w_1, ..., x_n, y_n, z_n, w_n) \in \mathbb{R}^{4n}$$

For clarity we will also use X_r as an abbreviation for (x_r, y_r, z_r, w_r), and let D be the open unit disk in $\mathbb{R}^4$. Let $1 \leq r \leq n$, consider the following cells for $S(n\rho)$:

Type A.

Cells $a_{r,1}$ generated by

$$\{(X_1, ..., X_{r-1}, x_r, 0, 0, 0, 0, ..., 0\} \in S(n\rho) \,|\, x_r \in [0, 1]\}.$$

The cell $a_{r,1}$ has dimension $4r - 4$. It is Q_8-free.

Type B.

Cells $b_{r,1}, b_{r,2}, b_{r,3}$ generated respectively by

$$\{(X_1, ..., X_{r-1}, x_r, y_r, 0, 0, 0, ..., 0\} \in S(n\rho) \,|\, x_r, y_r \in [0, 1]\},$$

$$\{(X_1, ..., X_{r-1}, x_r, 0, z_r, 0, 0, ..., 0\} \in S(n\rho) \,|\, x_r, z_r \in [0, 1]\},$$

$$\{(X_1, ..., X_{r-1}, x_r, 0, 0, w_r, 0, ..., 0\} \in S(n\rho) \,|\, x_r, w_r \in [0, 1]\}.$$

The cells $b_{r,1}, b_{r,2}, b_{r,3}$ have dimension $4r - 3$. They are Q_8-free.

Type C.

Cells $c_{r,1}, c_{r,2}, c_{r,3}, c_{r,4}$ generated respectively by

$$\{(X_1, ..., X_{r-1}, x_r, y_r, z_r, 0, 0, ..., 0\} \in S(n\rho) \,|\, x_r, y_r, z_r \in [0, 1]\},$$

$$\{(X_1, ..., X_{r-1}, x_r, y_r, 0, w_r, 0, ..., 0\} \in S(n\rho) \,|\, x_r, y_r, w_r \in [0, 1]\},$$

$$\{(X_1, ..., X_{r-1}, x_r, 0, z_r, w_r, 0, ..., 0\} \in S(n\rho) \,|\, x_r, z_r, w_r \in [0, 1]\},$$

$$\{(X_1, ..., X_{r-1}, 0, y_r, z_r, w_r, 0, ..., 0\} \in S(n\rho) \,|\, y_r, z_r, w_r \in [0, 1]\}.$$

The cells $c_{r,1}, c_{r,2}, c_{r,3}, c_{r,4}$ have dimension $4r - 2$. They are Q_8-free.

Type D.

Cells $d_{r,1}, d_{r,2}$ generated by

$$\{(X_1, ..., X_{r-1}, x_r, y_r, z_r, w_r, 0, ..., 0\} \in S(n\rho) \,|\, x_r, y_r, z_r, w_r \in [0, 1]\},$$

$$\{(X_1, ..., X_{r-1}, x_r, y_r, z_r, w_r, 0, ..., 0\} \in S(n\rho)|\, x_r, y_r, z_r, -w_r \in [0, 1]\}.$$

The cells $d_{r,1}, d_{r,2}$ have dimension $4r - 1$. They are Q_8-free.

Similarly it is straightforward to check that these cells give a Q_8-equivariant CW decomposition for $S(n\rho)$. By identifying $\mathbb{R}^{4n}$ with $\mathbb{C}^{2n}$, we use the rule as in the $G = D_{2p}$ case to determine induced orientation of the boundary: the induced orientation followed by the outward normal direction should make up together the standard orientation of $\mathbb{C}^{2n}$. With this rule we derive the following differentials:

Lemma 11. *With respect to the CW-structure and orientations described above, the Q_8-equivariant cell chain complex of $S(n\rho)$ in the sense of Bredon [Bre67] has differentials*

$da_{1,0} = 0$

For $1 < r \leq n$,

$da_{r>1,0} = (1 + i + j + ij + (-1) + (-i) + (-j) + (-ij))(d_{r,1} - d_{r,2})$

In the rest, for $1 \leq r \leq n$,

$db_{r,1} = ia_{r,1} - a_{r,1}$
$db_{r,2} = ja_{r,1} - a_{r,1}$
$db_{r,3} = (ij)a_{r,1} - a_{r,1}$
$dc_{r,1} = b_{r,1} - b_{r,2} - jb_{r,3}$
$dc_{r,2} = b_{r,1} - b_{r,3} + ib_{r,2}$
$dc_{r,3} = b_{r,2} - b_{r,3} - (ij)b_{r,1}$
$dc_{r,4} = -(j)b_{r,3} - (ij)b_{r,1} - (i)b_{r,2}$
$dd_{r,1} = c_{r,1} - c_{r,2} + c_{r,3} - c_{r,4}$
$dd_{r,2} = c_{r,1} - jc_{r,2} + (-ij)c_{r,3} - (-i)c_{r,4}.$

Proof. This is done by the same method as the previous Lemma 4. We again give an example on how to determining the incidence coefficients: we look at the cell $c_{r,1}$ for some $r \geq 1$. The generating cell of $c_{r,1}$ is given by

$$\{(X_1, ..., X_{r-1}, x_r, y_r, z_r, 0, 0, ..., 0\} \in S(n\rho) \,|\, x_r, y_r, z_r \in [0, 1]\} \tag{II.14}$$

The cell is of dimension $4r - 2$ and we consider cells of dimension $4r - 3$ to which $c_{r,1}$ attaches, and they are $b_{r,1}, b_{r,2}$ and $b_{r,3}$. It remains to determine the incidence numbers between $c_{r,1}$ and these cells. Consider the equivariant cell $b_{r,3}$ generated by

$$\{(X_1, ..., X_{r-1}, x_r, 0, 0, w_r, 0, ..., 0\} \in S(n\rho) \,|\, x_r, w_r \in [0, 1]\}.$$

Since $x_r, y_r, z_r \in [0, 1]$ in the generator of $c_{r,1}$, it is only attached to the j-orbit of $b_{r,3}$. This orbit is given by

$$\{(jX_1, ..., jX_{r-1}, 0, w_r, x_r, 0, 0, ..., 0\} \in S(n\rho) \,|\, x_r, w_r \in [0, 1]\}$$

since $j(x_r + w_r(ij)) = (w_r i + x_r j)$.

We could use the basis

$$(e_1, ie_1, ..., e_{2r-2}, ie_{2r-2}, e_{2r-1}, ie_{2r-1}) \tag{II.15}$$

to determine the orientation of the generator of $c_{r,1}$ (identify it by an orientation preserving homeomorphism with the unit disk in $\mathbb{C}^{2r-1}$ since z_r is determined by $X_1, ..., X_{r-1}, x_r, y_r$).

Similarly the induced orientation of $jb_{r,3}$ as subspace is

$$(e_2, -ie_2, -e_1, ie_1, ..., e_{2r-2}, -ie_{2r-2}, -e_{2r-3}, ie_{2r-3}, -ie_{2r-1}) \tag{II.16}$$

On a point of $jb_{r,3}$ that $c_{r,1}$ attaches, by the rules set above, the induced orientetation is given by

$$(e_1, ie_1, ..., e_{2r-2}, ie_{2r-2}, ie_{2r-1}) \tag{II.17}$$

since juxtaposing with outward normal direction $-e_{2r-1}$ (since in (II.14) we have $x_r \geq 0$) gives the same orientation as in (II.14). Comparing orientations (II.16) and (II.17) gives that the incidence number between $c_{r,1}$ and $jb_{r,3}$ is -1, i.e.,

$$dc_{r,1} = ... - jb_{r,3} + ...$$

Other incidence numbers are computed by the same method. $\square$

With this, we can calculate the Q_8-equivariant homology and cohomology of $S(n\rho)$ with coefficient $\underline{\mathbb{Z}}$. As an example we explicitly compute the homology here. Order the cells in the order as listed above ($b_{r,2}$ comes after $b_{r,1}$ for example), by Lemma 11, the chain complex is

$$\mathbb{Z}^2 \xrightarrow{d_3} \mathbb{Z} \to ... \to \mathbb{Z}^3 \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_4} \mathbb{Z}^2 \xrightarrow{d_3} \mathbb{Z}^4 \xrightarrow{d_2} \mathbb{Z}^3 \xrightarrow{d_1} \mathbb{Z} \to 0.$$

where the differentials are given by the following matrices:

$$d_1 = 0, d_2 = \begin{bmatrix} 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix}, d_3 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}, d_4 = \begin{bmatrix} 8 \\ -8 \end{bmatrix}.$$

Taking homology we get 4-periodic result

$$H_q^{Q_8}(S(n\rho), \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & 0 < q < 4n-1, q \equiv 1 \bmod 4 \\ \mathbb{Z}/8 & 0 < q < 4n-1, q \equiv 3 \bmod 4 \\ 0 & q > 0 \text{ even} \end{cases}$$

When $n \to \infty$, we recover the group homology of Q_8 with coefficient in $\mathbb{Z}$.

Since Borel (co)homology is complex stable, when suspended by $S^{k\alpha_1 + \ell\alpha_2 + m\alpha_3}$, we may assume that $k, \ell, m = 0, 1$. We may use the Borel (co)homology spectral sequence to compute the (co)homology of these spaces.

$$H_p(Q_8, \widetilde{H}_q(X)) \Rightarrow \widetilde{H}^{Q_8}_{p+q}(EG_+ \wedge X).$$

By a symmetry up to automorphisms of Q_8, it suffices to consider three cases: suspension by

$X = S^{\alpha_1}, S^{\alpha_1+\alpha_2}, S^{\alpha_1+\alpha_2+\alpha_3}$. By looking at the top homology class, which represents orientations, the action of Q_8 on $\widetilde{H}_*(X) \cong \mathbb{Z}$ is either a trivial action which corresponds to $X = S^{\alpha_1+\alpha_2+\alpha_3}$, or to a twisted action which corresponds to $X = S^{\alpha_1}$ or $X = S^{\alpha_1+\alpha_2}$. Therefore, the spectral sequence collapses, and the computations reduces to calculate the group (co)homology of Q_8 with twisted coefficients.

To do this, one may either compute directly, using the universal space $S(\infty\gamma)_+$, then determine the top class, or suspending by S^{α_1} and calculate directly. Using the first method for homology, we get the following chain complex after tensor with twisted coefficients over G:

$$... \to \mathbb{Z}^3 \xrightarrow{d_1'} \mathbb{Z} \xrightarrow{d_4'} \mathbb{Z}^2 \xrightarrow{d_3'} \mathbb{Z}^4 \xrightarrow{d_2'} \mathbb{Z}^3 \xrightarrow{d_1'} \mathbb{Z} \to 0.$$

The twisted coefficients $\mathbb{Z}$ here has a nontrivial action by $j, ij, -j, -ij \in Q_8$, so the differentials are

$$d_1' = \begin{bmatrix} 0 & -2 & -2 \end{bmatrix}, d_2' = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, d_3' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & -1 \\ -1 & -1 \end{bmatrix}, d_4' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence we conclude that

$$\widetilde{H}_n^{Q_8}(S(m\gamma)_+ \wedge S^{\alpha_1}, \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z}/2 & 0 \le n \le 4m,\, n \equiv 1, 2, 3 \text{ mod } 4 \\ 0 & 0 \le n \le 4m,\, n \equiv 0 \text{ mod } 4 \end{cases}$$

$$\widetilde{H}_n^{Q_8}(S(m\gamma)_+ \wedge S^{\alpha_1+\alpha_2}, \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z}/2 & 0 \le n \le 4m+1,\, n \equiv 0, 2, 3 \text{ mod } 4 \\ 0 & 0 \le n \le 4m+1,\, n \equiv 1 \text{ mod } 4 \end{cases}$$

$$\widetilde{H}_n^{Q_8}(S(m\gamma)_+ \wedge S^{\alpha_1+\alpha_2+\alpha_3}, \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z} & n = 3 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & 0 < n \le 4m+2,\, n \equiv 1 \text{ mod } 4 \\ \mathbb{Z}/8 & 2 < n < 4m+2,\, n \equiv 3 \text{ mod } 4 \\ 0 & n \text{ even} \end{cases}$$

Remark 12. *All these results could be more concisely summarized as the following:*

$$H^*(Q_8, \mathbb{Z}^+) : \mathbb{Z}, 0, \mathbb{Z}/2 \oplus \mathbb{Z}/2, 0, \mathbb{Z}/8, 0, \mathbb{Z}/2 \oplus \mathbb{Z}/2, 0, \mathbb{Z}/8, ...$$

$$H_*(Q_8, \mathbb{Z}^+) : \mathbb{Z}, \mathbb{Z}/2 \oplus \mathbb{Z}/2, 0, \mathbb{Z}/8, 0, \mathbb{Z}/2 \oplus \mathbb{Z}/2, 0, \mathbb{Z}/8, 0, ...$$

$$H^*(Q_8, \mathbb{Z}^-) : 0, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/2, ...$$

$$H_*(Q_8, \mathbb{Z}^-) : \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/2, 0, ...$$

With these we may read off the edge maps in the Lyndon-Hochschild-Serre spectral sequences for the central extension

$$0 \to \mathbb{Z}/2 \to Q_8 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to 0.$$

For example, for

$$H^p(\mathbb{Z}/2 \oplus \mathbb{Z}/2; H^q(\mathbb{Z}/2, \mathbb{Z}^+)) \Rightarrow H^{p+q}(Q_8, \mathbb{Z}^+),$$

we have the following E_2 page

4	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$				
3	0	0	0	0			
2	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^4$	$(\mathbb{Z}/2)^5$		
1	0	0	0	0	0	0	
0	$\mathbb{Z}$	0	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^4$
	0	1	2	3	4	5	6

E_2 page for $H^*(Q_8, \mathbb{Z}^+)$

Clearly there is no room for any d^2. And d^3 will wipe out $H^0(\mathbb{Z}/2 \oplus \mathbb{Z}/2, H^2(\mathbb{Z}/2, \mathbb{Z}^+))$, and two $\mathbb{Z}/2$ summands in $E_3^{4,0}$ (since the answer is a $\mathbb{Z}/8$).

In conclusion, the edge map

$$H^*(\mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{Z}^+) \to H^*(Q_8, \mathbb{Z}^+)$$

is given by

$$\mathbb{Z}[u^2, v^2, w]/(2u^2, 2v^2, 2w, u^4v^2 + u^2v^4 = w^2) \to \\ \mathbb{Z}[u^2, v^2, \pi]/(2u^2, 2v^2, 8\pi, u^4, v^4, u^2v^2 = 4\pi)$$

$$u \mapsto u, v \mapsto v, w \mapsto 0.$$

To obtain a full computation of the $RO(G)$-graded coefficients, we will need to smash the sequence

$$S(n\rho)_+ \to S^0 \to S^{n\rho}$$

with $\Sigma^{k\alpha+\ell\beta+m\gamma}H\underline{\mathbb{Z}}$ for $k, \ell, m \in \mathbb{Z}$. Recall that we have assumed $k \geq \ell \geq 0$, the map

$$S(n\rho)_+ \to S^0$$

induces map on the level of $\mathbb{Z}/4$-fixed points of chains. When $m = 0$, it is

$$C_*(\Sigma^{k\alpha+\ell\beta}S(n\rho)_+, \underline{\mathbb{Z}})^{\mathbb{Z}/4} \to C_*(S^{k\alpha+\ell\beta}; \underline{\mathbb{Z}})^{\mathbb{Z}/2}, \tag{II.18}$$

since the center acts trivially on the target.

The main strategy to deal with suspensions by $S^{m\gamma}$ is to take $\mathbb{Z}/4$-fixed points of the chain complexes so that the $m\gamma$-suspension becomes an operation on the level of $\mathbb{Z}/2$-equivariant chain complexes.

Let $n \geq 0$. We take the $\mathbb{Z}/4 = \langle ij \rangle$-fixed points

$$C_*(S(n\rho)_+, \underline{\mathbb{Z}})^{\mathbb{Z}/4},$$

$$C_*(S(n\rho)_+, \underline{\mathbb{Z}^-})^{\mathbb{Z}/4}.$$

Let the generator of the quotient $C' := Q_8/\langle ij \rangle$ be σ. With Lemma 11, it is again routine to calculate the following differentials in the $\mathbb{Z}$ case for $1 \leq r \leq n$:

$db_{r,1} = 0$
$db_{r,2} = (\sigma - 1)a_{r,1}$
$db_{r,3} = (\sigma - 1)a_{r,0}$
$dc_{r,1} = b_{r,1} - b_{r,2} - \sigma b_{r,3}$
$dc_{r,2} = b_{r,1} + b_{r,2} - b_{r,3}$
$dc_{r,3} = -\sigma b_{r,1} + b_{r,2} - b_{r,3}$
$dc_{r,4} = -\sigma b_{r,1} - b_{r,2} - \sigma b_{r,3}$
$dd_{r,1} = c_{r,1} - c_{r,2} + c_{r,3} - c_{r,4}$
$dd_{r,2} = c_{r,1} - \sigma c_{r,2} + \sigma c_{r,3} - c_{r,4}$
$da_{r+1,1} = (4 + 4\sigma)(d_{r,1} - d_{r,2})$

Define $C(r)$ for $1 \leq r \leq n - 1$ to be the following chain complex:

$$0 \to \mathbb{Z}[C'] \xrightarrow{1+\sigma} \mathbb{Z}[C'] \xrightarrow{1-\sigma} \mathbb{Z}[C'] \xrightarrow{4+4\sigma} \mathbb{Z}[C'] \xrightarrow{1-\sigma} \mathbb{Z}[C'] \xrightarrow{1+\sigma} \mathbb{Z}[C'] \to 0,$$

where the free $\mathbb{Z}[C']$-modules are generated by

$$c_{r+1,2}, b_{r+1,2}, -a_{r,1}, d_{r,2}, c_{r,2} - c_{r,3}, b_{r,2} - b_{r,3}.$$

Define

$$C(n): \quad 0 \to [d_{n,2}] \xrightarrow{1-\sigma} [c_{n,2} - c_{n,3}] \xrightarrow{1+\sigma} [b_{n,2} - b_{n,3}] \to 0,$$

and

$$C(0): \quad 0 \to [c_{1,2}] \xrightarrow{1+\sigma} [b_{1,2}] \xrightarrow{1-\sigma} [-a_{1,1}] \to 0.$$

(Note that all the bottom $\mathbb{Z}[C']$'s are at degree 0).

These complexes are connected by chain maps f_r (in fact, they are also differentials in the chain complex) for $1 \leq r \leq n-1$:

$$f_r : C(r)[-5] \to C(r+1), \quad [c_{r+1,2}] \xrightarrow{1-\sigma} [b_{r+1,2} - b_{r+1,3}],$$

and the chain map

$$f_0 : C(0)[-2] \to C(1), \quad [c_{1,2}] \xrightarrow{1-\sigma} [b_{1,2} - b_{1,3}].$$

If we quotient out, for each $1 \leq r \leq n$, two acyclic complexes

$$0 \to c_{r,1} \to b_{r,1} - b_{r,2} - \sigma b_{r,3} \to 0$$

$$0 \to d_{r,1} \to c_{r,1} - c_{r,2} + c_{r,3} - c_{r,4} \to 0$$

in $C_*(S(n\rho)_+, \underline{\mathbb{Z}})^{\mathbb{Z}/4}$ and then take the cokernel, the result could be written a totalization of the following double complex:

$$\Theta^+_{n,0} := \mathrm{Tot}(C(0) \xrightarrow{f_0[2]} C(1)[2] \xrightarrow{f_1[7]} C(2)[7] \to ... \to C(n)[5n-3]).$$

With $\underline{\mathbb{Z}^-}$ coefficient, similar calculations define

$$\Theta^-_{n,0} := \mathrm{Tot}(C^-(0) \xrightarrow{f_0^-[2]} C^-(1)[2] \xrightarrow{f_1^-[7]} C^-(2)[7] \to ... \to C^-(n)[5n-3]),$$

where the C^- chains and f^- chain maps are differed from their counterparts by changing signs of σ in all differentials (note that the chain maps f_r's were noted as differentials). As an example,

$$C^-(0): 0 \to [c_{1,2}] \xrightarrow{1-\sigma} [b_{1,2}] \xrightarrow{1+\sigma} [-a_{1,1}] \to 0.$$

For example, when $n = 2$, the chain complex $\Theta^+_{2,0}$ could be visualized as

$$
\begin{tikzcd}
\circ \arrow[r, "1-\sigma"] & \circ \arrow[r, "1+\sigma"] & \circ & & & & & \\
& \circ \arrow[r, "1+\sigma"] \arrow[ur, "1-\sigma"] & \circ \arrow[r, "1-\sigma"] & \circ \arrow[r, "4+4\sigma"] & \circ \arrow[r, "1-\sigma"] & \circ \arrow[r, "1+\sigma"] & \circ & \\
& & & & & \circ \arrow[r, "1+\sigma"] \arrow[ur, "1-\sigma"] & \circ \arrow[r, "1-\sigma"] & \circ
\end{tikzcd}
$$

where each circle represents a $\mathbb{Z}[C']$.

Let γ' be the sign representation of C', it is useful to also introduce the following notations for $m \geq 0$:

$$A_s^+ = C_*(S^{s\gamma'}),$$

$$A_s^- = C_*(S^{s\gamma'}/S^{\gamma'})[-1].$$

These chain complexes are given explicitly by

$$A_s^+ : \mathbb{Z}[C'] \xrightarrow{1+(-1)^{s-1}\sigma} \mathbb{Z}[C'] \to ... \to \mathbb{Z}[C'] \xrightarrow{1-\sigma} \mathbb{Z}[C'] \xrightarrow{\text{aug}} \mathbb{Z}$$

$$A_s^- : \mathbb{Z}[C'] \xrightarrow{1+(-1)^{s}\sigma} \mathbb{Z}[C'] \to ... \to \mathbb{Z}[C'] \xrightarrow{1+\sigma} \mathbb{Z}[C'] \xrightarrow{\text{aug}} \mathbb{Z}^-.$$

We have the following decomposition:

Lemma 13. *If $k \geq \ell \geq 0$, then*

$$C_*(S^{k\alpha+\ell\beta}; \underline{\mathbb{Z}})^{\mathbb{Z}/2} = \bigoplus_{s=\ell}^{k} A_\ell^{(-1)^s}[s] \oplus \bigoplus_{s=0}^{\ell-1} (A_s^{(-1)^s}[s] \oplus A_s^{(-1)^{s+1}}[s+1]).$$

Proof. Smashing $S^{k\alpha}, S^{\ell\beta}$ together, we have the standard CW structure of $S^{k\alpha+\ell\beta}$. Choose a generator of the top cohomology class, and map it by differentials of the chain complex, until it hits a cell which is not free (coming from $S^{\ell\beta}$ given $k \leq \ell$), then take the cokernel of this subcomplex, which turns out to be a direct sum. Then the result follows by induction. □

As an illustration, when $k = 7, \ell = 5$, the $(\ker\gamma)$-fixed point is decomposed as the direct sum of the blobs in Figure II.1. A square represents a copy of $\mathbb{Z}$.

To get the result, we compute the cofiber of the map (II.18), then smash the chain complex with the chain complex of $S^{m\gamma}$, and finally take (co)homology. For this purpose, let $A_s^+(m), A_s^-(m)$ respectively be the result of smashing A_s^+, A_s^- with $S^{m\gamma}$. Then we have

$$A_s^{\pm}(m) = \begin{cases} A_{s+m}^{\pm} & s+m \geq 0 \\ (A_{-m-s}^{\pm})^\vee & s+m < 0 \end{cases} \tag{II.19}$$

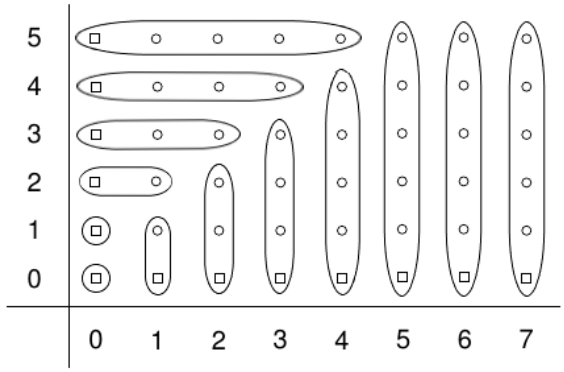

Figure II.1: $C_*(S^{k\alpha+\ell\beta};\underline{\mathbb{Z}})^{\mathbb{Z}/2}$ when $k=7,\ell=5$

The homology of the $\mathbb{Z}/2$-fixed points of these chain complexes are given in the following proposition.

Proposition 14. *Taking homology of the $\mathbb{Z}/2$-fixed points, we have*

$$H_q(A_s^+)^{\mathbb{Z}/2} = \begin{cases} \mathbb{Z} & q=s,\ s \textit{ even} \\ \mathbb{Z}/2 & 0\le q<s,\ q \textit{ even} \\ 0 & \textit{else} \end{cases}$$

$$H_q((A_s^+)^\vee)^{\mathbb{Z}/2} = \begin{cases} \mathbb{Z} & q=-s,\ s \textit{ even} \\ \mathbb{Z}/2 & -s\le q\le 0,\ q \textit{ odd} \\ 0 & \textit{else} \end{cases}$$

$$H_q(A_s^-)^{\mathbb{Z}/2} = \begin{cases} \mathbb{Z} & q=s,\ s \textit{ odd} \\ \mathbb{Z}/2 & 0\le q<s,\ q \textit{ odd} \\ 0 & \textit{else} \end{cases}$$

$$H_q((A_s^-)^\vee)^{\mathbb{Z}/2} = \begin{cases} \mathbb{Z} & q=-s,\ s \textit{ odd} \\ \mathbb{Z}/2 & -s\le q\le -2,\ q \textit{ even} \\ 0 & \textit{else} \end{cases}$$

Proof. See [Lew88] and [**?**]. □

Now let $\Theta^+_{n,m}$ and $\Theta^-_{n,m}$ respectively be the result of smashing $\Theta^+_{n,0},\Theta^-_{n,0}$ with $S^{m\gamma}$. When $m\le 2$, $\Theta^+_{n,m}$ is the totalization of the following double chain complex:

$$C(0)_m \xrightarrow{f_0[2-m]} C(1)[2-m] \xrightarrow{f_1[7-m]} C(2)[7-m] \to \dots \to C(n)[5n-m-3],$$

where $C(0)_m$ is

$$0 \to \mathbb{Z}[C'] \xrightarrow{1+(-1)^0\sigma} \mathbb{Z}[C'] \xrightarrow{1+(-1)^1\sigma} \mathbb{Z}[C'] \to ... \xrightarrow{1+(-1)^{1-m}\alpha} \mathbb{Z}[C'] \to 0.$$

When $m > 2$, $\Theta^+_{n,m}$ is the totalization of the following double chain complex:

$$C(0)_m[2-m] \xrightarrow{\widetilde{f_0}[2-m]} C(1)[2-m] \xrightarrow{f_1[7-m]} ... \to C(n)[5n-m-3],$$

where $C(0)_m$ is

$$0 \to \mathbb{Z}[C'] \xrightarrow{1+(-1)^{m-3}\sigma} \mathbb{Z}[C'] \xrightarrow{1+(-1)^{m-4}\sigma} ... \xrightarrow{1+(-1)^0\sigma} \mathbb{Z}[C'] \to 0$$

and $\widetilde{f_0} : C(0)_m \to C(1)$ is given by $1-\sigma$ at the bottom degree of the both chain complexes.

As an example, the chain complex $\Theta^+_{2,-2}$ could be presented as

$$\begin{tikzcd}
\circ \arrow[r, "1-\sigma"] & \circ \arrow[r, "1+\sigma"] & \circ & & & & & & & \\
& \circ \arrow[r, "1+\sigma"] \arrow[ur, "1-\sigma"] & \circ \arrow[r, "1-\sigma"] & \circ \arrow[r, "4+4\sigma"] & \circ \arrow[r, "1-\sigma"] & \circ \arrow[r, "1+\sigma"] & \circ & & & \\
& & & & & \circ \arrow[r, "1+\sigma"] \arrow[ur, "1-\sigma"] & \circ \arrow[r, "1-\sigma"] & \circ \arrow[r, "1+\sigma"] & \circ \arrow[r] & \square
\end{tikzcd} \tag{II.20}$$

Similarly, when $m \leq 2$, $\Theta^-_{n,m}$ is the totalization of the following double chain complex:

$$C^-(0)_m \xrightarrow{f_0^-[2-m]} C^-(1)[2-m] \xrightarrow{f_1^-[7-m]} C(2)[7-m] \to ... \to C^-(n)[5n-m-3],$$

where $C^-(0)_m$ is

$$0 \to \mathbb{Z}[C'] \xrightarrow{1-(-1)^0\sigma} \mathbb{Z}[C'] \xrightarrow{1-(-1)^1\sigma} \mathbb{Z}[C'] \to ... \xrightarrow{1-(-1)^{1-m}\alpha} \mathbb{Z}[C'] \to 0.$$

When $m > 2$, $\Theta^-_{n,m}$ is the totalization of the following double chain complex:

$$C^-(0)_m[2-m] \xrightarrow{\widetilde{f_0^-}[2-m]} C^-(1)[2-m] \xrightarrow{f_1^-[7-m]} ... \to C^-(n)[5n-m-3],$$

where $C^-(0)_m$ is

$$0 \to \mathbb{Z}[C'] \xrightarrow{1-(-1)^{m-3}\sigma} \mathbb{Z}[C'] \xrightarrow{1-(-1)^{m-4}\sigma} ... \xrightarrow{1-(-1)^0\sigma} \mathbb{Z}[C'] \to 0$$

and $\widetilde{f_0^-} : C^-(0)_m \to C^-(1)$ is given by $1+\sigma$ at the bottom degree of the both chain complexes.

As an example, the chain complex $\Theta^-_{2,5}$ could be presented as

$$\begin{tikzcd}
\circ \arrow[r, "1-\sigma"] & \circ \arrow[r, "1+\sigma"] & \circ & & & & \\
& \circ \arrow[r, "1+\sigma"] \arrow[ur, "1-\sigma"] & \circ \arrow[r, "1-\sigma"] & \circ \arrow[r, "4+4\sigma"] & \circ \arrow[r, "1-\sigma"] & \circ \arrow[r, "1+\sigma"] & \circ \\
& & \square \arrow[r, "1+\sigma"] & \circ \arrow[r, "1-\sigma"] & \circ \arrow[r, "1+\sigma"] & \circ \arrow[ur, "1-\sigma"] &
\end{tikzcd}$$

Proposition 15. *When $m \leq 0$, we have*

$$H_q((\Theta^+_{n,m})^{\mathbb{Z}/2}) = \begin{cases} \mathbb{Z} & q = 0 \textit{ and } m \textit{ even} \\ \mathbb{Z}/2 & 0 \leq q \leq -m,\ q \equiv m+1 \textit{ mod 2} \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & -m \leq q \leq 4n-m-1,\ q \equiv -m+1 \textit{ mod 4} \\ \mathbb{Z}/8 & -m \leq q \leq 4n-m-1,\ q \equiv -m+3 \textit{ mod 4} \\ 0 & \textit{else} \end{cases}$$

When $m > 0$, we have

$$H_q((\Theta^+_{n,m})^{\mathbb{Z}/2}) = \begin{cases} \mathbb{Z} & q = 0 \textit{ and } m \textit{ even} \\ \mathbb{Z}/2 & q = m-2, \textit{ or} \\ & 0 \leq q \leq 3-m \textit{ and } q \equiv -m+3 \textit{ mod 2} \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & m+2-4n \leq q \leq m-6,\ q \equiv m-2 \textit{ mod 4} \\ \mathbb{Z}/8 & m-4n \leq q \leq m-4,\ q \equiv m \textit{ mod 4} \\ 0 & \textit{else} \end{cases}$$

And when $m \leq 0$, we have

$$H_q((\Theta^-_{n,m})^{\mathbb{Z}/2}) = \begin{cases} \mathbb{Z} & q = 0 \textit{ and } m \textit{ odd} \\ \mathbb{Z}/2 & 0 \leq q \leq -m,\ q \equiv m \textit{ mod 2, or} \\ & -m \leq q \leq 4n-m-1,\ q+m \equiv 0,1,2 \textit{ mod 4} \\ 0 & \textit{else} \end{cases}$$

Finally, when $m > 0$, we present the homology of $(\Theta^-_{n,m})^{\mathbb{Z}/2})$ as a sum of two chain complexes

$$H_*((\Theta^-_{n,m})^{\mathbb{Z}/2}) \cong H_*((\Theta^-_{n,2})^{\mathbb{Z}/2})[2-m] \oplus H_*(C^-(0)_m[2-m])^{\mathbb{Z}/2}$$

where

$$H_q((\Theta^-_{n,2})^{\mathbb{Z}/2}) = \begin{cases} \mathbb{Z}/2 & -1 \le q \le 4n-3,\ q \equiv 0,2,3 \textit{ mod 4}, \\ 0 & \textit{else} \end{cases}$$

and

$$H_q(C^-(0)_m[2-m])^{\mathbb{Z}/2} = \begin{cases} \mathbb{Z} & q = 0 \textit{ and } m \textit{ odd} \\ \mathbb{Z}/2 & -1 \le q \le 2-m,\ q \equiv m+1 \textit{ mod 2}, \\ 0 & \textit{else} \end{cases}$$

Proof. As seen from the above definition, the chain complexes in these have fewer than three copies of $\mathbb{Z}$ in each dimension, also the differentials are simple. Thus we can proceed by direct computation. □

With all the ingredient described, we may write down the first case of the main result.

Theorem 16. *For $k \ge \ell \ge 0$ and $n \ge 0$, as a $\mathbb{Z}/2$-equivariant chain complex,*

$$C^{Q_8}_*(S^{k\alpha+\ell\beta+m\gamma+n\rho})^{\mathbb{Z}/4} = \bigoplus_{s=\ell}^{k-1} A_\ell^{(-1)^s}(m)[s] \oplus \bigoplus_{s=0}^{\ell-1} A_s^{(-1)^s}(m)[s]$$
$$\oplus A_s^{(-1)^{s+1}}(m)[s+1]) \oplus \Theta^{(-1)^{(k+1)(\ell+1)}}_{n,-\ell-m}[\ell].$$

The homology of all the chain complexes involved, are computed in Proposition 14 and Proposition 15.

□

By Spanier-Whitehead duality, it suffices to furthermore consider the case $n < 0$. Essentially, this means n and k, ℓ have different signs. If, say, $k, \ell < 0$ and $n, m > 0$, we can flip all the signs and compute the cohomology instead.

So here we assume $k \ge \ell \ge 0$, and $n < 0$. The cofiber sequence now looks like

$$S(-n\rho)_+ \to S^0 \to S^{-n\rho}.$$

Take the dual of this sequence, we have

$$S^{n\rho} \to S^0 \to DS(-n\rho)_+.$$

The dual of $S(-n\rho)_+$ is $\Sigma^{n\rho+1} S(-n\rho)_+$, hence

$$S^{n\rho} \to S^0 \to \Sigma^{n\rho+1} S(-n\rho)_+.$$

Define now for $n < 0$:

$$\Theta^{\pm}_{n,m} := \mathrm{Hom}(\Theta^{\pm}_{-n,m}, \mathbb{Z}),$$

so that $\Theta^{+}_{n,m}$ and $\Theta^{-}_{n,m}$ are still results of smashing $\Theta^{+}_{n,0}, \Theta^{-}_{n,0}$ with $S^{m\gamma}$. Their homology is recorded in the following proposition whose proof is analogous to Proposition 15.

Proposition 17. *Let $n < 0$. When $m \leq 0$, we have*

$$H_q((\Theta^{+}_{n,m})^{\mathbb{Z}/2}) = \begin{cases} \mathbb{Z} & q = 0 \textit{ and } m \textit{ even} \\ \mathbb{Z}/2 & m-1 \leq q \leq -1,\ q \equiv m \textit{ mod 2} \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & m+4n \leq q \leq m-1,\ q \equiv m-2 \textit{ mod 4} \\ \mathbb{Z}/8 & m+4n \leq q \leq m-1,\ q \equiv m \textit{ mod 4} \\ 0 & \textit{else} \end{cases}$$

When $m > 0$, we have

$$H_q((\Theta^{+}_{n,m})^{\mathbb{Z}/2}) = \begin{cases} \mathbb{Z} & q = 0 \textit{ and } m \textit{ even} \\ \mathbb{Z}/2 & q = -m+1, \textit{ or} \\ & -m+2 \leq q \leq -1 \textit{ and } q \equiv m \textit{ mod 2} \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & 5-m \leq q \leq -4n-m-3,\ q \equiv -m+1 \textit{ mod 4} \\ \mathbb{Z}/8 & 3-m \leq q \leq -4n-m-1,\ q \equiv -m+3 \textit{ mod 4} \\ 0 & \textit{else} \end{cases}$$

And when $m \leq 0$, we have

$$H_q((\Theta^{-}_{n,m})^{\mathbb{Z}/2}) = \begin{cases} \mathbb{Z} & q = 0 \textit{ and } m \textit{ odd} \\ \mathbb{Z}/2 & m-1 \leq q \leq 1,\ q \equiv m+1 \textit{ mod 2, or} \\ & m+4n \leq q \leq m-1,\ q-m \equiv 1,2,3 \textit{ mod 4} \\ 0 & \textit{else} \end{cases}$$

Finally, when $m > 0$, we present the homology of $(\Theta^{-}_{n,m})^{\mathbb{Z}/2}$ as a sum of two chain complexes

$$H_*((\Theta^{-}_{n,m})^{\mathbb{Z}/2}) \cong H_*((\Theta^{-}_{n,2})^{\mathbb{Z}/2})[2-m] \oplus H_*(C^{-}(0)_m[2-m])^{\mathbb{Z}/2}$$

where

$$H_q((\Theta^{-}_{n,2})^{\mathbb{Z}/2}) = \begin{cases} \mathbb{Z}/2 & 4n+2 \leq q \leq 0,\ q \equiv 0,1,3 \textit{ mod 4,} \\ 0 & \textit{else} \end{cases}$$

and

$$H_q(C^-(0)_m[2-m])^{\mathbb{Z}/2} = \begin{cases} \mathbb{Z} & q = 0 \text{ and } m \text{ odd} \\ \mathbb{Z}/2 & m-3 \leq q \leq 0,\ q \equiv m \text{ mod } 2, \\ 0 & \text{else} \end{cases}$$

□

Suspend by $S^{k\alpha+\ell\beta}$, the connecting map connects at the top degree, and after taking the cofiber, we obtain the next case of our main result:

Theorem 18. *For $k \geq \ell \geq 0$ and $n < 0$, as a $\mathbb{Z}/2$-equivariant chain complex,*

$$C_*^{Q_8}(S^{k\alpha+\ell\beta+m\gamma+n\rho})^{\mathbb{Z}/4} = \bigoplus_{s=\ell}^{k-1} A_\ell^{(-1)^s}(m)[s] \oplus \bigoplus_{s=0}^{\ell-1} A_s^{(-1)^s}(m)[s] \oplus A_s^{(-1)^{s+1}}(m)[s+1]) \oplus \Theta_{n,\ell-m}^{(-1)^{(k+1)(\ell+1)}}[\ell].$$

The homology of all the chain complexes involved, are computed in Proposition 14 and Proposition 17.

□

Finally, we complete our discussion by adding the cohomology results. The cohomology of duals of the chain complexes Θ's and A's are easily derived from Proposition 14, 15 and 17 using universal coefficients theorem. What is new is the decomposition of cochain complexes. The answer is the follows:

Theorem 19. *For $k \geq \ell \geq 0$ and $n \geq 0$, as a $\mathbb{Z}/2$-equivariant chain complex,*

$$C^*_{Q_8}(S^{k\alpha+\ell\beta+m\gamma+n\rho})^{\mathbb{Z}/4} = \bigoplus_{s=\ell}^{k-1} (A_\ell^{(-1)^s}(m)[s])^\vee \oplus \bigoplus_{s=0}^{\ell-1} (A_s^{(-1)^s}(m)[s])^\vee \oplus (A_s^{(-1)^{s+1}}(m)[s+1])^\vee \oplus \Theta_{-n,-\ell-m}^{(-1)^{(k+1)(\ell+1)}}[\ell].$$

For $k \geq \ell \geq 0$ and $n < 0$, as a $\mathbb{Z}/2$-equivariant chain complex,

$$C^*_{Q_8}(S^{k\alpha+\ell\beta+m\gamma+n\rho})^{\mathbb{Z}/4} = \bigoplus_{s=\ell}^{k-1} (A_\ell^{(-1)^s}(m)[s])^\vee \oplus \bigoplus_{s=0}^{\ell-1} (A_s^{(-1)^s}(m)[s])^\vee \oplus (A_s^{(-1)^{s+1}}(m)[s+1])^\vee \oplus \Theta_{-n,\ell-m}^{(-1)^{(k+1)(\ell+1)}}[\ell].$$

□

CHAPTER III

Equivariant Complex Cobordism and Formal Group Laws

3.1 Homotopical Equivariant Complex Cobordism MU_G

Let M be a smooth manifold of even dimension. An almost complex structure on a smooth real manifold M of even dimension is a complex structure on its tangent bundle TM. A smooth real manifold M has a stable almost complex structure if there is a $k \geq 0$ such that $M \oplus \mathbb{R}^k$ admits a almost complex structure.

Let $MU(n)$ be the Thom space of the universal bundle $\gamma_n : EU(n) \to BU(n)$. They assemble to complex cobordism spectrum MU, with structure maps given by classifying maps for $\gamma_n \oplus \mathbb{C}$.

The complex cobordism MU is a well-studied generalized cohomology theory. By the works of Quillen [Qui69], MU is the universal theory for complex oriented cohomology theories, and it supports a universal formal group law. Based on the calculations of $\pi_* MU$ by Milnor and Novikov [Mil60, Nov62], Brown and Peterson [BP66] constructed the p-local Brown Peterson spectrum BP for a prime p and proved analogous results to [Qui69] where one replaces MU by the p-local BP and 1-dimensional formal group laws by 1-dimensional p-typical formal group laws. Furthermore, we can produce Morava E-theories and Morava K-theories, which are some main interests of study in chromatic homotopy theory.

Hill, Hopkins and Ravenel's solution for the Kervaire invariant one problem [HHR16] uses the Real cobordism theory $MU_{\mathbb{R}}$. There is an equivariant version MU_G of complex cobordism, which is first defined by tom Dieck [tD70]. Fix a complete universe $\mathcal{U}$. Let $BU^G(n)$ be the Grassmanian of complex n-dimensional linear subspaces of $\mathcal{U}$. Let γ_n^G denote the tautological complex n-plane bundle over $BU^G(n)$. For a real representation V, define $T(V)$ be the Thom space of $\gamma_{|V|}^G$, where $|V|$ is the real dimension of V. Apply spectrification functor, the result is the homotopical equivariant complex cobordism MU_G.

Because of transversality issues, the coefficient rings of MU_G's are different from the natural cobordism rings of weakly stably complex G-manifolds. The homotopy cobordism rings are of much more fundamental homotopy-theoretical interest. We expect MU_G to play the same key role in equivariant stable homotopy theory.

The coefficient ring $(MU_G)_*$ for $G = \mathbb{Z}/p$ is described in [Kri99]. The explicit calculation with generators and relations is first given by [Str01] for $G = \mathbb{Z}/2$, and later generalized in [Hu21] for primary cyclic groups. The case when G is finite is studied in [AK15], and the case when $G = S^1$ is studied in [Sin01]. However, the picture for nonabelian groups is still largely unknown.

Another aspect of complex cobordism concerns the evenness and freeness properties of the coefficient ring. However, recently both the geometric and the homotopical evenness conjectures are proven to be false, respectively in [Sam22] and [Kri21b].

3.2 The case $G = \Sigma_3$

We will use α to denote the sign representation of Σ_3, and γ for the two dimensional irreducible representation. For simplicity, when there is no confusion we will abbreviate MU_{Σ_3} as MU.

3.2.1 The coefficients of $S^{\infty\alpha} \wedge MU$.

By the Tate diagram, there is a "Tate square"

$$\begin{tikzcd} MU \arrow[r] \arrow[d] & S^{\infty\gamma} \wedge MU \arrow[d] \\ F(S(\infty\gamma)_+, MU) \arrow[r] & S^{\infty\gamma} \wedge F(S(\infty\gamma)_+, MU) \end{tikzcd}$$

for MU_G. Smash the square with $S^{\infty\alpha} = \widetilde{E\mathscr{F}(\mathbb{Z}/3)}$ we see that $S^{\infty\alpha} \wedge MU_{\Sigma_3}$ is the homotopy pullback of the diagram

$$\begin{tikzcd} & \widetilde{E\mathscr{F}[\Sigma_3]} \wedge MU \arrow[d] \\ S^{\infty\alpha} \wedge F(S(\infty\gamma)_+, MU) \arrow[r] & \widetilde{E\mathscr{F}[\Sigma_3]} \wedge F(S(\infty\gamma)_+, MU) \end{tikzcd} \tag{III.1}$$

The upper right corner is the geometric fixed point, and is calculated by tom Dieck [tD70] as

$$(\widetilde{E\mathscr{F}[\Sigma_3]} \wedge MU)_* = \Phi^{\Sigma_3} MU_* = MU_*[u_\alpha, u_\alpha^{-1}, u_\gamma, u_\gamma^{-1}, b_i^\alpha, b_i^\gamma]. \tag{III.2}$$

Here the generator b_i^α has degree $2(i-1)$, and b_i^γ has dimension $2(i-2)$ for $i \in \mathbb{N} = \{1, 2, \ldots\}$.

Note that $S(\infty\gamma)$ is the homotopy pushout of

$$\begin{tikzcd} \Sigma_3 \times_{\mathbb{Z}/2} E\mathbb{Z}/2 \arrow[r] \arrow[d] & E\Sigma_3 \\ \Sigma_3/(\mathbb{Z}/2). & \end{tikzcd} \tag{III.3}$$

Apply the functor $F(-, MU_{\Sigma_3})$, taking fixed point after smashing with $S^{\infty\alpha}$, we see that $S^{\infty\alpha} \wedge F(S(\infty\gamma)_+, MU)$ is the homotopy pullback of the following diagram

$$\begin{tikzcd} & (S^{\infty\alpha} \wedge F((E\Sigma_3)_+, MU))^{\Sigma_3} \arrow[d] \\ \Phi^{\mathbb{Z}/2} MU \arrow[r] & \widehat{MU}_{\mathbb{Z}/2} \end{tikzcd} \tag{III.4}$$

Proposition 20. *The vertical arrow of (III.4) induces isomorphism on coefficients.*

Proof. The bottom right corner $\widehat{MU}_{\mathbb{Z}/2}$ is the $\mathbb{Z}/2$-fixed point of the Tate cohomology. Consider the commutative diagram

$$\begin{tikzcd} \mathbb{Z}/3 \arrow[r, "\nu", "\subset"'] \arrow[d, "\lhd"'] & S^1 \arrow[r, "\iota"] \arrow[d, "\lambda"'] & S^1 \times S^1 \arrow[d, "\mu"'] \\ \Sigma_3 \arrow[r, "\gamma_{\mathbb{R}}", "\subset"'] & O(2) \arrow[r, "\kappa"] & U(2) \end{tikzcd} \tag{III.5}$$

where κ is the complexification, λ, μ is the inclusion of maximal torus, and $\iota(z) = (z, z^{-1})$.

Taking the MU-cohomology of classifying spaces, first we have

$$MU^*(B(S^1 \times S^1)) = MU_*[[u_+, u_-]]$$

where $u_+, u_- \in MU^2(\mathbb{C}P^\infty)$ are the Euler classes of the two factors. In these terms, μ^* is injective, and its image has generators

$$u_\alpha = u_+ +_F u_-,\ u_\gamma = u_+ u_-.$$

One can of course instead of u_α also use $u_+ + u_-$, but the advantage of this notation is that u_γ can be considered as the Euler class of the identity representation γ, while u_α is the Euler class of the determinant representation α.

Wilson [Wil84] proved that κ^* is onto, and computed

$$MU^*(BO(2)) = MU^*[[u_\alpha, u_\gamma]]/([2]u_\alpha, u_\gamma - \widetilde{u}_\gamma).$$

In the image of μ^*, we have

$$\widetilde{u}_\gamma = i(u_+)i(u_-)$$

where i denotes the formal inverse. Since $u_\gamma - \widetilde{u}_gamma$ restricts trivially to S^1 via $\mu \circ \iota$, the class is divisible by $u_\alpha u_\gamma$ in $MU^*BU(2)$. This can also be verified directly algebraically, since it is equal in $MU_*[[u_+, u_-]]$ to

$$u_+(u_- - i(u_+)) + (u_+ - i(u_-))i(u_+).$$

When restricting to Σ_3, we additionally have the relation

$$0 = \{3\}u_\gamma := [2]u_+[2]u_- - u_+u_-, \tag{III.6}$$

since the inclusion $\Sigma_3 \subset U(2)$ factors through $\mathbb{Z}/2 \wr \mathbb{Z}/3$, where $[3]u_+ = [3]u_- = 0$. On the other hand, it is important to know that despite the notation, the series $\{3\}u_\gamma$ defined in (III.6) is a power series in both u_α, u_γ.

We have

$$H^*(B\Sigma_3; \mathbb{Z}) = \mathbb{Z}[u_\alpha, u_\gamma]/(2u_\alpha, 3u_\gamma), \tag{III.7}$$

so the Atiyah-Hirzebruch spectral sequence collapses to E_2. Since the right hand side maps to $MU^*B\Sigma_3$ by $\gamma_{\mathbb{R}}^*$, and on the level of associated graded objects with respect to the Atiyah-Hirzebruch filtration, it induces an isomorphism, we conclude that

$$MU^*B\Sigma_3 = MU_*[[u_\alpha, u_\gamma]]/([2]u_\alpha, \{3\}u_\gamma). \tag{III.8}$$

Observe that

$$[2]u_\alpha \equiv 2u_\alpha \mod (u_\alpha^2),$$

$$\{3\}u_\gamma \equiv 3u_\gamma \mod (u_\gamma^2, u_\gamma u_\alpha).$$

Thus,

$$(\{3\}u_\gamma)u_\alpha - ([2]u_\alpha)u_\gamma \equiv u_\alpha u_\gamma \mod (u_\alpha^2 u_\gamma, u_\alpha u_\gamma^2),$$

and thus, the relations of (III.8) imply

$$u_\alpha u_\gamma = 0. \tag{III.9}$$

In particular, the relations (III.8) imply the relation

$$u_\gamma - \widetilde{u}_\gamma = 0.$$

As a result, smashing with $S^{\infty\alpha}$ kills the u_α part, and the vertical map induces isomorphism on homotopy groups. $\square$

Now by diagram (III.4), the fixed points of the lower left corner of (III.1) are just $\Phi^{\mathbb{Z}/2}MU_{\mathbb{Z}/2}$, whose coefficients, by [tD70], are

$$(\Phi^{\mathbb{Z}/2}MU)_* = MU_*[u_\alpha, u_\alpha^{-1}, b_i^\alpha, i \in \mathbb{N}].$$

Now the coefficients of the lower right corner of (III.1) are obtained from the coefficients of (III.4) by inverting u_γ, thus, by inverting u_γ in the coefficients of the lower left corner of (III.4), which gives 0. In other words, the coefficients of the lower right corner of (III.1) are 0, and we obtain

Theorem 21. *We have an isomorphism of rings*

$$(S^{\infty\alpha} \wedge MU_{\Sigma_3})_* \cong (\Phi^{\mathbb{Z}/2}MU)_{*\times} (\Phi^{\Sigma_3}MU)_*. \tag{III.10}$$

□

3.2.2 The coefficients of $F(S(\infty\alpha)_+, MU)$.

The spectrum $F(S(\infty\alpha)_+, MU)^{\Sigma_3}$ is the homotopy pullback of the diagram

$$\begin{tikzcd} & F(E\mathbb{Z}/2_+, \Phi^{\mathbb{Z}/3}MU)^{\mathbb{Z}/2} \arrow[d] \\ F((B\Sigma_3)_+, MU) \arrow[r] & F(E\mathbb{Z}/2_+, \widehat{MU}_{\mathbb{Z}/3})^{\mathbb{Z}/2} \end{tikzcd} \tag{III.11}$$

(which arises from the "Tate square"

$$\begin{tikzcd} MU \arrow[r] \arrow[d] & S^{\infty\gamma} \wedge MU \arrow[d] \\ F(S(\infty\gamma)_+, MU) \arrow[r] & S^{\infty\gamma} \wedge F(S(\infty\gamma)_+, MU) \end{tikzcd}$$

by applying $F(S(\infty\alpha)_+, -)$ and taking Σ_3-fixed points.)

We first calculate the top right corner of (III.11). We notice that $\mathbb{Z}/2$ acts on $(\Phi^{\mathbb{Z}/3}MU)_*$ by a permutation representation, with

$$\widehat{H}^0(\mathbb{Z}/2, (\Phi^{\mathbb{Z}/3}MU)_*) = MU_*[u_\gamma, u_\gamma^{-1}, b_{2i}^\gamma]/2.$$

In this situation, it is formal that the MU-Borel cohomology spectral sequence collapses, and we have:

The coefficient ring $(F(E\mathbb{Z}/2_+, \Phi^{\mathbb{Z}/3}MU)^{\mathbb{Z}/2})_*$ is the pullback of rings

$$\begin{tikzcd} & MU_*[u_\gamma, u_\gamma^{-1}, b_{2i}^\gamma][[u_\alpha]]/[2]u_\alpha \arrow[d] \\ ((\Phi^{\mathbb{Z}/3}MU)_*)^{\mathbb{Z}/2} \arrow[r] & MU_*[u_\gamma, u_\gamma^{-1}, b_{2i}^\gamma]/2 \end{tikzcd}$$

where $u_\alpha \mapsto 0$ by the vertical arrow. (Note that the lower left corner of this diagram means the algebraic fixed points in the category of rings.) $\square$

Now in the coefficients of the lower right corner of (III.11), (III.9) is in effect, so we obtain

$$F(E\mathbb{Z}/2_+, \widehat{MU}_{\mathbb{Z}/3})_* = ((\widehat{MU}_{\mathbb{Z}/3})_*)^{\mathbb{Z}/2}. \tag{III.12}$$

For the same reason, $F(B\Sigma_3, MU)_* = F(E\mathbb{Z}/2_+, F(E\mathbb{Z}/3_+, MU)^{\mathbb{Z}/3})_*$ can be rewritten as the (algebraic) pullback of rings

$$\begin{tikzcd} & (MU^*B\mathbb{Z}/3)^{\mathbb{Z}/2} \arrow[d, "res"] \\ MU^*B\mathbb{Z}/2 \arrow[r, two heads, "res"] & MU_*. \end{tikzcd} \tag{III.13}$$

(Also note that at the upper right corner, we have algebraic fixed points in the category of rings.)

But now by commutation of limits, the pullback of rings

$$\begin{tikzcd} & ((\Phi^{\mathbb{Z}/3}MU)_*)^{\mathbb{Z}/2} \arrow[d] \\ (MU^*B\mathbb{Z}/3)^{\mathbb{Z}/2} \arrow[r] & ((\widehat{MU}_{\mathbb{Z}/3})_*)^{\mathbb{Z}/2} \end{tikzcd} \tag{III.14}$$

is

$$((MU_{\mathbb{Z}/3})_*)^{\mathbb{Z}/2}. \tag{III.15}$$

(Again, this means algebraic fixed points in the category of rings.) Thus, we obtain

Theorem 22. *The coefficient ring $F(S(\infty\alpha)_+, MU_{\Sigma_3})_*$ is the limit of the diagram of rings*

$$\begin{tikzcd} & & MU_*[u_\gamma, u_\gamma^{-1}, b_{2i}^\gamma][[u_\alpha]]/[2]u_\alpha \arrow[d] \\ & ((MU_{\mathbb{Z}/3})_*)^{\mathbb{Z}/2} \arrow[r] \arrow[d, "res"] & MU_*[u_\gamma, u_\gamma^{-1}, b_{2i}^\gamma]/2 \\ MU^*B\mathbb{Z}/2 \arrow[r, two heads, "res"] & MU_* & \end{tikzcd} \tag{III.16}$$

□

It is worth pointing out that by [Kri99, Str01, Hu21], the ring $(MU_{\mathbb{Z}/3})_*$ is now completely known, and the action of $\mathbb{Z}/2$ is explicit:

Theorem 23. *[Hu21] Let p be a prime. For $1 \leq \alpha \leq p-1$, let α^{-1} be the inverse of α in $(\mathbb{Z}/p)^\times$. (Namely, we choose the representative in $\mathbb{Z}$ such that $1 \leq \alpha^{-1} \leq p-1$.) In $\mathbb{Z}$, write $\alpha \cdot \alpha^{-1} = 1 + k_\alpha p$.*

The ring $(MU_{\mathbb{Z}/p})_*$ is isomorphic to

$$MU_*[u, b_{i,j}^{(\alpha)}, \lambda_\alpha, q_j \mid \alpha \in (\mathbb{Z}/p)^\times, i, j \geq 0]/\sim$$

where the relations are:

$$b_{0,0}^{(1)} = u, b_{0,1}^{(1)} = 1, b_{0,j}^{(1)} = 0$$

for $j \geq 2$,

$$b_{i,j}^{(\alpha)} - a_{i,j}^{(\alpha)} = ub_{i,j+1}^{(\alpha)}$$

where $a_{i,j}^{(\alpha)}$ is the coefficient of $x^i u^j$ in $x +_F [\alpha]u$,

$$q_0 = 0, q_j - r_j = uq_{j+1}$$

where r_j is the coefficient of u^j in $[p]u$, and

$$\lambda_1 = 1, \lambda_\alpha b_{0,1}^\alpha = 1 + k_\alpha q_1.$$

□

Note that the relations imply that $\lambda_\alpha u_\alpha = u$ (where $u_\alpha = b_{0,0}^{(\alpha)}$).

One can, in fact, be more explicit about (III.15). In [Hu21], Theorem 23 is derived from [Kri99] analogously as the result of [Str01]. The algebra, however, is substantially more complicated. One major difference between the presentations of [Str01] and [Hu21] in the case of $p = 2$, which helps with the generalization, is that the elements q_j in [Hu21] (and Theorem 23) are chosen in such a way that their relations do not involve the elements $b_{i,j}^{(\alpha)}$.

3.2.3 The coefficients of MU_{Σ_3}.

Now MU_{Σ_3} is the homotopy pullback of the diagram

$$\begin{tikzcd} & S^{\infty\alpha} \wedge MU \arrow[d] \\ F(S(\infty\alpha)_+, MU) \arrow[r] & S^{\infty\alpha} \wedge F(S(\infty\alpha)_+, MU). \end{tikzcd} \tag{III.17}$$

The coefficients of the upper right and lower left corners are known by Theorem 21 and Theorem 22. The coefficients of the lower right corner are obtained by inverting u_α in (III.16). We see, however, that after inverting u_α, the middle row of (III.16) becomes an isomorphism.

Also, we can consider an analogous diagram to (III.17) using just the lower leftmost term of (III.16):

$$\begin{tikzcd} & (S^{\infty\alpha} \wedge MU)_*^{\mathbb{Z}/2} \arrow[d] \\ F(B\mathbb{Z}/2_+, MU)_* \arrow[r] & \widehat{MU}_*^{\mathbb{Z}/2}. \end{tikzcd}$$

This produces $(MU_{\mathbb{Z}/2})_*$ by [Kri99], which is used in [Str01] and Theorem 23 for $p = 2$.

Thus, we need to calculate the algebraic pullback of rings corresponding to the uppermost right corner of (III.16) with the corresponding parts of diagram (III.17). This diagram has the form

$$\begin{tikzcd} & MU_*[u_\gamma, u_\gamma^{-1}, u_\alpha, u_\alpha^{-1}, b_i^\alpha, b_i^\gamma] \arrow[d] \\ MU_*[u_\gamma, u_\gamma^{-1}, b_{2i}^\gamma][[u_\alpha]]/[2]u_\alpha \arrow[r] & u_\alpha^{-1} MU_*[u_\gamma, u_\gamma^{-1}, b_{2i}^\gamma][[u_\alpha]]/[2]u_\alpha. \end{tikzcd} \tag{III.18}$$

In particular, we need to calculate the vertical map (III.18). As in [Kri99], we have

$$b_i^\alpha \mapsto \text{coeff}_{x^i}(x +_F u_\alpha). \tag{III.19}$$

Thus, we need to determine where the elements b_{2i+1}^γ map. To this end, we consider the MU^*-cohomology of $B\mathbb{Z}/2 \wr S^1$. Writing

$$MU^*(S^1 \times S^1) = MU_*[[u_+, u_-]],$$

the Serre spectral sequence collapses to E_2. Denoting the Euler class of the map $\mathbb{Z}/2 \wr S^1 \to \mathbb{Z}/2$

by w, we need a relation of the form

$$0 = w(u_+ + u_- + HOT).$$

The relation can be detected by inflation associated with the map

$$(\mathbb{Z}/2 \ltimes S^1) \times S^1 \to \mathbb{Z}/2 \wr S^1 \tag{III.20}$$

where on the left hand side, an element α of the first copy of S^1 maps to (α, α^{-1}), and an element β of the second copy of S^1 maps to (β, β).

Lemma 24. *This inflation in MU^*-cohomology is injective (and also remains injective when inverting the Euler class w of the projection to $\mathbb{Z}/2$).*

Proof. Restricting to the $S^1 \times S^1$-subgroups, the inflation on MU^*-cohomology can be written as

$$MU_*[[u, v]] \to MU_*[[x, z]] \tag{III.21}$$

where

$$u \mapsto x +_F z, \ v \mapsto i(x) +_F z$$

where $i(x)$ is the formal inverse. Clearly, this is injective. To deduce the statement of the Lemma, by the Serre spectral sequence (which collapses both in the source and the target), it suffices to show that (III.21) induces an injection on the $\mathbb{Z}/2$-Tate cohomology $\widetilde{H}\mathbb{Z}/2$. However, explicitly, on $\widetilde{H}\mathbb{Z}/2$, we get the map

$$MU_*[[uv]] \mapsto MU_*[[xi(x), z]]$$

where

$$uv \mapsto (x +_F z)(i(x) +_F z),$$

which is injective. $\square$

Now we know the cohomology of the classifying space of the source of (III.20). Explicitly, since the first factor is $O(2)$, we can write

$$MU^*B((\mathbb{Z}/2 \ltimes S^1) \times S^1) = MU_*[[u_\alpha, u_\gamma, z]]/([2]u_\alpha, u_\gamma - \widetilde{u}_\gamma).$$

The inflation map is

$$w \mapsto u_\alpha, \ u_+ \mapsto u_+ +_F z, \ u_- \mapsto u_- +_F z.$$

Further, in the target of the inflation, we have the relation

$$(u_+ +_F z)(u_- +_F z) - (i(u_+) +_F z)(i(u_-) +_F z).$$

Thus, in $MU^*B(\mathbb{Z}/2 \wr S^1)$, we obtain the relation

$$u_+u_- - (u_+ +_F w)(u_- +_F w), \tag{III.22}$$

which is of the required form. Thus, we have proved

Theorem 25. *The ring $MU^*B(\mathbb{Z}/2 \wr S^1)$ is isomorphic to the quotient of*

$$MU_*[[u_\alpha, u_\gamma, w]]/[2]w$$

by the relation (III.22) (where, as usual, we write $u_\alpha = u_+ +_F u_-$, $u_\gamma = u_+u_-$).

□

To see what this has to do with the elements b^γ_{2i+1} in (III.18), we note that we have

$$MU_*[u_\gamma, u_\gamma^{-1}, u_\alpha, u_\alpha^{-1}, b_i^\alpha, b_i^\gamma] = \Phi^{\mathbb{Z}/2}(MU \wedge (BU \times BU)_+)[u_\gamma, u_\gamma^{-1}]$$

where $\mathbb{Z}/2$ acts by interchanging the BU coordinates. By the same method as in [Kri99], we then obtain a map from this ring to $MU^*B(\mathbb{Z}/2 \wr S^1)$ given by

$$u_\alpha \mapsto w,\ b_i^\alpha \mapsto \text{coeff}_{x^i}(x +_F w),\ u_\gamma \mapsto u_\gamma,\ b_i^\gamma \mapsto \text{coeff}_{x^i}(x +_F u_+)(x +_F u_-). \tag{III.23}$$

In more detail, start with the composition

$$MU \wedge BU \to (\Phi^{S^1} MU)^{S^1} \to (\widehat{MU}_{S^1})^{S^1} \tag{III.24}$$

where we work over the universe containing only the trivial and standard representation of S^1, and the first map (III.24) comes from the tom Dieck calculation [tD70]. Smashing two copies of (III.24), we obtain a (naively) $\mathbb{Z}/2$-equivariant map, and taking its homotopy fixed points gives the required map (III.23). (To be even more precise, in the target, we have to compose with another map, completing, on the level of Borel cohomology, the smash product, and then inverting the Euler classes.)

Lemma 26. *The map (III.23) is injective.*

Proof. The proof proceeds in the same way as the proof of Lemma 24, once we prove that the map induced by (III.24) on coefficients is injective. This map is

$$b_i \mapsto \mathrm{coeff}_{x^i}(x +_F u) \in MU_*[[u]].$$

We must show that the images of the b_i's are algebraically independent. To this end, note that for $i \geq 1$, the lowest term of the power series in u to which b_i maps is $a_{1,i}u$. If there is an algebraic relation between these elements, it must remain valid after dividing by u. But if those elements are algebraically dependent, they are also algebraically dependent modulo u, which means that the $a_{1,i}$'s are algebraically dependent over MU_*. This is well known not to be the case. (In fact, the coefficients of the series

$$\int_0^x \frac{dt}{\sum_{i\geq 0} a_{i1}t^i}$$

are the coefficients of the universal logarithm, which are algebraically independent by Lazard's theorem.) $\square$

To see what happens to the b_i^α's, we can enhance the relation (III.22) by adding formally another formal variable t, thus obtaining

$$(u_+ +_F t)(u_- +_F t) - (u_+ +_F w +_F t)(u_- +_F w +_F t). \tag{III.25}$$

Note that by the Serre spectral sequence, this relation must in fact follow from (III.22), but it is more convenient for our purposes. In effect, translating back via (III.23), we obtain

$$\sum_{j\geq 1} b_j^\gamma t^j = \sum_{j\geq 1} b_j^\gamma (u_\alpha +_F t)^j, \tag{III.26}$$

valid in the bottom right term of (III.18). Note that examining the t^{j-1} coefficient of (III.26), and using $[2]u_\alpha = 0$, for j odd, we obtain an expression containing a summand of $b_j^\gamma u_\alpha$ and possibly $b_k^\gamma u_\alpha$ with some additional coefficients for $k < j$, modulo higher powers of u_α. Working by induction on j, we can eliminate the summands $b_k^\gamma u_\alpha$ with k odd modulo higher powers of u_α, and then repeat the procedure, ultimately expressing $b_j^\gamma u_\alpha$ as a power series in u_α (in powers ≥ 2) whose coefficients are polynomials in the b_i's with i even. In particular, this gives recursive relations in the lower right corner of (III.18) of the form

$$b_{2i+1}^\gamma = \sum_{j\geq 1} c_j u_\alpha^j, \tag{III.27}$$

where c_j are polynomials with coefficients in the b_{2k}^γ's.

For example, we have

$$\begin{aligned}
&b_1^\gamma = -b_2^\gamma u_\alpha - 2x_1 b_2^\gamma u_\alpha^2 \\
&+(-6x_1^2 b_2^\gamma - 3x_2 b_2^\gamma + 5b_4^\gamma)u_\alpha^3 + \\
&(-40x_1^3 b_2^\gamma + x_1x_2 b_2^\gamma - 43x_3 b_2^\gamma + 54x_1 b_4^\gamma)u_\alpha^4 + ...,
\end{aligned}$$

also

$$\begin{aligned}
&b_3^\gamma = 2x_1 b_2^\gamma + (6x_1^2 b_2^\gamma + 3x_2 b_2^\gamma - 6b_4^\gamma)u_\alpha \\
&+(42x_1^3 b_2^\gamma - 3x_1x_2 b_2^\gamma - 33x_3 b_2^\gamma - 58x_1 b_4^\gamma)u_\alpha^2 + ...,
\end{aligned}$$

and

$$b_5^\gamma = -2x_1^3 b_2^\gamma + 2x_1x_2 b_2^\gamma + 4x_3 b_2^\gamma + 4x_1 b_4^\gamma + ... \,.$$

This lets us calculate the pullback of rings (III.18) by the same method as in [Hu21, Str01]. The answer is the ring

$$\begin{aligned}
&R = MU_*[u_\gamma, u_\gamma^{-1}, u_\alpha, b_{i,j}^\alpha, q_j, b_{2i}^\gamma, b_{2i+1,j}^\gamma]/ \\
&(b_{i,j}^\alpha - a_{ij} = b_{i,j+1}^\alpha u_\alpha, q_0 u_\alpha, q_j - r_j = q_{j+1}u_\alpha, b_{2i+1,j}^\gamma - c_j = b_{2i+1,j+1}^\gamma u_\alpha)
\end{aligned} \tag{III.28}$$

Note that this maps canonically to $MU_*[u_\gamma, u_\gamma^{-1}, b_{2i}^\gamma]/2$ by mapping via the vertical arrow in (III.18) (which we determined explicitly), and taking the constant term of the applicable u_α-series. In summary, we have our main result:

Theorem 27. *The ring $(MU_{\Sigma_3})_*$ is the limit of the diagram of rings*

$$\begin{tikzcd}
 & & R \arrow[d] \\
 & ((MU_{\mathbb{Z}/3})_*)^{\mathbb{Z}/2} \arrow[r] \arrow[d, two heads, "res"] & MU_*[u_\gamma, u_\gamma^{-1}, b_{2i}^\gamma]/2 \\
(MU_{\mathbb{Z}/2})_* \arrow[r, two heads, "res"] & MU_* &
\end{tikzcd} \tag{III.29}$$

where the rightmost vertical arrow is described above.

□

3.3 Connections to equivariant formal group laws

An $RO(G)$-graded equivariant cohomology theory for a compact Lie group G is called complex oriented if it satisfies the Thom isomorphism with respect to all G-equivariant complex bundles. For an abelian compact Lie grouip A, the theory of A-equivariant formal group laws is established. It was conjectured by Greenlees that the Lazard ring for A-equivariant formal group laws is isomorphic to the stable equivariant cobordism ring MU_*^G. This was recently proved for $G = \mathbb{Z}/2$ by Hanke and Wiemeler, and in full generality by Hausmann. In this chapter we give another proof of this conjecture for finite cyclic groups.

If A is an abelian compact Lie group, an A-equivariant formal group law over a commutative ring k is

(1). a complete topological Hopf k-algebra R with

(2). a homomorphism $\theta : R \to k^{A^*}$ of topological Hopf k-algebras so that the topology on R is defined by the finite intersections of kernels of its components $\theta_\alpha : R \to k$ for $\alpha \in A^*$.

(3). an element $x(\epsilon) \in R$ which is (i) regular and (ii) generates the kernel of the ϵth component of θ; equivalently, $x(\epsilon)$ gives an exact sequence

$$0 \to R \xrightarrow{x(\epsilon)} R \to k \to 0.$$

If A is finite, axiom (2) shows that the topology on R is defined by the single ideal ker β. Since θ is a map of Hopf algebras it follows that θ_ϵ is the counit of R.

The element x_ϵ is called the coordinate of the formal group law, since in geometric terms it is a function whose vanishing defines the identity of the group. If the coordinate is not specified, the resulting structure represents an equivariant formal group. Indeed, by axiom 1, R may be viewed as the ring of functions on a group object G in the category of formal schemes over k.

The k-module structure of every equivariant formal group law is topologically free, and we may therefore express the structure maps of R in terms of the basis. There is an action of A^* on R via $\ell_\alpha(r) = (\theta_{\alpha^{-1}} \otimes 1)\Delta(r)$. Thus the element $x(\epsilon)$ determines elements $x(\alpha)$ for $\alpha \in A^*$ by the formula $x(\alpha) = \ell_\alpha(x(\epsilon))$. A complex complete A-universe is a countably infinite dimensional complex representation of A in which every simple representation occurs infinitely often. Then we have the following result:

Theorem 28. *If we choose a complete A flag $F = V_1 \subset V_2 \subset V_3 \subset ...$ in a complete universe, then an equivariant formal group law R has an additive topological k-basis $1, x(V_1), x(V_2), ...$ where $x(V^n) = x(\alpha_1)...x(\alpha_n)$ if $V_n = \alpha_1 \oplus ... \oplus \alpha_n$.*

Note that if A is the trivial group, the definition reduces to the usual concept of a (non-equivariant, commutative, one dimensional) formal group law.

Note that the set of A-equivariant formal group laws over k is a functor of the ring k. This functor is represented by a ring L_A, the equivariant Lazard ring for A-equivariant formal group laws. The ring L_A may be constructed by giving generators for each of the structure constants, and imposing relations to ensure that the axioms of definition hold. The A-equivariant formal group law over k corresponding to a ring homomorphism $f : L_A \to k$ is the one with structure constants given by the image of the corresponding generators of L_A.

A G-equivariant cohomology theory E is *complex stable* if there are suspension isomorphisms

$$\sigma_V : \tilde{E}^n(X) \xrightarrow{\cong} \tilde{E}^{n+|V|}(\Sigma^V X)$$

for all complex representations V, where $|V|$ is the real dimension of V.

Given an A-equivariant formal group law we may define the Euler class of a one dimensional representation α by

$$u_\alpha = \theta_\epsilon(x(\alpha)).$$

The Euler class u_α is the value of the coordinate $x(\alpha)$ at the identity. Note that by definition we have $u_\epsilon = 0$.

We also have

$$u_k = \epsilon(\alpha^{-\ell})(x_{\alpha^{\ell-k}}),$$

$$u_0 = 0.$$

Thus, inductively we can compute,

$$\begin{aligned} u_k &= \epsilon(\alpha^{-k})(x) = (\epsilon(\alpha^{-1}) \otimes \epsilon(\alpha^{1-k}))(\Delta(x)) \\ &= u_{k-1} + u \cdot (\sum_{i=0}^{k-1} \overline{a_{1,i}}(\prod_{j=0}^{i-1} u_{k-1-j})). \end{aligned} \tag{III.30}$$

Thus, referring to (III.30), we obtain a relation among the elements $u, \overline{a_{i,j}} \in A$ given by

$$u_n = 0 \tag{III.31}$$

which has the form

$$nu \mod (u^2). \tag{III.32}$$

We will also need the following result: let p be a prime factor of n, and define $m = n/p$. Similar computation gives

$$\begin{aligned} u_{km} &= \epsilon(\alpha^{-km})(x) = (\epsilon(\alpha^{-m}) \otimes \epsilon(\alpha^{m-km}))(\Delta(x)) \\ &= (\epsilon(\alpha^{-m}) \otimes \epsilon(\alpha^{m-km}))(\sum_{i,j\geq 0} \overline{a_{i,j}} x_i \otimes x_j) \\ &= u_m + u_{(k-1)m} + \sum_{i=1}^{m} \sum_{j=1}^{(k-1)m} \overline{a_{i,j}}(\prod_{s=0}^{i-1} u_{m-s})(\prod_{t=0}^{j-1} u_{(k-1)m-t}). \end{aligned} \tag{III.33}$$

The relation (III.31) therefore has an alternative form

$$pu_m \mod (u_m^2). \tag{III.34}$$

Another set of relations is obtained as follows. If $(A, R, \Delta, \epsilon, x_L)$ is a $\mathbb{Z}/n$-equivariant formal group law, then $R^\wedge_{(x)} = A[[x]]$ (since x is regular, $R/(x) = A$ implies $R/(x^n) \cong A[x]/(x^n)$). Thus, applying the completion map

$$R \to A[[x]],$$

the coproduct Δ maps to a non-equivariant formal group law on A. By Lazard's theorem, we obtain an expression of the coefficients $a_{i,j}$ as polynomials of $\overline{a_{i,j}}$ and u. Note also that, by the definition of $\overline{a_{i,j}}$, with this identification, we have

$$\overline{a_{i.j}} \equiv a_{i,j} \mod (u) \tag{III.35}$$

(since $x_L \equiv x \mod (u)$). Thus, the relations

$$r(a_{i,j}) \tag{III.36}$$

in the Lazard ring give, by substitution, a set of relations among the $\overline{a_{i,j}}$'s and u. (Recall that modulo decomposables, the relations among the $a_{i.j}$'s say that they are all multiples of the Lazard generators x_{i+j}, which in turn, modulo indecomposables, is a linear combination of the $a_{i,j}$'s with coefficients in $\mathbb{Z}$.)

Now we state the main result:

Theorem 29. *(1) There exists a $\mathbb{Z}/n$-equivariant formal group law*

$$(A, R, \Delta, \epsilon, x_L)$$

where A is the quotient ring of $\mathbb{Z}[u, \overline{a_{i,j}}]$ modulo the relations (III.31), (III.36). Furthermore, this $\mathbb{Z}/n$-equivariant formal group law is universal in the sense that for any $\mathbb{Z}/n$-equivariant formal group law $(A', R', \Delta', \epsilon', x'_L)$, there exists unique ring homomorphisms $A \to A'$, $R \to R'$, which carries Δ to Δ', ϵ to ϵ', and x_L to x'_L.

(2) There is an isomorphism $A \cong (MU_{\mathbb{Z}/n})_*$.

The theorem will be proved by induction. For clarity we will sometimes write A_n for the ring A in the statement above. Throughout we use the complete flag $\{V_i\}_{i\geq 0}$ defined above.

Proof. First, we observe that if we have a homomorphism of rings $f : A \to A'$, there exists at most one equivariant formal group law $(A', R', \Delta', \epsilon', x'_L)$ over A' such that the coproduct formula holds with $\overline{a_{i,j}}$ replaced by $f(\overline{a_{i,j}})$, and other notations replaced. Applying f to the coefficients of the computations of the coproduct formula, we get formulas for all euler classes $u'_k \in A'$ in terms of the images of $\overline{a_{i,j}}$'s. Now

$$\epsilon'(\alpha^{-1})(x_k) = \prod_{i=1}^{k} u'_i \in A'.$$

Therefore,

$$\epsilon'(\alpha^{-\ell})(x_k) = \prod_{i=1}^{k} u'_{i+\ell}.$$

This determines ϵ' by linear extension. Since

$$\Delta'(x') = \sum_{i,j} f(\overline{a_{i,j}})\, x'_i \otimes x'_j,$$

the following formula determines elements

$$x'_L = (\epsilon'(L) \otimes 1)\Delta'(x_1).$$

For example when $p = 2, L = \alpha$ this gives

$$x'_\alpha = u' + x'_1 + u' f(\overline{a_{1,1}})x'_1 + u' f(\overline{a_{1,2}})x'_2 + ...).$$

It holds a priori that

$$x'_\alpha x'_1 = x'_2,$$

$$x'_{\alpha^2} x'_2 = x'_3,$$

$$...$$

To calculate $x'_{\alpha^\ell} x'_k$ for any $\ell \in \mathbb{Z}/n$ and $k \geq 1$, the recipe is to rotate the above formula for $x'_{\alpha^{\ell-k}}$ by α^k. This gives x'_{α^ℓ} in terms of a new basis

$$\{V_{k+i}/V_k\}_{i \geq 1} = \{\alpha^k V_i\}_{i \geq 1}.$$

Now since x'_k corresponds to V_k and

$$(V_{k+i}/V_k) \oplus V_k \cong V_{k+i},$$

we have $x'_{\alpha^\ell} x'_k$ in original basis $\{V_i\}_{i \geq 1}$.

Similarly, we can compute $x'_{\alpha^\ell} x'_m$ for any ℓ, m. Thus, by induction, the product in R' is determined.

Now by Axiom (2),

$$\begin{aligned}\Delta'(x'_{\alpha^\ell}) &= \Delta'(\epsilon'(\alpha^\ell) \otimes Id) \circ \Delta'(x')) \\ &= ((\epsilon'(\alpha^\ell) \otimes Id) \circ \Delta') \otimes (\epsilon') \otimes Id) \circ \Delta') \circ \Delta'(x') \\ &= \sum_{i,j} f(\overline{a_{i,j}}) \prod_{k=1}^{i} x'_{\alpha^{\ell+k-1}} \otimes x'_j,\end{aligned}$$

(since $(\epsilon'(\alpha^\ell) \otimes Id) \circ \Delta'(x'_i) = \prod_{k=1}^{i} x'_{\alpha^{\ell+k-1}}$). Thus Δ' is also determined.

Note that we do not yet know that A actually supports a $\mathbb{Z}/n$-equivariant formal group law. However, we have the following

Lemma 30. *The ring $A_{\mathbb{Z}/n}$ has u_m-torsion of order less or equal to 1, i.e. for every $z \in A_{\mathbb{Z}/n}$, if $u_m^2 z = 0$, then $u_m z = 0$.*

Proof. It will be convenient to introduce the polynomial generators x_k of the (non-equivariant) Lazard ring L. We also join another formal generator u_m to A. Then we can write

$$A = \mathbb{Z}[\overline{a_{i,j}}, x_k, u, u_m]/(r_{i,j}, u_m = [m]u, [p]u_m)$$

where the relations

$$r_{i,j} = a_{i,j} - q_{i,j}(x_k)$$

are given by thinking of $a_{i,j}$ as a polynomial in the $\overline{a_{i',j'}}$'s, and

$$[p]u_m, \quad u_m = [m]u$$

are results from computations (III.30), and

$$[p]u_m$$

is (III.34) of form

$$pu_m \mod u_m^2.$$

Suppose

$$u_m^2 \mid Q = c \cdot [p]u_m + \sum c_{i,j} r_{i,j} + d \cdot [m]u$$

for some $c_{i,j}, c, d \in \mathbb{Z}[\overline{a_{i,j}}, x_k, u, u_m]$. Then, factoring out u_m and considering non-equivariant formal group law theory (namely the algebraic independence of the relations $r_{i,j}$), we conclude that

$$u_m \mid c_{i,j},\ u_m \mid d$$

for all i, j. Thus, we may write

$$\mathrm{coeff}_{u_m}(Q) = cp + \sum \frac{c_{i,j}}{u_m} r_{i,j} + \frac{d}{u_m}. \tag{III.37}$$

Then assuming $u_m \nmid c$, factoring out u_m, (III.37) would again contradict the algebraic independence of the relations $r_{i,j}$ of the classical (non-equivariant) Lazard ring. Therefore, $u_m \mid c$, and therefore the relation Q can be divided by u_m, as claimed. $\square$

Recall that we choose $p \mid n$ and let $m = n/p$. Let $\mathscr{F} = \mathscr{F}(\mathbb{Z}/m)$ be the family of subgroups contained in $\mathbb{Z}/m$. Denote the universal space for this family by $E\mathbb{Z}/p$ (since $E\mathbb{Z}/p$ is a model for $E\mathscr{F}(\mathbb{Z}/m)$) and consider the cofiber $\widetilde{E\mathbb{Z}/p}$ of $E\mathbb{Z}/p_+ \to S^0$, we have a pullback square for equivariant complex cobordism

$$\begin{tikzcd} MU_{\mathbb{Z}/n} \arrow[r] \arrow[d] & \widetilde{E\mathbb{Z}/p} \wedge MU_{\mathbb{Z}/n} \arrow[d] \\ F(E\mathbb{Z}/p_+, MU_{\mathbb{Z}/n}) \arrow[r] & \widetilde{E\mathbb{Z}/p} \wedge F(E\mathbb{Z}/p_+, MU_{\mathbb{Z}/n}). \end{tikzcd} \tag{III.38}$$

On the other hand, by [Str01], we have a pullback square for $A = A_{\mathbb{Z}/n}$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & u_m^{-1}A \arrow[d] \\ A^{\wedge}_{u_m} \arrow[r] & u_m^{-1}A^{\wedge}_{u_m}. \end{tikzcd} \tag{III.39}$$

The coefficient of the square (III.38) is

$$\begin{tikzcd} (MU_{\mathbb{Z}/n})_* \arrow[r] \arrow[d] & u_m^{-1}(MU_{\mathbb{Z}/n})_* \arrow[d] \\ (MU_{\mathbb{Z}/n})^{\wedge}_{*u_m} \arrow[r] & u_m^{-1}(MU_{\mathbb{Z}/n})^{\wedge}_{*u_m}. \end{tikzcd} \tag{III.40}$$

Now we shall argue that the top row and the left hand column actually define $\mathbb{Z}/n$-equivariant formal group laws on the respective target rings, and the rings are isomorphic to the corresponding coefficients in the square (III.40).

In the case of the bottom left corner, we start with the case when n is a prime. This is due to the fact that if we are allowed to sum infinite power series in u, then the $\overline{a_{i,j}}$'s can also be expressed as power series in the $a_{i,j}$'s (rather than just vice versa). Under this correspondence, the relation (III.31) just becomes $[p]u = 0$. Thus,

$$A^{\wedge}_{(u)} \cong L[[u]]/[p]u$$

where L is the non-equivariant Lazard ring.

In the general case, use induction assumption we have

$$A_{\mathbb{Z}/n}/u_m \cong A_{\mathbb{Z}/m} \cong (MU_{\mathbb{Z}/m})_* \cong (MU_{\mathbb{Z}/n})_*/u_m$$

(the first isomorphism follows from our definition and the last isomorphism comes computations for equivariant complex cobordism). We can use Borel cohomology spectral sequence to compute the associated graded ring of $((MU_{\mathbb{Z}/n})_*)^{\wedge}_{u_m}$

$$E_2^{s,t} = H^s(\mathbb{Z}/p, \pi_t(MU_{\mathbb{Z}/n})) \Rightarrow \pi_* F(E\mathbb{Z}/p_+, MU_{\mathbb{Z}/n}).$$

It collapses since the coefficient concentrates in even degrees and there are no p-torsions, and gives the associated graded ring as

$$(MU_{\mathbb{Z}/m})_*[[u_m]]/[p]u_m.$$

Denote $(MU_{\mathbb{Z}/n})_*$ by S and denote u_m by ω:

$$\omega^r S/\omega^{r+1} S \cong \omega^r S^{\wedge}_{\omega}/\omega^{r+1} S^{\wedge}_{\omega} \cong S/(\omega, p).$$

Compare the short exact sequence

$$0 \to pS/\omega S \to S/\omega S \to S/(\omega, p) \cong \omega^r S/\omega^{r+1} S \to 0$$

with

$$0 \to \ker q \to A/\omega A \xrightarrow{q} \omega^r A/\omega^{r+1} A \to 0$$

and $[p]\omega/\omega \in \ker q$ maps to $p \in pS/\omega S$, hence the map

$$\ker q \to pS/\omega S$$

is surjective and

$$\omega^r A/\omega^{r+1} A \to \omega^r S/\omega^{r+1} S$$

is an isomorphism for all $r \geq 1$. Thus,

$$A^{\wedge}_{u_m} \cong ((MU_{\mathbb{Z}/n})_*)^{\wedge}_{u_m}.$$

On $u_m^{-1}A$, we are inverting a certain collection of euler classes, namely those of representations α^k's for $k \mid m$. We know by the Chinese Remainder Theorem that we must put

$$R = \prod_{\alpha \in \widehat{G}, u_\alpha \text{ inverted}} A[[x_\alpha]].$$

Now the relations (III.36) give an x-completed coproduct on $A[[x]]$, which we denote by $F(y, z)$ (i.e. we really have $y = x \otimes 1$, $z = 1 \otimes x$). The equivariant formal group laws axioms (2) and (3) imply that the coproduct on x_α must be

$$\prod_{\beta+\gamma=\alpha} \Delta(y_\beta, z_\gamma)$$

where $y_\alpha = x_\alpha \otimes 1$, $z_\alpha = 1 \otimes x_\alpha$. The relation (III.31) implies that this $\mathbb{Z}/n$-equivariant formal group laws definition is consistent, as well.

Moreover, since the $\mathbb{Z}/n$-equivariant formal group laws just defined on the corners of the diagram (III.39) are both induced from maps from the pullback A, they coincided when pushed forward to $u_,^{-1}(A^{\wedge}_{(u_m)})$.

By general universal algebra, the compatible $\mathbb{Z}/n$-equivariant formal group laws on the three remaining corners of (III.39) define a $\mathbb{Z}/n$-equivariant formal group law on A, which is induced, in the above sense, by $Id : A \to A$. It follows from the similar formal argument the $\mathbb{Z}/n$-equivariant formal group law on A is universal.

Additionally, by the explicit computation just performed, the limit diagram (III.39) (and hence the pullback) coincide with the corresponding terms of [Kri99], and thus, $A_{\mathbb{Z}/n} \cong (MU_{\mathbb{Z}/n})_*$.

□

3.4 Conclusion

By some algebraic tools, the structure of $(MU_{\Sigma_3})_*$ in Theorem 27 can be calculated explicitly. More details could be found in the appendix of [HKL21].

There are no satisfactory definitions for equivariant formal group laws for nonabelian groups yet, due to the reason that representations are not necessarily 1-dimensional. Hence we need to consider all the equivariant classifying spaces $BU_G(n)$ for $n > 0$, with structure maps induced by direct sum and tensor product of vector bundles. Schwede's splitting [Sch22] suggests it might be sufficient to model the E-cohomology of BU_G, which is the 0-connected component of the infinite loop space of G-equivariant K-theory KU_G. However, it is still conjectured that, with suitable definitions, there should be an isomorphism between $(MU_G)_*$ and the Lazard ring L_G. Such calculations may give an idea for giving a suitable definition.

On the other hand, nonabelian groups also arise naturally in many questions. In chromatic homotopy theory, the groups Q_8 arises as Sylow 2-subgroup for the Morava stabilizer group at prime 2, which is a fundamental object of the subject. The recent study on triangulation conjecture of manifolds also involves Q_8-actions. In [Man16], Manolescu looked at $Pin(2)$-equivariant Seiberg-Witten Floer homology. The group $Pin(2)$ contains Q_8 as a finite subgroup. It is the author's hope to apply these calculations to study other questions.

APPENDIX A

Equivariant Stable Homotopy Theory

1.1 The Category of G-Spectra

Let G be an arbitrary finite group, which we fix throughout this appendix. Furthermore, we will restrict to compactly generated weak Hausdorff topological spaces.

A G-space is a space X together with a continuous left action of the group G. A G-equivariant map (or simply equivariant map) $f : X \to Y$ of G-spaces commutes with the G-actions on X and Y. A pointed G-space is equipped with a G-fixed basepoint, and a pointed map between pointed G-spaces should respect basepoints.

Now, we define the category Top^G to have G-spaces as objects and equivariant maps as morphisms. Similarly, we define the category Top_*^G to have pointed G-spaces as objects and pointed equivariant maps as morphisms. However, we define the category T_*^G to have pointed G-spaces as its objects but continuous maps as its morphisms. Both Top_*^G and T_*^G are closed symmetric monoidal categories under smash product with the 0-sphere S^0 as the unit object. The group acts diagonally on the smash product and the adjunction reads as

$$Top_*^G(X \wedge Y, Z) \cong Top_*^G(X, F(Y, Z))$$

where $F(X, Y)$ denotes the pointed space of based maps from Y to Z.

An orthogonal G-spectrum X consists of the following data:

- pointed spaces X_n for $n \geq 0$, with a continuous left action by $O(n) \times G$, where $O(n)$ is the orthogonal group of $\mathbb{R}^n$.
- based structure maps $\sigma_n : X^n \wedge S^1 \to X_{n+1}$ that are G-equivariant with respect to the trivial action on S^1.
- the iterated structure map $\sigma^m : X_n \wedge S^m \to X_{n+m}$ is $O(n) \times O(m)$-equivariant.

We will usually abbreviate the notion as G-spectra. A morphism $f : X \to Y$ between two G-spectra is a collection of $O(n) \times G$-equivariant based maps $f_n : X_n \to Y_n$, which are compatible with the structure maps in the sense that $f_{n+1} \circ \sigma_n = \sigma_n \circ (f_n \wedge S^1)$ for $n \geq 0$. Hence we have a category Sp_G of orthogonal G-spectra.

Example 31. *Suspension spectra. Every pointed G-space A gives rise to a suspension spectrum $\Sigma^\infty A$ via*

$$(\Sigma^\infty A)_n = A \wedge S^n.$$

The orthogonal group acts through the action on S^n, the group G acts through the action on A, and the structure maps are the canonical homeomorphism

$$(A \wedge S^n) \wedge S^1 \cong A \wedge S^{n+1}.$$

For example, the sphere spectrum $\mathbb{S}$ is isomorphic to the suspension spectrum $\Sigma^\infty S^0$ (where G necessarily acts trivially on S^0).

There are other definitions of G-spectra. They are not the same, however they give equivalent categories. Hence it is one's favor to use different models. One of the other models is the theory of Lewis-May spectra, which we will also use in the calculations. The lecture notes [Sch16] gives a detailed account of orthogonal spectra. For treatment of Lewis-May spectra, we refer to [LMSM86].

1.2 Equivariant Homotopy Groups

Let X be a G-spectrum and V a representation of G. The loop spectrum $\Omega^V X$ is defined by

$$(\Omega^V X)_n = \Omega^V(X_n) = \text{map}(S^V, X_n).$$

Here $\text{map}(-,-)$ denotes the based mapping space of non-equivariant based maps. The group $O(n)$ acts through its action on X_n and G acts by conjugation. The structure map is given by the composition

$$\text{map}(S^V, X_n) \wedge S^1 \to \text{map}(S^V, X_n \wedge S^1) \to \text{map}(S^V, X_{n+1})$$

where the first map sends $\varphi \wedge t$ to $v \mapsto \varphi(v) \wedge t$.

The suspension spectrum $S^V \wedge X$ is defined similarly by

$$(S^V \wedge X)_n = S^V \wedge X_n$$

The group $O(n)$ acts on X_n and G acts diagonally. The structure map is the composite

$$(S^V \wedge X)_n \wedge S^1 = S^V \wedge X_n \wedge S^1 \to S^V \wedge X_{n+1}$$

Let ρ_G be the regular representation of G. The 0-th equivariant homotopy group $\pi_0^G(X)$ of an orthogonal G-spectra is defined as

$$\pi_0^G(X) = \text{colim}_n[S^{n\rho_G}, X(n\rho_G)]^G$$

where $[-,-]^G$ means the homotopy class of based G-maps. The colimit is taken along stabilization at regular representations.

We can define general homotopy groups: if k is a positive integer, we define

$$\pi_k^G(X) = \pi_0^G(\Omega^k X)$$

if k is negative, we define

$$\pi_k^G(X) = \pi_0^G(X \wedge S^{-k})$$

It is not hard to see that they are indeed abelian groups, as there are trivial representations in $n\rho_G$'s.

A morphism $f : X \to Y$ is a $\underline{\pi}_*$-isomorphism if the induced map $\pi_k^H(f) : \pi_k^H(X) \to \pi_k^H(Y)$ is an isomorphism for all integers k and all subgroups H of G. We define the G-equivariant stable homotopy category $Ho(Sp_G)$ as the category obtained from Sp_G via formally inverting all $\underline{\pi}_*$-isomorphisms.

As a rough summary, the category Sp_G, together with the stable category $Ho(Sp_G)$, enjoys the following nice properties [HHR16]:

- The functor Σ^∞ admits a right adjoint Ω^∞. And they induce adjoint functors $R\Omega^\infty \vdash L\Sigma^\infty$ passing to homotopy categories:

$$L\Sigma^\infty : Ho(Top_*^G) \to Ho(Sp_G)$$

$$R\Omega^\infty : Ho(Sp_G) \to Ho(Top_*^G)$$

- Both Sp_G and $Ho(Sp_G)$ are closed symmetric monoidal categories under smash product. The unit object is the sphere spectrum $\mathbb{S}$. And the functor $L\Sigma^\infty$ is symmetric monoidal.

- The functor $L\Sigma^\infty$ extends to a fully faithful, symmetric monoidal embedding of the Spanier-Whitehead category into $Ho(Sp_G)$.

- The objects S^V are invertible in $Ho(Sp_G)$ under the smash product.

- Arbitrary coproducts exist in $Ho(Sp_G)$ and can be computed by levelwise wedges.

- (homotopy presentation) Up to weak equivalence every object X is presentable in Sp_G as a homotopy colimit
$$...S^{-V_n} \wedge X_{V_n} \to S^{-V_{n+1}} \wedge X_{V_{n+1}} \to ...$$
in which $\{V_n\}$ is a fixed increasing sequence of representations eventually containing every finite dimensional representation of G, and each X_{V_n} is weakly equivalent to a suspension spectrum of a G-CW complex.

1.3 Change of groups and the Wirthmüller Isomorphism

Suppose that H is a subgroup of G, the restriction functor $\mathrm{res}^G_H : Sp_G \to Sp_H$ simply pulls back G-action to H-action. This restriction functor has both left adjoint and right adjoint, respectively called induced spectrum functor and coinduced spectrum functor.

Suppose that Y is an H-orthogonal spectrum. The induced G-spectrum, denoted by $G \ltimes_H Y$, is defined by
$$(G \ltimes_H Y)_n = G \ltimes_H Y_n,$$
with induced action by the orthogonal group and induced structure maps. The coinduced G-spectrum, denoted by $\mathrm{map}^H(G, Y)$, is defined by
$$(\mathrm{map}^H(G, Y))_n = \mathrm{map}^H(G, Y_n)$$
with induced action by the orthogonal group and induced structure maps.

At each level, we can define a G-map $G \ltimes_H Y_n \to \mathrm{map}^H(G, Y_n)$ by
$$\Psi_{Y_n}(g \ltimes y)(\gamma) = \begin{cases} \gamma g y & \text{if } \gamma g \in H \\ * & \text{if } \gamma g \notin H \end{cases}$$
They assemble to a morphism of orthogonal G-spectrum $\Psi_Y : G \ltimes_H Y \to \mathrm{map}^H(G, Y)$. Wirthmüller Theorem states that for finite group G, it is a $\underline{\pi}_*$-isomorphism of spectra.

Theorem 32. *(Wirthmüller) Let H be a subgroup of a finite group G, and Y an orthogonal H-spectrum. Then the morphism*
$$\Psi_Y : G \ltimes_H Y \to \mathit{map}^H(G, Y)$$
is a $\underline{\pi}_*$-isomorphism.

1.4 The Tate Square

A family $\mathscr{F}$ of subgroups of a finite group G is a collection of subgroups which is closed under subgroups and conjugations. Given a family $\mathscr{F}$, there is a universal space $E\mathscr{F}$ characterized by

$$E\mathscr{F}^H \simeq \begin{cases} * & \text{when } H \in \mathscr{F} \\ \varnothing & \text{else} \end{cases}$$

If the family $\mathscr{F} = \{e\}$, then $E\mathscr{F}$ is the universal space EG.

We can form a cofiber sequence

$$E\mathscr{F}_+ \to S^0 \to \widetilde{E\mathscr{F}}.$$

Hence $\widetilde{E\mathscr{F}}$ is characterized by

$$\widetilde{E\mathscr{F}}^H \simeq \begin{cases} * & \text{when } H \in \mathscr{F} \\ S^0 & \text{else} \end{cases}$$

Let X be a genuine G-spectrum. Smashing X with the cofiber sequence above, as well as applying the functor $F(E\mathscr{F}, -)$, we obtain the Tate diagram

$$\begin{tikzcd} E\mathscr{F}_+ \wedge X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \widetilde{E\mathscr{F}} \wedge X \arrow[d] \\ E\mathscr{F}_+ \wedge F(E\mathscr{F}_+, X) \arrow[r] & F(E\mathscr{F}_+, X) \arrow[r] & \widetilde{E\mathscr{F}} \wedge F(E\mathscr{F}_+, X) \end{tikzcd}$$

The left vertical map is an equivalence. As a result the Tate square (the square on the right) is a homotopy pullback, and it would be a homotopy pullback of rings if X is a ring spectrum.

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