Hilbert's Inequality, Generalized Factorials, and Partial Factorizations of Generalized Binomial Products

by

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Dedicated to my mom, my dad, my brother, and my teachers.

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ABSTRACT

This dissertation treats three topics in number theory. The first topic concerns the problem of determining the optimal constant in the Montgomery–Vaughan weighted generalization of Hilbert's inequality. The second topic presents a further generalization of Bhargava's generalized factorials in the ring $\mathbb{Z}$. We define invariants associated to all pairs $(S, \mathfrak{b})$ of a nonempty subset S of $\mathbb{Z}$ and a nontrivial proper ideal $\mathfrak{b}$ in $\mathbb{Z}$ and use them to construct generalized factorials. The third topic is asymptotics of partial factorizations of products of generalized binomial coefficients constructed using generalized factorials from the second topic.

CHAPTER 1

Introduction

1.1 Topics covered

This dissertation consists of three topics in number theory.

(1) The Montgomery–Vaughan weighted generalization of Hilbert's inequality

(2) Generalized factorials and binomial coefficients allowing composite bases

(3) Asymptotics of partial factorizations of products of generalized binomial coefficients

These three topics are treated in Chapters 2, 3, and 4, respectively. The next three sections 1.2, 1.3, and 1.4 discuss these chapters.

1.2 The Montgomery–Vaughan weighted generalization of Hilbert's inequality

Chapter 2 concerns the problem of determining the optimal absolute constant in the Montgomery–Vaughan weighted generalization of Hilbert's inequality.

1.2.1 Hilbert's inequality

In a lecture on integral equations held in summer 1907, Hilbert introduced an example of a bounded linear operator from ℓ^2 to ℓ^2 whose row and column sums are divergent. The linear operator is given by the infinite matrix

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where the (m, n)-entry is $\frac{1}{m+n}$. Hilbert demonstrated the boundedness of this operator by proving the following bound for a real bilinear form:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{x_m y_n}{m+n} \leq \pi \left(\sum_{m=1}^{\infty} x_m^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} y_n^2\right)^{\frac{1}{2}} \tag{1.2.1}$$

for all vectors $[x_1, x_2, x_3, \dots]$ and $[y_1, y_2, y_3, \dots]$ of real numbers. This result is known as Hilbert's double series theorem.

Hilbert's proof was published in Weyl's dissertation [15]. It is based on the identity

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} t f(t)^2 \, dt = S + T,$$

where

$$f(t) := \sum_{n=1}^{N} (-1)^n \left(x_n \sin(nt) - y_n \cos(nt)\right),$$

$$S := \sum_{m=1}^{N}\sum_{n=1}^{N}\frac{x_m y_n}{m+n}, \quad \text{and} \quad T := \sum_{m=1}^{N}\sum_{\substack{n=1 \\ n\neq m}}^{N}\frac{x_m y_n}{m-n}.$$

From this identity, Hilbert derived (1.2.1) and a similar bound for the bilinear form T:

$$\left|\sum_{m=1}^{N}\sum_{\substack{n=1 \\ n\neq m}}^{N}\frac{x_m y_n}{m-n}\right| \leq c_0 \left(\sum_{m=1}^{N} x_m^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{N} y_n^2\right)^{\frac{1}{2}} \tag{1.2.2}$$

with the absolute constant $c_0 = 2\pi$. Schur [12] later determined the optimal value of c_0 to be π.

The coefficient matrix of the bilinear form on the left side of (1.2.2) is skew-symmetric. The following equivalence is well-known.

Proposition 1.2.1. *Let $A = [a_{mn}]$ be an $N \times N$ matrix with complex entries such that $A^\top = -A$. Let c be a nonnegative real number. Then the inequality*

$$\left|\sum_{m=1}^{N}\sum_{n=1}^{N} a_{mn} x_m y_n\right| \leq c\left(\sum_{m=1}^{N} x_m^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{N} y_n^2\right)^{\frac{1}{2}} \tag{1.2.3}$$

holds for all vectors $[x_1, \ldots, x_N]$ and $[y_1, \ldots, y_N]$ in $\mathbb{R}^N$ if and only if the inequality

$$\left|\sum_{m=1}^{N}\sum_{n=1}^{N} a_{mn} z_m \overline{z_n}\right| \leq c\sum_{n=1}^{N} |z_n|^2 \tag{1.2.4}$$

holds for all vectors $[z_1, \ldots, z_N] \in \mathbb{C}^N$.

From Proposition 1.2.1, we see that (1.2.2) is equivalent to

$$\left|\sum_{m=1}^{N}\sum_{\substack{n=1\\ n\neq m}}^{N} \frac{z_m \overline{z_n}}{m-n}\right| \leq c_0 \sum_{n=1}^{N} |z_n|^2 . \tag{1.2.5}$$

Let H be the $N \times N$ matrix with entries given by

$$(H)_{mn} = \begin{cases} \frac{1}{m-n} & \text{if } m \neq n, \\ 0 & \text{if } m = n. \end{cases}$$

That is,

$$H = \begin{bmatrix} 0 & -1 & -\frac{1}{2} & \cdots & -\frac{1}{N-1} \\ 1 & 0 & -1 & \cdots & -\frac{1}{N-2} \\ \frac{1}{2} & 1 & 0 & \cdots & -\frac{1}{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N-1} & \frac{1}{N-2} & \frac{1}{N-3} & \cdots & 0 \end{bmatrix}.$$

Let $\langle \cdot, \cdot \rangle$ be the inner product on the complex vector space $\mathbb{C}^N$ defined by

$$\langle \mathbf{z}, \mathbf{w} \rangle = \sum_{n=1}^{N} z_n \overline{w_n},$$

where $\mathbf{z} = [z_1, \ldots, z_N]^\top$ and $\mathbf{w} = [w_1, \ldots, w_N]^\top$ are column vectors in $\mathbb{C}^N$. In vector notation, (1.2.5) can be rewritten as the following inequality involving the sesquilinear form $(\mathbf{z}, \mathbf{w}) \mapsto \langle \mathbf{z}, H\mathbf{w} \rangle$:

$$|\langle \mathbf{z}, H\mathbf{z} \rangle| \leq C_0 \langle \mathbf{z}, \mathbf{z} \rangle .$$

1.2.2 The Montgomery–Vaughan generalization of Hilbert's inequality

While H. L. Montgomery was visiting the Institute for Advanced Study during 1971–1972, Selberg showed him a proof of the following result.

Theorem 1.2.2. *Let δ be a positive real number. Let $(\lambda_k)_{k=-\infty}^{\infty}$ be a sequence of real numbers such that $\lambda_{k+1} - \lambda_k \geq \delta$ for all k. Then for any sequence $(z_1, \ldots, z_N) \in \mathbb{C}^N$,*

$$\left| \sum_{m=1}^{N} \sum_{\substack{n=1 \\ n \neq m}}^{N} \frac{z_m \overline{z_n}}{\lambda_m - \lambda_n} \right| \leq \frac{\pi}{\delta} \sum_{n=1}^{N} |z_n|^2 . \tag{1.2.6}$$

Theorem 1.2.2 generalizes Hilbert's inequality (1.2.5) with the optimal constant $c_0 = \pi$. If the frequencies λ_k form an arithmetic progression with a common difference of δ (i.e., $\lambda_k = \lambda_0 + k\delta$ for all k), then (1.2.6) yields (1.2.5) with $c_0 = \pi$.

For frequencies λ_k that are more irregularly spaced, Selberg had a more complicated proof of a more sensitive inequality.

Theorem 1.2.3. *Let $(\lambda_k)_{k=-\infty}^{\infty}$ be a strictly increasing sequence of real numbers. Let $\delta_k := \min\{\lambda_k - \lambda_{k-1}, \lambda_{k+1} - \lambda_k\}$. Then for any sequence $(z_1, \ldots, z_N) \in \mathbb{C}^N$,*

$$\left| \sum_{m=1}^{N} \sum_{\substack{n=1 \\ n \neq m}}^{N} \frac{z_m \overline{z_n}}{\lambda_m - \lambda_n} \right| \leq c_1 \sum_{n=1}^{N} \frac{|z_n|^2}{\delta_n}, \tag{1.2.7}$$

where c_1 is the absolute constant $\frac{3}{2}\pi$.

Theorem 1.2.3 is of particular interest when applied with the frequencies $\lambda_n = \log n$ for $n \geq 1$. One obtains a mean-value theorem for Dirichlet series: For sequences $(a_n)_{n=1}^{\infty}$ of complex numbers such that $\sum_{n=1}^{\infty} n |a_n|^2 < \infty$ and $T \in \mathbb{R}$,

$$\int_0^T \left| \sum_{n=1}^{\infty} a_n n^{-it} \right|^2 dt = \sum_{n=1}^{\infty} |a_n|^2 (T + O(n)). \tag{1.2.8}$$

Selberg never published any of his work on Hilbert's inequality.

Around the same time, it became apparent that the form

$$\sum_{m=1}^{N}\sum_{\substack{n=1\\ n\neq m}}^{N}\frac{z_m\overline{z_n}}{\sin\left(\pi\left(x_m-x_n\right)\right)},$$

where $x_1,\ldots,x_N$ are distinct real numbers modulo 1, is related to the large sieve. During 1973–1974, Montgomery and Vaughan [9], [10] used Selberg's method to prove

Theorem 1.2.4 (Montgomery and Vaughan [10])**.** *Let $x_1,\ldots,x_N$ be real numbers, distinct modulo* 1*. Let $d_n := \min_{m\neq n}\|x_m-x_n\|$, where $\|x\| := \min_{k\in\mathbb{Z}}|x-k|$. Let $d := \min_n d_n$. Then for any sequence $(z_1,\ldots,z_N)\in\mathbb{C}^N$,*

$$\left|\sum_{m=1}^{N}\sum_{\substack{n=1\\ n\neq m}}^{N}\frac{z_m\overline{z_n}}{\sin\left(\pi\left(x_m-x_n\right)\right)}\right|\leq\frac{1}{d}\sum_{n=1}^{N}|z_n|^2 \tag{1.2.9}$$

and

$$\left|\sum_{m=1}^{N}\sum_{\substack{n=1\\ n\neq m}}^{N}\frac{z_m\overline{z_n}}{\sin\left(\pi\left(x_m-x_n\right)\right)}\right|\leq\frac{3}{2}\sum_{n=1}^{N}\frac{|z_n|^2}{d_n}. \tag{1.2.10}$$

They recovered Theorems 1.2.2 and 1.2.3 from Theorem 1.2.4 by a limiting argument. G. L. Watson pointed out to Vaughan in 1974 that the converse is also true.

In the paper [9], they applied the weighted form (1.2.10) with the x_n's being the nonzero Farey fractions of a given order to prove several important applications in number theory, one of which is an improvement of the Brun–Titchmarsh theorem without an error term: If q and r are positive integers with $\gcd(q,r)=1$ and x and y are positive real numbers with $y>q$, then

$$\pi(x+y,q,r)-\pi(x,q,r)<\frac{2y}{\phi(q)\log(y/q)}, \tag{1.2.11}$$

where $\pi(t,q,r)$ is the number of prime numbers $p \le t$ with $p \equiv r \pmod q$ and $\phi(q)$ is the number of positive integers $s \le q$ with $\gcd(s,q) = 1$.

Denote by $\bar{c}_1$ the minimum of all absolute constants c_1 for which (1.2.7) holds. Chapter 2 is motivated by the problem of determining $\bar{c}_1$.

By substituting $\lambda_k = k$ in (1.2.7) and comparing with Schur's result, we obtain the lower bound $\bar{c}_1 \ge \pi$. If $\bar{c}_1 = \pi$, then (1.2.7) would imply (1.2.6), and it is widely believed to be the case.

The currently known best upper bound for $\bar{c}_1$ is due to Preissmann [11].

Theorem 1.2.5 (Preissmann [11])**.** *We have* $\frac{\bar{c}_1}{\pi} \le \sqrt{1 + \frac{2}{3}\sqrt{\frac{6}{5}}} = 1.31540\ldots$.

According to Montgomery [7, p. 557], Selberg (unpublished) said that he had shown that $\bar{c}_1 \le 3.2$ (i.e., $\frac{\bar{c}_1}{\pi} \le 1.01859\ldots$). However, it seems that no trace remains of his argument.

1.2.3 Main results of Chapter 2

Chapter 2 studies an auxiliary family of bounds for real quadratic forms parametrized by $0 \le \alpha \le 2$. For $0 \le \alpha \le 2$, let $\overline{C}(\alpha)$ be the minimum of all constants $C(\alpha)$ for which the inequality

$$\sum_{m=1}^{N} \sum_{\substack{n=1 \\ n \ne m}}^{N} \frac{\delta_m^{2-\alpha} \delta_n^{\alpha} t_m t_n}{(\lambda_m - \lambda_n)^2} \le C(\alpha) \sum_{n=1}^{N} t_n^2 \tag{1.2.12}$$

holds for all choices of a positive integer N, real numbers $\lambda_1 < \cdots < \lambda_N$,

$$\delta_n := \min_{m \ne n} |\lambda_m - \lambda_n|,$$

and nonnegative real numbers $t_1, \ldots, t_N$. Let $\overline{C}(\alpha) = \infty$ if there is no such real number $C(\alpha)$.

The value $\overline{C}\left(\frac{1}{2}\right)$ is relevant to the generalized Hilbert inequality (1.2.7). We prove the following inequality between $\bar{c}_1$ and $\overline{C}\left(\frac{1}{2}\right)$.

Theorem 1.2.6. *We have* $\bar{c}_1 \leq \sqrt{\frac{\pi^2}{3} + 2\overline{C}\left(\frac{1}{2}\right)}$.

The previous approaches to get an upper bound for $\bar{c}_1$ in [10], [11], and [13] rely on an upper bound for $\overline{C}\left(\frac{1}{2}\right)$ and Theorem 1.2.6. Montgomery and Vaughan [10] first showed that $\overline{C}\left(\frac{1}{2}\right)$ is finite. Specifically, they proved $\overline{C}\left(\frac{1}{2}\right) \leq \frac{17}{2}$. The same bound has been used in [8], but the best known upper bound for $\overline{C}\left(\frac{1}{2}\right)$ is due to Preissmann [11].

Theorem 1.2.7 (Preissmann [11])**.** *We have* $\overline{C}\left(\frac{1}{2}\right) \leq \frac{\pi^2}{3} + \frac{\pi^2}{3}\sqrt{\frac{6}{5}}$.

By means of Theorem 1.2.6, Theorem 1.2.7 implies Theorem 1.2.5. Another immediate consequence of Theorem 1.2.6 is that the lower bound $\bar{c}_1 \geq \pi$ implies $\overline{C}\left(\frac{1}{2}\right) \geq \frac{\pi^2}{3}$. (This lower bound has been pointed out in [8, p. 36].) Moreover, the conjecture that $\bar{c}_1 = \pi$ would follow if $\overline{C}\left(\frac{1}{2}\right) = \frac{\pi^2}{3}$.

We prove that the graph of $\overline{C}(\alpha)$ is symmetric about $\alpha = 1$ and is log-convex. The function $\overline{C}(\alpha)$ is weakly decreasing on $0 \leq \alpha \leq 1$ (and is weakly incresing on $1 \leq \alpha \leq 2$) and is finite-valued only for $\frac{1}{2} \leq \alpha \leq \frac{3}{2}$.

Theorem 1.2.8. (1) *For real numbers* $0 \leq \alpha \leq 2$*, we have* $\overline{C}(\alpha) = \overline{C}(2-\alpha) > 0$.

(2) *For real numbers* $0 \leq \alpha_1 < \alpha_2 \leq 2$ *and* $0 < \theta < 1$*, we have*

$$\overline{C}\left(\theta\alpha_1 + (1-\theta)\alpha_2\right) \leq \overline{C}\left(\alpha_1\right)^{\theta} \overline{C}\left(\alpha_2\right)^{1-\theta}.$$

(3) *For real numbers* $0 \leq \alpha_1 < \alpha_2 \leq 1$*, we have* $\overline{C}\left(\alpha_1\right) \geq \overline{C}\left(\alpha_2\right)$*. Therefore the minimum of* $\overline{C}(\alpha)$ *for* $0 \leq \alpha \leq 2$ *is attained at* $\alpha = 1$.

(4) *For real numbers* $0 \leq \alpha < \frac{1}{2}$*, we have* $\overline{C}(\alpha) = \infty$.

We determine the minimum value of $\overline{C}(\alpha)$.

Theorem 1.2.9. *We have* $\overline{C}(1) = \frac{\pi^2}{3}$.

A main result of Chapter 2 is a new lower bound for $\overline{C}\left(\frac{1}{2}\right)$.

Theorem 1.2.10. *We have* $\overline{C}\left(\frac{1}{2}\right) \geq 0.35047\pi^2$.

From Theorem 1.2.10, we deduce that any upper bound for $\overline{c}_1$ obtainable by Theorem 1.2.6 cannot be smaller than 3.19497. It follows that this method of using Theorem 1.2.6 is incapable of proving $\overline{c}_1 = \pi$.

In Chapter 2, we also prove a generalized Hilbert inequality with the constant π.

Theorem 1.2.11. *Let* $(\lambda_k)_{k=-\infty}^{\infty}$ *be a strictly increasing sequence of real numbers. Denote by* δ_k *the minimum between* $\lambda_k - \lambda_{k-1}$ *and* $\lambda_{k+1} - \lambda_k$. *Then for any sequence* $(z_1, \ldots, z_N)$ *of complex numbers,*

$$\left|\sum_{m=1}^{N} \sum_{\substack{n=1 \\ n \neq m}}^{N} \frac{z_m \overline{z_n}}{\lambda_m - \lambda_n}\right| \leq \pi \left(\sum_{n=1}^{N} \frac{|z_n|^2}{\delta_n^{\frac{3}{4}}}\right)^{\frac{5}{6}} \left(\sum_{n=1}^{N} \frac{|z_n|^2}{\delta_n^{\frac{9}{4}}}\right)^{\frac{1}{6}}.$$

1.3 Generalized factorials and binomial coefficients allowing composite bases

Chapter 3 is about a generalized version of Bhargava's theory of factorial ideals based on $\mathfrak{p}$-orderings of a set S for all prime ideals $\mathfrak{p}$ in a Dedekind ring R. We treat the ring $\mathbb{Z}$ and generalize Bhargava's theory to $\mathfrak{b}$-orderings of a nonempty subset S of $\mathbb{Z}$ for all nontrivial proper ideals $\mathfrak{b}$ in $\mathbb{Z}$. We define generalized factorials $[k]!_{S,\mathcal{T}}$, where $\mathcal{T} \subseteq \mathcal{B} := \{b \in \mathbb{Z} : b \geq 2\}$ which corresponds to the set of all nontrivial proper ideals of $\mathbb{Z}$. We treat in detail the special case $[k]!_{\mathbb{Z},\mathcal{B}}$ and its associated binomial coefficients $\left[{k \atop \ell}\right]_{\mathbb{Z},\mathcal{B}}$.

1.3.1 Background: generalized factorials and generalized binomial coefficients

There have been many studies of generalized notions of factorials and binomial coefficients. Our interest lies in number systems that have three sequences of integers:

generalized positive integers, generalized factorials, and generalized binomial coefficients. Our most general setting is in a commutative ring R with some additional structures, but our attention in this thesis is in the special case $R = \mathbb{Z}$.

Firstly, we consider a generalized notion of positive integers. For $n = 1, 2, 3, \dots$, the nth generalized positive integer (or simply generalized integer) is denoted by $[n]$. In the case $R = \mathbb{Z}$, we want our generalized positive integers to be positive integers; so we will always restrict ourself to the condition that $[n] \in \mathbb{N}$ for all $n \in \mathbb{N}$, where $\mathbb{N} := \{1, 2, 3, \dots\}$ is the set of all positive integers.

Secondly, we consider the generalized factorials corresponding to a sequence of generalized integers. For $k = 0, 1, 2, \dots$, the factorial of k corresponding to $\mathcal{N} := ([n])_{n=1}^{\infty}$ is denoted by $[k]!_{\mathcal{N}}$ and satisfies the relation

$$[k]!_{\mathcal{N}} = \prod_{n=1}^{k} [n].$$

In the case $R = \mathbb{Z}$, this relation and the condition that $[n] \in \mathbb{N}$ for all $n \in \mathbb{N}$ imply that $[k]!_{\mathcal{N}} \in \mathbb{N}$ for all $k \in \mathbb{N} \cup \{0\}$.

Thirdly, we consider the generalized binomial coefficients corresponding to a sequence of generalized integers. For integers k and ℓ such that $0 \le \ell \le k$, the generalized k choose ℓ corresponding to $\mathcal{N} := ([n])_{n=1}^{\infty}$ is denoted by $\begin{bmatrix} k \\ \ell \end{bmatrix}_{\mathcal{N}}$ and satisfies the relation

$$\begin{bmatrix} k \\ \ell \end{bmatrix}_{\mathcal{N}} [\ell]!_{\mathcal{N}} = \prod_{j=1}^{\ell} [k - j + 1].$$

In the case $R = \mathbb{Z}$, these generalized binomial coefficients are not necessarily integers for a general sequence $\mathcal{N}$ of positive integers, as illustrated by the following example.

Example 1.3.1. Consider the sequence $\mathcal{N}_1 := ([n])_{n=1}^{\infty}$ given by $[1] = 2$ and $[n] = 1$

for $n \geq 2$. The generalized factorials corresponding to $\mathcal{N}_1$ are given by

$$[k]!_{\mathcal{N}_1} = \begin{cases} 1 & \text{if } k = 0, \\ 2 & \text{if } k \geq 1, \end{cases}$$

and the generalized binomial coefficients corresponding to $\mathcal{N}_1$ are given by

$$\begin{bmatrix} k \\ \ell \end{bmatrix}_{\mathcal{N}_1} = \begin{cases} 1 & \text{if } \ell = 0 \text{ or } \ell = k, \\ \frac{1}{2} & \text{if } 0 < \ell < k. \end{cases}$$

So a problem arises.

Problem. Characterize the sequences $\mathcal{N}$ of positive integers such that the generalized binomial coefficients corresponding to $\mathcal{N}$ are all integers.

In 1989, Knuth and Wilf [5] studied the notion of regularly divisible sequences: a sequence $(C_n)_{n=1}^{\infty}$ of positive integers is said to be *regularly divisible* if $\gcd(C_m, C_n) = C_{\gcd(m,n)}$ for all positive integers m and n. In fact, sequences with this property were already studied by Ward [14] in 1936. They proved

Theorem 1.3.2 (Ward [14], Knuth and Wilf [5])**.** *The generalized binomial coefficients corresponding to a regularly divisible sequence are all integers.*

Example 1.3.3. Let $a \in \mathbb{N}$ and $p \in \mathbb{N} \cup \{0\}$. Then the sequence $\mathcal{A} := (A_n)_{n=1}^{\infty}$ given by $A_n = an^p$ is regularly divisible, because

$$\gcd(A_m, A_n) = \gcd(am^p, an^p) = a \gcd(m,n)^p = A_{\gcd(m,n)}$$

for all positive integers m and n. The generalized factorials corresponding to $\mathcal{A}$ are given by

$$[k]!_{\mathcal{A}} = a^k (k!)^p,$$

and the generalized binomial coefficients corresponding to $\mathcal{A}$ are given by

$$\begin{bmatrix} k \\ \ell \end{bmatrix}_{\mathcal{A}} = \binom{k}{\ell}^p,$$

where $k!$ and $\binom{k}{\ell}$ are the usual factorial of k and k choose ℓ respectively.

Example 1.3.4. The Fibonacci sequence $\mathcal{F} := (F_n)_{n=1}^{\infty}$, defined recursively by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$, is regularly divisible. The generalized binomial coefficients corresponding to $\mathcal{F}$ are integers that satisfy the boundary conditions

$$\begin{bmatrix} k \\ 0 \end{bmatrix}_{\mathcal{F}} = \begin{bmatrix} k \\ k \end{bmatrix}_{\mathcal{F}} = 1$$

for all $k \geq 0$ and the recurrence

$$\begin{bmatrix} k \\ \ell \end{bmatrix}_{\mathcal{F}} = F_{\ell+1} \begin{bmatrix} k-1 \\ \ell \end{bmatrix}_{\mathcal{F}} + F_{k-\ell-1} \begin{bmatrix} k-1 \\ \ell-1 \end{bmatrix}_{\mathcal{F}}$$

for all $0 < \ell < k$, where $F_0 := 0$.

Proposition 1.3.5. *Let I be a set of indices. For each $i \in I$, let $\mathcal{C}_i := (C_{i,n})_{n=1}^{\infty}$ be a sequence of positive integers, and suppose that the generalized binomial coefficients corresponding to $\mathcal{C}_i$ are all integers. Assume that the sequence $\mathcal{D} := \left(\prod_{i \in I} C_{i,n}\right)_{n=1}^{\infty}$ exists. Then for integers $0 \leq \ell \leq k$, we have*

$$\begin{bmatrix} k \\ \ell \end{bmatrix}_{\mathcal{D}} = \prod_{i \in I} \begin{bmatrix} k \\ \ell \end{bmatrix}_{\mathcal{C}_i}.$$

Hence the generalized binomial coefficients corresponding to $\mathcal{D}$ are all integers.

However, direct products of regularly divisible sequences are not necessarily regularly divisible.

Example 1.3.6. The sequence $(E_n)_{n=1}^{\infty}$ given by $E_n = nF_n$ is the direct product of two regularly divisible sequences, namely $(n)_{n=1}^{\infty}$ and $(F_n)_{n=1}^{\infty}$. However, $(E_n)_{n=1}^{\infty}$ is not regularly divisible, because

$$\gcd(E_2, E_3) = \gcd(2, 6) = 2 \neq 1 = E_1 = E_{\gcd(2,3)}.$$

1.3.2 Previous work: Bhargava's theory of generalized factorials

Beginning in 1997, Bhargava developed a theory of generalized factorials in a class of commutative rings R that he called Dedekind-type rings. These rings are quotients of Dedekind domains and include all Dedekind domains. Bhargava's generalized factorials are associated to nonempty sets S of elements of R and to the set of all prime ideals of R,

$$\mathrm{Spec}\,(R) = \{\mathfrak{p} : \mathfrak{p} \text{ is a prime ideal in } R\}. \tag{1.3.1}$$

Bhargava's generalized factorials $k!_S$ are ideals in R.

For each prime ideal $\mathfrak{p}$ in R and a nonempty subset S of R, he assigned an associated $\mathfrak{p}$-sequence $(\nu_k(S,\mathfrak{p}))_{k=0}^{\infty}$ of S in which $\nu_k(S,\mathfrak{p})$ is a power of $\mathfrak{p}$. He constructed the associated $\mathfrak{p}$-sequence using $\mathfrak{p}$-orderings of S. The generalized factorials of S, denoted $k!_S$, are defined as in [2, Definition 7] by

$$k!_S := \prod_{\mathfrak{p}\in\mathrm{Spec}(R)} \nu_k(S,\mathfrak{p}).$$

We can write

$$\nu_k(S,\mathfrak{p}) = \mathfrak{p}^{\alpha_k(S,\mathfrak{p})}, \tag{1.3.2}$$

where $\alpha_k(S,\mathfrak{p}) \in \mathbb{N}\cup\{\infty\}$, with the conventions $\mathfrak{p}^0 = R$ and $\mathfrak{p}^\infty = (0)$.

Bhargava showed his factorials have many applications to many problems in commutative algebra, to finding rings of integer-valued polynomials on a set S, and to finding good bases for suitable function spaces, see also [3].

Bhargava originally developed his generalized factorials for the ring of integers $\mathbb{Z}$, in which case $\mathrm{Spec}(\mathbb{Z}) = \{(p) : p \text{ is a prime number}\}$, which we may identify with $\mathcal{P} := \{2, 3, 5, \dots\}$, the set of all prime numbers. Bhargava [2] gave details for the case $R = \mathbb{Z}$. In this thesis we treat the case $R = \mathbb{Z}$, and we describe Bhargava's theory in this case, following [2].

Bhargava's important idea is the construction of $\mathfrak{p}$-orderings of S for any fixed prime ideal $\mathfrak{p}$. We describe it for the case $R = \mathbb{Z}$. Let $p \in \mathcal{P}$ be the prime number that generates $\mathfrak{p} \in \mathrm{Spec}(\mathbb{Z})$. A *$p$-ordering of S* is any sequence $\mathbf{a} = (a_i)_{i=0}^{\infty}$ of elements of S that can be formed recursively as follows:

- $a_0 \in S$ is chosen arbitrarily;
- Given $a_j \in S$, $j = 0, \ldots, i-1$, the next element $a_i \in S$ is chosen so that it minimizes the highest power of p dividing the product $\prod_{j=0}^{i-1}(a_i - a_j)$.

We note that:

(1) This construction does not give a unique p-ordering of S if $|S| > 1$.

(2) A p-ordering of S does not need to include all the elements of S.

We define $\nu_i(S, p, \mathbf{a})$ to be the highest power of p dividing $\prod_{j=0}^{i-1}(a_i - a_j)$. That is, we may write

$$\nu_i(S, p, \mathbf{a}) = p^{\alpha_i(S,p,\mathbf{a})}, \tag{1.3.3}$$

where

$$\alpha_i(S, p, \mathbf{a}) := \mathrm{ord}_p\left(\prod_{j=0}^{i-1}(a_i - a_j)\right) \tag{1.3.4}$$

and $\mathrm{ord}_p(\cdot)$ is the additive p-adic valuation given by

$$\mathrm{ord}_p(k) := \sup\left\{\alpha \in \mathbb{N} : p^{\alpha} \text{ divides } k\right\}. \tag{1.3.5}$$

Bhargava calls the sequence $(\nu_i(S, p, \mathbf{a}))_{i=0}^{\infty}$ the *associated p-sequence of S corresponding to* $\mathbf{a}$. Bhargava [2, Theorem 5] showed

Theorem 1.3.7 (Bhargava [2])**.** *The associated p-sequence of S is independent of the choice of p-ordering.*

Therefore one may write $\nu_i(S,p) = \nu_i(S,p,\mathbf{a})$ as an invariant of S and p and call $(\nu_i(S,p))_{i=0}^{\infty}$ the associated p-sequence of S.

Bhargava used this invariant to define his generalized factorials. The *factorial function of S*, denoted $k!_S$, is defined by

$$k!_S := \prod_p \nu_k(S,p). \tag{1.3.6}$$

Thus Bhargava's theory produces factorials via their prime factorizations.

In the special case $S = \mathbb{Z}$, Bhargava showed that the generalized factorials agree with the usual factorials. To do this, Bhargava [2, Proposition 6] showed

Theorem 1.3.8 (Bhargava [2])**.** *The natural ordering $0, 1, 2, \ldots$ of the nonnegative integers forms a p-ordering of $\mathbb{Z}$ for all primes p simultaneously.*

From Theorem 1.3.8, Bhargava deduces that

$$\nu_k(\mathbb{Z},p) = w_p\left(\prod_{j=0}^{k-1}(k-j)\right) = w_p(k!),$$

where $w_p(a)$ denotes the highest power of p dividing a (i.e., $w_p(a) = p^{\mathrm{ord}_p(a)}$).

Therefore

$$k!_{\mathbb{Z}} = \prod_p w_p(k!) = k!. \tag{1.3.7}$$

Bhargava also treated generalized binomial coefficients. Bhargava [2, Theorem 8] showed

Theorem 1.3.9. *For any nonnegative integers k and ℓ, $(k+\ell)!_S$ is a multiple of $k!_S \ell!_S$.*

In other words, the generalized binomial coefficients

$$\binom{k+\ell}{k}_S := \frac{(k+\ell)!_S}{k!_S \ell!_S}$$

is always an integer.

1.3.3 Main results of Chapter 3: generalized factorials allowing composite bases

We generalize Bhargava's theory of $\mathfrak{p}$-orderings for prime ideals $\mathfrak{p}$ in the ring $R = \mathbb{Z}$ to treat $\mathfrak{b}$-orderings for nontrivial proper ideals $\mathfrak{b}$ in $\mathbb{Z}$. The set of all nontrivial proper ideals of $\mathbb{Z}$ may be identified with the set

$$\mathcal{B} := \{b \in \mathbb{Z} : b \geq 2\} = \mathbb{N} \backslash \{0, 1\} \tag{1.3.8}$$

by the positive generators of the ideals. Here $\mathbb{N} := \{0, 1, 2, \dots\}$ is the set of all nonnegative integers.

Definition 1.3.10. Let S be a nonempty subset of the ring of integers $\mathbb{Z}$. For $b \in \mathcal{B}$, a sequence $\mathbf{a} = (a_i)_{i=0}^{\infty}$ of elements of S is an *admissible b-ordering of S* if

$$\sum_{j=0}^{i-1} \operatorname{ord}_b (a_i - a_j) = \min_{s \in S} \sum_{j=0}^{i-1} \operatorname{ord}_b (s - a_j) \tag{1.3.9}$$

for all $i = 1, 2, 3, \dots$, where $\operatorname{ord}_b(k)$ is defined for $k \in \mathbb{Z}$ by

$$\operatorname{ord}_b(k) := \sup \{\alpha \in \mathbb{N} : b^{\alpha} \text{ divides } k\}. \tag{1.3.10}$$

We note a conceptual difference in the quantities that are being minimized in Bhargava's theory and in Definition 1.3.10. In Bhargava's theory, the quantity $\operatorname{ord}_p \left(\prod_{j=0}^{i-1} (a_i - a_j) \right)$ is minimized at each step. In Definition 1.3.10, we minimize $\sum_{j=0}^{i-1} \operatorname{ord}_b (a_i - a_j)$. If b is a prime, then

$$\operatorname{ord}_b \left(\prod_{j=0}^{i-1} (a_i - a_j) \right) = \sum_{j=0}^{i-1} \operatorname{ord}_b (a_i - a_j),$$

but this equality does not hold in general for composite b.

Given any initial value $a_0 \in S$, one can find an admissible b-ordering with that initial value using the recurrence (1.3.9). So there will be more than one admissible b-orderings of S, unless S is a singleton.

Definition 1.3.11. Let $b \in \mathcal{B}$. Let S be a nonempty subset of the ring $\mathbb{Z}$. Let $\mathbf{a} = (a_i)_{i=0}^{\infty}$ be an admissible b-ordering of S. The *associated b-sequence of S with respect to* $\mathbf{a}$, denoted $(\alpha_i(S, b, \mathbf{a}))_{i=0}^{\infty}$, is defined by

$$\alpha_i(S, b, \mathbf{a}) := \sum_{j=0}^{i-1} \operatorname{ord}_b (a_i - a_j). \tag{1.3.11}$$

We note that:

(1) $\alpha_i(S, b, \mathbf{a}) \in \mathbb{N} \cup \{\infty\}$.

(2) If S is finite, then $\alpha_i(S, b, \mathbf{a}) = \infty$ for all $i \geq |S|$.

A main result of Chapter 3 is that all associated b-sequences of a given set S are the same.

Theorem 1.3.12 (Well-definedness of the associated b-sequence of S)**.** *Let $b \in \mathcal{B}$. Let S be a nonempty subset of the ring $\mathbb{Z}$. Let $\mathbf{a}_1$ and $\mathbf{a}_2$ be admissible b-orderings of S. Then $\alpha_i (S, b, \mathbf{a}_1) = \alpha_i (S, b, \mathbf{a}_2)$ for all $i = 0, 1, 2, \ldots$.*

This result generalizes Bhargava's Theorem 1.3.7, in which b is assumed to be prime. Bhargava's proofs, as presented in [1] and [2], do not extend to the case of composite bases b.

Theorem 1.3.12 provides the well-definedness of the associated b-sequence of S. We write $(\alpha_i(S, b))_{i=0}^{\infty}$ for *the associated b-sequence of S*, which is given by $\alpha_i(S, b) := \alpha_i(S, b, \mathbf{a})$ for any admissible b-ordering $\mathbf{a}$ of S.

The generalized factorials $[k]!_{S,\mathcal{T}}$ associated to a nonempty subset S of the ring $\mathbb{Z}$ and a set of allowed bases (or generalized prime numbers) $\mathcal{T} \subseteq \mathcal{B} := \{2, 3, 4, \ldots\}$ are defined for $k = 0, 1, 2, \ldots$ by

$$[k]!_{S,\mathcal{T}} := \prod_{b \in \mathcal{T}} b^{\alpha_k(S,b)}. \tag{1.3.12}$$

The special case $\mathcal{T} = \mathcal{P}$ agrees with Bhargava's generalized factorials [2], which give the usual factorial function as the case $(S, \mathcal{T}) = (\mathbb{Z}, \mathcal{P})$. That is, $[k]!_{\mathbb{Z},\mathcal{P}} = k!$.

Proposition 1.3.13 (Ordering)**.** (1) *Let $S_1 \subseteq S_2$ be nonempty subsets of the ring $\mathbb{Z}$. Let $\mathcal{T} \subseteq \mathcal{B}$. Then for integers $0 \leq k < |S_1|$,*

$$[k]!_{S_2,\mathcal{T}} \quad \textit{divides} \quad [k]!_{S_1,\mathcal{T}}.$$

(2) *Let S be a nonempty subset of the ring $\mathbb{Z}$. Let $\mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \mathcal{B}$. Then for integers $0 \leq k < |S|$,*

$$[k]!_{S,\mathcal{T}_1} \quad \textit{divides} \quad [k]!_{S,\mathcal{T}_2}.$$

Now, we define generalized positive integers $[n]_{S,\mathcal{T}}$. For positive integers $n < |S|$, the *nth generalized positive integer associated to S and $\mathcal{T}$* is $[n]_{S,\mathcal{T}} := \frac{[n]!_{S,\mathcal{T}}}{[n-1]!_{S,\mathcal{T}}}$.

Theorem 1.3.14. *Let S be a nonempty subset of the ring $\mathbb{Z}$. Let $\mathcal{T} \subseteq \mathcal{B}$. Then for positive integers $n < |S|$, the generalized positive integer $[n]_{S,\mathcal{T}}$ is an integer.*

For integers $0 \leq \ell \leq k < |S|$, the *generalized binomial coefficient* $\begin{bmatrix} k \\ \ell \end{bmatrix}_{S,\mathcal{T}}$ is defined by

$$\begin{bmatrix} k \\ \ell \end{bmatrix}_{S,\mathcal{T}} := \frac{[k]!_{S,\mathcal{T}}}{[\ell]!_{S,\mathcal{T}}[k-\ell]!_{S,\mathcal{T}}}. \tag{1.3.13}$$

Theorem 1.3.15. *Let S be a nonempty subset of the ring $\mathbb{Z}$. Let $\mathcal{T} \subseteq \mathcal{B}$. Then for integers $0 \leq \ell \leq k < |S|$, the generalized binomial coefficient $\begin{bmatrix} k \\ \ell \end{bmatrix}_{S,\mathcal{T}}$ is an integer.*

This result generalizes Bhagava's Theorem 1.3.9.

We treat in detail the important case $(S, \mathcal{T}) = (\mathbb{Z}, \mathcal{B})$, in which both S and $\mathcal{T}$ are maximal.

Theorem 1.3.16. *The natural ordering $0, 1, 2, \ldots$ of the nonnegative integers forms an admissible b-ordering of $S = \mathbb{Z}$ for all $b \in \mathcal{B}$ simultaneously.*

Theorem 1.3.16 generalizes Bhargava's Theorem 1.3.8. We next show

Theorem 1.3.17. *For* $k = 0, 1, 2, \ldots,$ *the generalized factorial of* k *associated to* $S = \mathbb{Z}$ *and* $\mathcal{T} = \mathcal{B}$ *is*

$$[k]!_{\mathbb{Z},\mathcal{B}} = \prod_{b=2}^{k} b^{\gamma(k,b)}, \tag{1.3.14}$$

where

$$\gamma(k,b) := \sum_{i=1}^{\infty} \left\lfloor \frac{k}{b^i} \right\rfloor. \tag{1.3.15}$$

Theorem 1.3.17 is analogous to de Polignac's formula for $k!$ (also known as Legendre's formula), which states that

$$\operatorname{ord}_p(k!) = \sum_{i=1}^{\infty} \left\lfloor \frac{k}{p^i} \right\rfloor \tag{1.3.16}$$

for all $p \in \mathcal{P}$. The right side of (1.3.16) is $\gamma(k,p)$.

Theorem 1.3.18. *For* $n = 1, 2, 3, \ldots,$ *the* n*th generalized positive integer associated to* $S = \mathbb{Z}$ *and* $\mathcal{T} = \mathcal{B}$ *is*

$$[n]_{\mathbb{Z},\mathcal{B}} = \prod_{b=2}^{n} b^{\operatorname{ord}_b(n)}, \tag{1.3.17}$$

where $\operatorname{ord}_b(n)$ *is the maximal* $\alpha \in \mathbb{N}$ *such that* b^α *divides* n*.*

Theorem 1.3.18 is analogous to the prime factorization of positive integers:

$$n = \prod_{p \in \mathcal{P}} p^{\operatorname{ord}_p(n)}.$$

We prove formulas for generalized binomial coefficients $\left[{k \atop \ell}\right]_{\mathbb{Z},\mathcal{B}}$.

Theorem 1.3.19. *Let* $k \geq \ell$ *be nonnegative integers. Then:*

(1) *We have*

$$\begin{bmatrix} k \\ \ell \end{bmatrix}_{\mathbb{Z},\mathcal{B}} = \prod_{b=2}^{k} b^{\beta(k,\ell,b)}, \tag{1.3.18}$$

where

$$\beta(k,\ell,b) := \sum_{i=1}^{\infty} \left(\left\lfloor \frac{k}{b^i} \right\rfloor - \left\lfloor \frac{\ell}{b^i} \right\rfloor - \left\lfloor \frac{k-\ell}{b^i} \right\rfloor \right). \tag{1.3.19}$$

(2) *For $b \in \mathcal{B}$,*

$$\beta(k,\ell,b) = \frac{1}{b-1}\left(d_b(\ell) + d_b(k-\ell) - d_b(k)\right), \tag{1.3.20}$$

where $d_b(j)$ is the sum of the base-b digits of j.

Theorem 1.3.19 explicitly shows that these generalized binomial coefficients are integers, as covered by Theorem 1.3.15. In Subsection 3.7.1, we give tables of values of the generalized integers $[n]_{\mathbb{Z},\mathcal{B}}$. The tables show that the sequence $([n]_{\mathbb{Z},\mathcal{B}})_{n=1}^{\infty}$ of generalized integers is not regularly divisible. For example,

$$\gcd\left([4]_{\mathbb{Z},\mathcal{B}}, [6]_{\mathbb{Z},\mathcal{B}}\right) = \gcd(16,36) = 4 \quad \text{but} \quad [\gcd(4,6)]_{\mathbb{Z},\mathcal{B}} = [2]_{\mathbb{Z},\mathcal{B}} = 2.$$

This example shows that Theorem 1.3.15 is not a special case of Theorem 1.3.2.

Finally we obtain

Corollary 1.3.20. *Let $\overline{\overline{G}}_n$ be the product of the generalized binomial coefficients associated to $\mathbb{Z}$ and $\mathcal{B}$ in the nth row of Pascal's triangle:*

$$\overline{\overline{G}}_n := \prod_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\mathbb{Z},\mathcal{B}}. \tag{1.3.21}$$

Then for $n = 1, 2, 3, \ldots$,

$$\overline{\overline{G}}_n := \prod_{b=2}^{n} b^{\overline{\nu}(n,b)}, \tag{1.3.22}$$

where

$$\overline{\nu}(n,b) := \frac{2}{b-1} S_b(n) - \frac{n-1}{b-1} d_b(n) \tag{1.3.23}$$

and $S_b(n) := \sum_{j=1}^{n-1} d_b(j)$.

These numbers $\overline{\overline{G}}_n$ will be further studied in Chapter 4.

1.4 Asymptotics of partial factorizations of products of generalized binomial coefficients

Chapter 4 is about partial factorizations of generalized binomial products. To describe the results in this chapter we first review previous work on this subject.

1.4.1 Previous work: products of binomial coefficients

The product of binomial coefficients on the nth row of Pascal's triangle

$$\overline{G}_n := \prod_{k=0}^{n} \binom{n}{k} \tag{1.4.1}$$

was studied by Lagarias and Mehta [6]. The number $\overline{G}_n$ is the reciprocal of the product of all the nonzero unreduced Farey fractions of order n. They determined that $\log \overline{G}_n = \frac{1}{2}n^2 + O(n \log n)$ for $n \geq 2$. They also determined the prime factorization

$$\overline{G}_n = \prod_{p} p^{\nu_p(\overline{G}_n)},$$

where $\nu_p(\overline{G}_n) = \mathrm{ord}_p(\overline{G}_n)$ is the additive p-adic valuation of $\overline{G}_n$. They expressed $\nu_p(\overline{G}_n)$ in terms of sums of the base-p digits of positive integers up to n.

Theorem 1.4.1. *Let p be a prime number. Then for integers $n \geq 1$,*

$$\nu_p(\overline{G}_n) = \frac{2}{p-1} S_p(n) + \frac{n-1}{p-1} d_p(n), \tag{1.4.2}$$

where $d_p(j)$ is the sum of the base-p digits of j and

$$S_p(n) := \sum_{j=1}^{n-1} d_p(j).$$

Du and Lagarias [4] studied the partial factorization of $\overline{G}_n$ with primes up to x:

$$G(n, x) = \prod_{p \leq x} p^{\nu_p(\overline{G}_n)}.$$

They particularly considered $x = \alpha n$ for $0 < \alpha \leq 1$ and studied the asymptotic behavior of $\log G(n, \alpha n)$ as $n \to \infty$. The asymptotic estimates depend on prime number theory. They gave unconditional results and results depending on the Riemann hypothesis with a better error term.

Theorem 1.4.2. *Let* $G(n,x) = \prod_{p \le x} p^{\nu_p(\overline{G}_n)}$. *Then for all* $n \geq 2$ *and all* $\frac{1}{n} \le \alpha \le 1$,

$$\log G(n, \alpha n) = f_G(\alpha) n^2 + R_G(n, \alpha n), \tag{1.4.3}$$

where $f_G(\alpha)$ *is a function given for* $\alpha > 0$ *by*

$$f_G(\alpha) = \frac{1}{2} + \frac{1}{2}\alpha^2 \left\lfloor \frac{1}{\alpha} \right\rfloor^2 + \frac{1}{2}\alpha^2 \left\lfloor \frac{1}{\alpha} \right\rfloor - \alpha \left\lfloor \frac{1}{\alpha} \right\rfloor, \tag{1.4.4}$$

with $f_G(0) = 0$ *and* $R(n, \alpha n)$ *is a remainder term.*

(1) *Unconditionally there is a positive constant* c *such that for all* $n \geq 4$, *and all* $0 < \alpha \le 1$ *the remainder term satisfies*

$$R_G(n, \alpha n) = O\left(\frac{1}{\alpha} n^2 \exp\left(-c\sqrt{\log n}\right)\right). \tag{1.4.5}$$

The implied constant in the O*-notation does not depend on* α.

(2) *Conditionally on the Riemann hypothesis, for all* $n \geq 4$ *and all* $0 < \alpha \le 1$, *the remainder term satisfies*

$$R_G(n, \alpha n) = O\left(\frac{1}{\alpha} n^{7/4} (\log n)^2\right), \tag{1.4.6}$$

The implied constant in the O*-notation does not depend on* α.

The limit scaling function $f_G(\alpha)$ is pictured below in the (α, β)-plane, $0 \le \alpha \le 1$. Here $f_G(0) = 0$ and $f_G(1) = \frac{1}{2}$.

The Riemann hypothesis is related to the rate of convergence of $\frac{\log G(n,\alpha n)}{n^2}$ to $f_G(\alpha)$ as $n \to \infty$; it shows a power-savings remainder term $O\left(n^{-1/4}(\log n)^2\right)$. The paper [4] suggested that a converse result may hold, that a power-savings remainder term would imply a zero-free region for the Riemann zeta function for $\sigma > 1 - \delta$ for some $\delta > 0$.

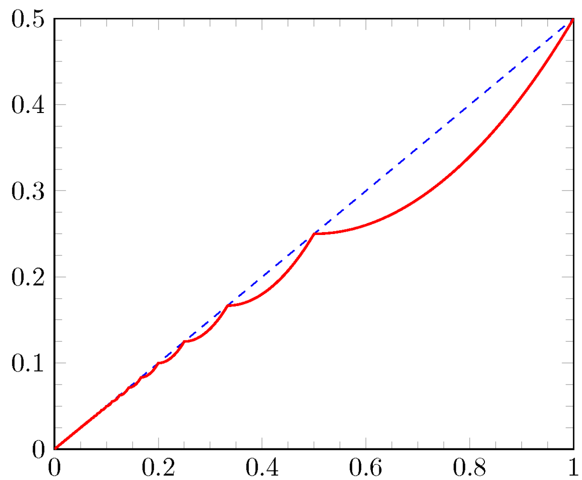

Figure 1.1: The graph $\beta = f_G(\alpha)$, $0 \le \alpha \le 1$ (solid red) in the (α, β)-plane. The line segment $\beta = \frac{1}{2}\alpha$, $0 \le \alpha \le 1$ is shown in dashed blue.

The analysis of [4] depends on obtaining estimates for the auxiliary functions

$$A(n,x) := \sum_{p \le x} \frac{2}{p-1} S_p(n) \log p$$

and

$$B(n,x) := \sum_{p \le x} \frac{n-1}{p-1} d_p(n) \log p.$$

In what follows, $H_n := \sum_{j=1}^{n} \frac{1}{j}$ is the nth harmonic number, and

$$\gamma := \lim_{n \to \infty} (H_n - \log n) = 0.57721\ldots$$

is Euler's constant.

For $B(n,x)$: they showed that for $0 < \alpha \le 1$,

$$B(n, \alpha n) = f_B(\alpha) n^2 + R_B(n, \alpha n), \tag{1.4.7}$$

where $f_B(\alpha)$ is a function given for $\alpha > 0$ by

$$f_B(\alpha) = 1 - \gamma + \left(H_{\lfloor \frac{1}{\alpha} \rfloor} - \log \frac{1}{\alpha} \right) - \alpha \left\lfloor \frac{1}{\alpha} \right\rfloor, \tag{1.4.8}$$

with $f_B(0) = 0$, and $R_B(n, \alpha n)$ is a remainder term.

For $A(n, x)$: they showed that for $0 < \alpha \le 1$,

$$A(n, \alpha n) = f_A(\alpha) n^2 + R_A(n, \alpha n), \tag{1.4.9}$$

where $f_A(\alpha)$ is a function given for $\alpha > 0$ by

$$f_A(\alpha) = \frac{3}{2} - \gamma + \left(H_{\lfloor \frac{1}{\alpha} \rfloor} - \log \frac{1}{\alpha} \right) + \frac{1}{2}\alpha^2 \left\lfloor \frac{1}{\alpha} \right\rfloor^2 + \frac{1}{2}\alpha^2 \left\lfloor \frac{1}{\alpha} \right\rfloor - 2\alpha \left\lfloor \frac{1}{\alpha} \right\rfloor, \tag{1.4.10}$$

with $f_A(0) = 0$, and $R_A(n, \alpha n)$ is a remainder term.

The estimates for the remainder terms $R_B(n, \alpha n)$ and $R_A(n, \alpha n)$ are similar to those of Theorems 1.4.2.

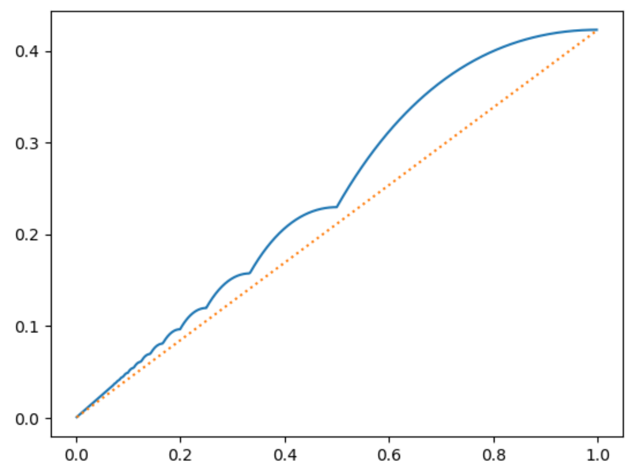

Figure 1.2: The graph $\beta = f_B(\alpha)$, $0 \le \alpha \le 1$ (solid blue) in the (α, β)-plane. The line segment $\beta = (1 - \gamma)\alpha$, $0 \le \alpha \le 1$ is shown in dotted orange.

1.4.2 Main results of Chapter 4: products of generalized binomial coefficients

The new work in Chapter 4 starts from the observation that the formula (1.4.2) for $\nu_p(\overline{G}_n)$ makes sense when replacing p by any integer base $b \ge 2$. For integers $b \ge 2$ and $n \ge 1$, let

$$\overline{\nu}(n, b) := \frac{2}{b-1} S_b(n) + \frac{n-1}{b-1} d_b(n),$$

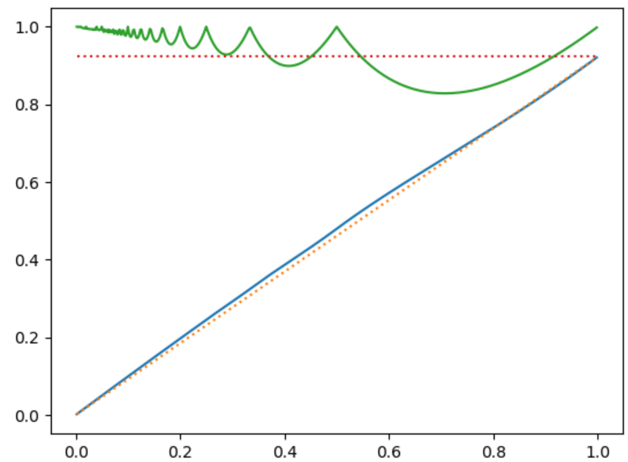

Figure 1.3: The graph $\beta = f_A(\alpha)$, $0 \leq \alpha \leq 1$ (solid blue) in the (α, β)-plane. The line segment $\beta = \left(\frac{3}{2} - \gamma\right)\alpha$, $0 \leq \alpha \leq 1$ is shown in dotted orange. Superimposed are the graph $\beta = f'_A(\alpha)$, $0 \leq \alpha \leq 1$ shown in solid green and the line segment $\beta = \frac{3}{2} - \gamma$, $0 \leq \alpha \leq 1$ shown in dotted red.

where $d_b(j)$ is the sum of the base-b digits of j and

$$S_b(n) := \sum_{j=1}^{n-1} d_b(j).$$

We prove that $\overline{\nu}(n, b)$ is always a nonnegative integer. Moreover, $\overline{\nu}(n, b) = 0$ infinitely often, exactly when $n + 1$ has one nonzero digit in base b.

We define the generalized binomial products $\overline{\overline{G}}_n$ by the formula

$$\overline{\overline{G}}_n := \prod_{b=2}^{n} b^{\overline{\nu}(n,b)}. \tag{1.4.11}$$

It is shown in Chapter 3 (specifically Corollary 1.3.20) that the product in (1.4.11) can be interpreted as a product of generalized binomial coefficients:

$$\overline{\overline{G}}_n = \prod_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\mathbb{Z},\mathcal{B}}.$$

Chapter 4 determines asymptotic estimates for the analogous partial factorization of $\overline{\overline{G}}_n$ which includes all bases b up to x:

$$\overline{\overline{G}}(n, x) := \prod_{2 \leq b \leq x} b^{\overline{\nu}(n,b)}. \tag{1.4.12}$$

The main result of Chapter 4 is as follows.

Theorem 1.4.3. *Let* $\overline{\overline{G}}(n,x) = \prod_{b=2}^{\lfloor x \rfloor} b^{\overline{\nu}(n,b)}$. *Then for integers* $n \geq 2$ *and real* $\alpha \in \left[\frac{1}{\sqrt{n}}, 1\right]$,

$$\log \overline{\overline{G}}(n,\alpha n) = f_{\overline{\overline{G}}}(\alpha)\, n^2 \log n + g_{\overline{\overline{G}}}(\alpha)\, n^2 + O\left(n^{3/2}\log n\right), \tag{1.4.13}$$

in which:

(a) $f_{\overline{\overline{G}}}(\alpha)$ *is a function with* $f_{\overline{\overline{G}}}(0) = 0$ *and defined for* $\alpha > 0$ *by*

$$f_{\overline{\overline{G}}}(\alpha) = \frac{1}{2} + \frac{1}{2}\alpha^2 \left\lfloor \frac{1}{\alpha} \right\rfloor^2 + \frac{1}{2}\alpha^2 \left\lfloor \frac{1}{\alpha} \right\rfloor - \alpha \left\lfloor \frac{1}{\alpha} \right\rfloor; \tag{1.4.14}$$

(b) $g_{\overline{\overline{G}}}(\alpha)$ *is a function with* $g_{\overline{\overline{G}}}(0) = 0$ *and defined for* $\alpha > 0$ *by*

$$\begin{aligned} g_{\overline{\overline{G}}}(\alpha) \;=\; & \left(\frac{1}{2}\gamma - \frac{3}{4}\right) - \frac{1}{2}\left(H_{\lfloor \frac{1}{\alpha} \rfloor} - \log\frac{1}{\alpha}\right) \\ & + \left(\log\frac{1}{\alpha}\right)\left(-\frac{1}{2} - \frac{1}{2}\alpha^2 \left\lfloor \frac{1}{\alpha} \right\rfloor \left\lfloor \frac{1}{\alpha} + 1 \right\rfloor + \alpha \left\lfloor \frac{1}{\alpha} \right\rfloor\right) \\ & - \frac{1}{4}\alpha^2 \left\lfloor \frac{1}{\alpha} \right\rfloor \left\lfloor \frac{1}{\alpha} + 1 \right\rfloor + \alpha \left\lfloor \frac{1}{\alpha} \right\rfloor. \end{aligned} \tag{1.4.15}$$

Moreover, for integers $n \geq 2$ *and real* $\alpha \in \left[\frac{1}{n}, \frac{1}{\sqrt{n}}\right]$,

$$\log \overline{\overline{G}}(n,\alpha n) = O\left(n^{3/2}\log n\right). \tag{1.4.16}$$

We observe three features of this theorem.

1. The first limit scaling function $f_{\overline{\overline{G}}}(\alpha)$ in Theorem 1.4.3 is the same as the limit scaling function $f_G(\alpha)$ obtained in Theorem 1.4.2.

2. The formula (1.4.13) has a secondary term with a new limit scaling function $g_{\overline{\overline{G}}}(\alpha)$.

3. The remainder terms in (1.4.13) and (1.4.16) have a power-savings estimate which is unconditional, while the Riemann hypothesis is needed to obtain a power-savings remainder term in Theorem 1.4.2.

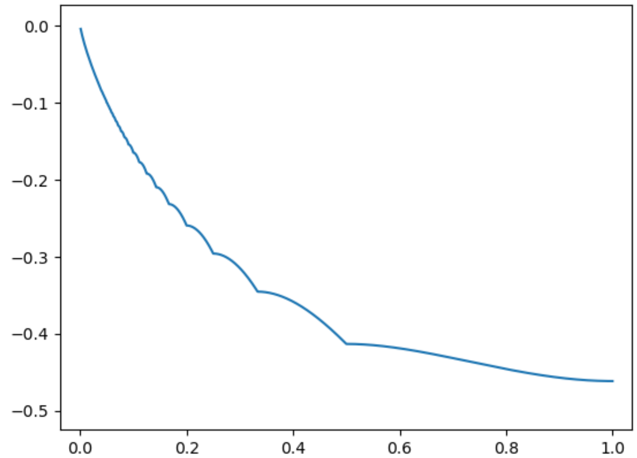

Figure 1.4: The graph $\beta = g_{\overline{G}}(\alpha)$, $0 \le \alpha \le 1$ in the (α, β)-plane.

The analysis depends on obtaining estimates for the auxiliary functions

$$\overline{A}(n,x) := \sum_{2 \le b \le x} \frac{2}{b-1} S_b(n) \log b$$

and

$$\overline{B}(n,x) := \sum_{2 \le b \le x} \frac{n-1}{b-1} d_b(n) \log b.$$

In what follows, $J_n := \sum_{j=1}^{n} \frac{\log j}{j}$, and

$$\gamma_1 := \lim_{n\to\infty} \left(J_n - \frac{1}{2}(\log n)^2 \right) = -0.07281\ldots$$

is the first Stieltjes constant.

Theorem 1.4.4. *Let* $\overline{B}(n,x) = \sum_{b=2}^{\lfloor x \rfloor} \frac{n-1}{b-1} d_b(n) \log b$. *Then for integers* $n \ge 2$ *and real* $\alpha \in \left[\frac{1}{\sqrt{n}}, 1\right]$,

$$\overline{B}(n,\alpha n) = f_{\overline{B}}(\alpha) n^2 \log n + g_{\overline{B}}(\alpha) n^2 + O\left(n^{3/2} \log n\right), \tag{1.4.17}$$

in which:

(a) *$f_{\overline{B}}(\alpha)$ is a function with $f_{\overline{B}}(0) = 0$ and defined for $\alpha > 0$ by*

$$f_{\overline{B}}(\alpha) = (1-\gamma) + \left(H_{\lfloor \frac{1}{\alpha} \rfloor} - \log \frac{1}{\alpha}\right) - \alpha \left\lfloor \frac{1}{\alpha} \right\rfloor ; \tag{1.4.18}$$

(b) *$g_{\overline{B}}(\alpha)$ is a function with $g_{\overline{B}}(0) = 0$ and defined for $\alpha > 0$ by*

$$\begin{aligned} g_{\overline{B}}(\alpha) &= (\gamma + \gamma_1 - 1) - \left(H_{\lfloor \frac{1}{\alpha} \rfloor} - \log \frac{1}{\alpha}\right) - \left(J_{\lfloor \frac{1}{\alpha} \rfloor} - \frac{1}{2}\left(\log \frac{1}{\alpha}\right)^2\right) \\ &\quad + \left(\log \frac{1}{\alpha}\right)\left(-1 + \alpha \left\lfloor \frac{1}{\alpha} \right\rfloor\right) + \alpha \left\lfloor \frac{1}{\alpha} \right\rfloor . \end{aligned} \tag{1.4.19}$$

Moreover, for integers $n \geq 2$ and real $\alpha \in \left[\frac{1}{n}, \frac{1}{\sqrt{n}}\right]$,

$$\overline{B}(n, \alpha n) = O\left(n^{3/2} \log n\right). \tag{1.4.20}$$

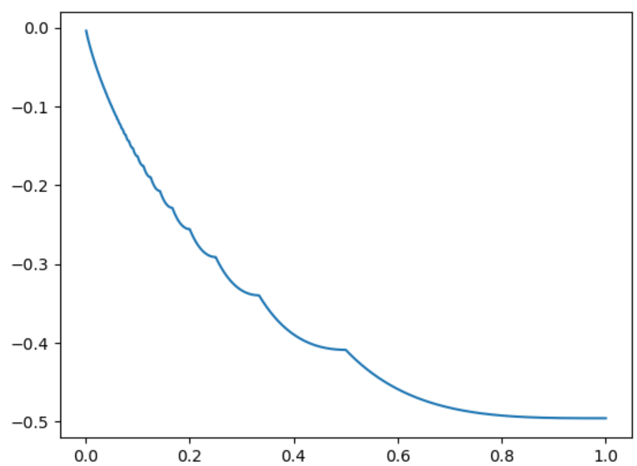

Figure 1.5: The graph $\beta = g_{\overline{B}}(\alpha)$, $0 \leq \alpha \leq 1$ in the (α, β)-plane.

Theorem 1.4.5. *Let $\overline{A}(n, x) = \sum_{b=2}^{\lfloor x \rfloor} \frac{2}{b-1} S_b(n) \log b$. Then for integers $n \geq 2$ and real $\alpha \in \left[\frac{1}{\sqrt{n}}, 1\right]$,*

$$\overline{A}(n, \alpha n) = f_{\overline{A}}(\alpha) n^2 \log n + g_{\overline{A}}(\alpha) n^2 + O\left(n^{3/2} \log n\right), \tag{1.4.21}$$

in which:

(a) $f_{\overline{A}}(\alpha)$ *is a function with* $f_{\overline{A}}(0) = 0$ *and defined for* $\alpha > 0$ *by*

$$f_{\overline{A}}(\alpha) = \left(\frac{3}{2} - \gamma\right) + \left(H_{\lfloor \frac{1}{\alpha} \rfloor} - \log\frac{1}{\alpha}\right) + \frac{1}{2}\alpha^2 \left\lfloor \frac{1}{\alpha} \right\rfloor^2 + \frac{1}{2}\alpha^2 \left\lfloor \frac{1}{\alpha} \right\rfloor - 2\alpha \left\lfloor \frac{1}{\alpha} \right\rfloor; \tag{1.4.22}$$

(b) $g_{\overline{A}}(\alpha)$ *is a function with* $g_{\overline{A}}(0) = 0$ *and defined for* $\alpha > 0$ *by*

$$\begin{aligned} g_{\overline{A}}(\alpha) \quad = \quad & \left(\frac{3}{2}\gamma + \gamma_1 - \frac{7}{4}\right) - \frac{3}{2}\left(H_{\lfloor \frac{1}{\alpha} \rfloor} - \log\frac{1}{\alpha}\right) - \left(J_{\lfloor \frac{1}{\alpha} \rfloor} - \frac{1}{2}\left(\log\frac{1}{\alpha}\right)^2\right) \\ & + \left(\log\frac{1}{\alpha}\right)\left(-\frac{3}{2} - \frac{1}{2}\alpha^2 \left\lfloor \frac{1}{\alpha} \right\rfloor \left\lfloor \frac{1}{\alpha} + 1 \right\rfloor + 2\alpha \left\lfloor \frac{1}{\alpha} \right\rfloor\right) \\ & - \frac{1}{4}\alpha^2 \left\lfloor \frac{1}{\alpha} \right\rfloor \left\lfloor \frac{1}{\alpha} + 1 \right\rfloor + 2\alpha \left\lfloor \frac{1}{\alpha} \right\rfloor. \end{aligned} \tag{1.4.23}$$

Moreover, for integers $n \geq 2$ *and real* $\alpha \in \left[\frac{1}{n}, \frac{1}{\sqrt{n}}\right]$,

$$\overline{A}(n, \alpha n) = O\left(n^{3/2} \log n\right). \tag{1.4.24}$$

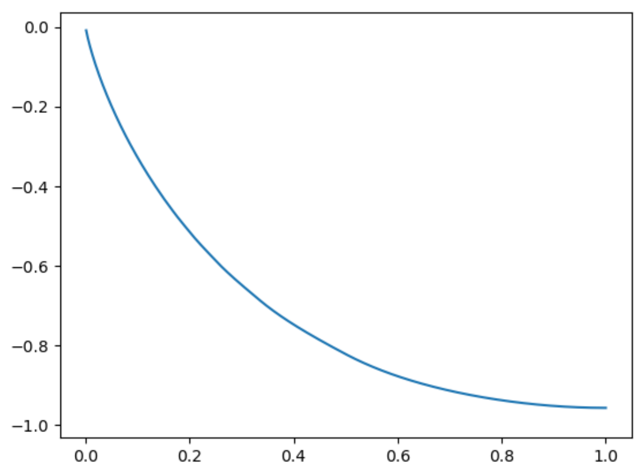

Figure 1.6: The graph $\beta = g_{\overline{A}}(\alpha)$, $0 \leq \alpha \leq 1$ in the (α, β)-plane.

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CHAPTER 2

On the Montgomery–Vaughan Weighted Generalization of Hilbert's Inequality

2.0 Abstract

This chapter studies the problem of determining the optimal constant in the Montgomery–Vaughan weighted generalization of Hilbert's inequality. We consider an approach pursued by previous authors via a parametric family of inequalities. We obtain upper and lower bounds for the constants in inequalities in this family. A lower bound at $\alpha = \frac{1}{2}$ indicates that the method in its current form cannot achieve any value below 3.19497, so cannot achieve the conjectured constant π. The problem of determining the optimal constant remains open.

2.1 Introduction

In this paper, we study a parametric family of inequalities, given in (2.1.8) below, that can yield an upper bound on the optimal constant in the Montgomery–Vaughan weighted generalization of Hilbert's inequality (2.1.3). The inequality (2.1.3) is important in the theory of the large sieve; see [8] and [5].

2.1.1 History of the problem

Let N denote a positive integer, and let $z_1, \ldots, z_N$ denote complex numbers. Hilbert's inequality states that

$$\left| \sum_{m=1}^{N} \sum_{\substack{n=1 \\ n\neq m}}^{N} \frac{z_m \overline{z_n}}{m-n} \right| \leq c_0 \sum_{n=1}^{N} |z_n|^2, \tag{2.1.1}$$

where c_0 is the absolute constant 2π. Hilbert's proof was published in Weyl's dissertation [15, § 15]. In 1911, Schur [13] obtained (2.1.1) with $c_0 = \pi$ and demonstrated that this absolute constant is best possible. Hardy, Littlewood, and Pólya [3, pp. 235–236] gave an account of Hilbert's proof. Schur's proof is also reproduced in [3, Theorem 294].

In 1974, Montgomery and Vaughan [9] established a generalization: If $\delta > 0$ and $(\lambda_k)_{k=-\infty}^{\infty}$ is a sequence of real numbers such that $\lambda_{k+1} - \lambda_k \geq \delta$ for all k, then

$$\left| \sum_{m=1}^{N} \sum_{\substack{n=1 \\ n\neq m}}^{N} \frac{z_m \overline{z_n}}{\lambda_m - \lambda_n} \right| \leq \frac{\pi}{\delta} \sum_{n=1}^{N} |z_n|^2 . \tag{2.1.2}$$

Schur's bound is included in (2.1.2) as the case $\lambda_{k+1} - \lambda_k = \delta$. In the same paper, Montgomery and Vaughan also established a weighted form:

$$\left| \sum_{m=1}^{N} \sum_{\substack{n=1 \\ n\neq m}}^{N} \frac{z_m \overline{z_n}}{\lambda_m - \lambda_n} \right| \leq c_1 \sum_{n=1}^{N} \frac{|z_n|^2}{\delta_n}, \tag{2.1.3}$$

where $\lambda_{k+1} > \lambda_k$ for all k and $\delta_k := \min\{\lambda_k - \lambda_{k-1}, \lambda_{k+1} - \lambda_k\}$ and c_1 is the absolute constant $\frac{3\pi}{2}$. Denote by $\overline{c}_1$ the minimum of all absolute constants c_1 for which (2.1.3) holds. Montgomery and Vaughan [9] have raised the

Problem. Determine $\overline{c}_1$.

By setting $\lambda_k = k$ in (2.1.3) and comparing with Schur's result, we see that

$$\overline{c}_1 \geq \pi. \tag{2.1.4}$$

If $\overline{c}_1 = \pi$, then (2.1.3) would contain (2.1.2), and it is widely believed to be the case. In 1984, Preissmann [11] proved that

$$\overline{c}_1 \leq \pi\sqrt{1 + \frac{2}{3}\sqrt{\frac{6}{5}}} = (1.31540\ldots)\pi < \frac{4\pi}{3}. \tag{2.1.5}$$

Preissmann's proof is based on that of Montgomery and Vaughan. Selberg (unpublished) said that he had shown that $\overline{c}_1 \leq 3.2$ (which is $(1.01859\ldots)\pi < \frac{54\pi}{53}$), but it seems that no trace remains of his argument; cf. [5, p. 557] and [6, p. 145].

In 1981, Graham and Vaaler [1] constructed extreme majorants and minorants of the functions

$$E(\beta, x) := \begin{cases} e^{-\beta x} & \text{if } x \geq 0, \\ \\ 0 & \text{if } x < 0, \end{cases}$$

where β is an arbitrary positive real number, and used them to prove that

$$\frac{1}{\delta\left(e^{\beta/\delta} - 1\right)} \sum_{n=1}^{N} |z_n|^2 \leq \sum_{m=1}^{N} \sum_{n=1}^{N} \frac{z_m \overline{z_n}}{\beta + 2\pi i\left(\lambda_m - \lambda_n\right)} \leq \frac{e^{\beta/\delta}}{\delta\left(e^{\beta/\delta} - 1\right)} \sum_{n=1}^{N} |z_n|^2. \tag{2.1.6}$$

The inequality (2.1.6) includes (2.1.2) as the limiting case $\beta \to 0^+$. In 1999, Montgomery and Vaaler [7] established a generalization of (2.1.3):

$$\left| \sum_{m=1}^{N} \sum_{\substack{n=1 \\ n \neq m}}^{N} \frac{z_m \overline{z_n}}{\beta_m + \beta_n + i\left(\lambda_m - \lambda_n\right)} \right| \leq c_2 \sum_{n=1}^{N} \frac{|z_n|^2}{\delta_n}, \tag{2.1.7}$$

where $\beta_1, \ldots, \beta_N$ are nonnegative real numbers and c_2 is the absolute constant $84 = (26.73803\ldots)\pi$, which is not optimal. Their proof involves the theory of H^2 functions in a half-plane and a maximal theorem of Hardy and Littlewood.

In 2005, Li [4] posed a question about the finite Hilbert transformation associated with a polynomial and proved that if the question always has an affirmative answer, then $\overline{c}_1 = \pi$.

2.1.2 Main results: Parametric family of inequalities

We study the following parametric family of inequalities. For $0 \leq \alpha \leq 2$, let $\overline{C}(\alpha)$ be the minimum of all constants $C(\alpha)$ for which the inequality

$$\sum_{m=1}^{N} \sum_{\substack{n=1 \\ n \neq m}}^{N} \frac{\delta_m^{2-\alpha} \delta_n^{\alpha} t_m t_n}{(\lambda_m - \lambda_n)^2} \leq C(\alpha) \sum_{n=1}^{N} t_n^2 \tag{2.1.8}$$

holds for all choices of a positive integer N, a strictly increasing sequence $(\lambda_k)_{k=-\infty}^{\infty}$ of real numbers,

$$\delta_k := \min \{\lambda_k - \lambda_{k-1}, \lambda_{k+1} - \lambda_k\},$$

and nonnegative real numbers $t_1, \ldots, t_N$. Let $\overline{C}(\alpha) = \infty$ if there is no such real number $C(\alpha)$.

The value $\overline{C}\left(\frac{1}{2}\right)$ is relevant to the generalized Hilbert inequality (2.1.3). In Section 2.3, we shall prove the following inequality between $\bar{c}_1$ and $\overline{C}\left(\frac{1}{2}\right)$.

Theorem 2.1.1. *We have* $\bar{c}_1 \leq \sqrt{\frac{\pi^2}{3} + 2\overline{C}\left(\frac{1}{2}\right)}$.

The previous approaches to get an upper bound for $\bar{c}_1$ in [9], [11], and [14] rely on an upper bound for $\overline{C}\left(\frac{1}{2}\right)$ and Theorem 2.1.1. Montgomery and Vaughan [9] first showed that $\overline{C}\left(\frac{1}{2}\right)$ is finite. Specifically, they proved $\overline{C}\left(\frac{1}{2}\right) \leq \frac{17}{2} = (0.86123\ldots)\pi^2$. The same bound has been used in [7] to prove (2.1.7), but the currently known best upper bound for $\overline{C}\left(\frac{1}{2}\right)$ is due to Preissmann [11].

Theorem 2.1.2 (Preissmann)**.** *We have* $\overline{C}\left(\frac{1}{2}\right) \leq \frac{\pi^2}{3} + \frac{\pi^2}{3}\sqrt{\frac{6}{5}} = (0.69848\ldots)\pi^2$.

By means of Theorem 2.1.1, Theorem 2.1.2 implies (2.1.5). Another immediate consequence of Theorem 2.1.1 is that (2.1.4) implies $\overline{C}\left(\frac{1}{2}\right) \geq \frac{\pi^2}{3}$. (This lower bound has been pointed out in [7, p. 36].) Moreover, the conjecture that $\bar{c}_1 = \pi$ would follow if $\overline{C}\left(\frac{1}{2}\right) = \frac{\pi^2}{3}$.

In Section 2.4, we shall prove the following properties of $\overline{C}(\alpha)$.

Theorem 2.1.3. (1) *For real numbers* $0 \le \alpha \le 2$, *we have* $\overline{C}(\alpha) = \overline{C}(2-\alpha) > 0$.

(2) *For real numbers* $0 \le \alpha_1 < \alpha_2 \le 2$ *and* $0 < \theta < 1$, *we have*

$$\overline{C}\left(\theta\alpha_1 + (1-\theta)\alpha_2\right) \le \overline{C}\left(\alpha_1\right)^{\theta} \overline{C}\left(\alpha_2\right)^{1-\theta}.$$

(3) *For real numbers* $0 \le \alpha_1 < \alpha_2 \le 1$, *we have* $\overline{C}\left(\alpha_1\right) \ge \overline{C}\left(\alpha_2\right)$. *Therefore the minimum of* $\overline{C}(\alpha)$ *for* $0 \le \alpha \le 2$ *is attained at* $\alpha = 1$.

(4) *For real numbers* $0 \le \alpha < \frac{1}{2}$, *we have* $\overline{C}(\alpha) = \infty$.

Also in Section 2.4, we determine the minimum value.

Theorem 2.1.4. *We have* $\overline{C}(1) = \frac{\pi^2}{3}$.

In Section 2.5, we shall prove a new lower bound for $\overline{C}\left(\frac{1}{2}\right)$.

Theorem 2.1.5. *We have* $\overline{C}\left(\frac{1}{2}\right) \ge (0.35047)\pi^2$.

From Theorem 2.1.5, we deduce that any upper bound for $\overline{c}_1$ obtainable by Theorem 2.1.1 cannot be smaller than $3.19497 = (1.01699\ldots)\pi$. This method of using Theorem 2.1.1 is incapable of proving $\overline{c}_1 = \pi$.

2.1.3 Main results: Weighted inequalities

We prove upper bounds on the Hilbert sesquilinear form involving different weights.

Theorem 2.1.6. *Let* $(\lambda_k)_{k=-\infty}^{\infty}$ *be a strictly increasing sequence of real numbers. Denote by* δ_k *the minimum between* $\lambda_k - \lambda_{k-1}$ *and* $\lambda_{k+1} - \lambda_k$. *Then for any positive real number* D *and any sequence* $(z_1, \ldots, z_N)$ *of complex numbers,*

$$\left| \sum_{m=1}^{N} \sum_{\substack{n=1 \\ n\ne m}}^{N} \frac{z_m \overline{z_n}}{\lambda_m - \lambda_n} \right| \le \pi \sum_{n=1}^{N} \frac{|z_n|^2}{\delta_n} \sqrt{\frac{D}{3\delta_n} + \frac{2}{3}\sqrt{\frac{\delta_n}{D}}}. \tag{2.1.9}$$

Theorem 2.1.6 has a multiplicative version.

Theorem 2.1.7. *Let* $(\lambda_k)_{k=-\infty}^{\infty}$ *be a strictly increasing sequence of real numbers. Denote by* δ_k *the minimum between* $\lambda_k - \lambda_{k-1}$ *and* $\lambda_{k+1} - \lambda_k$*. Then for any sequence* $(f_1, \ldots, f_N)$ *of positive real numbers and any sequence* $(z_1, \ldots, z_N)$ *of complex numbers,*

$$\left| \sum_{m=1}^{N} \sum_{\substack{n=1 \\ n \neq m}}^{N} \frac{z_m \overline{z_n}}{\lambda_m - \lambda_n} \right| \leq \pi \left(\sum_{n=1}^{N} \frac{f_n \left|z_n\right|^2}{\delta_n} \right)^{\frac{1}{2}} \left(\sum_{n=1}^{N} \frac{\left|z_n\right|^2}{f_n \sqrt{\delta_n}} \right)^{\frac{1}{3}} \left(\sum_{n=1}^{N} \frac{\left|z_n\right|^2}{f_n \delta_n^2} \right)^{\frac{1}{6}}. \tag{2.1.10}$$

In Section 2.6, we will prove Theorem 2.1.6 and then deduce Theorem 2.1.7 from it. As an immediate consequence of Theorem 2.1.7:

Corollary 2.1.8. *Let* $(\lambda_k)_{k=-\infty}^{\infty}$ *be a strictly increasing sequence of real numbers. Denote by* δ_k *the minimum between* $\lambda_k - \lambda_{k-1}$ *and* $\lambda_{k+1} - \lambda_k$*. Then for any sequence* $(z_1, \ldots, z_N)$ *of complex numbers,*

$$\left| \sum_{m=1}^{N} \sum_{\substack{n=1 \\ n \neq m}}^{N} \frac{z_m \overline{z_n}}{\lambda_m - \lambda_n} \right| \leq \pi \left(\sum_{n=1}^{N} \frac{\left|z_n\right|^2}{\delta_n^{\frac{3}{4}}} \right)^{\frac{5}{6}} \left(\sum_{n=1}^{N} \frac{\left|z_n\right|^2}{\delta_n^{\frac{9}{4}}} \right)^{\frac{1}{6}}. \tag{2.1.11}$$

Proof. Substitute $f_n = \delta_n^{\frac{1}{4}}$ in Theorem 2.1.7. □

It is clear that the right side of (2.1.11) is less than or equal to that of (2.1.2).

2.2 Preliminaries

2.2.1 Eigenvalues of generalized weighted Hilbert matrices

Let us consider $N \times N$ matrices $H = [h_{mn}]$ with entries given by

$$h_{mn} := \begin{cases} \frac{w_m w_n}{\lambda_m - \lambda_n} & \text{if } m \neq n, \\ 0 & \text{if } m = n, \end{cases} \tag{2.2.1}$$

where $(\lambda_k)_{k=-\infty}^{\infty}$ is a strictly increasing sequence of real numbers and $w_1, \dots, w_N$ are positive real numbers. Since H is skew-Hermitian (i.e., iH is Hermitian), all its eigenvalues are purely imaginary. Let $[u_1, \dots, u_N]^\top$ be an eigenvector of H, and let $i\mu$ be its associated eigenvalue. That is,

$$\sum_{\substack{n=1\\n\neq m}}^{N} \frac{w_m w_n u_n}{\lambda_m - \lambda_n} = i\mu u_m$$

for all $m = 1, \dots, N$.

It is well known (see, e.g., [6, § 7.4]) that the numerical radius of a normal matrix is the same as its spectral radius (and its operator norm). Thus, if $i\mu$ has the largest modulus among all eigenvalues of H, then

$$\left| \sum_{m=1}^{N} \sum_{\substack{n=1\\n\neq m}}^{N} \frac{w_m w_n z_m \overline{z_n}}{\lambda_m - \lambda_n} \right| \leq |\mu| \sum_{n=1}^{N} |z_n|^2 \tag{2.2.2}$$

for all complex numbers $z_1, \dots, z_N$. On replacing z_n by $\frac{z_n}{w_n}$, we see that (2.2.2) is equivalent to

$$\left| \sum_{m=1}^{N} \sum_{\substack{n=1\\n\neq m}}^{N} \frac{z_m \overline{z_n}}{\lambda_m - \lambda_n} \right| \leq |\mu| \sum_{n=1}^{N} \frac{|z_n|^2}{w_n^2}. \tag{2.2.3}$$

One may obtain the generalized Hilbert inequality (2.1.3) with some constant c_1 from (2.2.3) by giving an upper bound for the sizes of eigenvalues of H in the case that $w_n^2 = \delta_n = \min\{\lambda_n - \lambda_{n-1}, \lambda_{n+1} - \lambda_n\}$. A key result to that end is:

Lemma 2.2.1. *Let $[u_1, \dots, u_N]^\top$ be an eigenvector of H, and let $i\mu$ be its associated eigenvalue. Then the identity*

$$\mu^2 |u_m|^2 = \sum_{\substack{n=1\\n\neq m}}^{N} \frac{w_m^2 w_n^2 |u_n|^2}{(\lambda_m - \lambda_n)^2} + 2 \sum_{\substack{n=1\\n\neq m}}^{N} \frac{w_m^3 w_n \Re(\overline{u_m} u_n)}{(\lambda_m - \lambda_n)^2} \tag{2.2.4}$$

holds for all $m = 1, \dots, N$.

Proof. See Preissmann and Lévêque [12, Lemma 5 (b)]. □

2.2.2 A weighted spacing lemma and Shan's method

The goal of this subsection is to prove:

Lemma 2.2.2. *Let $(\lambda_k)_{k=-\infty}^{\infty}$ be a strictly increasing sequence of real numbers. Denote by δ_k the minimum between $\lambda_k - \lambda_{k-1}$ and $\lambda_{k+1} - \lambda_k$. Then for real numbers $\sigma > 1$ and integers ℓ, we have*

$$\sum_{\substack{k=-\infty \\ k\neq\ell}}^{\infty} \frac{\delta_k}{|\lambda_k - \lambda_\ell|^\sigma} \leq \frac{2\zeta(\sigma)}{\delta_\ell^{\sigma-1}}. \tag{2.2.5}$$

One can show that equality holds in (2.2.5) if and only if the sequence $(\lambda_{k+1} - \lambda_k)_{k=-\infty}^{\infty}$ is constant, but we shall not treat it here.

Lemma 2.2.2 is a direct consequence of Lemme 1 of Preissmann [11]. We present a proof using a method of Shan [14], who independently derived Lemma 2.2.2. The work of Shan, done at the same time as that of Preissmann, is obscure and hard to obtain. Peng Gao (private communication) translated Shan's argument, which appears in [10, pp. 590–595]. The next three lemmas are an exposition of Shan's method.

Let f be a real-valued function, defined on the interval $[1, \infty)$. We will assume that f satisfies some (or all) of the following four conditions:

(a) $f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$ for all $0 \leq \theta \leq 1$ and $1 \leq x \leq y$.

(b) $f(x) \geq f(y)$ for all $1 \leq x \leq y$.

(c) $f(x) \geq 0$ for all $x \geq 1$.

(d) The series $\sum_{j=1}^{\infty} f(j)$ converges.

We note that (c) follows from (b) and (d), since (b) implies $f(x) \geq \lim_{k\to\infty} f(k)$ and (d) implies $\lim_{k\to\infty} f(k) = 0$.

Lemma 2.2.3. *Assume that* $f:[1,\infty)\to\mathbb{R}$ *satisfies* (a) *and* (b)*. Let* $(a_n)_{n=1}^{\infty}$ *be a sequence of real numbers such that* $a_n \geq 1$ *for all* n*. Set* $\lambda_n := \sum_{m=1}^{n} a_m$*. Then for positive integers* N*, we have*

$$\sum_{n=1}^{N} a_n f(\lambda_n) \leq \sum_{j=1}^{\lfloor \lambda_N \rfloor} f(j) + \{\lambda_N\} f(\lfloor \lambda_N \rfloor + 1),$$

where $\{x\} = x - \lfloor x \rfloor$ *denotes the fractional part of* x*.*

Proof. By the convexity of f, we have

$$f(\lambda_n) \leq (1 - \{\lambda_n\}) f(\lfloor \lambda_n \rfloor) + \{\lambda_n\} f(\lfloor \lambda_n \rfloor + 1). \tag{2.2.6}$$

Moreover, since $a_n \geq 1$ and f is weakly decreasing, it follows that

$$(a_n - 1) f(\lambda_n) \leq (a_n - 1) f(\lfloor \lambda_n \rfloor). \tag{2.2.7}$$

On summing (2.2.6) and (2.2.7), we obtain

$$a_n f(\lambda_n) \leq (a_n - \{\lambda_n\}) f(\lfloor \lambda_n \rfloor) + \{\lambda_n\} f(\lfloor \lambda_n \rfloor + 1). \tag{2.2.8}$$

Now, we consider the first term on the right side of (2.2.8) and note that $\lambda_n = \lambda_{n-1} + a_n \geq \lambda_{n-1} + 1$:

$$\begin{aligned}
(a_n - \{\lambda_n\}) f(\lfloor \lambda_n \rfloor) &= (\lfloor \lambda_n \rfloor - \lfloor \lambda_{n-1} \rfloor - 1) f(\lfloor \lambda_n \rfloor) + (1 - \{\lambda_{n-1}\}) f(\lfloor \lambda_n \rfloor) \\
&\leq \sum_{j=\lfloor \lambda_{n-1} \rfloor + 2}^{\lfloor \lambda_n \rfloor} f(j) + (1 - \{\lambda_{n-1}\}) f(\lfloor \lambda_{n-1} \rfloor + 1) \\
&= \sum_{j=\lfloor \lambda_{n-1} \rfloor + 1}^{\lfloor \lambda_n \rfloor} f(j) - \{\lambda_{n-1}\} f(\lfloor \lambda_{n-1} \rfloor + 1).
\end{aligned}$$

On inserting this in (2.2.8), we get

$$a_n f(\lambda_n) \leq \sum_{j=\lfloor \lambda_{n-1} \rfloor + 1}^{\lfloor \lambda_n \rfloor} f(j) - \{\lambda_{n-1}\} f(\lfloor \lambda_{n-1} \rfloor + 1) + \{\lambda_n\} f(\lfloor \lambda_n \rfloor + 1). \tag{2.2.9}$$

The result follows by summing (2.2.9) over $n = 1, \ldots, N$; the resulting sum on the right side is a telescoping sum. □

In what follows, we consider

$$F_N(\mathbf{x}) := \sum_{n=1}^{N} \min\{x_n, x_{n+1}\} f\left(\sum_{m=1}^{n} x_m\right), \tag{2.2.10}$$

where $\mathbf{x} = (x_n)_{n=1}^{\infty}$ is a sequence of positive real numbers with $x_1 \geq 1$.

Lemma 2.2.4. *Assume that $f : [1, \infty) \to \mathbb{R}$ satisfies* (a)–(c). *Let $\mathbf{a} = (a_n)_{n=1}^{\infty}$ be a sequence of positive real numbers with $a_1 \geq 1$. Suppose that $\nu \geq 2$ is an integer such that $a_{\nu-1} > a_\nu$. Let $0 < \varepsilon \leq a_{\nu-1} - a_\nu$. Define $\mathbf{b} = (b_n)_{n=1}^{\infty}$ by*

$$b_n := \begin{cases} a_n & \text{for } n \neq \nu, \\ a_\nu + \varepsilon & \text{for } n = \nu. \end{cases}$$

Then for positive integers N, we have

$$F_N(\mathbf{a}) \leq F_N(\mathbf{b}). \tag{2.2.11}$$

Proof. If $N \leq \nu - 2$, then (2.2.11) is an identity. So let us assume that $N \geq \nu - 1$. Put $\lambda_n := \sum_{m=1}^{n} a_m$. It follows from the definition of b_n that

$$\min\{b_n, b_{n+1}\} - \min\{a_n, a_{n+1}\} \begin{cases} = \varepsilon & \text{if } n = \nu - 1, \\ \geq 0 & \text{if } n = \nu, \\ = 0 & \text{otherwise,} \end{cases}$$

$$\sum_{m=1}^{n} b_m = \begin{cases} \lambda_n & \text{for } n \leq \nu - 1, \\ \lambda_n + \varepsilon & \text{for } n \geq \nu. \end{cases}$$

By the nonnegativity of f, $\min\{b_\nu, b_{\nu+1}\} f(\lambda_\nu + \varepsilon) \geq \min\{a_\nu, a_{\nu+1}\} f(\lambda_\nu + \varepsilon)$. So

$$F_N(\mathbf{b}) - F_N(\mathbf{a}) \geq \varepsilon f(\lambda_{\nu-1}) + \sum_{n=\nu}^{N} \min\{a_n, a_{n+1}\} \left(f(\lambda_n + \varepsilon) - f(\lambda_n)\right). \tag{2.2.12}$$

By the convexity of f, it follows that

$$\frac{f(\lambda_n+\varepsilon)-f(\lambda_n)}{\varepsilon} \geq \frac{f(\lambda_n)-f(\lambda_{n-1})}{a_n}$$

for all $n \geq 2$. So (2.2.12) implies that

$$\begin{aligned} F_N(\mathbf{b})-F_N(\mathbf{a}) &\geq \varepsilon f(\lambda_{\nu-1})+\varepsilon\sum_{n=\nu}^{N}\frac{\min\{a_n,a_{n+1}\}}{a_n}\left(f(\lambda_n)-f(\lambda_{n-1})\right) \\ &\geq \varepsilon f(\lambda_{\nu-1})+\varepsilon\sum_{n=\nu}^{N}\left(f(\lambda_n)-f(\lambda_{n-1})\right) \\ &= \varepsilon f(\lambda_N) \geq 0. \end{aligned}$$

Hence $F_N(\mathbf{a}) \leq F_N(\mathbf{b})$. □

We now prove an upper bound for $F_N(\mathbf{a})$ that depends only on f.

Lemma 2.2.5. *Assume that* $f:[1,\infty)\to\mathbb{R}$ *satisfies* (a)–(d). *Let* $\mathbf{a}=(a_n)_{n=1}^{\infty}$ *be a sequence of positive real numbers with* $a_1 \geq 1$. *Then for positive integers* N, *we have*

$$F_N(\mathbf{a}) \leq \sum_{j=1}^{\infty} f(j). \tag{2.2.13}$$

By taking $a_n = 1$ for all n and letting $N\to\infty$, we see that (2.2.13) is sharp.

Proof. Define a sequence $\overline{\mathbf{a}} = (\overline{a}_n)_{n=1}^{\infty}$ by $\overline{a}_n := \max\{a_m : m=1,\ldots,n\}$. Then $\overline{a}_{n+1} \geq \overline{a}_n$ for all n and $\overline{a}_1 = a_1 \geq 1$. Let N be a positive integer. By applying Lemma 2.2.4, with $\varepsilon = a_{\nu-1}-a_\nu$, as many times as we need, we see that

$$F_N(\mathbf{a}) \leq F_N(\overline{\mathbf{a}}) = \sum_{n=1}^{N}\overline{a}_n f\left(\overline{\lambda}_n\right), \tag{2.2.14}$$

where $\overline{\lambda}_n := \sum_{m=1}^{n}\overline{a}_m$.

By Lemma 2.2.3 and the nonnegativity of f, the right side of (2.2.14) is

$$\sum_{n=1}^{N}\overline{a}_n f\left(\overline{\lambda}_n\right) \leq \sum_{j=1}^{\lfloor\overline{\lambda}_N\rfloor} f(j)+\left\{\overline{\lambda}_N\right\} f\left(\lfloor\overline{\lambda}_N\rfloor+1\right) \leq \sum_{j=1}^{\infty} f(j). \tag{2.2.15}$$

The result (2.2.13) follows by combining (2.2.14) and (2.2.15). □

We are now ready to prove Lemma 2.2.2.

Proof of Lemma 2.2.2. Let ℓ be an integer. Define sequences $\mathbf{a} = (a_n)_{n=1}^{\infty}$ and $\mathbf{b} = (b_n)_{n=1}^{\infty}$ by

$$a_n := \frac{\lambda_{\ell+n} - \lambda_{\ell+n-1}}{\delta_\ell} \quad \text{and} \quad b_n := \frac{\lambda_{\ell-n+1} - \lambda_{\ell-n}}{\delta_\ell},$$

for all n. Then $\mathbf{a}$ and $\mathbf{b}$ are sequences of positive real numbers with

$$a_1 = \frac{\lambda_{\ell+1} - \lambda_\ell}{\delta_\ell} \geq 1 \quad \text{and} \quad b_1 = \frac{\lambda_\ell - \lambda_{\ell-1}}{\delta_\ell} \geq 1.$$

We have

$$\min\{a_n, a_{n+1}\} = \frac{\delta_{\ell+n}}{\delta_\ell} \quad \text{and} \quad \min\{b_n, b_{n+1}\} = \frac{\delta_{\ell-n}}{\delta_\ell},$$

$$\sum_{m=1}^{n} a_m = \frac{\lambda_{\ell+n} - \lambda_\ell}{\delta_\ell} \quad \text{and} \quad \sum_{m=1}^{n} b_m = \frac{\lambda_\ell - \lambda_{\ell-n}}{\delta_\ell}.$$

Let $\sigma > 1$. Applying Lemma 2.2.5 with $f(x) = \frac{1}{x^\sigma}$, we obtain

$$\begin{aligned}
\delta_\ell^{\sigma-1} \sum_{\substack{k=\ell-N \\ k\neq \ell}}^{\ell+N} \frac{\delta_k}{|\lambda_k - \lambda_\ell|^\sigma} &= \delta_\ell^{\sigma-1} \sum_{n=1}^{N} \left(\frac{\delta_{\ell+n}}{(\lambda_{\ell+n} - \lambda_\ell)^\sigma} + \frac{\delta_{\ell-n}}{(\lambda_\ell - \lambda_{\ell-n})^\sigma} \right) \\
&= F_N(\mathbf{a}) + F_N(\mathbf{b}) \\
&\leq 2 \sum_{j=1}^{\infty} f(j) = 2\zeta(\sigma).
\end{aligned}$$

The result (2.2.5) follows by letting $N \to \infty$. □

2.3 Proofs of Theorems 2.1.1 and 2.1.2

2.3.1 Proof of Theorem 2.1.1

Proposition 2.3.1. *Let N be a positive integer. Let $(\lambda_k)_{k=-\infty}^{\infty}$ be a strictly increasing sequence of real numbers. Denote by δ_k the minimum between $\lambda_k - \lambda_{k-1}$ and $\lambda_{k+1} - \lambda_k$. Assume that c_3 is a positive constant such that the inequality*

$$\sum_{m=1}^{N} \sum_{\substack{n=1 \\ n\neq m}}^{N} \frac{\delta_m^{\frac{3}{2}} \delta_n^{\frac{1}{2}} t_m t_n}{(\lambda_m - \lambda_n)^2} \leq c_3 \sum_{n=1}^{N} t_n^2 \tag{2.3.1}$$

holds for all nonnegative real numbers $t_1, \ldots, t_N$. *Then the inequality* (2.1.3) *holds for all complex numbers* $z_1, \ldots, z_N$ *with the constant* $c_1 = \sqrt{\frac{\pi^2}{3} + 2c_3}$.

Proof. Suppose that (2.3.1) holds. Let $[u_1, \ldots, u_N]^\top$ be a unit eigenvector of $H = [h_{mn}]$, where h_{mn} are given by (2.2.1) with $w_n = \sqrt{\delta_n}$, and let $i\mu$ be the eigenvalue associated with this eigenvector. On applying Lemma 2.2.1 and summing (2.2.4) over m, we get

$$\mu^2 = \sum_{m=1}^{N} \sum_{\substack{n=1 \\ n \neq m}}^{N} \frac{\delta_m \delta_n |u_n|^2}{(\lambda_m - \lambda_n)^2} + 2 \sum_{m=1}^{N} \sum_{\substack{n=1 \\ n \neq m}}^{N} \frac{\delta_m^{\frac{3}{2}} \delta_n^{\frac{1}{2}} \Re\left(\overline{u_m} u_n\right)}{(\lambda_m - \lambda_n)^2} \leq S + 2T, \tag{2.3.2}$$

where S and T are given by

$$S := \sum_{m=1}^{N} \sum_{\substack{n=1 \\ n \neq m}}^{N} \frac{\delta_m \delta_n |u_n|^2}{(\lambda_m - \lambda_n)^2} \quad \text{and} \quad T := \sum_{m=1}^{N} \sum_{\substack{n=1 \\ n \neq m}}^{N} \frac{\delta_m^{\frac{3}{2}} \delta_n^{\frac{1}{2}} |u_m| |u_n|}{(\lambda_m - \lambda_n)^2}.$$

On one hand, by Lemma 2.2.2, we obtain

$$S = \sum_{n=1}^{N} \delta_n |u_n|^2 \left(\sum_{\substack{m=1 \\ m \neq n}}^{N} \frac{\delta_m}{(\lambda_m - \lambda_n)^2} \right) \leq \sum_{n=1}^{N} \delta_n |u_n|^2 \left(\frac{\pi^2}{3\delta_n} \right) = \frac{\pi^2}{3}. \tag{2.3.3}$$

On the other hand, substituting $t_n = |u_n|$ in (2.3.1) gives

$$T \leq c_3. \tag{2.3.4}$$

It follows from (2.3.2), (2.3.3), and (2.3.4) that

$$|\mu| \leq \sqrt{S + 2T} \leq \sqrt{\frac{\pi^2}{3} + 2c_3}. \tag{2.3.5}$$

By the argument preceding (2.2.3), we deduce from (2.2.3) and (2.3.5) that (2.1.3) holds with $c_1 = \sqrt{\frac{\pi^2}{3} + 2c_3}$. □

One weak point in the proof of Proposition 2.3.1 is the bound in (2.3.2), where we disregard cancellation between terms.

Proof of Theorem 2.1.1. Since (2.3.1) holds with $c_3 = \overline{C}\left(\frac{1}{2}\right)$, it follows by Proposition 2.3.1 that (2.1.3) holds with $c_1 = \sqrt{\frac{\pi^2}{3} + 2\overline{C}\left(\frac{1}{2}\right)}$. Hence the result follows. □

2.3.2 Proof of Theorem 2.1.2

Lemma 2.3.2. *Let $(\lambda_k)_{k=-\infty}^{\infty}$ be a strictly increasing sequence of real numbers. Denote by δ_k the minimum between $\lambda_k - \lambda_{k-1}$ and $\lambda_{k+1} - \lambda_k$. Then for distinct integers ℓ and m, we have*

$$\sum_{\substack{k=-\infty \\ k\neq\ell \\ k\neq m}}^{\infty} \frac{\delta_k}{(\lambda_k - \lambda_\ell)^2 (\lambda_k - \lambda_m)^2} \leq \frac{\pi^2 (\delta_\ell + \delta_m)}{3\delta_\ell \delta_m (\lambda_\ell - \lambda_m)^2} - \frac{3(\delta_\ell + \delta_m)}{(\lambda_\ell - \lambda_m)^4}. \tag{2.3.6}$$

Proof. See Preissmann [11, Lemme 6]. □

Proof of Theorem 2.1.2. Let

$$U := \sum_{m=1}^{N} \sum_{\substack{n=1 \\ n\neq m}}^{N} \frac{\delta_m^{\frac{3}{2}} \delta_n^{\frac{1}{2}} t_m t_n}{(\lambda_m - \lambda_n)^2} \quad \text{and} \quad V := \sum_{n=1}^{N} t_n^2.$$

By Cauchy's inequality,

$$\begin{aligned} U^2 &= \left(\sum_{n=1}^{N} t_n \sum_{\substack{m=1 \\ m\neq n}}^{N} \frac{\delta_m^{\frac{3}{2}} \delta_n^{\frac{1}{2}} t_m}{(\lambda_m - \lambda_n)^2} \right)^2 \\ &\leq \left(\sum_{n=1}^{N} t_n^2 \right) \left(\sum_{n=1}^{N} \left(\sum_{\substack{m=1 \\ m\neq n}}^{N} \frac{\delta_m^{\frac{3}{2}} \delta_n^{\frac{1}{2}} t_m}{(\lambda_m - \lambda_n)^2} \right)^2 \right) = V(S+T), \end{aligned}$$

where

$$S := \sum_{n=1}^{N} \sum_{\substack{m=1 \\ m\neq n}}^{N} \frac{\delta_m^3 \delta_n t_m^2}{(\lambda_m - \lambda_n)^4} \quad \text{and} \quad T := \sum_{n=1}^{N} \sum_{\substack{\ell=1 \\ \ell\neq n}}^{N} \sum_{\substack{m=1 \\ m\neq n \\ m\neq\ell}}^{N} \frac{\delta_\ell^{\frac{3}{2}} \delta_m^{\frac{3}{2}} \delta_n t_\ell t_m}{(\lambda_\ell - \lambda_n)^2 (\lambda_m - \lambda_n)^2}.$$

Applying Lemma 2.2.2 with $\sigma = 4$, we obtain

$$S = \sum_{m=1}^{N} \delta_m^3 t_m^2 \left(\sum_{\substack{n=1 \\ n\neq m}}^{N} \frac{\delta_n}{(\lambda_n - \lambda_m)^4} \right) \leq \sum_{m=1}^{N} \delta_m^3 t_m^2 \left(\frac{\pi^4}{45\delta_m^3} \right) = \frac{\pi^4}{45} V.$$

Applying Lemma 2.3.2, we obtain

$$T = \sum_{\ell=1}^{N} \sum_{\substack{m=1 \\ m\neq\ell}}^{N} \delta_\ell^{\frac{3}{2}} \delta_m^{\frac{3}{2}} t_\ell t_m \left(\sum_{\substack{n=1 \\ n\neq\ell \\ n\neq m}}^{N} \frac{\delta_n}{\left(\lambda_n - \lambda_\ell\right)^2 \left(\lambda_n - \lambda_m\right)^2} \right)$$

$$\leq \sum_{\ell=1}^{N} \sum_{\substack{m=1 \\ m\neq\ell}}^{N} \delta_\ell^{\frac{3}{2}} \delta_m^{\frac{3}{2}} t_\ell t_m \left(\frac{\pi^2 \left(\delta_\ell + \delta_m\right)}{3\delta_\ell \delta_m \left(\lambda_\ell - \lambda_m\right)^2} \right) = \frac{2\pi^2}{3} U.$$

So $U^2 \leq V\left(\frac{\pi^4}{45} V + \frac{2\pi^2}{3} U\right)$. Solving this gives $U \leq \left(\frac{\pi^2}{3} + \frac{\pi^2}{3}\sqrt{\frac{6}{5}}\right) V$. □

2.4 Proofs of Theorems 2.1.3 and 2.1.4

2.4.1 Proof of Theorem 2.1.3

For real numbers $0 \leq \alpha \leq 2$ and positive integers N, let $\overline{C}(\alpha, N)$ be the minimum of all constants $C(\alpha, N)$ for which the inequality

$$\sum_{m=1}^{N} \sum_{\substack{n=1 \\ n\neq m}}^{N} \frac{\delta_m^{2-\alpha} \delta_n^{\alpha} t_m t_n}{\left(\lambda_m - \lambda_n\right)^2} \leq C(\alpha, N) \sum_{n=1}^{N} t_n^2 \tag{2.4.1}$$

holds for all choices of a strictly increasing sequence $(\lambda_k)_{k=-\infty}^{\infty}$ of real numbers,

$$\delta_k := \min\left\{\lambda_k - \lambda_{k-1}, \lambda_{k+1} - \lambda_k\right\},$$

and nonnegative real numbers $t_1, \ldots, t_N$.

Proposition 2.4.1. (1) *For real numbers* $0 \leq \alpha \leq 2$, *we have* $\overline{C}(\alpha, 1) = 0$ *and* $\overline{C}(\alpha, 2) = 1$.

(2) *For real numbers* $0 \leq \alpha \leq 2$ *and positive integers* N, *we have* $\overline{C}(\alpha, N) \leq \overline{C}(\alpha, N+1)$.

(3) *For real numbers* $0 \leq \alpha \leq 2$ *and positive integers* N, *we have* $0 \leq \overline{C}(\alpha, N) \leq N - 1$.

(4) *For real numbers* $0 \le \alpha \le 2$*, we have* $\overline{C}(\alpha) = \lim_{N\to\infty} \overline{C}(\alpha, N)$*.*

Proof. (1) If $N = 1$, the left side of (2.4.1) is 0. So $\overline{C}(\alpha, 1) = 0$. If $N = 2$, the left side of (2.4.1) is $2t_1t_2$. So $\overline{C}(\alpha, 2) = 1$.

(2) Let $t_1, \ldots, t_N$ be nonnegative real numbers, and let $t_{N+1} = 0$. Then

$$\sum_{m=1}^{N} \sum_{\substack{n=1 \\ n\neq m}}^{N} \frac{\delta_m^{2-\alpha}\delta_n^{\alpha} t_m t_n}{(\lambda_m - \lambda_n)^2} = \sum_{m=1}^{N+1} \sum_{\substack{n=1 \\ n\neq m}}^{N+1} \frac{\delta_m^{2-\alpha}\delta_n^{\alpha} t_m t_n}{(\lambda_m - \lambda_n)^2} \le \overline{C}(\alpha, N+1) \sum_{n=1}^{N+1} t_n^2 = \overline{C}(\alpha, N+1) \sum_{n=1}^{N} t_n^2.$$

So $\overline{C}(\alpha, N) \le \overline{C}(\alpha, N+1)$.

(3) We have

$$\sum_{m=1}^{N} \sum_{\substack{n=1 \\ n\neq m}}^{N} \frac{\delta_m^{2-\alpha}\delta_n^{\alpha} t_m t_n}{(\lambda_m - \lambda_n)^2} \le \sum_{m=1}^{N} \sum_{\substack{n=1 \\ n\neq m}}^{N} t_m t_n \le \sum_{m=1}^{N} \sum_{\substack{n=1 \\ n\neq m}}^{N} \frac{t_m^2 + t_n^2}{2} = (N-1) \sum_{n=1}^{N} t_n^2.$$

So $\overline{C}(\alpha, N) \le N - 1$. On the other hand, from (2) and (1), we have $\overline{C}(\alpha, N) \ge \overline{C}(\alpha, 1) = 0$.

(4) Since (2.4.1) holds with $C(\alpha, N) = \overline{C}(\alpha)$, it follows that $\overline{C}(\alpha, N) \le \overline{C}(\alpha)$ for all N. Hence $\lim_{N\to\infty} \overline{C}(\alpha, N) \le \overline{C}(\alpha)$. On the other hand, by (2), $\lim_{N\to\infty} \overline{C}(\alpha, N) = \sup_N \overline{C}(\alpha, N)$. So (2.1.8) holds with $C(\alpha) = \lim_{N\to\infty} \overline{C}(\alpha, N)$. Hence $\overline{C}(\alpha) \le \lim_{N\to\infty} \overline{C}(\alpha, N)$. □

Proposition 2.4.2. (1) *For real numbers* $0 \le \alpha \le 2$ *and integers* $N \ge 2$*, we have* $\overline{C}(\alpha, N) = \overline{C}(2 - \alpha, N) \ge 1$*.*

(2) *For real numbers* $0 \le \alpha_1 < \alpha_2 \le 2$ *and* $0 < \theta < 1$*, and for positive integers* N*, we have*

$$\overline{C}\left(\theta\alpha_1 + (1-\theta)\alpha_2, N\right) \le \overline{C}\left(\alpha_1, N\right)^{\theta} \overline{C}\left(\alpha_2, N\right)^{1-\theta}.$$

(3) *For real numbers* $0 \le \alpha_1 < \alpha_2 \le 1$ *and positive integers* N*, we have* $\overline{C}\left(\alpha_1, N\right) \ge \overline{C}\left(\alpha_2, N\right)$*.*

(4) *For real numbers* $0 \leq \alpha < \frac{1}{2}$ *and integers* $N \geq 2$, *we have* $\overline{C}(\alpha, N) \gg N^{\frac{1}{2}-\alpha}$.

Proof. (1) The left side of (2.4.1) is unchanged on replacing α by $2-\alpha$. It follows that $\overline{C}(\alpha, N) = \overline{C}(2-\alpha, N)$. In addition, by Proposition 2.4.1, we see that $\overline{C}(\alpha, N) \geq \overline{C}(\alpha, 2) = 1$.

(2) Let $\alpha = \theta\alpha_1 + (1-\theta)\alpha_2$. Apply Hölder's inequality:

$$\sum_{m=1}^{N}\sum_{\substack{n=1\\ n\neq m}}^{N} \frac{\delta_m^{2-\alpha}\delta_n^{\alpha} t_m t_n}{(\lambda_m - \lambda_n)^2} \leq \left(\sum_{m=1}^{N}\sum_{\substack{n=1\\ n\neq m}}^{N} \frac{\delta_m^{2-\alpha_1}\delta_n^{\alpha_1} t_m t_n}{(\lambda_m - \lambda_n)^2}\right)^{\theta} \left(\sum_{m=1}^{N}\sum_{\substack{n=1\\ n\neq m}}^{N} \frac{\delta_m^{2-\alpha_2}\delta_n^{\alpha_2} t_m t_n}{(\lambda_m - \lambda_n)^2}\right)^{1-\theta}$$

$$\leq \overline{C}\left(\alpha_1, N\right)^{\theta} \overline{C}\left(\alpha_2, N\right)^{1-\theta} \sum_{n=1}^{N} t_n^2.$$

So $\overline{C}(\alpha, N) \leq \overline{C}\left(\alpha_1, N\right)^{\theta} \overline{C}\left(\alpha_2, N\right)^{1-\theta}$.

(3) Let $\theta = \frac{2-\alpha_1-\alpha_2}{2(1-\alpha_1)}$. Then $0 < \theta < 1$ and $\alpha_2 = \theta\alpha_1 + (1-\theta)\left(2-\alpha_1\right)$. By (2), we have

$$\overline{C}\left(\alpha_2, N\right) = \overline{C}\left(\theta\alpha_1 + (1-\theta)\left(2-\alpha_1\right), N\right) \leq \overline{C}\left(\alpha_1, N\right)^{\theta} \overline{C}\left(2-\alpha_1, N\right)^{1-\theta}.$$

The last quantity is equal to $\overline{C}\left(\alpha_1, N\right)$ by (1).

(4) We choose $\lambda_k = k$ for $k \leq 1$ and $\lambda_{2+\ell} = 2 + \frac{\ell}{N}$ for $\ell \geq 0$. Then $\delta_k = 1$ for $k \leq 1$ and $\delta_{2+\ell} = \frac{1}{N}$ for $\ell \geq 0$. Choose $t_1 = \sqrt{\frac{N+1}{2N}}$ and $t_n = \frac{1}{\sqrt{2N}}$ for $2 \leq n \leq N$. So $\sum_{n=1}^{N} t_n^2 = 1$, and (2.4.1) yields

$$C(\alpha, N) \geq \sum_{m=1}^{N}\sum_{\substack{n=1\\ n\neq m}}^{N} \frac{\delta_m^{2-\alpha}\delta_n^{\alpha} t_m t_n}{(\lambda_m - \lambda_n)^2} \geq \sum_{n=2}^{N} \frac{\delta_1^{2-\alpha}\delta_n^{\alpha} t_1 t_n}{(\lambda_1 - \lambda_n)^2} = \sum_{n=2}^{N} \frac{\sqrt{N+1}}{2N^{\alpha+1}\left(1+\frac{n-2}{N}\right)^2}.$$

The last quantity is $\gg N^{\frac{1}{2}-\alpha}$ for $N \geq 2$. Hence $\overline{C}(\alpha, N) \gg N^{\frac{1}{2}-\alpha}$ for $N \geq 2$. □

Proof of Theorem 2.1.3. The result follows as we let $N \to \infty$ in Proposition 2.4.2. □

2.4.2 Proof of Theorem 2.1.4

Proposition 2.4.3. *Let* $(\lambda_k)_{k=-\infty}^{\infty}$ *be a strictly increasing sequence of real numbers. Denote by* δ_k *the minimum between* $\lambda_k - \lambda_{k-1}$ *and* $\lambda_{k+1} - \lambda_k$. *Then for any sequence* $(t_1, \ldots, t_N)$ *of nonnegative real numbers,*

$$\sum_{m=1}^{N} \sum_{\substack{n=1 \\ n \neq m}}^{N} \frac{\delta_m \delta_n t_m t_n}{(\lambda_m - \lambda_n)^2} \leq \frac{\pi^2}{3} \sum_{n=1}^{N} t_n^2. \tag{2.4.2}$$

Proof. By the inequality of arithmetic and geometric means,

$$\sum_{m=1}^{N} \sum_{\substack{n=1 \\ n \neq m}}^{N} \frac{\delta_m \delta_n t_m t_n}{(\lambda_m - \lambda_n)^2} \leq \sum_{m=1}^{N} \sum_{\substack{n=1 \\ n \neq m}}^{N} \frac{\delta_m \delta_n \left(t_m^2 + t_n^2\right)}{2\left(\lambda_m - \lambda_n\right)^2} = \sum_{n=1}^{N} \delta_n t_n^2 \left(\sum_{\substack{m=1 \\ m \neq n}}^{N} \frac{\delta_m}{(\lambda_m - \lambda_n)^2} \right).$$

By Lemma 2.2.2, the right side above is $\leq \sum_{n=1}^{N} \delta_n t_n^2 \left(\frac{\pi^2}{3\delta_n}\right) = \frac{\pi^2}{3} \sum_{n=1}^{N} t_n^2$. ☐

Proof of Theorem 2.1.4. Proposition 2.4.3 shows $\overline{C}(1) \leq \frac{\pi^2}{3}$. Now taking $\lambda_n = n$ and $t_n = \frac{1}{\sqrt{N}}$ in (2.4.1) yields

$$\overline{C}(\alpha, N) \geq \frac{2}{N} \sum_{n=1}^{N-1} \frac{N-n}{n^2} = 2 \sum_{n=1}^{N-1} \frac{1}{n^2} - \frac{2}{N} \sum_{n=1}^{N-1} \frac{1}{n}.$$

Letting $N \to \infty$ gives $\overline{C}(\alpha) \geq \frac{\pi^2}{3}$ for all $0 \leq \alpha \leq 2$. Hence $\overline{C}(1) = \frac{\pi^2}{3}$. ☐

2.5 Proof of Theorem 2.1.5

Let M denote a positive integer, and let $x_1, \ldots, x_M$ denote real numbers, distinct modulo 1. Put

$$d_m := \min_{n \neq m} \|x_n - x_m\|,$$

where $\|x\| = \min_{k \in \mathbb{Z}} |x - k|$ denotes the distance between x and a nearest integer. In the case that $M = 1$, we let $d_1 := 1$. Let $\tau_1, \ldots, \tau_M$ denote nonnegative real numbers.

Lemma 2.5.1. *The inequality* (2.3.1) *holds (for all N, λ_n, δ_n, and t_n) if and only if the inequality*

$$\frac{1}{3}\sum_{m=1}^{M} d_m^2\tau_m^2 + \sum_{m=1}^{M}\sum_{\substack{n=1\\ n\neq m}}^{M} \frac{d_m^{\frac{3}{2}} d_n^{\frac{1}{2}}\tau_m\tau_n}{\sin^2\left(\pi\left(x_m - x_n\right)\right)} \leq \frac{c_3}{\pi^2}\sum_{m=1}^{M}\tau_m^2 \tag{2.5.1}$$

holds for all positive integer M, distinct real numbers $x_1, \ldots, x_M$ modulo 1,

$$d_m := \min\left\{|x_n - x_m - k| : k \in \mathbb{Z}\right\} \setminus \{0\}, \tag{2.5.2}$$

and nonnegative real numbers $\tau_1, \ldots, \tau_M$.

Proof. ($\Rightarrow$) Suppose that (2.3.1) holds. Let $x_1, \ldots, x_M$ be real numbers, distinct modulo 1. By symmetry in $x_1, \ldots, x_M$, we may assume without loss of generality that $x_1 < \cdots < x_M < x_1 + 1$. Let d_m be given by (2.5.2). Let $\tau_1, \ldots, \tau_M$ be nonnegative real numbers. Let K be a positive integer. We apply (2.3.1) with $N = KM$. For integers k and m with $1 \leq m \leq M$, put $\lambda_{kM+m} = k + x_m$. Then $\delta_{kM+m} = d_m$. If $0 \leq k < K$, put $t_{kM+m} = \tau_m$. On inserting into (2.3.1), we obtain

$$2\sum_{m=1}^{M}\sum_{k=1}^{K-1}\frac{(K-k)d_m^2\tau_m^2}{k^2} + \sum_{m=1}^{M}\sum_{\substack{n=1\\ n\neq m}}^{M}\sum_{\substack{k\in\mathbb{Z}\\ |k|<K}}\frac{(K-|k|)d_m^{\frac{3}{2}}d_n^{\frac{1}{2}}\tau_m\tau_n}{\left(x_m - x_n - k\right)^2} \leq c_3 K\sum_{m=1}^{M}\tau_m^2. \tag{2.5.3}$$

Now, since the series

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6} \quad \text{and} \quad \sum_{k\in\mathbb{Z}}\frac{1}{(x-k)^2} = \frac{\pi^2}{\sin^2(\pi x)}$$

converge, it follows that they are (C, 1) summable to the same values (see, e.g., [2, p. 10]), which is to say that

$$\lim_{K\to\infty}\frac{1}{K}\sum_{k=1}^{K-1}\frac{K-k}{k^2} = \frac{\pi^2}{6} \quad \text{and} \quad \lim_{K\to\infty}\frac{1}{K}\sum_{\substack{k\in\mathbb{Z}\\ |k|<K}}\frac{K-|k|}{(x-k)^2} = \frac{\pi^2}{\sin^2(\pi x)}.$$

Hence, dividing (2.5.3) by $\pi^2 K$ and letting $K \to \infty$ gives (2.5.1).

($\Leftarrow$) Suppose that (2.5.1) holds. Let $(\lambda_k)_{k=-\infty}^{\infty}$ be a strictly increasing sequence of real numbers, and let $\delta_k := \min\left\{\lambda_k - \lambda_{k-1}, \lambda_{k+1} - \lambda_k\right\}$. Let $t_1, \ldots, t_N$ be nonnegative real numbers. Let $0 < \varepsilon < \frac{1}{2(\lambda_N - \lambda_0)}$. We apply (2.5.1) with $M = N$. For positive integers $n \le N$, put $x_n = \varepsilon\lambda_n$ and $\tau_n = t_n$. Then $d_n \ge \varepsilon\delta_n$, and (2.5.1) implies

$$\frac{\varepsilon^2}{3}\sum_{n=1}^{N}\delta_n^2 t_n^2 + \sum_{m=1}^{N}\sum_{\substack{n=1\\ n\neq m}}^{N}\frac{\varepsilon^2\delta_m^{\frac{3}{2}}\delta_n^{\frac{1}{2}}t_m t_n}{\sin^2\left(\pi\varepsilon\left(\lambda_m-\lambda_n\right)\right)} \le \frac{c_3}{\pi^2}\sum_{n=1}^{N}t_n^2.$$

On multiplying by π^2 and letting $\varepsilon \to 0^+$, we obtain (2.3.1). □

Lemma 2.5.2. *For positive real numbers $B < 1$ and positive integers L, we have*

$$\sum_{\ell=1}^{L}\frac{L+1-\ell}{\sin^2\left(\frac{\pi\ell B}{L}\right)} = \frac{L^3}{6B^2} - \frac{L^2\log L}{\pi^2 B^2} + O_B\left(L^2\right). \tag{2.5.4}$$

Proof. From the identity $\frac{\pi^2}{\sin^2(\pi x)} = \sum_{k\in\mathbb{Z}}\frac{1}{(x-k)^2}$, we see that if $0 < x \le B$, then

$$\frac{\pi^2}{\sin^2(\pi x)} - \frac{1}{x^2} = \sum_{n=1}^{\infty}\left(\frac{1}{(n+x)^2} + \frac{1}{(n-x)^2}\right) < \sum_{n=1}^{\infty}\left(\frac{1}{n^2} + \frac{1}{(n-B)^2}\right).$$

Hence, for $0 < x \le B$, we have $\frac{1}{\sin^2(\pi x)} = \frac{1}{\pi^2 x^2} + O_B(1)$. Applying this estimate to each term on the left side of (2.5.4), we obtain

$$\begin{aligned}\sum_{\ell=1}^{L}\frac{L+1-\ell}{\sin^2\left(\frac{\pi\ell B}{L}\right)} &= \sum_{\ell=1}^{L}\frac{L^2(L+1-\ell)}{\pi^2\ell^2 B^2} + O_B\left(\sum_{\ell=1}^{L}(L+1-\ell)\right)\\ &= \frac{L^2(L+1)}{\pi^2 B^2}\sum_{\ell=1}^{L}\frac{1}{\ell^2} - \frac{L^2}{\pi^2 B^2}\sum_{\ell=1}^{L}\frac{1}{\ell} + O_B\left(L^2\right).\end{aligned}$$

Since $\sum_{\ell=1}^{L}\frac{1}{\ell^2} = \frac{\pi^2}{6} + O\left(\frac{1}{L}\right)$ and $\sum_{\ell=1}^{L}\frac{1}{\ell} = \log L + O(1)$, the result (2.5.4) follows. □

Proof of Theorem 2.1.5. To prove a lower bound for $\overline{C}\left(\frac{1}{2}\right)$, we apply (2.5.1) with particular sets of values. Let K be a positive integer. Let A and B be positive real numbers such that $(K+1)A + B = 1$. Let $L \ge \frac{B}{A}$ be an integer. We apply (2.5.1) with $M = K + L + 1$. Choose $x_k = kA$ for $1 \le k \le K$ and $x_{K+\ell+1} = (K+1)A + \frac{\ell B}{L}$ for $0 \le \ell \le L$. Then $d_k = A$ for $1 \le k \le K$ and $d_{K+\ell+1} = \frac{B}{L}$ for $0 \le \ell \le L$. Choose $\tau_k = \frac{1}{\sqrt{K}}$ for $1 \le k \le K$ and $\tau_{K+\ell+1} = \frac{u}{\sqrt{L+1}}$ for $0 \le \ell \le L$ where u is a nonnegative real number to be chosen later. Then (2.5.1) implies

$$\frac{A^2}{3}+\frac{u^2B^2}{3L^2}+\frac{2A^2}{K}\sum_{k=1}^{K-1}\frac{K-k}{\sin^2(\pi kA)}+\frac{2u^2B^2}{L^2(L+1)}\sum_{\ell=1}^{L}\frac{L+1-\ell}{\sin^2\left(\frac{\pi\ell B}{L}\right)} \tag{2.5.5}$$
$$+u\sqrt{\frac{AB}{KL(L+1)}}\left(A+\frac{B}{L}\right)\sum_{k=1}^{K}\sum_{\ell=0}^{L}\frac{1}{\sin^2\left(\pi\left(kA+\frac{\ell B}{L}\right)\right)}\leq\frac{c_3}{\pi^2}\left(1+u^2\right).$$

We observe that

$$\begin{aligned}\lim_{L\to\infty}\frac{1}{L}\sum_{k=1}^{K}\sum_{\ell=0}^{L}\frac{1}{\sin^2\left(\pi\left(kA+\frac{\ell B}{L}\right)\right)}&=\sum_{k=1}^{K}\int_0^1\frac{dx}{\sin^2(\pi(kA+Bx))}\\&=\frac{1}{\pi B}\sum_{k=1}^{K}(\cot(\pi(K+1-k)A)+\cot(\pi kA))\\&=\frac{2}{\pi B}\sum_{k=1}^{K}\cot(\pi kA).\end{aligned}$$

Now we let $L\to\infty$ in (2.5.5) and use the above estimate and Lemma 2.5.2, obtaining

$$\frac{A^2}{3}+\frac{2A^2}{K}\sum_{k=1}^{K-1}\frac{K-k}{\sin^2(\pi kA)}+\frac{u^2}{3}+\frac{2u}{\pi}\sqrt{\frac{A^3}{BK}}\sum_{k=1}^{K}\cot(\pi kA)\leq\frac{c_3}{\pi^2}\left(1+u^2\right).$$

That is,

$$g(u):=\frac{\kappa_0+\kappa_1u+\frac{u^2}{3}}{1+u^2}\leq\frac{c_3}{\pi^2}, \tag{2.5.6}$$

where κ_0 and κ_1 depend on A, B, and K and are given by

$$\kappa_0:=\frac{A^2}{3}+\frac{2A^2}{K}\sum_{k=1}^{K-1}\frac{K-k}{\sin^2(\pi kA)}\quad\text{and}\quad\kappa_1:=\frac{2}{\pi}\sqrt{\frac{A^3}{BK}}\sum_{k=1}^{K}\cot(\pi kA).$$

We find that $g(u)$ is maximized on $u\geq 0$ at

$$u=u_0:=\frac{1}{\kappa_1}\left(\frac{1}{3}-\kappa_0+\sqrt{\left(\frac{1}{3}-\kappa_0\right)^2+\kappa_1^2}\right).$$

On inserting $u=u_0$ in (2.5.6), we get

$$G_K(A):=\frac{1}{2}\left(\frac{1}{3}+\kappa_0+\sqrt{\left(\frac{1}{3}-\kappa_0\right)^2+\kappa_1^2}\right)\leq\frac{c_3}{\pi^2}.$$

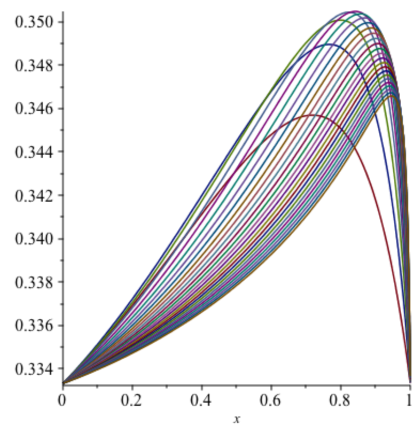

Figure 2.1: The graphs $y = G_K\left(\frac{x}{K+1}\right)$, $0 < x < 1$ for $K = 1, \ldots, 25$ in the (x, y)-plane.

Figure 2.1 shows the plot of $G_K\left(\frac{x}{K+1}\right)$ for $K = 1, \ldots, 25$ and $0 < x < 1$. We find

$$G_5(0.14) > 0.35047.$$

By Lemma 2.5.1, this gives the lower bound $\frac{c_3}{\pi^2} \geq 0.35047$ for any absolute constant c_3 such that (2.3.1) holds. Since (2.3.1) holds with $c_3 = \overline{C}\left(\frac{1}{2}\right)$, the result follows. □

2.6 Proofs of Theorems 2.1.6 and 2.1.7

Proof of Theorem 2.1.6. Let $[u_1, \ldots, u_N]^\top$ be an eigenvector of $H = [h_{mn}]$, where h_{mn} are given by (2.2.1) with

$$w_n = \sqrt{\delta_n}\left(\frac{D}{3\delta_n} + \frac{2}{3}\sqrt{\frac{\delta_n}{D}}\right)^{-\frac{1}{4}},$$

and let $i\mu$ be the purely imaginary eigenvalue associated with this eigenvector. We apply Lemma 2.2.1. On multiplying (2.2.4) by $\frac{|u_m|}{w_m}$ and summing over m, we get

$$\begin{aligned}\mu^2\sum_{m=1}^{N}\frac{|u_m|^3}{w_m} &= \sum_{m=1}^{N}\sum_{\substack{n=1\\n\neq m}}^{N}\frac{w_m w_n^2\,|u_m|\,|u_n|^2}{(\lambda_m-\lambda_n)^2}+2\sum_{m=1}^{N}\sum_{\substack{n=1\\n\neq m}}^{N}\frac{w_m^2 w_n\,|u_m|\,\Re\left(\overline{u_m}u_n\right)}{(\lambda_m-\lambda_n)^2}\\ &\le \sum_{m=1}^{N}\sum_{\substack{n=1\\n\neq m}}^{N}\frac{w_m w_n^2\,|u_m|\,|u_n|^2}{(\lambda_m-\lambda_n)^2}+2\sum_{m=1}^{N}\sum_{\substack{n=1\\n\neq m}}^{N}\frac{w_m^2 w_n\,|u_m|^2\,|u_n|}{(\lambda_m-\lambda_n)^2}\\ &= 3\sum_{m=1}^{N}\sum_{\substack{n=1\\n\neq m}}^{N}\frac{w_m w_n^2\,|u_m|\,|u_n|^2}{(\lambda_m-\lambda_n)^2}.\end{aligned}$$

Now, let us apply Hölder's inequality with the last quantity:

$$\begin{aligned}&3\sum_{m=1}^{N}\sum_{\substack{n=1\\n\neq m}}^{N}\frac{w_m w_n^2\,|u_m|\,|u_n|^2}{(\lambda_m-\lambda_n)^2}\\ &\le 3\left(\sum_{m=1}^{N}\sum_{\substack{n=1\\n\neq m}}^{N}\frac{\delta_n w_m^3\,|u_m|^3}{\delta_m^2\,(\lambda_m-\lambda_n)^2}\right)^{\frac{1}{3}}\left(\sum_{m=1}^{N}\sum_{\substack{n=1\\n\neq m}}^{N}\frac{\delta_m w_n^3\,|u_n|^3}{\delta_n^{\frac{1}{2}}\,(\lambda_m-\lambda_n)^2}\right)^{\frac{2}{3}}\\ &\le \pi^2\left(\sum_{m=1}^{N}\frac{w_m^3|u_m|^3}{\delta_m^3}\right)^{\frac{1}{3}}\left(\sum_{n=1}^{N}\frac{w_n^3|u_n|^3}{\delta_n^{\frac{3}{2}}}\right)^{\frac{2}{3}},\end{aligned}$$

where we use Lemma 2.2.2. By the AM–GM inequality, the right side above is less than or equal to

$$\pi^2\left(\frac{D}{3}\sum_{m=1}^{N}\frac{w_m^3|u_m|^3}{\delta_m^3}+\frac{2}{3D^{\frac{1}{2}}}\sum_{n=1}^{N}\frac{w_n^3|u_n|^3}{\delta_n^{\frac{3}{2}}}\right)=\pi^2\sum_{m=1}^{N}\frac{w_m^3}{\delta_m^2}\left(\frac{D}{3\delta_m}+\frac{2}{3}\sqrt{\frac{\delta_m}{D}}\right)|u_m|^3.$$

Combining the above, we obtain

$$\mu^2\sum_{m=1}^{N}\frac{|u_m|^3}{w_m}\le\pi^2\sum_{m=1}^{N}\frac{w_m^3}{\delta_m^2}\left(\frac{D}{3\delta_m}+\frac{2}{3}\sqrt{\frac{\delta_m}{D}}\right)|u_m|^3=\pi^2\sum_{m=1}^{N}\frac{|u_m|^3}{w_m}.$$

Hence $\mu\le\pi$, and (2.1.9) follows. □

Proof of Theorem 2.1.7. Let $f_1,\dots,f_N$ be positive real numbers. By Cauchy's inequality, the right side of (2.1.9) is

$$\pi \sum_{n=1}^{N} \frac{|z_n|^2}{\delta_n} \sqrt{\frac{D}{3\delta_n} + \frac{2}{3}\sqrt{\frac{\delta_n}{D}}} \leq \pi \left(\sum_{n=1}^{N} \frac{f_n |z_n|^2}{\delta_n} \right)^{\frac{1}{2}} \left(\sum_{n=1}^{N} \frac{|z_n|^2}{f_n \delta_n} \left(\frac{D}{3\delta_n} + \frac{2}{3}\sqrt{\frac{\delta_n}{D}} \right) \right)^{\frac{1}{2}}$$

$$= \pi \left(\sum_{n=1}^{N} \frac{f_n |z_n|^2}{\delta_n} \right)^{\frac{1}{2}} \left(\frac{D}{3} \sum_{n=1}^{N} \frac{|z_n|^2}{f_n \delta_n^2} + \frac{2}{3\sqrt{D}} \sum_{n=1}^{N} \frac{|z_n|^2}{f_n \sqrt{\delta_n}} \right)^{\frac{1}{2}}.$$

The result (2.1.10) follows on taking

$$D = \left(\sum_{n=1}^{N} \frac{|z_n|^2}{f_n \sqrt{\delta_n}} \right)^{\frac{2}{3}} \left(\sum_{n=1}^{N} \frac{|z_n|^2}{f_n \delta_n^2} \right)^{-\frac{2}{3}}.$$

$\square$

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CHAPTER 3

Generalized Factorials allowing Composite Bases

3.0 Abstract

This chapter presents a generalized version of Bhargava's theory of factorial ideals based on $\mathfrak{p}$-orderings of a nonempty subset S of a Dedekind ring R for all prime ideals $\mathfrak{p}$ in R. We treat the ring $R = \mathbb{Z}$ and generalize Bhargava's theory to $\mathfrak{b}$-orderings of a nonempty subset S of $\mathbb{Z}$ for all nontrivial proper ideals $\mathfrak{b}$ in $\mathbb{Z}$. We define generalized factorials $[k]!_{S,\mathcal{T}}$, where $\mathcal{T}$ is a subset of $\mathcal{B} := \{b \in \mathbb{Z} : b \geq 2\}$ which corresponds to the set of all nontrivial proper ideals of $\mathbb{Z}$. We treat in detail the special case $(S, \mathcal{T}) = (\mathbb{Z}, \mathcal{B})$ and compute its associated binomial coefficients $\left[{k \atop \ell}\right]_{\mathbb{Z},\mathcal{B}}$.

3.1 Introduction

3.1.1 Bhargava's generalized factorials

Beginning in 1997, Bhargava developed a theory of generalized factorials for a class of commutative rings R that he called Dedekind rings. These rings are quotients of Dedekind domains and include all Dedekind domains. Bhargava's generalized factorials are associated to nonempty sets S of elements of R and to the set of all prime ideals of R,

$$\operatorname{Spec}(R) = \{\mathfrak{p} : \mathfrak{p} \text{ is a prime ideal in } R\}. \tag{3.1.1}$$

Bhargava's generalized factorials $k!_S$ are ideals in R.

For each prime ideal $\mathfrak{p}$ in R and a nonempty subset S of R, he assigned an associated $\mathfrak{p}$-sequence $(\nu_k(S,\mathfrak{p}))_{k=0}^{\infty}$ of S in which $\nu_k(S,\mathfrak{p})$ is a power of $\mathfrak{p}$. He constructed the associated $\mathfrak{p}$-sequence using $\mathfrak{p}$-orderings of S. The generalized factorials of S, denoted $k!_S$, are defined as in [3, Definition 7] by

$$k!_S := \prod_{\mathfrak{p}\in \operatorname{Spec}(R)} \nu_k(S,\mathfrak{p}).$$

We can write

$$\nu_k(S,\mathfrak{p}) = \mathfrak{p}^{\alpha_k(S,\mathfrak{p})}, \tag{3.1.2}$$

where $\alpha_k(S,\mathfrak{p}) \in \mathbb{N}\cup\{\infty\}$, with the conventions $\mathfrak{p}^0 = R$ and $\mathfrak{p}^\infty = (0)$.

Bhargava showed his factorials have many applications to many problems in commutative algebra, to finding rings of integer-valued polynomials on a set S, and to finding good bases for suitable function spaces, see also [4].

Bhargava originally developed his generalized factorials for the ring of integers $\mathbb{Z}$, in which case $\operatorname{Spec}(\mathbb{Z}) = \{(p) : p \text{ is a prime number}\}$, which we may identify with $\mathcal{P} := \{2, 3, 5, \dots\}$, the set of all prime numbers. Bhargava [3] gave details for the case $R = \mathbb{Z}$. In this chapter we treat the case $R = \mathbb{Z}$, and we describe Bhargava's theory in this case, following [3].

Bhargava's key idea is the construction of $\mathfrak{p}$-orderings of S for any fixed prime ideal $\mathfrak{p}$. We describe it for the case $R = \mathbb{Z}$. Let $p \in \mathcal{P}$ be the prime number that generates $\mathfrak{p} \in \operatorname{Spec}(\mathbb{Z})$. A *p-ordering of S* is any sequence $\mathbf{a} = (a_i)_{i=0}^{\infty}$ of elements of S that can be formed recursively as follows:

(i) $a_0 \in S$ is chosen arbitrarily;

(ii) Given $a_j \in S$, $j = 0, \dots, i-1$, the next element $a_i \in S$ is chosen so that it minimizes the highest power of p dividing the product $\prod_{j=0}^{i-1}(a_i - a_j)$.

We note that:

(1) This construction does not give a unique p-ordering of S if $|S| > 1$.

(2) A p-ordering of S does not need to include all the elements of S.

Bhargava defines $\nu_i(S, p, \mathbf{a})$ to be the highest power of p dividing $\prod_{j=0}^{i-1}(a_i - a_j)$. That is, we may write

$$\nu_i(S, p, \mathbf{a}) = p^{\alpha_i(S,p,\mathbf{a})}, \tag{3.1.3}$$

where

$$\alpha_i(S, p, \mathbf{a}) := \mathrm{ord}_p\left(\prod_{j=0}^{i-1}(a_i - a_j)\right) \tag{3.1.4}$$

and $\mathrm{ord}_p(\cdot)$ is the additive p-adic valuation given by

$$\mathrm{ord}_p(k) := \sup\left\{\alpha \in \mathbb{N} : p^{\alpha} \text{ divides } k\right\}. \tag{3.1.5}$$

Bhargava calls the sequence $(\nu_i(S, p, \mathbf{a}))_{i=0}^{\infty}$ the *associated p-sequence of S corresponding to the p-ordering* $\mathbf{a}$. Bhargava [3, Theorem 5] showed

Theorem 3.1.1 (Bhargava [3])**.** *The associated p-sequence of S is independent of the choice of p-ordering.*

Therefore one may write $\nu_i(S, p) = \nu_i(S, p, \mathbf{a})$ as an invariant under the choice of p-ordering $\mathbf{a}$ and call $(\nu_i(S, p))_{i=0}^{\infty}$ the associated p-sequence of S.

Bhargava used this invariant to define his generalized factorials. The *factorial function associated to S*, denoted $k!_S$, is defined by

$$k!_S := \prod_p \nu_k(S, p). \tag{3.1.6}$$

Thus Bhargava's theory produces factorials via their prime factorizations.

In the special case $S = \mathbb{Z}$, Bhargava showed that the generalized factorials agree with the usual factorials. To do this, Bhargava [3, Proposition 6] showed

Theorem 3.1.2 (Bhargava [3]). *The natural ordering* $0, 1, 2, \dots$ *of the nonnegative integers forms a* p*-ordering of* $\mathbb{Z}$ *for all primes* p *simultaneously.*

From Theorem 3.1.2, Bhargava deduces that

$$\nu_k(\mathbb{Z}, p) = w_p\left(\prod_{j=0}^{k-1}(k-j)\right) = w_p(k!),$$

where $w_p(a)$ denotes the highest power of p dividing a (i.e., $w_p(a) = p^{\mathrm{ord}_p(a)}$).

Therefore

$$k!_{\mathbb{Z}} = \prod_p w_p(k!) = k!. \tag{3.1.7}$$

Bhargava also treated generalized binomial coefficients. Bhargava [3, Theorem 8] showed

Theorem 3.1.3 (Bhargava [3]). *For any nonnegative integers* k *and* ℓ*,* $(k+\ell)!_S$ *is a multiple of* $k!_S \ell!_S$*.*

In other words, the generalized binomial coefficients

$$\binom{k+\ell}{k}_S := \frac{(k+\ell)!_S}{k!_S \ell!_S}$$

are always integers.

3.2 Main results of this chapter

3.2.1 A generalization of Bhargava's theory in the ring $R = \mathbb{Z}$

We generalize Bhargava's theory of $\mathfrak{p}$-orderings for prime ideals $\mathfrak{p}$ in the ring $R = \mathbb{Z}$ to treat $\mathfrak{b}$-orderings for nontrivial proper ideals $\mathfrak{b}$ in $\mathbb{Z}$. The set of all nontrivial proper ideals of $\mathbb{Z}$ may be identified with the set

$$\mathcal{B} := \{b \in \mathbb{Z} : b \geq 2\} = \mathbb{N} \backslash \{0, 1\} \tag{3.2.1}$$

by the positive generators of the ideals. Here $\mathbb{N} := \{0, 1, 2, \dots\}$ is the set of all nonnegative integers.

Definition 3.2.1. Let $b \in \mathcal{B}$. Let S be a nonempty subset of the ring of integers $\mathbb{Z}$. We call a sequence $\mathbf{a} = (a_i)_{i=0}^{\infty}$ of elements of S an *admissible b-ordering of S* if for all $i = 1, 2, 3, \ldots,$

$$\sum_{j=0}^{i-1} \mathrm{ord}_b \left(a_i - a_j\right) = \min_{s \in S} \sum_{j=0}^{i-1} \mathrm{ord}_b \left(s - a_j\right), \tag{3.2.2}$$

where $\mathrm{ord}_b(k)$ is defined for $k \in \mathbb{Z}$ by

$$\mathrm{ord}_b(k) := \sup \left\{\alpha \in \mathbb{N} : b^\alpha \text{ divides } k\right\}. \tag{3.2.3}$$

Given any initial value $a_0 \in S$, one can find an admissible b-ordering with that initial value using the recurrence (3.2.2). There will be more than one admissible b-ordering of S, unless S is a singleton.

Definition 3.2.2. Let $b \in \mathcal{B}$. Let S be a nonempty subset of the ring $\mathbb{Z}$. Let $\mathbf{a} = (a_i)_{i=0}^{\infty}$ be an admissible b-ordering of S. The *associated b-sequence of S corresponding to* $\mathbf{a}$, denoted $(\alpha_i(S, b, \mathbf{a}))_{i=0}^{\infty}$, is defined by

$$\alpha_i(S, b, \mathbf{a}) := \sum_{j=0}^{i-1} \mathrm{ord}_b \left(a_i - a_j\right). \tag{3.2.4}$$

We note that:

(1) $\alpha_i(S, b, \mathbf{a}) \in \mathbb{N} \cup \{\infty\}$.

(2) If S is finite, then $\alpha_i(S, b, \mathbf{a}) = \infty$ for all $i \geq |S|$.

A main result of this chapter is that all associated b-sequences of a given set S are the same.

Theorem 3.2.3 (Well-definedness of the associated b-sequence of S)**.** *Let $b \in \mathcal{B}$. Let S be a nonempty subset of the ring $\mathbb{Z}$. Let $\mathbf{a}_1$ and $\mathbf{a}_2$ be admissible b-orderings of S. Then $\alpha_i\left(S, b, \mathbf{a}_1\right) = \alpha_i\left(S, b, \mathbf{a}_2\right)$ for all $i = 0, 1, 2, \ldots$.*

Bhargava proved Theorem 3.2.3 for all $b \in \mathcal{P}$, where $\mathcal{P} := \{2, 3, 5, \dots\}$ is the set of all primes. Bhargava's proofs, as presented in [2] and [3], do not extend to the case of composite bases b. We complete the proof of Theorem 3.2.3 in Subsection 3.5.3.

Theorem 3.2.3 provides the well-definedness of the associated b-sequence of S.

Definition 3.2.4. Let $b \in \mathcal{B}$. Let S be a nonempty subset of the ring $\mathbb{Z}$. We write $(\alpha_i(S, b))_{i=0}^{\infty}$ for *the associated b-sequence of S*, which is defined by

$$\alpha_i(S, b) := \alpha_i(S, b, \mathbf{a}) \tag{3.2.5}$$

for any admissible b-ordering $\mathbf{a}$ of S.

3.2.2 Generalized factorials and generalized positive integers

We now define generalized factorials associated to a nonempty subset S of the ring $\mathbb{Z}$ and a set of allowed bases (or generalized prime numbers) $\mathcal{T} \subseteq \mathcal{B} := \{2, 3, 4, \dots\}$. Here $\mathcal{B}$ corresponds to the set of all nontrivial proper ideals of the ring $\mathbb{Z}$.

Definition 3.2.5. Let S be a nonempty subset of the ring $\mathbb{Z}$. Let $\mathcal{T} \subseteq \mathcal{B}$. For $k = 0, 1, 2, \dots$, the *generalized factorial of k associated to S and $\mathcal{T}$*, denoted $[k]!_{S,\mathcal{T}}$, is defined by

$$[k]!_{S,\mathcal{T}} := \prod_{b \in \mathcal{T}} b^{\alpha_k(S,b)}. \tag{3.2.6}$$

We note that:

(1) If $\mathcal{T} = \varnothing$, then the product on the right side of (3.2.6) is empty; so $[k]!_{S,\varnothing} = 1$.

(2) If $\mathcal{T} \neq \varnothing$, then $[k]!_{S,\mathcal{T}}$ is a (finite) positive integer if and only if $k < |S|$.

(3) From (3.2.5) and (3.2.4), we see that $\alpha_0(S, b) = 0$ for all $b \in \mathcal{B}$; so $[0]!_{S,\mathcal{T}} = 1$.

(4) $[1]!_{S,\mathcal{T}} = 1$ as long as S is not contained in a single congruence class modulo b for any $b \in \mathcal{T}$.

Example 3.2.6. The special case $\mathcal{T} = \mathcal{P}$ agrees with Bhargava's generalized factorials [3]. It contains the usual factorial function as the special case $(S, \mathcal{T}) = (\mathbb{Z}, \mathcal{P})$:

$$[k]!_{\mathbb{Z},\mathcal{P}} = k!.$$

Proposition 3.2.7 (Ordering)**.** (1) *Let $S_1 \subseteq S_2$ be nonempty subsets of the ring $\mathbb{Z}$. Let $\mathcal{T} \subseteq \mathcal{B}$. Then for integers $0 \leq k < |S_1|$,*

$$[k]!_{S_2,\mathcal{T}} \quad \textit{divides} \quad [k]!_{S_1,\mathcal{T}}.$$

(2) *Let S be a nonempty subset of the ring $\mathbb{Z}$. Let $\mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \mathcal{B}$. Then for integers $0 \leq k < |S|$,*

$$[k]!_{S,\mathcal{T}_1} \quad \textit{divides} \quad [k]!_{S,\mathcal{T}_2}.$$

We prove Proposition 3.2.7 in Subsection 3.5.3.

Now, we define generalized positive integers $[n]_{S,\mathcal{T}}$.

Definition 3.2.8. Let S be a nonempty subset of the ring $\mathbb{Z}$. Let $\mathcal{T} \subseteq \mathcal{B}$. For positive integers $n < |S|$, the *nth generalized positive integer associated to S and $\mathcal{T}$*, denoted $[n]_{S,\mathcal{T}}$, is defined by

$$[n]_{S,\mathcal{T}} := \frac{[n]!_{S,\mathcal{T}}}{[n-1]!_{S,\mathcal{T}}}. \tag{3.2.7}$$

Theorem 3.2.9. *Let S be a nonempty subset of the ring $\mathbb{Z}$. Let $\mathcal{T} \subseteq \mathcal{B}$. Then for positive integers $n < |S|$, the generalized positive integer $[n]_{S,\mathcal{T}}$ is an integer.*

We prove Theorem 3.2.9 in Subsection 3.5.3.

3.2.3 Generalized binomial coefficients

Definition 3.2.10. Let S be a nonempty subset of the ring $\mathbb{Z}$. Let $\mathcal{T} \subseteq \mathcal{B}$. For integers $0 \leq \ell \leq k < |S|$, the *generalized binomial coefficient k choose ℓ associated* *to* S *and* $\mathcal{T}$, denoted $\left[{k \atop \ell}\right]_{S,\mathcal{T}}$, is defined by

$$\left[{k \atop \ell}\right]_{S,\mathcal{T}} := \frac{[k]!_{S,\mathcal{T}}}{[\ell]!_{S,\mathcal{T}}[k-\ell]!_{S,\mathcal{T}}}. \tag{3.2.8}$$

Theorem 3.2.11. *Let* S *be a nonempty subset of the ring* $\mathbb{Z}$*. Let* $\mathcal{T} \subseteq \mathcal{B}$*. Then for integers* $0 \leq \ell \leq k < |S|$*, the generalized binomial coefficient* $\left[{k \atop \ell}\right]_{S,\mathcal{T}}$ *is an integer.*

We prove Theorem 3.2.11 in Subsection 3.5.3.

3.2.4 The special case $(S,\mathcal{T}) = (\mathbb{Z},\mathcal{B})$: generalized factorials

We treat in detail the case $(S,\mathcal{T}) = (\mathbb{Z},\mathcal{B})$, in which both S and $\mathcal{T}$ are maximal.

Theorem 3.2.12. *The natural ordering* $0, 1, 2, \ldots$ *of the nonnegative integers forms an admissible* b*-ordering of* $S = \mathbb{Z}$ *for all* $b \in \mathcal{B}$ *simultaneously.*

We prove Theorem 3.2.12 in Section 3.6.

Theorem 3.2.13. *For* $k = 0, 1, 2, \ldots,$ *the generalized factorial of* k *associated to* $S = \mathbb{Z}$ *and* $\mathcal{T} = \mathcal{B}$ *is*

$$[k]!_{\mathbb{Z},\mathcal{B}} = \prod_{b=2}^{k} b^{\gamma(k,b)}, \tag{3.2.9}$$

where

$$\gamma(k,b) := \sum_{i=1}^{\infty} \left\lfloor \frac{k}{b^i} \right\rfloor. \tag{3.2.10}$$

Theorem 3.2.13 is analogous to Legendre's formula (also known as de Polignac's formula), which states that

$$\mathrm{ord}_p(k!) = \sum_{i=1}^{\infty} \left\lfloor \frac{k}{p^i} \right\rfloor \tag{3.2.11}$$

for all $p \in \mathcal{P}$. The right side of (3.2.11) is $\gamma(k,p)$. We prove Theorem 3.2.13 in Section 3.6.

Theorem 3.2.14. *For* $n = 1, 2, 3, \ldots,$ *the* n*th generalized positive integer associated to* $S = \mathbb{Z}$ *and* $\mathcal{T} = \mathcal{B}$ *is*

$$[n]_{\mathbb{Z},\mathcal{B}} = \prod_{b=2}^{n} b^{\mathrm{ord}_b(n)}, \tag{3.2.12}$$

where $\operatorname{ord}_b(n)$ *is the maximal* $\alpha \in \mathbb{N}$ *such that* b^α *divides* n*.*

Theorem 3.2.14 is analogous to the prime factorization of positive integers:

$$n = \prod_{p \in \mathcal{P}} p^{\operatorname{ord}_p(n)}.$$

We prove Theorem 3.2.14 in Section 3.6.

Knuth and Wilf [6] considered the notion of generalized binomial coefficients $\begin{bmatrix} k \\ \ell \end{bmatrix}_{\mathcal{C}}$ defined by a sequence $\mathcal{C} = (C_n)_{n=1}^{\infty}$ of positive integers by

$$\begin{bmatrix} k \\ \ell \end{bmatrix}_{\mathcal{C}} = \prod_{j=1}^{\ell} \frac{C_{k-j+1}}{C_j}$$

for integers $0 \leq \ell \leq k$. They showed that if $\mathcal{C}$ satisfies the condition

$$\gcd\left(C_m, C_n\right) = C_{\gcd(m,n)}$$

for all positive integers m and n, in which case $\mathcal{C}$ is said to be *regularly divisible*, then the generalized binomial coefficients are all integers. Here, the sequence $\mathcal{C}_1 = ([n]_{\mathbb{Z},\mathcal{B}})_{n=1}^{\infty}$ is not regularly divisible, since

$$\gcd\left([4]_{\mathbb{Z},\mathcal{B}}, [6]_{\mathbb{Z},\mathcal{B}}\right) = \gcd(16, 36) = 4 \quad \text{but} \quad [\gcd(4,6)]_{\mathbb{Z},\mathcal{B}} = [2]_{\mathbb{Z},\mathcal{B}} = 2.$$

However, the generalized binomial coefficients

$$\begin{bmatrix} k \\ \ell \end{bmatrix}_{\mathcal{C}_1} = \begin{bmatrix} k \\ \ell \end{bmatrix}_{\mathbb{Z},\mathcal{B}}$$

are all integers by Theorem 3.2.11.

3.2.5 The special case $(S, \mathcal{T}) = (\mathbb{Z}, \mathcal{B})$: generalized binomial coefficients

Theorem 3.2.15. *Let* $k \geq \ell$ *be nonnegative integers. Then:*

(1) *We have*

$$\begin{bmatrix} k \\ \ell \end{bmatrix}_{\mathbb{Z},\mathcal{B}} = \prod_{b=2}^{k} b^{\beta(k,\ell,b)}, \tag{3.2.13}$$

where

$$\beta(k,\ell,b) := \sum_{i=1}^{\infty}\left(\left\lfloor \frac{k}{b^i}\right\rfloor - \left\lfloor \frac{\ell}{b^i}\right\rfloor - \left\lfloor \frac{k-\ell}{b^i}\right\rfloor\right). \tag{3.2.14}$$

(2) *For* $b \in \mathcal{B}$*,*

$$\beta(k,\ell,b) = \frac{1}{b-1}\left(d_b(\ell) + d_b(k-\ell) - d_b(k)\right), \tag{3.2.15}$$

where $d_b(j)$ *is the sum of the base-*b *digits of* j*.*

We prove Theorem 3.2.15 in Section 3.6.

Corollary 3.2.16. *Let* $\overline{\overline{G}}_n$ *be the product of the generalized binomial coefficients associated to* $S = \mathbb{Z}$ *and* $\mathcal{T} = \mathcal{B}$ *in the* n*th row of Pascal's triangle:*

$$\overline{\overline{G}}_n := \prod_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\mathbb{Z},\mathcal{B}}. \tag{3.2.16}$$

Then for $n = 1, 2, 3, \ldots,$

$$\overline{\overline{G}}_n = \prod_{b=2}^{n} b^{\overline{\nu}(n,b)}, \tag{3.2.17}$$

where

$$\overline{\nu}(n,b) := \frac{2}{b-1}S_b(n) - \frac{n-1}{b-1}d_b(n) \tag{3.2.18}$$

and $S_b(n) := \sum_{j=1}^{n-1} d_b(j)$*.*

We prove Corollary 3.2.16 in Section 3.6.

3.3 Preliminaries

We derive identities for generalized factorials and generalized binomial coefficients which will be used in Subsection 3.5.3 and Section 3.6.

Proposition 3.3.1. *Let* S *be a nonempty subset of the ring* $\mathbb{Z}$*. Let* $\mathcal{T} \subseteq \mathcal{B}$*. Then for positive integers* $n < |S|$*,*

$$[n]_{S,\mathcal{T}} = \prod_{b\in\mathcal{T}} b^{\alpha_n(S,b)-\alpha_{n-1}(S,b)}. \tag{3.3.1}$$

Proof. This follows from Definitions 3.2.8 and 3.2.5. □

Proposition 3.3.2. *Let S be a nonempty subset of the ring $\mathbb{Z}$. Let $\mathcal{T} \subseteq \mathcal{B}$. Then for integers $0 \leq \ell \leq k < |S|$,*

$$\begin{bmatrix} k \\ \ell \end{bmatrix}_{S,\mathcal{T}} = \prod_{b \in \mathcal{T}} b^{\alpha_k(S,b) - \alpha_\ell(S,b) - \alpha_{k-\ell}(S,b)}. \tag{3.3.2}$$

Proof. This follows from Definitions 3.2.10 and 3.2.5. □

3.4 Bhargava's theory in the ring $D[[t]]$

3.4.1 t-orderings of an arbitrary subset of $D[[t]]$

In what follows, we let D be an integral domain. Let $U \neq \varnothing$ be a subset of $D[[t]]$, the ring of formal power series over D. We define a valuation $\mathrm{ord}_t : D[[t]] \to \{0, 1, 2, \dots\} \cup \{\infty\}$ by $\mathrm{ord}_t(0) = \infty$ and for nonzero $f(t) = \sum_{i=0}^{\infty} d_i t^i$,

$$\mathrm{ord}_t(f(t)) = \min \{i \in \mathbb{N} : d_i \neq 0\}.$$

That is,

$$\mathrm{ord}_t(f(t)) = \sup \{\alpha \in \mathbb{N} : t^\alpha \text{ divides } f(t)\}.$$

Proposition 3.4.1. *The function ord_t satisfies*

$$\mathrm{ord}_t(f(t) \pm g(t)) \geq \min \{\mathrm{ord}_t(f(t)), \mathrm{ord}_t(g(t))\}, \tag{3.4.1}$$

$$\mathrm{ord}_t(f(t)g(t)) = \mathrm{ord}_t(f(t)) + \mathrm{ord}_t(g(t)). \tag{3.4.2}$$

Proof. If $f(t) = 0$ or $g(t) = 0$, then (3.4.1) and (3.4.2) are clearly true. Now, suppose that $\alpha := \mathrm{ord}_t(f(t))$ and $\beta := \mathrm{ord}_t(g(t))$ are finite. So $f(t) = \sum_{i=0}^{\infty} d_i t^i$ where $d_j = 0$ for all $j < \alpha$ and $d_\alpha \neq 0$, and $g(t) = \sum_{i=0}^{\infty} e_i t^i$ where $e_k = 0$ for all $k < \beta$ and $e_\beta \neq 0$. The coefficient of t^i in $f(t) \pm g(t)$ is given by $[t^i]\,(f(t) \pm g(t)) = d_i \pm e_i$, which is zero if $i < \min\{\alpha, \beta\}$. So $\mathrm{ord}_t(f(t) \pm g(t)) \geq \min\{\alpha, \beta\}$. The coefficient of t^i in $f(t)g(t)$ is given by $[t^i]\,(f(t)g(t)) = \sum_{j=0}^{i} d_j e_{i-j}$, which is zero if $i < \alpha + \beta$. Moreover, $\left[t^{\alpha+\beta}\right](f(t)g(t)) = d_\alpha e_\beta \neq 0$. So $\mathrm{ord}_t(f(t)g(t)) = \alpha + \beta$. □

Definition 3.4.2. A *t-ordering* of U is a sequence $\mathbf{f} := (f_i(t))_{i=0}^{\infty}$ of formal power series in U that is formed as follows:

- Choose any formal power series $f_0(t)$ in U.
- Suppose that $f_j(t),\ j = 0, 1, 2, \ldots, k-1$ are chosen. Choose $f_k(t)$ in U that minimizes
$$\sum_{j=0}^{k-1} \mathrm{ord}_t\left(f_k(t) - f_j(t)\right).$$

In general, a t-ordering of U is not unique. But we will prove

Theorem 3.4.3. *The sequence $(\alpha_k(U, \mathbf{f}))_{k=0}^{\infty}$ defined by*

$$\alpha_k(U, \mathbf{f}) := \sum_{j=0}^{k-1} \mathrm{ord}_t\left(f_k(t) - f_j(t)\right) \tag{3.4.3}$$

is independent of the choice of t-ordering $\mathbf{f} = (f_i(t))_{i=0}^{\infty}$ of U.

To prove this, we consider polynomials $p(x;t)$ in x with coefficients in $D[[t]]$.

Definition 3.4.4. We say that a polynomial $p(x;t)$ with coefficients in $D[[t]]$ is *t-primitive* if $\mathrm{ord}_t\left([x^i]\,p(x;t)\right) = 0$ for some i.

Theorem 3.4.5. *Let $p(x;t)$ be a t-primitive polynomial in x of degree k. Let $\mathbf{f}$ be a t-ordering of U. Then*

$$\min\left\{\mathrm{ord}_t(p(f(t);t)) : f(t) \in U\right\} \leq \alpha_k(U, \mathbf{f}).$$

In the next Subsection 3.4.2, we prove Theorems 3.4.5 and 3.4.3.

3.4.2 Proofs of Theorems 3.4.5 and 3.4.3

Lemma 3.4.6. *For $U \subseteq D[[t]]$ and a t-ordering $\mathbf{f} = (f_i(t))_{i=0}^{\infty}$ of U, the associated t-sequence $(\alpha_k(U, \mathbf{f}))_{k=0}^{\infty}$ is weakly increasing.*

Proof. Let $U \subseteq D[[t]]$, and let $\mathbf{f} = (f_i(t))_{i=0}^{\infty}$ be a t-ordering of U. Let k be a nonnegative integer. From the definition (3.5.5), we have

$$\alpha_k(U, \mathbf{f}) = \sum_{j=0}^{k-1} \mathrm{ord}_t \left(f_k(t) - f_j(t) \right),$$

$$\alpha_{k+1}(U, \mathbf{f}) = \sum_{j=0}^{k} \mathrm{ord}_t \left(f_{k+1}(t) - f_j(t) \right).$$

From Definition 3.4.2, we have

$$\sum_{j=0}^{k-1} \mathrm{ord}_t \left(f_k(t) - f_j(t) \right) \leq \sum_{j=0}^{k-1} \mathrm{ord}_t \left(f_{k+1}(t) - f_j(t) \right).$$

Hence $\alpha_k(U, \mathbf{f}) \leq \alpha_{k+1}(U, \mathbf{f}) - \mathrm{ord}_t \left(f_{k+1}(t) - f_k(t) \right) \leq \alpha_{k+1}(U, \mathbf{f})$. □

Lemma 3.4.7. *Suppose that $(f_i(t))_{i=0}^{\infty}$ is a t-ordering of U. For $j = 0, 1, 2, \ldots, k$, let $c_j(t)$ be formal power series in $D[[t]]$, and let*

$$p_j(x; t) := c_j(t) \prod_{i=0}^{j-1} \left(x - f_i(t) \right), \tag{3.4.4}$$

$$p(x; t) := \sum_{j=0}^{k} p_j(x; t). \tag{3.4.5}$$

Then

$$\min \left\{ \mathrm{ord}_t \left(p_j(f(t); t) \right) : f(t) \in U \right\} = \mathrm{ord}_t \left(c_j(t) \right) + \sum_{i=0}^{j-1} \mathrm{ord}_t \left(f_j(t) - f_i(t) \right), \tag{3.4.6}$$

$$\min \left\{ \mathrm{ord}_t (p(f(t); t)) : f(t) \in U \right\} \leq \min \left\{ \mathrm{ord}_t \left(p_j(f(t); t) \right) : f(t) \in U \right\} \tag{3.4.7}$$

for all $j = 0, 1, 2, \ldots, k$.

Proof. We prove (3.4.6) first. Using Proposition 3.4.1, we see that

$$\operatorname{ord}_t\left(p_j(f(t);t)\right) = \operatorname{ord}_t\left(c_j(t)\right) + \sum_{i=0}^{j-1} \operatorname{ord}_t\left(f(t) - f_i(t)\right).$$

The right-hand side attains its minimum value over $f(t) \in S$ at $f(t) = f_j(t)$. Hence (3.4.6) follows.

We prove (3.4.7) by induction on j. Denote by μ the left-hand side of (3.4.7). Then

$$\mu \leq \operatorname{ord}_t\left(p\left(f_0(t);t\right)\right) = \operatorname{ord}_t\left(c_0(t)\right) = \operatorname{ord}_t\left(p_0(f(t);t)\right)$$

for any $f(t) \in U$. Hence (3.4.7) is true for $j = 0$. Now, suppose that (3.4.7) is true for $j = 0, 1, 2, \ldots, \ell - 1$, where $1 \leq \ell \leq k$. By Proposition 3.4.1, we can rewrite (3.4.6) with j replaced by ℓ as

$$\min\left\{\operatorname{ord}_t\left(p_\ell(f(t);t)\right) : f(t) \in U\right\} = \operatorname{ord}_t\left(p_\ell\left(f_\ell(t);t\right)\right). \tag{3.4.8}$$

From the identity

$$p_\ell\left(f_\ell(t);t\right) = p\left(f_\ell(t);t\right) - \sum_{j=0}^{\ell-1} p_j\left(f_\ell(t);t\right)$$

and Proposition 3.4.1, we deduce that

$$\begin{aligned}\operatorname{ord}_t\left(p_\ell\left(f_\ell(t);t\right)\right) &\geq \min\left\{\operatorname{ord}_t\left(p\left(f_\ell(t);t\right)\right)\right\} \cup \left\{\operatorname{ord}_t\left(p_j\left(f_\ell(t);t\right)\right) : 0 \leq j \leq \ell - 1\right\} \\ &\geq \min\left\{\mu\right\} \cup \left\{\operatorname{ord}_t\left(p_j(f(t);t)\right) : f(t) \in U,\ 0 \leq j \leq \ell - 1\right\}.\end{aligned}$$

The last quantity is μ by the induction hypothesis. From (3.4.8), we conclude that (3.4.7) is true for $j = \ell$. This completes the proof. □

Proof of Theorem 3.4.5. Let $\mathbf{f} = (f_i(t))_{i=0}^{\infty}$ be a t-ordering of U. We show that

$$\min\left\{\operatorname{ord}_t(p(f(t);t)) : f(t) \in U\right\} \leq \alpha_k(U, \mathbf{f}).$$

The polynomial $p(x;t)$ can be decomposed into the form (3.4.5), where $p_j(x;t)$ are given by (3.4.4) and $c_j(t) \in D[[t]]$ are uniquely determined by $p(x;t)$ and $\mathbf{f}$. Since $p(x;t)$ is t-primitive, it follows that there is $j_0 \in \{0,\ldots,k\}$ such that $\operatorname{ord}_t(c_{j_0}(t)) = 0$. Applying Lemma 3.4.7 with $j = j_0$, we obtain by (3.4.7) and (3.4.6) respectively

$$\begin{aligned}
\min\{\operatorname{ord}_t(p(f(t);t)) : f(t) \in U\} &\leq \min\{\operatorname{ord}_t(p_{j_0}(f(t);t)) : f(t) \in U\} \\
&= \operatorname{ord}_t(c_{j_0}(t)) + \sum_{i=0}^{j_0-1} \operatorname{ord}_t(f_{j_0}(t) - f_i(t)) \\
&= \sum_{i=0}^{j_0-1} \operatorname{ord}_t(f_{j_0}(t) - f_i(t)) \\
&= \alpha_{j_0}(U,\mathbf{f}).
\end{aligned}$$

The last quantity is $\leq \alpha_k(S,\mathbf{f})$ by Lemma 3.4.6. $\square$

Proposition 3.4.8. *Let* $\mathbf{f} = (f_i(t))_{i=0}^{\infty}$ *be a* t*-ordering of* U*. Then*

$$\alpha_k(U,\mathbf{f}) = \max_{p(x;t)} \min\{\operatorname{ord}_t(p(f(t);t)) : f(t) \in U\}, \tag{3.4.9}$$

where the maximum runs over all t*-primitive polynomials* $p(x;t)$ *of degree* k*. Moreover, if*

$$q_k(x;t) := \prod_{j=0}^{k-1} (x - f_j(t)),$$

then

$$\alpha_k(U,\mathbf{f}) = \min\{\operatorname{ord}_t(q_k(f(t);t)) : f(t) \in U\}. \tag{3.4.10}$$

Proof. Taking the maximum over all t-primitive polynomials $p(x;t)$ of degree k in Theorem 3.4.5, we obtain

$$\alpha_k(U,\mathbf{f}) \geq \max_{p(x;t)} \min\{\operatorname{ord}_t(p(f(t);t)) : f(t) \in U\}.$$

Hence it suffices to prove the second assertion. By the definition of t-orderings,

$$\begin{aligned}\min\{\operatorname{ord}_t(q_k(f(t);t)) : f(t)\in U\} &= \min_{f(t)\in U}\sum_{j=0}^{k-1}\operatorname{ord}_t(f(t)-f_j(t))\\ &= \sum_{j=0}^{k-1}\operatorname{ord}_t(f_k(t)-f_j(t))\\ &= \alpha_k(U,\mathbf{f}).\end{aligned}$$

This completes the proof. □

Proof of Theorem 3.4.3. From Proposition 3.4.8, since the right side of (3.4.9) does not depend on the choice of t-ordering $\mathbf{f}$, the result follows. □

Since we have proved that $\alpha_k(U,\mathbf{f})$ does not depend on $\mathbf{f}$, we will refer to it as $\alpha_k(U)$ from now on.

Definition 3.4.9. We call $(\alpha_k(U))_{k=0}^{\infty} = (\alpha_k(U,\mathbf{f}))_{k=0}^{\infty}$ the *associated t-sequence* of U.

3.4.3 Properties of the associated t-sequence

Theorem 3.4.10. *For nonnegative integers k and ℓ, we have*

$$\alpha_{k+\ell}(U) \geq \alpha_k(U) + \alpha_\ell(U).$$

Proof. Let $\mathbf{f} = (f_i(t))_{i=0}^{\infty}$ be a t-ordering of U. Applying Proposition 3.4.8, we obtain

$$\begin{aligned}\alpha_k(U)+\alpha_\ell(U) &= \min\{\operatorname{ord}_t(q_k(f(t);t)) : f(t)\in U\} + \min\{\operatorname{ord}_t(q_\ell(f(t);t)) : f(t)\in U\}\\ &\leq \min\{\operatorname{ord}_t(q_k(f(t);t)) + \operatorname{ord}_t(q_\ell(f(t);t)) : f(t)\in U\}\\ &= \min\{\operatorname{ord}_t(q_k(f(t);t)q_\ell(f(t);t)) : f(t)\in U\}\\ &\leq \max_{p(x;t)}\min\{\operatorname{ord}_t(p(f(t);t)) : f(t)\in U\}\\ &= \alpha_{k+\ell}(U),\end{aligned}$$

where the maximum runs over all t-primitive polynomials $p(x;t)$ of degree $k+\ell$. □

Theorem 3.4.11. *Suppose that $U_1 \subseteq U_2 \subseteq D[[t]]$. Then $\alpha_k(U_1) \geq \alpha_k(U_2)$ for every nonnegative integer k.*

Proof. Let $p(x;t)$ be a t-primitive polynomial of degree k. Since $U_1 \subseteq U_2$, it follows that

$$\min\{\operatorname{ord}_t(p(f(t);t)) : f(t) \in U_1\} \geq \min\{\operatorname{ord}_t(p(f(t);t)) : f(t) \in U_2\}.$$

Taking the maximum over all t-primitive polynomials $p(x;t)$ of degree k and applying Proposition 3.4.8, we obtain $\alpha_k(U_1) \geq \alpha_k(U_2)$. □

Theorem 3.4.12. *Let $g_i(t)$, $i = 0, 1, 2, \ldots, n$ be formal power series in U. Then*

$$\sum_{i=0}^{n-1} \sum_{j=i+1}^{n} \operatorname{ord}_t(g_i(t) - g_j(t)) \geq \sum_{k=1}^{n} \alpha_k(U). \tag{3.4.11}$$

Proof. If $g_i(t) = g_j(t)$ for some $0 \leq i < j \leq n$, then the left side of (3.5.6) is ∞ and (3.5.6) is true. Now, assume that $g_1(t), \ldots, g_n(t)$ are pairwise distinct. Without loss of generality, we may assume that $(g_i(t))_{i=0}^n$ is a t-ordering of $V := \{g_i(t) : i = 0, \ldots, n\}$. So for $k = 1, \ldots, n$,

$$\alpha_k(V) = \sum_{j=0}^{k-1} \operatorname{ord}_t(g_k(t) - g_j(t)).$$

On the other hand, since $V \subseteq U$, Theorem 3.4.11 yields $\nu_k(V) \geq \nu_k(U)$. Thus

$$\sum_{j=0}^{k-1} \operatorname{ord}_t(g_k(t) - g_j(t)) \geq \alpha_k(U). \tag{3.4.12}$$

The result follows by summing (3.4.12) over $k = 1, \ldots, n$. □

3.5 Property C

3.5.1 Mapping to $D[[t]]$

Let R be a commutative ring and $\mathfrak{b}$ a nonzero proper ideal of R. The crucial property for our argument will be the existence of an injective map $\varphi_{\mathfrak{b}} : R \to D[[t]]$ that has the following property.

Property C. *The map* $\varphi_{\mathfrak{b}} : R \to D[[t]]$ *satisfies* Property C $(R, \mathfrak{b}, D)$ *(or simply* Property C*) if*

$$\varphi_{\mathfrak{b}}(a_1) \equiv \varphi_{\mathfrak{b}}(a_2) \pmod{t^k} \quad \textit{if and only if} \quad a_1 \equiv a_2 \pmod{\mathfrak{b}^k} \tag{3.5.1}$$

for all $k \geq 1$*.*

For $a \in R$, denote by $\mathrm{ord}_{\mathfrak{b}}(a)$ the supremum of all nonnegative integers k such that $a \in \mathfrak{b}^k$; i.e.,

$$\mathrm{ord}_{\mathfrak{b}}(a) := \sup\left\{k \in \mathbb{N} : a \in \mathfrak{b}^k\right\}.$$

Here $\mathfrak{b}^0 = R$.

Proposition 3.5.1. *Assume that* $\varphi_{\mathfrak{b}} : R \to D[[t]]$ *satisfies Property C. Then*

$$\mathrm{ord}_t\left(\varphi_{\mathfrak{b}}(a_1) - \varphi_{\mathfrak{b}}(a_2)\right) = \mathrm{ord}_{\mathfrak{b}}(a_1 - a_2). \tag{3.5.2}$$

Proof. The result readily follows from the definitions of ord_t and $\mathrm{ord}_{\mathfrak{b}}$. □

We note that the quantity on the right side of (3.5.2) is independent of the map $\varphi_{\mathfrak{b}}$ and the integral domain D.

3.5.2 $\mathfrak{b}$-orderings of an arbitrary subset of R

As in the previous section, we let R be a commutative ring and let $\mathfrak{b}$ be a nonzero proper ideal of R. Let S be an arbitrary subset of R. Given that there exists a map $\varphi_{\mathfrak{b}} : R \to D[[t]]$ satisfying Property C, we can define $\mathfrak{b}$-orderings of $S \subseteq R$ analogous to t-orderings of $U \subseteq D[[t]]$.

Definition 3.5.2. A $\mathfrak{b}$-*ordering* of S is a sequence $\mathbf{a} := (a_i)_{i=0}^{\infty}$ of elements of S that is formed as follows:

- Choose any element a_0 of S.

- Suppose that a_j, $j = 0, 1, 2, \ldots, k-1$ are chosen. Choose a_k in S that minimizes

$$\sum_{j=0}^{k-1} \operatorname{ord}_{\mathfrak{b}} (a_k - a_j).$$

Lemma 3.5.3. *Assume that $\varphi_{\mathfrak{b}} : R \to D[[t]]$ satisfies Property C. Suppose that $\mathbf{a} := (a_i)_{i=0}^{\infty}$ is a $\mathfrak{b}$-ordering of S. Then $(\varphi_{\mathfrak{b}}(a_i))_{i=0}^{\infty}$ is a t-ordering of $\varphi_{\mathfrak{b}}(S) := \{\varphi_{\mathfrak{b}}(s) : s \in S\}$ and*

$$\alpha_k (\varphi_{\mathfrak{b}} (S)) = \sum_{j=0}^{k-1} \operatorname{ord}_{\mathfrak{b}} (a_k - a_j). \tag{3.5.3}$$

Proof. The first assertion follows from Definitions 3.4.2 and 3.5.2 and Proposition 3.5.1. Hence the associated t-sequence of $\varphi_{\mathfrak{b}}(S)$ is given by

$$\alpha_k (\varphi_{\mathfrak{b}} (S)) = \sum_{j=0}^{k-1} \operatorname{ord}_t (\varphi_{\mathfrak{b}} (a_k) - \varphi_{\mathfrak{b}} (a_j)). \tag{3.5.4}$$

By Proposition 3.5.1, the right side of (3.5.4) is equal to $\sum_{j=0}^{k-1} \operatorname{ord}_{\mathfrak{b}} (a_k - a_j)$. □

Corollary 3.5.4. *Assume that there is a map $\varphi_{\mathfrak{b}} : R \to D[[t]]$ satisfying Property C. Then the sequence $(\alpha_k (S, \mathfrak{b}, \mathbf{a}))_{k=0}^{\infty}$ defined by*

$$\alpha_k (S, \mathfrak{b}, \mathbf{a}) := \sum_{j=0}^{k-1} \operatorname{ord}_{\mathfrak{b}} (a_k - a_j) \tag{3.5.5}$$

is independent of the choice of $\mathfrak{b}$-ordering $\mathbf{a} = (a_i)_{i=0}^{\infty}$ of S.

Proof. The left side of (3.5.3) is independent of $\mathbf{a}$. □

Definition 3.5.5. On the assumption that there exists a map $\varphi_{\mathfrak{b}} : R \to D[[t]]$ satisfying Property C, we call $(\alpha_k (S, \mathfrak{b}))_{k=0}^{\infty} = (\alpha_k (S, \mathfrak{b}, \mathbf{a}))_{k=0}^{\infty}$ the *associated $\mathfrak{b}$-sequence* of S.

Lemma 3.5.6. *Assume that $\varphi_{\mathfrak{b}} : R \to D[[t]]$ satisfies Property C. Then for nonnegative integers k, we have*

$$\alpha_k (S, \mathfrak{b}) = \alpha_k (\varphi_{\mathfrak{b}} (S)).$$

Ring	$D[[t]]$	R
Subset	U	S
Base	$tD[[t]]$	$\mathfrak{b}$
Valuation	ord_t	$\text{ord}_{\mathfrak{b}}$
Orderings	$\mathbf{f} = (f_i(t))_{i=0}^{\infty}$	$\mathbf{a} = (a_i)_{i=0}^{\infty}$
Invariant	$(\alpha_k(U))_{k=0}^{\infty}$	$(\alpha_k(S, \mathfrak{b}))_{k=0}^{\infty}$

Table 3.1: List of notations

Proof. The result follows from Definition 3.5.5 and Equations (3.5.5) and (3.5.3) □

Corollary 3.5.7. *Assume that there is a map $\varphi_{\mathfrak{b}} : R \to D[[t]]$ satisfying Property C. Then for nonnegative integers k and ℓ, we have*

$$\alpha_{k+\ell}(S, \mathfrak{b}) \geq \alpha_k(S, \mathfrak{b}) + \alpha_\ell(S, \mathfrak{b}).$$

Proof. Applying Theorem 3.4.10 with $U = \varphi_{\mathfrak{b}}(S)$, we obtain

$$\alpha_{k+\ell}(\varphi_{\mathfrak{b}}(S)) \geq \alpha_k(\varphi_{\mathfrak{b}}(S)) + \alpha_\ell(\varphi_{\mathfrak{b}}(S)).$$

By Lemma 3.5.6, the above is $\alpha_{k+\ell}(S, \mathfrak{b}) \geq \alpha_k(S, \mathfrak{b}) + \alpha_\ell(S, \mathfrak{b})$. □

Corollary 3.5.8. *Assume that there is a map $\varphi_{\mathfrak{b}} : R \to D[[t]]$ satisfying Property C. Let s_i, $i = 0, 1, 2, \ldots, n$ be elements of S. Then*

$$\sum_{i=0}^{n-1} \sum_{j=i+1}^{n} \text{ord}_{\mathfrak{b}}(s_i - s_j) \geq \sum_{k=1}^{n} \alpha_k(S, \mathfrak{b}). \tag{3.5.6}$$

Proof. Applying Theorem 3.4.12 with $U = \varphi_{\mathfrak{b}}(S)$ and $g_i(t) = \varphi_{\mathfrak{b}}(s_i)$, we obtain

$$\sum_{i=0}^{n-1} \sum_{j=i+1}^{n} \text{ord}_t(\varphi_{\mathfrak{b}}(s_i) - \varphi_{\mathfrak{b}}(s_j)) \geq \sum_{k=1}^{n} \alpha_k(\varphi_{\mathfrak{b}}(S)).$$

By Proposition 3.5.1 and Lemma 3.5.6, the above is (3.5.6). □

Corollary 3.5.9. *Assume that there is a map $\varphi_{\mathfrak{b}} : R \to D[[t]]$ satisfying Property C. Then the associated $\mathfrak{b}$-sequence $(\alpha_k(S, \mathfrak{b}))_{k=0}^{\infty}$ is weakly increasing.*

Proof. Applying Lemma 3.4.6 with $U = \varphi_{\mathfrak{b}}(S)$, we see that the associated t-sequence $(\alpha_k(\varphi_{\mathfrak{b}}(S)))_{k=0}^{\infty}$ is weakly increasing, and this sequence is the associated $\mathfrak{b}$-sequence $(\alpha_k(S, \mathfrak{b}))_{k=0}^{\infty}$ by Lemma 3.5.6. □

Corollary 3.5.10. *Assume that there is a map $\varphi_{\mathfrak{b}} : R \to D[[t]]$ satisfying Property C. Suppose that $S_1 \subseteq S_2 \subseteq R$. Then $\alpha_k(S_1, \mathfrak{b}) \geq \alpha_k(S_2, \mathfrak{b})$ for every nonnegative integer k.*

Proof. Applying Lemma 3.4.11 with $U_1 = \varphi_{\mathfrak{b}}(S_1)$ and $U_2 = \varphi_{\mathfrak{b}}(S_2)$, we obtain

$$\alpha_k(\varphi_{\mathfrak{b}}(S_1)) \geq \alpha_k(\varphi_{\mathfrak{b}}(S_2)).$$

By Lemma 3.5.6, the above is $\alpha_k(S_1, \mathfrak{b}) \geq \alpha_k(S_2, \mathfrak{b})$. □

3.5.3 The case $R = \mathbb{Z}$

Thoughtout this section, we consider the case that $R = \mathbb{Z}$. Since $\mathbb{Z}$ is a principal ideal domain, it follows that the ideal $\mathfrak{b}$ is principal. Let $b \in \mathbb{Z}$ be the positive generator of $\mathfrak{b}$. Since $\mathfrak{b}$ is a proper ideal of $\mathbb{Z}$, it follows that $b \geq 2$. We choose $D = \mathbb{Z}$ and define $\varphi_{\mathfrak{b}} : \mathbb{Z} \to \mathbb{Z}[[t]]$ by

$$\varphi_{\mathfrak{b}}(a) = f_{a,b}(t) := \sum_{k=0}^{\infty} d_k t^k, \tag{3.5.7}$$

where

$$d_k = d_k(a, b) := \left\lfloor \frac{a}{b^k} \right\rfloor - b \left\lfloor \frac{a}{b^{k+1}} \right\rfloor. \tag{3.5.8}$$

If $a \geq 0$, then the d_k's are the digits of a in base b.

Proposition 3.5.11. *The map $\varphi_{\mathfrak{b}} : \mathbb{Z} \to \mathbb{Z}[[t]]$ defined by* (3.5.7) *and* (3.5.8) *satisfies Property $C(\mathbb{Z}, \mathfrak{b}, \mathbb{Z})$.*

Proof. ($\Rightarrow$) Suppose that $\varphi_{\mathfrak{b}}(a_1) \equiv \varphi_{\mathfrak{b}}(a_2) \pmod{t^k}$. That is, $d_\ell(a_1, b) = d_\ell(a_2, b)$ for all $\ell = 0, \ldots, k-1$. From the identity

$$a - b^k \left\lfloor \frac{a}{b^k} \right\rfloor = \sum_{\ell=0}^{k-1} d_\ell(a, b) b^\ell,$$

we see that $a_1 - b^k \left\lfloor \frac{a_1}{b^k} \right\rfloor = a_2 - b^k \left\lfloor \frac{a_2}{b^k} \right\rfloor$. We deduce that $a_1 \equiv a_2 \pmod{\mathfrak{b}^k}$.

($\Leftarrow$) Suppose that $a_1 \equiv a_2 \pmod{\mathfrak{b}^k}$. That is, $a_1 - a_2 = b^k q$ for some $q \in \mathbb{Z}$. So for $\ell = 0, \ldots, k-1$,

$$\begin{aligned} d_\ell(a_1, b) &= \left\lfloor \frac{a_1}{b^\ell} \right\rfloor - b \left\lfloor \frac{a_1}{b^{\ell+1}} \right\rfloor \\ &= \left\lfloor \frac{a_2 + b^k q}{b^\ell} \right\rfloor - b \left\lfloor \frac{a_2 + b^k q}{b^{\ell+1}} \right\rfloor \\ &= \left(\left\lfloor \frac{a_2}{b^\ell} \right\rfloor + b^{k-\ell} q \right) - b \left(\left\lfloor \frac{a_2}{b^{\ell+1}} \right\rfloor + b^{k-\ell-1} q \right) \\ &= \left\lfloor \frac{a_2}{b^\ell} \right\rfloor - b \left\lfloor \frac{a_2}{b^{\ell+1}} \right\rfloor = d_\ell(a_2, b). \end{aligned}$$

Hence $\varphi_{\mathfrak{b}}(a_1) \equiv \varphi_{\mathfrak{b}}(a_2) \pmod{t^k}$. □

Proposition 3.5.11 implies that all the corollaries in Section 3.5.2 hold for the case that $R = D = \mathbb{Z}$.

Proof of Theorem 3.2.3. The result follows from Proposition 3.5.11 and Corollary 3.5.4 with $\mathfrak{b} = b\mathbb{Z}$. □

Proof of Proposition 3.2.7. (1) This is true because

$$[k]!_{S,\mathcal{T}_2} = [k]!_{S,\mathcal{T}_2 \setminus \mathcal{T}_1} [k]!_{S,\mathcal{T}_1}.$$

(2) It follows from Proposition 3.5.11 and Corollary 3.5.10 that $\alpha_k(S_1, b) \geq \alpha(S_2, b)$ for all $b \in \mathcal{B}$. Since

$$\frac{[k]!_{S_1,\mathcal{T}}}{[k]!_{S_2,\mathcal{T}}} = \prod_{b \in \mathcal{T}} b^{\alpha_k(S_1,b) - \alpha(S_2,b)},$$

the result follows. □

Proof of Theorem 3.2.9. It follows from Proposition 3.5.11 and Corollary 3.5.9 that $\alpha_n(S,b) \geq \alpha_{n-1}(S,b)$ for all $b \in \mathcal{B}$. The result follows from (3.3.1). □

Proof of Theorem 3.2.11. It follows from Proposition 3.5.11 and Corollary 3.5.7 that $\alpha_k(S,b) \geq \alpha_\ell(S,b) + \alpha_{k-\ell}(S,b)$ for all $b \in \mathcal{B}$. The result follows from (3.3.2). □

3.6 The case $(S, \mathcal{T}) = (\mathbb{Z}, \mathcal{B})$

Proof of Theorem 3.2.12. The proof is by induction: if $0, 1, 2, \ldots, k-1$ is a b-ordering for the first $k-1$ steps, then at the kth step we need to pick $a_k \in \mathbb{Z}$ to minimize

$$Q := \sum_{j=0}^{k-1} \operatorname{ord}_b\left(a_k - j\right).$$

Write

$$\operatorname{ord}_b\left(x\right) = \sum_{\substack{i\geq 1\\ b^i | x}} 1. \tag{3.6.1}$$

Then

$$\begin{aligned} Q &= \sum_{j=0}^{k-1} \sum_{\substack{i\geq 1\\ a_k \equiv j \bmod b^i}} 1 \\ &= \sum_{i\geq 1} \sum_{\substack{j=0\\ j\equiv a_k \bmod b^i}}^{k-1} 1 \\ &\geq \sum_{i\geq 1} \left\lfloor \frac{k}{b^i} \right\rfloor, \end{aligned}$$

and there is equality if $a_k = k$. So at the kth step we choose $a_k = k$, and the claim follows by induction. □

Proof of Theorem 3.2.13. This follows from the proof of Theorem 3.2.12. Note that $\gamma(k,b) = 0$ if $b > k$. □

Proof of Theorem 3.2.14. The result follows by noting that

$$\left\lfloor \frac{n}{b^i} \right\rfloor - \left\lfloor \frac{n-1}{b^i} \right\rfloor = \begin{cases} 1 & \text{if } b^i \mid n, \\ 0 & \text{otherwise.} \end{cases}$$

So from (3.6.1),

$$\operatorname{ord}_b(n) = \sum_{i \geq 1} \left(\left\lfloor \frac{n}{b^i} \right\rfloor - \left\lfloor \frac{n-1}{b^i} \right\rfloor \right).$$

□

Proof of Theorem 3.2.15. (1) follows from the definition and Theorem 3.2.13.

(2) follows from (1) and the identity

$$d_b(n) = n - (b-1) \sum_{i=1}^{\infty} \left\lfloor \frac{n}{b^i} \right\rfloor.$$

□

Proof of Corollary 3.2.16. We have

$$\begin{aligned} \overline{\overline{G}}_n &= \prod_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\mathbb{Z},\mathcal{B}} \\ &= \prod_{k=0}^{n} \prod_{b=2}^{n} b^{\beta(n,k,b)} \\ &= \prod_{b=2}^{n} b^{\sum_{k=0}^{n} \beta(n,k,b)}. \end{aligned}$$

From (3.2.15),

$$\begin{aligned} \sum_{k=0}^{n} \beta(n,k,b) &= \sum_{k=1}^{n-1} \frac{1}{b-1} \left(d_b(k) + d_b(n-k) - d_b(n) \right) \\ &= \overline{\nu}(n,b). \end{aligned}$$

□

3.7 Appendix: tables of values for the special case $(S, \mathcal{T}) = (\mathbb{Z}, \mathcal{B})$

3.7.1 Generalized positive integers

n	$[n]_{\mathbb{Z},\mathcal{B}}$			n	$[n]_{\mathbb{Z},\mathcal{B}}$			n	$[n]_{\mathbb{Z},\mathcal{B}}$
1	1	=	1	21	441	=	$3^2 \times 7^2$	41	41
2	2	=	2	22	484	=	$2^2 \times 11^2$	42	$2^4 \times 3^4 \times 7^4$
3	3	=	3	23	23	=	23	43	43
4	16	=	2^4	24	$1,327,104$	=	$2^{14} \times 3^4$	44	$2^7 \times 11^3$
5	5	=	5	25	625	=	5^4	45	$3^7 \times 5^3$
6	36	=	$2^2 \times 3^2$	26	676	=	$2^2 \times 13^2$	46	$2^2 \times 23^2$
7	7	=	7	27	$6,561$	=	3^8	47	47
8	256	=	2^8	28	$43,904$	=	$2^7 \times 7^3$	48	$2^{25} \times 3^5$
9	81	=	3^4	29	29	=	29	49	7^4
10	100	=	$2^2 \times 5^2$	30	$810,000$	=	$2^4 \times 3^4 \times 5^4$	50	$2^3 \times 5^7$
11	11	=	11	31	31	=	31	51	$3^2 \times 17^2$
12	$3,456$	=	$2^7 \times 3^3$	32	$2,097,152$	=	2^{21}	52	$2^7 \times 13^3$
13	13	=	13	33	$1,089$	=	$3^2 \times 11^2$	53	53
14	196	=	$2^2 \times 7^2$	34	$1,156$	=	$2^2 \times 17^2$	54	$2^4 \times 3^{14}$
15	225	=	$3^2 \times 5^2$	35	$1,225$	=	$5^2 \times 7^2$	55	$5^2 \times 11^2$
16	$32,768$	=	2^{15}	36	$362,797,056$	=	$2^{11} \times 3^{11}$	56	$2^{14} \times 7^4$
17	17	=	17	37	37	=	37	57	$3^2 \times 19^2$
18	$17,496$	=	$2^3 \times 3^7$	38	$1,444$	=	$2^2 \times 19^2$	58	$2^2 \times 29^2$
19	19	=	19	39	$1,521$	=	$3^2 \times 13^2$	59	59
20	$16,000$	=	$2^7 \times 5^3$	40	$10,240,000$	=	$2^{14} \times 5^4$	60	$2^{13} \times 3^6 \times 5^6$

Table 3.2: $[n]_{\mathbb{Z},\mathcal{B}}$ decimal for $1 \leq n \leq 40$ and factored for $1 \leq n \leq 60$

3.7.2 Generalized factorials

k	$[k]!_{\mathbb{Z},\mathcal{B}}$		
0	1	=	1
1	1	=	1
2	2	=	2
3	6	=	2×3
4	96	=	$2^5 \times 3$
5	480	=	$2^5 \times 3 \times 5$
6	$17,280$	=	$2^7 \times 3^3 \times 5$
7	$120,960$	=	$2^7 \times 3^3 \times 5 \times 7$
8	$30,965,760$	=	$2^{15} \times 3^3 \times 5 \times 7$
9	$2,508,226,560$	=	$2^{15} \times 3^7 \times 5 \times 7$
10	$250,822,656,000$	=	$2^{17} \times 3^7 \times 5^3 \times 7$
11	$2,759,049,216,000$	=	$2^{17} \times 3^7 \times 5^3 \times 7 \times 11$
12	$9,535,274,090,496,000$	=	$2^{24} \times 3^{10} \times 5^3 \times 7 \times 11$
13	$123,958,563,176,448,000$	=	$2^{24} \times 3^{10} \times 5^3 \times 7 \times 11 \times 13$
14	$24,295,878,382,583,808,000$	=	$2^{26} \times 3^{10} \times 5^3 \times 7^3 \times 11 \times 13$
15	$5,466,572,636,081,356,800,000$	=	$2^{26} \times 3^{12} \times 5^5 \times 7^3 \times 11 \times 13$
16	$179,128,652,139,113,899,622,400,000$	=	$2^{41} \times 3^{12} \times 5^5 \times 7^3 \times 11 \times 13$
17	$3,045,187,086,364,936,293,580,800,000$	=	$2^{41} \times 3^{12} \times 5^5 \times 7^3 \times 11 \times 13 \times 17$
18	$53,278,593,263,040,925,392,489,676,800,000$	=	$2^{44} \times 3^{19} \times 5^5 \times 7^3 \times 11 \times 13 \times 17$
19	$1,012,293,271,997,777,582,457,303,859,200,000$	=	$2^{44} \times 3^{19} \times 5^5 \times 7^3 \times 11 \times 13 \times 17 \times 19$

Table 3.3: $[k]!_{\mathbb{Z},\mathcal{B}}$ decimal and factored for $0 \leq k \leq 19$

k	$[k]!_{\mathbb{Z},\mathcal{B}}$
20	$2^{51} \times 3^{19} \times 5^8 \times 7^3 \times 11 \times 13 \times 17 \times 19$
21	$2^{51} \times 3^{21} \times 5^8 \times 7^5 \times 11 \times 13 \times 17 \times 19$
22	$2^{53} \times 3^{21} \times 5^8 \times 7^5 \times 11^3 \times 13 \times 17 \times 19$
23	$2^{53} \times 3^{21} \times 5^8 \times 7^5 \times 11^3 \times 13 \times 17 \times 19 \times 23$
24	$2^{67} \times 3^{25} \times 5^8 \times 7^5 \times 11^3 \times 13 \times 17 \times 19 \times 23$
25	$2^{67} \times 3^{25} \times 5^{12} \times 7^5 \times 11^3 \times 13 \times 17 \times 19 \times 23$
26	$2^{69} \times 3^{25} \times 5^{12} \times 7^5 \times 11^3 \times 13^3 \times 17 \times 19 \times 23$
27	$2^{69} \times 3^{33} \times 5^{12} \times 7^5 \times 11^3 \times 13^3 \times 17 \times 19 \times 23$
28	$2^{76} \times 3^{33} \times 5^{12} \times 7^8 \times 11^3 \times 13^3 \times 17 \times 19 \times 23$
29	$2^{76} \times 3^{33} \times 5^{12} \times 7^8 \times 11^3 \times 13^3 \times 17 \times 19 \times 23 \times 29$
30	$2^{80} \times 3^{37} \times 5^{16} \times 7^8 \times 11^3 \times 13^3 \times 17 \times 19 \times 23 \times 29$

Table 3.4: $[k]!_{\mathbb{Z},\mathcal{B}}$ factored for $20 \leq k \leq 30$

3.7.3 Generalized binomial coefficients

$k\backslash\ell$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	16	24	16	1						
5	1	5	40	40	5	1					
6	1	36	90	480	90	36	1				
7	1	7	126	210	210	126	7	1			
8	1	256	896	10,752	3,360	10,752	896	256	1		
9	1	81	10,368	24,192	54,432	54,432	24,192	10,368	81	1	
10	1	100	4,050	345,600	151,200	1,088,640	151,200	345,600	4,050	100	1

Table 3.5: $\left[{k \atop \ell}\right]_{\mathbb{Z},\mathcal{B}}$ decimal for $0 \le \ell \le k \le 10$

$k\backslash\ell$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	2^4	$2^3 \cdot 3$	2^4	1			
5	1	5	$2^3 \cdot 5$	$2^3 \cdot 5$	5	1		
6	1	$2^2 \cdot 3^2$	$2 \cdot 3^2 \cdot 5$	$2^5 \cdot 3 \cdot 5$	$2 \cdot 3^2 \cdot 5$	$2^2 \cdot 3^2$	1	
7	1	7	$2 \cdot 3^2 \cdot 7$	$2 \cdot 3 \cdot 5 \cdot 7$	$2 \cdot 3 \cdot 5 \cdot 7$	$2 \cdot 3^2 \cdot 7$	7	1
8	1	2^8	$2^7 \cdot 7$	$2^9 \cdot 3 \cdot 7$	$2^5 \cdot 3 \cdot 5 \cdot 7$	$2^9 \cdot 3 \cdot 7$	$2^7 \cdot 7$	2^8
9	1	3^4	$2^7 \cdot 3^4$	$2^7 \cdot 3^3 \cdot 7$	$2^5 \cdot 3^5 \cdot 7$	$2^5 \cdot 3^5 \cdot 7$	$2^7 \cdot 3^3 \cdot 7$	$2^7 \cdot 3^4$
10	1	$2^2 \cdot 5^2$	$2 \cdot 3^4 \cdot 5^2$	$2^9 \cdot 3^3 \cdot 5^2$	$2^5 \cdot 3^3 \cdot 5^2 \cdot 7$	$2^7 \cdot 3^5 \cdot 5 \cdot 7$	$2^5 \cdot 3^3 \cdot 5^2 \cdot 7$	$2^9 \cdot 3^3 \cdot 5^2$

Table 3.6: $\left[{k \atop \ell}\right]_{\mathbb{Z},\mathcal{B}}$ factored for $0 \le \ell \le k \le 10$, $\ell \le 7$

3.7.4 Generalized binomial products

n	$\overline{\overline{G}}_n$		
0	1	$=$	1
1	1	$=$	1
2	2	$=$	2
3	9	$=$	3^2
4	$6,144$	$=$	$2^{11} \times 3$
5	$40,000$	$=$	$2^6 \times 5^4$
6	$5,038,848,000$	$=$	$2^{11} \times 3^9 \times 5^3$
7	$34,306,448,400$	$=$	$2^4 \times 3^6 \times 5^2 \times 7^6$
8	$20,436,839,713,048,300,093,440$	$=$	$2^{53} \times 3^3 \times 5 \times 7^5$
9	$1,222,959,700,798,803,745,499,202,453,504$	$=$	$2^{38} \times 3^{32} \times 7^4$
10	$487,579,439,713,294,378,598,400,000,000,000,000,000$	$=$	$2^{41} \times 3^{25} \times 5^{17} \times 7^3$

Table 3.7: $\overline{\overline{G}}_n$ decimal and factored for $0 \leq n \leq 10$

n	$\overline{\overline{G}}_n$
11	$2^{24} \times 3^{18} \times 5^{14} \times 7^2 \times 11^{10}$
12	$2^{84} \times 3^{44} \times 5^{11} \times 7 \times 11^9$
13	$2^{60} \times 3^{34} \times 5^8 \times 11^8 \times 13^{12}$
14	$2^{62} \times 3^{24} \times 5^5 \times 7^{25} \times 11^7 \times 13^{11}$
15	$2^{36} \times 3^{42} \times 5^{30} \times 7^{22} \times 11^6 \times 13^{10}$
16	$2^{235} \times 3^{30} \times 5^{25} \times 7^{19} \times 11^5 \times 13^9$
17	$2^{194} \times 3^{18} \times 5^{20} \times 7^{16} \times 11^4 \times 13^8 \times 17^{16}$
18	$2^{204} \times 3^{125} \times 5^{15} \times 7^{13} \times 11^3 \times 13^7 \times 17^{15}$
19	$2^{160} \times 3^{106} \times 5^{10} \times 7^{10} \times 11^2 \times 13^6 \times 17^{14} \times 19^{18}$
20	$2^{249} \times 3^{87} \times 5^{62} \times 7^7 \times 11 \times 13^5 \times 17^{13} \times 19^{17}$
21	$2^{198} \times 3^{108} \times 5^{54} \times 7^{44} \times 13^4 \times 17^{12} \times 19^{16}$
22	$2^{189} \times 3^{87} \times 5^{46} \times 7^{39} \times 11^{41} \times 13^3 \times 17^{11} \times 19^{15}$
23	$2^{136} \times 3^{66} \times 5^{38} \times 7^{34} \times 11^{38} \times 13^2 \times 17^{10} \times 19^{14} \times 23^{22}$
24	$2^{405} \times 3^{137} \times 5^{30} \times 7^{29} \times 11^{35} \times 13 \times 17^9 \times 19^{13} \times 23^{21}$
25	$2^{338} \times 3^{112} \times 5^{118} \times 7^{24} \times 11^{32} \times 17^8 \times 19^{12} \times 23^{20}$
26	$2^{321} \times 3^{87} \times 5^{106} \times 7^{19} \times 11^{29} \times 13^{49} \times 17^7 \times 19^{11} \times 23^{19}$
27	$2^{252} \times 3^{270} \times 5^{94} \times 7^{14} \times 11^{26} \times 13^{46} \times 17^6 \times 19^{10} \times 23^{18}$
28	$2^{372} \times 3^{237} \times 5^{82} \times 7^{90} \times 11^{23} \times 13^{43} \times 17^5 \times 19^9 \times 23^{17}$
29	$2^{296} \times 3^{204} \times 5^{70} \times 7^{82} \times 11^{20} \times 13^{40} \times 17^4 \times 19^8 \times 23^{16} \times 29^{28}$
30	$2^{336} \times 3^{287} \times 5^{174} \times 7^{74} \times 11^{17} \times 13^{37} \times 17^3 \times 19^7 \times 23^{15} \times 29^{27}$

Table 3.8: $\overline{\overline{G}}_n$ factored for $11 \leq n \leq 30$

3.8 Bibliography

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CHAPTER 4

Partial Factorizations of Generalized Binomial Products

4.0 Abstract

This chapter studies an integer sequence $\overline{\overline{G}}_n$ analogous to $\overline{G}_n = \prod_{k=0}^{n} \binom{n}{k}$, the product of the elements in the nth row of Pascal's triangle. It is known that the exponent $\nu_p(\overline{G}_n)$ in the prime factorization $\overline{G}_n = \prod_{p \le n} p^{\nu_p(\overline{G}_n)}$ can be expressed in terms of the base-p digits of the positive integers up to n. These radix statistics make sense for all bases $b \ge 2$. This chapter studies the asymptotics of the binomial product $\overline{\overline{G}}_n = \prod_{2 \le b \le n} b^{\overline{\nu}(n,b)}$, arising from Chapter 3, and its partial factorizations $\overline{\overline{G}}(n,x) = \prod_{2 \le b \le x} b^{\overline{\nu}(n,b)}$, which are also integers. It shows that $\log \overline{\overline{G}}(n, \alpha n)$ is well approximated by $f_{\overline{\overline{G}}}(\alpha) n^2 \log n + g_{\overline{\overline{G}}}(\alpha) n^2$ as $n \to \infty$ for scaling functions $f_{\overline{\overline{G}}}(\alpha)$ and $g_{\overline{\overline{G}}}(\alpha)$ defined for $0 \le \alpha \le 1$. The main results are deduced from the asymptotic study of functions $\overline{B}(n,x)$ and $\overline{A}(n,x)$ that are weighted sums of base-b radix statistics of n (and smaller integers) over $b \in [2,x]$. Unconditional estimates with power-savings remainder terms are derived for $\overline{B}(n,x)$ and $\overline{A}(n,x)$.

4.1 Introduction

Let $\overline{G}_n$ be the product of the binomial coefficients in the nth row of Pascal's triangle,

$$\overline{G}_n := \prod_{k=0}^{n} \binom{n}{k} = \frac{(n!)^{n+1}}{\prod_{k=0}^{n}(k!)^2}. \tag{4.1.1}$$

This sequence arises as the reciprocal of the product of all the nonzero unreduced Farey fractions of order n, see Lagarias and Mehta [20]. Its asymptotic growth is

$$\log \overline{G}_n = \frac{1}{2}n^2 - \frac{1}{2}n \log n + O(n), \tag{4.1.2}$$

by Stirling's formula. The number $\overline{G}_n$ is n-smooth (i.e., having no prime factor larger than n), and we may write its prime factorization as

$$\overline{G}_n = \prod_{p \le n} p^{\nu_p(\overline{G}_n)}, \tag{4.1.3}$$

where $\nu_p(a)$ denotes the additive p-adic valuation of a. The quantities $\nu_p(\overline{G}_n)$ are known to equal an expression $\overline{\nu}(n,p)$ defined purely in terms of the base-p digits of the positive integers up to n, given by

$$\nu_p(\overline{G}_n) = \frac{2}{p-1}S_p(n) - \frac{n-1}{p-1}d_p(n), \tag{4.1.4}$$

where $d_p(n)$ is the sum of the base-p digits of n and $S_p(n) := \sum_{j=1}^{n-1} d_p(j)$. (See [13, Theorem 5.1].) The left side of (4.1.4) is a nonnegative integer, while examples show the two terms on the right side are sometimes not integers. Du and Lagarias [13] studied the sizes of partial factorizations of $\overline{G}_n$:

$$G(n,x) := \prod_{p \le x} p^{\nu_p(\overline{G}_n)}, \tag{4.1.5}$$

where $1 \le x \le n$. They showed for $0 < \alpha \le 1$ and $n \ge 2$ the estimate

$$\log G(n, \alpha n) = f_G(\alpha)\, n^2 + O\left(\frac{1}{\alpha} n^2 \exp\left(-c\sqrt{\log n}\right)\right), \tag{4.1.6}$$

where the limit scaling function $f_G(\alpha)$ is given by

$$f_G(\alpha) = \frac{1}{2} + \frac{1}{2}\alpha^2 \left\lfloor \frac{1}{\alpha} \right\rfloor^2 + \frac{1}{2}\alpha^2 \left\lfloor \frac{1}{\alpha} \right\rfloor - \alpha \left\lfloor \frac{1}{\alpha} \right\rfloor \tag{4.1.7}$$

and $c > 0$ is an absolute constant. The remainder term estimate in (4.1.6) was improved to the power-savings estimate $O\left(\frac{1}{\alpha} n^{7/4} (\log n)^2\right)$ conditional on the Riemann hypothesis.

This chapter studies the asymptotics of an integer sequence $\overline{\overline{G}}_n$ defined in Chapter 3. The sequence $\overline{\overline{G}}_n$ is given as a product of generalized binomial coefficients:

$$\overline{\overline{G}}_n := \prod_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\mathbb{Z},\mathcal{B}}.$$

From Corollary 3.2.16, $\overline{\overline{G}}_n$ can be written in generalized prime factorization as

$$\overline{\overline{G}}_n = \prod_{b=2}^{n} b^{\overline{\nu}(n,b)}, \tag{4.1.8}$$

where the exponent $\overline{\nu}(n,b)$ is defined purely in terms of the base-b digits of the positive integers up to n, by the formula

$$\overline{\nu}(n,b) := \frac{2}{b-1} S_b(n) - \frac{n-1}{b-1} d_b(n) \tag{4.1.9}$$

generalizing the formula (4.1.4) for $\nu_p(\overline{G}_n)$.

In fact, it follows immediately from the proof of Corollary 3.2.16 that all $\overline{\nu}(n,b)$ are nonnegative integers. So the integer $\overline{\overline{G}}_n$ is n-smooth.

In this chapter, we characterize all positive integers n for which $\overline{\nu}(n,b) = 0$; see Theorem 4.2.2. The quantities $\overline{\nu}(n,b)$ are generally not equal to the maximal power of b dividing $\overline{G}_n$; in fact they can sometimes be larger than, and sometimes be smaller than, this quantity.

The main purpose of this chapter is to study the asymptotics of the partial factorizations of $\overline{\overline{G}}_n$, defined by

$$\overline{\overline{G}}(n,x) := \prod_{2 \le b \le x} b^{\overline{\nu}(n,b)}, \tag{4.1.10}$$

parallel to those in [13]. Here $\overline{\overline{G}}(n,x)$ is an integer sequence in n for fixed x because all $\overline{\nu}(n,b)$ are nonnegative integers. We have $\overline{\overline{G}}_n = \overline{\overline{G}}(n,n)$, and we have the stabilization

$$\overline{\overline{G}}(n,x) = \overline{\overline{G}}(n,n) = \overline{\overline{G}}_n \quad \text{for} \quad x \geq n.$$

The main results of this chapter determine the growth rate of the integer sequence $\overline{\overline{G}}_n$ and more generally the growth behavior of $\log \overline{\overline{G}}(n,x)$ for all $n \geq 1$. The overall approach of the proofs have parallels to that in [13] but have some significant differences, as given in Section 4.1.2.

There are a number of reasons for interest in the study of integer sequences like $\overline{\overline{G}}(n,x)$. The binomial products $\overline{G}_n$ single out prime bases as special via their prime factorizations (4.1.3). But the definition (4.1.9) (compare (4.1.4)) makes sense for all integers $b \geq 2$. Heuristically, we may expect that a sum over the primes p in an interval I (weighted by prime gaps Δp) can be approximated by a smoother sum over the integers $b \in I$:

$$\sum_{p \in I} f(p)\Delta p \approx \sum_{b \in I} f(b).$$

This is analogous to the Euler–Maclaurin formula which approximates a sum over the integers $b \in I$ by an integral over I:

$$\sum_{b \in I} f(b) \approx \int_I f(x)\, dx.$$

The sequence $\overline{\overline{G}}(n,x)$ is defined to be a product over all integers $b \in [2,x]$. So $\log \overline{\overline{G}}(n,x)$ can be viewed as a "discrete" integral related to $\log G(n,x)$ as a sum (without prime gap weights Δp). The asymptotic study in this chapter gives a motivating example of the effect of this kind of discrete approximation. Finally, the sequence $\overline{\overline{G}}_n$ arises as a special case of products of generalized binomial coefficients studied in Chapter 3; see Corollary 3.2.16.

4.1.1 Main results: Asymptotics of $\overline{\overline{G}}_n$ and $\overline{\overline{G}}(n,x)$

We obtain the following result for the the sequence $\overline{\overline{G}}_n$.

Theorem 4.1.1. *Let $\overline{\overline{G}}_n = \prod_{b=2}^{n} b^{\overline{\nu}(n,b)}$. Then for integers $n \geq 2$,*

$$\log \overline{\overline{G}}_n = \frac{1}{2} n^2 \log n + \left(\frac{1}{2}\gamma - \frac{3}{4}\right) n^2 + O\left(n^{3/2} \log n\right), \tag{4.1.11}$$

where γ is Euler's constant.

This result is proved in Section 4.4. Although we have shown in Chapter 3 that $\overline{\overline{G}}_n$ is a product of ratios of generalized factorials, we do not know if $\overline{\overline{G}}_n$ can be written as a (nice) product of ratios of the usual factorials. Thus we do not have Stirling's formula available to directly estimate the size of $\overline{\overline{G}}_n$, nor do we have combinatorial identities and recursion formulas available in dealing with binomial coefficients.

We note that compared to [13] there are two main terms in the asymptotics, rather then one. The leading order term has the same constant $\frac{1}{2}$ as for $\log \overline{G}_n$ in (4.1.2), while Euler's constant appears in the second leading order term. This result will be used as an initial condition to obtain estimates for partial factorizations $\overline{\overline{G}}(n,x)$.

The main result of the chapter determines the size of the partial factorization function $\overline{\overline{G}}(n,x)$ in the range $1 \leq x \leq n$. It establishes the following limiting behavior as $n \to \infty$, taking $x = \alpha n$ where α is a scaling parameter.

Theorem 4.1.2. *Let $\overline{\overline{G}}(n,x) = \prod_{b=2}^{\lfloor x \rfloor} b^{\overline{\nu}(n,b)}$. Then for integers $n \geq 2$ and real $\alpha \in \left[\frac{1}{\sqrt{n}}, 1\right]$,*

$$\log \overline{\overline{G}}(n, \alpha n) = f_{\overline{\overline{G}}}(\alpha)\, n^2 \log n + g_{\overline{\overline{G}}}(\alpha)\, n^2 + O\left(n^{3/2} \log n\right), \tag{4.1.12}$$

in which:

(a) $f_{\overline{\overline{G}}}(\alpha)$ is a function with $f_{\overline{\overline{G}}}(0) = 0$ and defined for $\alpha > 0$ by

$$f_{\overline{\overline{G}}}(\alpha) = \frac{1}{2} + \frac{1}{2}\alpha^2 \left\lfloor \frac{1}{\alpha} \right\rfloor^2 + \frac{1}{2}\alpha^2 \left\lfloor \frac{1}{\alpha} \right\rfloor - \alpha \left\lfloor \frac{1}{\alpha} \right\rfloor; \tag{4.1.13}$$

(b) $g_{\overline{\overline{G}}}(\alpha)$ *is a function with* $g_{\overline{\overline{G}}}(0) = 0$ *and defined for* $\alpha > 0$ *by*

$$\begin{aligned} g_{\overline{\overline{G}}}(\alpha) &= \left(\frac{1}{2}\gamma - \frac{3}{4}\right) - \frac{1}{2}\left(H_{\lfloor \frac{1}{\alpha} \rfloor} - \log\frac{1}{\alpha}\right) \\ &+ \left(\log\frac{1}{\alpha}\right)\left(-\frac{1}{2} - \frac{1}{2}\alpha^2\left\lfloor\frac{1}{\alpha}\right\rfloor\left\lfloor\frac{1}{\alpha}+1\right\rfloor + \alpha\left\lfloor\frac{1}{\alpha}\right\rfloor\right) \\ &- \frac{1}{4}\alpha^2\left\lfloor\frac{1}{\alpha}\right\rfloor\left\lfloor\frac{1}{\alpha}+1\right\rfloor + \alpha\left\lfloor\frac{1}{\alpha}\right\rfloor. \end{aligned} \tag{4.1.14}$$

Moreover, for integers $n \geq 2$ *and real* $\alpha \in \left[\frac{1}{n}, \frac{1}{\sqrt{n}}\right]$,

$$\log \overline{\overline{G}}(n, \alpha n) = O\left(n^{3/2}\log n\right). \tag{4.1.15}$$

Theorem 4.1.2 follows from Theorem 4.7.1, taking $\alpha = \frac{x}{n}$. The theorem implies that $f_{\overline{\overline{G}}}(\alpha)$ can be defined as a limit function

$$f_{\overline{\overline{G}}}(\alpha) := \lim_{n\to\infty} \frac{1}{n^2 \log n} \log \overline{\overline{G}}(n, \alpha n).$$

In fact

$$f_{\overline{\overline{G}}}(\alpha) = f_G(\alpha), \tag{4.1.16}$$

where

$$f_G(\alpha) := \lim_{n\to\infty} \frac{1}{n^2} \log G(n, \alpha n)$$

is the limit function given in [13, Theorem 1.1].

We note an alternate form for $f_{\overline{\overline{G}}}(\alpha)$ given by

$$f_{\overline{\overline{G}}}(\alpha) = \frac{1}{2}\alpha^2\left(\left\lfloor\frac{1}{\alpha}\right\rfloor + \left\{\frac{1}{\alpha}\right\}^2\right), \tag{4.1.17}$$

where $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x. It is pictured in Figure 4.1.

Some properties of the limit function follow from [13, Lemma 4.2], since $f_{\overline{\overline{G}}}(\alpha) = f_G(\alpha)$.

(i) The function $f_{\overline{\overline{G}}}(\alpha)$ is continuous on $[0, \infty)$. It is differentiable everywhere except at $\alpha = \frac{1}{j}$ for $j = 1, 2, 3, \ldots$, where

$$\lim_{h\to 0^+} \frac{f_{\overline{\overline{G}}}(\frac{1}{j}+h) - f_{\overline{\overline{G}}}(\frac{1}{j})}{h} = 0 \quad \text{and} \quad \lim_{h\to 0^-} \frac{f_{\overline{\overline{G}}}(\frac{1}{j}+h) - f_{\overline{\overline{G}}}(\frac{1}{j})}{h} = 1.$$

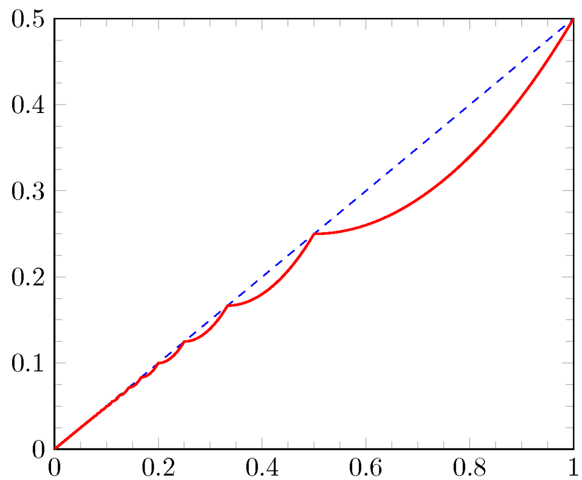

Figure 4.1: The graph $\beta = f_{\overline{\overline{G}}}(\alpha)$, $0 \le \alpha \le 1$ (solid red) in the (α, β)-plane. The line segment $\beta = \frac{1}{2}\alpha$, $0 \le \alpha \le 1$ is shown in dashed blue.

At $\alpha = 0$, we have

$$\lim_{h \to 0^+} \frac{f_{\overline{\overline{G}}}(h) - f_{\overline{\overline{G}}}(0)}{h} = \frac{1}{2}.$$

(ii) It satisfies

$$f_{\overline{\overline{G}}}(\alpha) \le \frac{1}{2}\alpha \quad \text{for} \quad \alpha \ge 0. \tag{4.1.18}$$

Equality holds if and only if $\alpha = \frac{1}{j}$ for some positive integer j or $\alpha = 0$.

(iii) $f_{\overline{\overline{G}}}(\alpha)$ is strictly increasing and is piecewise quadratic on $(0, 1]$. For integers $j \ge 1$ and real $\alpha \in \left[\frac{1}{j+1}, \frac{1}{j}\right]$,

$$f_{\overline{\overline{G}}}(\alpha) = \frac{1}{2} - j\alpha + \frac{j(j+1)}{2}\alpha^2. \tag{4.1.19}$$

Furthermore, $f_{\overline{\overline{G}}}(\alpha) = \frac{1}{2}$ for $\alpha \ge 1$.

The limit function $g_{\overline{\overline{G}}}(\alpha)$ is pictured in Figure 4.2. Some properties of this limit function are:

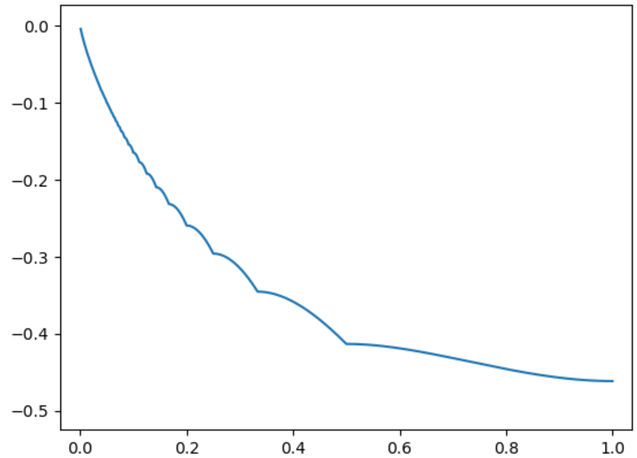

Figure 4.2: The graph $\beta = g_{\overline{\overline{G}}}(\alpha)$, $0 \le \alpha \le 1$ in the (α, β)-plane.

(i) The function $g_{\overline{\overline{G}}}(\alpha)$ is continuous on $[0, \infty)$. It is real-analytic on $(0, \infty)$ except at $\alpha = \frac{1}{j}$ for $j = 1, 2, 3, \ldots$. It is differentiable at $\alpha = 1$ with $g'_{\overline{\overline{G}}}(1) = 0$.

(ii) $g_{\overline{\overline{G}}}(\alpha)$ is strictly decreasing on $[0, 1]$. It has $g_{\overline{\overline{G}}}(\alpha) = \frac{1}{2}\gamma - \frac{3}{4} \approx -0.46139$ for $\alpha \ge 1$.

4.1.2 Results: Asymptotics of $\overline{A}(n)$ and $\overline{B}(n)$

To obtain Theorem 4.1.2, we determine the asymptotics of similar sums as in [13]. Taking the logarithms of both sides of the product formula (4.1.10) and substituting $\overline{\nu}(n, b)$ using (4.1.9) yields

$$\log \overline{\overline{G}}(n, x) = \overline{A}(n, x) - \overline{B}(n, x), \tag{4.1.20}$$

where

$$\overline{A}(n, x) = \sum_{2 \le b \le x} \frac{2}{b-1} S_b(n) \log b \tag{4.1.21}$$

and

$$\overline{B}(n, x) = \sum_{2 \le b \le x} \frac{n-1}{b-1} d_b(n) \log b. \tag{4.1.22}$$

The functions $\overline{B}(n,x)$ and $\overline{A}(n,x)$ are weighted sums of the base-b digits of n (and smaller positive integers) over $b \in [2,x]$.

The determination of the asymptotics for the sums $\overline{A}(n,x)$ and $\overline{B}(n,x)$ are given later in the chapter, from which we obtain the asymptotics for $\log \overline{\overline{G}}(n,x)$ via (4.1.20). On the one hand, the computations in this chapter are more involved than those of [13] in order to obtain a secondary term and a power-savings remainder term. On the other hand, many sums over the integers $b \in [2,x]$ here are easier to handle than sums over the primes $p \in [2,x]$ in [13], and the Riemann hypothesis is not needed.

The proofs first obtain estimates for the case $x = n$, setting

$$\overline{A}(n) := \overline{A}(n,n) = \sum_{b=2}^{n} \frac{2}{b-1} S_b(n) \log b, \tag{4.1.23}$$

$$\overline{B}(n) := \overline{B}(n,n) = \sum_{b=2}^{n} \frac{n-1}{b-1} d_b(n) \log b. \tag{4.1.24}$$

We determine the asymptotic for $\overline{B}(n)$ first. Then we estimate $\overline{A}(n)$ by applying the estimates for $\overline{B}(j)$ for $1 \le j \le n$.

Theorem 4.1.3. *Let $\overline{B}(n) := \sum_{b=2}^{n} \frac{n-1}{b-1} d_b(n) \log b$. Then for integers $n \ge 2$,*

$$\overline{B}(n) = (1-\gamma) n^2 \log n + (\gamma + \gamma_1 - 1)\, n^2 + O\left(n^{3/2} \log n\right), \tag{4.1.25}$$

where γ is Euler's constant and γ_1 is the first Stieltjes constant.

The result involves the Stieltjes constants $\gamma_0 = \gamma$ and $\gamma_1 \approx -0.07282$. The *Stieltjes constants* γ_n appear in the Laurent expansion of the Riemann zeta function at $s = 1$:

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (s-1)^n. \tag{4.1.26}$$

Here $\gamma_0 = \gamma \approx 0.57721$ is Euler's constant (see the survey [19] for more about γ), and more generally

$$\gamma_n := \lim_{m\to\infty} \left(\sum_{k=1}^{m} \frac{(\log k)^n}{k} - \frac{(\log m)^{n+1}}{n+1} \right). \tag{4.1.27}$$

We prove Theorem 4.1.3 in a similar fashion to [13]. We first show that the main contribution in the sum $\overline{B}(n)$ comes from the terms with $b > \sqrt{n}$. Then we observe that n has at most two digits in those bases b and that the value of $d_b(n)$ follows some simple pattern. The rest is a straightforward calculation.

We also deduce a corresponding result for $\overline{A}(n)$.

Theorem 4.1.4. *Let* $\overline{A}(n) := \sum_{b=2}^{n} \frac{2}{b-1} S_b(n) \log b$. *Then for integers* $n \geq 2$,

$$\overline{A}(n) = \left(\frac{3}{2} - \gamma\right) n^2 \log n + \left(\frac{3}{2}\gamma + \gamma_1 - \frac{7}{4}\right) n^2 + O\left(n^{3/2} \log n\right), \tag{4.1.28}$$

where γ *is Euler's constant and* γ_1 *is the first Stieltjes constant.*

In [13], the asymptotic of $\log \overline{G}_n$ can be computed directly by Stirling's formula. They proved an analogue of Theorem 4.1.4 from the asymptotic of $\log \overline{G}_n$ and an analogue of Theorem 4.1.3. Here we first prove Theorem 4.1.3 then use it to prove Theorem 4.1.4 and the asymptotic of $\log \overline{\overline{G}}_n$ (Theorem 4.1.1), via the relation

$$\log \overline{\overline{G}}_n = \overline{A}(n) - \overline{B}(n), \tag{4.1.29}$$

which is a special case of (4.1.20), taking $x = n$.

4.1.3 Results: Asymptotics of $\overline{A}(n, x)$ and $\overline{B}(n, x)$

We first determine asymptotics for $\overline{B}(n, \alpha n)$ for $0 \leq \alpha \leq 1$, by bootstrapping the result for $\overline{B}(n) = \overline{B}(n, n)$ decreasing x from $x = n$. In what follows $H_m := \sum_{j=1}^{m} \frac{1}{j}$ and $J_m := \sum_{j=1}^{m} \frac{\log j}{j}$.

Theorem 4.1.5. *Let* $\overline{B}(n, x) = \sum_{b=2}^{\lfloor x \rfloor} \frac{n-1}{b-1} d_b(n) \log b$. *Then for integers* $n \geq 2$ *and real* $\alpha \in \left[\frac{1}{\sqrt{n}}, 1\right]$,

$$\overline{B}(n, \alpha n) = f_{\overline{B}}(\alpha) n^2 \log n + g_{\overline{B}}(\alpha) n^2 + O\left(n^{3/2} \log n\right), \tag{4.1.30}$$

in which:

(a) $f_{\overline{B}}(\alpha)$ *is a function with* $f_{\overline{B}}(0) = 0$ *and defined for* $\alpha > 0$ *by*

$$f_{\overline{B}}(\alpha) = (1-\gamma) + \left(H_{\lfloor \frac{1}{\alpha} \rfloor} - \log\frac{1}{\alpha}\right) - \alpha\left\lfloor \frac{1}{\alpha} \right\rfloor; \tag{4.1.31}$$

(b) $g_{\overline{B}}(\alpha)$ *is a function with* $g_{\overline{B}}(0) = 0$ *and defined for* $\alpha > 0$ *by*

$$\begin{aligned} g_{\overline{B}}(\alpha) &= (\gamma + \gamma_1 - 1) - \left(H_{\lfloor \frac{1}{\alpha} \rfloor} - \log\frac{1}{\alpha}\right) - \left(J_{\lfloor \frac{1}{\alpha} \rfloor} - \frac{1}{2}\left(\log\frac{1}{\alpha}\right)^2\right) \\ &\quad + \left(\log\frac{1}{\alpha}\right)\left(-1 + \alpha\left\lfloor \frac{1}{\alpha} \right\rfloor\right) + \alpha\left\lfloor \frac{1}{\alpha} \right\rfloor. \end{aligned} \tag{4.1.32}$$

Moreover, for integers $n \geq 2$ *and real* $\alpha \in \left[\frac{1}{n}, \frac{1}{\sqrt{n}}\right]$,

$$\overline{B}(n, \alpha n) = O\left(n^{3/2}\log n\right). \tag{4.1.33}$$

The functions $f_{\overline{B}}(\alpha)$ and $g_{\overline{B}}(\alpha)$ as given in (4.1.31) and (4.1.32) appear to be combinations of functions with jump discontinuities at $\alpha = \frac{1}{j}$, $j = 1, 2, 3, \ldots$. However, it can be shown that $f_{\overline{B}}(\alpha)$ and $g_{\overline{B}}(\alpha)$ are continuous on $[0, \infty)$.

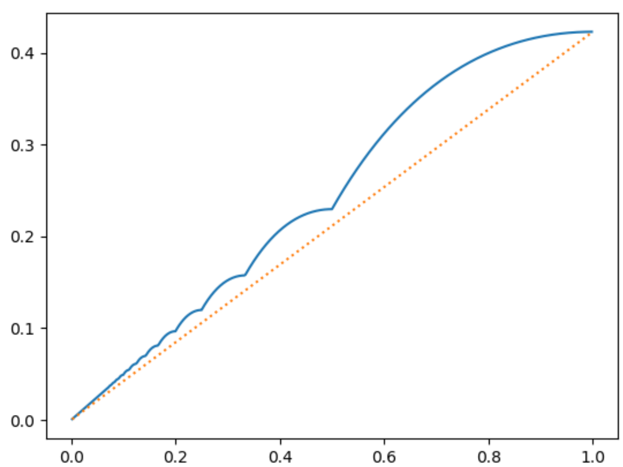

Figure 4.3: The graph $\beta = f_{\overline{B}}(\alpha)$, $0 \leq \alpha \leq 1$ (solid blue) in the (α, β)-plane. The line segment $\beta = (1-\gamma)\alpha$, $0 \leq \alpha \leq 1$ is shown in dotted orange.

The limit function $f_{\overline{B}}(\alpha)$ is pictured in Figure 4.3. It is the same limit function as $f_B(\alpha)$ in Theorem 1.5 in [13]. Some properties of this limit function are:

(i) The function $f_{\overline{B}}(\alpha)$ is continuous on $[0,\infty)$. It is differentiable everywhere except at $\alpha = \frac{1}{j}$ for $j = 1, 2, 3, \ldots$, where

$$\lim_{h\to 0^+} \frac{f_{\overline{B}}(\frac{1}{j}+h) - f_{\overline{B}}(\frac{1}{j})}{h} = 0 \quad \text{and} \quad \lim_{h\to 0^-} \frac{f_{\overline{B}}(\frac{1}{j}+h) - f_{\overline{B}}(\frac{1}{j})}{h} = 1.$$

At $\alpha = 0$, we have

$$\lim_{h\to 0^+} \frac{f_{\overline{B}}(h) - f_{\overline{B}}(0)}{h} = \frac{1}{2}.$$

(ii) $f_{\overline{B}}(\alpha)$ is strictly increasing on $[0,\infty)$. It has $f_{\overline{B}}(1) = 1 - \gamma \approx 0.42278$.

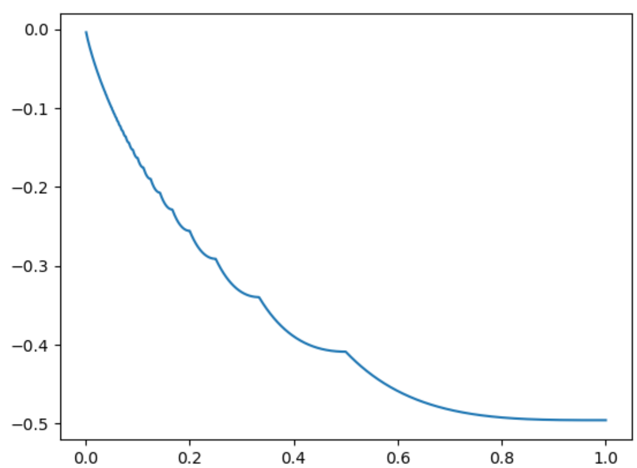

Figure 4.4: The graph $\beta = g_{\overline{B}}(\alpha)$, $0 \le \alpha \le 1$ in the (α, β)-plane.

The limit function $g_{\overline{B}}(\alpha)$ is pictured in Figure 4.4. Some properties of this limit function are:

(i) The function $g_{\overline{B}}(\alpha)$ is continuous on $[0,\infty)$. It is real-analytic on $(0,\infty)$ except at $\alpha = \frac{1}{j}$ for $j = 1, 2, 3, \ldots$. It is differentiable at $\alpha = 1$ with $g'_{\overline{B}}(1) = 0$.

(ii) $g_{\overline{B}}(\alpha)$ is strictly decreasing on $[0,1]$ and is strictly increasing on $[1,\infty)$. Its minimum on $[0,\infty)$ is attained at $\alpha = 1$ with $g_{\overline{B}}(1) = \gamma + \gamma_1 - 1 \approx -0.49560$.

Next, we obtain the asymptotics of $\overline{A}(n,x)$ using a recursion (4.6.5), starting from $\overline{A}(n,n)$ and working downward. The recursion involves a function $\overline{C}(n,x)$ studied in Subsection 4.2.4. This approach is different from that used in [13], which started from $A(x,x)$ and worked upward using estimates for $B(y,x)$ for $x < y < n$.

Theorem 4.1.6. *Let* $\overline{A}(n,x) = \sum_{b=2}^{\lfloor x \rfloor} \frac{2}{b-1} S_b(n) \log b$*. Then for integers* $n \geq 2$ *and real* $\alpha \in \left[\frac{1}{\sqrt{n}}, 1\right]$,

$$\overline{A}(n,\alpha n) = f_{\overline{A}}(\alpha) n^2 \log n + g_{\overline{A}}(\alpha) n^2 + O\left(n^{3/2} \log n\right), \tag{4.1.34}$$

in which:

(a) $f_{\overline{A}}(\alpha)$ *is a function with* $f_{\overline{A}}(0) = 0$ *and defined for* $\alpha > 0$ *by*

$$f_{\overline{A}}(\alpha) = \left(\frac{3}{2} - \gamma\right) + \left(H_{\lfloor \frac{1}{\alpha} \rfloor} - \log \frac{1}{\alpha}\right) + \frac{1}{2}\alpha^2 \left\lfloor \frac{1}{\alpha} \right\rfloor^2 + \frac{1}{2}\alpha^2 \left\lfloor \frac{1}{\alpha} \right\rfloor - 2\alpha \left\lfloor \frac{1}{\alpha} \right\rfloor; \tag{4.1.35}$$

(b) $g_{\overline{A}}(\alpha)$ *is a function with* $g_{\overline{A}}(0) = 0$ *and defined for* $\alpha > 0$ *by*

$$\begin{aligned} g_{\overline{A}}(\alpha) \;=\; & \left(\frac{3}{2}\gamma + \gamma_1 - \frac{7}{4}\right) - \frac{3}{2}\left(H_{\lfloor \frac{1}{\alpha} \rfloor} - \log \frac{1}{\alpha}\right) - \left(J_{\lfloor \frac{1}{\alpha} \rfloor} - \frac{1}{2}\left(\log \frac{1}{\alpha}\right)^2\right) \\ & + \left(\log \frac{1}{\alpha}\right)\left(-\frac{3}{2} - \frac{1}{2}\alpha^2 \left\lfloor \frac{1}{\alpha} \right\rfloor \left\lfloor \frac{1}{\alpha} + 1 \right\rfloor + 2\alpha \left\lfloor \frac{1}{\alpha} \right\rfloor\right) \\ & - \frac{1}{4}\alpha^2 \left\lfloor \frac{1}{\alpha} \right\rfloor \left\lfloor \frac{1}{\alpha} + 1 \right\rfloor + 2\alpha \left\lfloor \frac{1}{\alpha} \right\rfloor. \end{aligned} \tag{4.1.36}$$

Moreover, for integers $n \geq 2$ *and real* $\alpha \in \left[\frac{1}{n}, \frac{1}{\sqrt{n}}\right]$,

$$\overline{A}(n,\alpha n) = O\left(n^{3/2} \log n\right). \tag{4.1.37}$$

The functions $f_{\overline{A}}(\alpha)$ and $g_{\overline{A}}(\alpha)$ as given in (4.1.35) and (4.1.36) appear to be combinations of functions with jump discontinuities at $\alpha = \frac{1}{j}$, $j = 1, 2, 3, \ldots$. However, it can be shown that $f_{\overline{A}}(\alpha)$ and $g_{\overline{A}}(\alpha)$ are continuous on $[0, \infty)$.

The limit function $f_{\overline{A}}(\alpha)$ is pictured in Figure 4.5. It is the same limit function as $f_A(\alpha)$ in Theorem 1.6 in [13]. Some properties of this limit function are:

(i) The function $f_{\overline{A}}(\alpha)$ is continuously differentiable on $(0,\infty)$ with

$$f'_{\overline{A}}(\alpha) = \frac{2}{\alpha} f_{\overline{\overline{G}}}(\alpha).$$

At $\alpha = 0$, we have

$$\lim_{h\to 0^+} \frac{f_{\overline{A}}(h) - f_{\overline{A}}(0)}{h} = 1 = \lim_{\alpha\to 0^+} f'_{\overline{A}}(\alpha).$$

(ii) $f_{\overline{A}}(\alpha)$ is strictly increasing on $[0,\infty)$. It has $f_{\overline{A}}(1) = \frac{3}{2} - \gamma \approx 0.92278$.

(iii) It satisfies the relation

$$f_{\overline{\overline{G}}}(\alpha) = f_{\overline{A}}(\alpha) - f_{\overline{B}}(\alpha).$$

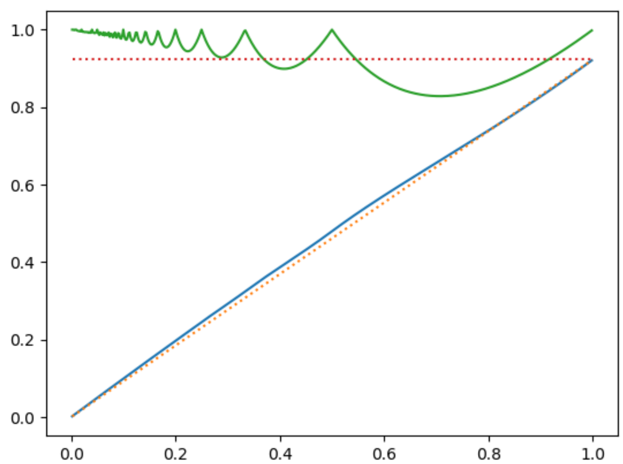

Figure 4.5: The graph $\beta = f_{\overline{A}}(\alpha)$, $0 \le \alpha \le 1$ (solid blue) in the (α,β)-plane. The line segment $\beta = \left(\frac{3}{2} - \gamma\right)\alpha$, $0 \le \alpha \le 1$ is shown in dotted orange. Superimposed are the graph $\beta = f'_{\overline{A}}(\alpha)$, $0 \le \alpha \le 1$ shown in solid green and the line segment $\beta = \frac{3}{2} - \gamma$, $0 \le \alpha \le 1$ shown in dotted red.

The limit function $g_{\overline{A}}(\alpha)$ is pictured in Figure 4.6. Some properties of this limit function are:

(i) The function $g_{\overline{A}}(\alpha)$ is continuously differentiable on $(0,\infty)$ with

$$g'_{\overline{A}}(\alpha) = -\frac{2}{\alpha}\left(\log\frac{1}{\alpha}\right) f_{\overline{\overline{G}}}(\alpha).$$

(ii) $g_{\overline{A}}(\alpha)$ is strictly decreasing on $[0,1]$ and is strictly increasing on $[1,\infty)$. Its minimum on $[0,\infty)$ is attained at $\alpha = 1$ with $g_{\overline{A}}(1) = \frac{3}{2}\gamma + \gamma_1 - \frac{7}{4} \approx -0.95699$.

(iii) It satisfies the relation

$$g_{\overline{\overline{G}}}(\alpha) = g_{\overline{A}}(\alpha) - g_{\overline{B}}(\alpha).$$

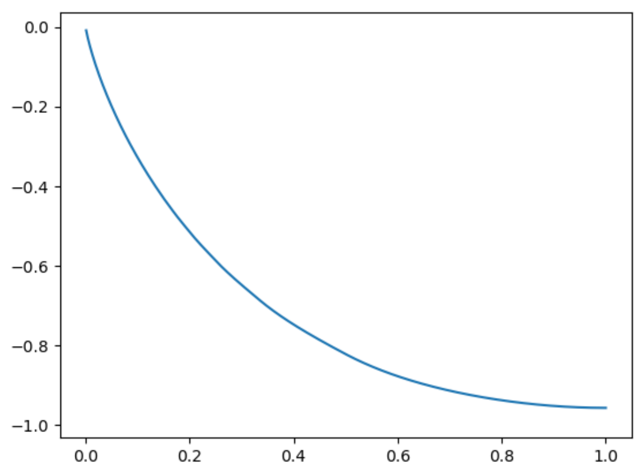

Figure 4.6: The graph $\beta = g_{\overline{A}}(\alpha)$, $0 \leq \alpha \leq 1$ in the (α, β)-plane.

We obtain Theorem 4.1.2 as a corollary of Theorems 4.1.5 and 4.1.6 by substituting their estimates into the relation (4.1.20).

4.1.4 Discussion

We compare and contrast the main results in this chapter with those for binomial products in [13].

1. Each of $\log \overline{\overline{G}}(n,x)$, $\overline{A}(n,x)$, and $\overline{B}(n,x)$ has a main term $f(\alpha)n^2 \log n$ and a secondary term $g(\alpha)n^2$, where $\alpha = \frac{x}{n}$. The main term scaling functions $f(\alpha)$ match those in [13]. The secondary term scaling functions $g(\alpha)$ are new scaling functions whose salient feature is that they are strictly decreasing on $[0,1]$.

2. There is a power-savings remainder term $O(n^{3/2+\epsilon})$, which is provable *unconditionally.* Parallel results of [13] had power-savings remainder term conditional on the Riemann hypothesis. They argued (but did not prove rigorously) that the existence of a power-savings remainder term in their results would imply the existence of a zero-free region for the Riemann zeta function of the form $\Re(s) > 1 - \delta$ for some $\delta \in (0, 1/2]$ depending on the amount of power saving. For $\overline{\overline{G}}(n, x)$, the averaging over all integers $b \in [2, x]$ led to an unconditional power saving in the remainder term.

4.1.5 Related work

Lagarias and Mehta [20] studied radix expansion statistics radix expansion statistics of integers which hold the integer n fixed, while varying across different radix bases up to n e.g. statistics $A(n, x)$ and $B(n, x)$. The work [21] studied analogous statistics for Farey fractions. Du and Lagarias [13] studied the statistics $A(n, x)$ and $B(n, x)$ for products of binomial coefficients. The motivation of [13] was study of prime number distribution from a novel direction.

There has been a great deal of study of the radix statistics $d_b(n)$ and $S_b(n)$ for a fixed base $b \geq 2$ and letting n vary. Work on $d_b(n)$ has mainly been probabilistic, for random integers in an initial interval $[1, n]$, which is surveyed by Chen et al [8]. One has for all $n \geq 1$,

$$\mathbb{E}[d_b(k) : 0 \leq k \leq n-1] \leq \frac{b-1}{2} \log_b n, \tag{4.1.38}$$

a result which is close to sharp when $n = b^k$ for some integer $k \geq 1$. We have $d_b(n) \leq (b-1)\log_b(n+1$, see Lemma 4.2.4. It implies

$$\overline{B}(n) \leq \sum_{b=2}^{n} \frac{n-1}{b-1} \left(\frac{(b-1)\log(n+1)}{\log b} \right) \log b = (n-1)^2 \log(n+1).$$

Work on the smoothed function $S_b(n)$ studying asymptotics of the as $n \to \infty$ started with Bush [7] in 1940, followed by Bellman and Shapiro [3], and Mirsky [22], who in 1949 showed that for fixed $b \geq 2$, the asymptotic formula

$$S_b(n) = \frac{b-1}{2} n \log_b(n) + O(n).$$

In 1952 Drazin and Griffith [10] deduced an inequality implying

$$S_b(n) \leq \frac{b-1}{2} n \log_b n, \tag{4.1.39}$$

for all $b \geq 2$ and $n \geq 1$, see Lemma 4.2.4. This inequality is sharp: equality holds for $n = b^k$ for $k \geq 1$, see [20, Theorem 5.8]. Using Drazin and Griffith's inequality (4.1.39) for $S_b(n)$ we have

$$\overline{A}(n) \leq \overline{A}^*(n) := \sum_{b=2}^{n} \frac{2}{b-1} \left(\frac{(b-1) n \log n}{2 \log b} \right) \log b = n(n-1) \log n. \tag{4.1.40}$$

A formula of Trollope [26] in 1968 gave an exact formula for $S_b(n)$ for base $b = 2$. Notable work of Delange [9] obtained exact formulas for $S_b(n)$ for all $b \geq 2$, which exhibited an oscillating term in the asymptotics. We mention later work of Flajolet et al [14] and Grabner and Hwang [15]. Recently Drmota and Grabner [11] surveyed this topic.

We mention other inequalities for the function $S_b(n)$. In 2011, Allaart [1, Equation (4)] showed an approximate convexity inequality for binary expansions: For integers ℓ and m with $0 \leq \ell \leq m$,

$$S_2(m+\ell) + S_2(m-\ell) - 2S_2(m) \leq \ell. \tag{4.1.41}$$

Allaart [2, Theorem 3] proved a generalization to any base b:

$$S_b(m+\ell) + S_b(m-\ell) - 2S_b(m) \leq \left\lfloor \frac{b+1}{2} \right\rfloor \ell. \tag{4.1.42}$$

Allaart [2, Theorem 1] also showed a superadditivity inequality:

$$S_b(\ell+m) \geq S_b(\ell) + S_b(m) + \ell. \tag{4.1.43}$$

4.1.6 Contents of this chapter

Section 4.2 collects facts about digit sums and provides estimates for a wide variety of sums needed in later estimates. In particular, we estimate the function

$$\overline{C}(n,x) := \sum_{1 \le b \le x} \left\lfloor \frac{n}{b} \right\rfloor \log b. \tag{4.1.44}$$

Section 4.3 estimates $\overline{B}(n)$, proving Theorem 4.1.3.

Section 4.4 estimates $\overline{A}(n)$, proving Theorem 4.1.4. Theorem 4.1.1 for $\overline{\overline{G}}(n)$ then follows.

Section 4.5 estimates $\overline{B}(n,x)$, proving Theorem 4.5.1. Theorem 4.1.5 for $\overline{B}(n,\alpha n)$ then follows.

Section 4.6 estimates $\overline{A}(n,x)$, proving Theorem 4.6.1. Theorem 4.1.6 for $\overline{A}(n,\alpha n)$ then follows.

Section 4.7 estimates $\overline{\overline{G}}(n,x)$, proving Theorem 4.7.1. Theorem 4.1.2 for $\overline{\overline{G}}(n,\alpha n)$ then follows.

Section 4.8 presents concluding remarks. These include an interpretation of $\overline{\overline{G}}_n$ as a product of generalized binomial coefficients, treated in Chapter 3.

4.2 Preliminaries

The first subsection establishes properties of radix expansion statistics $\overline{\nu}(n,b)$, and derives inequalities on the size of $d_b(n)$ and $S_b(n)$. The next four subsections estimate four families of sums for an integer n and a real number x, treated as step functions: the harmonic numbers $H(x) = \sum_{b=1}^{\lfloor x \rfloor} \frac{1}{b}$, the sums $J(x) = \sum_{b=1}^{\lfloor x \rfloor} \frac{\log b}{b}$, the sums $\overline{C}(n,x) = \sum_{b=1}^{\lfloor x \rfloor} \lfloor \frac{n}{b} \rfloor \log b$, and $L_i(n) = \sum_{b=2}^{n} b(\log b)^i$ for $i \ge 1$.

4.2.1 Radix expansion statistics

Fix an integer $b \geq 2$. Let n be a positive integer. Then n can be written uniquely as

$$n = \sum_{i=0}^{k} a_i(b,n) b^i, \tag{4.2.1}$$

where $a_i(b,n) \in \{0,1,2,\ldots,b-1\}$ are the *base-b digits* of n and the *top* digit a_k is positive. We say that n has $k+1$ digits in base b. One has $b^k \leq n < b^{k+1}$. Hence the number of base-b digits of n is

$$\left\lfloor \frac{\log n}{\log b} \right\rfloor + 1.$$

Each base-b digit of n can also be expressed in terms of the floor function:

$$a_i(b,n) = \left\lfloor \frac{n}{b^i} \right\rfloor - b \left\lfloor \frac{n}{b^{i+1}} \right\rfloor . \tag{4.2.2}$$

Note that (4.2.2) also defines $a_i(b,n)$ to be 0 for all $i > \frac{\log n}{\log b}$. The following two statistics of the base-b digits of numbers will show up frequently in this chapter.

Definition 4.2.1. (1) The *sum of digits function* $d_b(n)$ is given by

$$d_b(n) := \sum_{i=0}^{\left\lfloor \frac{\log n}{\log b} \right\rfloor} a_i(b,n) = \sum_{i=0}^{\infty} a_i(b,n), \tag{4.2.3}$$

where $a_i(b,n)$ is given by (4.2.2).

(2) The *running digit sum function* $S_b(n)$ is given by

$$S_b(n) := \sum_{j=1}^{n-1} d_b(j). \tag{4.2.4}$$

Theorem 4.2.2. *Let $b \geq 2$ be an integer, and let the radix expansion statistic $\overline{\nu}(n,b)$ given for $n \geq 1$ by*

$$\overline{\nu}(n,b) = \frac{2}{b-1} S_b(n) - \frac{n-1}{b-1} d_b(n) \tag{4.2.5}$$

Then:

(1) For all $n \geq 1$, $\overline{\nu}(n,b)$ is a nonnegative integer.

(2) $\overline{\nu}(n,b) = 0$ if and only if $n = ab^k + b^k - 1$ for some $a \in \{1,2,3,\ldots,b-1\}$ and integer $k \geq 0$.

Proof. To show (1), we substitute (4.2.2) into (4.2.3) and obtain

$$d_b(n) = \sum_{i=0}^{\infty} \left\lfloor \frac{n}{b^i} \right\rfloor - b \sum_{i=0}^{\infty} \left\lfloor \frac{n}{b^{i+1}} \right\rfloor = n - (b-1) \sum_{i=1}^{\infty} \left\lfloor \frac{n}{b^i} \right\rfloor . \tag{4.2.6}$$

We then substitute (4.2.6) into (4.2.4) and obtain

$$S_b(n) = \sum_{j=1}^{n-1} j - (b-1) \sum_{j=1}^{n-1} \sum_{i=1}^{\infty} \left\lfloor \frac{j}{b^i} \right\rfloor = \frac{n(n-1)}{2} - (b-1) \sum_{i=1}^{\infty} \sum_{j=1}^{n-1} \left\lfloor \frac{j}{b^i} \right\rfloor . \tag{4.2.7}$$

Now, we substitute (4.2.6) and (4.2.7) into (4.2.5) and obtain

$$\begin{aligned} \overline{\nu}(n,b) &= \left(\frac{n(n-1)}{b-1} - 2 \sum_{i=1}^{\infty} \sum_{j=1}^{n-1} \left\lfloor \frac{j}{b^i} \right\rfloor \right) - \left(\frac{n(n-1)}{b-1} - (n-1) \sum_{i=1}^{\infty} \left\lfloor \frac{n}{b^i} \right\rfloor \right) \\ &= \sum_{i=1}^{\infty} \left((n-1) \left\lfloor \frac{n}{b^i} \right\rfloor - 2 \sum_{j=1}^{n-1} \left\lfloor \frac{j}{b^i} \right\rfloor \right) \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{n-1} \left(\left\lfloor \frac{n}{b^i} \right\rfloor - \left\lfloor \frac{j}{b^i} \right\rfloor - \left\lfloor \frac{n-j}{b^i} \right\rfloor \right). \end{aligned} \tag{4.2.8}$$

The last quantity (4.2.8) expresses $\overline{\nu}(n,b)$ as the sum of integers, which are all non-negative due to the identity valid for all real x and y,

$$\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor + \lfloor \{x\} + \{y\} \rfloor \geq \lfloor x \rfloor + \lfloor y \rfloor,$$

see Graham et al [16, Section 3.1, page 70]. Hence $\overline{\nu}(n,b)$ is a nonnegative integer.

We show (2). We prove the 'only if' part first. Suppose that n is a positive integer not of the form $ab^k + b^k - 1$, where $1 \leq a \leq b-1$ and $k \geq 0$. Then $cb^\ell \leq n \leq (c+1)b^\ell - 2$ for some $c \in \{1,2,3,\ldots,b-1\}$ and positive integer ℓ. We show $\overline{\nu}(n,b)$ is positive. We see that the double sum in (4.2.8), is greater than or equal to the summand with $(i,j) = \left(\ell, b^\ell - 1\right)$. It follows that

$$\overline{\nu}(n,b) \geq \left\lfloor \frac{n}{b^\ell} \right\rfloor - \left\lfloor \frac{b^\ell - 1}{b^\ell} \right\rfloor - \left\lfloor \frac{n - b^\ell + 1}{b^\ell} \right\rfloor = c - 0 - (c-1) = 1.$$

Thus, if $\overline{\nu}(n,b)=0$, then n must be of the form ab^k+b^k-1 with $1\le a\le b-1$ and $k\ge 0$.

Conversely, suppose that n is of the form ab^k+b^k-1 with $1\le a\le b-1$ and $k\ge 0$. Suppose that j is an integer with $1\le j\le n-1$. For $i\le k-1$, we have $a_i(b,j)\le b-1=a_i(b,n)$. For $i\ge k$, we also have $a_i(b,j)\le a_i(b,n)$ because $j<n$. Hence

$$a_i(b,n-j)=a_i(b,n)-a_i(b,j)$$

for all $i\ge 0$. Summing over $i\ge 0$, we obtain

$$d_b(n-j)=d_b(n)-d_b(j).$$

Summing over $1\le j\le n-1$, we obtain

$$S_b(n)=(n-1)d_b(n)-S_b(n),$$

which implies

$$\overline{\nu}(n,b)=\frac{2}{b-1}S_b(n)-\frac{n-1}{b-1}d_b(n)=0.$$

This completes the proof. □

Remark 4.2.3. In general, $\overline{\nu}(n,b)$ does not equal the largest integer k such that b^k divides $\overline{G}_n$, which we denote by $\nu_b(\overline{G}_n)$. Moreover $\overline{\nu}(n,b)$ can be larger or smaller than $\nu_b(\overline{G}_n)$. For example, $\overline{\nu}(4,4)=3>2=\nu_4(\overline{G}_4)$, while $\overline{\nu}(6,4)=1<2=\nu_4(\overline{G}_6)$.

We establish inequalities on the size of $d_b(n)$ and $S_b(n)$.

Lemma 4.2.4. *For integers $b\ge 2$ and $n\ge 1$, we have*

$$1\le d_b(n)\le\frac{(b-1)\log(n+1)}{\log b}, \tag{4.2.9}$$

$$0\le S_b(n)\le\frac{(b-1)n\log n}{2\log b}. \tag{4.2.10}$$

Proof. The lower bound in (4.2.9) follows from the observation that $d_b(n)$ is greater than or equal to the top (base-b) digit of n, which is at least 1. The lower bound in (4.2.10) then follows from the positivity of $d_b(j)$.

The upper bound in (4.2.10) is a result of Drazin and Griffith [10, Theorem 1]. To prove the upper bound in (4.2.9), we apply Theorem 4.2.2:

$$0 \le (b-1)\overline{\nu}(n,b) = 2S_b(n) - (n-1)d_b(n) = 2S_b(n+1) - (n+1)d_b(n).$$

On replacing n by $n+1$ in (4.2.10), we obtain $S_b(n+1) \le \frac{(b-1)(n+1)\log(n+1)}{2\log b}$. Hence

$$d_b(n) \le \frac{2}{n+1} S_b(n+1) \le \frac{(b-1)\log(n+1)}{\log b},$$

as desired. □

4.2.2 The harmonic numbers H_n

For positive real numbers $x \ge 1$, we consider the step function

$$H(x) := \sum_{1 \le b \le x} \frac{1}{b}.$$

At integer values $n = \lfloor x \rfloor$ we write $H(x) = H_{\lfloor x \rfloor} = H_n$, the n-th harmonic number.

Lemma 4.2.5. *For positive integers n, we have*

$$H_n = \log n + \gamma + \frac{1}{2n} + O\left(\frac{1}{n^2}\right), \tag{4.2.11}$$

where $\gamma \approx 0.57721$ is Euler's constant.

Proof. This standard result appears in Tenenbaum [25, Chapter I.0, Theorem 5]. □

The restriction to integer n is needed in Lemma 4.2.5 because for positive real numbers x, one has

$$H(x) - \log x - \gamma = \Omega_{\pm}\left(\frac{1}{x}\right).$$

Indeed, using the Euler–Maclaurin summation formula [23, Theorem B.5], one can show that for real numbers $x \geq 1$,

$$H(x) = \log x + \gamma + \frac{1 - 2\{x\}}{2x} + O\left(\frac{1}{x^2}\right).$$

Hence, $\limsup_{x\to\infty} x(H(x) - \log x - \gamma) = \frac{1}{2}$ and $\liminf_{x\to\infty} x(H(x) - \log x - \gamma) = -\frac{1}{2}$.

4.2.3 Estimates: $J(x)$

For real numbers $x \geq 1$, we consider the step function

$$J(x) := \sum_{1 \leq b \leq x} \frac{\log b}{b}. \tag{4.2.12}$$

At integer values $n = \lfloor x \rfloor$ we write $J(x) = J_{\lfloor x \rfloor} = J_n$. The asymptotics of this step function of x involve the first Stieltjes constant γ_1, defined in Section 4.1.2.

Lemma 4.2.6. *For real numbers $x \geq 1$, we have*

$$J(x) = \frac{1}{2}(\log x)^2 + \gamma_1 + O\left(\frac{\log(x+1)}{x}\right), \tag{4.2.13}$$

where $\gamma_1 \approx -0.0728158$ is the first Stieltjes constant.

Proof. By partial summation, we obtain

$$J(x) = \sum_{1 \leq b \leq x} \frac{\log b}{b} = (\log x) H(x) - \int_1^x \frac{H(u)}{u}\, du. \tag{4.2.14}$$

It is well-known that $H(u) = \log u + \gamma + R(u)$, where the remainder $R(u) \ll \frac{1}{u}$, for $u \geq 1$. (See [23, Corollary 1.15].) On inserting this in (4.2.14) and rearranging, we get

$$\begin{aligned} J(x) &= \frac{1}{2}(\log x)^2 + (\log x) R(x) - \left(\int_1^\infty \frac{R(u)}{u}\, du - \int_x^\infty \frac{R(u)}{u}\, du\right) \\ &= \frac{1}{2}(\log x)^2 + c + O\left(\frac{\log(x+1)}{x}\right), \end{aligned}$$

where $c := -\int_1^\infty \frac{R(u)}{u}\, du$. By taking $x \to \infty$, we see that

$$c = \lim_{x\to\infty}\left(J(x) - \frac{1}{2}(\log x)^2\right) = \lim_{x\to\infty}\left(\sum_{b \leq x} \frac{\log b}{b} - \frac{1}{2}(\log x)^2\right) = \gamma_1,$$

the first Stieltjes constant, according to (4.1.27). ∎

4.2.4 Estimates: $\overline{C}(n,x)$

For real numbers $n \ge 1$ and $x \ge 1$, let

$$\overline{C}(n,x) := \sum_{1 \le b \le x} \left\lfloor \frac{n}{b} \right\rfloor \log b. \tag{4.2.15}$$

Here, $\overline{C}(n,x)$ is a nonnegative step function of the real variable x, viewing n as fixed. This function stabilizes for $x \ge n$:

$$\overline{C}(n,x) = \overline{C}(n,n) \quad \text{for} \quad x \ge n. \tag{4.2.16}$$

Proposition 4.2.7. *(1) For real numbers $n \ge 2$, we have*

$$\overline{C}(n,n) = \frac{1}{2} n (\log n)^2 + (\gamma - 1) n \log n + (1-\gamma) n + O\left(\sqrt{n} \log n\right).$$

(2) For real numbers $n \ge 2$ and x such that $1 \le x \le n$, we have

$$\overline{C}(n,n) - \overline{C}(n,x) = \int_x^n \left\lfloor \frac{n}{u} \right\rfloor \log u \, du + O\left(\frac{n \log n}{x}\right). \tag{4.2.17}$$

In addition,

$$\int_x^n \left\lfloor \frac{n}{u} \right\rfloor \log u \, du = \left(H_{\lfloor \frac{n}{x} \rfloor} - \frac{x}{n} \left\lfloor \frac{n}{x} \right\rfloor\right)(n \log n - n) - \left(J_{\lfloor \frac{n}{x} \rfloor} - \frac{x}{n} \left\lfloor \frac{n}{x} \right\rfloor \log \frac{n}{x}\right) n. \tag{4.2.18}$$

To prove Proposition 4.2.7 we use the following identity.

Lemma 4.2.8. *For real numbers $n \ge 1$ and $x \ge 1$, we have*

$$\begin{aligned} &\overline{C}(n,n) + \overline{C}(n,x) - \overline{C}\left(n, \frac{n}{x}\right) \\ &= (\log n) \sum_{1 \le b \le x} \left\lfloor \frac{n}{b} \right\rfloor - \lfloor x \rfloor \log\left(\left\lfloor \frac{n}{x} \right\rfloor !\right) - nH(x) + \lfloor x \rfloor + \sum_{1 \le b \le x} \int_1^{\frac{n}{b}} \frac{\{u\}}{u} \, du. \end{aligned} \tag{4.2.19}$$

Proof. By partial summation, we have the identity

$$\sum_{1 \le k \le t} \log \frac{t}{k} = t - 1 - \int_1^t \frac{\{u\}}{u} \, du,$$

for any $t > 0$. On setting $t = \frac{n}{b}$ in this identity and summing over positive integers $b \le x$, we obtain

$$\sum_{1 \le b \le x} \sum_{1 \le k \le \frac{n}{b}} \log \frac{n}{bk} = nH(x) - \lfloor x \rfloor - \sum_{1 \le b \le x} \int_1^{\frac{n}{b}} \frac{\{u\}}{u}\, du. \tag{4.2.20}$$

The double sum on the left side of (4.2.20) is equal to

$$\sum_{1 \le b \le x} \sum_{1 \le k \le \frac{n}{b}} (\log n - \log b - \log k) = (\log n) \sum_{1 \le b \le x} \left\lfloor \frac{n}{b} \right\rfloor - \overline{C}(n,x) - \sum_{1 \le k \le n} \sum_{\substack{1 \le b \le x \\ b \le \frac{n}{k}}} \log k.$$

On substituting the right side into (4.2.20) and rearranging, we obtain

$$\sum_{1 \le k \le n} \sum_{\substack{1 \le b \le x \\ b \le \frac{n}{k}}} \log k = (\log n) \sum_{1 \le b \le x} \left\lfloor \frac{n}{b} \right\rfloor - \overline{C}(n,x) - nH(x) + \lfloor x \rfloor + \sum_{1 \le b \le x} \int_1^{\frac{n}{b}} \frac{\{u\}}{u}\, du. \tag{4.2.21}$$

The double sum on the left side of (4.2.21) is equal to

$$\sum_{\frac{n}{x} < k \le n} \sum_{1 \le b \le \frac{n}{k}} \log k + \sum_{1 \le k \le \frac{n}{x}} \sum_{1 \le b \le x} \log k = \left(\overline{C}(n,n) - \overline{C}\left(n, \frac{n}{x}\right)\right) + \lfloor x \rfloor \log \left(\left\lfloor \frac{n}{x} \right\rfloor !\right).$$

On inserting the right side into (4.2.21) and rearranging, we get (4.2.19). □

Proof of Proposition 4.2.7. (1) On substituting $x = \sqrt{n}$ in Lemma 4.2.8, two of the terms on the left side cancel and we get

$$\begin{aligned} \overline{C}(n,n) = (\log n) \sum_{1 \le b \le \sqrt{n}} \left\lfloor \frac{n}{b} \right\rfloor - \lfloor \sqrt{n} \rfloor \log \left(\lfloor \sqrt{n} \rfloor !\right) - nH\left(\sqrt{n}\right) + \lfloor \sqrt{n} \rfloor \\ + \sum_{1 \le b \le \sqrt{n}} \int_1^{\frac{n}{b}} \frac{\{u\}}{u}\, du. \end{aligned} \tag{4.2.22}$$

Now, we estimate each term on the right of (4.2.22). For the first term, we use $\lfloor t \rfloor = t + O(1)$, obtaining

$$\begin{aligned} (\log n) \sum_{1 \le b \le \sqrt{n}} \left\lfloor \frac{n}{b} \right\rfloor &= (\log n) n H_{\lfloor \sqrt{n} \rfloor} + O\left(\sqrt{n} \log n\right) \\ &= n (\log n) \left(\log \lfloor \sqrt{n} \rfloor + \gamma + O\left(\frac{1}{\lfloor \sqrt{n} \rfloor}\right)\right) + O\left(\sqrt{n} \log n\right) \\ &= \frac{1}{2} n (\log n)^2 + \gamma n \log n + O\left(\sqrt{n} \log n\right), \end{aligned}$$

where we used Lemma 4.2.5 to estimate $H_{\lfloor\sqrt{n}\rfloor}$. For the second term, Stirling's formula gives

$$\lfloor\sqrt{n}\rfloor \log\left(\lfloor\sqrt{n}\rfloor!\right) = \frac{1}{2}n\log n - n + O\left(\sqrt{n}\log n\right).$$

For the third term, the harmonic number estimate in Lemma 4.2.5 gives

$$nH\left(\sqrt{n}\right) = \frac{1}{2}n\log n + \gamma n + O\left(\sqrt{n}\right).$$

The last two terms are negligible:

$$\lfloor\sqrt{n}\rfloor + \sum_{1\le b\le\sqrt{n}} \int_1^{\frac{n}{b}} \frac{\{u\}}{u}\,du \le \sqrt{n} + \sum_{1\le b\le\sqrt{n}} \int_1^{n} \frac{1}{u}\,du = O\left(\sqrt{n}\log n\right).$$

Substituting these estimates into the right side of (4.2.22) yields

$$\overline{C}(n,n) = \frac{1}{2}n(\log n)^2 + (\gamma-1)n\log n + (1-\gamma)n + O\left(\sqrt{n}\log n\right).$$

(2) We will prove that for $2 \le x \le n$

$$\begin{aligned}\overline{C}(n,n) - \overline{C}(n,x) &= \left(H_{\lfloor\frac{n}{x}\rfloor} - \frac{x}{n}\left\lfloor\frac{n}{x}\right\rfloor\right)(n\log n - n)\\ &\quad - \left(J_{\lfloor\frac{n}{x}\rfloor} - \frac{x}{n}\left\lfloor\frac{n}{x}\right\rfloor\log\frac{n}{x}\right)n + O\left(\frac{n\log n}{x}\right).\end{aligned} \tag{4.2.23}$$

and then deduce (4.2.18).

First, we prove (4.2.23). We replace x by $\frac{n}{x}$ in Lemma 4.2.8, and rearrange a term to obtain

$$\begin{aligned}\overline{C}(n,n) - \overline{C}(n,x) =& \overline{C}\left(n,\frac{n}{x}\right) + (\log n)\sum_{1\le b\le\frac{n}{x}}\left\lfloor\frac{n}{b}\right\rfloor - \left\lfloor\frac{n}{x}\right\rfloor\log(\lfloor x\rfloor!) - nH_{\lfloor\frac{n}{x}\rfloor}\\ &+ \left(\left\lfloor\frac{n}{x}\right\rfloor + \sum_{1\le b\le\frac{n}{x}}\int_1^{\frac{n}{b}}\frac{\{u\}}{u}\,du\right).\end{aligned} \tag{4.2.24}$$

We estimate the terms on the right side of (4.2.24). For the first term, using $\lfloor t \rfloor = t + O(1)$, we see that

$$\overline{C}\left(n, \frac{n}{x}\right) = \sum_{1 \le b \le \frac{n}{x}} \left\lfloor \frac{n}{b} \right\rfloor \log b = nJ_{\lfloor \frac{n}{x} \rfloor} + O\left(\log\left(\left\lfloor \frac{n}{x} \right\rfloor !\right) + 1\right).$$

Using the bounds

$$0 \le \log\left(\left\lfloor \frac{n}{x} \right\rfloor !\right) \le \left\lfloor \frac{n}{x} \right\rfloor \log \left\lfloor \frac{n}{x} \right\rfloor \le \frac{n \log n}{x},$$

we obtain the estimate

$$\overline{C}\left(n, \frac{n}{x}\right) = nJ_{\lfloor \frac{n}{x} \rfloor} + O\left(\frac{n \log n}{x}\right). \tag{4.2.25}$$

For the second term, again using $\lfloor t \rfloor = t + O(1)$ we obtain

$$(\log n) \sum_{1 \le b \le \frac{n}{x}} \left\lfloor \frac{n}{b} \right\rfloor = n(\log n) H_{\lfloor \frac{n}{x} \rfloor} + O\left(\frac{n \log n}{x}\right). \tag{4.2.26}$$

For the third term we assert

$$\left\lfloor \frac{n}{x} \right\rfloor \log(\lfloor x \rfloor !) = n(\log n) \frac{x}{n} \left\lfloor \frac{n}{x} \right\rfloor - n\frac{x}{n} \left\lfloor \frac{n}{x} \right\rfloor \left(1 + \log \frac{n}{x}\right) + O\left(\frac{n \log n}{x}\right). \tag{4.2.27}$$

This estimate follows using Stirling's formula with remainder in the form, for $x \ge 2$,

$$\log\left(\lfloor x \rfloor !\right) = x \log x - x + O(\log x), \tag{4.2.28}$$

which yields

$$\left\lfloor \frac{n}{x} \right\rfloor \log(\lfloor x \rfloor !) = \left\lfloor \frac{n}{x} \right\rfloor (x \log x - x) + O\left(\frac{n \log x}{x}\right),$$

and (4.2.27) follows. For the final term we have, for $n \ge 2$ and $2 \le x \le n$,

$$\left\lfloor \frac{n}{x} \right\rfloor + \sum_{1 \le b \le \frac{n}{x}} \int_1^{\frac{n}{b}} \frac{\{u\}}{u}\, du \le \frac{n}{x} + \sum_{1 \le b \le \frac{n}{x}} \int_1^{n} \frac{1}{u}\, du = O\left(\frac{n \log n}{x}\right). \tag{4.2.29}$$

On inserting (4.2.25), (4.2.26), (4.2.27), and (4.2.29) into (4.2.24) and rearranging, we obtain (4.2.23).

Next, we prove (4.2.18). By the substitution $v = \frac{n}{u}$, we get

$$\int_x^n \left\lfloor \frac{n}{u} \right\rfloor \log u \, du = n \int_1^{\frac{n}{x}} \frac{\lfloor v \rfloor}{v^2} \log \frac{n}{v} \, dv. \tag{4.2.30}$$

The integral has a closed form quadrature:

$$\frac{d}{dv}\left(\frac{1}{v} - \frac{1}{v}\log\frac{n}{v}\right) = \frac{1}{v^2}\log\frac{n}{v}$$

valid on unit intervals $b \le v < b+1$ where $\lfloor v \rfloor = b$. By partial summation, the right side of (4.2.30) is then equal to

$$\begin{aligned} & n\left\lfloor \frac{n}{x} \right\rfloor \left(\frac{x}{n} - \frac{x}{n}\log x\right) - n \sum_{1 \le b \le \frac{n}{x}} \left(\frac{1}{b} - \frac{1}{b}\log\frac{n}{b}\right) \\ = & \, x(1 - \log x)\left\lfloor \frac{n}{x} \right\rfloor + (n\log n - n)H_{\lfloor \frac{n}{x} \rfloor} - nJ_{\lfloor \frac{n}{x} \rfloor}. \end{aligned}$$

We obtain

$$\begin{aligned} \int_x^n \left\lfloor \frac{n}{u} \right\rfloor \log u \, du &= x(1 - \log x)\left\lfloor \frac{n}{x} \right\rfloor + (n\log n - n)H_{\lfloor \frac{n}{x} \rfloor} - nJ_{\lfloor \frac{n}{x} \rfloor} \\ &= \left(H_{\lfloor \frac{n}{x} \rfloor} - \frac{x}{n}\left\lfloor \frac{n}{x} \right\rfloor\right)(n\log n - n) - \left(J_{\lfloor \frac{n}{x} \rfloor} - \frac{x}{n}\left\lfloor \frac{n}{x} \right\rfloor \log\frac{n}{x}\right) n, \end{aligned}$$

completing the proof. □

Lemma 4.2.9. *For real numbers $n \ge 2$, we have*

$$\overline{C}\left(n, \sqrt{n}\right) = \frac{1}{8}n(\log n)^2 + \gamma_1 n + O\left(\sqrt{n}\log n\right).$$

Proof. By using the estimate $\lfloor t \rfloor = t + O(1)$, we see that

$$\overline{C}\left(n, \sqrt{n}\right) = \sum_{1 \le b \le \sqrt{n}} \left\lfloor \frac{n}{b} \right\rfloor \log b = nJ\left(\sqrt{n}\right) + O\left(\log\left(\lfloor \sqrt{n} \rfloor !\right)\right). \tag{4.2.31}$$

By applying Lemma 4.2.6 with $x = \sqrt{n}$, we obtain

$$J\left(\sqrt{n}\right) = \frac{1}{8}(\log n)^2 + \gamma_1 + O\left(\frac{\log n}{\sqrt{n}}\right).$$

Moreover, we have

$$\log\left(\lfloor \sqrt{n} \rfloor !\right) \le \lfloor \sqrt{n} \rfloor \log \lfloor \sqrt{n} \rfloor \le \frac{1}{2}\sqrt{n}\log n.$$

Inserting these estimates back into (4.2.31) yields the lemma. □

4.2.5 Estimates: $L_i(n)$

For positive integers $i \geq 1$ and $n \geq 2$, we set

$$L_i(n) := \sum_{b=2}^{n} b(\log b)^i. \tag{4.2.32}$$

We give formulas for all $i \geq 1$ but will only need the cases $i = 1, 2$ in the sequel.

Lemma 4.2.10. *For integers $i \geq 1$ and $n \geq 2$, we have*

$$L_i(n) = \int_1^n u(\log u)^i \, du + \theta_i(n) n (\log n)^i, \tag{4.2.33}$$

where $0 \leq \theta_i(n) \leq 1$. In particular,

$$L_1(n) = \frac{1}{2} n^2 \log n - \frac{1}{4} n^2 + O\left(n \log n\right), \tag{4.2.34}$$

$$L_2(n) = \frac{1}{2} n^2 (\log n)^2 - \frac{1}{2} n^2 \log n + \frac{1}{4} n^2 + O\left(n (\log n)^2\right). \tag{4.2.35}$$

Proof. The function $u(\log u)^i$, $1 \leq u \leq n$ is increasing. We have lower and upper bounds

$$\int_1^n u(\log u)^i \, du \leq \sum_{b=2}^{n} b(\log b)^i = L_i(n),$$

$$\int_1^n u(\log u)^i \, du \geq \sum_{b=1}^{n-1} b(\log b)^i = L_i(n) - n(\log n)^i.$$

Thus the assertion (4.2.33) follows.

The assertions (4.2.34) and (4.2.35) follow from the first assertion with the formulas

$$\int_1^n u \log u \, du = \left[\frac{1}{2} u^2 \log u - \frac{1}{4} u^2 \right]_{u=1}^{n} = \frac{1}{2} n^2 \log n - \frac{1}{4} n^2 + \frac{1}{4},$$

$$\begin{aligned} \int_1^n u(\log u)^2 \, du &= \left[\frac{1}{2} u^2 (\log u)^2 - \frac{1}{2} u^2 \log u + \frac{1}{4} u^2 \right]_{u=1}^{n} \\ &= \frac{1}{2} n^2 (\log n)^2 - \frac{1}{2} n^2 \log n + \frac{1}{4} n^2 - \frac{1}{4}, \end{aligned}$$

completing the proof. ∎

Remark 4.2.11. It can be shown by induction on i that

$$\int_1^n u(\log u)^i\, du = n^2 \sum_{k=0}^{i} \frac{(-1)^k k!}{2^{k+1}} \binom{i}{k} (\log n)^{i-k} + \frac{(-1)^{i+1} i!}{2^{i+1}}. \tag{4.2.36}$$

4.3 Estimates for $\overline{B}(n)$

In this section we obtain the estimates for $\overline{B}(n) = \sum_{b=2}^{n} \frac{n-1}{b-1} d_b(n) \log b$ given in Theorem 4.1.3.

4.3.1 Digit sum identity and preliminary reduction

Our estimate for $\overline{B}(n)$ will be derived using the observation that n has exactly 2 digits in base b when $\sqrt{n} < b \le n$.

Lemma 4.3.1. *Let j and n be positive integers. Denote by $I(j,n)$ the interval $\left(\frac{n}{j+1}, \frac{n}{j}\right] \cap (\sqrt{n}, n]$. Then*

1. *$I(j,n)$ is empty unless $j < \sqrt{n}$.*

2. *If $b \in I(j,n)$ is an integer, then $d_b(n) = n - j(b-1)$, in consequence,*

$$\frac{n-1}{b-1} d_b(n) \log b = (n-1) \left(\frac{n \log b}{b-1} - j \log b \right). \tag{4.3.1}$$

Proof. (1) Suppose that $x \in I(j,n)$. Then $\sqrt{n} < x \le \frac{n}{j}$, and hence $j < \sqrt{n}$.

(2) Since $\frac{n}{j+1} < b \le \frac{n}{j}$, it follows that $\left\lfloor \frac{n}{b} \right\rfloor = j$. Since $b > \sqrt{n}$, it follows that $\left\lfloor \frac{n}{b^i} \right\rfloor = 0$ for all $i \ge 2$. From (4.2.6), we have

$$d_b(n) = n - (b-1) \sum_{i=1}^{\infty} \left\lfloor \frac{n}{b^i} \right\rfloor = n - j(b-1),$$

and (4.3.1) follows by multiplying by $\frac{n-1}{b-1} \log b$. This completes the proof. □

We split the sum $B(n)$ into three parts, the third part being a cutoff term removing all $2 \le b \le \sqrt{n}$, and the first two parts using the digit sum identity (4.3.1). applied to the range $\sqrt{n} < b \le n$.

Lemma 4.3.2. *(1) For integers $n \geq 2$, we have*

$$\overline{B}(n) = \overline{B}_1(n) - \overline{B}_2(n) + \overline{B}_R(n), \tag{4.3.2}$$

in which $\overline{B}_1(n)$, $\overline{B}_2(n)$, and $\overline{B}_R(n)$ are defined by

$$\overline{B}_1(n) := n(n-1) \sum_{\sqrt{n} < b \leq n} \frac{\log b}{b-1}, \tag{4.3.3}$$

$$\overline{B}_2(n) := (n-1) \sum_{j=1}^{\lfloor \sqrt{n} \rfloor} j \left(\sum_{\frac{n}{j+1} < b \leq \frac{n}{j}}{}' \log b \right), \tag{4.3.4}$$

where the prime in the inner sum in (4.3.4) *means only $b > \sqrt{n}$ are included, and*

$$\overline{B}_R(n) := \sum_{2 \leq b \leq \sqrt{n}} \frac{n-1}{b-1} d_b(n) \log b. \tag{4.3.5}$$

(2) For integers $n \geq 2$ the remainder term $\overline{B}_R(n)$ satisfies

$$0 \leq \overline{B}_R(n) \leq \frac{3}{2} n^{3/2} \log n. \tag{4.3.6}$$

Proof. (1) Recall that $\overline{B}(n) = \sum_{b=2}^{n} \frac{n-1}{b-1} d_b(n) \log b$. The remainder term $\overline{B}_R(n)$ first cuts off the terms with $2 \leq b \leq \sqrt{n}$ in the sum. The other two terms $\overline{B}_1(n)$ and $\overline{B}_2(n)$ are obtained by applying the decomposition (4.3.1) of Lemma 4.3.1 to each index $b \in (\sqrt{n}, n]$ term by term.

(2) From (4.2.9), it follows that $0 \leq \frac{n-1}{b-1} d_b(n) \log b \leq (n-1)\log(n+1)$. Summing from $b = 2$ to $\lfloor \sqrt{n} \rfloor$, we obtain

$$0 \leq \overline{B}_R(n) \leq \left(\lfloor \sqrt{n} \rfloor - 1 \right) (n-1) \log(n+1) \leq \left(\sqrt{n} \right) (n) \left(\frac{3}{2} \log n \right) = \frac{3}{2} n^{3/2} \log n$$

as desired. □

The sums $\overline{B}_1(n)$ and $\overline{B}_2(n)$ are of comparable sizes, on the order of $n^2 \log n$. We estimate them separately.

4.3.2 Estimate for $\overline{B}_1(n)$

Lemma 4.3.3. *Let* $\overline{B}_1(n) = n(n-1)\sum_{\sqrt{n}<b\le n} \frac{\log b}{b-1}$. *Then for integers* $n \ge 2$, *we have*

$$\overline{B}_1(n) = \frac{3}{8}n^2(\log n)^2 + O\left(n^{3/2}\log n\right). \tag{4.3.7}$$

Proof. We rewrite the sum $\frac{\overline{B}_1(n)}{n(n-1)}$ as

$$\frac{\overline{B}_1(n)}{n(n-1)} = \sum_{\sqrt{n}<b\le n} \frac{\log b}{b-1} = \sum_{\sqrt{n}<b\le n} \frac{\log b}{b} + \sum_{\sqrt{n}<b\le n} \frac{\log b}{b(b-1)}. \tag{4.3.8}$$

The contribution from the last sum in (4.3.8) is negligible:

$$0 \le \sum_{\sqrt{n}<b\le n} \frac{\log b}{b(b-1)} \le (\log n)\sum_{b>\sqrt{n}} \frac{1}{b(b-1)} = \frac{\log n}{\lfloor\sqrt{n}\rfloor} \le \frac{2\log n}{\sqrt{n}}. \tag{4.3.9}$$

We use Lemma 4.2.6 to estimate the first sum on the right in (4.3.8) and obtain

$$\begin{aligned}\sum_{\sqrt{n}<b\le n} \frac{\log b}{b} &= J(n) - J\left(\sqrt{n}\right)\\ &= \frac{1}{2}(\log n)^2 - \frac{1}{2}\left(\log\sqrt{n}\right)^2 + O\left(\frac{\log n}{\sqrt{n}}\right)\\ &= \frac{3}{8}(\log n)^2 + O\left(\frac{\log n}{\sqrt{n}}\right).\end{aligned} \tag{4.3.10}$$

On inserting (4.3.9) and (4.3.10) into (4.3.8), we obtain

$$\frac{\overline{B}_1(n)}{n(n-1)} = \frac{3}{8}(\log n)^2 + O\left(\frac{\log n}{\sqrt{n}}\right).$$

On multiplying by $n(n-1)$, we obtain (4.3.7) as desired. □

4.3.3 Estimate for $\overline{B}_2(n)$

Lemma 4.3.4. *Let*

$$\overline{B}_2(n) := (n-1)\sum_{j=1}^{\lfloor\sqrt{n}\rfloor} j\left(\sum_{\frac{n}{j+1}<b\le\frac{n}{j}}{}^{'} \log b\right),$$

where the prime in the inner sum means only $b > \sqrt{n}$ are included. Then for integers $n \geq 2$,

$$\overline{B}_2(n) = \frac{3}{8}n^2(\log n)^2 + (\gamma - 1)n^2 \log n + (1 - \gamma - \gamma_1)\, n^2 + O\left(n^{3/2} \log n\right), \quad (4.3.11)$$

where γ is Euler's constant and γ_1 is the first Stieltjes constant.

Proof. We have

$$\frac{\overline{B}_2(n)}{n-1} = \sum_{j=1}^{\lfloor \sqrt{n} \rfloor} \Bigg(\sideset{}{'}\sum_{\frac{n}{j+1} < b \leq \frac{n}{j}} \left\lfloor \frac{n}{b} \right\rfloor \log b \Bigg) = \sum_{\sqrt{n} < b \leq n} \left\lfloor \frac{n}{b} \right\rfloor \log b = \overline{C}(n, n) - \overline{C}\left(n, \sqrt{n}\right).$$

Applying Proposition 4.2.7 and Lemma 4.2.9 to estimate $\overline{C}(n, n)$ and $\overline{C}\left(n, \sqrt{n}\right)$, we obtain

$$\begin{aligned} \frac{\overline{B}_2(n)}{n-1} &= \left(\frac{1}{2}n(\log n)^2 + (\gamma - 1)n \log n + (1 - \gamma)n\right) - \left(\frac{1}{8}n(\log n)^2 + \gamma_1 n\right) \\ &\quad + O\left(\sqrt{n} \log n\right) \\ &= \frac{3}{8}n(\log n)^2 + (\gamma - 1)n \log n + (1 - \gamma - \gamma_1)\, n + O\left(\sqrt{n} \log n\right). \end{aligned}$$

On multiplying by $(n - 1)$, we obtain (4.3.11) as desired. □

4.3.4 Proof of Theorem 4.1.3

We combine the results on $\overline{B}_1(n)$ and $\overline{B}_2(n)$ to estimate $\overline{B}(n)$.

Proof of Theorem 4.1.3. We estimate $\overline{B}(n)$. We start from the Lemma 4.3.2 decomposition $\overline{B}(n) = \overline{B}_1(n) - \overline{B}_2(n) + \overline{B}_R(n)$. By Lemma 4.3.2 (2) we have $\overline{B}_R(n) = O(n^{3/2} \log n)$, which is absorbed in the remainder term estimate in the theorem statement. Substituting the formulas of Lemma 4.3.3 and Lemma 4.3.4, we obtain

$$\begin{aligned}\overline{B}(n) &= \overline{B}_1(n) - \overline{B}_2(n) + \overline{B}_R(n)\\ &= \frac{3}{8}n^2(\log n)^2 + O\left(n^{3/2}\log n\right)\\ &\quad - \left(\frac{3}{8}n^2(\log n)^2 + (\gamma-1)n^2\log n + (1-\gamma-\gamma_1)n^2 + O\left(n^{3/2}\log n\right)\right)\\ &= (1-\gamma)n^2\log n + (\gamma+\gamma_1-1)n^2 + O\left(n^{3/2}\log n\right),\end{aligned}$$

as asserted. □

4.4 Estimates for $\overline{A}(n)$ and $\overline{\overline{G}}_n$

In this section we derive asymptotics for $\overline{A}(n) = \sum_{b=2}^{n} \frac{2}{b-1} S_b(n)\log b$ given in Theorem 4.1.4 and deduce the estimate for $\log \overline{\overline{G}}_n$ given in Theorem 4.1.1. In the case of binomial products $\overline{G}_n$ treated in [13] an asymptotic for $A(n)$ was obtained from the relation $\log \overline{G}_n = A(n) - B(n)$ and the existence of a good estimate for $\log \overline{G}_n$ coming from its expression as a product of factorials. Here we do not have a corresponding direct estimate for $\log \overline{\overline{G}}_n$, so we must estimate $\overline{A}(n)$ directly. The proof details have some parallel with those for $\overline{B}(n)$.

4.4.1 Preliminary reduction

Recall that $\overline{A}(n) = \sum_{b=2}^{n} \frac{2}{b-1} S_b(n)\log b$.

Lemma 4.4.1. *For integers $n \geq 2$, we have*

$$\overline{A}(n) = \overline{A}_1(n) + O\left(n(\log n)^2\right), \tag{4.4.1}$$

where

$$\overline{A}_1(n) := \sum_{b=2}^{n}\sum_{j=2}^{n} \frac{2\log b}{b} d_b(j). \tag{4.4.2}$$

Proof. We rewrite the sum (4.1.23) that defines $\overline{A}(n)$ using the identity

$$\frac{1}{b-1} = \frac{1}{b} + \frac{1}{b(b-1)} \tag{4.4.3}$$

and obtain

$$\overline{A}(n) = \sum_{b=2}^{n} \frac{2\log b}{b} S_b(n) + \sum_{b=2}^{n} \frac{2\log b}{b(b-1)} S_b(n). \tag{4.4.4}$$

Since $S_b(n) = \sum_{j=1}^{n-1} d_b(j)$, the first sum on the right in (4.4.4) is

$$\sum_{b=2}^{n} \frac{2\log b}{b} S_b(n) = \sum_{b=2}^{n} \sum_{j=1}^{n-1} \frac{2\log b}{b} d_b(j) = \overline{A}_1(n) - \sum_{b=2}^{n} \frac{2\log b}{b} \left(d_b(n) - 1\right). \tag{4.4.5}$$

By Lemma 4.2.4,

$$0 \le d_b(n) - 1 \le \frac{(b-1)\log(n+1)}{\log b} - 1 < \frac{b\log(n+1)}{\log b}.$$

So the last sum in (4.4.5) satisfies, for $n \ge 2$,

$$0 \le \sum_{b=2}^{n} \frac{2\log b}{b} \left(d_b(n) - 1\right) < \sum_{b=2}^{n} 2\log(n+1) \le 2n\log n.$$

Hence

$$\sum_{b=2}^{n} \frac{2\log b}{b} S_b(n) = \overline{A}_1(n) + O(n\log n). \tag{4.4.6}$$

Now, we treat the last sum in (4.4.4). We apply Lemma 4.2.4 to bound $S_b(n)$, obtaining

$$\sum_{b=2}^{n} \frac{2\log b}{b(b-1)} S_b(n) \le \sum_{b=2}^{n} \frac{n\log n}{b} \ll n(\log n)^2. \tag{4.4.7}$$

On inserting (4.4.6) and (4.4.7) into (4.4.4), we obtain (4.4.1) as desired. □

4.4.2 Estimate for $\overline{A}_1(n)$ reduction

Lemma 4.4.2. *(1) For integers $n \ge 2$, the sum $\overline{A}_1(n)$ given by (4.4.2) can be rewritten as*

$$\overline{A}_1(n) = \overline{A}_{11}(n) + \overline{A}_{12}(n) - \overline{A}_R(n), \tag{4.4.8}$$

where

$$\overline{A}_{11}(n) := \sum_{j=2}^{n} \frac{2}{j-1}\overline{B}(j), \tag{4.4.9}$$

$$\overline{A}_{12}(n) := \sum_{j=2}^{n} \sum_{b=j+1}^{n} \frac{2j\log b}{b}, \tag{4.4.10}$$

$$\overline{A}_{R}(n) := \sum_{j=2}^{n} \sum_{b=2}^{j} \frac{2\log b}{b(b-1)} d_b(j), \tag{4.4.11}$$

and $\overline{B}(n)$ *is given by* (4.1.24).

(2) For integers $n \geq 2$*, we have*

$$\overline{A}_{R}(n) \leq 3n(\log n)^2. \tag{4.4.12}$$

Proof. (1) We start from (4.4.2) and interchange the order of summation, obtaining

$$\begin{aligned}
\overline{A}_1(n) &= \sum_{b=2}^{n} \sum_{j=2}^{n} \frac{2\log b}{b} d_b(j) \\
&= \sum_{j=2}^{n} \sum_{b=2}^{n} \frac{2\log b}{b} d_b(j) \\
&= \sum_{j=2}^{n} \sum_{b=2}^{j} \frac{2\log b}{b} d_b(j) + \sum_{j=2}^{n} \sum_{b=j+1}^{n} \frac{2\log b}{b} d_b(j). \qquad (4.4.13)
\end{aligned}$$

Recall that $\overline{B}(j) = \sum_{b=2}^{j} \frac{j-1}{b-1} d_b(j) \log b$. We next use the identity (4.4.3) to rewrite the first sum on the right in (4.4.13):

$$\sum_{j=2}^{n} \sum_{b=2}^{j} \frac{2\log b}{b} d_b(j) = \sum_{j=2}^{n} \sum_{b=2}^{j} \frac{2\log b}{b-1} d_b(j) - \sum_{j=2}^{n} \sum_{b=2}^{j} \frac{2\log b}{b(b-1)} d_b(j) = \overline{A}_{11}(n) - \overline{A}_R(n).$$

Finally, we note that $d_b(j) = j$ for $j < b$; so the second sum on the right in (4.4.13) is

$$\sum_{j=2}^{n} \sum_{b=j+1}^{n} \frac{2\log b}{b} d_b(j) = \sum_{j=2}^{n} \sum_{b=j+1}^{n} \frac{2j\log b}{b} = \overline{A}_{12}(n).$$

(2) We first bound $\overline{A}_R(n)$ by

$$0 \leq \overline{A}_R(n) \leq \sum_{j=2}^{n} \sum_{b=2}^{n} \frac{2\log b}{b(b-1)} d_b(j) = \sum_{b=2}^{n} \frac{2\log b}{b(b-1)} \left(S_b(n+1) - 1\right)$$

Applying Lemma 4.2.4, to bound the last quantity, we obtain for $n \geq 2$,

$$\overline{A}_R(n) < \sum_{b=2}^{n} \frac{(n+1)\log(n+1)}{b} \leq 3n(\log n)^2,$$

as asserted. □

4.4.3 Estimates for $\overline{A}_{11}(n)$ and $\overline{A}_{12}(n)$

Lemma 4.4.3. *For integers $n \geq 2$, we have*

$$\overline{A}_{11}(n) = (1-\gamma)n^2 \log n + \left(\frac{3}{2}\gamma + \gamma_1 - \frac{3}{2}\right) n^2 + O\left(n^{3/2} \log n\right). \qquad (4.4.14)$$

Proof. We start from (4.4.9) and use the identity (4.4.3) to rewrite $\overline{A}_{11}(n)$:

$$\overline{A}_{11}(n) = \sum_{j=2}^{n} \frac{2}{j-1} \overline{B}(j) = \sum_{j=2}^{n} \frac{2}{j} \overline{B}(j) + \sum_{j=2}^{n} \frac{2}{j(j-1)} \overline{B}(j). \qquad (4.4.15)$$

From Theorem 4.1.3, it follows that $\overline{B}(j) \ll j(j-1)\log j$ for $j \geq 2$. As a result, the contribution from the last sum in (4.4.15) is negligible:

$$\sum_{j=2}^{n} \frac{2}{j(j-1)} \overline{B}(j) \ll \sum_{j=2}^{n} \log j \leq \sum_{j=2}^{n} \log n < n \log n. \qquad (4.4.16)$$

Now, we estimate the first sum on the right of (4.4.15) using Theorem 4.1.3:

$$\begin{aligned}
\sum_{j=2}^{n} \frac{2}{j} \overline{B}(j) &= 2(1-\gamma) \sum_{j=2}^{n} j \log j + 2(\gamma + \gamma_1 - 1) \sum_{j=2}^{n} j + O\left(\sum_{j=2}^{n} \sqrt{j} \log j \right) \\
&= 2(1-\gamma) \sum_{j=2}^{n} j \log j + 2(\gamma + \gamma_1 - 1) \left(\frac{n^2+n}{2} - 1 \right) + O\left(\sum_{j=2}^{n} \sqrt{n} \log n \right) \\
&= 2(1-\gamma) \sum_{j=2}^{n} j \log j + (\gamma + \gamma_1 - 1) n^2 + O\left(n^{3/2} \log n\right).
\end{aligned}$$

We use Lemma 4.2.10 to estimate $\sum_{j=2}^{n} j \log j$ and obtain

$$\sum_{j=2}^{n} \frac{2}{j} \overline{B}(j) = (1-\gamma)n^2 \log n + \left(\frac{3}{2}\gamma + \gamma_1 - \frac{3}{2}\right) n^2 + O\left(n^{3/2} \log n\right). \qquad (4.4.17)$$

On inserting (4.4.16) and (4.4.17) into (4.4.15), we obtain (4.4.14) as desired. □

Lemma 4.4.4. *For integers $n \geq 2$, we have*

$$\overline{A}_{12}(n) = \frac{1}{2}n^2 \log n - \frac{1}{4}n^2 + O\left(n(\log n)^2\right).$$

Proof. We can rewrite (4.4.10) in terms of $J(x) = \sum_{1 \leq b \leq x} \frac{\log b}{b}$ as

$$\overline{A}_{12}(n) = \sum_{j=2}^{n} 2j(J(n) - J(j)).$$

For $2 \leq j \leq n$, it follows from Lemma 4.2.6 that

$$J(n) - J(j) = \frac{1}{2}(\log n)^2 - \frac{1}{2}(\log j)^2 + O\left(\frac{\log n}{j}\right).$$

Hence

$$\begin{aligned}\overline{A}_{12}(n) &= \sum_{j=2}^{n} j(\log n)^2 - \sum_{j=2}^{n} j(\log j)^2 + O\bigg(\sum_{b=2}^{n} \log n\bigg) \\ &= \left(\frac{1}{2}n^2 + \frac{1}{2}n - 1\right)(\log n)^2 - \sum_{j=2}^{n} j(\log j)^2 + O\left(n \log n\right).\end{aligned}$$

We use Lemma 4.2.10 to estimate $\sum_{j=2}^{n} j(\log j)^2$ and obtain

$$\overline{A}_{12}(n) = \frac{1}{2}n^2 \log n - \frac{1}{4}n^2 + O\left(n(\log n)^2\right),$$

as desired. □

4.4.4 Proofs of Theorems 4.1.4 and 4.1.1

We derive the estimate for $\overline{A}(n)$ in Theorem 4.1.4 and that for $\overline{\overline{G}}_n$ in Theorem 4.1.1.

Proof of Theorem 4.1.4. By Lemma 4.4.1 and Lemma 4.4.2,

$$\overline{A}(n) = \overline{A}_1(n) + O\left(n(\log n)^2\right) = \overline{A}_{11}(n) + \overline{A}_{12}(n) + O\left(n(\log n)^2\right).$$

Inserting the estimates of Lemma 4.4.3 for $\overline{A}_{11}(n)$ and Lemma 4.4.4 for $\overline{A}_{12}(n)$ yields

$$\overline{A}(n) = \left(\frac{3}{2} - \gamma\right) n^2 \log n + \left(\frac{3}{2}\gamma + \gamma_1 - \frac{7}{4}\right) n^2 + O\left(n^{3/2} \log n\right),$$

as required. □

Proof of Theorem 4.1.1. The estimate for $\overline{\overline{G}}_n$ follows from the relation $\log \overline{\overline{G}}_n = \overline{A}(n) - \overline{B}(n)$ using the estimates of Theorem 4.1.3 for $\overline{B}(n)$ and Theorem 4.1.4 for $\overline{A}(n)$. □

4.5 Estimates for the generalized partial factorization sums $\overline{B}(n, x)$

We derive estimates for $\overline{B}(n, x)$ in the interval $1 \le x \le n$ starting from the asymptotic estimates for $\overline{B}(n) = \overline{B}(n, n)$. Let $H_m = \sum_{k=1}^{m} \frac{1}{k}$ denote the m-th harmonic number.

Theorem 4.5.1. *Let $\overline{B}(n, x) = \sum_{b=2}^{\lfloor x \rfloor} \frac{n-1}{b-1} d_b(n) \log b$. Then for integers $n \ge 2$ and real $x \in [\sqrt{n}, n]$,*

$$\overline{B}(n, x) = \overline{B}_0(n, x)\, n^2 \log n + \overline{B}_1(n, x)\, n^2 + O\left(n^{3/2} \log n\right), \tag{4.5.1}$$

where the functions $\overline{B}_0(n, x)$ and $\overline{B}_1(n, x)$ only depend on $\frac{x}{n}$ and are given by

$$\overline{B}_0(n, x) := (1 - \gamma) + \left(H_{\lfloor \frac{n}{x} \rfloor} - \log \frac{n}{x}\right) - \frac{x}{n} \left\lfloor \frac{n}{x} \right\rfloor \tag{4.5.2}$$

and

$$\begin{aligned} \overline{B}_1(n, x) := (\gamma + \gamma_1 - 1) - \left(H_{\lfloor \frac{n}{x} \rfloor} - \log \frac{n}{x}\right) - \left(J_{\lfloor \frac{n}{x} \rfloor} - \frac{1}{2}\left(\log \frac{n}{x}\right)^2\right) \\ - \log \frac{n}{x} + \left(\log \frac{n}{x}\right) \frac{x}{n} \left\lfloor \frac{n}{x} \right\rfloor + \frac{x}{n} \left\lfloor \frac{n}{x} \right\rfloor. \end{aligned} \tag{4.5.3}$$

Moreover, for integers $n \ge 2$ and real $x \in [1, \sqrt{n}]$,

$$\overline{B}(n, x) = O\left(n^{3/2} \log n\right). \tag{4.5.4}$$

Remark 4.5.2. The functions $\overline{B}_0(n, x)$ and $\overline{B}_1(n, x)$ above are functions of a single variable $\alpha := \frac{x}{n}$ having $0 \le \alpha \le 1$. That is, the answer displays a *scale invariance*, in terms of the variables x and n. However various intermediate parts of the proof involve terms in n and x that are not scale invariant.

4.5.1 Preliminary reduction

We write

$$\overline{B}(n,x) = \overline{B}(n) - \overline{B}^c(n,x), \tag{4.5.5}$$

where $\overline{B}^c(n,x)$ is the complement sum

$$\overline{B}^c(n,x) := \sum_{x<b\le n} \frac{n-1}{b-1} d_b(n) \log b. \tag{4.5.6}$$

The sum $\overline{B}(n)$ can be estimated by Theorem 4.1.3. To estimate $\overline{B}^c(n,x)$, we break it into two parts.

Lemma 4.5.3. *For integers $n \ge 2$ and real numbers x such that $\sqrt{n} \le x \le n$, we have*

$$\overline{B}^c(n,x) = \overline{B}^c_{11}(n,x) - (n-1)\left(\overline{C}(n,n) - \overline{C}(n,x)\right), \tag{4.5.7}$$

where $\overline{C}(n,x)$ is given by (4.2.15) *and*

$$\overline{B}^c_{11}(n,x) := n(n-1) \sum_{x<b\le n} \frac{\log b}{b-1}. \tag{4.5.8}$$

Proof. Recall from (4.2.6) that $d_b(n) = n - (b-1)\sum_{i=1}^{\infty} \left\lfloor \frac{n}{b^i} \right\rfloor$. Since $x > \sqrt{n}$, if $b > x$, then $b^2 > x^2 \ge n$, and hence $\left\lfloor \frac{n}{b^i} \right\rfloor = 0$ for all $i \ge 2$. In this circumstance, $d_b(n) = n - (b-1)\left\lfloor \frac{n}{b} \right\rfloor$ for $b > x$. Inserting this formula into the definition (4.5.6), we obtain

$$\begin{aligned} \overline{B}^c(n,x) &= n(n-1) \sum_{x<b\le n} \frac{\log b}{b-1} - (n-1) \sum_{x<b\le n} \left\lfloor \frac{n}{b} \right\rfloor \log b \\ &= \overline{B}^c_{11}(n,x) - (n-1)\left(\overline{C}(n,n) - \overline{C}(n,x)\right) \end{aligned} \tag{4.5.9}$$

as required. □

4.5.2 Estimate for $\overline{B}^c_{11}(n,x)$

Lemma 4.5.4. *For real numbers $n \geq 2$ and x such that $1 \leq x \leq n$, we have*

$$\overline{B}^c_{11}(n,x) = \frac{1}{2}n^2(\log n)^2 - \frac{1}{2}n^2(\log x)^2 + O\left(\frac{n^2 \log n}{x}\right). \tag{4.5.10}$$

Proof. We start from (4.5.8) and use the identity (4.4.3) to rewrite $\frac{1}{n(n-1)}\overline{B}^c_{11}(n,x)$:

$$\frac{1}{n(n-1)}\overline{B}^c_{11}(n,x) = \sum_{x<b\leq n} \frac{\log b}{b} + \sum_{x<b\leq n} \frac{\log b}{b(b-1)}. \tag{4.5.11}$$

The contribution from the last sum in (4.5.11) is negligible:

$$0 \leq \sum_{x<b\leq n} \frac{\log b}{b(b-1)} < (\log n) \sum_{b>x} \frac{1}{b(b-1)} = \frac{\log n}{\lfloor x \rfloor} < \frac{2\log n}{x}. \tag{4.5.12}$$

The first sum on the right in (4.5.11) can be estimated using Lemma 4.2.6:

$$\sum_{x<b\leq n} \frac{\log b}{b} = J(n) - J(x) = \frac{1}{2}(\log n)^2 - \frac{1}{2}(\log x)^2 + O\left(\frac{\log n}{x}\right). \tag{4.5.13}$$

On inserting (4.5.12) and (4.5.13) into (4.5.11), we obtain

$$\frac{1}{n(n-1)}\overline{B}^c_{11}(n,x) = \frac{1}{2}(\log n)^2 - \frac{1}{2}(\log x)^2 + O\left(\frac{\log n}{x}\right).$$

On multiplying by $n(n-1)$, we obtain

$$\overline{B}^c_{11}(n,x) = \frac{1}{2}n^2(\log n)^2 - \frac{1}{2}n^2(\log x)^2 - \left(\frac{1}{2}n(\log n)^2 - \frac{1}{2}n(\log x)^2\right) + O\left(\frac{n^2 \log n}{x}\right).$$

Since $e^t = 1 + t + \frac{t^2}{2} + \cdots > t$ for $t > 0$, it follows that $\log \frac{n}{x} < \frac{n}{x}$ and

$$0 \leq \frac{1}{2}n(\log n)^2 - \frac{1}{2}n(\log x)^2 = \frac{1}{2}n\left(\log \frac{n}{x}\right)\log(nx) < \frac{1}{2}n\left(\frac{n}{x}\right)(2\log n) = \frac{n^2 \log n}{x}.$$

Hence (4.5.10) follows. ∎

4.5.3 Estimate for $\overline{B}^c(n,x)$

We obtain an asymptotic estimate for $\overline{B}^c(n,x)$.

Proposition 4.5.5. *For integers $n \geq 2$ and real numbers x such that $\sqrt{n} \leq x \leq n$, we have*

$$\overline{B}^c(n,x) = n^2\left(J\left(\frac{n}{x}\right) - \frac{1}{2}\left(\log\frac{n}{x}\right)^2\right) - (n^2\log n - n^2)\left(H_{\lfloor\frac{n}{x}\rfloor} - \log\frac{n}{x}\right)$$
$$+ n^2\left(1 - \left\lfloor\frac{n}{x}\right\rfloor\frac{x}{n}\right)\log\frac{n}{x} + (n^2\log n - n^2)\left\lfloor\frac{n}{x}\right\rfloor\frac{x}{n} + O\left(n^{3/2}\log n\right). \tag{4.5.14}$$

Proof. We have

$$\overline{B}^c(n,x) = \overline{B}^c_{11}(n,x) - (n-1)\left(\overline{C}(n,n) - \overline{C}(n,x)\right). \tag{4.5.15}$$

We suppose $\sqrt{n} \leq x \leq n$. From Lemma 4.5.4 we obtain

$$\overline{B}^c_{11}(n,x) = \frac{1}{2}n^2\left((\log n)^2 - (\log x)^2\right) + O\left(n^{3/2}\log n\right). \tag{4.5.16}$$

Now Proposition 4.2.7(2) gives for $2 \leq x \leq n$,

$$\overline{C}(n,n) - \overline{C}(n,x) = \left(H_{\lfloor\frac{n}{x}\rfloor} - \frac{x}{n}\left\lfloor\frac{n}{x}\right\rfloor\right)(n\log n - n) - \left(J_{\lfloor\frac{n}{x}\rfloor} - \frac{x}{n}\left\lfloor\frac{n}{x}\right\rfloor\log\frac{n}{x}\right)n$$
$$+ O\left(\frac{n\log n}{x}\right).$$

Substituting these estimates into (4.5.15), and assuming $x \geq \sqrt{n}$ yields

$$\overline{B}^c(n,x) = \frac{1}{2}n^2\left((\log n)^2 - (\log x)^2\right) + n\left(H_{\lfloor\frac{n}{x}\rfloor} - \frac{x}{n}\left\lfloor\frac{n}{x}\right\rfloor\right)(n\log n - n)$$
$$-n\left(J_{\lfloor\frac{n}{x}\rfloor} - \frac{x}{n}\left\lfloor\frac{n}{x}\right\rfloor\log\frac{n}{x}\right) + O\left(n^{3/2}(\log n)^2\right). \tag{4.5.17}$$

In this formula we also replaced a factor $(n-1)$ by n, introducing an error of $O\left(n(\log n)^2\right)$ absorbed in the remainder term. Our goal is to simplify this expression to obtain (4.5.14). We rewrite (4.5.17) as

$$\overline{B}^c(n,x) = n^2\left(J_{\lfloor\frac{n}{x}\rfloor} - \frac{1}{2}\left(\log\frac{n}{x}\right)^2\right) + (n^2\log n - n^2)\left(H_{\lfloor\frac{n}{x}\rfloor} - \log\frac{n}{x}\right) + \overline{B}^c_{21}(n,x)$$
$$+ O\left(n^{3/2}(\log n)^2\right), \tag{4.5.18}$$

where we define

$$\begin{aligned}\overline{B}^c_{21}(n,x) \quad := \quad & \left(\frac{1}{2}n^2\left(\log\frac{n}{x}\right)^2 - n^2(\log n)\left(\log\frac{n}{x}\right) + n^2\left(\log\frac{n}{x}\right)\right) \\ & + \frac{1}{2}n^2\left((\log n)^2 - (\log x)^2\right) \\ & - \frac{x}{n}\left\lfloor\frac{n}{x}\right\rfloor\left(n^2\log n - n^2\right) - \left(\frac{x}{n}\left\lfloor\frac{n}{x}\right\rfloor\right)n^2\log\frac{n}{x}. \qquad (4.5.19)\end{aligned}$$

Expanding $\log\frac{n}{x} = \log n - \log x$ in the first two terms in (4.5.19) gives

$$\frac{1}{2}\left(\log\frac{n}{x}\right)^2 - n^2(\log n)\left(\log\frac{n}{x}\right) = -\frac{1}{2}n^2(\log n)^2 + \frac{1}{2}n^2(\log x)^2,$$

which cancels the next two terms appearing in (4.5.19). Rearranging the remaining uncancelled terms results in

$$\overline{B}^c_{21}(n,x) = \frac{x}{n}\left\lfloor\frac{n}{x}\right\rfloor\left(n^2\log n - n^2\right) + \left(1 - \frac{x}{n}\left\lfloor\frac{n}{x}\right\rfloor\right)n^2\log\frac{n}{x}, \qquad (4.5.20)$$

which when substituted in (4.5.18) yields (4.5.14). □

4.5.4 Proof of Theorem 4.5.1

We obtain an estimate of $\overline{B}(n,x)$.

Proof of Theorem 4.5.1. For $n \geq x \geq 1$ we have the decomposition

$$\overline{B}(n,x) = B(n) - \overline{B}^c(n,x). \qquad (4.5.21)$$

By Theorem 4.1.3 we have for $n \geq 2$,

$$\overline{B}(n) = (1-\gamma)n^2\log n + (\gamma + \gamma_1 - 1)\,n^2 + O\left(n^{3/2}\log n\right).$$

By Proposition 4.5.5 we have for $n \geq 2$ and $\sqrt{n} \leq x \leq n$,

$$\begin{aligned}\overline{B}^c(n,x) \quad := \quad & n^2\left(J_{\lfloor\frac{n}{x}\rfloor} - \frac{1}{2}\left(\log\frac{n}{x}\right)^2\right) - \left(n^2\log n - n^2\right)\left(H_{\lfloor\frac{n}{x}\rfloor} - \log\frac{n}{x}\right) \\ & + n^2\left(1 - \left\lfloor\frac{n}{x}\right\rfloor\frac{x}{n}\right)\log\frac{n}{x} + \left(n^2\log n - n^2\right)\left\lfloor\frac{n}{x}\right\rfloor\frac{x}{n} + O\left(n^{3/2}\log n\right).\end{aligned}$$

We obtain for $n \geq 2$ and $\sqrt{n} \leq x \leq n$,

$$\overline{B}(n,x) = \overline{B}_0(n,x)\, n^2 \log n + \overline{B}_1(n,x)\, n^2 + O\left(n^{3/2} \log n\right), \qquad (4.5.22)$$

with

$$\overline{B}_0(n,x) = (1-\gamma) + \left(H_{\lfloor \frac{n}{x} \rfloor} - \log \frac{n}{x}\right) - \left\lfloor \frac{n}{x} \right\rfloor \frac{x}{n}$$

and

$$\overline{B}_1(n,x) = (\gamma + \gamma_1 - 1) - \left(H_{\lfloor \frac{n}{x} \rfloor} - \log \frac{n}{x}\right) - \left(J_{\lfloor \frac{n}{x} \rfloor} - \frac{1}{2}\left(\log \frac{n}{x}\right)^2\right) - \left(1 - \left\lfloor \frac{n}{x} \right\rfloor \frac{x}{n}\right) \log \frac{n}{x} + \left\lfloor \frac{n}{x} \right\rfloor \frac{x}{n},$$

which is (4.5.1).

Finally, for integers $n \geq 2$ and real $x \in [1, \sqrt{n}]$, we have

$$\begin{aligned} \overline{B}(n,x) &= \sum_{2 \leq b \leq x} \frac{n-1}{b-1} d_b(n) \log b \\ &\leq \sum_{2 \leq b \leq x} (n-1) \log(n+1) \\ &< x(n-1)\log(n+1) \\ &< 2n^{3/2} \log n, \end{aligned}$$

where the bound of Lemma 4.2.4 for $d_b(n)$ was used in the first inequality. We have obtained (4.5.4). □

4.5.5 Proof of Theorem 4.1.5

Proof of Theorem 4.1.5. We estimate $\overline{B}(n, \alpha n)$. The theorem follows on choosing $x = \alpha n$ in Theorem 4.5.1 and simplifying. Now $\overline{B}_0(n,x) = f_{\overline{B}}(\alpha)$ is a function of $\alpha = \frac{x}{n}$, with

$$f_{\overline{B}}(\alpha) = (1-\gamma) + \left(H_{\lfloor \frac{1}{\alpha} \rfloor} - \log \frac{1}{\alpha}\right) - \alpha \left\lfloor \frac{1}{\alpha} \right\rfloor$$

Similarly $\overline{B}_1(n,x) = g_{\overline{B}}(\alpha)$ is a function of α with

$$\begin{aligned} g_{\overline{B}}(\alpha) &= (\gamma+\gamma_1-1) - \left(H_{\lfloor\frac{1}{\alpha}\rfloor} - \log\frac{1}{\alpha}\right) - \left(J_{\lfloor\frac{1}{\alpha}\rfloor} - \frac{1}{2}\left(\log\frac{1}{\alpha}\right)^2\right) \\ &\quad + \left(\alpha\left\lfloor\frac{1}{\alpha}\right\rfloor - 1\right)\log\frac{1}{\alpha} + \left\lfloor\frac{1}{\alpha}\right\rfloor\alpha. \end{aligned}$$

We allow $\frac{1}{\sqrt{n}} \le \alpha \le 1$, and for $n \ge 2$ the remainder term in the estimate is $O\left(n^{3/2}\log n\right)$, independent of α in this range. For the range $x \in [1, \sqrt{n}]$ we use the final estimate (4.5.4). □

Remark 4.5.6. The function $f_{\overline{B}}(\alpha)$ has $f_{\overline{B}}(1) = 1-\gamma$, and has $\lim_{\alpha\to 0^+} f_{\overline{B}}(\alpha) = 0$ since $H_{\lfloor\frac{1}{\alpha}\rfloor} - \log\frac{1}{\alpha} \to \gamma$ as $\alpha \to 0^+$. Various individual terms in the formulas for $f_{\overline{B}}(\alpha)$ and $g_{\overline{B}}(\alpha)$ are discontinuous at points $\alpha = \frac{1}{k}$ for $k \ge 1$. The function $f_{\overline{B}}(\alpha)$ was shown to be continuous on $[0,1]$ in [13]; the function $g_{\overline{B}}(\alpha)$ can be checked to be continuous.

4.6 Estimates for the generalized partial factorization sums $\overline{A}(n,x)$

The main goal of this section is to prove the following theorem.

Theorem 4.6.1. *Let $\overline{A}(n,x) = \sum_{b=2}^{\lfloor x\rfloor} \frac{2}{b-1} S_b(n)\log b$. Then for integers $n \ge 2$ and real $x \in [\sqrt{n}, n]$,*

$$\overline{A}(n,x) = \overline{A}_0(n,x)n^2\log n + \overline{A}_1(n,x)n^2 + O\left(n^{3/2}\log n\right), \tag{4.6.1}$$

where the functions $\overline{A}_0(n,x)$ and $\overline{A}_1(n,x)$ only depend on $\frac{x}{n}$ and are given by

$$\overline{A}_0(n,x) := \left(\frac{3}{2}-\gamma\right) + \left(H_{\lfloor\frac{n}{x}\rfloor} - \log\frac{n}{x}\right) + \frac{1}{2}\left(\frac{x}{n}\right)^2\left\lfloor\frac{n}{x}\right\rfloor\left\lfloor\frac{n}{x}+1\right\rfloor - 2\left(\frac{x}{n}\right)\left\lfloor\frac{n}{x}\right\rfloor \tag{4.6.2}$$

and

$$\overline{A}_1(n,x) := \left(\frac{3}{2}\gamma + \gamma_1 - \frac{7}{4}\right) - \frac{3}{2}\left(H_{\lfloor \frac{n}{x} \rfloor} - \log\frac{n}{x}\right) - \left(J_{\lfloor \frac{n}{x} \rfloor} - \frac{1}{2}\left(\log\frac{n}{x}\right)^2\right)$$
$$- \frac{3}{2}\log\frac{n}{x} - \frac{1}{2}\left(\log\frac{n}{x}\right)\left(\frac{x}{n}\right)^2 \left\lfloor \frac{n}{x} \right\rfloor \left\lfloor \frac{n}{x} + 1 \right\rfloor + 2\left(\log\frac{n}{x}\right)\frac{x}{n}\left\lfloor \frac{n}{x} \right\rfloor$$
$$- \frac{1}{4}\left(\frac{x}{n}\right)^2 \left\lfloor \frac{n}{x} \right\rfloor \left\lfloor \frac{n}{x} + 1 \right\rfloor + 2\left(\frac{x}{n}\right)\left\lfloor \frac{n}{x} \right\rfloor. \tag{4.6.3}$$

Moreover, for integers $n \geq 2$ *and real* $x \in [1, \sqrt{n}]$*,*

$$\overline{A}(n,x) = O\left(n^{3/2}\log n\right). \tag{4.6.4}$$

We derive estimates for $\overline{A}(n,x)$ starting from $\overline{A}(n,n)$ and working downward, via a recursion in Lemma 4.6.2 below.

4.6.1 Estimates for the complement sum $\overline{A}(n,n) - \overline{A}(n,x)$

First, we show that $\overline{A}(n,n) - \overline{A}(n,x)$ can be written in terms of known quantities, namely $\overline{B}_{11}^c(n,x)$ and $\overline{C}(j,j) - \overline{C}(j,x)$.

Lemma 4.6.2. *For integers* $n \geq 2$ *and real numbers* x *such that* $\sqrt{n-1} \leq x \leq n$*, we have*

$$\overline{A}(n,n) - \overline{A}(n,x) = \overline{B}_{11}^c(n,x) - 2\sum_{x \leq j < n}\left(\overline{C}(j,j) - \overline{C}(j,x)\right), \tag{4.6.5}$$

where $\overline{C}(n,x)$ *and* $\overline{B}_{11}^c(n,x)$ *are given by* (4.2.15) *and* (4.5.8)*, respectively.*

Proof. From (4.1.21), we have

$$\overline{A}(n,n) = \overline{A}(n,x) + \sum_{x < b \leq n} \frac{2}{b-1} S_b(n) \log b \tag{4.6.6}$$

Observe that for positive integers $b > x$ and $j \leq n-1$, we have $b^2 > x^2 \geq n-1 \geq j$, and hence $\left\lfloor \frac{j}{b^i} \right\rfloor = 0$ for all $i \geq 2$. From (4.2.7), if $b > x$, then

$$S_b(n) = \frac{n(n-1)}{2} - (b-1)\sum_{j=1}^{n-1}\left\lfloor \frac{j}{b} \right\rfloor.$$

On inserting this into (4.6.6), we obtain

$$\begin{aligned}\overline{A}(n,n)-\overline{A}(n,x) &= n(n-1)\sum_{x<b\le n}\frac{\log b}{b-1}-2\sum_{j=1}^{n-1}\sum_{x<b\le n}\left\lfloor\frac{j}{b}\right\rfloor\log b\\ &= \overline{B}^{c}_{11}(n,x)-2\sum_{j=1}^{n-1}\left(\overline{C}(j,n)-\overline{C}(j,x)\right).\end{aligned}$$

From (4.2.16), if $1\le j<n$, then $\overline{C}(j,n)=\overline{C}(j,j)$. Hence

$$\overline{A}(n,n)-\overline{A}(n,x)=\overline{B}^{c}_{11}(n,x)-2\sum_{1\le j<n}\left(\overline{C}(j,j)-\overline{C}(j,x)\right).$$

From (4.2.16), if $1\le j<x$, then $\overline{C}(j,x)=\overline{C}(j,j)$. Hence (4.6.5) follows. □

The next lemma gives an estimate for the sum of values of a dilated floor function. We will use this estimate to prove the main Lemma 4.6.4 below.

Lemma 4.6.3. *For real numbers t and u such that $1\le u\le t$, we have*

$$\sum_{j=1}^{\lfloor t\rfloor}\left\lfloor\frac{j}{u}\right\rfloor=t\left\lfloor\frac{t}{u}\right\rfloor-\frac{1}{2}u\left\lfloor\frac{t}{u}\right\rfloor^2-\frac{1}{2}u\left\lfloor\frac{t}{u}\right\rfloor+O\left(\frac{t}{u}\right).$$

Proof. We write $\left\lfloor\frac{j}{u}\right\rfloor=\sum_{1\le k\le\frac{j}{u}}1$ and interchange the order of summation, obtaining

$$\sum_{j=1}^{\lfloor t\rfloor}\left\lfloor\frac{j}{u}\right\rfloor=\sum_{1\le j\le t}\left(\sum_{1\le k\le\frac{j}{u}}1\right)=\sum_{1\le k\le\frac{t}{u}}\left(\sum_{uk\le j\le t}1\right).$$

The inner sum on the right counts the number of integers from $\lceil uk\rceil$ to $\lfloor t\rfloor$. Hence the above is

$$\sum_{j=1}^{\lfloor t\rfloor}\left\lfloor\frac{j}{u}\right\rfloor=\sum_{1\le k\le\frac{t}{u}}\left(\lfloor t\rfloor-\lceil uk\rceil+1\right)=\left(\lfloor t\rfloor+1\right)\left\lfloor\frac{t}{u}\right\rfloor-\sum_{k=1}^{\left\lfloor\frac{t}{u}\right\rfloor}\lceil uk\rceil.$$

By using the estimate $\lceil v\rceil=v+O(1)$, we obtain

$$\begin{aligned}\sum_{j=1}^{\lfloor t\rfloor}\left\lfloor\frac{j}{u}\right\rfloor &= \left(\lfloor t\rfloor+1\right)\left\lfloor\frac{t}{u}\right\rfloor-\sum_{k=1}^{\left\lfloor\frac{t}{u}\right\rfloor}uk+O\left(\frac{t}{u}\right)\\ &= t\left\lfloor\frac{t}{u}\right\rfloor-\frac{1}{2}u\left\lfloor\frac{t}{u}\right\rfloor^2-\frac{1}{2}u\left\lfloor\frac{t}{u}\right\rfloor+O\left(\frac{t}{u}\right)\end{aligned}$$

as desired. □

The following lemma gives an estimate for the complement sum $\overline{A}(n,n)-\overline{A}(n,x)$.

Lemma 4.6.4. *For integers $n \geq 2$ and real numbers x such that $\sqrt{n-1} \leq x \leq n$, we have*

$$\overline{A}(n,n)-\overline{A}(n,x)=\int_x^n \left(\left\lfloor\frac{n}{u}\right\rfloor+\left\{\frac{n}{u}\right\}^2\right) u\log u\,du+O\left(\frac{n^2\log n}{x}\right). \tag{4.6.7}$$

Proof. We start from (4.6.5) and apply Proposition 4.2.7 to estimate each term $\big(\overline{C}(j,j)-\overline{C}(j,x)\big)$:

$$\begin{aligned}\overline{A}(n,n)-\overline{A}(n,x)&=\overline{B}^c_{11}(n,x)-2\sum_{x\leq j<n}\big(\overline{C}(j,j)-\overline{C}(j,x)\big)\\&=\overline{B}^c_{11}(n,x)-2\sum_{x\leq j<n}\int_x^j\left\lfloor\frac{j}{u}\right\rfloor\log u\,du+O\bigg(\frac{1}{x}\sum_{j=1}^n j\log j\bigg).\end{aligned} \tag{4.6.8}$$

Now, we estimate each term on the right of (4.6.6). To simplify the error term, by Lemma 4.2.10, we have

$$\frac{1}{x}\sum_{j=1}^n j\log j=O\left(\frac{n^2\log n}{x}\right). \tag{4.6.9}$$

The first term can be estimated by Lemma 4.5.4:

$$\begin{aligned}\overline{B}^c_{11}(n,x)&=\frac{1}{2}n^2(\log n)^2-\frac{1}{2}n^2(\log x)^2+O\left(\frac{n^2\log n}{x}\right)\\&=\int_x^n\frac{n^2\log u}{u}\,du++O\left(\frac{n^2\log n}{x}\right).\end{aligned} \tag{4.6.10}$$

For the second term, we observe that $\left\lfloor\frac{j}{u}\right\rfloor=0$ for $0<j<u$. Hence

$$\begin{aligned}-2\sum_{x\leq j<n}\int_x^j\left\lfloor\frac{j}{u}\right\rfloor\log u\,du&=-2\sum_{x\leq j<n}\int_x^n\left\lfloor\frac{j}{u}\right\rfloor\log u\,du\\&=-2\int_x^n\left(\sum_{x\leq j<n}\left\lfloor\frac{j}{u}\right\rfloor\right)\log u\,du.\end{aligned} \tag{4.6.11}$$

The inner sum on the right of (4.6.11) can be estimated using Lemma 4.6.3. If $1 \le j < x$ and $u \ge x$, then $0 < j < u$, and then $\left\lfloor \frac{j}{u} \right\rfloor = 0$. Hence $\sum_{1 \le j < x} \left\lfloor \frac{j}{u} \right\rfloor = 0$ and

$$\begin{aligned}\sum_{x \le j < n} \left\lfloor \frac{j}{u} \right\rfloor &= \sum_{j=1}^{n} \left\lfloor \frac{j}{u} \right\rfloor - \left\lfloor \frac{n}{u} \right\rfloor \\ &= n \left\lfloor \frac{n}{u} \right\rfloor - \frac{1}{2} u \left\lfloor \frac{n}{u} \right\rfloor^2 - \frac{1}{2} u \left\lfloor \frac{n}{u} \right\rfloor + O\left(\frac{n}{u}\right) \\ &= n \left\lfloor \frac{n}{u} \right\rfloor - \frac{1}{2} u \left\lfloor \frac{n}{u} \right\rfloor^2 - \frac{1}{2} u \left\lfloor \frac{n}{u} \right\rfloor + O\left(\frac{n}{x}\right).\end{aligned}$$

On inserting this into (4.6.11), we obtain

$$\begin{aligned}-2 \sum_{x \le j < n} \int_x^j \left\lfloor \frac{j}{u} \right\rfloor \log u \, du = \int_x^n \left(-2n \left\lfloor \frac{n}{u} \right\rfloor + u \left\lfloor \frac{n}{u} \right\rfloor^2 + u \left\lfloor \frac{n}{u} \right\rfloor \right) \log u \, du \\ + O\left(\frac{n^2 \log n}{x} \right). \qquad (4.6.12)\end{aligned}$$

On inserting (4.6.9), (4.6.10), and (4.6.12) into (4.6.8), we obtain

$$\begin{aligned}\overline{A}(n,n) - \overline{A}(n,x) &= \int_x^n \left(\frac{n^2}{u} - 2n \left\lfloor \frac{n}{u} \right\rfloor + u \left\lfloor \frac{n}{u} \right\rfloor^2 + u \left\lfloor \frac{n}{u} \right\rfloor \right) \log u \, du + O\left(\frac{n^2 \log n}{x} \right) \\ &= \int_x^n \left(\left(\frac{n}{u} - \left\lfloor \frac{n}{u} \right\rfloor \right)^2 + \left\lfloor \frac{n}{u} \right\rfloor \right) u \log u \, du + O\left(\frac{n^2 \log n}{x} \right) \\ &= \int_x^n \left(\left\lfloor \frac{n}{u} \right\rfloor + \left\{ \frac{n}{u} \right\}^2 \right) u \log u \, du + O\left(\frac{n^2 \log n}{x} \right)\end{aligned}$$

as desired. □

The next lemma shows that the main term in (4.6.7) can be written in the form $f\left(\frac{x}{n}\right) n^2 \log n + g\left(\frac{x}{n}\right) n^2$.

Lemma 4.6.5. *For real numbers n and x such that $0 < x \le n$, we have*

$$\begin{aligned}\int_x^n \left(\left\lfloor \frac{n}{u} \right\rfloor + \left\{ \frac{n}{u} \right\}^2 \right) u \log u \, du = n^2 (\log n) \int_1^{\frac{n}{x}} \frac{\lfloor v \rfloor + \{v\}^2}{v^3} \, dv \\ - n^2 \int_1^{\frac{n}{x}} \frac{\lfloor v \rfloor + \{v\}^2}{v^3} \log v \, dv. \qquad (4.6.13)\end{aligned}$$

Proof. By the substitution $v = \frac{n}{u}$, we see that

$$\int_x^n \left(\left\lfloor \frac{n}{u} \right\rfloor + \left\{ \frac{n}{u} \right\}^2 \right) u \log u \, du = \int_{\frac{n}{x}}^1 \left(\lfloor v \rfloor + \{v\}^2 \right) \frac{n}{v} \left(\log \frac{n}{v} \right) \left(-\frac{n}{v^2} \right) dv$$
$$= n^2 \int_1^{\frac{n}{x}} \frac{\lfloor v \rfloor + \{v\}^2}{v^3} \log \frac{n}{v} \, dv.$$

Since $\log \frac{n}{v} = \log n - \log v$, we obtain (4.6.13) as desired. □

To evaluate the integrals on the right of (4.6.13), we use the following lemma.

Lemma 4.6.6. *Suppose that f is a twice differentiable function with continuous second derivative on the interval $[1, \infty)$. Then for real numbers $\beta \geq 1$,*

$$\frac{1}{2} \int_1^\beta \left(\lfloor v \rfloor + \{v\}^2 \right) f''(v) \, dv = \int_1^\beta f(v) \, dv - \sum_{b=2}^{\lfloor \beta \rfloor} f(b)$$
$$- \{\beta\} f(\beta) + \frac{1}{2} \left(\lfloor \beta \rfloor + \{\beta\}^2 \right) f'(\beta) - \frac{1}{2} f'(1). \tag{4.6.14}$$

Proof. By the Euler–Maclaurin summation formula (cf. [23, Theorem B.5]),

$$\sum_{b=2}^{\lfloor \beta \rfloor} f(b) = \int_1^\beta f(v) \, dv - \left(\{\beta\} - \frac{1}{2} \right) f(\beta) - \frac{1}{2} f(1) + \frac{1}{2} \left(\{\beta\}^2 - \{\beta\} + \frac{1}{6} \right) f'(\beta)$$
$$- \frac{1}{12} f'(1) - \frac{1}{2} \int_1^\beta \left(\{v\}^2 - \{v\} + \frac{1}{6} \right) f''(v) \, dv.$$

Rearranging the terms, we obtain

$$\frac{1}{2} \int_1^\beta \left(\{v\}^2 - \{v\} + \frac{1}{6} \right) f''(v) \, dv = \int_1^\beta f(v) \, dv - \sum_{b=2}^{\lfloor \beta \rfloor} f(b) - \left(\{\beta\} - \frac{1}{2} \right) f(\beta)$$
$$+ \frac{1}{2} \left(\{\beta\}^2 - \{\beta\} + \frac{1}{6} \right) f'(\beta)$$
$$- \frac{1}{2} f(1) - \frac{1}{12} f'(1). \tag{4.6.15}$$

On the other hand, we use integration by parts to see that

$$\frac{1}{2}\int_1^\beta \left(v-\frac{1}{6}\right) f''(v)\,dv = \frac{1}{2}\left(v-\frac{1}{6}\right) f'(v)\Big|_{v=1}^{\beta} - \frac{1}{2}\int_1^\beta f'(v)\,dv$$
$$= -\frac{1}{2}f(\beta)+\frac{1}{2}\left(\beta-\frac{1}{6}\right)f'(\beta)+\frac{1}{2}f(1)-\frac{5}{12}f'(1). \tag{4.6.16}$$

Adding (4.6.15) and (4.6.16), we obtain (4.6.14). □

We apply Lemma 4.6.6 to evaluate the integrals on the right of (4.6.13).

Lemma 4.6.7. *For real numbers α such that $0<\alpha\le 1$, we have*

$$\int_1^{\frac{1}{\alpha}} \frac{\lfloor v\rfloor + \{v\}^2}{v^3}\,dv = \frac{3}{2} - \left(H_{\lfloor\frac{1}{\alpha}\rfloor} - \log\frac{1}{\alpha}\right) - \alpha\left(\left\{\frac{1}{\alpha}\right\}+\frac{1}{2}\right) - \alpha^2\left(\frac{1}{2}\left\{\frac{1}{\alpha}\right\}^2 - \frac{1}{2}\left\{\frac{1}{\alpha}\right\}\right) \tag{4.6.17}$$

and

$$\int_1^{\frac{1}{\alpha}} \frac{\lfloor v\rfloor + \{v\}^2}{v^3}\log v\,dv = \frac{7}{4} - \frac{3}{2}\left(H_{\lfloor\frac{1}{\alpha}\rfloor} - \log\frac{1}{\alpha}\right) - \left(J\left(\frac{1}{\alpha}\right) - \frac{1}{2}\left(\log\frac{1}{\alpha}\right)^2\right)$$
$$-\left(\alpha\log\frac{1}{\alpha}\right)\left(\left\{\frac{1}{\alpha}\right\}+\frac{1}{2}\right) - \alpha\left(\frac{3}{2}\left\{\frac{1}{\alpha}\right\}+\frac{1}{4}\right)$$
$$-\left(\alpha^2\log\frac{1}{\alpha}\right)\left(\frac{1}{2}\left\{\frac{1}{\alpha}\right\}^2 - \frac{1}{2}\left\{\frac{1}{\alpha}\right\}\right)$$
$$-\alpha^2\left(\frac{1}{4}\left\{\frac{1}{\alpha}\right\}^2 - \frac{1}{4}\left\{\frac{1}{\alpha}\right\}\right). \tag{4.6.18}$$

Proof. For (4.6.17), apply Lemma 4.6.6 with $f(v)=\frac{1}{v}$ and $\beta=\frac{1}{\alpha}$:

$$\int_1^{\frac{1}{\alpha}} \frac{\lfloor v\rfloor + \{v\}^2}{v^3}\,dv = \log\frac{1}{\alpha} - \left(H_{\lfloor\frac{1}{\alpha}\rfloor} - 1\right) - \alpha\left\{\frac{1}{\alpha}\right\} - \frac{1}{2}\alpha^2\left(\left\lfloor\frac{1}{\alpha}\right\rfloor + \left\{\frac{1}{\alpha}\right\}^2\right) + \frac{1}{2}.$$

Replacing $\lfloor\frac{1}{\alpha}\rfloor + \{\frac{1}{\alpha}\}^2$ by $\frac{1}{\alpha} - \{\frac{1}{\alpha}\} + \{\frac{1}{\alpha}\}^2$ and rearranging the terms, we obtain (4.6.17).

For (4.6.18), apply Lemma 4.6.6 with $f(v) = \frac{3}{2v} + \frac{\log v}{v}$ and $\beta = \frac{1}{\alpha}$:

$$\int_1^{\frac{1}{\alpha}} \frac{\lfloor v \rfloor + \{v\}^2}{v^3} \log v \, dv = \left(\frac{3}{2} \log \frac{1}{\alpha} + \frac{1}{2} \left(\log \frac{1}{\alpha} \right)^2 \right) - \left(\frac{3}{2} H_{\lfloor \frac{1}{\alpha} \rfloor} + J\left(\frac{1}{\alpha} \right) - \frac{3}{2} \right)$$
$$- \alpha \left\{ \frac{1}{\alpha} \right\} \left(\frac{3}{2} + \log \frac{1}{\alpha} \right)$$
$$- \alpha^2 \left(\left\lfloor \frac{1}{\alpha} \right\rfloor + \left\{ \frac{1}{\alpha} \right\}^2 \right) \left(\frac{1}{4} + \frac{1}{2} \log \frac{1}{\alpha} \right) + \frac{1}{4}.$$

Replacing $\lfloor \frac{1}{\alpha} \rfloor + \{ \frac{1}{\alpha} \}^2$ by $\frac{1}{\alpha} - \{ \frac{1}{\alpha} \} + \{ \frac{1}{\alpha} \}^2$ and rearranging the terms, we obtain (4.6.18). □

4.6.2 Proof of Theorem 4.6.1

We combine results in the previous subsection to obtain an estimate for $\overline{A}(n, x)$ as stated in Theorem 4.6.1.

Proof of Theorem 4.6.1. Combining Theorem 4.1.4 and Lemma 4.6.4, which estimate $\overline{A}(n, n)$ and $\overline{A}(n, n) - \overline{A}(n, x)$ respectively, we obtain an estimate for $\overline{A}(n, x)$:

$$\overline{A}(n, x) = \left(\frac{3}{2} - \gamma \right) n^2 \log n + \left(\frac{3}{2}\gamma + \gamma_1 - \frac{7}{4} \right) n^2 - \int_x^n \left(\left\lfloor \frac{n}{u} \right\rfloor + \left\{ \frac{n}{u} \right\}^2 \right) u \log u \, du$$
$$+ O\left(n^{3/2} \log n \right).$$

The integral on the right can be evaluated using Lemma 4.6.5:

$$\begin{aligned} \overline{A}(n, x) = & \left(\frac{3}{2} - \gamma \right) n^2 \log n + \left(\frac{3}{2}\gamma + \gamma_1 - \frac{7}{4} \right) n^2 \\ & - n^2 (\log n) \int_1^{\frac{n}{x}} \frac{\lfloor v \rfloor + \{v\}^2}{v^3} \, dv + n^2 \int_1^{\frac{n}{x}} \frac{\lfloor v \rfloor + \{v\}^2}{v^3} \log v \, dv \\ & + O\left(n^{3/2} \log n \right) \\ = & \overline{A}_0(n, x) n^2 \log n + \overline{A}_1(n, x) n^2 + O\left(n^{3/2} \log n \right), \end{aligned}$$

where the functions $\overline{A}_0(n, x)$ and $\overline{A}_1(n, x)$ are given by

$$\overline{A}_0(n, x) := \left(\frac{3}{2} - \gamma \right) - \int_1^{\frac{n}{x}} \frac{\lfloor v \rfloor + \{v\}^2}{v^3} \, dv, \tag{4.6.19}$$

$$\overline{A}_1(n,x) := \left(\frac{3}{2}\gamma + \gamma_1 - \frac{7}{4}\right) + \int_1^{\frac{n}{x}} \frac{\lfloor v \rfloor + \{v\}^2}{v^3} \log v \, dv. \tag{4.6.20}$$

It remains to show that (4.6.19) is equivalent to (4.6.2) and that (4.6.20) is equivalent to (4.6.3). To that end, we apply Lemma 4.6.7 with $\alpha = \frac{x}{n}$ to evaluate the integrals in (4.6.19) and (4.6.20). We obtain

$$\overline{A}_0(n,x) = \left(H_{\lfloor \frac{n}{x} \rfloor} - \log\frac{n}{x} - \gamma\right) + \frac{x}{n}\left(\left\{\frac{n}{x}\right\} + \frac{1}{2}\right) - \left(\frac{x}{n}\right)^2 \left(\frac{1}{2}\left\{\frac{n}{x}\right\}^2 - \frac{1}{2}\left\{\frac{n}{x}\right\}\right),$$

$$\begin{aligned}\overline{A}_1(n,x) \;&=\; -\frac{3}{2}\left(H_{\lfloor \frac{n}{x} \rfloor} - \log\frac{n}{x} - \gamma\right) - \left(J\left(\frac{n}{x}\right) - \frac{1}{2}\left(\log\frac{n}{x}\right)^2 - \gamma_1\right) \\ &\quad - \left(\frac{x}{n}\log\frac{n}{x}\right)\left(\left\{\frac{n}{x}\right\} + \frac{1}{2}\right) - \frac{x}{n}\left(\frac{3}{2}\left\{\frac{n}{x}\right\} + \frac{1}{4}\right) \\ &\quad - \left(\frac{x}{n}\right)^2\left(\log\frac{n}{x}\right)\left(\frac{1}{2}\left\{\frac{n}{x}\right\}^2 - \frac{1}{2}\left\{\frac{n}{x}\right\}\right) - \left(\frac{x}{n}\right)^2\left(\frac{1}{4}\left\{\frac{n}{x}\right\}^2 - \frac{1}{4}\left\{\frac{n}{x}\right\}\right).\end{aligned}$$

Replacing $\left\{\frac{n}{x}\right\}$ by $\frac{n}{x} - \left\lfloor\frac{n}{x}\right\rfloor$ and rearranging the terms, we obtain the formulas (4.6.2) and (4.6.3).

Finally, for integers $n \geq 2$ and real $x \in [1, \sqrt{n}]$, we have

$$\begin{aligned}\overline{A}(n,x) &= \sum_{2 \leq b \leq x} \frac{2}{b-1} S_b(n) \log b \\ &\leq \sum_{2 \leq b \leq x} n \log n < xn \log n \\ &\leq n^{3/2} \log n,\end{aligned}$$

where the bound of Lemma 4.2.4 for $S_b(n)$ was used in the first inequality. We have obtained (4.6.4). □

4.6.3 Proof of Theorem 4.1.6

Proof of Theorem 4.1.6. The result for the range $x \in [\sqrt{n}, n]$ follows from Theorem 4.6.1 on choosing $x = \alpha n$ and simplifying. For the range $x \in [1, \sqrt{n}]$ we use the final estimate (4.6.4). □

Remark 4.6.8. The function $f_{\overline{A}}(\alpha)$ has $f_{\overline{A}}(1) = \frac{3}{2} - \gamma$, and has $\lim_{\alpha\to 0^+} f_{\overline{A}}(\alpha) = 0$ since $H_{\lfloor \frac{1}{\alpha} \rfloor} - \log\frac{1}{\alpha} \to \gamma$ as $\alpha \to 0^+$.

4.7 Estimates for partial factorizations $\overline{\overline{G}}(n,x)$

We deduce asymptotics of $\overline{\overline{G}}(n,x)$.

Theorem 4.7.1. *Let* $\overline{\overline{G}}(n,x) = \prod_{b=2}^{\lfloor x \rfloor} b^{\overline{\nu}(n,b)}$. *Then for integers* $n \geq 2$ *and real* $x \in [\sqrt{n}, n]$,

$$\log \overline{\overline{G}}(n,x) = \overline{\overline{G}}_0(n,x)\, n^2 \log n + \overline{\overline{G}}_1(n,x) n^2 + O\left(n^{3/2}\log n\right), \tag{4.7.1}$$

where the functions $\overline{\overline{G}}_0(n,x)$ *and* $\overline{\overline{G}}_1(n,x)$ *only depend on* $\frac{x}{n}$ *and are given by*

$$\overline{\overline{G}}_0(n,x) := \frac{1}{2} + \frac{1}{2}\left(\frac{x}{n}\right)^2 \left\lfloor \frac{n}{x} \right\rfloor \left\lfloor \frac{n}{x} + 1 \right\rfloor - \frac{x}{n}\left\lfloor \frac{n}{x} \right\rfloor \tag{4.7.2}$$

and

$$\begin{aligned}\overline{\overline{G}}_1(n,x) := &\left(\frac{1}{2}\gamma - \frac{3}{4}\right) - \frac{1}{2}\left(H_{\lfloor \frac{n}{x} \rfloor} - \log\frac{n}{x}\right) \\ &- \frac{1}{2}\log\frac{n}{x} - \frac{1}{2}\left(\log\frac{n}{x}\right)\left(\frac{x}{n}\right)^2 \left\lfloor \frac{n}{x} \right\rfloor \left\lfloor \frac{n}{x} + 1 \right\rfloor + \left(\log\frac{n}{x}\right)\frac{x}{n}\left\lfloor \frac{n}{x} \right\rfloor \\ &- \frac{1}{4}\left(\frac{x}{n}\right)^2 \left\lfloor \frac{n}{x} \right\rfloor \left\lfloor \frac{n}{x} + 1 \right\rfloor + \frac{x}{n}\left\lfloor \frac{n}{x} \right\rfloor.\end{aligned} \tag{4.7.3}$$

Moreover, for integers $n \geq 2$ *and real* $x \in [1, \sqrt{n}]$,

$$\log \overline{\overline{G}}(n,x) = O\left(n^{3/2}\log n\right). \tag{4.7.4}$$

Proof. Recall from (4.1.20) the identity

$$\log \overline{\overline{G}}(n,x) = \overline{A}(n,x) - \overline{B}(n,x).$$

The result (4.7.1) follows for the range $x \in [\sqrt{n}, n]$ by inserting the formulas (4.6.1) in Theorem 4.6.1 and (4.5.1) in Theorem 4.5.1. The formula (4.7.4) in the range $x \in [1, \sqrt{n}]$ follows from the corresponding range bounds in Theorems 4.6.1 and 4.5.1. □

4.7.1 Proof of Theorem 4.1.2

Proof of Theorem 4.1.2. The theorem follows from Theorem 4.7.1 on choosing $x = \alpha n$ and simplifying. The O-constant in the remainder term is absolute for the range $\frac{1}{\sqrt{n}} \leq \alpha \leq 1$. Here $\frac{n}{x} = \frac{1}{\alpha}$. □

4.8 Concluding remarks

Viewing the general definition of generalized binomial products (4.1.8) as a kind of integration operation (over $b \geq 2$) the smoothing aspect of the integration operation is evident in the existence of unconditional estimates giving a power-savings remainder term; the Riemann hypothesis is not needed.

A large class of limit functions may occur in problems of this sort, generalizing the limit function $f_G(\alpha)$ in [13] given by (4.1.7). This chapter exhibited a new limit scaling functions $g_{\overline{\overline{G}}}(\alpha)$. It may be of interest to determine the class of of such scaling functions obtained by iterated integral constructions of this kind.

This definition (4.1.8) did not provide any hint whether the product possesses a sub-factorization into analogues of binomial coefficients. In Chapter 3 of this thesis we showed that the sequence $\overline{\overline{G}}_n$ can alternatively be defined as a product of generalized binomial coefficients $\binom{n}{k}_{\mathbb{N}}$ which are integers, which themselves can be written in terms of generalized factorials of a new kind

$$\binom{n}{k}_{\mathbb{N}} = \frac{[n]!_{\mathbb{N}}}{[k]!_{\mathbb{N}}[n-k]!_{\mathbb{N}}}.$$

These new factorials can themselves be factorized into a product of generalized integers

$$[n]_{\mathbb{N}} = \prod_{k=1}^{n} [k]_{\mathbb{N}}.$$

These generalized integers $[n]_{\mathbb{N}}$ are not monotonically increasing but have an internal structure driven by the prime factorization of n.

The factorials $[n]!_{\mathbb{N}}$ above also have many of the properties of the generalized factorials of Bhargava ([4], [5], [6]). They seem to not be included in Bhargava's theory of P-orderings, but we showed in Chapter 3 that they can be covered by a generalization of this theory.

These generalized factorials also fit in the general framework of Knuth and Wilf [18] treating generalized factorials and binomial coefficients as products of generalized integers (denoted C_n in their paper). The sequence of generalized integers $[n]_{\mathbb{N}}$ is not a regularly divisible sequence as defined in [18]; see the remark at the end of Subsection 3.2.4.

4.9 Bibliography

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