Representation Theory of Curried Algebras and Non-primality of Certain Symmetric Ideals

by

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Dedication

To my family, teachers, and friends.

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ABSTRACT

This thesis consists of two separate parts. The first part (Chapter II to Chapter VI) is about the representation theory of curried algebras, and the second part (Chapter VII) gives a proof of non-primality of certain symmetric ideals.

In Chapter II, we review some preliminary background on the representation theory of curried algebras. We summarize results from [SS2] and [SS3], and present definitions that will be used later, such as triangular categories, the Brauer category, etc. We end the chapter by summarizing comparison theorems associated to modules of a curried algebras over a standard $\mathbf{FB}$-module.

In Chapter III, we explain the representation theory of certain inverse monoids, namely the Rook monoid R_n and $\mathscr{P}_n$-monoid. Representation theory of R_n has been extensively studied: for instance, see [Mu1], [Mu2], [CP1], [CP2], [Sol], or [St]. On the other hand, to our knowledge, the monoid $\mathscr{P}_n$ or its representation theory has not been studied. Curried algebras give a clear reason to study the representation theory of $\mathscr{P}_n$. In Chapter IV we will show that in a Tannakian sense, actions of $\mathbf{N} \times \mathscr{P}_n$ for all $n \in \mathbf{Z}_{\geq 0}$ can be identified as a curried symplectic Lie algebra $\underline{\mathfrak{sp}}(\mathbf{V})$ associated to the standard module $\mathbf{V}$. Later in Chapter V we will relate modules over $\mathscr{P}_n$ with those of decorated Brauer categories. Keeping these in mind, one of the main goals of Chapter III is to translate the representation theory of $\mathscr{P}_n$ (which naturally arises from the study of curried algebras) to those of Rook monoid R_n. In particular, we show that although R_n and $\mathscr{P}_n$ are not isomorphic as monoids, the algebras kR_n and $k\mathscr{P}_n$ generated by the two monoids are isomorphic as rings. In addition, we present new results on a tensor product structure associated to Rook monoids, such as Littlewood-Richardson theorem for Rook monoids. Proofs that involve properties of inverse monoids will be given in Appendix A.

Chapter IV gives the definition of a B-category. In short, B-categories are categories equipped with combinatorial structures that give a nice theory of curried algebras associated to a standard module. In that setting, we extend the theory of curried algebras of the standard $\mathbf{FB}$-module, first introduced in [SS3]. In particular we show that if $\mathfrak{C}$ is a B-category, then a $\mathfrak{C}$-module $M = (M_n)_{n \geq 0}$ is a module over a curried general linear algebra of the standard module $\mathbf{V}$ if and only if each degree n piece M_n has additional $\mathscr{P}_n$-module structure, compatible with the intrinsic combinatorial structure of M_n obtained from the action of $\mathbf{End}_{\mathfrak{C}}([n])$.

Then in Chapter V, we link earlier discussions from Chapter III through Chapter IV. We define a decorated diagram category, which contains additional information on morphisms. We then give

the comparison theorem, giving an equivalence of categories between modules over curried general linear algebras associated to the standard $\mathbf{OB}$-module and modules over a certain decorated diagram category. An important implication of this is that the original comparison theorem associated to the standard $\mathbf{FB}$-module from [SS3] can be extended to any $\mathfrak{C}$-module for any B-category $\mathfrak{C}$.

Chapter VI discusses thoughts on curried exceptional algebras. In particular, we give a curried exceptional algebra $\mathfrak{g}_2$ associated to the standard module $\mathbf{V}$. This finishes the first part of the thesis.

Chapter VII stands in its own, independent from preceding chapters. In this chapter, we give an answer to the question from [NS] about the explicit proof of non-primality of certain $\mathfrak{S}$-ideals (see 1.2.1 for the definition). We end the chapter by stating some additional conjectures associated to those ideals.

CHAPTER I

Introduction

This thesis is separated into two separate parts, first on the representation theory of curried algebras, and second on the non-primality of certain symmetric ideals.

1.1 First Part: Curried Algebras

1.1.1 Basic Definitions

We first outline the notion of curried algebras first introduced in [SS3]. Details and precise definitions can be found in [SS2] and [SS3]. We also give a summary of notions we need in Subsection 2.3.1.

Let V be a finite-dimensional vector space over a base field k. Then, a module M is a $\mathfrak{gl}(V)$-representation if we have a linear map

$$a : \mathfrak{gl}(V) \otimes M \to M,$$

with an additional condition from Jacobi identity. Since $\mathfrak{gl}(V) \simeq V \otimes V^*$ as vector spaces, we see that giving such a map is equivalent to giving a map (we will keep naming the map as a)

$$a : V \otimes M \to V \otimes M,$$

together with Jacobi condition. The process of converting the map $\underline{\mathfrak{gl}}(V) \otimes M \to M$ to the equivalent map $V \otimes M \to V \otimes M$ is called "currying." This has merit as it makes sense even in a situation when duals do not exist. To be specific, let V be an arbitrary object of a non-rigid monoidal category $\mathfrak{C}$. Then, an element corresponding to $\mathfrak{gl}(V) \simeq V \otimes V^*$ does not exist in $\mathfrak{C}$, or at least, we cannot define $\mathfrak{gl}(V)$ as $V \otimes V^*$, since $\mathfrak{C}$ is non-rigid. However, we can define when $M \in \mathfrak{C}$ is a "$\mathfrak{gl}(V)$-representation", via currying. We say that an object $M \in \mathfrak{C}$ is a $\underline{\mathfrak{gl}}(V)$-module if there exists a map

$$a : V \otimes M \to V \otimes M$$

with some additional properties. Using this, although $\underline{\mathfrak{gl}}(V)$ is nonexistent as an object in $\mathfrak{C}$ we can define $\underline{\mathfrak{gl}}(V)$ in a Tannakian sense. We underline $\underline{\mathfrak{gl}}$ to emphasize that $\underline{\mathfrak{gl}}(V)$ only exists virtually. We can similarly define various curried algebras (symplectic, orthogonal, periplectic, etc) associated to an object V of an arbitrary monoidal category V.

1.1.2 B-category and Curried Algebras of the Standard Module V

Now, we introduce a certain combinatorial category that will be used later (see Chapter IV for details). A category $\mathfrak{C}$ is a **B-category** (where B stands for bijections) if the following two conditions hold. First, objects of $\mathfrak{C}$ are finite sets. Second, morphisms between objects $A, B \in \mathfrak{C}$ exist if and only if $|A| = |B|$, and if so, A and B are isomorphic, and $\mathbf{Hom}(A, B) \simeq \mathbf{End}([|A|])$ is a finite monoid. For instance, categories $\mathbf{FB}$ (objects are finite sets and morphisms are bijections between finite sets) and $\mathbf{OB}$ (objects are finite sets and $\mathbf{Hom}(A, B)$ consists of a single isomorphism between A and B if $|A| = |B|$) satisfy such conditions.

Now, consider the category $\mathbf{Mod}_{\mathfrak{C}}$ of $\mathfrak{C}$-modules. Equip $\mathbf{Mod}_{\mathfrak{C}}$ with a Day convolution product. The category $\mathbf{Mod}_{\mathfrak{C}}$ is not necessarily rigid, but we can still consider various curried algebras associated to an object $M \in \mathbf{Mod}_{\mathfrak{C}}$. Informally, B-categories are categories where various curried algebras associated to the standard module $\mathbf{V}$ can be defined. We will define the standard module $\mathbf{V}$ and its associated curried algebras later in Section 2.4 (for $\mathbf{FB}$-modules) and in Chapter IV (for any B-categories).

1.1.3 Goal of the First Part

One primarily goal of this part of the thesis is to extend the discussion from [SS3]. We explain what this means in more detail. In [SS3], curried algebras associated to the standard $\mathbf{FB}$-module $\mathbf{V}$ were analyzed in detail, and connections between representations of such curried algebras and modules over associated diagram categories were found. We give a framework that enables to extend such analysis to an arbitrary B-category. In particular, we give a comparison theorem between curried symplectic algebra associated to the standard $\mathbf{OB}$-modules with modules of the decorated Brauer category. We give some new results on the representation theory of certain inverse monoids that we needed to prove the comparison theorem.

1.2 Second Part: Non-primality of Symmetric Ideals

Let k be a field of characteristic zero, and let $R = k[x_1, x_2, \cdots]$ be the polynomial ring in countably many variables x_i. The ring R is equipped with a natural action of the infinite symmetric group $\mathfrak{S} = \bigcup_{n \in \mathbf{N}} \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symmetric group of degree n. The ring R is not noetherian, but the size of $\mathfrak{S}$ bridges the gap: in [Coh1] Cohen showed that R is equivariantly noetherian with respect to the natural action of $\mathfrak{S}$. Equivariant Noetherianity of R leads us to consider $\mathfrak{S}$-equivariant commutative algebra on R. One is interested in translating the classical notion to the realm of equivariant commutative algebra. See [NS] for basic notions of G-equivariant commutative algebra. As prime ideals are one of the most fundamental notions in commutative algebra and algebraic geometry, we are interested in $\mathfrak{S}$-prime ideals:

Definition 1.2.1. *A $\mathfrak{S}$-**ideal** of R is an ideal closed under the action of $\mathfrak{S}$. A $\mathfrak{S}$-ideal $\mathfrak{p}$ is $\mathfrak{S}$-**prime** if $f \cdot \sigma(g) \in \mathfrak{p}$ for all $\sigma \in \mathfrak{S}$ implies $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$.*

In fact, $\mathfrak{S}$-prime ideals of R are completely classified in [NS], and one of the main steps of the classification of $\mathfrak{S}$-primes in [NS] is the following:

Theorem 1.2.2. *For any positive integer N, the ideal $I(N) := \langle (x_i - x_j)^N \rangle$ is $\mathfrak{S}$-prime if and only if N is odd.*

Although Theorem 1.2.2 tells us that $I(2n)$ is not $\mathfrak{S}$-prime, the proof in [NS] does not provide $f, g \notin I(2n)$ such that $f\sigma(g) \in I(2n)$ for all σ in $\mathfrak{S}$. The goal of this part is to provide such an explicit pair.

1.3 Outline

The first part consists of Chapter II through Chapter VI, and deals with the representation theory of Curried Lie algebras and related topics. Here, we start by reviewing backgrounds of curried Lie algebras, following [SS2] and [SS3] . In Chapter III, we analyze the representation theory of two inverse monoids, Rook monoid R_n and a monoid $\mathscr{P}_n$. Representation theory of R_n is well-known, and those of $\mathscr{P}_n$ is (at least superficially) something that we first introduce. We note that the representation theory of $\mathscr{P}_n$ is the one that naturally arises when we analyze the curried algebras associated with the standard module. We show that kR_n and $k\mathscr{P}_n$ are isomorphic as algebras for any field k, eliminating the need to differentiate the two at least module-theoretically.

Then we define a B-category in Chapter IV, and within that framework, Chapter V connects the representation theory of standard curried algebras associated to a B-category. In particular, we limit our attention to the category $\mathbf{OB}$, and we give the comparison theorem between the curried Lie algebras associated to the standard $\mathbf{OB}$-module $\mathbf{V}$ and modules over a decorated diagram category.

In Chapter VII, we shift gears and give a constructive proof of Theorem 1.2.2. We end the chapter by giving some unresolved conjectures associated to $\mathfrak{S}$-ideal $I(N) = \langle (x_i - x_j)^N \rangle$.

CHAPTER II

Backgrounds on Curried Algebras

In this chapter, we give backgrounds on representation theory of curried algebras that will be used later. First, we give the definition of a triangular category. Later in Chapter V, we will show that ordered Brauer category $\mathfrak{B}_{\mathbf{OB}}$ has a triangular structure, so $\mathfrak{B}_{\mathbf{OB}}$ satisfies all the properties we will soon list at Section 2.1.

Then, we review the definition and properties of Brauer category, and after that we define the notion of curried algebras of an object M from a tensor category $\mathfrak{C}$. We end this chapter by relating the two notions: we give representations of curried algebras associated to the standard $\mathbf{FB}$-module $\mathbf{V}$, and link those to certain modules of Brauer category. In Chapter IV, we extend the argument to a broader setting. Throughout the chapter, we follow [SS2] and [SS3].

2.1 Triangular Category

Informally, a triangular category is a category modeled from a triangular decomposition of a semisimple complex Lie algebra $\mathfrak{g}$. Recall that if $\mathfrak{g}$ is a semisimple complex Lie algebra, then we can decompose $\mathfrak{g}$ into three parts $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where elements of $\mathfrak{n}_-$ has negative weights, those of $\mathfrak{h}$ zero weight, and $\mathfrak{n}_+$ positive weights. Triangular category has a notion of "upwards" morphism (resp. "downwards" morphism), which replaces a role of $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$). For a category $\mathfrak{C}$, we denote by $|\mathfrak{C}|$ the set of isomorphism classes in $\mathfrak{C}$.

Before going further, we first summarize the notion of representations of categories and finiteness conditions on them. Let k be a field (or more generally a commutative ring), and $\mathfrak{C}$ be an essentially small k-linear category. A k-linear functor $M : \mathfrak{C} \to \mathbf{Vec}_k$ (or if k is a commutative ring, category of k-modules) is called a **representation of** $\mathfrak{C}$ or $\mathfrak{C}$-module. A natural transformation $\phi : M \to N$ of functors is called a **morphism** of $\mathfrak{C}$-representations. We denote the category of $\mathfrak{C}$-representations as $\mathbf{Mod}_{\mathfrak{C}}$.

Definition 2.1.1 (Finiteness conditions)**.** *Let M be a $\mathfrak{C}$-module. Then,*

- *The module M is **finitely generated** if there exists a finite set S, consisting of elements in various $M(x)$'s, such that M is generated by S.*

- M *is* ***pointwise finite*** *if* $M(x)$ *is a finite dimensional vector space for all* $x \in \mathfrak{C}$*.*
- M *is* ***noetherian*** *if every submodule of* M *is finitely generated. The category* $\mathfrak{C}$ *is* ***noetherian*** *if every finitely generated* M *is noetherian.*

One of the main goals of representation stability is to show finiteness properties of certain modules of interest, such as **FI**-modules.

Definition 2.1.2 (Pullback functor)**.** *Let* $f : \mathfrak{C} \to \mathfrak{D}$ *be a* k*-linear functor. We define a pullback functor* $f^* : \textbf{\textit{Mod}}_{\mathfrak{D}} \to \textbf{\textit{Mod}}_{\mathfrak{C}}$ *via* $f^* : M \mapsto M \cdot f$*.*

From the above definition, we see that if M is pointwise finite module, $f^*(M)$ is also pointwise finite. As (co)limits in module categories are computed pointwise, f^* has both a left adjoint functor, denoted as $f_!$, and a right adjoint functor, denoted as f_*. See [SS2] for details.

Definition 2.1.3 (Upward and Downward category)**.** *Let* $\mathfrak{C}$ *be a category such that* $\leq$ *is a partial order in* $|\mathfrak{C}|$*. The pair* $(\mathfrak{C}, \leq)$ *is* ***upward*** *if the partial order* $\leq$ *respects well with non-zero morphism. That is, if there is a non-zero morphism* $x \to y$*, we have* $x \leq y$*. If there is no confusion, we simply say that* $\mathfrak{C}$ *is* ***upward****. The pair* $(\mathfrak{C}, \leq)$ *is* ***downward*** *if* $x \geq y$ *whenever there is nonzero morphism* $x \to y$*.*

Definition 2.1.4. *A tensor category* $\mathfrak{B}$ *is* ***triangular*** *if there is a pair* $(\mathfrak{U}, \mathfrak{D})$ *of wide subcategories of* $\mathfrak{B}$ *satisfying the following triangular axioms:*

(T0) The category $\mathfrak{B}$ *is an essentially small* k*-linear category where all its Hom-spaces are finite dimensional.*

(T1) For all $x \in \mathfrak{B}$*, we have* $\mathbf{End}_{\mathfrak{U}}(x) = \mathbf{End}_{\mathfrak{D}}(x)$ *and the endormorphism ring is semisimple.*

(T2) There exists an admissible partial order $\leq$ *satisfying the following condition:*

(a) For any representative x *of* $|\mathfrak{B}|$*, we only have finitely many representatives* $y \in |\mathfrak{B}|$ *such that* $y \leq x$*.*

(b) The pair $(\mathfrak{U}, \leq)$ *is upward, and the pair* $(\mathfrak{D}, \leq)$ *is downward.*

(T3) For $x, z \in \mathfrak{B}$*, we have the following Poincare-Birkhoff-Witt isomorphism*

$$\bigoplus_{y \in |\mathfrak{B}|} \Big(\mathbf{Hom}_{\mathfrak{U}}(y, z) \otimes_{\mathbf{End}_{\mathfrak{U}(y)}} \mathbf{Hom}_{\mathfrak{D}}(x, y) \Big) \xrightarrow{\sim} \mathbf{Hom}_{\mathfrak{B}}(x, z)$$

Let $\mathfrak{B}$ be a triangular category with upward category $\mathfrak{U}$ and downward category $\mathfrak{D}$. Set $\mathfrak{M} = \mathfrak{U} \cap \mathfrak{D}$. Essentially, we can think of $\mathfrak{U}$ as an upper triangular parabolic subalgebra, $\mathfrak{D}$ a lower-triangular parabolic subalgebra, and $\mathfrak{M}$ the Levi factor. Just as in such semisimple complex Lie algebras, triangular categories have the following notion of weights. Weights give the notion of highest weights to a triangular category.

Definition 2.1.5 (Weights on a triangular category)**.** *Let $\mathfrak{C}$ be a triangular category, with $\mathfrak{U}$ and $\mathfrak{D}$ being upward and downward categories, and let $\mathfrak{M} = \mathfrak{U} \cap \mathfrak{D}$ be the Levi subcategory. Define the* ***set of weights*** *to be the set Λ of isomorphism classes of simple $\mathfrak{M}$-modules. An element $\lambda \in \Lambda$ is called a* ***weight****.*

Example 2.1.6. *Let $\mathfrak{B}$ a Brauer category (see the next section for the precise definition). Since $\mathfrak{B}$ has a triangular structure, there is a notion weights in $\mathfrak{B}$. Via associated Schur-Weyl duality we can show the equivalence of categories between the category $\textbf{Mod}_{\mathfrak{B}}$ of $\mathfrak{B}$-modules and a version of parabolic category $\mathfrak{O}$ associated to the infinite rank symplectic Lie algebra. There is also the weight in category $\mathfrak{O}$, and the two notions of weight match via Schur-Weyl duality. Details can be found in [SS4]*

In addition to notions of weights, we have the following proposition that allows us to convert finiteness result of a $\mathfrak{B}$-module to that of $\mathfrak{M}$ (or $\mathfrak{D}$ or $\mathfrak{U}$) modules:

Proposition 2.1.7. *Let $\mathfrak{B}$ be a triangular category with $\mathfrak{U}, \mathfrak{D}$, and $\mathfrak{M} = \mathfrak{U} \cap \mathfrak{D}$ being upper, lower, and Levi subcategories, with the following commutative diagram consisting of inclusion functors.*

$$\begin{tikzcd} \mathfrak{M} \arrow[r, "j'"] \arrow[d, "i'"] \arrow[dr, "k"] & \mathfrak{U} \arrow[d, "i"] \\ \mathfrak{D} \arrow[r, "j"] & \mathfrak{B} \end{tikzcd}$$

Then, we have the following properties:

(a) For any $\mathfrak{D}$-module M, the map $j'_!(i')^*M \to i^*j_!M$ is an isomorphism.

(b) For any $\mathfrak{U}$-module M, the map $j^*i_*M \to i'_*(j')^*M$ is an isomorphism

(c) The map j' takes injectives to injectives.

(d) The functors i_* and $j_!$ are exact.

In particular, we can use the above proposition to prove the following local noetherianity result:

Corollary 2.1.8. *If $\textbf{Mod}_{\mathfrak{U}}$(or $\textbf{Mod}_{\mathfrak{D}}$) is locally noetherian, then so is $\textbf{Mod}_{\mathfrak{B}}$.*

Note that the above corollary translates local noetherianity of relatively complicated category $\textbf{Mod}_{\mathfrak{B}}$ (the category is equipped with more morphisms) to that of $\textbf{Mod}_{\mathfrak{U}}$.

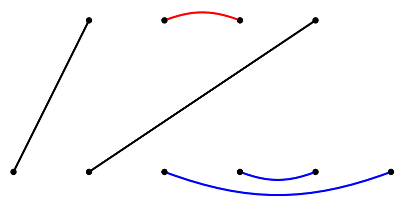
2.2 Brauer Category

2.2.1 Diagram Categories

Diagram categories are extensions of diagram algebras, first introduced in [Br] to extend the classical Schur-Weyl duality to those of orthogonal and symplectic groups. In this section, we limit our attention to Brauer category, but there are many more diagram categories, such as Brauer-like categories, partition category, Temperley-Lieb categories, etc. Later in Chapter V we will introduce a variant of Brauer category, which we denote as a *decorated Brauer category*.

Definition 2.2.1. *A **Brauer diagram** from a finite set S to another finite set T is a perfect matching on $S \amalg T$: in terms of diagrams, we can express such a matching by drawing $|S|$ and $|T|$ number of vertices in two rows. By convention, we draw S below T. Then, we connect points by three kinds of edges vertical edge connecting a vertex in S row to a vertex in T row, S-edges connecting two vertices in S, and T-edges connecting two vertices in T.*

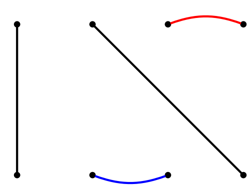
Example 2.2.2. *The following is a Brauer diagram from $S = [6]$ to $T = [4]$:*



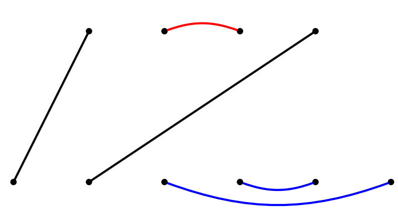
Vertical edges are colored in black, T-edges in red, and S-edges in blue.

Let α be the Brauer diagram from S to T, and β from T to U. Then, we define *the composition* $\alpha \bullet \beta$ to be the Brauer diagram from S to U defined by the following. The vertical edges of $\alpha \bullet \beta : S \to U$ are compositions of edges of α and those of β, and S-edges are those of α, and U-edges are those of β.

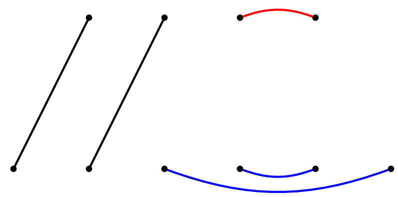
For example, let β be the following Brauer diagram from $S = [4]$ to $T = [4]$



and let $\alpha : T = [6] \to U = [4]$



Then, $\alpha \bullet \beta$ is the following Brauer diagram from $U = [4]$ to $S = [6]$



We denote $c(\alpha, \beta)$ to be the number of cycles: the above example has 1 cycle consisting of $2 \to 3$ in the second row of α and $2 \to 3$ in the first row of β.

Definition 2.2.3. *Fix a field k and a constant $\delta \in k$. Then, **Brauer category** over k with a parameter δ, denoted as $\mathfrak{B}(\delta)$ is a category with finite sets as objects, and $\mathbf{Hom}_{\mathfrak{B}}(S, T)$ for two finite sets S, T is the free k-module on the Brauer diagrams from S to T. For $\alpha \in \mathbf{Hom}_{\mathfrak{B}}(S, T)$ and $\beta \in \mathbf{Hom}_{\mathfrak{B}}(T, U)$, the composition $\beta \cdot \alpha$ is defined to be $\delta^{c(\beta,\alpha)}(\beta \bullet \alpha)$, and we extend this linearly to define compositions for general morphisms.*

Note that the endomorphism algebra of $\mathfrak{B}$ is the classical Brauer algebra in [Br].

Definition 2.2.4 (Triangular structure). *A Brauer diagram $\alpha : S \to T$ is **upward** if it contains no S-edges and **downward** if it contains no T-edges. Let $\mathfrak{U}$ be the wide subcategory of $\mathfrak{B}$ where $\mathbf{Hom}_{\mathfrak{U}}(S, T)$ is the subspace of $\mathbf{Hom}_{\mathfrak{B}}(S, T)$ generated by upward diagrams, and $\mathfrak{D}$ be the subcategory of $\mathfrak{B}$ where $\mathbf{Hom}$-sets are generated by downward diagrams.*

We can show that $\mathfrak{B}$ is a triangular category, with the above upward and downward structure. See [SS2] for the proof.

2.3 Curried Algebras

2.3.1 Curried General Linear Algebra

Let V be a finite-dimensional vector space. Then, a module M is a representation of the general linear Lie algebra $\mathfrak{gl}(V)$ if it is equipped with a linear map $\mu : \mathfrak{gl}(V) \to \mathbf{End}(M)$ preserving the Lie bracket $[\cdot, \cdot]$. That is, for all $X, Y \in \mathfrak{gl}(V)$, we have $\mu([X, Y]) = [\mu(X), \mu(Y)]$, where $[X, Y] := XY - YX$ is the Lie bracket. Since $\mathfrak{gl}(V)$ is isomorphic to the tensor product $V \otimes V^*$, giving μ is equivalent to giving a linear map $a : V \otimes M \to V \otimes M$ satisfying the following (see [SS3] for the proof):

Proposition 2.3.1. *Let V be a finite dimensional vector space. The map $\mu : \mathbf{GL}(V) \to \mathbf{End}(M)$ gives M a structure of a $\mathfrak{gl}(V)$ representation if and only if the map $a : V \otimes M \to V \otimes M$ satisfies $[a_1, a_2] = \tau(a_1 - a_2)$, where three maps*

$$a_1, a_2, \tau : V \otimes V \otimes M \to \tau : V \otimes V \otimes M$$

are defined as follows. The map

$$\tau : v_1 \otimes v_2 \otimes m \mapsto v_2 \otimes v_1 \otimes m$$

switches the first two tensor factors, and

$$a_2 = \mathbf{1} \otimes a, \text{ and } a_1 = \tau \cdot a_2 \cdot \tau.$$

Using Proposition 2.3.1, we define the curried general linear Lie algebra associated to an object V of a symmetric tensor category $\mathfrak{C}$.

Definition 2.3.2. *Let V be an object of a symmetric tensor category $\mathfrak{C}$. Then, the* ***curried general Lie algebra*** *on V, denoted as $\underline{\mathfrak{gl}}(V)$ is defined by the following. A* ***representation*** *of $\underline{\mathfrak{gl}}(V)$ is an object $M \in \mathfrak{C}$ equipped with a morphism*

$$a : V \otimes M \to V \otimes M,$$

satisfying $[a_1, a_2] = \tau(a_1, a_2)$, where τ is a symmetric structure on $\mathfrak{C}$, and $a_2 = \mathbf{1} \otimes a$, and $a_1 = \tau \cdot a_2 \cdot \tau$.

Now fix an object $V \in \mathfrak{C}$. We give some examples of curried general representations:

Example 2.3.3 (Trivial representation)**.** *For any $M \in \mathfrak{C}$, take $a : V \otimes M \to V \otimes M$ to be the zero map. Then, (a, M) satisfies the condition in Definition 2.3.2. We denote this as the* ***trivial representation*** *on M.*

Example 2.3.4 (Tensor products of two $\underline{\mathfrak{gl}}(V)$-modules)**.** *Assume that (M, a_M) and (N, a_N) are $\underline{\mathfrak{gl}}(V)$-modules. Embed $\mathbf{End}(V \otimes M)$ and $\mathbf{End}(V \otimes N)$ to $\mathbf{End}(V \otimes M \otimes N)$. Then, $a_M + a_N$ gives $\underline{\mathfrak{gl}}(V)$ module structure to $M \otimes N$. Note that when $\mathfrak{C}$ equals to the category of finite dimensional vector spaces, this construction is equivalent to giving a $\mathfrak{gl}(V)$-module structure to $M \otimes N$, where M, N are $\mathfrak{gl}(V)$-modules.*

2.3.2 Other Curried Algebras

It is possible to extend the analysis from the Subsection 2.3.1 to other Lie algebras. As an example, we give the definition of the curried symplectic algebra. Other examples (Witt algebra, Weyl Lie algebra and many more) can be found in [SS3]. Later in Chapter VI, we define the curried exceptional Lie algebra $\mathfrak{g}_2$.

Let V be a finite dimensional vector space, and let $\mathfrak{sp}(V \oplus V^*)$ be the associated symplectic Lie algebra. We can be decompose $\mathfrak{sp}(V \oplus V^*)$ into

$$\mathfrak{sp}(V \oplus V^*) = \mathbf{Div}^2(V^*) \oplus \mathfrak{gl}(V) \oplus \mathbf{Div}^2(V). \tag{II.1}$$

Let M be an arbitrary vector space, and let

$$\mu : \mathfrak{sp}(V \oplus V^*) \otimes M \to M \tag{II.2}$$

be an arbitrary linear map. By the decomposition (II.1), we see that giving a μ is equivalent to giving the following three maps

$$a : V \otimes M \to V \otimes M, \qquad b : \mathbf{Div}^2(V) \otimes M \to M, \qquad b' : M \to \mathbf{Sym}^2(V) \otimes M.$$

We are interested in finding conditions on three maps a, b, b' that (via equivalence) makes the pair (μ, M) a $\mathfrak{sp}(V \oplus V^*)$-representation.

Proposition 2.3.5. *Let μ and (a, b, b') be defined as above. Then, μ gives M a $\mathfrak{sp}(V \oplus V^*)$-module structure if and only if a, b, b' satisfies the following conditions:*

(a) [$\underline{\mathfrak{gl}}(V)$-module structure] The map a defines a $\underline{\mathfrak{gl}}(V)$ structure on M, as in Definition 2.3.2

(b) [Multiplication is commutative] Let $b_2 : \mathbf{Div}^2(V) \otimes \mathbf{Div}^2(V) \otimes M \to \mathbf{Div}^2(V) \otimes M$ be the map $\mathbf{1} \otimes b$, and $b_1 = \tau b_2 \tau$. Then, $bb_2 = bb_1$ as maps $\mathbf{Div}^2(V) \otimes \mathbf{Div}^2(V) \otimes M \to M$

(b') [Comultiplication is cocomutative] Similarly define b'_1 and b'_2. e have $b'_1 b' = b_2 b'$ as maps $M \to \mathbf{Sym}^2(V) \otimes \mathbf{Sym}^2(V) \otimes M$.

(c) The maps b, b' are maps of $\underline{\mathfrak{gl}}(V)$-modules.

(d) Let Δ be a comultiplication map and m be a multiplication map. We then have $b'b - b_1 b'_2 = (m \otimes \mathbf{1})(\mathbf{1} \otimes a)(\Delta \otimes \mathbf{1})$as maps $\mathbf{Div}^2(V) \otimes M \to \mathbf{Sym}^2(V) \otimes M$.

Using the above proposition, we can define a curried symplectic algebra on any object V of a tensor category $\mathfrak{C}$:

Definition 2.3.6 (Curried Symplectic Algebra)**.** *Let $\mathfrak{C}$ be a tensor category and $V \in \mathbf{ob}(\mathfrak{C})$. An arbitrary object M of $\mathfrak{C}$ is a $\underline{\mathfrak{sp}}(V \oplus V^*)$**-module** if M is equipped with maps*

$$a : V \otimes M \to V \otimes M, \qquad b : \mathbf{Div}^2(V) \otimes M \to M, \qquad b' : M \to \mathbf{Sym}^2(V) \otimes M.$$

satisfying conditions (a)- (d) in Proposition 2.3.5.

We will now give a specific example of curried algebras associated to the category $\mathbf{FB}$. We first review properties of $\mathbf{FB}$-modules and operations on $\mathbf{FB}$-modules, and introduce standard curried structures on the category $\mathbf{FB}$. Later in Chapter IV, we will give a more general setting where standard curried structures on the category of $\mathbf{FB}$ can carry over.

2.4 Curried Representations on a Standard FB-module and Brauer Categories

2.4.1 Category FB and FB-modules

Definition 2.4.1 (Category $\mathbf{FB}$ and $\mathbf{FB}$-modules)**.** *Let $\mathbf{FB}$ be the category with finite sets as objects and bijections between finite sets as morphisms. We define a $\mathbf{FB}$-module to be a functor $M : \mathbf{FB} \to$ **Vec^{fin}**, the category of finite dimensional vector spaces. Equivalently, $\mathbf{FB}$-modules are sequences $(M_n)_{n\geq 0}$, where for each n, M_n is a $\mathfrak{S}_n$-representation. We call this sequence model. We then equip a tensor structure to $\mathbf{FB}$ by the following. For two $\mathbf{FB}$ modules M and N, we define $M \otimes N$ to be*

$$(M \otimes N)(S) = \bigoplus_{T\subset S} M(T) \otimes N(S \setminus T),$$

*where S is a finite set. The **standard** $\mathbf{FB}$**-module**, denoted as $\mathbf{V}$ is the $\mathbf{FB}$-module that is $\mathbf{C}$ on degree 1 and 0 on all other degree.*

Let M be an $\mathbf{FB}$-module. An (m, n)**-operation** on M is a map ϕ that assigns to every finite set S and two tuples $\underline{x}, \underline{y}$ of elements of S (where $|\underline{x}| = m$ and $|\underline{y}| = n$) with distinct coordinates, a natural linear map

$$\phi^S_{\underline{x},\underline{y}} : M(S \setminus y) \to M(S \setminus x),$$

such that for each bijection $f : S \to T$, the following diagram commutes:

$$\begin{array}{ccc} M(S\setminus y) & \xrightarrow{\phi^S_{x,y}} & M(S\setminus x) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ M(T\setminus f(y)) & \xrightarrow{\phi^T_{f(x),f(y)}} & M(T\setminus f(x)) \end{array}$$

Definition 2.4.2. *Let ϕ be an operation on a $\mathbf{FB}$-module M. Then,*

*(a) The operation ϕ is **symmetric** if $\phi_{\underline{x},\underline{y}}$ is invariant under permutations of $\underline{x}$ and those of $\underline{y}$. Giving a symmetric (m, n)-operation is equivalent to giving a map*

$$a : \mathbf{Sym}^n(V) \otimes M \to \mathbf{Sym}^m(V) \otimes M,$$

where V is the standard $\mathbf{FB}$-module *(b) We say ϕ is skew-symmetric if $\phi_{\underline{x},\underline{y}}$ respects to signature of permutations on $\underline{x}$ and $\underline{y}$.*

*(c) Lastly, we say ϕ is **simple** if $\phi^S_{\underline{x},\underline{y}} = 0$ whenever $\underline{x}, \underline{y}$ intersect non-trivially.*

2.4.2 Curried General Linear Algebra of Standard $\mathbb{FB}$ modules

Now we define curried general linear algebras associated to a $\mathbf{FB}$-module M. Note in particular that $\mathbf{FB}$ is not a rigid category: we do not have a nice notion of duality in $\mathbf{FB}$-modules, since $\mathbf{FB}$ is only defined on non-negative degrees. If anything, the dual M^* of M should only be nonzero in non-positive degrees, but from the definition of $\mathbf{FB}$-modules, the sequential model of $\mathbf{FB}$-modules do have *negative degrees*. In short, it is impossible to define $\mathfrak{gl}(M)$ as the tensor product $M \otimes M^*$, because the dual M^* of M is nonexistent. Instead, we use Definition 2.3.2, and define the curried general linear algebra $\underline{\mathfrak{gl}}(M)$ associated to M. For simplicity, we limit ourselves to an $\mathbf{FB}$-module that is **C** in degree 1 and 0 in all other degrees. We call such $\mathbf{FB}$-module **the standard $\mathbb{FB}$-module**, and denote by

$$\mathbf{V} = (0, \mathbf{C}, 0, \cdots).$$

Now, let M be an arbitrary $\mathbf{FB}$-module, and consider a map of $\mathbf{FB}$-modules

$$a : \mathbf{V} \otimes M \to \mathbf{V} \otimes M.$$

Let S be a finite set, and pick $y \in S$ and $m \in M(S \setminus y)$. Then, the map a maps an element $t^y \otimes m \in \mathbf{V} \otimes M$ to

$$t^y \otimes \tau^{S \setminus y}(m) + \sum_{x \in S \setminus y} \left(t^x \otimes \sigma^S_{x,y}(m) \right),$$

where τ is 0-operation and σ is a simple 1-operation on M. Whether (M, μ) is a $\underline{\mathfrak{gl}}(V)$-representation. The following theorem classifies $\underline{\mathfrak{gl}}(\mathbf{V})$ module with respect to maps τ and σ:

Theorem 2.4.3. *Let a be the map defined above. The map a defines a $\underline{\mathfrak{gl}}(\mathbf{V})$ module structure on M if and only if the following two conditions hold:*

- *The operations τ, σ commute with themselves and each other*
- *Given a finite set S and $x, y, z \in S$ distinct, we have $\sigma^{S \setminus x}_{y,z} \sigma^{S \setminus z}_{x,y} = \sigma^{S \setminus y}_{x,z}$.*

The theorem was originally proven in [SS3]. Later we will show that the proof can be extended to a broader setting.

We have the following canonical $\underline{\mathfrak{gl}}(\mathbf{V})$-module structure, called as the δ-standard structure.

Definition 2.4.4 (δ-standard $\underline{\mathfrak{gl}}(\mathbf{V})$-structure)**.** *Let δ be an element of the base field k. The δ-**standard** $\underline{\mathfrak{gl}}(\mathbf{V})$-**structure** on an $\mathbf{FB}$-module M is the representation of $\underline{\mathfrak{gl}}(\mathbf{V})$ on M defined by*

$$\tau := \delta\mathbf{1} \qquad \textit{and} \qquad \sigma^S_{x,y} := (i^S_{x,y})_*,$$

where $i^S_{x,y} : S \setminus \{y\} \to S \setminus \{x\}$ is the following bijection

$$i^S_{x,y}(z) = \begin{cases} y & \textit{if } z = x \\ z & \textit{else} \end{cases}. \tag{II.3}$$

We can also give a sequential model for the δ-standard $\underline{\mathfrak{gl}}(\mathbf{V})$ structure:

Proposition 2.4.5 (Sequential Model)**.** *Giving a δ-standard $\underline{\mathfrak{gl}}(\mathbf{V})$-module is giving a sequence (M_n) as above where $\mathfrak{A}_n$ acts trivially and the generator of the first $\mathbf{N}$ acts by the structure constant δ.*

The equivalence between the two models is given by the following lemma.

Lemma 2.4.6. *Let $\mathfrak{A}_n$ be the monoid generated by n commuting idempotents $e_1, \cdots, e_n$. Then, giving a $\underline{\mathfrak{gl}}(\mathbf{V})$-module M is equivalent to giving a sequence $(M_n)_{n\geq 0}$ where M_n is a representation of monoid $\mathbf{N} \times (\mathfrak{S}_n \ltimes \mathfrak{A}_n)$.*

Again for the proof of equivalence, see [SS3]. In Chapter V, we will present an analogous result for $\mathbf{OB}$-modules.

2.4.3 Connection Between $\underline{\mathfrak{sp}}$-modules and Modules of Brauer Category

We end the section by giving concrete connections between the modules of Brauer category and the category of $\mathbf{FB}$-modules. Let $\mathbf{V}$ be the standard $\mathbf{FB}$-representation.

Definition 2.4.7 (Standard δ-representation)**.** *Fix $\delta \in k$. Let M be a representation of $\underline{\mathfrak{sp}}(\mathbf{V} \oplus \mathbf{V}^*)$, given by three maps (a, b, b') (see Definition 2.3.6 for details). Then, M is δ-**standard** if the map a induces a δ-standard representation, defined by Definition 2.4.4. We denote by $\textbf{\textit{Rep}}_\delta(\underline{\mathfrak{sp}}(\mathbf{V} \oplus \mathbf{V}^*))$ the category of δ-standard representations.*

Let $\mathfrak{B} = \mathfrak{B}(\delta)$ be the Brauer category with parameter δ, and M be a $\mathfrak{B}$-module i.e., a functor $\mathfrak{B} \to \mathbf{Vec}$. Recall that objects of $\mathfrak{B}$ are finite sets with morphisms being the vector space spanned by Brauer diagrams. We can thus embed the category $\mathbf{FB}$ into $\mathfrak{B}$. Via the inclusion $\mathbf{FB} \subset \mathfrak{B}$, of categories, we can regard M as an $\mathbf{FB}$-module. Indeed, it turns out that an $\mathfrak{B}$-module M can be thought as a $\mathbf{FB}$-module with additional data, given by $(0, 2)$-operation β and $(2, 0)$-operation

β' together with some compatibility conditions. The converse is also true- i.e., if we have an $\mathbf{FB}$-module M with $(0,2)$-map β and $(2,0)$-map β' with corresponding compatibility conditions, we can give a unique $\mathfrak{B}$-module structure on M that extends $\mathbf{FB}$-module structure.

In that respect, we have the following important comparison theorem between the category of standard representations and the category of Brauer category:

Theorem 2.4.8. *Let $\delta \in k$, where char$(k) \neq 2$ or $\delta = 0$. We then have the following natural isomorphism of categories*

$$\mathbf{Mod}_{\mathfrak{B}(\delta)} \simeq \mathbf{Rep}_{\delta/2}(\underline{\mathfrak{sp}}(\mathbf{V} \oplus \mathbf{V}^*)).$$

Details regarding the relationship between $\mathfrak{B}$-modules and $\mathbf{FB}$-modules, together with the proof of Theorem 2.4.8 can be found in [SS3]. Later in Chapter V, we will present analogous comparison theorems corresponding to the category $\mathbf{OB}$.

2.4.4 References

Proofs regarding triangular categories can be found in [SS2], and those regarding curried algebras can be found in [SS2]. Note also that there are many other curried structures associated to the standard $\mathbf{FB}$-module $\mathbf{V}$. See [SS3] for details. One can also find some related discussions from [SS2] (triangular categories, diagram categories, etc) or [SS4] (Schur-Weyl duality).

CHAPTER III

Representation Theory of Inverse Monoids

In this chapter, we discuss modules over two specific inverse monoids, namely Rook monoid R_n and a monoid $\mathscr{P}_n$, which will be defined in Example 3.2.5. We show the following two main results. First, we prove that although two monoids R_n and $\mathscr{P}_n$ are not isomorphic as monoids, monoid algebras kR_n and $k\mathscr{P}_n$ generated by the two monoids are isomorphic. In particular, they are Morita equivalent so there is no need to differentiate the two if we are only interested in the representation theory of the two. Second, we give a tensor structure on a sequential category $\mathfrak{R} = \{(M_n)_{n\geq 0}\}$ of sequences of R_n-modules, and identify $\mathfrak{R}$ by another category via Schur-Weyl duality. In Chapter V, we show that the sequential category $\mathfrak{R}$ is almost the same as the category of curried representations associated to $\mathbf{OB}$-modules.

3.1 Basic Definitions and Properties of Monoids

We first review the structural theory of inverse monoids and analyze representation theory of certain combinatorial monoids. A **monoid** M is a set that is closed under an associative binary operation with an element. From now on, we reserve 1 to be the identity element of M. An element $m \in M$ is a **unit** if there exists an inverse element $m^{-1} \in M$ such that $mm^{-1} = m^{-1}m = 1$. The submonoid G of units of M forms a group, and we call the group **group of units** of M. An element $e \in M$ is an **idempotent** if $e^2 = e$. If $X \subset M$ is a subset, we define $E(X)$ to be the set of idempotents in X. For $e \in E(M)$, let G_e be the group of units of monoid eMe with a binary operation inherited from that of M. It is easy to see that G_e is a group with identity e. We call G_e the **maximal subgroup of** M **at** e. We record some properties of monoid that we will use later:

Proposition 3.1.1 (Properties of monoid)**.**

(a) The maximal subgroup G_e is the unique subgroup with identity e, maximal with respect to containment

(b) There is a natural partial order on $E(M)$ via the following: for $e, f \in E(M)$, we set $e \leq f$ if $ef = fe = e$.

(c) ***Green's relation****: For $m, n \in M$, we denote $m \mathscr{J} n$ if $MmM = MnM$. This is an equivalence relation.*

For the proof, see [St].

Definition 3.1.2. *We define J_m to be the $\mathscr{J}$-class of $m \in M$. That is,*

$$J_m = \{m' \in M : m \mathscr{J} m'\}.$$

Properties of monoids from this section will be used later in Appendix A.

3.2 Inverse Monoids

In general, the representation theory of arbitrary finite monoids is extremely difficult. In particular, if M is an arbitrary monoid, we do not usually expect the monoid ring kM generated by M to be semisimple, even if we assume the field k to be of characteristic zero and $|M| < \infty$. However, if we limit ourselves to **inverse monoids**, we do have a nice representation theory.

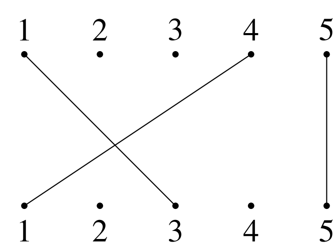
Definition 3.2.1. *A monoid M is an* ***inverse monoid*** *if for all $m \in M$, there exists a unique element m^*, called the* ***inverse*** *of m, satisfying the following conjugacy property: $mm^*m = m$ and $m^*mm^* = m^*$.*

Example 3.2.2 (Trivial Example)**.** *Any group is an inverse monoid, with $g^{-1} = g^*$*

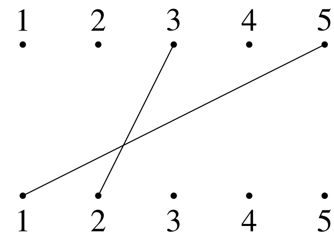
Now we define a Rook monoid R_n, which is one of two important monoids we will focus on:

Example 3.2.3 (Rook Monoid, or Symmetric Inverse Monoid)**.** *Let X be a set. The symmetric inverse monoid R_X is the inverse monoid of all partial injective functions from X to itself, with respect to usual composition of partial functions. When $[n] = \{1, 2, \cdots, n\}$ for example, $R_n = R_{[}n]$ is a function from a subset A of size n to a subset $B \subset [n]$ with same cardinality. In this case, we can think R_n as a $n \times n$ matrix with 0 and 1 entries, with at most one 1's in each row and column (where the name "Rook" comes from since in such a matrix the positions of 1's correspond to the legal placement of rooks on $n \times n$ chessboard).*

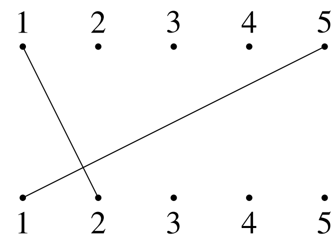
The following is a diagrammatic representation of R_n: as mentioned above, an element m of R_n is a function from a subset A of $[n]$ to another subset B of the same cardinality. For instance, fix $n = 5$ and x be the map $1 \to 3, 4 \to 1$, and $5 \to 5$:



and let y be the map $3 \to 2$ and $5 \to 1$:



Then, $x \cdot y$ sends $1 \to 2$ and $5 \to 1$:



Rook monoids are especially important due to the following Cayley's theorem on finite inverse monoids:

Theorem 3.2.4 (Fundamental Embedding Theorem)**.** *Any finite inverse monoid M is isomorphic to an inverse submonoid of some rook monoid R.*

We give another inverse monoid, which we denote by $\mathscr{P}_n$ monoid.

Example 3.2.5 ($\mathscr{P}_n$-monoid)**.** *We obtain another "pairing" of $[n]$ to itself via filling the unmatched pairs by the following process:*

(1) Choose a partial permutation $U \to V$, where $U, V \subset [n]$. We use straight lines to pair U with V.

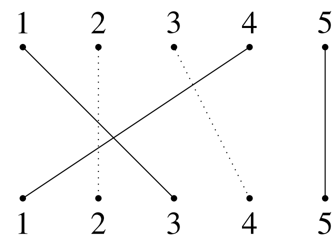
(2) Then, pair $[n] \setminus U$ with $[n] \setminus V$ via the unique order-preserving isomorphism $[n] \setminus U$ with $[n] \setminus V$. To differentiate this pairing with those in (1), we use dotted lines.

(3) The composition law between (straight/dotted) lines are as follows: composing the same type of lines give the same type. The composition of a dotted line and a straight line gives a straight line.

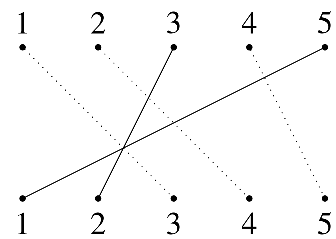
Denote by $\mathscr{P}_n$ the set of all such permutations (together with additional information) of the set $[n]$.

Equivalently, any element of $\mathscr{P}_n$ can be expressed by (U, ϕ), where $U \subset [n]$ and $\phi : U \to [n]$ is injective function ("straight" lines). The rest is filled by order-preserving bijection.

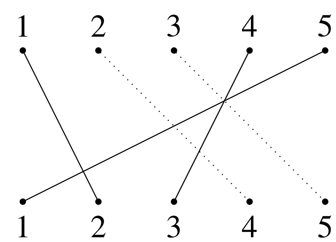
We give a diagrammatic representation of $\mathscr{P}_n$. Again fix $n = 5$ and let x be the following:



and let y be the followimg map:



Then the composition $x \cdot y$ is the following:



We emphasize that dotted lines must preserve orders: in particular, no two dotted lines cannot cross each other. Straight lines, on the other hand, do not have such restrictions. If we compose a dotted order-preserving line with a straight line (not necessarily order-preserving), the composed line loses order-preserving property, so it becomes a straight line.

Remark 3.2.6. *In Chapter V, we will see that the category of curried symplectic representations of standard* $\mathbf{OB}$*-module is (almost) equivalent to the category of sequences of* $\mathscr{P}_n$ *monoids.*

From the construction of $\mathscr{P}_n$ modules, we see that $|R_n| = |\mathscr{P}_n|$, since order-preserving map from a complement $[n] \setminus U$ to $[n] \setminus V$ is unique. To be specific, we have

$$|R_n| = |\mathscr{P}_n| = \sum_{k=0}^{n} \binom{n}{k}^2 k!$$

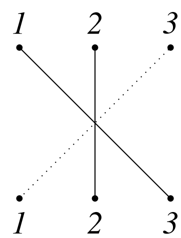
Remark 3.2.7. *Using the fundamental embedding theorem of inverse monoids, we see that we can embed* $\mathscr{P}_n$ *to a Rook monoid* R_N *for some* N*. In fact, one can choose* $N = 2n$*: we can realize* $\mathscr{P}_n$ *as* $2n \times 2n$ *square matrix, constructed by* 2×2 *block matrices, by the following process. First pick* *any $A \in \mathscr{P}_n$. Then, we record the partial permutations of $[n]$ given by A, disregarding whether the lines are dotted or not. Then, for dotted lines we assign 2×2 identity matrix*

$$\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and for straight lines, we assign the idempotent

$$\mathbf{e} := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

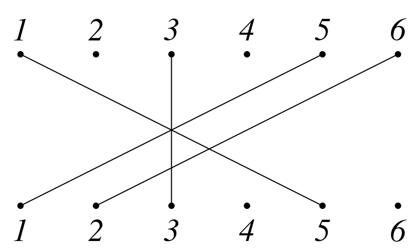
For instance, the following element of $\mathscr{P}_3$



gets mapped to 6×6 matrix

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{e} \\ \mathbf{0} & \mathbf{e} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

or equivalently the following element of rook monoid R_6:



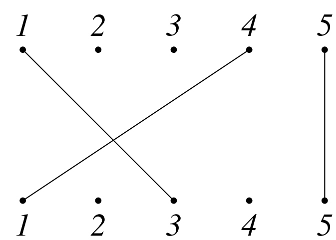
Any inverse monoid M has the following natural partial order:

Lemma 3.2.8 (Partial order in Inverse Monoids)**.** *Let M be an inverse monoids, and m, n be two elements of M. Then, we give the* ***natural partial order*** *on M by $m \leq n$ if any of the following equivalent conditions hold:*

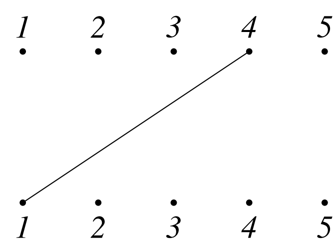
- $m = ne$ *for some idempotent* $e \in E(M)$
- $m = fn$ *for some* $f \in E(M)$
- $m = nm^*m$ *where* m^* *is an inverse of m*
- $m = mm^*n$.

For the proof, see [St].

Example 3.2.9 (Natural order in Rook Monoid)**.** *If* m, n *are elements of* R_n *i.e., they are* $n \times n$ *matrices with at most one 1's in each column and row, then* $m \leq n$ *if we can remove some number of* 1*'s from* n *and obtain* m. *For instance, let* n *be the following element of* R_n *(represented diagrammatically):*

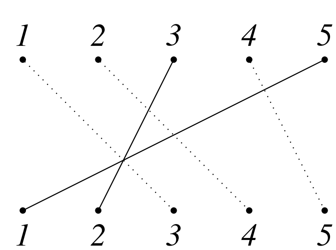


and m *be the following, obtained from* n *by removing lines connecting top* 1 *to bottom* 3*, and top* 5 *to bottom* 5*:*

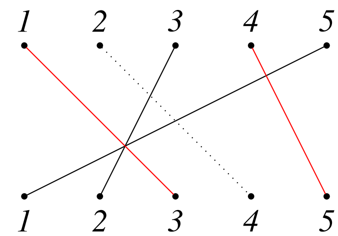


Then, we see that $m < n$.

Example 3.2.10 (Natural order in $\mathscr{P}_n$–monoid)**.** *Similar to Rook monoids, the natural order of* $\underline{\mathfrak{gl}}(V_{\mathbf{OB}})$ *is given by the following diagrammatic rule: if* $m \leq n$*, then we can remove certain number of dotted lines of* n *and make* m. *For instance, assume that* n *is represented by the following diagram:*

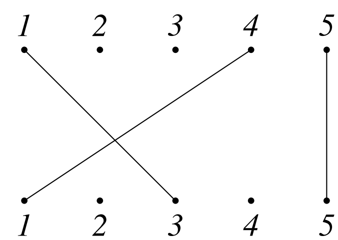


Then, any element that is obtained by replacing dotted lines by straight lines are less than n. For instance, the following

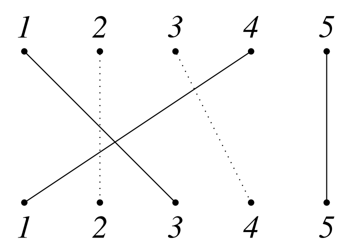


is less than n (red lines are the straight lines that replace dotted ones).

Remark 3.2.11. *It is quite confusing, but for Rook monoids, if $m < n$, then n has more straight lines. However, in $\mathscr{P}_n$ monoid, $m < n$ means that n has less straight lines. One may ask why we have chosen such a convention: we could have replaced the roles of straight and dotted lines in diagrammatic representations of $\mathscr{P}$ to make the number of straight lines and partial inequality to be consistent. There is a reason for this seemingly confusing decision. First, we do have a natural bijection $f : R_n \to \mathscr{P}_n$ by keeping all the straight lines, and filling the remaining unmatched pairs by unique-order preserving bijection- i.e., by dotted lines. For example if $x \in R_5$ is the following,*



then $f(x) \in \mathscr{P}_5$ is



There is another fundamental reason behind the choice, which we will presenter later in Appendix A.

Remark 3.2.12. *Note in particular that the bijection $f : R_n \to \mathscr{P}_n$ from Remark 3.2.11 is not a monoid homomorphism: in general $f(xy) \neq f(x)f(y)$. However, we do have that*

$$\text{If the domain of } x = \text{ the range of } y, \text{ then } f(xy) = f(x)f(y). \qquad \text{(III.1)}$$

Later, we will use the condition (III.1).

We have the following decomposition on the ring kM generated by a finite inverse monoid M.

Theorem 3.2.13. *Let M be a finite inverse monoid and $e_1, \cdots, e_s$ be idempotent representatives of the $\mathscr{J}$-class of M. Then we have an isomorphism*

$$kM \simeq \prod_{i=1}^{s} M_{n_i}(kG_{e_i})$$

of rings, where $n_i = |E(J_{e_i})|$ and for each i,

$$G_e = \{m \in M | m^* m = mm^* = e\}$$

is the maximal subgroup at e.

The proof will be given Appendix A. Using Theorem 3.2.13, we can show the following theorem on the semisimplicity of finite inverse monoids:

Theorem 3.2.14 (Semisimplicity of Finite Inverse Monoids)**.** *Let k be a field of characteristic zero and M be a finite inverse monoid. Then, the ring kM generated by M is semisimple.*

Remark 3.2.15. *Note in particular that the isomorphism of rings appearing in Theorem 3.2.13 is independent of the characteristic of the base field k.*

3.3 Representation Theory of R_n and $\mathscr{P}_n$

Now, we limit our attention to R_n and $\mathscr{P}_n$. In this section, we show that although the two monoids are not isomorphic as monoids, their monoid algebras are isomorphic. So in particular, the monoid algebras kR_n and $k\mathscr{P}_n$ are Morita equivalent.

It is straightforward to check that R_n and $\mathscr{P}_n$ are not isomorphic as monoids if $n > 1$:

Proposition 3.3.1. *IF $n > 1$, two monoids R_n and $\mathscr{P}_n$ are* ***not isomorphic*** *as monoids.*

Proof. Observe that the zero-matrix $0 \in R_n$ satisfies the following:

$$\text{for any } A \in R_n, O \cdot A = A \cdot O = O. \tag{III.2}$$

On the other hand, there is no such element in $\mathscr{P}_n$ satisfying (III.2): as monoid of square matrices, $\mathscr{P}_n$ does not have an element $P \in \mathscr{P}_n$ where $PA = AP = P$ for any $A \in \mathscr{P}_n$. This is also easy to observe when we consider $\mathscr{P}_n$ as permutations of $[n]$ while keeping track of some additional information (i.e., whether lines are straight or dotted). By forgetting the additional information altogether, we have a surjective map $\mathscr{P}_n \to \mathscr{P}_n$ of monoids. Since $\mathfrak{S}_n$ is a symmetric group with more than 1 elements, no element of $\mathfrak{S}_n$ satisfies (III.2). Hence, $\mathscr{P}_n$ does not have a zero element. $\square$

Although R_n and $\mathscr{P}_n$ are not isomorphic as monoids, for any field k, the rings $k[R_n]$ and $k[\mathscr{P}_n]$ generated by two monoids are isomorphic.

Lemma 3.3.2. *Let k be any field. Then, we have the following isomorphism of rings*

$$\Phi_n : k[R_n] \simeq k[\mathscr{P}_n]$$

Proof. Let's first analyze idempotents of two monoids R_n and $\mathscr{P}_n$, with respect to partial orders we described in Example 3.2.9 and Example 3.2.10. Recall that for any inverse monoid M and elements $m, n \in M$, we have $m \leq n$ if and only if there exists an idempotent e of M satisfying $m = n \cdot e$. In particular, natural partial orders of R_n and $\mathscr{P}_n$ respect well with rank (see Appendix A, Definition 1.3.1 and paragraphs following that for the definition and properties).

Following tables summarize idempotent representatives, number of $\mathscr{J}$-classes of each idempotent representative e, and maximal subgroup G_e corresponding to an idempotent class e (see Appendix A for details).

First we give a table for R_n:

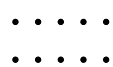
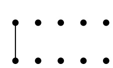
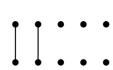
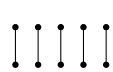
rank	e	# of class	G_e	Algebra
0		$\binom{n}{0}$	$\mathfrak{S}_0$	$M_{\binom{n}{0}}(k\mathfrak{S}_0)$
1		$\binom{n}{1}$	$\mathfrak{S}_1$	$M_{\binom{n}{1}}(k\mathfrak{S}_1)$
2		$\binom{n}{2}$	$\mathfrak{S}_2$	$M_{\binom{n}{2}}(k\mathfrak{S}_2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n		$\binom{n}{n}$	$\mathfrak{S}_n$	$M_{\binom{n}{n}}(k\mathfrak{S}_n)$

Table III.1: Idempotent classes associated to R_n.

For now, the last "Algebra" column gives matrix algebras corresponding to e, appearing in the decomposition of $k[R_n]$ (See Theorem 3.2.13 and Appendix A)

Now, we give the corresponding table for $\mathscr{P}_n$:

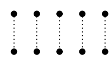
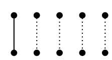
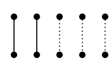
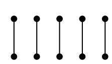
rank	e	# of class	G_e	Algebra
0		$\binom{n}{0}$	$\mathfrak{S}_0$	$M_{\binom{n}{0}}(k\mathfrak{S}_0)$
1		$\binom{n}{1}$	$\mathfrak{S}_1$	$M_{\binom{n}{1}}(k\mathfrak{S}_1)$
2		$\binom{n}{2}$	$\mathfrak{S}_2$	$M_{\binom{n}{2}}(k\mathfrak{S}_2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n		$\binom{n}{n}$	$\mathfrak{S}_n$	$M_{\binom{n}{n}}(k\mathfrak{S}_n)$

Table III.2: Idempotent classes associated to $\mathscr{P}$.

Note that the two tables are quite the same: the only difference is that in Table III.2, we draw dotted lines for all the non-connected pairs. Using Theorem 3.2.13, the two rings kR_n and $k\mathscr{P}_n$ are decomposed as

$$kR_n = \prod_{i=0}^{n} M_{\binom{n}{i}}(k\mathfrak{S}_i), \tag{III.3}$$

and

$$k\mathfrak{P}_n = \prod_{i=0}^{n} M_{\binom{n}{i}}(k\mathfrak{S}_i). \tag{III.4}$$

$\square$

To summarize, using Theorem 3.2.13 we obtain the following Corollary:

Corollary 3.3.3. *If M is either R_n or $\mathscr{P}_n$, we have the following decomposition of monoid algebra for any field k:*

$$kM = \prod_{i=0}^{n} M_{\binom{n}{i}}(k[\mathfrak{S}_i]).$$

Furthermore, as each component $M_{\binom{n}{i}}(k\mathfrak{S}_i)$ of the product is Morita equivalent to $\mathfrak{S}_i$, the ring kM as a whole is Morita equivalent to

$$\prod_{i=0}^{n} k[\mathfrak{S}_i.]$$

In particular, if $k = \mathbf{C}$, the ring kM is semisimple.

Note that both the above isomorphism and Morita equivalence do not depend on the characteristic of the base field k.

When we further assume the characteristic of the field k to be zero, then we know that for each i, the group ring $k[\mathfrak{S}_i]$ is semisimple. Hence, if M is either R_n or $\mathscr{P}_n$, the ring kM is semisimple. Furthermore, from the explicit description of irreducible representations of symmetric groups, we conclude that irreducible representations of kM are completely determined by Young diagrams of length at most n, where Young diagrams of length exactly $i \leq n$ correspond to irreducible representations of each $M_{\binom{n}{i}}(k\mathfrak{S}_i)$ (or its Morita equivalent ring $k\mathfrak{S}_i$). We summarize this as the following theorem:

Theorem 3.3.4. *When k is a field of characteristic zero, the ring kM is semisimple, and all the irreducible representations of kM are indexed by*

$$\{\lambda | \lambda \text{ is a Young diagram of length at most } n.\}$$

3.3.1 Tensor Product on kR_n

Just as in sequences of $\mathfrak{S}_n$– modules (for details, see [SS1]), we can think of sequences $(M_n)_{n\in\mathbf{N}}$ of R_n-modules (or equivalently $\mathscr{P}_n$-modules), where R_n acts on nth piece M_n. Consider the category $\mathfrak{R}$ of sequences of R_n modules (resp. category $\mathfrak{P}$ of sequences of $\mathscr{P}_n$–modules). We can endow the following tensor product structure in each category:

Definition 3.3.5 (Tensor product structure). *Let $M = (M_i), N = (N_j)$ be sequences of R_n-modules. Then, degree n piece of the tensor product $M \otimes N$ is given by*

$$(M \otimes N)_n := \sum_{i+j=n} (\mathbf{Ind}_{R_i \times R_j}^{R_n} M_i \otimes N_j),$$

where $\mathbf{Ind}_{R_i \times R_j}^{R_{i+j}} M_i \otimes N_j$ *is defined as*

$$\mathbf{Ind}_{R_i \times R_j}^{R_{i+j}} M_i \otimes N_j = k[R_{i+j=n}] \otimes_{k[R_i \times R_j]} (M_i \otimes N_j).$$

Tensor category structure on $\mathscr{P}$ is given similarly.

From Corollary 3.3.3, we know that $\mathfrak{R}$ and $\mathfrak{P}$ are equivalent as abelian categories. One natural question immediately follows: does the equivalence extends to tensor category structures? The following lemma gives an affirmative answer.

Lemma 3.3.6. *The two tensor categories $\mathfrak{R}$ and $\mathfrak{P}$, where the tensor structure is given by induced representations in each degree, are equivalent.*

The proof will be given later in Appendix A.

Hence, there is no need to differentiate the two, at least when we are interested in tensor categories $\mathfrak{R}$ and $\mathfrak{P}$. This viewpoint is extremely useful as Rook monoids and their representations were extensively studied, but as far as we know, those of $\mathscr{P}_n$–monoids were not. At the same time, as we will soon see, $\mathscr{P}_n$–monoids naturally arise as a model for curried representations of standard $\mathbf{OB}$–modules, and thus the monoids contain combinatorial descriptions of objects of interest in representation stability. Lemma 3.3.2 together with Lemma 3.3.6 give connections between the two.

If the characteristic of the base field k is zero, the ring $k[R_n]$ is semisimple, and the irreducible representations of $k[R_n]$ are classified by Theorem 3.3.4. We have the following version of the Littlewood-Richardson theorem:

Theorem 3.3.7 (Littlewood-Richardson Theorem for Rook Monoids)**.** *Let k be the field of characteristic zero, and $M = M_\mu, N = N_\nu$ be irreducible representations of R_m and R_n, corresponding to Young diagrams λ, μ of length i, j, where $i \leq m, j \leq n$. We then have the following decomposition of induced representation:*

$$\mathbf{Ind}_{k[R_m \times R_n]}^{k[R_{m+n}]} M_\mu \otimes N_\nu = \sum_\lambda c_{\mu\nu}^\lambda P_\lambda,$$

where P_λ's are irreducible representations of R_{m+n} (thus are classified by Young tableaux of length at most $m+n$) and $c_{\mu\nu}^\lambda$ are non-negative integers and $c_{\mu\nu}^\lambda = 0$ unless $|\lambda| = |\mu| + |\nu|$ and $\mu, \nu \in \lambda$. If $|\lambda| = |\mu| + |\nu|$ and $\mu, \nu \in \lambda$, then $c_{\mu\nu}^\lambda$ equals to the usual Littlewood-Richardson coefficient (see [Ma] for details about the proof of the usual Littlewood-Richardson theorem).

The main idea of the proof of Theorem 3.3.7 is the following:

Proposition 3.3.8. *Let ι be the ring homomorphism*

$$\left(\prod_i M_{\binom{m}{i}}(k[\mathfrak{S}_i])\right) \bigotimes \left(\prod_j M_{\binom{n}{j}}(k[\mathfrak{S}_j])\right) \simeq k[R_m] \otimes k[R_m] \simeq k[R_m \times R_n] \to k[R_{m+n}],$$

where the last map $k[R_m \times R_n] \to k[R_{m+n}]$ is the map induced by the injection $R_m \times R_n \to R_{m+n}$. Then

$$\iota\left(M_{\binom{m}{i}}(k[\mathfrak{S}_i]) \otimes M_{\binom{n}{j}}(k[\mathfrak{S}_j])\right) \subset M_{\binom{m+n}{i+j}}(k[\mathfrak{S}_{i+j}]),$$

where

$$k[R_{m+n}] = \prod_l M_{\binom{m+n}{l}}(k[\mathfrak{S}_l]).$$

For now, we will prove Theorem 3.3.7, assuming Proposition 3.3.8. We will postpone the proof of Proposition until Appendix A.

proof of Theorem 3.3.7. Let μ, ν be Young diagrams of length at most m and n, and M_μ, N_ν be the associated irreducible representations of R_m and R_n. For an arbitrary Young diagram λ of length at most $m+n$, let P_λ be the corresponding irreducible representation of R_{m+n}. Now consider the multiplicity of

$$\mathbf{Hom}_{k[R_{m+n}]}\left(\mathbf{Ind}\left(M_\mu \otimes N_\nu\right), P_\lambda\right) \simeq \mathbf{Hom}_{k[R_m \times R_n]}\left(M_\mu \otimes N_\nu, P_\lambda\right), \tag{III.5}$$

which equals to the constant $c^\lambda_{\mu\nu}$. The only component of

$$k[R_m \times R_n] \simeq \left(\prod_i M_{\binom{m}{i}}(k[\mathfrak{S}_i])\right) \bigotimes \left(\prod_j M_{\binom{n}{j}}(k[\mathfrak{S}_j])\right)$$

that acts nontrivially on $M_\mu \otimes N_\nu$ is the term

$$M_{\binom{m}{|\mu|}}(k[\mathfrak{S}_{|\mu|}]) \otimes M_{\binom{m}{|\nu|}}(k[\mathfrak{S}_{|\nu|}])$$

which from Proposition 3.3.8, gets mapped to

$$M_{\binom{m+n}{|\mu|+|\nu|}}(k[\mathfrak{S}_{|\mu|+|\nu|}]) \subset k[R_{m+n}].$$

Since P_λ is an irreducible representation associated with the Young diagram λ, the only component of the ring

$$k[R_{m+n}] = \prod_l M_{\binom{m+n}{l}}(k[\mathfrak{S}_l])$$

that acts nontrivially on P_λ is the term

$$M_{\binom{m+n}{|\lambda|}}(k[\mathfrak{S}_{|\lambda|}).$$

Hence, $c^\lambda_{\mu\nu} = 0$ whenever $|\lambda| \neq |\mu| + |\nu|$. If $|\lambda| = |\mu| + |\nu|$, then we are in the usual setting of Littlewood-Richardson, and the multiplicity of (III.5) is given by the Littlewood-Richardson coefficient $c^\lambda_{\mu\nu}$, which completes the proof. $\square$

3.4 Schur-Weyl Duality

We fix the base field to be $k = \mathbf{C}$ throughout this section.

3.4.1 Equivalence from Classical Schur-Weyl Duality

We first review the classical Schur-Weyl equivalence. For details see [SS1] or [Ma]. Recall that Rep($\mathfrak{S}$) is defined to be the category of sequences of $\mathfrak{S}_n$-representations.

Definition 3.4.1 (Category Rep($\mathfrak{S}$))**.** *The category Rep($\mathfrak{S}$) is defined to be the following category:*

(a) The objects of Rep($\mathfrak{S}$) are sequences $(V_n)_{n\in\mathbf{N}}$ where each ***degree*** *n* ***piece*** *V_n is a $\mathfrak{S}_n$-module.*

(b) A morphism $f : V \to W$ is a sequence $(f_n)_{n\in\mathbf{N}}$ of $\mathfrak{S}_n$ equivariant morphisms.

A representation M of $\mathbf{GL}(\infty)$ is **polynomial** if it appears as a subquotient of direct sum of representations of the form $(\mathbf{C}^\infty))^{\otimes k}$. Then, we define the category Rep$^{\text{pol}}(\mathbf{GL})$ to be the category of polynomial representations of $\mathbf{GL}(\infty)$.

Classical Schur-Weyl duality between representations of symmetric groups and polynomial representations of general linear groups gives the following equivalence between the tensor categories:

Theorem 3.4.2. *Let F be a functor defined by*

$$F : \textit{Rep}^{pol}(\mathbf{GL}) \to \textit{Rep}(\mathfrak{S}), \qquad F : V \mapsto (V_{[n]})_{n\geq 0}),$$

where $V_{[n]}$ is the weight space for the weight $1^n = (1, \cdots, 1, 0, 0, \cdots)$. Then, F gives an equivalence between tensor categories Rep($\mathfrak{S}$) and Rep$^{pol}(\mathbf{GL})$

Sketch of the proof. We can check that F satisfies the following:

(a) The functor F sends irreducible representation V_λ of Rep$^{\text{pol}}(\mathbf{GL})$, indexed by Young diagram λ to irreducible representation M_λ of $\mathfrak{S}_n$.

(b) F commutes with direct sums.

Hence, F gives an equivalence between tensor categories, as two categories $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$ and $\mathrm{Rep}(\mathfrak{S})$ are both semisimple and the rule for decomposing tensor products is the same. $\square$

3.4.2 Identification of $\mathfrak{R}$ via Graded Schur-Weyl Duality

Let $\mathfrak{C}$ be a category. Recall that $V = (V_n)_{n\geq 0}$ is a **graded $\mathfrak{C}$-representation** if each V_n is a $\mathfrak{C}$-module. We let $\mathrm{Rep}_{\mathbf{N}}(\mathfrak{C})$ to be the category of graded $\mathfrak{C}$-representations. If $\mathfrak{C}$ is a tensor category, then we can equip the category $\mathrm{Rep}_{\mathbf{N}}(\mathfrak{C})$ with Day convolution and make it into a tensor category: for $V, W \in \mathrm{Rep}_{\mathbf{N}}(\mathfrak{C})$, we define the tensor product $V \otimes W$ to be

$$(V \otimes W)_n := \bigoplus_{i+j=n} V_i \otimes V_j$$

Let $\mathrm{Rep}_{\mathbf{N}}(\mathfrak{S})$ be the category of graded $\mathrm{Rep}(\mathfrak{S})$-modules, and $\mathrm{Rep}^{\mathrm{pol}}_{\mathbf{N}}(\mathbf{GL})$ be the graded $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$-modules. Then, the following is an immediate consequence from Theorem 3.4.2:

Corollary 3.4.3. *The category $Rep_{\mathbf{N}}(\mathfrak{S})$ the category $Rep^{pol}_{\mathbf{N}}(\mathbf{GL})$ are equivalent as tensor categories.*

Observe that we can identify $\mathrm{Rep}_{\mathbf{N}}(\mathfrak{S})$ as a category of sequences of the form $(M_m)_{m\geq 0}$, where for each fixed m, we have $M_m = (M_{m,n})_{n\geq 0} \in (\mathfrak{S})$. Hence M can be identified as $\mathbf{N} \times \mathbf{N}$ array

$$\begin{array}{ccccc} M_{0,0} & M_{0,1} & \cdots & M_{0,n} & \cdots \\ M_{1,0} & M_{1,1} & \cdots & M_{1,n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ M_{n,0} & M_{n,1} & \cdots & M_{n,n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \tag{III.6}$$

of vector spaces, where for each m, n, the symmetric group $\mathfrak{S}_n$ acts on $M_{m,n}$.

Now we can identify $\mathfrak{R}$ as a subcategory of $\mathrm{Rep}_{\mathbf{N}}(\mathfrak{S})$ from the following: let $M = (M_m)_{m\geq 0} \in \mathfrak{R}$. Recall that $\mathbf{C}[R_m]$ is Morita equivalent to $\prod_{i=0}^{m} \mathbf{C}[\mathfrak{S}_i]$. Hence, for each fixed m we can decompose a R_m-module M_m by

$$M_m = \prod_{i=0}^{m} M_{m,n},$$

where $M_{m,n}$ is a $\mathfrak{S}_n$-representation. Hence, the category $\mathfrak{R}$ can be thought as the lower-triangular subcategory of $\mathrm{Rep}_{\mathbf{N}}(\mathfrak{S})$ consisting of $\mathbf{N} \times \mathbf{N}$ arrays of the form

$$\begin{array}{cccccc} M_{0,0} & 0 & 0 & \cdots & 0 & \cdots \\ M_{1,0} & M_{1,1} & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ M_{n,0} & M_{n,1} & \cdots & M_{n,n} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \tag{III.7}$$

It is easy to check that the embedding $\mathfrak{R} \hookrightarrow \mathrm{Rep}_{\mathbf{N}}(\mathfrak{S})$ is compatible with tensor product structures of two categories, as both tensor products are Day convolutions.

Now using Corollary 3.4.3, we can identify $\mathfrak{R}$ as a subcategory of $\mathrm{Rep}^{\mathrm{pol}}_{\mathbf{N}}(\mathbf{GL})$. Under the identification, the category $\mathfrak{R}$ (or to be precise, the subcategory of $\mathrm{Rep}^{\mathrm{pol}}_{\mathbf{N}}(\mathbf{GL})$ that is equivalent to $\mathfrak{R}$)is the subcategory of $\mathrm{Rep}^{\mathrm{pol}}_{\mathbf{N}}(\mathbf{GL})$ consisting of elements $M = (M_n)_{n\geq 0}$ where for each n, the weight of M_n is at most n. We summarize this by the following theorem:

Theorem 3.4.4. *The category $\mathfrak{R}$ is equivalent to the subcategory of*

$$\mathit{Rep}^{\mathit{pol}}_{\mathbf{N}}(\mathbf{GL}) = \{(M_n)_{n\geq 0} : M_n \in \mathit{Rep}^{\mathit{pol}}(\mathbf{GL})\}$$

where for each $n \geq 0$, the weight of M_n is at most n.

There is another way of embedding $\mathfrak{R}$ to $\mathrm{Rep}_{\mathbf{N}}(\mathfrak{S})$, by the "shift functor" F shifting up all the columns of (III.7), and eliminating all the zeros appearing above the diagonal. That is, if m is the following element of $\mathfrak{R}$

$$\begin{array}{cccccc} M_{0,0} & 0 & 0 & \cdots & 0 & \cdots \\ M_{1,0} & M_{1,1} & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ M_{n,0} & M_{n,1} & \cdots & M_{n,n} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

then $F(m)$ is the following element

$$\begin{array}{cccccc} M_{0,0} & M_{1,1} & M_{2,2} & \cdots & M_{n,n} & \cdots \\ M_{1,0} & M_{2,1} & M_{3,2} & \cdots & M_{n+1,n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ M_{n,0} & M_{n+1,1} & \cdots & \cdots & M_{2n,n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

of $\mathrm{Rep}_{\mathbf{N}}(\mathfrak{S})$. It is easy to check that the map $F : \mathfrak{R} \to \mathrm{Rep}_{\mathbf{N}}(\mathfrak{S})$ gives an equivalence of categories between $\mathfrak{R}$ and $\mathrm{Rep}_{\mathbf{N}}(\mathfrak{S})$. However, note that the shift functor F does not extend to the tensor structures of the two categories. As tensor categories, the two categories $\mathfrak{R}$ and $\mathrm{Rep}_{\mathbf{N}}(\mathfrak{S})$ are not equivalent: at least, the functor F we just mentioned does not give such an equivalence.

CHAPTER IV

B-Category

4.1 Definition of B-categories

In this section, we define a B-category, a category that carries a standard curried structure. Recall that in Section 2.4, we discussed representations of curried algebras associated to the standard $\mathbf{FB}$-module $\mathbf{V}$. Essentially, B-category is a category with a standard module $\mathbf{V}$, and we can define curried structures associated to the standard module $\mathbf{V}$.

4.1.1 Definition

Definition 4.1.1 (B-categories)**.** *Let $\mathfrak{C}$ be the category, where $\mathbf{Obj}(\mathfrak{C})$ is the family of finite sets. We say $\mathfrak{C}$ is a **B-category**, where "B" stands for bijection, if it satisfies*

(a) Any two $S, T \in \mathbf{Obj}(\mathfrak{C})$ with same cardinality are isomorphic

(b) $\mathbf{Hom}(S, T)$ is a finite set, and if $|S| \neq |T|$, it is an empty set.

Example 4.1.2. *The category $\mathbf{FB}$ we introduced in Chapter II is a B-category.*

Example 4.1.3. *Let $\mathbf{OB}$ be the category of finite sets where $\mathbf{Hom}(S, T) = \{1\}$ if $S = T$ and 0 otherwise. Then the category $\mathbf{OB}$ is also a B-category.*

Note that many authors have defined the category $\mathbf{OB}$ to be the category of totally ordered finite sets with order-preserving bijections. It is not hard to see that the two definitions are equivalent.

Remark 4.1.4. *We have a forgetful functor*

$$\mathbf{Mod}_{\mathbf{FB}} \to \mathbf{Mod}_{\mathbf{OB}},$$

between $\mathbf{Mod}_{\mathbf{FB}}$ and $\mathbf{Mod}_{\mathbf{OB}}$, forgetting the $\mathfrak{S}_n$-structure for each V_n.

If $\mathfrak{C}$ is a B-category, then the category $\mathbf{Mod}_{\mathfrak{C}}$ of functors from $\mathfrak{C}$ to the category of vector spaces is equivalent to the category of sequences $(M_n)_{n\geq 0}$ of vector spaces, where in each degree n, the vector space M_n is equipped with an action of $\mathbf{End}([n])$. For instance, we have $\mathbf{End}_{\mathbf{FB}}([n]) = \mathfrak{S}_n$ and $\mathbf{End}_{\mathbf{OB}}([n]) = 1$.

For two $\mathfrak{C}$-modules M and N, we define their **shuffle tensor product** (or, **Day convolution**) by

$$(M \otimes N)(S) := \bigoplus_{S=A\bigsqcup B} M(A) \otimes N(B),$$

where the sum is over all partitions of S into two disjoint sets A and B, with induced orders. This gives the category $\mathfrak{C}$ a monoidal category structure.

From now on until the end of this chapter, we fix a B-category $\mathfrak{C}$.

Definition 4.1.5. *Let $\mathfrak{C}$ be a B-category. Then the* ***standard*** $\mathfrak{C}$ ***module*** *is the $\mathfrak{C}$-module $\mathbf{V}$ that is k in degree 1 and 0 in other degrees.*

For a finite set S with n elements, then $\mathbf{V}^{\otimes n}(S)$ is the k-vector space with basis given by all permutations of the elements of S, and $\mathbf{V}^{\otimes n}(T) = 0$ if $|T| \neq n$. We have a natural permutation action of $\mathfrak{S}_n$ on $\mathbf{V}^{\otimes n}$. The n**-th symmetric power** $\mathbf{Sym}^n(\mathbf{V})$ of the standard $\mathbf{OB}$-module is defined to be $\mathfrak{S}_n$ coinvariants of $\mathbf{V}^{\otimes n}$, and we define the symmetric algebra $\mathbf{Sym}(\mathbf{V})$ as the infinite direct sum of $\mathbf{Sym}^n$'s:

$$\mathbf{Sym}(\mathbf{V}) = \bigoplus_{n\geq 0} \mathbf{Sym}^n(\mathbf{V})$$

The symmetric algebra $\mathbf{Sym}^n$ admits a multiplication and a comultiplication map

$$m : \mathbf{Sym}(\mathbf{V}) \otimes \mathbf{Sym}(\mathbf{V}) \to \mathbf{Sym}(\mathbf{V}), \qquad \Delta : \mathbf{Sym}(\mathbf{V}) \to \mathbf{Sym}(\mathbf{V}) \otimes \mathbf{Sym}(\mathbf{V}),$$

given by

$$m : t^A \otimes t^B \mapsto t^{A\cup B}, \qquad \Delta : t^S \mapsto \sum_{S=A\sqcup B} t^A \otimes t^B.$$

4.1.2 Operations on $\mathfrak{C}$–modules

Let S be a finite set. We define operations on $\mathfrak{C}$-modules similar to those of $\mathbf{FB}$-modules (See Subsection 2.4.1). That is, an **operation** on a $\mathfrak{C}$-module M is a natural linear map

$$\phi^S_{\underline{x},\underline{y}} : M(S \setminus \underline{y}) \to M(S \setminus x).$$

The naturality means that for any order-preserving bijection $i : S \to T$, the diagram

$$\begin{tikzcd} M(S \setminus y) \arrow[r, "\phi^S_{x,y}"] \arrow[d, "f"] & M(S \setminus x) \arrow[d, "f"] \\ M(T \setminus f(y)) \arrow[r, "\phi^T_{f(x),f(y)}"] & M(T \setminus f(x)) \end{tikzcd}$$

commutes.

Definition 4.1.6. *Let M be an $\mathfrak{C}$–module. Then (m,n)–operation ϕ on M is a* ***symmetric*** (m,n) ***operation*** *if it is induced by a map of $\mathfrak{C}$–modules*

$$a : \mathbf{Sym}^n(\mathbf{V}) \otimes M \to \mathbf{Sym}^m(\mathbf{V}) \otimes M.$$

That is, for a finite set S and a subset B of S, the map

$$t^B \otimes x \mapsto \sum_{A \subset S, |A| = m} t^A \otimes \phi^S_{A,B}(x)$$

is a map of $\mathfrak{C}$-modules.

We say that ϕ is **symmetric** if it is given by a map

$$a : \mathbf{Sym}^n(\mathbf{V}) \otimes M \to \mathbf{Sym}^m(\mathbf{V}) \otimes M$$

of $\mathfrak{C}$-modules. An operation ϕ is **simple** if $\phi^S_{\underline{x},\underline{y}} = 0$ whenever $\underline{x}, \underline{y}$ intersect non-trivially.

4.2 Curried Algebras Associated to the Standard $\mathfrak{C}$-module

4.2.1 Curried $\underline{\mathfrak{gl}}$-structure on a B-category

Let $\mathbf{V}$ be the standard $\mathfrak{C}$–module and M be an arbitrary $\mathfrak{C}$–module, equipped with a map

$$a : \mathbf{V} \otimes M \to \mathbf{V} \otimes M$$

of $\mathfrak{C}$–modules For a finite set S, an element $y \in S$, and $m \in M(S \setminus y)$, we can write

$$a(t^y \otimes m) = t^y \otimes \tau^{S \setminus y}(m) + \sum_{x \in S \setminus y} t^x \otimes \sigma^S_{x,y}(m) \tag{IV.1}$$

in terms of bases. Here, τ is $(0,0)$-operation and σ is a simple $(1,1)$-operation. We then have the following classification for $\underline{\mathfrak{gl}}(\mathbf{V})$-modules:

Theorem 4.2.1. *The map*

$$a : \mathbf{V} \otimes M \to \mathbf{V} \otimes M \qquad t^y \otimes m \mapsto t^y \otimes \tau^{S\setminus y}(m) + \sum_{x \in S\setminus y} t^x \otimes \sigma^S_{x,y}(m)$$

defines a $\underline{\mathfrak{gl}}(\mathbf{V})$-module structure on an $\mathfrak{C}$-module M if and only if

(a) The operations τ and σ commute with themselves and each other

(b) Given a finite set S and three distinct elements $x, y, z \in S$, we have $\sigma^{S\setminus x}_{y,z}\sigma^{S\setminus z}_{x,y} = \sigma^{S\setminus y}_{x,z}$.

Proof. Let $S \in \mathbf{Obj}(\mathfrak{C})$, and M be an arbitrary $\mathfrak{C}$-module. Let ϕ be an $(1,1)$-operation on M that corresponds to a. That is, we have

$$a(t^y \otimes m) = \sum_{x\in S} t^x \otimes \phi^S_{x,y}(m).$$

Comparing this with (IV.1), we see that $\sigma, \tau = \phi[0], \phi[1]$.

Now, for two distinct elements $y, z \in S$, we have

$$\begin{aligned} a_1(a_2(t^y \otimes t^z \otimes m)) &= \sum_{w\in S\setminus y}\sum_{x \in S\setminus w} t^x \otimes t^w \otimes \phi^{S\setminus w}_{x,y}(\phi^{S\setminus y}_{w,z}(m)) \\ a_2(a_1(t^y \otimes t^z \otimes m)) &= \sum_{x\in S\setminus z}\sum_{w \in S\setminus x} t^x \otimes t^w \otimes \phi^{S\setminus x}_{w,z}(\phi^{S\setminus z}_{x,y}(m)) \\ \tau(a_1(t^y \otimes t^z \otimes m)) &= \sum_{w\in S\setminus z} t^z \otimes t^w \otimes \phi^{S\setminus z}_{w,y}(m) \\ \tau(a_2(t^y \otimes t^z \otimes m)) &= \sum_{x\in S\setminus y} t^x \otimes t^y \otimes \phi^{S\setminus y}_{x,z}(m). \end{aligned} \tag{IV.2}$$

Recall that by definition, the map a gives a $\underline{\mathfrak{gl}}(V)$-module structure on M if and only if $[a_1, a_2] = \tau(a_1 - a_2)$. Applying this to (IV.2), we see that a gives $\underline{\mathfrak{gl}}(\mathbf{V})$-module structure if and only if both the conditions (a) and (b) from Theorem 4.2.1 hold. □

Let M be the $\underline{\mathfrak{gl}}(\mathbf{V})$–module. Given $[n]$ and two elements $x, y \in [n]$, define $\rho_{x,y}$ to be the composition

$$M([n]) \to M([n] \cup \{\infty\} \setminus x) \to M([n] \cup \{\infty\} \setminus y) \to M([n]) \tag{IV.3}$$

Via the map from (IV.3), each $\sigma_{x,y}$ acts on $M([n])$. Now consider a map

$$\{\sigma_{x,y} : x \neq y \in [n]\} \to \mathscr{P}_n, \qquad \sigma_{x,y} \mapsto (U := \{x\}, \phi : \{x\} \to \{y\}).$$

See Example 3.2.5 for the definition of $\mathscr{P}_n$ monoid and how to represent elements of $\mathscr{P}_n$ as pairs of $S \subset [n]$ and an injection from S to $[n]$. It is easy to show that elements of the form $(x, \phi : x \to y)$ generate the monoid $\mathscr{P}_n$, so we can decompose any element of $\mathscr{P}_n$ by the product of such cycles. This is analogous to the fact that the group $\mathfrak{S}_n$ is generated by the product of cycles. Hence, we see that for each $n \geq 0$, the degree n component $M([n])$ carries a representation of the monoid $\mathbf{N} \times \mathscr{P}_n$, where the generator $\mathbf{N}$ corresponds to τ operator in Proposition 4.2.1. Hence, we see that if $M \in \mathbf{Mod}_{\mathfrak{C}}$ is a $\underline{\mathfrak{gl}}(\mathbf{V})$-module if degree n component of $M = (M_n)_{n \geq 0}$ has an action of $\mathscr{P}_n$ that is compatible with the $\mathfrak{C}$-structural action of $\mathbf{End}([n])$.

For instance, let $M = (M_n)$ be an $\mathbf{OB}$-module. Then, since $\mathbf{OB}$-structure of M_n is trivial (as $\mathbf{End}_{\mathbf{OB}}([n]) = 1$), we see that M is a $\underline{\mathfrak{gl}}(\mathbf{V}_{\mathbf{OB}})$-module if and only if each M_n has an $\mathbf{N} \times \mathscr{P}_n$ action on each degree. We summarize this by the following

Theorem 4.2.2. *Giving a $\underline{\mathfrak{gl}}(\mathbf{V})$–structure on an $\mathbf{OB}$-module M is equivalent to giving a sequence M_n of $N \times \mathscr{P}_n$–modules.*

Remark 4.2.3. *It is not hard to show that $\mathscr{P}_n$ can be generated by*

$$\{(x, \phi : x \to y), |x - y| = 1\}$$

together with order-preserving isomorphisms. These elements can be thought of as Coxeter generators of the monoid $\mathscr{P}_n$.

If, on the other hand, we have an $\mathbf{FB}$-module $M = (M_n)$, then for each n, the symmetric group $\mathfrak{S}_n$ acts on M_n. Thus we have an action of $\mathfrak{S}_n$ and the action of $\mathbf{N} \times \mathscr{P}_n$. See [SS3], Remark 4.7 for how the two actions interact. As a consequence, the additional $\mathbf{N} \times \mathscr{P}_n$-structure gives the following classification from [SS3]

Theorem 4.2.4. *An $\mathbf{FB}$-module $M = (M_n)_{n \geq 0}$ is a $\underline{\mathfrak{gl}}(\mathbf{V})$-module if and only if $\mathbf{N} \times (\mathfrak{S}_n \ltimes \mathfrak{A}_n)$ acts on M_n, where $\mathfrak{A}_n$ is a monoid generated by n commuting idempotents*

Remark 4.2.5. *As $\mathscr{P}_n$ is a submonoid of $\mathfrak{S}_n \ltimes \mathfrak{A}_n$, we have a natural forgetful functor*

$$F_{\textit{forget}} : \underline{\mathfrak{gl}}(\mathbf{V}_{\mathbf{FB}})\text{-}\textit{modules} \to \underline{\mathfrak{gl}}(\mathbf{V}_{\mathbf{OB}})\text{-}\textit{modules}$$

4.2.2 The Symplectic Algebra on the Category of $\mathfrak{C}$-modules

Let $\mathfrak{C}$ be a B-category, and let M be a $\mathfrak{C}$-module with three maps

$$a : \mathbf{V} \otimes M \to \mathbf{V} \otimes M, \quad b : \mathbf{Div}^2(\mathbf{V}) \otimes M, \quad b' : M \to \mathbf{Sym}^2(\mathbf{V}) \otimes M.$$

Let σ and τ be the simple $(1,1)$ and $(0,0)$-operations corresponding to a in (IV.1). Let β and β' be the symmetric $(0,2)$– and $(2,0)$–operations corresponding to b and b'. Thus we have

$$b(t^{\{x,y\}} \otimes m) = \beta^S_{x,y}(m) \quad b'(n) = \sum_{\{x,y\} subset S} t^{\{x,y\}} \otimes (\beta')^S_{x,y}(n)$$

for $m \in M(S \setminus \{x,y\})$ and $n \in M(S)$. We can extend the proof for those of standard $\mathbf{FB}$ modules in [SS3] and show the following (we omit the proof):

Proposition 4.2.6. *The triple (a, b, b') defines a representation of $\underline{\mathfrak{sp}}(\mathbf{V} \otimes \mathbf{V}^*)$ on M if and only if the following conditions hold for all totally ordered finite sets S:*

(a) σ, τ, β and β' pairwise commute and each commutes with itself

(b) Given x, y, z distinct we have $\sigma^{S\setminus x}_{y,z}\sigma^{S\setminus z}_{x,y} = \sigma^{S\setminus y}_{x,z}$.

(c) Given $x, y, z \in S$, we have $(\beta')^S_{x,y}\beta^S_{y,z} = \sigma^{S\setminus y}_{x,z}$.

(d) Given x, y distinct, we have $(\beta')^S_{x,y}\beta^S_{x,y} = 2\tau^{S\setminus\{x,y\}}$.

Remark 4.2.7. *The condition (b) in Proposition 4.2.6 gives a $\mathbf{N} \times \mathscr{P}_n$-action on each degree n component M_n of $\mathfrak{C}$-module M. In addition to that, for each n, we have an upward and downward map $M_n \to M_{n+2}$ and $M_{n+2} \to M_n$, and conditions (c) and (d) give compatibility conditions of upward and downward maps.*

We end this chapter by noting that representations of many different standard curried algebras (Witt, Weyl, etc), associated to the category $\mathbf{FB}$ were studied in [SS3]. The analysis can be extended to any B-category $\mathfrak{C}$. As in the case for standard $\underline{\mathfrak{gl}}(\mathbf{V})$-representations and standard $\underline{\mathfrak{sp}}(\mathbf{V})$-representations, M is a standard Witt (or Weyl, etc) module if M is equipped with additional structures associated to curried algebras.

CHAPTER V

Diagram Categories and Standard Curried Structures

In Section 2.4, we showed that the category of δ-standard curried $\underline{\mathfrak{gl}}(\mathbf{V})$-modules in $\mathbf{FB}$ is equivalent to a module over a Brauer category $\mathfrak{B}(\delta)$. In this chapter, we give an analogous result for the standard $\mathbf{OB}$-modules. We first define *decorated Brauer category*, which replaces the role of the Brauer category in $\mathbf{FB}$-setting. We then show that the decorated Brauer category has a triangular structure, just like the standard Brauer category. Then we present the comparison theorem. Note that some of the proofs in this chapter are almost identical to their counterparts on $\mathbf{FB}$-setting (see [SS3] for details). This is not a coincidence. Quite the identical analysis can go through even if we replace $\mathbf{FB}$ by an arbitrary B-category $\mathfrak{C}$: as $\mathfrak{C}$ is a B-category, we have $\mathbf{V} \in \mathfrak{C}$ and the category of curried $\underline{\mathfrak{gl}}(\mathbf{V})$-modules can be thought as a category of sequences of the form $(M_n)_{n\geq 0}$, where for each n, the degree n component M_n has following two structures. First, we have an action of $\mathbf{End}([n])$, which can be thought of as the inherited structure from $\mathfrak{C}$-module structure. In addition, we have the action of the monoid $\mathfrak{P}_n$ on M_n, which can be thought of as the curried $\underline{\mathfrak{gl}}(\mathbf{V})$ structure. The two actions are patched by the compatibility condition, depending on the actual definition of $\mathfrak{C}$. Extending this, we get the holistic picture that can be applied to any curried algebras over an arbitrary B-category $\mathfrak{B}$.

Throughout this chapter, we denote by k the base field (or more generally, commutative ring).

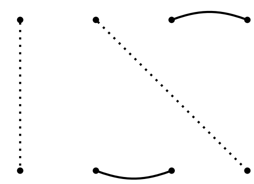
5.1 Decorated Brauer category

5.1.1 Definition

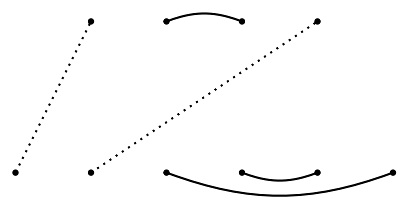
Let M be any monoid. A **decorated Brauer diagram** from a finite set S to another finite set T is a perfect matching on $S \sqcup T$, with an assignment $l \mapsto m_l \in M$ for each pair on $S \sqcup T$. Then, we define compositions of two diagrams analogous to those of usual Brauer diagrams we introduced earlier in Chapter II, where we further assert that decorations of compositions are given by multiplication on M.

For example, let β be the following decorated Brauer diagram from $S = [4]$ to $T = [4]$. We decorate the matching by drawing lines either by dotted or black, where dotted lines correspond to

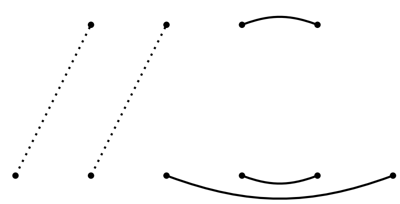
1_M, and black to the idempotent e.



and let $\alpha : T = [6] \to U = [4]$



Since the structure of M governs the decoration, we see that black lines *infect* others: if we compose any line with a black line, the composed line would turn into a black line. Hence, for the given diagrams α and β above, their composition $\alpha \bullet \beta : [4] \to [6]$ is the following decorated Brauer diagram



Definition 5.1.1. *Fix a field (or a commutative ring) k and let $\delta \in k$. For a monoid M, we define* ***decorated Brauer category*** *over k with parameter δ with decoration M, denoted as $\mathfrak{B}_M$ to be the following category:*

(a) Objects of $\mathfrak{B}_M$ are finite sets

(b) For $S, T \in \mathbf{Obj}(\mathfrak{B}_M)$, the set $\mathbf{Hom}_{\mathfrak{B}}(S, T)$ is the free k–module on the decorated Brauer diagrams from S to T.

(c) Composition of morphisms is defined similarly to those of the usual Brauer category while keeping the record of decorations M attached to the diagram.

Remark 5.1.2. *The standard Brauer category $\mathfrak{B}$ can be thought as decorated Brauer category $\mathfrak{B}_{G_1}$ with trivial decoration $G_1 = \langle 1 \rangle$ with a trivial group. For any $\mathfrak{B}_M$, there is a forgetful functor from $\mathfrak{B}_M$ to the standard Brauer category $\mathfrak{B} = \mathfrak{B}_{G_1}$* forgetting M. *In general, homomorphism $f : M \to N$ between two monoids induces a functor $F : \mathfrak{B}_M \to \mathfrak{B}_N$ between the corresponding decorated Brauer categories.*

From now on, we fix the monoid $M = \langle 1, e = e^2 \rangle$.

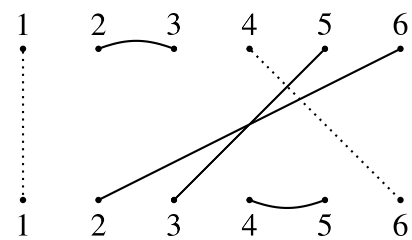
Definition 5.1.3. *An* ***ordered Brauer Category*** $\mathfrak{B}_{\mathbf{OB}}$ *is a category with the following objects and morphisms. The objects of* $\mathfrak{B}_{\mathbf{OB}}$ *are totally ordered finite sets.* $\mathbf{Hom}_{\mathfrak{B}_{\mathbf{OB}}}(S, T)$ *is a free k–module on perfect matchings on $S \sqcup T$ decorated by $\mathfrak{M}$ that satisfy the following conditions:*

(a) Horizontal edges can only be decorated by e

(b) Vertical edges can either be decorated by 1 or e.

(c) If $S_1 \subset S$ and $T_1 \subset T$ are subsets of $S, T \subset \mathbf{Obj}(\mathfrak{B}_{\mathbf{OB}})$ that are paired by lines decorated by e, the restriction $S_1 \to T_1$ is order-preserving.

For instance, see the following diagram with $S = T = [6]$. Again dotted lines correspond to 1 and black lines to e.



The condition (a) from Definition 5.1.3 enforces that all the horizontal lines should be black. From condition (b), there is no such restriction on vertical lines, but from condition (c), no two dotted lines can cross each other and permute the order. Black lines, on the other hand, can freely cross any other black lines or dotted lines. We see that compositions between morphisms are well defined, and $\mathfrak{B}_{\mathbf{OB}}$ is indeed a well-defined category.

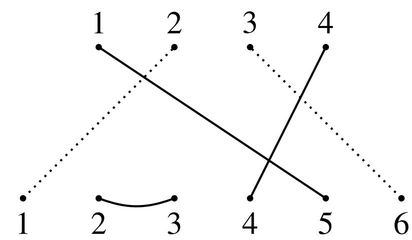
We equip the following triangular structure to $\mathfrak{B}_{\mathbf{OB}}$. See Definition 2.2.4 for the triangular structure of the standard Brauer category.

Definition 5.1.4 (Triangular structure on $\mathfrak{B}_{\mathbf{OB}}$). *A decorated ordered Brauer diagram $S \to T$ is* ***upward*** *if it contains no horizontal edges in S and* ***downward*** *if it contains no horizontal edges in T. Then denote by $\mathfrak{U}$ the wide subcategory of $\mathfrak{B}_{\mathbf{OB}}$ where $\mathbf{Hom}_{\mathfrak{U}}(S, T)$ is the subspace of $\mathbf{Hom}_{\mathfrak{B}_{\mathbf{OB}}}(S, T)$ generated by upward diagrams, and $\mathfrak{D}$ the subcategory of $\mathfrak{B}_{\mathbf{OB}}$ where $\mathbf{Hom}$-sets are generated by downward diagrams.*

Diagrammatically, upward diagrams look like upward trapezoids with no horizontal connections on the top row, where downward diagrams look like downward trapezoids. For instance, a decorated Brauer diagram

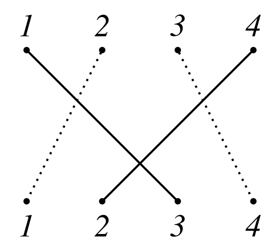
$$f_U : [4] \to [6], \tag{V.1}$$

defined by

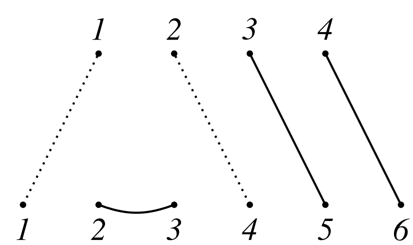


is an upward diagram.

Remark 5.1.5. *Let S be a totally ordered set, and $x, y \in S$ be distinct elements. Define $\eta^S_{x,y}$ to be an upward diagram $S \setminus \{x, y\} \to S$, where we connect bottom x and y by a black line, and connect $S \setminus \{x, y\}$ to itself by the unique order-preserving isomorphism. Observe that any upward diagram $f_U : [n] \to [m]$ can be obtained by composing an element $x \in \mathscr{P}_n$ together with elements of the form $\eta^S_{x,y}$. For instance, the upward diagram f defined in* (V.1) *can be obtained by composing the following element*



of $\mathscr{P}_4$ with



Proposition 5.1.6. *With the above definitions, $\mathfrak{B}_{\mathbf{OB}}$ is a triangular category (see Definition 2.1.4 for the definition of triangular category).*

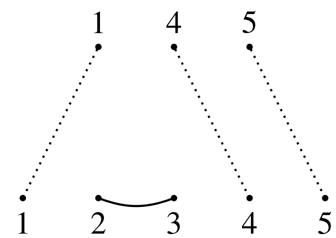
Proof. The proof of (T0), (T2), and (T3) is identical to those of standard Brauer category, so we will only present the proof of (T1), i.e., that the endomorphism ring $\mathbf{End}_{\mathfrak{U}}([n])$ is semisimple. By definition of $\mathfrak{U}$, the endomorphism algebra $\mathbf{End}_{\mathfrak{U}}([n])$ is generated by decorated diagrams from $[n]$ to $[n]$ with no horizontal maps. That is, we have

$$\mathbf{End}_{\mathfrak{U}}([n]) = k\mathscr{P}_n$$

where $\mathscr{P}_n$ is the finite inverse monoid we defined in Example 3.2.5. We have shown the semisimplicity of the ring $k\mathscr{P}_n$ in Corollary 3.3.3 (in general, the ring generated by any finite inverse monoid is semisimple), so the proof is complete. $\square$

5.1.2 Connection to the Ordered Brauer Category $\mathfrak{B}_{\mathbf{OB}}$

We now connect the standard $\underline{\mathfrak{sp}}(\mathbf{V_{OB}})$-modules with $\mathfrak{B}_{\mathbf{OB}}$. Let $\mathfrak{B} = \mathfrak{B}_{\mathbf{OB}}(\delta)$ be the ordered Brauer category from Definition 5.1.3. Recall that earlier in Proposition 5.1.6, we have shown that $\mathfrak{B}$ is a triangular category. Let $x, y \in [n]$ be distinct elements. We have a morphism $\eta^{[n]}_{x,y} : [n] \setminus \{x, y\} \to [n]$ in $\mathfrak{B}$ corresponding to the diagram with a unique horizontal straight line (so the unique dotted line corresponds to the idempotent e of M) between x and y in the target. We fill other pairs by order-preserving vertical dotted lines, all decorated by the identity e. For instance, if $n = 5$, $x = 2$, and $y = 3$, the diagram corresponding to $\eta_{x,y}$ is given by



For each x, y, such $\eta_{x,y}$ induces a linear map

$$\beta^{[n]}_{x,y} : M(S \setminus \{x, y\}) \to M(S).$$

Straightforward computation shows that that $\beta_{x,y}$ is a symmetric $(0, 2)$–operation on M. Similarly, the opposite diagram, where there is a unique horizontal straight line at the top connecting x and y, gives a morphism $(\eta')^{[n]}_{x,y}$, and this, in turn, induces a linear map

$$(\beta')^{[n]}_{x,y} : M(S) \to M(S \setminus \{x, y\}).$$

Using the composition rule for $\mathfrak{B}$, it is straightforward to check that the maps β and β' satisfy the following

Lemma 5.1.7. *The maps β and β' satisfies the following properties:*

(a) β and β' commute with themselves and with each other

(b) Let $x, y, z \in [n]$ be distinct. Then, $(\beta')^S_{x,y}(\beta)^S_{x,y} = (\{x\}, x \mapsto z) \in \mathscr{P}([n] \setminus y)$.

(c) Let $x, y \in [n]$ be distinct. Then $(\beta')^S_{x,y}\beta^S_{x,y} = \delta$.

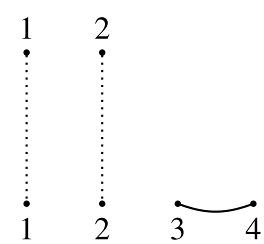
Note that the map $(\{x\}, x \mapsto z) \in \mathscr{P}([n] \setminus y)$ in condition (b) can be thought as $i_{x,z}^{S\setminus y}$, together with decoration, where the map i is defined earlier in Chapter II, Equation II.3.

We embed the category $\mathbf{OB}$ into $\mathfrak{B}_{\mathbf{OB}}$ as a wide subcategory, where the morphisms of $\mathbf{OB}$ are the unique order-preserving isomorphism from $[n]$ to itself, consisting only of dotted lines (i.e., decorated by 1). Then the following shows that η, η' together with the morphisms in $\mathbf{OB}$ (i.e., the order-preserving bijections) generate the full $\mathfrak{B}_{\mathbf{OB}}$:

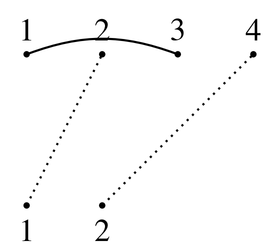
Proposition 5.1.8. *The category* $\mathfrak{B}_{\mathbf{OB}}$ *is generated by* η, η'*, and morphisms of* $\mathbf{OB}$*.*

Proof. Pick an element $f : S \to T$ in $\mathfrak{B}_{\mathbf{OB}}$, where S, T are totally ordered sets. We first note that we can disregard all the horizontal lines, and limit ourselves to the case when $f : S \to T$ is one-to-one. After constructing such a diagram, we can compose it with appropriate η and η' that give correct horizontal lines.

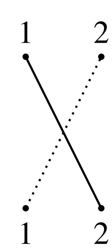
So limit ourselves for $f \in \mathscr{P}_n$, as $f : S \to T$ without any horizontal lines correspond exactly to an element of $\mathscr{P}_n$ with $n = |S| = |T|$. Now pick an element $m \in [n]$ where $m + 1$ is also in $[n]$. By composing with appropriate η and η''s we can construct an element $(\{m\}, m \mapsto m + 1) \in \mathscr{P}_n$, which we will denote as $m^{\uparrow}$. For simplicity, consider the case $n = 2$ and $m = 1$, and let $\eta : [2] \to [4]$ be the following diagram



and $\eta' : [4] \to [2]$ be



Then, we see that their composition $\eta'\eta : [2] \to [2]$ is the desired $m^{\uparrow}$:



We similarly define $m^{\downarrow}$ to be $(\{m\}, x \mapsto m-1)$, which can be generated by composing appropriate η and η'. General case for $n \geq 2$ is similar.

Now it is straightforward to see that we can generate any element $x \in \mathscr{P}_n$ by composing $m^{\uparrow}$ and $m^{\downarrow}$'s. In fact, $m^{\uparrow}$ and $m^{\downarrow}$'s are the Coxeter generators of $\mathscr{P}_n$ introduced in Remark 4.2.3. $\square$

Using Lemma 5.1.7 and Proposition 5.1.8, we see that the operations β, β' completely determine the $\mathfrak{B}_{\mathbf{OB}}$ structure on M:

Proposition 5.1.9. *Let M be an* $\mathbf{OB}$*–module equipped with a symmetric $(0,2)$–operations β and a symmetric $(2,0)$–operations β' satisfying the three conditions of Lemma 5.1.7 above. Then M carries a unique $\mathfrak{B}_{\mathbf{OB}}$–structure inducing β and β'*

Proof. We proceed as following:

1. First, we show that giving a $\mathfrak{U}$-structure on M gives a self-commuting symmetric $(0,2)$-operation. A similar argument shows that giving a $\mathfrak{D}$-structure on M gives rise to a self-commuting symmetric $(2,0)$-operation.

2. Then we show that giving $(0,2)$-operation β and $(2,0)$-operation β' that commute with themselves and with each other gives $\mathfrak{U}$ and $\mathfrak{D}$-structure.

3. Lastly, we check compatibility conditions between $\mathfrak{U}$-structure and $\mathfrak{D}$-structure.

For (1), assume that M is equipped with $\mathfrak{U}$-structure. For any totally ordered finite set S and two distinct elements $x, y \in S$, we define $\beta^S_{x,y}$ to be the action of $\eta^S_{x,y}$ on M (i.e., x and y is connected by horizontal black line decorated by the idempotent e). By construction, β is symmetric and commutes with itself. A similar argument shows the analogous result for $\mathfrak{D}$-structure. Now for (2), assume that M is equipped with such operations β and β'. We construct the full $\mathfrak{U}$-structure using the two operations. Pick any $\mathfrak{U}$-morphism $f : S \to T$, where S, T are totally ordered finite sets. We can factor f by a $\mathscr{P}_n$ map $\sigma : S \to f(S)$ (where $n = |S|$), followed by morphisms of the form $\eta^U_{x,y}$, for distinct x, y's. Note that as in the proof of Proposition 5.1.8, the $\mathscr{P}_n$-map σ can be generated by compositions of maps η and η'.

Now we define $M_f : M(S) \to M(T)$ to be the composition $M_\sigma : M(S) \to M(f(S))$ (where σ itself can be factored into a composition of order-preserving bijection, η, and η') with the corresponding composition of maps given by β, β' coming from the factorization. By the commutativity of β and β', the order of the factorization does not affect the result and as it is symmetric the order of elements x, y at each stage does not affect the result. The functoriality $M_{g \cdot f} = M_g \cdot M_f$ for another $\mathfrak{U}$-morphism $g : T \to U$ follows from the naturality condition. Again, similar arguments shows the corresponding result for $\mathfrak{D}$.

Now we check the compatibility. Let $\mathscr{U}$ (resp. $\mathscr{D}$) be the class of morphisms in $\mathfrak{U}$ (resp. $\mathfrak{D}$) isomorphic to $\eta^S_{x,y}$ for some totally order set S and two elements $x, y \in S$. We see that $\mathscr{U}$ generates $\mathfrak{U}$, and $\mathscr{D}$ generates $\mathfrak{D}$. Now using Proposition 3.4 from [SS3], we only need to show that any arbitrary pair $(\phi, \psi) \in \mathscr{U} \times \mathscr{D}$, where $\phi \cdot \psi$ is defined, is compatible. Let $\phi = \eta^S_{a,b}$, and $\psi = {\eta'}^S_{c,d}$. Let $n := |\{a, b\} \cup \{c, d\}|$. There are three cases to consider for each $n = 0, 1, 2$. For each case, the same argument as in the proof of [SS3], Proposition 5.4 goes through. We briefly outline the argument from [SS3]) below.

(a) If $n = 0$, the intersection is an empty set. From the commutativity of η, η' we have

$$(\eta')^S_{c,d}\eta^{S\setminus\{c,d\}}_{a,b} = \eta^{S\setminus\{c,d\}}_{a,b}(\eta')^{S\setminus\{a,b\}}_{c,d}$$

which corresponds to composition laws of ordered Brauer diagrams.

(b) If $n = 1$, and if we set $b = c$ without loss of generality, we have $(\eta')^S_{b,d}\eta^S_{a,b} = i^{S\setminus b}_{a,d} := (\{a\} : a \mapsto d)$. This also fits with composition laws of ordered Brauer diagrams. In essence, the map $i^{S\setminus b}_{a,d}$ corresponds to a black line, where permuting orders is allowed. Hence, the argument from the usual Brauer diagrams is applicable.

(c) If $n = 2$ and $\{a, b\} = \{c, d\}$, we have a cycle and $(\eta')^S_{a,b}\eta^S_{a,b} = \delta$ corresponds to the composition law for cycles in Brauer diagrams.

$\square$

5.1.3 The Comparison Theorem

Recall that in Chapter II, we defined δ-standard $\underline{\mathfrak{gl}}(\mathbf{V_{FB}})$-structure on an $\mathbf{FB}$-module M, via

$$\tau := \delta \cdot 1, \qquad \sigma^S_{x,y} = (i^S_{x,y})_*,$$

where the map i is defined in Equation (II.3). With this, we defined a notion of δ-standard $\underline{\mathfrak{sp}}(\mathbf{V_{FB}} \oplus \mathbf{V^*_{FB}})$ module structure: a $\underline{\mathfrak{sp}}(\mathbf{V_{FB}} \oplus \mathbf{V^*_{FB}})$-module M with structure maps (a, b, b') is δ-standard if (M, a) is δ-standard.

On the other hand, in the $\mathbf{OB}$ setting, we cannot use such i to define δ-standard $\underline{\mathfrak{gl}}(\mathbf{V_{OB}})$-structure: unless $|x - y| \leq 1$, the map $i^{[n]}_{x,y}$ is not order-preserving, so $i_{x,y}$ is not a well-defined morphism in $\mathbf{OB}$. However, if an $\mathbf{OB}$-module M is also an $\underline{\mathfrak{sp}}(\mathbf{V_{OB}} \oplus \mathbf{V^*_{OB}})$-module, we can define "virtual" i, using property (b) from Lemma 5.1.7: we define

$$i^{S\setminus y}_{x,z} := (\beta')^S_{x,y}(\beta)^S_{x,y}. \tag{V.2}$$

Then, we have the following definition for δ-standard representation for fixed $\delta \in k$.

Definition 5.1.10. *Let M be an $\mathbf{OB}$-module, equipped with a pair (a, b, b') of $\underline{\mathfrak{sp}}(\mathbf{V_{OB}} \oplus \mathbf{V^*_{OB}})$-structure maps. Then, we say that M is δ-standard if the representation of $\underline{\mathfrak{gl}}(\mathbf{V_{OB}})$ is virtually δ-standard. That is, the $\underline{\mathfrak{gl}}(\mathbf{V_{OB}})$-structure on M given by the map $a : \mathbf{V_{OB}} \otimes M \to \mathbf{V_{OB}} \otimes M$ equals to the structure given by*

$$\tau = \delta \cdot 1, \qquad \sigma^S_{x,y} = (i^S_{x,y})_*$$

for any finite totally ordered set S, and i is a map defined by (V.2). *We define $\mathbf{Rep}_\delta(\underline{\mathfrak{sp}}(\mathbf{V} \oplus \mathbf{V}^*))$ to be the category of δ-standard representations.*

We emphasize that we cannot define a δ-standard $\underline{\mathfrak{gl}}(\mathbf{V_{OB}})$-structure on a $\underline{\mathfrak{gl}}(\mathbf{V_{OB}})$-module M. What we can define is a *virtual* δ-standard $\underline{\mathfrak{gl}}(\mathbf{V_{OB}})$-structure, when the module M is $\underline{\mathfrak{sp}}(\mathbf{V_{OB}} \oplus \mathbf{V^*_{OB}})$-module. We will discuss about "virtual" standard structures in detail in the next section.

Now we present the fundamental comparison theorem:

Theorem 5.1.11. *Assume $\delta \in 2k$. Then we have a natural isomorphism of categories*

$$\mathbf{Mod}_{\mathfrak{B}_{\mathbf{OB}}(\delta)} \cong \mathbf{Rep}_{\delta/2}(\underline{\mathfrak{sp}}(\mathbf{V} \oplus \mathbf{V}^*)).$$

Proof. This follows immediately from comparing Propositions 4.2.6 and 5.1.9. $\square$

CHAPTER VI

Curried Exceptional Lie Algebras

In this chapter, we define curried exceptional Lie algebras $\mathfrak{g}_2$ and $\mathfrak{e}_6$.

6.1 Curried Representation of an Exceptional Lie Algebra of type $\mathfrak{g}_2$

Let $\mathfrak{g}_2$ be the exceptional complex Lie algebra. Recall that the dimension of $\mathfrak{g}_2$ is 14. We can decompose $\mathfrak{g}_2$ into the following:

$$\mathfrak{g}_2 = \left(\bigwedge^2 V \otimes V\right)^* \oplus \left(\bigwedge^2 V\right)^* \oplus V^* \oplus \mathfrak{gl}(V) \oplus V \oplus \bigwedge^2 V \oplus \left(\bigwedge^2 V \otimes V\right),$$

where V is a two-dimensional vector space. Hence, for an arbitrary vector space M, giving a linear map

$$\mu : \mathfrak{g}_2 \otimes M \to M$$

is equivalent to giving the following seven linear maps:

$$a : \mathfrak{gl}(V) \otimes M \to M,$$

$$b : V \otimes M \to M, \qquad c : \bigwedge^2 V \otimes M \to M, \qquad d : \left(\bigwedge^2 V \otimes V\right) \otimes M \to M,$$

$$b' : M \to V \otimes M, \qquad c' : M \to \bigwedge^2 V \otimes M, \qquad d' : M \to \left(\bigwedge^2 V \otimes V\right) \otimes M.$$

Now our goal is to determine the relationships of the seven maps a, b, b', c, c', d, d' when the map $\mu : \mathfrak{g}_2 \otimes M \to M$ gives $\mathfrak{g}_2$-module structure on M. Denote by $m : V \otimes V \to \bigwedge^2 V$ the multiplication map

$$m : v_1 \otimes v_2 \mapsto v_1 \bigwedge v_2,$$

and $\Delta : \bigwedge^2 V \to V \otimes V$ by a co-multiplication map

$$v_1 \bigwedge v_2 \mapsto v_1 \otimes v_2 - v_2 \otimes v_1.$$

As usual, $\tau : V' \otimes V \to V \otimes V$ is the permutation map $v_1 \otimes v_2 \mapsto v_2 \otimes v_1$.

Proposition 6.1.1. *Let μ and $a, \cdots, d'$ as above. Then, μ defines a $\mathfrak{g}_2$ representation if and only if the maps $a, \cdots, d'$ satisfy the following five conditions:*

1. $a = bb'\tau - b'b : V' \otimes V \otimes M \to M$.
2. $c = b^2\Delta$ *as a map from $\bigwedge^2 V \otimes M$ to M, and $b^2 - b^2\tau_{12} = cm : V^{\otimes 2} \otimes M \to M$, where $\tau_{12} : V^{\otimes 2} \to V^{\otimes 2}$ is a permutation map permuting first and second V's.*
3. *Analogous dual statement for c', b', Δ', m'.*
4. $d = cb - bc\tau : \bigwedge^2 V \otimes V \otimes M \to M$. *Here, τ permutes $\bigwedge^2 V$ and V.*
5. *Analogous dual statement for d', c', b'.*

Proof. We give the proof that if μ defines a $\mathfrak{g}_2$ representation, then we have conditions 1, 2 and 4. The proof of 3 and 5 assuming that μ defines a $\mathfrak{g}_2$ representation is almost the same, and the proof is completely reversible so proving the other direction is not hard. (DO THE PROOF) □

Here, note that maps b, c, d give maps of V, $\bigwedge^2 V$, and $\bigwedge^2 \otimes V$ into $\mathbf{End}(M)$. Furthermore, if $\mathfrak{g}_2$ embeds into $\mathbf{End}(M)$, the maps induced by b, c, d are also injections. Now, let $\mathfrak{C}$ be an arbitrary linear tensor category over the field of characteristic zero. We want the category $\mathfrak{C}$ to be linear i.e., that all the Hom-sets are vector spaces and all composition operations are bilinear, because we need to define wedge products. Let V be an object of $\mathfrak{C}$. The symmetric group $\mathfrak{S}_n$ acts on $V^{\otimes n}$, and as $\mathfrak{C}$ is a linear category, we have the antisymmetrization map $p_A : V^{\otimes n} \to V^{\otimes n}$ given by

$$p_A(v_1 \otimes \cdots \otimes v_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma)\sigma(v_1 \otimes \cdots \otimes v_n).$$

Then, we define the nth antisymmetric tensor power $\wedge^n V$ be the cokernel of the map $(1 - p_A)$. Note that we can also define $\wedge^n V$ when the characteristic of the base field is $p > 0$, but for simplicity, we limit ourselves to characteristic zero cases. Then, we have the following definition:

Definition 6.1.2. *Let V, M be an object of $\mathfrak{C}$, and let μ and $a, \cdots, d'$ as above. Then, M is a $\mathfrak{g}(V)$-module if $a, \cdots, d'$ satisfy the following:*

1. *The maps b, c, d, b', c', d' give embeddings of $V, \bigwedge^2 V, \bigwedge^2 \otimes V \cdots$ to $\mathbf{End}(M)$.*

2. $a = [b, b']$

3. $c = b^2\Delta : \bigwedge^2 V \otimes M \to M$*, and* $b^2 - b^2\tau_{12} = cm : V^{\otimes 2} \otimes M \to M$*, where* $\tau_{12} : V^{\otimes 2} \to V^{\otimes 2}$ *is the symmetric structure on the tensor category* $\mathfrak{C}$*.*

4. *Analogous (dual) condition for* c', b', Δ', m'*.*

5. $d = cb - bc\tau : \bigwedge^2 V \otimes V \otimes M \to M$*. Here, 4* τ *is the symmetric structure that permutes* $\bigwedge^2 V$ *and* V*.*

6. *Analogous (dual) condition for* d', c', b'*.*

Note that the above definition makes sense even When $\mathfrak{C}$ is a non rigid tensor category.

Recall that in decomposition of $\mathfrak{g}_2$ as

$$\mathfrak{g}_2 = \left(\bigwedge^2 V \otimes V\right)^* \oplus \left(\bigwedge^2 V\right)^* \oplus V^* \oplus \mathfrak{gl}(V) \oplus V \oplus \bigwedge^2 V \oplus \left(\bigwedge^2 V \otimes V\right),$$

the dimension of V is 2. We have an analogous condition on $\mathfrak{g}(V)$ on an arbitrary arbitrary linear tensor category:

Proposition 6.1.3. *If* $V \in \mathfrak{C}$ *such that* $\bigwedge^3 V \neq 0$*, then none of* M *in* $\mathfrak{C}$ *can be equipped with* $\mathfrak{g}(V)$*-module structure.*

Proof. Assume $\bigwedge^3 V \neq 0$, and M is a $\mathfrak{g}(V)$-module. Then, the conditions from Definition 6.1.2 give the following commutative diagram:

$$\begin{tikzcd}
\bigwedge^2 V \otimes V \arrow[r, "d"] & \mathbf{End}(M) \\
\bigwedge^2 V \otimes V \arrow[u, "\mathbf{1}"] \arrow[ur, "cb - bc\tau_{12}"] & \\
V \otimes V \otimes V \arrow[u, "m \otimes \mathbf{1}"] \arrow[uur, bend right, "\phi"'] &
\end{tikzcd}$$

where $\phi := cmb - bcm\tau$. As d is injective and the diagram is commutative, we see that $\ker(\phi) = \ker(m \otimes \mathbf{1}) = \mathbf{Sym}^2(V) \otimes V$. However, rewriting $c = b^2 - b^2\tau_{12}$ we get $\phi = b^3((123) - (213) - (312) + (321))$. Thus, we see that kernel of ϕ contains (at least) $\bigwedge^3 V$. Hence, we see that $\bigwedge^3 V \subset \mathbf{Sym}^2(V) \otimes V$. We see that this can only happen if and only if $\bigwedge^3 V = 0$. $\square$

Note that the above result matches well with the classical result when $\mathfrak{C}$ is the category of complex vector spaces: at least for finite-dimensional except Lie algebras, we have $\mathfrak{g}_2$, but we do not have $\mathfrak{g}_3$: here $\bigwedge^3 V \neq 0$.

Remark 6.1.4. *Note that we have a short exact sequence*

$$0 \to S_{(2,1)}V \to \bigwedge^2 V \otimes V \to \bigwedge^3 V \to 0,$$

where $S_{(2,1)}$ is a Schur functor associated to a partition $(2,1)$. Hence, if $\bigwedge^3 V = 0$, we have $S_{(2,1)}(V) = \bigwedge^2 V \otimes V$.

Note also that

$$V^{\otimes 3} = \bigwedge^3 V \oplus \mathbf{Sym}^3(V) \oplus S_{(2,1)}V.$$

Here, we see that $\bigwedge^3 V \subset \ker \phi$, and $\mathbf{Sym}^3(V) \subset \ker(m \otimes \mathbf{1})$. Hence, the only component of $V^{\otimes 3}$ (or of $\bigwedge^2 \otimes V$) that can possibly act via ϕ is $S_{(2,1)}V$.

Definition 6.1.5. *Let V be an object in a tensor category $\mathfrak{C}$. We define $\mathfrak{g}'(V)$ to be*

$$\mathfrak{g}'(V) := \left(S_{(2,1)}V\right)^* \oplus \left(\bigwedge^2 V\right)^* \oplus V^* \oplus \mathfrak{gl}(V) \oplus V \oplus \bigwedge^2 V \oplus \left(S_{(2,1)}V\right).$$

CHAPTER VII

Non-Primality of Certain Symmetric Ideals

This chapter answers the question from [NS] about the explicit proof of non-primality of specific $\mathfrak{S}$–ideal. As a consequence, the chapter is independent from the preceding chapters.

7.1 Main Results

The goal of this chapter is to prove the following theorem:

Theorem 7.1.1. *If $f = g = (x_1 - x_2)^{2n-1}$, then for any $\sigma \in \mathfrak{S}$, we have $f \cdot \sigma(g) \in I(2n)$.*

Theorem 7.1.1 implies that $I(2n)$ is not $\mathfrak{S}$-prime for any n as $(x_1 - x_2)^{2n-1} \notin I(2n)$, because any nonzero element of a homogeneous ideal $I(2n)$ is of degree $\geq 2n$.

For $n = 1$, we can directly show that Theorem 7.1.1 holds:

Example 7.1.2. *Let $f = g = x_1 - x_2$, and $\sigma \in \mathfrak{S}$ be any permutation. Then, as any nonzero element of $I(2)$ has degree at least 2, we see that $f = g \notin I(2)$. However, note that $f \cdot \sigma(g) \in I(2)$ from the following identity:*

$$(X - Z)^2 - (Y - Z)^2 - (X - W)^2 + (Y - W)^2 = -2(X - Y)(Z - W). \qquad \text{(VII.1)}$$

If σ maps x_1 to x_i and x_2 to x_j, set $X = x_1$, $Y = x_2$, $Z = x_i$, and $W = x_j$, and we see that $f\sigma(g) \in I(2)$ from (VII.1).

7.1.1 Open Problems

In §7.3, we will see that Theorem 7.1.1 is related to a contraction of an ideal I in $k[\delta, x, y]$ to $k[x, y]$. See Theorem 7.3.6. We then have two conjectures that generalize Theorem 7.3.6: Conjecture 7.3.7 explicitly gives monomial generators of the contracted ideal, and Conjecture 7.4.1 generalizes Theorem 7.3.6 for more variables. See §7.4 for details.

7.1.2 Outline

In §7.3, we give a proof of Theorem 7.1.1. In § 7.4 we discuss about conjectures that generalize our main results.

7.2 Preliminary Linear Algebra

In this section, we will discuss two results that will play key roles in the proof of the Theorem 7.1.1.

7.2.1 Determinant of a Matrix with Binomial Entries

Proposition 7.2.1. *Let A be the following lower-triangular unipotent $N \times N$ matrix:*

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ \binom{N}{1} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \binom{N}{N-2} & \binom{N}{N-3} & \cdots & 1 & 0 \\ \binom{N}{N-1} & \binom{N}{N-2} & \cdots & \binom{N}{1} & 1 \end{bmatrix}. \tag{VII.2}$$

For $1 \leq m \leq N$, let A_m be the following lower-left submatrix of A

$$A_m = \begin{bmatrix} \binom{N}{N-m} & \binom{N}{N-m-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \binom{N}{N-2} & \binom{N}{N-3} & \cdots & \binom{N}{N-m-2} & \binom{N}{N-m-1} \\ \binom{N}{N-1} & \binom{N}{N-2} & \cdots & \binom{N}{N-m-1} & \binom{N}{N-m} \end{bmatrix}. \tag{VII.3}$$

Then, A_m is an invertible matrix.

Proof. Let A_k be the $k \times k$ submatrix of the matrix above. Note that $\binom{n}{m} = e_m(1, \cdots, 1)$ where e_m is the elementary symmetric polynomial

$$e_m(x_1, \cdots, x_n) := \sum_{1 \leq i_1 \leq \cdots \leq i_m \leq n} X_{i_1} \cdots X_{i_m}.$$

Then, we see that the determinant of A_k equals the determinant of the $k \times k$ matrix $B = (b_{i,j})$, evaluated at $(1, \cdots, 1)$, where all the entries $b_{i,j}$ are the elementary symmetric polynomials, corresponding to $(i, j)^{th}$ entry of A_k. Using Jacobi–Trudi identity (see [Ma, pp. 40–41] or [FH, pp.455]), we see that $\det(B) = s$, where s is a Schur polynomial. Then the value of s at $(1, \cdots, 1)$ is gives the dimension of the irreducible representation of GL_n with highest weight λ: that is, if $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k \geq 0)$ and $s = s_\lambda$ is a Schur polynomial associated to the partition λ, we have the following closed-form expression

$$s_\lambda(1, \cdots, 1) = \prod_{i<j} \frac{\lambda_i - \lambda_j + j - i}{j - i} > 0. \qquad \square$$

Remark 7.2.2. *The above proposition also applies to the matrix whose* (i, j)*th entry is* $\binom{n}{m+i-j}$. *The same logic as above applies, so* $\det(A)$ *is a value of the corresponding Schur polynomial at* $(1, \cdots, 1)$. *Hence,* $\det(A)$ *is positive and in particular, nonzero.*

7.2.2 Anti-diagonal Transposition

For $n \times n$ matrix A, we define A^τ to be an anti-diagonal transposition,

$$A^\tau := JA^TJ,$$

where J is the following matrix:

$$J = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

Note that for any two matrices A and B, we have

$$(A+B)^\tau = A^\tau + B^\tau,$$

and

$$(AB)^\tau = B^\tau A^\tau.$$

We need the following lemma:

Lemma 7.2.3. *If $A = (a_{i,j})_{1\le i,j\le n}, B = (b_{i,j})_{1\le i,j\le n}$ are $n \times n$ matrices satisfying:*

- *For any $1 \le m \le n$, $m \times m$ "upper-right" submatrix*

$$A_m = \begin{bmatrix} a_{1,n-m+1} & a_{1,n-m+2} & \cdots & a_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,n-m+1} & a_{m,n+m+2} & \cdots & a_{m,n} \end{bmatrix}$$

  of A is nonsingular

- *All the anti-diagonal entries of B are zero*

- *$B^\tau = -B$.*

Then there exists a unique strictly lower-triangular $n \times n$ matrix X satisfying $AX - (AX)^\tau = B$.

Proof. Since X is a strictly lower triangular matrix, write

$$X = \begin{bmatrix} | & | & \cdots & | & | \\ X_1 & X_2 & \ddots & X_{n-1} & X_n \\ | & | & \cdots & | & | \end{bmatrix},$$

where

$$X_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_{i+1,i} \\ \vdots \\ x_{n,i} \end{bmatrix}$$

is a column vector with i zeros from the top. Specifically, $X_n = 0$. Also write

$$A = \begin{bmatrix} — & A_1 & — \\ — & A_2 & — \\ \vdots & \ddots & \vdots \\ — & A_n & — \end{bmatrix},$$

where A_j's are horizontal $1 \times n$ vectors. Then, we see that

$$AX = \begin{bmatrix} A_1X_1 & A_1X_2 & \cdots & A_1X_{n-1} & 0 \\ A_2X_1 & A_2X_2 & \cdots & A_2X_{n-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{n-1}X_1 & A_{n-1}X_2 & \cdots & A_{n-1}X_{n-1} & 0 \\ A_nX_1 & A_nX_2 & \cdots & A_nX_{n-1} & 0 \end{bmatrix},$$

and

$$(AX)^\tau = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ A_nX_{n-1} & A_{n-1}X_{n-1} & \cdots & A_2X_{n-1} & A_1X_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_nX_2 & A_{n-1}X_2 & \cdots & A_2X_2 & A_1X_2 \\ A_nX_1 & A_{n-1}X_1 & \cdots & A_2X_1 & A_1X_1 \end{bmatrix},$$

Hence, we see that

$$AX-(AX)^\tau = \begin{bmatrix} A_1X_1 & A_1X_2 & \cdots & A_1X_{n-1} & 0 \\ A_2X_1 - A_nX_{n-1} & A_2X_2 - A_{n-1}X_{n-1} & \cdots & 0 & -A_1X_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{n-1}X_1 - A_nX_2 & 0 & \cdots & -(A_2X_2 - A_{n-1}X_{n-1}) & -A_1X_2 \\ 0 & -(A_{n-1}X_1 - A_nX_2) & \cdots & -(A_2X_1 - A_nX_{n-1}) & -A_1X_1 \end{bmatrix}.$$

Thus, if we let

$$B = \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{n-1,n} & 0 \\ b_{2,2} & b_{2,2} & \cdots & 0 & -b_{1,1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n-1,1} & 0 & \cdots & -b_{2,2} & -b_{1,2} \\ 0 & -b_{n-1,1} & \cdots & -b_{1,2} & -b_{1,1} \end{bmatrix}$$

we only need to check that there exists $X_1 \cdots X_{n-1}$ so that strictly upper anti-diagonal entries of $C := AX - (AX)^\tau$ and those of B match.

If we look at ith column ($0 < i < n$) of C and that of B, we require that

$$\begin{bmatrix} — & A_1 & — \\ \vdots & \vdots & \vdots \\ — & A_{n-i} & — \end{bmatrix} \begin{bmatrix} | \\ X_i \\ | \end{bmatrix} - \begin{bmatrix} — & 0 & — \\ \vdots & \vdots & \vdots \\ — & 0 & — \\ — & A_{n-i+1} & — \end{bmatrix} \begin{bmatrix} | \\ X_{i+1} \\ | \end{bmatrix} + \cdots + \begin{bmatrix} — & A_{n-i+1} & — \\ — & 0 & — \\ \vdots & \vdots & \vdots \\ — & 0 & — \end{bmatrix} \begin{bmatrix} | \\ X_{n-1} \\ | \end{bmatrix} = \begin{bmatrix} b_{1,i} \\ b_{2,i} \\ \vdots \\ b_{n-i,i} \end{bmatrix}$$

Here, recall that X_i is a column vector with i consecutive zeros from the top. Hence,

$$\begin{bmatrix} — & A_1 & — \\ \vdots & \vdots & \vdots \\ — & A_{n-i} & — \end{bmatrix} \begin{bmatrix} | \\ X_i \\ | \end{bmatrix} = \begin{bmatrix} \mathbf{0} \mid A_i, \end{bmatrix} \begin{bmatrix} | \\ X_i \\ | \end{bmatrix}$$

where A_i is $i \times i$ upper right submatrix of A. Hence, if we write X_i' to be $(n-i) \times 1$ column vector obtained by removing consecutive zeros from X_i, we see that (in a block matrix form),

$$\begin{bmatrix} A_{n-1} & * & * & \cdots & * & * \\ 0 & A_{n-2} & * & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & A_1 \end{bmatrix} \begin{bmatrix} X_1' \\ X_2' \\ \vdots \\ X_{n-1}' \end{bmatrix} = \vec{b},$$

where $\vec{b}$ is $n(n-1)/2 \times 1$ column vector consisting of strictly upper anti-diagonal entries from B. Note that the coefficient matrix of the above equation is nonsingular, since its determinant equals to $\det(A_1) \cdot \cdots \cdot \det(A_{n-1})$, which is nonzero by assumption. This completes the proof. □

7.3 Proof of Theorem 7.1.1

7.3.1 Suitable Change of Variables

Let $x_{i,j} := x_i - x_j$. We show the following is sufficient to prove Theorem 7.1.1:

Proposition 7.3.1. *We have*

$$(x_{1,2})^{2n-1}(x_{3,4})^{2n-1} \in \langle (x_{1,3})^{2n}, (x_{1,4})^{2n}, (x_{2,3})^{2n}, (x_{2,4})^{2n} \rangle.$$

Proof of Theorem 7.1.1, assuming Proposition 7.3.1. If σ sends $x_1 \to x_i$ and $x_2 \to x_j$, set $x_3 = x_i$, and $x_4 = x_j$. Then we see that

$$f \cdot \sigma(g) = (x_{1,2})^{2n-1}(x_{i,j})^{2n-1} \in \langle (x_{1,i})^{2n}, (x_{1,j})^{2n}, (x_{2,i})^{2n}, (x_{2,j})^{2n} \rangle \subset I(2n).$$

Hence $f \cdot \sigma(g) \in I(2n)$ for all σ. $\square$

To prove the above proposition, we will make a change of variables. Set $S = k[\delta, x, y]$, and define

$$J(2n) := \langle \delta^{2n}, (\delta + x)^{2n}, (\delta + y)^{2n}, (\delta + x + y)^{2n} \rangle.$$

If we write $x = x_{1,2}$, $y = -x_{3,4}$, and $\delta = x_{2,4}$, we see that

$$(x_{1,2})^{2n-1}(x_{3,4})^{2n-1} = -(xy)^{2n-1},$$

and

$$\langle (x_{1,3})^{2n}, (x_{1,4})^{2n}, (x_{2,3})^{2n}, (x_{2,4})^{2n} \rangle = J(2n).$$

Hence, Proposition 7.3.1 follows from the following lemma:

Lemma 7.3.2. *We have* $(xy)^{2n-1} \in J(2n)$.

7.3.2 The Proof

Finally, we will complete the proof of Theorem 7.1.1 by showing Lemma 7.3.2.

Proof of Lemma 7.3.2. We need to find homogeneous polynomials $f_\delta, f_{\delta+x}, f_{\delta+y}, f_{\delta+x+y} \in k[\delta, x, y]$ of degree $2n-2$ such that

$$(xy)^{2n-1} = f_\delta \delta^{2n} + f_{\delta+x}(\delta + x)^{2n} + f_{\delta+y}(\delta + y)^{2n} + f_{\delta+x+y}(\delta + x + y)^{2n} \tag{VII.4}$$

Divide both sides by δ^{4n-2}. We then get

$$\left(\frac{x}{\delta}\right)^{2n-1} \cdot \left(\frac{y}{\delta}\right)^{2n-1} = f'_\delta + f'_{\delta+x} \cdot \left(1 + \frac{x}{\delta}\right)^{2n} + f'_{\delta+y} \cdot \left(1 + \frac{y}{\delta}\right)^{2n} + f'_{\delta+x+y} \cdot \left(1 + \frac{x}{\delta} + \frac{y}{\delta}\right)^{2n},$$

where f'_* are polynomials of degree at most $2n-2$ obtained by dehomogenization of f_*'s:

$$f'_*\left(\frac{x}{\delta}, \frac{y}{\delta}\right) = f_*\left(1, \frac{x}{\delta}, \frac{y}{\delta}\right)$$

We will find the equations between the coefficients, and although such comparison is possible without dehomogenization, it is more convenient to eliminate one of the variables. Then set $s = 1 + x/\delta$, and $t = y/\delta$. Define a, b, c, d to be polynomials in s and t obtained from $f'_\delta, \cdots, f'_{\delta+x+y}$ after the change of variables. Then, we are reduced to show that there exist polynomials a, b, c, d in

s, t with degrees $\leq 2n-2$ such that

$$(s-1)^{2n-1}t^{2n-1} = a + b \cdot s^{2n} + c \cdot (1+t)^{2n} + d \cdot (s+t)^{2n}. \qquad \text{(VII.5)}$$

Note that we have two gradings, obtained by dehomogenization: one from s–degree and another from t-degree. Such will allow us to obtain the relationships between the coefficients of the terms, and we will show that there exist a (unique) pair of coefficients that satisfy the relationships between the coefficients of a, b, c, d that we will soon derive.

Put

$$b = \sum_{i=0}^{2n-2} b_i s^i,$$

where $b_i = b_i(t)$ is a polynomial in t with degree $\leq i$ for $0 \leq i \leq 2n-2$. Similarly write $a = \sum a_i s^i$, $c = \sum c_i s^i$, and $d = \sum d_i s^i$. Expand the term $d(s+t)^{2n}$ in (VII.5), and compare the coefficients of s^k: i.e., polynomial in t- from $k = 0$ to $k = 4n-2$.

We now limit our attention to terms with s–degree $\geq 2n$. Observe that such terms can only appear from $b \cdot s^{2n}$ and $d \cdot (s+t)^{2n}$. As the terms from the expansion $c \cdot (1+t)^{2n}$ cannot have such terms as the degrees of b (and thus s–degrees of those) is at most $2n-2$. Thus, we get

$$\begin{aligned} d_{2n-2}t^{2n-2}\binom{2n}{2n-2} + \cdots + d_1 t\binom{2n}{1} + d_0 &= -b_0 \text{ (for } s^{2n}) \\ &\vdots \\ d_{2n-2}t\binom{2n}{1} + d_{2n-3} &= -b_{2n-3} \text{ (for } s^{4n-3}) \\ d_{2n-2} &= -b_{2n-2} \text{ (for } s^{4n-2}). \end{aligned}$$

Multiply the first equation by $1/(s^{2n})$, the second by $t/s^{2n+1}, \cdots$, and the last equation by t^{2n}/s^{4n-2}. We then get the following equation:

$$\begin{bmatrix} 1 & \binom{2n}{1} & \cdots & \binom{2n}{2n-2} \\ 0 & 1 & \cdots & \binom{2n}{2n-3} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \cdot \begin{bmatrix} d_0 \\ d_1 t \\ \vdots \\ d_{2n-2}t^{2n-2} \end{bmatrix} = - \begin{bmatrix} b_0 \\ b_1 t \\ \vdots \\ b_{2n-2}t^{2n-2} \end{bmatrix}. \qquad \text{(VII.6)}$$

Thus, we see that the d's uniquely determine the b's.

Now, compare the terms with s-degree from zero to to $2n-1$ from the equation (VII.5). Now the terms we have come from the remaining terms of $d \cdot (s+t)^{2n}$, from a and $c \cdot (1+t)^{2n}$, and lastly from the expansion of $(s-1)^{2n-1}t^{2n-1}$, where the coefficients are entirely determined by binomial coefficients. We write the terms in that order:

$$\begin{aligned} d_0 t^{2n} &= -a_0 - c_0(1+t)^{2n} + (-1)^{2n-1}\binom{2n-1}{2n-1}t^{2n-1} \\ \sum_{i=2n-1}^{2n} d_{i-(2n-1)}\binom{2n}{i}t^i &= -a_1 - c_1(1+t)^{2n} + (-1)^{2n-2}\binom{2n-1}{2n-2}t^{2n-1} \\ &\vdots \\ \sum_{i=2}^{2n} d_{i-2}\binom{2n}{i}t^i &= -a_{2n-2} - c_{2n-2}(1+t)^{2n} + (-1)^{1}\binom{2n-1}{1}t^{2n-1} \\ \sum_{i=1}^{2n-1} d_{i-1}\binom{2n}{i}t^i &= -0 - 0 + (-1)^{0}\binom{2n-1}{0}t^{2n-1} \end{aligned} \tag{VII.7}$$

Write

$$c_k = c_{k,0} + \cdots + c_{k,2n-2-k}t^{2n-2-k}, \text{ and } d_k = d_{k,0} + d_{k,1}t + \cdots + d_{k,2n-2-k}t^{2n-2-k}.$$

Here, note that c_k and d_k are polynomials in t of degree $2n-2-k$, and $c_{i,j}$ and $d_{i,j}$ are scalar coefficients, of polynomials c_i and d_i respectively. Recall that $\deg a_i \le i$, so a_i's can be thought as the remainder of $(RHS)-(LHS)$ with respect to t^i. Hence, if we consider the quotient of (LHS) and (RHS) of the equation (VII.7) with respect to the divisor t^i, we get the following equations system of $2n-2$ linear equations involving c_{ij}'s and d_{ij}'s:

$$\Lambda C + D\Lambda = B. \tag{VII.8}$$

Here, Λ is an upper triangular unipotent matrix

$$\Lambda = \begin{bmatrix} 1 & \binom{2n}{1} & \binom{2n}{2} & \cdots & \binom{2n}{2n-3} & \binom{2n}{2n-2} \\ 0 & 1 & \binom{2n}{1} & \cdots & \binom{2n}{2n-4} & \binom{2n}{2n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \binom{2n}{1} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \tag{VII.9}$$

and B is a diagonal matrix

$$B = \textbf{Diag}\left[(-1)^{2n-1}\binom{2n-1}{2n-1}, (-1)^{2n-2}\binom{2n-1}{2n-2}, \cdots, (-1)^0\binom{2n-1}{0}\right], \tag{VII.10}$$

and C, D are the following strictly lower triangular matrices

$$C, D = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ c_{0,0} & 0 & 0 & \cdots & 0 \\ c_{0,1} & c_{1,0} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{0,2n-2} & c_{1,2n-3} & c_{2,2n-4} & \cdots & c_{2n-2,0} \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ d_{0,0} & 0 & 0 & \cdots & 0 \\ d_{0,1} & d_{1,0} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{0,2n-2} & d_{1,2n-3} & d_{2,2n-4} & \cdots & d_{2n-2,0} \end{bmatrix}. \tag{VII.11}$$

Using Proposition 7.2.1, we see that Λ, B satisfies the condition of Lemma 7.2.3. Hence by Lemma 7.2.3, there is a pair (C, D) with $D = -C^\tau$ that satisfies (VII.8). This completes the proof. □

Remark 7.3.3. *Conceptually, the reason why we expect an equation symmetric in C (the coefficient matrix of c) and D(those of d) is that once we apply different change of variables and dehomogenizations to* (VII.4)*, we can actually interchange c and d. Consider the (well-defined linear) map ϕ*

induced by

$$\delta \mapsto \delta + x, \tag{VII.12}$$

$$\delta + x \mapsto \delta, \tag{VII.13}$$

$$\delta + y \mapsto \delta + x + y, \tag{VII.14}$$

$$\delta + x + y \mapsto \delta + y, \tag{VII.15}$$

and dehomogenize after applying ϕ. Then, c and d are mapped to d and c. As a consequence we expect to observe some symmetry, i.e., one that we see from (VII.8).

Example 7.3.4. *By unraveling, we can compute explicit polynomials a, b, c, d satisfying $(xy)^{2n-1} = a\delta^{2n} + \cdots + d(\delta + x + y)^{2n}$ for any specific values of n. For instance, when $2n = 4$, we have*

$$\begin{bmatrix} a(\delta, x, y) \\ \\ b(\delta, x, y) \\ \\ c(\delta, x, y) \\ \\ d(\delta, x, y) \end{bmatrix} = \begin{bmatrix} \frac{1}{5}\delta^2 + \frac{3}{5}\delta x + \frac{3}{5}\delta y + \frac{3}{2}xy \\ \\ -\frac{1}{5}\delta^2 + \frac{1}{5}\delta x + \frac{2}{5}x^2 - \frac{3}{5}\delta y + \frac{9}{10}xy \\ \\ -\frac{1}{5}\delta^2 - \frac{3}{5}\delta x + \frac{1}{5}\delta y + \frac{9}{10}xy + \frac{2}{5}y^2 \\ \\ \frac{1}{5}\delta^2 - \frac{1}{5}\delta x - \frac{2}{5}x^2 - \frac{1}{5}\delta y + \frac{7}{10}xy - \frac{2}{5}y^2 \end{bmatrix},$$

and we can check that indeed,

$$a\delta^4 + b(\delta + x)^4 + c(\delta + y)^4 + d(\delta + x + y)^4 = (xy)^3.$$

Remark 7.3.5. *We can replace k, a field of characteristic zero, with any field of characteristic p such that the statement of Theorem 7.2.1 holds.*

In fact, we can show the following general result:

Theorem 7.3.6. *Let N be a positive integer. Then the contracted ideal $\mathfrak{J}(N) := J(N) \cap k[x, y]$ contains $\langle x^{2N-1}, y^{2N-1}, x^{2i+1}y^{2j+1} | i + j = N - 2\rangle$.*

Proof. Without loss of generality assume $i \leq j$. We can show that $x^{2i+1}y^{2j+1} \in J(N)$ with the same dehomogenization and comparison of coefficients, and we see that

$$x^{2N-1} = ((\delta + x) - \delta)^{2N-1} \in \langle \delta^N, (\delta + x)^N \rangle$$

from binomial expansion. □

Note that Theorem 7.1.1 is a special case of Theorem 7.3.6. Furthermore, we have the following conjecture that the contracted ideal $\mathfrak{J}(N)$ is a monomial ideal:

Conjecture 7.3.7. *In fact,* $\mathfrak{J}(N)$ *equals to* $\langle x^{2N-1}, y^{2N-1}, x^{2i+1}y^{2j+1} | i + j = N - 2\rangle$.

7.4 Open Problems

In this section, we give more conjectures that generalize Theorem 7.1.1 and Theorem 7.3.6.

7.4.1 More Variables

Fix $r, n \in \mathbf{N}$, and for a subset S of $[r] := \{1, 2, \cdots, r\}$, define $t_S = \sum_{i \in S} t_i$, and $t_{S,0} = t_0 + t_S$. Here, t_0 is a special variable, that takes a role of δ in (VII.4). We then define $I_r^{(d)} \subset R := K[t_0, \cdots, t_r]$ to be

$$I_r^{(d)} := \langle (t_{S,0})^d : S \subset [r] \rangle.$$

Conjecture 7.4.1. *For any* $r \geq 2$ *and* $n \geq 1$, *we have*

$$(t_1 \cdots t_r)^{2n-1} \in I_r^{(nr)}.$$

For $r = 1$, the above conjecture reduces to $t_1^{2n-1} \in \langle t_0^n, (t_0 + t_1)^n \rangle$, which follows easily from the binomial expansion of $t^{2n-1} = ((t + \delta) - \delta)^{2n-1}$. Our theorem gives a positive answer for $r = 2$. We have verified the conjecture for some small values of $r \geq 3$ by computer.

7.4.2 Betti Tables

We also observe some patterns from Betti tables of the family of ideals $I_r^{(d)}$ with fixed r. Let $\mathbf{F}$ be the minimal free resolution of $I_r^{(d)}$, and write

$$F_i = \bigoplus_{j \in \mathbf{Z}} R(-j)^{\beta_{i,j}}.$$

When $r = 1$, we expect $\beta_{i,j} = 0$ for all i, j except for $(i, j) = (0, 0), (d - 1, 1)$, and $(2d - 2, 2)$. The nonzero values of $\beta_{i,j}$'s are given by $\beta_{0,0} = \beta_{2d-2,2} = 1, \beta_{d-1,1} = 2$. See the following Betti table, where all the omitted rows are zero rows, and entries with $\cdot$ is a zero entry:

$i \backslash j$	**0**	**1**	**2**
0	1	$\cdot$	$\cdot$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$d-1$	$\cdot$	2	$\cdot$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2d-2$	$\cdot$	$\cdot$	1

For For $r = 2$, we have the following conjectured Betti table obtained for some small values we tested by computer:

$i \backslash j$	**0**	**1**	**2**	**3**
0	1	$\cdot$	$\cdot$	$\cdot$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$d-1$	$\cdot$	4	$\cdot$	$\cdot$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2d-3$	$\cdot$	$\cdot$	d	$\cdot$
$2d-2$	$\cdot$	$\cdot$	3	d

Lastly, we give the Betti table for $r = 3$:

$i \backslash j$	**0**	**1**	**2**	**3**	**4**
0	1	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$d-1$	$\cdot$	8	$\cdot$	$\cdot$	$\cdot$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2d-4$	$\cdot$	$\cdot$	$(d-1)d/2$	$\cdot$	$\cdot$
$2d-3$	$\cdot$	$\cdot$	$4d$	$(d-1)(d+1)$	$\cdot$
$2d-2$	$\cdot$	$\cdot$	6	$4d$	$d(d-1)/2$

There are several patterns we can observe: why do we expect each entry of the Betti table to stabilize? For instance, why do we expect to get "6" at the lower-left vertex of the triangle (i.e., $\beta_{2d-2,3} = 6$ for $r = 3$)? More fundamentally, why do we see $r \times r$ right triangle at the lower right corner? Should we expect to observe similar patterns for larger values of r?

APPENDIX A

Equivalence of Tensor Categories Associated to Finite Inverse Monoids

In this appendix, we give proofs of Lemma 3.3.6 and Proposition 3.3.8. Before giving the proofs of Lemma 3.3.6 and Proposition 3.3.8, we first summarize basic theory regarding the representations of finite inverse monoids and review the proof of the decomposition of ring algebra kM associated with a finite inverse monoid M. Throughout this chapter, we fix the base field k with arbitrary characteristic, and a monoid M will always be a finite inverse monoid.

1.1 Notations

For a subset $X \subset M$ of a finite inverse monoid M, we define

$$E(X) := \{x \in X : x^2 = x\}$$

to be the set of idempotents inside the subset X. Recall that in Definition 3.1.2, we defined the $\mathscr{J}$-class of $m \in M$ to be the set

$$J_m := \{m' \in M : Mm'M = MmM\}.$$

For an idempotent $e \in E(M)$, the **maximal subgroup of** M **at** e is the unique maximal subgroup G_e of M, with identity e. Since M is an inverse monoid, can show that G_e equals to

$$G_e = \{m \in M : m^*m = mm^* = e\}.$$

1.2 Decomposition of a Monoid Algebra

1.2.1 Basic Definitions

Recall that a **groupoid** is a small category G where every morphism is invertible. That is, for any morphism $f : x \to y$ between two objects $x, y \in \mathbf{Obj}(G)$, there is a unique map $f^{-1} : y \to x$ such that $f^{-1}f = 1_x$ and $ff^{-1} = 1_y$. Thus, the endomorphism monoid G_x for every object $x \in \mathbf{Obj}(G)$

is a group. For $x \in \mathbf{Obj}(G)$, we let $\overline{x} := \{y \in \mathbf{Obj}(G) :$ there exists $f : x \to y\}$ to be the isomorphism class of x. Lastly, we let $\mathbf{Arr}(G)$ be the set of arrows of G.

Definition 1.2.1. *The* ***groupoid algebra*** kG *of a groupoid* G *is the algebra defined by the following:*

(a) The underlying k*-vector space of* kG *is* $k[\mathbf{Arr}(G)]$

(b) For $f, g \in \mathbf{Arr}(G)$ *the product* $f \cdot g$ *is given by*

$$fg := \begin{cases} f \cdot g & \textit{if domain of } f \textit{ equals to the range of } g \\ 0, & \textit{otherwise} \end{cases}.$$

The groupoid algebra kG of a finite groupoid G can be decomposed into the product of matrix algebras.

Theorem 1.2.2. *Let* G *be a finite groupoid, and* $x_1, \cdots, x_s$ *be distinct representatives of isomorphism classes of* $\mathbf{Obj}(G)$*, and for each* $i = 1, \cdots, s$*, let* $n_i := |\overline{x_i}|$ *be the number of elements of* G *isomorphic to* c_i*. Then the groupoid algebra* kG *can be decomposed as*

$$kG \simeq \prod_{i=1}^{s} M_{n_i}(kG_{x_i}). \tag{A.1}$$

Proof. For each $i = 1, \cdots, s$, let $e_i := \sum_{x \in \overline{x_i}} 1_c$ be the sum of all identity morphisms of $\overline{x_i}$. Then from the definition of multiplication for groupoid algebra, we have

$$e_i \cdot e_j = \begin{cases} e_i & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, all the e_i's are central in kG i.e., for any $x \in G$, we have $e_i x = x e_i$, as e_i's are sums of identity morphisms. Thus e_i's are **orthogonal central idempotents** of kG. Note further that the sum

$$\sum_{i=1}^{s} e_i = 1_G,$$

of all e_i's equals to the identity.

Let H_i be the subgroupoid of G restricted to the class x_i. That is, $\mathbf{Obj}(H_i) = \overline{x_i}$ and $\mathbf{Arr}(H_i)$ consisting of arrows between objects of $\overline{x_i}$. Then, let $\pi_i : kG \to kH_i$ be the the map

$$\pi_i(x) = e_i \cdot g \cdot e_i.$$

It is easy to see that the map π_i projects the algebra kG onto subalgebra kH_i:

$$\pi_i(kG) = kH_i$$

Since the set $\{e_1, \cdots, e_s\}$ forms an orthogonal set of central idempotents that sums up to 1_G, and G is a finite groupoid by assumption, we can decompose kG into the direct product of kH_i's:

$$\pi : kG \simeq \prod_{i=1}^{s} kH_i,$$

where π equals to $(\pi_1, \cdots, \pi_s)$. Now, we only need to show that kH_i is isomorphic to $M_{n_i}(kG_{c_i})$ for each i. By definition of groupoid algebra, the algebra kH_i is generated by the arrows of H_i, and the category H_i consists of objects isomorphic to the representative c_i (and there are $n_i := |\overline{x_i}|$ of them). Since we defined G_{c_i} to be the endomorphism group of the representative x_i, we have

$$kH_i \simeq M_{n_i}(kG_{c_i}),$$

which completes the proof. $\square$

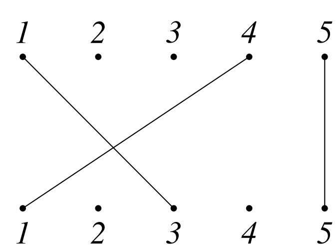
1.2.2 Groupoid Associated to an Inverse Monoid

Let M be an inverse monoid. The **groupoid associated to** M is the groupoid $G(M)$ with $\mathbf{Obj}(G(M)) = E(M)$ and $\mathbf{Arr}(G(M)) = \{[x] | x \in M\}$, a set canonically isomorphic to M itself. For each $[x]$, we define the **domain** $d([x])$ of $[x]$ to be x^*x and the **range** $r(x) := xx^*$. Two arbitrary arrows $[x], [y] \in G(M)$ can be composed if and only if $d([x]) = r([y])$, and if so, we define the composition $[x][y]$ by

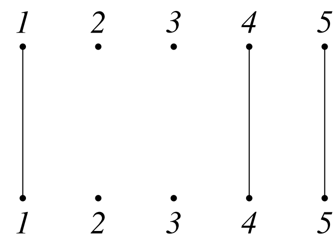
$$[x][y] := [xy]. \tag{A.2}$$

Example 1.2.3. *If $M = R_n$ is a Rook monoid, domain $d([x])$ of $[x] \in \mathbf{Arr}(G(M))$ (resp. range $r([x])$ of $[x]$) is an idempotent from the domain $d(x)$ to itself (resp. $r(x)$ to itself). Here, we consider x to be the partial map from $\{1, \cdots, n\}$ to itself.*

For example, let $n = 5$, and x be the following partial map:



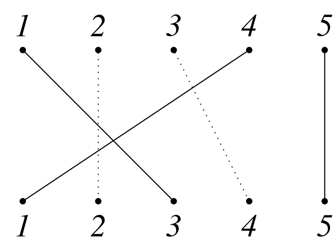
Then, we see that $d([x]) = x^*x$ is the following:



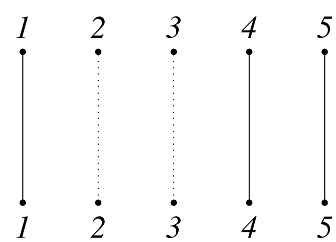
The range $r([x]) = xx^*$, on the other hand, is an idempotent sending $\{1, 3, 5\}$ to itself. Thus, we can identify the domain and range of $[x]$ by domain and range of x (by regarding x as a partial map). Then the condition $d([x]) = r([y])$ is equivalent to $d(x) = r(y)$.

We have similar descriptions on domains and ranges for $M = \mathscr{P}_n$:

Example 1.2.4. *If $M = \mathscr{P}_n$, domain $d([y])$ of $[y]$ (resp. range $r([y])$ of $[y]$) is an idempotent concentrated on the domain of straight lines of $[y]$ (resp. range of straight lines of $[y]$). We explain what this means by an example. Again let $n = 5$, and $y \in \mathscr{P}_5$ be the following:*



Then, the domain $d([y])$ is the following:



Now we relate two groupoids $G(R_n)$ and $G(\mathscr{P}_n)$, via the map $f : R_n \to \mathscr{P}_n$ from Remark 3.2.11. If we set $y = f(x) \in \mathscr{P}_n$ for $x \in R_n$, the domain $d([y]) = d([f(x)])$ of y equals to $f(d([x]))$. For a concrete example, pick $x \in R_5$ to be the partial map from Example 1.2.3, then the image $f(x)$ of x is the element y in Example 1.2.4. Observe that $d([f(x)]) = f(d([x])$. Similarly, f also commutes with the range map r. That is, $r[y] = r[f(x)] = f(r[x])$. Then, using the composition law (A.2) together with III.1 from Remark 3.2.12 we get

Proposition 1.2.5. *The bijection $f : R_n \to \mathscr{P}_n$ induces a a functor $F : G(R_n) \to G(\mathscr{P}_n)$, which gives an isomorphism between two groupoids.*

We have the following two lemmas regarding the groupoid $G(M)$ associated to an inverse monoid M:

Lemma 1.2.6. *The groupoid $G(M)$ is finite, and an automorphism group $G(M)_e$ at an element $e \in \mathbf{Obj}(G(M)) = E(M)$ is isomorphic to the maximal subgroup G_e.*

Lemma 1.2.7. *Let e, f be idempotents of M. Then, e and f are isomorphic in $G(M)$ if and only if $e \mathscr{J} f$ (in other words, $MeM = MfM$).*

The following theorem gives an explicit isomorphism between $k[M]$ and the associated groupoid algebra $k[G(M)]$.

Theorem 1.2.8. *Let M be a finite inverse monoid and $G = G(M)$ be the groupoid associated to M. For a field k, we have an isomorphism $\alpha : k[M] \to k[G]$ via*

$$\alpha : y \mapsto \sum_{x \leq y} [x].$$

See [St] for the proofs of Lemmas 1.2.6, 1.2.7 and Theorem 1.2.8.

Using Theorem 1.2.8 together with Proposition 1.2.5, we get the following corollary on two rings $k[R_n]$ and $k[\mathscr{P}_n]$.

Corollary 1.2.9. *We have an isomorphism between rings $k[R_n]$ and $k[\mathscr{P}_n]$.*

1.2.3 The Decomposition of Monoid Algebra

Now we are ready to show the following theorem regarding the decomposition of $k[M]$:

Theorem 1.2.10. *Let M b a finite inverse monoid and k be any field. Let $e_1, \cdots, e_s$ be idempotent representatives of the $\mathscr{J}$-class of M, and $n_i = |E(J_{e_i})|$ i.e., the number of idempotents $e \in M$ satisfying $e \mathscr{J} e_i$. Then we have an isomorphism*

$$kM \simeq \prod_{i=1}^{s} M_{n_i}(kG_{e_i})$$

of algebras.

Proof. Using Theorem 1.2.8 we reduce the problem into into the decomposition of the ring $kG(M)$, and the decomposition of $kG(M)$ into a product of matrix algebras is given in Theorem 1.2.2. We use Lemmas 1.2.6 and 1.2.7 and identify idempotent representatives $\{e_i\}$ with representatives $\{x_i\}$ appearing in Theorem 1.2.2. $\square$

Using Maschke's Theorem, we have

Corollary 1.2.11. *If characteristic of k does not divide the product*

$$\prod_i |G_{e_i}|,$$

then the ring $k[M]$ is semisimple.

In particular, when $\mathrm{char}(k) = 0$, the ring $k[M]$ is semisimple.

Remark 1.2.12. *Theorem 1.2.10 was first proved by Munn for Rook monoids (See [Mu1] and [Mu2]). We followed [St] that works for any finite inverse monoid M. For alternate proofs, see the original proof [Mu1] and [Mu2], or [Sol] where the name "Rook monoid" first appeared.*

1.3 Partial Order of an Inverse Monoid

Throughout this section, we limit ourselves into two monoids R_n or $\mathscr{P}_n$. Using Theorem 1.2.2 and information from Tables III.1 and III.2, we have the following decomposition of algebra $G(M)$ when M equals to either R_n or $\mathscr{P}_n$ (i.e., the groupoid algebra version of Corollary 3.3.3):

$$\pi : kG(M) = \prod_{i=0}^{n} kH_i = \prod_{i=0}^{n} M_{\binom{n}{i}}(k[\mathfrak{S}_i]), \tag{A.3}$$

where H_i is the subgroupoid of G restricted to each class of representatives of $G(M)$. Since $\mathbf{Obj}(G(M)) = E(M)$, we see that H_i is the subgroupoid associated to each idempotent representative e_i of M.

As inverse monoids, R_n and $\mathscr{P}_n$ are equipped with natural partial orders (see Lemma 3.2.8, and Examples 3.2.9 and 3.2.10). The goal of this section is to relate natural partial orders of R_n and $\mathscr{P}_n$ with the decomposition (A.3).

Definition 1.3.1. *Let x be an element of R_n (or $\mathscr{P}_n$). The **rank** of x, denoted as $\mathbf{rank}(x)$ is the number of straight lines of x.*

Recall that any finite inverse monoids can be embedded into submonoid of $n \times n$ square matrices. In particular, we have seen how to realize an element of R_n as $n \times n$ matrix consisting of 1's and 0's (where the name "Rook" monoids comes from), and an element of $\mathscr{P}_n$ as $2n \times 2n$ matrix (see Remark 3.2.7).

Example 1.3.2. *For $x \in R_n$, the rank of x equals to the usual matrix rank of x, when x is considered as a $n \times n$ Rook matrix.*

Example 1.3.3. *For $x \in \mathscr{P}_n$, $\mathbf{rank}(x)$ does not equal to the matrix rank of x as a $2n \times 2n$ Rook matrix. In fact, it is $2n$ minus the rank of corresponding Rook matrix: i.e., $\mathbf{rank}(x) = 2n - rank_{mat}(X)$, where X is $2n \times 2n$ Rook matrix corresponding to x.*

The natural partial order of R_n respects well with ranks: let x, y be elements of R_n satisfying $x \leq y$. Then we have $\mathbf{rank}(x) \leq \mathbf{rank}(y)$. Note however that the converse does not hold: two elements x and y are not comparable in general, but we can always compare $\mathbf{rank}(a)$ and $\mathbf{rank}(b)$ as they are nonnegative integers! For $\mathscr{P}_n$ the inequality is reversed: if $x \leq y$ for two elements $x, y \in \mathscr{P}_n$, then $\mathbf{rank}(x) \geq \mathbf{rank}(y)$. See Remark 3.2.11.

Lemma 1.3.4. *Let M be either R_n or $\mathscr{P}_n$. The subalgebra kH_i, appearing in the decomposition* (A.3) *of an algebra $G(M)$, consists of elements of the form*

$$\sum_{\mathbf{rank}(m)=i} c_m[m],$$

where each coefficient c_m is an element of k.

Proof. Let $e_0, \cdots, e_n$ be idempotent representatives of $\mathbf{Obj}(G(M)) = E(M)$, with $\mathbf{rank}(e_i) = i$ for each $i = 0, \cdots, n$. Then, from the definition of the associated subgroupoid H_i, the subalgebra $kH_i \subset kG(M)$ is generated by morphisms between elements of the idempotent class $\overline{e_i}$. Since idempotent e is an element of $\overline{e_i}$ if and only if $\mathbf{rank}(e) = \mathbf{rank}(e_i)$, we have

$$\overline{e_i} = \{e \in E(M) : \mathbf{rank}(e) = i\}.$$

If $e, f \in \overline{e_i}$ have the same rank $i \leq n$, and $[m] : e \to f$ is a morphism between them, we have $m^*m = e$ and $mm^* = f$ from definitions of domain and range of $[m] \in \mathbf{Arr}(G(M))$. Hence, $\mathbf{rank}(m)$ equals to $\mathbf{rank}(e) = \mathbf{rank}(f)$. Since $k[H_i]$ is an algebra generated by such $[m]$'s, we have

$$k[H_i] = \left\{ \sum_{\mathbf{rank}(m)=i} c_i[m] \right\}$$

$\square$

Remark 1.3.5. *Note also that kH_i equals to the projection of $kG(M)$ under the map π_i, where the map π_i given by*

$$\pi_i : x \mapsto \left(\sum_{e \in e_i} 1_e \right) \cdot x \cdot \left(\sum_{e \in e_i} 1_e \right).$$

If we write $x \in kG(M)$ as

$$x = \sum_{m \in M} c_m[m] = \sum_{j=0}^{n} \left(\sum_{\mathbf{rank}(m)=j} c_m[m] \right).$$

we see that

$$\pi_i(x) = \sum_{\mathbf{rank}(m)=i} c_m[m].$$

Thus, we can consider π_i as filtration of $kG(M)$ with respect to rank. Extending this viewpoint, the isomorphism $\pi = (\pi_0, \cdots, \pi_n)$ can be thought of as rearranging the linear factors of

$$x = \sum_{m \in M} c_m[m]$$

with respect to the rank of each m.

1.4 Proof of Lemma 3.3.6

Now we are ready to prove Lemma 3.3.6. For each $n \in \mathbf{N}$, let $\alpha_n : kR_n \to kG(R_n)$ be the isomorphism

$$\alpha_n : y \mapsto \sum_{x \leq y} [x].$$

from Theorem 1.2.8.

For each $m, n \in \mathbf{N}$, we have a natural injection $\iota : R_m \times R_n \to R_{m+n}$ where for each pair $(x, y) \in R_m \times R_n$, we construct an element of R_{m+n} by mapping first m dots through x and last n dots by y. We denote $\iota(x, y)$ by $x|y$: we can think of the bar $|$ as an impenetrable wall between x and y where lines of x and that of y cannot pass through. See Example 1.4.1 for an explicit example. Similarly, we have an injection $\iota' : G(R_m) \times G(R_n) \to G(R_{m+n})$ between groupoids, defined by

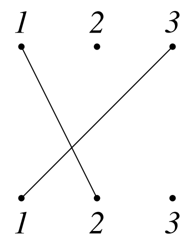
$$\iota' : [x] \times [y] \mapsto [x|y].$$

Then, we define

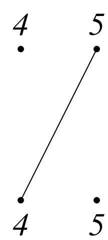
$$i^R_{m,n} : k[R_m \times R_n] \to k[R_{m+n}], \text{ and } i^G_{m,n} : k[G(R_m) \times G(R_n)] \to k[G(R_{m+n})]$$

to be the maps induced by ι and ι', respectively. For simplicity, we omit subscripts and/or superscripts if such omission does not make a confusion.

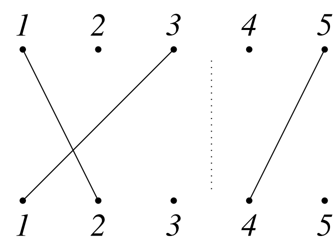
Example 1.4.1 (Example of the map ι)**.** *Let $m = 3$ and $n = 2$. If $x \in R_3$ is given by*



and $y \in R_2$ is given by (we labeled two dots as 4 and 5 instead of 1 and 2 as we will write y next to x)



then $x|y$ is the following element inside of R_{3+2}



Remark 1.4.2. *Note in particular that the rank is additive under ι: for each $(x, y) \in R_m \times R_n$, we have*

$$\mathbf{rank}(x|y) = \mathbf{rank}(x) + \mathbf{rank}(y).$$

Lemma 1.4.3. *The following diagram on ring morphisms is commutative:*

$$\begin{tikzcd} k[R_m \times R_n] \arrow[r, "i^R"] \arrow[d, "\alpha"] & k[R_{m+n}] \arrow[d, "\alpha"] \\ k[G(R_m) \times G(R_n)] \arrow[r, "i^G"] & k[G(R_{m+n})] \end{tikzcd}$$

Proof. On each $(x, y) \in R_m \times R_n$, we have

$$i^G(\alpha(x, y)) = i^G(\sum_{x' \le x} \sum_{y' \le y} [x'] \times [y']) = \sum_{x' \le x, y' \le y} [x'|y'].$$

Now,

$$\alpha(i^R(x,y)) = \alpha(x|y) = \sum_{z \leq x|y} [z].$$

Since $z \leq x|y$ means that z can be obtained from $x|y$ by removing some lines from $x|y$ - or in other word, from removing some (possibly zero) straight lines from x and some from y, we see that $\sum_{x' \leq x, y' \leq y} [x'|y'] = \sum_{z \leq x|y} [z]$, and we are done. $\square$

We have a corresponding commutativity theorem on $\mathscr{P}$-monoids. We omit the proof as it is the same as that of Lemma 1.4.3.

Lemma 1.4.4. *The following diagram on ring morphisms is commutative:*

$$\begin{tikzcd}
k[\mathscr{P}_m \times \mathscr{P}_n] \arrow[r, "i^{\mathscr{P}}"] \arrow[d, "\alpha"] & k[\mathscr{P}_{m+n}] \arrow[d, "\alpha"] \\
k[G(\mathscr{P}_m) \times G(\mathscr{P}_n)] \arrow[r, "i^G"] & k[G(\mathscr{P}_{m+n})]
\end{tikzcd}$$

Using Lemmas 1.4.3 and 1.4.4, we can replace rings $k[R_*]$ (resp $k[\mathscr{P}_*]$) by $k[G(R_*)]$ (resp $k[G(\mathscr{P}_*)]$), even when we consider those rings together with morphisms $k[R_m \times R_n] \to k[R_{m+n}]$ (resp. $k[\mathscr{P}_m \times \mathscr{P}_n \to k[\mathscr{P}_{m+n}]$) for $m, n \in \mathbf{N}$.

Now we are ready to give the proof of Lemma 3.3.6, which we restate below:

Theorem 1.4.5. *The two tensor categories $\mathfrak{R}$ and $\mathfrak{P}$, where the tensor structure is given by induced representations in each degree, are equivalent.*

Proof. It suffices to show that for each $k[R_m]$-module M and $k[R_n]$-module N, we have

$$\mathbf{Ind}_{k[R_m \times R_n]}^{k[R_{m+n}]} M \otimes N \simeq \mathbf{Ind}_{k[\mathscr{P}_m \times \mathscr{P}_n]}^{k[\mathscr{P}_{m+n}]} M \otimes N. \tag{A.4}$$

From definitions of $\mathbf{Ind}$ functor in $\mathfrak{R}$ and in $\mathfrak{P}$, we see that (A.4) is equivalent to

$$(M \otimes N) \otimes_{k[R_m \times R_n]} k[R_{m+n}] \simeq (M \otimes N) \otimes_{k[\mathscr{P}_m \times \mathscr{P}_n]} k[\mathscr{P}_{m+n}]. \tag{A.5}$$

Let $F'_n := k[R_n] \to k[\mathscr{P}_n] : \alpha_n^{-1} \cdot F_n \cdot \alpha_n$ be the canonical isomorphism between $k[R_n]$ and $k[\mathscr{P}_n]$ obtained by the following composition of maps:

$$k[R_n] \xrightarrow{\alpha_n} k[G(R_n)] \xrightarrow{F_n} k[G(\mathscr{P}_n)] \xrightarrow{\alpha_n^{-1}} k[\mathscr{P}_n)]$$

Then, (A.5) is an isomorphism if

$$\begin{array}{ccc} k[R_m] \otimes k[R_n] & \xrightarrow{F'_m \otimes F'_n} & k[\mathscr{P}_m] \otimes k[\mathscr{P}_n] \\ \downarrow i & & \downarrow i \\ k[R_{m+n}] & \xrightarrow{F'_{m+n}} & k[\mathscr{P}_{m+n}] \end{array}$$

is a commutative diagram, where the vertical maps are the maps induced by injections $R_m \times R_n \to R_{m+n}$ and $\mathscr{P}_m \times \mathscr{P}_n \to \mathscr{P}_{m+n}$.

Using Lemmas 1.4.3 and 1.4.4, the above diagram is commutative if and only if the following commutes

$$\begin{array}{ccc} k[G(R_m)] \otimes k[G(R_n)] & \xrightarrow{F_m \otimes F_n} & k[G(\mathscr{P}_m)] \otimes k[G(\mathscr{P}_n)] \\ \downarrow i & & \downarrow i \\ k[G(R_{m+n})] & \xrightarrow{F_{m+n}} & k[G(\mathscr{P}_{m+n})] \end{array}$$

At the level of generators $[x] \otimes [y]$, we see that

$$F_{m+n} \cdot i([x] \otimes [y]) = F_{m+n}([x|y]) = [f_{m+n}(x|y)] = [f_m(x)|f_n(y)],$$

since $f_{m+n}(x|y) = f_m(x)|f_n(y)$: we can easily see from the uniqueness that dotted (unique) order-preserving lines obtained from applying f_{m+n} to $x|y$ cannot pass the wall "|". On the other hand, we have

$$i \cdot (F_m \otimes F_n)([x] \otimes [y]) = i\left([f_m(x)] \otimes [f_n(y)]\right) = [f_m(x)|f_n(y)].$$

Extending by linearity, we are done. □

1.5 Proof of Proposition 3.3.8

We end the Appendix by giving the proof of Proposition 3.3.8 (which we needed to show Littlewood-Richardson for Rook monoids), which we restate below:

Theorem 1.5.1. *Let ι be the ring homomorphism*

$$\iota : k[R_m \times R_n] \to k[R_{m+n}],$$

induced by the injection $R_m \times R_n \to R_{m+n}$. Then

$$\iota\left(M_{\binom{m}{i}}(k[\mathfrak{S}_i]) \otimes M_{\binom{n}{j}}(k[\mathfrak{S}_j])\right) \subset M_{\binom{m+n}{i+j}}(k[\mathfrak{S}_{i+j}]),$$

where

$$k[R_{m+n}] = \prod_l M_{\binom{m+n}{l}}(k[\mathfrak{S}_l]).$$

Proof. As in the proof of Theorem 1.4.5, we use Lemma 1.4.3 and replace monoid algebras by corresponding groupoid algebras. Then the statement reduces down to the following: for $(x, y) \in R_m \times R_n$ with $\mathbf{rank}(x) = i$ and $\mathbf{rank}(y) = j$ we have $\mathbf{rank}(x|y) = i + j$, and we have seen this earlier in Remark 1.4.2. $\square$

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