Rigid Inner Forms Over Function Fields

by

Peter Emil Dillery

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in the University of Michigan
2022

Doctoral Committee:

Professor Tasho Kaletha, Chair
Professor Alexander Bertoloni-Meli
Professor Victoria Booth
Professor Stephen DeBacker

Peter Emil Dillery
dillery@umich.edu
ORCID iD: 0000-0002-3894-5419

DEDICATION

For my parents.

ACKNOWLEDGMENTS

First and foremost, I thank my advisor Tasho Kaletha. From my study of basic algebraic number theory to when I submitted my first paper, he has been extremely helpful and supportive during my graduate studies. His guidance was essential to the research making up this thesis, and to my development as a mathematician—I could not ask for a better advisor.

I thank David Schwein and Alexander Bertoloni-Meli for many helpful and productive mathematical discussions, which I hope will continue long into the future. The University of Michigan and in particular the math department is a wonderful place to do mathematics and be a graduate student—without my fellow students from the "loud office" (as well as Will Dana, Aleksander Horawa, Ryan Kohl, and Carsten Peterson), the last five years would have been far emptier. Thank you also to Kelsi Archibald, Molly Long, Teresa Stokes, and Doreen Fussman for all of their help in making sure that the only problems I got stuck on at the department were purely mathematical, to Mattias Jonsson for his guidance, and to Kartik Prasanna for the work he does for the number theory group at Michigan.

Finally, I gratefully acknowledge the support of NSF grant DMS-1840234 and the Rackham Predoctoral Fellowship.

TABLE OF CONTENTS

ABSTRACT

We generalize the concept of rigid inner forms, defined by Kaletha in [Kal16] and [Kal18], to the setting of a local or global function field F in order to study endoscopy over F and state conjectures regarding representations of an arbitrary connected reductive group G over F. To do this, we define for such G a new cohomology set $H^1(\mathcal{E}, Z \to G) \subset H^1_{\text{fpqc}}(\mathcal{E}, G)$, where $\mathcal{E}$ is an fpqc A-gerbe over F attached to a class in $H^2_{\text{fppf}}(F, A)$ for an explicit profinite commutative group scheme A depending only on F (not on G), and extend the classical Tate-Nakayama duality theorem (locally), Tate's global duality (cf. [Tat66]) result for tori, and their reductive analogues to these new expanded cohomology sets.

We define a relative transfer factor for an endoscopic datum serving a connected reductive group G over local F, and use rigid inner forms to extend this to an absolute transfer factor, enabling the statement of endoscopic conjectures relating stable virtual characters and $\dot{s}$-stable virtual characters for a semisimple $\dot{s}$ associated to a tempered (local) Langlands parameter. Using global rigid inner forms, a localization map from the local gerbe to its global counterpart allows us to organize sets of local rigid inner forms into coherent families, allowing for a definition of global L-packets and a conjectural formula for the multiplicity of an automorphic representation π in the discrete spectrum of G in terms of these L-packets. We also show that, for a connected reductive group G over a global function field F, the adelic transfer factor $\Delta_{\mathbb{A}}$ for the ring of adeles $\mathbb{A}$ of global F serving an endoscopic datum for G decomposes as the product of the normalized local transfer factors.

CHAPTER 1

Introduction

1.1 Motivation

The purpose of this paper is to generalize the theory of *rigid inner forms*, introduced in [Kal16] and [Kal18] for local and global fields of characteristic zero, to local function fields. Rigid inner forms allow one to study the representation theory of a connected reductive group G over a local field F by working simultaneously with all inner forms of G—in particular, they allow locally for an unambiguous statement of the endoscopic Langlands conjectures for arbitrary connected reductive groups over F, and globally for a construction of a pairing involving the L-packet for a global L-parameter giving a conjectural multiplicity formula for an automorphic representation in the discrete spectra of such groups.

The idea of studying all the inner forms of G simultaneously for endoscopy was first suggested by Adams-Barbasch-Vogan in [ABV92]; generally speaking, given a tempered Langlands parameter $\varphi\colon W_F' \to {}^LG$, we should have a subset of representations of inner forms of G, denoted by Π_φ, and a bijective map to some set of representations related to S_φ, the centralizer of φ in $\widehat{G}$. A fundamental question encountered when treating all inner forms at the same time is when two inner forms should be declared "the same". Since we are concerned with representation theory, a natural requirement of isomorphisms of inner forms is that an automorphism of an inner form G' of G should preserve the conjugacy classes of $G'(F)$ as well as the representations of $G'(F)$.

In order to ensure that automorphisms of inner twists satisfy the above requirements, Vogan in [Vog93] expanded the data of an inner twist to that of a *pure inner twist*, which gives the desired rigidity. A pure inner twist is a triple (G', ψ, x), where $\psi\colon G \to G'$ is an inner form of G, and $x \in Z^1(F, G)$ is a 1-cocycle such that $\mathrm{Ad}(x(\sigma)) = \psi^{-1} \circ {}^\sigma\psi$ for all σ in Γ. However, not every inner twist can be enriched to a pure inner twist, since in general $H^1(F, G) \to H^1(F, G_{\mathrm{ad}})$ need not be surjective. The question then becomes: How does one rigidify the notion of inner twists in a way that includes all of them?

The concept of rigid inner forms introduced by Kaletha in [Kal16] answers this question when

F is of characteristic zero. Again we take tuples (G', ψ, z), where now z is a 1-cocycle in a new cohomology set, denoted by $H^1(u \to W, Z \to G)$, where Z is some finite central F-subgroup of G. The cohomology set $H^1(u \to W, Z \to G)$ carries a canonical surjective map to $H^1(F, G/Z)$, which means that such tuples encompass all inner forms of G. Moreover, rigid inner forms are rigid enough so that their automorphisms preserve both desired representation-theoretic properties discussed above. We also have an embedding $H^1(F, G) \hookrightarrow H^1(u \to W, Z \to G)$, connecting rigid inner twists to Vogan's pure inner twists.

Assume that F is a finite extension of $\mathbb{Q}_p$ for some p, so that the theory of [Kal16] applies. The following is a short account of the conjectures enabled by rigid inner forms:

We first record the conjectures coming from Vogan's pure inner twists. Fix $\varphi\colon W_F' \to {}^LG$ a tempered Langlands parameter with centralizer $S_\varphi \subset \widehat{G}$, as well as G^*, a quasi-split pure inner form of G. After fixing a Whittaker datum $\mathfrak{w}$ for G^*, we have a conjectural map $\iota_{\mathfrak{w}}$ and subset $\Pi_\varphi^{\text{pure}}$ of the irreducible tempered representations of the pure inner forms of G^* making the following diagram commute:

$$\begin{array}{ccc} \Pi_\varphi^{\text{pure}} & \xrightarrow{\iota_{\mathfrak{w}}} & \text{Irr}(\pi_0(S_\varphi)) \\ \downarrow & & \downarrow \\ H^1(F, G^*) & \longrightarrow & \pi_0(Z(\widehat{G})^\Gamma)^*, \end{array}$$

where the left arrow sends a pure inner form representation (G', ψ, x, π) to the class $[x]$, the lower arrow is the Kottwitz pairing (see [Kot86]), and the right-hand arrow sends an irreducible representation to its central character. Moreover, the map $\iota_{\mathfrak{w}}$ provides the correct virtual characters which are needed for the endoscopic character identities for a choice of semisimple element $s \in S_\varphi(\mathbb{C})$. However, there need not be a quasi-split pure inner form of our general connected reductive G.

Now we will see the conjectures allowed by replacing the notion of pure inner forms with rigid inner forms. In addition to the Langlands parameter φ with centralizer S_φ, let Z be a fixed finite central F-subgroup of G. The isogeny $G \to G/Z := \overline{G}$ dualizes to an isogeny $\widehat{\overline{G}} \to \widehat{G}$; let S_φ^+ denote the preimage of S_φ under this isogeny. Then, after fixing a Whittaker datum $\mathfrak{w}$ for G^*, a quasi-split rigid inner form of G (which always exists), we conjecture the existence of a subset Π_φ of $\Pi_\varphi^{\text{temp}}$, the tempered representations of the rigid inner forms of G^*, and a bijective map $\iota_{\mathfrak{w}}$ making the following diagram commute

$$\begin{array}{ccc} \Pi_\varphi & \xrightarrow{\iota_{\mathfrak{w}}} & \text{Irr}(\pi_0(S_\varphi^+)) \\ \downarrow & & \downarrow \\ H^1(u \to W, Z \to G^*) & \longrightarrow & \pi_0(Z(\widehat{\overline{G}})^+)^* \end{array} \tag{1.1}$$

where the left map sends a representation of a rigid inner twist to the corresponding class in

$H^1(u \to W, Z \to G^*)$, the right map sends a representation to its central character, and the bottom map is an extension of the duality isomorphism $H^1(F, G) \xrightarrow{\sim} \pi_0(Z(\widehat{G})^\Gamma)^*$ defined by Kottwitz in [Kot86]; here $Z(\widehat{\overline{G}})^+$ denotes the preimage of $Z(\widehat{G})^\Gamma$ in $Z(\widehat{\overline{G}})$.

We now turn to endoscopy. Choosing a semisimple $s \in S_\varphi(\mathbb{C})$, along with the data of φ, gives rise to an endoscopic datum $\mathfrak{e} = (H, \mathcal{H}, \eta, s)$ for G; for simplicity we will assume that $\mathcal{H} = {}^L H$. Rigid inner forms allow us to define, given a fixed quasi-split rigid inner twist (G^*, ψ, z) of G, a ($\mathfrak{w}$-normalized) absolute transfer factor $\Delta'[\dot{\mathfrak{e}}, \psi, z, \mathfrak{w}]$ for pairs of related strongly regular semisimple elements of $H(F)$ and $G(F)$—this was only previously possible for quasi-split G. The fact that we have replaced $\mathfrak{e}$ by $\dot{\mathfrak{e}}$ corresponds to the necessity of replacing s by a preimage $\dot{s}$ in $S_\varphi^+(\mathbb{C})$, on which this factor depends. This absolute transfer factor allows for the formulation of endoscopic virtual character identities for the images $\iota_{\mathfrak{w}}(\dot{\pi})$ of representations $\dot{\pi} \in \Pi_\varphi$ of rigid inner twists of G in the set $\text{Irr}(\pi_0(S_\varphi^+))$.

If we want to generalize these conjectures to connected reductive groups over a local function field F, a natural question that arises is whether or not an analogue of the theory of rigid inner forms can be developed in this new situation. There are nontrivial obstacles to a direct translation of the theory established in [Kal16]. Notably, the cohomology set $H^1(u \to W, Z \to G)$ is defined using the cohomology of a group extension

$$0 \to u \to W \to \Gamma \to 0$$

corresponding to a canonical class in $H^2(F, u)$ for a special profinite commutative affine group u (where Γ denotes the absolute Galois group of F). The group u will not be smooth in positive characteristic, and so it is no longer true that $H^2(F, u) = H^2(\Gamma, u(F^s))$ (where F^s is a separable closure of F), and therefore there is no way of choosing a corresponding group extension in this situation.

We remedy this deficiency by working instead with the fppf cohomology group $H^2_{\text{fppf}}(F, u)$, which may be computed using the Čech cohomology related to the fpqc cover $\text{Spec}(\overline{F}) \to \text{Spec}(F)$. Classes in the group $\check{H}^2_{\text{fppf}}(F, u)$ correspond to isomorphism classes of u-gerbes over $\text{Spec}(F)$, which means that for a canonical class in $H^2_{\text{fppf}}(F, u)$ we get a corresponding u-gerbe $\mathcal{E}$, whose role will replace that of W in [Kal16]. With the gerbe $\mathcal{E}$ in hand, we investigate its cohomology in a way that parallels the cohomology of the group W in [Kal16], culminating in the construction of a cohomology set $H^1(\mathcal{E}, Z \to G)$ that is the analogue of $H^1(u \to W, Z \to G)$ discussed above. In particular, we will have a Tate-Nakayama type isomorphism for $H^1(\mathcal{E}, Z \to G)$ that will be used to construct a canonical pairing

$$H^1(\mathcal{E}, Z \to G) \times \pi_0(Z(\widehat{\overline{G}})^+) \to \mathbb{C}^*$$

extending the positive-characteristic analogue (see [Tha11]) of the Kottwitz pairing in characteristic zero alluded to above.

Note that if F is a finite extension of $\mathbb{Q}_p$, then u is smooth, and in this case our gerbe $\mathcal{E}$ may be replaced by a group extension of Γ by $u(\overline{F})$ using the comparison isomorphism $H^2_{\text{fppf}}(F, u) \xrightarrow{\sim} H^2_{\text{étale}}(F, u) = H^2(\Gamma, u(\overline{F}))$. This then recovers the group W used in [Kal16], cf. the discussion of *Galois gerbes* in [LR87].

The definition of the cohomology set $H^1(\mathcal{E}, Z \to G)$ allows for a completely analogous definition of rigid inner forms, which, when combined with a construction of the relative local transfer factor for local functions fields, allows for the definition of an absolute transfer factor for an endoscopic datum $\mathfrak{e}$ associated to an arbitrary connected reductive group over F. The development of the local theory culminates in a statement of the above conjectures in the setting of local function fields.

Moving beyond local fields to a global function field F, global rigid inner forms both allow us to relate the adelic transfer factor $\Delta_{\mathbb{A}}$ serving an endoscopic datum for G to the normalized transfer factors serving the localizations of this datum and give precise information about the global L-packet Π_{φ} for a tempered discrete homomorphism $\varphi\colon L_F \to {}^LG$, where L_F is the conjectural Langlands dual group of F. Previously, such descriptions were only possible in the case when G is quasi-split.

In light of the above local discussion, one can ask the natural question: How does one describe the global L-packet Π_{φ} for a tempered discrete homomorphism $\varphi\colon L_F \to {}^LG$ using the local L-packets for the localizations φ_v, and how can one use the two horizontal maps of (1.1) to obtain information about these L-packets (namely, how they relate to the discrete spectrum of G)? The key to this problem is organizing families of representations of local rigid inner forms of G_{F_v} into so-called *coherent families*, which is to say, finding a notion of a *global rigid inner form* corresponding to a *global gerbe* $\mathcal{E}_{\dot{V}}$ which localizes in an appropriate way to such a family. Moreover, in order to show that the family of homomorphisms $\{H^1(\mathcal{E}_v, Z \to G) \to \pi_0(Z(\widehat{\overline{G}})^{+,v})^*\}_v$ corresponding to a family of rigid inner forms behaves in a reasonable manner (such as having a well-defined product over all places), one would like a homomorphism

$$H^1(\mathcal{E}_{\dot{V}}, Z \to G) \to [\pi_0(Z(\widehat{\overline{G}})^+)]^*$$

that equals the product of all of the local homomorphisms (note that if, as in the local case, $Z(\widehat{\overline{G}})^+$ is the preimage of $Z(\widehat{G})^{\Gamma}$, then we have maps $\pi_0(Z(\widehat{\overline{G}})^+) \to \pi_0(Z(\widehat{\overline{G}})^{+,v})$ for all v, so this product statement makes sense).

The combination of our local gerbe construction (now denoted by $\mathcal{E}_v$ for a place v) and the construction of the global Galois gerbe $\mathcal{E}_{\dot{V}}$ for number fields in [Kal18] gives a blueprint for the construction of the global gerbe for function fields described in the above paragraph (and thus of global rigid inner forms). As in the local case, the gerbe $\mathcal{E}_{\dot{V}}$ will be banded by a canonically-defined profinite group denoted by $P_{\dot{V}}$ defined in an identical way as for the characteristic-zero analogue [Kal18]. We will then extract the gerbe via proving the existence of a canonical class in $H^2_{\text{fppf}}(F, P_{\dot{V}})$; unlike in the local case, this existence result requires significant work—in particular, we must study gerbes over $\text{Spec}(\mathbb{A})$ and generalize the notion of *complexes of tori*, as in [KS99], to Čech cohomology of the covers $\overline{F}/F$ and $\overline{\mathbb{A}}/\mathbb{A}$, where $\overline{\mathbb{A}} := \overline{F} \otimes_F \mathbb{A}$.

Once the canonical class is established, we use the geometry of $G_{\mathcal{E}_{\dot{V}}}$-torsors on $\mathcal{E}_{\dot{V}}$ to define the cohomology sets $H^1(\mathcal{E}_{\dot{V}}, Z \to G)$ which provides the global analogue of the sets $H^1(\mathcal{E}_v, Z \to G)$, and to define a duality result for this cohomology set (analogous to local Tate-Nakayama duality) which, among other properties, gives the homomorphism $H^1(\mathcal{E}_{\dot{V}}, Z \to G) \to [\pi_0(Z(\widehat{\overline{G}})^+)]^*$ described above. Our constructions also provide us with morphisms of gerbes $\mathcal{E}_v \to \mathcal{E}_{\dot{V}}$ which allow us to localize these cohomology sets.

Using the above construction, one can then define a *coherent family* of rigid inner forms

$$\{(G_{F_v}, (\mathscr{T}_v, \overline{h}_v))\}_v$$

for a fixed inner quasi-split inner twist $G^* \xrightarrow{\psi} G$ to be one such that each torsor $\mathscr{T}_v$ is the localization (defined appropriately using the localization functors described above) of a global torsor $\mathscr{T}$ with $[\mathscr{T}] \in H^1(\mathcal{E}_{\dot{V}}, Z \to G^*)$ (for some appropriate choice of Z). Given a such family, we can then define the global L-packet Π_φ for some a fixed φ via

$$\Pi_\varphi := \{\pi = \otimes_v' \pi_v \mid (G_{F_v}, \psi, (\mathscr{T}_v, \bar{h}_v), \pi_v) \in \Pi_{\varphi_v}, \iota_{\varphi_v}((G_{F_v}, \psi, (\mathscr{T}_v, \bar{h}_v), \pi_v)) = 1 \text{ for almost all } v\},$$

as desired. We show that this consists of irreducible tempered admissible representations of $G(\mathbb{A})$ in Lemma 9.4.1 using a torsor-theoretic analogue of a result by Taïbi ([Taï18, Proposition 6.1.1]), and hence is well-defined. Moreover, given such a π, we can then give a conjectural description of the multiplicity of π in the discrete spectrum of G by defining (for each φ) a pairing

$$\langle -, - \rangle \colon \mathcal{S}_\varphi \times \Pi_\varphi \to \mathbb{C},$$

where $\mathcal{S}_\varphi$ is a finite group closely related to the centralizer of φ in $\widehat{G}$, which is defined as a product over all places of two factors involving the local pairings and (conjectural) local bijections $\iota_{\varphi_v, \mathfrak{w}_v}$. The key to proving that such a product formula is well-defined is precisely the fact that our representation π arises from a coherent family of representations of local rigid inner forms. Once we know that such a pairing exists, we have for each π and L-packet Π_φ containing π an integer

$$m(\varphi, \pi) := |\mathcal{S}_\varphi|^{-1} \sum_{x \in \mathcal{S}_\varphi} \langle x, \pi \rangle,$$

and, furthermore, we conjecture:

Conjecture 1.1.1 (Kottwitz, [Kot84]) *The multiplicity of π in the discrete spectrum of G is given by the sum*

$$\sum_{\varphi} m(\varphi, \pi),$$

where the sum is over all φ such that $\pi \in \Pi_\varphi$.

Since local rigid inner forms were the vital ingredient for proving the existence of a normalized local transfer factor $\Delta_v = \Delta[\mathfrak{w}_v, \dot{\mathfrak{e}}_v, \mathfrak{z}_v, \psi, (\mathscr{T}_v, \bar{h}_v)]$ serving a fixed endoscopic datum for G_{F_v} (depending on a quasi-split rigid inner form $(\psi, (\mathscr{T}_v, \bar{h}_v))$ of G_{F_v} and a Whittaker datum $\mathfrak{w}_v$ for it), one can use global rigid inner forms to relate the global adelic transfer factor $\Delta_{\mathbb{A}}$ defined in [LS87] (for number fields, but which is easily translated to a global function field) serving a global endoscopic datum to the transfer factors Δ_v serving the localizations of that datum. Indeed, using the relationship between the local and global pairings described above, one obtains (Proposition 9.3.1) a product formula

$$\Delta_{\mathbb{A}}(\gamma_1, \delta) = \prod_{v \in V} \langle \text{loc}_v(\mathscr{T}_{\text{sc}}), \dot{y}'_v \rangle \cdot \Delta[\mathfrak{w}_v, \dot{\mathfrak{e}}_v, \mathfrak{z}_v, \psi, (\mathscr{T}_v, \bar{h}_v)](\gamma_{1,v}, \delta_v)$$

which expresses the value of $\Delta_{\mathbb{A}}$ at a pair of adelic elements (γ_1, δ) as a product of each Δ_v at the localizations of these elements, along with some auxiliary factors $\langle \text{loc}_v(\mathscr{T}_{\text{sc}}), \dot{y}'_v \rangle$ which are harmless and only necessary for technical reasons. Of course, one must take each Δ_v to arise from the localizations of the same global rigid inner form and the local Whittaker data to be the localizations of the same global Whittaker datum $\mathfrak{w}$, even though such a datum is not used to define the left-hand side of the above equation.

1.2 Overview

We now summarize the structure of this thesis. The first two chapters should be viewed as establishing background results. The goal of Chapter 2 is to obtain a concrete interpretation of torsors on gerbes, beginning by recalling the basic theory of fibered categories, stacks, and gerbes, progressing to a characterization of torsors on gerbes, and concluding by investigating the analogue of the inflation-restriction sequence in group cohomology in the setting of gerbes. Following this, Chapter 3 discusses fundamental properties of Čech cohomology; one of its main focuses is comparing the Čech cohomology of group schemes with respect to fpqc covers to the fppf cohomology of these groups schemes, with the goal of determining when these two cohomologies coincide. It also proves certain cohomological vanishing results of certain covers of rings, defines an unbalanced cup product, and concludes with some miscellaneous results about adelic Čech cohomology.

Chapter 4 constructs the local gerbe and proves a duality result for the resulting cohomology sets $H^1(\mathcal{E}, Z \to G)$: We construct the local pro-algebraic group u, investigate its cohomology, and then define the cohomology set $H^1(\mathcal{E}, Z \to S)$ for an F-torus S, where $\mathcal{E}$ is a u-gerbe associated to a canonical cohomology class in $H^2(F, u)$ and discuss basic functoriality properties of the cohomology group $H^1(\mathcal{E}, Z \to S)$ using our insight from Chapter 2. An analogue of the classical Tate-Nakayama isomorphism is constructed for $H^1(\mathcal{E}, Z \to S)$ in §4.4. Once the situation for tori is established, we then define $H^1(\mathcal{E}, Z \to G)$ for a general connected reductive group G and extend all of the previous results to this new situation. There is not much to do here: the bulk of the work is just direct translation of the results in [Kal16], §3 and §4 to fppf cohomology, using basic theorems about the structure theory of connected reductive groups over local function fields (see [Deb06], [Tha08], [Tha11]).

In order to apply Chapter 4 to the local Langlands conjectures, it is necessary to recall and define the (relative) local transfer factor corresponding to an endoscopic datum for a reductive group over a local function field—we do this in Chapter 5. This section is entirely self-contained for expository purposes, and in many cases is just a direct exposition of the constructions stated in [LS87]; the only aspects of the arguments loc. cit. that require minor adjustment are those concerning the Δ_I and Δ_{III_1} factors, but we include a discussion of all of the factors for completeness.

The final local chapter is Chapter 6, where we define rigid inner forms for local function fields and then use them to define an absolute local transfer factor for an endoscopic datum associated to an arbitrary connected reductive group over F. Once this is done, we give a brief summary of the conjectures stemming from our constructions. This section closely parallels §5 in [Kal16]; in many cases, we follow the arguments verbatim, substituting Galois-cohomological calculations with analogous computations in Čech cohomology.

The first section in which we focus on a global function field is F is Chapter 7, where we begin by proving an analogue of global Tate duality for the groups $H^2_{\text{fppf}}(F, Z)$, where Z is a finite multiplicative F-group scheme. After that, we define a projective system of multiplicative group schemes $\{P_{E,\dot{S}_E,n}\}$ whose limit gives the pro-algebraic group $P_{\dot{V}}$ that will band our global gerbe. Once $P_{\dot{V}}$ is defined, we show that its first fppf cohomology group over F vanishes using local and global class field theory and that its second fppf cohomology group contains a canonical class. Constructing such a global class is considerably more difficult than in the local case, and requires utilizing a Čech-cohomological analogue of complexes of tori.

Once the global canonical class is defined, we can construct the global gerbe $\mathcal{E}_{\dot{V}}$, whose cohomology is studied in Chapter 8, building towards proving a duality result for the cohomology sets $H^1(\mathcal{E}_{\dot{V}}, Z \to G)$, where Z is a finite central subgroup of G. We also prove a result concerning the localizations of torsor on $\mathcal{E}_{\dot{V}}$ which will be used in Chapter 9 to prove that global L-packets consist of irreducible, tempered, admissible representations.

Finally, in Chapter 9 we develop endoscopy, defining the adelic transfer factor for function fields and coherent families of rigid inner forms. We relate our local constructions to global endoscopy, including the adelic transfer factor and the multiplicity formula. In Appendix A, we establish complexes of tori in the setting of Čech cohomology and prove several results analogous to those in the appendices of [KS99] (that used Galois cohomology) which are used in the proof of the existence of a canonical class in Chapter 7.

1.3 Notation and conventions

In Chapters 4 through 6, we will use F to denote a local field of characteristic $p > 0$. In Chapters 7 through 9, we will use F to denote a global field of characteristic $p > 0$, and its completion at a place v will be denoted by F_v. For an arbitrary algebraic group G over F, G° denotes the identity component. For a connected reductive group G over F, $Z(G)$ denotes the center of G, and for H a subgroup of G, $N_G(H), Z_G(H)$ denote the normalizer and centralizer group schemes of H in G, respectively. We will denote by $\mathscr{D}(G)$ the derived subgroup of G, by G_{ad} the quotient $G/Z(G)$, and if G is semisimple, we denote by G_{sc} the simply-connected cover of G; if G is not semisimple, G_{sc} denotes $\mathscr{D}(G)_{\text{sc}}$. If T is a maximal torus of G, denote by T_{sc} its preimage in G_{sc}. For local and global F we fix an algebraic closure $\overline{F}$ of F, which contains a separable closure of F, denoted by F^s. For E/F a Galois extension, we denote the Galois group of E over F by $\Gamma_{E/F}$, and we set $\Gamma_{F^s/F} =: \Gamma$.

For global F, we denote by V the set of all places of F, and for E/F a finite extension and $S \subseteq V$, we denote by S_E the preimage of S in V_E, the set of all places of E. We call a subset of V *full* if it equals S_F for some subset S of places of $\mathbb{F}_p(t)$ (after choosing an embedding $\mathbb{F}_p(t) \to F$). For a finite subset $S \subset V$, we set $\mathbb{A}_S := \prod_{v \in S} F_v \times \prod_{v \notin S} O_{F_v}$, and set $\mathbb{A}_{E,S} := \mathbb{A}_{E,S_E}$.

We call an affine, commutative algebraic group over a ring R *multiplicative* if it is Cartier dual to an étale R-group scheme. In this paper, whenever we discuss a general group scheme over R, it will always be assumed to be affine. For Z a multiplicative group over F, we denote by $X^*(Z), X_*(Z)(= X_*(Z^{\circ}))$ the character and co-character modules of Z, respectively, viewed as Γ-modules. For two R-schemes X, Y and R-algebra S, we set $X \times_{\text{Spec}(R)} Y =: X \times_R Y$, or by $X \times Y$ if R is understood, and set $X \times_R \text{Spec}(S) =: X_S$. We also set $X(\text{Spec}(S)) =: X(S)$, the set of R-morphisms $\{\text{Spec}(S) \to X\}$; when X is a variety over $\mathbb{C}$ (for us, this will be a Langlands dual group $\widehat{G}$ for a connected reductive group G over F), we frequently abuse notation and write X to mean $X(\mathbb{C})$. For a morphism $f\colon A \to B$ of multiplicative group schemes over R, we use $f^\sharp$ to denote both induced morphisms $X_*(A) \to X_*(B)$ and $X^*(B) \to X^*(A)$. Also, given a morphism $f\colon U \to V$ of two objects in a stack $\mathcal{C}$ and sheaf $\mathcal{F}$ on $\mathcal{C}$, we also use the symbol $f^\sharp$ to denote the induced morphism $\mathcal{F}(V) \to \mathcal{F}(U)$; there will be no danger of confusing these two notations.

CHAPTER 2

Gerbe-Theoretic Preliminaries

2.1 Basics of fibered categories and stacks

The purpose of this subsection is to briefly review the theory of fibered categories and stacks that will be used later in the paper. For a comprehensive treatment, see for example [Ols16], Chapter 3. Let $\mathcal{C}$ denote a category which has finite fibered products. In the later sections, this will be the category Sch/S of schemes over a fixed scheme S, but for now we will allow it to be arbitrary. Let $\mathscr{X} \xrightarrow{\pi} \mathcal{C}$ be a morphism of categories (i.e., a functor).

Definition 2.1.1 *For $X, Y \in Ob(\mathscr{X})$ denote by U, V (respectively) the objects $\pi(X), \pi(Y)$ in $\mathcal{C}$ (i.e., X and Y **lie above** or **lift** U and V); we say that a morphism $f\colon Y \to X$ in $\mathscr{X}$ is **strongly cartesian** if for every pair of a morphism $g\colon Z \to X$ in $\mathscr{X}$ and morphism $h\colon \pi(Z) \to V$ in $\mathcal{C}$ such that $\pi(g) = \pi(f) \circ h$, there is a unique $\tilde{h}\colon Z \to Y$ such that $f \circ \tilde{h} = g$ and $\pi(\tilde{h}) = h$. In this case, we say that $\tilde{h}$ **lifts** h.*

We continue working with a fixed $\mathscr{X} \xrightarrow{\pi} \mathcal{C}$.

Definition 2.1.2 *For a fixed $U \in Ob(\mathcal{C})$, we define a category $\mathscr{X}(U)$ as follows; its objects will be given by the set $\{X \in Ob(\mathscr{X}) \colon \pi(X) = U\}$ and its morphisms will be those morphisms $X \xrightarrow{f} X'$ such that $\pi(f) = id_U$. We call this the **fiber category over** U, or just the **fiber over** U. We say that $\mathscr{X} \to \mathcal{C}$ is **fibered in groupoids** if for all $U \in Ob(\mathcal{C})$, $\mathscr{X}(U)$ is a groupoid (recall that a category is a **groupoid** if all morphisms are isomorphisms). We will denote the group $Aut_{\mathscr{X}(U)}(X)$ simply by $Aut_U(X)$ for ease of notation.*

Definition 2.1.3 *We say that $\mathscr{X} \xrightarrow{\pi} \mathcal{C}$ is a **fibered category** over $\mathcal{C}$ if for every $U \in Ob(\mathcal{C})$, morphism $V \xrightarrow{f} U$ in $\mathcal{C}$, and $X \in \mathscr{X}(U)$, there is an object $Y \in \mathscr{X}(V)$ and strongly cartesian morphism $\tilde{f}\colon Y \to X$ such that $\pi(\tilde{f}) = f$. One checks that if we have another strongly cartesian $Y' \xrightarrow{\tilde{f}'} X$ satisfying the above property, then there is a unique isomorphism $Y' \to Y$ making all the obvious diagrams commute. We define a **morphism of fibered categories** from $\mathscr{X} \xrightarrow{\pi} \mathcal{C}$ to $\mathscr{X}' \xrightarrow{\pi'} \mathcal{C}$ to be a functor $f\colon \mathscr{X} \to \mathscr{X}'$ such that $\pi = \pi' \circ f$.*

Lemma 2.1.4 *If $\mathscr{X} \to \mathcal{C}$ is a fibered category, then $\mathscr{X}$ also has finite fibered products.*

Proof. Since we assume that $\mathcal{C}$ has finite fibered products, this follows from Lemma I.4.33.4 in [Stacks]. □

In all that follows, given a fibered category $\mathscr{X} \to \mathcal{C}$, for every $U \in \text{Ob}(\mathcal{C})$, $X \in \mathscr{X}(U)$, and morphism $V \xrightarrow{f} U$ in $\mathcal{C}$, we choose some $Y \to X$ satisfying the conditions in the above definition, and will denote this by $f^*X \to X$. One checks that for any morphism $X \xrightarrow{\varphi} Y$ in $\mathscr{X}(U)$, a morphism $f\colon V \to U$ induces a canonical morphism $f^*X \to f^*Y$ in $\mathscr{X}(V)$, which we will denote by $f^*\varphi$.

Definition 2.1.5 *Given a fibered category $\mathscr{X} \xrightarrow{\pi} \mathcal{C}$ and $X, Y \in \mathscr{X}(U)$, we may define a presheaf (of sets), denoted by $\underline{Hom}(X,Y)$, on the category $\mathcal{C}/U$ (the category of pairs (V, g) where $V \in Ob(\mathcal{C})$ and $g\colon V \to U$, morphisms given in the obvious way) by setting*

$$\underline{Hom}(X,Y)(V \xrightarrow{f} U) := Hom_{\mathscr{X}(V)}(f^*X, f^*Y),$$

and for a morphism $(W \xrightarrow{g} U) \xrightarrow{h} (V \xrightarrow{f} U)$, we define the restriction map to be

$$Hom_{\mathscr{X}(V)}(f^*X, f^*Y) \xrightarrow{h^*} Hom_{\mathscr{X}(W)}(h^*(f^*X), h^*(f^*Y)) \cong Hom_{\mathscr{X}(W)}(g^*X, g^*Y),$$

where the first map above sends φ to $h^*\varphi$, and the second map is the canonical isomorphism induced by the canonical identifications $h^*(f^*X) \cong g^*X$, $h^*(f^*Y) \cong g^*Y$. For the remainder of this paper, it will be harmless to make such identifications, and we do so without comment. If $\mathscr{X} \to \mathcal{C}$ is fibered in groupoids and $Y = X$, we denote the above presheaf by $\underline{Aut}_U(X)$—this is a presheaf of groups. It will play an important role in what follows.

We will now assume that we may endow $\mathcal{C}$ with the structure of a site, denoted by $\mathcal{C}_{\text{fpqc}}$, so that it makes sense to talk about sheaves on $\mathcal{C}_{\text{fpqc}}$.

Definition 2.1.6 *We say that a fibered category is a **prestack** (over $\mathcal{C}_{fpqc}$) if for all $U \in Ob(\mathcal{C})$ and $X, Y \in \mathscr{X}(U)$, the presheaf $\underline{Hom}(X,Y)$ is a sheaf on $(\mathcal{C}/U)_{fpqc}$.*

Definition 2.1.7 *Fix $U \in Ob(\mathcal{C})$, a covering $\{V_i \xrightarrow{h_i} U\}_{i \in I}$ of V (here I denotes the indexing set), and a subset $\{X_i \in \mathscr{X}(V_i)\}_{i \in I}$ of $Ob(\mathscr{X})$. The fibered product $V_{ij} := V_i \times_U V_j$ has two projections; we will denote the one to V_i by p_1 and the one to V_j by p_2. We say that this subset, together with a collection of isomorphisms $\{f_{ij}\colon p_1^*X_i \xrightarrow{\sim} p_2^*X_j\colon f_{ij} \in Hom(\mathscr{X}(V_{ij}))\}_{i,j \in I}$ is a* *descent datum (for this fixed covering of U) if the following diagram commutes for all $i, j, k \in I$:*

$$\begin{array}{ccccc} p_{12}^*p_1^*X_i & \xrightarrow{p_{12}^*f_{ij}} & p_{12}^*p_2^*X_j & = & p_{23}^*p_1^*X_j \\ \| & & & & \downarrow{\scriptstyle p_{23}^*f_{jk}} \\ p_{13}^*p_1^*X_i & \xrightarrow{p_{13}^*f_{ik}} & p_{13}^*p_2^*X_k & = & p_{23}^*p_2^*X_k, \end{array}$$

*where the equalities denote the canonical isomorphisms discussed above, p_{ij} denotes the projection $V_{ijk} := V_i \times_U V_j \times_U V_k \to V_{ij}$, and analogously for the other projections. Given another descent datum $\{Y_i \in \mathscr{X}(V_i)\}_{i\in I}$, $\{g_{ij}\}_{i,j\in I}$, we say that it is **isomorphic** to our above datum if there are isomorphisms $\phi_i \colon X_i \to Y_i$ in $\mathscr{X}(V_i)$ which for all i, j satisfy $p_2^*\phi_j^{-1} \circ g_{ij} \circ p_1^*\phi_i = f_{ij}$.*

Continuing the notation of the above definition, note that if $X \in \mathscr{X}(U)$, then we get a descent datum for free via setting $X_i := h_i^*X$ and $f_{ij} \colon p_1^*h_i^*X \to p_2^*h_j^*X$ the canonical isomorphism between these two pullbacks to V_{ij} of X. We denote this descent datum by X_{canon}.

Definition 2.1.8 *We say that a descent datum $\{X_i\}_{i\in I}$, $\{f_{ij}\}_{i,j\in I}$ for U with respect to the cover $\{V_i \to U\}$ is **effective** if there is an object $X \in \mathscr{X}(U)$ such that $\{X_i\}_{i\in I}$, $\{f_{ij}\}_{i,j\in I}$ is isomorphic to X_{canon}. We say that a prestack $\mathscr{X} \to \mathcal{C}_{fpqc}$ is a **stack** if all descent data (for all objects of $\mathcal{C}$ and their covers) are effective. We define a **morphisms of stacks over $\mathcal{C}_{fpqc}$** to be a morphism between their underlying fibered categories.*

The following proposition shows that whether or not a morphism between two stacks over $\mathcal{C}_{\text{fpqc}}$ is an equivalence can be checked over a cover of $\mathcal{C}_{\text{fpqc}}$. We will assume that $\mathcal{C}$ has a final object U and that our cover consists of one element $U_0 \to U$ (this will be our general situation for the rest of the paper). It is easy to check that if $\mathscr{X} \to \mathcal{C}_{\text{fpqc}}$ is a stack, then restricting $\mathscr{X}$ to the full subcategory of all objects lying above an object in $\mathcal{C}/U_0$ is a stack over $(\mathcal{C}/U_0)_{\text{fpqc}}$. We denote this stack by $\mathscr{X}_{U_0}$. This may also be viewed as the fibered product of categories $\mathscr{X} \times_{\mathcal{C}} (\mathcal{C}/U_0)$, for the definition of this, see e.g. [Stacks] I.4.31. We set $U_1 := U_0 \times_U U_0$.

Proposition 2.1.9 *Let $U_0 \to U$ be a cover of $\mathcal{C}_{fpqc} = (\mathcal{C}/U)_{fpqc}$, and $\phi \colon \mathscr{X} \to \mathscr{X}'$ be a morphism of stacks over $\mathcal{C}_{fpqc}$; we have an induced morphism of stacks over $(\mathcal{C}/U_0)_{fpqc}$, denoted by $\phi_{U_0} \colon \mathscr{X}_{U_0} \to \mathscr{X}'_{U_0}$. Then ϕ is an equivalence of categories if and only if ϕ_{U_0} is.*

Proof. One direction is trivial. For the other, if X' is an object of $\mathscr{X}'$, then we may find an object $\tilde{X}$ of $\mathscr{X}$ and f a morphism in $\mathscr{X}'(U_0)$ such that $\phi(\tilde{X}) \xrightarrow{f,\sim} X'_{U_0}$ (where we are denoting the pullback of X' to U_0 by X'_{U_0}). We may also find objects $\tilde{X}_1$, $\tilde{X}_2$ in $\mathscr{X}(U_1)$ and morphisms f_i in $\mathscr{X}'(U_1)$ with $\phi(\tilde{X}_i) \xrightarrow{f_i,\sim} p_i^*(X'_{U_0})$ for $i = 1, 2$, which, since ϕ_{U_0} is an equivalence, are such that we have isomorphisms $\tilde{X}_i \xrightarrow{\tilde{f}_i,\sim} p_i^*\tilde{X}$ with $p_i^*f \circ \phi_{U_0}(\tilde{f}_i) = f_i$ as well as an isomorphism $h \colon \tilde{X}_1 \to \tilde{X}_2$ such that $f_2 \circ \phi(h) \circ f_1^{-1}$ is the canonical identification $p_1^* X'_{U_0} \cong p_2^* X'_{U_0}$. It is straightforward to check that $\mathscr{D} := \{\tilde{X}\}, \{\tilde{f}_2 \circ h \circ \tilde{f}_1^{-1}, \tilde{f}_1 \circ h^{-1} \circ \tilde{f}_2^{-1}\}$ is a descent datum on $\mathscr{X}$, and hence (since $\mathscr{X}$ is a stack) there is some $X \in \mathscr{X}(U)$ with X_{canon} isomorphic to $\mathscr{D}$ as descent data. Then since $\mathscr{X}'$ is a prestack, the local isomorphism $\phi(X)_{U_0} \xrightarrow{\sim} X'_{U_0}$ induced by f and the isomorphism of descent data glues to an isomorphism $\phi(X) \xrightarrow{\sim} X'$, as desired. The analogous argument for morphisms is similar, and left as an exercise. □

2.2 Basics of gerbes

Let R be a ring (we will assume all of our rings are commutative with 1), and let $\mathbf{A}$ be a fixed commutative R-group scheme (recall that all group schemes in this paper are assumed to be affine). Denote by $(\text{Sch}/R)_{\text{fpqc}}$ the site of schemes over $\text{Spec}(R)$ equipped with the fpqc topology. Recall that for a site $\mathcal{C}$ and $\mathscr{G}$ a group sheaf on $\mathcal{C}$, *a $\mathscr{G}$-torsor* $\mathscr{T}$ is a sheaf on $\mathcal{C}$ equipped with a right group (sheaf) action $\mathscr{T} \times \mathscr{G} \to \mathscr{T}$ (satisfying the usual group action axioms) such that for every object X of $\mathcal{C}$, there is some cover $\{Y_i \to X\}$ such that $\mathscr{T}_{Y_i} := \mathscr{T} \times_{\mathcal{C}} (\mathcal{C}/Y_i)$ is ($\mathscr{G}$-equivariantly) isomorphic to the trivial $\mathscr{G}_{Y_i}$-torsor $\mathscr{G}_{Y_i}$, that is, the group sheaf $\mathscr{G}_{Y_i}$ equipped with the right translation action.

We begin with a result that says it is harmless to identify G-torsors for a group scheme G over R with torsors for the associated group sheaf on $(\text{Sch}/R)_{\text{fpqc}}$.

Proposition 2.2.1 *Let G be an fpqc group scheme over R, with $\underline{G}$ the associated sheaf on $(Sch/R)_{fpqc}$. For every $\underline{G}$-torsor P on $(Sch/R)_{fpqc}$, P is representable (as a torsor) by a G-torsor $T \to Spec(R)$.*

Proof. To begin with, let $\mathcal{V} = \{V_i \to \text{Spec}(R)\}$ be an fpqc cover of $\text{Spec}(R)$ trivializing P. Choosing trivializations $h_i \colon P_{V_i} \xrightarrow{\sim} \underline{G}_{V_i}$ (as $\underline{G}_{V_i}$-torsors) gives an element $x = (x_{ij}) \in \prod_{i,j} G(V_i \times_F V_j)$ satisfying the 1-cocycle condition. This furnishes us with an fpqc descent datum of torsors on the site $(\text{Sch}/R)_{\text{fpqc}}$ via the cover $\{V_i \to \text{Spec}(R)\}$, objects $\{G_{V_i}\}$ (with trivial right G_{V_i}-action), and isomorphisms $m_{x_{ij}} \colon p_1^*(G_{V_j}) \xrightarrow{\sim} p_2^*(G_{V_i})$ of $G_{V_{ij}}$-torsors given by left-translation by x_{ij}. Now, since the morphisms $G_{V_i} \to V_i$ are quasi-affine (indeed, they are the base change of the affine morphism $G \to \text{Spec}(R)$), by [Stacks], Lemma II.35.35.1, this descent datum is effective, and hence we get an R-scheme T with a G-action such that $\underline{T} \xrightarrow{\sim} P$ as fpqc G-sheaves. Now, since $G \to \text{Spec}(R)$ is fpqc and T is isomorphic to G after an fpqc base-change, the scheme T is also fpqc over R, and we have a section $T \xrightarrow{\Delta} T \times_R T$ given by the diagonal, showing that T is trivialized over an fpqc cover of $\text{Spec}(R)$. □

Remark 2.2.2 *We will frequently use this proposition without comment in order to identify G-torsors over R and $\underline{G}$-torsors on $(Sch/R)_{fpqc}$. Because of this, it is harmless to abuse notation and denote the sheaf $\underline{G}$ by G.*

We denote by $\check{H}^1_{\text{fpqc}}(R, G)$ the pointed set of isomorphism classes of G-torsors over R.

Definition 2.2.3 *A stack $\mathcal{E} \xrightarrow{\pi} (Sch/R)_{fpqc}$ fibered in groupoids is called a **gerbe** if every object U of $(Sch/R)_{fpqc}$ has a cover $\{V_i \to U\}$ such that every V_i has a lift in $\mathcal{E}$, and for any two objects $X, Y \in Ob(\mathcal{E}(U))$, there is a cover $\{V_i \xrightarrow{f_i} U\}$ such that f_i^*X and f_i^*Y are isomorphic in $\mathcal{E}(V_i)$ for all i.*

In the setting of the above definition, we will frequently omit the topology on Sch/R and just write $\mathcal{E} \xrightarrow{\pi} \text{Sch}/R$ to mean that Sch/R has the fpqc topology.

Example 2.2.4 *The **classifying stack of $\mathbf{A}$ over** R, denoted by $B_R\mathbf{A} \to Sch/R$, has fiber category $B_R\mathbf{A}(U)$, for $U \in Ob(Sch/R)$ an R-scheme, the category of all $\mathbf{A}_U$ torsors T with morphisms being isomorphisms of $\mathbf{A}_U$ torsors. For $V \xrightarrow{f} U$ in Sch/R and T, S fixed $\mathbf{A}_U, \mathbf{A}_V$-torsors (respectively), a morphism $(V, S) \to (U, T)$ lifting f is an isomorphism of $\mathbf{A}_V$-torsors $S \to f^*T$. One verifies easily that this is a gerbe over Sch/R.*

Definition 2.2.5 *As we discussed in §2.1, for any $X \in \mathcal{E}(U)$, the functor on Sch/U given by sending $V \xrightarrow{f} U$ to $Aut_U(f^*X)$ defines a sheaf of groups on $(Sch/U)_{fpqc}$, denoted by $\underline{Aut}_U(X)$. We call our gerbe $\mathcal{E}$ **abelian** if this group sheaf is abelian for all X.*

Lemma 2.2.6 *If $\mathcal{E}$ is an abelian gerbe, then the sheaves $\underline{Aut}_U(X)$, as X varies through all objects of $\mathcal{E}$, glue to define an abelian group sheaf on Sch/R, called the **band** of $\mathcal{E}$ and denoted by $Band(\mathcal{E})$. Moreover, we have for any $X \in \mathcal{E}(U)$ an isomorphism $Band(\mathcal{E})\big|_U \xrightarrow{h_X} \underline{Aut}_U(X)$ of sheaves on $(Sch/U)_{fpqc}$ such that for any $X, Y \in \mathcal{E}(U)$ and isomorphism $\varphi \colon X \to Y$ in $\mathcal{E}(U)$, the following diagram commutes*

$$\begin{array}{ccc} Band(\mathcal{E})\big|_U & = & Band(\mathcal{E})\big|_U \\ \downarrow h_X & & \downarrow h_Y \\ \underline{Aut}_U(X) & \xrightarrow{f \mapsto \varphi \circ f \circ \varphi^{-1}} & \underline{Aut}_U(Y) \end{array}$$

Proof. This is Lemma I.8.11.8 in [Stacks]. □

In fact, following the setup of the above lemma, even if X and Y are not isomorphic in $\mathcal{E}(U)$, since they are locally isomorphic (by the definition of a gerbe), we may find a cover $\{V_i \to U\}$ such that the pullbacks of X and Y to each V_i are isomorphic via some ϕ_i, so that we get an isomorphism $\underline{\text{Aut}}_U(X)\big|_{V_i} \xrightarrow{\sim} \underline{\text{Aut}}_U(Y)\big|_{V_i}$ for all i of sheaves on $(\text{Sch}/V_i)_{\text{fpqc}}$ which is independent of the choice of ϕ_i in view of the above lemma, and hence glues to a *canonical* isomorphism $\underline{\text{Aut}}_U(X) \xrightarrow{\sim} \underline{\text{Aut}}_U(Y)$ of sheaves on $(\text{Sch}/U)_{\text{fpqc}}$ (which is the same as $h_Y \circ h_X^{-1}$). Because of this observation, it is harmless to identify $\text{Band}(\mathcal{E})\big|_U$ with $\underline{\text{Aut}}_U(X)$ for some $X \in \mathcal{E}(U)$ via h_X, which we will do in what follows.

For the rest of this paper, all gerbes will be assumed to be abelian, and when we refer to a "gerbe," we always mean an abelian gerbe.

If we fix a ring homomorphism $R \to R'$ and abelian sheaf $\mathscr{F}$ on Sch/R, then $\check{H}^i(R'/R, \mathscr{F}) = \check{H}^i(\mathrm{Spec}(R') \to \mathrm{Spec}(R), \mathscr{F})$ denotes the ith cohomology group of the complex

$$\mathscr{F}(R') \to \mathscr{F}(R' \otimes_R R') \to \mathscr{F}(R' \otimes_R R' \otimes_R R') \to \dots,$$

where the differentials are given by the alternating sum of the $n+1$ natural maps $\mathscr{F}((R')^{\bigotimes_R n}) \to \mathscr{F}((R')^{\bigotimes_R (n+1)})$. One can make an identical definition (using fibered products of schemes instead of tensor products) if the cover of $\mathrm{Spec}(R)$ is not affine. To ease notation, we set $U_n := \mathrm{Spec}((R')^{\bigotimes_R (n+1)})$ (the ring R' will always be clear from the context, so we omit it from this piece of notation).

Convention 2.2.7 *A simplifying convention we will use in this paper is that, when discussing an abelian R-group scheme $\mathbf{A}$ and an fpqc cover $U_0 \to$ Spec(R), we will always assume that $\check{H}^1_{fpqc}(U_n, \mathbf{A}_{U_n}) = 0$ for all $n \geq 0$. Equivalently, every $\mathbf{A}_W$-torsor over W has a W-trivialization (see Remark 2.2.2) for $W = U_n$. If $R = F$ a field and $\mathbf{A}$ is of finite-type, this condition holds for $U_0 =$ Spec$(\overline{F})$, see [Ros19], §2.9.*

Definition 2.2.8 *We call a pair $(\mathcal{E}, \theta)$ of a gerbe $\mathcal{E}$ and an isomorphism $\theta \colon \mathbf{A} \xrightarrow{\sim}$ Band$(\mathcal{E})$ an **$\mathbf{A}$-gerbe**. In practice, θ will be a way for us to identify automorphisms of objects in $\mathcal{E}$ with elements of $\mathbf{A}$ in a manner that does not depend on isomorphism classes in the fibers; we will frequently omit explicit mention of the map θ. For $X \in \mathcal{E}(V)$, we denote the isomorphism $h_X \circ \theta_U$ from Lemma 2.2.6 by θ_X. Any morphism of stacks over $(Sch/F)_\tau$ between two gerbes $\mathcal{E}$ and $\mathcal{E}'$ induces a morphism of group schemes over R between the corresponding bands. If both can be given the structure of $\mathbf{A}$-gerbes, then we say that such a morphism of Sch$/F$ categories between two $\mathbf{A}$-gerbes is a **morphism of $\mathbf{A}$-gerbes** if it is the identity on bands (via the identifications of both bands with $\mathbf{A}$). By [Ols16], Lemma 12.2.4, any morphism of $\mathbf{A}$-gerbes is an equivalence, and so we will also call such a functor an **equivalence of $\mathbf{A}$-gerbes**.*

Example 2.2.9 *The gerbe $B_R\mathbf{A} \to$ Sch$/R$ may be canonically given the structure of an $\mathbf{A}$-gerbe, since for an abelian group sheaf $\mathbf{A}$ and $\mathbf{A}$-torsor T, the automorphism sheaf defined by T is canonically isomorphic to $\mathbf{A}$.*

We say that an $\mathbf{A}$-gerbe $\mathcal{E}$ is *split* over the cover $V \to \mathrm{Spec}(R)$ if $\mathcal{E}_V := \mathcal{E} \times_{\mathrm{Sch}/R} \mathrm{Sch}/V$ is equivalent as an $\mathbf{A}_V$-gerbe to $B_V(\mathbf{A}_V)$. The following result is a useful alternative characterization of an $\mathbf{A}$-gerbe $\mathcal{E}$ splitting over a cover $V \to \mathrm{Spec}(R)$:

Proposition 2.2.10 *The gerbe $\mathcal{E} \to Sch/R$ is split over $V \to Spec(R)$ if and only if there is an object $X \in \mathcal{E}(V)$.*

Proof. It is clear that if an **A**-gerbe $\mathcal{E}$ is split over V, we have such an object. For the other direction, see [Vis15], Remark 2.4. □

Fact 2.2.11 *Gerbes are closely related to Čech 2-cocycles of **A** with respect to covers of Sch/R, and in this sense are natural analogues of the group extensions that arise in the study of 2-cocycles from Galois cohomology. Indeed, let $(\mathcal{E}, \theta)$ be an **A**-gerbe over Sch/R, and take some $U_0 \to Spec(R)$ a cover such that we have some $X \in \mathcal{E}(U_0)$ with $p_1^*X \xrightarrow{\varphi, \sim} p_2^*X$ for some φ an isomorphism in $\mathcal{E}(U_1)$ (because of Convention 2.2.7, for any $X \in \mathcal{E}(U_0)$, we can always find a φ). We extract a Čech 2-cocycle $c \in \mathbf{A}(U_2)$ in the following manner: φ defines an automorphism of q_1^*X over U_2 via the composition*

$$d\varphi := (p_{13}^*\varphi)^{-1} \circ (p_{23}^*\varphi) \circ (p_{12}^*\varphi) \in Aut_{U_2}(q_1^*X),$$

*and we set $c = \theta_{q_1^*X}(c) \in \mathbf{A}(U_2)$. Then c is a Čech 2-cocycle, whose class in $\check{H}^2(U_0 \to Spec(R), \mathbf{A})$ is independent of the choice of φ and X (see [Moe02], §3). We denote by $[\mathcal{E}] \in \check{H}^2(U_0 \to Spec(R), \mathbf{A})$ the Čech cohomology class obtained from $\mathcal{E}$ as above, and call $[\mathcal{E}]$ the **Čech class corresponding to** $\mathcal{E}$.*

Corollary 2.2.12 *We have a well-defined map from the set of **A**-gerbes split over V to the group $\check{H}^2(V \to Spec(F), \mathbf{A})$ defined by $\mathcal{E} \mapsto [\mathcal{E}]$. Moreover, if $(\mathcal{E}, \theta)$ is equivalent to $(\mathcal{E}', \theta')$, then $[\mathcal{E}] = [\mathcal{E}']$.*

Proof. The first statement is immediate. The second statement is a straightforward exercise using Lemma 2.2.6 and pullbacks in fibered categories. □

2.3 Some explicit gerbes

In this subsection we show that the map from Corollary 2.2.12 is surjective and discuss other fundamental properties of gerbes. We fix an affine fpqc cover $U_0 \to \text{Spec}(R)$ and group **A** as in the previous subsection.

Definition 2.3.1 *Fix a Čech 2-cocycle a of **A** taking values in the cover $U_0 \to Spec(R)$, that is to say, $a \in \mathbf{A}(U_2)$. Then we may define an **A**-gerbe as follows: take the fibered category $\mathcal{E}_a \to Sch/R$ whose fiber over V is defined to be the category of pairs (T, ψ), where T is a (right) $\mathbf{A}_{V \times_R U_0}$-torsor on $V \times_R U_0$ with **A**-action m (in the fpqc topology), along with an isomorphism of $\mathbf{A}_{V \times_R U_1}$-torsors*

*$\psi\colon p_2^*T \xrightarrow{\sim} p_1^*T$, called a **twisted gluing map**, satisfying the following "twisted gluing condition" on the $\boldsymbol{A}_{V\times_R U_2}$-torsor q_1^*T:*

$$(p_{12}^*\psi)\circ(p_{23}^*\psi)\circ(p_{13}^*\psi)^{-1} = m_a,$$

*where m_a denotes the automorphism of the torsor q_1^*T given by right-translation by a. A morphism $(T,\psi_T)\to(S,\psi_S)$ in $\mathcal{E}_a$ lifting the morphism of R-schemes $V\xrightarrow{f}V'$ is a morphism of $\boldsymbol{A}_{V\times U_0}$-torsors $T\xrightarrow{h}f^*S$ satisfying, on $V\times_R U_1$, the relation $f^*\psi_S\circ p_2^*h = p_1^*h\circ\psi_T$. We will call such a pair (T,ψ) in $\mathcal{E}_a(V)$ an a-**twisted torsor over** V when $\boldsymbol{A}$ is understood. We call $\mathcal{E}_a$ the **gerbe corresponding to** a.*

When working with the fibered category $\mathcal{E}_a\to\mathrm{Sch}/R$ there is an obvious canonical choice of pullbacks. Indeed, for $(T,\psi)\in\mathcal{E}_a(U)$ and $f\colon V\to U$, we set $f^*(T,\psi):=(f^*T,f^*\psi)$, and the strongly cartesian morphism $f^*(T,\psi)\to(T,\psi)$ to be the one induced by the identity. We always work with this choice of pullbacks.

Proposition 2.3.2 *The category $\mathcal{E}_a\to Sch/R$ may be canonically given the structure of an $\boldsymbol{A}$-gerbe $(\mathcal{E}_a,\theta)$, with an object $X\in\mathcal{E}_a(U_0)$ and an isomorphism $\varphi\colon p_1^*X\to p_2^*X$ satisfying $\theta_{q_1^*X}^{-1}(d\varphi)=a\in\boldsymbol{A}(U_2)$. In particular, $\mathcal{E}_a$ is split over U_0 and $[\mathcal{E}_a]=[a]$.*

Proof. If we prove that there is such an object $X\in\mathcal{E}_a(U_0)$, it will follow immediately that $\mathcal{E}_a$ defines a gerbe. Moreover, we have that $\mathrm{Band}(\mathcal{E}_a)$ is canonically isomorphic to $\mathbf{A}$, since (for $V=\mathrm{Spec}(R)$, the general case is identical) any automorphism of an a-twisted torsor (T,ψ) is given by a unique element $x\in\mathbf{A}(U_0)$, and since ψ is a morphism of $\mathbf{A}_{U_1}$-torsors commuting with this chosen automorphism, we in fact have that $x\in\mathbf{A}(R)$ (using fpqc descent, cf. the proof of Lemma 2.4.7 below). All that's left to show is the existence of X and φ. This follows from the following lemma (which is important in its own right). □

Lemma 2.3.3 *We have a canonical section $x: Sch/U_0\to\mathcal{E}_a$ such that the two pullbacks x_1 and x_2 to Sch/U_1 are isomorphic via $\varphi\colon x_1\xrightarrow{\sim}x_2$ satisfying $d\varphi:=(p_{13}^*\varphi)^{-1}\circ(p_{23}^*\varphi)\circ(p_{12}^*\varphi)=\iota_a$ as a natural transformation from $q_1^*x: Sch/U_2\to\mathcal{E}_a$ to itself, where we are using ι_a to denote the natural transformation from the identity functor $(\mathcal{E}_a)_{U_2}\to(\mathcal{E}_a)_{U_2}$ to itself given by the automorphism $\theta_Z(a_V)\colon Z\xrightarrow{\sim}Z$ for all $Z\in\mathcal{E}_{U_2}(V)$.*

Proof. Define the a-twisted torsor on $(\mathrm{Sch}/U_0)_{\text{fpqc}}$ to be (as an $\mathbf{A}_{U_0}$-torsor) $\mathbf{A}_{U_0}$; we will define the twisted gluing map after a short discussion. The gluing map should be an isomorphism of $\mathbf{A}_{U_1}$-torsors: $\psi:\tilde{p_2}^*(\mathbf{A}_{U_0})\to\tilde{p_1}^*(\mathbf{A}_{U_0})$, where $\tilde{p_2}:U_0\times U_1\to U_0\times U_0$ is $\mathrm{id}_{U_0}\times p_2$ and $\tilde{p_1}:U_0\times U_1\to U_0\times U_0$ is $\mathrm{id}_{U_0}\times p_1$. We have that $U_0\times U_1=U_2$, and $U_0\times U_0=U_1$, and then

$\tilde{p_1}$ equals p_{12}, $\tilde{p_2}$ equals p_{13}. So, giving ψ reduces to giving a morphism of $\mathbf{A}_{U_0 \times U_1} = \mathbf{A}_{U_2}$-torsors $p_{13}^*(\mathbf{A}_{U_1}) \to p_{12}^*(\mathbf{A}_{U_1})$. Both sides are canonically equal to $\mathbf{A}_{U_2}$, because $\mathbf{A}$ is a sheaf on Sch$/R$ so its value on a U_1-object only depends on the map to Spec(R), which is the same regardless of the map from U_2 to U_1. So we may take ψ to be m_a, which makes sense since $a \in \mathbf{A}(U_2)$; this is $\mathbf{A}$-equivariant since $\mathbf{A}$ is commutative. We need to check that ψ satisfies the twisted cocycle condition.

The above paragraph relied on the equalities $U_0 \times U_1 = U_2$ and $U_0 \times U_0 = U_1$. Continuing these identifications, $\tilde{p_{12}} : U_0 \times U_2 \to U_0 \times U_1$ is the map $U_3 \to U_2$ given by q_{123}, and similarly $\tilde{p_{13}} = q_{124}, \tilde{p_{23}} = q_{134}$. Whence, $\tilde{p_{13}}^*(\psi^{-1}) \circ \tilde{p_{12}}^*(\psi) \circ \tilde{p_{23}}^*(\psi) = (q_{124}^* m_a^{-1}) \circ (q_{123}^* m_a) \circ (q_{134}^* m_a) = q_{234}^* m_a$, since a is a Čech 2-cocycle. Take $\tilde{q_1}^*(\mathbf{A}_{U_0}), \tilde{q_1} = \mathrm{id}_{U_0} \times q_1$. By construction, after identifying $\tilde{q_1}\mathbf{A}$ with $\mathbf{A}_{U_3}$, we see that the left multiplication map $m_{a_{U_2,r_2}}$, where a_{U_2,r_2} denotes the image of a in $\mathbf{A}(U_0 \times U_2) = \mathbf{A}(U_3)$ via the map $r_2 \colon U_0 \times U_2 \to U_2$ which projects onto the second factor, equals $q_{234}^* m_a$, as desired. This a-twisted $\mathbf{A}$-torsor on $(\mathrm{Sch}/U_0)_{\mathrm{fpqc}}$ induces an a_V-twisted $\mathbf{A}$-torsor on each $(\mathrm{Sch}/V)_{\mathrm{fpqc}}$, $V \to U$, via pullback, giving our map x, which one easily checks is a functor.

We now need to define a natural transformation $\varphi : x_1 \xrightarrow{\sim} x_2$ between the two pullbacks of x to U_1. It's enough (by taking pullbacks) to define a morphism of a-twisted torsors

$$\varphi \colon \mathbf{A}_{(U_1 \xrightarrow{p_1} U_0) \times U_0} \to \mathbf{A}_{(U_1 \xrightarrow{p_2} U_0) \times U_0},$$

which we can take to be translation by a, via the same identifications as above. We will verify shortly that $p_1^* \varphi \circ \psi_{U_1 \xrightarrow{p_1} U_0} = \psi_{U_1 \xrightarrow{p_2} U_0} \circ p_2^* \varphi$. The same argument showing that $d\psi = m_a$ gives that $d\varphi = m_a$, which is ι_a, by the definition of the inertial action on $\mathcal{E}_a$.

We now justify our above claim that φ is a morphism of a-twisted torsors. For $V \xrightarrow{f} U_0$, the gluing map ψ_V is

$$(\mathbf{A}_{V \times U_0}) \times_{\mathrm{id} \times p_2} (V \times U_1) \to (\mathbf{A}_{V \times U_0}) \times_{\mathrm{id} \times p_1} (V \times U_1),$$

given by left translation by $a \in \mathbf{A}(U_2) \xrightarrow{(f \times \mathrm{id})^\sharp} \mathbf{A}(V \times U_1)$. As such, we first look at $\psi_{U_1 \xrightarrow{p_1} U_0}$. This is the map on $\mathbf{A}_{U_3}$ given by left translation by the image of a in $\mathbf{A}(U_1 \times U_1)$ via $\mathbf{A}(U_2) \xrightarrow{(p_1 \times \mathrm{id})^\sharp} \mathbf{A}(U_3)$, which is evidently $p_{134}(a)$.

We also have the map

$$\varphi : \mathbf{A}_{(U_1 \xrightarrow{p_1} U_0) \times U_0} \to \mathbf{A}_{(U_1 \xrightarrow{p_2} U_0) \times U_0}$$

which is also left translation by $a \in \mathbf{A}(U_2)$. Thus, $p_1^* \varphi$ is the map

$$\varphi : \mathbf{A}_{(U_1 \xrightarrow{p_1} U_0) \times U_0} \times_{\mathrm{id} \times p_1} (U_1 \times U_1) \to \mathbf{A}_{(U_1 \xrightarrow{p_2} U_0) \times U_0} \times_{\mathrm{id} \times p_1} (U_1 \times U_1),$$

which is left translation by the image of a in $\mathbf{A}(U_3)$ via $U_3 \xrightarrow{\mathrm{id}\times p_1} U_2$, which is $p_{123}(a)$.

On the other hand, the map

$$p_2^*\varphi = \varphi\colon \mathbf{A}_{(U_1 \xrightarrow{p_1} U_0)\times U_0} \times_{\mathrm{id}\times p_2} (U_1 \times U_1) \to \mathbf{A}_{(U_1 \xrightarrow{p_2} U_0)\times U_0} \times_{\mathrm{id}\times p_2} (U_1 \times U_1),$$

corresponds on $\mathbf{A}_{U_3}$ to translation by $(\mathrm{id}\times p_2)^\sharp(a) = p_{124}(a)$, and, finally, we have

$$\psi_{U_1 \xrightarrow{p_2} U_0}\colon \mathbf{A}_{(U_1 \xrightarrow{p_2} U_0)\times U_0} \times_{\mathrm{id}\times p_2} (U_1 \times U_1) \to \mathbf{A}_{(U_1 \xrightarrow{p_2} U_0)\times U_0} \times_{\mathrm{id}\times p_1} (U_1 \times U_1)$$

given by $(p_2 \times \mathrm{id})^\sharp(a) = p_{234}(a)$. The desired equality holds since $p_{234}(a) \cdot p_{124}(a) = p_{134}(a) \cdot p_{123}(a)$, since a is a 2-cocycle. □

We now give a basic functoriality result:

Construction 2.3.4 *Let $\boldsymbol{A} \xrightarrow{f} \boldsymbol{B}$ be an R-morphism of commutative group schemes and $a, b \in \boldsymbol{A}(U_2), \boldsymbol{B}(U_2)$ two Čech 2-cocycles such that $[f(a)] = [b]$ in $\check{H}^2(U_0 \to \mathit{Spec}(R), \boldsymbol{B})$. Then for any $x \in \boldsymbol{B}(U_1)$ satisfying $d(x)\cdot b = f(a)$, we may define a morphism of Sch$/R$-stacks $\mathcal{E}_a \xrightarrow{\phi_{a,b,x}} \mathcal{E}_b$.*

For any $V \in Ob(Sch/R)$, given a a-twisted torsor (T,ψ) over V, we define a b-twisted torsor (T',ψ') over V as follows. Define the $\boldsymbol{B}_{V\times_R U_0}$ torsor T' to be $T \times^{\boldsymbol{A}_{V\times U_0}, f} \boldsymbol{B}_{V\times U_0}$, and take the gluing map to be $\psi' := \overline{m_{x^{-1}} \circ \psi}$, where $\overline{m_{x^{-1}} \circ \psi}$ denotes the isomorphism of contracted products

$$p_2^*(T \times^{\boldsymbol{A}_{V\times U_0}, f} \boldsymbol{B}_{V\times U_0}) = (p_2^*T) \times^{\boldsymbol{A}_{V\times U_1}, f} \boldsymbol{B}_{V\times U_1} \to (p_1^*T) \times^{\boldsymbol{A}_{V\times U_1}, f} \boldsymbol{B}_{V\times U_1} = p_1^*(T \times^{\boldsymbol{A}_{V\times U_0}, f} \boldsymbol{B}_{V\times U_0})$$

induced by $(m_{x^{-1}} \circ \psi) \times id_{\boldsymbol{B}}$ (and we are implicitly identifying x with its image in $\boldsymbol{B}(V \times_R U_1)$).

We compute that

$$(p_{12}^*\psi') \circ (p_{23}^*\psi') \circ (p_{13}^*\psi')^{-1} = m_{p_{12}(x)^{-1} p_{23}(x)^{-1} p_{13}(x)\cdot f(a)} = m_b,$$

so that $\phi_{a,b,x}((T,\psi)) := (T',\psi')$ indeed defines an element of $\mathcal{E}_b(V)$. From here, one checks that any morphism $\varphi\colon (S,\psi_S) \to (T,\psi_T)$ of a-twisted torsors induces a morphism of the corresponding b-twisted torsors by means of the map on contracted products induced by $\varphi \times id$, giving the desired morphism of stacks.

Note that the above morphism does in general depend on the choice of x; indeed, any two such morphisms differ by post-composing by an automorphism of $\mathcal{E}_b$ determined by a Čech 1-cocycle z with respect to the cover $U_0 \to Spec(R)$.

Proposition 2.3.5 *Suppose that $\mathcal{E} \to Sch/R$ and $\mathcal{E}' \to Sch/R$ are two $\boldsymbol{A}$-gerbes split over U_0. Then $[\mathcal{E}] = [\mathcal{E}']$ in $\check{H}^2(U_0 \to Spec(R), \boldsymbol{A})$ if and only if $\mathcal{E}$ is $\boldsymbol{A}$-equivalent to $\mathcal{E}'$.*

Proof. We already know the "if" direction from Corollary 2.2.12. Let $\mathcal{E}$ be a $\mathbf{A}$-gerbe with $X \in \mathcal{E}(U_0)$ and $\varphi\colon p_1^*X \xrightarrow{\sim} p_2^*X$ in $\mathcal{E}(U_1)$. By Definition 2.2.8, it's enough to construct a (Sch/R)-morphism $\mathcal{E} \to \mathcal{E}'$ which is the identity on bands. If we show that $\mathcal{E}$ is $\mathbf{A}$-equivalent to $\mathcal{E}_a$ for $a \in \mathbf{A}(U_2)$ giving $d\varphi$, then the result will follow from applying Construction 2.3.4 to cohomologous cocycles (with $f = \mathrm{id}_{\mathbf{A}}$, in the notation of the construction).

At the level of objects, send $Y \in \mathcal{E}(V \xrightarrow{f} \mathrm{Spec}(R))$ to the sheaf $\underline{\mathrm{Isom}}_{\mathcal{E}(V\times U_0)}(\tilde{p}_2^*X, \tilde{p}_1^*Y)$, on $\mathrm{Sch}/(V \times_F U_0)$, where $\tilde{p}_i$ is the ith projection of $V \times U_0$. We claim that this sheaf is an a-twisted torsor over V. First, it is easy to see that the above sheaf is an $\mathbf{A}_{V\times U_0}$-torsor, by means of the action of the band of $\mathcal{E}$ on either side of the isomorphism (it doesn't matter which by Lemma 2.2.6). We need to define an isomorphism of $\mathbf{A}_{V\times U_1}$-torsors

$$\psi\colon p_2^*[\underline{\mathrm{Isom}}_{\mathcal{E}(V\times U_0)}(\tilde{p}_2^*X, \tilde{p}_1^*Y)] \xrightarrow{\sim} p_1^*[\underline{\mathrm{Isom}}_{\mathcal{E}(V\times U_0)}(\tilde{p}_2^*X, \tilde{p}_1^*Y)]$$

satisfying the twisted gluing condition with respect to a. We may take this to be the isomorphism obtained by pre-composing by $\tilde{p}_{2,U_1}^*\varphi$ (after making appropriate canonical identifications which we leave to the reader, where $\tilde{p}_{i,U_1}$ is the ith projection for $V \times U_1$). This defines our equivalence on the level of objects.

At the level of morphisms, for $Y \xrightarrow{\tilde{f}} Z$ lifting $V \xrightarrow{f} W$, we have an induced morphism $\tilde{p}_1^*Y \to \tilde{p}_1^*Z$, and post-composing by this map gives a morphism

$$(\underline{\mathrm{Isom}}_{\mathcal{E}(V\times U_0)}(\tilde{p}_2^*X, \tilde{p}_1^*Y), \psi_Y) \to (\underline{\mathrm{Isom}}_{\mathcal{E}(V\times U_0)}(\tilde{p}_2^*X, \tilde{p}_1^*Z), \psi_Z),$$

which is a morphism of a-twisted torsors. The induced morphism of bands is the identity by definition of the $\mathbf{A}$-action on each $\underline{\mathrm{Isom}}_{\mathcal{E}(V\times U_0)}(\tilde{p}_2^*X, \tilde{p}_1^*Y)$. □

For $\mathcal{E}$ an $\mathbf{A}$-gerbe as above, we call a choice of equivalence $\mathcal{E} \to \mathcal{E}_a$ for any a such that $[a] = [\mathcal{E}]$ a (choice of) *normalization* of $\mathcal{E}$.

Remark 2.3.6 *The above equivalence $\mathcal{E} \to \mathcal{E}_a$ sends X to the isomorphism class of the **trivial a-twisted torsor**, which we define to be the a-twisted torsor over U_0 given by $x(U_0)$, where x is the section constructed in Lemma 2.3.3.*

2.4 Torsors on gerbes

We continue with the notation of the previous subsections of this chapter.

Definition 2.4.1 *For a stack $\mathscr{X} \xrightarrow{\pi} Sch/R$, we give $\mathscr{X}$ the structure of a site $\mathscr{X}_{fpqc}$ via the **fpqc topology on** $\mathscr{X}$. First, recall that $\mathscr{X}$ has finite fibered products, by Lemma 2.1.4; to define this* *topology, for* $X \in Ob(\mathscr{X})$ *say that a collection of morphisms* $\{X_i \xrightarrow{f_i} X\}$ *in* $\mathscr{X}$ *is a cover if and only if* $\{\pi(X_i) \xrightarrow{\pi(f_i)} \pi(X)\}$ *is a cover in* Sch/R*. This endows* $\mathscr{X}$ *with the structure of a site such that* $\mathscr{X} \xrightarrow{\pi} Sch/R$ *is a morphism of sites. We will frequently abbreviate* $\mathscr{X}_{fpqc}$ *to just* $\mathscr{X}$*.*

We begin this subsection with an important result concerning torsors on gerbes which will be crucial for our later cohomological constructions. In what follows, we fix a finite type fpqc R-group scheme G. If $\mathcal{E} \xrightarrow{\pi} \mathrm{Sch}/R$ is a gerbe, we denote by $G_{\mathcal{E}}$ the corresponding group sheaf on $\mathcal{E}$ with the induced fpqc topology. For $\mathscr{F}$ is a sheaf (of sets) on an $\mathbf{A}$-gerbe $(\mathcal{E}, \theta)$, we have a morphism of sheaves on $\mathcal{E}$ denoted by

$$\iota \colon \mathbf{A}_{\mathcal{E}} \times_{\mathcal{E}} \mathscr{F} \to \mathscr{F},$$

called the *inertial action*, which for an object X of $\mathcal{E}(U)$ and $a \in \mathbf{A}_{\mathcal{E}}(X) = \mathbf{A}(U)$ is defined by the automorphism $\mathscr{F}(X) \xrightarrow{\theta_X(a)^\sharp} \mathscr{F}(X)$. This gives an action of the group sheaf $\mathbf{A}_{\mathcal{E}}$ on the sheaf $\mathscr{F}$, see [Shi19], 2.3.

Lemma 2.4.2 *Assume that* G *is abelian. If* $\mathscr{T}$ *is a* $G_{\mathcal{E}}$*-torsor on the* $\mathbf{A}$*-gerbe* $\mathcal{E} \xrightarrow{\pi} Sch/R$ *split over* U_0*, then there is a unique map* $\phi \in Hom_R(\mathbf{A}, G)$ *such that the inertial action* $\iota \colon \mathbf{A}_{\mathcal{E}} \times_{\mathcal{E}} \mathscr{T} \to \mathscr{T}$ *is induced by* $\phi_{\mathcal{E}} := \pi^*\phi \colon \mathbf{A}_{\mathcal{E}} \to G_{\mathcal{E}}$*. We denote this homomorphism by* $Res(\mathscr{T})$*.*

Proof. If such a map exists, uniqueness is clear. For $V \to \mathrm{Spec}(R)$, $X \in \mathcal{E}(V)$, and $x \in \mathbf{A}_{\mathcal{E}}(X) = \mathbf{A}(V)$, the induced automorphism of sheaves $\iota_x \colon \mathscr{T}\big|_{\mathcal{E}/X} \to \mathscr{T}\big|_{\mathcal{E}/X}$ is $G_{\mathcal{E}}\big|_{\mathcal{E}/X}$-equivariant, since the $G_{\mathcal{E}}$-action $\mathscr{T} \times_{\mathcal{E}} G_{\mathcal{E}} \to \mathscr{T}$ is a morphism of sheaves on $\mathcal{E}$ and ι_x is induced by an automorphism of X (by definition). It follows that ι_x must be given by right-translation by a unique element $g_x \in G_{\mathcal{E}}(X) = G(V)$, defining a map $\mathbf{A}_{\mathcal{E}}(X) = \mathbf{A}(V) \xrightarrow{\Phi_X} G(V) = G_{\mathcal{E}}(X)$. Moreover, if $X \xrightarrow{\nu} Y$ is a morphism in $\mathcal{E}$, lifting $V \xrightarrow{f} U$, then the square

$$\begin{array}{ccc} \mathbf{A}(U) & \xrightarrow{\Phi_X} & G(U) \\ {\scriptstyle f^\sharp}\uparrow & & \uparrow{\scriptstyle f^\sharp} \\ \mathbf{A}(V) & \xrightarrow{\Phi_Y} & G(V) \end{array}$$

commutes because of the commutativity of the squares

$$\begin{array}{ccc} \mathbf{A}_{\mathcal{E}}\big|_{\mathcal{E}/X} \times \mathscr{T}\big|_{\mathcal{E}/X} & \xrightarrow{\iota_X} & \mathscr{T}\big|_{\mathcal{E}/X} \\ {\scriptstyle \nu^\sharp}\uparrow & & \uparrow{\scriptstyle \nu^\sharp} \\ \mathbf{A}_{\mathcal{E}}\big|_{\mathcal{E}/Y} \times \mathscr{T}\big|_{\mathcal{E}/Y} & \xrightarrow{\iota_Y} & \mathscr{T}\big|_{\mathcal{E}/Y} \end{array} \qquad \begin{array}{ccc} \mathscr{T}\big|_{\mathcal{E}/X} \times G_{\mathcal{E}}\big|_{\mathcal{E}/X} & \longrightarrow & \mathscr{T}\big|_{\mathcal{E}/X} \\ {\scriptstyle \nu^\sharp}\uparrow & & \uparrow{\scriptstyle \nu^\sharp} \\ \mathscr{T}\big|_{\mathcal{E}/Y} \times G_{\mathcal{E}}\big|_{\mathcal{E}/Y} & \longrightarrow & \mathscr{T}\big|_{\mathcal{E}/Y}. \end{array}$$

It follows that the above maps glue across all objects in $\mathcal{E}$ to define a homomorphism of group sheaves $\Phi_{\mathcal{E}}\colon \mathbf{A}_{\mathcal{E}} \to G_{\mathcal{E}}$. This defines a homomorphism $\bar{\Phi}\colon \mathbf{A}_{U_0} \to G_{U_0}$ via pulling back by a section $s\colon \mathrm{Sch}/U_0 \to \mathcal{E}$, and the above argument shows that the two pullbacks to $\mathbf{A}_{U_1}$ coincide by setting $\nu = \varphi\colon p_1^*X \xrightarrow{\sim} p_2^*X$ any isomorphism in $\mathcal{E}(U_1)$ for $X := s(U_0)$, showing that $\bar{\Phi}$ descends to an F-homomorphism Φ, whose pullback by π is $\Phi_{\mathcal{E}}$. □

We first record a characterization of sheaves on the site $B_R\mathbf{A} \to \mathrm{Sch}/R$ (with the induced fpqc topology). Consider the category of sheaves on $B_R\mathbf{A}$, as well as the category of sheaves on Sch/R equipped with an $\mathbf{A}$-action, where we require morphisms in this latter category to be $\mathbf{A}$-equivariant. There is a canonical section $s\colon \mathrm{Sch}/R \to B_R\mathbf{A}$ sending $U \to \mathrm{Spec}(R)$ to the trivial $\mathbf{A}_U$-torsor $\mathbf{A}_U$. Define the map between the above two categories to be the one which sends a sheaf $\mathscr{F}$ on $B_R\mathbf{A}$ to the sheaf $s^*\mathscr{F}$ on Sch/R with $\mathbf{A}$-action given by

$$\mathbf{A} \times_R s^*\mathscr{F} \xrightarrow{s^*\iota} s^*\mathscr{F},$$

and sends the morphism of sheaves $\mathscr{F} \xrightarrow{f} \mathscr{F}'$ to s^*f, where in the definition of the action we are making the identification $s^*(\mathbf{A}_{\mathcal{E}}) = \mathbf{A}$.

Proposition 2.4.3 *The above map defines an equivalence of categories.*

Proof. See [Shi19], Remark 2.6. □

Definition 2.4.4 *For our fixed G and $\mathcal{E} \to \mathit{Sch}/R$ a gerbe, define the fibered category $\textbf{\textit{Tors}}(G, \mathcal{E})$ over $(\mathit{Sch}/R)_{\mathit{fpqc}}$, where the fiber over $U \in \mathit{Ob}(\mathit{Sch}/R)$ is the category of $G_{\mathcal{E}_U}$-torsors on $\mathcal{E}_U$, with a morphism from $\mathscr{T}$ to $\mathscr{S}$ lying above $f : V \to U$ given by a morphism of $G_{\mathcal{E}_V}$-torsors $\mathscr{T} \to f^*\mathscr{S}$. Here $f^*\mathscr{S}$ denotes the pullback of the $G_{\mathcal{E}_U}$-torsor $\mathscr{S}$ to $\mathcal{E}_V$ via the morphism $\mathcal{E}_V := \mathcal{E} \times_{\mathit{Sch}/R} (\mathit{Sch}/V) \to \mathcal{E} \times_{\mathit{Sch}/R} (\mathit{Sch}/U) =: \mathcal{E}_U$ induced by the functor $\mathit{Sch}/V \to \mathit{Sch}/U$ sending $W \to V$ to $W \to V \xrightarrow{f} U$.*

Proposition 2.4.5 *The fibered category $\textbf{\textit{Tors}}(G, \mathcal{E}) \to (\mathit{Sch}/R)_{\mathit{fpqc}}$ is a stack.*

Proof. Our above construction is clearly a fibered category, and the remaining conditions, namely that the isomorphism functor associated to the fiber over $U \in \mathrm{Ob}(\mathrm{Sch}/R)$ is a sheaf and that all descent data from $(\mathrm{Sch}/R)_{\mathrm{fpqc}}$ are effective, follow from (respectively) gluing of morphisms of torsors and gluing of torsors on stacks over $(\mathrm{Sch}/R)_{\mathrm{fpqc}}$ with the induced fpqc topology, which follow easily from the discussion in [Stacks], §I.7.26 (with our stack being $\mathcal{E} \to (\mathrm{Sch}/R)_{\mathrm{fpqc}}$). □

We now introduce the category of a-twisted G-torsors on the site $(\mathrm{Sch}/R)_{\mathrm{fpqc}}$, corresponding to a Čech 2-cocycle $a \in \mathbf{A}(U_2)$, whose purpose is to give a concrete interpretation of the above stack in the case where $\mathcal{E} = \mathcal{E}_a$. This definition is a generalization of Definition 1.2.1 in [Căl00].

Definition 2.4.6 *An a-**twisted** G-**torsor over** R is a quadruple (T, ψ, m, n) consisting of a G_{U_0}-torsor $m : T \times G_{U_0} \to T$ over U_0, an $\mathbf{A}_{U_0}$-action $n : \mathbf{A}_{U_0} \times_{U_0} T \to T$ which commutes with m, and an $\mathbf{A}$-equivariant isomorphism of G_{U_1}-torsors $\psi : p_2^* T \to p_1^* T$ satisfying the **twisted cocycle condition***

$$(p_{12}^* \psi) \circ (p_{23}^* \psi) \circ (p_{13}^* \psi)^{-1} = n_a$$

*on $q_1^* T$. We occasionally abbreviate the quadruple (T, ψ, m, n) by (T, ψ) (in such cases there will be no ambiguity regarding the associated actions). A morphism $h : (T, \psi_T, m_T, n_T) \to (S, \psi_S, m_S, n_S)$ of a-twisted G-torsors over R is an $\mathbf{A}$-equivariant morphism of G_{U_0}-torsors over U_0, $h : T \to S$, satisfying $\psi_S \circ p_2^* h = p_1^* h \circ \psi_T$. We get an associated fibered category over Sch$/R$, denoted by **Tors**$_a(G, \mathbf{A}, R)$, by letting the fiber over V be all a_V**-twisted-torsors over** V, where a_V is the image of a in $\mathbf{A}(V \times U_2)$, defined identically as above after replacing U_0, U_1 by $V \times U_0$ and $V \times U_1 = (V \times U_0) \times_V (V \times U_0)$.*

The following lemma provides a different way to interpret some aspects of the above definition.

Lemma 2.4.7 *Assume that G is abelian. For a G_{U_0}-torsor T, having a G_{U_0}-equivariant $\mathbf{A}_{U_0}$-action on T is equivalent to requiring that the $\mathbf{A}_{U_0}$-action be induced by a group homomorphism $\mathbf{A}_{U_0} \to G_{U_0}$, and insisting further that there is a twisted gluing map giving T (along with the two given actions) the structure of an a-twisted G-torsor implies that this homomorphism is defined over R.*

Proof. For $V \to U_0$, if we fix $x \in \mathbf{A}(V)$, then $n_x \colon T_V \xrightarrow{\sim} T_V$ is an automorphism of G_V-torsors, and is thus right-translation m_{g_x} by some unique $g_x \in G(V)$, and the assignment $a \mapsto g_x$ is functorial in V by uniqueness of g_x, and hence we get a group homomorphism $\mathbf{A}_{U_0} \xrightarrow{f} G_{U_0}$ giving the $\mathbf{A}$-action.

This homomorphism f descends to a morphism $\mathbf{A} \to G$ because $p_1^* f$ is induced by the $\mathbf{A}_{U_1}$-action on $p_1^* T$ and $p_2^* f$ by the $\mathbf{A}_{U_1}$-action on $p_2^* T$, and we have an $\mathbf{A}_{U_1}$-equivariant morphism of G_{U_1}-torsors $\psi \colon p_2^* T \xrightarrow{\sim} p_1^* T$, which means that if $x \in \mathbf{A}(U_1)$ induces the automorphism m_{g_x} on $p_2^* T$, then since the diagram

$$\begin{array}{ccc} p_2^* T & \xrightarrow{\psi} & p_1^* T \\ \downarrow{\scriptstyle (p_2^* n)_x} & & \downarrow{\scriptstyle (p_1^* n)_x} \\ p_2^* T & \xrightarrow{\psi} & p_1^* T \end{array}$$

commutes and ψ is G_{U_1}-equivariant, the right-hand translation $(p_1^* n)_x$ equals $\psi \circ (p_2^* n)_x \circ \psi^{-1} = \psi \circ m_{g_x} \circ \psi^{-1} = m_{g_x}$, giving the result (by fpqc descent of morphisms). $\square$

Proposition 2.4.8 *The fibered category **Tors**$_a(G, \mathbf{A}, R) \to (Sch/R)_{fpqc}$ is a stack.*

Proof. The isomorphism functor on $V \in \mathrm{Ob}(\mathrm{Sch}/R)$ associated to the fiber category over V is evidently a sheaf, by gluing of morphism of sheaves (again, see [Stacks], §I.7.26), and if the equivariance conditions hold on an fpqc cover, they hold on V. Thus, all that remains to check is effectivity of descent data. This follows because of gluing of G-torsors on $(\mathrm{Sch}/U_0)_{\text{fpqc}}$ with the fpqc topology, and the $\mathbf{A}$-action on compatible torsors defined on any cover $\{V_i \to V\}$ extends to a $\mathbf{A}$-action of the glued torsor on V by gluing of morphisms (using $\mathbf{A}$-equivariance of morphisms in $\mathbf{Tors}_a(G, \mathbf{A}, R)$). Again, the commutation relations can be checked locally. □

The next fundamental result shows that the above two notions of torsors actually coincide. We begin with a lemma that addresses the case when $\mathcal{E} = B_R\mathbf{A}$.

Lemma 2.4.9 *There is an equivalence of categories* $\eta : \boldsymbol{Tors}(G, B_R\boldsymbol{A}) \to \boldsymbol{Tors}_{e_{\mathbf{A}}}(G, \boldsymbol{A}, R)$.

Proof. If we start with the data of an object (T, ψ) in $\mathbf{Tors}_{e_{\mathbf{A}}}(G, \mathbf{A}, R)$, the map ψ furnishes T with a descent datum (of torsors, not just sheaves) with respect to the fpqc cover $U_0 \to \mathrm{Spec}(R)$. By gluing of fpqc sheaves (see [Stacks], §I.7.26) such an object then gives a G-torsor over R with G-equivariant $\mathbf{A}$-action. By Proposition 2.4.3, this defines a sheaf $\mathscr{T}$ on $B_R\mathbf{A}$, so all we need to do is define the $G_{\mathcal{E}}$-action, $\tilde{m} : \mathscr{T} \times G_{\mathcal{E}} \to \mathscr{T}$.

Denote by $s\colon \mathrm{Sch}/R \to B_R\mathbf{A}$ the canonical section. Denote by $\mathcal{C}$ the (categorical) image of this embedding of categories. We may define a morphism of sheaves

$$\mathscr{T}\big|_{\mathcal{C}} \times G_{\mathcal{E}}\big|_{\mathcal{C}} \to \mathscr{T}\big|_{\mathcal{C}} \tag{2.1}$$

by applying π^* to the action $T \times G \to T$.

For an arbitrary $\mathbf{A}$-torsor over V, say X, we may find an fpqc cover $\{V_i \xrightarrow{f_i} V\}$ such that we have isomorphisms of $\mathbf{A}_{V_i}$-torsors $h_{X_i}\colon X_i := f_i^* X \xrightarrow{\sim} \mathbf{A}_{V_i}$, and, if $s_{X_i}\colon \mathrm{Sch}/V_i \to \mathcal{E}$ denotes the embedding of categories induced by X_i, we can define an action

$$\mathscr{T}\big|_{s_{X_i}(\mathrm{Sch}/V_i)} \times G_{\mathcal{E}}\big|_{s_{X_i}(\mathrm{Sch}/V_i)} \to \mathscr{T}\big|_{s_{X_i}(\mathrm{Sch}/V_i)} \tag{2.2}$$

by conjugating our above action (2.1) by $h_{X_i}\colon s_{X_i}(\mathrm{Sch}/V_i) \xrightarrow{\sim} \mathcal{C}/\mathbf{A}_{V_i}$ (where we identify h_{X_i} with the induced equivalence between the embedded categories). To check that this glues to give a morphism of sheaves $\mathscr{T} \times G_{\mathcal{E}} \to \mathscr{T}$, it's enough to check that the action defined in (2.2) is independent of the choice of h_{X_i}, which is equivalent to showing that the action in (2.1) is equivariant under the inertial action. This follows because the G-action on T is $\mathbf{A}$-equivariant.

We have thus constructed a map $\mathbf{Tors}_{e_{\mathbf{A}}}(G, \mathbf{A}, R) \to \mathbf{Tors}(G, B_R\mathbf{A})$ which is the inverse of the map $\mathbf{Tors}(G, B_R\mathbf{A}) \to \mathbf{Tors}_{e_{\mathbf{A}}}(G, \mathbf{A}, R)$ obtained by pulling back by the section s. □

Proposition 2.4.10 *For $\mathcal{E} = \mathcal{E}_a$, there is a canonical equivalence of categories $\eta : \textbf{\textit{Tors}}(G, \mathcal{E}) \to \textbf{\textit{Tors}}_a(G, \textbf{\textit{A}}, R)$.*

Proof. The argument largely follows that in [Lie04], §2.1.3 (where we replace the action via a character χ by the inertial action). Let $x : \mathrm{Sch}/U_0 \to \mathcal{E}$ be the section constructed in Lemma 2.3.3; let X be the corresponding lift of U_0. This same lemma also tells us that the two pullbacks of x to U_1, the maps x_1 and x_2, are isomorphic via φ; this means that for every $V \xrightarrow{f} U_1$, we have an isomorphism $\varphi_V : (p_1 \circ f)^* X \to (p_2 \circ f)^* X$ in $\mathcal{E}(V)$.

Let $\mathscr{T} \in \mathbf{Tors}(G, \mathcal{E})(\mathrm{Spec}(R))$ (the argument is identical for a $G_{\mathcal{E}_U}$-torsor). Then define the G-torsor over U_0 to be $T := x^* \mathscr{T}$ (sending $V \xrightarrow{f} U_0$ to $\mathscr{T}(f^* X)$). We know that $\mathbf{A}_{\mathcal{E}}$ acts on $\mathscr{T}$ via the inertial action, denoted by $\iota : \mathbf{A}_{\mathcal{E}} \times \mathscr{T} \to \mathscr{T}$. As such, we get an $\mathbf{A}$-action on T via taking $x^*\iota$ (using that $x^* \mathbf{A}_{\mathcal{E}} = \mathbf{A}$). Similarly, we can set ψ to be the U_1-sheaf isomorphism $p_2^* x^* \mathscr{T} \to p_1^* x^* \mathscr{T}$ induced by the natural transformation $\varphi : x \circ p_1 \xrightarrow{\sim} x \circ p_2$. One sees that ψ satisfies the twisted cocycle condition, since the map from $(q_1^* x)(\mathrm{Sch}/U_2)$ to itself given by the natural transformation of $q_1^* x$:

$$d\varphi = (p_{13}^* \varphi)^{-1} \circ (p_{23}^* \varphi) \circ (p_{12}^* \varphi)$$

equals ι_a, so that the induced map $q_1^* T \to q_1^* T$ is exactly translation by a. Note that ψ is $\mathbf{A}$-equivariant for our $\mathbf{A}$-action, since for $z \in \mathbf{A}(U_0)$, we can identify z with $\theta_{p_1^* X}(z), \theta_{p_2^* X}(z) \in \mathrm{Aut}_{U_1}(p_1^* X), \mathrm{Aut}_{U_1}(p_2^* X)$, and then $\varphi_{U_0} \circ \theta_{p_2^* X}(z) = \theta_{p_1^* X}(z) \circ \varphi_{U_0}$, as $\theta_{p_1^* X}(z) = \varphi_{U_0} \circ \theta_{p_2^* X}(z) \circ \varphi_{U_0}^{-1}$ (by Lemma 2.2.6).

We take $m : T \times G_{U_0} \to T$ to be the pullback of the $G_{\mathcal{E}}$-action $\tilde{m}$ on $\mathscr{T}$ by x. Fixing $V \xrightarrow{f} U$, since $\tilde{m} : \mathscr{T} \times G_{\mathcal{E}} \to \mathscr{T}$ is a morphism of sheaves on $\mathcal{E}$, it commutes with the restriction maps $\varphi_V^\sharp$, giving the G-equivariance of ψ. One checks via an identical argument that m commutes with the $\mathbf{A}_{U_0}$-action (since it acts via the band of $\mathcal{E}$), and that if $\mathscr{T} \to \mathscr{S}$ is a morphism in $\mathbf{Tors}(G, \mathcal{E})(U_0)$, the induced maps $\mathscr{T}(f^* X) \to \mathscr{S}(f^* X)$ give a morphism in $\mathbf{Tors}_a(G, \mathbf{A}, R)(U_0)$ We thus obtain our functor η (after applying the above construction with U_0 replaced by an arbitrary $V \to U_0$, which proceeds identically as above).

Since both $\mathbf{Tors}(G, \mathcal{E})$ and $\mathbf{Tors}_a(G, \mathbf{A}, R)$ are stacks over $(\mathrm{Sch}/R)_{\mathrm{fpqc}}$, it's enough to check that η is locally an equivalence, by Proposition 2.1.9 (where we are using that we are working with the fpqc sites). By base-changing to U_0, we may assume that a is a 1-coboundary; one checks easily (using an argument similar to the one used in Construction 2.3.4) that if a is cohomologous to b, then $\mathbf{Tors}_a(G, \mathbf{A}, R)$ and $\mathbf{Tors}_b(G, \mathbf{A}, R)$ are equivalent, and we know from Construction 2.3.4 that $\mathcal{E}_b$ and $\mathcal{E}_a$ are isomorphic (not just equivalent). Hence, we may assume that $a = e_{\mathbf{A}}$, and $\mathcal{E} = B_R \mathbf{A}$, and now we may apply Lemma 2.4.9. □

The following two results follow immediately from the above proof, pulling back functors between the categories $\mathbf{Tors}(G, \mathcal{E})$ (with varying G and/or $\mathcal{E}$) by the section x:

Corollary 2.4.11 *Let $G \xrightarrow{f} H$ be a morphism of R-group schemes, giving the usual functor*

$$\textit{\textbf{Tors}}(G, \mathcal{E}_a) \to \textit{\textbf{Tors}}(H, \mathcal{E}_a),$$

which sends $\mathscr{T}$ to $\mathscr{T} \times^{G_{\mathcal{E}}, f_{\mathcal{E}}} H_{\mathcal{E}}$. Then this corresponds via the equivalence η to the functor

$$\textit{\textbf{Tors}}_a(G, \boldsymbol{A}, R) \to \textit{\textbf{Tors}}_a(H, \boldsymbol{A}, R)$$

sending (T, ψ, m, n) to the H_{U_0}-torsor $T \times^{G,f} H$, with $\boldsymbol{A}$-action induced by $n \times id$; when G is abelian, this is the same as replacing the homomorphism $\boldsymbol{A} \to G$ giving n with its post-composition by f. The new gluing map $\tilde{\psi}$ is obtained by applying $- \times^{G,f} H$ and taking the morphism induced by $\psi \times id$.

Corollary 2.4.12 *Let $\phi_{a,b,x} \colon \mathcal{E}_a \to \mathcal{E}_b$ be the morphism of stacks over R defined in Construction 2.3.4 between the $\boldsymbol{A}$-gerbe $\mathcal{E}_a$ corresponding to the Čech 2-cocycle $a \in \boldsymbol{A}(U_2)$, the $\boldsymbol{B}$-gerbe $\mathcal{E}_b$, corresponding to the Čech 2-cocycle $b \in \boldsymbol{B}(U_2)$, induced by a homomorphism $\boldsymbol{A} \xrightarrow{h} \boldsymbol{B}$ such that $[h(a)] = [b] \in \check{H}^2(U_0 \to \mathit{Spec}(R), \boldsymbol{B})$ and $x \in \boldsymbol{B}(U_1)$ such that $d(x) \cdot b = h(a)$. Then the functor*

$$\textit{\textbf{Tors}}(G, \mathcal{E}_b) \to \textit{\textbf{Tors}}(G, \mathcal{E}_a)$$

induced by pullback by $\phi_{a,b,x}$ corresponds via η to the functor

$$\textit{\textbf{Tors}}_b(G, \boldsymbol{B}, R) \to \textit{\textbf{Tors}}_a(G, \boldsymbol{A}, R)$$

sending the object (T, ψ, m, n) to the a-twisted G-torsor with underlying G_{U_0}-torsor T, $\boldsymbol{A}$-action given by mapping to $\boldsymbol{B}$ by h, and gluing map $\tilde{\psi}$ given by translating ψ by x.

2.5 Inverse limits of gerbes

In this section we present a few elementary results concerning inverse limits of gerbes. We keep all of the previous notation and conventions of this chapter. The new assumptions of this subsection are as follows: We have a system $\{u_n\}_{n \in \mathbb{N}}$ of fpqc commutative groups over R with transition maps $p_{n+1,n} \colon u_{n+1} \to u_n$ (defined over R) which are epimorphisms. We also assume that we have systems of elements $\{u_n \in u_n(U_2)\}$ and $\{x_n \in u_n(U_1)\}$ such that u_n are Čech 2-cocycles and $u_n \cdot dx_n = p_{n+1,n}(a_{n+1})$. This gives rise to a system of gerbes $\{\mathcal{E}_n := \mathcal{E}_{u_n} \to \mathrm{Sch}/R\}_{n \in \mathbb{N}}$ (abbreviated as just $\{\mathcal{E}_n\}$) with morphisms of Sch/R-categories $\pi_{n+1,n} \colon \mathcal{E}_{n+1} \to \mathcal{E}_n$, where $\pi_{n+1,n} := \phi_{a_{n+1}, u_n, x_n}$, see Construction 2.3.4. We will make the additional assumption that the projection maps $u_{n+1}(U_m) \to u_n(U_m)$ are surjective for all n, m (this will be the case with every fpqc cover and system of groups considered in this paper).

Definition 2.5.1 *Define the **inverse limit** of the system $\{\mathcal{E}_n\}$, denoted by $\varprojlim_n \mathcal{E}_n \to Sch/R$, to be the category with fiber over $U \in Ob(Sch/R)$ given by the systems $(X_n)_{n\in\mathbb{N}}$ with $X_n \in \mathcal{E}_n(U)$ such that $\pi_{n+1,n}(X_{n+1}) = X_n$ for all n, and morphisms $(X_n) \to (Y_n)$ given by a system of morphisms $\{f_n : X_n \to Y_n\}$ such that $\pi_{n+1,n} f_{n+1} = f_n$ for all n. We call such a system of morphisms **coherent**. It is clear that we have a compatible system of canonical morphisms of Sch/F-categories $\pi_m : \varprojlim_n \mathcal{E}_n \to \mathcal{E}_m$ for all m.*

It will turn out that the category $\mathcal{E} := \varprojlim_n \mathcal{E}_n \to (\text{Sch}/R)_{\text{fpqc}}$ is canonically a $u := \varprojlim_n u_n$-gerbe, split over U_0. Denote the projection map $u \to u_n$ by p_n. Note that we have maps $\check{H}^i(U_0 \to \text{Spec}(R), u_{n+1}) \to \check{H}^i(U_0 \to \text{Spec}(R), u_n)$ induced by $p_{n+1,n}$, and thus also a map

$$\check{H}^i(U_0 \to \text{Spec}(R), u) \to \varprojlim_n \check{H}^i(U_0 \to \text{Spec}(R), u_n) \tag{2.3}$$

for all $i \geq 0$. Recall from Proposition 2.3.5 that the fpqc u-gerbe $\mathcal{E}$ corresponds to a class in $\check{H}^2(U_0 \to \text{Spec}(R), u)$. We give one preliminary result to show that our Convention 2.2.7 applies for the group u if it applies for each u_i:

Lemma 2.5.2 *Using the notation as above, $\check{H}^1_{fpqc}(U_n, u_{U_n}) = 0$ for all $n \geq 0$.*

Proof. Let V denote U_n for $n \geq 1$, and let P be an fpqc u_V-torsor over V (cf. Remark 2.2.2). Then for all n we obtain a $u_{n,V}$-torsor by taking $P_n := P \times^{u_V, p_n} u_{n,V}$. Moreover, by Convention 2.2.7, we have an isomorphism of $u_{1,V}$-torsors $P_1 \xrightarrow{h_1} u_{1,V}$. Similarly, we have a trivialization $h_2 : P_2 \xrightarrow{\sim} u_{2,V}$, and the induced isomorphism $P_1 = P_2 \times^{u_{2,V}, p_{2,1}} u_{1,V} \to u_{1,V}$ differs from h_1 by post-composing by an automorphism of the trivial $u_{1,V}$-torsor $u_{1,V}$ which must be translation by some $y_1 \in u_1(V)$, which we may lift to $\tilde{y}_1 \in u_2(V)$. We may then replace h_2 by its post-composition with translation by $\tilde{y}_1$ to assume that, via $p_{2,1}$, it induces h_1. Proceeding inductively in this manner, we obtain trivializations $h_n : P_n \xrightarrow{\sim} u_{n,1}$ such that h_{n-1} is induced by h_n via $p_{n,n-1}$, as above. This allows us to define a morphism of u-torsors $h : P \to u$ by applying $\varprojlim_n$ to the (u-equivariant) composition $P \xrightarrow{\text{id} \times e_{u_{n,V}}} P_n \xrightarrow{h_n} u_{n,V}$ (where $e_{u_{n,V}} : V \to u_{n,V}$ is the identity section), which is automatically an isomorphism. □

The main result of this subsection is:

Proposition 2.5.3 *With the setup as above, the category $\mathcal{E} := \varprojlim_n \mathcal{E}_n \to Sch/R$ can be given the structure of a u-gerbe, split over U_0. Moreover, the map (2.3) sends the class in $\check{H}^2(U_0 \to Spec(R), u)$ corresponding to $\mathcal{E}$ to the element $([u_n]) \in \varprojlim_n \check{H}^2(U_0 \to Spec(R), u_n)$.*

Proof. We will construct an object $X \in \mathcal{E}(U_0)$ and an isomorphism $\tilde{\varphi}\colon p_1^*X \xrightarrow{\sim} p_2^*X$.

We do this inductively; for $n = 1$, construct the a-twisted u_{1,U_0}-torsor (T_1, ψ_1) by setting, as in the proof of Lemma 2.3.3, $T_1 = u_{1,U_1}$, ψ_1 given by translation by a_1, and $\varphi_1\colon p_1^*(T_1, \psi_1) \to p_2^*(T_1, \psi_1)$ given by translation by a_1. Repeat this construction for u_2; one then checks after lifting x_1 to $\tilde{x}_1 \in u_2(U_1)$ (viewed as a 0-cochain with respect to the fpqc cover $U_1 \xrightarrow{p_1} U_0$), we may translate ψ_2 by $d\tilde{x}_1^{-1} = p_{12}(\tilde{x}_1)^{-1}p_{13}(\tilde{x}_1) \in u_2(U_2)$ to get a new a_2-twisted gluing map $\tilde{\psi}_2$ such that $\pi_{2,1}(T_2, \tilde{\psi}_2) = (T_1, \psi_1)$; also replace φ_2 with $\tilde{\varphi}_2$ defined by replacing a_2 by $p_{12}(\tilde{x}_1)^{-1}p_{13}(\tilde{x}_1)p_{23}(\tilde{x}_1)^{-1} \cdot a_2$, so that $\tilde{\varphi}_2\colon p_1^*(T_2, \tilde{\psi}_2) \to p_2^*(T_2, \tilde{\psi}_2)$ satisfies $\pi_{2,1}\tilde{\varphi}_2 = \varphi_1$.

Now for $n = 3$, we again start with (T_3, ψ_3) and φ_3 as above. We may then pick lifts $\tilde{x_1}^{(3)}, \tilde{x_2}^{(3)}$ of $\tilde{x}_1, x_2$ (respectively) in $u_3(U_1)$ and replace ψ_3 by $\tilde{\psi}_3 :=$ translation by $a_3 \cdot (d\tilde{x}_1^{(3)})^{-1} \cdot (d\tilde{x}_2^{(3)})^{-1}$ (the differential applied to the elements viewed as 0-cochains with respect to the fpqc cover $U_1 \xrightarrow{p_1} U_0$, as in the previous paragraph), and φ_3 by translation by

$$p_{12}(\tilde{x}_1^{(3)})^{-1}p_{13}(\tilde{x}_1^{(3)})p_{23}(\tilde{x}_1^{(3)})^{-1}p_{12}(\tilde{x}_2^{(3)})^{-1}p_{13}(\tilde{x}_2^{(3)})p_{23}(\tilde{x}_2^{(3)})^{-1}a_3,$$

which we call $\tilde{\varphi}_3$. Proceeding inductively in this manner, we get a system $X := ((T_n, \tilde{\psi}_n))_n$ of coherent lifts of U_0 and a coherent system of isomorphisms $(\tilde{\varphi}_n)$, which by definition lift to give an isomorphism $\tilde{\varphi}\colon p_1^*X \to p_2^*X$. This shows that $\mathcal{E}$ is a gerbe, split over U_0.

The band of $\mathcal{E}$ is canonically isomorphic to u, since for $U \to \text{Spec}(R)$, any automorphism of the coherent system $(X_n)_n$, $X_n \in \mathcal{E}_n(U)$, is given by a compatible system of automorphisms $X_n \xrightarrow{\sim} X_n$; since for each n we have a canonical identification of the band of $\mathcal{E}_n$ with u_n, and the compatibility hypothesis exactly says that we have a coherent system of elements with respect to the projective system $\{u_n(U)\}$ for any such system of automorphisms. This finishes the proof of the first claim.

For the second claim, we may use the lift X and isomorphism $\tilde{\varphi}\colon p_1^*X \to p_2^*X$ constructed above to compute the class $[\mathcal{E}] \in \check{H}^2(U_0 \to \text{Spec}(R), u)$ (see Fact 2.2.11). It is clear from our above construction that, via the natural projection map $u = \text{Band}(\mathcal{E}) \to \text{Band}(\mathcal{E}_n) = u_n$, the differential of $\tilde{\varphi}$ maps to the differential of $\tilde{\varphi}_n$, which one checks gives translation by an element that is cohomologous to u_n, as desired. □

We conclude this subsection with a couple of results concerning inverse limits of Čech classes. To ease notation for Čech cohomology, assume $U_0 = \text{Spec}(S)$ for a ring S.

Lemma 2.5.4 *The natural map $\check{H}^i(S/R, u) \to \varprojlim_n \check{H}^i(S/R, u_n)$ is surjective for all i.*

Proof. For $i = 0$ this is trivial, so assume $i \geq 1$. Let $\{x_n\}$ be a sequence of Čech i-cochains representing the classes in $\check{H}^i(S/R, u_n)$. By assumption, there is some $a_{2,1} \in u_1(S^{\bigotimes_R i})$ such that $d(a_{2,1})p_{2,1}(x_2) = x_1$. We may lift $a_{2,1}$ to $\tilde{a}_{2,1} \in u_2(S^{\bigotimes_R i})$ by assumption, and then replacing x_2 by

$d(\tilde{a}_{2,1})x_2$ gives a cohomologous element in $u_2(S^{\bigotimes_R (i+1)})$ whose projection to u_1 is x_1. Continuing this procedure inductively gives a i-cocycle in $u(S^{\bigotimes_R (i+1)})$ whose image in each $\check{H}^i(S/R, u_n)$ is $[x_n]$. □

The following result characterizes when the above surjections are isomorphisms.

Proposition 2.5.5 *Fix $i \geq 1$; if we have $\varprojlim_n^{(1)} \check{H}^{i-1}(S/R, u_n) = 0$ and $\varprojlim_n^{(1)} B^{i-1}(n) = 0$, where $B^{i-1}(n) \in C^{i-1}(S/R, u_n)$ is the subgroup of $(i-2)$-coboundaries (the group of (-1)-coboundaries is defined to be trivial), then the natural map $\check{H}^i(S/R, u) \to \varprojlim_n \check{H}^i(S/R, u_n)$ is injective.*

Proof. We denote the differential $u_k(S^{\bigotimes_R i}) \to u_k(S^{\bigotimes_R (i+1)})$ (which is a group homomorphism) by $d^{(k)}$. First, note that since $\varprojlim_k^{(1)} \check{H}^{i-1}(S/R, u_k) = 0$, the natural map

$$\varprojlim_k [u_k(S^{\bigotimes_R i})/B^{i-1}(k)] \to \varprojlim_k [(u_k(S^{\bigotimes_R i})/B^{i-1}(k))/(\check{H}^{i-1}(S/R, u_k))]$$

is surjective. Moreover, the natural map $u(S^{\bigotimes_R i}) = \varprojlim_k u_k(S^{\bigotimes_R i}) \to \varprojlim_k [u_k(S^{\bigotimes_R i})/B^{i-1}(k)]$ is surjective, since we assume that $\varprojlim_n^{(1)} B^{i-1}(n) = 0$.

Now by left-exactness of the inverse-limit functor, we have the exact sequence

$$1 \longrightarrow \varprojlim_k \frac{u_k(S^{\bigotimes_R i})/B^{i-1}(k)}{\check{H}^{i-1}(S/R, u_k)} \xrightarrow{\varprojlim d^{(k)}} u(S^{\bigotimes_R (i+1)}) \longrightarrow \varprojlim_k \frac{u_k(S^{\bigotimes_R (i+1)})}{d^{(k)}(u_k(S^{\bigotimes_R i}))}.$$

In particular, if $x \in u(S^{\bigotimes_R (i+1)})$ is such that its image in $\varprojlim_k \frac{u_k(S^{\bigotimes_R (i+1)})}{d^{(k)}(u_k(S^{\bigotimes_R i}))}$ is zero (which is the hypothesis of the Proposition), then it lies in the image of $\bar{d} := \varprojlim d^{(k)}$. But now the diagram

$$\begin{array}{ccc}
u(S^{\bigotimes_R i}) & \xrightarrow{d} & \\
\downarrow & & \\
\varprojlim_k [u_k(S^{\bigotimes_R i})/B^{i-1}(k)] & \longrightarrow & u(S^{\bigotimes_R (i+1)}) \\
\downarrow & \nearrow_{\bar{d}} & \\
\varprojlim_k \frac{u_k(S^{\bigotimes_R i})/B^{i-1}(k)}{\check{H}^{i-1}(S/R, u_k)} & &
\end{array}$$

commutes, and since the vertical composition is surjective and such an x lies in the image of the lower-diagonal map, it lies in the image of the upper-diagonal map, giving the desired result. □

2.6 Twisted cocycles

In this section, we introduce the notion of twisted cocycles, which facilitate computations involving torsors on gerbes. We continue with the notation and conventions of the previous subsections, but now we specify that $R = F$ a field and set $U_0 = \mathrm{Spec}(\overline{F})$. Fix a Čech 2-cocycle $a \in \mathbf{A}(U_2)$. For a gerbe $\mathcal{E}$ over F and finite type group scheme G over F, denote the pointed set of isomorphism classes of $G_{\mathcal{E}}$-torsors on $\mathcal{E}$ by $H^1(\mathcal{E}, G)$.

Let $\mathcal{A}$ be the category of monomorphisms $Z \to G$ defined over F, where G is either a commutative algebraic group of finite type over F or a connected reductive group defined over F, and Z is a finite multiplicative group defined over F (usually thought of as a subgroup of G) whose image in G is central. We define the set of morphisms $\mathcal{A}(Z_1 \to G_1, Z_2 \to G_2)$ to be the set of commutative diagrams

$$\begin{array}{ccc} Z_1 & \longrightarrow & Z_2 \\ \downarrow & & \downarrow \\ G_1 & \longrightarrow & G_2, \end{array}$$

where the horizontal maps are morphisms of algebraic groups defined over F. Set $\mathcal{T} \subset \mathcal{A}$ (resp. $\mathcal{R} \subset \mathcal{A}$) to be the subcategory where $[Z \to G]$ belongs to $\mathcal{T}$ (resp. $\mathcal{R}$) if G is a torus (resp. a connected reductive group).

For G abelian with finite F-subgroup Z we define the set $H^1(\mathcal{E}, Z \to G)$ to be the group of isomorphism classes of $G_{\mathcal{E}}$-torsors on $\mathcal{E}$ such that $\mathrm{Res}(\mathscr{T})$ factors through Z. For arbitrary $[Z \to G] \in \mathcal{A}$, we define $H^1(\mathcal{E}, Z \to G)$ to be the pointed set of all isomorphism classes of $G_{\mathcal{E}}$-torsors on $\mathcal{E}$ such that the inertial action is induced by an F-homomorphism $\phi \colon \mathbf{A} \to Z$; note that this agrees with our previous definition if G is abelian, and define the set $H^1_{\mathrm{bas}}(\mathcal{E}, G_{\mathcal{E}})$ to be $\varinjlim_Z H^1(\mathcal{E}, Z \to G)$, where the direct limit is over all finite central subgroups of G (for arbitrary G). If $\mathscr{T}$ is a $G_{\mathcal{E}}$-torsor whose isomorphism class lies in $H^1(\mathcal{E}, Z \to G)$, we say that $\mathscr{T}$ is Z-*twisted*. The map $[Z \to G] \mapsto H^1(\mathcal{E}, Z \to G)$ defines a functor from $\mathcal{A}$ to the category of pointed sets (abelian groups if G is abelian).

Definition 2.6.1 *An a-**twisted Čech 1-cocycle valued in** G (or just an a-**twisted cocycle** if G is understood) is a pair (x, ϕ), where $\phi \colon u \to Z(G)$ is an F-homomorphism and $x \in G(U_1)$ satisfies $dx = \phi(a)$. We say that (x, ϕ) and (y, ϕ') are **equivalent** if $\phi = \phi'$ and there exists $z \in G(U_0)$ such that $p_1(z)^{-1} y p_2(z) = x$ (in this case, we say that z **realizes the equivalence** of (x, ϕ) and (y, ϕ')). This clearly defines an equivalence relation. We denote the set of all a-twisted cocycles by $Z^1(\mathcal{E}_a, G_{\mathcal{E}_a})$, and the set of all equivalence classes by $H^{1,*}(\mathcal{E}_a, G_{\mathcal{E}_a})$. For some fixed finite central Z in G, we say that an a-twisted cocycle (x, ϕ) is an a-**twisted** Z-**cocycle** if ϕ factors through Z. We denote the set of all a-twisted Z-cocycles of G for a fixed Z by $Z^1(\mathcal{E}_a, Z \to G)$. If (x, ϕ)*

is in $Z^1(\mathcal{E}_a, Z \to G)$*, then evidently its whole equivalence class is as well. Denote the set of equivalence classes of* a*-twisted* Z*-cocycles by* $H^{1,*}(\mathcal{E}_a, Z \to G)$*, and the set* $\varinjlim_Z H^{1,*}(\mathcal{E}_a, Z \to G)$ *by* $H^{1,*}_{bas}(\mathcal{E}_a, G_{\mathcal{E}_a})$*, where the direct limit is over all finite central* F*-subgroups.*

We get the following expected result:

Proposition 2.6.2 *For* G *a finite-type* F*-group and* Z *a finite central* F*-subgroup, we have a canonical bijection from* $H^1(\mathcal{E}_a, Z \to G)$ *to* $H^{1,*}(\mathcal{E}_a, Z \to G)$ *which is functorial in* $[Z \to G]$*. Taking direct limits, this induces a canonical bijection* $H^1_{bas}(\mathcal{E}_a, G_{\mathcal{E}_a}) \to H^{1,*}_{bas}(\mathcal{E}_a, G_{\mathcal{E}_a})$*, functorial in the group* G*. If* G *is abelian, we also have a canonical bijection* $H^1(\mathcal{E}_a, G_{\mathcal{E}_a}) \to H^{1,*}(\mathcal{E}_a, G_{\mathcal{E}_a})$*.*

Proof. Let $\mathscr{T}$ be a Z-twisted $G_{\mathcal{E}}$-torsor. Set $\phi := \mathrm{Res}(\mathscr{T})$; it remains to construct the appropriate $x \in G(U_1)$. Let $X := s(\mathrm{Spec}(\overline{F}))$, where s denotes the canonical section $\mathrm{Sch}/\overline{F} \to \mathcal{E}_a$ constructed in Lemma 2.3.3, and let $\varphi \in \mathcal{E}_a(U_1)$ be the isomorphism $p_1^*X \to p_2^*X$ from the same Lemma. Setting $T := s^*\mathscr{T}$ gives a $G_{\overline{F}}$-torsor—choose a $\overline{F}$-trivialization h of T. Taking $p_1^*h \circ \varphi^\sharp \circ p_2^*h^{-1}$ defines an automorphism $G_{U_1} \to G_{U_1}$ which is given by left-translation by a unique element $x \in G(U_1)$, and this x satisfies $dx = \phi(a)$, as desired (we leave the details to the reader, cf. the proof of Proposition 2.4.10). Note that choosing a different h gives an equivalent twisted cocycle.

Moreover, given any isomorphism $\Psi \colon \mathscr{T}_1 \to \mathscr{T}_2$, fixing trivializations $h_i \colon T_i \to G_{\overline{F}}$ as above gives the isomorphism $h_1 \circ s^*\Psi \circ h_2^{-1} \colon G_{\overline{F}} \xrightarrow{\sim} G_{\overline{F}}$, which is left-translation by a unique $y \in G(\overline{F})$, which realizes the equivalence between the twisted cocycles obtained using h_1 and h_2. Thus, we have a canonical well-defined map $H^1(\mathcal{E}_a, Z \to G) \to H^{1,*}(\mathcal{E}_a, Z \to G)$. The fact that this is a bijection is immediate from Proposition 2.4.10. Functoriality in $[Z \to G] \in \mathcal{A}$ is trivial. The proof of the last statement follows by replacing Z by G in the above argument for abelian G. □

We thus get a concrete interpretation of $H^1(\mathcal{E}_a, Z \to G)$ and $H^1_{\mathrm{bas}}(\mathcal{E}_a, G_{\mathcal{E}_a})$; in light of the above results, we denote $H^{1,*}(\mathcal{E}_a, Z \to G)$ simply by $H^1(\mathcal{E}_a, Z \to G)$ and $H^{1,*}(\mathcal{E}_a, G_{\mathcal{E}_a})$ by $H^1(\mathcal{E}_a, G_{\mathcal{E}_a})$ (we make this latter identification only for abelian G)—the above identifications are implicit in this notation.

To extend this to an arbitrary $\mathbf{A}$-gerbe $\mathcal{E}$ split over $\overline{F}$, we need the following result:

Proposition 2.6.3 *Let* $(\mathcal{E}, \theta)$ *be an arbitrary* $\mathbf{A}$*-gerbe and* $a \in \mathbf{A}(U_2)$ *such that* $[a] = [\mathcal{E}]$ *in* $\check{H}^2(\overline{F}/F, \mathbf{A})$*. If* $\check{H}^1(\overline{F}/F, \mathbf{A}) = 0$*, we have a canonical functorial bijection between* $H^1(\mathcal{E}, G_{\mathcal{E}})$ *and* $H^1(\mathcal{E}_a, G_{\mathcal{E}_a})$*.*

Proof. We have an equivalence of $\mathbf{A}$-gerbes $\eta_a \colon \mathcal{E} \to \mathcal{E}_a$ for $a \in \mathbf{A}(U_2)$ representing $[\mathcal{E}] \in \check{H}^2(\overline{F}/F, \mathbf{A})$. This means we have a quasi-inverse $\nu_a \colon \mathcal{E}_a \to \mathcal{E}$ of $\mathbf{A}$-gerbes, so that pullback by ν_a and η_a induce the claimed bijection; if Ψ is the natural isomorphism $\nu_a \circ \eta_a \xrightarrow{\sim} \mathrm{id}_{\mathcal{E}}$, then

$\Psi^\sharp$ gives an isomorphism from $\eta_a^*(\nu_a^*\mathscr{T})$ to $\mathscr{T}$. To check that the above map is independent of the choice of ν_a, it's enough to show that if $\eta\colon \mathcal{E} \to \mathcal{E}$ is an auto-equivalence of **A**-gerbes, then the induced map $H^1(\mathcal{E}, G_{\mathcal{E}}) \to H^1(\mathcal{E}, G_{\mathcal{E}})$ is the identity. This is the content of the following lemma, which will be useful later. $\square$

Lemma 2.6.4 *If* **A** *is such that* $\check{H}^1(\overline{F}/F, \mathbf{A}) = 0$*, then for any* **A***-gerbe* $\mathcal{E}$ *split over* $\overline{F}$ *and* **A***-equivalence* $\eta\colon \mathcal{E} \to \mathcal{E}$*, the induced map* $\eta^*\colon H^1(\mathcal{E}, G_{\mathcal{E}}) \to H^1(\mathcal{E}, G_{\mathcal{E}})$ *is the identity for any* F*-group scheme* G*.*

Proof. The first step is to extract a Čech 1-cocycle from η. Let $Y \in \mathcal{E}(U)$; note that for any morphism $f\colon V \to U$, we have a unique isomorphism ϕ_f making the diagram

$$\begin{array}{ccc} \eta(f^*Y) & \xrightarrow{\phi_f} & f^*\eta(Y) \\ & \searrow & \downarrow \\ & & \eta(Y) \end{array}$$

commute. This means, for an object $X \in \mathcal{E}(\mathrm{Spec}(\overline{F}))$, we have canonical identifications $p_k^*\eta(X) \xrightarrow{\sim} \eta(p_k^*X)$, $p_{ij}^*\eta(p_k^*X) \xrightarrow{\sim} \eta(p_{ij}^*p_k^*X)$, and (combining the previous two) $p_{ij}^*p_k^*\eta(X) \xrightarrow{\sim} \eta(p_{ij}^*p_k^*X)$ for all $1 \leq i, j \leq 3$, $1 \leq k \leq 2$. We make these identifications without comment in what follows.

Picking an isomorphism $\varphi\colon p_1^*X \xrightarrow{\sim} p_2^*X$ in $\mathcal{E}(U_1)$, these identifications allow us to view the isomorphism $\eta(\varphi)$ as an isomorphism from $p_1^*\eta(X)$ to $p_2^*\eta(X)$. Choosing an isomorphism $h\colon X \xrightarrow{\sim} \eta(X)$ in $\mathcal{E}(\mathrm{Spec}(\overline{F}))$ (possible because of Convention 2.2.7), the map $[p_1^*h^{-1} \circ \eta(\varphi)^{-1} \circ p_2^*h] \circ \varphi$ lies in $\mathrm{Aut}_{U_1}(p_1^*X)$ and thus (via $\theta_{p_1^*X}^{-1}$) gives an element $x \in \mathbf{A}(U_1)$. We claim that x is a Čech 1-cocycle. This follows from repeated use of Lemma 2.2.6 and the fact that, on q_1^*X, we may use the above identifications and the fact that η is the identity on bands to deduce that $\theta^{-1}_{q_1^*\eta(X)}(d\eta(\varphi)) = \theta^{-1}_{q_1^*X}(d\varphi) \in \mathbf{A}(U_2)$. It is important to note that the 1-cocycle x does *not* depend on the choice of φ, since if φ' is obtained by precomposing φ by an automorphism y of p_1^*X, then the extra y cancels out, again using Lemma 2.2.6 and the fact that η is the identity on bands.

With this in hand, since we assume that $\check{H}^1(\overline{F}/F, \mathbf{A}) = 0$, we get that $x = dy$ for some $y \in \mathbf{A}(\overline{F})$. We will show that any $G_{\mathcal{E}}$-torsor $\mathscr{T}$ is isomorphic to $\eta^*\mathscr{T}$, which gives the result. It's enough to construct a 2-isomorphism $\mu\colon \mathrm{id}_{\mathcal{E}} \xrightarrow{\sim} \eta$, since then $\mu^\sharp$ will give the desired isomorphism of $G_{\mathcal{E}}$-torsors (for any choice of G). This will just consist of a compatible system of isomorphisms $X \xrightarrow{\mu_X} \eta(X)$ in $\mathcal{E}(U)$ for every $X \in \mathcal{E}(U)$. The argument will be similar to the proof of Lemma 2.4.9; we will first construct such a system of isomorphisms on $\mathcal{E}_{\overline{F}}$, which we will descend to a system of isomorphisms on $\mathcal{E}$ using the fact that $\mathcal{E} \to (\mathrm{Sch}/F)_{\mathrm{fpqc}}$ is a stack.

We first define this system of isomorphisms on the embedded subcategory $\mathcal{C} := s(\mathrm{Sch}/\overline{F}) \subset \mathcal{E}$, where s is the section induced by X. For $f^*X \in \mathcal{E}(V)$, μ_{f^*X} is given by f^*h post-composed with $\theta_{f^*\eta(X)}(y_V^{-1}) \in \mathrm{Aut}_V(f^*\eta(X))$. It is a straightforward exercise to verify that for any object $Z \in \mathcal{E}(W \xrightarrow{g} \mathrm{Spec}(\overline{F}))$ such that we have a (non-canonical) isomorphism $Z \xrightarrow{\lambda,\sim} g^*X$ in $\mathcal{E}(W)$, the isomorphism $Z \to \eta(Z)$ in $\mathcal{E}(W)$ given by $\eta(\lambda^{-1}) \circ \mu_{g^*X} \circ \lambda$ is independent of the choice of λ, and so we set $\mu_Z := \eta(\lambda^{-1}) \circ \mu_{g^*X} \circ \lambda$. By taking common refinements of fpqc covers (since $\mathcal{E} \to (\mathrm{Sch}/F)_{\mathrm{fpqc}}$ is a gerbe), this implies that μ_X induces a natural isomorphism $\mathrm{id}\big|_{\mathcal{E}_{\overline{F}}} \xrightarrow{\bar{\mu}} \eta\big|_{\mathcal{E}_{\overline{F}}}$.

To show that $\bar{\mu}$ descends to $\mathcal{E}$, we need to show (by gluing of morphisms) that $p_1^*(\bar{\mu}) = p_2^*(\bar{\mu})$ on $\mathcal{E}_{U_1}$. Let $Y \in \mathcal{E}(V \xrightarrow{f} U_1)$; there is an fpqc cover $\{V_i \xrightarrow{f_i} V\}$ such that we have isomorphisms $f_i^*Y \xrightarrow{\Psi_{i,1}} f_i^*f^*p_1^*X$ in $\mathcal{E}(V_i)$, as well as isomorphisms $\{\Psi_{i,2}\}$ defined analogously. For each i, we have the following diagram

$$\begin{array}{ccccccc} f_i^*Y & \xrightarrow{\Psi_{i,1}} & f_i^*f^*p_1^*X & \xrightarrow{f_i^*f^*p_1^*h} & f_i^*f^*p_1^*\eta(X) & \xrightarrow{\eta(\Psi_{i,1})^{-1}} & f_i^*\eta(Y) \\ & \searrow^{\Psi_{i,2}} & \downarrow{\scriptstyle \Psi_{i,1,2}} & & \downarrow{\scriptstyle \eta(\Psi_{i,1,2})} & \nearrow_{\eta(\Psi_{i,2})^{-1}} & \\ & & f_i^*f^*p_2^*X & \xrightarrow{f_i^*f^*p_2^*h} & f_i^*f^*p_2^*\eta(X), & & \end{array}$$

where we have made the canonical identifications mentioned at the beginning of the proof in several places and $\Psi_{i,1,2} := \Psi_{i,2} \circ \Psi_{i,1}^{-1}$. The diagram does not commute because of the middle square. Indeed, starting at the top-left corner, going right then down then left yields $f_i^*f^*p_2^*h^{-1} \circ \eta(\Psi_{i,12}) \circ f_i^*f^*p_1^*h = \theta_{f_i^*Y}(x_{V_i}^{-1}) \circ \Psi_{i,1,2}$, where x_{V_i} denotes the image of $x \in \mathbf{A}(U_1) \to \mathbf{A}(V_i \xrightarrow{f \circ f_i} U_1)$ (using that x does not depend on the choice of φ, see beginning of the proof). But now replacing $f_i^*f^*p_k^*h$ with $\bar{\mu}_{f_i^*f^*p_k^*X}$ for $k = 1, 2$ serves to replace the above composition with $\theta_{f_i^*Y}(p_{1,V_i}(y)^{-1}) \circ \theta_{f_i^*Y}(p_{2,V_i}(y)) \circ \theta_{f_i^*Y}(x_{V_i}^{-1}) \circ \Psi_{i,1,2} = \theta_{f_i^*Y}(d(y_{V_i}) \cdot x_{V_i}^{-1}) \circ \Psi_{i,1,2} = \Psi_{i,1,2}$, where p_{k,V_i} for $k = 1, 2$ denotes the map $V_i \xrightarrow{p_k \circ f \circ f_i} \mathrm{Spec}(\overline{F})$, since $dy = x$ by construction. This gives the main result, since if $(p_1^*\bar{\mu})_Y$ and $(p_2^*\bar{\mu})_Y$ coincide on an fpqc cover of Y, they coincide on Y as well. □

The above proof also gives a useful blueprint for constructing isomorphisms of $G_{\mathcal{E}}$-torsors. We give one application here, using it to explain how to explicitly construct an isomorphism of $G_{\mathcal{E}_a}$-torsors $\mathscr{T}_1 \to \mathscr{T}_2$ given an equivalence between their corresponding (class of) a-twisted cocycles (x_1, ϕ), (x_2, ϕ) coming from trivializations h_1, h_2, realized by the element $y \in G(\overline{F})$. Namely, we first define the map $\mathscr{T}_1\big|_{\mathcal{C}} \to \mathscr{T}_2\big|_{\mathcal{C}}$ on the category $\mathcal{C} := s(\mathrm{Sch}/\overline{F})$ by taking $h_2^{-1} \circ \theta_{\mathrm{Spec}(\overline{F})}(y)^{\sharp} \circ h_1$, and then extend this to all of $\mathcal{E}_{\overline{F}}$ by conjugating by fpqc-local isomorphisms to objects in $\mathcal{C}$ (as in the above proof, cf. also the proof of Lemma 2.4.9). The fact that $dy \in \mathbf{A}(U_1)$ is 1-cocycle implies that this isomorphism descends to an isomorphism of $G_{\mathcal{E}_a}$-torsors $\mathscr{T}_1 \to \mathscr{T}_2$.

The punchline of this entire subsection is the following result:

Corollary 2.6.5 *For any $\boldsymbol{A}$-gerbe $\mathcal{E}$ split over $\overline{F}$ and $a \in \boldsymbol{A}(U_2)$ with $[a] = [\mathcal{E}] \in \check{H}^2(\overline{F}/F, \boldsymbol{A})$, if $\check{H}^1(\overline{F}/F, \boldsymbol{A}) = 0$, then we have a canonical functorial bijection between $H^1(\mathcal{E}, Z \to G)$ and $H^{1,*}(\mathcal{E}_a, Z \to G)$ for any $[Z \to G]$ in $\mathcal{A}$.*

2.7 Inflation-restriction

We continue with the notation of the previous sections; in particular, $\mathcal{E} \xrightarrow{\pi} \mathrm{Sch}/F$ is a fixed $\mathbf{A}$-gerbe split over $\overline{F}$. In this section, we discuss the analogue of the inflation-restriction exact sequence in the setting of gerbes. Again G will be a fixed finite type F-group scheme. Our goal is to define a functorial "inflation-restriction" sequence for any $[Z \to G] \in \mathcal{A}$:

$$0 \longrightarrow \check{H}^1(\overline{F}/F, G) \xrightarrow{\text{Inf}} H^1(\mathcal{E}, Z \to G) \xrightarrow{\text{Res}} \mathrm{Hom}_F(\mathbf{A}, G) \xrightarrow{tg} \check{H}^2(\overline{F}/F, G),$$

where the H^2-term is to be ignored if G is non-abelian. In order to define this sequence, we may assume that $\mathcal{E} = \mathcal{E}_a$ for some $a \in \mathbf{A}(U_2)$, due to Corollary 2.6.5, and take $H^1(\mathcal{E}_a, Z \to G)$ to be equivalence classes of a-twisted Z-cocycles valued in G. This makes computations significantly simpler.

We take the first map, called *inflation*, to be the one induced by sending the 1-cocycle $x \in G(U_1)$ to $(x, 0) \in Z^1(\mathcal{E}_a, G_{\mathcal{E}_a})$, we take the second map, called *restriction*, to be the one that sends the a-twisted cocycle (a, ϕ) to ϕ, and we take the third map, called *transgression* to be the one that sends $\phi \in \mathrm{Hom}_F(\mathbf{A}, Z(G))$ to $[\phi(a)] \in \check{H}^2(\overline{F}/F, G)$. We leave it to the reader to check that these maps are well-defined.

Proposition 2.7.1 *The image of the class $[\mathscr{T}] \in H^1(\mathcal{E}, Z \to G)$ under the restriction map defined above equals the unique F-homomorphism $\boldsymbol{A} \to Z(G)$ inducing the inertial action on $\mathscr{T}$ (see Lemma 2.4.2).*

Proof. We leave this as an exercise, using the proof of Proposition 2.4.10 for the case $\mathcal{E} = \mathcal{E}_a$. □

Proposition 2.7.2 *The above maps define a functorial exact sequence of pointed sets (groups if G is abelian, where the H^2 term is to be ignored if G is non-abelian):*

Proof. Clearly the image of the first set is contained in the fiber over identity of the second map. Conversely, if we have some (x, ϕ) with $\phi = 0$, then the twisted cocycle condition on $x \in G(U_1)$ is just the usual cocycle condition, and hence $[x] \in \check{H}^1(\overline{F}/F, G)$ maps to $(x, 0)$. Taking a twisted cocycle (x, ϕ) already gives an element $x \in G(U_1)$ such that $dx = \phi(a)$, so that evidently $[\phi(a)] = 0$ in $\check{H}^2(\overline{F}/F, G)$. Finally, if $\phi \in \mathrm{Hom}_F(\mathbf{A}, Z(G))$ is such that $\phi(a) = dx$ for $x \in G(U_1)$, then (x, ϕ) defines a twisted cocycle, completing the proof. We leave functoriality in G as an exercise. □

For $[Z \to G]$ in $\mathcal{A}$, denote by $G \xrightarrow{\pi} \overline{G}$ the quotient of G by Z. The following version of the long exact sequence in fpqc cohomology will be useful later:

Lemma 2.7.3 *For $[Z \to G] \in \mathcal{A}$ we have an exact sequence of pointed sets (abelian groups if G is abelian):*

$$\overline{G}(F) \longrightarrow H^1(\mathcal{E}, Z_{\mathcal{E}}) \longrightarrow H^1(\mathcal{E}, G_{\mathcal{E}}) \longrightarrow H^1(\mathcal{E}, \overline{G}_{\mathcal{E}})$$

Proof. Again, we may work with a-twisted cocycles. The first map is defined to be the composition $\overline{G}(F) \xrightarrow{\delta} H^1(\overline{F}/F, Z) \xrightarrow{\text{Inf}} H^1(\mathcal{E}_a, Z_{\mathcal{E}_a})$ from the short exact sequence of fppf group schemes associated to $Z \to G$, and the second and third maps come from functoriality. The first map lands in the kernel of the second because the composition of the first two maps may be factored as $\overline{G}(\overline{F}) \xrightarrow{\delta} \check{H}^1(\overline{F}/F, Z) \to \check{H}^1(\overline{F}/F, G) \xrightarrow{\text{Inf}} H^1(\mathcal{E}_a, G_{\mathcal{E}_a})$. Moreover, if $(x, \phi) \in Z^1(\mathcal{E}_a, Z_{\mathcal{E}_a})$ has trivial image in $H^1(\mathcal{E}_a, G_{\mathcal{E}_a})$, then $\phi = 0$ and hence (x, ϕ) lies in the image of the inflation map $\check{H}^1(\overline{F}/F, Z) \to H^1(\mathcal{E}_a, Z_{\mathcal{E}_a})$, and again the composition $\check{H}^1(\overline{F}/F, Z) \to H^1(\mathcal{E}_a, Z_{\mathcal{E}_a}) \to H^1(\mathcal{E}_a, G_{\mathcal{E}_a})$ factors as $\check{H}^1(\overline{F}/F, Z) \to \check{H}^1(\overline{F}/F, G) \hookrightarrow H^1(\mathcal{E}_a, G_{\mathcal{E}_a})$, giving the other containment.

For exactness at the second spot, if (x, ϕ) is such that $[\pi(x, \phi)] = 0$, then $\pi \circ \phi = 0$, and so by basic properties of quotients, this happens if and only if ϕ factors through Z. Given this, the class maps to the identity if and only if (using centrality) we have that $x \in G(U_1)$ is such that there is some $z \in Z(U_1)$ with $x = p_1(g)^{-1} z p_2(g)$, and this z necessarily satisfies $dz = dx = \phi(a)$ (since it's cohomologous to x). This holds if and only if $[(x, \phi)] = [(z, \phi)]$ in $H^1(\mathcal{E}_a, G_{\mathcal{E}_a})$, and $(z, \phi) \in Z^1(\mathcal{E}_a, Z_{\mathcal{E}_a})$, as desired. □

One checks easily (using Construction 2.3.4) that if $\mathcal{E}$ is an $\mathbf{A}$-gerbe split over $\overline{F}$ and $\mathcal{E}'$ is a $\mathbf{B}$-gerbe split over $\overline{F}$, and we have a morphism $\nu \colon \mathcal{E} \to \mathcal{E}'$ of categories over Sch$/F$ inducing the map $f \in \text{Hom}_F(\mathbf{A}, \mathbf{B})$, then the following diagram also commutes for any finite type G:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \check{H}^1(\overline{F}/F, G) & \longrightarrow & H^1(\mathcal{E}, G_{\mathcal{E}}) & \longrightarrow & \text{Hom}_F(\mathbf{A}, Z(G)) & \longrightarrow & \check{H}^2(\overline{F}/F, G) \\ & & \| & & \uparrow \nu^* & & \uparrow f^* & & \| \\ 0 & \longrightarrow & \check{H}^1(\overline{F}/F, G) & \longrightarrow & H^1(\mathcal{E}', G_{\mathcal{E}'}) & \longrightarrow & \text{Hom}_F(\mathbf{B}, Z(G)) & \longrightarrow & \check{H}^2(\overline{F}/F, G). \end{array}$$

CHAPTER 3

Results on Čech Cohomology

3.1 Derived-to-Čech comparison

Fix a commutative ring R. The category of abelian sheaves on $(\mathrm{Sch}/R)_{\mathrm{fppf}}$ is an abelian category with enough injectives, and for an abelian R-group scheme $\mathbf{A}$ we may thus define the cohomology groups $H^i((\mathrm{Sch}/R)_{\mathrm{fppf}}, \mathbf{A})$ for $i \geq 0$ by taking the derived functors of the global section functor on this abelian category, viewing $\mathbf{A}$ as a sheaf on $(\mathrm{Sch}/R)_{\mathrm{fppf}}$. We will denote $H^i((\mathrm{Sch}/R)_{\mathrm{fppf}}, \mathbf{A})$ by $H^i(R, \mathbf{A})$, or sometimes by $H^i_{\mathrm{fppf}}(R, \mathbf{A})$ when we want to emphasize our use of the fppf topology.

For any abelian fppf group scheme A over R with pro-fppf cover S/R , the Grothendieck spectral sequence gives us a spectral sequence

$$E_2^{p,q} = \check{H}^p(S/R, \underline{H}^q_{\mathrm{fppf}}(A)) \Rightarrow H^{p+q}_{\mathrm{fppf}}(R, A),$$

where $\underline{H}^q_{\mathrm{fppf}}(A))$ denotes the fppf-sheafification of the presheaf on Sch$/R$ sending U to $H^q(U, A_U)$ (see [Stacks, 03AV]). We have the following result:

Proposition 3.1.1 *([Stacks, 03AV]) If $H^i_{fppf}(S^{\otimes_R n}, A) = 0$ for all $n, i \geq 1$, then the above spectral sequence induces a canonical isomorphism $\check{H}^i(S/R, A) \xrightarrow{\sim} H^i_{fppf}(R, A)$ for all i.*

Remark 3.1.2 *Strictly speaking, Lemmas 21.10.6 and 21.10.7 in [Stacks], 03AV are stated in the setting of an fppf cover S/R, but taking the direct limit of spectral sequences gives us the result for pro-fppf covers (rings S which are a direct limit of fppf covers, such as $\overline{F}$ for $R = F$ a field).*

The following result states how the above isomorphisms behave with respect to connecting homomorphisms.

Proposition 3.1.3 *Under the comparison isomorphisms of Proposition 3.1.1, we have $\check{\delta} = \delta$, where δ is the usual connecting homomorphism arising from the derived functor formalism.*

Proof. This is [Ros19], Proposition E.2.1. □

In light of Proposition 3.1.1, the following result is relevant:

Proposition 3.1.4 *For G a finite type commutative group scheme over $R = F$ a field and $S = \overline{F}$, we have $H^i_{fppf}(\overline{F}^{\bigotimes_F n}, G) = 0$ for all $i > 0$ and all $j \geq 0$.*

Proof. This is [Ros19], Lemma 2.9.4. □

3.2 Čech cohomology over $O_{F,S}$

Fix a global function field F of characteristic $p > 0$, a finite set S of places of F, and an F-torus T which is unramified outside S. Let $O_{F,S}$ denote the elements of F whose valuation is non-negative at all places outside S, and for a finite Galois extension K/F, denote by $O_{K,S}$ the elements of K whose valuation is non-negative at all places outside S_K, the set of all places of K lying above S. We set $O_S := \varinjlim_{K/F} O_{K,S}$, where K/F ranges over all finite Galois extensions which are unramified outside of S. Denote by F_S the maximal field extension of F which is unramified outside S, and denote its Galois group over F by Γ_S; note that $F_S = \text{Frac}(O_S)$. Since T is defined over the subring $O_F \subset F$, it is also defined over $O_{F,S}$ for any set of places S; it thus makes sense to ease notation by denoting the corresponding O_F- or $O_{F,S}$-scheme also by T.

For all $q > 0$, it is a basic fact of fppf cohomology ([Čes16, Lemma 2.1]) that for a commutative group scheme $\mathscr{G}$ on $O_{F,S}$ which is locally of finite presentation, we have $H^q_{\text{fppf}}(O_S, \mathscr{G}) = \varinjlim_{K/F} H^q_{\text{fppf}}(O_{K,S}, \mathscr{G})$, with the transition maps induced by pullback of fppf sheaves (the same is true if we replace “fppf" by “étale").

We begin with the following commutative-algebraic lemma:

Lemma 3.2.1 *For K/F a finite Galois extension unramified outside S and $n \geq 2$, the natural injection $O_{K,S}^{\bigotimes_{O_{F,S}} n} \to \prod_{\Gamma^{n-1}_{K/F}} O_{K,S}$ is an isomorphism.*

Proof. By induction, it is enough to prove the result for $n = 2$. First, note that $O_{K,S}/O_{F,S}$ is finite étale by assumption (since K/F is unramified outside of S). In particular, $O_{K,S}$ is finitely-generated and torsion-free as an $O_{F,S}$-module, and both rings are Dedekind domains which are integrally closed in their fields of fractions. By base-change, we get a finite étale extension $O_{K,S} \otimes_{O_{F,S}} O_{K,S}/O_{K,S}$, which is still finitely-generated, locally free, and torsion-free as an $O_{K,S}$-module (this last fact follows from using the injection $O_{K,S} \otimes_{O_{F,S}} O_{K,S} \hookrightarrow \prod_{\Gamma_{K/F}} O_{K,S}$, under which $O_{K,S}$ maps into the diagonally-embedded copy, which clearly acts on the product without torsion).

We are thus in the setting of [Con18, Theorem 1.3], which says that the composition

$$O_{K,S} \otimes_{O_{F,S}} O_{K,S} \hookrightarrow K \otimes_{O_{F,S}} O_{K,S} \xrightarrow{\sim} \prod_i K_i,$$

where each K_i is some finite separable extension of K and the last isomorphism comes from the fact that $K \otimes_{O_{F,S}} O_{K,S}$ is finite étale over K a field, maps $O_{K,S} \otimes_{O_{F,S}} O_{K,S}$ isomorphically onto the product of integral closures of $O_{K,S}$ in each K_i. It is thus enough to show that we have an isomorphism $K \otimes_{O_{F,S}} O_{K,S} \xrightarrow{\sim} \prod_{\Gamma_{K/F}} K$.

Choose an element $\alpha \in O_{K,S}$ such that $K = F(\alpha)$; since $O_{K,S}$ is the integral closure of $O_{F,S}$ inside K, we know that the minimal polynomial of α over F, denoted by f, lies in $O_{F,S}[x]$, and so the desired result follows from the series of elementary manipulations

$$K \otimes_{O_{F,S}} O_{K,S} \xrightarrow{\sim} \frac{F[x]}{(f)} \otimes_{O_{F,S}} O_{K,S} \xrightarrow{\sim} F \otimes_{O_{F,S}} (\frac{O_{F,S}[x]}{(f)} \otimes_{O_{F,S}} O_{K,S}) \xrightarrow{\sim} F \otimes_{O_{F,S}} (\prod_{\Gamma_{K/F}} O_{K,S}),$$

followed by commuting the tensor product with the (finite) product and applying the canonical isomorphism $F \otimes_{O_{F,S}} O_{K,S} \xrightarrow{\sim} K$. We leave it to the reader to check that the isomorphism $O_{K,S} \otimes_{O_{F,S}} O_{K,S} \xrightarrow{\sim} \prod_{\Gamma_{K/F}} O_{K,S}$ constructed in the above proof agrees with the injection in the statement of the Lemma. □

Corollary 3.2.2 *We have a canonical isomorphism $\check{H}^i(O_{K,S}/O_{F,S}, G) \xrightarrow{\sim} H^i(\Gamma_{K/F}, G(O_{K,S}))$ for any commutative $O_{F,S}$-group G. Taking the direct limit also gives a canonical isomorphism $\check{H}^i(O_S/O_{F,S}, G) \xrightarrow{\sim} H^i(\Gamma_S, G(O_S))$.*

Proof. All that one must check is that the isomorphism of Lemma 3.2.1 preserves cocycles and coboundaries, which is straightforward. □

According to §3.1, in order to compare the Čech cohomology groups $\check{H}^i(O_S/O_{F,S}, T)$ with $H^i_{\text{fppf}}(O_{F,S}, T)$, we need to prove some cohomological vanishing results. The first result involves étale cohomology:

Lemma 3.2.3 *We have that $H^i_{et}(O_S, T_{O_S}) = 0$ for all $i > 0$.*

Proof. Since we assume that T is unramified outside S, it is enough to prove the result for $T = \mathbb{G}_m$. For $i = 1$, the result follows from the above paragraph, using the fact that for $F \subset K \subset F_S$ a finite subextension, we have $H^1_{\text{et}}(O_{K,S}, \mathbb{G}_m) = \text{Pic}(\text{Spec}(O_{K,S})) = \text{Cl}(O_{K,S})$ (the first equality comes from [Mil06, Proposition II.2.1]) and that $\varinjlim_{K/F} \text{Cl}(O_{K,S}) = 0$, where the limit is over all finite subextensions, by the proof of [NSW08, Proposition 8.3.6].

For $i = 2$, first note that $H^2_{\text{et}}(O_{K,S}, \mathbb{G}_m) = \text{Br}(O_{K,S})$, and then by [Poo17, 6.9.2], we have an exact sequence

$$0 \to \text{Br}(O_{K,S}) \to \bigoplus_{v \in S_K} \text{Br}(K_v) \xrightarrow{\sum \text{inv}_v} \mathbb{Q}/\mathbb{Z},$$

where K_v denotes the completion of K at v. Taking the direct limit of the first two terms shows that we have an injective map $\mathrm{Br}(O_S) \hookrightarrow \bigoplus_{v \in S_{F_S}} \mathrm{Br}(F_S \cdot F_v)$. Note that the field extension $F_S \cdot F_v$ contains F_v^{nr}, the maximal unramified extension of F_v, using the fact that F_S/F contains all finite extensions of the constant field of F. Moreover, the valuation ring $O_{F_S \cdot F_v}$ of this field is Henselian, as it is the direct limit of the Henselian rings O_{K_v}([Hoc10, pp. 56]), and the previous sentence implies that it has algebraically closed residue field. We may then deduce from the proof of [Mil06, Proposition I.A.1] that $\mathrm{Br}(F_S \cdot F_v) = 0$, giving the desired result.

Finally, for $i \geq 3$, we have that for any K/F a finite Galois extension, we have $H^i_{\mathrm{et}}(O_{K,S}, \mathbb{G}_m) = 0$, by [Mil06], Remark II.2.2. Taking the direct limit gives the desired result. □

Corollary 3.2.4 *We have canonical isomorphisms* $H^i(\Gamma_S, T(O_S)) \xrightarrow{\sim} H^i_{et}(O_{F,S}, T)$ *for all* $i \geq 1$.

Proof. This follows immediately from combining Lemma 3.2.3 with the spectral sequence

$$H^p(\Gamma_S, H^q_{\mathrm{et}}(O_S, T_{O_S})) \Rightarrow H^{p+q}_{\mathrm{et}}(O_{F,S}, T)$$

from [Poo17, Theorem 6.7.5]. □

Lemma 3.2.5 *We have that*

$$H^i_{\mathit{fppf}}(O_S^{\bigotimes_{O_{F,S}} n}, T) = 0$$

for all $n, i \geq 1$.

Proof. It suffices to prove the result for $T = \mathbb{G}_m$ since we assume that T is unramified outside S. Moreover, it is enough to show that

$$\varinjlim_{K/F} H^i_{\mathrm{fppf}}(O_{K,S}^{\bigotimes_{O_{F,S}} n}, \mathbb{G}_m) = 0,$$

where the limit is over all finite subextensions of F inside F_S. By Lemma 3.2.1, we have a canonical identification

$$\mathrm{Spec}(O_{K,S}^{\bigotimes_{O_{F,S}} n}) = \coprod_{\underline{\sigma} \in \Gamma_{K/F}^{n-1}} \mathrm{Spec}(O_{K,S}),$$

as well as a canonical isomorphism $H^i_{\mathrm{fppf}}(\coprod_{\underline{\sigma}} \mathrm{Spec}(O_{K,S}), \mathbb{G}_m) \xrightarrow{\sim} \prod_{\underline{\sigma}} H^i_{\mathrm{fppf}}(O_{K,S}, \mathbb{G}_m)$. Also, if K'/K is finite and contained in F_S, then the natural map

$$H^i_{\mathrm{fppf}}(O_{K,S}^{\bigotimes_{O_{F,S}} n}, \mathbb{G}_m) \to H^i_{\mathrm{fppf}}(O_{K',S}^{\bigotimes_{O_{F,S}} n}, \mathbb{G}_m)$$

corresponds via this isomorphism to diagonally embedding each factor of

$\prod_{\underline{\sigma}\in\Gamma_{K/F}^{n-1}} H^i_{\mathrm{fppf}}(O_{K,S}, \mathbb{G}_m)$ into some subset of the factors of $\prod_{\underline{\sigma}\in\Gamma_{K'/F}^{n-1}} H^i_{\mathrm{fppf}}(O_{K',S}, \mathbb{G}_m)$ (by means of the pullback map $H^i_{\mathrm{fppf}}(O_{K,S}, \mathbb{G}_m) \to H^i_{\mathrm{fppf}}(O_{K',S}, \mathbb{G}_m)$).

Hence, if $\alpha \in H^i_{\mathrm{fppf}}(O_{K,S}^{\bigotimes_{O_{F,S}} n}, \mathbb{G}_m)$ is any element, then to show that α vanishes in some $H^i_{\mathrm{fppf}}(O_{K',S}^{\bigotimes_{O_{F,S}} n}, \mathbb{G}_m)$ for large K', it is enough to show that $\varinjlim_{K/F} H^i_{\mathrm{fppf}}(O_{K,S}, \mathbb{G}_m) = 0$ for all i, thus reducing the result to the case $n = 1$, which follows from combining Lemmas 3.2.3 and 3.2.5 with Corollary 3.2.4. □

Recall from §3.1 that for any abelian fppf group scheme A over $O_{F,S}$ and pro-fppf cover $R/O_{F,S}$, the Grothendieck spectral sequence gives us a spectral sequence

$$E_2^{p,q} = \check{H}^p(R/O_{F,S}, \underline{H}^q_{\mathrm{fppf}}(A)) \Rightarrow H^{p+q}_{\mathrm{fppf}}(O_{F,S}, A).$$

Corollary 3.2.6 *The above spectral sequence induces a canonical isomorphism*

$$\check{H}^i(O_S/O_{F,S}, T) \xrightarrow{\sim} H^i_{\mathit{fppf}}(O_{F,S}, T)$$

for all i.

Proof. Combine Lemma 3.2.5 with Proposition 3.1.1. □

We now move to the realm of possibly non-étale extensions, in order to handle the cohomology of non-smooth finite F-groups. For R an $\mathbb{F}_p$-algebra, let $R^{\mathrm{perf}} := \varinjlim R$, where the direct limit is over successively higher powers of the Frobenius homomorphism. For $R = O_{F,S}$, the ring $O_{F,S}^{\mathrm{perf}}$ is obtained by adjoining all p-power roots of elements of $O_{F,S}$ (in a fixed algebraic closure $\overline{F}/F$). We begin by recalling an elementary lemma on the splitting of primes in rings of integers of purely inseparable extensions:

Lemma 3.2.7 *Let F'/F be a purely inseparable extension and $\mathfrak{p} \subset O_F$. Then $\mathfrak{p} \cdot O_{F'} = (\mathfrak{p}')^{[F':F]}$ for some prime $\mathfrak{p}'$ of $O_{F'}$.*

Proof. It is evidently enough to prove this in the case when $[F' : F] = p$, which we now assume. We claim that $O_{F'} = O_F^{(p)}$, the extension of O_F obtained by adjoining all p-power roots. There is an obvious inclusion of O_F-algebras $O_{F'} \hookrightarrow O_F^{(p)}$ because $F' = F^{(p)}$. The morphism of smooth projective curves $X' \to X$ corresponding to the inclusion $F \to F'$ is purely inseparable of degree p, so by [Stacks], 0CCV, we obtain an isomorphism of O_F-algebras $O_{F'} \xrightarrow{\sim} O_F^{(p)}$, giving the claim. The claim implies that, at the level of local rings, a uniformizer $\varpi \in O_{F,\mathfrak{p}}$ has a pth root in $O_{F',\mathfrak{p}'}$ for any prime $\mathfrak{p}'$ above $\mathfrak{p}$, giving the desired result. □

Denote by F_m the field extension of F obtained by adjoining all p^m-power roots; note that by the proof of the above lemma, this is a finite, purely inseparable extension. We have the following characterization of the perfect closure O_S^{perf}:

Lemma 3.2.8 *The canonical map*

$$\varinjlim_m O_{S_m} \to O_S^{perf}$$

is an isomorphism, where S_m denotes the preimage of S in Spec(O_{F_m}).

Proof. For the inclusion of the right-hand side into the left-hand side, note that if $x \in \overline{F}$ is such that $x^{p^m} \in O_{E,S}$ for some finite (Galois) $E \subset F_S$, then $x \in E' := E \cdot F_m$, which is unramified over F_m outside of S_m, and so $x \in O_{E',S_m} \subset O_{S_m}$. For the other inclusion, consider a finite Galois extension K' of the finite purely inseparable extension $F' := F_m/F$ with $S' := S_m$. We may factor K'/F as a tower $K'/K/F$, where K/F is the separable (Galois) closure of F in K' and K'/K is purely inseparable. Note that $K \cdot F' = K'$; one containment is clear, and the other follows from the fact that K and F' are linearly disjoint and $[K' : F'] = [K : F]$.

We want to show that K/F is unramified outside S; this follows because for any prime $\mathfrak{p}$ of O_F, we know from Lemma 3.2.7 that $\mathfrak{p}$ splits as $(\mathfrak{p}')^{[F':F]}$ in $O_{F'}$, and if $\mathfrak{p}'$ is a prime of $O_{F',S'}$, then it factors in $O_{K'}$ as $\mathfrak{P}'_1 \cdot \cdots \cdot \mathfrak{P}'_r$, which means that $\mathfrak{p}$ splits in $O_{K'}$ as $(\mathfrak{P}'_1 \cdot \cdots \cdot \mathfrak{P}'_r)^{[F':F]}$. Since $[F' : F] = [K' : K]$, we know that $\mathfrak{p}$ must not ramify in O_K, or else the ramification degree would be too large. Now for any element $x \in O_{K',S'}$, we have that $x^{p^m} \in K$ and is integral over $O_{F,S}$, and hence lies in $O_{K,S}$, showing that $O_{K',S'} \subseteq O_{K,S}^{(p^m)} \subset O_S^{\mathrm{perf}}$, giving the other inclusion. □

With these results in hand, we are ready to prove that passing to the perfection of O_S allows us to compute the Čech cohomology of multiplicative $O_{F,S}$-group schemes.

Lemma 3.2.9 *For A a multiplicative F-group (which, as for tori, has a canonical model over $\mathbb{Z}$ so it makes sense to treat it as an O_F-scheme) split over O_S, the groups $H^i_{fppf}((O_S^{perf})^{\bigotimes_{O_{F,S}} n}, A)$ vanish for all $i, n \geq 0$.*

Proof. It is enough to prove the result for $A = \mathbb{G}_m$ and $A = \mu_m$. We focus on the former first: Note that we may use the smoothness of $\mathbb{G}_m$ and [Ros19, Lemma 2.2.9] to replace $(O_S^{\mathrm{perf}})^{\bigotimes_{O_{F,S}} n}$ by $[(O_S^{\mathrm{perf}})^{\bigotimes_{O_{F,S}} n}]_{\mathrm{red}}$. We now have $[(O_S^{\mathrm{perf}})^{\bigotimes_{O_{F,S}} n}]_{\mathrm{red}} = (O_S^{\mathrm{perf}})^{\bigotimes_{O_{F,S}^{\mathrm{perf}}} n}$, so it's enough to show that the groups

$$H^j_{\mathrm{fppf}}((O_S^{\mathrm{perf}})^{\bigotimes_{O_{F,S}^{\mathrm{perf}}} n}, \mathbb{G}_m)$$

all vanish. By Lemma 3.2.8, we have

$$(O_S^{\mathrm{perf}})^{\bigotimes_{O_{F,S}^{\mathrm{perf}}} n} = \varinjlim_m O_{S_m}^{\bigotimes_{O_{F_m,S_m}} n},$$

and hence it's enough to show that

$$\varinjlim_m H^j_{\text{fppf}}(O_{S_m}^{\bigotimes_{O_{F_m,S_m}} n}, \mathbb{G}_m) = 0$$

for all $j, n \geq 1$. Now the result follows from Lemma 3.2.5, which shows that each term in the direct limit is zero.

We now prove the μ_m-case. For $i > 1$, we immediately deduce that $H^i_{\text{fppf}}((O_S^{\text{perf}})^{\bigotimes_{O_{F,S}} n}, \mu_m)$ vanishes from the long exact sequence in fppf cohomology and the $\mathbb{G}_m$-case.

For $i = 1$, since $H^1_{\text{fppf}}((O_S^{\text{perf}})^{\bigotimes_{O_{F,S}} n}, \mathbb{G}_m) = 0$, we have from the long exact sequence in fppf cohomology that $H^1_{\text{fppf}}((O_S^{\text{perf}})^{\bigotimes_{O_{F,S}} n}, \mu_m)$ is the quotient

$$((O_S^{\text{perf}})^{\bigotimes_{O_{F,S}} n})^* / (((O_S^{\text{perf}})^{\bigotimes_{O_{F,S}} n})^*)^m.$$

We know that O_S^* is n'-divisible for n' coprime to p by [NSW08, Proposition 8.3.4], and hence so is the group $(O_S^{\bigotimes_{O_{F,S}} n})^*$ (using Lemma 3.2.1). Now $O_S^{\text{perf},*}$ is $\mathbb{N}$-divisible, since it is obtained from O_S by adjoining all p-power roots, and once again this implies that $((O_S^{\text{perf}})^{\bigotimes_{O_{F,S}} n})^* = (O_S^{\bigotimes_{O_{F,S}} n})^{\text{perf},*}$ is as well. □

We immediately obtain:

Corollary 3.2.10 *For A as above, we have a canonical isomorphisms*

$$\check{H}^i(O_S^{\textit{perf}}/O_{F,S}, A) \xrightarrow{\sim} H^i_{\textit{fppf}}(O_{F,S}, A)$$

for all i. Moreover, for an F-torus T unramified outside S, the natural map

$$\check{H}^i(O_S/O_{F,S}, T) \to \check{H}^i(O_S^{\textit{perf}}/O_{F,S}, T)$$

induced by the inclusion $O_S \to O_S^{\textit{perf}}$ is an isomorphism.

In the global case, the ring O_S^{perf} will have the role that $\overline{F}$ plays locally. We conclude with a useful result concerning the finite-level Čech cohomology of an $O_{F,S}$-torus T split over $O_{E,S}$. We first recall the following result from [Mor72]:

Proposition 3.2.11 *([Mor72, Theorem 3.2]) Let $S/R/O_{F,S}$ be two fppf covers of $O_{F,S}$; set*

$$\Sigma := [\bigcup_i R^{\bigotimes_{O_{F,S}} i}] \cup [\bigcup_i S^{\bigotimes_{O_{F,S}} i}] \cup [\bigcup_i S^{\bigotimes_R i}].$$

If $\mathscr{F}$ is a sheaf on $(Sch/O_{F,S})_{fppf}$ such that $H^1_{fppf}(A, \mathscr{F}) = 0$ for all $A \in \Sigma$, then we have an exact sequence

$$0 \to \check{H}^2(R/O_{F,S}, \mathscr{F}) \to \check{H}^2(S/O_{F,S}, \mathscr{F}) \to \check{H}^2(S/R, \mathscr{F}).$$

We now obtain:

Lemma 3.2.12 *Let E/F be a finite Galois extension, let E'/E be a finite purely inseparable extension, and $S \subset V$ a finite set of places such that $Cl(O_{E,S})$ is trivial. Then if T is an $O_{F,S}$-torus split over $O_{E,S}$, the natural map $\check{H}^2(O_{E,S}/O_{F,S}, T) \to \check{H}^2(O_{E',S}/O_{F,S}, T)$ is an isomorphism.*

Proof. We leave it to the reader to check that the Σ-condition of Proposition 3.2.11 is satisfied (since everything in Σ is an $O_{E,S}$-algebra, we may replace T with $\mathbb{G}_m$ for this condition and use the fact that $O_{E,S}$ and $O_{E',S}$ are principal ideal domains, along with [Ros19, Lemma 2.2.9]). It thus suffices to show that the group $\check{H}^2(O_{E',S}/O_{E,S}, \mathbb{G}_m)$ vanishes. Note that, for any n, $\mathbb{G}_m(O_{E',S}^{\bigotimes_{O_{E,S}} n}) = \mathbb{G}_m([O_{E',S}^{\bigotimes_{O_{E,S}} n}]_{\text{red}})$, and now $[O_{E',S}^{\bigotimes_{O_{E,S}} n}]_{\text{red}} = O_{E',S}^{\bigotimes_{O_{E',S}} n} = O_{E,S}$, so our Čech cohomology computations on this cover reduce to that of the trivial cover $O_{E',S}/O_{E',S}$, giving the desired vanishing. □

3.3 Čech cohomology over $\mathbb{A}$

In this subsection we prove some basic results that allow us to do Čech cohomology on (covers of) the adele ring $\mathbb{A}$ of our global function field F. Let G a multiplicative F-group scheme with fixed O_{F,S_0}-model $\mathcal{G}$ for a finite subset of places $\Sigma_0 \subset V$. We begin with some basic results about local fields:

Lemma 3.3.1 *Let $F' = F_m/F$ be a finite, purely inseparable extension. Then F' and F_v are linearly disjoint over F inside $\overline{F_v}$ (recall that we have fixed such an algebraic closure).*

Proof. Suppose that we know the result for $F' = F_1$. Then, proceeding by induction, F_{m-1} and F_v are linearly disjoint, the valuation v extends uniquely to a valuation v' on F_{m-1}, and $F_{m-1} \cdot F_v$ is the completion of F_{m-1} with respect to v'. Thus, F_m/F_{m-1} is of degree p, and we may replace F_v by $(F_{m-1})_{v'}$ and use the $m = 1$ case to deduce that $(F_{m-1})_{v'} = F_{m-1} \cdot F_v$ and F_m are linearly disjoint over F_{m-1}, which implies the desired result.

Thus, we may assume that $F' = F_1$. Note that the extension $F' \cdot F_v/F_v$ is either degree 1 or degree p, since $[F' \cdot F_v \colon F_v] = [F' \colon F_v \cap F] \mid p$, and F' and F_v are linearly disjoint if and only if this degree equals p. Hence, it's enough to show that $F' \cap F_v = F$. Thus, suppose that $x \in F_v$ is such that $x^p \in F$. If $F(x) \neq F$, then $F(x) = F'$, so that F_v contains all pth roots of F; in particular, $\varpi^{1/p} \in F_v$, where $\varpi \in O_{F,v}$ (the localization of O_F at v) is a v-adic uniformizer, which is clearly false. □

Now let K/F be a finite (not necessarily separable) field extension with completion K_w for $w \mid v$. The following result is important for our adelic Čech cohomology:

Lemma 3.3.2 *For any n, the natural map $O_{K_w}^{\bigotimes_{O_{F_v}} n} \to K_w^{\bigotimes_{F_v} n}$ is injective.*

Proof. The ring O_{K_w} is finite and torsion-free over the principal ideal domain O_{F_v}, and is thus free as an O_{F_v}-module. We may thus pick a basis (which is also an F_v-basis for K_w) which allows us to view the map in question as the natural map

$$(O_{F_v}^{\bigoplus m})^{\bigotimes_{O_{F_v}} n} \to (F_v^{\bigoplus m})^{\bigotimes_{F_v} n},$$

which may be rewritten as the obvious inclusion

$$O_{F_v}^{\bigoplus mn} \hookrightarrow F_v^{\bigoplus^{mn}},$$

giving the result. □

We can now prove our first adelic result. Let K/F be a finite field extension; note that the equality $\mathbb{A}_K = K \otimes_F \mathbb{A}$ implies that $\mathbb{A}_K^{\bigotimes_{\mathbb{A}} n} = (K^{\bigotimes_F n}) \otimes_F \mathbb{A}$. Let V denote the set of all places of F, let $\mathbb{A}_{K,v}$ denote the F_v-algebra $K \otimes_F F_v$, and let $O_{K,v}$ denote the O_{F_v}-algebra $O_K \otimes_{O_F} O_{F_v}$.

Proposition 3.3.3 *For any finite (not necessarily Galois) extension K/F, we have a canonical identification*

$$\mathbb{A}_K^{\bigotimes_{\mathbb{A}} n} \xrightarrow{\sim} \prod\nolimits'_{v \in V} \mathbb{A}_{K,v}^{\bigotimes_{F_v} n},$$

where the restriction is with respect to the image of the map $O_{K,v}^{\bigotimes_{O_{F_v}} n} \to \mathbb{A}_{K,v}^{\bigotimes_{F_v} n}$ (in fact, the proof will imply that this map is an inclusion).

Proof. Identifying $\mathbb{A}_K^{\bigotimes_{\mathbb{A}} n}$ with $(K^{\bigotimes_F n}) \otimes_F \mathbb{A}$, we claim that it may be identified further with the restricted product

$$\prod_{v \in V}{}' (K^{\bigotimes_F n} \otimes_F F_v), \tag{3.1}$$

where the restricted product is with respect to the image of the homomorphisms

$$O_K^{\bigotimes_{O_F} n} \otimes_{O_F} O_{F_v} \to K^{\bigotimes_F n} \otimes_F F_v, \tag{3.2}$$

via the isomorphism defined on simple tensors by sending $x \otimes (a_v)_v$ to $(x \otimes a_v)_v$, proving the Proposition. The substance of this claim is that this morphism is well-defined, i.e., that for all but finitely many v, the element $x \otimes a_v$ actually lies in the image of $O_K^{\bigotimes_{O_F} n} \otimes_{O_F} O_{F_v}$. To this end, it suffices to show that we have an isomorphism

$$O_K \otimes_{O_F} O_{F_v} \xrightarrow{\sim} \prod_{w|v} O_{K_w}$$

for any $v \in V_F$ (the analogous decomposition for $K \otimes_F F_v$ is clear). In fact, once this is done, the injectivity of the maps in (3.2) follows (using Lemma 3.3.2), and it is straightforward to verify that the claimed identification of $(K^{\bigotimes_F n}) \otimes_F \mathbb{A}$ with the ring in (3.1) is indeed an isomorphism. Letting K' be the maximal Galois subextension of K, we already know that $O_{K'} \otimes_{O_F} O_{F_v}$ is isomorphic to $\prod_{w'|v} O_{K'_{w'}}$, and so we're left with the ring $O_K \otimes_{O_{K'}} [\prod_{w'|v} O_{K'_{w'}}]$.

We claim that the natural map $O_K \otimes_{O_{K'}} O_{K'_{w'}} \to O_{K_w}$ (for w the unique extension of w' to K) is an isomorphism. For surjectivity, note that by the proof of Lemma 3.2.7, we have $O_K = O_{K'}^{(1/p^m)}$, where p^m is $[K\colon K']$. We know that $O_{K'_{w'}}$ spans O_{K_w} over $O_{K'}^{(1/p^m)}$, since the ring $O_{K'_{w'}} \cdot O_{K'}^{(1/p^m)}$ is finitely-generated over the complete discrete valuation ring $O_{K'_{w'}}$, using that $O_{K'}^{(1/p^m)}$ is finite over $O_{K'}$ by the finiteness of the relative Frobenius morphism (by [Stacks], OCC6, using that $O_{K'}$ is of finite type over $\mathbb{F}_q$, being the coordinate ring of an affine open subscheme of a smooth curve over $\mathbb{F}_q$), and hence is complete as a topological ring, contains O_K, and thus must be the w-adic completion O_{K_w}. Injectivity immediately follows from the linear disjointness given by Lemma 3.3.1. □

We immediately obtain:

Corollary 3.3.4 *For any finite (not necessarily Galois) extension K/F, the ring $\mathbb{A}_K^{\bigotimes_{\mathbb{A}} n}$ may be canonically identified as the direct limit over any cofinal system of finite subsets Σ of V of products as follows:*

$$\mathbb{A}_K^{\bigotimes_{\mathbb{A}} n} = \varinjlim_{\Sigma} [\prod_{v \in \Sigma} \mathbb{A}_{K,v}^{\bigotimes_{F_v} n} \times \prod_{v \notin \Sigma} O_{K,v}^{\bigotimes_{O_{F_v}} n}].$$

This allows to decompose groups of adelic Čech cochains:

Corollary 3.3.5 *For any finite (not necessarily Galois) extension K/F, we have a canonical identification*

$$G(\mathbb{A}_K^{\bigotimes_{\mathbb{A}} n}) = \varinjlim_{\Sigma_0 \subset \Sigma} [\prod_{v \in \Sigma} G(\mathbb{A}_{K,v}^{\bigotimes_{F_v} n}) \times \prod_{v \notin \Sigma} \mathcal{G}(O_{K,v}^{\bigotimes_{O_{F_v}} n})].$$

Proof. This is immediate from the our Corollary 3.3.4 and [Čes16, Lemma 2.4]. □

In fact, since the natural map $O_{K,v}^{\bigotimes_{O_{F_v}} n} \to \mathbb{A}_{K,v}^{\bigotimes_{F_v} n}$ is injective (and so the same is true for $\mathcal{G}(O_{K,v}^{\bigotimes_{O_{F_v}} n}) \to G(\mathbb{A}_{K,v}^{\bigotimes_{F_v} n})$), we actually get a restricted product decomposition

$$G(\mathbb{A}_K^{\bigotimes_{\mathbb{A}} n}) = \textstyle\prod'_{v\in V} G(\mathbb{A}_{K,v}^{\bigotimes_{F_v} n}),$$

where the restriction is with respect to the subgroups $\mathcal{G}(O_{K,v}^{\bigotimes_{O_{F_v}} n})$. If $\overline{\mathbb{A}} := \overline{F} \otimes_F \mathbb{A}$, we immediately obtain:

Corollary 3.3.6 *We have a canonical identification*

$$G(\overline{\mathbb{A}}^{\bigotimes_{\mathbb{A}} n}) \xrightarrow{\sim} \varinjlim_{K/F} \textstyle\prod'_{v\in V} G(\mathbb{A}_{K,v}^{\bigotimes_{F_v} n}),$$

where the direct limit is over all finite extensions K/F.

We give one more result which will be useful for Čech-cohomological computations:

Proposition 3.3.7 *For K/F a finite extension, the above restricted product decomposition of $G(\mathbb{A}_K^{\bigotimes_{\mathbb{A}} n})$ identifies the subgroup of Čech n-cocycles inside $G(\mathbb{A}_K^{\bigotimes_{\mathbb{A}} n})$ with elements of the kernel of the map*

$$\textstyle\prod'_{v\in V} G(\mathbb{A}_{K,v}^{\bigotimes_{F_v} n}) \longrightarrow \prod'_{v\in V} G(\mathbb{A}_{K,v}^{\bigotimes_{F_v} n+1})$$

given by the Čech differentials with respect to the cover $\mathbb{A}_{K,v}/F_v$ on the $G(\mathbb{A}_{K,v}^{\bigotimes_{F_v} n})$-factors and the Čech differentials with respect to the cover $O_{K,v}/O_{F_v}$ on the $\mathcal{G}(O_{K,v}^{\bigotimes_{O_{F_v}} n})$-factors (note that these differentials land in the desired restricting subgroups, so this is well-defined).

Proof. It's enough to check that the restricted product identifications are compatible with the three inclusion maps $p_i \colon \mathbb{A}_K^{\bigotimes_{\mathbb{A}} n} \to \mathbb{A}_K^{\bigotimes_{\mathbb{A}} n+1}$, $p_i^v \colon \mathbb{A}_{K,v}^{\bigotimes_{F_v} n} \to \mathbb{A}_{K,v}^{\bigotimes_{F_v} n+1}$, and $p_i^{v,\circ} \colon O_{K,v}^{\bigotimes_{O_{F_v}} n} \to O_{K,v}^{\bigotimes_{O_{F_v}} n+1}$ for $1 \leq i \leq n+1$, which is straightforward. □

We now move on from examining adelic tensor products and look at some cohomological results concerning covers of $\mathbb{A}$, analogous to the results proved in the previous subsection for covers of $O_{F,S}$. Set $\overline{\mathbb{A}}_v := \overline{F} \otimes_F F_v$. For notational convenience, the symbol H^i will denote H^i_{fppf}.

Lemma 3.3.8 *For M a multiplicative F-group scheme, we have $H^n(\overline{\mathbb{A}}_v^{\bigotimes_{F_v} k}, M) = 0$ for all $n, k \geq 1$.*

Proof. For E'/F a finite algebraic extension with Galois and purely inseparable subextensions E, F' respectively, note that we have a sequence of isomorphisms

$$(E' \otimes_F F_v)^{\bigotimes_{F_v} k} \xrightarrow{\sim} [F' \otimes_F (E \otimes_F F_v)]^{\bigotimes_{F_v} k} \xrightarrow{\sim} [\prod_{w|v} E'_{w'}]^{\bigotimes_{F_v} k} \xrightarrow{\sim} \prod_{w_1,\ldots,w_k|v} \bigotimes_{F_v}^{i=1,\ldots,k} E'_{w'_i},$$

where $E'_{w'}$ is the completion of E' with respect to the unique extension w' of the valuation w on E to the purely inseparable extension E', for all $w \mid v$ in V_E, and in the third term above, F_v is embedded into the direct product diagonally. and so we obtain an identification

$$H^n((E' \otimes_F F_v)^{\bigotimes_{F_v} k}, M) \xrightarrow{\sim} \prod_{w_1,\ldots,w_n|v_F} H^n((E'_{w'_i})^{\bigotimes_{F_v} k}, M).$$

Moreover, for K'/E' two such extensions, the inductive map $(E' \otimes_F F_v)^{\bigotimes_{F_v} k} \to (K' \otimes_F F_v)^{\bigotimes_{F_v} k}$ gets translated to the map on the corresponding products defined by the product over all k-tuples $(w_1, \ldots, w_k)$ of the maps

$$\bigotimes_{F_v}^{j=1,\ldots,k} E'_{w'_j} \to \prod_{\tilde{w}_1,\ldots\tilde{w}_k;\tilde{w}_j|w_j\forall j} \bigotimes_{F_v}^{j=1,\ldots,k} K'_{\tilde{w}'_j}$$

given in the obvious way. The upshot is that it's enough to show that each direct limit

$$\varinjlim_{K'/F} H^n(K'_{(w_K)'}/F_v, M)$$

vanishes, where $\{w_K\}$ is a coherent system of places lifting v (equivalent to fixing a place $\dot{v}$ on F^{sep} lifting v). But each direct limit of this form is isomorphic to $H^n(\overline{F_v}, M)$, which we know vanishes. □

Fix an embedding $\overline{F} \to \overline{F_v}$, which is equivalent to picking a place $\dot{v} \in V_{F^{\text{sep}}}$ lying above v. Note that we have a homomorphism of F_v-algebras $h\colon \overline{F_v} \to \overline{\mathbb{A}}_v$ defined as follows: For any E'/F a finite algebraic extension, we may define a ring homomorphism

$$E' \cdot F_v \to E' \otimes_F F_v$$

by writing $E' = E(x^{1/p^m})$ for $x \in F$, where E/F is a finite Galois extension, and then using linear disjointness to write $E' = F(x^{1/p^m}) \otimes_F E$. Note that, even more than this, $F(x^{1/p^m})$ and $E \cdot F_v$ are linearly disjoint over F inside $\overline{F_v}$, so that our desired ring homomorphism may be obtained from any homomorphism $E \cdot F_v \to E \otimes_F F_v$ by applying the functor $F(x^{1/p^m}) \otimes_F -$. But such a homomorphism may be obtained via the composition

$$E \cdot F_v \to \prod_{w|v} E_w \xrightarrow{\sim} E \otimes_F F_v,$$

where the first map is the diagonal embedding induced by the fixed embedding $E \cdot F_v \to \overline{F_v}$ and a choice of section $\Gamma_{E/F}/\Gamma_{E/F}^{v_E} \to \Gamma_{E/F}$, where $v_E := \dot{v}\big|_E$ and $\Gamma_{E/F}^{v_E}$ is the decomposition group of v_E, and the second map is the usual isomorphism from basic number theory. If we pick our sections to come from a section $\Gamma_F/\Gamma_F^{\dot{v}} \to \Gamma_F$, then it is clear that these homomorphisms splice to give the desired map h.

Corollary 3.3.9 *For any $k \in \mathbb{N}$ and multiplicative F-group M, the map h induces an isomorphism, called the "Shapiro isomorphism,"*

$$S_v^k \colon \check{H}^k(\overline{F_v}/F_v, M) \to \check{H}^k(\overline{\mathbb{A}}_v/F_v, M).$$

Proof. Note that for any finite algebraic field extension E'/F, the extension of rings $F_v \to F_v \otimes_F E'$ is fppf. Thus, we get a natural map

$$\check{H}^k(\overline{\mathbb{A}}_v/F_v, M) \xrightarrow{\sim} \varinjlim_{E'/F} \check{H}^k((E' \otimes_F F_v)/F_v, M) \to H^k(F_v, M)$$

via the natural comparison homomorphism $\check{H}^k_{\text{fppf}}(F_v, M) \to H^k(F_v, M)$ (from [Stacks, Lemma 03AX]). By taking the direct limit of the spectral sequence from [Stacks, Lemma 03AZ], we deduce that the above map $\check{H}^k(\overline{\mathbb{A}}_v/F_v, M) \to H^k(F_v, M)$ is an isomorphism, since the cohomology groups $H^j(\overline{\mathbb{A}}_v^{\bigotimes_{F_v} m}, M)$ vanish for all $j, m \geq 1$ by Lemma 3.3.8. Now the commutative diagram

$$\begin{array}{ccc}
\check{H}^k(\overline{F_v}/F_v, M) & \xrightarrow{S_v^k} & \check{H}^k(\overline{\mathbb{A}}_v/F_v, M) \\
& \searrow^{\sim} \quad \swarrow_{\sim} & \\
& \check{H}^k_{\text{fppf}}(F_v, M) & \\
& \downarrow \wr & \\
& H^k(F_v, M) &
\end{array}$$

implies that S_v^k is an isomorphism. □

We conclude this subsection by discussing the independence of S_v^2 on the section $\Gamma_F/\Gamma_F^{\dot{v}} \to \Gamma_F$ used to construct h.

Lemma 3.3.10 *Let s_v and s'_v be two choices of sections, M a multiplicative F-group, and $\dot{S}^2_v$, $\dot{S}'^2_v$ the corresponding Shapiro homomorphisms $M(\overline{F_v}^{\bigotimes_{F_v} 3}) \to M(\overline{\mathbb{A}}_v^{\bigotimes_{F_v} 3})$. Then the induced maps on Čech cohomology from $\check{H}^2(\overline{F_v}/F_v, M)$ to $\check{H}^2(\overline{\mathbb{A}}_v/F_v, M)$ are the same.*

Proof. Since the Shapiro homomorphisms are constructed via the direct limit over finite algebraic extensions, it's enough to prove that, for any fixed $x \in M((E'_v)^{\bigotimes_{F_v} 3})$ a 2-cocycle, E'/F a finite extension of fields, there is a 1-cochain $c \in M((E' \otimes_F F_v)^{\bigotimes_{F_v} 2})$ such that $dc = \dot{S}^2_v(x) \cdot \dot{S}'^2_v(x)^{-1}$, and that if we have a inductive system $\{x_{E'}\}_{E'}$ of such 2-cocycles, as E'/F ranges over an exhaustive tower of finite extensions, then the system $\{c_{E'}\}_{E'}$ is also inductive. We will construct each $c_{E'}$ explicitly using x (it will be useful later to have an explicit cochain to work with).

As above, we let E/F (resp. F'/F) denote the maximal Galois (resp. purely inseparable) subextension of E'/F, set $E_v := E_{v_E}$, and denote the extension of v_E to E' by v'. For $w \mid v$ in V_E, denote by $r_w, \bar{r}_w$ the corresponding isomorphisms $E'_{v'} \xrightarrow{\sim} E'_{w'}$ (induced by applying $F' \otimes_F -$ to the isomorphisms $E_v \xrightarrow{\sim} E_w$ defined by our sections). We define

$$c \in \prod_{w_{i_1}, w_{i_2} | v_F} M(E'_{w'_{i_1}} \otimes_{F_v} E'_{w'_{i_2}})$$

to be given on the (w_{i_1}, w_{i_2})-factor by

$$(r_{w_{i_1,1}} \cdot \bar{r}_{w_{i_1,3}} \otimes r_{w_{i_2,2}})(x) \cdot (\bar{r}_{w_{i_1,2}} \otimes r_{w_{i_2,1}} \cdot \bar{r}_{w_{i_2,3}})(x)^{-1},$$

where $r_{w_{i_j},k}$ denotes that the source is the kth tensor factor of $(E'_{v'})^{\bigotimes_{F_v} 3}$, $1 \leq k \leq 3$. It is clear that such a system of 1-cochains $\{c_{E'}\}$ is inductive if the system $\{x_{E'}\}$ is. We will do an involved Čech computation. Recall that $\dot{S}^2_v$, $\dot{S}'^2_v$ are group homomorphisms

$$M((E'_{v'})^{\bigotimes_{F_v} 3}) \to \prod_{w_{i_1}, w_{i_2}, w_{i_3} | v_F} M(E'_{w'_{i_1}} \otimes_{F_v} E'_{w'_{i_2}} \otimes_{F_v} E'_{w'_{i_2}}).$$

To show that $dc = \dot{S}^2_v(x) \cdot \dot{S}'^2_v(x)^{-1}$, we may focus on a fixed $(w_{i_1}, w_{i_2}, w_{i_3})$-factor of the right-hand side. In this factor, the differential of c is given by the six-term product

$$(1 \otimes r_{w_{i_2,1}} \cdot \bar{r}_{w_{i_2,3}} \otimes r_{w_{i_3,2}})(x) \cdot (r_{w_{i_1,1}} \cdot \bar{r}_{w_{i_1,3}} \otimes 1 \otimes r_{w_{i_3,2}})(x)^{-1} \cdot (r_{w_{i_1,1}} \cdot \bar{r}_{w_{i_1,3}} \otimes r_{w_{i_2,2}} \otimes 1)(x)$$

$$\cdot (1 \otimes \bar{r}_{w_{i_2,2}} \otimes r_{w_{i_3,1}} \cdot \bar{r}_{w_{i_3,3}})(x)^{-1} \cdot (\bar{r}_{w_{i_1,2}} \otimes 1 \otimes r_{w_{i_3,1}} \cdot \bar{r}_{w_{i_3,3}})(x) \cdot (\bar{r}_{w_{i_1,2}} \otimes r_{w_{i_2,1}} \cdot \bar{r}_{w_{i_2,3}} \otimes 1)(x)^{-1}.$$

The key fact here is that $x \in M(E_v^{\bigotimes_{F_v} 3})$ is a 2-cocycle, not just a 2-cochain. Thus, we have that the factor $(1 \otimes r_{w_{i_2,1}} \cdot \bar{r}_{w_{i_2,3}} \otimes r_{w_{i_3,2}})(x)$ equals

$$(\bar{r}_{w_{i_1,1}} \otimes r_{w_{i_2,3}} \otimes \bar{r}_{w_{i_3,2}})(x) \cdot (\bar{r}_{w_{i_1,2}} \otimes r_{w_{i_2,1}} \cdot \bar{r}_{w_{i_2,3}} \otimes 1)(x) \cdot (\bar{r}_{w_{i_1,2}} \otimes \bar{r}_{w_{i_2,1}} \otimes r_{w_{i_3,3}})(x)^{-1}. \quad (3.3)$$

To see this, note that

$$(1 \otimes \mathrm{id}_1 \otimes \mathrm{id}_2 \otimes \mathrm{id}_3)(x) \cdot (\mathrm{id}_1 \otimes 1 \otimes \mathrm{id}_2 \otimes \mathrm{id}_3)(x)^{-1} \cdot (\mathrm{id}_1 \otimes \mathrm{id}_2 \otimes 1 \otimes \mathrm{id}_3)(x) \cdot (\mathrm{id}_1 \otimes \mathrm{id}_2 \otimes \mathrm{id}_3 \otimes 1)(x)^{-1} = 1$$

(inside the group $M(E_v^{\bigotimes_{F_v} 3})$), and now applying $(\mathrm{id}_2 \otimes \mathrm{id}_1 \cdot \mathrm{id}_4 \otimes \mathrm{id}_3) \circ (r_{w_{i_2}} \otimes \bar{r}_{w_{i_1}} \otimes r_{w_{i_3}} \otimes \bar{r}_{w_{i_2}})$ to the above expression gives the desired equality. We will leave the checking of similar equalities to the reader throughout the proof. Note that the second factor in (3.3) cancels with the last factor in the main equation. Next, we may rewrite the first term of (3.3) as

$$(1 \otimes \bar{r}_{w_{i_2,2}} \otimes r_{w_{i_3,1}} \cdot \bar{r}_{w_{i_3,3}})(x) \cdot (\bar{r}_{w_{i_1,1}} \otimes \bar{r}_{w_{i_2,2}} \otimes \bar{r}_{w_{i_3,3}})(x)^{-1} \cdot (\bar{r}_{w_{i_1,1}} \otimes 1 \otimes r_{w_{i_3,2}} \cdot \bar{r}_{w_{i_3,3}})(x). \quad (3.4)$$

We may also replace $(r_{w_{i_1,1}} \cdot \bar{r}_{w_{i_1,3}} \otimes r_{w_{i_2,2}} \otimes 1)(x)$ from the main equation by the expression

$$(\bar{r}_{w_{i_1,2}} \otimes r_{w_{i_2,1}} \otimes r_{w_{i_3,3}})(x) \cdot (r_{w_{i_1,1}} \cdot \bar{r}_{w_{i_1,2}} \otimes 1 \otimes r_{w_{i_3,3}})(x)^{-1} \cdot (r_{w_{i_1,1}} \otimes r_{w_{i_2,2}} \otimes r_{w_{i_3,3}})(x),$$

reducing us to showing the equality

$$\begin{aligned} &(\bar{r}_{w_{i_1,1}} \otimes 1 \otimes r_{w_{i_3,2}} \cdot \bar{r}_{w_{i_3,3}})(x) \cdot (r_{w_{i_1,1}} \cdot \bar{r}_{w_{i_1,2}} \otimes 1 \otimes r_{w_{i_3,3}})(x)^{-1} \\ \cdot &(\bar{r}_{w_{i_1,2}} \otimes 1 \otimes r_{w_{i_3,1}} \cdot \bar{r}_{w_{i_3,3}})(x) \cdot (r_{w_{i_1,1}} \cdot \bar{r}_{w_{i_1,3}} \otimes 1 \otimes r_{w_{i_3,2}})(x)^{-1} = 1. \end{aligned} \quad (3.5)$$

Replacing the third factor of (3.5) by the expression

$$(r_{w_{i_1,1}} \cdot \bar{r}_{w_{i_1,3}} \otimes 1 \otimes \bar{r}_{w_{i_3,2}})(x) \cdot (r_{w_{i_1,1}} \otimes 1 \otimes r_{w_{i_3,2}} \cdot \bar{r}_{w_{i_3,3}})(x)^{-1} \cdot (r_{w_{i_1,1}} \cdot \bar{r}_{w_{i_1,2}} \otimes 1 \otimes \bar{r}_{w_{i_3,3}})(x)$$

reduces (3.5) to the equality

$$(\bar{r}_{w_{i_1,1}} \otimes 1 \otimes r_{w_{i_3,2}} \cdot \bar{r}_{w_{i_3,3}})(x) \cdot (r_{w_{i_1,1}} \cdot \bar{r}_{w_{i_1,2}} \otimes 1 \otimes r_{w_{i_3,3}})(x)^{-1}$$

$$\cdot (r_{w_{i_1,1}} \otimes 1 \otimes r_{w_{i_3,2}} \cdot \bar{r}_{w_{i_3,3}})(x)^{-1} \cdot (r_{w_{i_1,1}} \cdot \bar{r}_{w_{i_1,2}} \otimes 1 \otimes \bar{r}_{w_{i_3,3}})(x) = 1,$$

which follows easily from the fact that x is a 2-cocycle. □

3.4 The unbalanced cup product

For S/R a fixed finite flat extension (not necessarily étale), S'/R a Galois extension contained in S, and G a commutative R-group scheme, define the group $C^n(S/R, G)$ to be $G(S^{\bigotimes_R (n+1)})$, and $C^{n,1}(S/R, S', G)$ to be the subgroup $G(S^{\bigotimes_R n} \otimes_R S')$. Our goal is to define an unbalanced cup product

$$C^{n,1}(S/R, S', G) \times C^{-1}_{\text{Tate}}(\Gamma', H(S')) \xrightarrow{\underset{S'/R}{\sqcup}} C^{n-1}(S/R, J)$$

for two commutative R-group schemes G, H and R-pairing $P \colon G \times H \to J$ for J another commutative R-group scheme, where as above $C^{-1}_{\text{Tate}}(\Gamma', H(S')) = H(S')$ and $\Gamma' := \text{Aut}_{R\text{-alg}}(S')$.

We have a homomorphism of R-algebras $\lambda \colon S^{\bigotimes_R n} \otimes_R S' \to \prod_{\Gamma'} S^{\bigotimes_R n}$ defined on simple tensors by

$$a_{i,0} \otimes \cdots \otimes a_{i,n} \mapsto (a_{i,0} \otimes \cdots \otimes a_{i,n-1} \, {}^{\sigma}a_{i,n})_{\sigma \in \Gamma'}.$$

Moreover, for any R-group scheme J, we have a canonical identification $J(\prod_{\Gamma'} S^{\bigotimes_R n}) \to \prod_{\Gamma'} J(S^{\bigotimes_R n})$; we then define, for $a \in G(S^{\bigotimes_R n} \otimes_R S')$ and $b \in H(S')$,

$$a \underset{S'/R}{\tilde{\sqcup}} b = \lambda(a \cup b^{(0)}) \in \prod_{\Gamma'} J(S^{\bigotimes_R n}).$$

In the above formula we are using the fppf cup product as defined in [Sha64], §3, and $b^{(0)}$ denotes the element $b \in H(S')$ viewed as a 0-cochain.

We now apply the group homomorphism $N \colon \prod_{\Gamma'} J(S^{\bigotimes_R n}) \to J(S^{\bigotimes_R n})$ obtained by taking the sum of all elements on the left-hand side, and the resulting pairing

$$C^{n,1}(S/R, S', G) \times C^{-1}_{\text{Tate}}(\Gamma', H(S')) \to J(S^{\bigotimes_R n})$$

is $\mathbb{Z}$-bilinear. Indeed,

$$(a + a') \underset{S'/R}{\tilde{\sqcup}} b = \lambda[(a + a') \cup b^{(0)}] = \lambda(a \cup b^{(0)} + a' \cup b^{(0)}) = \lambda(a \cup b^{(0)}) + \lambda(a' \cup b^{(0)}).$$

This will be our desired pairing, denoted by $a \underset{S'/R}{\sqcup} b$.

We will now prove some basic properties of this pairing. The first order of business is to show that this agrees with the analogous pairing from [Kal16] in the case that S/R is also finite Galois, with $\text{Aut}_{R\text{-alg}}(S) =: \Gamma$. There is a simple way to compare Čech cohomology and Galois cohomology in such a case: for any commutative R-group G, there is a comparison homomorphism

$$C^n(S/R, G) \to C^n(\Gamma, G(S))$$

given as follows: We have a homomorphism of R-algebras

$$t\colon S^{\bigotimes_F(n+1)} \to \prod_{\underline{\sigma}\in\Gamma^n} S_{\underline{\sigma}}, \tag{3.6}$$

induced by the map on simple tensors

$$a_0 \otimes \cdots \otimes a_n \mapsto (a_0\, {}^{\sigma_1}a_1\, {}^{(\sigma_1\sigma_2)}a_2 \ldots {}^{(\sigma_1\ldots\sigma_n)}a_n)_{(\sigma_1,\ldots,\sigma_n)}.$$

We immediately get a homomorphism $c : G(S^{\bigotimes_R(n+1)}) \to G(\prod_{\underline{\sigma}\in\Gamma^n} S_{\underline{\sigma}}) = C^n(\Gamma, G(S))$, where the last equality is the obvious identification. Passing to cohomology, this induces a homomorphism $\check{H}^n(S/R, G) \to H^n(\Gamma, G(S))$. Note that all of these maps are isomorphisms if S/R is a finite Galois extension of fields. This comparison map also respects our special subgroups; that is to say, the homomorphism $G(S^{\bigotimes_R(n+1)}) \to \prod_{\Gamma^n} G(S)$ maps $C^{n,1}(S/R, S', G)$ into $C^{n,1}(\Gamma, \Gamma', G(S))$, which again is an isomorphism when R and S are fields.

Proposition 3.4.1 *When S/R is finite Galois, the unbalanced cup product $a \underset{S'/R}{\sqcup} b$ agrees with the unbalanced cup-product from [Kal16] after applying the comparison homomorphism* (3.6).

Proof. Recall that the pairing from [Kal16] sends $a \in C^{n,1}(\Gamma, \Gamma', G(S))$ and $b \in H(S')$ to the $(n-1)$-cochain

$$(\sigma_1, \ldots, \sigma_{n-1}) \mapsto P[\Sigma_{\sigma\in\Gamma'} a(\sigma_1, \ldots, \sigma_{n-1}, \sigma) \otimes {}^{\sigma_1\ldots\sigma_{n-1}\sigma}b] \in J(S),$$

where we are abusing notation and using P to denote the map $G(S) \otimes_{\mathbb{Z}} H(S) \to J(S)$ induced by the pairing P.

In the Čech setting, the point $a \cup b^{(0)}$ corresponds to the R-algebra homomorphism $R[J] \to R[G] \otimes_F R[H] \to S^{\bigotimes_R n+1}$ given by post-composing $P^\sharp$ by the map determined by a and $1 \otimes b$ (identifying the points with their ring homomorphisms). Then the map λ sends this point to the map $R[J] \to R[G] \otimes_F R[H] \to \prod_{\Gamma'} S^{\bigotimes_R n}$ given by post-composing $P^\sharp$ by the map $R[G] \otimes_F R[H] \to \prod_{\sigma\in\Gamma'} S^{\bigotimes_R n}$ determined by $\lambda \circ a$ and $\lambda \circ (1 \otimes b)$. It is straightforward to verify that via the composition $\prod_{\Gamma'} J(S^{\bigotimes_R n}) \to \prod_{\Gamma'} \prod_{\Gamma^{n-1}} J(S) \to \prod_{\Gamma^{n-1}} J(S)$, we obtain the $(n-1)$-cochain of Γ valued in $J(S)$ sending $(\sigma_1, \ldots, \sigma_{n-1})$ to $\sum_{\sigma\in\Gamma'} P(a(\sigma_1, \ldots, \sigma_{n-1}, \sigma), {}^{\sigma_1\ldots\sigma_{n-1}\sigma}b) \in J(S)$, as desired. □

We now prove an elementary result stating how this map behaves with respect to Čech differentials.

Lemma 3.4.2 *For $a \in C^{n,1}(S/R, S', J)$, we have $N\lambda(da) = (\#\Gamma')(-1)^{n+1}a + d(N\lambda a)$.*

Proof. First, note that d maps $C^{n,1}(S/R, S', J)$ into $C^{n+1,1}(S/R, S', J)$, so the statement makes sense. Start with the $S^{\bigotimes_R n} \otimes_R S'$-point $a\colon R[J] \to S^{\bigotimes_R n} \otimes_R S'$. Applying the differential yields the sum as i ranges from 0 to $n+1$ of $(-1)^i$ times the $S^{\bigotimes_R (n+1)} \otimes_R S'$-point $p_{\hat{i}} \circ a\colon R[J] \to S^{\bigotimes_R (n+1)} \otimes_R S'$, where $0 \leq i \leq n+1$. Note that $\lambda \circ p_{\hat{i}} \circ a$ equals $(\lambda \circ a)_i := p_{\hat{i}} \circ \lambda \circ a$ for $i \neq n+1$ and $\lambda \circ p_{\widehat{n+1}} \circ a = (a)_{\sigma \in \Gamma'}$. We conclude that

$$N\lambda(da) = (\#\Gamma')(-1)^{(n+1)}a + N[\sum_{0 \leq i \leq n} (-1)^i (\lambda \circ a)_i]$$

which equals $(\#\Gamma')(-1)^{n+1}a + d(N\lambda a)$. □

We now reach the key property of our unbalanced cup product.

Proposition 3.4.3 *For $a \in C^{n,1}(S/R, S', G)$ and $b \in C^{-1}_{Tate}(\Gamma', H(S'))$, we have*

$$d(a \underset{S'/R}{\sqcup} b) = (da) \underset{S'/R}{\sqcup} b + (-1)^n (a \cup db).$$

Proof. The left-hand side equals $d[N(\lambda(a \cup b^{(0)}))] = (\#\Gamma')(-1)^n(a \cup b^{(0)}) + N\lambda(d(a \cup b^{(0)}))$, by the above lemma. This in turn equals $(\#\Gamma')(-1)^n(a \cup b^{(0)}) + N\lambda[(da) \cup b^{(0)} + (-1)^n(a \cup db^{(0)})]$ (by [Sha64], §3). Thus, the desired equality reduces to

$$(\#\Gamma')(a \cup b^{(0)}) + N\lambda(a \cup db^{(0)}) = a \cup db.$$

Now, $db^{(0)} = -p_1(b) + p_2(b) \in H(S' \otimes_R S')$, so that $\lambda(a \cup db^{(0)}) = \lambda(a \cup -p_1(b)) + \lambda(a \cup p_2(b))$, and $\lambda(a \cup -p_1(b)) = (a \cup -b^{(0)})_\sigma$, so all we need to show is $N\lambda(a \cup p_2(b)) = a \cup db$. Note that $\lambda(a \cup p_2(b)) = (a \cup {}^\sigma b^{(0)})_\sigma$, so applying N gives the desired result. □

The setting we will be concerned with in this paper is the case where $G = \mathbb{G}_m$, $H = \underline{X}_*(J)$ is the étale R-group scheme associated to the cocharacter module of an R-torus J, and the pairing $P\colon \mathbb{G}_m \times \underline{X}_*(J) \to J$ is the canonical one; we switch to multiplicative rather than additive notation for our abelian groups here. The following two elementary results will be used repeatedly in what follows, so we record them here:

Lemma 3.4.4 *For $f \in X_*(J)$ and $x \in \mathbb{G}_m(S^{\bigotimes_R n})$, we have $f \cup x = [p_1^* f](x)$. Furthermore, if we take two multiplicative groups M, N, both split over S, and look at the R-pairing $M \times \underline{Hom}(M, N) \to N$, then for $\phi \in Hom(M, N)$ defined over R, we have $x \cup \phi = \phi \cup x = \phi(x)$ for all $x \in M(S^{\bigotimes_R n})$.*

Proof. This is a straightforward computation. □

Lemma 3.4.5 *If $g \in X_*(J)^\Gamma$, then for $f \in X_*(\mathbb{G}_m)$ and $x \in C^{n,1}(S/R, S, \mathbb{G}_m)$, we have*

$$x \underset{S/R}{\sqcup} (g \circ f) = (x \underset{S/R}{\sqcup} f) \cup g.$$

Proof. We have that $x \cup (g \circ f) = x \cup (f \cup g) = (x \cup f) \cup g = g(x \cup f)$, where we are using the fact that $f \cup g = g \circ f$, and that since $g \in X_*(J)^\Gamma$, we have by Lemma 3.4.4 that $(x \cup f) \cup g = g(x \cup f)$. Thus, $\lambda[x \cup (g \circ f)] = \lambda[g(x \cup f)] = (\prod_\sigma g)[\lambda(x \cup f)]$, where this last equality follows from the fact that g is defined over R. Finally, applying N gives that $x \underset{S/R}{\sqcup} (g \circ f) = N(\prod_\sigma g)[\lambda(x \cup f)] = g(x \underset{S/R}{\sqcup} f) = (x \underset{S/R}{\sqcup} f) \cup g$, where this last equality comes once again from Lemma 3.4.4. $\square$

CHAPTER 4

The Local Gerbe

4.1 The multiplicative pro-algebraic group u

For a finite Galois extension E/F, we consider the algebraic group $R_{E/F}[n] := \text{Res}_{E/F}\mu_n$, which is a multiplicative F-group with character group $X^*(R_{E/F}[n]) = \mathbb{Z}/n\mathbb{Z}[\Gamma_{E/F}]$ with $\Gamma_{E/F}$ acting by left-translation. We have the diagonal embedding $\mu_n \to R_{E/F}[n]$ induced by the Γ-homomorphism $\mathbb{Z}/n\mathbb{Z}[\Gamma_{E/F}] \to \mathbb{Z}/n\mathbb{Z}$ defined by $[\gamma] \mapsto 1$. The kernel of this homomorphism will be denoted by $\mathbb{Z}/n\mathbb{Z}[\Gamma_{E/F}]_0$, and is the character group of the multiplicative F-group $R_{E/F}[n]/\mu_n$, which will denote by $u_{E/F,n}$. Note that $u_{E/F,n}$ is smooth if and only if n is coprime to the characteristic of F.

If K/F is a finite Galois extension containing E and m is a multiple of n, then the injective morphism of Γ-modules $\mathbb{Z}/n\mathbb{Z}[\Gamma_{E/F}] \to \mathbb{Z}/m\mathbb{Z}[\Gamma_{K/F}]$ induced by the inclusion $\mathbb{Z}/n\mathbb{Z} \hookrightarrow \mathbb{Z}/m\mathbb{Z}$ and the map

$$[\gamma] \mapsto \sum_{\substack{\sigma\in\Gamma_{K/F} \\ \sigma\mapsto\gamma}} [\sigma]$$

induces an epimorphism $R_{K/F}[m] \to R_{E/F}[n]$. This maps $R_{K/F}[m]_0$ to $R_{E/F}[n]_0$ and thus induces an epimorphism $u_{K/F,m} \to u_{E/F,n}$. We define the pro-algebraic multiplicative group u to be the limit

$$u := \varprojlim u_{E/F,n}$$

taken over the index category $\mathcal{I}$ whose objects are tuples $(E/F, n)$ as n ranges through $\mathbb{N}$ and E/F ranges over all finite Galois extensions of F, and where there is at most one morphism $(K/F, m) \to (E/F, n)$ in $\mathcal{I}$ and it exists if and only if $E \subset K$ and $n \mid m$. For every $(E/F, n)$, the canonical map $u \to u_{E/F,n}$ is an epimorphism. Note that u is a commutative affine group scheme over F; when taking the cohomology of u, we view it as a commutative fpqc group sheaf on $(\text{Sch}/F)_{\text{fpqc}}$ (and thus also a sheaf on $(\text{Sch}/F)_{\text{fppf}}$).

For a finite multiplicative algebraic group Z over F, any F-homomorphism $u \to Z$ factors through an F-homomorphism $u_{E/F,n} \to Z$ for some $(E/F, n) \in \mathcal{I}$. We also have the "evaluation at e" map $\delta_e : \mu_n \to u_{E/F,n}$, which is induced by the corresponding morphism of character groups from $\mathbb{Z}/n\mathbb{Z}[\gamma_{E/F}]_0$ to $\mathbb{Z}/n\mathbb{Z}$ sending $\sum_{\gamma \in \Gamma_{E/F}} c_\gamma[\gamma]$ to c_e. It's easy to check that, for E splitting Z, we have an isomorphism

$$\mathrm{Hom}_F(u_{E/F,n}, Z) \to \mathrm{Hom}(\mu_n, Z)^{N_{E/F}}, \qquad f \mapsto f \circ \delta_e, \tag{4.1}$$

where the superscript $N_{E/F}$ denotes the kernel of the norm map and for two algebraic F-groups A, B, $\mathrm{Hom}(A, B)$ denotes the abelian group $\mathrm{Hom}_{F^s}(A_{F^s}, B_{F^s})$, which carries a natural Γ-action.

Before we analyze the cohomology of u, it's necessary to recall some facts about the cohomology of profinite groups. By [RZ00], 2.2, the left-exact functor $\varprojlim$ from the abelian category of inverse systems of abelian profinite groups with continuous transition maps to the abelian category of abelian profinite groups is exact. As a consequence, its associated first derived functor, $\varprojlim^{(1)}$, sends everything to the trivial group.

Proposition 4.1.1 *We have the following results about $H^1(F, u)$ and $H^2(F, u)$:*

1. *The projective systems $\{H^1(F, \mu_n)\}, \{H^1(F, R_{E/F}[n])\}, \{H^1(F, u_{E/F,n})\}, \{H^2(F, \mu_n)\}$ (all indexed by $\mathcal{I}$) can be given the structure of projective systems of abelian profinite groups with continuous transition maps, such that, for all n, the associated long exact sequence in cohomology associated to the short exact sequence of group sheaves*

$$0 \longrightarrow \mu_n \longrightarrow R_{E/F}[n] \longrightarrow u_{E/F,n} \longrightarrow 0,$$

   consists entirely of continuous maps, up until the map $H^2(F, \mu_n) \to H^2(F, R_{E/F}[n])$ (we have not specified a topology on the right-hand group);

2. *We have a canonical isomorphism $H^1(F, u) = \varprojlim H^1(F, u_{E/F,n})$;*

3. *We have a canonical isomorphism $H^2(F, u) = \varprojlim H^2(F, u_{E/F,n})$.*

Proof. First we fix $(E/F, n) \in \mathcal{I}$. We know from Hilbert's Theorem 90 that $H^1(F, \mu_n) = F^*/F^{*,n}$, from Shapiro's lemma that $H^1(F, R_{E/F}[n]) = E^*/E^{*,n}$, and from local class field theory that $H^2(F, \mu_n) = \mathbb{Z}/n\mathbb{Z}$, all of which carry the natural structure of a profinite group (we don't need to identify $H^2(F, \mu_n)$ with anything; just give it the discrete topology). Under these correspondences, the map $H^1(F, \mu_n) \to H^1(F, R_{E/F}[n])$ corresponds to the obvious map $F^*/F^{*,n} \to E^*/E^{*,n}$ (which is evidently continuous), and so we have a short exact sequence of groups

$$0 \longrightarrow E^*/(F^* \cdot E^{*,n}) \longrightarrow H^1(F, u_{E/F,n}) \longrightarrow C_n \longrightarrow 0,$$

where C_n is the image of $H^1(F, u_{E/F,n}) \to H^2(F, \mu_n)$. The first and third terms in the sequence have natural profinite topologies, since the image of $F^*/F^{*,n}$ in $E^*/E^{*,n}$ is a closed subgroup. Then $H^1(F, u_{E/F,n})$ carries a unique structure of a profinite group realizing C_n as a topological quotient of $H^1(F, u_{E/F,n})$ by the open (closed) subgroup $E^*/(F^* \cdot E^{*,n})$ with the subspace topology, see [RZ00], 2.2.1. It's trivial to check that all lower-degree maps in the long exact sequence are continuous.

Now we look at the transition maps in the corresponding projective systems (so that $(E/F, n)$ is no longer fixed). The ones for $\{H^1(F, \mu_n)\}$ correspond to the quotient maps $F^*/F^{*,m} \to F^*/F^{*,n}$, which are clearly continuous, the ones for $\{H^1(F, R_{E/F}[n])\}$ correspond to the (quotient) norm maps $K^*/K^{*,m} \to E^*/E^{*,n}$, which are continuous, and all $\{H^2(F, \mu_n)\}$ are finite. For $n \mid m$ and $K/E/F$, we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E^*/F^*E^{*,n} & \longrightarrow & H^1(F, u_{E/F,n}) & \longrightarrow & C_n & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle N_{K/E}} & & \uparrow{\scriptstyle p_{m,n}} & & \uparrow & & \\ 0 & \longrightarrow & K^*/F^*K^{*,m} & \longrightarrow & H^1(F, u_{K/F,m}) & \longrightarrow & C_m & \longrightarrow & 0; \end{array}$$

it's a straightforward exercise in profinite abelian groups to show that if the left and right vertical homomorphisms are continuous, then so is the middle one (where, again, the middle groups are equipped with the unique profinite topology discussed above). This completes (1).

For (2) and (3), by [Stacks], I.21.22.2, we have the (canonical) short exact sequences

$$0 \longrightarrow \varprojlim{}^{(1)} H^0(F, u_{E/F,n}) \longrightarrow H^1(F, u) \longrightarrow \varprojlim H^1(F, u_{E/F,n}) \longrightarrow 0;$$

$$0 \longrightarrow \varprojlim{}^{(1)} H^1(F, u_{E/F,n}) \longrightarrow H^2(F, u) \longrightarrow \varprojlim H^2(F, u_{E/F,n}) \longrightarrow 0,$$

and in both cases the left-hand terms vanish: the first vanishes because it's an inverse system of finite groups, and the second because we proved in (1) that $\{H^1(F, u_{E/F,n})\}$ is a system of profinite groups with continuous transition maps. □

The following result will be important when discussing the uniqueness of our constructions. When taking inverse limits of the groups $u_{E/F,n}$ (and computing any cohomology groups) we may replace the category $\mathcal{I}$ with any co-final subcategory $\{E_k/F, n_k\}$ in $\mathcal{I}$, which we do in what follows by taking a tower $F = E_0 \subset E_1 \subset E_2 \subset \dots$ of finite Galois extensions of F with the property that $\cup E_k = F^s$ and a co-final sequence $\{n_k\} \subset \mathbb{N}^\times$. We set $R_k := R_{E_k/F}[n_k]$ and $u_k := u_{E_k/F,n_k}$.

Lemma 4.1.2 *We have $H^i(U_n, u) = 0$ for all $i > 0$, $n \geq 0$.*

Proof. First note that $H^i(U_n, u_k) = 0$ for any $i, k > 0$, $n \geq 0$, by Proposition 3.1.4. Thus, the result is clear if we can show that $H^i(U_n, u) = \varprojlim H^i(U_n, u_k)$ for all $i > 0$, $n \geq 0$. Using the same short exact sequence for inverse limits and cohomology used in the proof of Proposition 4.1.1, it's enough to show that $\varprojlim^{(1)} H^j(U_n, u_k) = 0$ for all $j \geq 0$.

For $j \geq 1$ this is immediate, since all the groups in the system are zero, by above. Thus, all that's left is showing $\varprojlim^{(1)} H^0(U_n, u_k) = 0$ for all n. For $k > l$, the transition map $R_k(U_n) \to R_l(U_n)$ is identified (via splitting the R_j's) with the map

$$\prod_{\gamma \in \Gamma_{E_k/F}} [\mu_{n_k}(U_n)]_\gamma \to \prod_{\sigma \in \Gamma_{E_l/F}} [\mu_{n_l}(U_n)]_\sigma$$

given by raising all coordinates to the n_k/n_l-power and then mapping all Galois-preimage coordinates to their image coordinate (and taking their product). This map is clearly surjective, and since all $H^1(U_n, \mu_{n_j})$ are zero, the long exact sequence in cohomology tells us that $H^0(U_n, R_j)$ surjects onto $H^0(U_n, u_j)$ for all j. Finally, since the square

$$\begin{array}{ccc} H^0(U_n, R_l) & \twoheadrightarrow & H^0(U_n, u_l) \\ \uparrow\!\!\uparrow & & \uparrow \\ H^0(U_n, R_k) & \twoheadrightarrow & H^0(U_n, u_k) \end{array}$$

commutes, the right vertical maps are all surjective, and so the inverse system $\{H^0(U_n, u_k)\}_k$ satisfies the Mittag-Leffler condition, giving the result. □

Corollary 4.1.3 *We have canonical isomorphisms $\check{H}^p(\overline{F}/F, u) \to H^p(F, u)$ for all $p \in \mathbb{N}$.*

Proof. This is an immediate consequence of combining Lemma 4.1.2 with Proposition 3.1.1. □

Next, we prove the basic result about the cohomology of u.

Theorem 4.1.4 *We have $H^1(F, u) = 0$ and a canonical isomorphism $H^2(F, u) = \widehat{\mathbb{Z}}$.*

Proof. As above, we fix a co-final subcategory $\{(E_k, n_k)\}$ of $\mathcal{I}$. By Proposition 4.1.1, $H^i(F, u) = \varprojlim H^i(F, u_{E_k/F, n_k})$ for $i = 1, 2$.

The argument for $i = 2$ is identical to that in [Kal16], with a few minor adjustments—we have the functorial isomorphism

$$H^2(F, u_k) \cong H^0(F, \underline{X}^*(u_k))^* = H^0(\Gamma, X^*(u_k))^* \cong \left[\frac{n_k}{(n_k, [E_k : F])}\mathbb{Z}/n_k\mathbb{Z}\right]^* \cong \mathbb{Z}/(n_k, [E_k : F])\mathbb{Z},$$

where for an abelian group M, M^* denotes the group $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$, $\underline{X}^*(u_k)$ denotes the étale group scheme associated to the Γ-module $X^*(u_k)$, and the first isomorphism is given by the analogue of Poitou-Tate duality for fppf cohomology of finite group schemes over a local field of positive characteristic, see for example [Mil06], III.6.10. For $k > l$, the transition map $H^2(p) : H^2(F, u_k) \to H^2(F, u_l)$ is translated by this isomorphism to the natural projection map

$$\mathbb{Z}/(n_k, [E_k : F])\mathbb{Z} \to \mathbb{Z}/(n_l, [E_l : F])\mathbb{Z}.$$

We may then set $n_k = [E_k : F]$ for all k, giving $(n_k, [E_k : F]) = n_k$, settling the case $i = 2$.

For $i = 1$, by the long exact sequence in cohomology, we have the exact sequence

$$H^1(F, R_k) \longrightarrow H^1(F, u_k) \longrightarrow H^2(F, \mu_{n_k}),$$

and, by Proposition 4.1.1, these are all profinite groups, and the maps in the above sequence are continuous; whence, this sequence remains exact after taking the inverse limit, and so it's enough to show that $\varprojlim H^1(F, R_k) = 0$, $\varprojlim H^2(F, \mu_{n_k}) = 0$. To show that the latter is zero, it's enough to find, for every l fixed, some $k > l$ such that the transition map $H^2(F, \mu_{n_k}) \to H^2(F, \mu_{n_l})$ is zero. For this, note that, at the level of character modules, the map $p^\sharp_{k,l} : X^*(R_l) \to X^*(R_k)$ induces a map on quotients by the subgroups $X^*(R_l)_0$, $X^*(R_k)_0$ (respectively) that's identified with the map $\mathbb{Z}/n_l\mathbb{Z} \to \mathbb{Z}/n_k\mathbb{Z}$ sending $[1]$ to $[(\frac{n_k}{n_l})^2]$, and so we may choose k so that n_k/n_l is a multiple of n_l.

It remains to show that $\varprojlim H^1(F, R_k) = 0$, which is the same as showing $\varprojlim E_k^*/E_k^{*,n_k} = 0$. Consider the short exact sequence induced by the valuation map v:

$$0 \longrightarrow O_k^\times/(O_k^\times)^{n_k} \longrightarrow E^*/E^{*,n_k} \xrightarrow{\ v\ } \mathbb{Z}/n_k\mathbb{Z} \longrightarrow 0,$$

where $O_k^\times$ denotes the units of O_{E_k}. Note that $\{O_k^\times/(O_k^\times)^{n_k}\}$ is a projective system with continuous transition maps induced by N_{E_k/E_l} since the norm map preserves unit groups and n_k-powers (and $n_l \mid n_k$ for $l < k$ by construction).

As in the proof of Proposition 4.1.1, varying k in the above short exact sequence gives three projective systems of profinite abelian groups, with continuous morphisms between the systems. Whence, the sequence stays exact after we take the inverse limit of each system. We claim that the inverse limit of the right-hand terms is zero. Fix $l \in \mathbb{N}$: we know from basic number theory that if π_k is a uniformizer of E_k, then $v_l(N_{E_k/E_l}(\pi_k)) = f_{E_k/E_l}$, where f_{E_k/E_l} denotes the degree of the associated extension of residue fields. Whence, we may choose $k >> l$ so that $n_l \mid f_{E_k/E_l}$, and so the transition map $\mathbb{Z}/n_k\mathbb{Z} \to \mathbb{Z}/n_l\mathbb{Z}$ is zero, giving the claim.

It's thus enough to show that $\varprojlim O_k^\times/(O_k^\times)^{n_k} = 0$. We get a new short exact sequence of *profinite* groups

$$0 \longrightarrow (O_k^\times)^{n_k} \longrightarrow O_k^\times \longrightarrow O_k^\times/(O_k^\times)^{n_k} \longrightarrow 0,$$

where the left-hand term is profinite since it's a closed subgroup of $O_k^\times$, being the image of a compact group under a continuous homomorphism.

Taking the inverse limit of each term, we get a surjection $\varprojlim O_k^\times \twoheadrightarrow \varprojlim O_k^\times/(O_k^\times)^{n_k}$, so we only need to show $\varprojlim O_k^\times = 0$. This follows from local class field theory because our transition maps are norms and for k fixed the universal reciprocity map $\Psi : E_k^* \to \Gamma_{E_k}$ is injective for E_k any local field (see [FV02], IV.6.2). □

Combining the above result with Corollary 4.1.3 immediately gives:

Corollary 4.1.5 *We have $\check{H}^1(\overline{F}/F, u) = 0$ and a canonical identification $\check{H}^2(\overline{F}/F, u) \xrightarrow{\sim} \widehat{\mathbb{Z}}$. In particular, the natural map $\check{H}^p(\overline{F}/F, u) \to \varprojlim_k \check{H}^p(\overline{F}/F, u_k)$ is an isomorphism for $p = 1, 2$.*

We denote by $\alpha \in H^2(F, u)$ the element corresponding to $-1 \in \mathbb{Z}$. For any multiplicative algebraic group Z defined over F, we obtain a map

$$\alpha^* \colon \mathrm{Hom}_F(u, Z) \to H^2(F, Z) \tag{4.2}$$

via taking the image of α under the map $H^2(F, u) \to H^2(F, Z)$ induced by $\phi \in \mathrm{Hom}_F(u, Z)$.

Proposition 4.1.6 *If Z is any finite multiplicative algebraic group defined over F, then α^* is surjective. If Z is also split, then α^* is also injective.*

The identical proof as in [Kal16], Proposition 3.1 works here, with the only difference being the replacement of the classical local Poitou-Tate with the version for finite groups schemes over local fields of positive characteristic, which does not affect the rest of the argument.

4.2 Basic properties of $H^1(\mathcal{E}, Z \to G)$

Fix a u-gerbe $(\mathcal{E}, \theta)$ split over $\overline{F}$ corresponding to the class $\alpha \in H^2(F, u)$, where by "corresponding" we mean $[\mathcal{E}] \in \check{H}^2(\overline{F}/F, u) \xrightarrow{\sim} \varprojlim_n \check{H}^2(\overline{F}/F, u_n) = \widehat{\mathbb{Z}}$ maps to α (see Corollary 4.1.3, and Proposition 4.1.1). This subsection closely follows §3 in [Kal16].

Given $[Z \to G]$ in $\mathcal{A}$, recall that we have defined the cohomology set $H^1(\mathcal{E}, Z \to G)$ to be the subset of $H^1(\mathcal{E}, G_{\mathcal{E}})$ consisting of elements whose image under the map $H^1(\mathcal{E}, G_{\mathcal{E}}) \xrightarrow{\text{Res}} \mathrm{Hom}_F(\mathbf{A}, G)$ is an F-homomorphism $u \to G$ which factors through $Z \hookrightarrow G$, and that this construction is functorial in $[Z \to G]$. For any other choice of $\mathbf{A}$-gerbe $\mathcal{E}'$ with $[\mathcal{E}'] = -1$, we know from Corollary 2.5.5 that $[\mathcal{E}] = [\mathcal{E}'] \in \check{H}^2(\overline{F}/F, u)$, and hence by Proposition 2.3.5 we have a u-equivalence $\eta\colon \mathcal{E} \to \mathcal{E}'$, which (via pullback) induces a map $H^1(\mathcal{E}', G_{\mathcal{E}'}) \to H^1(\mathcal{E}, G_{\mathcal{E}})$, and it is straightforward to verify that this map further gives rise to a map $H^1(\mathcal{E}', Z \to G) \to H^1(\mathcal{E}, Z \to G)$ for any $[Z \to G] \in \mathcal{A}$, which by Lemma 2.6.4, is independent of the choice of η.

Lemma 4.2.1 *The transgression map Hom$_F(u, Z) \to H^2(F, G)$ can be taken to be the composition of the map α^* defined in* (4.2) *and the natural map $H^2(F, Z) \to H^2(F, G)$.*

Proof. We may work with a-twisted cocycles valued in G for a choice of $a \in \mathbf{A}(U_2)$ with $[a] = \alpha$. By the functoriality of our inflation-restriction sequence, we may replace G by Z, and we are reduced to showing that the transgression map $\mathrm{Hom}_F(u, Z) \to H^2(F, Z)$ equals the map α^*. Recall that α^* is defined by mapping a homomorphism to the image of α under the induced map $H^2(F, u) \to H^2(F, Z)$. By construction, the image of $f \in \mathrm{Hom}_F(u, Z)$ under the transgression map is the class $[f(a)] \in H^2(F, Z)$, which is exactly the statement of the lemma, since $[a] = \alpha$. □

Remark 4.2.2 *The above proof does not use anything specific about the group u; the result holds when u is replaced by any commutative F-group scheme $\mathbf{A}$, a by a Čech 2-cocycle c, and $\mathcal{E}$ by $\mathcal{E}_c$. We will use this in the rest of this chapter without comment.*

For $[Z \to G]$ in $\mathcal{A}$, recall that $G \xrightarrow{\pi} \overline{G} := G/Z$.

Lemma 4.2.3 *There is a group homomorphism $b\colon H^1(\mathcal{E}, Z \to G) \to H^1(F, \overline{G})$.*

Proof. The inflation-restriction sequence on $\mathcal{E}$ for the F-group $\overline{G}$ identifies $H^1(F, \overline{G})$ with the kernel of the restriction map $H^1(\mathcal{E}, (\overline{G})_{\mathcal{E}}) \to \mathrm{Hom}_F(u, \overline{G})$. Since the square

$$\begin{array}{ccc} H^1(\mathcal{E}, G_{\mathcal{E}}) & \xrightarrow{\mathrm{Res}} & \mathrm{Hom}_F(u, G) \\ \downarrow & & \downarrow \\ H^1(\mathcal{E}, (\overline{G})_{\mathcal{E}}) & \xrightarrow{\mathrm{Res}} & \mathrm{Hom}_F(u, \overline{G}) \end{array}$$

commutes, it's clear that since $H^1(\mathcal{E}, Z \to G)$ is killed by the right-down composition, its image in $H^1(\mathcal{E}, (\overline{G})_{\mathcal{E}})$ lies inside the inflation of $H^1(F, \overline{G})$. This gives our map. □

The following is the most important proposition of the section, and will be used extensively in the next section.

Proposition 4.2.4 *Let $[Z \to G] \in \mathcal{A}$. Put $\overline{G} = G/Z$. Then we have the commutative diagram with* *exact rows and columns (where the right-most H^2-terms are to be ignored if G is non-abelian):*

$$\begin{array}{ccccccccc}
 & & \overline{G}(F) & = & \overline{G}(F) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & H^1(F,Z) & \xrightarrow{\textit{Inf}} & H^1(\mathcal{E}_a, Z \to Z) & \xrightarrow{\textit{Res}} & \textit{Hom}_F(u,Z) & & \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & H^1(F,G) & \xrightarrow{\textit{Inf}} & H^1(\mathcal{E}_a, Z \to G) & \xrightarrow{\textit{Res}} & \textit{Hom}_F(u,Z) & \xrightarrow{tg} & H^2(F,G) \\
 & & \| & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle \alpha^*} & & \| \\
 & & H^1(F,G) & \longrightarrow & H^1(F,\overline{G}) & \longrightarrow & H^2(F,Z) & \longrightarrow & H^2(F,G) \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

Proof. We may work with a-twisted cocycles for an appropriate choice of a. The second and third rows come from the already-established inflation-restriction result, the fourth row and left column come from the long exact sequence for fppf cohomology associated to the short exact sequence $0 \longrightarrow Z \longrightarrow G \longrightarrow \overline{G} \longrightarrow 0$, and the middle column is from Lemma 2.7.3 and Lemma 4.2.3. It follows from the same lemmas that the middle column is exact, except for possibly the surjectivity of b, which we will show later in the proof. The commutativity of all squares is obvious, except for the bottom right one, which is exactly Lemma 4.2.1, and the bottom middle one, which we will show now.

The map $H^1(\mathcal{E}_a, G_{\mathcal{E}_a}) \to H^1(\mathcal{E}_a, \overline{G}_{\mathcal{E}_a})$ sends the class of the a-twisted cocycle (x,ϕ) to the class of $(\pi(x), \pi \circ \phi)$. Since we assume that ϕ factors through Z, the class $[(\pi(x), \pi \circ \phi)]$ is the class of $[\pi(x), 0]$, where $\pi(x) \in \overline{G}(U_1)$ is an actual 1-cocycle (because $\pi(\phi(a)) = e_{\overline{G}}$). We want to look at the image of the class $[\pi(x)]$ under the connecting homomorphism $\delta\colon H^1(F,\overline{G}) \to H^2(F,Z)$ (computed as in Proposition 3.1.3).

To compute $\delta([\pi(x)])$, we first lift $\pi(x)$ to $G(U_1)$; the natural element to pick here is $x \in G(U_1)$. Then $\delta([\pi(x)])$ is exactly $[dx] \in H^1(\overline{F}/F, Z)$, which, by assumption, equals $[\phi(a)]$, which gives the desired commutativity, since the class $[\text{Res}[(x,\phi)](a)] = [\phi(a)]$ is exactly the element of $H^2(F,Z)$ obtained by following the square in the other direction, see the proof of Lemma 4.2.1.

The last thing to show is the surjectivity of b. If G is abelian, this follows immediately from the surjectivity of α^*, using the commutativity of the bottom right and middle squares and the four-lemma. We will address the non-abelian case in Proposition 4.5.6 (we will not investigate this construction for non-abelian G until that section anyway, so there is no danger of circularity). □

4.3 Extending Tate-Nakayama

Let S be an F-torus and E/F a finite Galois extension. As in [Kal16], §4, the goal of this section is to extend the notion of the classical Tate-Nakayama isomorphism

$$X_*(S)_{\Gamma,\text{tor}} = H^{-1}_{\text{Tate}}(\Gamma_{E/F}, X_*(S)) \xrightarrow{\sim} H^1(\Gamma, S)$$

to the setting of our cohomology group $H^1(\mathcal{E}, Z \to S)$. Some new notation: for an affine F-group scheme G, we will denote by $F[G]$ the coordinate ring of G. Let $H^1(\mathcal{E})$ denote the functor from $\mathcal{T}$ to AbGrp which sends $[Z \to S]$ to the group $H^1(\mathcal{E}, Z \to S)$.

Following [Kal16], we will first construct a functor $\overline{Y}_{+,\text{tor}} \colon \mathcal{T} \to$ AbGrp which extends the functor $S \mapsto X_*(S)_{\Gamma,\text{tor}}$, as well as a morphism of functors from $\overline{Y}_{+,\text{tor}}$ to the functor $[Z \to S] \mapsto \text{Hom}_F(u, Z)$. Then we will construct a unique isomorphism of functors

$$\overline{Y}_{+,\text{tor}} \to H^1(\mathcal{E})$$

on $\mathcal{T}$ which for objects $[1 \to S] \in \mathcal{T}$ coincides with the Tate-Nakayama isomorphism, and such that the composition $\overline{Y}_{+,\text{tor}}(Z \to S) \to H^1(\mathcal{E}, Z \to S) \to \text{Hom}_F(u, Z)$ equals the morphism alluded to above.

We start by defining the functor $\overline{Y}_{+,\text{tor}}$, which is just a summary of §4.1 in [Kal16].

For $[Z \to S] \in \mathcal{T}$, we set $\overline{S} := S/Z$. Then if $Y := X_*(S)$ and $\overline{Y} := X_*(\overline{S})$, we have an injective morphism of Γ-modules $Y \to \overline{Y}$.

We then have an isomorphism of Γ-modules

$$\overline{Y}/Y \to \text{Hom}(\mu_n, Z) \qquad \bar{\lambda} \mapsto [x \mapsto (n\lambda)(x)],$$

for any $n \in \mathbb{N}$ such that $[\overline{Y} \colon Y] \mid n$, where for $\lambda \in \overline{Y}$, we identify $n\lambda$ with an element of Y. Take any finite Galois extension E/F which splits S, and take $I \subset \mathbb{Z}[\Gamma_{E/F}]$ to be the augmentation ideal. Set $\overline{Y}_+ := \overline{Y}/IY$, and $\overline{Y}_+^N$ the quotient of $\overline{Y}^N$ by IY, where the superscript N denotes the kernel of the norm map $N_{E/F}$.

Then we have $\overline{Y}_+^N = \overline{Y}_{+,\text{tor}}$ (see [Kal16], Fact 4.1), and the natural map $\overline{Y}_+^N \to [\overline{Y}/Y]^N$ post-composed with the isomorphisms $[\overline{Y}/Y]^N \xrightarrow{\sim} \text{Hom}(\mu_n, Z)^N \xrightarrow{\sim} \text{Hom}_F(u_{E/F,n}, Z)$ (this second isomorphism comes from (4.1)) gives a homomorphism $\overline{Y}_+^N \to \text{Hom}_F(u, Z)$. For varying E/F and n, these homomorphisms are compatible and splice to a homomorphism $\overline{Y}_{+,\text{tor}} \to \text{Hom}_F(u, Z)$.

Given a morphism $[Z_1 \to S_1] \to [Z_2 \to S_2]$ in $\mathcal{T}$, the induced morphism $\overline{S_1} \to \overline{S_2}$ induces a Γ-morphism $X_*(\overline{S_1})_{+,\text{tor}} \to X_*(\overline{S_2})_{+,\text{tor}}$, showing that the assignment $[Z \to S] \mapsto \overline{Y}_{+,\text{tor}}$ is functorial in $[Z \to S]$.

4.4 Construction of the isomorphism

We are now ready to construct the isomorphism of functors on $\mathcal{T}$ from $\overline{Y}_{+,\text{tor}}$ to $H^1(\mathcal{E})$.

Choose an increasing tower E_k of finite Galois extensions of F and cofinal sequence $\{n_k\}$ in $\mathbb{N}^\times$, with associated prime-to-p sequence $\{n'_k\}$. Choose a sequence of 2-cocycles c_k representing the canonical classes in each $H^2(\Gamma_{E_k/F}, E_k^*)$ as in [Kal16], §4.4, which we will identify with their corresponding Čech 2-cocycles, and maps $l_k \colon (F^s)^* \to (F^s)^*$ satisfying $l_k(x)^{n'_k} = x$ and $l_{k+1}(x)^{n'_{k+1}/n'_k} = l_k(x)$ for all $x \in (F^s)^*$. For K/F a finite Galois extension, we may also view l_k as a map on Čech-cochains $C^n(K/F, \mathbb{G}_m) \to C^n(F^s/F, \mathbb{G}_m)$ by identifying the left-hand side with $\prod_{\underline{\sigma} \in \Gamma^n_{K/F}} K^*_{\underline{\sigma}}$, applying l_k to each coordinate, and then mapping by t^{-1} to $L^{\bigotimes_F (n+1)}$, where L/F is some finite Galois extension containing all the chosen n'_k-roots of the entries of x.

Denote $u_{E_k/F,n_k}$ by u_k and $R_{E_k/F}[n_k]$ by R_k. Recall the homomorphism $\delta_e \colon \mu_{n_k} \to R_k$ inducing a homomorphism $\delta_e \colon \mu_{n_k} \to u_k$ that is killed by the norm map for the group $\Gamma_{E_k/F}$ acting on $\text{Hom}(\mu_{n_k}, u_k)$.

Following [Kal16], §4.5, we define

$$\xi_k = d[(l_k c_k)^{(1/p^{m_k})}] \underset{E_k/F}{\sqcup} \delta_e \in C^2(\overline{F}/F, u_k),$$

where for an n-cochain $x \in \mathbb{G}_m(\overline{F}^{\bigotimes_F (n+1)})$, we choose for every p-power $p^{m_k} := n_k/n'_k$ a p^{m_k}-root of x, denoted by $x^{(1/p^{m_k})}$, satisfying $(x^{(1/p^{m_{k+1}})})^{p^{m_{k+1}}/p^{m_k}} = x^{(1/p^{m_k})}$ and if $x \in \overline{F} \otimes_F \overline{F} \otimes_F \cdots \otimes_F \overline{F} \otimes_F E$ for E/F a finite Galois extension, then $x^{(1/p^{m_k})} \in \overline{F} \otimes_F \overline{F} \otimes_F \cdots \otimes_F \overline{F} \otimes_F E$ as well (it is a straightforward exercise in purely inseparable extensions of fields to prove that such a choice of roots exists). For ease of notation, denote $(l_k c_k)^{(1/p^{m_k})}$ by $\widetilde{l_k c_k}$, which we view as a Čech 2-cochain valued in $\mathbb{G}_m(U_2)$.

To ensure that the above definition makes sense, we need to verify that $l_k c_k \in C^{2,1}(F^s/F, E_k, \mathbb{G}_m)$ and $(l_k c_k)^{(1/p^{m_k})} \in C^{2,1}(\overline{F}/F, E_k, \mathbb{G}_m)$. The first inclusion follows from looking at the corresponding Galois n-cochain, as in [Kal16], and the second inclusion follows from the first and the construction of the $(-)^{(1/p_{m_k})}$-maps.

Define the torus $S_{E_k/F}$ to be the quotient of $\text{Res}_{E_k/F}(\mathbb{G}_m)$ by the diagonally-embedded $\mathbb{G}_m$; it's clear that u_k is the subgroup $S_{E_k/F}[n_k]$ of n_k-torsion points. Define

$$\alpha'_k = (l_k c_k \underset{E_k/F}{\sqcup} \delta_{e,k})^{-1} \cdot p'_{k+1,k}(l_{k+1} c_{k+1} \underset{E_{k+1}/F}{\sqcup} \delta_{e,k+1}) \in C^1(F^s/F, S_{E_k/F})$$

and

$$\alpha_k = (\widetilde{l_k c_k} \underset{E_k/F}{\sqcup} \delta_{e,k})^{-1} \cdot p_{k+1,k}(\widetilde{l_{k+1}c_{k+1}} \underset{E_{k+1}/F}{\sqcup} \delta_{e,k+1}) \in C^1(\overline{F}/F, S_{E_k/F}),$$

where by $p_{k+1,k}$ we mean the map from $S_{E_{k+1}/F}$ to $S_{E_k/F}$ induced by the homomorphism of Γ-modules $\mathbb{Z}[\Gamma_{E_k/F}]_0 \to \mathbb{Z}[\Gamma_{E_{k+1}/F}]_0$ defined by $[\gamma] \mapsto (n_{k+1}/n_k)\sum_{\sigma\mapsto\gamma}[\sigma]$, similarly with $p'_{k+1,k}$. By $\delta_{e,k}$ we mean the extension of $\delta_e \colon \mu_{n_k} \to u_k$ to the map $\mathbb{G}_m \to S_{E_k/F}$ defined on Γ-modules by $\mathbb{Z}[\Gamma_{E_k/F}] \to \mathbb{Z}$ the evaluation at $[e]$ map. Note that this is not in general Γ-equivariant, but is still killed by the norm $N_{E_k/F}$.

Lemma 4.4.1 *1. The cochain α_k takes values in u_k and the equality $d\alpha_k = p_{k+1,k}(\xi_{k+1})\xi_k^{-1}$ holds in $C^2(\overline{F}/F, u_k)$.*

2. The element $([\xi_k])$ of $\varprojlim H^2(F, u_k)$ is equal to the canonical class α.

Proof. We start by proving (1). To show that $\alpha_k \in u_k(\overline{F}\otimes_F \overline{F}) = S_{E_k/F}[n_k](\overline{F}\otimes_F\overline{F})$, it's enough to show that $\alpha_k^{p^{m_k}} \in S_{E_k/F}[n'_k](\overline{F}\otimes_F\overline{F})$. By construction,

$$\alpha_k^{p^{m_k}} = (l_k c_k \underset{E_k/F}{\sqcup} \delta_{e,k})^{-1} \cdot p_{k+1,k}([\widetilde{l_{k+1}c_{k+1}}]^{p^{m_k}} \underset{E_{k+1}/F}{\sqcup} \delta_{e,k+1}) = \alpha'_k,$$

since $p_{k+1,k}$ is $p'_{k+1,k}$ pre-composed with the $p^{m_{k+1}}/p^{m_k}$-power map on $S_{E_{k+1}/F}$. Thus, it's enough to show that $\alpha'_k \in S_{E_k/F}[n'_k](F^s\otimes_F F^s)$, which follows from Lemma 4.5 in [Kal16].

To show the second part of (1), we note by Proposition 3.4.3 that

$$d(\widetilde{l_k c_k} \underset{E_k/F}{\sqcup} \delta_{e,k}) = d(\widetilde{l_k c_k}) \underset{E_k/F}{\sqcup} \delta_e = \xi_k,$$

since $\delta_{e,k}$ is killed by $N_{E_k/F}$. As $p_{k+1,k}$ is defined over F, Lemmas 3.4.4 and 3.4.5 give us the equality

$$p_{k+1,k}(\widetilde{l_{k+1}c_{k+1}} \underset{E_{k+1}/F}{\sqcup} \delta_{e,k+1}) = \widetilde{l_{k+1}c_{k+1}} \underset{E_{k+1}/F}{\sqcup} p_{k+1,k}\circ\delta_{e,k+1}.$$

Note that $p_{k+1,k}\circ\delta_{e,k+1}\colon \mathbb{G}_m \to S_{E_k/F}$ equals $(n_{k+1}/n_k)\delta_{e,k}$, and so it is killed by $N_{E_k/F}$ (and hence by $N_{E_{k+1}/F}$), and so Proposition 3.4.3 and Lemma 3.4.5, together with the above equality, imply that

$$d[p_{k+1,k}(\widetilde{l_{k+1}c_{k+1}} \underset{E_{k+1}/F}{\sqcup} \delta_{e,k+1})] = (d\widetilde{l_{k+1}c_{k+1}}) \underset{E_{k+1}/F}{\sqcup} p_{k+1,k}\circ\delta_e = p_{k+1,k}[(d\widetilde{l_{k+1}c_{k+1}}) \underset{E_{k+1}/F}{\sqcup} \delta_e],$$

and this last term is exactly $p_{k+1,k}(\xi_{k+1})$.

It remains to prove (2). As in the analogous part of the proof of Lemma 4.5 in [Kal16], it's enough to show that under the isomorphism $H^2(F,u_k) \to H^0(\Gamma, X^*(u_k))^* \to \mathbb{Z}/(n_k,[E_k:F])\mathbb{Z}$ used in the proof of Theorem 4.1.4, the class of ξ_k maps to the element -1. Consider the cup product of ξ_k with the element $\frac{n_k}{(n_k,[E_k:F])} \in \frac{n_k}{(n_k,[E_k:F])}\mathbb{Z}/n_k\mathbb{Z} \cong H^0(\Gamma, X^*(u_k))$, which we denote by $\chi \in H^0(\Gamma, X^*(u_k))$. We have by Lemmas 3.4.4 and 3.4.5 that $\xi_k \cup \chi = \chi(\xi_k) = \widetilde{dl_k c_k} \underset{E_k/F}{\sqcup} \chi \circ \delta_e$.

Note that $\chi \circ \delta_e \colon \mu_{n_k} \to \mathbb{G}_m$ is fixed by $\Gamma_{E/F}$, and so by Lemma 3.4.5, we get that

$$\widetilde{dl_k c_k} \underset{E_k/F}{\sqcup} \chi \circ \delta_e = (\widetilde{dl_k c_k} \underset{E_k/F}{\sqcup} \mathrm{id}_{\mathbb{G}_m}) \cup (\chi \circ \delta_e).$$

Since the E_k/F-norm of $\mathrm{id}_{\mathbb{G}_m}$ is the $[E_k\colon F]$-power map on $\mathbb{G}_m$, it follows from Proposition 3.4.3 that $\widetilde{dl_k c_k} \underset{E_k/F}{\sqcup} \mathrm{id}_{\mathbb{G}_m}$ is cohomologous to $(\widetilde{l_k c_k} \cup [E_k\colon F] \cdot \mathrm{id}_{\mathbb{G}_m})^{-1}$. Thus (by basic properties of the cup product), we have that $\xi_k \cup \chi$ is cohomologous to

$$([E_k\colon F] \cdot \widetilde{l_k c_k})^{-1} \cup (\chi \circ \delta_e),$$

where $\chi \circ \delta_e \colon \mathbb{G}_m \to \mathbb{G}_m$ is interpreted as the extension of $\chi \circ \delta_e \colon \mu_{n_k} \to \mu_{n_k}$ to the map induced by the group homomorphism $\mathbb{Z} \to \mathbb{Z}$ given by $1 \mapsto [E_k\colon F]$.

If $z \in X^*(R_k)$ is the character generating $H^0(\Gamma_{E/F}, X^*(R_k))$, then by construction $\chi = \frac{n_k}{(n_k,[E_k\colon F])} z$ and $z \circ \delta_e = \mathrm{id}_{\mu_{n_k}}$. Viewing $z \circ \delta_e$ as the map $\mathrm{id}_{\mathbb{G}_m}$, we can factor δ_e through R_k, and get by $\mathbb{Z}$-bilinearity that

$$([E_k\colon F] \cdot \widetilde{l_k c_k})^{-1} \cup (\chi \circ \delta_e) = \frac{n_k}{(n_k, [E_k\colon F])}([E_k\colon F] \cdot \widetilde{l_k c_k})^{-1}.$$

Since $\frac{n_k}{(n_k,[E_k\colon F])} \cdot [E_k\colon F] = n_k \cdot \frac{[E_k\colon F]}{(n_k,[E_k\colon F])}$ and by design $\widetilde{l_k c_k}$ is an n_kth root of c_k, we get that $[\xi \cup \chi]$ is $-\frac{[E_k\colon F]}{(n_k,[E_k\colon F])}$ times the class $[c_k]$, which thus has invariant equal to $-1/(n_k, [E_k\colon F])$. This exactly gives that ξ_k sends χ to $-1/(n_k, [E_k\colon F])$ under the pairing of used in the proof of Theorem 4.1.4, giving the result. □

For fixed $k \in \mathbb{N}$, denote by $\mathcal{E}_k := \mathcal{E}_{\xi_k}$ the u_k-gerbe corresponding to the Čech 2-cocycle ξ_k. For any fixed k we have a morphism of F-stacks $\pi_{k+1,k} \colon \mathcal{E}_{k+1} \to \mathcal{E}_k$ given by $\phi_{\xi_{k+1},\xi_k,\alpha_k}$, obtained by combining Lemma 4.4.1 with Construction 2.3.4. In fact, the systems $(\xi_k)_k$ and $(\alpha_k)_k$, along with the groups u_k and gerbes $\mathcal{E}_k$, exactly satisfy the assumptions made in §2.5, our subsection on inverse limits of gerbes. Thus, as a consequence of Proposition 2.5.3, we may take $\mathcal{E} := \varprojlim_k \mathcal{E}_k$ to be the gerbe used to define the groups $H^1(\mathcal{E}, Z \to S)$ for $[Z \to S]$ in $\mathcal{T}$.

We are now ready to begin describing the Tate-Nakayama isomorphism. For a fixed $[Z \to S]$ in $\mathcal{T}$, let k be large enough so that E_k splits S and $|Z|$ divides n_k. Let $\bar{\lambda} \in \overline{Y}^{N_{E_k/F}}$, and $\phi_{\bar{\lambda},k} \in \mathrm{Hom}_F(u_k, Z)$ be its image under the isomorphism

$$[\overline{Y}/Y]^{N_{E_k/F}} \to \mathrm{Hom}(\mu_{n_k}, Z)^{N_{E_k/F}} \to \mathrm{Hom}_F(u_k, Z).$$

Define a ξ_k-twisted S-torsor on $\overline{F}$ as follows. Take the trivial $S_{\overline{F}}$-torsor $S_{\overline{F}}$, with u_k-action induced by the homomorphism $u_k \xrightarrow{\phi_{\bar{\lambda},k}} S_{\overline{F}}$ and gluing map $S_{\overline{F}\otimes_F\overline{F}} \xrightarrow{\sim} S_{\overline{F}\otimes_F\overline{F}}$ given by left-translation by $z_{k,\bar{\lambda}} := \widetilde{l_k c_k} \underset{E_k/F}{\sqcup} n_k\bar{\lambda} \in S(\overline{F}\otimes_F \overline{F})$, where we view $n_k\bar{\lambda}$ as an element of $X_*(S)$ (this makes sense since $|Z|$ divides n_k). This gluing map is trivially S- and hence u_k-equivariant.

Lemma 4.4.2 *The above $S_{\overline{F}}$-torsor with the specified u_k -action and gluing map defines a ξ_k-twisted S-torsor, which we will denote by $Z_{k,\bar{\lambda}}$. Moreover, for every k, we have the equality of ξ_{k+1}-twisted S-torsors*

$$\pi^*_{k+1,k}Z_{k,\bar{\lambda}} = Z_{k+1,\bar{\lambda}}.$$

Proof. For the first statement, we just need to check that the above $S_{\overline{F}}$-torsor is ξ_k-twisted with respect to translation by $z_{\bar{\lambda},k}$ on $S_{\overline{F}\otimes_F\overline{F}}$. Since u_k acts via $\phi_{\bar{\lambda},k}$, this is the same as showing that $d(z_{\bar{\lambda},k}) = \phi_{\bar{\lambda},k}(\xi_k)$. Since $\bar{\lambda}$ is killed by $N_{E_k/F}$, so is $n_k\bar{\lambda}$, and hence by Proposition 3.4.3 we have $d(\widetilde{l_k c_k} \underset{E_k/F}{\sqcup} n_k\bar{\lambda}) = (d\widetilde{l_k c_k}) \underset{E_k/F}{\sqcup} n_k\bar{\lambda}$.

Moreover, $\phi_{\bar{\lambda},k}$ is such that $\phi_{\bar{\lambda},k} \circ \delta_e = n_k\bar{\lambda}$, and so by Lemma 3.4.5, since $\phi_{\bar{\lambda},k}$ is defined over F, we obtain

$$(d\widetilde{l_k c_k}) \underset{E_k/F}{\sqcup} n_k\bar{\lambda} = \phi_{\bar{\lambda},k}[(d\widetilde{l_k c_k}) \underset{E_k/F}{\sqcup} \delta_e] = \phi_{\bar{\lambda},k}(\xi_k),$$

as desired. We thus get our ξ_k-twisted S-torsor $Z_{\bar{\lambda},k}$.

We now want to compare the pullback $\pi^*_{k+1,k}Z_{\bar{\lambda},k}$ to $Z_{\bar{\lambda},k+1}$. As $S_{\overline{F}}$-torsors, these are both trivial, so it's enough to show that the u_{k+1}-actions coincide, and that the difference of the two gluing maps is the identity in $S(\overline{F}\otimes_F\overline{F})$. By Corollary 2.4.12, the u_{k+1}-action on $\pi^*_{k+1,k}Z_{\bar{\lambda},k}$ is given by the homomorphism $u_{k+1} \xrightarrow{\phi_{\bar{\lambda},k}\circ p_{k+1,k}} S_{\overline{F}}$ and the u_{k+1}-action on $Z_{\bar{\lambda},k}$ is given by $\phi_{\bar{\lambda},k+1}$. One checks easily that $\phi_{\bar{\lambda},k+1} = \phi_{\bar{\lambda},k} \circ p_{k+1,k}$, so the u_{k+1}-actions coincide.

Corollary 2.4.12 also tells us that the twisted gluing map for $\pi^*_{k+1,k}Z_{\bar{\lambda},k}$ is left-translation on $S_{\overline{F}}$ by $\phi_{\bar{\lambda},k}(\alpha_k)\cdot z_{\bar{\lambda},k} \in S(\overline{F}\otimes_F\overline{F})$, and for $Z_{\bar{\lambda},k+1}$ is left-translation by $z_{\bar{\lambda},k+1}$. We want to look at

$$z_{\bar{\lambda},k}\cdot\phi_{\bar{\lambda},k}(\alpha_k)\cdot z^{-1}_{\bar{\lambda},k+1} = \phi_{\bar{\lambda},k}(\alpha_k)\cdot(\widetilde{l_{k+1}c_{k+1}} \underset{E_{k+1}/F}{\sqcup} n_{k+1}\bar{\lambda})^{-1}\cdot\widetilde{l_k c_k} \underset{E_k/F}{\sqcup} n_k\bar{\lambda}.$$

Recall (since $p_{k+1,k}$ is defined over F) that

$$\alpha_k = (\widetilde{l_k c_k} \underset{E_k/F}{\sqcup} \delta_{e,k})^{-1}\cdot(\widetilde{l_{k+1}c_{k+1}} \underset{E_{k+1}/F}{\sqcup} p_{k+1,k}\circ\delta_{e,k+1}),$$

and since the extension of $\phi_{\bar{\lambda},k}$ to $S_{E_k/F}$ (see [Kal20], page 3), which we will also denote by $\phi_{\bar{\lambda},k}$, is defined over F, we may pull it inside both cup products to obtain

$$\phi_{\bar{\lambda},k}(\alpha_k) = (\widetilde{l_k c_k} \underset{E_k/F}{\sqcup} \phi_{\bar{\lambda},k}\circ\delta_{e,k})^{-1}\cdot(\widetilde{l_{k+1}c_{k+1}} \underset{E_{k+1}/F}{\sqcup} \phi_{\bar{\lambda},k}\circ p_{k+1,k}\circ\delta_{e,k+1}).$$

Since $\phi_{\bar{\lambda},k} \circ p_{k+1,k} = \phi_{\bar{\lambda},k+1}$, the above is exactly $z_{\bar{\lambda},k}^{-1} \cdot z_{\bar{\lambda},k+1}$, so we are done. □

Again choosing $k \in \mathbb{N}$ such that E_k splits S and $|Z|$ divides n_k, we may define an $S_{\mathcal{E}}$-torsor on $\mathcal{E}$ by pulling back $Z_{k,\bar{\lambda}}$ (identifying this ξ_k-twisted S-torsor with an $S_{\mathcal{E}_k}$-torsor on $\mathcal{E}_k$ as in Proposition 2.4.10) to $\mathcal{E}$ via the projection map $\pi_k \colon \mathcal{E} \to \mathcal{E}_k$. By the above lemma, this does not depend on the choice of k, and so we denote this torsor simply by $Z_{\bar{\lambda}}$. We are now in a position to prove the main result. The statement and proof largely follow the analogous result in [Kal16], which is that paper's Theorem 4.8.

Theorem 4.4.3 *The assignment $\bar{\lambda} \mapsto Z_{\bar{\lambda}}$ induces an isomorphism*

$$\iota \colon \overline{Y}_{+,tor} \to H^1(\mathcal{E})$$

of functors $\mathcal{T} \to AbGrp$. This isomorphism coincides with the Tate-Nakayama isomorphism for objects $[1 \to S]$ in $\mathcal{T}$ and lifts the morphism from $\overline{Y}_{+,tor}$ to $Hom_F(u, -)$ described earlier in the subsection. Moreover, ι is the unique isomorphism between these two functors satisfying the above two properties.

Proof. This assignment is clearly additive in $\bar{\lambda}$, and so it defines a group homomorphism from $\overline{Y}^N$ to $H^1(\mathcal{E}, Z \to S)$ for any object $[Z \to S]$ of $\mathcal{T}$. Moreover, any morphism $[Z \to S] \xrightarrow{h} [Z' \to S']$ in $\mathcal{T}$ induces the morphism $H^1(\mathcal{E}, Z \to S) \to H^1(\mathcal{E}, Z' \to S')$ sending the class of $\pi_k^* Z_{\bar{\lambda},k}$ (for suitable k, as discussed above) to that of $\pi_k^*(Z_{\bar{\lambda},k} \times^{h,S} S')$, and so it is enough to show that $Z_{\bar{\lambda},k} \times^{h,S} S$ is isomorphic to $Z_{h^\sharp(\bar{\lambda}),k}$ as ξ_k-twisted S'-torsors. Note that $Z_{\bar{\lambda},k} \times^{h,S} S'$ is evidently trivial as an $S'_{\overline{F}}$-torsor, and has u_k-action given by $h \circ \phi_{\bar{\lambda},k}$, whereas $Z_{h^\sharp(\bar{\lambda}),k}$ has u_k-action given by $\phi_{h^\sharp \bar{\lambda},k} = h \circ \phi_{\bar{\lambda},k}$, since if $\phi_{\bar{\lambda},k} \circ \delta_e = n_k \bar{\lambda}$, then $h \circ (\phi_{\bar{\lambda},k} \circ \delta_e) = h \circ n_k \bar{\lambda} = h^\sharp \bar{\lambda}$. Finally, one checks by a similar argument that $h(z_{\bar{\lambda},k}) = z_{h^\sharp \bar{\lambda},k}$, giving the desired equality of torsors, and hence that the assignment of the theorem gives a morphism of functors from $\overline{Y}^N$ to $H^1(\mathcal{E})$.

We need to check that for $[Z \to S]$ in $\mathcal{T}$ fixed, the homomorphism $\overline{Y}^N \to H^1(\mathcal{E}, Z \to S)$ descends to the quotient $\overline{Y}_{+,\text{tor}} = \overline{Y}^N / IY$. To this end, suppose that $\bar{\lambda} \in \overline{Y}^N$ lies in Y. Then (choosing k large enough) by §4.3, $\phi_{\bar{\lambda},k}$ is trivial, and moreover

$$z_{\bar{\lambda},k} = \widetilde{l_k c_k} \underset{E_k/F}{\sqcup} n_k \bar{\lambda} = c_k \underset{E_k/F}{\sqcup} \bar{\lambda}.$$

Note that $c_k \in \mathbb{G}_m(E_k \otimes_F E_k)$, and hence by Proposition 3.4.1, this unbalanced cup product may be computed using the definition given in [Kal16], working with Galois cohomology. By [Kal16], §4.3, this coincides with the usual cup product in finite Tate cohomology with respect to the group $\Gamma_{E_k/F}$, and thus yields the image of $\bar{\lambda}$ induced by the Tate-Nakayama isomorphism $X_*(S)^{N_k} \to [X_*(S)/IX_*(S)]^{N_k} \xrightarrow{\sim} H^1(\Gamma_{E_k/F}, S(E_k)) = H^1(F, S)$. As a consequence, if $\bar{\lambda} \in IY$, then

$z_{\bar{\lambda},k} = 1$, and so $Z_{\bar{\lambda},k}$ is given by the trivial $S_{\overline{F}}$-torsor with trivial u_k-action and gluing map equal to the identity, thus yielding the trivial ξ_k-twisted S-torsor on $\mathcal{E}_k$, as desired.

The argument of the above paragraph also shows that if we take $[1 \to S] \in \mathcal{T}$, then $\overline{Y}_{+,\text{tor}}[1 \to S] = Y/IY$ and the homomorphism $Y/IY \to H^1(\mathcal{E}, 1 \to S) = H^1(F, S)$ is exactly the Tate-Nakayama isomorphism. For the morphism of functors on $\mathcal{T}$ from $\overline{Y}_{+,\text{tor}}$ to $\text{Hom}_F(u, -)$ sending $\bar{\lambda}$ to $\phi_{\bar{\lambda},k} \circ p_k$, we have already discussed that the image of $\pi_k^* Z_{\bar{\lambda},k}$ under the restriction morphism $H^1(\mathcal{E}, Z \to S) \to \text{Hom}_F(u, Z)$ equals $\phi_{\bar{\lambda},k} \circ p_k$, giving the desired compatibility of morphisms of functors to $\text{Hom}_F(u, -)$.

The final thing to show is that for $[Z \to S]$ fixed, the assignment of the theorem yields an isomorphism from $\overline{Y}_{+,\text{tor}}$ to $H^1(\mathcal{E}, Z \to S)$. As in [Kal16], consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(F,S) & \longrightarrow & H^1(\mathcal{E}, Z \to S) & \longrightarrow & \text{Hom}_F(u,Z) & \longrightarrow & H^2(F,S) \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & Y_{\Gamma,\text{tor}} & \longrightarrow & \overline{Y}_{+,\text{tor}} & \longrightarrow & \varinjlim [\overline{Y}/Y]^{N_k} & \longrightarrow & \varinjlim Y^{\Gamma}/N_k(Y), \end{array}$$

where the top horizontal sequence is just inflation-restriction, the first lower-horizontal map is induced by the inclusion $X_*(S) \to X_*(\overline{S})$, the second is induced by the maps $\overline{Y}_{+,\text{tor}} = \overline{Y}^{N_k}/I_k Y \to [\overline{Y}/Y]^{N_k}$, and the third is induced by the maps $[\overline{Y}/Y]^{N_k} \to Y^{\Gamma}/N_k(Y)$ given by $[\bar{\lambda}] \mapsto [N_k(\bar{\lambda})]$. It's a straightforward exercise in group cohomology to check that the bottom horizontal sequence is exact. The first vertical map is the Tate-Nakayama isomorphism, the second vertical map is the assignment $\bar{\lambda} \mapsto Z_{\bar{\lambda}}$, the third vertical map is induced by the system of maps $[\overline{Y}/Y]^{N_k} \to \text{Hom}_F(u_k, Z) \to \text{Hom}_F(u, Z)$ discussed in §4.3, and the final vertical map is induced by the system of negative Tate-Nakayama isomorphisms $H^0_{\text{Tate}}(\Gamma_{E_k/F}, Y) \xrightarrow{\sim} H^2(\Gamma_{E_k/F}, S(E_k)) \xrightarrow{\text{Inf}} H^2(F, S)$.

We claim that this diagram commutes; the first square commutes by our above discussion of the compatibility with the Tate-Nakayama isomorphism, and the middle square commutes by compatibility between the two morphisms of functors to $\text{Hom}_F(u, -)$. Thus, we only need to show that the right-hand square commutes. It's enough to do this for a sufficiently large fixed k and u replaced by u_k, because any $\phi \colon u \to Z$ factors through some $\phi_k \colon u_k \to Z$, and then $\phi(\alpha) = \phi_k(p_k(\alpha)) = [\phi_k(\xi_k)]$ in $H^2(F, Z)$, since $[p_k(\alpha)] = [\xi_k]$ in $\check{H}^2(\overline{F}/F, u_k)$ (by construction). Fix $\bar{\lambda} \in \overline{Y}$ whose norm lies in Y. Then its image in $\text{Hom}_F(u_k, Z)$ is $\phi_{\bar{\lambda},k}$, which, by Lemma 4.2.1, maps under the transgression map to the image of the class $[\phi_{\bar{\lambda},k}(\xi_k)] \in H^2(F, Z)$ in $H^2(F, S)$, which equals the class of $(\widetilde{dl_k c_k}) \underset{E_k/F}{\sqcup} n_k \bar{\lambda}$, since we may pull $\phi_{\bar{\lambda},k}$ inside the cup product defining ξ_k by Lemma 3.4.5.

On the other hand, if we take $N_k(\bar{\lambda}) \in Y^{\Gamma} = Y^{\Gamma_{E_k/F}}$, then its image under the Tate-Nakayama map $Y^{\Gamma_{E_k/F}} \to H^2(\Gamma_{E_k/F}, S(E_k))$ is obtained by taking the cup product with the class $[c_k] \in H^2(\Gamma_{E_k/F}, E_k^*) \xrightarrow{\text{Inf}} H^2(F, \mathbb{G}_m)$. I.e., we obtain the class of the cocycle $(c_k \cup N_k(\bar{\lambda}))^{-1}$ in $H^2(F, S)$.

By Proposition 3.4.3, $(\widetilde{dl_k c_k}) \underset{E_k/F}{\sqcup} n_k\bar{\lambda}$ is cohomologous to $[\widetilde{l_k c_k} \cup d(n_k\bar{\lambda})]^{-1}$, which, since $N_k(\bar{\lambda}) \in Y$, equals $(c_k \cup N_k(\bar{\lambda}))^{-1}$, giving the claim.

The first and third vertical maps are isomorphisms, and the last vertical map is injective, and so by the five-lemma we get that the second vertical map is an isomorphism. The uniqueness of ι satisfying the two properties of the theorem follows from the argument for the analogous result in the characteristic zero case in [Kal16], §4.2. □

4.5 Extending to reductive groups

In order to apply the above cohomological results to the local Langlands correspondence, it is necessary to extend the above constructions to connected reductive groups over a local function field F. We use the same notation as above; $\mathcal{E}$ will always be a u-gerbe split over $\overline{F}$ with $[\mathcal{E}] = \alpha$. We start by briefly recalling non-abelian Čech cohomology and some fundamental cohomological results on reductive algebraic groups over F a local function field.

For a general ring R, we may define Čech cohomology sets $\check{H}^0(U_0 \to \text{Spec}(R), G)$ and $\check{H}^1(U_0 \to \text{Spec}(R), G)$ for an arbitrary (possibly non-abelian) R-group scheme G, using the conventions of [Gir71] III.3.6, which agree with our previous Čech cohomology conventions if G is abelian. Namely, we define differentials from $G(U_0)$ to $G(U_1)$ and from $G(U_1)$ to $G(U_2)$, given (respectively) by

$$g \mapsto p_1(g)^{-1}p_2(g), \quad g \mapsto p_{12}(g)p_{23}(g)p_{13}(g)^{-1}. \tag{4.3}$$

We may then take $\check{H}^0(U_0 \to \text{Spec}(F), G)$ to be the fiber over the identity of the degree-zero differential, and $\check{H}^1(U_0 \to \text{Spec}(R), G)$ to be the pointed set consisting of the fiber over the identity of the degree-one differential modulo the equivalence relation given by declaring a and b equivalent if there exists $g \in G(U_0)$ with $a = p_1(g)^{-1}bp_2(g)$. It is clear that $\check{H}^1(U_0 \to \text{Spec}(R), G)$ classifies isomorphism classes of G-torsors over R which are trivialized over U_0.

Theorem 4.5.1 *For any simply-connected reductive group G over a local field F, $H^1(F, G) = 0$.*

This is [Ser95], Theorem 5.

Theorem 4.5.2 *Let G be a semisimple group over F a local field, and let C denote the kernel of the central isogeny $G_{sc} \to G$. Then the natural map $H^1(F, G) \to H^2(F, C)$ is a bijection, thus endowing $H^1(F, G)$ with the canonical structure of an abelian group.*

This is Theorem 2.4 in [Tha08].

The arguments in [Kal16] which extend the Tate-Nakayama isomorphism of §4.4 to reductive groups rely heavily on the existence of elliptic/fundamental maximal tori (see [Kot86], §10), and their corresponding cohomological properties.

Theorem 4.5.3 *Every semisimple algebraic group over a local function field F contains a maximal F-torus T which is anisotropic over F.*

This follows from §2.4 in [Deb06]. It follows immediately that every reductive group G contains a maximal F-torus which is F-anisotropic modulo $Z(G)^\circ$; this will be an elliptic maximal torus.

Moreover, we have the following result for G a connected reductive group over F, implied by the proof of Lemma 10.2 in [Kot86] and Theorem 4.5.2:

Theorem 4.5.4 *If T is an elliptic maximal torus of G, then $H^1(F, T) \to H^1(F, G)$ is surjective.*

We also have the following, which is a generalization of Theorem 1.2 in [Kot86]; it concerns the functor $\mathcal{A}$ from the category of connected reductive F-groups to abelian groups, defined by $\mathcal{A}(G) = \pi_0(Z(\widehat{G})^\Gamma)^*$, where $\widehat{G}$ denotes a Langlands dual group of G. Recall that Tate-Nakayama duality gives us an isomorphism $H^1(F, T) \xrightarrow{\sim} \pi_0(\widehat{T}^\Gamma)^*$ for any F-torus T (this will be reviewed in more detail in §5.1).

Theorem 4.5.5 *There is a unique extension of the above isomorphism of functors to an isomorphism of functors on the category of reductive F-groups, given by a natural transformation*

$$\alpha_G \colon H^1(F, G) \to \mathcal{A}(G).$$

This is [Tha11], Theorem 2.1.

We are now ready to extend our previous constructions on $\mathcal{T}$ to the category $\mathcal{R}$. For the most part, the arguments from [Kal16] carry over verbatim, since most depend on the structure theory of reductive groups, in particular the part of the theory that deals with character and cocharacter modules, which is uniform for local fields of any characteristic. The purpose of the remainder of this section is to summarize those results and fill in certain arguments which are different in the case of a local function field.

Proposition 4.5.6 *Proposition 4.2.4 holds for $[Z \to G]$ in $\mathcal{R}$, ignoring the $H^2(F, G)$ terms.*

Proof. Everything from the proof of 4.2.4 holds, except for the use of the five-lemma to give the surjectivity of $H^1(\mathcal{E}, Z \to G) \to H^1(F, \overline{G})$. Instead, we may use the analogous argument used in [Kal16], Proposition 3.6, using the existence of an elliptic maximal torus in G and replacing the use of Lemma 10.2 from [Kot86] with Theorem 4.5.4, its analogue for local function fields. □

Proposition 4.5.7 *(Analogue of Corollary 3.7 in [Kal16])*

1. *If G possesses anisotropic maximal tori, then the map $H^1(\mathcal{E}, Z \to G) \to \mathit{Hom}_F(u, Z)$ defined above is surjective.*

2. *If $S \subset G$ is an elliptic maximal torus, then the map*

$$H^1(\mathcal{E}, Z \to S) \to H^1(\mathcal{E}, Z \to G)$$

is surjective.

Proof. The same proof as in [Kal16] works, again replacing the use of Lemma 10.2 from [Kot86] with Theorem 4.5.4. □

Let $[Z \to G] \in \mathcal{R}$. We need to extend the functor $\overline{Y}_{+,\text{tor}}$ defined in §4.3. Following [Kal16], $\overline{Y}_{+,\text{tor}}[Z \to G]$ is taken to be the limit over all maximal F-tori S of G of the following colimit:

$$\varinjlim \frac{[X_*(S/Z)/X_*(S_{\text{sc}})]^N}{I(X_*(S)/X_*(S_{\text{sc}}))},$$

where the colimit is taken over the set of Galois extensions E/F splitting S and the superscript N denotes the kernel of the norm map. We need to explain what the limit maps are between the above objects for varying S. For two such tori S_1, S_2, picking $g \in G(F^s)$ such that $\text{Ad}(g)(S_1)_{F^s} = (S_2)_{F^s}$ induces an isomorphism

$$\text{Ad}(g)\colon X_*(S_1/Z)/X_*((S_1)_{\text{sc}}) \to X_*(S_2/Z)/X_*((S_2)_{\text{sc}})$$

which is independent of the choice of g, by Lemma 4.2 in [Kal16], and is thus Γ-equivariant. It follows that these maps may be used to define the desired limit maps for varying maximal F-tori in G.

We now extend the isomorphism of functors $\overline{Y}_{+,\text{tor}} \xrightarrow{\sim} H^1(\mathcal{E})$ on $\mathcal{T}$ given in Theorem 4.4.3 to the category $\mathcal{R}$. The strategy will be as follows: we will show that Lemmas 4.9 and 4.10 from [Kal16] hold in our setting, and then the result will follow from the proof of Theorem 4.11 in [Kal16], using the existence of elliptic maximal tori, as argued above, Proposition 4.5.7, and the aforementioned lemmas. As in §4.4, we work with the specific choice of $\mathcal{E}$ given by $\varprojlim_k \mathcal{E}_{\xi_k}$ for ξ_k as in §4.4; by the uniqueness of $H^1(\mathcal{E}, Z \to G)$ up to canonical isomorphism, this will prove the result for an arbitrary choice of $\mathcal{E}$.

Lemma 4.5.8 *(Analogue of Lemma 4.9 in [Kal16]) Let $[Z \to G] \in \mathcal{R}$ and $S \subset G$ a maximal torus. The fibers of the composition*

$$\overline{Y}_{+,tor}[Z \to S] \to H^1(\mathcal{E}, Z \to S) \to H^1(\mathcal{E}, Z \to G)$$

are torsors under the image of $X_*(S_{sc})_{\Gamma,tor}$ in $\overline{Y}_{+,tor}[Z \to S]$.

Proof. The argument of [Kal16] works here, replacing Theorem 1.2 of [Kot86] with the analogue for local function fields, namely Theorem 2.1 from [Tha11]. $\square$

Lemma 4.5.9 *(Analogue of Lemma 4.10 in [Kal16]) Let $[Z \to G] \in \mathcal{R}$, and let $S_1, S_2 \subset G$ be maximal tori defined over F. Let $g \in G(\overline{F})$ with $Ad(g)(S_1)_{\overline{F}} = (S_2)_{\overline{F}}$. If $\bar{\lambda}_i \in \overline{Y}_i^N$ are such that $\bar{\lambda}_2 = Ad(g)\bar{\lambda}_1$, then the images of $\iota_{[Z\to S_1]}(\bar{\lambda}_1)$ and $\iota_{[Z\to S_2]}(\bar{\lambda}_2)$ in $H^1(\mathcal{E}, Z \to G)$ coincide.*

Proof. This argument will require more substantial adjustments, so we recall some details of the argument in [Kal16]. If $P_i^\vee := X_*(S_{i,\text{ad}})$, the isogeny $S_i/Z \to S_i/(Z \cdot Z(\mathscr{D}(G)))$ provides an injection $\overline{Y}_i \to P_i^\vee \oplus X_*(G/Z \cdot \mathscr{D}(G))$; we write $\bar{\lambda}_i = p_1 + z$ according to this decomposition, and so $\bar{\lambda}_2 = p_2 + z$, with $p_2 = \text{Ad}(g)p_1$. As in [Kal16], we choose k large enough so that $n_k p_1 \in Q_1^\vee := X_*(S_{1,\text{sc}})$ and $n_k z \in X_*(Z(G)^\circ)$ [via the isogeny $Z(G)^\circ \to G/Z \cdot \mathscr{D}(G)$].

Our goal will be to show that $z_{\bar{\lambda}_2,k} = p_1(x) z_{\bar{\lambda}_1,k} p_2(x)^{-1}$ for some $x \in G_{\text{sc}}(\overline{F})$ (recall from §2.6 that this is what it means for two twisted Čech cocycles to be equivalent). We have that $\phi_{\bar{\lambda}_1,k} = \phi_{\bar{\lambda}_2,k}$ and $\widetilde{l_k c_k} \underset{E_k/F}{\sqcup} n_k z \in Z(G)^\circ(U_2)$, and hence by decomposing $n_k\bar{\lambda}_i = n_k p_i + n_k z$ we see that it's enough to show that $a_2 = p_1(x) a_1 p_2(x)^{-1}$ for some $x \in G_{\text{sc}}(\overline{F})$, where $a_i := \widetilde{l_k c_k} \underset{E_k/F}{\sqcup} n_k p_i$ (this will show that the classes of $\iota_{[Z\to S_1]}(\bar{\lambda}_1)$ and $\iota_{[Z\to S_2]}(\bar{\lambda}_1)$ are equal in $H^1(\mathcal{E}_{a_k}, Z \to G)$, and hence have the same pullback to $H^1(\mathcal{E}, Z \to G)$).

The image of $a_1 \in S_{1,\text{sc}}(U_1)$ in $S_{1,\text{ad}}$ is equal to $c_k \cup p_1$ (the usual Galois cohomology cup product), and is thus a Galois 1-cocycle, so we can twist the Γ-structure on G_{sc} using it, obtaining the twisted structure G^1_{sc}. By basic descent theory (see, for example, §4.5 in [Poo17]), we have an $\overline{F}$-group isomorphism

$$\phi\colon (G_{\text{sc}})_{\overline{F}} \xrightarrow{\sim} (G^1_{\text{sc}})_{\overline{F}}$$

satisfying $p_1^*\phi^{-1} \circ p_2^*\phi = \text{Ad}(a_1)$ on $(G_{\text{sc}})_{U_1}$.

We claim now that $p_1^*\phi(a_2 \cdot a_1^{-1})$ is a cocycle in $G^1_{\text{sc}}(U_1)$. It's enough to check that the differential post-composed with the group isomorphism $q_1^*\phi^{-1}$ sends this element to the identity in $G_{\text{sc}}(U_2)$.

One computes (using the non-abelian Čech differential formulas, see equation (4.3)) that

$$q_1^*\phi^{-1}(dp_1^*\phi(a_2 \cdot a_1^{-1})) = q_1^*\phi^{-1}[p_{12}^*p_1^*\phi(p_{12}(a_2 \cdot a_1^{-1})) \cdot p_{23}^*p_1^*\phi(p_{23}(a_2 \cdot a_1^{-1})) \cdot (p_{13}^*p_1^*\phi(p_{13}(a_2 \cdot a_1^{-1})))^{-1}].$$

Rewriting each composition of pullbacks in the usual way, this may be rewritten as:

$$q_1^*\phi^{-1}[q_1^*\phi(p_{12}(a_2 \cdot a_1^{-1})) \cdot q_2^*\phi(p_{23}(a_2 \cdot a_1^{-1})) \cdot (q_1^*\phi(p_{13}(a_2 \cdot a_1^{-1})))^{-1}].$$

Now distributing $q_1^*\phi^{-1}$ to each term (since ϕ is a morphism of group sheaves) gives:

$$p_{12}(a_2 \cdot a_1^{-1}) \cdot (q_1^*\phi^{-1} \circ q_2^*\phi)(p_{23}(a_2 \cdot a_1^{-1})) \cdot (p_{13}(a_2 \cdot a_1^{-1}))^{-1}.$$

Since $(q_1^*\phi^{-1} \circ q_2^*\phi) = p_{12}^*(p_1^*\phi^{-1} \circ p_2^*\phi) = p_{12}^*\mathrm{Ad}(a_1)$, the above element becomes

$$p_{12}(a_2)p_{12}(a_1)^{-1}p_{12}(a_1)p_{23}(a_2)[p_{23}(a_1)^{-1}p_{12}(a_1)^{-1}p_{13}(a_1)]p_{13}(a_2)^{-1}.$$

The bracketed terms all lie in $S_{1,\mathrm{sc}}(U_2)$ and hence may be rearranged to give $da_1^{-1} \in Z(G_{\mathrm{sc}})(U_2)$. By centrality, this may then be moved to the front, yielding $da_2 \in Z(G_{\mathrm{sc}})(U_2)$, giving us $da_2 \cdot da_1^{-1}$. However, we know that

$$da_1 = d\widetilde{l_k c_k} \underset{E_k/F}{\sqcup} n_k p_1 = d\widetilde{l_k c_k} \underset{E_k/F}{\sqcup} n_k p_2 = da_2,$$

because the images of p_1 and p_2 under $P_i^\vee \to P_i^\vee/Q_i^\vee \to \mathrm{Hom}(\mu_n, Z(G_{\mathrm{sc}}))$ coincide, showing the cocycle claim.

Since G^1_{sc} is simply-connected, Theorem 4.5.1 tells us that $p_1^*\phi(a_2 \cdot a_1^{-1}) = d(\phi(x))$, some $x \in G_{\mathrm{sc}}(\overline{F})$. One computes easily (using a similar but simpler calculation) as above that

$$a_2 \cdot a_1^{-1} = p_1^*\phi^{-1}d(\phi(x)) = p_1(x)^{-1}a_1p_2(x)a_1^{-1},$$

as desired. □

We are now ready to prove the main result of the section.

Theorem 4.5.10 *(Theorem 4.11 in [Kal16]) The isomorphism ι of Theorem 4.4.3 extends to an isomorphism*

$$\iota\colon \overline{Y}_{+,tor} \to H^1(\mathcal{E})$$

of functors $\mathcal{R} \to$ Sets which lifts the morphism of functors on $\mathcal{R}$ from $\overline{Y}_{+,tor} \to \mathrm{Hom}_F(u, -)$.

Proof. We define the map in this proof for a fixed $[Z \to G] \in \mathcal{R}$; the fact that this map satisfies the statement of the theorem follows from the proof of the analogous result in [Kal16] (the arguments loc. cit. work in our setting because of the above lemmas). Defining this isomorphism of functors will first require defining, for a fixed elliptic maximal torus S of G defined over F, a bijection

$$\varinjlim \frac{[X_*(S/Z)/X_*(S_{\mathrm{sc}})]^N}{I(X_*(S)/X_*(S_{\mathrm{sc}}))} \xrightarrow{\sim} H^1(\mathcal{E}, Z \to G).$$

For E splitting S, we have an exact sequence

$$\frac{X_*(S_{\mathrm{sc}})^N}{IX_*(S_{\mathrm{sc}})} \longrightarrow \frac{[X_*(S/Z)]^N}{IX_*(S)} \longrightarrow \frac{[X_*(S/Z)/X_*(S_{\mathrm{sc}})]^N}{I(X_*(S)/X_*(S_{\mathrm{sc}}))} \longrightarrow \frac{X_*(S_{\mathrm{sc}})^\Gamma}{N(X_*(S_{\mathrm{sc}}))},$$

where the last map sends an element represented by $x \in X_*(S/Z)$ to $N(x)$, which gives an isomorphism

$$\overline{Y}_{+,\text{tor}}[Z \to S]/(X_*(S_{\text{sc}})^N/IX_*(S_{\text{sc}})) \to \varinjlim \frac{[X_*(S/Z)/X_*(S_{\text{sc}})]^N}{I(X_*(S)/X_*(S_{\text{sc}}))},$$

since H^0_{Tate} vanishes for an elliptic maximal torus of a simply-connected semisimple group (in any characteristic).

Note that we also have a bijection

$$\overline{Y}_{+,\text{tor}}[Z \to S]/(X_*(S_{\text{sc}})^N/IX_*(S_{\text{sc}})) \to H^1(\mathcal{E}, Z \to G)$$

induced by the composition $\overline{Y}_{+,\text{tor}}[Z \to S] \xrightarrow{\sim} H^1(\mathcal{E}, Z \to S) \twoheadrightarrow H^1(\mathcal{E}, Z \to G)$, where the first map is from Theorem 4.4.3 and the surjectivity of the second map is from Proposition 4.5.7. The induced bijection is an immediate consequence of Lemma 4.5.8. We thus obtain the desired bijection.

For this to be well-defined across the inverse limit, we need to check that if S_1, S_2 are two elliptic maximal F-tori in G and we take $g \in G(F^s)$ such that $\text{Ad}(g)(S_1)_{F^s} = (S_2)_{F^s}$, then an element $\bar{\lambda} \in \varinjlim \frac{[X_*(S_1/Z)/X_*((S_1)_{\text{sc}})]^N}{I(X_*(S_1)/X_*((S_1)_{\text{sc}})}$ maps to the same element in $H^1(\mathcal{E}, Z \to G)$ as its isomorphic image (via $\text{Ad}(g)$) in the same direct limit with S_2 instead of S_1.

This follows because, by what we did above, we may lift $\bar{\lambda}$ to $\dot{\lambda} \in \frac{[X_*(S_1/Z)]^N}{IX_*(S_1)} = \overline{Y}_{+,\text{tor}}[Z \to S_1]$ and then map to $H^1(\mathcal{E}, Z \to G)$ via $H^1(\mathcal{E}, Z \to S_1)$, and may analogously lift the image of $\bar{\lambda}$ in $\varinjlim \frac{[X_*(S_2/Z)/X_*((S_2)_{\text{sc}})]^N}{I(X_*(S_2)/X_*((S_2)_{\text{sc}})}$ to $\text{Ad}(g)\dot{\lambda} \in \frac{[X_*(S_2/Z)]^N}{IX_*(S_2)}$ and then map to $H^1(\mathcal{E}, Z \to G)$ via $H^1(\mathcal{E}, Z \to S_2)$. Now Lemma 4.5.9 implies that these images coincide. □

Corollary 4.5.11 *The isomorphism of functors constructed in Theorem 4.5.10 is unique satisfying the hypotheses.*

Proof. This follows from the discussion in [Kal16], §4.2, which relies on the existence of elliptic maximal tori and Corollary 3.7 loc. cit, both of which we have established in our situation. □

We conclude by citing one more result of [Kal16] that holds here, which will be used in Chapter 6.

Proposition 4.5.12 *Let G be a connected reductive group defined over F, let Z be the center of $\mathscr{D}(G)$, and set $\overline{G} = G/Z$. Then both natural maps*

$$H^1(\mathcal{E}, Z \to G) \to H^1(F, \overline{G}) \to H^1(F, G_{ad})$$

are surjective. If G is split, then the second map is bijective and the first map has trivial kernel.

Proof. See the proof of Corollary 3.8 in [Kal16], replacing the use of Theorem 1.2 in [Kot86] with [Tha11], Theorem 2.1. □

CHAPTER 5

The Relative Local Transfer Factor

In order to apply the concepts we have developed, we need to define the local transfer factor, as defined in [LS87], for reductive groups over local function fields. For expository purposes, we make this section entirely self-contained.

5.1 Notation and preliminaries

We will always take G to be a connected reductive group defined over F, a local field of characteristic $p > 0$. Let G^* be a quasi-split group over F such that we have $\psi\colon G \xrightarrow{\sim} G^*$ satisfying $\psi^{-1} \circ {}^{\sigma}\psi = \mathrm{Ad}(u_\sigma)$ for some $u_\sigma \in G_{\mathrm{ad}}(F^s)$ for all σ in Γ. That is to say, G^* is a *quasi-split inner form of* G over F. One important difference that emerges here in the positive characteristic case is that such a u_σ need not have a lift in $G(F^s)$, due to the potential non-smoothness of $Z(G)$. Such lifts are useful for computational purposes, and so to combat the smoothness issue we give an equivalent characterization of inner forms in the fppf language.

Again for G^* a quasi-split group over F, we say that G^* is a quasi-split inner form of G if there is an isomorphism $\psi\colon G_{F^s} \xrightarrow{\sim} G^*_{F^s}$ satisfying $p_1^*\psi^{-1} \circ p_2^*\psi = \mathrm{Ad}(\bar{u})$ for some $\bar{u} \in G_{\mathrm{ad}}(F^s \otimes_F F^s)$. Since $H^1(\overline{F} \otimes_F \overline{F}, Z(G)) = 0$ (by Proposition 3.1.4), we may always lift $\bar{u}$ to an element $u \in G(\overline{F} \otimes_F \overline{F})$. Recall that p_i denotes the ith projection map from Spec$\overline{F} \times_F$ Spec$\overline{F}$ to Spec$\overline{F}$. We will frequently treat inner forms using this approach, as it enables computations using the Čech cohomology of the fpqc cover Spec$(\overline{F}) \to$ Spec(F) (see, for example, §5.3.3).

We fix some *dual group* $\widehat{G}$ corresponding to G, in the sense of [Kot84], §1.5, and define ${}^LG := \widehat{G}(\mathbb{C}) \rtimes W_F$ the associated L-group of G, where W_F denotes the absolute Weil group of F. This is a topological group, where $\widehat{G}(\mathbb{C})$ is given the analytic topology in the usual way. Associated to $\widehat{G}$ is a Γ-equivariant bijection $\Psi(G)^\vee \to \Psi(\widehat{G})$ of based root data (see [Kot84], §1.1), and we define a bijection $\Psi(G^*)^\vee \xrightarrow{\psi} \Psi(G)^\vee \to \Psi(\widehat{G})$, which, along with the data of $\widehat{G}$ with its given Γ-action, also defines a dual group for G^*—note that this new bijection is still Γ-equivariant precisely because G and G^* are inner forms.

Definition 5.1.1 *We call a tuple $(H, \mathcal{H}, s, \eta)$ an* ***endoscopic datum*** *for G if H is a quasi-split reductive group defined over F with a choice of dual group $\widehat{H}$, $\mathcal{H}$ is a split extension of W_F by $\widehat{H}(\mathbb{C})$, and $\eta\colon \mathcal{H} \to {}^L G$ is a map such that:*

1. *The conjugation action by W_F on $\widehat{H}$ induced by a section $W_F \to \mathcal{H}$ and any Γ-splitting of $\widehat{H}$ coincides with the L-group W_F-action on $\widehat{H}$;*
2. *The element s lies in $Z(\widehat{H})(\mathbb{C})$;*
3. *The map η is a morphism of W_F-extensions which restricts to an isomorphism of algebraic groups $\widehat{H} \xrightarrow{\sim} Z_{\widehat{G}}(\eta(s))^\circ$;*
4. *We have $s \in Z(\widehat{H})^\Gamma \cdot \eta^{-1}(Z(\widehat{G}))$.*

This is formulated slightly differently from the exposition in [LS87], §1.2; it is easily checked that this definition is equivalent to the one given there. An *isomorphism of endoscopic data* from $(H, \mathcal{H}, s, \eta)$ to $(H', \mathcal{H}', s', \eta')$ is an element $g \in \widehat{G}(\mathbb{C})$ such that $g\eta(\mathcal{H})g^{-1} = \eta'(\mathcal{H}')$, thus inducing an isomorphism $\beta\colon \mathcal{H} \xrightarrow{\eta'^{-1} \circ \mathrm{Ad}(g) \circ \eta} \mathcal{H}'$, which we further require to satisfy that $\beta(s)$ and s' are equal modulo $Z(\widehat{H'})^{\Gamma,\circ} \cdot \eta'^{-1}(Z(\widehat{G}))$. One checks that this agrees with the analogous definition in [LS87].

Fix an endoscopic datum $(H, \mathcal{H}, s, \eta)$ for G. If we fix two Borel pairs $(B_G, T_G), (\mathscr{B}_G, \mathscr{T}_G)$ in $G_{F^s}, \widehat{G}$ (respectively), then the bijection of based root data gives an isomorphism $\widehat{T_G} \to \mathscr{T}_G$. The associated isomorphism $X_*(T_G) \to X^*(\mathscr{T}_G)$ transports the coroot system $R^\vee$ of T_G to the root system of $\mathscr{T}_G$ mapping the B_G-simple coroots to the $\mathscr{B}_G$-simple roots, and identifies the Weyl group $W(G_{F^s}, T_G)$ with the Weyl group $W(\widehat{G}, \mathscr{T}_G)$. Moreover, if $(\mathscr{T}_H, \mathscr{B}_H)$ is a pair in $\widehat{H}$, then we may find $g \in \widehat{G}(\mathbb{C})$ such that $(\mathrm{Ad}(g) \circ \eta)(\mathscr{T}_H) = \mathscr{T}_G$ and $\mathrm{Ad}(g) \circ \eta$ maps $\mathscr{B}_H$ into $\mathscr{B}_G$. This means that if we fix a pair (T_H, B_H) in H_{F^s}, then we have an isomorphism $\widehat{T_H} \to \mathscr{T}_H \to \mathscr{T}_G \to \widehat{T_G}$, inducing an isomorphism $T_H \to T_G$. This isomorphism transports $R_H, R_H^\vee, W(H_{F^s}, T_H)$ into $R, R^\vee, W(G_{F^s}, T_G)$.

Suppose that we fix such a T_H, T_G, but now require that they are defined over F. An F-isomorphism $T_H \to T_G$ is called *admissible* if it is obtained as in the above paragraph (this is not unique—we chose four Borel subgroups in the above construction). We sometimes also call this an *admissible embedding* of T_H in G. Such an embedding is unique up to conjugacy by an element of the set $\tilde{\mathfrak{A}}(T_G)$, defined by

$$\tilde{\mathfrak{A}}(T_G) = \{\bar{g} \in G_{\mathrm{ad}}(F^s)\colon \mathrm{Ad}(\bar{g}^{-1}\,{}^\sigma(\bar{g}))\big|_{(T_G)_{F^s}} = \mathrm{id}_{(T_G)_{F^s}}\ \forall \sigma \in \Gamma\}.$$

Another way of describing this set is those points $\bar{g} \in G_{\mathrm{ad}}(F^s)$ such that $\mathrm{Ad}(\bar{g})\big|_{(T_G)_{F^s}}$ is defined over F. Note that given such a $\bar{g}$, we may always find some $g \in G(F^s)$ inducing the same automorphism of T_G. Indeed, if $g \in G(\overline{F})$ is such that $\mathrm{Ad}(g)\big|_{(T_G)_{\overline{F}}}$ is defined over F, then we may find a point $g' \in G(F^s)$ such that $\mathrm{Ad}(g) = \mathrm{Ad}(g')$ on T_G—this follows from the fact that $N_G(T_G)/T_G$ is étale. Thus, such an embedding is also unique up to conjugacy by an element of the set

$$\mathfrak{A}(T_G) = \{g \in G(F^s) \colon g^{-1} \cdot {}^{\sigma}g \in T_G(F^s)\ \forall \sigma \in \Gamma\}.$$

Given any $g \in \mathfrak{A}(T_G)$, we may also find a point in $G_{\mathrm{sc}}(F^s)$ inducing the same map on T_G, where G_{sc} denotes the simply connected cover of $\mathscr{D}(G)$. To see this, first note that there is no harm in assuming that G is semisimple. Suppose that $\mathrm{Ad}(g)$ sends T to T', where T and T' are two maximal F-tori. Then we may take the preimages $(T_{\mathrm{sc}})_{\overline{F}}, (T'_{\mathrm{sc}})_{\overline{F}}$ in $(G_{\mathrm{sc}})_{\overline{F}}$, and fix a preimage $\tilde{g} \in G_{\mathrm{sc}}(\overline{F})$ of g, so that $\mathrm{Ad}(\tilde{g}) \colon (T_{\mathrm{sc}})_{\overline{F}} \xrightarrow{\sim} (T'_{\mathrm{sc}})_{\overline{F}}$. This isomorphism is defined over F^s, i.e., we get a descent to an isomorphism $(T_{\mathrm{sc}})_{F^s} \xrightarrow{\sim} (T'_{\mathrm{sc}})_{F^s}$, which is given by $\mathrm{Ad}(x)$ for some $x \in G_{\mathrm{sc}}(F^s)$, again using that the Weyl group scheme is étale; then x satisfies $\mathrm{Ad}(x)\big|_{T_{F^s}} = \mathrm{Ad}(\tilde{g})\big|_{T_{F^s}}$, as desired.

We call an element $\gamma \in G(\overline{F})$ *strongly regular* if it is semisimple and its centralizer is a maximal torus (there is a notion of strong regularity for non-semisimple elements but we will not need it here); denote the subset of strongly regular F-points of G by $G_{\mathrm{sr}}(F)$. We call an element $\gamma_H \in H(F)$ *strongly G-regular* if it is the preimage of a strongly regular $\gamma_G \in G(F)$ under an admissible isomorphism. In such a case, γ_H is itself strongly regular in H, and the admissible isomorphism between centralizers $T_H \xrightarrow{\sim} T_G$ sending γ_H to γ_G is unique; denote this subset of $H(F)$ by $H_{G-\mathrm{sr}}(F)$, and call such a pair of elements γ_H, γ_G *related*.

Lemma 5.1.2 *Let T_H be the centralizer of $\gamma_H \in H_{G-sr}(F)$. Then there exists an admissible embedding $T_H \hookrightarrow G^*$.*

Proof. By assumption we already have an admissible isomorphism $T_H \to T_G$, where T_G is a maximal F-torus of G. It is easy to see that it then suffices to find an admissible embedding of T_G into G^*. We can always do this, since G^* is quasi-split and F is a non-archimedean local field, see for example [Kal19], Lemma 3.2.2. □

5.1.1 The Tits section

We need to discuss the *Tits section*, which is a (non-multiplicative) map $n \colon W(G_{F^s}, T_{F^s}) \hookrightarrow N_G(T)(F^s)$. To do this, we must fix a Borel subgroup B of G_{F^s} (corresponding to a root basis Δ) and a basis $\{X_\alpha\}$ of the root space $\mathfrak{g}_\alpha \subset \mathrm{Lie}(G_{F^s})$ for each $\alpha \in \Delta$. Let G_α be the Levi subgroup of $\mathscr{D}(G_{F^s})$ corresponding to the root α; then there is a unique embedding $\zeta_\alpha \colon SL_2 \to G_\alpha$ which (on Lie algebras) sends $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ to X_α and such that the image of $\zeta_\alpha(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix})$ in $W(G_{F^s}, T_{F^s})$

is the reflection r_α defined by α (see [KS12], §2.1). We then map r_α to the image of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ under ζ_α. We may then lift any element of $W(G_{F^s}, T_{F^s})$ by considering their reduced expression in terms of Δ.

5.1.2 Duality results

We recall Langlands' reinterpretation of Tate-Nakayama duality. Let T be an F-torus; the usual Tate-Nakayama duality theorem gives a perfect $\mathbb{Z}$-pairing

$$H^1(F, T) \times H^1(\Gamma, X^*(T)) \to \mathbb{Q}/\mathbb{Z},$$

see for example [Mil06], I.2.4. Consider the short exact sequence of abelian groups

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \longrightarrow 1.$$

Tensoring this sequence over $\mathbb{Z}$ with $X_*(\widehat{T}) = X^*(T)$ preserves exactness, and thus yields the exact sequence

$$0 \longrightarrow X^*(T) \longrightarrow \text{Lie}(\widehat{T}) \longrightarrow \widehat{T}(\mathbb{C}) \longrightarrow 1,$$

which then gives a canonical identification $H^1(\Gamma, X^*(T)) \xrightarrow{\sim} \pi_0(\widehat{T}^\Gamma)$, and hence a perfect pairing

$$H^1(F, T) \times \pi_0(\widehat{T}^\Gamma) \to \mathbb{Q}/\mathbb{Z}. \tag{5.1}$$

Returning to the setting of a connected reductive group G, note that if T is any maximal F-torus of G, for any maximal torus $\mathscr{T}$ of $\widehat{G}$, we have an isomorphism $\mathscr{T} \to \widehat{T}$ which is unique up to precomposing with conjugation by an element of $N_{\widehat{G}}(\mathscr{T})(\mathbb{C})$, so we get a canonical embedding $Z(\widehat{G}) \hookrightarrow \widehat{T}$, which clearly also does not depend on the choice of $\mathscr{T}$ (any two such tori are $\widehat{G}(\mathbb{C})$-conjugate). Denote $\widehat{T}/Z(\widehat{G})$ by $\widehat{T}_{\text{ad}}$. Assume for the moment that G is semisimple. One checks using the basic theory of (co)character groups and root systems that (via the above embedding) $X^*(Z(\widehat{G}))$ corresponds to the quotient $X_*(T)/\mathbb{Z}R(G_{F^s}, T_{F^s})^\vee$ of $X^*(\widehat{T}) = X_*(T)$. Whence, we have a canonical identification of $X^*(\widehat{T}_{\text{ad}})$ with $X_*(T_{\text{sc}})$, where T_{sc} is the preimage of T in G_{sc}, giving a Γ-isomorphism $\widehat{T_{\text{sc}}} \xrightarrow{\sim} \widehat{T}_{\text{ad}}$. For general G, one checks easily that a similar argument yields a canonical isomorphism $\widehat{T_{\text{sc}}} \xrightarrow{\sim} \widehat{T}_{\text{ad}}$, where now T_{sc} denotes the preimage of the maximal F-torus $T \cap \mathscr{D}(G) \subset \mathscr{D}(G)$ in G_{sc}, the simply connected cover of $\mathscr{D}(G)$. We conclude that Tate-Nakayama then gives a perfect pairing

$$H^1(F, T_{\text{sc}}) \times \pi_0(\widehat{T}_{\text{ad}}^\Gamma) \to \mathbb{Q}/\mathbb{Z}.$$

We may replace $\mathbb{Q}/\mathbb{Z}$ by $\mathbb{C}^*$ by means of the embedding $\mathbb{Q}/\mathbb{Z} \xrightarrow{\exp} \mathbb{C}^*$.

Recall, for an F-torus T split over E/F a finite Galois extension, we have the classical Tate isomorphism $H^{-1}_{\text{Tate}}(\Gamma_{E/F}, X_*(T)) \xrightarrow{\sim} H^1(F, T)$ induced by taking the cup product with the canonical class (see [Tat66]). The following useful duality result generalizes this to finite multiplicative group schemes over F.

Proposition 5.1.3 *Let T be an F-torus and S the quotient of T by a finite F-subgroup Z. Choose E/F a finite Galois extension splitting T and set $\Gamma := \Gamma_{E/F}$. Choose E large enough so that $|Z|$ and $|H^1(\Gamma, X^*(T))|$ divide $[E\colon F]$ (for finiteness of the latter, see [Mil06], III.6). We have a canonical isomorphism*

$$H^{-2}_{\mathit{Tate}}(\Gamma, X_*(S)/X_*(T)) \xrightarrow{\sim} H^1(F, Z)$$

which is compatible with the Tate isomorphism $H^{-1}_{\mathit{Tate}}(\Gamma, X_*(T)) \xrightarrow{\sim} H^1(F, T)$.

Proof. Cohomology in negative degrees will always be Tate cohomology, and we omit the “Tate" notation in such cases. We have an exact sequence of character groups

$$0 \longrightarrow X^*(S) \longrightarrow X^*(T) \longrightarrow X^*(Z) \longrightarrow 0$$

which, by applying the functor $\text{Hom}(-, \mathbb{Z})$, yields the short exact sequence (of Γ-modules)

$$0 \longrightarrow X_*(T) \longrightarrow X_*(S) \xrightarrow{\delta} \text{Ext}^1_{\mathbb{Z}}(X^*(Z), \mathbb{Z}) \longrightarrow 0.$$

By basic homological algebra, we have a canonical isomorphism (as Γ-modules)

$$\text{Ext}^1_{\mathbb{Z}}(X^*(Z), \mathbb{Z}) \cong \text{Hom}_{\mathbb{Z}}(X^*(Z), \mathbb{Q}/\mathbb{Z}).$$

We make these identifications in what follows without comment. For an abelian group M, we set $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) =: M^*$. We have the obvious identifications $H^{-1}(\Gamma, \mathbb{Q}/\mathbb{Z}) = \mathbb{Z}/[E\colon F]\mathbb{Z}$, and $H^0_{\text{Tate}}(\Gamma, \mathbb{Z}) = \mathbb{Z}/[E\colon F]\mathbb{Z}$. By Proposition 7.1 and Exercise 3 (respectively) in [Bro82], we have the following duality pairings of Γ-modules induced by the cup product and these identifications:

$$H^{-2}(\Gamma, X^*(Z)^*) \times H^1(\Gamma, X^*(Z)) \to \mathbb{Z}/[E\colon F]\mathbb{Z},$$

$$H^{-1}(\Gamma, X_*(T)) \times H^1(\Gamma, X^*(T)) \to \mathbb{Z}/[E\colon F]\mathbb{Z}.$$

Note that the group $H^1(\Gamma, X^*(Z))$ is $|Z|$-torsion, so that

$$H^1(\Gamma, X^*(Z))^* = \text{Hom}_{\mathbb{Z}}(H^1(\Gamma, X^*(Z)), \mathbb{Z}/[E\colon F]\mathbb{Z}),$$

analogously for $H^1(\Gamma, X^*(T))$.

As a consequence, we have a canonical isomorphism

$$H^{-2}(\Gamma, X_*(S)/X_*(T)) \xrightarrow{\sim} H^1(\Gamma, X^*(Z))^* \xrightarrow{\sim} H^1(F, Z),$$

where the second isomorphism comes from the Poitou-Tate duality pairing for finite commutative group schemes over arbitrary local fields, see [Mil06], Theorem III.6.10, and is induced by the cup-product followed by the invariant map. We now get a commutative diagram

$$\begin{array}{ccc} H^{-2}(\Gamma, X^*(Z)^*) & \longrightarrow & H^{-1}(\Gamma, X_*(T)) \\ \downarrow \wr & & \downarrow \wr \\ H^1(\Gamma, X^*(Z))^* & \longrightarrow & H^1(\Gamma, X^*(T))^* \\ \downarrow \wr & & \downarrow \wr \\ H^1(F, Z) & \longrightarrow & H^1(F, T), \end{array}$$

where the top square commutes by the functoriality of the cup product in Tate cohomology and the bottom square commutes by the discussion in [Mil06], §III.6; see in particular the diagram used in the proof of Lemma 6.11 loc. cit. The right-hand column equals the classical Tate isomorphism discussed in [Tat66], again by the functoriality of the cup product in Tate cohomology. □

Remark 5.1.4 *This remark concerns how the above discussion relates to the Tate-Nakayama pairing involving $\pi_0(\widehat{T}^\Gamma)$ discussed earlier. Identifying $H^1(\Gamma, X^*(T)) = H^1(\Gamma, X_*(\widehat{T}))$ with $\widehat{T}(\mathbb{C})^\Gamma/(\widehat{T}(\mathbb{C})^\Gamma)^\circ$ as above, we note that there is a natural pairing*

$$H^{-1}(\Gamma, X_*(T)) \times \frac{\widehat{T}(\mathbb{C})^\Gamma}{(\widehat{T}(\mathbb{C})^\Gamma)^\circ} = H^{-1}(\Gamma, X^*(\widehat{T})) \times \frac{\widehat{T}(\mathbb{C})^\Gamma}{(\widehat{T}(\mathbb{C})^\Gamma)^\circ} \to \mathbb{C}^* \tag{5.2}$$

given by evaluating an element on a character. One checks that the following diagram commutes:

$$\begin{array}{ccc} H^1(F, T) \times \pi_0(\widehat{T}^\Gamma) & \longrightarrow & \mathbb{C}^* \\ \downarrow f \times id & & \| \\ H^{-1}(\Gamma, X^*(\widehat{T})) \times \pi_0(\widehat{T}^\Gamma) & \longrightarrow & \mathbb{C}^*, \end{array}$$

where the top pairing is the one from (5.1)*, the bottom pairing is as in* (5.2)*, and we are using f to denote the isomorphism $H^1(F, T) \to H^{-1}(\Gamma, X_*(T))$ constructed above.*

We conclude this subsection by recalling Langlands duality for tori, which is the following result:

Theorem 5.1.5 *For an F-torus T, F a local field, we have a canonical isomorphism*

$$H^1_{cts}(W_F, \widehat{T}(\mathbb{C})) \xrightarrow{\sim} \mathit{Hom}_{cts}(T(F), \mathbb{C}^*).$$

This isomorphism induces a pairing

$$H^1_{cts}(W_F, \widehat{T}(\mathbb{C})) \times T(F) \to \mathbb{C}^*$$

which is functorial with respect to F-morphisms of tori and respects restriction of scalars.

For the proof, see [Lan97], Theorem 2.a and [Bor79], §9 and §10.

5.2 Setup

This section completely follows §2 of [LS87] and §2 of [KS12]; its purpose is to explain why the results proved therein still work in our section.

5.2.1 The splitting invariant

Fix a connected reductive F-group G which we assume to be quasi-split over F, and an F-splitting $(B_0, T_0, \{X_\alpha\})$, along with an arbitrary maximal F-torus T in G. Assume further that G is semisimple and simply-connected. For a root $\alpha \in R := R(G_{F^s}, T_{F^s})$, we take $\Gamma_\alpha, \Gamma_{\pm\alpha}$ to be the stabilizers of α and $\{\alpha, -\alpha\}$, respectively, with $F_\alpha \supset F_{\pm\alpha}$ the corresponding fixed fields. An *a-data* $\{a_\alpha\}_{\alpha\in R}$ for the Γ-action on R is an element $a_\alpha \in F_\alpha^*$ for each $\alpha \in R$ satisfying $\sigma(a_\alpha) = a_{\sigma\alpha}$ for all $\sigma \in \Gamma$ and $a_{-\alpha} = -a_\alpha$. It is easy to check that a-data exist for our Γ action on R above; fix such a datum $\{a_\alpha\}_{\alpha\in R}$. Our goal is to define the *splitting invariant* $\lambda_{\{a_\alpha\}}(T) \in H^1(F, T)$.

We first choose a Borel subgroup B of G_{F^s} containing T, and take some $h \in G(F^s)$ such that h conjugates the pair $((B_0)_{F^s}, (T_0)_{F^s})$ to (B_{F^s}, T_{F^s}). Denote by σ_T the action of $\sigma \in \Gamma$ on T_{F^s} and its transport to $(T_0)_{F^s}$ via $\mathrm{Ad}(h)^{-1}$. For ease of notation, let Ω denote the absolute Weyl group of $W(G_{F^s}, (T_0)_{F^s})$, with Tits section $n\colon \Omega \to N_G(T_0)(F^s)$. We then have (as automorphisms of the root system $R(G, T_0)$)

$$\sigma_T = \omega_T(\sigma) \rtimes \sigma_{T_0} \in \Omega \rtimes \Gamma,$$

where $\omega_T(\sigma) := n(h \cdot \sigma(h)^{-1}) \in N_G(T_0)(F^s)$. We may view our a-data $\{a_\alpha\}_{\alpha\in R}$ as an a-data for the (transported) action of Γ on $R(G, T_0)$, and denote it also by $\{a_\alpha\}_\alpha$.

For any automorphism ζ of $R(G, T_0)$, we define the element $x(\zeta) \in T_0(F^s)$ by

$$x(\zeta) = \prod_{\alpha\in R(\zeta)} \alpha^\vee(a_\alpha),$$

where $R(\zeta) = \{\alpha \in R(G, T_0) | \alpha > 0, \zeta^{-1}\alpha < 0\}$ where the ordering on $R(G, T_0)$ is from the base Δ corresponding to the Borel subgroup B_0.

Then the function

$$m(\sigma) := x(\sigma_T)n(\omega_T(\sigma))$$

is a 1-cocycle of Γ in $N_G(T_0)(F^s)$ and

$$t(\sigma) := hm(\sigma)\sigma(h)^{-1}$$

is a 1-cocycle of Γ in $T(F^s)$, whose class we take to be the splitting invariant $\lambda_{\{a_\alpha\}}(T) \in H^1(F, T)$—for a proof, see [LS87] §2.3, which as [KS12] explains, works in any characteristic. The same references show that $\lambda_{\{a_\alpha\}}(T)$ is independent of the choice of h and the Borel subgroup of G_{F^s} containing T_{F^s}. However, it does depend on the F-splitting of G.

5.2.2 χ-data and L-embeddings

The following discussion is essentially a summary of §2.4-2.6 in [LS87]. To more closely align with [LS87],§2.5, we replace F^s by a finite Galois extension L and denote $\Gamma_{L/F}$ by Γ and $W_{L/F}$, the relative Weil group, by W. We will fix an arbitrary Γ-module X which is finitely-generated and free over $\mathbb{Z}$, along with a finite subset Γ-stable subset $\mathscr{R} \subset X$ closed under inversion. Any Γ-set is also a W-set by means of inflation along the surjection $W \to \Gamma$. Set $\Gamma' := \Gamma \times \mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z}$ acts on X by inversion. As in §5.2.1, for $\lambda \in \mathscr{R}$ we define $\Gamma_{+\lambda}$ (resp. $\Gamma_{\pm\lambda}$) to be the stabilizer of $\{\lambda\}$ (resp. $\{\pm\lambda\}$), with corresponding fixed field $F_\lambda \subset L$ (resp. $F_{\pm\lambda}$). The reason we want to work in this increased generality is to allow our theory to encompass the actions of Γ on the character groups of tori in $\widehat{G}$, a Langlands dual of the connected reductive F-group G. Define a *gauge* on $\mathscr{R}$ to be a function $p \colon \mathscr{R} \to \{\pm 1\}$ such that $p(-\lambda) = -p(\lambda)$.

Definition 5.2.1 *We say that a collection of continuous characters $\{\chi_\lambda \colon F_\lambda^* \to \mathbb{C}^*\}_{\lambda \in \mathscr{R}}$ is a χ-**data** if it satisfies $\chi_{-\lambda} = \chi_\lambda^{-1}$ and $\chi_\lambda \circ \sigma^{-1} = \chi_{\sigma\lambda}$ for all $\sigma \in \Gamma$, and if $[F_\lambda \colon F_{\pm\lambda}] = 2$, χ_λ extends the quadratic character $F_{\pm\lambda}^* \to \{\pm 1\}$ associated to the quadratic extension F_λ that we obtain from local class field theory.*

It is straightforward to check that we can always find a χ-data; fix such a χ-data $\{\chi_\lambda\}_{\lambda \in \mathscr{R}}$.

Assume for the moment that Γ' acts transitively on $\mathscr{R}$; fix $\lambda \in \mathscr{R}$, set $\Gamma_\pm := \Gamma_{\pm\lambda}$, and choose representatives $\sigma_1, \ldots \sigma_n$ for $\Gamma_\pm \setminus \Gamma$. We set $W_+ := W_{L/F_+}, W_\pm = W_{L/F_\pm}$. We may view the character χ_λ as a (continuous) character on W_+, by taking $\chi_\lambda \circ \mathbf{a}_{L/F_+}$, where $\mathbf{a}_{L/F_+} \colon W_+ \to F_+^*$ is the Artin reciprocity map.

Define a gauge p on $\mathscr{R}$ by $p(\lambda') = 1$ if and only if $\lambda' = \sigma_i^{-1}\lambda$ for some $1 \leq i \leq n$. Choose $w_1, \ldots, w_n \in W$ such that w_i maps to σ_i under the surjection $W \to \Gamma$. If $W_\pm$ (resp. W_+) denotes the stabilizer of $\{\pm\lambda\}$ (resp. $\{\lambda\}$) under the inflated W-action, then the w_i are representatives for the quotient $W_\pm \setminus W$. For $w \in W$, define $u_i(w) \in W_\pm$ by

$$w_i w = w u_i(w), \qquad i = 1, \dots, n.$$

Choose representatives $v_0 \in W_+$ and $v_1 \in W_\pm$ for $W_+ \setminus W_\pm$ if $[F_\lambda \colon F_{\pm\lambda}] = 2$, and otherwise just pick some $v_0 \in W_+$. For $u \in W_\pm$ we define $v_0(u) \in W_+$ by $v_0 \cdot u = v_0(u) \cdot v_{i'}$, where $i' = 0$ or 1 depending on if $W_+ = W_\pm$ or not. For $w \in W$ we set

$$r_p(w) = \prod_{i=1,\dots,n} [\chi_\lambda(v_0(u_i(w))) \otimes \lambda_i] \in \mathbb{C}^* \otimes_{\mathbb{Z}} X,$$

where $\lambda_i := \sigma_i^{-1}\lambda$ and we view $\mathbb{C}^* \otimes_{\mathbb{Z}} X$ as a Γ-module (and thus a W-module) via the trivial action on the first tensor factor. We view r_p as a 1-cochain of W valued in $\mathbb{C}^* \otimes_{\mathbb{Z}} X$. We have the following result, which will be used when we look at the uniqueness of our L-embeddings:

Lemma 5.2.2 *Suppose $\{\xi_\lambda\}_{\lambda \in \mathscr{R}}$ satisfies the conditions of a χ-data, except that for λ with $[F_\lambda \colon F_{\pm\lambda}] = 2$ we require that ξ_λ is trivial on $F_{\pm\lambda}^*$ rather than extending the quadratic character. Then*

$$c(w) = \prod_{i=1,\dots,n} [\xi_\lambda(v_0(u_i(w))) \otimes \lambda_i] \in \mathbb{C}^* \otimes_{\mathbb{Z}} X$$

is a 1-cocycle of W in $\mathbb{C}^* \otimes_{\mathbb{Z}} X$ whose cohomology class does not depend on any choices.

Proof. This is [LS87] Corollary 2.5.B, which follows from Lemma 2.5.A loc. cit. These results, along with the auxiliary Lemma 2.4.A, are proved in a purely group-cohomological setting, and thus the same proofs work verbatim. □

If the action of Γ' is not transitive, then we define r_p and c for each of the Γ'-orbits on $\mathscr{R}$ and take the product of these functions over all such orbits; the resulting functions on W are again denoted by r_p and c.

We now take G a connected reductive group defined over F with maximal F-torus T with root system $R := R(G_{F^s}, T_{F^s})$ and a Langlands dual group $\widehat{G}$. In addition, we fix a Γ-stable splitting $(\mathscr{B}, \mathscr{T}, \{\mathbf{X}\})$ of $\widehat{G}$. We shall attach to a χ-data $\{\chi_\alpha\}_{\alpha \in R}$ for T a canonical $\widehat{G}$-conjugacy class of *admissible embeddings* ${}^L T \to {}^L G$; recall that a homomorphism of W-extensions $\xi \colon {}^L T \to {}^L G$ is called an admissible embedding if the map $\widehat{T} \to \mathscr{T}$ induced by ξ corresponds to the isomorphism coming from the pair $(\mathscr{B}, \mathscr{T})$ and a choice of Borel subgroup B of G_{F^s} containing T_{F^s}. We replace F^s by a finite Galois extension L/F splitting T; there is no harm in doing this for the purposes of constructing such an admissible embedding. The $\widehat{G}$-conjugacy class of such an embedding is independent of the choice of B and splitting of $\widehat{G}$.

Fix a Borel subgroup B of G_{F^s} containing T_{F^s} as above, giving an isomorphism $\widehat{T} \xrightarrow{\xi} \mathscr{T}$. It is clear that such an embedding $\xi\colon {}^LT \to {}^LG$, is determined by its values on W (via the canonical splitting $W \to \widehat{T} \rtimes W$). As in §5.2.1, we may use ξ to transport the Γ-action on $\widehat{T}$ to $\mathscr{T}$, and for $\gamma \in \Gamma$ will denote this automorphism of $\mathscr{T}$ by σ_T. We have that $w \in W$ transports via ξ to an action on $\mathscr{T}$ given by

$$\omega_T(\sigma) \rtimes w,$$

where $w \mapsto \sigma \in \Gamma$ and $\omega_T(\sigma) \in W(\widehat{G}, \mathscr{T})$.

Our goal will be to construct a homomorphism $\xi\colon W \to {}^LG$ giving rise to our desired embedding. As explained in [LS87], it's enough that each $\mathrm{Ad}(\xi(w))$ acts on $\mathscr{T}$ as σ_T, where $w \mapsto \sigma \in \Gamma$. First, we note that our χ-data for the action of Γ on R yields a χ-data for the ξ-transported action of Γ on $R(\widehat{G}, \mathscr{T})^\vee$; we define a gauge p on the Γ-set $R(\widehat{G}, \mathscr{T})^\vee$ by setting $p(\beta^\vee) = 1$ if and only if β is a root of $\mathscr{T}$ in $\mathscr{B}$, and (along with our transported χ-data) get an associated 1-cochain $r_p\colon W \to \mathbb{C}^* \otimes_{\mathbb{Z}} X_*(\mathscr{T})$, which we view as a 1-cochain $r_p\colon W \to \mathscr{T}(\mathbb{C})$ using the canonical pairing. Let $n\colon W(\widehat{G}, \mathscr{T}) \to N_{\widehat{G}}(\mathscr{T})(\mathbb{C})$ denote the Tits section associated to our splitting of $\widehat{G}$. Finally, for $w \in W$ we set

$$\xi(w) = [r_p(w) \cdot n(\omega_T(\sigma))] \rtimes w \in {}^LG.$$

We claim that this map satisfies the desired properties.

The verification that this map works comes down to a 2-cocycle arising from the Tits section. For $w \in W$, set $n(w) := n(\omega_T(\sigma)) \rtimes w$; we have for $w_1, w_2 \in W$ the equality

$$n(w_1)n(w_2)n(w_1w_2)^{-1} = t(\sigma_1, \sigma_2),$$

where $w_i \mapsto \sigma_i$ and t is a 2-cocycle of Γ valued in $\mathscr{T}(\mathbb{C})$. We have the following crucial identity:

Lemma 5.2.3 *In our above situation, the differential of $r_p^{-1} \in C^1(W, \mathscr{T}(\mathbb{C}))$ equals Inf(t) $\in Z^2(W, \mathscr{T}(\mathbb{C}))$ (where the above groups are given the ξ-transported W-action).*

Proof. After applying Lemma 2.1.A in [LS87], this reduces to a special case of Lemma 2.5.A loc. cit., which is proved in an purely group-cohomological setting. The proof of Lemma 2.1.A in [LS87] is root-theoretic, and therefore works in our setting as well. □

With the above lemma in hand, it is straightforward to check that our $\xi\colon W \to {}^LG$ defined above is a homomorphism that induces an admissible embedding $\xi\colon {}^LT \to {}^LG$. We conclude this section with a discussion of how the admissible embedding ξ depends on the choices we have made during its construction.

Fact 5.2.4 *Suppose that we replace our* Γ*-splitting by the* $g \in \widehat{G}^{\Gamma}$*-conjugate* $(\mathscr{B}^g, \mathscr{T}^g, \{X^g\})$ *(see [Kot84], 1.7). If* $Ad(g)^{\sharp}\colon X_*(\mathscr{T}) \to X_*(\mathscr{T}^g)$ *is the induced isomorphism of cocharacter groups, then for* $\lambda \in X_*(\mathscr{T})$ *the trivial equality* ${}^{\sigma_T}(Ad(g^{-1})^{\sharp}\lambda) = Ad(g^{-1})^{\sharp}({}^{\sigma_T}\lambda)$ *gives that for* $w \in W$*,* $r_{p^g}(w) = g r_p(w) g^{-1}$*. One checks that* $n(w)$ *is also replaced by* $g n(w) g^{-1}$*, and so the embedding* ξ *is replaced by* $Ad(g) \circ \xi$*, which is in the same* $\widehat{G}^{\Gamma}$*-conjugacy class as* ξ*.*

Fact 5.2.5 *The conjugacy class of* ξ *is also independent of our choice of Borel subgroup* $T_{F^s} \subset B \subset G_{F^s}$*. If* B' *is another such subgroup, we may find* $v \in N_G(T)(F^s)$ *such that* $vBv^{-1} = B$*, and denote the corresponding admissible embedding by* ξ'*. Transporting* $Ad(v)\big|_T$ *to* $W(\widehat{G}, \mathscr{T})$ *using* ξ*, we obtain an element* $\mu \in W(\widehat{G}, \mathscr{T})$*. Then it is proved in [LS87], Lemma 2.6.A (the proof of which relies on Lemmas 2.1.A and 2.3.B loc. cit.—we have already discussed the former. The latter depends on torus normalizers, root theory,* a*-data, and the Tits section, which may be dealt with over* F^s*, so the proof loc. cit. works verbatim) that we have the equality*

$$Ad(g^{-1}) \circ \xi = \xi',$$

where $g \in N_{\widehat{G}}(\mathscr{T})(\mathbb{C})$ *acts on* $\mathscr{T}$ *as* μ*, giving the claim.*

Fact 5.2.6 *For dependence on the* χ*-data* $\{\chi_\alpha\}$ *for the* Γ*-action on* $R(G_{F^s}, T_{F^s})$*, we fix another* χ*-data* $\{\chi'_\alpha\}$*, and we write* $\chi'_\alpha = \zeta_\alpha \cdot \chi_\alpha$*, where* ζ_α *is a character of* F_α*. The set* $\{\zeta_\alpha\}_{\alpha \in R}$ *then satisfies the hypotheses of Lemma 5.2.2 (where, in the notation of that lemma,* $\mathscr{R} = X_*(\mathscr{T})$ *with* ξ*-transported* Γ*-action); we then obtain a 1-cocycle* $c \in Z^1(W, \mathscr{T}(\mathbb{C}))$ *whose class* $[c] \in H^1(W, \mathscr{T}(\mathbb{C}))$ *is independent of any choices made in the construction of* c *from* $\{\zeta_\alpha\}$*. Then it's immediate from the construction of* c *that the embedding* ξ *is replaced by* $t \rtimes w \mapsto c(w) \cdot \xi(t \rtimes w)$*.*

Fact 5.2.7 *Finally, suppose that we take another* F*-torus* T'*, and take* $g \in G(F^s)$ *such that* $Ad(g)$ *is an* F*-isomorphism from* T *to* T'*. Note that* $Ad(g)$ *identifies a* χ*-data* $\{\chi_\alpha\}$ *for* T *with* χ*-data* $\{\chi'_\beta\}$ *for* T'*, since the induced map on character groups is* Γ*-equivariant; take* $\{\chi'_\beta\}$ *to be the* χ*-data for* T' *used to construct any admissible* L*-embeddings. The map* $Ad(g)$ *extends to an isomorphism of* L*-groups* $\lambda_g\colon {}^LT \to {}^LT'$*. Let* ξ *be the embedding* ${}^LT \to {}^LG$ *constructed above, determined by a choice of Borel subgroup* B *containing* T_{F^s}*. Then we have the equality of admissible embeddings* $\xi \circ \lambda_g = \xi'$*, where* ξ' *is the admissible embedding* ${}^LT' \to {}^LG$ *constructed above corresponding to the* χ*-data* $\{\chi'_\beta\}$ *and the Borel subgroup* gBg^{-1} *containing* $(T')_{F^s}$*. We conclude that the* $\widehat{G}$*-conjugacy class of embeddings* ${}^LT \to {}^LG$ *attached to the* χ*-data* $\{\chi_\alpha\}$ *for* T *is equivalent to the class of embeddings* ${}^LT' \to {}^LG$ *attached to* $\{\chi'_\beta\}$ *for* T' *via* λ_g*.*

5.3 The local transfer factor

We construct one factor at a time, following [LS87], §3 and [KS12], §3. Recall that G is a fixed connected reductive group over F a local field of positive characteristic and $\psi\colon G_{F^s} \to G^*_{F^s}$ is a quasi-split inner form of G. We fix an endoscopic datum $(H, \mathcal{H}, \eta, s)$ of G, which may also be viewed as an endoscopic datum for G^*, since we are taking the dual group of G^* to be $\widehat{G}$ with bijection of based root data given by $\Psi(G^*)^\vee \xrightarrow{\psi} \Psi(G)^\vee \to \Psi(\widehat{G})$. Let $\gamma_H, \bar{\gamma}_H \in H_{G-\mathrm{sr}}(F)$ with corresponding images $\gamma_G, \bar{\gamma}_G \in G_{\mathrm{sr}}(F)$. Denote by $T_H, \bar{T}_H$ the centralizers in H of $\gamma_H, \bar{\gamma}_H$ respectively; these are maximal F-tori. By Lemma 5.1.2, we may fix two admissible embeddings $T_H \xrightarrow{\sim} T \hookrightarrow G^*$, $\bar{T}_H \xrightarrow{\sim} \bar{T} \hookrightarrow G^*$. Recall that such embeddings are unique up to conjugation by elements of $\mathfrak{A}(T), \mathfrak{A}(\bar{T})$—denote by $\gamma, \bar{\gamma} \in T(F), \bar{T}(F)$ the images of $\gamma_H, \bar{\gamma}_H$ under the above embeddings.

Set $R := R(G^*_{F^s}, T_{F^s}), \bar{R} = R(G^*_{F^s}, \bar{T}_{F^s})$, similarly with $R^\vee, \bar{R}^\vee$. Fix a- and χ-data for the standard Γ actions on R and $\bar{R}$—these may also be viewed as data for the Γ-action on $R^\vee, \bar{R}^\vee$, and data for the Γ-action on $R((G^*_{\mathrm{sc}})_{F^s}, (T_{\mathrm{sc}})_{F^s}), R((G^*_{\mathrm{sc}})_{F^s}, (\bar{T}_{\mathrm{sc}})_{F^s})$, where G^*_{sc} denotes the simply-connected cover of $\mathscr{D}(G^*)$, and T_{sc} denotes the preimage of $T \cap \mathscr{D}(G^*)$ in this group (analogously for $\bar{T}$). If we replace the embedding $T_H \to G^*$ by a $\mathfrak{A}(T)$-conjugate $T_H \to T'$, then we may view the a- and χ-data as data for $R(G^*_{F^s}, T'_{F^s})$. Our goal will be to define a value

$$\Delta(\gamma_H, \gamma_G; \bar{\gamma}_H, \bar{\gamma}_G) \in \mathbb{C}$$

which will be constructed purely from the admissible embeddings, the map ψ, and the a- and χ-data, but which only depends on the four inputs. As such, we need to examine the following two things:

1. How Δ changes when we replace the admissible embeddings $T_H \to G^*$, $\bar{T}_H \to G^*$ by $\mathfrak{A}(T), \mathfrak{A}(\bar{T})$-conjugates, and use the translated a- and χ-data;

2. How Δ changes when we keep the admissible embeddings the same but pick different a- and χ-data.

In light of these observations, we may fix Γ-splittings $(\mathscr{B}, \mathscr{T}, \{X\}), (\mathscr{B}_H, \mathscr{T}_H, \{X^H\})$ of $\widehat{G}$, $\widehat{H}$, respectively, that give rise to our admissible embeddings $T_H \to T$, $\bar{T}_H \to \bar{T}$, since choosing different splittings only serves to conjugate the admissible embeddings by $\mathfrak{A}(T)$, $\mathfrak{A}(\bar{T})$, which is included in condition (1). Implicit in the construction of the admissible embedding $T_H \to G^*$ is also the choice of $g \in \widehat{G}(\mathbb{C})$ such that $\mathrm{Ad}(g)[\eta(\mathscr{T}_H)] = \mathscr{T}$ and $\mathrm{Ad}(g)[\eta(\mathscr{B}_H)] \subset \mathscr{B}$; thus, if we replace the endoscopic datum by $(H, \mathcal{H}, \mathrm{Ad}(g) \circ \eta, s)$, then $\gamma_H, \bar{\gamma}_H \in H(F)$ are still strongly G-regular, and so if we carry out the construction of Δ for this datum, the admissible embeddings and a- and χ-data are unaffected, and hence our value of Δ will be the same. If we choose a different $g \in \widehat{G}(\mathbb{C})$ satisfying the above properties, it again only serves to replace our admissible embeddings with $\mathfrak{A}$-conjugates. The upshot is that we may assume that η carries $\mathscr{T}_H$ to $\mathscr{T}$ and $\mathscr{B}_H$ into $\mathscr{B}$.

Suppose we have a fixed admissible embedding $T_H \xrightarrow{f} T$, dual to $\widehat{T_H} \xrightarrow{\hat{f}} \widehat{T}$. Recall that we have our element $s \in \widehat{H}(\mathbb{C})$ from the endoscopic datum. Let B_H be a Borel subgroup containing $(T_H)_{F^s}$ which is used to induce f (there is no such unique B_H in general). Since by assumption $s \in Z(\widehat{H})(\mathbb{C})$, it lies in $\mathscr{T}_H(\mathbb{C})$ and its preimage under the map $\widehat{T}_H \xrightarrow{\sim} \mathscr{T}_H$ induced by B_H (and our fixed $(\mathscr{B}_H, \mathscr{T}_H)$) is independent of choice of B_H. We conclude that the image of s in $\widehat{T}(\mathbb{C})$, denoted by s_T, only depends on the choice of admissible embedding $T_H \to T$. In the definition of an endoscopic datum, it is assumed that $s \in Z(\widehat{H})^{\Gamma} \cdot \eta^{-1}(Z(\widehat{G}))$, and hence the preimage of s in $\widehat{T_H}(\mathbb{C})$ lies in $\iota(Z(\widehat{H}))^{\Gamma} \cdot \widehat{f}^{-1}(\iota(Z(\widehat{G})))$, where we have pedantically denoted the canonical embeddings $Z(\widehat{H}) \to \widehat{T_H}, Z(\widehat{G}) \to \widehat{T}$ by ι, and have also used the fact that $Z(\widehat{H}) \to \widehat{T_H}$ is canonical to obtain Γ-equivariance. This implies (since $\widehat{f}$ is Γ-equivariant) that s_T lies in $\widehat{T}_{\text{ad}}^{\Gamma}$, and we set $\mathbf{s}_T$ to be its image in $\pi_0(\widehat{T}_{\text{ad}}^{\Gamma})$.

We make the assumption throughout this section that for any endoscopic datum, $\mathcal{H} = {}^L H$ with embedding $\widehat{H} \to {}^L H$ the canonical embedding; this assumption will only be necessary in §5.3.4. We will discuss how to deal with general $\mathcal{H}$ in §5.4.

5.3.1 The factor Δ_I

We set

$$\Delta_I(\gamma_H, \gamma_G) := \langle \lambda_{\{a_\alpha\}}(T_{\text{sc}}), \mathbf{s}_T \rangle,$$

where we view the a-data for T as an a-data for T_{sc}, the pairing $\langle -, - \rangle$ is from Tate-Nakayama duality, and $\lambda_{\{a_\alpha\}}(T_{\text{sc}})$ is the splitting invariant associated to the maximal F-torus $T_{\text{sc}} \hookrightarrow G^*_{\text{sc}}$, a fixed F-splitting $\mathscr{S}$ of G^*_{sc}, and the a-data $\{a_\alpha\}$.

Lemma 5.3.1 *The value*

$$\frac{\Delta_I(\gamma_H, \gamma_G)}{\Delta_I(\bar{\gamma}_H, \bar{\gamma}_G)}$$

is independent of the splitting $\mathscr{S}$.

Proof. Suppose that we replace $\mathscr{S} = (B, S, \{X_\alpha\})$ by another F-splitting $\mathscr{S}' = (B', S', \{X'_\alpha\})$ of G^*_{sc}. It will be necessary to use fppf cohomology here, since these two splittings need not be $G^*(F^s)$-conjugate. Accordingly, take $z \in G^*_{\text{sc}}(\overline{F})$ such that $z\mathscr{S}'z^{-1} = \mathscr{S}$ and $p_1(z)p_2(z)^{-1} \in Z_{\text{sc}}(\overline{F} \otimes_F \overline{F}) := Z(G^*_{\text{sc}})(\overline{F} \otimes_F \overline{F})$. Then if B_T is a fixed Borel subgroup containing $(T_{\text{sc}})_{F^s}$ and $h \in G^*_{\text{sc}}(F^s)$ carries (B, S) to $(B_T, (T_{\text{sc}})_{F^s})$, then hz carries (B', S') to $(B_T, (T_{\text{sc}})_{F^s})$, and for all

$\sigma \in \Gamma$, we have $n_{S'}(\omega_T(\sigma)) = \mathrm{Ad}(z^{-1})n_S(\omega_T(\sigma)) \in N_{G^*_{\mathrm{sc}}}(S')(F^s)$ (notation as in the definition of the splitting invariant, where $n_S, n_{S'}$ denote the Tits sections corresponding to $\mathscr{S}, \mathscr{S}'$), similarly for $x(\sigma)$. We need to be careful here, since we defined the splitting invariant in terms of a Galois cocycle and it is not in general true that $z \in G^*_{\mathrm{sc}}(F^s)$. However, recall the definition of the splitting invariant: the cocycle m is still a Galois cocycle for us, since $x(\sigma) \in G^*_{\mathrm{sc}}(F^s)$ and $n(\omega_T(\sigma)) \in N_{G^*_{\mathrm{sc}}}(F^s)$, and we may view it as a Čech cocycle $m \in G^*_{\mathrm{sc}}(\overline{F} \otimes_F \overline{F})$. Then we may set

$$\lambda_{\{a_\alpha\}}(T) := p_1(h)mp_2(h)^{-1} \in T_{\mathrm{sc}}(\overline{F} \otimes_F \overline{F}),$$

and get the same definition as in §5.2.1. However, this modified definition allows us to compute that if $c' \in T_{\mathrm{sc}}(\overline{F} \otimes_F \overline{F})$ is the cocycle used to defined the splitting invariant for $\mathscr{S}'$, then $m' = p_1(z)^{-1}mp_1(z) \in G^*_{\mathrm{sc}}(\overline{F} \otimes_F \overline{F})$, and so we have:

$$c' = p_1(h)p_1(z)p_1(z)^{-1}mp_1(z)p_2(z)^{-1}p_2(h)^{-1} = p_1(z)p_2(z)^{-1}(p_1(h)mp_1(h)^{-1}),$$

and we conclude that $\lambda_{\{a_\alpha\}}$ computed with respect to $\mathscr{S}'$ differs from the one computed with respect to $\mathscr{S}$ by left-translation by the class $\mathbf{z}_T$ in $H^1(F, T)$ represented by $p_1(z)p_2(z)^{-1}$. Whence, to prove the lemma, it's enough to show that

$$\langle \mathbf{z}_T, \mathbf{s}_T \rangle = \langle \mathbf{z}_{\bar{T}}, \mathbf{s}_{\bar{T}} \rangle.$$

Replace F^s with a finite Galois extension L/F splitting T_{sc}, and set $\Gamma := \Gamma_{L/F}$. By Proposition 5.1.3, we have the following commutative diagram with exact columns

$$\begin{array}{ccc}
H^1(F, Z_{\mathrm{sc}}) & \xrightarrow{\sim} & H^{-2}(\Gamma, X_*(T_{\mathrm{ad}})/X_*(T_{\mathrm{sc}})) \\
\downarrow & & \downarrow \\
H^1(F, T_{\mathrm{sc}}) & \xrightarrow{\sim} & H^{-1}(\Gamma, X_*(T_{\mathrm{sc}})) \\
\downarrow & & \downarrow \\
H^1(F, T_{\mathrm{ad}}) & \xrightarrow{\sim} & H^{-1}(\Gamma, X_*(T_{\mathrm{ad}})),
\end{array}$$

with horizontal isomorphisms induced by Tate-Nakayama duality, as discussed in §5.1.2. From here, one may deduce the result from the argument in the proof of Lemma 3.2.A in [LS87], which looks at the images of $\mathbf{z}_T, \mathbf{z}_{\bar{T}}$ in the right-hand column and then uses group-cohomological calculations, along with the alternative characterization of the Tate-Nakayama pairing that we discussed in Remark 5.1.4 (replacing the use of duality results loc. cit. with our Proposition 5.1.3). □

We now discuss how Δ_I changes under conjugation by $\mathfrak{A}(T_{\mathrm{sc}})$ and another choice of a-data.

Lemma 5.3.2 *The factor Δ_I satisfies:*

1. *If $T_H \to T$ is replaced by its conjugate under $g \in \mathfrak{A}(T_{sc})$, with corresponding transported a-data, then $\Delta_I(\gamma_H, \gamma_G)$ is multiplied by $\langle \boldsymbol{g}_T, \boldsymbol{s}_T \rangle^{-1}$, where $\boldsymbol{g}_T$ is the class of $\sigma \mapsto g\sigma(g)^{-1}$ in $H^1(F, T_{sc})$.*

2. *Suppose that the a-data $\{a_\alpha\}$ is replaced by $\{a'_\alpha\}$. Set $b_\alpha = a'_\alpha/a_\alpha$. Then the term $\Delta_I(\gamma_H, \gamma_G)$ is multiplied by the sign*

$$\prod_\alpha sgn_{F_\alpha/F_{\pm\alpha}}(b_\alpha),$$

*where the product is taken over a set of representatives for the symmetric Γ-orbits (the orbit of α is **symmetric** if it contains $-\alpha$, otherwise it is **asymmetric**) in R that lie outside $R(H_{F^s}, (T_H)_{F^s})$.*

Proof. Part (1) is the analogue of Lemma 3.2.B in [LS87], and the proof loc. cit. works in our situation, since all elements of $\mathfrak{A}(T_{\mathrm{sc}})$ are separable points, the construction of the splitting invariant only uses separable points, and the Tate-Nakayama duality pairing for tori works the same way in positive characteristic.

For (2), we first note that the expression $\mathrm{sgn}_{F_\alpha/F_{\pm\alpha}}(b_\alpha)$ makes sense, since b_α is fixed by $\Gamma_{\pm\alpha}$, and thus lies in $F_{\pm\alpha}$. Our result is exactly [KS12], Lemma 3.4.1, which is proved without assumptions on the characteristic of F. □

5.3.2 The factor Δ_{II}

We define

$$\Delta_{II}(\gamma_H, \gamma_G) = \prod \chi_\alpha \left(\frac{\alpha(\gamma) - 1}{a_\alpha} \right), \tag{5.3}$$

where the product is over representatives α for the orbits of Γ in R the lie outside $R(H_{F^s}, (T_H)_{F^s})$. This is easily checked to be independent of the representatives chosen.

Lemma 5.3.3 *The factor $\Delta_{II}(\gamma_H, \gamma_G)$ is unaffected by replacing the admissible embedding $T_H \to T$ by an $\mathfrak{A}(T)$-conjugate (and the transporting the χ- and a-data accordingly). Moreover, replacing the a-data $\{a_\alpha\}$ by a different data $\{a'_\alpha\}$ serves to multiply $\Delta_{II}(\gamma_H, \gamma_G)$ by*

$$\prod_\alpha sgn_{F_\alpha/F_{\pm\alpha}}(b_\alpha)^{-1},$$

where $b_\alpha = a'_\alpha/a_\alpha$ and the product is over representatives for the symmetric orbits outside $R(H_{F^s}, (T_H)_{F^s})$.

Proof. The arguments in [LS87], Lemmas 3.3.B and 3.3.C are purely root-theoretic and work verbatim here. □

It remains to check the dependency of Δ_{II} on the χ-data. Suppose the χ-data $\{\chi_\alpha\}$ are replaced by $\{\chi'_\alpha\}$, and set $\zeta_\alpha := \chi'_\alpha/\chi_\alpha$. Note that ζ_α restricts to the trivial character on $F^*_{\pm\alpha}$. To analyze this dependency, we will need to introduce some new notation, following [LS87], §3.3. Let $\mathcal{O}$ be a symmetric orbit of Γ on R, with a gauge q, $X^{\mathcal{O}}$ the free abelian group on the elements $\mathcal{O}_+ = \{\alpha \in \mathcal{O} \colon q(\alpha) = 1\}$,with inherited Γ-action, and X^α the $\mathbb{Z}$-submodule generated by some $\alpha \in \mathcal{O}_+$, which is preserved by $\Gamma_{\pm\alpha}$, and so $X^{\mathcal{O}} = \mathrm{Ind}^{\Gamma}_{\Gamma_{\pm\alpha}}(X^\alpha)$. We obtain a corresponding $F_{\pm\alpha}$-torus T^α which is one-dimensional, anisotropic, and split over F_α, and corresponding F-torus $T^{\mathcal{O}}$ which satisfies $T^{\mathcal{O}} = \mathrm{Res}_{F_{\pm\alpha}/F} T^\alpha$.

We have a natural Γ-homomorphism $X^{\mathcal{O}} \to X^*(T)$ which induces a morphism of F-tori $T \to T^{\mathcal{O}}$ that maps $T(F)$ into $T^\alpha(F_{\pm\alpha})$; denote by γ^α the image of γ in $T^\alpha(F_{\pm\alpha})$. Note that the norm map $T^\alpha(F_\alpha) \to T^\alpha(F_{\pm\alpha})$ is surjective, since we have the exact sequence of $F_{\pm\alpha}$-tori

$$0 \longrightarrow T' \longrightarrow \mathrm{Res}_{F_\alpha/F_{\pm\alpha}}(T^\alpha_{F_\alpha}) \xrightarrow{\text{Norm}} T^\alpha \longrightarrow 0,$$

where T' is a split $F_{\pm\alpha}$-torus, and so taking the long exact sequence in cohomology (along with Hilbert 90) gives the desired surjectivity. Whence, we may write

$$\gamma_\alpha = \delta^\alpha \overline{\delta^\alpha},$$

where $\delta^\alpha \in T^\alpha(F_\alpha)$ and the bar denotes the map from $T^\alpha(F_\alpha)$ to itself induced by the unique automorphism of $F_\alpha/F_{\pm\alpha}$.

If $\mathcal{O}$ is an asymmetric Γ-orbit in R, then $X^{\pm\mathcal{O}}$ is defined to be the free abelian group on $\mathcal{O}$ with inherited Γ-action and X^α is the subgroup generated by some $\alpha \in \mathcal{O}$, which again carries a $\Gamma_{\pm\alpha} = \Gamma_\alpha$-action. We get a corresponding split 1-dimensional F_α-torus T^α and F-torus $T^{\pm\mathcal{O}}$, with $T^{\mathcal{O}} = \mathrm{Res}_{F_\alpha/F} T^\alpha$. We again obtain a map $T \to T^{\pm\mathcal{O}}$, inducing a map $T(F) \to T^\alpha(F_\alpha)$; denote the image of γ under this map by γ^α. We are now ready to state how Δ_{II} changes when we alter the χ-data.

Lemma 5.3.4 *If the χ-data $\{\chi_\alpha\}$ are replaced by $\{\chi'_\alpha\}$, with $\zeta_\alpha = \chi'_\alpha/\chi_\alpha$, then $\Delta_{II}(\gamma_H, \gamma_G)$ is multiplied by*

$$\prod_{aysmm} \zeta_\alpha(\gamma^\alpha) \cdot \prod_{symm} \zeta_\alpha(\delta^\alpha),$$

where $\prod_{asymm}$ denotes the product over representatives α for pairs $\pm\mathcal{O}$ of asymmetric orbits of R outside H, and to make sense of $\zeta_\alpha(\gamma^\alpha)$, we are using the canonical isomorphism $T^\alpha \xrightarrow{\sim} \mathbb{G}_m$ given on character groups by $1 \mapsto \alpha$, and $\prod_{symm}$ is the product over representatives α for the symmetric *orbits of R outside H, and to make sense of $\zeta_\alpha(\delta^\alpha)$ we are using the canonical isomorphism $T^\alpha_{F_\alpha} \xrightarrow{\sim} \mathbb{G}_m$ given on character groups by $1 \mapsto \alpha$.*

Proof. This is Lemma 3.3.D in [LS87], the proof of which (along with the proof of Lemma 3.3.A loc. cit.) carries over to our setting verbatim. □

5.3.3 The factor Δ_{III_1} (or Δ_1)

The construction of this factor is the only part of the construction of the relative local transfer factor that involves fppf cohomology rather than Galois cohomology. For the moment, we will assume that G is quasi-split over F, with $\psi = \mathrm{id}$; the construction of Δ_1 in this case can be done using Galois cohomology, but in order to match more closely with the general case, we work in the setting of fppf cohomology. By construction, the admissible embedding $T_H \to G$ is obtained by first taking $T_H \xrightarrow{\sim} T_G$ determined by γ_H, γ_G and then conjugating an embedding $(T_G)_{F^s} \to G_{F^s}$ induced by a choice of Borel subgroup containing $(T_G)_{F^s}$ and $(\mathscr{B}, \mathscr{T})$ by some appropriate $g \in G_{\mathrm{sc}}(\overline{F})$. As a consequence, we see that γ_G and γ are conjugate by some $h \in G_{\mathrm{sc}}(\overline{F})$ such that $p_1(h)p_2(h)^{-1} \in T_{\mathrm{sc}}(\overline{F} \otimes_F \overline{F})$. We then set $v = p_1(h)p_2(h)^{-1}$ and denote the class of v in $H^1(F, T_{\mathrm{sc}})$ by $\mathrm{inv}(\gamma_H, \gamma_G)$; this class is independent of the choice of h, since if we choose any other $h' \in G_{\mathrm{sc}}(\overline{F})$ with $h'\gamma_G h'^{-1} = \gamma$, then $h^{-1}h' \in T_{\mathrm{sc}}(\overline{F})$, since γ is strongly regular. We then set

$$\Delta_1(\gamma_H, \gamma_G) = \langle \mathrm{inv}(\gamma_H, \gamma_G), \mathbf{s}_T \rangle^{-1}.$$

Now we return to the setting of a general connected reductive group G over F with $\psi \colon G_{F^s} \to G^*_{F^s}$ the quasi-split inner form of G over F with the assumptions stated in the beginning of §5.3. In particular, we have two pairs of elements γ_H, γ_G and $\bar{\gamma}_H, \bar{\gamma}_G$. As in the quasi-split case, we may find $h, \bar{h} \in G^*_{\mathrm{sc}}(\overline{F})$ such that

$$h\psi(\gamma_G)h^{-1} = \gamma, \; \bar{h}\psi(\bar{\gamma}_G)\bar{h}^{-1} = \bar{\gamma}.$$

One could take $h, \bar{h} \in G^*_{\mathrm{sc}}(F^s)$, but since we will be using these elements to construct fppf Čech cocycles, we want to view them as $\overline{F}$-points anyway. Further, let $u \in G^*_{\mathrm{sc}}(\overline{F} \otimes_F \overline{F})$ be such that $p_1^*\psi \circ p_2^*\psi^{-1} = \mathrm{Ad}(u)$ on $G^*_{\overline{F} \otimes_F \overline{F}}$; the existence of such a u is the reason we need to use fppf cohomology to define the Δ_{III_1} factor. We then obtain two (Čech) cochains,

$$v := p_1(h)up_2(h)^{-1} \in T_{\mathrm{sc}}(\overline{F} \otimes_F \overline{F}), \; \bar{v} := p_1(\bar{h})up_2(\bar{h})^{-1} \in \bar{T}_{\mathrm{sc}}(\overline{F} \otimes_F \overline{F});$$

we have that $v \in T_{\mathrm{sc}}(\overline{F} \otimes_F \overline{F})$ because (since γ, γ_G are F-points)

$$v\gamma v^{-1} = p_1(h)(p_1^*\psi \circ p_2^*\psi^{-1}(p_2(\psi(\gamma_G))))p_1(h)^{-1} = p_1(h)p_1(\psi(\gamma_G))p_1(h)^{-1} = \gamma,$$

similarly for $\bar{v}$.

By construction, we have $dv = d\bar{v} = du \in Z_{\text{sc}}(\overline{F} \otimes_F \overline{F})$, where recall that $Z_{\text{sc}} := Z(G^*_{\text{sc}})$, and by d we are denoting the Čech differential. We have an embedding $Z_{\text{sc}} \to T_{\text{sc}} \times \bar{T}_{\text{sc}}$ defined by $i^{-1} \times j$, where i and j denote the obvious inclusions. Set

$$U(T, \bar{T}) = U := \frac{T_{\text{sc}} \times \bar{T}_{\text{sc}}}{Z_{\text{sc}}},$$

which is an F-torus. We have the following easy lemma:

Lemma 5.3.5 *The image of $(v, \bar{v}) \in T_{sc}(\overline{F} \otimes_F \overline{F}) \times \bar{T}_{sc}(\overline{F} \otimes_F \overline{F}) = (T_{sc} \times \bar{T}_{sc})(\overline{F} \otimes_F \overline{F})$ in $U(\overline{F} \otimes_F \overline{F})$ is a 1-cocycle, whose cohomology class, denoted by*

$$\mathit{inv}\left(\frac{\gamma_H, \gamma_G}{\bar{\gamma}_H, \bar{\gamma}_G}\right) \in H^1(F, U), \tag{5.4}$$

is independent of the choices of $u, h, \bar{h}$.

Proof. The fact the above defines a 1-cocycle is trivial, since

$$U(\overline{F} \otimes_F \overline{F}) = \frac{T_{\text{sc}}(\overline{F} \otimes_F \overline{F}) \times \bar{T}_{\text{sc}}(\overline{F} \otimes_F \overline{F})}{Z_{\text{sc}}(\overline{F} \otimes_F \overline{F})},$$

using the fact that $H^1(\overline{F} \otimes_F \overline{F}, Z_{\text{sc}}) = 0$, and the construction of v, $\bar{v}$, and U. Replacing u by u' satisfies $u' = uz$, $z \in Z_{\text{sc}}(\overline{F})$, and so the new element $(v', \bar{v}') \in T_{\text{sc}} \times \bar{T}_{\text{sc}}$ is equivalent to $(v, \bar{v})$ modulo Z_{sc}. Replacing h by $h' = ht$, where $t \in T_{\text{sc}}(\overline{F})$, gives $v' = d(t) \cdot v \in T_{\text{sc}}(\overline{F} \otimes_F \overline{F})$, and so the image of $(v', \bar{v})$ in U differs from the image of $(v, \bar{v})$ by $(d(t), 1)$, a coboundary, similarly with the element $\bar{h}$. □

Note that if G is quasi-split and π denotes the quotient map defining U, then

$$\left(\frac{\gamma_H, \gamma_G}{\bar{\gamma}_H, \bar{\gamma}_G}\right) = \pi[(\text{inv}(\gamma_H, \gamma_G)^{-1}, \text{inv}(\bar{\gamma}_H, \bar{\gamma}_G))]. \tag{5.5}$$

Now let $\hat{T}_{\text{sc}}$ denote the torus dual to $T_{\text{ad}} = T/Z(G)$, and set $\widehat{Z}_{\text{sc}} := Z(\widehat{G}_{\text{sc}})$. The homomorphism $X_*(T) \to X_*(T_{\text{ad}})$ induces a morphism of $\hat{T}_{\text{sc}} \to \widehat{T} \hookrightarrow \widehat{G}$ (using an isomorphism $\widehat{T} \to \mathscr{T}$ giving our admissible embedding) which factors through $\mathscr{D}(\widehat{G}) \cap \widehat{T}$ by dimension and root system considerations. From this, one obtains $\hat{T}_{\text{sc}} \to \mathscr{D}(\widehat{G})$ which further factors through an embedding $\hat{T}_{\text{sc}} \to \mathscr{D}(\widehat{G})_{\text{sc}}$ that identifies $\hat{T}_{\text{sc}}$ with a maximal torus of $\widehat{G}_{\text{sc}}$, giving an embedding $\widehat{Z}_{\text{sc}} \hookrightarrow \hat{T}_{\text{sc}}$ which is canonical (because of centrality, this does not depend on our initial embedding of $\widehat{T}$ in $\widehat{G}$). The same result holds for $\hat{\bar{T}}_{\text{sc}}$.

With this in hand, we set

$$\widehat{U} := \frac{\hat{T}_{\text{sc}} \times \hat{\bar{T}}_{\text{sc}}}{\widehat{Z}_{\text{sc}}},$$

where now $\widehat{Z}_{\text{sc}}$ is embedded diagonally. The $\mathbb{Q}$-pairing $\mathbb{Q}R^\vee \times \mathbb{Q}R \to \mathbb{Q}$ gives a pairing $X^*(\hat{T}_{\text{sc}}) \times X^*(T_{\text{sc}}) \to \mathbb{Q}$ which, together with the analogue for $\bar{T}$, yields a $\mathbb{Q}$-pairing between $X^*(\hat{T}_{\text{sc}} \times \hat{\bar{T}}_{\text{sc}})$ and $X^*(T_{\text{sc}} \times \bar{T}_{\text{sc}})$, which further induces a perfect $\mathbb{Z}$-pairing between $X^*(\widehat{U})$ and $X^*(U)$, identifying $\widehat{U}$ with the dual of U, see [LS87], §3.4.

Take the projection of $\eta(s) \in \mathscr{T}(\mathbb{C})$ in $\mathscr{T}_{\text{ad}}(\mathbb{C})$, and then pick an arbitrary preimage $\tilde{s}$ of this projection in $\mathscr{T}_{\text{sc}}(\mathbb{C})$. We have isomorphisms $\hat{T}_{\text{sc}} \to \mathscr{T}_{\text{sc}}$, $\hat{\bar{T}}_{\text{sc}} \to \mathscr{T}_{\text{sc}}$ induced by choices of isomorphisms $\widehat{T}, \widehat{\bar{T}} \to \mathscr{T}$ giving our admissible embeddings, and the respective preimages of $\tilde{s}$, denoted by $\tilde{s}_T, \tilde{s}_{\bar{T}}$, only depend on choice of $\tilde{s}$ and the admissible isomorphisms $T_H \to T, \bar{T}_H \to \bar{T}$. We then set $s_U := (\tilde{s}_T, \tilde{s}_{\bar{T}}) \in \widehat{U}(\mathbb{C})$. Note that a different choice of $\tilde{s}$ corresponds to replacing $\tilde{s}_T, \tilde{s}_{\bar{T}}$ by $\tilde{s}_T z_T, \tilde{s}_{\bar{T}} z_{\bar{T}}$, where $z \in \widehat{Z}_{\text{sc}}(\mathbb{C})$ and $z_T, z_{\bar{T}}$ denote the images of z under the canonical embeddings of $\widehat{Z}_{\text{sc}}$ in $\hat{T}_{\text{sc}}, \hat{\bar{T}}_{\text{sc}}$. Thus, s_U is independent of the choice of $\tilde{s}$. Then one can show that $s_U \in \widehat{U}^\Gamma$, see for example the discussion of the Δ_{III_1} factor in [Kal16], proof of Proposition 5.6. Hence, it makes sense to define $\mathbf{s}_U$ to be the image of s_U in $\pi_0(\widehat{U}^\Gamma)$. We then set

$$\Delta_{III_1}(\gamma_H, \gamma_G; \bar{\gamma}_H, \bar{\gamma}_G) := \langle \text{inv}\left(\frac{\gamma_H, \gamma_G}{\bar{\gamma}_H, \bar{\gamma}_G}\right), \mathbf{s}_U \rangle. \tag{5.6}$$

By what we have done, it is clear that if G is quasi-split over F, then

$$\Delta_{III_1}(\gamma_H, \gamma_G; \bar{\gamma}_H, \bar{\gamma}_G) = \langle \text{inv}(\gamma_H, \gamma_G), \mathbf{s}_T \rangle^{-1} \langle \text{inv}(\bar{\gamma}_H, \bar{\gamma}_G), \mathbf{s}_{\bar{T}} \rangle.$$

Lemma 5.3.6 *If $T_H \to T$ and $\bar{T}_H \to \bar{T}$ are replaced by their g- and $\bar{g}$-conjugates, $g, \bar{g} \in \mathfrak{A}(T_{sc}), \mathfrak{A}(\bar{T}_{sc})$, then $\Delta_{III_1}(\gamma_H, \gamma_G; \bar{\gamma}_H, \bar{\gamma}_G)$ is multiplied by*

$$\langle \boldsymbol{g}_T, \boldsymbol{s}_T \rangle \langle \boldsymbol{g}_{\bar{T}}, \boldsymbol{s}_{\bar{T}} \rangle^{-1},$$

where $\boldsymbol{g}_T$ is the class of the 1-cocycle $p_1(g)p_2(g)^{-1} \in T_{sc}(\overline{F} \otimes_F \overline{F})$, analogously for $\boldsymbol{g}_{\bar{T}}$.

Proof. Denote the $g^{-1}, \bar{g}^{-1}$-conjugates of $T, \bar{T}$ by $T', \bar{T}'$. One checks that v as defined above is replaced by $p_1(g)^{-1} v p_2(g) \in T'_{\text{sc}}(\overline{F} \otimes_F \overline{F})$ and conjugating this element by $p_1(g)$ yields the element $v(p_1(g)p_2(g)^{-1})^{-1}$, analogously for $\bar{T}$ and $\bar{v}$. Similarly, $\tilde{s}_T, \tilde{s}_{\bar{T}}$ can be taken to be $\text{Ad}(g)\tilde{s}_{T'}$ (by $\text{Ad}(g)$, we mean the induced dual map $\hat{T'}_{\text{sc}} \to \hat{T}_{\text{sc}}$) and $\text{Ad}(\bar{g})\tilde{s}_{\bar{T}'}$. The functoriality of the Tate-Nakayama pairing then gives the result. $\square$

5.3.4 The factor Δ_{III_2}

To construct this factor, we will fix Borel subgroups $B \supset T_{F^s}$, $B_H \supset (T_H)_{F^s}$ which (along with our fixed $(\mathscr{B}, \mathscr{T}), (\mathscr{B}_H, \mathscr{T}_H))$ determine the admissible isomorphism $T_H \to T$; note that our χ- and a-data also serve the Γ-action on $R(H_{F^s}, (T_H)_{F^s}) \subset R$. Then, according to §5.2.2, we obtain from our χ-data $\{\chi_\alpha\}$ (viewed as a χ-data for T and for T_H) admissible embeddings $\xi_T\colon\ {}^LT \to {}^LG$ extending the map $\widehat{T} \to \mathscr{T}$ and $\xi_{T_H}\colon\ {}^LT_H \to {}^LH$ extending $T_H \to \mathscr{T}_H$. We then obtain

$$\eta \circ \xi_{T_H} = a \cdot \xi_T,$$

where we view ξ_T as a map on ${}^LT_H$ by means of the isomorphism ${}^LT_H \to {}^LT$ induced by the admissible isomorphism $T_H \to T$ and a is a 1-cocycle in $\mathscr{T}(\mathbb{C})$ for the $\widehat{T}$-transported W_F-action. Its class **a** in $H^1(W_F, \widehat{T}(\mathbb{C}))$ (after applying the fixed isomorphism $\mathscr{T} \to \widehat{T}$ to a) is independent of the choice of B_H and B, as well as the Γ-splittings $(\mathscr{B}, \mathscr{T}, \{X\})$ and $(\mathscr{B}_H, \mathscr{T}_H, \{X^H\})$ by Facts 5.2.5 and 5.2.4 from §5.2, respectively.

Suppose now that $T_H \to T$ (and the corresponding data) is replaced with a $g \in \mathfrak{A}(T_{\text{sc}})$-conjugate $T' = \text{Ad}(g^{-1})T$ with admissible embedding $\xi_{T'}$. Then Fact 5.2.7 from §5.2 shows that the induced isomorphism $\lambda_g\colon\ {}^LT' \to {}^LT$ satisfies $\xi_T \circ \lambda_g = \xi_{T'}$, and so it follows that the class **a** is the image of **a**$' \in H^1(W_F, \widehat{T}'(\mathbb{C}))$ under the isomorphism $H^1(W_F, \widehat{T}'(\mathbb{C})) \xrightarrow{\text{Ad}(g)} H^1(W_F, \widehat{T}(\mathbb{C}))$. The dependence on the χ-data will be addressed later.

We then set

$$\Delta_{III_2}(\gamma_H, \gamma_G) := \langle \mathbf{a}, \gamma \rangle,$$

where the above pairing comes from Langlands duality for tori, as in Theorem 5.1.5. By the functoriality of the pairing (Theorem 5.1.5) and our above remarks on the cocycle **a**, it is immediate that this number does not change if the admissible embedding $T_H \to T$ (and corresponding data) is changed by a $\mathfrak{A}(T_{\text{sc}})$-conjugate.

Lemma 5.3.7 *Suppose that the χ-data $\{\chi_\alpha\}$ is replaced by $\{\chi'_\alpha\}$, with $\zeta_\alpha := \chi'_\alpha/\chi_\alpha$. Then $\Delta_{III_2}(\gamma_H, \gamma_G)$ is multiplied by*

$$\prod_{aysmm} \zeta_\alpha(\gamma^\alpha)^{-1} \cdot \prod_{symm} \zeta_\alpha(\delta^\alpha)^{-1},$$

where γ^α and δ^α are defined as in 5.3.2.

Proof. This result is Lemma 3.5.A in [LS87]. The proof loc. cit. depends on our Lemma 5.2.2 (which is Corollary 2.5 loc. cit.) as well as the general discussion of our §5.2.2, Galois-cohomological computations similar to the ones done in our §5.3.2, and the fact that the pairing coming from Langlands duality for tori is functorial and respects restriction of scalars. All of these facts/techniques are unchanged in our setting, and therefore the same argument works. □

5.3.5 The factor Δ_{IV}

We denote the (normalized) absolute value on F by $|\cdot|$. For our $\gamma \in T(F)$, we set

$$D_{G^*}(\gamma) := |\prod_{\alpha \in R}(\alpha(\gamma) - 1)|^{1/2}. \tag{5.7}$$

Note that this is well-defined because $\prod_{\alpha \in R}(\alpha(\gamma) - 1) \in F$. Then we set

$$\Delta_{IV}(\gamma_H, \gamma_G) := D_{G^*}(\gamma) \cdot D_H(\gamma_H)^{-1}.$$

This is clearly unchanged if the admissible embedding is replaced by a $\mathfrak{A}(T_{\text{sc}})$-conjugate.

5.3.6 The local transfer factor

We are now ready to define the absolute transfer factor for quasi-split connected reductive groups G over F a local function field and the relative transfer factor for arbitrary connected reductive groups over F. Fix two pairs γ_G, γ_H, $\bar{\gamma}_H, \bar{\gamma}_G$ as in the beginning of §5.3.

For quasi-split G over F, we set

$$\Delta_0(\gamma_H, \gamma_G) = \Delta_I(\gamma_H, \gamma_G)\Delta_{II}(\gamma_H, \gamma_G)\Delta_1(\gamma_H, \gamma_G)\Delta_{III_2}(\gamma_H, \gamma_G)\Delta_{IV}(\gamma_H, \gamma_G).$$

For general G, we set

$$\Delta(\gamma_H, \gamma_G; \bar{\gamma}_H, \bar{\gamma}_G) := \frac{\Delta_I(\gamma_H, \gamma_G)}{\Delta_I(\bar{\gamma}_H, \bar{\gamma}_G)} \cdot \frac{\Delta_{II}(\gamma_H, \gamma_G)}{\Delta_{II}(\bar{\gamma}_H, \bar{\gamma}_G)} \cdot \frac{\Delta_{III_2}(\gamma_H, \gamma_G)}{\Delta_{III_2}(\bar{\gamma}_H, \bar{\gamma}_G)} \cdot \frac{\Delta_{IV}(\gamma_H, \gamma_G)}{\Delta_{IV}(\bar{\gamma}_H, \bar{\gamma}_G)} \cdot \Delta_{III_1}(\gamma_H, \gamma_G; \bar{\gamma}_H, \bar{\gamma}_G). \tag{5.8}$$

We have the following results that discuss the dependence of Δ_0, Δ on the admissible embeddings and χ- and a-data.

Theorem 5.3.8 *The factor $\Delta(\gamma_H, \gamma_G; \bar{\gamma}_H, \bar{\gamma}_G)$ is independent of the choice of admissible embeddings, a-data, and χ-data.*

Proof. If the admissible embeddings are replaced by $g^{-1} \in \mathfrak{A}(T_{\text{sc}})$ and $\bar{g}^{-1} \in \mathfrak{A}(\bar{T}_{\text{sc}})$-conjugate embeddings (with translated a- and χ-data), $\Delta_I(\gamma_H, \gamma_G)$ is multiplied by $\langle \mathbf{g}_T, \mathbf{s}_T \rangle^{-1}$ by Lemma 5.3.2 (similarly for $\bar{\gamma}_H, \bar{\gamma}_G$), $\Delta_{II}(\gamma_H, \gamma_G)$ is unchanged, Δ_{III_1} is multiplied by $\langle \mathbf{g}_T, \mathbf{s}_T \rangle \langle \mathbf{g}_{\bar{T}}, \mathbf{s}_{\bar{T}} \rangle^{-1}$ by Lemma 5.3.6, and Δ_{III_2}, Δ_{IV} are unaffected. Thus, Δ is unaffected.

If we change the a- and χ-data to $\{a'_\alpha\}$, $\{\chi'_\alpha\}$ with $b_\alpha := a'_\alpha/a_\alpha$ and $\zeta_\alpha := \chi'_\alpha/\chi_\alpha$, then the change in $\Delta_I(\gamma_H, \gamma_G)$ induced by the new a-data cancels with the change in $\Delta_{II}(\gamma_H, \gamma_G)$ induced by the new a-data, by Lemmas 5.3.2 and 5.3.3. The change in $\Delta_{II}(\gamma_H, \gamma_G)$ induced by the new χ-data is cancelled by the change in $\Delta_{III_2}(\gamma_H, \gamma_G)$ induced by the new χ-data, by Lemmas 5.3.4 and 5.3.7. All the other factors are unaffected. □

Note that by Lemma 5.3.1, $\Delta(\gamma_H, \gamma_G; \bar{\gamma}_H, \bar{\gamma}_G)$ is also independent of the F-splitting chosen for G^*_{sc} in the construction of the splitting invariant used to define Δ_I.

Corollary 5.3.9 *The factor $\Delta_0(\gamma_H, \gamma_G)$ only depends on the chosen F-splitting of G^*_{sc}.*

Proof. This is immediate after using the above proof and replacing Lemma 5.3.6 with the observation that conjugating the admissible embedding $T_H \to T$ by $g^{-1} \in \mathfrak{A}(T_{\mathrm{sc}})$ serves to multiply $\Delta_1(\gamma_H, \gamma_G)$ by $\langle \mathbf{g}_T, \mathbf{s}_T \rangle$, cancelling the corresponding new factor from $\Delta_I(\gamma_H, \gamma_G)$. □

5.4 Addendum: z-pairs

We continue with the same notation as §5.3. In particular, G is a connected reductive group over F with quasi-split inner twist G^* and endoscopic datum $\mathfrak{e}$. Our goal in this section is to extend the definition of the (relative) transfer factor Δ to the case where $\widehat{H} \to \mathcal{H}$ is not necessarily equal to the canonical embedding $\widehat{H} \to {}^L H$. To do this, we need to introduce the concept of a z-pair.

Definition 5.4.1 *A **z-pair** $\mathfrak{z} = (H_\mathfrak{z}, \eta_\mathfrak{z})$ for the endoscopic datum $\mathfrak{e}$ is an F-group $H_\mathfrak{z}$ that is an extension of H by an induced central torus such that $\mathscr{D}(H_\mathfrak{z})$ is simply-connected, and a map $\eta_\mathfrak{z} \colon \mathcal{H} \to {}^L H_\mathfrak{z}$ that is an L-embedding extending the embedding $\widehat{H} \to \widehat{H}_\mathfrak{z}$ dual to $H_\mathfrak{z} \to H$. We call an element of $H_\mathfrak{z}(F)$ **strongly G-regular semisimple** if its image in $H(F)$ is strongly G-regular and semisimple, as we defined above; this set will be denoted by $H_{\mathfrak{z},G-sr}(F)$.*

The following result explains the usefulness of this concept:

Proposition 5.4.2 *A z-pair $(H_\mathfrak{z}, \eta_\mathfrak{z})$ for $\mathfrak{e}$ always exists.*

Proof. The group $H_\mathfrak{z}$ without the data of $\eta_\mathfrak{z}$ is called a *z-extension* of H. Such a z-extension exists in any characteristic, using [MS89], Proposition 3.1; although the proposition loc. cit. is stated for local fields of characteristic zero, the proof works in the local function field setting as well. Once we have such an extension, Lemma 2.2.A in [KS99] shows that we can find an $\eta_\mathfrak{z}$ satisfying the desired properties (the proof loc. cit. does not depend on the characteristic of F either). □

We will now discuss how to extend the relative transfer factor to a function

$$\Delta\colon H_{\mathfrak{z},G-\mathrm{sr}}(F) \times G_{\mathrm{sr}}(F) \times H_{\mathfrak{z},G-\mathrm{sr}}(F) \times G_{\mathrm{sr}}(F) \to \mathbb{C},$$

satisfying all the desired properties enjoyed by the factor Δ defined above. This discussion is taken from the proof of Proposition 5.6 in [Kal16]. Let $\gamma_{\mathfrak{z}}, \bar{\gamma}_{\mathfrak{z}} \in H_{\mathfrak{z},G-\mathrm{sr}}(F)$ with images $\gamma_H, \bar{\gamma}_H$ in $H_{G-\mathrm{sr}}(F)$, related to $\gamma_G, \bar{\gamma}_G \in G_{\mathrm{sr}}(F)$. The factors $\Delta_I(\gamma_{\mathfrak{z}}, \gamma_G)$, $\Delta_{II}(\gamma_{\mathfrak{z}}, \gamma_G)$, $\Delta_{III_1}(\gamma_{\mathfrak{z}}, \gamma_G; \bar{\gamma}_{\mathfrak{z}}, \bar{\gamma}_G)$, and $\Delta_{IV}(\gamma_{\mathfrak{z}}, \gamma_G)$ are all defined to be the same factors with $\gamma_{\mathfrak{z}}, \bar{\gamma}_{\mathfrak{z}}$ replaced by their images $\gamma_H, \bar{\gamma}_H$. It remains to define $\Delta_{III_2}(\gamma_{\mathfrak{z}}, \gamma_G)$. Consider the following diagram:

$$\begin{array}{ccccccc} {}^L H_{\mathfrak{z}} & \hookleftarrow & {}^L T_{H_{\mathfrak{z}}} & \dashleftarrow & {}^L T_{H_{\mathfrak{z}}} & \hookleftarrow & {}^L T_H \\ \uparrow{\scriptstyle \eta_{\mathfrak{z}}} & & & & & \nearrow{\scriptstyle {}^L\phi_{\gamma_H,\gamma}} & \\ \mathcal{H} & \xhookrightarrow{\eta} & {}^L G & \hookleftarrow & {}^L T, & & \end{array}$$

where we are denoting the centralizer of $\gamma_{\mathfrak{z}}$ by $T_{H_{\mathfrak{z}}}$, the map ${}^L T \to {}^L G$ is the one corresponding to a choice of χ-data for T, as discussed in §5.3.4 and §5.2.2, we are denoting the choice of admissible embedding $T_H \to T$ by $\phi_{\gamma_H,\gamma}$, and the embedding ${}^L T_{H_{\mathfrak{z}}} \hookrightarrow {}^L H_{\mathfrak{z}}$ is obtained by transporting the χ-data to T_H and then to $T_{H_{\mathfrak{z}}}$ via the projection $T_{H_{\mathfrak{z}}} \to T_H$ (this makes sense because $H_{\mathfrak{z}}$ is a central extension of H, so that T_H and $T_{H_{\mathfrak{z}}}$ have the same root systems). The dotted arrow is the unique L-homomorphism extending the identity on $\widehat{T_{H_{\mathfrak{z}}}}$ and making the diagram commute; its restriction to W_F gives a 1-cocycle $a\colon W_F \to \widehat{T_{H_{\mathfrak{z}}}}(\mathbb{C})$; for an explanation of why such an L-homomorphism exists, as well as the fact that this is a cocycle, see [KS99], §4.4. We then set $\Delta_{III_2}(\gamma_{\mathfrak{z}}, \gamma_G) := \langle a, \gamma_{\mathfrak{z}} \rangle$, where as in §5.3.4 the pairing is from Langlands duality for tori.

We then define $\Delta(\gamma_{\mathfrak{z}}, \gamma_G; \bar{\gamma}_{\mathfrak{z}}, \bar{\gamma}_G)$ identically as in §5.3, except with our new Δ_{III_2} factor. We may also use this to define an analogous factor $\Delta_0(\gamma_{\mathfrak{z}}, \gamma_G)$ in the quasi-split case, where we simply replace the Δ_{III_2} factor in the definition given in §5.3 with the factor we defined above (and take the image of $\gamma_{\mathfrak{z}}$ in $H(F)$ to define the other Δ_i-factors).

Proposition 5.4.3 *The above factor does not depend on the choice of admissible embeddings, χ-data, or a-data.*

Proof. This is Theorem 4.6.A in [KS99]. In view of the proof of Theorem 5.3.8, it suffices to check that $\Delta(\gamma_{\mathfrak{z}}, \gamma_G; \bar{\gamma}_{\mathfrak{z}}, \bar{\gamma}_G)$ is unaffected by changing the χ-data for T. Verifying this comes down to examining the new Δ_{III_2}-factor, which is not affected by the characteristic of F, so the proof loc. cit. works in our situation as well. □

CHAPTER 6

Applications to the Local Langlands Conjectures

This section applies the theory we have constructed in order to state the local Langlands conjectures for connected reductive groups over local fields of positive characteristic. Again, in this section F is a local field of characteristic $p > 0$, G is a connected reductive group over F, and $\mathcal{E}$ is a u-gerbe split over $\overline{F}$ with $[\mathcal{E}] = \alpha \in \check{H}^2(\overline{F}/F, u)$. Recall that one of our goals is to generalize the notion of *rigid inner forms*, introduced in [Kal16], in order to work with the representations of all inner forms of G simultaneously.

6.1 Rigid inner twists

In order to assign to inner twists of G the "correct" automorphism group (i.e., one such that automorphisms preserve F-conjugacy classes and F-representations), we need to refine the data of an inner twist to that of a rigid inner twist. For a $G_{\mathcal{E}}$-torsor $\mathscr{T}$, we denote the $(G_{\text{ad}})_{\mathcal{E}}$-torsor $\mathscr{T} \times^{G_{\mathcal{E}}} (G_{\text{ad}})_{\mathcal{E}}$ by $\overline{\mathscr{T}}$.

Definition 6.1.1 *1. A* ***rigid inner twist*** *of G is a triple $(\xi, \mathscr{T}, \bar{h})$ of an inner twist $\xi \colon G \to G'$, a Z-twisted $G_{\mathcal{E}}$-torsor $\mathscr{T}$ for some finite central Z, and an isomorphism of $(G_{ad})_{\mathcal{E}}$-torsors $\bar{h} \colon \overline{\mathscr{T}}_{\overline{F}} \to (G_{ad})_{\mathcal{E},\overline{F}}$ which satisfies $p_1^*\bar{h} \circ p_2^*\bar{h}^{-1} \colon (G_{ad})_{\mathcal{E},U_1} \to (G_{ad})_{\mathcal{E},U_1}$ is left-translation by $\bar{x} \in G_{ad}(U_1)$ such that $Ad(\bar{x}) = p_1^*\xi^{-1} \circ p_2^*\xi$. If we demand that $\mathscr{T}$ is Z-twisted for some fixed finite central Z, then we say further that the rigid inner twist is a Z*-***rigid inner twist***.*

2. *An* ***isomorphism of rigid inner twists*** *$(f, \Psi) \colon (\xi_1, \mathscr{T}_1, \bar{h}_1) \to (\xi_2, \mathscr{T}_2, \bar{h}_2)$ is a pair consisting of an isomorphism $f \colon G_1 \to G_2$ defined over F and an isomorphism of $G_{\mathcal{E}}$-torsors $\Psi \colon \mathscr{T}_1 \to \mathscr{T}_2$; note that such an isomorphism induces an isomorphism $\bar{h}_2 \circ \Psi \circ \bar{h}_1^{-1} \colon (G_{ad})_{\mathcal{E},\overline{F}} \to (G_{ad})_{\mathcal{E},\overline{F}}$, giving an element $\bar{\delta} \in G_{ad}(\overline{F})$ which we require to satisfy $\xi_2^{-1} \circ f \circ \xi_1 = Ad(\bar{\delta})$.*

Denote by $RI(G, \mathcal{E})$ (resp. $RI_Z(G, \mathcal{E})$) the category whose objects are rigid inner twists of G (resp. Z-rigid inner-twists of G) and morphisms are isomorphisms of rigid inner twists. It is clear that the natural functor $RI_Z(G,\mathcal{E}) \to RI(G,\mathcal{E})$ is fully faithful and $RI(G,\mathcal{E}) = \varinjlim RI_Z(G,\mathcal{E})$, where the colimit is taken over all finite central Z. Note that for every inner twist $\psi\colon G \to G'$, there exists a $Z(\mathscr{D}(G))$-twisted $G_{\mathcal{E}}$-torsor $\mathscr{T}$ and trivialization $\bar{h}$ such that $(\psi, \mathscr{T}, \bar{h})$ is a rigid inner twist, by Proposition 4.5.12. For computational purposes, we reformulate the above definition in the case $\mathcal{E} = \mathcal{E}_a$ for $[a] = \alpha \in \check{H}^2(\overline{F}/F, u)$:

Definition 6.1.2 *1. For $a \in u(U_2)$ such that $[a] = \alpha \in \check{H}^2(\overline{F}/F, u)$, an a-**normalized rigid inner twist** of G is a pair $(\xi, (x,\phi))$ of an inner twist $\xi\colon G \to G'$ and $(x,\phi) \in Z^1(\mathcal{E}_a, Z \to G)$ for some finite central Z such that the image of (x,ϕ) in $Z^1(F, G_{ad})$, denoted by $\bar{x}$, satisfies Ad$(\bar{x}) = p_1^*\xi^{-1} \circ p_2^*\xi$. If we demand that ϕ factors through some fixed finite central Z, then we say further that the a-normalized rigid inner twist is an a-**normalized** Z-**rigid inner twist**.*

*2. An **isomorphism of** a-**normalized rigid inner twists** $(f,\delta)\colon (\xi_1, (x_1,\phi_1)) \to (\xi_2, (x_2,\phi_2))$ for $\phi_1 = \phi_2$, is a pair consisting of an isomorphism $f\colon G_1 \to G_2$ defined over F and $\delta \in G(\overline{F})$ such that $\xi_2^{-1} \circ f \circ \xi_1 = Ad(\delta)$ and $x_1 = p_1(\delta)^{-1} x_2 p_2(\delta)$.*

Denote by $RI(G,a)$ (resp. $RI_Z(G,a)$) the category whose objects are a-normalized rigid inner twists of G (resp. a-normalized Z-rigid inner twists of G) and morphisms are isomorphisms of a-normalized rigid inner twists.

We claim that, for $\mathcal{E} = \mathcal{E}_a$, the isomorphism classes of the category $RI(G, \mathcal{E}_a)$ are in canonical bijection with those of $RI(G,a)$. Let $s\colon \mathrm{Sch}/\overline{F} \to \mathcal{E}_a$ denote the section constructed in Lemma 2.3.3. Then if $(\xi, \mathscr{T}, \bar{h})$ is a Z-rigid inner twist, by the proof of Proposition 2.6.2, after setting $\phi := \mathrm{Res}(\mathscr{T})$, choosing a trivialization h of $s^*\mathscr{T}$ *lifting* $\bar{h}$, by which we mean is such that the diagram

$$\begin{array}{ccc} s^*\mathscr{T} & \xrightarrow{h} & G_{\overline{F}} \\ \downarrow & & \downarrow \\ s^*\overline{\mathscr{T}} & \xrightarrow{s^*\bar{h}} & (G_{\mathrm{ad}})_{\overline{F}} \end{array}$$

commutes (such an h evidently always exists), gives an a-twisted Z-cocycle (x,ϕ) valued in G, and by construction we also have that $\mathrm{Ad}(\bar{x}) = p_1^*\xi^{-1} \circ p_2^*\xi$. Thus, we have a way of associating to a Z-rigid inner twist an a-normalized Z-rigid inner twist.

Moreover, given any isomorphism (f, Ψ) between the Z-rigid inner twists $(\xi_1, \mathscr{T}_1, \bar{h}_1)$ and $(\xi_2, \mathscr{T}_2, \bar{h}_2)$, choices h_i of $\overline{F}$-trivializations lifting $s^*\bar{h}_i$ give an automorphism $h_2 \circ s^*\Psi \circ h_1^{-1}\colon G_{\overline{F}} \xrightarrow{\sim} G_{\overline{F}}$ which is left-translation by a unique $\delta \in G(\overline{F})$. Then we may define an isomorphism $(\xi_1, (x_1, \phi_1)) \to (\xi_2, (x_2, \phi_2))$ between the corresponding twisted cocycles (obtained using h_1 and h_2) given by (f, δ).

By Proposition 2.6.2 (see also Proposition 2.4.10), every a-normalized Z-rigid inner twist is isomorphic to the image of some Z-rigid inner twist under the above map. By the discussion following the proof of Lemma 2.6.4, if the image of two rigid inner twists are isomorphic as a-normalized rigid inner twists, then they are isomorphic as rigid inner twists (using that the condition on δ and $\bar{\delta}$ is the same).

A similar argument using Lemma 2.6.3 shows that for arbitrary $\mathcal{E}$, we have a canonical bijection between isomorphism classes in $RI(G, \mathcal{E})$ and $RI(G, \mathcal{E}_a)$, and hence also between classes in $RI(G, \mathcal{E})$ and isomorphism classes in $RI(G, a)$. Everything said above applies if we restrict ourselves to Z-rigid inner forms for some fixed Z as well.

We have the following important fact about automorphisms of rigid inner forms:

Proposition 6.1.3 *The automorphism group of a fixed a-normalized rigid inner twist $(\xi, (x, \phi))$ for $\xi \colon G_{F^s} \to (G')_{F^s}$ is canonically isomorphic to $G'(F)$ by the map $(f, \delta) \mapsto \xi(\delta)$.*

Proof. One computes the 0-differential of $\xi(\delta)$ to be $p_1^*\xi(p_1\delta^{-1}) \cdot p_2^*\xi(p_2\delta)$, and post-composing with $p_1^*\xi^{-1}$ yields

$$p_1\delta^{-1} \cdot x \cdot p_2\delta \cdot x^{-1} = e,$$

giving $\xi(\delta) \in G'(F)$, showing that the above map is well-defined. From here it is straightforward to check that it defines an isomorphism. □

Corollary 6.1.4 *The automorphism group of a fixed rigid inner twist $(\xi, \mathscr{T}, \bar{h})$ for $\xi \colon G_{F^s} \to (G')_{F^s}$ is canonically isomorphic to $G'(F)$.*

Proof. Fix a section $s \colon (\mathrm{Sch}/\overline{F}) \to \mathcal{E}$, as well as a trivialization h of $s^*\mathscr{T}$ lifting $\bar{h}$ (terminology as above). Note that any two choices of h differ by precomposing by an automorphism of $s^*\mathscr{T}$ induced by an automorphism of $\mathscr{T}$ given by right-translation by some $z \in Z(G)(\overline{F})$. The map $h \circ s^*\Psi \circ h^{-1}$ is left-translation by an element $\delta \in G(\overline{F})$, and any different choice of h yields the same δ, by the $G_{\mathcal{E}}$-equivariance of Ψ. We may thus define our desired isomorphism to send (f, Ψ) to $\xi(\delta)$, which lies in $G'(F)$, by the proof of Proposition 6.1.3. It is straightforward to verify that the element δ does not depend on the choice of section s. □

We now define rational and stable conjugacy of elements of rigid inner forms. Let $(\xi_1, \mathscr{T}_1, \bar{h}_1)$ and $(\xi_2, \mathscr{T}_2, \bar{h}_2)$ be two Z-rigid inner twists for some fixed Z corresponding to the groups G_1, G_2, and let $\delta_i \in G_{i,\mathrm{sr}}(F)$ for $i = 1, 2$. We say that $(G_1, \xi_1, \mathscr{T}_1, \bar{h}_1, \delta_1)$ and $(G_2, \xi_2, \mathscr{T}_2, \bar{h}_2, \delta_2)$ are *rationally conjugate* if there exists an isomorphism $(f, \Psi) \colon (\xi_1, \mathscr{T}_1, \bar{h}_1) \to (\xi_2, \mathscr{T}_2, \bar{h}_2)$ such that $f(\delta_1) = \delta_2$. We say that they are *stably conjugate* if $\xi_1^{-1}(\delta_1)$ is $G(\overline{F})$-conjugate to $\xi_2^{-1}(\delta_2)$. The arguments used in §5.1 show that the latter condition is equivalent to $\xi_1^{-1}(\delta_1)$ being $G(F^s)$-conjugate to $\xi_2^{-1}(\delta_2)$ (this centers on the fact that the Weyl group scheme of a maximal torus in an algebraic group is étale). Define rational and stable conjugacy identically for elements of a-normalized rigid inner twists.

We need the following lemma:

Lemma 6.1.5 *Assume that G is quasi-split. For any $(G_1, \xi_1, \mathscr{T}_1, \bar{h}_1, \delta_1)$ (resp. $(G_1, \xi_1, (x_1, \phi_1), \delta_1)$) as above, there exists $\delta \in G_{sr}(F)$ such that $(G_1, \xi_1, \mathscr{T}_1, \bar{h}_1, \delta_1)$ (resp. $(G_1, \xi_1, (x_1, \phi_1), \delta_1)$) is stably conjugate to $(G, id_G, G_{\mathcal{E}}, id_{\overline{F}}, \delta)$ (resp. $(G, id_G, (e, 0), \delta)$).*

It is evidently enough to generalize Corollary 2.2 of [Kot82] to our setting, which says:

Lemma 6.1.6 *Let G be a quasi-split reductive group over F and $i : T_{F^s} \to G_{F^s}$ be an embedding over F^s of an F-torus T into G such that $i(T_{F^s})$ is a maximal torus of G_{F^s} and such that ${}^{\sigma}i$ is conjugate under $G(F^s)$ to i for all $\sigma \in \Gamma$. Then some $G(F^s)$-conjugate of i is defined over F.*

Proof. The proof of this result in [Kot82] depends on first proving the following result (Lemma 2.1 loc. cit.): Let $w \colon \Gamma \to W(G_{F^s}, T_{F^s})$ be a 1-cocycle of Γ in the absolute Weyl group of T, and choose an arbitrary lift $n_\sigma \in N_G(T)(F^s)$ of $w(\sigma)$ for all $\sigma \in \Gamma$. Then we may use it to twist T, obtaining an F-torus ${}^*T$ which is an F^s-form of T, and to twist the F-variety G/T, obtaining the F-variety ${}^*(G/T)$ which is an F^s-form of G/T. The claim is then that ${}^*(G/T)(F) \neq \emptyset$. As in [Kot82], this will follow if we can find some $t \in T_{\text{sr}}(F^s)$ and $g \in G(F^s)$ such that $gtg^{-1} \in G(F)$. We will view $({}^*T)_{F^s}$ as a subtorus of G_{F^s} via the isomorphism $({}^*T)_{F^s} \xrightarrow{\phi} T_{F^s}$ coming from its construction as an F^s-form of T.

To this end, we know by unirationality that ${}^*T(F)$ is Zariski-dense in $({}^*T)_{\overline{F}}$, and also that the locus of strongly regular elements in $T(\overline{F})$ forms a Zariski-open subset of $T_{\overline{F}}$, by [Ste65], Theorem 1.3.a, and hence there is some element $t \in ({}^*T)(F)$ that lies in $T_{\text{sr}}(\overline{F})$; such a point necessarily lies in $T(F^s)$, since ϕ maps ${}^*T(F^s)$ into $T(F^s)$. Then [BS68], 8.6 (which is a generalization of Theorem 1.7 in [Ste65] to imperfect fields) shows that we may find a point in $G_{\text{sr}}(F)$ which is $G(\overline{F})$-conjugate to t, which we know is equivalent to $G(F^s)$-conjugacy. This gives the claim; with this in hand, the argument in [Kot82], Lemma 2.1, carries over verbatim to show that ${}^*(G/T)(F) \neq \emptyset$.

Now we prove the main lemma, following [Kot82]. We may assume that $i(T_{F^s})$ is defined over F, with F-descent denoted by T', by conjugating by an appropriate element of $G(F^s)$. Choose $n_\sigma \in N_G(T)(F^s)$ such that $\text{Ad}(n_\sigma) \circ i = {}^{\sigma}i$ with image $w(\sigma) \in W(G_{F^s}, T'_{F^s})$ independent of choice of n_σ. Now apply the above claim to the F-torus T' and the cocycle $\sigma \mapsto w(\sigma)$, thus obtaining $\bar{g} \in {}^*(G/T')(F) \subset (G/T')(F^s) = G(F^s)/T(F^s)$ (containment via the defining isomorphism of the twisted form). This last equality comes from the fact that for every $t \in (G/T')(F^s)$, if $\pi \colon G_{F^s} \to (G/T)_{F^s}$ denotes the canonical quotient map, the (scheme-theoretic) fiber $\pi^{-1}(t) \hookrightarrow G_{F^s}$ is a T_{F^s}-torsor, which is split over F^s and thus contains an F^s-point.

The upshot is that we have some $g \in G(F^s)$ which satisfies $g^{-1}\,{}^{\sigma}g n_\sigma \in i(T_{F_s})(F^s)$ for all $\sigma \in \Gamma$, which means that $\mathrm{Ad}(g) \circ i$ is defined over F. □

We continue to assume that G is quasi-split. For any $(G_1, \xi_1, \mathscr{T}_1, \bar{h}_1, \delta_1)$, there exists $\delta \in G_{\mathrm{sr}}(F)$ such that $(G_1, \xi_1, \mathscr{T}_1, \bar{h}_1, \delta_1)$ is stably conjugate to $(G, \mathrm{id}_G, G_{\mathcal{E}}, \mathrm{id}_{\overline{F}}, \delta)$, by the above lemma. As in [Kal16], we now fix $\delta \in G_{\mathrm{sr}}(F)$ and consider the category $\mathcal{C}_Z(\delta, \mathcal{E})$ whose objects are points $(G_1, \xi_1, \mathscr{T}_1, \bar{h}_1, \delta_1)$ which are stably conjugate to $(G, \mathrm{id}_G, G_{\mathcal{E}}, \mathrm{id}_{\overline{F}}, \delta)$ such that $(\xi_1, \mathscr{T}_1, \bar{h}_1)$ is a Z-rigid inner twist and whose morphisms $(G_1, \xi_1, \mathscr{T}_1, \bar{h}_1, \delta_1) \to (G_2, \xi_2, \mathscr{T}_2, \bar{h}_2, \delta_2)$ are isomorphisms of rigid inner twists (f, Ψ) such that $f(\delta_1) = \delta_2$. We interpret this category as the stable conjugacy class of $(G, \mathrm{id}_G, G_{\mathcal{E}}, \mathrm{id}_{\overline{F}}, \delta)$, and it is clear that the isomorphism classes within $\mathcal{C}_Z(\delta, \mathcal{E})$ give the rational conjugacy classes within this stable conjugacy class. We define the category $\mathcal{C}_Z(\delta, a)$ using a-normalized Z-rigid inner twists completely analogously. By our previous discussion, it is clear that the isomorphism classes of $\mathcal{C}_Z(\delta, \mathcal{E}_a)$ are in canonical bijection with those of $\mathcal{C}_Z(\delta, a)$, as are the isomorphism classes of $\mathcal{C}_Z(\delta, \mathcal{E})$.

Set $S := Z_G(\delta)$, a maximal torus. We will now define a map from the isomorphism classes of $\mathcal{C}_Z(\delta, \mathcal{E})$ to $H^1(\mathcal{E}, Z \to S)$, denoted by $\mathrm{inv}(-, \delta)$. The simplest way to do this for general $\mathcal{E}$ is to first define it for isomorphism classes in $\mathcal{C}_Z(\delta, a)$ for a representing $[\mathcal{E}]$, invoke the canonical bijection between the isomorphism classes in $\mathcal{C}_Z(\delta, \mathcal{E})$ and those of $\mathcal{C}_Z(\delta, a)$ and then check that for a cohomologous to a', the diagram

$$\begin{array}{ccc} \mathrm{Isom}[\mathcal{C}_Z(\delta, a)] & \longrightarrow & H^1(\mathcal{E}_a, Z \to S) \\ \downarrow & & \downarrow \\ \mathrm{Isom}[\mathcal{C}_Z(\delta, a')] & \longrightarrow & H^1(\mathcal{E}_{a'}, Z \to S) \end{array} \tag{6.1}$$

commutes, where $\mathrm{Isom}[\mathcal{C}_Z(\delta, a)]$ denotes the set of isomorphism classes in $\mathcal{C}_Z(\delta, a)$, and the vertical arrows are the canonical bijections induced by any choice of $y \in u(U_1)$ such that $dy \cdot a = a'$ (cf. Construction 2.3.4). This last condition ensures that the map we define is canonical.

Fix $(G_1, \xi_1, (x_1, \phi_1), \delta_1) \in \mathcal{C}_Z(\delta, a)$, and choose $g \in G(F^s)$ such that $\xi_1(g\delta g^{-1}) = \delta_1$. The map sending this element to the a-twisted cocycle $(p_1(g)^{-1} x_1 p_2(g), \phi_1)$ gives a map $\mathcal{C}_Z(\delta, a) \to Z^1(\mathcal{E}_a, Z \to S)$, since translating by g does not affect the differential of x_1. This induces a map $\mathrm{inv}(-, \delta) \colon \mathcal{C}_Z(\delta, a) \to H^1(\mathcal{E}_a, Z \to S)$, which does not depend on the choice of g, by construction of the equivalence relation defined on a-twisted cocycles. The following result shows that the cohomology set $H^1(\mathcal{E}_a, Z \to S)$ parametrizes the rational classes within the stable class of δ.

Proposition 6.1.7 *The map inv$(-, \delta)$ induces a bijection from the isomorphisms classes of $\mathcal{C}_Z(\delta, a)$ to $H^1(\mathcal{E}_a, Z \to S)$.*

Proof. First note that if $(G_1, \xi_1, (x_1, \phi_1), \delta_1) \in \mathcal{C}_Z(\delta, a)$ and $(G_2, \xi_2, (x_2, \phi_2), \delta_2) \in \mathcal{C}_Z(\delta, a)$ are isomorphic via (f, g) then if we take g_i satisfying $\xi_1(g_i \delta_1 g_i^{-1}) = \delta_i$, we have $\phi_1 = \phi_2$ (by definition) and $g_1^{-1} g^{-1} g_2 \in S(\overline{F})$, since

$$\mathrm{Ad}(g_1^{-1} g^{-1} g_2)\delta = \mathrm{Ad}(g_1^{-1})(\xi_1^{-1} \circ f^{-1} \circ \xi_2)(\mathrm{Ad}(g_2)(\delta)) = \mathrm{Ad}(g_1^{-1})(\xi_1^{-1}(\delta_1)) = \delta,$$

giving that $[(p_1(g_1)^{-1} x_1 p_2(g_1), \varphi_1)] = [(p_1(g_2)^{-1} x_2 p_2(g_2), \varphi_1)]$ in $H^1(\mathcal{E}_a, Z \to S)$. This shows that the invariant map is constant on isomorphism classes.

For injectivity, we note that if $[(p_1(g_1)^{-1} x_1 p_2(g_1), \phi)] = [(p_1(g_2)^{-1} x_2 p_2(g_2), \phi)]$ in $H^1(\mathcal{E}_a, Z \to S)$, then if we take $g \in S(\overline{F})$ realizing this equivalence of cocycles, the (fppf descent of the) map $G_1 \to G_2$ defined by $\xi_2 \circ \mathrm{Ad}(g_2 g g_1^{-1}) \circ \xi_1^{-1}$ defines an isomorphism from $(G_1, \xi_1, (x_1, \phi_1), \delta_1)$ to $(G_2, \xi_2, (x_2, \phi_2), \delta_2)$ in $\mathcal{C}_Z(\delta)$.

For surjectivity, if we fix $[(x, \phi)] \in H^1(\mathcal{E}_a, Z \to S)$, then since $dx \in Z(G)$, we may twist G by x to obtain G^x, with the usual isomorphism $\xi \colon G \xrightarrow{\sim} G^x$ satisfying $p_1^* \xi^{-1} \circ p_2^* \xi = \mathrm{Ad}(x)$, and then (since x commutes with δ) the tuple $(G^x, \xi, (x, \phi), \xi(\delta))$ lies in $\mathcal{C}_Z(\delta, a)$ and trivially maps to $(x, \phi) \in Z^1(\mathcal{E}_a, Z \to S)$. □

Lemma 6.1.8 *The diagram* (6.1) *commutes.*

Proof. If $y \in u(U_1)$ is such that $dy \cdot a = a'$, then the map $\mathcal{C}_Z(\delta, a) \to \mathcal{C}_Z(\delta, a')$ may be defined by sending $(G_1, \xi_1, (x_1, \phi_1), \delta_1)$ to $(G_1, \xi_1, (x_1 \cdot \phi_1(y), \phi_1), \delta_1)$. This maps to the equivalence class of the a'-twisted cocycle $(p_1(g)^{-1} \phi_1(y) x p_2(g), \phi_1)$ in $H^{1,*}(\mathcal{E}_{a'}, Z \to G)$. Going the other direction, the class of $(G_1, \xi_1, (x_1, \phi_1), \delta_1)$ maps to the equivalence class of the a-twisted cocycle $(p_1(g)^{-1} x p_2(g), \phi_1)$, which then maps to the class of $(p_1(g)^{-1} \phi_1(y) x p_2(g), \phi_1)$, by the centrality of Z. □

Because of the above result, in the context of the invariant map it will be harmless to denote $\mathcal{C}_Z(\delta, \mathcal{E})$ for a choice of $\mathcal{E}$ simply by $\mathcal{C}_Z(\delta)$, and for computational purposes to identify $\mathcal{C}_Z(\delta)$ with $\mathcal{C}_Z(\delta, a)$ for a choice of a. Note that if $Z \to S$ factors through another finite central $Z' \to S$, then we have a canonical functor $\iota_{Z,Z'} \colon \mathcal{C}_Z(\delta) \to \mathcal{C}_{Z'}(\delta)$ which is fully faithful. Moreover, the two invariant maps to $H^1(\mathcal{E}, Z \to S), H^1(\mathcal{E}, Z' \to S)$ commute with the natural inclusion $H^1(\mathcal{E}, Z \to S) \to H^1(\mathcal{E}, Z' \to S)$; thus, the invariant map does not depend on the choice of Z.

The last thing we do in this subsection is define a representation of a rigid inner form.

Definition 6.1.9 *A* ***representation*** *of a rigid inner twist of G is a tuple* $(G_1, \xi_1, \mathscr{T}_1, \bar{h}_1, \pi_1)$, *where* $(\xi_1, \mathscr{T}_1, \bar{h}_1)$ *is a rigid inner twist of G and* π_1 *is an admissible representation of* $G_1(F)$. *An* ***isomorphism of representations*** *of rigid inner twists* $(G_1, \xi_1, \mathscr{T}_1, \bar{h}_1, \pi_1) \to (G_2, \xi_2, \mathscr{T}_2, \bar{h}_2, \pi_2)$ *is an isomorphism of rigid inner twists* $(f, \Psi) \colon (\xi_1, \mathscr{T}_1, \bar{h}_1) \to (\xi_2, \mathscr{T}_2, \bar{h}_2)$ *such that the* $G_1(F)$-*representations* π_1 *and* $\pi_2 \circ f$ *are isomorphic.*

One verifies easily that two representations $(G_1, \xi_1, \mathscr{T}_1, \bar{h}_1, \pi_1)$ and $(G_1, \xi_1, \mathscr{T}_1, \bar{h}_1, \pi_2)$ are isomorphic in the above sense if and only if π_1 and π_2 are isomorphic as $G_1(F)$-representations.

6.2 Local transfer factors and endoscopy

Let $[Z \to G] \in \mathcal{R}$ and let $\widehat{G}$ be a complex Langlands dual group for G. We have an isogeny $G \to \overline{G}$ which dualizes to an isogeny $\widehat{\overline{G}} \to \widehat{G}$, inducing a homomorphism $Z(\widehat{\overline{G}}) \to Z(\widehat{G})$. Identifying these complex varieties with their $\mathbb{C}$-points, we define $Z(\widehat{\overline{G}})^+ \subset Z(\widehat{\overline{G}})$ to be the preimage of $Z(\widehat{G})^\Gamma$ under this isogeny. We thus obtain a functor $\mathcal{R} \to$ FinAbGrp by sending G to $\pi_0(Z(\widehat{\overline{G}})^+)^*$; this can be seen as an analogue of functor introduced in Theorem 1.2 in [Kot86].

Proposition 6.2.1 *We have a functorial isomorphism*

$$Y_{+,tor}(Z \to G) \xrightarrow{\sim} \pi_0(Z(\widehat{\overline{G}})^+)^*.$$

Proof. We describe what the construction of this map is; the proof that this construction indeed is a functorial isomorphism is identical to the one given in [Kal16], Proposition 5.3.

Recall that for $[Z \to G] \in \mathcal{R}$, the group $Y_{+,\mathrm{tor}}(Z \to G)$ is an inverse limit as S ranges over all maximal F-tori of G of groups of the form

$$\varinjlim \frac{(X_*(\bar{S})/X_*(S_{\mathrm{sc}}))^N}{I(X_*(S)/X_*(S_{\mathrm{sc}}))},$$

where each direct limit is over all finite Galois extensions of F splitting S. For a fixed S, we have a commutative square of multiplicative groups corresponding to the commutative square of character groups:

$$\begin{array}{ccc} Z(\widehat{\overline{G}}) & \longrightarrow & Z(\widehat{G}) \\ \downarrow & & \downarrow \\ \widehat{\bar{S}} & \longrightarrow & \widehat{S} \end{array} \qquad\qquad \begin{array}{ccc} \frac{X_*(\bar{S})}{X_*(S_{\mathrm{sc}})} & \longleftarrow & \frac{X_*(S)}{X_*(S_{\mathrm{sc}})} \\ \uparrow & & \uparrow \\ X_*(\bar{S}) & \longleftarrow & X_*(S). \end{array}$$

Under the canonical embedding $Z(\widehat{G}) \to \widehat{S}$, the subgroup inclusion $Z(\widehat{G})^\Gamma \subset Z(\widehat{G})$ corresponds at the level of character groups to the quotient map

$$X_*(S)/X_*(S_{\mathrm{sc}}) \to [X_*(S)/X_*(S_{\mathrm{sc}})]/I[X_*(S)/X_*(S_{\mathrm{sc}})],$$

and it follows that the subgroup $Z(\widehat{\overline{G}})^+ \subset Z(\widehat{\overline{G}})$ has character group

$$X^*(Z(\widehat{\overline{G}})^+) = [X_*(\bar{S})]/[IX_*(S) + X_*(S_{\text{sc}})].$$

Finally, passing to the component group corresponds to taking the torsion subgroup, which (for a Galois extension splitting S) contains $[X_*(\bar{S})/X_*(S_{\text{sc}})]^N/I[X_*(S)/X_*(S_{\text{sc}})]$. This gives a natural inclusion

$$\frac{(X_*(\bar{S})/X_*(S_{\text{sc}}))^N}{I(X_*(S)/X_*(S_{\text{sc}}))} \hookrightarrow \pi_0(Z(\widehat{\overline{G}})^+)^*,$$

since we have the obvious identification $X^*(\pi_0(Z(\widehat{\overline{G}})^+)) = \pi_0(Z(\widehat{\overline{G}})^+)^*$. These maps glue for varying Galois extensions of F, and then induce an isomorphism on the direct limit over all extensions E (see [Kal16], Proposition 5.3). □

The analogue of Corollary 5.4 in [Kal16] makes precise our earlier statement comparing this new functor to the one defined in [Kot86], Theorem 1.2:

Corollary 6.2.2 *There is a perfect pairing*

$$H^1(\mathcal{E}, Z \to G) \times \pi_0(Z(\widehat{\overline{G}})^+) \to \mathbb{Q}/\mathbb{Z},$$

which is functorial in $[Z \to G] \in \mathcal{R}$. Moreover, if Z is trivial then this pairing coincides with the one stated in Theorem 4.5.5.

We now recall the notion of a *refined endoscopic datum*, introduced in [Kal16], §5. As before, assume that we have some fixed finite central $Z \to G$, and denote G/Z by $\overline{G}$. First, let $(H, \mathcal{H}, s, \eta)$ be an endoscopic datum for G. We may always replace this datum with an equivalent $\widehat{G}(\mathbb{C})$-conjugate datum $(H, \mathcal{H}, s', \eta')$ such that $s' \in Z(\widehat{H})^\Gamma$ without affecting the value of the transfer factors Δ, Δ_0 involving $(H, \mathcal{H}, s, \eta)$ (see the beginning §5.3). We will always assume that our endoscopic datum has this form.

Choices of maximal tori in $\widehat{H}$, $\widehat{G}$, H, and G give embeddings $Z_{F^s} \to Z(H)_{F^s}$ which differ by pre- and post-composing with inner automorphisms induced by $G(F^s)$, $H(F^s)$, and hence are all the same, meaning that we have a canonical F-embedding $Z \hookrightarrow H$ (the Γ-equivariance follows from the fact that the maps $\widehat{T} \to \mathscr{T}$ for T maximal in G, $\mathscr{T}$ maximal in $\widehat{G}$, are Γ-equivariant up to the action of the Weyl group—analogously for H). It thus makes sense to define $\overline{H} := H/Z$, which gives rise to the isogeny $\widehat{\overline{H}} \to \widehat{H}$.

As above, we define $Z(\widehat{\overline{H}})^+$ to be the preimage of $Z(\widehat{H})^\Gamma$ in $Z(\widehat{\overline{H}})$, and declare that $(H, \mathcal{H}, \dot{s}, \eta)$ is a *refined endoscopic datum* if $H, \mathcal{H}$, and η are defined as for an endoscopic datum, and $\dot{s} \in Z(\widehat{\overline{H}})^+$ is such that $(H, \mathcal{H}, s, \eta)$ is an endoscopic datum, where s is the image of $\dot{s}$ under the map

$Z(\widehat{\overline{H}})^+ \to Z(\widehat{H})^\Gamma$. An isomorphism of two refined endoscopic data $(H, \mathcal{H}, \dot{s}, \eta), (H', \mathcal{H}', \dot{s}', \eta')$ is an element $\dot{g} \in \widehat{\overline{G}}(\mathbb{C})$ such that its image g in $\widehat{G}(\mathbb{C})$ satisfies $g\eta(\mathcal{H})g^{-1} = \eta'(\mathcal{H}')$, inducing $\beta\colon \mathcal{H} \to \mathcal{H}'$ and the restriction $\beta\colon \widehat{H} \to \widehat{H}'$, which (by basic properties of central isogenies) lifts uniquely to a map $\bar{\beta}\colon \widehat{\overline{H}} \to \widehat{\overline{H'}}$, and such that the images of $\bar{\beta}(\dot{s})$ and $\dot{s}'$ in $\pi_0(Z(\widehat{\overline{H'}})^+)$ coincide. It is clear that every endoscopic datum lifts to a refined endoscopic datum, and that every isomorphism of refined endoscopic data induces an isomorphism of the associated endoscopic data.

Let $\dot{\mathfrak{e}} = (\mathcal{H}, H, \eta, \dot{s})$ be a refined endoscopic datum for G with associated endoscopic datum $\mathfrak{e} = (\mathcal{H}, H, \eta, s)$, which is also an endoscopic datum for G^*. Let $\mathfrak{z} = (H_{\mathfrak{z}}, \eta_{\mathfrak{z}})$ be a z-pair for $\mathfrak{e}$. As discussed in Chapter 5, have two functions

$$\Delta[\mathfrak{e}, \mathfrak{z}]\colon H_{\mathfrak{z}, G-\text{sr}}(F) \times G^*_{\text{sr}}(F) \times H_{\mathfrak{z}, G-\text{sr}}(F) \times G^*_{\text{sr}}(F) \to \mathbb{C},$$

$$\Delta[\mathfrak{e}, \mathfrak{z}, \psi]\colon H_{\mathfrak{z}, G-\text{sr}}(F) \times G_{\text{sr}}(F) \times H_{\mathfrak{z}, G-\text{sr}}(F) \times G_{\text{sr}}(F) \to \mathbb{C},$$

where the first equation makes sense because strongly G-regular elements of $H(F)$ are strongly G^*-regular via choices of admissible embeddings $T_H \xrightarrow{\sim} T$, $T_{\bar{H}} \xrightarrow{\sim} \bar{T}$, as in our discussion of the local transfer factor. As in [Kal16], we have added terms in the brackets to show what each factor depends on. We set the above function to zero if either of the pairs of inputs consist of two elements which are not related.

For our arbitrary G, we say that an *absolute transfer factor* is a function

$$\Delta[\mathfrak{e}, \mathfrak{z}]_{\text{abs}}\colon H_{\mathfrak{z}, G-\text{sr}} \times G_{\text{sr}}(F) \to \mathbb{C},$$

which is nonzero for any pair $(\gamma_{\mathfrak{z}}, \delta)$ of related elements and satisfies the relation

$$\Delta[\mathfrak{e}, \mathfrak{z}]_{\text{abs}}(\gamma_{\mathfrak{z},1}, \delta_1) \cdot \Delta[\mathfrak{e}, \mathfrak{z}]_{\text{abs}}(\gamma_{\mathfrak{z},2}, \delta_2)^{-1} = \Delta[\mathfrak{e}, \mathfrak{z}](\gamma_{\mathfrak{z},1}, \delta_1; \gamma_{\mathfrak{z},2}, \delta_2).$$

By Chapter 5, if G is quasi-split, setting $\Delta[\mathfrak{e}, \mathfrak{z}] = \Delta_0$ (and zero if the pair is unrelated) satisfies these properties. As we noted in Corollary 5.3.9, this function is not unique, depending on a choice of F-splitting of G_{sc}. Our next goal will be to use the notions of refined endoscopic data and Z-rigid inner forms to construct an absolute transfer factor in the non quasi-split case which is associated to some splitting of the quasi-split inner form G^*, extending the absolute transfer factor in the quasi-split case. This follows the corresponding construction in [Kal16], §5.3.

We return to the setting of arbitrary G connected reductive over F with quasi-split inner form $\psi\colon G_{F^s} \to G^*_{F^s}$ and fixed Z-rigid inner form $(\xi, \mathscr{T}, \bar{h})$, $\xi := \psi^{-1}$, for some fixed finite central Z defined over F. Let $\delta' \in G(F)$ and $\gamma_{\mathfrak{z}} \in H_{\mathfrak{z}, G-\text{sr}}(F)$ be related elements, and let γ_H be the image of $\gamma_{\mathfrak{z}}$ in $H(F)$. Then, by Lemma 6.1.5, we may find $\delta \in G^*(F)$ such that $\dot{\delta}' := (G, \xi, \mathscr{T}, \bar{h}, \delta')$

lies in $\mathcal{C}_Z(\delta)$; note that by strong regularity, the induced isomorphism of centralizers $\mathrm{Ad}(g) \circ \psi \colon Z_G(\delta')_{F^s} \to Z_{G^*}(\delta)_{F^s}$, some $g \in G^*(F^s)$, is defined over F.

Let S_H denote the centralizer of γ_H in H, and S denote the centralizer of δ in G^*. Since γ_H and δ' are related, we have an admissible isomorphism $S_H \to Z_G(\delta')$ sending γ_H to δ'. Post-composing this map with the F-isomorphism $Z_G(\delta') \to S$ gives an admissible isomorphism $\phi_{\gamma_H,\delta} \colon S_H \to S$ which sends γ_H to δ, and is unique with these properties. This isomorphism identifies the canonically embedded copies of Z in both of the tori, and therefore induces an isomorphism $\bar{\phi}_{\gamma_H,\delta} \colon \overline{S}_H \to \overline{S}$. If $[\widehat{\overline{S}_H}]^+$ denotes the preimage of $\widehat{S_H}^{\Gamma}$ under the isogeny $\widehat{\overline{S}_H} \to \widehat{S_H}$, then the canonical ($\Gamma$-equivariant) embeddings $Z(\widehat{H}) \hookrightarrow \widehat{S_H}$, $Z(\widehat{\overline{H}}) \hookrightarrow \widehat{\overline{S}_H}$ induce a canonical embedding $Z(\widehat{\overline{H}})^+ \hookrightarrow [\widehat{\overline{S}_H}]^+$. If the group $[\widehat{\overline{S}}]^+$ is defined analogously, we have that $\bar{\phi}_{\gamma_H,\delta}^{-1}$ dualizes to a map $[\widehat{\overline{S}_H}]^+ \to [\widehat{\overline{S}}]^+$ (since $\bar{\phi}_{\gamma_H,\delta}$ is defined over F) which further induces an embedding

$$Z(\widehat{\overline{H}})^+ \hookrightarrow [\widehat{\overline{S}}]^+.$$

We thus obtain from $\dot{s} \in Z(\widehat{\overline{H}})^+$ associated to our refined endoscopic datum an element $\dot{s}_{\gamma_H,\delta} \in [\widehat{\overline{S}}]^+$. Then we set

$$\Delta[\dot{\mathfrak{e}}, \mathfrak{z}, \psi, (\mathscr{T}, \bar{h})]_{\text{abs}}(\gamma_{\mathfrak{z}}, \delta') := \Delta[\mathfrak{e}, \mathfrak{z}]_{\text{abs}}(\gamma_{\mathfrak{z}}, \delta) \cdot \langle \text{inv}(\delta, \dot{\delta}'), \dot{s}_{\gamma_H,\delta} \rangle^{-1}, \tag{6.2}$$

where the pairing $\langle -, - \rangle$ is as in Corollary 6.2.2 with $G = S$.

It is clear that we could have replaced $(\xi, \mathscr{T}, \bar{h})$ with any a-normalized Z-rigid inner twist $(\xi, (y, \phi^*))$ in its isomorphism class from the start, and defined the transfer factor using the invariant of the corresponding class of $(G, \xi, (y, \phi^*)), \delta')$ in $\mathcal{C}_Z(\delta, a)$. The last main goal of this paper will be to prove that (6.2) defines an absolute transfer factor on G. In light of the above discussion, it is enough to work entirely with a-normalized Z-rigid inner twists for some fixed choice of $a \in u(U_2)$ with $[a] = \alpha$. In this context, $\dot{\delta}'$ will denote the element $(G, \xi, (y, \phi^*), \delta') \in \mathcal{C}_Z(\delta, a)$, and we denote the function from (6.2) by $\Delta[\dot{\mathfrak{e}}, \mathfrak{z}, \psi, (y, \phi^*)]$.

Before we prove this, we discuss the dependency of this factor on Z. Let Z' be another finite central F-subgroup of G which contains F, viewed also as a finite central F-subgroup of G^*. We denote by $(y, (\phi')^*) \in Z^1(F, Z' \to G)$ the image of (y, ϕ^*) under the natural inclusion map, so that $(\phi^*)'$ is $\phi^* \colon u \to Z$ post-composed with the inclusion map, defining a Z'-rigid inner twist $(\xi, (y, (\phi')^*))$. As with Z, we have a canonical F-embedding $Z' \hookrightarrow H$ which commutes with our embedding of Z and the inclusion map, and we set $\overline{H}' := H/Z'$. Now we have an isogeny $\overline{H} \to \overline{H}'$ which dualizes to an isogeny $\widehat{\overline{H}'} \to \widehat{\overline{H}}$, inducing a canonical surjection $Z(\widehat{\overline{H}'})^+ \to Z(\widehat{\overline{H}})^+$. Choose a preimage $\ddot{s}$ in $Z(\widehat{\overline{H}'})^+$ of $\dot{s}$, giving a refined endoscopic datum $\ddot{\mathfrak{e}} := (H, \mathcal{H}, \ddot{s}, \eta)$. Note that the point $\ddot{\delta}' := (G, \xi, (y, (\phi')^*), \delta')$ equals $\iota_{Z,Z'}(\dot{\delta}') \in \mathcal{C}_{Z'}(\delta)$. As we discussed in §6.1, we then have that

$$\mathrm{inv}(\delta, \iota_{Z,Z'}(\dot{\delta}')) = i(\mathrm{inv}(\delta, \dot{\delta}'))$$

in $H^1(\mathcal{E}_a, Z' \to S)$, where i is the natural map $H^1(\mathcal{E}_a, Z \to S) \to H^1(\mathcal{E}_a, Z' \to S)$. One checks easily that $\ddot{s}_{\gamma_H,\delta}$ maps to $\dot{s}_{\gamma_H,\delta}$ under the dual surjection $\widehat{\overline{S}}' \to \widehat{\overline{S}}$. The functoriality of the pairing from Corollary 6.2.2 then gives us that

$$\langle i(\mathrm{inv}(\delta, \dot{\delta})), \ddot{s}_{\gamma_H,\delta}\rangle = \langle \mathrm{inv}(\delta, \dot{\delta}), \dot{s}_{\gamma_H,\delta}\rangle.$$

Since this factor is the only part of $\Delta[\ddot{\mathfrak{e}}, \mathfrak{z}, \psi, (y, (\phi')^*)]_{\mathrm{abs}}$ that depends on Z, we see that

$$\Delta[\dot{\mathfrak{e}}, \mathfrak{z}, \psi, (y, \phi^*)]_{\mathrm{abs}}(\gamma_{\mathfrak{z}}, \delta') = \Delta[\ddot{\mathfrak{e}}, \mathfrak{z}, \psi, (y, (\phi')^*))]_{\mathrm{abs}}(\gamma_{\mathfrak{z}}, \delta'). \tag{6.3}$$

Proposition 6.2.3 *The value of $\Delta[\dot{\mathfrak{e}}, \mathfrak{z}, \psi, (y, \phi^*)]_{abs}(\gamma_{\mathfrak{z}}, \delta')$ does not depend on the choice of δ, and the function $\Delta[\dot{\mathfrak{e}}, \mathfrak{z}, \psi, (y, \phi^*)]_{abs}$ is an absolute transfer factor. Moreover, this function does not change if we replace $\dot{\mathfrak{e}}$ by an equivalent refined endoscopic datum, or if we replace $(G, \xi, (y, \phi^*))$ by an isomorphic (a-normalized) Z-rigid inner twist of G^*.*

Proof. We follow the proof of [Kal16], Proposition 5.6. For the independence of the choice of δ let $\delta_0 \in G^*_{\mathrm{sr}}(F)$ be another element such that $(G, \xi, (y, \phi^*), \delta') \in \mathcal{C}_Z(\delta_0)$ and $\mathrm{Ad}(g') \circ \psi$, for some $g' \in G^*(F^s)$, induces an F-isomorphism $Z_G(\delta') \to Z_{G^*}(\delta_0)$. By taking the composition $(\mathrm{Ad}(g') \circ \psi) \circ (\psi^{-1} \circ \mathrm{Ad}(g^{-1}))$, we see that δ and δ_0 are conjugate by an element $c \in \mathfrak{A}(S) \subset G^*(F^s)$, notation as in Chapter 5. Similarly, the element realizing the stable conjugacy of δ and δ' may be chosen to lie in $G^*(F^s)$. From here, the same argument used in [Kal16] for the corresponding part of the proof of Proposition 5.6 works in our setting—we can still use Galois cohomology and our analysis of the local transfer factor in Chapter 5 lines up exactly with that of [LS87], §3.

As is remarked in [Kal16], invariance under isomorphisms of rigid inner twists is immediate from the fact that $\mathrm{inv}(\delta, \dot{\delta}')$ depends only on the isomorphism class of $\dot{\delta}'$ in $\mathcal{C}_Z(\delta)$. Similar to our justification of the fact that our function is independent of choice of δ, our discussion in Chapter 5 can be substituted for §3 of [LS87] and then the corresponding argument in [Kal16], Proposition 5.6 carries over verbatim to show that our function is invariant under isomorphisms of refined endoscopic data.

The only work we need to do here is to show that $\Delta[\dot{\mathfrak{e}}, \mathfrak{z}, \psi, (y, \phi^*)]_{\mathrm{abs}}$ is indeed an absolute transfer factor. This means that we need to show that

$$\Delta[\dot{\mathfrak{e}}, \mathfrak{z}, \psi, (y, \phi^*)]_{\mathrm{abs}}(\gamma_{\mathfrak{z},1}, \delta_1') \cdot \Delta[\dot{\mathfrak{e}}, \mathfrak{z}, \psi, (y, \phi^*)]_{\mathrm{abs}}(\gamma_{\mathfrak{z},2}, \delta_2')^{-1} = \Delta[\mathfrak{e}, \mathfrak{z}, \psi](\gamma_{\mathfrak{z},1}, \delta_1'; \gamma_{\mathfrak{z},2}, \delta_2').$$

We emphasize that we still follow the corresponding argument in [Kal16], Proposition 5.6, closely. Replacing $\dot{\mathfrak{e}}$ by an appropriate refined endoscopic datum as in our construction of $\ddot{\mathfrak{e}}$ above, we may assume, using the identity (6.3), that Z contains $Z(\mathscr{D}(G))$. Choose $\delta_1, \delta_2 \in G^*(F)$ which are stably conjugate to δ_1', δ_2'. It's enough to show that

$$\frac{\langle \text{inv}(\delta_1, \dot{\delta_1}'), \dot{s}_{\gamma_1,\delta_1}\rangle^{-1}}{\langle \text{inv}(\delta_2, \dot{\delta_2}'), \dot{s}_{\gamma_2,\delta_2}\rangle^{-1}} = \frac{\Delta[\mathfrak{e}, \mathfrak{z}, \psi](\gamma_{\mathfrak{z},1}, \delta_1'; \gamma_{\mathfrak{z},2}, \delta_2')}{\Delta[\mathfrak{e}, \mathfrak{z}](\gamma_{\mathfrak{z},1}, \delta_1; \gamma_{\mathfrak{z},2}, \delta_2)},$$

where we are using γ_i to denote the image of $\gamma_{\mathfrak{z},i}$ in $H_{G-\text{sr}}(F)$. To simplify the right-hand side, note that in the definition of the bottom factor, we may choose our admissible embeddings $Z_H(\gamma_i) \hookrightarrow G^*$ to be the unique ones from $Z_H(\gamma_i)$ to G^* that map γ_i to δ_i. Then, as in the definition of the factor Δ_1 in the quasi-split case (see §5.3.3), we have that $\gamma_{G^*} = \gamma$, and hence we can take $h = \text{id}$ and so $\text{inv}(\gamma_i, \delta_i) = 0 \in H^1(F, Z_{G^*}(\delta_i))$, giving $\Delta_1(\gamma_1, \delta_1; \gamma_2, \delta_2) = 1$. All of the $\Delta_I, \Delta_{II}, \Delta_{III_2}$, and Δ_{IV} factors of the numerator and denominator of the right-hand side coincide, and so all we're left with is

$$\Delta_{III_1}(\gamma_1, \delta_1'; \gamma_2, \delta_2') := \langle \text{inv}\left(\frac{\gamma_1, \delta_1'}{\gamma_2, \delta_2'}\right), \mathbf{s}_U\rangle, \tag{6.4}$$

where all the notation is as defined in §5.3.3.

Set $Z_H(\gamma_i) := S_i^H$, $Z_G(\delta_i') := S_i'$, and $Z_{G^*}(\delta_i) := S_i$; these are all maximal F-tori. Set

$$V := \frac{S_1 \times S_2}{Z(G^*)},$$

where $Z(G^*) \hookrightarrow S_1 \times S_2$ via $i^{-1} \times j$. The homomorphism $S_1 \times S_2 \to V$ defines a morphism $[Z \times Z \to S_1 \times S_2] \to [(Z \times Z)/Z \to V]$ in the category $\mathcal{T}$. We claim that the image in $H^1(\mathcal{E}_a, (Z \times Z)/Z \to V)$ of the element

$$(\text{inv}(\delta_1, \dot{\delta_1}')^{-1}, \text{inv}(\delta_2, \dot{\delta_2}')) \in H^1(\mathcal{E}_a, Z \times Z \to S_1 \times S_2),$$

where $\text{inv}(\delta_i, \dot{\delta_i}')$ is defined as in §6.1, lies inside $H^1(F, V)$ (embedded in $H^1(\mathcal{E}_a, (Z\times Z)/Z \to V)$ via the “inflation" map).

It is clear that the restriction maps $H^1(\mathcal{E}_a, Z \to S_i) \to \text{Hom}_F(u, Z)$ factor as a composition of the maps $H^1(\mathcal{E}_a, Z \to S_i) \to H^1(\mathcal{E}_a, Z \to G^*)$ and $H^1(\mathcal{E}_a, Z \to G^*) \xrightarrow{\text{Res}} \text{Hom}_F(u, Z)$. Moreover, the image of $\text{inv}(\delta_i, \delta_i')$ in $H^1(\mathcal{E}_a, Z \to G^*)$ is the class of the twisted a-cocycle $(p_1(g_i)yp_2(g_i)^{-1}, \phi^*)$, where $g_i \in G^*(\overline{F})$ is such that $\text{Ad}(g_i)\psi(\delta_i') = \delta_i$, which is just the class of the twisted cocycle $(y, \phi^*) \in Z^1(\mathcal{E}_a, Z \to G)$. This means that the image of $(\text{inv}(\delta_1, \dot{\delta_1}')^{-1}, \text{inv}(\delta_2, \dot{\delta_2}'))$ in $\text{Hom}_F(u, Z \times Z) = \text{Hom}_F(u, Z) \times \text{Hom}_F(u, Z)$ equals $(\text{Res}((y, \phi^*))^{-1}, \text{Res}((y, \phi^*))) = (-\phi^*, \phi^*)$ which is zero in $\text{Hom}_F(u, (Z \times Z)/Z)$. Whence, the exact sequence

$$H^1(F, V) \to H^1(\mathcal{E}_a, (Z \times Z)/Z \to V) \to \text{Hom}_F(u, Z)$$

gives the claim.

Recall from §5.3.3 that $U := ((S_1)_{\text{sc}} \times (S_2)_{\text{sc}})/Z_{\text{sc}}$ where Z_{sc} embeds via $i^{-1} \times j$ (here we are taking our admissible embeddings $S_i^H \to G^*$ to be the unique ones that send γ_i to δ_i); there is an obvious homomorphism $U \to V$. We now claim that the image of $\text{inv}(\gamma_1, \delta_1'/\gamma_2, \delta_2') \in H^1(F, U)$ in $H^1(F, V)$ coincides with the image of $(\text{inv}(\delta_1, \dot{\delta_1}')^{-1}, \text{inv}(\delta_2, \dot{\delta_2}'))$. From the rigidifying element $(y, \phi^*) \in Z^1(\mathcal{E}_a, Z \to G^*)$, $y \in G^*(\overline{F} \otimes_F \overline{F})$, $\psi^{-1} \colon Z \to G^*$, we extract the Čech 1-cochain y, which we will factor as $\bar{u} \cdot z$ with $\bar{u} \in \mathscr{D}(G^*)(\overline{F} \otimes_F \overline{F})$ and $z \in Z(G^*)(\overline{F} \otimes_F \overline{F})$; we can do this because the central isogeny decomposition for G^* is surjective on $\overline{F} \otimes_F \overline{F}$-points, owing to the fact that $H^1(\overline{F} \otimes_F \overline{F}, Z(\mathscr{D}(G^*))) = 0$. Let $u \in G^*_{\text{sc}}(\overline{F} \otimes_F \overline{F})$ be a lift of $\bar{u}$. By construction (see §5.3.3, using the fact that $\text{Ad}(u) = \text{Ad}(\bar{u}) = \text{Ad}(y) = p_1^*\psi \circ p_2^*\psi^{-1}$ on $G^*_{\overline{F} \otimes_F \overline{F}}$), we have the equality

$$\text{inv}\left(\frac{\gamma_1, \delta_1'}{\gamma_2, \delta_2'}\right) = ([p_1(g_1)up_2(g_1)^{-1}]^{-1}, p_1(g_2)up_2(g_2)^{-1}) \in U(\overline{F} \otimes_F \overline{F}), \tag{6.5}$$

whose image in $V(\overline{F} \otimes_F \overline{F})$ coincides with the image of $([p_1(g_1)yp_2(g_1)^{-1}]^{-1}, p_1(g_2)yp_2(g_2)^{-1})$, because, by design, $y = \bar{u} \cdot z$ for $z \in Z(G^*)(\overline{F} \otimes_F \overline{F})$. This gives the claim.

Since the pairing from Corollary 6.2.2 is functorial and extends the Tate-Nakayama pairing for tori, our desired equality

$$\frac{\langle \text{inv}(\delta_1, \dot{\delta_1}'), \dot{s}_{\gamma_1, \delta_1} \rangle^{-1}}{\langle \text{inv}(\delta_2, \dot{\delta_2}'), \dot{s}_{\gamma_2, \delta_2} \rangle^{-1}} = \langle \text{inv}\left(\frac{\gamma_1, \delta_1'}{\gamma_2, \delta_2'}\right), \mathbf{s}_U \rangle, \tag{6.6}$$

will follow from our above calculations if we produce an element of $[\widehat{\overline{V}}]^+$ whose image in $[\widehat{\overline{S_1}}]^+ \times [\widehat{\overline{S_2}}]^+$ via the map $\widehat{\overline{V}} \to \widehat{\overline{S_1}} \times \widehat{\overline{S_2}}$ dual to the projection map $\overline{S_1} \times \overline{S_2} \to \overline{V}$, where $\overline{V} := \frac{V}{(Z \times Z)/Z}$, is equal to $(\dot{s}_{\gamma_1, \delta_1}, \dot{s}_{\gamma_2, \delta_2})$ and whose image in $[\widehat{\overline{U}}]^+$ maps to s_U under the isogeny $[\widehat{\overline{U}}]^+ \to \widehat{U}^\Gamma$, where $\overline{U}$ is formed from the object $[Z(G^*_{\text{sc}}) \to U] \in \mathcal{T}$. Indeed, if we find such an element v, then we have the diagram

$$\begin{array}{ccc}
\text{inv}\left(\frac{\gamma_1, \delta_1'}{\gamma_2, \delta_2'}\right) \in H^1(F, U) & & s_U \in \widehat{U}^\Gamma \\
\downarrow & & \uparrow \\
\pi((\text{inv}(\delta_1, \dot{\delta}_1')^{-1}, \text{inv}(\delta_2, \dot{\delta}_2))) \in H^1(F, V) & & v \in [\widehat{\overline{V}}]^+ \\
\pi\uparrow & & \downarrow \\
(\text{inv}(\delta_1, \dot{\delta_1}')^{-1}, \text{inv}(\delta_2, \dot{\delta_2}')) \in H^1(\mathcal{E}_a, Z \times Z \to S_1 \times S_2) & & (\dot{s}_{\gamma_1, \delta_1}, \dot{s}_{\gamma_2, \delta_2}) \in [\widehat{\overline{S_1}}]^+ \times [\widehat{\overline{S_2}}]^+,
\end{array}$$

where the top pair of elements are the inputs of the pairing in the right-hand side of our main desired equality, the bottom pair of elements are the inputs of the pairing in the left-hand side of that equality, and by functoriality their pairings both equal the pairing of the two elements in the middle line.

The argument for the fact that we can find such an element of $[\widehat{\overline{V}}]^+$ is identical to the corresponding argument in [Kal16], proof of Proposition 5.6. □

6.3 The local Langlands conjectures

We now use our constructions to discuss the Langlands correspondence for an arbitrary connected reductive group defined over a local function field F. This section is a summary of §5.4 in [Kal16].

Let G^* be a connected, reductive, and quasi-split group over F with finite central F-subgroup Z which is an inner form of our fixed arbitrary connected reductive group G. Fix a *Whittaker datum* $\mathfrak{w}$ for G^*, which recall is a $G^*(F)$-conjugacy class of pairs (B, ζ_B) consisting of an F-Borel subgroup $B \subset G^*$ and a non-degenerate character $\zeta_B \colon B_u(F) \to \mathbb{C}^*$, where the subscript u denotes the unipotent radical. We may view the group Z as a finite central F-subgroup of G, also denoted by Z, with $\overline{G} := G/Z$ as before.

Definition 6.3.1 *Given a quasi-split connected reductive group G^* over F, we write $\Pi^{rig}(G^*)$ for the set of isomorphism classes of irreducible admissible representations of rigid inner twists of G^* (see Definition 6.1.9). Define the subsets $\Pi^{rig}_{unit}(G^*), \Pi^{rig}_{temp}(G^*), \Pi^{rig}_2(G^*)$ to be those representations which are unitary, tempered, and essentially square-integrable.*

Let $\varphi \colon W'_F \to {}^LG$ be a *tempered Langlands parameter*, which means that it's a homomorphism of W_F-extensions that is continuous on W_F, restricts to a morphism of algebraic groups on $SL_2(\mathbb{C})$, and sends W_F to a set of semisimple elements of ${}^LG$ that project onto a bounded subset of $\widehat{G}(\mathbb{C})$. Setting $S_\varphi = Z_{\widehat{G}}(\varphi)$, and S_φ^+ its preimage in $\widehat{\overline{G}}$, we have an inclusion $Z(\widehat{\overline{G}})^+ \subset S_\varphi^+$ which induces a map $\pi_0(Z(\widehat{\overline{G}})^+) \to \pi_0(S_\varphi^+)$ with central image. Denote by Irr$(\pi_0(S_\varphi^+))$ the set of irreducible representations of the finite group $\pi_0(S_\varphi^+)$.

Conjecture 6.3.2 *There is a finite subset $\Pi_\varphi \subset \Pi^{rig}_{temp}(G^*)$ and a commutative diagram*

$$\begin{array}{ccc} \Pi_\varphi & \xrightarrow{\iota_{\mathfrak{w}}} & Irr(\pi_0(S_\varphi^+)) \\ \downarrow & & \downarrow \\ H^1(\mathcal{E}, Z \to G^*) & \longrightarrow & \pi_0(Z(\widehat{\overline{G}})^+)^* \end{array}$$

in which the top map is a bijection, the bottom map is given by the pairing of Corollary 6.2.2, the right map assigns to each irreducible representation the restriction of its central character to $\pi_0(Z(\widehat{\overline{G}}))^+$, and the left map sends a representation $(G_1, \xi_1, \mathscr{T}_1, \bar{h}, \pi)$ to the class of $\mathscr{T}_1$. We also *expect that there is a unique element $(G, id_G, G_{\mathcal{E}}, id_{\overline{F}}, \pi)$ of Π_φ such that π is $\mathfrak{w}$-generic and the map $\iota_{\mathfrak{w}}$ identifies this element with the trivial irreducible representation, see [Sh90], §9.*

For $\dot{\pi} := (G_1, \xi_1, \mathscr{T}_1, \bar{h}_1, \pi_1) \in \Pi_\varphi$, denote by $\langle -, \dot{\pi}\rangle$ the conjugation-invariant function on $\pi_0(S_\varphi^+)$ given by the trace of the irreducible representation $\iota_{\mathfrak{w}}(\dot{\pi})$. We let $\Theta_{\dot{\pi}}$ denote the distribution on $G_2(F)$ for any isomorphic rigid inner twist $(G_2, \xi_2, \mathscr{T}_2, \bar{h}_2)$ given by transporting the Harish-Chandra character Θ_{π_1} associated to π_1 to $G_2(F)$ via any choice of isomorphism of rigid inner twists—note that by Corollary 6.1.4 this distribution does not depend on the choice of isomorphism, justifying the notation. For a fixed rigid inner twist $(\xi, \mathscr{T}, \bar{h})\colon G^* \to G$ enriching our inner twist $\psi^{-1}\colon G^*_{F^s} \xrightarrow{\sim} G_{F^s}$, we define the virtual character

$$S\Theta_{\varphi,\xi,(\mathscr{T},\bar{h})} = e(G) \sum_{\dot{\pi}\in\Pi_\varphi, \dot{\pi}\mapsto[\mathscr{T}]} \langle 1, \dot{\pi}\rangle \Theta_{\dot{\pi}} \tag{6.7}$$

and for semisimple $\dot{s} \in S_\varphi^+(\mathbb{C})$ we set

$$\Theta^{\dot{s}}_{\varphi,\mathfrak{w},\xi,(\mathscr{T},\bar{h})} = e(G) \sum_{\dot{\pi}\in\Pi_\varphi, \dot{\pi}\mapsto[\mathscr{T}]} \langle \dot{s}, \dot{\pi}\rangle \Theta_{\dot{\pi}}. \tag{6.8}$$

Here $e(G)$ denotes the sign defined in [Kot83]; we expect $S\Theta_{\varphi,\xi,(\mathscr{T},\bar{h})}$ to be a stable distribution on $G(F)$, as defined in [Lan83], I.4.

The element $\dot{s}$ also defines a refined endoscopic datum $\dot{\mathfrak{e}}$ as follows: Let $s \in S_\varphi(\mathbb{C})$ be the image of $\dot{s}$, set $\widehat{H} = Z_{\widehat{G}}(s)^\circ$, set $\mathcal{H} = \widehat{H}(\mathbb{C}) \cdot \varphi(W_F)$, and take $\eta\colon \mathcal{H} \to {}^LG$ to be the natural inclusion, and define $\dot{\mathfrak{e}} = (H, \mathcal{H}, \eta, \dot{s})$. Take also a z-pair $(H_{\mathfrak{z}}, \eta_{\mathfrak{z}})$ corresponding to the endoscopic datum $\mathfrak{e}$ associated to the refined datum $\dot{\mathfrak{e}}$, which induces a tempered Langlands parameter $\varphi_{\mathfrak{z}} := \eta_{\mathfrak{z}} \circ \varphi$.

According to §5.5 in [KS12], we may define a *Whittaker normalization* of the absolute transfer factor for quasi-split groups, denoted by $\Delta'[\mathfrak{e}, \mathfrak{z}, \mathfrak{w}]\colon H_{\mathfrak{z},G-\mathrm{sr}}(F) \times G^*_{\mathrm{sr}}(F) \to \mathbb{C}$ associated to our Whittaker datum $\mathfrak{w}$. We briefly describe this normalization: using the notation of Chapter 5, we set

$$\Delta'[\mathfrak{e}, \mathfrak{z}, \mathfrak{w}] := \epsilon_L(V, \psi_F)(\Delta_I\Delta_1)^{-1}\Delta_{II}\Delta_{IV},$$

where $\epsilon_L(V, \psi_F)$ is a 4th root of unity associated to a specific virtual representation V of Γ (and thus of W_F) coming from $\mathfrak{e}$ and $\mathfrak{w}$, together with a choice of additive character $\psi_F\colon F \to \mathbb{C}^*$; for details, see [KS99], §5.3. The important takeaway is that the construction of the normalization factor $\epsilon_L(V, \psi_F)$ can be done uniformly for all non-archimedean local fields. One deduces from the arguments in [KS99] §5.3 that this still defines an absolute transfer factor for related strongly regular elements of $H_{\mathfrak{z}}$ and G^* which depends only on $\mathfrak{w}$.

As a consequence, we may combine this normalization with our new absolute transfer factor (6.2) to obtain a normalized absolute transfer factor for general connected reductive groups over

F; we use the same notation as in our transfer factor formula (6.2). We then set

$$\Delta'[\dot{\mathfrak{e}}, \mathfrak{z}, \mathfrak{w}, \psi, (\mathscr{T}, \bar{h})](\gamma_{\mathfrak{z}}, \delta') = \Delta'[\mathfrak{e}, \mathfrak{z}, \mathfrak{w}](\gamma_{\mathfrak{z}}, \delta)\langle \mathrm{inv}(\delta, \dot{\delta}'), \dot{s}_{\gamma,\delta}\rangle. \tag{6.9}$$

Note that we have switched the sign of $\langle \mathrm{inv}(\delta, \dot{\delta}'), \dot{s}_{\gamma,\delta}\rangle$ so that our formula agrees with the sign changes in the factors defining $\Delta'[\mathfrak{e}, \mathfrak{z}, \mathfrak{w}]$.

Then if $f^{\dot{\mathfrak{e}}}$ and f are smooth compactly supported functions on $H_{\mathfrak{z}}(F)$ and $G(F)$ respectively, whose orbital integrals are $\Delta'[\dot{\mathfrak{e}}, \mathfrak{z}, \mathfrak{w}, \psi, (\mathscr{T}, \bar{h})]$-matching (as in [KS99], 5.5), we then expect to have the equality

$$S\Theta_{\varphi_{\mathfrak{z}},\mathrm{id},(G_{\mathcal{E}},\mathrm{id})}(f^{\dot{\mathfrak{e}}}) = \Theta^{\dot{s}}_{\varphi,\mathfrak{w},\xi,(\mathscr{T},\bar{h})}(f).$$

CHAPTER 7

The Global Canonical Class

This chapter concerns the construction of the pro-algebraic group $P_{\dot{V}}$, which will be a global analogue of the local group u, as well as an analogue of the local canonical class. For a fixed a finite Galois extension E/F of a global function field F and $S \subset V$ a finite set of places of F, we have two common conditions that we want S to satisfy:

Conditions 7.0.1 *1. S contains all of the places that ramify in E*

2. Every ideal class of E contains an ideal with support in S_E, ie., the group Cl$(O_{E,S})$ is trivial.

As in the previous section, we use H^i as a short-hand for H^i_{fppf}.

7.1 Tate duality for finite multiplicative Z

The goal of this subsection is to construct an analogue of the global Tate duality isomorphism from [Tat66] for the cohomology group $H^2_{\text{fppf}}(F, Z) = \check{H}^2(\overline{F}/F, Z)$, where Z is a finite multiplicative group over F. Temporarily fix a finite set of places $S \subset V$ a multiplicative group M over $O_{F,S}$ split over E; denote $X^*(M)$ by X, and $X_*(M)(= X_*(M^\circ))$ by Y.

For $v \in S$ a fixed place, we denote by $\text{Res}_{E,v}(M)$ the multiplicative $O_{F,S}$-group split over the finite étale extension $O_{E,S}$ determined by the $\Gamma_{E/F}$-module $X \otimes_{\mathbb{Z}} \mathbb{Z}[\{v\}_E] =: X[\{v\}_E]$ (via the correspondence between finitely-generated $\text{Aut}_{O_{F,S}}(O_{E,S}) = \Gamma_{E/F}$-modules over $\mathbb{Z}$ and multiplicative $O_{F,S}$-groups split over $O_{E,S}$ given by [Gil21, §17]). We set $\text{Res}_{E,S}(M) := \prod_{v \in S} \text{Res}_{E,v}(M)$, another multiplicative $O_{F,S}$-group split over $O_{E,S}$, with character group $X[S_E]$. Note that we have an embedding $M \hookrightarrow \text{Res}_{E/S}(M)$ via the augmentation map on characters $X[S_E] \to X$; denote the character module of the cokernel of this embedding by $X[S_E]_0$ (the kernel of the augmentation map).

Global Tate duality for tori (as in [Tat66]) shows that there exists a class

$$\alpha_3(E, S) \in H^2(\Gamma_{E/F}, \frac{\text{Res}_{E,S}(\mathbb{G}_m)}{\mathbb{G}_m}(O_{E,S}))$$

such that cup product with this class induces for all $i \in \mathbb{Z}$ an isomorphism

$$\widehat{H}^{i-2}(\Gamma_{E/F}, Y[S_E]_0) \xrightarrow{\sim} \widehat{H}^{i}(\Gamma_{E/F}, T(O_{E,S})),$$

where to make sense of the relevant cup product pairing, we are making the identifications

$$Y \otimes_{\mathbb{Z}} \mathbb{Z}[S_E]_0 = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}[S_E]_0) = \mathrm{Hom}_{O_{E,S}\text{-gp}}(\frac{\mathrm{Res}_{E,S}(\mathbb{G}_m)}{\mathbb{G}_m}, T).$$

We no longer fix T and S as above. Let Z be a finite multiplicative group defined over F, set $A = X^*(Z)$ and $A^\vee = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$. As mentioned above, our temporary goal is to construct a functorial isomorphism

$$\Theta\colon \varinjlim_{E', S'} \widehat{H}^{-1}(\Gamma_{E'/F}, A^\vee[S'_{E'}]_0) \xrightarrow{\sim} H^2(F, Z),$$

where the limit is over all finite subsets $S' \subset V$ and finite Galois extensions E'/F. Choose a finite Galois extension E/F splitting Z and a finite full subset $S \subset V$ such that S satisfies Conditions 7.0.1 with respect to E and the following additional condition:

Conditions 7.1.1 *For each $w \in V_E$, there exists $w' \in S_E$ such that Stab$(w, \Gamma_{E/F})$ = Stab$(w', \Gamma_{E/F})$.*

It is straightforward to check that such a pair (E, S) always exists, and that if $S \subseteq S'$ is finite and full, then it also satisfies Conditions 7.0.1 and 7.1.1 (with respect to E).

Note that for n a multiple of exp(Z), we have a functorial isomorphism

$$\Phi_{E,S,n}\colon A^\vee[S_E]_0 \xrightarrow{\sim} \mathrm{Hom}_{O_{E,S}}(\frac{\mathrm{Res}_{E,S}(\mu_n)}{\mu_n}, Z) = \mathrm{Hom}_{O_S}(\frac{\mathrm{Res}_{E,S}(\mu_n)}{\mu_n}, Z), \tag{7.1}$$

which sends the map $g \in A^\vee[S_E]_0$ to the homomorphism induced by the map $A \to (\mathbb{Z}/n\mathbb{Z})[S_E]_0$ defined by

$$a \mapsto \sum_{w \in S_E} ng(w)(a) \cdot [w], \tag{7.2}$$

where $g(w)$ denotes the $A^\vee$-coefficient of $[w]$ in g.

Fix a cofinal sequence $\{n_i\}$ in $\mathbb{N}^\times$ and denote the associated cofinal prime-to-p sequence by $n'_i := n_i/p^{m_i}$. Identifying $\mathrm{Res}_{E/S}(\mathbb{G}_m)(O_S)$ with $\mathrm{Maps}(S_E, O_S^\times)$ in the obvious way, we may pick functions

$$k'_i\colon \mathrm{Maps}(S_E, O_S^\times) \to \mathrm{Maps}(S_E, O_S^\times)$$

such that $k'_i(x)^{n'_i} = x$ and $k'_{i+1}(x)^{n'_{i+1}/n'_i} = k'_i(x)$. Under the bijection between Čech cochains in

$\mathrm{Res}_{E/S}(\mathbb{G}_m)(O_S^{\bigotimes_{O_{F,S}} n})$ and $C^{n-1}(\Gamma_S, \mathrm{Res}_{E/S}(\mathbb{G}_m)(O_S))$ (via Lemma 3.2.1) this also defines an analogous map

$$k_i' \colon \mathrm{Res}_{E/S}(\mathbb{G}_m)(O_S^{\bigotimes_{O_{F,S}} n}) \to \mathrm{Res}_{E/S}(\mathbb{G}_m)(O_S^{\bigotimes_{O_{F,S}} n})$$

for all n. In the above we are using the fact that $O_S^\times$ is n-divisible for n coprime to p (see [NSW08, Proposition 8.3.4]).

As in the local case, we want to extend this to p-power roots. First note that the map

$$\mathrm{Res}_{E,S}(\mathbb{G}_m)(O_{E,S}^{\bigotimes_{O_{F,S}} n}) \to \frac{\mathrm{Res}_{E,S}(\mathbb{G}_m)}{\mathbb{G}_m}(O_{E,S}^{\bigotimes_{O_{F,S}} n})$$

is surjective, since $H^1(O_{E,S}^{\bigotimes_{O_{F,S}} n}, \mathbb{G}_m) = 0$, by combining the proof of Lemma 3.2.1 with the fact that $H^1(O_{E,S}, \mathbb{G}_m) = 0$, since $O_{E,S}$ is a principal ideal domain. It follows that we may lift a cocycle representing $\alpha_3(E,S) \in \check{H}^2(O_{E,S}/O_{F,S}, \frac{\mathrm{Res}_{E,S}(\mathbb{G}_m)}{\mathbb{G}_m})$ to an element $c_{E,S} \in \mathrm{Res}_{E/S}(\mathbb{G}_m)(O_{E,S}^{\bigotimes_{O_{F,S}} 3})$. We may then take

$$k_i'(c_{E,S}) \in C^{2,2}(O_S/O_{F,S}, O_{E,S}, \mathrm{Res}_{E/S}(\mathbb{G}_m)) := \mathrm{Res}_{E/S}(\mathbb{G}_m)(O_S \otimes_{O_{F,S}} O_{E,S} \otimes_{O_{F,S}} O_{E,S}),$$

and the right-hand side may be interpreted explicitly as

$$\prod_{w \in S_E} (O_S \otimes_{O_{F,S}} O_{E,S} \otimes_{O_{F,S}} O_{E,S})_w^*.$$

As in the local case, it is straightforward to check that for every $x \in O_S \otimes_{O_{F,S}} O_{E,S} \otimes_{O_{F,S}} O_{E,S}$ and power p^{m_i}, we may find a p^{m_i}th root $x^{(1/p^m)} \in O_S^{\mathrm{perf}} \otimes_{O_{F,S}} O_{E,S} \otimes_{O_{F,S}} O_{E,S}$ such that the resulting system of roots satisfies $(x^{(1/p^{m_{i+1}})})^{p^{m_{i+1}}/p^{m_i}} = x^{(1/p^{m_i})}$. Applying this across all $w \in S_E$, we may define an analogous map

$$(-)^{(1/p^{m_i})} \colon \mathrm{Res}_{E/S}(\mathbb{G}_m)(O_S \otimes_{O_{F,S}} O_{E,S} \otimes_{O_{F,S}} O_{E,S}) \to \mathrm{Res}_{E/S}(\mathbb{G}_m)(O_S^{\mathrm{perf}} \otimes_{O_{F,S}} O_{E,S} \otimes_{O_{F,S}} O_{E,S}).$$

We then set $\alpha_{p,i}(E,S)$ to be the image of $(k_i'(c_{E,S}))^{(1/p^{m_i})}$ in $[\mathrm{Res}_{E,S}(\mathbb{G}_m)/\mathbb{G}_m]((O_S^{\mathrm{perf}})^{\bigotimes_{O_{F,S}} 3})$. We then obtain

$$d\alpha_{p,i}(E,S) \in Z^{3,2}(O_S^{\mathrm{perf}}/O_{F,S}, O_{E,S}, \frac{\mathrm{Res}_{E,S}(\mu_{n_i})}{\mu_{n_i}}),$$

and define the map

$$\Theta_{E,S} \colon \widehat{H}^{-1}(\Gamma_{E/F}, A^\vee[S_E]_0) \to \check{H}^2(O_S^{\mathrm{perf}}/O_{F,S}, Z),$$

$$g \mapsto d\alpha_i(E,S) \underset{O_{E,S}/O_{F,S}}{\sqcup} g,$$

the pairing

$$\underline{A^\vee[S_E]_0} \times [\frac{\mathrm{Res}_{E,S}(\mu_{n_i})}{\mu_{n_i}}]_{O_S^{\mathrm{perf}}} \to Z_{O_S^{\mathrm{perf}}}$$

is given by (7.1) and we choose n_i so that it is divisible by $\exp(Z)$. One checks that this map does not depend on the choice of n_i.

As in [Kal18], we have the following important lemma which connects the above map to the global Tate duality pairing for tori discussed above (whose corresponding isomorphisms for various tori and Tate cohomology groups will all be denoted by "TN", for *Tate-Nakayama*):

Lemma 7.1.2 *Let T be a torus defined over F and split over E, and let $Z \to T$ be an injection with cokernel $\bar{T}$, all viewed as $O_{F,S}$ groups in the usual way. We write $Y = X_*(T)$ and $\bar{Y} = X_*(\bar{T})$. Then the following diagram commutes, and its columns are exact.*

$$\begin{array}{ccccc}
\widehat{H}^{-1}(\Gamma_{E/F}, Y[S_E]_0) & \xrightarrow{TN} & \check{H}^1(O_{E,S}/O_{F,S}, T) & \xrightarrow{\sim} & \check{H}^1(O_S^{perf}/O_{F,S}, T) \\
\downarrow & & \downarrow & & \downarrow \\
\widehat{H}^{-1}(\Gamma_{E/F}, \bar{Y}[S_E]_0) & \xrightarrow{TN} & \check{H}^1(O_{E,S}/O_{F,S}, \bar{T}) & \xrightarrow{\sim} & \check{H}^1(O_S^{perf}/O_{F,S}, \bar{T}) \\
\downarrow & & & & \downarrow \\
\widehat{H}^{-1}(\Gamma_{E/F}, A^\vee[S_E]_0) & & \xrightarrow{\Theta_{E,S}} & & \check{H}^2(O_S^{perf}/O_{F,S}, Z) \\
\downarrow & & & & \downarrow \\
\widehat{H}^{0}(\Gamma_{E/F}, Y[S_E]_0) & \xrightarrow{-TN} & \check{H}^2(O_{E,S}/O_{F,S}, T) & \longrightarrow & \check{H}^2(O_S^{perf}/O_{F,S}, T) \\
\downarrow & & \downarrow & & \downarrow \\
\widehat{H}^{0}(\Gamma_{E/F}, \bar{Y}[S_E]_0) & \xrightarrow{-TN} & \check{H}^2(O_{E,S}/O_{F,S}, \bar{T}) & \longrightarrow & \check{H}^2(O_S^{perf}/O_{F,S}, \bar{T})
\end{array}$$

Proof. The right-hand isomorphisms on the first two lines follow from the fact that all T-torsors over $O_{F,S}$ are trivial over $O_{E,S}$. The right-hand column is exact because, applying the isomorphisms $\check{H}^i(O_S^{\mathrm{perf}}/O_{F,S}, M) \xrightarrow{\sim} H^i(O_{F,S}, M)$ for $i = 1, 2$ and $M = T, \bar{T}, Z$, the resulting two-column diagram commutes, by functoriality of the Čech-to-derived comparison maps and in [Ros19, Proposition E.2.1]. From here, the identical argument as in [Kal18] gives the result, using the fundamental properties of the unbalanced cup product on fppf cohomology discussed in §3.4. □

Corollary 7.1.3 *The map $\Theta_{E,S}$ is a functorial injection which is independent of the choices of $c_{E,S}$, k_i, and $(-)^{(1/p^{m_i})}$.*

Proof. As in the proof of [Kal18, Proposition 3.2.4], we may choose $\bar{Y}$ to be a free $\mathbb{Z}[\Gamma_{E/F}]$-module, which implies that the connecting homomorphism of the left-hand column is injective, and $\Theta_{E,S}$ is the restriction of "$-$TN", which is an isomorphism that does not depend on the choices of $c_{E,S}$, k_i, or $(-)^{(1/p^{m_i})}$. □

Recall the local analogue of $\Theta_{E,S}$ which, if $\dot{v} \in S_{F_S}$ with restriction to F (and to E, by abuse of notation) denoted by v and $c_v \in \mathbb{G}_m(E_v^{\bigotimes_{F_v} 3})$ represents the canonical class of $H^2(\Gamma_{E_v/F_v}, E_v^*)$, is defined by

$$\Theta_{E_v,n_i} \colon \widehat{H}^{-1}(\Gamma_{E_v/F_v}, A^\vee) \to \check{H}^2(\overline{F_v}/F_v, Z_{F_v}),$$

$$g \mapsto d\alpha_v \underset{E_v/F_v}{\sqcup} \Phi_{n_i}(g),$$

where $\alpha_v \in \overline{F_v} \otimes_{F_v} \overline{F_v} \otimes_{F_v} E_v$ is an n_ith-root of c_v, chosen in an analogous way to $c_{E,S}$ above. This is also a functorial injection, independent of the choices of i, c_v, and α_v.

To compare this local construction to the above global analogue, first note that we have a homomorphism of Γ_{E_v/F_v}-modules $A^\vee[S_E]_0 \to A^\vee$ given by mapping onto the v-factor, as well as an $O_{F,S}$-algebra homomorphism $(O_S^{\text{perf}})^{\bigotimes_{O_{F,S}} 3} \to \overline{F_v}^{\bigotimes_{F_v} 3}$ determined by $\dot{v}$, giving a group homomorphism $Z((O_S^{\text{perf}})^{\bigotimes_{O_{F,S}} 3}) \to Z(\overline{F_v}^{\bigotimes_{F_v} 3})$. Then [Kal18, Lemma 3.2.6] shows that the resulting square

$$\begin{array}{ccc} \widehat{H}^{-1}(\Gamma_{E/F}, A^\vee[S_E]_0) & \xrightarrow{-\text{TN}} & \check{H}^2(O_S^{\text{perf}}/O_{F,S}, Z) \\ \downarrow & & \downarrow \\ \widehat{H}^{-1}(\Gamma_{E_v/F_v}, A^\vee) & \xrightarrow{-\text{TN}} & \check{H}^2(\overline{F_v}/F_v, Z_{F_v}) \end{array}$$

commutes, where to obtain the right-hand vertical map we are using the fact that the homomorphism $Z((O_S^{\text{perf}})^{\bigotimes_{O_{F,S}} 3}) \to Z(\overline{F_v}^{\bigotimes_{F_v} 3})$ preserves Čech cocycles and cochains, which is straightforward to check.

Following [Kal18, §3.2], we now collect some basic functoriality properties of the map $\Theta_{E,S}$. The proofs are identical to the proofs loc. cit, so we state the results and refer to [Kal18].

Lemma 7.1.4 *The natural map $\check{H}^2(O_S^{perf}/O_S, Z) \to \check{H}^2(\overline{F}/F, Z)$ is injective.*

Proof. The proof of [Kal18, Lemma 3.2.7] works verbatim here, replacing $H^i(\Gamma_S, M(O_S))$ with $\check{H}^i(O_S^{\text{perf}}/O_{F,S}, M)$ for $M = T, \bar{T}, Z$ and $i = 1, 2$. □

Let K/F be a finite Galois extension containing E, and let S' be a finite set of places of F satisfying Conditions 7.0.1 and 7.1.1 with respect to K/F. Then [Kal18] defines two maps on Tate cohomology, the first from $\widehat{H}^{-1}(\Gamma_{E/F}, A^\vee[S_E]_0)$ to $\widehat{H}^{-1}(\Gamma_{E/F}, A^\vee[S'_E]_0)$ induced by the inclusion $S \subseteq S'$, and the second map, denoted by !, from $\widehat{H}^{-1}(\Gamma_{E/F}, A^\vee[S'_E]_0)$ to $\widehat{H}^{-1}(\Gamma_{K/F}, A^\vee[S'_K]_0)$ given by choosing a section $s \colon S'_E \to S'_K$ of the natural projection $S'_K \to S'_E$ and then defining $s_! \colon \mathbb{Z}[S'_E]_0 \to \mathbb{Z}[S'_K]_0$ by sending $[w]$ to $[s(w)]$; it is shown in [Kal18, Lemma 3.1.7] that passing to -1-degree Tate cohomology (and tensoring with $A^\vee$) gives the claimed well-defined homomorphism. Now we have the analogue of [Kal18, Lemma 3.2.8]:

Lemma 7.1.5 *Let K and S' be as above. Then both of the above maps are injective, and fit into the commutative diagrams:*

$$\begin{array}{ccc} \widehat{H}^{-1}(\Gamma_{E/F}, A^\vee[S_E]_0) & \xrightarrow{\Theta_{E,S}} & \check{H}^2(O_S^{perf}/O_{F,S}, Z) \\ \downarrow & & \downarrow{\scriptstyle Inf} \\ \widehat{H}^{-1}(\Gamma_{E/F}, A^\vee[S'_E]_0) & \xrightarrow{\Theta_{E,S'}} & \check{H}^2(O_{S'}^{perf}/O_{F,S'}, Z) \end{array} \qquad \begin{array}{ccc} \widehat{H}^{-1}(\Gamma_{E/F}, A^\vee[S'_E]_0) & \xrightarrow{\Theta_{E,S'}} & \check{H}^2(O_{S'}^{perf}/O_{F,S'}, Z) \\ \downarrow{\scriptstyle !} & & \| \\ \widehat{H}^{-1}(\Gamma_{K/F}, A^\vee[S'_K]_0) & \xrightarrow{\Theta_{K,S'}} & \check{H}^2(O_{S'}^{perf}/O_{F,S'}, Z). \end{array}$$

Proof. The proof of [Kal18, Lemma 3.2.8] works verbatim here, replacing the diagram of Lemma 3.2.5 loc. cit. with the diagram from our Lemma 7.1.2. □

We then get the main result of this subsection, which characterizes the cohomology group $H^2(F, Z)$:

Proposition 7.1.6 *The maps $\Theta_{E,S}$ splice to a functorial isomorphism*

$$\Theta\colon \varinjlim_{E} \widehat{H}^{-1}(\Gamma_{E/F}, A^\vee[S_E^{(E)}]_0) \to H^2(F, Z),$$

where the limit is over all finite Galois extensions E/F splitting Z and $S^{(E)}$ denotes an arbitrary choice of places of V satisfying Conditions 7.0.1 and 7.1.1 for E/F such that if $K/E/F$, we have $S^{(E)} \subset S^{(K)}$ (by Lemma 7.1.5, the above map does not depend on the choices of the $S^{(E)}$'s).

Proof. This proof closely follows the proof of [Kal18, Corollary 3.2.9]. It is enough to prove the result with $H^2(F, Z)$ replaced by $\check{H}^2(\overline{F}/F, Z)$. By Corollary 7.1.3 and Lemmas 7.1.4 and 7.1.5, we obtain a functorial injective homomorphism Θ as claimed, which is independent of the choices of (appropriately chosen) $S^{(E)}$, so all that remains to prove is surjectivity.

For any $h \in \check{H}^2(\overline{F}/F, Z)$, we may find E'/F finite such that $h \in \check{H}^2(E'/F, Z)$; denote the Galois closure of F in E' by E, so that $E' = E \cdot F_m$ for some unique $m \in \mathbb{N}$. Moreover, since $E'^{\bigotimes_F 3} = \varinjlim_{S^{(E)}} O_{E',S^{(E)}}^{\bigotimes_{O_{F,S^{(E)}}} 3}$, where the direct limit is over all finite $S^{(E)} \subset V$ satisfying the required conditions with respect to E/F, there is some finite $S^{(E)}$ satisfying the required conditions with respect to E/F such that we can find $h_{E',S^{(E)}} \in Z(O_{E',S^{(E)}}^{\bigotimes_{O_{F,S^{(E)}}} 3})$ with image in $Z(E'^{\bigotimes_F 3}) \to \check{H}^2(E'/F, Z)$ equal to h. We may enlarge $S^{(E)}$ even further to assume that $h_{E',S^{(E)}} \in Z^2(O_{E',S^{(E)}}/O_{F,S^{(E)}}, Z)$, since the Čech differential on $Z(O_{E',S^{(E)}}^{\bigotimes_{O_{F,S^{(E)}}} 3})$ is the same as that of $Z(E'^{\bigotimes_F 3})$, and we may use finitely many elements of F and E' to encode the fact that $dh_{E',S^{(E)}} = 1$ in $Z(E'^{\bigotimes_F 4})$. Denote by $\bar{h}_{E',S^{(E)}}$ the image of $h_{E',S^{(E)}}$ in $\check{H}^2(O_{E',S^{(E)}}/O_{F,S^{(E)}}, Z)$.

Once we have such an $\bar{h}_{E',S^{(E)}}$, choose an $O_{F,S}$-torus $Z \hookrightarrow T$ with $\bar{T} := T/Z$ such that $\bar{Y} = X_*(\bar{T})$ is free over $\Gamma_{E/F}$, and denote the image of $\bar{h}_{E',S^{(E)}}$ in $\check{H}^2(O_{E',S^{(E)}}/O_{F,S^{(E)}}, T)$ by $\bar{h}_{E',S^{(E)},T}$. Note that we have a commutative diagram of isomorphisms from Lemma 3.2.12:

$$\begin{array}{ccc} \check{H}^2(O_{E,S^{(E)}}/O_{F,S^{(E)}}, T) & \xrightarrow{\sim} & \check{H}^2(O_{E',S^{(E)}}/O_{F,S^{(E)}}, T) \\ \downarrow & & \downarrow \\ \check{H}^2(O_{E,S^{(E)}}/O_{F,S^{(E)}}, \bar{T}) & \xrightarrow{\sim} & \check{H}^2(O_{E',S^{(E)}}/O_{F,S^{(E)}}, \bar{T}), \end{array}$$

and so we may pick a (unique) preimage, denoted by $\bar{h}_{E,S^{(E)},T}$, of $\bar{h}_{E',S^{(E)},T}$ in $\check{H}^2(O_{E,S}/O_{F,S}, T)$, and by the commutativity of the diagram, the image of $\bar{h}_{E,S^{(E)},T}$ in $\check{H}^2(O_{E,S^{(E)}}/O_{F,S^{(E)}}, \bar{T})$ is zero. We may thus lift -TN$^{-1}(\bar{h}_{E,S^{(E)},T}) \in \widehat{H}^0(\Gamma_{E/F}, Y[S_E^{(E)}]_0)$ to some $g \in \widehat{H}^{-1}(\Gamma_{E/F}, A^\vee[S_E^{(E)}]_0)$, and then the same argument as in [Kal18, Corollary 3.2.9] shows that

$$\Theta_{E,S^{(E)}}(g) \in \check{H}^2(O_{S^{(E)}}^{\text{perf}}/O_{F,S^{(E)}}, Z)$$

has image in $\check{H}^2(\overline{F}/F, Z)$ equal to h, as desired (even though we need to take the image of $\bar{h}_{E',S^{(E)},T}$ in $\check{H}^2(O_{E,S}/O_{F,S}, T)$, the argument of [Kal18] uses that the image of their $\Theta_{E,S^{(E)}}(g)$ in $\check{H}^2(\overline{F}/F, T)$ is the same as that of h, which is still true for our g obtained via the above adjustment for non-separability). □

7.2 The groups $P_{E,\dot{S}_E,n}$

Let E/F be a finite Galois extension, $S \subset V$ a finite full set of places, and $\dot{S}_E \subseteq S_E$ a set of lifts for the places in S. When working with a multiplicative $O_{F,S}$-group M, we will frequently work with $\check{H}^2(O_S^{\text{perf}}/O_{F,S}, M)$ rather than $H^2_{\text{fppf}}(O_{F,S}, Z)$; these two groups are canonically isomorphic by Corollary 3.2.10. We assume that the pair $(S, \dot{S}_E)$ satisfies the following:

Conditions 7.2.1 *1. S contains all places that ramify in E.*

2. Every ideal class of E contains an ideal with support in S_E (i.e., Cl$(O_{E,S}) = 0$).

3. For every $w \in V_E$, there exists $w' \in S_E$ with Stab$(w, \Gamma_{E/F}) =$ Stab$(w', \Gamma_{E/F})$.

4. For every $\sigma \in \Gamma_{E/F}$, there exists $\dot{v} \in \dot{S}_E$ such that $\sigma\dot{v} = \dot{v}$.

Pairs $(S, \dot{S}_E)$ satisfying these conditions always exist, and if $(S', \dot{S}'_E)$ contains $(S, \dot{S}_E)$ (in the obvious sense) and the latter satisfies these conditions, then so does the former. For notational ease, denote the group $\frac{\text{Res}_{E,S}(\mu_n)}{\mu_n}$ introduced in the previous subsection by $\bar{R}_{E,S}[n]$. For a fixed $n \in \mathbb{N}$, we first set $P_{E,S,n}$ to be the multiplicative $O_{F,S}$-group split over $O_{E,S}$ given by character $\Gamma_{E/F}$-module consisting of elements of $\mathbb{Z}/n\mathbb{Z}[\Gamma_{E/F} \times S_E]$ killed by both augmentation maps, denoted by $\mathbb{Z}/n\mathbb{Z}[\Gamma_{E/F} \times S_E]_{0,0}$.

We then define the multiplicative group $P_{E,\dot{S}_E,n}$ to correspond to the $\Gamma_{E/F}$-submodule of elements $x \in \mathbb{Z}/n\mathbb{Z}[\Gamma_{E/F} \times S_E]_{0,0}$ such that $x[(\sigma, w)] = 0$ if $w \notin \sigma(\dot{S}_E)$. As another piece of notation, we set $X^*(P_{E,S,n}) =: M_{E,S,n}$ and $X^*(P_{E,\dot{S}_E,n}) =: M_{E,\dot{S}_E,n}$. We have the following purely character-theoretic lemma from [Kal18]:

Lemma 7.2.2 *Let A be a $\mathbb{Z}[\Gamma_{E/F}]$-module which is finite as an abelian group.*

1. *If exp(A) divides n, then we may define a homomorphism*

$$\Psi_{E,S,n}\colon \textit{Hom}(A, M_{E,S,n})^{\Gamma} \to \widehat{Z}^{-1}(\Gamma_{E/F}, A^{\vee}[S_E]_0), H \mapsto h,$$

   where $h := \sum_{w \in S_E} h_w[w]$, with $h_w\colon A \to \mathbb{Q}/\mathbb{Z}$ defined by $h_w(a) = H(a)[(e, w)]$ (identifying $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$ with $\mathbb{Z}/n\mathbb{Z}$ via multiplying the left-hand side by n and taking the residue modulo n). Furthermore the above map is an isomorphism of finite abelian groups, functorial in A, which restricts to an isomorphism

$$\textit{Hom}(A, M_{E,\dot{S}_E,n})^{\Gamma} \to A^{\vee}[\dot{S}_E]_0 \cap \widehat{Z}^{-1}(\Gamma_{E/F}, A^{\vee}[S_E]_0).$$

2. *For $n \mid m$, the isomorphisms $\Psi_{E,S,n}$ and $\Psi_{E,S,m}$ are compatible with the natural inclusion $M_{E,\dot{S}_E,n} \to M_{E,\dot{S}_E,m}$. Setting $M_{E,S} := \varinjlim_n M_{E,S,n}$, we thus obtain an isomorphism*

$$\Psi_{E,S}\colon \textit{Hom}(A, M_{E,S})^{\Gamma} \to \widehat{Z}^{-1}(\Gamma_{E/F}, A^{\vee}[S_E]_0).$$

3. *The map*

$$A^{\vee}[\dot{S}_E]_0 \cap \widehat{Z}^{-1}(\Gamma_{E/F}, A^{\vee}[S_E]_0) \to \widehat{H}^{-1}(\Gamma_{E/F}, A^{\vee}[S_E]_0)$$

   is surjective.

Proof. See [Kal18, Lemma 3.3.2.]. □

Now for fixed $n \in \mathbb{N}$ and A a $\mathbb{Z}[\Gamma_{E/F}]$-module which is finite as an abelian group with corresponding $O_{F,S}$-group Z such that exp(A) divides n, we obtain a map

$$\Theta^{P}_{E,\dot{S}_E,n}\colon \mathrm{Hom}(P_{E,\dot{S}_E,n}, Z)^{\Gamma} \xrightarrow{\Theta_{E,S} \circ \Psi_{E,S,n}} \check{H}^2(O_S^{\mathrm{perf}}/O_{F,S}, Z), \tag{7.3}$$

note that this map is functorial in the group Z. Now for $A = M_{E,\dot{S}_E,n}$, we have the canonical element id of the left-hand side of (7.3), and we define $\xi_{E,\dot{S}_E,n} \in \check{H}^2(O_S^{\text{perf}}/O_{F,S}, P_{E,\dot{S}_E,n})$ to be its image.

Note that for n dividing m, the natural inclusion of character modules $M_{E,\dot{S}_E,n} \to M_{E,\dot{S}_E,m}$ induces a surjection of $O_{F,S}$-groups $P_{E,\dot{S}_E,m} \to P_{E,\dot{S}_E,n}$. We have the following result about how the elements $\xi_{E,\dot{S}_E,n}$ change as one varies n:

Lemma 7.2.3 *For $n \mid m$, the induced map $\check{H}^2(O_S^{perf}/O_{F,S}, P_{E,\dot{S}_E,m}) \to \check{H}^2(O_S^{perf}/O_{F,S}, P_{E,\dot{S}_E,n})$ maps $\xi_{E,\dot{S}_E,m}$ to $\xi_{E,\dot{S}_E,n}$.*

Proof. After invoking the functoriality of $\Theta_{E,S}$, the argument is purely character-theoretic, and thus the proof of the analogous result (Lemma 3.3.3) in [Kal18] carries over verbatim to this setting. □

We will now see how the groups $P_{E,\dot{S}_E,n}$ behave when we vary the field extension E/F. For $(S', \dot{S}'_K)$ satisfying Conditions 7.2.1 with respect to the finite Galois extension K/F and $m \in \mathbb{N}$, we write

$$(E, \dot{S}_E, n) < (K, \dot{S}'_K, m)$$

when K contains E, $S \subseteq S'$, and $\dot{S}_E \subseteq (\dot{S}'_K)_E$. Note that given E, $(S, \dot{S}_E)$, and K, one can always find such a pair $(S', \dot{S}'_K)$. For $(E, \dot{S}_E, n) < (K, \dot{S}'_K, m)$, we may define a homomorphism of $\Gamma_{K/F}$-modules from $M_{E,\dot{S}_E,n}$ to $M_{K,\dot{S}'_K,m}$ (with inflated action on the left-hand side) given by

$$\sum_{(\sigma,w)\in\Gamma_{E/F}\times S_E} a_{\sigma,w}[(\sigma, w)] \mapsto \sum_{(\gamma,u)} a_{\bar{\gamma},u_E}[(\gamma, u)],$$

where the right-hand sum is over all pairs (γ, u) in $\Gamma_{K/F} \times S'_K$ such that $\gamma^{-1}u \in \dot{S}'_K \cap S_K$, and $\bar{\gamma}$ denotes the image of γ in $\Gamma_{E/F}$. Again, we get the following result from [Kal18] (Lemma 3.3.4):

Lemma 7.2.4 *For any $\Gamma_{E/F}$-module A which is a finite abelian group with exp$(A) \mid n$, the following diagram commutes*

$$\begin{array}{ccc} \mathit{Hom}(A, M_{E,\dot{S}_E,n})^\Gamma & \xrightarrow{\Theta^P_{E,\dot{S}_E,n}} & \check{H}^2(O_S^{perf}/O_{F,S}, Z) \\ \downarrow & & \downarrow \mathit{Inf} \\ \mathit{Hom}(A, M_{K,\dot{S}'_K,n})^\Gamma & \xrightarrow{\Theta^P_{K,\dot{S}'_K,n}} & \check{H}^2(O_{S'}^{perf}/O_{F,S'}, Z_{O_{F,S'}}), \end{array}$$

where the left-hand vertical homomorphism is induced by the map from $M_{E,\dot{S}_E,n}$ to $M_{K,\dot{S}'_K,n}$ discussed in the above paragraph.

According to [Kal18, Lemma 3.3.5], we get the hoped-for coherence between the canonical classes $\xi_{E,\dot{S}_E,n}$ discussed above:

Lemma 7.2.5 *The homomorphism* $\check{H}^2(O_{S'}^{perf}/O_{F,S'}, P_{K,\dot{S}'_K,n}) \to \check{H}^2(O_{S'}^{perf}/O_{F,S'}, (P_{E,\dot{S}_E,n})_{O_{F,S'}})$ *maps* $\xi_{K,\dot{S}'_K,n}$ *to the image of* $\xi_{E,\dot{S}_E,n}$ *under the inflation map*

$$\check{H}^2(O_S^{perf}/O_S, P_{E,\dot{S}_E,n}) \to \check{H}^2(O_{S'}^{perf}/O_{S'}, (P_{E,\dot{S}_{E,n}})_{O_{F,S'}}).$$

Moreover, it is straightforward to check that for $n \mid m$, the following square commutes:

$$\begin{array}{ccc} M_{E,\dot{S}_E,n} & \longrightarrow & M_{E,\dot{S}_E,m} \\ \downarrow & & \downarrow \\ M_{K,\dot{S}'_K,n} & \longrightarrow & M_{K,\dot{S}'_K,m}. \end{array}$$

Fix a system of quadruples $(E_i, S_i, \dot{S}_{E_i}, n_i)_{i\in\mathbb{N}}$ such that $(S_i, \dot{S}_i)$ satisfies Conditions 7.2.1 with respect to the finite Galois extension E_i/F, the E_i form an exhaustive tower of finite Galois extensions of F, the S_i form an exhaustive tower of finite subsets of V, the n_i form a cofinal system in $\mathbb{N}^\times$, we have the containment $\dot{S}_i \subseteq (\dot{S}_{i+1})_{E_i}$ for all i, and $n_i \mid n_{i+1}$ for all i. Such a system evidently exists. Note that $\dot{V} := \varprojlim_i \dot{S}_i$ is a subset of $V_{F^{\text{sep}}}$ of lifts of V, and the group

$$P_{\dot{V}} := \varprojlim_i P_{E_i,\dot{S}_i,n_i}$$

is a profinite algebraic group independent of the choice of system $(n_i)_{i\in\mathbb{N}}$ which carries the natural structure of a $\varinjlim_i O_{F,S_i} = F$-scheme. Note that for any finite F-group Z, we obtain from the maps $\Theta^P_{E_i,\dot{S}_i,n_i}$ (and Lemma 7.2.4) a homomorphism

$$\Theta^P_{\dot{V}} \colon \text{Hom}(P_{\dot{V}}, Z)^\Gamma \to \check{H}^2(\overline{F}/F, Z)(= H^2(F, Z)),$$

which factors through the homomorphisms

$$\text{Hom}(P_{E_i,\dot{S}_i,n_i}, Z)^\Gamma \xrightarrow{\Theta^P_{E_i,\dot{S}_i,n_i}} \check{H}^2(O^{\text{perf}}_{S_i}/O_{F,S_i}, Z) \to \check{H}^2(\overline{F}/F, Z) \qquad (7.4)$$

for all sufficiently large i, and hence is surjective, since we may choose i with $\exp(Z) \mid n_i$ and invoke Lemma 7.2.2 and Proposition 7.1.6 to deduce the surjectivity of the map (7.4) for all sufficiently large $j > i$.

From [Kal18, Lemma 3.3.6], we have the following alternative characterization of $\text{Hom}_F(P_{\dot{V}}, Z)$ for Z a finite multiplicative F-group:

Lemma 7.2.6 *Let Z be a finite multiplicative F-group and $A = X^*(Z)$. Let $A^\vee[\dot{V}]_0$ denote the kernel of the augmentation map $A^\vee[\dot{V}] \to A^\vee$. Then we have a natural isomorphism*

$$\mathit{Hom}_F(P_{\dot{V}}, Z) \xrightarrow{\sim} A^\vee[\dot{V}]_0.$$

We conclude this subsection by discussing some local-global compatibility regarding $P_{\dot{V}}$ and its local analogues u_v. For a fixed place $v \in \dot{V}$, recall the multiplicative F_v-groups

$$u_{E_v/F_v,n} := \frac{\mathrm{Res}_{E_v/F_v}(\mu_n)}{\mu_n}, u_v := \varprojlim_{E_v/F_v,n} u_{E_v/F_v,n},$$

where the former groups are finite and the latter group is profinite, see §4.1. For Z a finite multiplicative F_v-group with $\exp(Z) \mid n$, denote the isomorphism

$$\mathrm{Hom}_{F_v}(u_{E_v/F_v,n}, Z) \to \widehat{Z}^{-1}(\Gamma_{E_v/F_v}, A^\vee)$$

by $\Psi_{E_v,n}$—these are the local analogues of our maps $\Psi_{E,S,n}$.

We now define a localization map

$$\mathrm{loc}_v^P \colon u_v \to (P_{\dot{V}})_{F_v}$$

for a fixed $v \in \dot{V}$. Fix E/F a finite Galois extension along with a triple $(S, \dot{S}_E, n)$ such that $(S, \dot{S}_E)$ satisfies Conditions 7.2.1 with respect to E/F. Then if $I_{E_v/F_v,n}$ denotes the character group of $u_{E_v/F_v,n}$ (which is just $\mathbb{Z}/n\mathbb{Z}[\Gamma_{E_v/F_v}]_0$), we may define a map

$$\mathrm{loc}_v^{M_{E,\dot{S}_E,n}} \colon M_{E,\dot{S}_E,n} \to I_{E_v/F_v,n},$$

given by

$$H = \sum_{(\sigma,w)\in\Gamma_{E/F}\times S_E} c_{\sigma,w}[(\sigma,w)] \mapsto \sum_{(\sigma,v),\sigma\in\Gamma_{E_v/F_v}} c_{\sigma,v}[\sigma] := H_v.$$

This is a well-defined homomorphism of Γ_{E_v/F_v}-modules, and hence induces a morphism of F_v-group schemes $\mathrm{loc}_v^{P_{E,\dot{S}_E,n}} \colon u_{E_v/F_v,n} \to (P_{E,\dot{S}_E,n})_{F_v}$. It is clear that these morphism glue as we range over all 4-tuples $(E_i, S_i, \dot{S}_i, n_i)$, so that we get an induced homomorphism of profinite F_v-groups $\mathrm{loc}_v^P \colon u_v \to (P_{\dot{V}})_{F_v}$, as desired.

For a finite F-group Z, there is a local analogue of the map $\Theta_{\dot{V}}^P \colon \mathrm{Hom}_F(P_{\dot{V}}, Z) \to \check{H}^2(\overline{F}/F, Z)$ constructed above, which we denote by

$$\Theta_v \colon \mathrm{Hom}_{F_v}(u_v, Z) \to \check{H}^2(\overline{F_v}/F_v, Z)$$

and is defined by $\varprojlim_i (\Theta_{(E_i)_v,n_i} \circ \Psi_{(E_i)_v,n_i})$ (see our §7.1 for the definition of the $\Theta_{E_v,n}$-maps). The following result, once again from [Kal18], shows that these local maps agree with the global map $\Theta^P_{\dot{V}}$ after localization:

Lemma 7.2.7 *For E/F finite Galois splitting Z, $(S, \dot{S}_E)$ satisfying Conditions 7.2.1 with respect to E, $n \in \mathbb{N}$ a multiple of $exp(Z)$, and $\dot{v} \in \dot{V}$ (with $\dot{v}_F, \dot{v}_E =: v$, by abuse of notation), the following diagram commutes*

$$\begin{array}{ccc} Hom_F(P_{E,\dot{S}_E,n}, Z) & \xrightarrow{\Theta^P_{E,\dot{S}_E,n}} & \check{H}^2(\overline{F}/F, Z) \\ \Big\downarrow loc_v^{P_{E,\dot{S}_E,n}} & & \Big\downarrow \\ Hom_{F_v}(u_{E_v/F_v,n}, Z_{F_v}) & \xrightarrow{\Theta^u_{E_v,n}} & \check{H}^2(\overline{F_v}/F_v, Z_{F_v}), \end{array}$$

where the right vertical map is induced by the inclusion $\overline{F} \to \overline{F_v}$ determined by $\dot{v}$.

Recall from Lemma 7.2.3 that elements $\xi_i := \xi_{E_i,\dot{S}_i,n_i}$ form a coherent system in the projective system of groups $\{\check{H}^2(O^{\text{perf}}_{S_i}/O_{F,S_i}, P_{E_i,\dot{S}_i,n_i})\}_i$. We also have (by Lemma 7.1.4), for all i, injective homomorphisms

$$\check{H}^2(O^{\text{perf}}_{S_i}/O_{F,S_i}, P_{E_i,\dot{S}_i,n_i}) \to \check{H}^2(\overline{F}/F, P_{E_i,\dot{S}_i,n_i}),$$

and hence the element $(\xi_i)_i$ may be viewed as an element of $\varprojlim_i \check{H}^2(\overline{F}/F, P_{E_i,\dot{S}_i,n_i})$. Let $\xi_v \in \check{H}^2(\overline{F_v}/F_v, u_v) \xrightarrow{\sim} \widehat{\mathbb{Z}}$ denote the canonical class obtained by taking the preimage of $-1 \in \widehat{\mathbb{Z}}$. We may now deduce the final result of this subsection:

Corollary 7.2.8 *For $\dot{v} \in \dot{V}$, consider the maps*

$$\check{H}^2(\overline{F}/F, P_{\dot{V}}) \to \check{H}^2(\overline{F_v}/F_v, (P_{\dot{V}})_{F_v}) \leftarrow \check{H}^2(\overline{F_v}/F_v, u_v),$$

where the left map is induced by the inclusion $\overline{F} \to \overline{F_v}$ determined by $\dot{v}$ and the right map is loc_v^P. If $\tilde{\xi} \in H^2(\overline{F}/F, P_{\dot{V}})$ is any preimage of (ξ_i) (which exists by Lemma 2.5.4) then the images of $\tilde{\xi}$ and ξ_v in the middle term are equal.

Proof. We claim first that the natural homomorphism

$$\check{H}^2(\overline{F_v}/F_v, (P_{\dot{V}})_{F_v}) \to \varprojlim_i \check{H}^2(\overline{F_v}/F_v, P_{E_i,\dot{S}_i,n_i})$$

is an isomorphism. To simplify notation, set $P_i := (P_{E_i,\dot{S}_i,n_i})_{F_v}$; by Lemma 2.5.5, it suffices to show that $\varprojlim^{(1)}_i \check{H}^1(\overline{F_v}/F_v, P_i) = 0$ and $\varprojlim^{(1)}_i B^1(i) = 0$ (using the notation of the previously cited lemma). By [RZ00], 2.2, this vanishing would follow if we knew that these systems consisted of profinite groups with continuous transition maps. For $B^1(i) := d(P_i(\overline{F_v}))$, this trivially follows because each P_i is finite over F_v, and so this is a system of finite groups. Moreover, the system $\{H^1(F_v, P_i)\}$ has this property as well, since by [Mil06], Theorem 6.10, each $H^1(F_v, P_i)$ is Pontryagin dual to a discrete torsion group (and therefore profinite), and the transition maps are continuous, since they come from dualizing morphisms of discrete torsion groups.

The isomorphism we just proved implies that the map $\check{H}^2(\overline{F}/F, P_{\dot{V}}) \to \check{H}^2(\overline{F_v}/F_v, (P_{\dot{V}})_{F_v})$ factors as the composition

$$\check{H}^2(\overline{F}/F, P_{\dot{V}}) \to \varprojlim_i \check{H}^2(\overline{F}/F, P_{E_i,\dot{S}_i,n_i}) \to \varprojlim_i \check{H}^2(\overline{F_v}/F_v, (P_{E_i,\dot{S}_i,n_i})_{F_v}),$$

where the second map is the inverse limit of the obvious maps for each i. It is thus enough to show that, for each i, the map $\mathrm{loc}_v^{P_{E_i,\dot{S}_i,n_i}}$ sends $\xi_{(E_i)_v,n_i} \in \check{H}^2(\overline{F_v}/F_v, u_{(E_i)_v/F_v,n_i})$ to the image of $\xi_{E_i,\dot{S}_i,n_i}$ under the map

$$\check{H}^2(\overline{F}/F, P_{E_i,\dot{S}_i,n_i}) \to \check{H}^2(\overline{F_v}/F_v, (P_{E_i,\dot{S}_i,n_i})_{F_v}).$$

Once we have reached this step, we get the desired result from the proof of [Kal18, Corollary 3.8], which may be followed verbatim here. □

7.3 The vanishing of $H^1(F, P_{\dot{V}})$ and $H^1(F_v, (P_{\dot{V}})_{F_v})$

In the local case, an instrumental property of the groups u_v was that $H^1(F, u_v) = 0$; our goal in this subsection is to prove the analogue for $P_{\dot{V}}$ and its localizations.

The following alternative characterization of $M_{E,\dot{S}_E,n}$ will be useful: As a $\Gamma_{E/F}$-module, $M_{E,\dot{S}_E,n}$ is canonically isomorphic to the subgroup of elements

$$x = \sum_{(\sigma,v)\in\Gamma_{E/F}\times S} a_{\sigma,v}[(\sigma,v)] \in \mathbb{Z}/n\mathbb{Z}[\Gamma_{E/F}\times S]$$

such that $\sum_{\sigma\in\Gamma_{E/F}^{\dot{v}}} a_{\theta\sigma,v} = 0$ for all $\theta\in\Gamma_{E/F}$, $v \in S$ (where $\dot{v} \in \dot{S}_E$ denotes the unique lift of v to $\dot{S}_E$) and $\sum_{v\in S} a_{\theta,v} = 0$ for all $\theta \in \Gamma_{E/F}$, with $\Gamma_{E/F}$-action given by

$$\gamma \cdot (\sum_{(\sigma,v)\in\Gamma_{E/F}\times S} a_{\sigma,v}[(\sigma,v)]) := \sum_{(\sigma,v)\in\Gamma_{E/F}\times S} a_{\sigma,v}[(\tau\sigma,v)].$$

The proposed identification is given by

$$\sum_{(\sigma,w)\in\Gamma_{E/F}\times S_E} a_{\sigma,w}[(\sigma,w)] \mapsto \sum_{(\sigma,v)\in\Gamma_{E/F}\times S} a_{\sigma,\sigma\dot{v}}[(\sigma,v)].$$

As a consequence, we get the exact sequence of $\Gamma_{E/F}$-modules

$$0 \to M_{E,\dot{S}_E,n} \to \frac{1}{n}\mathbb{Z}/\mathbb{Z}[\Gamma_{E/F}\times S]_0 \to \frac{1}{n}\mathbb{Z}/\mathbb{Z}[S_E]_0 \to 0,$$

where we identify $\mathbb{Z}/n\mathbb{Z}$ with $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$ via $\bar{1} \mapsto \overline{1/n}$ (as above), the middle term denotes the kernel of the augmentation map $\frac{1}{n}\mathbb{Z}/\mathbb{Z}[\Gamma_{E/F}, S] \to \frac{1}{n}\mathbb{Z}/\mathbb{Z}[S]$, and the second map is defined by

$$\sum_{(\sigma,v)\in\Gamma_{E/F}\times S} a_{\sigma,v}[(\sigma,v)] \mapsto \sum_{(\theta,v)\in\Gamma_{E/F}\times S} \Big(\sum_{\tau\in\Gamma^{\dot{v}}_{E/F}} a_{\theta\tau,v}\Big)[\theta\dot{v}];$$

for a proof of exactness, see [Kal18, Lemma 3.4.2].

Set $\mu_n^S := \prod_{v\in S}\mu_n$. At the level of $O_{F,S}$-groups, the above exact sequences identifies $P_{E,\dot{S}_E,n}$ with the quotient

$$\frac{\mathrm{Res}_{E/F}(\mu_n^S)/\mu_n^S}{\mathrm{Res}_{E,S}(\mu_n)/\mu_n},$$

where the embedding $\mathrm{Res}_{E,S}(\mu_n)/\mu_n \hookrightarrow \mathrm{Res}_{E/F}(\mu_n^S)/\mu_n^S$ is induced by the embedding

$$\mathrm{Res}_{E,S}(\mu_n) \to \mathrm{Res}_{E/F}(\mu_n^S)$$

given on the direct factor $\mathrm{Res}_{E,v}(\mu_n) = \mathrm{Res}_{E^{d,\dot{v}}/F}(\mu_n)$ by taking inclusion into the vth factor $\mu_n \to \mu_n^S$, applying $\mathrm{Res}_{E^{d,\dot{v}}/F}(-)$, and then applying the natural map $\mathrm{Res}_{E^{d,\dot{v}}/F}(\mu_n^S) \hookrightarrow \mathrm{Res}_{E/F}(\mu_n^S)$.

Under these identifications, the transition maps $P_{K,\dot{S}'_K,m} \to P_{E,\dot{S}_E,n}$ become (after making the above identification), at the level of character groups, the maps

$$M_{E,\dot{S}_E,n} \to M_{K,\dot{S}'_K,m}, \sum_{(\sigma,v)\in\Gamma_{E/F}\times S} a_{\sigma,v}[(\sigma,v)] \mapsto \sum_{(\gamma,v)\in\Gamma_{K/F}\times S} a_{\bar{\gamma},v}[(\gamma,v)],$$

which is well-defined because $S \subseteq S'$. At the level of F-groups, this is the map

$$\frac{\mathrm{Res}_{K/F}(\mu_m^{S'})/\mu_m^{S'}}{\mathrm{Res}_{K,S'}(\mu_m)/\mu_m} \to \frac{\mathrm{Res}_{E/F}(\mu_n^S)/\mu_n^S}{\mathrm{Res}_{E,S}(\mu_n)/\mu_n}$$

induced by the homomorphism $\mathrm{Res}_{K/F}(\mu_m^{S'}) \to \mathrm{Res}_{E/F}(\mu_n^S)$ defined by composing the norm map $\mathrm{Res}_{K/F}(\mu_m^{S'}) \to \mathrm{Res}_{E/F}(\mu_m^{S'})$ with $\mathrm{Res}_{E/F}(-)$ applied to the composite map $\mu_m^{S'} \to \mu_n^S$ given by the projection map $\mu_m^{S'} \to \mu_m^S$ and (on each component) the m/n-power map $\mu_m \to \mu_n$. Note the similarity to the local transition maps $u_{K_v/F_v,m} \to u_{E_v/F_v,n}$ defined in §4.1. The transition maps fit into a commutative diagram of $\Gamma_{K/F}$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_{E,\dot{S}_E,n} & \longrightarrow & \frac{1}{n}\mathbb{Z}/\mathbb{Z}[\Gamma_{E/F} \times S]_0 & \longrightarrow & \frac{1}{n}\mathbb{Z}/\mathbb{Z}[S_E]_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_{K,\dot{S}'_K,m} & \longrightarrow & \frac{1}{m}\mathbb{Z}/\mathbb{Z}[\Gamma_{K/F} \times S']_0 & \longrightarrow & \frac{1}{m}\mathbb{Z}/\mathbb{Z}[S'_K]_0 & \longrightarrow & 0, \end{array}$$

where the middle map is induced by the map $\mathrm{Res}_{K/F}(\mu_m^{S'}) \to \mathrm{Res}_{E/F}(\mu_n^S)$ as described above, and the right-most map is defined by

$$\sum_{(\sigma,w)\in\Gamma_{E/F}\times S_E} a_{\sigma,w}[(\sigma,w)] \mapsto \sum_{(\gamma,u)\in\Gamma_{K/F}\times S_K} (\#\Gamma^u_{K/E}) a_{\bar{\gamma},u_E}[(\gamma,u)].$$

The following result is a key first step in the argument for the desired cohomological vanishing; it is a simpler version of [Kal18, Lemma 3.4.3]:

Lemma 7.3.1 *Given $(E, \dot{S}_E, n)$, there exists $(K, \dot{S}'_K, m) > (E, \dot{S}_E, n)$ such that for all subgroups $\Delta \subseteq \Gamma_{K/F}$, the transition map*

$$\frac{1}{n}\mathbb{Z}/\mathbb{Z}[S_E]_0 \to \frac{1}{m}\mathbb{Z}/\mathbb{Z}[S'_K]_0$$

is zero.

Proof. In our situation we can strengthen the result by insisting that $m = n$; choose K/F such that $\#\Gamma^u_{K/E}$ is a multiple of n for all places $u \in S_K$ and take S'_K and $\dot{S}'_K \subset S'_K$ satisfying Conditions 7.2.1 with respect to K/F such that $S \subset S'$ and $(\dot{S}'_K)_E \subseteq \dot{S}$. Then any $\xi \in \frac{1}{n}\mathbb{Z}/\mathbb{Z}[S_E]_0$ has trivial image in $\frac{1}{n}\mathbb{Z}/\mathbb{Z}[S'_K]_0$, because for all $u \in S_K$ we have $(\#\Gamma_{K/E,u}) \cdot \frac{1}{n}\mathbb{Z}/\mathbb{Z} = 0$. □

We may now deduce some preliminary cohomological vanishing:

Lemma 7.3.2 *The following colimits over $(E, \dot{S}_E, n)$ vanish.*

1. $\varinjlim H^1(\Gamma, \frac{1}{n}\mathbb{Z}/\mathbb{Z}[\Gamma_{E/F} \times S]_0) = 0$*;*
2. $\varinjlim H^1(\Gamma_v, \frac{1}{n}\mathbb{Z}/\mathbb{Z}[\Gamma_{E/F} \times S]_0) = 0$ *for all $v \in V$.*

Proof. The beginning of this argument follows the proof of [Kal18, Lemma 3.4.4]. Since the inclusion $\frac{1}{n}\mathbb{Z}/\mathbb{Z}[\Gamma_{E/F} \times S]_0 \hookrightarrow \frac{1}{n}\mathbb{Z}/\mathbb{Z}[\Gamma_{E/F} \times S]$ has a $\Gamma_{E/F}$-equivariant splitting given by choosing an arbitrary place of S, the inclusion also induces an inclusion at the level of cohomology groups, so we may prove the result for the modules $\frac{1}{n}\mathbb{Z}/\mathbb{Z}[\Gamma_{E/F} \times S]$ instead. Now, as a Γ-module, $\frac{1}{n}\mathbb{Z}/\mathbb{Z}[\Gamma_{E/F} \times S]$ is isomorphic to $\prod_S \frac{1}{n}\mathbb{Z}/\mathbb{Z}[\Gamma_{E/F}]$, and under this identification the transition map $\prod_S \frac{1}{n}\mathbb{Z}/\mathbb{Z}[\Gamma_{E/F}] \to \prod_{S'} \frac{1}{m}\mathbb{Z}/\mathbb{Z}[\Gamma_{K/F}]$ which is obtained by taking the maps $\frac{1}{n}\mathbb{Z}/\mathbb{Z}[\Gamma_{E/F}] \to \frac{1}{m}\mathbb{Z}/\mathbb{Z}[\Gamma_{K/F}]$ and then including S into S' to determine the direct factors. Thus, we may further replace the system of modules $\{\frac{1}{n}\mathbb{Z}/\mathbb{Z}[\Gamma_{E/F} \times S]\}$ by $\{\frac{1}{n}\mathbb{Z}/\mathbb{Z}[\Gamma_{E/F}]\}$.

After making this reduction (which is identical to the one done in the proof of [Kal18, Lemma 3.4.4]), we may use the same argument as that in [Kal18, Lemma 3.4.4] to deduce that the first system has vanishing colimit. We now turn to the system $\varinjlim H^1(\Gamma_v, \frac{1}{n}\mathbb{Z}/\mathbb{Z}[\Gamma_{E/F}])$. The Mackey formula and Shapiro's lemma tell us that

$$H^1(\Gamma_v, \frac{1}{n}\mathbb{Z}/\mathbb{Z}[\Gamma_{E/F}]) = \bigoplus_{w|v} H^1(\Gamma_{E_w}, \frac{1}{n}\mathbb{Z}/\mathbb{Z}),$$

where the sum runs over all places $w \in V_E$ lying above v. Identifying each $H^1(\Gamma_{E_w}, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$ with $\mathrm{Hom}(\Gamma_{E_w}, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$, the transition map

$$\bigoplus_{w|v} \mathrm{Hom}(\Gamma_{E_w}, \frac{1}{n}\mathbb{Z}/\mathbb{Z}) \to \bigoplus_{u|v} \mathrm{Hom}(\Gamma_{K_u}, \frac{1}{m}\mathbb{Z}/\mathbb{Z})$$

is given by the maps

$$\mathrm{Hom}(\Gamma_{E_w}, \frac{1}{n}\mathbb{Z}/\mathbb{Z}) \to \bigoplus_{u|w} \mathrm{Hom}(\Gamma_{K_u}, \frac{1}{m}\mathbb{Z}/\mathbb{Z})$$

induced by the inclusions $\Gamma_{K_u} \hookrightarrow \Gamma_{E_w}$ (and $\frac{1}{n}\mathbb{Z}/\mathbb{Z} \hookrightarrow \frac{1}{m}\mathbb{Z}/\mathbb{Z}$). For a fixed homomorphism $f_w \in \mathrm{Hom}(\Gamma_{E_w}, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$, the kernel H_{f_w} of f_w is an open normal subgroup of Γ_{E_w}, and so if K/E is a large enough finite Galois extension, we have $\Gamma_{K_u} \subseteq H_{f_w}$ for all $u \mid w$ places of K. Note that, given such a K as in the previous sentence, this property also holds for any $K'/K/F$ finite Galois and $\tilde{u} \mid w$ a place of K'. Thus, given $(f_w) \in \bigoplus_{w|v} \mathrm{Hom}(\Gamma_{E_w}, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$, we may look at all f_w as w ranges over $\{v\}_E \subseteq V_E$ to find a finite Galois extension K/F such that for any $w \in V_E$ with $f_w \neq 0$ and $u \in V_K$ with $u \mid w$, we have $\Gamma_{K_u} \subseteq H_{f_w}$. This means that the image of $(f_w)_w$ in $\bigoplus_{u|v} \mathrm{Hom}(\Gamma_{K_u}, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$ is trivial, showing that the second colimit in the statement of the proposition vanishes. □

Proposition 7.3.3 *For any $v \in \dot{V}$, we have $H^1(F_v, (P_{\dot{V}})_{F_v}) = 0$.*

Proof. The first thing to note is that $H^1(F_v, (P_{\dot{V}})_{F_v}) = \varprojlim_i H^1(F_v, (P_{E_i,\dot{S}_i,n_i})_{F_v})$, since the derived inverse limit $\varprojlim_i^{(1)} H^0(F_v, (P_{E_i,\dot{S}_i,n_i})_{F_v}) = 0$, because the system $H^0(F_v, (P_{E_i,\dot{S}_i,n_i})_{F_v})$ consists of finite groups. Thus, local Poitou-Tate duality gives

$$H^1(F_v,(P_{\dot{V}})_{F_v}) = \varprojlim_i (H^1(\Gamma_v, M_{E_i,\dot{S}_i,n_i})^*) = (\varinjlim_i H^1(\Gamma_v, M_{E_i,\dot{S}_i,n_i}))^*,$$

where the second equality holds by the universal property of colimits. Now we have the exact sequence

$$0 \to C_i \to H^1(\Gamma_v, M_{E_i,\dot{S}_i,n_i}) \to H^1(\Gamma_v, \frac{1}{n_i}\mathbb{Z}/\mathbb{Z}[\Gamma_{E_i/F} \times S_i]_0),$$

where C_i is a subquotient of $\frac{1}{n_i}\mathbb{Z}/\mathbb{Z}[(S_i)_{E_i}]_0$, and the colimits of the outer two terms are zero, by Lemmas 7.3.1 and 7.3.2, giving the result. □

For the next result, we need to recall a result from global class field theory. Let $\overline{C}$ denote the inverse limit $\varprojlim_{K/F} \overline{C_K}$, where $\overline{C_K}$ is the profinite completion of the idéle class group of the finite Galois extension K/F, and the limit is over all such extensions. For fixed K/F finite Galois and $n \in \mathbb{N}$, note that we have $\overline{C}[n]^{\Gamma_K} = \overline{C_K}[n]$.

Corollary 7.3.4 *The completed universal norm group*

$$\overline{N} := \varprojlim_{K/F} N_{K/F}(\overline{C_K})$$

is trivial (viewed as a subgroup of $\overline{C_F}$*).*

Proof. For any such K/F, we have the exact sequence

$$0 \to N_{K/F}(C_K) \to C_F \xrightarrow{(-,K/F)} \Gamma_{K/F}^{\text{ab}} \to 0.$$

Since the group $N_{K/F}(C_K)$ is open of finite index in C_F, the inverse limit over all open subgroups of C_F of finite index may be taken over all open subgroups of finite index which lie in $N_{K/F}(C_K)$, and for any such subgroup U, we get the exact sequence

$$0 \to \frac{N_{K/F}(C_K)}{U} \to \frac{C_F}{U} \to \Gamma_{K/F}^{\text{ab}} \to 0,$$

which after applying the (left-exact) functor $\varprojlim(-)$ yields the exact sequence

$$0 \to N_{K/F}(C_K)^{\wedge} \to \overline{C_F} \to \Gamma_{K/F}^{\text{ab}} \to 0;$$

note that surjectivity is preserved because the kernels are all finite groups. Now since C_K is dense in $\overline{C_K}$, we have that $N_{K/F}(\overline{C_K}) = N_{K/F}(C_K)^{\wedge}$ inside $\overline{C_F}$, by continuity of the norm map, yielding the short exact sequence

$$0 \to N_{K/F}(\overline{C_K}) \to \overline{C_F} \to \Gamma_{K/F}^{\text{ab}} \to 0,$$

which, after applying the inverse limit over all finite Galois K/F, yields the exact sequence

$$0 \to \overline{N} \to \overline{C_F} \to \Gamma^{\text{ab}}$$

so it's enough to show that the completed universal residue map $\overline{C_F} \to \Gamma^{\text{ab}}$ is injective, which is a basic fact of global class field theory (see e.g. [NSW08, Proposition 8.1.26]). □

We move on to a slightly more involved vanishing result:

Lemma 7.3.5 *The following colimit over $(E, \dot{S}_E, n)$ vanishes:*

$$\varinjlim H^2(\Gamma, \frac{1}{n}\mathbb{Z}/\mathbb{Z}[\Gamma_{E/F} \times S]_0) = 0.$$

Proof. As in the proof of Lemma 7.3.2, it is enough to show that the colimit $\varinjlim H^2(\Gamma_E, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$ vanishes, with the transition maps given by the restriction homomorphism. For (E, n) fixed, by [NSW08, Theorem 8.4.4] (with $S = V_E$), we have an isomorphism

$$H^2(\Gamma_E, \frac{1}{n}\mathbb{Z}/\mathbb{Z}) \xrightarrow{\sim} (\widehat{H}^0(\Gamma_E, \overline{C}[n]))^\vee,$$

where recall that $\widehat{H}^0(\Gamma_E, \overline{C}[n]) := \varprojlim_{K/E} \widehat{H}^0(\Gamma_{K/E}, \overline{C}[n]^{\Gamma_K})$, with transition maps given by the projections

$$\frac{\overline{C_E}[n]}{N_{K'/E}(\overline{C_{K'}}[n])} \to \frac{\overline{C_E}[n]}{N_{K/E}(\overline{C_K}[n])};$$

recall that for M a locally-compact Hausdorff topological group, $M^\vee$ denotes $\text{Hom}_{\text{cts}}(M, \mathbb{R}/\mathbb{Z})$.

We claim that the natural map

$$\overline{C_E}[n] \to \widehat{H}^0(\Gamma_E, \overline{C}[n])$$

is an isomorphism. To see this, note that it suffices to show that

$$\varprojlim_{K/E} N_{K/E}(\overline{C_K}[n]) = \varprojlim{}^{(1)} N_{K/E}(\overline{C_K}[n]) = 0.$$

For the first vanishing, note that we have a natural inclusion $N_{K/E}(\overline{C_K}[n]) \hookrightarrow N_{K/E}(\overline{C_K})[n]$, and so we also get an inclusion

$$\varprojlim N_{K/E}(\overline{C_K}[n]) \hookrightarrow (\varprojlim N_{K/E}(\overline{C_K}))[n] = 0,$$

where the last term equals zero by Corollary 7.3.4. Thus, to prove the claim it suffices to show that $\varprojlim^{(1)} N_{K/E}(\overline{C_K}[n]) = 0$, which follows from the fact that the system $\{N_{K/E}(\overline{C_K}[n])\}$ may be given the structure of a system of profinite groups with continuous transition maps. In conclusion, we obtain an isomorphism

$$H^2(\Gamma_E, \frac{1}{n}\mathbb{Z}/\mathbb{Z}) \xrightarrow{\sim} \overline{C_E}[n]^\vee.$$

From here, we have reduced the proposition to showing that the direct limit $\varinjlim \overline{C_E}[n]^\vee$ vanishes, where the transition maps are induced by the maps $\overline{C_K}[m] \to \overline{C_E}[n]$ given by $N_{K/E}$ composed with the m/n-power map. If $(f) \in \varinjlim \overline{C_E}[n]^\vee$, then choosing a representative $f \in \overline{C_E}[n]^\vee$, we have for any $(K, m) > (E, n)$ a factorization

$$\begin{array}{ccc} \overline{C_K}[m] & \xrightarrow{f'} & \frac{1}{n}\mathbb{Z}/\mathbb{Z} \\ \downarrow{\scriptstyle \cdot(m/n)\circ N_{K/E}} & & \| \\ \overline{C_E}[n] & \xrightarrow{f} & \frac{1}{n}\mathbb{Z}/\mathbb{Z}, \end{array}$$

where $f' \in \overline{C_K}[m]^\vee$ also represents (f). But now since f is continuous and $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$ is finite, the kernel of f is an open subgroup of $\overline{C_E}[n]$, and since the norm groups $N_{K/E}(\overline{C_K}) \subseteq \overline{C_E}$ shrink to the identity, there is some large enough $(K, m) > (E, n)$ such that $N_{K/E}(\overline{C_K}) \subseteq \ker(f)$ (using the finite intersection property). This shows that the image of f in $\overline{C_K}[m]^\vee$ is zero, giving the desired result. □

Before we prove the main vanishing result, we need a result about the Čech cohomology of $P_{\dot{V}}$, which is the analogue of Corollary 4.1.3:

Lemma 7.3.6 *For all p, the natural map $\check{H}^p(\overline{?}/?, P_{\dot{V}}) \to H^p(?, P_{\dot{V}})$ is an isomorphism for $? = F, F_v$.*

Proof. By [Stacks], 03F7, It's enough to show that the groups $H^j(\overline{?}^{\bigotimes_? n}, P_{\dot{V}})$ vanish for all $j, n \geq 1$. Since this is true for $P_{\dot{V}}$ replaced by any P_i, the short exact sequence

$$0 \to \varprojlim^{(1)} H^{j-1}(\overline{?}^{\bigotimes_? n}, P_i) \to H^j(\overline{?}^{\bigotimes_? n}, P_{\dot{V}}) \to \varprojlim H^j(\overline{?}^{\bigotimes_? n}, P_i) \to 0$$

reduces the lemma to showing that the derived inverse limit $\varprojlim^{(1)} H^0(\overline{?}^{\bigotimes_? n}, P_i)$ vanishes for all n. This is immediate from the fact that the transition maps $P_{i+1}(\overline{?}^{\bigotimes_? n}) \to P_i(\overline{?}^{\bigotimes_? n})$ are all surjective. □

Proposition 7.3.7 *We have $H^1(F, P_{\dot{V}}) = 0$.*

Proof. Since we have a natural localization map $\check{H}^1(\overline{F}/F, P_{\dot{V}}) \to \check{H}^1(\overline{F_v}/F_v, P_{\dot{V}})$ for all $v \in \dot{V}$, the isomorphisms from Lemma 7.3.6 give a localization map $H^1(F, P_{\dot{V}}) \to H^1(F_v, P_{\dot{V}})$ for all $v \in \dot{V}$. We get an exact sequence

$$0 \to \ker^1(F, P_{\dot{V}}) \to H^1(F, P_{\dot{V}}) \to \prod_{\dot{v} \in \dot{V}} H^1(F_{\dot{v}}, P_{\dot{V}}),$$

and so Lemma 7.3.2 implies that it's enough to show that $\ker^1(F, P_{\dot{V}}) = 0$. Since the natural map $H^1(F, P_{\dot{V}}) \to \varprojlim_i H^1(F, P_{E_i, \dot{S}_i, n_i})$ is an isomorphism, we have a natural isomorphism

$$\ker^1(F, P_{\dot{V}}) \xrightarrow{\sim} \varprojlim_i \ker^1(F, P_{E_i, \dot{S}_i, n_i}),$$

so it's enough to show that the right-hand side vanishes.

For i fixed, [Čes16, Lemma 4.4] tells us that we have a perfect pairing of finite abelian groups

$$\ker^1(F, P_{E_i, \dot{S}_i, n_i}) \times \ker^2(F, \underline{M_{E_i, \dot{S}_i, n_i}}) \to \mathbb{Q}/\mathbb{Z},$$

where $\underline{M_{E_i, \dot{S}_i, n_i}}$ is the étale F-group scheme associated to the Γ-module $M_{E_i, \dot{S}_i, n_i}$ (and is Cartier dual to the finite flat F-group scheme $P_{E_i, \dot{S}_i, n_i}$). Thus, it's enough to show that

$$\varprojlim_i (\ker^2(\Gamma, M_{E_i, \dot{S}_i, n_i}))^* = (\varinjlim_i \ker^2(\Gamma, M_{E_i, \dot{S}_i, n_i}))^* = 0,$$

which we will do by showing that the direct limit $\varinjlim_i \ker^2(\Gamma, M_{E_i, \dot{S}_i, n_i})$ vanishes, for which we will use an easier version of the analogous argument in [Kal18], proof of Proposition 3.4.6.

For any (E, S, n), the long exact sequence in cohomology gives the exact sequence

$$H^1(\Gamma, \frac{1}{n}\mathbb{Z}/\mathbb{Z}[\Gamma_{E/F} \times S]_0) \to H^1(\Gamma, \frac{1}{n}\mathbb{Z}/\mathbb{Z}[S_E]_0) \to H^2(\Gamma, M_{E, \dot{S}_E, n}) \to H^2(\Gamma, \frac{1}{n}\mathbb{Z}/\mathbb{Z}[\Gamma_{E/F} \times S]_0),$$

and so applying the (exact) functor $\varinjlim(-)$ and Lemmas 7.3.2, 7.3.5, we get an isomorphism

$$\varinjlim_i H^1(\Gamma, \frac{1}{n_i}\mathbb{Z}/\mathbb{Z}[(S_i)_{E_i}]_0) \xrightarrow{\sim} \varinjlim_i H^2(\Gamma, M_{E_i, \dot{S}_i, n_i}).$$

We have the inductive system of short exact sequence

$$0 \to \frac{1}{n_i}\mathbb{Z}/\mathbb{Z}[(S_i)_{E_i}]_0 \to \frac{1}{n_i}\mathbb{Z}/\mathbb{Z}[(S_i)_{E_i}] \to \frac{1}{n_i}\mathbb{Z}/\mathbb{Z} \to 0,$$

where the induced transition map from $\frac{1}{n_i}\mathbb{Z}/\mathbb{Z}$ to $\frac{1}{n_{i+1}}\mathbb{Z}/\mathbb{Z}$ is the natural inclusion followed by multiplication by $[E_{i+1}\colon E_i]$. It follows that after taking direct limits, the right-hand term in the short exact sequence vanishes and we get an isomorphism

$$\varinjlim_i H^1(\Gamma, \frac{1}{n_i}\mathbb{Z}/\mathbb{Z}[(S_i)_{E_i}]_0) \xrightarrow{\sim} \varinjlim_i H^1(\Gamma, \frac{1}{n_i}\mathbb{Z}/\mathbb{Z}[(S_i)_{E_i}]).$$

For i fixed, we have an isomorphism

$$H^1(\Gamma, \frac{1}{n_i}\mathbb{Z}/\mathbb{Z}[(S_i)_{E_i}]) \xrightarrow{\sim} \bigoplus_{v\in S_i} H^1(\Gamma_{E_i}\cdot \Gamma^{\dot{v}}_{F^{\mathrm{sep}}/F}, \frac{1}{n_i}\mathbb{Z}/\mathbb{Z})$$

by Shapiro's Lemma, and for $v \in S_i$, this isomorphism translates the transition map to the map given on the v-factor by

$$H^1(\Gamma_{E_i}\cdot \Gamma^{\dot{v}}_{F^{\mathrm{sep}}/F}, \frac{1}{n_i}\mathbb{Z}/\mathbb{Z}) \xrightarrow{\cdot(\#\Gamma^{\dot{v}_{E_{i+1}}}_{E_{i+1}/E_i})\circ \mathrm{Res}} H^1(\Gamma_{E_{i+1}}\cdot \Gamma^{\dot{v}}_{F^{\mathrm{sep}}/F}, \frac{1}{n_{i+1}}\mathbb{Z}/\mathbb{Z}).$$

Thus, if $(x) \in \varinjlim_i H^1(\Gamma, \frac{1}{n_i}\mathbb{Z}/\mathbb{Z}[(S_i)_{E_i}])$ is fixed with representative $x \in H^1(\Gamma, \frac{1}{n_i}\mathbb{Z}/\mathbb{Z}[(S_i)_{E_i}])$, some i fixed, we may choose $k > i$ large enough so that $\#\Gamma^{\dot{v}_{E_{i+1}}}_{E_k/E_i} = \#\Gamma_{(E_k)_{\dot{v}}/(E_i)_{\dot{v}}}$ is divisible by n_i for all $\dot{v} \in \dot{S}_i$, guaranteeing that the image of x in $H^1(\Gamma, \frac{1}{n_k}\mathbb{Z}/\mathbb{Z}[(S_k)_{E_k}])$ is zero, which shows that

$$\varinjlim_i H^2(\Gamma, M_{E_i,\dot{S}_i,n_i}) \xrightarrow{\sim} \varinjlim_i H^1(\Gamma, \frac{1}{n_i}\mathbb{Z}/\mathbb{Z}[(S_i)_{E_i}]_0) \xrightarrow{\sim} \varinjlim_i H^1(\Gamma, \frac{1}{n_i}\mathbb{Z}/\mathbb{Z}[(S_i)_{E_i}]) = 0.$$

This gives the desired result, since $\varinjlim_i \ker^2(\Gamma, M_{E_i,\dot{S}_i,n_i})$ injects into $\varinjlim_i H^2(\Gamma, M_{E_i,\dot{S}_i,n_i})$. $\square$

7.4 The canonical class

The purpose of this final subsection is to show that there is a canonical element $\xi \in \check{H}^2(\overline{F}/F, P_{\dot{V}})$ lifting the element $(\xi_i) \in \varprojlim_i \check{H}^2(\overline{F}/F, P_{E_i,\dot{S}_i,n_i})$ constructed above. For notational convenience, set $P := P_{\dot{V}}$, $P_i := P_{E_i,\dot{S}_i,n_i}$ and $M_i := M_{E_i,\dot{S}_i,n_i}$, and denote the projection $P \to P_i$ by p_i. Whenever we work with an embedding $\overline{F} \to \overline{F_v}$ for $v \in V$, we assume it is the one induced by $\dot{v} \in \dot{V}$ unless otherwise specified. We begin by proving some basic results about some Čech cohomology groups that associated to $P_{\dot{V}}$.

Lemma 7.4.1 *The natural maps*

$$\check{H}^k(\overline{\mathbb{A}}_v/F_v, P) \to \varprojlim_i \check{H}^k(\overline{\mathbb{A}}_v/F_v, P_i)$$

are isomorphisms for $k = 0, 1, 2$.

Proof. The case $k = 0$ is trivial, so we only need to focus on $k = 1, 2$. By Lemma 2.5.5, it's enough to show that $\varprojlim_i^{(1)} \check{H}^k(\overline{\mathbb{A}}_v/F_v, P_i) = 0$ for $k = 0, 1$ and that $\varprojlim_i^{(1)} B^1(i) = 0$. The vanishing of $\varprojlim_i^{(1)} \check{H}^0(\overline{\mathbb{A}}_v/F_v, P_i)$ follows from the fact that $\check{H}^0(\overline{\mathbb{A}}_v/F_v, P_i) = P_i(F_v)$, and the system $\{P_i(F_v)\}$ consists of finite groups. The vanishing of $\varprojlim_i^{(1)} B^1(i)$ comes from the fact that the system $\{B^1(i)\}$ has surjective transition maps: On Čech 0-cochains the transition maps $P_{i+1}(\overline{\mathbb{A}}_v) \to P_i(\overline{\mathbb{A}}_v)$ are all surjective by Lemma 3.3.8, and since the Čech differentials are compatible with F-homomorphisms (in our case, the transition maps $P_{i+1} \to P_i$), this surjectivity carries over to the group of 1-coboundaries.

It remains to show that the derived inverse limit $\varprojlim_i^{(1)} \check{H}^1(\overline{\mathbb{A}}_v/F_v, P_i)$ vanishes. The proof of Corollary 3.3.9 shows that the groups $\check{H}^1(\overline{\mathbb{A}}_v/F_v, P_i)$ are (compatibly) isomorphic to $H^1(F_v, P_i)$, and so it's enough to show that $\varprojlim_i^{(1)} H^1(F_v, P_i) = 0$, which follows from the fact that the system $\{H^1(F_v, P_i)\}_i$ may be given the structure of a system of profinite groups with continuous transition maps, as we showed in the proof of Corollary 7.2.7. □

Combining Lemma 7.4.1 with the proof of Corollary 3.3.9 gives an isomorphism

$$\check{H}^1(\overline{\mathbb{A}}_v/F_v, P) \xrightarrow{\sim} \varprojlim_i \check{H}^1(\overline{\mathbb{A}}_v/F_v, P_i) \xrightarrow{\sim} \varprojlim_i \check{H}^1(\overline{F_v}/F_v, P_i) \xrightarrow{\sim} \check{H}^1(\overline{F_v}/F_v, P),$$

and so Lemma 7.3.6 lets us identify $\check{H}^1(\overline{\mathbb{A}}_v/F_v, P)$ with $H^1(F_v, P)$ as well. The local canonical class $\xi_v \in \check{H}^2(\overline{F_v}/F_v, u_v) = H^2(F_v, u_v)$ maps via $S_v^2 \circ \mathrm{loc}_v$ to a class in $\check{H}^2(\overline{\mathbb{A}}_v/F_v, P)$ (notation as in §3.3).

The goal is to construct a canonical class $x \in \check{H}^2(\overline{\mathbb{A}}/\mathbb{A}, P)$ such that for each $\dot{v} \in \dot{V}$ and $v := \dot{v}_F$, its image in $\check{H}^2(\overline{\mathbb{A}}_v/F_v, P)$ (via the ring homomorphism $\overline{F} \otimes_F \mathbb{A} \xrightarrow{\iota \otimes \pi_v} \overline{F_v} \otimes_F F_v$, where $\iota \colon \overline{F} \to \overline{F_v}$ is our fixed inclusion, and π_v is projection onto the vth-factor) equals $S_v^2(\mathrm{loc}_v(\xi_v))$. We will proceed by constructing a Čech-theoretic analogue of the construction in [Kal18, §3.5]. Fix $\dot{\xi}_v \in u_v(\overline{F_v}^{\bigotimes_{F_v} 3})$ a Čech 2-cocycle representing ξ_v, and let $\Gamma_{\dot{v}} \subseteq \Gamma$ denote the decomposition group of $\dot{v} \in \dot{V}$; choose a (set-theoretic) section $\Gamma/\Gamma_{\dot{v}} \to \Gamma$—recall from §3.3 that this is equivalent to fixing a compatible system of diagonal embeddings

$$E \cdot F_v \to \prod_{w \in V_E, w|v} E_w$$

as we range over all finite Galois extensions E/F (which are the identity $E \cdot F_v \to E_{\dot{v}_E}$ on the $\dot{v}_E$-factor), and thus (as explained in §3.3) an explicit realization of the Shapiro map $h \colon G(\overline{F_v}^{\bigotimes_{F_v} 3}) \to G(\overline{\mathbb{A}_v}^{\bigotimes_{F_v} 3})$ at the level of Čech 2-cochains for any multiplicative F-group scheme G, which is functorial in G (with respect to F-homomorphisms) and compatible with the Čech differentials on both sides.

As we range through all i, these maps $S^2_{v,i}\colon P_i(\overline{F_v}^{\bigotimes_{F_v} 3}) \to P_i(\overline{\mathbb{A}_v}^{\bigotimes_{F_v} 3})$ splice to give a group homomorphism

$$\dot{S}^2_v\colon P(\overline{F_v}^{\bigotimes_{F_v} 3}) \to P(\overline{\mathbb{A}_v}^{\bigotimes_{F_v} 3}),$$

and we set $\dot{x}_v := \dot{S}^2_v(\mathrm{loc}_v(\dot{\xi}_v)) \in Z^2(\overline{\mathbb{A}}_v/F_v, P)$.

Note that for fixed i, we have $p_i(\dot{x}_v) = 1 \in P_i(\overline{\mathbb{A}_v}^{\bigotimes_{F_v} 3})$ for all $\dot{v} \in \dot{V}$ such that $v \notin S_i$. Indeed, the functoriality of the Shapiro maps implies that $p_i \circ \dot{S}^2_v \circ \mathrm{loc}_v = S^2_{v,i} \circ p_i \circ \mathrm{loc}_v$ on $P_i(\overline{F_v}^{\bigotimes_{F_v} 3})$, and now $p_i \circ \mathrm{loc}_v\colon u_v \to P_{F_v} \to (P_i)_{F_v}$ is trivial for $v \notin S_i$, since it is induced by the direct limit over $j \in \mathbb{N}$ (with $\dot{v}_{E_j} \in \dot{S}_j$) of $\Gamma_{\dot{v}}$-module homomorphisms

$$\frac{1}{n_i}\mathbb{Z}/\mathbb{Z}[\Gamma_{E_i/F} \times (S_i)_{E_i}]_{0,0} \to \frac{1}{n_j}\mathbb{Z}/\mathbb{Z}[\Gamma_{E_j/F} \times (S_j)_{E_j}]_{0,0} \to X^*(u_{n_j,E_j/F_v}),$$

where the kernel of the second map contains all elements whose $(\sigma, \dot{v}_{E_j})$-coefficients $c_{\sigma,\dot{v}_{E_j}}$ are zero for all $\sigma \in \Gamma_{E_j/F}$, and the image of the first map lands in the subgroup of elements whose coefficients $c_{\sigma,w}$ are zero for all $w \in (S_j)_{E_j}$ which do not lie above an element of $(S_i)_{E_i}$, which is the case for $\dot{v}_{E_j}$, since $\dot{v}_{E_i} \in (S_i)_{E_i}$ means that $v \in S_i$, which is not the case, giving our desired triviality.

The above paragraph implies that the element $\prod_{v\in V} p_i(\dot{x}_v) \in \prod_{v \in V} P_i(\overline{\mathbb{A}}_v^{\bigotimes_{F_v} 3})$ is trivial in all but finitely-many v-coordinates, so we may view $\prod_{v\in V} p_i(\dot{x}_v)$ as an element of

$$\bigoplus_{v\in S_i} Z^2(\overline{\mathbb{A}}_v/F_v, P_i).$$

By viewing $\prod_{v\in V} p_i(\dot{x}_v)$ as an element of the restricted product $\prod'_v P_i(\mathbb{A}_{L,v}^{\bigotimes_{F_v} 3})$ for some sufficiently large finite extension L/F (possible because $\overline{\mathbb{A}}_v = \varinjlim \mathbb{A}_{K,v}$ over all finite extensions K/F and each P_i/F is of finite type), where this product is restricted with respect to the subgroups $P_i(O_{L,v}^{\bigotimes_{O_{F_v}} 3})$, cf. §3.3), we obtain by Proposition 3.3.7 an element of $Z^2(\overline{\mathbb{A}}/\mathbb{A}, P_i)$. It is straightforward to check that as we vary across all $i \in \mathbb{N}$, these elements describe an element of the projective system $\{Z^2(\overline{\mathbb{A}}/\mathbb{A}, P_i)\}_i$, giving an element $\dot{x} \in Z^2(\overline{\mathbb{A}}/\mathbb{A}, P)$.

Following [Kal18], we will now show that the class of $\dot{x}$ in $\check{H}^2(\overline{\mathbb{A}}/\mathbb{A}, P)$ is independent of the choice of local canonical class representatives $\dot{\xi}_v$ and section $\Gamma/\Gamma_{\dot{v}} \to \Gamma$. Showing independence on the choices of $\dot{\xi}_v$ follows easily from the analogous argument in [Kal18, §3.5, pp. 318] after replacing the group cohomological differentials loc. cit. with Čech differentials.

As a consequence of Lemma 3.3.10 , for any $v \in V$ we may find $c_v \in P(\overline{\mathbb{A}}_v \otimes_{F_v} \overline{\mathbb{A}}_v)$ such that $dc_v = \dot{S}_v(\mathrm{loc}_v(\dot{\xi}_v)) \cdot \dot{S}'_v(\mathrm{loc}_v(\dot{\xi}_v))^{-1}$. Moreover, we claim that we may choose c_v such that for i

fixed and $v \notin S_i$, we have $p_i(c_v) = 1$ in $P_i(\overline{\mathbb{A}}_v \otimes_{F_v} \overline{\mathbb{A}}_v)$. Indeed, by the construction in the proof of Lemma 3.3.10, we may take

$$c_v = (r_1 \cdot \bar{r}_3 \otimes r_2)(\mathrm{loc}_v(\dot{\xi}_v)) \cdot (\bar{r}_2 \otimes r_1 \cdot \bar{r}_3)(\mathrm{loc}_v(\dot{\xi}_v))^{-1},$$

where $r \colon \overline{F_v} \to \overline{\mathbb{A}}_v$ is the direct limit of the maps $E'_{(\dot{v}_E)'} \to \prod_{w|v} E'_{w'}$ defined on the w-coordinate by the isomorphism $r_w \colon E'_{(\dot{v}_E)'} \xrightarrow{\sim} E'_{w'}$ determined by the section s, similarly with $\bar{r}$, where as in the proof of Lemma 3.3.10, the subscript i in r_i denotes that its source is the ith tensor factor of $(E'_{v'})^{\bigotimes_{F_v} 3}$. Since the maps $r, \bar{r}$ are F_v-homomorphisms, they commute with the projections p_i, and hence

$$p_i(c_v) = (r_1 \cdot \bar{r}_3 \otimes r_2)(p_i\mathrm{loc}_v(\dot{\xi}_v)) \cdot (\bar{r}_2 \otimes r_1 \cdot \bar{r}_3)(p_i\mathrm{loc}_v(\dot{\xi}_v))^{-1},$$

giving the existence of such a c_v.

As a result, the element $\tilde{c} := \prod_v c_v \in \prod_{v \in V} P(\overline{\mathbb{A}}_v \otimes_{F_v} \overline{\mathbb{A}}_v)$ has projection $p_i(\tilde{c})$ with all but finitely-many trivial coordinates, and hence has well-defined image in $P_i(\overline{\mathbb{A}} \otimes_{\mathbb{A}} \overline{\mathbb{A}})$ (using Corollary 3.3.6), and setting $c := \varprojlim_i p_i(\tilde{c})$ gives an element of $P(\overline{\mathbb{A}} \otimes_{\mathbb{A}} \overline{\mathbb{A}})$ which satisfies $\dot{S}^2_v(\mathrm{loc}_v(\dot{\xi}_v)) \cdot \dot{S}'^2_v(\mathrm{loc}_v(\dot{\xi}_v))^{-1} = dc$, concluding the argument for why the class $[\dot{x}] \in \check{H}^2(\overline{\mathbb{A}}/\mathbb{A}, P)$ is canonical.

The final key step in constructing a canonical class in $\check{H}^2(\overline{F}/F, P)$ is showing that there is a unique element of $\check{H}^2(\overline{F}/F, P)$ whose image in $\check{H}^2(\overline{\mathbb{A}}/\mathbb{A}, P)$ is the class $x := [\dot{x}]$, and whose image in $\varprojlim_i \check{H}^2(\overline{F}/F, P_i)$ is (ξ_i), which we turn to now. The argument will use complexes of tori, following the analogous one in [Kal18, §3.5]. The first result that makes this possible can be taken directly from [Kal18] (Lemma 3.5.1 loc. cit.).

Lemma 7.4.2 *For each i, there exists an isogeny of tori $f_i \colon T_i \to U_i$ defined over F with kernel equal to P_i. Moreover, we have the commutative diagram*

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K_i & \longrightarrow & K'_i & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & P_{i+1} & \longrightarrow & T_{i+1} & \longrightarrow & U_{i+1} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & P_i & \longrightarrow & T_i & \longrightarrow & U_i & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1, & &
\end{array}$$

where K_i and K'_i are tori.

For any i, consider the double complex of abelian groups $K^{p,q} =$

$$\begin{array}{ccccccc} T_i(\overline{\mathbb{A}}) & \longrightarrow & T_i(\overline{\mathbb{A}} \otimes_{\mathbb{A}} \overline{\mathbb{A}}) & \longrightarrow & T_i(\overline{\mathbb{A}} \otimes_{\mathbb{A}} \overline{\mathbb{A}} \otimes_{\mathbb{A}} \overline{\mathbb{A}}) & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ U_i(\overline{\mathbb{A}}) & \longrightarrow & U_i(\overline{\mathbb{A}} \otimes_{\mathbb{A}} \overline{\mathbb{A}}) & \longrightarrow & U_i(\overline{\mathbb{A}} \otimes_{\mathbb{A}} \overline{\mathbb{A}} \otimes_{\mathbb{A}} \overline{\mathbb{A}}) & \longrightarrow & \dots . \end{array}$$

Note that the complex with jth term $C^j := H^0(K^{j,\bullet})$ ($j \geq 0$) is exactly the Čech complex of P_i with respect to the fpqc cover $\overline{\mathbb{A}}/\mathbb{A}$, and so the low-degree exact sequence for the spectral sequence associated to a double complex gives an injective map

$$\check{H}^1(\overline{\mathbb{A}}/\mathbb{A}, P_i) \hookrightarrow H^1(\overline{\mathbb{A}}/\mathbb{A}, T_i \to U_i).$$

Moreover, lemma A.3.1 tells us that we may canonically identify $H^1(\overline{\mathbb{A}}/\mathbb{A}, T_i \to U_i)$ with the group $H^1(\mathbb{A}^{\text{sep}}/\mathbb{A}, T_i \to U_i)$, and the majority of the results we will be using in this section, developed in Appendix A, are stated for the latter group. This identification, along with the analogous one for the groups $\bar{H}^i(\overline{\mathbb{A}}/\mathbb{A}, T_i \to U_i)$ (Lemma A.3.2) will be implicit in what follows, in order to apply our results from Appendix A.

Since the kernels of $T_{i+1} \to T_i$ and $U_{i+1} \to U_i$ are tori, combining Corollary 3.3.6 with Lemma 3.3.8 tells us that the maps $T_{i+1}(\overline{\mathbb{A}}^{\bigotimes_{\mathbb{A}} n}) \to T_i(\overline{\mathbb{A}}^{\bigotimes_{\mathbb{A}} n})$ and $U_{i+1}(\overline{\mathbb{A}}^{\bigotimes_{\mathbb{A}} n}) \to U_i(\overline{\mathbb{A}}^{\bigotimes_{\mathbb{A}} n})$ are surjective for all n (this is also the case when $\overline{\mathbb{A}}$ is replaced by $\mathbb{A}^{\text{sep}}$, by smoothness). It follows that the induced map

$$C^j(\overline{\mathbb{A}}/\mathbb{A}, T_{i+1} \to U_{i+1}) \to C^j(\overline{\mathbb{A}}/\mathbb{A}, T_i \to U_i)$$

(where $C^j(\overline{\mathbb{A}}/\mathbb{A}, T \to U)$ is the group of j-cochains for the corresponding total complex) is surjective for any j, and so the system $\{C^j(\overline{\mathbb{A}}/\mathbb{A}, T_i \to U_i)\}_{i \geq 0}$ satisfies the Mittag-Lefler condition. Replacing $\overline{\mathbb{A}}$ by $\mathbb{A}^{\text{sep}}$ in order to use group cohomology (Lemma A.3.3), it follows from [NSW08, Theorem 3.5.8] that we obtain the exact sequence

$$1 \to \varprojlim^{(1)} H^1(\overline{\mathbb{A}}/\mathbb{A}, T_i \to U_i) \to H^2(\overline{\mathbb{A}}/\mathbb{A}, T \to U) \to \varprojlim H^2(\overline{\mathbb{A}}/\mathbb{A}, T_i \to U_i) \to 1, \tag{7.5}$$

where the middle term denotes the cohomology of the complex with jth term

$$C^j(\overline{\mathbb{A}}/\mathbb{A}, T \to U) := \varprojlim C^j(\overline{\mathbb{A}}/\mathbb{A}, T_i \to U_i) = C^j(\overline{\mathbb{A}}/\mathbb{A}, T) \oplus C^{j-1}(\overline{\mathbb{A}}/\mathbb{A}, U),$$

where $T = \varprojlim_i T_i$ and $U := \varprojlim_i U_i$ are pro-tori over F (note that, using left-exactness of inverse limits, the kernel of $T \to U$ is our group P). Once again, the low-degree exact sequence for double complexes gives us a map

$$\check{H}^2(\overline{\mathbb{A}}/\mathbb{A}, P) \to H^2(\overline{\mathbb{A}}/\mathbb{A}, T \to U),$$

which need not be injective. We also have the natural map $\check{H}^2(\overline{F}/F, P) \to \check{H}^2(\overline{\mathbb{A}}/\mathbb{A}, P)$.

We have the following analogue of [Kal18, Proposition 3.5.2]:

Proposition 7.4.3 *There exists a unique element of $\check{H}^2(\overline{F}/F, P)$ whose image in $\varprojlim \check{H}^2(\overline{F}/F, P_i)$ equals the canonical system (ξ_i), and whose image in $H^2(\overline{\mathbb{A}}/\mathbb{A}, T \to U)$ coincides with the image of the class $x \in \check{H}^2(\overline{\mathbb{A}}/\mathbb{A}, P)$ there.*

Proof. If $\tilde{\xi} \in \check{H}^2(\overline{F}/F, P)$ is any preimage of $(\xi_i) \in \varprojlim \check{H}^2(\overline{F}/F, P_i)$ and $\tilde{\xi}_{\mathbb{A}}$ denotes its image in $\check{H}^2(\overline{\mathbb{A}}/\mathbb{A}, P)$, the images of x and $\tilde{\xi}_{\mathbb{A}}$ in $\varprojlim H^2(\overline{\mathbb{A}}/\mathbb{A}, T_i \to U_i)$ via the composition

$$\check{H}^2(\overline{\mathbb{A}}/\mathbb{A}, P) \to H^2(\overline{\mathbb{A}}/\mathbb{A}, T \to U) \to \varprojlim H^2(\overline{\mathbb{A}}/\mathbb{A}, T_i \to U_i)$$

coincide by the identical argument in [Kal18, Proposition 3.5.2], replacing the use of [KS99, Theorem C.1.B] loc. cit. with our Proposition A.3.6 and the use of Corollary 3.3.8 loc. cit. with our Corollary 7.2.8. To finish the proof of the Proposition, we need the following analogue of [Kal18, Lemma 3.5.3]:

Lemma 7.4.4 *The natural map*

$$\varprojlim^{(1)} \check{H}^1(\overline{F}/F, P_i) \to \varprojlim^{(1)} H^1(\overline{\mathbb{A}}/\mathbb{A}, T_i \to U_i)$$

is an isomorphism.

Proof. We have the tautological short exact sequence of topological groups (see §A.3 for the definition of the topologies, replacing F^{sep} and $\mathbb{A}^{\text{sep}}$ in that section with $\overline{F}$ and $\overline{\mathbb{A}}$ via the canonical identifications)

$$1 \to H^1(\overline{F}/F, T_i \to U_i)/\text{ker}^1(\overline{F}/F, T_i \to U_i) \to H^1(\overline{\mathbb{A}}/\mathbb{A}, T_i \to U_i) \to \text{cok}^1(\overline{F}/F, T_i \to U_i) \to 1, \tag{7.6}$$

and by Corollary A.4.7, the group $\text{cok}^1(\overline{F}/F, T_i \to U_i)$ is compact as a topological group. Since the projective system $\{\text{cok}^1(\overline{F}/F, T_i \to U_i)\}_{i \geq 0}$ consists of compact, locally profinite groups, it is a system of profinite groups, and we thus get that $\varprojlim^{(1)} \text{cok}^1(\overline{F}/F, T_i \to U_i) = 0$.

As in [Kal18], the next step is to show that $\varprojlim \text{cok}^1(\overline{F}/F, T_i \to U_i)$ also vanishes. By Proposition A.4.8, the compact group $\text{cok}^1(\overline{F}/F, T_i \to U_i)$ is Pontryagin dual to the discrete group $H^1(W_F, \widehat{U}_i \to \widehat{T}_i)_{\text{red}}/\text{ker}^1(W_F, \widehat{U}_i \to \widehat{T}_i)_{\text{red}}$, which by Proposition A.4.2 and Lemma A.4.10 is canonically isomorphic to $H^1(\Gamma, \widehat{U}_i \to \widehat{T}_i)_{\text{red}}/\ker^1(\Gamma, \widehat{U}_i \to \widehat{T}_i)_{\text{red}}$. Using Lemma A.4.10, we may further identify the group $H^1(\Gamma, \widehat{U}_i \to \widehat{T}_i)_{\text{red}}$ with $H^2(\Gamma, X^*(U_i) \to X^*(T_i)) = H^1(\Gamma, M_i)$ (this last identification comes from the five-lemma), and compatibly (with respect to the first identification) identify the group $\ker^1(\Gamma, \widehat{U}_i \to \widehat{T}_i)_{\text{red}}$ with $\ker^2(\Gamma, X^*(U) \xrightarrow{f^*} X^*(T)) = \ker^1(\Gamma, M_i)$. Thus,

$$\varprojlim \text{cok}^1(\overline{F}/F, T_i \to U_i) = \varprojlim (H^1(\Gamma, M_i)/\ker^1(\Gamma, M_i)])^\vee = (\varinjlim [H^1(\Gamma, M_i)/\ker^1(\Gamma, M_i)])^\vee,$$

so the claim will follow from showing that $\varinjlim H^1(\Gamma, M_i)$ vanishes, which is an immediate consequence of Lemma 7.3.1 and Lemma 7.3.2 part (1).

Applying the functor $\varprojlim(-)$ to the short exact sequence (7.6), the vanishing results we just proved tell us that the map

$$\varprojlim^{(1)} [H^1(\overline{F}/F, T_i \to U_i)/\ker^1(\overline{F}/F, T_i \to U_i)] \to \varprojlim^{(1)} H^1(\overline{\mathbb{A}}/\mathbb{A}, T_i \to U_i)$$

is an isomorphism. But now since the system $\{\ker^1(\overline{F}/F, T_i \to U_i)\}_{i \geq 0}$ consists of finite groups (by Lemma A.4.8), it has vanishing $\varprojlim^{(1)}$, and hence the natural map

$$\varprojlim^{(1)} H^1(\overline{F}/F, T_i \to U_i) \to \varprojlim^{(1)} [H^1(\overline{F}/F, T_i \to U_i)/\ker^1(\overline{F}/F, T_i \to U_i)]$$

is also an isomorphism. The claim of the lemma then follows from the fact that the natural inclusion $\check{H}^1(\overline{F}/F, P_i) \to H^1(\overline{F}/F, T_i \to U_i)$ is an isomorphism, by the five-lemma. □

The short exact sequence (7.5) and the above lemma imply that we may modify $\tilde{\xi}$ by an element of $\varprojlim^{(1)} \check{H}^1(\overline{F}/F, P_i)$ to ensure that the images of $\tilde{\xi}_{\mathbb{A}}$ and x in $H^2(\overline{\mathbb{A}}/\mathbb{A}, T \to U)$ are equal, proving the existence claim of the proposition. Uniqueness follows from the fact that the composition

$$\varprojlim^{(1)} \check{H}^1(\overline{F}/F, P_i) \to \varprojlim^{(1)} H^1(\overline{\mathbb{A}}/\mathbb{A}, T_i \to U_i) \to H^2(\overline{\mathbb{A}}/\mathbb{A}, T \to U)$$

is injective (any two such ξ differ by an element of $\varprojlim^{(1)} \check{H}^1(\overline{F}/F, P_i)$ which has trivial image in $H^2(\overline{\mathbb{A}}/\mathbb{A}, T \to U)$). □

We may now define our canonical class of $\check{H}^2(\overline{F}/F, P)$:

Definition 7.4.5 *The canonical class $\xi \in \check{H}^2(\overline{F}/F, P)$ is the element whose existence and uniqueness is asserted in Proposition 7.4.3.*

As explained in [Kal18] in the remarks following Definition 3.5.4, the class ξ is independent of the choice of tower of isogenies $\{f_i \colon T_i \to U_i\}$.

CHAPTER 8

Cohomology of the Global Gerbe

8.1 Basic definitions

As in previous sections, we write H^i for H^i_{fppf}. Let $\xi \in \check{H}^2(\overline{F}/F, P_{\dot{V}})$ be the canonical class of Definition 7.4.5. By Propositions 2.3.2 and 2.3.5, ξ corresponds to an isomorphism class of (fpqc) $P_{\dot{V}}$-gerbes split over $\overline{F}$. Let $\mathcal{E}_{\dot{V}} \to (\text{Sch}/F)_{\text{fpqc}}$ be such a gerbe. We equip $\mathcal{E}_{\dot{V}}$ with the structure of a site by giving it the fpqc topology inherited from $(\text{Sch}/F)_{\text{fpqc}}$.

Recall for a finite central $Z \hookrightarrow G$ the subset $H^1(\mathcal{E}_{\dot{V}}, Z \to G) \subseteq H^1(\mathcal{E}_{\dot{V}}, G_{\mathcal{E}_{\dot{V}}})$ of isomorphism classes Z-twisted G-torsors on $\mathcal{E}_{\dot{V}}$. Note that since the target of a homomorphism $P_{\dot{V}} \to Z$ is finite, it always factors through the projection $P_{\dot{V}} \to P_{E_i, \dot{S}_i, n_i}$ for some i. For any other choice of $P_{\dot{V}}$-gerbe $\mathcal{E}'_{\dot{V}}$ split over $\overline{F}$ representing ξ, we have an isomorphism of $P_{\dot{V}}$-gerbes $h \colon \mathcal{E}_{\dot{V}} \to \mathcal{E}'_{\dot{V}}$, inducing an isomorphism $H^1(\mathcal{E}_{\dot{V}}, G_{\mathcal{E}_{\dot{V}}}) \to H^1(\mathcal{E}_{\dot{V}}, G_{\mathcal{E}'_{\dot{V}}})$ which, since $\check{H}^1(\overline{F}/F, P_{\dot{V}})$ vanishes by Proposition 7.3.7, is independent of the choice of h, by Lemma 2.6.4.

The inflation-restriction sequence gives us the commutative diagram

$$\begin{array}{ccccccc} H^1(F,G) & \longrightarrow & H^1(\mathcal{E}_{\dot{V}}, Z \to G) & \longrightarrow & \text{Hom}_F(P_{\dot{V}}, Z) & \longrightarrow & H^2(F,G) \\ \| & & \downarrow & & \downarrow \Theta^P_{\dot{V}} & & \| \\ H^1(F,G) & \longrightarrow & H^1(F, G/Z) & \longrightarrow & H^2(F,Z) & \longrightarrow & H^2(F,G), \end{array}$$

where the second vertical map comes from defining the $(G/Z)_{\mathcal{E}_{\dot{V}}}$-torsor $\mathscr{T} \times^{G_{\mathcal{E}_{\dot{V}}}} (G/Z)_{\mathcal{E}_{\dot{V}}}$, which, by construction, has trivial $P_{\dot{V}}$-action, and thus is the pullback of a G/Z-torsor T over F, whose class we take to be the image of $[\mathscr{T}]$. We have the following two direct translations of results from [Kal18]:

Lemma 8.1.1 *If G is either abelian or connected and reductive, then the map*

$$H^1(\mathcal{E}_{\dot{V}}, Z \to G) \to H^1(F, G/Z)$$

defined above is surjective.

Proof. The identical argument as in [Kal18, Lemma 3.6.1] works here, replacing the use of Lemma A.1 loc. cit. with [Tha19, Corollary 1.10] for connected reductive G. □

We get an analogue of [Kal18, Lemma 3.6.2]:

Lemma 8.1.2 *If G is connected and reductive, then for each $x \in H^1(\mathcal{E}_{\dot{V}}, Z \to G)$, there exists a maximal torus $T \subset G$ such that x is in the image of $H^1(\mathcal{E}_{\dot{V}}, Z \to T)$.*

Proof. One can use the same proof as for the corresponding result in [Kal18], once again replacing the use of Lemma A.1 loc. cit. with [Tha19, Corollary 1.10]. □

The next goal is to construct a localization map $\text{loc}_v \colon H^1(\mathcal{E}_{\dot{V}}, Z \to G) \to H^1(\mathcal{E}_v, Z \to G)$ for any $v \in V$, where $\mathcal{E}_v$ denotes the local gerbe defined in Chapter 3, such that the diagram

$$\begin{array}{ccccc} H^1(F, G) & \longrightarrow & H^1(\mathcal{E}_{\dot{V}}, Z \to G) & \longrightarrow & \text{Hom}_F(P_{\dot{V}}, Z) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(F_v, G) & \longrightarrow & H^1(\mathcal{E}_v, Z \to G) & \longrightarrow & \text{Hom}_{F_v}(u_v, Z) \end{array} \tag{8.1}$$

commutes, where (using Čech cohomology) the left-hand vertical map is induced by the inclusion $\overline{F} \to \overline{F_v}$ and the right-hand map is induced by the F-homomorphism $\text{loc}_v^P \colon u_v \to (P_{\dot{V}})_{F_v}$ defined in §**??**.

We have the category $(\mathcal{E}_{\dot{V}})_{F_v} := \mathcal{E} \times_{\text{Sch}/F} (\text{Sch}/F_v)$, which is an fpqc $(P_{\dot{V}})_{F_v}$-gerbe split over $\overline{F_v}$; recall that the objects of $(\mathcal{E}_{\dot{V}})_{F_v}$ are pairs (X, f), where $f \colon U \to \text{Spec}(F_v)$ is an F-morphism and X lies in $\mathcal{E}_{\dot{V}}(U)$ and morphisms are defined in the obvious way.

Fixing an isomorphism of $P_{\dot{V}}$-gerbes $(\mathcal{E}_{\dot{V}})_{F_v} \xrightarrow{\sim} \mathcal{E}_{x_v}$, where x_v denotes a Čech 2-cocycle representing the image of $\xi \in \check{H}^2(\overline{F}/F, P_{\dot{V}})$ in $\check{H}^2(\overline{F_v}/F_v, P_{\dot{V}})$, and an isomorphism of u_v-gerbes $\mathcal{E}_v \xrightarrow{\sim} \mathcal{E}_{\xi_v}$, where ξ_v is a Čech 2-cocycle representing the local canonical class $[\xi_v]$, the fact that $\text{loc}_v^P([\xi_v]) = [x_v]$ (by Corollary 7.2.8) implies, by the functoriality of gerbes given by Construction 2.3.4, that we have a (non-canonical) morphism of fibered categories over F_v from $\mathcal{E}_{\xi_v}$ to $\mathcal{E}_{x_v}$ which is the morphism loc_v^P on bands, and, via the composition

$$\mathcal{E}_v \xrightarrow{\sim} \mathcal{E}_{\xi_v} \to \mathcal{E}_{x_v} \xrightarrow{\sim} (\mathcal{E}_{\dot{V}})_{F_v} \to \mathcal{E}_{\dot{V}},$$

we obtain a functor $\mathcal{E}_v \to \mathcal{E}_{\dot{V}}$.

The functor $\mathcal{E}_v \to \mathcal{E}_{\dot{V}}$ defined above is highly non-canonical. However, the morphism $\mathcal{E}_{\xi_v} \to \mathcal{E}_{x_v}$ is unique up to post-composing by an automorphism of $\mathcal{E}_{x_v}$ determined by a Čech 1-cocycle of $P_{\dot{V}}$ valued in $\overline{F_v}$ (see §2.3), and since, by Proposition 7.3.3, the group $\check{H}^1(\overline{F_v}/F_v, P_{\dot{V}})$ is trivial, such a 1-cocycle is in fact a 1-coboundary. The same is true for the normalizing isomorphisms $\mathcal{E}_v \to \mathcal{E}_{\xi_v}$ and $(\mathcal{E}_{\dot{V}})_{F_v} \to \mathcal{E}_{x_v}$, using Proposition 7.3.3 and Corollary 7.3.7. We are now ready to define our main localization map.

We can use the functor $\mathcal{E}_v \to \mathcal{E}_{\dot{V}}$ to pull back any Z-twisted $G_{\mathcal{E}_{\dot{V}}}$-torsor to a Z_{F_v}-twisted $G_{\mathcal{E}_v}$-torsor, giving a map

$$\text{loc}_v \colon H^1(\mathcal{E}_{\dot{V}}, Z \to G) \to H^1(\mathcal{E}_v, Z \to G)$$

which, by combining the above paragraph with Proposition 2.6.3, is canonical. Note that this map is canonical up to finer equivalence classes of $G_{\mathcal{E}_v}$-torsors, where we replace isomorphism classes with classes whose elements are related via isomorphisms $\mathscr{T} \xrightarrow{\sim} \eta^* \mathscr{T}$ of $G_{\mathcal{E}_v}$-torsors induced by translation by $z^{-1} \in Z(\overline{F_v})$, where $\eta \colon \mathcal{E}_v \to \mathcal{E}_v$ is the automorphism of gerbes induced by the 1-coboundary $d(z)$. It is straightforward to check that this localization map makes the diagram (8.1) commute.

8.2 Tate-Nakayama duality for tori

We are now ready to discuss duality for tori. As in [Kal18], we define $\mathcal{T} \subset \mathcal{A}$ to be the full subcategory consisting of objects $[Z \to G]$ for which G is a torus, and for $v \in \dot{V}$, the category $\mathcal{T}_v$ the category of pairs $[Z \to T]$ where T is an F_v-torus and Z is finite (and defined over F_v), with morphisms given as in $\mathcal{A}$. Recall from §4.3 that associated to such a pair $[Z \to T] \in \mathcal{T}_v$ we have the group

$$\overline{Y}_{+v,\text{tor}}[Z \to T] := (X_*(T/Z)/[I_v X_*(T)])_{\text{tor}} = (X_*(T/Z)/[I_v X_*(T)])^{N_{E/F_v}},$$

where $I_v \subset \mathbb{Z}[\Gamma_v]$ denotes the augmentation ideal, E/F_v denotes a finite Galois extension splitting T, and the superscript N_{E/F_v} denotes the kernel of the norm map. Moreover, by Theorem 4.4.3, we have a canonical functorial isomorphism

$$\iota_v \colon \overline{Y}_{+v,\text{tor}}[Z \to T] \xrightarrow{\sim} H^1(\mathcal{E}_v, Z \to T)$$

which commutes with the maps of both groups to $\text{Hom}_{F_v}(u_v, Z)$.

Following [Kal18], the first step is to construct the global analogue of the groups $\overline{Y}_{+v,\text{tor}}[Z \to T]$, which is unchanged in the function field setting. For fixed $[Z \to T] \in \mathcal{T}$ we set $Y := X_*(T)$, $\overline{Y} := X_*(T/Z)$, and $A^\vee := \text{Hom}_{\mathbb{Z}}(X^*(Z), \mathbb{Q}/\mathbb{Z})$. We have a short exact sequence

$$0 \to Y \to \overline{Y} \to A^\vee \to 0,$$

due to the vanishing of the $\mathrm{Ext}^1_{\mathbb{Z}}$-functor for free abelian groups. For any i, the $\Gamma_{E_i/F}$-module $\mathbb{Z}[(S_i)_{E_i}]_0$ is a free abelian group, and thus we may tensor it with the above exact sequence, giving a new short exact sequence

$$0 \to Y[(S_i)_{E_i}]_0 \to \overline{Y}[(S_i)_{E_i}]_0 \to A^\vee[(S_i)_{E_i}]_0 \to 0,$$

and denote by $\overline{Y}[(S_i)_{E_i}, \dot{S}_i]_0 \subseteq \overline{Y}[(S_i)_{E_i}]_0$ the preimage of the subgroup $A^\vee[\dot{S}_i]$ under the above surjection; note that, by construction $\overline{Y}[(S_i)_{E_i}, \dot{S}_i]_0$ contains the image of $Y[(S_i)_{E_i}]_0$.

Choosing any section $s\colon (S_i)_{E_i} \to (S_{i+1})_{E_{i+1}}$ such that $s(\dot{S}_i) \subset \dot{S}_{i+1}$, we may define a map

$$s_!\colon \overline{Y}[(S_i)_{E_i}, \dot{S}_i]_0 \to \overline{Y}[(S_{i+1})_{E_{i+1}}, \dot{S}_{i+1}]_0$$

by

$$s_!(\sum_{w \in (S_i)_{E_i}} c_w[w]) = \sum_{w' \in (S_{i+1})_{E_{i+1}}, s((w')_{E_i}) = w'} c_{(w')_{E_i}}[w'].$$

The following result of [Kal18] (Lemma 3.7.1 loc. cit.) carries over verbatim to our situation:

Lemma 8.2.1 *The assignment $f \mapsto s_! f$ induces a well-defined homomorphism*

$$!\colon \frac{\overline{Y}[(S_i)_{E_i}, \dot{S}_i]_0}{I_{E_i/F}Y[(S_i)_{E_i}]_0} \to \frac{\overline{Y}[(S_{i+1})_{E_{i+1}}, \dot{S}_{i+1}]_0}{I_{E_{i+1}/F}Y[(S_{i+1})_{E_{i+1}}]_0}$$

which is independent of the choice of s.

Definition 8.2.2 *We define*

$$\overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,tor} := \varinjlim_i \frac{\overline{Y}[(S_i)_{E_i}, \dot{S}_i]_0}{I_{E_i/F}Y[(S_i)_{E_i}]_0}[tor],$$

with transition maps given by $!$. We also define

$$Y[V_{\overline{F}}]_{0,\Gamma,tor} := \varinjlim_i \frac{Y[(S_i)_{E_i}]_0}{I_{E_i/F}Y[(S_i)_{E_i}]_0}[tor],$$

with transition maps induced by $!$.

The above two groups fit into the short exact sequence

$$0 \to Y[V_{\overline{F}}]_{0,\Gamma,\mathrm{tor}} \to \overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,\mathrm{tor}} \to A^\vee[\dot{V}]_0 \to 0,$$

where the last term is as defined in Lemma 7.2.6.

For any $v \in V$ we can define a localization morphism

$$l_v \colon \overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,\text{tor}} \to \overline{Y}_{+v,\text{tor}}$$

as follows. For a fixed index i, choose a representative $\dot{\tau} \in \Gamma_{E_i/F}$ for each right coset $\tau \in \Gamma^{\dot{v}}_{E_i/F} \backslash \Gamma_{E_i/F}$ such that $\dot{\tau} = 1$ for the trivial coset, and then for $f = \sum_{w \in (S_i)_{E_i}} c_w[w] \in \overline{Y}[(S_i)_{E_i}, \dot{S}_i]_0$, set

$$l^i_v(f) = \sum_{\tau \in \Gamma^{\dot{v}}_{E_i/F} \backslash \Gamma_{E_i/F}} {}^{\dot{\tau}} c_{\tau^{-1}(\dot{v})} \in \overline{Y}.$$

With the construction of l^i_v in hand, the following result of [Kal18] (which is unchanged in our setting) shows that it provides the desired localization map:

Lemma 8.2.3 *The assignment $f \mapsto l^i_v(f)$ descends to a group homomorphism*

$$l^i_v \colon \frac{\overline{Y}[(S_i)_{E_i}, \dot{S}_i]_0}{I_{E_i/F} Y[(S_i)_{E_i}]_0} \to \frac{\overline{Y}}{I_v Y}$$

that is independent of the choices of representatives $\dot{\tau}$ and is compatible with the transition maps ! defined above.

Proof. See the proof of [Kal18, Lemma 3.7.2]. □

We may thus define the localization map l_v as the direct limit of the maps l^i_v. We can now give the statement of the global Tate-Nakayama isomorphism, following [Kal18, Theorem 3.7.3] in the number field case:

Theorem 8.2.4 *There exists a unique isomorphism*

$$\iota_{\dot{V}} \colon \overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,tor} \to H^1(\mathcal{E}_{\dot{V}}, Z \to T)$$

of functors $\mathcal{T} \to$ AbGrp that fits into the commutative diagram

$$\begin{array}{ccccc} Y[V_{\overline{F}}]_{0,\Gamma,tor} & \longrightarrow & \overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,tor} & \longrightarrow & A^\vee[\dot{V}]_0 \\ \downarrow{\scriptstyle TN} & & \downarrow{\scriptstyle \iota_{\dot{V}}} & & \downarrow \\ H^1(F,T) & \longrightarrow & H^1(\mathcal{E}_{\dot{V}}, Z \to T) & \longrightarrow & \mathit{Hom}_F(P_{\dot{V}}, Z), \end{array}$$

where TN denotes the colimit over i of the finite global Tate-Nakayama isomorphisms

$$H^{-1}(\Gamma_{E_i/F}, Y[(S_i)_{E_i}]_0) \to H^1(\Gamma_{E_i/F}, T(O_{E_i,S_i}))$$

first mentioned in Lemma 7.1.2 (these splice to give a well-defined map, by Lemma 3.1.2 and Corollary 3.1.8 from [Kal18]), and the right vertical arrow is the one from Lemma 7.2.6.

Moreover, for each $v \in \dot{V}$, the following diagram commutes

$$\begin{array}{ccc} \overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,tor} & \xrightarrow{l_v} & \overline{Y}_{+v,tor} \\ \downarrow \iota_{\dot{V}} & & \downarrow \iota_v \\ H^1(\mathcal{E}_{\dot{V}}, Z \to T) & \xrightarrow{loc_v} & H^1(\mathcal{E}_v, Z \to T). \end{array}$$

As in [Kal18], this theorem takes some work to prove. We start with some linear algebraic results which can be taken directly from [Kal18]. Although $\overline{Y}[(S_i)_{E_i}, \dot{S}_i]_0$ is not $\Gamma_{E_i/F}$-stable, it still makes sense to define the group $\overline{Y}[(S_i)_{E_i}, \dot{S}_i]_0^{N_{E_i/F}}$ as the intersection $\overline{Y}[(S_i)_{E_i}, \dot{S}_i]_0 \cap \overline{Y}[(S_i)_{E_i}]_0^{N_{E_i/F}}$.

Lemma 8.2.5 *We have the equality*

$$\frac{\overline{Y}[(S_i)_{E_i}, \dot{S}_i]_0^{N_{E_i/F}}}{I_{E_i/F}Y[(S_i)_{E_i}]_0} = \frac{\overline{Y}[(S_i)_{E_i}, \dot{S}_i]_0}{I_{E_i/F}Y[(S_i)_{E_i}]_0}[tor].$$

Proof. This is [Kal18, Lemma 3.7.6]. □

Lemma 8.2.6 *Every element of $\overline{Y}[(S_i)_{E_i}, \dot{S}_i]_0/I_{E_i/F}Y[(S_i)_{E_i}]_0$ has a representative supported on $\dot{S}_i$.*

Proof. This is [Kal18, Lemma 3.7.7]. □

Following the outline of [Kal18, §3.7], the first step is proving an analogous Tate-Nakayama isomorphism involving not $P_{\dot{V}}$, but the groups $P_{E_i,\dot{S}_i}$, which are defined as $\varprojlim_{n\in\mathbb{N}} P_{E_i,\dot{S}_i,n}$; note that an alternative description of $P_{\dot{V}}$ is as the limit $\varprojlim_i P_{E_i,\dot{S}_i}$; for more details, see [Kal18, §3.3]. Fix a triple $(E, S, \dot{S}_E)$ satisfying Conditions 7.2.1, since we will be focusing on only one fixed index i at first. Denote by $\mathcal{T}_E$ the full subcategory of objects $[Z \to T]$ of $\mathcal{T}$ such that T splits over E.

Note that $\check{H}^2(O_S^{\text{perf}}/O_S, P_{E,\dot{S}_E}) = H^2(O_{F,S}, P_{E,\dot{S}_E}) = \varprojlim_n H^2(O_{F,S}, P_{E,\dot{S}_E,n})$; the first equality is a straightforward exercise, and the second one follows from the vanishing of the derived limit $\varprojlim^{(1)} H^1(O_{F,S}, P_{E,\dot{S}_E,n})$ due to:

Lemma 8.2.7 *The groups $H^1(O_{F,S}, P_{E,\dot{S}_E,n})$ are finite for all finite n.*

Proof. Set $P = P_{E,\dot{S}_E,n}$. By [Čes16, Proposition 4.12] (and its proof), the natural map

$$H^1(O_{F,S}, P) \to H^1(\mathbb{A}_{F,S}, P) = \prod_{v\in S} H^1(F_v, P) \times \prod_{v\notin S} H^1(O_v, P)$$

has closed, discrete image and finite kernel, so it suffices to show that the image is finite. We claim that the right-hand side is compact—for this claim, it's enough by Tychonoff's theorem to prove that each $H^1(F_v, P)$ and $H^1(O_v, P)$ is compact. We showed this result for the former groups in Corollary 7.2.8; for the latter, note that [Čes16] (3.1.1) says that each subset $H^1(O_v, P) \subseteq H^1(F_v, P)$ may be canonically topologized so that this inclusion is open, and since $H^1(F_v, P)$ is profinite (and hence totally disconnected), it is also closed, and therefore compact. Now the result follows, since closed, discrete subspaces of compact spaces are finite. □

We thus have a canonical class $\xi_{E,\dot{S}_E} \in \check{H}^2(O_S^{\text{perf}}/O_S, P_{E,\dot{S}_E}) = \varprojlim_n \check{H}^2(O_S^{\text{perf}}/O_{F,S}, P_{E,\dot{S}_E,n})$ given by the inverse limit of the classes $\xi_{E,\dot{S}_E,n}$ defined just before our Lemma 7.2.3, which form a coherent system by that same lemma. By §2.3, the group $\check{H}^2(O_S^{\text{perf}}/O_S, P_{E,\dot{S}_E})$ is in bijective correspondence with isomorphism classes of $P_{E,\dot{S}_E}$-gerbes (over Sch$/O_{F,S}$) split over O_S^{perf}; fix such a gerbe $\mathcal{E}_{E,\dot{S}_E}$. For any $[Z \to T] \in \mathcal{T}_E$, the group $H^1(\mathcal{E}_{E,\dot{S}_E}, Z \to T)$ is defined identically as above. We have the usual inflation-restriction exact sequence

$$1 \to H^1(O_S, T) \to H^1(\mathcal{E}_{E,\dot{S}_E}, Z \to T) \to \text{Hom}_{O_{F,S}}(P_{E_i,\dot{S}_i}, Z) \to H^2(O_S, T),$$

where the last map is the composition of the direct limit of the maps $\Theta^P_{E,\dot{S}_E,n}$ defined by equation (7.3) with the natural map $H^2(O_{F,S}, Z) \to H^2(O_{F,S}, T)$.

Pick any 2-cochain $c_{E,S} \in [\text{Res}_{E/S}(\mathbb{G}_m)](O_{E,S}^{\bigotimes_{O_{F,S}} 3})$ lifting a choice of 2-cocycle

$$\overline{c_{E,S}} \in [\text{Res}_{E/S}(\mathbb{G}_m)/\mathbb{G}_m](O_{E,S}^{\bigotimes_{O_{F,S}} 3})$$

representing the Tate class discussed in §7.1, a cofinal system $\{n_i\}_{i \in \mathbb{N}}$ in $\mathbb{N}^\times$, as well as a system of compatible n_i-root maps

$$k_i \colon \text{Res}_{E/S}(\mathbb{G}_m)(O_S \otimes_{O_{F,S}} O_S \otimes_{O_{F,S}} O_{E,S}) \to \text{Res}_{E/S}(\mathbb{G}_m)(O_S^{\text{perf}} \otimes_{O_{F,S}} O_S^{\text{perf}} \otimes_{O_{F,S}} O_{E,S}),$$

as constructed in §7.1 (by "compatible," we mean in the sense discussed in §7.1). Recall that for id $\in \text{End}(M_{E,\dot{S}_E,n_i})^\Gamma$, we get that $\Psi_{E,S,n_i}(\text{id}) := \beta_i \in \text{Maps}(S_E, M^\vee_{E,\dot{S}_E,n_i})_0$ which is a -1-cocycle given by

$$\beta_i(w)\Big(\sum_{(\gamma,w) \in \Gamma_{E/F} \times S_E} c_{(\gamma,w)}[(\gamma, w)]\Big) = c_{(1,w)} \in \frac{1}{n_i}\mathbb{Z}/\mathbb{Z}.$$

Finally, we showed in §7.1 that the class $\xi_{E,\dot{S}_E,n_i}$ was represented by the *explicit* 2-cocycle

$$\dot{\xi}_{E,\dot{S}_E,n_i} := d(\overline{k_i(c_{E,S})}) \underset{O_{E,S}/O_{F,S}}{\sqcup} \beta_i,$$

where for $x \in [\mathrm{Res}_{E/S}(\mathbb{G}_m)](R)$, $\bar{x}$ denotes its image in $[\mathrm{Res}_{E/S}(\mathbb{G}_m)/\mathbb{G}_m](R)$.

Fact 3.2.3 from [Kal18] shows that for any finite Z, $A_Z^\vee := \mathrm{Hom}(X^*(Z), \mathbb{Q}/\mathbb{Z})$, and $n \mid m$ multiples of $\exp(Z)$, the map

$$\Phi_{E,S,n} \colon \mathrm{Maps}(S_E, A_Z^\vee)_0 \to \mathrm{Hom}(\frac{\mathrm{Res}_{E/S}(\mu_n)}{\mu_n}, Z)$$

constructed in §?? (and used to provide the pairing used in the above cup product) satisfies $\Phi_{E,S,m}(g) = \Phi_{E,S,n}(g) \circ (-)^{\frac{m}{n}}$, so that if $p_{i+1,i} \colon P_{E,\dot{S}_E,n_{i+1}} \to P_{E,\dot{S}_E,n_i}$ denotes the transition map (defined over $O_{F,S}$), we compute—using Lemmas 3.4.4 and 3.4.5 for Čech cup product computations—that

$$p_{i+1,i}(d(\overline{k_{i+1}(c_{E,S})}) \underset{O_{E,S}/O_{F,S}}{\sqcup} \beta_{i+1}) = p_{i+1,i}[d(\overline{k_{i+1}(c_{E,S})}) \underset{O_{E,S}/O_{F,S}}{\sqcup} \Phi_{E,S,n_{i+1}}(\beta_{i+1})] =$$

$$d(\overline{k_{i+1}(c_{E,S})}) \underset{O_{E,S}/O_{F,S}}{\sqcup} (p_{i+1,i} \circ \Phi_{E,S,n_{i+1}}(\beta_{i+1})) = d(\overline{k_{i+1}(c_{E,S})}) \underset{O_{E,S}/O_{F,S}}{\sqcup} (p_{i+1,i} \circ \Phi_{E,S,n_i}(\beta_{i+1}) \circ (-)^{\frac{n_{i+1}}{n_i}}),$$

and by functoriality in the argument Z this last expression can be rewritten as

$$d(\overline{k_{i+1}(c_{E,S})}) \underset{O_{E,S}/O_{F,S}}{\sqcup} (\Phi_{E,S,n_i}(p_{i+1,i}^\vee(\beta_{i+1})) \circ (-)^{\frac{n_{i+1}}{n_i}}) = d(\overline{k_{i+1}(c_{E,S})}) \underset{O_{E,S}/O_{F,S}}{\sqcup} (\Phi_{E,S,n_i}(\beta_i) \circ (-)^{\frac{n_{i+1}}{n_i}}).$$

Since the map $(-)^{\frac{n_{i+1}}{n_i}}$ is clearly defined over F, basic Čech cup product calculations (cf. §3.4) show that the above expression may be rewritten as

$$((-)^{\frac{n_{i+1}}{n_i}}[d(\overline{k_{i+1}(c_{E,S})})]) \underset{O_{E,S}/O_{F,S}}{\sqcup} (\Phi_{E,S,n_i}(\beta_i)) = d(\overline{k_i(c_{E,S})}) \underset{O_{E,S}/O_{F,S}}{\sqcup} (\Phi_{E,S,n_i}(\beta_i)) = \dot{\xi}_{E,\dot{S}_E,n_i},$$

showing that the system $\{\dot{\xi}_{E,\dot{S}_E,n_i}\}_i$ is a well-defined 2-cocycle valued in $P_{E,\dot{S}_E}((O_S^{\mathrm{perf}})^{\bigotimes_{O_{F,S}} 3})$, which we will denote by $\dot{\xi}_{E,\dot{S}_E}$. Note that the corresponding $P_{E,\dot{S}_E}$-gerbe $\mathcal{E}_{\dot{\xi}_{E,\dot{S}_E}}$ is split over O_S^{perf} and represents the canonical class in $\check{H}^2(O_S^{\mathrm{perf}}/O_S, P_{E,\dot{S}_E})$ discussed above (in the above notation, we can take $\mathcal{E}_{E,\dot{S}_E}$, which is not explicit, to be $\mathcal{E}_{\dot{\xi}_{E,\dot{S}_E}}$, which is explicit). It is straightforward to check that $\mathcal{E}_{\dot{\xi}_{E,\dot{S}_E}}$ with morphisms $\mathcal{E}_{\dot{\xi}_{E,\dot{S}_E}} \to \mathcal{E}_{\dot{\xi}_{E,\dot{S}_E,n_i}}$ induced by the projection maps $P_{E,\dot{S}_E} \to P_{E,\dot{S}_E,n_i}$ may be canonically identified with the inverse limit $\varprojlim_i \mathcal{E}_{\dot{\xi}_{E,\dot{S}_E,n_i}}$ of the explicit finite-level gerbes (cf. §2.5).

Lemma 8.2.8 *The pullback maps*

$$H^1(\mathcal{E}_{\dot{\xi}_{E,\dot{S}_E,n_i}}, T_{\mathcal{E}_{\dot{\xi}_{E,\dot{S}_E,n_i}}}) \to H^1(\mathcal{E}_{\dot{\xi}_{E,\dot{S}_E}}, T_{\mathcal{E}_{\dot{\xi}_{E,\dot{S}_E}}})$$

induce an isomorphism

$$\varinjlim_i H^1(\mathcal{E}_{\dot{\xi}_{E,\dot{S}_E,n_i}}, Z \to T) \xrightarrow{\sim} H^1(\mathcal{E}_{\dot{\xi}_{E,\dot{S}_E}}, Z \to T).$$

Proof. Using the equivalence of categories between $T_{\mathcal{E}_{\dot{\xi}_{E,\dot{S}_E}}}$-torsors and $\dot{\xi}_{E,\dot{S}_E}$-twisted T-torsors given by Proposition 2.4.10, it's enough to prove the corresponding statement for twisted torsors. Picking an O_S^{perf}-trivialization of any such torsor (X, ψ) (where, recall that X is a T-torsor over O_S^{perf} and ψ is the twisted gluing isomorphism $\psi\colon p_2^*X \to p_1^*X$), we may assume that $X = T_{O_S^{\text{perf}}}$ is the trivial torsor. The $P_{E,\dot{S}_E}$-action on X is defined by an $O_{F,S}$-homomorphism $\varphi\colon P_{E,\dot{S}_E} \to Z$, which factors through a homomorphism $\varphi_i\colon P_{E,\dot{S}_E,n_i} \to Z$ for some i, and our twisted gluing map ψ is equivalent to giving an element $x \in T(O_S^{\text{perf}} \otimes_{O_{F,S}} O_S^{\text{perf}})$ whose differential is $\varphi(\dot{\xi}_{E,\dot{S}_E}) = \varphi_i(\dot{\xi}_{E,\dot{S}_E,n_i})$. The data of φ_i and x thus defines a $\dot{\xi}_{E,\dot{S}_E,n_i}$-twisted T-torsor whose pullback is isomorphic to (X, ψ), as desired. □

For a fixed $[Z \to T]$ in $\mathcal{T}_E$, we set $\bar{T} := T/Z$, and recall our usual notation with cocharacter groups. Applying the (exact) functor $- \otimes_{\mathbb{Z}} \mathbb{Z}[S_E]_0$ to the exact sequence

$$0 \to Y \to \overline{Y} \to A^\vee \to 0$$

gives the short exact sequence

$$0 \to Y[S_E]_0 \to \overline{Y}[S_E]_0 \to A^\vee[S_E]_0 \to 0. \tag{8.2}$$

There is an obvious pairing of $O_{F,S}$-group schemes

$$\frac{\text{Res}_{E/S}(\mathbb{G}_m)}{\mathbb{G}_m} \times \underline{Y[S_E]_0} \to T, \tag{8.3}$$

where we are making the canonical identification of $\Gamma_{E/F}$-modules

$$Y[S_E]_0 = \text{Hom}(\frac{\text{Res}_{E/S}(\mathbb{G}_m)}{\mathbb{G}_m}, T).$$

Note that for i large enough so that $\exp(Z)$ divides n_i, for $g \in \overline{Y}[S_E]_0$, we have $n_i \cdot g \in Y[S_E]_0$ and the restriction of $n_i \cdot g$ to the subgroup $\text{Res}_{E/S}(\mu_{n_i})/\mu_{n_i}$ factors through the subgroup Z, and in fact gives the map

$$\frac{\text{Res}_{E/S}(\mu_{n_i})}{\mu_{n_i}} \to Z$$

given by $[g] \times - \to Z$, via the pairing $A^\vee[S_E]_0 \times \text{Res}_{E/S}(\mu_{n_i})/\mu_{n_i} \to Z$ induced by (7.1), where

$[g]$ denotes the image of g in $A^\vee[S_E]_0$ in the short exact sequence (8.2).

Define $A^\vee[\dot{S}_E]^{N_{E/F}}$ to be $A^\vee[S_E]_0^{N_{E/F}} \cap A^\vee[\dot{S}_E]_0$, which is in bijection with $\mathrm{Hom}(P_{E,\dot{S}_E}, Z)^\Gamma$ via the map $\Psi_{E,S}$ defined in Lemma 7.2.2. Following the linear algebraic situation for the group $P_{\dot{V}}$, define $\overline{Y}[S_E, \dot{S}_E]_0$ as the preimage of $A^\vee[\dot{S}_E]_0$ in $\overline{Y}[S_E]_0$, and set $\overline{Y}[S_E, \dot{S}_E]_0^{N_{E/F}} := \overline{Y}[S_E, \dot{S}_E]_0 \cap \overline{Y}[S_E]_0^{N_{E/F}}$. We are now ready to give the first version of the extended Tate-Nakayama isomorphism, which is the analogue of in [Kal18, Proposition 3.7.8]:

Proposition 8.2.9 *1. Given $\bar{\Lambda} \in \overline{Y}[S_E, \dot{S}_E]_0^{N_{E/F}}$ and i large enough so that exp(Z) divides n_i, we may define a $\dot{\xi}_{E,\dot{S}_E,n_i}$-twisted Čech 2-cocycle valued in T by the pair*

$$z_{\bar{\Lambda},i} := (\overline{k_i(c_{E,S})} \underset{O_{E,S}/O_{F,S}}{\sqcup} n_i\bar{\Lambda}, \Psi^{-1}_{E,S,n_i}([\bar{\Lambda}])),$$

where the unbalanced cup product is with respect to the pairing (8.3).

2. *The pullback $p^*_{i+1,i}(z_{\bar{\Lambda},i})$ coincides with the $\dot{\xi}_{E,\dot{S}_E,n_{i+1}}$-twisted cocycle $z_{\bar{\Lambda},i+1}$. Thus, pulling back any $z_{\Lambda,i}$ to $\mathcal{E}_{\dot{\xi}_{E,\dot{S}_E}}$ defines the same $\dot{\xi}_{E,\dot{S}_E}$-twisted cocycle, denoted by $z_{\bar{\Lambda}}$.*

3. *The assignment $\bar{\Lambda} \mapsto z_{\bar{\lambda}}$ defines an isomorphism*

$$\iota_{E,\dot{S}_E} \colon \frac{\overline{Y}[S_E, \dot{S}_E]_0^{N_{E/F}}}{I_{E/F}Y[S_E]_0} \to H^1(\mathcal{E}_{\dot{\xi}_{E,\dot{S}_E}}, Z \to T)$$

which is functorial in $[Z \to T] \in \mathcal{T}_E$ and makes the following diagram commute:

$$\begin{array}{ccc}
1 & & 1 \\
\downarrow & & \downarrow \\
\widehat{H}^{-1}(\Gamma_{E/F}, Y[S_E]_0) & \xrightarrow{TN} & H^1(O_{F,S}, T) \\
\downarrow & & \downarrow \\
\frac{\overline{Y}[S_E,\dot{S}_E]_0^{N_{E/F}}}{I_{E/F}Y[S_E]_0} & \xrightarrow{\iota_{E,\dot{S}_E}} & H^1(\mathcal{E}_{\dot{\xi}_{E,\dot{S}_E}}, Z \to T) \\
\downarrow & & \downarrow \\
A^\vee[\dot{S}_E]^{N_{E/F}} & \xrightarrow{\Psi^{-1}_{E,S}} & \mathit{Hom}(P_{E,\dot{S}_E}, Z)^\Gamma \\
\downarrow & & \downarrow \\
\widehat{H}^0(\Gamma_{E/F}, Y[S_E]_0) & \xrightarrow{-TN} & H^2(O_{F,S}, T).
\end{array}$$

Proof. The proof will follow the same outline as the analogous one in [Kal18]. Proving the first claim just means showing, for fixed large enough i, the equality

$$d(\overline{k_i(c_{E,S})} \underset{O_{E,S}/O_{F,S}}{\sqcup} n_i\bar{\Lambda}) = \Psi^{-1}_{E,S,n_i}([\bar{\Lambda}])(\dot{\xi}_{E,\dot{S}_E,n_i}).$$

Viewing $n_i\bar{\Lambda}$ as a -1-cochain, we see that $d(n_i\bar{\Lambda}) = 0$, since by construction $\bar{\Lambda}$ is killed by $N_{E/F}$. Hence, it follows from Proposition 3.4.3 that

$$d(\overline{k_i(c_{E,S})} \underset{O_{E,S}/O_{F,S}}{\sqcup} n_i\bar{\Lambda}) = d[\overline{k_i(c_{E,S})}] \underset{O_{E,S}/O_{F,S}}{\sqcup} n_i\bar{\Lambda}, \tag{8.4}$$

and now since $d[\overline{k_i(c_{E,S})}]$ lies in the subgroup $[\text{Res}_{E/S}(\mu_{n_i})/\mu_{n_i}]((O_S^{\text{perf}})^{\bigotimes_{O_{F,S}} 3})$ and we know that the restriction of $n_i\bar{\Lambda}$ to $\text{Res}_{E/S}(\mu_{n_i})/\mu_{n_i}$ is equal to $\Phi_{E,S,n_i}([\bar{\Lambda}])$, we can rewrite the right-hand term of (8.4) as $d[\overline{k_i(c_{E,S})}] \underset{O_{E,S}/O_{F,S}}{\sqcup} \Phi_{E,S,n_i}([\bar{\Lambda}])$. By functoriality, this term can be rewritten as $d[\overline{k_i(c_{E,S})}] \underset{O_{E,S}/O_{F,S}}{\sqcup} \Phi_{E,S,n_i}(\Psi^{-1}_{E,S,n_i}([\bar{\Lambda}])^\vee(\beta_i))$, which again by functoriality may further be expressed as

$$d[\overline{k_i(c_{E,S})}] \underset{O_{E,S}/O_{F,S}}{\sqcup} \Psi^{-1}_{E,S,n_i}([\bar{\Lambda}]) \circ \Phi_{E,S,n_i}(\beta_i) = \Psi^{-1}_{E,S,n_i}([\bar{\Lambda}])(d[\overline{k_i(c_{E,S})}] \underset{O_{E,S}/O_{F,S}}{\sqcup} \Phi_{E,S,n_i}(\beta_i)),$$

where to obtain the above equality we are using the fact $\Psi^{-1}_{E,S,n_i}([\bar{\Lambda}])$ is $\Gamma_{E/F}$-fixed to apply Lemma 3.4.5. But now by definition this last term equals $\Psi^{-1}_{E,S,n_i}([\bar{\Lambda}])(\dot{\xi}_{E,\dot{S}_E,n_i})$, as desired.

We now move to the second claim of the proposition. The first step is noting that $p_{i+1,i} \circ \Psi^{-1}_{E,S,n_{i+1}}([\bar{\Lambda}]) = \Psi^{-1}_{E,S,n_i}([\bar{\Lambda}])$, since, as discussed in Lemma 7.2.2, the maps $\Psi_{E,S,n}$ are compatible with the projection maps for the system $\{P_{E,\dot{S}_E,n_i}\}_i$. Moreover, we have by the $\mathbb{Z}$-bilinearity of the unbalanced cup product and coherence of the system of maps $\{k_i\}_i$ that

$$\overline{k_{i+1}(c_{E,S})} \underset{O_{E,S}/O_{F,S}}{\sqcup} n_{i+1}\bar{\Lambda} = \overline{k_{i+1}(c_{E,S})} \underset{O_{E,S}/O_{F,S}}{\sqcup} (\frac{n_{i+1}}{n_i})[n_i\bar{\Lambda}] = \overline{k_i(c_{E,S})} \underset{O_{E,S}/O_{F,S}}{\sqcup} n_i\bar{\Lambda},$$

concluding the proof of the second claim.

It is clear that the map $\bar{\Lambda} \mapsto z_{\bar{\Lambda}}$ defines a functorial homomorphism from $\overline{Y}[S_E, \dot{S}_E]_0^{N_{E/F}}$ to $H^1(\mathcal{E}_{\dot{\xi}_{E,\dot{S}_E}}, Z \to T)$. Moreover, if $\bar{\Lambda}$ lies in the subgroup $Y[S_E]_0$, then we have, first of all, that $[\bar{\Lambda}]$ vanishes in $A^\vee[S_E]_0$, so that the homomorphism associated to $z_{\bar{\Lambda}}$ is trivial. By $\mathbb{Z}$-bilinearity and the fact that already $\bar{\Lambda} \in Y[S_E]_0$, the associated twisted cocycle (which is, by the previous line, an actual cocycle) is given by $\overline{c_{E,S}} \underset{O_{E,S}/O_{F,S}}{\sqcup} \bar{\Lambda}$, which, since $\overline{c_{E,S}}$ is valued in the finite étale (Galois) extension $O_{E,S}/O_{F,S}$, Proposition 3.4.1 and [Kal16, §4.3] tell us that (after applying the appropriate comparison isomorphisms) this cup product may be computed as the usual Galois-cohomological cup product $\overline{c_{E,S}} \cup \bar{\Lambda}$, which sends all of $I_{E/F}Y[S_E]_0$ to 1-coboundaries, showing that the above map induces a functorial homomorphism

$$\frac{\overline{Y}[S_E, \dot{S}_E]_0^{N_{E/F}}}{I_{E/F}Y[S_E]_0} \to H^1(\mathcal{E}_{\dot{\xi}_{E,\dot{S}_E}}, Z \to T),$$

as asserted. This argument also shows that the top square in the diagram of the proposition commutes. The commutativity of the middle square is by construction, and the final square commutes by the diagram in Lemma 7.1.2. Since all horizontal maps in the diagram apart from $i_{E,\dot{S}_E}$ are isomorphisms, it is an isomorphism as well by the five-lemma. □

The issue now is that, given our non-canonical explicit gerbe $\mathcal{E}_{\dot{\xi}_{E,\dot{S}_E}}$, it is not clear that such an isomorphism will be canonical, or even that the groups $H^1(\mathcal{E}_{E,\dot{S}_E}, Z \to T)$ are canonical. The following result addresses these concerns:

Proposition 8.2.10 *The group $H^1(\mathcal{E}_{E,\dot{S}_E}, Z \to T)$ is independent of the choice of gerbe $\mathcal{E}_{E,\dot{S}_E}$ up to unique isomorphism, and is equipped with a canonical functorial isomorphism $\iota_{E,\dot{S}_E}$ to the group*

$$\frac{\overline{Y}[S_E, \dot{S}_E]_0^{N_{E/F}}}{I_{E/F}Y[S_E]_0}$$

that fits into the commutative diagram of Proposition 8.2.9.

Proof. The map $\iota_{E,\dot{S}_E}$ is obtained by composing an isomorphism (which the proposition asserts is unique) $H^1(\mathcal{E}_{E,\dot{S}_E}, Z \to T) \to H^1(\mathcal{E}_{\dot{\xi}_{E,\dot{S}_E}}, Z \to T)$ induced by any isomorphism of $P_{E,\dot{S}_E}$-gerbes $\mathcal{E}_{E,\dot{S}_E} \xrightarrow{\sim} \mathcal{E}_{\dot{\xi}_{E,\dot{S}_E}}$ and then applying $i_{E,\dot{S}_E}$ from Proposition 8.2.9.

This proposition requires work to show, but all the necessary arguments are done in [Kal18, §3.7]. The main ingredient is Lemma 3.7.10 loc. cit., which is purely group-theoretic and carries over to our setting unchanged (in the statement of that Lemma, eliminate the use of S and replace $\mathbb{N}_S$ by $\mathbb{N}$). Once this result is known, [Kal18, Corollary 3.7.11] proves the proposition. The proof of this corollary relies on Lemma 3.7.9 loc. cit., which holds in our setting with $\mathbb{N}_S$ replaced by $\mathbb{N}$, Proposition 3.7.8 loc. cit., which is our Proposition 8.2.9, and the finiteness of $H^1(O_{F,S}, T)$, which is true in our setting as well. □

Note that, in particular, the isomorphism $\iota_{E,\dot{S}_E}$ does not depend on the choice of cochain $c_{E,S}$ lifting a representative of the canonical Tate class in $H^2(O_{F,S}, \mathrm{Res}_{E,S}(\mathbb{G}_m)/\mathbb{G}_m)$ which was used to construct the explicit gerbes $\mathcal{E}_{\dot{\xi}_{E,\dot{S}_{E,n_i}}}$ and the isomorphism $i_{E,\dot{S}_E}$ in Proposition 8.2.9.

In order to extend the isomorphism of Proposition 8.2.9 to $\mathcal{E}_{\dot{V}}$, we need to investigate what happens as we vary the extension E/F. As such, let K/F be a finite Galois extension containing E, and $(S'\dot{S}'_K)$ be a pair satisfying Conditions 7.2.1. We may assume that $S \subset S'$ and $\dot{S}_E \subset (\dot{S}'_K)_E$. Let $\mathcal{E}_{K,\dot{S}'_K}$ and $\mathcal{E}_{E,\dot{S}_E}$ be gerbes corresponding to the canonical classes

$\xi_{K,\dot{S}'_K} \in \check{H}^2(O_{S'}^{\text{perf}}/O_{F,S'}, P_{K,\dot{S}'_K})$ and $\xi_{E,\dot{S}_E} \in \check{H}^2(O_S^{\text{perf}}/O_{F,S}, P_{E,\dot{S}_E})$, respectively. The first step is to construct an inflation map

$$\text{Inf}\colon H^1(\mathcal{E}_{E,\dot{S}_E}, Z \to T) \to H^1(\mathcal{E}_{K,\dot{S}'_K}, Z \to T).$$

We begin by pulling back $\mathcal{E}_{E,\dot{S}_E}$, which we recall is a $P_{E,\dot{S}_E}$-gerbe over $O_{F,S}$ that is split over O_S^{perf}, to the $(P_{E,\dot{S}_E})_{O_{F,S'}}$-gerbe $\mathcal{E}_{E,\dot{S}_E} \times_{\text{Sch}/O_{F,S}} (\text{Sch}/O_{F,S'}) =: (\mathcal{E}_{E,\dot{S}_E})_{O_{F,S'}}$, which is split over $O_{F,S'} \cdot O_S^{\text{perf}}$ (taken inside $\overline{F}$), contained in $O_{S'}^{\text{perf}}$. It is straightforward to check that the Čech cohomology class in $\check{H}^2(O_{S'}^{\text{perf}}/O_{F,S'}, (P_{E,\dot{S}_E})_{O_{F,S'}})$ corresponding to $(\mathcal{E}_{E,\dot{S}_E})_{O_{F,S'}}$ is the image of $\xi_{E,\dot{S}_E}$ under the obvious morphism of Čech cohomology groups. We have a projection map $P_{K,\dot{S}'_K} \to (P_{E,\dot{S}_E})_{O_{F,S'}}$ given by the inverse limit of the finite-level projection maps, which on degree-2 Čech cohomology groups, by Lemma 7.2.5, sends $\xi_{K,\dot{S}'_K}$ to the image of $\xi_{E,\dot{S}_E}$. Using this equality of cocycles, picking normalizations of $\mathcal{E}_{E,\dot{S}_E}$ and $\mathcal{E}_{K,\dot{S}'_K}$ and using Construction 2.3.4 allows us to construct a (non-canonical) morphism of stacks over $O_{F,S'}$ from $\mathcal{E}_{K,\dot{S}'_K}$ to $(\mathcal{E}_{E,\dot{S}_E})_{O_{F,S'}}$. By pulling back torsors via the composition of functors

$$\mathcal{E}_{K,\dot{S}'_K} \to (\mathcal{E}_{E,\dot{S}_E})_{O_{F,S'}} \to \mathcal{E}_{E,\dot{S}_E},$$

we get the desired inflation map.

The map we just constructed from $H^1(\mathcal{E}_{E,\dot{S}_E}, Z \to T)$ to $H^1(\mathcal{E}_{K,\dot{S}'_K}, Z \to T)$ is evidently functorial in $[Z \to T] \in \mathcal{T}_E$, but (since we had to choose normalizations of gerbes as well as a 1-coboundary) it is not a priori clear that it is canonical. The following result addresses this issue, and is taken directly from [Kal18]:

Proposition 8.2.11 *The inflation map constructed above is independent of the choice of functor $\mathcal{E}_{K,\dot{S}'_K} \to \mathcal{E}_{E,\dot{S}_E}$, injective, functorial in $[Z \to T] \in \mathcal{T}_E$, and fits into the two commutative diagrams below:*

$$\begin{array}{ccc} H^1(\mathcal{E}_{E,\dot{S}_E}, Z \to T) & \xrightarrow{\text{Inf}} & H^1(\mathcal{E}_{K,\dot{S}'_K}, Z \to T) \\ \Big\uparrow{\scriptstyle \iota_{E,\dot{S}_E}} & & \Big\uparrow{\scriptstyle \iota_{K,\dot{S}'_K}} \\ \dfrac{\overline{Y}[S_E,\dot{S}_E]_0^{N_{E/F}}}{I_{E/F}Y[S_E]_0} & \xrightarrow{\quad ! \quad} & \dfrac{\overline{Y}[S'_K,\dot{S}'_K]_0^{N_{K/F}}}{I_{K/F}Y[S'_K]_0}, \end{array}$$

and

$$\begin{array}{ccc}
1 & & 1 \\
\downarrow & & \downarrow \\
H^1(O_{F,S},T) & \xrightarrow{\text{Inf}} & H^1(O_{F,S'},T) \\
\downarrow & & \downarrow \\
H^1(\mathcal{E}_{E,\dot{S}_E}, Z \to T) & \xrightarrow{\text{Inf}} & H^1(\mathcal{E}_{K,\dot{S}'_K}, Z \to T) \\
\downarrow & & \downarrow \\
\mathit{Hom}(P_{E,\dot{S}_E}, Z)^\Gamma & \longrightarrow & \mathit{Hom}(P_{K,\dot{S}'_K}, Z)^\Gamma.
\end{array}$$

Proof. The commutativity of the second diagram is by construction. For injectivity, note that the homomorphism $\mathrm{Hom}(P_{E,\dot{S}_E}, Z)^\Gamma \to \mathrm{Hom}(P_{K,\dot{S}'_K}, Z)^\Gamma$ is injective, since it's given by the homomorphism $\mathrm{Hom}(A, M_{E,\dot{S}_E})^\Gamma \to \mathrm{Hom}(A, M_{K,\dot{S}'_K})^\Gamma$ induced by the $M_{E,\dot{S}_E} \to M_{K,\dot{S}'_K}$ which is given as the direct limit of injective maps, and is thus itself injective. Moreover, the inflation map $H^1(O_{F,S},T) \to H^1(O_{F,S'},T)$ is injective by [Kal18, Lemma 3.1.10], which works in our setting via étale-to-group cohomology comparison discussed in §3.2. Now the desired injectivity follows from the second diagram and basic diagram-chasing. The rest of the proposition follows from the argument given in [Kal18] for the proof of Proposition 3.7.12 loc. cit. □

Recall the exhaustive tower of finite Galois extensions E_i/F and pairs $(S_i, \dot{S}_i)$ satisfying Conditions 7.2.1 and the inclusions $S_i \subset S_{i+1}$ and $\dot{S}_i \subset (\dot{S}_{i+1})_{E_i}$. For any $P_{E_i,\dot{S}_i}$-gerbe $\mathcal{E}_i$ over O_{F,S_i}, split over $O_{S_i}^{\mathrm{perf}}$, representing the Čech 2-cocycle $\xi_{E_i,\dot{S}_i}$, we first get the $(P_{E_i,\dot{S}_i})_F$-gerbe $(\mathcal{E}_i)_F \to \mathcal{E}_i$ over F, split over $\overline{F}$; note that the gerbe $(\mathcal{E}_i)_F$ corresponds to the Čech cohomology class given by the image of $\xi_{E_i,\dot{S}_i}$ in $\check{H}^2(\overline{F}/F, (P_{E_i,\dot{S}_i})_F)$. By construction of the canonical class ξ, the image of ξ in $\check{H}^2(\overline{F}/F, (P_{E_i,\dot{S}_i})_F)$ equals this image of $\xi_{E_i,\dot{S}_i}$. Thus, after normalizing the gerbes $\mathcal{E}_{\dot{V}}$ and $\mathcal{E}_i$ and choosing a coboundary, we get a functor $\mathcal{E}_{\dot{V}} \to \mathcal{E}_i$, and thus by pullback a group homomorphism

$$\mathrm{Inf}\colon H^1(\mathcal{E}_i, Z \to T) \to H^1(\mathcal{E}_{\dot{V}}, Z \to T).$$

Proposition 8.2.12 *The above inflation maps splice together to give a canonical isomorphism of functors $\mathcal{T} \to$ AbGrp:*

$$\varinjlim_i H^1(\mathcal{E}_i, Z \to T) \to H^1(\mathcal{E}_{\dot{V}}, Z \to T).$$

Proof. Following the structure of the proof of [Kal18, Proposition 3.7.13], the first step is showing that each inflation map is injective. This follows from an identical argument as in Proposition 8.2.11, replacing $O_{F,S'}$ with F and $\mathcal{E}_{K,\dot{S}'_K}$ with $\mathcal{E}_{\dot{V}}$, using that $P_{\dot{V}} \to P_{E_i,\dot{S}_i}$ is surjective and, again from [Kal18, Lemma 3.1.10], that the inflation map $H^1(O_{F,S_i},T) \to H^1(F,T)$ is injective.

Then the argument in the proof of [Kal18, Proposition 3.7.13] shows that each inflation map is independent of gerbe normalizations and choice of coboundary (and is thus canonical). From here, the rest of the argument in [Kal18] carries over verbatim to our situation (this argument uses Lemmas 3.7.10 and 3.1.10 loc. cit, which, as we have argued, are true in the global function field setting) □

We are now in a position to prove Theorem 8.2.4. We obtain the functorial isomorphism $\iota_{\dot{V}}$ by first (using Lemma 8.2.5) applying the functorial isomorphism $\overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,\text{tor}} \xrightarrow{\sim} \varinjlim_i \frac{\overline{Y}[(S_i)_{E_i}, \dot{S}_i]_0^{N_{E_i/F}}}{I_{E_i/F}Y[(S_i)_E]_0}$, then taking the functorial isomorphism

$$\varinjlim_i \iota_{E_i,\dot{S}_i} \colon \varinjlim_i \frac{\overline{Y}[(S_i)_{E_i}, \dot{S}_i]_0^{N_{E_i/F}}}{I_{E_i/F}Y[(S_i)_E]_0} \to \varinjlim_i H^1(\mathcal{E}_i, Z \to T),$$

which is canonical and well-defined by Proposition 8.2.11, and then finally applying the canonical identification $\varinjlim_i H^1(\mathcal{E}_i, Z \to T) \to H^1(\mathcal{E}_{\dot{V}}, Z \to T)$ of Proposition 8.2.12. Applying the direct limit functor to the diagram of Proposition 8.2.9 (and using Proposition 8.2.10) gives the commutativity of the first diagram in Theorem 8.2.4—the fact that we can apply the direct limit functor to this diagram is a consequence of Proposition 8.2.11. Now the uniqueness of $\iota_{\dot{V}}$ making the first diagram commute, as well as the commutativity of the second diagram, both follow from the abstract framework of [Kal18, Lemma 3.7.10], as explained in the proof of Theorem 3.7.3 loc. cit.

We conclude this subsection by collecting some local-to-global consequences of Theorem 8.2.4.

Corollary 8.2.13 *We have the following commutative diagram with exact bottom row*

$$\begin{array}{ccccc} H^1(\mathcal{E}_{\dot{V}}, Z \to T) & \xrightarrow{(loc_v)_v} & \bigoplus_{v\in\dot{V}} H^1(\mathcal{E}_v, Z \to T) & & \\ \uparrow{\scriptstyle \iota_{\dot{V}}} & & \uparrow{\scriptstyle (\iota_v)_v} & & \\ \overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,tor} & \xrightarrow{(l_v)_v} & \bigoplus_{v\in\dot{V}} \overline{Y}_{+v,tor} & \xrightarrow{\Sigma} & \frac{\overline{Y}}{IY}[tor]. \end{array}$$

Proof. This follows from the proof of [Kal18, Corollary 3.7.4] (the argument loc. cit. relies on that paper's analogue of Theorem 8.2.4 and arguments involving the bottom-row, which are purely Galois-cohomological and thus are unchanged in our setting). □

Corollary 8.2.14 *Let $[Z \to G] \in \mathcal{A}$ with connected reductive G and $x \in H^1(\mathcal{E}_{\dot{V}}, Z \to G)$. Then $loc_v(x)$ is the neutral element in $H^1(\mathcal{E}_v, Z \to G)$ for almost all $v \in \dot{V}$.*

Proof. As explained in [Kal18], this is a consequence of finding an element in $H^1(\mathcal{E}_{\dot{V}}, Z \to T)$ for some maximal torus T which maps to x (possible by Lemma 8.1.2), deducing the result for this element of $H^1(\mathcal{E}_{\dot{V}}, Z \to T)$ using the previous corollary, and then invoking the functoriality of our localization maps. □

8.3 Extending to reductive groups

Let $\mathcal{R}$ denote the full subcategory of $\mathcal{A}$ consisting of objects $[Z \to G]$ where G is a connected reductive group over F. In the corresponding section (§3.8) of [Kal18], it is necessary for duality theorems to replace the sets $H^1(\mathcal{E}_{\dot{V}}, Z \to G)$ with a quotient, denoted by $H^1_{\text{ab}}(\mathcal{E}_{\dot{V}}, Z \to G)$. However, in our case, due to the vanishing of $H^1(F, G)$ for all simply-connected (semi-simple) connected groups G over F (which is is an immediate consequence of [Tha08, Theorem 2.4]), this replacement will not be necessary for us.

The first step in extending Theorem 8.2.4 to $\mathcal{R}$ is defining an analogue of the linear algebraic data $\overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,\text{tor}}([Z \to T])$ for $[Z \to T] \in \mathcal{T}$. For a maximal F-torus T of G, define the abelian group

$$\varinjlim_{(E, S_E, \dot{S}_E)} \frac{[X_*(T/Z)/X_*(T_{\text{sc}})][S_E, \dot{S}_E]_0^{N_{E/F}}}{I_{E/F}([X_*(T)/X_*(T_{\text{sc}})][S_E]_0)},$$

where the colimit is over any cofinal system of triples $(E, S_E, \dot{S}_E)$, where E/F is a finite Galois extension splitting T and the pair $(S_E, \dot{S}_E)$ satisfies Conditions 7.2.1; the transition maps are given by the map ! defined in the previous section. The only term appearing in this colimit that we need to define is $[X_*(T/Z)/X_*(T_{\text{sc}})][S_E, \dot{S}_E]_0$, which we take to be those elements of $[X_*(T/Z)/X_*(T_{\text{sc}})][S_E]_0$ such that if $w \notin \dot{S}_E$, then $c_w \in X_*(T)/X_*(T_{\text{sc}})$ (as usual, the superscript $N_{E/F}$ denotes those elements which are killed by the E/F-norm).

Now for two such tori T_1, T_2, we can define a map

$$\varinjlim \frac{[X_*(T_1/Z)/X_*(T_{1,\text{sc}})][S_E, \dot{S}_E]_0^{N_{E/F}}}{I_{E/F}([X_*(T_1)/X_*(T_{1,\text{sc}})][S_E]_0)} \to \varinjlim \frac{[X_*(T_2/Z)/X_*(T_{2,\text{sc}})][S_E, \dot{S}_E]_0^{N_{E/F}}}{I_{E/F}([X_*(T_2)/X_*(T_{2,\text{sc}})][S_E]_0)} \tag{8.5}$$

as follows. By [Kal16, Lemma 4.2], for any $g \in G(F^{\text{sep}})$ such that $\text{Ad}(g)(T_1) = T_2$, we get an isomorphism $X_*(T_1/Z)/X_*(T_{1,\text{sc}}) \to X_*(T_2/Z)/X_*(T_{2,\text{sc}})$ which is independent of the choice of g, and is thus Γ-equivariant. It follows that $\text{Ad}(g)$ also induces the desired homomorphism (8.5) on direct limits. We then define a functor $\mathcal{R} \to \text{AbGrp}$ given by

$$\overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,\text{tor}}([Z \to G]) := \varinjlim [\varinjlim_{(E, S_E, \dot{S}_E)} \frac{[X_*(T/Z)/X_*(T_{\text{sc}})][S_E, \dot{S}_E]_0^{N_{E/F}}}{I_{E/F}([X_*(T)/X_*(T_{\text{sc}})][S_E]_0)}],$$

where the outer colimit is over all maximal F-tori T of G via the maps constructed above. It is clear that this extends the functor $\overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,\text{tor}}$ constructed in the previous section for $\mathcal{T} \subset \mathcal{R}$, so our notation is justified. In what follows, we will always take our colimits over the fixed cofinal system $(E_i, S_i, \dot{S}_i)$ constructed above (such a system eventually splits any F-torus T).

We can now prove an extended duality theorem:

Theorem 8.3.1 *The isomorphism of functors $\iota_{\dot{V}}$ from Theorem 8.2.4 extends to a unique isomorphism of functors (valued in pointed sets) on $\mathcal{R}$, also denoted by $\iota_{\dot{V}}$, from $\overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,tor}$ to $H^1(\mathcal{E}_{\dot{V}}, -)$.*

Proof. Fix $[Z \to G]$ in $\mathcal{R}$ and T a maximal F-torus of T. We claim that the fibers of the composition

$$\overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,\text{tor}}([Z \to T]) \xrightarrow{\iota_{\dot{V}}} H^1(\mathcal{E}_{\dot{V}}, Z \to T) \to H^1(\mathcal{E}_{\dot{V}}, Z \to G)$$

are torsors under the image of

$$Y[V_{\overline{F}}]_{0,\Gamma,\text{tor}}(T_{\text{sc}}) \to \overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,\text{tor}}([Z \to T]).$$

By twisting, it's enough to prove this for the fiber over the class of the trivial torsor in the pointed set $H^1(\mathcal{E}_{\dot{V}}, Z \to G)$. That the image of an element x lands in this class means that it lies in the subset $H^1(F, G)$ of the right-most term, and hence its image in the middle term lies in $H^1(F, T)$; this already means that $x \in Y[V_{\overline{F}}]_{0,\Gamma,\text{tor}}(T)$. Moreover, the image of x in $H^1(F, T)$ lies in the fiber over the neutral class for the map $H^1(F, T) \to H^1(F, G)$. We have the commutative diagram of pointed sets with exact rows

$$\begin{array}{ccccc} (G/T)(F) & \longrightarrow & H^1(F, T) & \longrightarrow & H^1(F, G) \\ \uparrow & & \uparrow & & \uparrow \\ (G_{\text{sc}}/T_{\text{sc}})(F) & \longrightarrow & H^1(F, T_{\text{sc}}) & \longrightarrow & H^1(F, G_{\text{sc}}), \end{array}$$

and since the natural map $G_{\text{sc}}/T_{\text{sc}} \to G/T$ is an isomorphism (of F-schemes, not groups), we may lift the image of x in $H^1(F, T)$ to an element $x_{\text{sc}} \in H^1(F, T_{\text{sc}})$. Now the claim is clear by the functoriality of Tate-Nakayama duality for tori.

The above claim immediately implies that we have an injective map

$$\frac{\overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,\text{tor}}([Z \to T])}{\text{Im}[Y[V_{\overline{F}}]_{0,\Gamma,\text{tor}}(T_{\text{sc}})]} \to H^1(\mathcal{E}_{\dot{V}}, Z \to G).$$

Arguments involving cocharacter modules (see [Kal18], proof of Theorem 3.8.1) show that the image $\text{Im}[Y[V_{\overline{F}}]_{0,\Gamma,\text{tor}}(T_{\text{sc}})]$ is exactly the kernel of the natural map

$$\overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,\text{tor}}([Z \to T]) \to \overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,\text{tor}}([Z \to G]),$$

and so, putting the above two observations together, we have a natural inclusion

$$\mathrm{Im}(\overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,\mathrm{tor}}([Z \to T]) \twoheadrightarrow \overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,\mathrm{tor}}([Z \to G])) \hookrightarrow H^1(\mathcal{E}_{\dot{V}}, Z \to G).$$

Now note that any two elements of $\overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,\mathrm{tor}}([Z \to G])$ lie in the image of the group $\overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,\mathrm{tor}}([Z \to T])$ for some maximal F-torus $T \subset G$. The analogous argument using elliptic maximal tori (over the local fields F_v) in the proof of [Kal18, Theorem 3.8.1] works for us, once we replace [PR94, Corollary 7.3] with [Tha13, Lemma 3.6.1], using that $H^2(F_v, T'_{\mathrm{sc}})$ vanishes for any F_v-anisotropic maximal torus T'_{sc} (by Tate-Nakayama duality), and the fact that the map $H^2(F, T'_{\mathrm{sc}}) \to \prod_{v \in V_F} H^2(F_v, T'_{\mathrm{sc}})$ is injective whenever there exists a place $v \in S$ such that $(T'_{\mathrm{sc}})_{F_v}$ is an F_v-anisotropic maximal torus in a connected semisimple group G_{sc} (see [PR94, Proposition 6.12], the proof of which works for function fields).

We now claim that if $x_i \in \overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,\mathrm{tor}}([Z \to T_i])$ for $i = 1, 2$ map to the same element in $\overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,\mathrm{tor}}([Z \to G])$, then their images $\iota_{\dot{V}}(x_i) \in H^1(\mathcal{E}_{\dot{V}}, Z \to T_i)$ map to the same element of $H^1(\mathcal{E}_{\dot{V}}, Z \to G)$. We show this explicitly: Choose j large enough so that E_j splits T_i, and such that $\exp(Z) \mid n_j$, x_i comes from $\bar{\Lambda}_i \in \overline{Y}_i[(S_j)_{E_j}, \dot{S}_j]_0^{N_{E_j/F}}$; choose also a lift $c_{E_j,S_j} \in [\mathrm{Res}_{E_j,S_j}(\mathbb{G}_m)/\mathbb{G}_m](O_{E_j,S}^{\bigotimes_{O_{F,S}} 3})$ of a 2-cocycle representative of the global Tate class (and n_j-root maps k_j as constructed in §7.1). Denote by $\dot{\mathcal{E}}_j$ the explicit gerbe $\mathcal{E}_{\xi_{E_j,\dot{S}_j}}$ defined in §8.2, similarly with $\dot{\mathcal{E}}_{j,n}$ for $n \in \mathbb{N}$. We know that $i_{\dot{V}}(x_i)$ is represented in $H^1(\dot{\mathcal{E}}_j, Z \to T_i)$ by the pullback of the twisted 2-cocycles $z_{\bar{\Lambda}_i,n_j}$ (defined in §8.2 — to get an element of $H^1(\mathcal{E}_{\dot{V}}, Z \to T_i)$, use the canonical inflation map from $H^1(\dot{\mathcal{E}}_j, Z \to T_i)$), denoted by $z_{\bar{\Lambda}_i}$. We want to show that the twisted 2-cocycles $z_{\bar{\Lambda}_i}$ give the same class of torsor in $H^1(\mathcal{E}_{\dot{V}}, Z \to G)$; since any choice of functor $\mathcal{E}_{\dot{V}} \to \dot{\mathcal{E}}_j$ factors through $(\dot{\mathcal{E}}_j)_{\overline{F}}$ (by construction), it is enough to show that $z_{\bar{\Lambda}_i}$ are equivalent as twisted 2-cocycles in $H^1((\dot{\mathcal{E}}_j)_{\overline{F}}, Z \to G)$; due to the fact that by construction both $z_{\bar{\Lambda}_i}$ are pulled back from $H^1((\dot{\mathcal{E}}_{j,n_j})_{\overline{F}}, Z \to G)$, it's enough to show the equality of $z_{\bar{\Lambda}_1,n_j}$ and $z_{\bar{\Lambda}_2,n_j}$ in the latter cohomology set. The next part of the argument is essentially the proof of Lemma 4.5.9.

It is clear that the images $[\bar{\Lambda}_i] \in \mathrm{Hom}(A, M_{E_j,S_j,n_j})^{\Gamma} = A^{\vee}[(S_j)_{E_j}]_0^{N_{E_j/F}}$ are equal, which is the first step to showing equality of twisted cocycles. Choose $g \in G(F^{\mathrm{sep}})$ such that $\mathrm{Ad}(g)\bar{\Lambda}_1 = \bar{\Lambda}_2 + M$ for $M \in X_*(T_{2,\mathrm{sc}}/Z)[(S_j)_{E_j}]_0$ (which exists by assumption). We have the $\Gamma_{E_j/F}$-equivariant injection $X_*(T_i/Z) \to X_*(T_{i,\mathrm{ad}}) \oplus X_*(G/(Z \cdot \mathscr{D}G))$ induced by the isogeny $T_i/Z \to T_i/(Z \cdot Z(\mathscr{D}G))$, and we write $\bar{\Lambda}_1 = q_1 + r$ according to this decomposition. Now since $\bar{\Lambda}_2 = \mathrm{Ad}(g)(\bar{\Lambda}_1) + M$, we get that the corresponding decomposition for $\bar{\Lambda}_2$ is given by $(\mathrm{Ad}(g)q_1 + M) + r$, since $M \in X_*(T_{2,\mathrm{sc}}/Z)[(S_j)_{E_j}]_0$, and the image of $X_*(T_{2,\mathrm{sc}}/Z)$ in $X_*(G/(Z \cdot \mathscr{D}G))$ is trivial, since the projection of $T_{2,\mathrm{sc}}$ to G factors through $\mathscr{D}G$. We may replace n_j with $n_{j'}$ for a sufficiently large $j' \in \mathbb{N}$ to assume that $n_j q_1 \in X_*(T_{1,\mathrm{sc}})[(S_j)_{E_j}]_0^{N_{E_j/F}}$ (possible because $T_{1,\mathrm{sc}} \to T_{1,\mathrm{ad}}$ is an isogeny), and that $n_j r \in X_*(Z(G)^{\circ})[(S_j)_{E_j}]_0$ (possible because $Z(G)^{\circ} \to G/(Z \cdot \mathscr{D}G)$ is an isogeny).

We are now ready to demonstrate the equality of the twisted cocycles $z_{\bar{\Lambda}_1,n_j}$ and $z_{\bar{\Lambda}_2,n_j}$ (or rather, their images in $H^1((\dot{\mathcal{E}}_{j,n_j})_{\overline{F}}, Z \to G)$). Recall (since we've already shown equality of the associated homomorphisms) that this means finding some $x \in G(\overline{F})$ such that

$$\overline{k_j(c_{E_j,S_j})} \underset{O_{E,S}/O_{F,S}}{\sqcup} n_j\bar{\Lambda}_2 = p_1(x) \cdot [\overline{k_j(c_{E_j,S_j})} \underset{O_{E,S}/O_{F,S}}{\sqcup} n_j\bar{\Lambda}_1] \cdot p_2(x)^{-1}$$

inside the group $G(\overline{F} \otimes_F \overline{F})$. Decomposing $\bar{\Lambda}_i$ as above and noting that $\overline{k_j(c_{E_j,S_j})} \underset{O_{E,S}/O_{F,S}}{\sqcup} n_j r \in Z(G)(\overline{F} \otimes_F \overline{F})$, this reduces to the same equality with $\bar{\Lambda}_1$ replaced by q_1 and $\bar{\Lambda}_2$ replaced by $q_2 := \text{Ad}(g)q_1 + M$. Following [Kal18], we set

$$c_i := \overline{k_j(c_{E_j,S_j})} \underset{O_{E,S}/O_{F,S}}{\sqcup} n_j q_i \in T_{i,\text{sc}}(\overline{F} \otimes_F \overline{F});$$

note that, by construction, $n_j q_i \in X_*(T_{i,\text{sc}})[(S_j)_{E_j}]_0^{N_{E_j/F}}$.

The image of c_i in $T_{i,\text{ad}}(\overline{F} \otimes_F \overline{F})$ is equal (by $\mathbb{Z}$-bilinearity of the unbalanced cup product, using that $q_i \in X_*(T_{1,\text{ad}})[(S_j)_{E_j}]_0$) to $\overline{c_{E_j,S_j}} \underset{E/F}{\sqcup} q_i$—here, since we are working with Čech cohomology with respect to $\overline{F}$, we have switched the unbalanced cup product notation. But now $\overline{c_{E_j,S_j}} \underset{E/F}{\sqcup} q_i = \overline{c_{E_j,S_j}} \cup q_i$ is a Čech 1-cocycle of $T_{i,\text{ad}}(\overline{F} \otimes_F \overline{F})$, so we may twist G_{sc} by c_1 to obtain the twisted $\overline{F}$-form G^1_{sc} with isomorphism

$$\phi\colon (G_{\text{sc}})_{\overline{F}} \to (G^1_{\text{sc}})_{\overline{F}}$$

such that $p_1^*\phi \circ p_2^*\phi^{-1} = \text{Ad}(c_1)$ on $(G_{\text{sc}})_{\overline{F} \otimes_F \overline{F}}$.

We claim that $p_1^*\phi(c_2 \cdot c_1^{-1})$ is a 1-cocycle in $G^1_{\text{sc}}(\overline{F} \otimes_F \overline{F})$; an identical computation as in the proof of Lemma 4.5.9 shows that the differential of $p_1^*\phi(c_2 \cdot c_1^{-1})$ post-composed with the isomorphism $q_1^*\phi^{-1}$ (where $q_1\colon \overline{F} \to \overline{F}^{\bigotimes_F 3}$ is inclusion into the first factor) gives $dc_2 \cdot dc_1^{-1}$, where, by our unbalanced cup product formulas,

$$dc_i = d[\overline{k_j(c_{E_j,S_j})}] \underset{E/F}{\sqcup} n_j q_i,$$

using that the E_j/F-norm of $n_j q_i$ vanishes. The cocycle claim is proven after we observe that, as explained in [Kal18], the inclusions $Z(G_{\text{sc}}) \to T_{i,\text{sc}}$ give maps

$$X_*(T_{i,\text{ad}})[(S_j)_{E_j}]_0 \to \frac{X_*(T_{i,\text{ad}})}{X_*(T_{i,\text{sc}})}[(S_j)_{E_j}]_0 \to \text{Hom}(\frac{\text{Res}_{E_j,S_j}(\mu_{n_j})}{\mu_{n_j}}, Z(G_{\text{sc}})),$$

under which the images of q_1 and q_2 coincide.

By the vanishing of $H^1(F, G^1_{\text{sc}}) = \check{H}^1(\overline{F}/F, G^1_{\text{sc}})$ (since G^1_{sc} is simply-connected and connected), there is some $x \in G_{\text{sc}}(\overline{F})$ such that

$$c_2 \cdot c_1^{-1} = p_1(x)^{-1} c_1 p_2(x) c_1^{-1},$$

and hence the image of x in $G(\overline{F})$ realizes the desired equivalence of twisted cocycles.

We may finally deduce the claim of the theorem. We showed above that for a maximal F-torus T of G, there is a natural inclusion

$$\text{Im}(\overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,\text{tor}}([Z \to T]) \to \overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,\text{tor}}([Z \to G])) \hookrightarrow H^1(\mathcal{E}_{\dot{V}}, Z \to G),$$

and, as we have shown, these images capture all elements of $\overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,\text{tor}}([Z \to G])$. Thus, for $x \in \overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,\text{tor}}([Z \to G]))$, we define $\iota_{\dot{V}}(x)$ to be the image of $\iota_{\dot{V}}(y) \in H^1(\mathcal{E}_{\dot{V}}, Z \to T)$ in $H^1(\mathcal{E}_{\dot{V}}, Z \to G)$, where $y \in \overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,\text{tor}}([Z \to T])$ maps to x. By our above argument, the induced map $\overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,\text{tor}}([Z \to G]) \to H^1(\mathcal{E}_{\dot{V}}, Z \to G)$ does not depend on the choice of preimage y. This map is evidently surjective, by Lemma 8.1.2, and is injective because of the above natural inclusion and the fact that any two elements of $\overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,\text{tor}}([Z \to G])$ both lie in the image of $\overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,\text{tor}}([Z \to T])$ for some T. By construction, these isomorphisms extend the isomorphism of functors $\iota_{\dot{V}}$ defined on the full subcategory $\mathcal{T}$, and are functorial with respect to morphisms $[Z \to T] \to [Z \to G]$ in $\mathcal{R}$ given by inclusions of maximal tori defined over F. Since every $x \in H^1(\mathcal{E}_{\dot{V}}, Z \to G)$ lies in the image of some $H^1(\mathcal{E}_{\dot{V}}, Z \to T)$, it follows that the extension of $\iota_{\dot{V}}$ to $\mathcal{R}$ also defines an isomorphism of functors on $\mathcal{R}$. □

To conclude this subsection, we state some local-global compatibilities that arise from Theorem 8.3.1. Note that the morphism of functors from $\mathcal{T}$ to AbGrp defined in the previous section, given by, for a fixed $v \in V$ and $[Z \to T] \in \mathcal{T}$, the map $l_v \colon \overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,\text{tor}}([Z \to T]) \to \overline{Y}_{+v,\text{tor}}([Z \to T])$, may be extended to a morphism of functors on $\mathcal{R}$ induced by (after picking a set of coset representatives for $\Gamma^{\dot{v}}_{E_i/F} \backslash \Gamma_{E_i/F}$) mapping $f \in [X_*(T/Z)/X_*(T_{\text{sc}})][S_E, \dot{S}_E]_0$ to an element of $X_*(T/Z)/X_*(T_{\text{sc}})$ via the same formula as in the tori case. We recall the functor $\overline{Y}_{+,\text{tor}} \colon \mathcal{R} \to$ AbGrp from §4.5.

Corollary 8.3.2 *We have a commutative diagram with exact bottom row:*

$$\begin{array}{ccccc} H^1(\mathcal{E}_{\dot{V}}, Z \to G) & \xrightarrow{(loc_v)_v} & \coprod_{v \in V} H^1(\mathcal{E}_v, Z \to G) & & \\ \uparrow{\scriptstyle \iota_{\dot{V}}} & & \uparrow{\scriptstyle (\iota_v)_v} & & \\ \overline{Y}[V_{\overline{F}}, \dot{V}]_{0,+,tor}([Z \to G]) & \xrightarrow{(l_v)_v} & \bigoplus_{v \in V} \overline{Y}_{+v,tor}([Z \to G]) & \xrightarrow{\Sigma} & \overline{Y}_{+,tor}([Z \to G]), \end{array}$$

where the symbol $\coprod$ denotes the subset of the direct product of pointed sets in which all but finitely *many coordinates equal the neutral element, and the map Σ makes sense since any maximal F_v-torus of G_{F_v} is $G(\overline{F_v})$-conjugate to the base-change T_{F_v} of a maximal F-torus T in G.*

Proof. The commutativity is an immediate consequence of Corollary 8.2.13, the functoriality of $\iota_{\dot{V}}$, and the fact that every $x \in H^1(\mathcal{E}_{\dot{V}}, Z \to G)$ lies in the image of some $H^1(\mathcal{E}_{\dot{V}}, Z \to T)$. The exactness of the bottom row is a straightforward character-theoretic argument. □

We also have the following analogue of [Kal18, Corollary 3.8.2]:

Corollary 8.3.3 *The image of*

$$H^1(\mathcal{E}_{\dot{V}}, Z \to G) \xrightarrow{(loc_v)_v} \coprod_{v \in V} H^1(\mathcal{E}_v, Z \to G)$$

consists precisely of those elements which map trivially under the composition

$$\coprod_{v \in \dot{V}} H^1(\mathcal{E}_v, Z \to G) \to \bigoplus_{v \in V} \overline{Y}_{+v,tor}([Z \to G]) \to \overline{Y}_{+,tor}([Z \to G]).$$

Proof. Unlike in [Kal18], where some work is needed, this is a trivial consequence of Corollary 8.3.2. □

8.4 Unramified localizations

Let G be a connected reductive group over F with finite central F-subgroup Z. Note that for any Z-twisted $G_{\mathcal{E}_{\dot{V}}}$-torsor $\mathscr{T}$ (denote the set of such torsors by $Z^1(\mathcal{E}_{\dot{V}}, Z \to G)$), we can pull $\mathscr{T}$ back to the $G_{\mathcal{E}_{\dot{V},\overline{F_v}}}$-torsor $\mathscr{T}_{\overline{F_v}}$, and then via picking gerbe normalizations and a 1-coboundary, we get a functor $\Phi\colon \mathcal{E}_v \to \mathcal{E}_{\dot{V}}$, and then $\text{loc}_v(\mathscr{T}) := \Phi^*(\mathscr{T}_{\overline{F_v}})$ is a Z-twisted $G_{\mathcal{E}_v}$-torsor, which depends on our choice of normalizations and coboundary up to replacing $\text{loc}_v(\mathscr{T})$ by the canonically-isomorphic (via translation by a^{-1}) torsor $\eta^*(\text{loc}_v(\mathscr{T}))$, where $\eta\colon \mathcal{E}_v \to \mathcal{E}_v$ is the automorphism induced by a 1-coboundary $d(a)$, for $a \in u_v(\overline{F_v})$.

Note that since $\text{Res}[\mathscr{T}] \in \text{Hom}_F(P_{\dot{V}}, Z)$ factors through P_{E_i,S_i,n_i} for some i, for all $v \notin S_i$ we have that $\text{Res}[\text{loc}_v(\mathscr{T})]$ is trivial, and hence $\text{loc}_v(\mathscr{T})$ is the pullback of some G-torsor over F_v via the projection $\mathcal{E}_v \xrightarrow{\pi} \text{Sch}/F_v$. The canonical inclusion $Z(O_{F_v^{\text{nr}}}) \to Z(\overline{F_v})$ is an equality for all but finitely many v (because Z is split over F_v^{nr} for all but finitely many v, and $O_{F_v^{\text{nr}}}$ contains all roots of unity in $\overline{F_v}$). Choose an $O_{F,S}$-model $\mathcal{G}$ of G for a some finite subset $S \subset V$; note that, for almost all v, the subgroups $\mathcal{G}(O_{F_v^{\text{nr}}})$ and $\mathcal{G}(O_{F_v^{\text{nr}}}^{\text{perf}})$ inside $G(F_v^{\text{sep}})$ and $G(\overline{F_v})$ (respectively) do not depend on the choice of model $\mathcal{G}$. Our goal in this subsection is to prove the following function field analogue of [Taï18, Proposition 6.1.1]:

Proposition 8.4.1 *Let $\mathscr{T} \in Z^1(\mathcal{E}_{\dot{V}}, Z \to G)$. For all but finitely many $v \in V$, the torsor $loc_v(\mathscr{T}) \in Z^1(\mathcal{E}_v, Z \to G)/d(Z)$ is inflated from a $\mathcal{G}$-torsor $\mathcal{T}_v$ over O_{F_v}. Here, we are using $Z^1(\mathcal{E}_v, Z \to G)/d(Z)$ to denote equivalence classes of $G_{\mathcal{E}_v}$-torsors with the equivalence relation given by $\mathscr{T} \sim \eta^*\mathscr{T}$ for $\eta\colon \mathcal{E}_v \to \mathcal{E}_v$ induced by $d(a)$ for $a \in u_v(\overline{F_v}) \mapsto z \in Z(\overline{F_v})$ (we can always assume that $z \in Z(O^{nr}_{F_v})$,by the above discussion).*

Moreover, choosing normalizations $\mathcal{E}_{\dot{\xi}}$ and $\mathcal{E}_{\xi_v}$ of the gerbes $\mathcal{E}_{\dot{V}}$ and $\mathcal{E}_v$ and viewing $\mathscr{T}$ as a torsor on $\mathcal{E}_{\dot{\xi}}$ (the choice of normalization and class $\dot{\xi}$ does not affect the class of $\mathscr{T}$ in $Z^1(\mathcal{E}_{\dot{\xi}}, Z \to G)/d(Z)$), we may canonically identify $\mathscr{T}$ with a $\dot{\xi}$-twisted G-torsor $(\mathcal{S}', Res(\mathscr{T}), \psi')$, where $\mathcal{S}'$ is a G-torsor over $\overline{F}$. Fix a $Z(\overline{F})$-orbit of trivializations $\mathcal{O} = \{\mathcal{S}' \xrightarrow{h'} \underline{G}\}$; then for all but finitely many v, for any $h \in \mathcal{O}$, we may choose the $\mathcal{G}$-torsors $\mathcal{T}_v$ over O_{F_v} such that the trivializations $h_{\overline{F_v}}$ on $\mathcal{S}_{\overline{F_v}}$ are induced by the pullback of a trivialization $h_v\colon \mathcal{T}_v \to \underline{G}$ over the ring $O^{perf}_{F_v^{nr}}$.

Proof. This proof is essentially a summary of [Taï18, §6.2] with some minor adjustments to accommodate the positive-characteristic situation. Let ξ_v denote a representative in $u_v(\overline{F_v}^{\bigotimes_{F_v} 3})$ of the local canonical class, $\dot{\xi} \in P_{\dot{V}}(\overline{F}^{\bigotimes_F 3})$ a representative of the global canonical class given in Definition 7.4.5. Pick a tower of resolutions by tori $(P_k \to T_k \to U_k)_k$ as in §7.4, and set $T := \varprojlim_k T_k$, $U := \varprojlim_k U_k$, which are pro-tori.

By construction of the global canonical class $[\dot{\xi}]$, the image of $[\dot{\xi}]$ in $H^2(\overline{\mathbb{A}}/\mathbb{A}, T \to U)$ coincides with the image of the adelic canonical class $[x] \in \check{H}^2(\overline{\mathbb{A}}/\mathbb{A}, P)$, which, unpacking the construction of $[x]$, is to say (by the definition of the differentials arising from the double complex associated to $T \to U$) that there is some $a \in T(\overline{\mathbb{A}} \otimes_{\mathbb{A}} \overline{\mathbb{A}})$ and $b \in U(\overline{\mathbb{A}})$ such that

$$\xi = [\prod_{v \in V} \dot{S}^2_v(\text{loc}_v(\xi_v))] \cdot d(a) \tag{8.6}$$

inside $T(\overline{\mathbb{A}}^{\bigotimes_{\mathbb{A}} 3})$ and $\overline{a} = db$ inside $U(\overline{\mathbb{A}} \otimes_{\mathbb{A}} \overline{\mathbb{A}})$, where recall that $\dot{S}^2_v(\text{loc}_v(\xi_v))$ denotes the image of $\text{loc}_v(\xi_v) \in P(\overline{F_v}^{\bigotimes_{F_v} 3})$ in $P(\overline{\mathbb{A}}_v^{\bigotimes_{F_v} 3})$ under a choice of Shapiro map (defined in §3.3—note that such a map is not canonical until one passes to cohomology). To make sense of the above product expression, we remind the reader that $P(\overline{\mathbb{A}}^{\bigotimes_{\mathbb{A}} 3}) = \varprojlim_i P_i(\overline{\mathbb{A}}^{\bigotimes_{\mathbb{A}} 3})$, and for a fixed i, all but finitely-many projections $p_i[\dot{S}^2_v(\text{loc}_v(\xi_v))]$ are trivial, and hence it makes sense to take this product in each $P_i(\overline{\mathbb{A}}^{\bigotimes_{\mathbb{A}} 3}) = \varinjlim_{K/F} \prod'_v P_i(\mathbb{A}_{K,v}^{\bigotimes_{F_v} 3})$ (by Corollary 3.3.6) and then take the inverse limit.

Recall that $\dot{v} \in V_{F^{\text{sep}}}$ determines a ring homomorphism $\text{pr}_{\dot{v}}\colon \overline{\mathbb{A}} \to \overline{F_v}$ defined by the direct limit of the the projection maps $\mathbb{A}_K = \prod'_{w \in V_K} K_w \to K_{\dot{v}_K}$ over all finite extensions K/F, where by $\dot{v}_K$ we mean the unique extension of $\dot{v}_{K'}$, where K' is the maximal Galois subextension of K/F, to a valuation on K. Restricting this ring homomorphism to the subring $\overline{\mathbb{A}}_v \subset \overline{\mathbb{A}}$ gives a homomorphism of F_v-algebras. It is straightforward to check that we may choose our section $\Gamma/\Gamma_v \to \Gamma$ (cf. the construction of the Shapiro maps in §3.3) such that, on k-cochains, we have

$\mathrm{pr}_{\dot{v}}\big|_{\overline{\mathbb{A}}_v} \circ \dot{S}_v^k = \mathrm{id}_{\overline{F_v}}$. We also have the projection map $\overline{\mathbb{A}} \xrightarrow{\mathrm{pr}_v} \overline{\mathbb{A}}_v$ defined the same way except using the direct limit of the project maps $\prod'_{w \in V_K} K_w \to \prod_{w|v} K_w$.

Applying $\mathrm{pr}_{\dot{v}}\big|_{\overline{\mathbb{A}}_v} \circ \mathrm{pr}_v$ to the equality (8.6), we see that, for a fixed $v \in V$, the image of $\dot{\xi}$ in $T(\overline{F_v}^{\bigotimes_{F_v} 3})$, denoted by $\mathrm{res}_v(\dot{\xi})$ is given by $\mathrm{loc}_v(\xi_v) \cdot d(a_v)$, where $a_v := \mathrm{pr}_{\dot{v}}(a) \in T(\overline{F_v} \otimes_{F_v} \overline{F_v})$. Although this equality is a priori taking place in $T(\overline{F_v}^{\bigotimes_{F_v} 3})$, since the image of $\dot{\xi}$ and $\mathrm{loc}_v(\xi_v)$ both lie in the subgroup $P(\overline{F_v}^{\bigotimes_{F_v} 3})$, we see that in fact $d(a_v) \in P(\overline{F_v}^{\bigotimes_{F_v} 3})$ and thus this equality takes place in P. Set $b_v := \mathrm{pr}_{\dot{v}}(b) \in U(\overline{F_v})$, and choose a lift $\tilde{b}_v \in T(\overline{F_v})$ of b_v, which is possible since the derived inverse limit $\varprojlim_i^1 P_i(\overline{F_v})$ vanishes, since it consists of surjective maps and thus satisfies the Mittag-Leffler condition. Define $a'_v := a_v/d(\tilde{b}_v)$, which lies in $P(\overline{F_v}^{\bigotimes_{F_v} 2})$ since its image under $T \to U$ equals $\mathrm{pr}_{\dot{v}}(\overline{a})/\mathrm{pr}_{\dot{v}}(db)$ (using that the isogenies $T_k \to U_k$ are defined over F, so they commute with Čech differentials), which is trivial by construction. We may replace a_v by a'_v and retain the equality

$$\mathrm{res}_v(\dot{\xi}) = \mathrm{loc}_v(\xi_v) \cdot d(a'_v).$$

Continuing to follow [Taï18], for $k \geq 0$ and $v \in V$, we denote by $a_{v,k}$ (resp. $b_{v,k}$, $\tilde{b}_{v,k}$, $a'_{v,k}$) the image of a_v (resp. b_v, $\tilde{b}_v$, a'_v) in $T_k(\overline{F_v}^{\bigotimes_{F_v} 2})$ (resp. $U_k(\overline{F_v})$, $T_k(\overline{F_v})$, $P_k(\overline{F_v}^{\bigotimes_{F_v} 2})$). We claim that there is a finite set of places S' of F such that for all $v \notin S$, the element $a'_{v,k}$ lies in the subgroup $P_k([O_{F_{\dot{v}}^{\mathrm{nr}}}^{\mathrm{perf}}]^{\bigotimes_{O_{F_v}} 2})$. Recall that

$$a_k \in T_k(\overline{\mathbb{A}} \otimes_{\mathbb{A}} \overline{\mathbb{A}}) = \varinjlim_{E/F} T_k(\mathbb{A}_E \otimes_{\mathbb{A}} \mathbb{A}_E) = \varinjlim_{E/F} (\varinjlim_{S} T_k(\mathbb{A}_{E,S} \otimes_{\mathbb{A}_S} \mathbb{A}_{E,S})),$$

where the outside limit is over all finite extensions E/F and the inside limit is over all finite sets of places of F. It follows that we may find K/F finite containing E_k and finite $S' \subset V$ containing S_k such that the maximal Galois subextension K'/F of K is unramified outside S', such that T_k is split over K', $a_k \in T_k(\mathbb{A}_{K,S'} \otimes_{\mathbb{A}_{S'}} \mathbb{A}_{K,S})$, and $b_k \in U_k(\mathbb{A}_{K,S'})$. It follows that, for $v \notin S'$, we have $a_{k,v} \in T_k(O_{K_{\dot{v}}} \otimes_{O_{F_v}} O_{K_{\dot{v}}})$, and moreover, $K'_{\dot{v}}/F_v$ is unramified, so that $a_{k,v} \in T_k(O_{F_{\dot{v}}^{\mathrm{nr}}}^{\mathrm{perf}} \otimes_{O_{F_v}} O_{F_{\dot{v}}^{\mathrm{nr}}}^{\mathrm{perf}})$. Since the group P_k is killed by the n_k-power map, there is a unique morphism $U_k \to T_k$ such that the composition $U_k \to T_k \to U_k$ is the n_k-power map. Since $b_{k,v} \in U_k(O_{K_{\dot{v}}})$ for all $v \notin S'$ and T_k and U_k are split over K, any preimage of $b_{k,v}$ lies in $T_k([O_{K_{\dot{v}}}^{(n'_k)}]^{(1/p^{m_k})})$, where $[O_{K_{\dot{v}}}^{(n'_k)}]^{(1/p^{m_k})}$ denotes the fppf extension of $O_{K_{\dot{v}}}$ given by the composition of two extensions defined as follows. If n'_k is the prime-to-p part of n_k with $n_k/n'_k = p^{m_k}$, then we first take the extension $O_{K_{\dot{v}}}^{(n'_k)}/O_{K_{\dot{v}}}$ obtained by adjoining all n'_k-roots of elements of $O_{K_{\dot{v}}}^{\times}$, which is finite étale, followed by the extension $[O_{K_{\dot{v}}}^{(n'_k)}]^{(1/p^{m_k})}$ defined by adjoining *all* p^{m_k}-power roots to $O_{K_{\dot{v}}}^{(n'_k)}$, which is finite flat.

We claim that the extension $[O_{K_{\dot{v}}}^{(n'_k)}]^{(1/p^{m_k})}/O_{F_v}$ lies in $O_{F_{\dot{v}}^{\mathrm{nr}}}^{\mathrm{perf}}$. Indeed, since $O_{K_{\dot{v}}^{\mathrm{nr}}}^{\mathrm{perf}} = O_{F_{\dot{v}}^{\mathrm{nr}}}^{\mathrm{perf}}$, it's enough to check that $[O_{K_{\dot{v}}}^{(n'_k)}]^{(1/p^{m_k})}$ lies in $O_{K_{\dot{v}}^{\mathrm{nr}}}^{\mathrm{perf}}$, which is clear since, as explained above, it factors as a finite étale extension of $O_{K_{\dot{v}}}$ followed by the extension obtained by adjoining all p^{m_k}-power roots. Thus, for any $v \notin S'$, we have $a'_{v,k} \in P_k([O_{K_{\dot{v}}}^{(n'_k)}]^{(1/p^{m_k})} \otimes_{O_{F_v}} [O_{K_{\dot{v}}}^{(n'_k)}]^{(1/p^{m_k})})$, and hence, since we showed in §7.4 that the image of $\mathrm{loc}_v(\xi_v)$ is trivial in P_k for all $v \notin S_k \subseteq S'$, we get the equality

$$\mathrm{res}_v(\xi_k) = d(a'_{v,k}) \in P_k(\overline{F_v}^{\bigotimes_{F_v} 3}),$$

where ξ_k denotes the image of $\dot{\xi}$ in $P(\overline{F_v}^{\bigotimes_{F_v} 3})$.

Let $\mathscr{T} \in Z^1(\mathcal{E}_{\dot{V}}, Z \to G)$, and choose normalizations of $\mathcal{E}_v$ and $\mathcal{E}_{\dot{V}}$, so that we may identify them with the explicit gerbes $\mathcal{E}_{\xi_v}$ and $\mathcal{E}_{\dot{\xi}}$, respectively. Recall that, after passing from $\mathcal{E}_{\dot{\xi}}$ to $\mathcal{E}_{\mathrm{res}_v(\dot{\xi})}$ (which we have explicitly identified with $(\mathcal{E}_{\dot{\xi}})_{\overline{F_v}}$), choosing different normalizations has the effect of twisting $\mathrm{loc}_v(\mathscr{T})$ by $d(z)$ for $z \in Z(O_{F_v^{\mathrm{nr}}})$ with $z = \mathrm{Res}([\mathscr{T}])(x)$ for some $x \in u_v(\overline{F_v})$, and thus does not affect the statement of the proposition. Changing the representatives $\dot{\xi}$ and ξ_v for the canonical classes has the same effect.

Having chosen normalizations, we may canonically identify $G_{\mathcal{E}_?}$-torsors on the gerbes $\mathcal{E}_?$ with ?-twisted G-torsors, for $? = \mathrm{res}_v(\dot{\xi}), \xi_v, \dot{\xi}$, by Proposition 2.4.10; write $\mathscr{T} = (\mathcal{S}', \mathrm{Res}(\mathscr{T}), \psi')$ under this identification. Choose k sufficiently large so that $\mathrm{Res}(\mathscr{T}) \in \mathrm{Hom}_F(P, Z)$ factors through P_k via $\varphi_k \in \mathrm{Hom}_F(P_k, Z)$, $\mathcal{S}'$ equals $j^*\mathcal{S}''$ for a $\mathcal{G}$-torsor $\mathcal{S}''$ over $O_{S_k}^{\mathrm{perf}}$, for $j\colon \mathrm{Spec}(\overline{F}) \to \mathrm{Spec}(O_{S_k}^{\mathrm{perf}})$, such that h equals $j^*h_{S_k}$ for an $O_{S_k}^{\mathrm{perf}}$-trivialization h_{S_k} of $\mathcal{S}''$, and such that the "twisted gluing isomorphism" $\psi'\colon p_2^*\mathcal{S}' \to p_1^*\mathcal{S}'$ is given by $j^*\psi$ for an isomorphism of $\mathcal{G}$-torsors

$$\psi\colon p_2^*\mathcal{S}'' \to p_1^*\mathcal{S}'';$$

choose $S' \supseteq S_k$ corresponding to k as in the above paragraphs. We have a morphism of gerbes $\mathcal{E}_{\xi_v} \to \mathcal{E}_{\mathrm{res}_v(\dot{\xi})}$ given at the level of objects by sending the ξ_v-twisted torsor (T', ψ) to the $\mathrm{res}_v(\dot{\xi})$-twisted torsor

$$(T' \times^{u_v, \mathrm{loc}_v} P_{\dot{V}}, m_{(a'_v)^{-1}} \circ \psi),$$

cf. Construction 2.3.4. Under this identification, pulling back by the morphism we just constructed sends the $\mathrm{res}_v(\dot{\xi})$-twisted G-torsor $(\mathcal{S}'_{\overline{F_v}}, \mathrm{Res}(\mathscr{T}), \psi')$ to the ξ_v-twisted G-torsor $(\mathcal{S}'_{\overline{F_v}}, \mathrm{Res}(\mathscr{T}) \circ \mathrm{loc}_v, m_{a'_v} \circ \psi')$. Note that, by construction, for any $v \notin S'$, the homomorphism $\mathrm{Res}(\mathscr{T}) \circ \mathrm{loc}_v$ on u_v is trivial, and hence $(\mathcal{S}'_{\overline{F_v}}, m_{a'_v} \circ \psi')$ gives a descent datum for a G-torsor $\mathcal{S}$ over F_v; we claim that the pair of $\mathcal{S}$ and the $\overline{F_v}$-trivialization induced by $h_{\overline{F_v}}$ descends further to a $\mathcal{G}$-torsor over O_{F_v} with an $O_{F_v^{\mathrm{nr}}}^{\mathrm{perf}}$-trivialization.

Define this new $\mathcal{G}$-torsor $\mathcal{T}$ as follows: we take the descent data with respect to the fpqc cover $O_{F_v^{\mathrm{nr}}}^{\mathrm{perf}}/O_{F_v}$ given by the torsor $\mathcal{S}''_{O_{F_v^{\mathrm{nr}}}^{\mathrm{perf}}}$, where this is well-defined since for $v \notin S'$, the ring $O_{F_v^{\mathrm{nr}}}^{\mathrm{perf}}$ is an $O_{S_k}^{\mathrm{perf}}$-algebra, and the gluing isomorphism given by $m_{\mathrm{Res}(\mathscr{T})(a'_v)} \circ \psi$, which is well-defined since $\mathrm{Res}(\mathscr{T})(a'_v) = \varphi_k(a'_{v,k})$, the morphism φ_k is defined over $O_{F,S'}$, and $a'_{v,k} \in P_k(O_{F_v^{\mathrm{nr}}}^{\mathrm{perf}} \otimes_{O_{F_v}} O_{F_v^{\mathrm{nr}}}^{\mathrm{perf}})$; this gives a well-defined gluing map by construction, and finishes the construction of $\mathcal{T}$—

by design, $h_v := (h_{S_k})_{O^{\text{perf}}_{F_v^{\text{nr}}}}$ trivializes $\mathcal{T}$ over $O^{\text{perf}}_{F_v^{\text{nr}}}$. The pullback of $\mathcal{T}$ is evidently equal to $\mathcal{S}$, since the descent datum giving $\mathcal{T}$ pulls back via the morphisms $\mathrm{Spec}(F_v) \to \mathrm{Spec}(O_{F_v})$ and $\mathrm{Spec}(\overline{F_v}) \to \mathrm{Spec}(O^{\text{perf}}_{F_v^{\text{nr}}})$ to the descent datum giving $\mathcal{S}$; similarly, h_v pulls back to $h_{\overline{F_v}}$. This proves the result. □

CHAPTER 9

Applications to Global Langlands

In this section, we use the above constructions to analyze an adelic transfer factor for a global function field F and state conjectures regarding the multiplicity of discrete automorphic representations in the discrete spectrum. In what follows, G will be a connected reductive group over F.

9.1 Adelic transfer factors for function fields

In this subsection, we follow [LS87, §6.3] to construct adelic transfer factors for connected reductive groups over a global function field F. Let $\psi\colon G_{F_s} \to G^*_{F_s}$ be a quasi-split inner form of G, with Langlands dual group $\widehat{G}^*$ and Weil-form ${}^LG^* := \widehat{G}^* \rtimes W_F$.

Definition 9.1.1 *A* ***global endoscopic datum*** *for G is a tuple $(H, \mathcal{H}, s, \xi)$ where H is a quasi-split connected reductive group over F, $\mathcal{H}$ is a split extension of W_F by $\widehat{H}$, $s \in Z(\widehat{H})$ is any element, and $\xi\colon \mathcal{H} \to {}^LG^*$ is an L-embedding such that:*

1. *The homomorphism $W_F \to \mathit{Out}(\widehat{H}) = \mathit{Out}(H)$ determined by $\mathcal{H}$ is the same as the homomorphism $W_F \to \Gamma \to \mathit{Out}(H)$ induced by the usual Γ-action on H.*

2. *The map ξ restricts to an isomorphism of algebraic groups over $\mathbb{C}$ from $\widehat{H}$ to $Z_{\widehat{G}^*}(t)^\circ$, where $t := \xi(s)$.*

3. *The first two conditions imply that we have a Γ-equivariant embedding $Z(\widehat{G}^*) \to Z(\widehat{H})$. We require that the image of s in $Z(\widehat{H})/Z(\widehat{G}^*)$, denoted by $\bar{s}$, is fixed by W_F and maps under the connecting homomorphism $H^0(W_F, Z(\widehat{H})/Z(\widehat{G}^*)) \to H^1(W_F, Z(\widehat{G}^*))$ to an element which is killed by the homomorphism $H^1(W_F, Z(\widehat{G}^*)) \to H^1(W_{F_v}, Z(\widehat{G}^*))$ for all $v \in V_F$ (such an an element is called* ***locally trivial****).*

Note that any global endoscopic datum $\mathfrak{e} = (H, \mathcal{H}, s, \xi)$ induces, for any place v of F, a local endoscopic datum given by $(H_{F_v}, \mathcal{H}_v, s_v, \xi_v)$, where $\mathcal{H}_v$ is the pullback of the two maps $\mathcal{H} \to W_F$

and $W_{F_v} \to W_F$ (which carries a natural splitting), $\xi_v\colon \mathcal{H}_v \to {}^L(G^*_{F_v})$ is induced by ξ and the natural map $\mathcal{H}_v \to \mathcal{H}$, which one checks is an L-embedding, and $s_v = s \in Z(\widehat{H})$. Following [Kal18], we will denote such a local endoscopic datum by $\mathfrak{e}_v = (H, \mathcal{H}, s_v, \xi)$. Fix a global endoscopic datum $(H, \mathcal{H}, s, \xi)$; we will temporarily assume that $\mathcal{H} = {}^L H$. Up to equivalence, a global endoscopic datum only depends on the image of s in $\pi_0([Z(\widehat{H})/Z(\widehat{G}^*)]^\Gamma)$. Recall that a strongly-regular semisimple element $\gamma_H \in H(F)$ with centralizer T_H (a maximal torus of H defined over F) is called *G-regular* if it is the preimage of a strongly-regular semisimple element $\gamma_G \in G(F)$ under an admissible isomorphism $T_H \to T_G := Z_G(\gamma_G)$. We'll need the following basic lemma:

Lemma 9.1.2 *There is an admissible embedding of T_G into G^*.*

Proof. This follows from Lemma 6.1.6, which is a generalization of [Kot82, Corollary 2.2]. Note that the Lemma loc. cit. is stated for a local function field F, but the proof holds verbatim for global function fields. □

It immediately follows that for any G-regular strongly-regular semisimple $\gamma_H \in H(F)$, we have an admissible embedding of T_H in G^* (which is not unique). We say that γ_H is a *related to* $\gamma_G \in G(\mathbb{A})$ if for all $v \in V$, the image of γ_H in $H(F_v)$ is an image (under an admissible embedding $(T_H)_{F_v} \to \widehat{G_{F_v}}$) of the element $\gamma_{G,v} \in G(F_v)$. If we fix an admissible embedding of T_H in G^*, with image a maximal F-torus denoted by T and image of γ_H denoted by $\gamma \in G^*(F)$, then the above condition means requiring that there exist $x_v \in G^*(F_v^{\mathrm{sep}})$ such that $\mathrm{Ad}(x_v) \circ \psi$ maps the maximal torus $T_{G,v}$ in G_{F_v} containing $\gamma_{G,v}$ to T_{F_v} (over F_v) and sends $\gamma_{G,v}$ to (the restriction of) γ.

Then for elements $\gamma_H, \bar{\gamma}_H$ related to $\gamma_G, \bar{\gamma}_G$ (respectively), we define

$$\mu_v = \mathrm{inv}\left(\frac{\gamma_H, \gamma_{G,v}}{\bar{\gamma}_H, \bar{\gamma}_{G,v}}\right), \tag{9.1}$$

which lies in the group $H^1(F_v, U)$, where $U = (T_{\mathrm{sc}} \times \bar{T}_{\mathrm{sc}})/Z_{\mathrm{sc}}$, where everything is as defined in §5.3.3.

We need the following analogue of [LS87, Lemma 6.3.A], whose proof we follow:

Lemma 9.1.3 *$\mu_v = 1$ for all but finitely many $v \in V$.*

Proof. Suppose that L/F is a finite Galois extension splitting T such that the map ψ is defined over L. Note that for all but finitely many v, the map ψ is defined over F_v, and that that since L splits T_H, for any $v \in V$, the completion L_v splits the maximal F_v-tori $(T_H)_{F_v}$, $T_{G,v}$, and T_{F_v}. It is straightforward to verify that, in this case, we have

$$\mu_v = \frac{\mathrm{inv}(\gamma_H, \gamma_{G,v})}{\mathrm{inv}(\bar{\gamma}_H, \bar{\gamma}_{G,v})},$$

where $\mathrm{inv}(\gamma_H, \gamma_{G,v})$ is defined by choosing some $h \in G^*(L_v)$ such that $\mathrm{Ad}(h)\psi(\gamma_{G,v}) = \gamma$ and then setting $\mathrm{inv}(\gamma_H, \gamma_{G,v}) := [p_1(h)p_2(h)^{-1}] \in \check{H}^1(L_v/F_v, T_{\mathrm{sc}})$, similarly with γ_H and $\gamma_{G,v}$ replaced by $\bar{\gamma}_H$ and $\bar{\gamma}_{G,v}$, where the above quotient takes place in the group $H^1(F_v, U)$ via the maps $H^1(F_v, T_{\mathrm{sc}}) \to H^1(F_v, U)$, similarly for $\bar{T}_{\mathrm{sc}}$, induced by the canonical maps $T_{\mathrm{sc}}, \bar{T}_{\mathrm{sc}} \to U$.

Note that for all but finitely many places v, the extension L/F is unramified at v, the image of γ in $T(F_v)$ lies in $T(O_{F_v})$, the map ψ is defined over F_v, the element $\gamma_{G,v}$ lies in $T_G(O_{F_v}) \subset \mathcal{G}(O_{F_v})$ for some fixed integral model $\mathcal{G}$ of G, and for each root $\alpha \in \Phi(G^*_{L'}, T_{L'})$, we have $\alpha(\gamma) \in O_{L_v}^{\times}$. Then Lemme 8.3 from [Lan83] (which is stated for p-adic local fields, but whose proof relies results from Bruhat-Tits theory that are stated for an arbitrary nonarchimedean local field, see [Tit79]) shows that $\gamma_{G,v}$ and γ are in fact conjugate under $G(O_{F_v})$ for all but finitely-many v. From here, the same argument as in the proof of [LS87, Lemma 6.3.A] shows that the class $\mathrm{inv}(\gamma_H, \gamma_{G,v}) \in \check{H}^1(L_v/F_v, T_{\mathrm{sc}}) = H^1(F_v, T_{\mathrm{sc}})$ (which is well-defined because G_{F_v} is quasi-split, see §5.3.3), vanishes. Of course, the same argument can be applied to show that the class $\mathrm{inv}(\bar{\gamma}_H, \bar{\gamma}_{G,v})$ vanishes, giving the result by the above paragraph. □

Note that a strongly G-regular $\gamma_H \in H(F)$ is related to $\gamma_G \in G(\mathbb{A})$ if and only if there exists $h \in G^*_{\mathrm{sc}}(\overline{\mathbb{A}})$ such that $h\psi(\gamma_G)h^{-1} = \gamma$. Now for any $u \in G^*_{\mathrm{sc}}(\overline{F} \otimes_F \overline{F})$ such that $\mathrm{Ad}(u) = p_1^*\psi \circ p_2^*\psi^{-1}$, we define $\mu_T \in \bar{H}^1(\overline{\mathbb{A}}/\mathbb{A}, T_{\mathrm{sc}})$ as the the image of $p_1(h)up_2(h)^{-1} \in T(\overline{\mathbb{A}} \otimes_{\mathbb{A}} \overline{\mathbb{A}})$ in $T(\overline{\mathbb{A}} \otimes_{\mathbb{A}} \overline{\mathbb{A}})/T(\overline{F} \otimes_F \overline{F})$. Identifying $\bar{H}^1(\overline{\mathbb{A}}/\mathbb{A}, T_{\mathrm{sc}})$ with $\bar{H}^1(\mathbb{A}^{\mathrm{sep}}/\mathbb{A}, T_{\mathrm{sc}})$ (notation as in §A.3), we get from our discussion in §A.3 a pairing $\bar{H}^1(\overline{\mathbb{A}}/\mathbb{A}, T_{\mathrm{sc}}) \times H^1(\Gamma, X^*(T)) \to \mathbb{Q}/\mathbb{Z}$. Identifying $X^*(T)$ with $X_*(\widehat{T})$, this determines a pairing

$$\bar{H}^1(\overline{\mathbb{A}}/\mathbb{A}, T_{\mathrm{sc}}) \times \pi_0(\widehat{T}_{\mathrm{ad}}^{\Gamma}) \to \mathbb{C}^*,$$

as explained in §5.1. Our element $s \in \pi_0([Z(\widehat{H})/Z(\widehat{G}^*)]^{\Gamma})$ determines an element $\mathbf{s}_T \in \pi_0(\widehat{T}_{\mathrm{ad}}^{\Gamma})$ via the canonical (Γ-equivariant) map $Z(\widehat{H}) \to \widehat{T}$, and we thus obtain a value $\langle \mu_T, \mathbf{s}_T \rangle \in \mathbb{C}^*$, which we denote by $d(\gamma_H, \gamma_G)$; it is clear that $d(\gamma_H, \gamma_G)$ is independent of the admissible embedding of T_H into G^*.

On the other hand, it follows from the above lemma and the isomorphism

$$H^2(\mathbb{A}, U) \xrightarrow{\sim} \bigoplus_{v \in V} H^2(F_v, U)$$

that the classes μ_v determine a well-defined element of $H^2(\mathbb{A}, U) \xrightarrow{\sim} \check{H}^2(\overline{\mathbb{A}}/\mathbb{A}, U)$; denote by μ_U the image of this class in $\bar{H}^2(\overline{\mathbb{A}}/\mathbb{A}, U)$. As explained in [LS87, §3.4], the global endoscopic datum determines an element $\mathbf{s}_{U,v} \in \pi_0(\widehat{U}^{\Gamma_{F_v}})$ for all v, as well as $\mathbf{s}_U \in \pi_0(\widehat{U}^{\Gamma})$. Then via the pairing of the above paragraph, we obtain a value

$$\langle \mu_U, \mathbf{s}_U \rangle = \prod_v \langle \mu_v, \mathbf{s}_{U,v} \rangle, \tag{9.2}$$

where the equality comes from the local-global compatibility of the Tate-Nakayama pairing for tori, see [Mil06, §4] for more details. We also have the equality

$$\langle \mu_U, \mathbf{s}_U \rangle = \frac{d(\bar{\gamma}_H, \bar{\gamma}_G)}{d(\gamma_H, \gamma_G)}.$$

Lemma 6.3.B in [LS87] discusses how the values $d(\gamma_H, \gamma_G)$ change as one varies the inputs—its proof also holds in our setting, and we record the result here:

Lemma 9.1.4 *1. $d(\gamma_H, \gamma_G) = d(\gamma'_H, \gamma_G)$ if γ'_H is stably-conjugate to γ_H in $H(F)$.*

2. $d(\gamma_H, \gamma_G) = d(\gamma_H, \gamma'_G)$ if γ'_G is $G(\mathbb{A})$-conjugate to γ_G.

3. $d(\gamma_H, \gamma_G) = d(\bar{\gamma}_H, \bar{\gamma}_G)$ if $\gamma_G, \bar{\gamma}_G \in G(F)$.

Fix a strongly G-regular $\bar{\gamma}_H \in H(F)$ which is related to $\bar{\gamma}_G \in G(F)$. If there are no such elements, we define $\Delta_{\mathbb{A}}(\gamma_H, \gamma_G)$ to be 0 for all $\gamma_H \in H(F)$, $\gamma_G \in G(\mathbb{A})$. Otherwise, we then define the adelic transfer for a strongly G-regular $\gamma_H \in H(F)$ and $\gamma_G \in G(\mathbb{A})$ by the quotient

$$\Delta_{\mathbb{A}}(\gamma_H, \gamma_G) := \frac{d(\bar{\gamma}_H, \bar{\gamma}_G)}{d(\gamma_H, \gamma_G)} \tag{9.3}$$

if γ_H is related to γ_G, and zero otherwise. It follows immediately from Lemma 9.1.4 that this factor is independent of the choice of the elements $\bar{\gamma}_H$ and $\bar{\gamma}_G$, the choice of γ_H up to stable conjugacy, the choice of γ_G up to $G(\mathbb{A})$-conjugacy, and thus equals 0 if γ_H is related to γ_G such that the $G(\mathbb{A})$-conjugacy class contains an element of $G(F)$.

We conclude this subsection by discussing local-global compatibility. Note that, if $\gamma_H \in H(F)$ is a strongly G-regular semisimple element which is related to $\gamma_G \in G(\mathbb{A})$, then for all $v \in V$, we have that the image of γ_H in $H(F_v)$, denoted by $\text{res}_v(\gamma_H)$, is strongly G_{F_v}-regular and is related to the element $\gamma_{G,v} \in G(F_v)$. We have the following result concerning the local transfer factor (see §5.3 for the definitions of the various component factors), where the local transfer factors are taken with respect to the local endoscopic data $(H, \mathcal{H}, s_v, \xi)$ coming from the fixed global endoscopic datum $(H, \mathcal{H}, s, \xi)$ as explained above:

Proposition 9.1.5 *([LS87, Theorem 6.4.A])*

1. For almost all v, the values $\Delta_i(res_v(\gamma_H), \gamma_{G,v})$ equal 1 for $i = I, II, III_2, IV$.

2. $\prod_v \Delta_i(\mathit{res}_v(\gamma_H), \gamma_{G,v}) = 1$ *for* $i = I, II, III_2, IV$.

Proof. We closely follow the analogous proof in [LS87]. As in §5.2.1, we may define, for the quasi-split simply-connected reductive group G^*_{sc} with maximal torus T_{sc}, a *global splitting invariant* $\lambda_{\{a_\alpha\}}(T_{\text{sc}}) \in H^1(F, T_{\text{sc}})$ which depends on an F-pinning of G^*_{sc} and a choice of a-data $\{a_\alpha\}$ for T. By the construction of the local splitting invariant, it is clear that $\lambda_{\{a_\alpha\}}(T_{\text{sc}})$ maps to the local splitting invariant $\lambda_{\{a_\alpha\}}(T_{F_v,\text{sc}})$ (where we are viewing the a-data $\{a_\alpha\}$ as an a-data for T_{F_v}) under the canonical map $H^1(F, T_{\text{sc}}) \to H^1(F_v, T_{F_v,\text{sc}})$. Since for all but finitely many v the image of $\lambda_{\{a_\alpha\}}(T_{\text{sc}})$ lands in the subgroup $H^1(O_{F_v}, T_{F_v,\text{sc}}) = 0$, it follows that $\langle \lambda_{\{a_\alpha\}}(T_{F_v,\text{sc}}), \mathbf{s}_{T,v} \rangle = \Delta_1(\text{res}_v(\gamma_H), \gamma_{G,v}) = 1$ for all but finitely many v.

Our above observation and the exact sequence

$$H^1(F, T_{\text{sc}}) \to H^1(\mathbb{A}, T_{\text{sc}}) = \check{H}^1(\overline{\mathbb{A}}/\mathbb{A}, T_{\text{sc}}) \to \bar{H}^1(\overline{\mathbb{A}}/\mathbb{A}, T_{\text{sc}})$$

(see [KS99, §D.1]) imply that the image $\bar{\lambda}$ of the element $(\langle \lambda_{\{a_\alpha\}}(T_{F_v,\text{sc}}), \mathbf{s}_{T,v} \rangle)_v \in H^1(\mathbb{A}, T_{\text{sc}}) = \bigoplus_v H^1(F_v, T_{F_v,\text{sc}})$ is trivial in $\bar{H}^1(\overline{\mathbb{A}}/\mathbb{A}, T_{\text{sc}})$, and so it follows by local-global compatibility of the Tate-Nakayama pairing that

$$\prod_v \langle \lambda_{\{a_\alpha\}}(T_{F_v,\text{sc}}), \mathbf{s}_{T,v} \rangle = \langle \bar{\lambda}, \mathbf{s}_T \rangle = 1,$$

as desired for the case $i = I$. The arguments for the remaining cases of $i = II$, III_2, and IV may be taken verbatim from the proof of [LS87, Theorem 6.4.A]. □

It follows from Lemma 9.1.3 that the value $\Delta_{III_1}(\text{res}_v(\gamma_H), \gamma_{G,v}, \text{res}_v(\bar{\gamma}_H), \bar{\gamma}_{G,v}) = \langle \mu_v, \mathbf{s}_{U,v} \rangle$, the remaining component of the local transfer factors, equals 1 for all but finitely v, and from the equality (9.2) the identity

$$\Delta_{\mathbb{A}}(\gamma_H, \gamma_G) = \prod_v \Delta_{III_1}(\text{res}_v(\gamma_H), \gamma_{G,v}, \text{res}_v(\bar{\gamma}_H), \bar{\gamma}_{G,v}).$$

We now use the above constructions to define a transfer factor for adelic elements of H. We call an element $\gamma \in H(\mathbb{A})$ *semisimple* if $\gamma_v \in H(F_v)$ is semisimple for all v, and we call it *strongly G-regular* if $\gamma \in H_{G-\text{sr}}(\mathbb{A}_{\overline{F}})$, where $H_{G-\text{sr}} \subset H_{\overline{F}}$ is the $\overline{F}$-scheme characterized by the Zariski open subset of strongly G-regular semisimple elements of the variety $H(\overline{F})$. Similarly, we call a semisimple element $\delta \in G(\mathbb{A})$ *strongly regular* if it lies in $G_{\text{sr}}(\mathbb{A}_{\overline{F}})$, where $G_{\text{sr}} \subset G_{\overline{F}}$ is the Zariski open subscheme characterized by the strongly regular elements of $G(\overline{F})$.

Definition 9.1.6 *For* $\gamma \in H_{G-sr}(\mathbb{A})$ *and* $\delta \in G_{sr}(\mathbb{A})$*, we set* $\Delta_{\mathbb{A}}(\gamma, \delta) = 0$ *if there is no strongly G-regular element of* $H(F)$ *which is related to an element of* $G(F)$*, and otherwise fix such a pair*

$\bar{\gamma}_H, \bar{\gamma}_G$ *and define*

$$\Delta_{\mathbb{A}}(\gamma, \delta) := \prod_v \Delta(\gamma_v, \delta_v, \bar{\gamma}_{H,v}, \bar{\gamma}_{G,v}). \tag{9.4}$$

This product is well-defined due to the following result:

Lemma 9.1.7 *In the notation of the above definition, the local transfer factor* $\Delta(\gamma_v, \delta_v, \bar{\gamma}_{H,v}, \bar{\gamma}_{G,v})$ *equals one for all but finitely many* v.

Proof. For all but finitely many v, the group G_{F_v} is quasi-split, in which case we may write

$$\Delta(\gamma_v, \delta_v, \bar{\gamma}_{H,v}, \bar{\gamma}_{G,v}) = \frac{\Delta(\gamma_v, \delta_v)}{\Delta(\bar{\gamma}_{H,v}, \bar{\gamma}_{G,v})}.$$

For a quasi-split connected reductive group over a local field, the (absolute) local transfer factor may be defined purely using Galois cohomology (cf. §5.3, 5.2.1). In such cases, the claim of the Lemma follows from the analogous fact in the characteristic-zero case, which is stated in [KS99, §7.3, pp. 109]. □

Remark 9.1.8 *It follows from Proposition 9.1.5 that the two formulas* (9.3) *and* (9.4) *given above for* $\Delta_{\mathbb{A}}$ *coincide when* $\gamma_H \in H(F)$*, so there is no notational ambiguity.*

Remark 9.1.9 *In the case that* $\mathcal{H} \neq {}^L H$ *in our global endoscopic datum, the formula for* $\Delta_{\mathbb{A}}$ *is slightly more complicated. To begin, we fix a* z*-pair* (H_1, ξ_{H_1}) *for the endoscopic datum* $\mathfrak{e} = (H, \mathcal{H}, s, \xi)$*, which always exist over fields of arbitrary characteristic. For any place* v *of* F*, this* z*-pair gives rise to a* z*-pair* $(H_{1,v}, \xi_{H_{1,v}})$ *for the local endoscopic datum* $\mathfrak{e}_v$*. We may then define the adelic transfer factor for pairs of elements* $\gamma_1 \in H_{1,G-sr}(\mathbb{A})$ *and* $\delta_v \in G_{sr}(\mathbb{A})$*, where* $\gamma_1 \in H_{1,G-sr}(\mathbb{A})$ *means that its image in* $H(\mathbb{A})$ *is* G*-strongly regular, using the relative local transfer factors for* z*-pairs as in §5.4:*

$$\Delta_{\mathbb{A}}(\gamma_1, \delta) := \prod_v \Delta(\gamma_{1,v}, \delta_v, \bar{\gamma}_{H,v}, \bar{\gamma}_{G,v}).$$

9.2 Endoscopic setup

This subsection is an analogue of [Kal18, §4.2, §4.3], which explain how to pass from global to local refined endoscopic data and discuss coherent families of local rigid inner twists; recall the notion of a refined endoscopic datum $(H, \mathcal{H}, \dot{s}, \xi)$ over a local function field F (defined in §6.2). A fixed global endoscopic datum $\mathfrak{e} = (H, \mathcal{H}, s, \xi)$ induces a canonical embedding $Z(G) \to Z(H)$, and we set $\bar{H} := H/Z_{\text{der}}$, where $Z_{\text{der}} := Z(\mathscr{D}(G^*))$, $Z_{\text{sc}} := Z(G^*_{\text{sc}})$, and $\bar{G}^* := G^*/Z_{\text{der}}$. Note that $\bar{G}^* = G^*_{\text{ad}} \times Z(G^*)/Z_{\text{der}}$ and $\widehat{\bar{G}^*} = \widehat{G^*}_{\text{sc}} \times Z(\widehat{G^*})^\circ$. We also set $Z := Z(G)$.

The L-embedding ξ induces an embedding $\widehat{\bar{H}} \to \widehat{\bar{G}^*}$ with image equal to $Z_{\widehat{\bar{G}^*}}(t)^\circ$, where recall that $t := \xi(s)$ (this is well-defined because $\widehat{\bar{G}^*}$ maps to $\widehat{G^*}$, which contains t). Then for $s_{\text{sc}} \in \widehat{G^*}_{\text{sc}}$ a fixed preimage of the image s_{ad} of s in $\widehat{G^*}_{\text{ad}}$ and a place $v \in V$, the third condition in the definition of a global endoscopic datum implies that we may find an element $y_v \in Z(\widehat{G^*})$ such that $s_{\text{der}} \cdot y_v \in Z(\widehat{H})^{\Gamma_v}$, where $s_{\text{der}} \in \mathscr{D}(\widehat{G^*})$ denotes the image of s_{sc}. We can then write $y_v = y'_v \cdot y''_v$ for $y'_v \in Z(\mathscr{D}(\widehat{G^*}))$ and $y''_v \in Z(\widehat{G^*})^\circ$, and we choose a lift $\dot{y}'_v \in \widehat{Z}_{\text{sc}}$ of y'_v. Then the element $(s_{\text{sc}} \cdot \dot{y}'_v, y''_v) =: \dot{s}_v$ lies in $\widehat{\bar{G}^*} = \widehat{G^*}_{\text{sc}} \times Z(\widehat{G^*})^\circ$, which, via the above L-embedding, belongs to the group $Z(\widehat{\bar{H}})^{+v}$, and $\dot{\mathfrak{e}}_v := (H, \mathcal{H}, \dot{s}_v, \xi)$ defines a local refined endoscopic datum at the place v. As noted [Kal18], we will show that the global objects coming from this collection $(\dot{\mathfrak{e}}_v)_v$ do not depend on the choices of s_{sc}, $\dot{y}'_v$, or y''_v, only on the equivalence class of the global endoscopic datum $\mathfrak{e}$.

We now discuss coherent families of local rigid inner twists. For an equivalence class Ψ of inner twists $G^*_{F^{\text{sep}}} \to G_{F^{\text{sep}}}$ (where two isomorphisms ψ, ψ' from G^* to G are equivalent if they differ by pre-composing with $\text{Ad}(g)$ for $g \in G^*_{\text{ad}}(F^{\text{sep}})$), base-changing to F_v for any $v \in V$ gives an equivalence class of Ψ_v of rigid inner twists $G^*_{F_v^{\text{sep}}} \to G_{F_v^{\text{sep}}}$. The class Ψ gives an element of $H^1(F, G^*_{\text{ad}})$ which by Lemma 8.1.1 has a preimage in the set $H^1(\mathcal{E}_{\dot{V}}, Z_{\text{sc}} \to G^*_{\text{sc}})$.

It follows that for every $\psi \in \Psi$, we can find a Z_{sc}-twisted $G^*_{\text{sc},\mathcal{E}_{\dot{V}}}$-torsor $\mathscr{T}_{\text{sc}}$ along with an isomorphism of $(G^*_{\text{ad}})_{\mathcal{E}_{\dot{V}}}$-torsors $\bar{h}\colon (\overline{\mathscr{T}_{\text{sc}}})_{\overline{F}} \xrightarrow{\sim} (\underline{(G^*_{\text{ad}})_{\mathcal{E}_{\dot{V}}}})_{\overline{F}}$, where $\overline{\mathscr{T}_{\text{sc}}} := \mathscr{T}_{\text{sc}} \times^{G^*_{\text{sc},\mathcal{E}_{\dot{V}}}} (G^*_{\text{ad}})_{\mathcal{E}_{\dot{V}}}$ and $\underline{(G^*_{\text{ad}})_{\mathcal{E}_{\dot{V}}}}$ denotes the trivial $(G^*_{\text{ad}})_{\mathcal{E}_{\dot{V}}}$-torsor, such that $p_1^*\bar{h} \circ p_2^*\bar{h}^{-1}$ is translation by $\bar{x} \in G^*_{\text{ad}}(\overline{F} \otimes_F \overline{F})$ which satisfies $\text{Ad}(\bar{x}) = p_1^*\psi^{-1} \circ p_2^*\psi$.

For each $v \in V$, we set $\mathscr{T}_v$ to be the Z-twisted $G^*_{\mathcal{E}_v}$-torsor given by $\text{loc}_v(\mathscr{T})$, where $\mathscr{T} := \mathscr{T}_{\text{sc}} \times^{G^*_{\text{sc},\mathcal{E}_{\dot{V}}}} G^*_{\mathcal{E}_{\dot{V}}}$, and loc_v is as defined at the beginning of §8.4; the $\overline{F}$-trivialization $\bar{h}$ evidently induces a $\overline{F_v}$-trivialization of $\overline{\mathscr{T}_v}$ (noting that $\overline{\mathscr{T}_{\text{sc}}} = \overline{\mathscr{T}}$), denoted by $\bar{h}_v$. Note that, by construction, the triple $(\psi, \mathscr{T}_v, \bar{h}_v)$ is a rigid inner twist over F_v; we thus get a collection $(\psi, \mathscr{T}_v, \bar{h}_v)_v$ of local rigid inner twists which depends on the definition of the localization maps loc_v (see §8.4), but only up to twisting torsors by $d(z)$ for an element $z \in Z_{\text{sc}}(\overline{F_v})$, which does not affect any associated cohomology sets. However, this family will in general depend on the choice of torsor $\mathscr{T}_{\text{sc}}$. Note that, in fact, since $\mathscr{T}$ is induced by the Z_{sc}-twisted G^*_{sc}-torsor $\mathscr{T}_{\text{sc}}$, it is actually Z_{der}-twisted, not just Z-twisted; that is, we may view $\mathscr{T}_v$ as an element of the set $H^1(\mathcal{E}_v, Z_{\text{der}} \to G^*)$.

9.3 Product decomposition of the adelic transfer factor

As in the previous section, Z denotes $Z(G)$. We will use the above results to show that the adelic transfer factor that we defined in §9.1 admits a decomposition in terms of the normalized local transfer factors constructed in §6.2, following [Kal18, §4.4]. We fix an equivalence class Ψ of inner twists $G^*_{F^{\mathrm{sep}}} \to G_{F^{\mathrm{sep}}}$, endoscopic datum $\mathfrak{e} = (H, \mathcal{H}, s, \xi)$ for G^*, and a z-pair $\mathfrak{z} = (H_1, \xi_1)$ for $\mathfrak{e}$. We assume that there exist strongly G-regular $\gamma_{1,0} \in H_1(F)$ and $\delta_0 \in G(F)$ such that $\gamma_{1,0}$ is related to δ_0 (so that, in particular, the image of $\gamma_{1,0}$ in $H(F)$, denoted by γ_0, is related to δ_0). As explained in §9.2, we can associate to $\mathfrak{e}$ the collection of refined local endoscopic data $(\dot{\mathfrak{e}}_v)_{v\in V}$ to the global z-pair $\mathfrak{z}$ a collection of local z-pairs $(\mathfrak{z}_v)_{v\in V}$, to the class Ψ a coherent family of local rigid inner twists $(\psi, \mathscr{T}_v, \bar{h}_v)_{v\in V}$, and to a fixed global Whittaker datum $\mathfrak{w}$ for G^*, a collection of local Whittaker data $(\mathfrak{w}_v)_{v\in V}$.

For any v, we can use the local Whittaker datum and z-pair to obtain from §6.2 the $\mathfrak{w}_v$-normalized local transfer factor

$$\Delta[\mathfrak{w}_v, \dot{\mathfrak{e}}_v, \mathfrak{z}_v, \psi, (\mathscr{T}_v, \bar{h}_v)] \colon H_{1,G-\mathrm{sr}}(F_v) \times G_{\mathrm{sr}}(F_v) \to \mathbb{C}.$$

This relates to the adelic transfer factor defined in §9.1 as follows:

Proposition 9.3.1 *For any $\gamma_1 \in H_{1,G-sr}(\mathbb{A})$ and $\delta \in G_{sr}(\mathbb{A})$, we have*

$$\Delta_{\mathbb{A}}(\gamma_1, \delta) = \prod_{v\in V} \langle loc_v(\mathscr{T}_{sc}), \dot{y}'_v\rangle \cdot \Delta[\mathfrak{w}_v, \dot{\mathfrak{e}}_v, \mathfrak{z}_v, \psi, (\mathscr{T}_v, \bar{h}_v)](\gamma_{1,v}, \delta_v).$$

In the above formula, $\dot{y}'_v \in \widehat{Z}_{sc}$ as in §9.2 and the pairing $\langle -, - \rangle \colon H^1(\mathcal{E}_v, Z_{sc} \to G^*_{sc}) \times \widehat{Z}_{sc} \to \mathbb{C}$ is from Corollary 6.2.2, which is well-defined since $\dot{y}'_v \in \widehat{Z}_{sc} = Z(\widehat{G^*_{sc}/Z_{sc}})^+$. For almost all $v \in V$, the corresponding factor in the product equals 1. For all v, the corresponding factor is independent of the choices of $\dot{y}'_v$ and y''_v made in §9.2.

Proof. The argument closely follows [Kal18, Proposition 4.4.1]; as in the proof of the result loc. cit., it follows from [LS87, Corollary 6.4.B] that the above product identity follows if we can show that the normalized factors $\langle \mathrm{loc}_v(\mathscr{T}_{\mathrm{sc}}), \dot{y}'_v\rangle \cdot \Delta[\mathfrak{w}_v, \dot{\mathfrak{e}}_v, \mathfrak{z}_v, \psi, (\mathscr{T}_v, \bar{h}_v)](\gamma_{1,v}, \delta_v)$ satisfy the following properties: First, that they are absolute transfer factors, and second, that their values at the F-rational pair $(\gamma_{1,0,v}, \delta_{0,v})$ equal 1 for all but finitely many v and have a product over all v that equals 1. The first property automatically holds for the above factors by Proposition 6.2.3 (the extra $\langle \mathrm{loc}_v(\mathscr{T}_{\mathrm{sc}}), \dot{y}'_v\rangle$-factor cancels out and thus makes no difference for this verification).

The same argument as in the proof of [Kal18, Proposition 4.4.1] (replacing the use of [LS87, Theorem 6.4.A] in the proof loc. cit. with our Proposition 9.1.5 and noting that the discussion of local and global ϵ-factors in [KS99], which in turn uses the construction of such factors in [Tat79], §3, works for local and global fields of arbitrary characteristic) reduces the second property above to showing that the terms

$$\langle \mathrm{loc}_v(\mathscr{T}_{\mathrm{sc}}), \dot{y}'_v\rangle^{-1} \langle \mathrm{inv}((G_{F_v}, \psi, (\mathscr{T}_v, \bar{h}_v), \delta_{0,v}), \delta^*_{0,v}), \dot{s}_{v,\gamma_0,\delta^*_0}\rangle \tag{9.5}$$

are equal to 1 for almost all v and have product over all v equal to 1, where $\delta_0^* \in G^*(F)$ is the image of γ_0 under a choice of admissible embedding of $T_{0,H}$ into G^* and $T_0 := Z_{G^*}(\delta_0^*)$, the map $\text{inv}(-, \delta_{0,v}^*)\colon C_{Z_{\text{der}}}(\delta_{0,v}^*) \to H^1(\mathcal{E}_v, Z_{\text{der}} \to T_0)$ is as defined in §6.1, the element $\dot{s}_{v,\gamma_0,\delta_0^*} \in \pi_0(\widehat{\bar{T}_0}^{+,v})$ is the image of $\dot{s}_v \in \pi_0(Z(\widehat{\bar{H}})^{+,v})$ under the composition $\hat{\varphi}\colon Z(\widehat{\bar{H}}) \to \widehat{\overline{T_{0,H}}} \to \widehat{\bar{T}_0}$ (recall that the bar indicates that we are quotienting out by Z_{der}) induced by our choice of admissible embedding of $T_{0,H}$ into G^*, and the right-hand pairing is from Corollary 6.2.2.

In order to work explicitly with the invariant at a place v, it will be convenient to fix an explicit Čech 2-cocycle ξ_v representing the canonical class in $\check{H}^2(\overline{F_v}/F_v, u_v)$ and replace the notion of Z_{der}-twisted torsors on the gerbe $\mathcal{E}_{\xi_v}$ with ξ_v-twisted 1-cocycles; we know by §6.2 that the invariant map and corresponding local transfer factor do not depend on such a choice, and hence we may do so without loss of generality.

By construction, the elements δ_0^* and δ_0 are stably conjugate, so that there exists $g \in G^*(\overline{F})$ such that $\psi(g\delta_0^* g^{-1}) = \delta_0$, and then $\text{inv}((G_{F_v}, \psi, (\mathscr{T}_v, \bar{h}_v), \delta_{0,v}), \delta_{0,v}^*) \in H^1(\mathcal{E}_v, Z_{\text{der}} \to T_0) = H^1(\mathcal{E}_{\xi_v}, Z_{\text{der}} \to T_0)$ is represented by the ξ_v-twisted (Čech) 1-cocycle

$$x_v := (p_1(g)^{-1} z_v p_2(g), \phi_v),$$

where (z_v, ϕ_v) is a choice of ξ_v-twisted 1-cocycle corresponding to the Z_{der}-twisted $G^*_{\mathcal{E}_v}$-torsor $\mathscr{T}_v$, as explained in §6.1. We may choose g so that it is the image of some $g_{\text{sc}} \in G^*_{\text{sc}}(\overline{F})$, and then we may lift the twisted cocycle x_v to the ξ_v-twisted cocycle $x_{v,\text{sc}} := (p_1(g_{\text{sc}})^{-1} z_{\text{sc},v} p_2(g_{\text{sc}}), \phi_{\text{sc},v})$, where $(z_{\text{sc},v}, \phi_{\text{sc},v}) \in Z^1(\mathcal{E}_{\xi_v}, Z_{\text{sc}} \to G^*_{\text{sc}})$ is a choice of ξ_v-twisted cocycle corresponding to the Z_{sc}-twisted G^*_{sc}-torsor $\text{loc}_v(\mathscr{T}_{\text{sc}})$ on $\mathcal{E}_v$.

Using the decomposition $\widehat{\bar{T}_0} = (\widehat{T_0})_{\text{sc}} \times Z(\widehat{G^*})^\circ$, we may use the notation of §9.2 to write $\dot{s}_{v,\gamma_0,\delta_0^*} = (\dot{y}_v' \hat{\varphi}(s_{\text{sc}}), y_v'')$. The functoriality of the pairing from Corollary 6.2.2 with respect to the morphism $[Z_{\text{sc}} \to T_{0,\text{sc}}] \to [Z_{\text{der}} \to T_0]$, then implies that

$$\langle \text{inv}((G_{F_v}, \psi, (\mathscr{T}_v, \bar{h}_v), \delta_{0,v}), \delta_{0,v}^*), \dot{s}_{v,\gamma_0,\delta_0^*} \rangle = \langle x_{v,\text{sc}}, \dot{y}_v' \hat{\varphi}(s_{\text{sc}}) \rangle.$$

By construction, the restriction of the character $\langle x_{v,\text{sc}}, - \rangle$ on $\pi_0(\widehat{\overline{T_{0,\text{sc}}}}^{+v})$ to $Z(\widehat{\overline{G^*_{\text{sc}}}})^{+v}$ equals the character $\langle (z_{\text{sc},v}, \phi_{\text{sc},v}) - \rangle$ by the functoriality of the pairing with respect to the morphism $[Z_{\text{sc}} \to T_{0,\text{sc}}] \to [Z_{\text{sc}} \to G^*_{\text{sc}}]$. It then follows by bilinearity that the expression (9.5) reduces to

$$\langle x_{v,\text{sc}}, \hat{\varphi}(s_{\text{sc}}) \rangle. \tag{9.6}$$

We have already fixed normalizations $\mathcal{E}_{\xi_v}$ of the gerbes $\mathcal{E}_v$ for all v—we now also fix a normalization $\mathcal{E}_{\dot{\xi}}$ of the gerbe $\mathcal{E}_{\dot{V}}$. Such a normalization identifies $\mathscr{T}_{\text{sc}}$ with a $\dot{\xi}$-twisted (Čech) 1-cocycle $(z_{\text{sc}}, \phi_{\text{sc}})$, where $z_{\text{sc}} \in G^*_{\text{sc}}(\overline{F} \otimes_F \overline{F})$, which by construction has image in $Z^1(\mathcal{E}_{\xi_v}, Z_{\text{sc}} \to G^*_{\text{sc}})$ equal to $(z_{\text{sc},v}, \phi_{\text{sc},v})$. We may thus define a global twisted 1-cocycle by the formula

$$x_{\text{sc}} := (p_1(g_{\text{sc}})^{-1} z_{\text{sc}} p_2(g_{\text{sc}}), \phi_{\text{sc}}) \in Z^1(\mathcal{E}_{\dot{\xi}}, Z_{\text{sc}} \to T_{0,\text{sc}}),$$

which satisfies $\text{loc}_v(x_{\text{sc}}) = x_{v,\text{sc}}$, where loc_v on twisted 1-cocycles is induced by the maps $u_v \to (P_{\dot{V}})_{F_v}$ and $G^*_{\text{sc}}(\overline{F} \otimes_F \overline{F}) \to G^*_{\text{sc}}(\overline{F_v} \otimes_{F_v} \overline{F_v})$ for a fixed v. It then follows from Corollary 8.2.14 that the class $[x_{\text{sc}}] \in H^1(\mathcal{E}_{\dot{\xi}}, Z_{\text{sc}} \to T_{0,\text{sc}})$ is such that $[\text{loc}_v(x_{\text{sc}})] = [x_{v,\text{sc}}] \in H^1(\mathcal{E}_{\xi_v}, Z_{\text{sc}} \to T_{0,\text{sc}})$ is trivial for all but finitely-many v, which shows that the expression (9.6), and thus also the expression (9.5), is 1 for all but finitely many v, as desired.

To finish proving the product identity, we first recall the functor $\overline{Y}_{+,\text{tor}} \colon \mathcal{R} \to \text{AbGrp}$. It follows from the proof of [Kal16, Proposition 5.3], (the proof of which is purely character-theoretic) that we have a functorial embedding

$$\overline{Y}_{+,\text{tor}}([Z \to G]) \hookrightarrow \pi_0([Z(\widehat{\overline{G}})^+])^*, \tag{9.7}$$

and it is straightforward to check that for any $[Z \to G] \in \mathcal{R}$, the diagram

$$\begin{array}{ccc} \bigoplus_v \overline{Y}_{+v,\text{tor}}([Z \to G]) & \xrightarrow{\Sigma} & Y_{+,\text{tor}}([Z \to G]) \\ \downarrow & & \downarrow \\ \bigoplus_v \pi_0([Z(\widehat{\overline{G}})^{+v}])^* & \longrightarrow & \pi_0([Z(\widehat{\overline{G}})^+])^* \end{array}$$

commutes, where the left-hand vertical map is the sum of the local embeddings

$$\overline{Y}_{+v,\text{tor}}([Z \to G]) \hookrightarrow \pi_0([Z(\widehat{\overline{G}})]^{+v})^*$$

and the lower horizontal map is induced by restricting characters on the groups $\pi_0([Z(\widehat{\overline{G}})]^{+v})$ to $\pi_0([Z(\widehat{\overline{G}})]^+)$.

If for each v we restrict the character $\langle \text{loc}_v([x_{\text{sc}}]), - \rangle$ on $\pi_0([\widehat{\overline{T_{0,\text{sc}}}}]^{+v})$ to $\pi_0([\widehat{\overline{T_{0,\text{sc}}}}]^+)$ and then take the product over all v (these characters are trivial for all but finitely-many v due to the above discussion and Corollary 8.3.2), we obtain the trivial character on $\pi_0([\widehat{\overline{T_{0,\text{sc}}}}]^+)$ via combining the above discussion with Corollary 8.3.3. By construction, we have that the image of $\hat{\varphi}(s_{\text{sc}}) \in \widehat{\overline{T}}_0$ in $\widehat{\overline{T_{0,\text{sc}}}}/([\widehat{\overline{T_{0,\text{sc}}}}]^{+,\circ})$ lies in $\pi_0([\widehat{\overline{T_{0,\text{sc}}}}]^+)$, which combines with the first part of this paragraph to give the equality $\langle x_{\text{sc}}, \hat{\varphi}(s_{\text{sc}}) \rangle = 1$, where the pairing is induced by the embedding (9.7) and Theorem 8.3.1, proving that the above product over all places equals 1, as desired. Finally, as in the number field case, the absence of $\dot{y}'_v$ and $\dot{y}''_v$ in the expression (9.5) implies that the product does not depend on the choice of such elements. Moreover, since $\langle x_{v,\text{sc}}, \hat{\varphi}(s_{\text{sc}}) \rangle$ only depends on the cohomology class of $x_{\text{sc},v}$, the product also does not depend on the choice of gerbe normalizations used to construct the torsors $\mathrm{loc}_v(\mathscr{T}_{\mathrm{sc}})$. □

9.4 The multiplicity formula for discrete automorphic representations

We use the same notation as in the previous subsection; in particular, $\bar{G}$ denotes G/Z_{der}. As in [Kal18, §4.5], fix an L-homomorphism $\varphi\colon L_F \to {}^LG^*$ with bounded image, where L_F is the hypothetical Langlands group of F. At each place $v \in V$, the parameter φ has a localization, which is a parameter $\varphi_v\colon L_{F_v} \to {}^LG^*$. The local conjecture ensures that there exists an L-packet Π_{φ_v} of tempered representations of rigid inner twists of G^* together with a bijection

$$\iota_{\varphi_v,\mathfrak{w}_v}\colon \Pi_{\varphi_v} \to \mathrm{Irr}(S^+_{\varphi_v}).$$

In the above setting, the set Π_{φ_v} consists of equivalence classes of tuples $(G'_v, \psi'_v, (\mathscr{T}'_v, \bar{h}'_v), \pi'_v)$, where $(\psi'_v, \mathscr{T}'_v, \bar{h}'_v)\colon G^*_{F_v} \to G'_v$ is a rigid inner twist over F_v and π'_v is an irreducible tempered representation of $G'(F_v)$. The group $S^+_{\varphi_v}$ is the preimage in $\widehat{\bar{G^*}}$ of $S_{\varphi_v} := Z_{\widehat{G^*}}(\varphi_v)$ and $\mathfrak{w}_v$ is a local Whittaker datum on which the bijection depends. As explained in [Kal18, §4.4], we may choose a global Whittaker datum $\mathfrak{w}$ for G^* and let $\mathfrak{w}_v$ be its localization at each place v.

Recall that we have fixed a quasi-split inner twist $\psi\colon G^*_{F^{\mathrm{sep}}} \to G_{F^{\mathrm{sep}}}$ of G; choose a coherent family of local rigid inner twists $(\psi, \mathscr{T}_v, \bar{h}_v)_v$ as in §9.2, and consider the subset $\Pi_{\varphi_v}(G) \subseteq \Pi_{\varphi_v}$ consisting of (isomorphism classes of) tuples $(G_{F_v}, \psi, (\mathscr{T}_v, \bar{h}_v), \pi_v)$. We then define the L-packet

$$\Pi_\varphi := \{\pi = \otimes'_v \pi_v \mid (G_{F_v}, \psi, \mathscr{T}_v, \bar{h}_v, \pi_v) \in \Pi_{\varphi_v}(G), \iota_{\varphi_v}((G_{F_v}, \psi, (\mathscr{T}_v, \bar{h}_v), \pi_v)) = 1 \text{ for almost all } v\}.$$

The following result is of crucial importance:

Lemma 9.4.1 *The set Π_φ consists of irreducible admissible tempered representations of $G(\mathbb{A})$.*

Proof. We may assume without loss of generality that we have picked a normalization of the gerbe $\mathcal{E}_{\dot{V}}$, which recall is a choice of representative $\dot{\xi}$ of the canonical class, as well as an isomorphism of $P_{\dot{V}}$-gerbes $\mathcal{E}_{\dot{V}} \to \mathcal{E}_{\dot{\xi}}$; we will nevertheless continuing using the notation $\mathcal{E}_{\dot{V}}$ for the explicit gerbe $\mathcal{E}_{\dot{\xi}}$. As in the proof of [Kal18, Lemma 4.5.1], everything is clear except for the fact that the representation π_v is unramified for almost all v. As explained in [Kal18], we may find a finite set S of places of F such that G^* and G have $O_{F,S}$-models $\mathcal{G}^*$, $\mathcal{G}$ (respectively), the inner twist isomorphism ψ is defined over $O_S \subset F^{\mathrm{sep}}$, the Whittaker datum $\mathfrak{w}_v$ is unramified for every $v \notin S$, the local parameter φ_v is unramified. We have the $G^*_{\mathcal{E}_{\dot{V}}}$-torsor $\mathscr{T}$ with fixed $\overline{F}$-trivialization $\overline{h}$ of $\overline{\mathscr{T}}$; note that if $s\colon \mathrm{Sch}/\overline{F} \to \mathcal{E}_{\dot{V}}$ is the canonical embedding of categories given by Lemma

2.3.2, we obtain a G^*-torsor $s^*\mathscr{T}$ over $\overline{F}$, and we may pick a trivialization of this torsor over $\overline{F}$ which is compatible with the trivialization $\bar{h}$ (see §6.1). Such a compatible trivialization is equivalent to picking a trivialization h of $\mathcal{S}$ over $\overline{F}$, where $(\mathcal{S}, \mathrm{Res}(\mathscr{T}), \psi_{\mathcal{S}})$ is the twisted $G^*_{\overline{F}}$-torsor corresponding to $\mathscr{T}$, such that the induced trivialization of the $G^*_{\mathrm{ad},\overline{F}}$-torsor $\mathcal{S} \times^{G^*} G^*_{\mathrm{ad}}$ over $\overline{F}$ associated to the twisted torsor $(\mathcal{S} \times^{G^*} G^*_{\mathrm{ad}}, 0, \bar{\psi}_{\mathcal{S}})$ equals the trivialization induced by $\bar{h}$.

We know from Proposition 8.4.1 that we may enlarge S to ensure that, for all $v \notin S$, the pair of each localization $\mathscr{T}_v$ and $\overline{F_v}$-trivialization h_v (induced by h) is the pullback of a $\mathcal{G}^*_{O_{F_v}}$-torsor T_v over O_{F_v} with trivialization $h_{O_{F_v}}$ over $O^{\mathrm{perf}}_{F_v^{\mathrm{nr}}}$. Note that a priori each $\mathscr{T}_v$ is a torsor on $\mathcal{E}_v$, not on Sch/F_v, but we may enlarge S to ensure that $\mathscr{T}_v$ is the pullback of a unique G^*-torsor over F_v, which we identify with $\mathscr{T}_v$ (see §8.4), so that this latter statement makes sense.

The cohomology set $\check{H}^1(O^{\mathrm{perf}}_{F_v^{\mathrm{nr}}}/O_{F_v}, \mathcal{G}^*)$ classifies isomorphism classes of $\mathcal{G}^*$-torsors over O_{F_v} which have a trivialization over the fpqc extension $O^{\mathrm{perf}}_{F_v^{\mathrm{nr}}}$. We have a natural injective map

$$\check{H}^1(O^{\mathrm{perf}}_{F_v^{\mathrm{nr}}}/O_{F_v}, \mathcal{G}^*) \to \check{H}^1_{\mathrm{fppf}}(O_{F_v}, \mathcal{G}^*),$$

where the latter set classifies isomorphism classes of $\mathcal{G}^*$-torsors over O_{F_v}. Moreover, the set $\check{H}^1_{\mathrm{fppf}}(O_{F_v}, \mathcal{G}^*)$ is trivial, by [Čes16, Corollary 2.9] (and Lang's theorem), giving the triviality of $\check{H}^1(O^{\mathrm{perf}}_{F_v^{\mathrm{nr}}}/O_{F_v}, \mathcal{G}^*)$. It follows that we may find an element $g \in G^*(O^{\mathrm{perf}}_{F_v^{\mathrm{nr}}}) = \mathcal{G}^*(O^{\mathrm{perf}}_{F_v^{\mathrm{nr}}})$ whose Čech differential coincides with the element of $\mathcal{G}^*(O^{\mathrm{perf}}_{F_v^{\mathrm{nr}}} \otimes_{O_{F_v}} O^{\mathrm{perf}}_{F_v^{\mathrm{nr}}})$ whose left-translation gives $p_1^* h_{O_{F_v}} \circ p_2^* h^{-1}_{O_{F_v}}$ on $\mathcal{G}^*_{O^{\mathrm{perf}}_{F_v^{\mathrm{nr}}} \otimes_{O_{F_v}} O^{\mathrm{perf}}_{F_v^{\mathrm{nr}}}}$. As a consequence, we get by fpqc descent that the morphism $f' := \psi_{O^{\mathrm{perf}}_{F_v^{\mathrm{nr}}}} \circ \mathrm{Ad}(g^{-1})$ descends to an O_{F_v}-morphism $f \colon \mathcal{G}^* \to \mathcal{G}$.

The element $g \in G^*(\overline{F_v})$ defines an F_v-trivialization of $\mathscr{T}_v$ by means of (the descent of) the composition

$$\Psi := \ell_g \circ h_v \colon (\mathscr{T}_v)_{\overline{F_v}} \to (\underline{G^*_{\mathcal{E}_v}})_{\overline{F_v}},$$

where ℓ_g denotes left-translation by g; by construction, this map descends to F_v. As a consequence, (f, Ψ) defines an isomorphism of rigid inner twists from $(\psi, \mathscr{T}_v, \bar{h}_v)$ to the trivial rigid inner twist $(\mathrm{id}_{G^*}, \underline{G^*_{\mathcal{E}_v}}, \mathrm{id})$. Choosing S large enough, the construction of Π_φ then implies that $\iota_{\varphi_v}((G^*, \mathrm{id}_{G^*}, \underline{G^*_{\mathcal{E}_v}}, \mathrm{id}, \pi_v \circ f)) = 1$, which means that the representation $\pi_v \circ f$ of $G^*(F_v)$ is $\mathfrak{w}_v$-generic. This latter fact implies, by [CS80], that the representation $\pi_v \circ f$ is unramified with respect to the hyperspecial subgroup $G^*(O_{F_v})$ of $G^*(F_v)$. The fact that the isomorphism f is defined over O_{F_v} then implies that π_v is unramified with respect to the subgroup $G(O_{F_v})$, as desired. □

As is conjectured in the number field case, we expect that every tempered discrete automorphic representation of $G(\mathbb{A})$ belongs to a set Π_φ for some discrete parameter φ. Moreover, for any such representation π, our framework allows for a conjectural description of the multiplicity of π in the discrete spectrum of G; to begin this description, we need some setup. First, note that we have a short-exact sequence of L_F-modules

$$1 \to Z(\widehat{G^*}) \to \widehat{G^*} \to (\widehat{G^*})_{\text{ad}} \to 1,$$

where the L_F-action is defined via ad $\circ\, \varphi$, which gives a connecting homomorphism

$$Z_{(\widehat{G^*})_{\text{ad}}}(\varphi) \to H^1(L_F, Z(\widehat{G^*})).$$

We then define S_φ^{ad} to be the kernel of the composition

$$Z_{(\widehat{G^*})_{\text{ad}}}(\varphi) \to H^1(L_F, Z(\widehat{G^*})) \to \prod_v H^1(L_{F_v}, Z(\widehat{G^*}))$$

and set $\mathcal{S}_\varphi := \pi_0(S_\varphi^{\text{ad}})$. We will construct a pairing

$$\langle -, - \rangle : \mathcal{S}_\varphi \times \Pi_\varphi \to \mathbb{C}$$

which yields an integer

$$m(\varphi, \pi) := |\mathcal{S}_\varphi|^{-1} \sum_{x \in \mathcal{S}_\varphi} \langle x, \pi \rangle.$$

We then expect (from [Kot84]) the multiplicity of π in the discrete spectrum of G to be given by

$$\sum_\varphi m(\varphi, \pi),$$

where the sum is over all equivalence classes (as in [Kot84, §10.4]) of φ such that $\pi \in \Pi_\varphi$.

The construction of the above pairing is identical to the number field in analogue in [Kal18], but we review it here for completeness. For some $s_{\text{ad}} \in S_\varphi^{\text{ad}}$, we choose a lift $s_{\text{sc}} \in S_\varphi^{\text{sc}}$ (the preimage of S_φ^{ad} in $(\widehat{G^*})_{\text{sc}}$). Then, as explained in [Kal18, §4.5], we obtain from s_{sc} an element $\dot{s}_v \in S_{\varphi_v}^+$ for each $v \in \dot{V}$, which we write as $(s_{\text{sc}} \cdot \dot{y}'_v, y''_v)$ for $y'_v \in Z((\widehat{G^*})_{\text{der}})$ and $y''_v \in Z(\widehat{G^*})^\circ$ via the decomposition $\widehat{\bar{G}^*} = (\widehat{\bar{G}^*})_{\text{sc}} \times Z(\widehat{G^*})^\circ$. Following [Kal18], we denote by

$$\langle (s_{\text{sc}} \cdot \dot{y}'_v, y''_v), (G_{F_v}, \psi, (\mathscr{T}_v, \bar{h}_v), \pi_v) \rangle$$

the character of the representation $\iota_{\varphi_v}((G_{F_v}, \psi, (\mathscr{T}_v, \bar{h}_v), \pi_v))$ of $\pi_0(S_{\varphi_v}^+)$ evaluated at $\dot{s}_v$. These values behave well after taking the product over all v in the following sense:

Proposition 9.4.2 *([Kal18, Proposition 4.5.2]) The value*

$$\langle loc_v(\mathscr{T}_{sc}), \dot{y}'_v \rangle^{-1} \cdot \langle (s_{sc} \cdot \dot{y}'_v, y''_v), (G_{F_v}, \psi, (\mathscr{T}_v, \bar{h}_v), \pi_v) \rangle$$

equals 1 *for all but finitely many* v*, where* $\mathscr{T}_{sc}$ *is as in §9.2, and the product*

$$\langle s_{ad}, \pi \rangle := \prod_{v \in V} \langle loc_v(\mathscr{T}_{sc}), \dot{y}'_v \rangle^{-1} \cdot \langle (s_{sc} \cdot \dot{y}'_v, y''_v), (G_{F_v}, \psi, (\mathscr{T}_v, \bar{h}_v), \pi_v) \rangle$$

is independent of the choices of $s_{sc}, \dot{y}'_v, y''_v$*, the torsor* $\mathscr{T}_{sc}$*, and the global Whittaker datum* $\mathfrak{w}$*. Moreover, the function* $s_{ad} \mapsto \langle s_{ad}, \pi \rangle$ *is the character of a finite-dimensional representation of* $\mathcal{S}_\varphi$*.*

Proof. This proof is identical to the proof of the analogous result in [Kal18], replacing the use of Corollary 3.7.5 loc. cit. with our Corollary 8.2.14 and the (conjectural) endoscopic character identities from [Kal16, §3.4], with the analogous identities from §6.3. □

Appendix A

(Complexes of Tori and Čech Cohomology)

This appendix gives an extension of the theory of *complexes of tori* developed in the appendices of [KS99] to the setting of local and global function fields.

A.1 Complexes of tori over local function fields—basic results

Suppose that we have a complex of commutative R-groups, which is concentrated in degrees 0 and 1, denoted by $G \xrightarrow{f} H$ (or, when both groups are R-tori, by $T \xrightarrow{f} U$). For any fpqc ring homomorphism $R \to S$, we obtain a double complex $K^{\bullet,\bullet}$ by taking the Čech complexes associated to G and H; that is, the double complex

$$\begin{array}{ccccccc}
G(S) & \longrightarrow & G(S \otimes_R S) & \longrightarrow & G(S \otimes_R S \otimes_R S) & \longrightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \\
H(S) & \longrightarrow & H(S \otimes_R S) & \longrightarrow & H(S \otimes_R S \otimes_R S) & \longrightarrow & \dots;
\end{array}$$

for our applications, it will always be the case that the Čech cohomology groups $\check{H}^i(S/R, G)$ compute the fppf cohomology $H^i(R, G)$ (although S/R itself need not be an fppf cover). As usual, we can associate to this double complex a new complex $L^\bullet$, whose degree-r term is given by

$$L^r(T^\bullet) = \bigoplus_{m+n=r} K^{m,n} = G(S^{\bigotimes_R r}) \oplus H(S^{\bigotimes_R r-1}),$$

with differentials defined by $(d_G \oplus f - d_H)$. Following [KS99], we call the elements of L^r *(Čech) r-hypercochains*, and the elements of the kernel of the rth differential *(Čech) r-hypercocycles*. Denote by $H^r(S/R, G \xrightarrow{f} H)$ the rth cohomology group of the complex $L^\bullet$. Note that, by fpqc descent, $H^0(S/R, G \xrightarrow{f} H) = \ker(G(R) \to H(R)) = \ker(f)(R)$, which will be useful when $\ker(f)$ is a finite-type F-group scheme whose cohomology we want to investigate.

The spectral sequences associated to a double complex give us the long exact sequence

$$\cdots \to H^r(S/R, G \xrightarrow{f} H) \to \check{H}^r(S/R, G) \to \check{H}^r(S/R, H) \to H^{r+1}(S/R, G \xrightarrow{f} H) \to \ldots, \tag{A.1}$$

where the first map sends $[(x, y)]$ to $[x]$, the last map sends $[x]$ to $[(0, x)]$, and the middle map is induced by f. They also give the long exact sequence

$$\cdots \to \check{H}^r(S/R, \ker(f)) \to H^r(S/R, G \xrightarrow{f} H) \to H^{r-1}(\text{cok}(f^{\bigotimes \bullet})) \to \check{H}^{r+1}(S/R, \ker(f)) \to \ldots, \tag{A.2}$$

where $\text{cok}(f^{\bigotimes \bullet})$ denotes the complex with degree-r term given by $\frac{H(S^{\bigotimes_R r})}{f(G(S^{\bigotimes_R r}))}$.

In the long exact sequence (A.2), the first map is given by $[x] \mapsto [(x, 0)]$, the middle map by $[(x, y)] \mapsto [\bar{y}]$, and the last map by the composition of the map $H^{r-1}(\text{cok}(f^{\bigotimes \bullet})) \to H^r(\text{im}(f^{\bigotimes \bullet}))$ defined by picking a preimage $x \in H(S^{\bigotimes_R r})$ of an r-cocycle $\bar{x} \in \frac{H(S^{\bigotimes_R r})}{f(G(S^{\bigotimes_R r}))}$ and then applying the Čech differential, and the map $H^r(\text{im}(f^{\bigotimes \bullet})) \to \check{H}^{r+1}(S/R, \ker(f))$ given by picking a preimage in $G(S^{\bigotimes_R (r+1)})$ of $x \in f(G(S^{\bigotimes_R (r+1)}))$ and then differentiating.

We now make the situation more concrete by setting $R = F$ a field; the following result is an immediate extension of the fact that, for a smooth finite type commutative F-group scheme G, the comparison map $\check{H}^i(F^{\text{sep}}/F, G) \to \check{H}^i(\overline{F}/F, G)$ is always an isomorphism:

Lemma A.1.1 *For all $i \geq 1$, the natural map $H^i(F^{sep}/F, T \xrightarrow{f} U) \to H^i(\overline{F}/F, T \xrightarrow{f} U)$ is an isomorphism.*

Proof. This follows immediately from the five-lemma, applied to the commutative diagram with exact rows induced by (A.1)

$$\begin{array}{ccccccccc}
\check{H}^{i-1}(F^{\text{sep}}/F, T) & \longrightarrow & \check{H}^{i-1}(F^{\text{sep}}/F, U) & \longrightarrow & H^i(F^{\text{sep}}/F, T \xrightarrow{f} U) & \longrightarrow & \check{H}^i(F^{\text{sep}}/F, T) & \longrightarrow & \check{H}^i(F^{\text{sep}}/F, T) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\check{H}^{i-1}(\overline{F}/F, T) & \longrightarrow & \check{H}^{i-1}(\overline{F}/F, U) & \longrightarrow & H^i(\overline{F}/F, T \xrightarrow{f} U) & \longrightarrow & \check{H}^i(\overline{F}/F, T) & \longrightarrow & \check{H}^i(\overline{F}/F, T),
\end{array}$$

where all vertical maps other than the one in consideration are isomorphisms, since T and U are tori (in particular, are smooth). □

We also have the following relation between Čech hypercohomology with respect to F^{sep}/F and Galois cohomology:

Lemma A.1.2 *For all i, we have a canonical isomorphism*

$$H^i(F^{sep}/F, T \xrightarrow{f} U) \xrightarrow{\sim} H^i(\Gamma, T(F^{sep}) \to U(F^{sep})),$$

where the latter group is as defined in [KS99], Appendix A.

Proof. This is immediate from applying the comparison isomorphisms discussed in §3.1. □

We now discuss a local Tate-Nakayama pairing in this context; there is not much work to do here, as we may simply follow [KS99]. We will now assume that $R = F$ is a local function field, $S = \overline{F}$ a fixed algebraic closure, $G \xrightarrow{f} H$ is a complex of F-tori, denoted by $T^{\bullet} := T \xrightarrow{f} U$, with dual complex of character modules (over Γ) $X^{*,\bullet} := X^*(U) \xrightarrow{-f^*} X^*(T)$, concentrated in degrees -1 and 0. The character groups are just Γ-modules, so the theory of [KS99], appendix A applies, giving us a double complex $K_*^{\bullet;\bullet}$ equal to

$$\begin{array}{ccccccc} X^*(U) & \longrightarrow & C^1(\Gamma, X^*(U)) & \longrightarrow & C^2(\Gamma, X^*(U)) & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ X^*(T) & \longrightarrow & C^1(\Gamma, X^*(T)) & \longrightarrow & C^2(\Gamma, X^*(T)) & \longrightarrow & \dots, \end{array}$$

where all vertical arrows are induced by f^*; the associated complex is $L_*^r(X^{*,\bullet}) = C^r(\Gamma, X^*(T)) \oplus C^{r+1}(\Gamma, X^*(U))$.

We have a pairing of abelian groups

$$\cup \colon L^r(T^{\bullet}) \times L_*^s(X^{*,\bullet}) \to \mathbb{G}_m(\overline{F}^{\bigotimes_F r+s})$$

defined by taking the sum of the pairing $T(\overline{F}^{\bigotimes_F r}) \times C^s(\Gamma, X^*(T)) \to \mathbb{G}_m(\overline{F}^{\bigotimes_F r+s})$ and $(-1)^{r-1}$ times the pairing $U(\overline{F}^{\bigotimes_F r-1}) \times C^{s+1}(\Gamma, X^*(U)) \to \mathbb{G}_m(\overline{F}^{\bigotimes_F r+s})$. It is straightforward to check that this cup product satisfies the identity $d(a \cup b) = (da) \cup b + (-1)^r (a \cup db)$ for all $x \in L^r(T^{\bullet})$, and thus induces a pairing

$$H^r(\overline{F}/F, T \xrightarrow{f} U) \times H^s(L_*^{\bullet}(X^{*,\bullet})) \to \check{H}^{r+s}(\overline{F}/F, \mathbb{G}_m) = H^{r+s}(F, \mathbb{G}_m).$$

Note that, via degree-shifting, there is a canonical isomorphism

$$H^s(L_*^{\bullet}(X^{*,\bullet})) \xrightarrow{\sim} H^{s+1}(\Gamma, X^*(U) \xrightarrow{f^*} X^*(T)),$$

where, as the notation indicates, we are now viewing the complex $X^*(U) \xrightarrow{f^*} X^*(T)$ as concentrated in degrees 0 and 1 and taking the cohomology of the corresponding total complex.

Remark A.1.3 *There is an apparent discrepancy between our use of cohomology with respect to the fqpc cover Spec$(\overline{F}) \to$ Spec(F) when dealing with the tori T, U, and our use of cohomology with respect to the fpqc cover Spec$(F^{sep}) \to$ Spec(F) implicit in our use of Γ-cohomology* *to treat the Cartier dual group schemes $\underline{X}^*(T), \underline{X}^*(U)$. However, we remind the reader that since for any F-torus S the Cartier dual group scheme $\underline{X}^*(S)$ is étale, the natural inclusion $\underline{X}^*(S)((F^{sep})^{\bigotimes_F n}) \to \underline{X}^*(S)(\overline{F}^{\bigotimes_F n})$ is an isomorphism, which means that we may canonically identify all groups of Čech cochains (and hence the cocycles and coboundaries) with respect to these two different covers.*

As such, for any $r \in \mathbb{Z}$, we may apply this identification and the invariant map $H^2(F, \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z}$ to obtain the *Tate-Nakayama pairing*

$$H^r(\overline{F}/F, T \xrightarrow{f} U) \times H^{3-r}(F, X^*(U) \xrightarrow{f^*} X^*(T)) \to \mathbb{Q}/\mathbb{Z}.$$

Note that for any F-torus S, we have $H^i(F, S) = 0$ for all $i \geq 3$, since $H^i(F, S) = H^i(\Gamma, S(F^{\text{sep}}))$, and the cohomological dimension of F is 2. This same reasoning also implies that $H^i(\Gamma, X^*(S)) = 0$ for all $i \geq 3$. Using the long exact sequence (A.1), we deduce that both of the groups in the above pairing are zero for $r \geq 4$ and negative r.

We now reach the analogue of [KS99, Lemma A.2.A]:

Lemma A.1.4 *The above pairing induces an isomorphism*

$$H^r(\overline{F}/F, T \xrightarrow{f} U) \to H^{3-r}(F, X^*(U) \xrightarrow{f^*} X^*(T))^*$$

for $r = 2, 3$. For $r = 2, 3$, the group $H^r(\overline{F}/F, T \xrightarrow{f} U)$ is finitely-generated, and is free for $r = 3$.

Proof. The identical proof of [KS99, Lemma A.2.A] works in our situation, using our long exact sequence (A.1). □

A.2 Pairing for $r = 1$

This section is primarily a summary of [KS99, §A.3]; when necessary, we explain why the arguments loc. cit. carry over to our double complex of Čech cochains valued in $\overline{F}$ rather than Galois cochains. The usual exponential exact sequences give a diagram of Γ-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X^*(U) & \longrightarrow & \text{Lie}(\widehat{U}) & \longrightarrow & \widehat{U} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow \hat{f} & & \\ 0 & \longrightarrow & X^*(T) & \longrightarrow & \text{Lie}(\widehat{T}) & \longrightarrow & \widehat{T} & \longrightarrow & 1, \end{array}$$

which gives a boundary map on hypercohomology

$$H^r(\Gamma, \widehat{U} \xrightarrow{\hat{f}} \widehat{T}) \to H^{r+1}(\Gamma, X^*(U) \xrightarrow{f^*} X^*(T)),$$

giving a pairing $H^r(\Gamma, T \xrightarrow{f} U) \times H^{2-r}(\Gamma, \widehat{U} \xrightarrow{\hat{f}} \widehat{T}) \to \mathbb{C}^\times$ (embedding $\mathbb{Q}/\mathbb{Z}$ into $\mathbb{C}^\times$ via the exponential map). As noted in [KS99], this pairing is insufficient for our purposes; we instead want to define a pairing involving the hypercohomology groups $H^r(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})$, where W_F denotes the Weil group of F.

Recall that the hypercohomology groups $H^r(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})$ are defined as follows: For any F-torus S, we set $C^0(W_F, \widehat{S}) = \widehat{S}(\mathbb{C})$ (with inflated W_F-action), $C^1(W_F, \widehat{S})$ the group of continuous 1-cocycles of W_F in $\widehat{T}(\mathbb{C})$, and all other cochain groups to be zero. We then define r-hypercochains with respect to the complex $\widehat{U} \xrightarrow{\hat{f}} \widehat{T}$ to be elements of

$$C^r(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T}) = C^r(W_F, \widehat{U}) \oplus C^{r-1}(W_F, \widehat{T}),$$

with the same differentials as in our previous total complexes, and corresponding cohomology groups $H^r(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})$.

To construct the desired pairing, we need to introduce one more homological construction. For K/F a finite Galois extension and $W_{K/F}$ the relative Weil group of K/F, we define the group $H_0(W_{K/F}, X_*(T) \xrightarrow{f_*} X_*(U))$ to be the kernel of $X_*(T) \oplus C_1(X_*(U)) \xrightarrow{f_* \oplus -\partial} X_*(U)$ modulo the image of

$$C_1(X_*(T)) \oplus C_2(X_*(U)) \xrightarrow{(\partial \oplus 0, f_* \oplus -\partial)} X_*(T) \oplus C_1(X_*(U)),$$

where $C_i(-)$ denotes the group of i-chains and ∂ is the usual differential from group homology. (with respect to the abstract group $W_{K/F}$). We then define $H_0(W_{K/F}, X_*(T) \xrightarrow{f_*} X_*(U))_0$ as the subgroup of elements whose $X_*(T)$- coordinates are killed by the K/F-norm. We then have maps

$$\phi \colon C_1(X_*(T)) \to T(K),$$

$$\psi \colon X_*(T)_0 \to \check{Z}^1(K/F, T) = Z^1(\Gamma_{K/F}, T(K))$$

which together induce, via (ψ, ϕ), a canonical isomorphism

$$H_0(W_{K/F}, X_*(T) \xrightarrow{f_*} X_*(U))_0 \xrightarrow{\sim} H^1(\Gamma_{K/F}, T(K) \xrightarrow{f} U(K)) = H^1(K/F, T \xrightarrow{f} U). \quad \text{(A.3)}$$

For the explicit construction of ϕ and ψ and the proof that they induce such an isomorphism, see [KS99, §A.3], (the constructions of the two maps are involved, and we omit summarizing them here). Note that since K/F is a finite Galois extension, we may work with group cohomology, so the arguments of [KS99] are unchanged in our new setting. Now since $\mathbb{C}^\times$ is divisible, we have an isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(H_0(W_{K/F}, X_*(T) \xrightarrow{f_*} X_*(U)), \mathbb{C}^\times) \xrightarrow{\sim} H^1_{\text{abs}}(W_{K/F}, \widehat{U} \xrightarrow{\hat{f}} \widehat{T}),$$

where the subscript "abs" means that we are viewing $W_{K/F}$ as an abstract group, and then restricting to subgroups, an isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(H_0(W_{K/F}, X_*(T) \xrightarrow{f_*} X_*(U))_0, \mathbb{C}^\times) \xrightarrow{\sim} H^1(W_{K/F}, \widehat{U} \xrightarrow{\hat{f}} \widehat{T}),$$

(for details on these isomorphisms, see [KS99, §A.3]) which, combined with the isomorphism (A.3), gives a pairing

$$H^1(K/F, T \xrightarrow{f} U) \times H^1(W_{K/F}, \widehat{U} \xrightarrow{\hat{f}} \widehat{T}) \to \mathbb{C}^\times.$$

Passing to direct limits gives a pairing

$$H^1(F^{\text{sep}}/F, T \xrightarrow{f} U) \times H^1(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T}) \to \mathbb{C}^\times, \tag{A.4}$$

and then applying our isomorphism from Lemma A.1.1 finally gives our desired pairing

$$H^1(\overline{F}/F, T \xrightarrow{f} U) \times H^1(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T}) \to \mathbb{C}^\times. \tag{A.5}$$

We now discuss some basic properties of this pairing. To match more closely with [KS99], we work with $H^1(F^{\text{sep}}/F, T \xrightarrow{f} U) = H^1(\Gamma_F, T(F^{\text{sep}}) \xrightarrow{f} U(F^{\text{sep}}))$, but we could just as well replace the left-hand group with $H^1(\overline{F}/F, T \xrightarrow{f} U)$ (cf. Lemma A.1.1). We have two exact sequences

$$\cdots \to H^0(F, U) \xrightarrow{j} H^1(F^{\text{sep}}/F, T \xrightarrow{f} U) \xrightarrow{i} H^1(F, T) \to \ldots,$$

$$\cdots \to H^0(W_F, \widehat{T}) \xrightarrow{\hat{j}} H^1(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T}) \xrightarrow{\hat{i}} H^1(W_F, \widehat{U}) \to \ldots,$$

from which we derive two compatibilities of pairings. First, we have $\langle j(u), \hat{z} \rangle = \langle u, \hat{i}(\hat{z}) \rangle^{-1}$, where the left-hand pairing is (A.4) and the right-hand pairing $U(F) \times H^1(W_F, \widehat{U}) \to \mathbb{C}^\times$ is given by Langlands duality for tori. Second, we have $\langle z, \hat{j}(\hat{t}) \rangle = \langle i(z), \hat{t} \rangle$, where the left-hand pairing is again from (A.4) and the right-hand pairing $H^1(F, T) \times \widehat{T}^{\Gamma_F} \to \mathbb{C}^\times$ comes from Tate-Nakayama duality.

We may endow $H^1(\overline{F}/F, T \xrightarrow{f} U)$ with a natural locally-profinite topology as follows. To see this, we first claim that the image $f(T(F)) \subseteq U(F)$ is closed. The scheme-theoretic image $f(T)$

is a closed subscheme of U by the closed orbit lemma, so that $f(T)(F)$ is closed in $U(F)$, which means that we can replace U by $f(T)$ to reduce to the case where f is (scheme-theoretically) surjective. We then may find an F-torus T' such that f factors as a composition $T \xrightarrow{f'} U' \xrightarrow{f''} U$ where the kernel of f' is a torus and f'' is an isogeny. Note that f'' is finite, and hence proper, which means that, at the F-rational level, the continuous map $U'(F) \to U(F)$ is proper (as a map of topological spaces), which means it's closed (since $U(F)$ is locally compact and Hausdorff), and so we can reduce further to the case where the kernel of $T \to U$ is a torus.

Note that, in this final case, the morphism $T \xrightarrow{f} U$ is smooth—indeed, quotient maps are always flat and surjective, and the smoothness of the kernel implies that we get a short-exact sequence at the level of tangent spaces at the identity. It then follows from the inverse function theorem for analytic manifolds ([Ser92, Theorem III.9.2], which again, is proved for all analytic manifolds over complete nonarchimedean fields) that f is open, and hence closed (since we are working with totally-disconnected Hausdorff topological spaces). In fact, the above argument shows that $f\colon T(F) \to U(F)$ is closed.

The closedness of $f(T(F))$ in $U(F)$ implies that the quotient $U(F)/f(T(F))$ has the canonical structure of a topological group. We then give $H^1(\overline{F}/F, T \xrightarrow{f} U)$ the unique locally-profinite topology such that the map

$$U(F)/[f(T(F))] \to H^1(\overline{F}/F, T \xrightarrow{f} U)$$

is an open immersion (note that $H^1(F, T)$ is always finite).

Proposition A.2.1 *Using the above topology, the pairing* (A.5) *induces a surjective homomorphism*

$$H^1(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T}) \to \mathit{Hom}_{cts}(H^1(\overline{F}/F, T \xrightarrow{f} U), \mathbb{C}^\times)$$

with kernel equal to the image of $(\widehat{T}^{\Gamma_F})^\circ$ *under the natural map*

$$\hat{j}\colon \widehat{T}^{\Gamma_F} \to H^1(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T}).$$

Proof. The proof proceeds identically as in the proof of [KS99, Lemma A.3.B], using our above compatibilities of pairings and the fact that Langlands duality for tori and Tate-Nakayama duality are unchanged in the local function field setting. □

We set $H^1(W_F, \widehat{U} \xrightarrow{f} \widehat{T})_{\text{red}}$ to be the quotient $H^1(W_F, \widehat{U} \xrightarrow{f} \widehat{T})/\hat{j}[(\widehat{T}^{\Gamma_F})^\circ]$. Note that the group $H^1(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})$ is redundant when f is an isogeny, by the following result:

Proposition A.2.2 *The canonical inclusion*

$$H^1(\Gamma, \widehat{U} \xrightarrow{\hat{f}} \widehat{T}) \to H^1(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})$$

is an isomorphism.

Proof. First note that for any finite extension K/F splitting U and T, we have an “inflation-restriction" sequence, given by the exact sequence

$$0 \to H^1(\Gamma_{K/F}, \widehat{U} \xrightarrow{\hat{f}} \widehat{T}) \to H^1(W_{K/F}, \widehat{U} \xrightarrow{\hat{f}} \widehat{T}) \to H^1(K^*, \widehat{U} \xrightarrow{\hat{f}} \widehat{T}),$$

where in the last term we are viewing K^* as a topological group. Indeed, suppose that we have a 1-hypercocycle $(\underline{u}, t) \in C^1(W_F, \widehat{U}) \oplus \widehat{T}(\mathbb{C})$ such that its restriction to K^* is a 1-coboundary; that is, we have $x \in \widehat{U}(\mathbb{C})$ such that $(\underline{u}, t) = (dx, f(x)^{-1})$. This means that for all $z \in F^*$, we have $\underline{u}(z) = {}^z x \cdot x^{-1} = 1$, so that $\underline{u}$ is trivial on K^*, and is therefore inflated from any 1-cocycle $\underline{\tilde{u}}$ of $\Gamma_{K/F}$ determined by picking a set-theoretic section $\Gamma_{K/F} \to W_{K/F}$. Since the $W_{K/F}$-action is inflated from $\Gamma_{K/F}$, the element $(\underline{\tilde{u}}, t)$ is a 1-hypercocycle of $\Gamma_{K/F}$ mapping to $(\underline{u}, t)$, as desired.

For a fixed K/F as above, fix $x \in H^1(W_{K/F}, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})$; to show that, for large enough L/F containing K, it lies in the image of the inflation map, it's enough to show that its image in $H^1(L^*, \widehat{U} \xrightarrow{\hat{f}} \widehat{T}) = \mathrm{Hom}_{\mathrm{cts}}(L^*, \ker(\hat{f}))$ is zero for large enough L. This is possible, since any continuous homomorphism $\chi \colon K^* \to \ker(\hat{f})$ has finite-index open kernel and the images of the norm maps $N_{L/K}(L^*)$ shrink to the identity as L/K varies over all finite Galois extensions of F containing K. □

A.3 Complexes of tori over global function fields—basic results

The last two subsections extend the content of [KS99], Appendix C, to a global function field F. We fix a 2-term complex of F-tori $T \xrightarrow{f} U$. Let $\mathbb{A}^{\mathrm{sep}} := F^{\mathrm{sep}} \otimes_F \mathbb{A}$. We first define $\bar{H}^i(\overline{\mathbb{A}}/\mathbb{A}, T \xrightarrow{f} U)$ to be the hypercohomology of the double complex

$$\begin{array}{ccccccc}
\frac{T(\overline{\mathbb{A}})}{T(\overline{F})} & \longrightarrow & \frac{T(\overline{\mathbb{A}} \otimes_{\mathbb{A}} \overline{\mathbb{A}})}{T(\overline{F} \otimes_F \overline{F})} & \longrightarrow & \frac{T(\overline{\mathbb{A}} \otimes_{\mathbb{A}} \overline{\mathbb{A}} \otimes_{\mathbb{A}} \overline{\mathbb{A}})}{T(\overline{F} \otimes_F \overline{F} \otimes_F \overline{F})} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\frac{U(\overline{\mathbb{A}})}{U(\overline{F})} & \longrightarrow & \frac{U(\overline{\mathbb{A}} \otimes_{\mathbb{A}} \overline{\mathbb{A}})}{U(\overline{F} \otimes_F \overline{F})} & \longrightarrow & \frac{U(\overline{\mathbb{A}} \otimes_{\mathbb{A}} \overline{\mathbb{A}} \otimes_{\mathbb{A}} \overline{\mathbb{A}})}{U(\overline{F} \otimes_F \overline{F} \otimes_F \overline{F})} & \longrightarrow & \cdots,
\end{array}$$

giving us a long exact sequence

$$\cdots \to H^i(\overline{F}/F, T \xrightarrow{f} U) \to H^i(\overline{\mathbb{A}}/\mathbb{A}, T \xrightarrow{f} U) \to \bar{H}^i(\overline{\mathbb{A}}/\mathbb{A}, T \xrightarrow{f} U) \to H^{i+1}(\overline{F}/F, T \xrightarrow{f} U) \to \dots \tag{A.6}$$

Let S be a finite set of places of F containing all places at which T and U are ramified. For every place v of F, we fix an algebraic closure $\overline{F_v}$ as well as an embedding $\overline{F} \hookrightarrow \overline{F_v}$. The following two results let us work in the group-cohomological setting:

Lemma A.3.1 *For all $i \geq 0$, the natural map $H^i(\mathbb{A}^{sep}/\mathbb{A}, T \xrightarrow{f} U) \to H^i(\overline{\mathbb{A}}/\mathbb{A}, T \xrightarrow{f} U)$ is an isomorphism, and the same is true with $\mathbb{A}$ replaced by F.*

Proof. Combining the proof of Lemma 3.3.8 with our results on adelic tensor products in §3.3 shows that $H^j((\mathbb{A}^{\text{sep}})^{\otimes_{\mathbb{A}} n}, M)$ vanishes for any F-torus M, $j, n \geq 1$, and hence the natural map $\check{H}^i(\mathbb{A}^{\text{sep}}/\mathbb{A}, M) \to H^i(\mathbb{A}, M)$ is an isomorphism. Since this is also true with $\mathbb{A}^{\text{sep}}$ replaced by $\overline{\mathbb{A}}$, the same argument in the proof of Lemma A.1.1 gives the result. The argument for F is the same. □

Corollary A.3.2 *For all $i \geq 0$, the natural map $\bar{H}^i(\mathbb{A}^{sep}/\mathbb{A}, T \xrightarrow{f} U) \to \bar{H}^i(\overline{\mathbb{A}}/\mathbb{A}, T \xrightarrow{f} U)$ is an isomorphism.*

Proof. This is an immediate consequence of combining Lemma A.3.1 with the long exact sequence (A.6) and applying the five-lemma. □

The next two results are left as straightforward exercises:

Lemma A.3.3 *For all i, we have a canonical isomorphism*

$$H^i(\mathbb{A}^{sep}/\mathbb{A}, T \xrightarrow{f} U) \to H^i(\Gamma_F, T(\mathbb{A}^{sep}) \xrightarrow{f} U(\mathbb{A}^{sep})).$$

Corollary A.3.4 *For all i, we have a canonical isomorphism*

$$\bar{H}^i(\mathbb{A}^{sep}/\mathbb{A}, T \xrightarrow{f} U) \to \bar{H}^i(\Gamma_F, T(\mathbb{A}^{sep})/T(F^{sep}) \xrightarrow{f} U(\mathbb{A}^{sep})/U(F^{sep})).$$

We now give an analogue of [KS99, Lemma C.1.A], which we need in order to work with restricted products. Note that the complex $T \xrightarrow{f} U$ is defined over the ring $O_{F,S}$. Let O_v denote the completion of O_F at a place v, and O_v^{nr} the ring of integers inside the maximal unramified extension F_v^{nr}/F_v.

Lemma A.3.5 *For any place $v \notin S$, the group $H^i(O_v^{nr}/O_v, T \xrightarrow{f} U)$ is equal to the kernel of $T(O_v) \xrightarrow{f} U(O_v)$ if $i = 0$, to the cokernel of the same map if $i = 1$, and is trivial if $i \geq 2$. Moreover, the natural map*

$$H^i(O_v^{nr}/O_v, T \xrightarrow{f} U) \to H^i(\overline{F_v}/F_v, T \xrightarrow{f} U)$$

is injective for all i.

Proof. To prove the first statement, using the long exact sequence (A.1), it's enough to show that $\check{H}^i(O_v^{\mathrm{nr}}/O_v, M) = 0$ for any F-torus M which is unramified at v for $i \geq 1$ (applying this result to T and U). We first claim that these groups may be identified with $H^i(O_v, M)$ under the natural Čech-to-fppf comparison map. As usual, it's enough to show that the fppf cohomology groups $H^j((O_v^{\mathrm{nr}})^{\bigotimes_{O_v} n}, M)$ vanish for all $j, n \geq 1$. Since O_v is the ring of integers in a nonarchimedean local field, for a fixed finite unramified extension E_w/F_v, we have the chain of identifications

$$O_w \otimes_{O_v} O_w \xrightarrow{\sim} O_w \otimes_{O_v} O_v[\varpi] \xrightarrow{\sim} O_w \otimes_{O_v} O_v[x]/(f) \xrightarrow{\sim} \prod_{\Gamma_{E_w/F_v}} O_w,$$

where $\varpi \in O_w$ and $f \in O_v[x]$. In the usual way, we are thus reduced to the case when $n = 1$; i.e., showing that the groups $H^i(O_v^{\mathrm{nr}}, M)$ vanish for all $i \geq 1$. This follows immediately from the fact that they are the direct limit of the groups $H^i(O_{E_w}, M)$, where E_w is as above, which all vanish by [Čes16, Corollary 2.9], using that O_{E_w} is a Henselian local ring with finite residue field k_w, and M_{k_w} is connected, being a k_w-torus. With the claim in hand, the result is immediate from the same Corollary, since O_v is a Henselian local ring with finite residue field k_v such that M_{k_v} is connected.

We now move on to the second statement. Using the first statement, we only need to show this for $i = 1$. As in the proof of [KS99, Lemma C.1.A], it's enough to show that any element $u \in U(O_v) \cap f(T(F_v))$ lies in $f(T(O_v))$. To this end, we may assume that f is surjective, and we may again factor f as the composition $T \xrightarrow{f'} U' \xrightarrow{f''} U$, where f' has a torus as its kernel and f'' is an isogeny. The argument of the proof of [KS99, Lemma C.1.A] proves the result for f', so that $U'(O_v) \cap f'(T(F_v)) = f'(T(O_v))$.

Note that f'' is proper as a morphism of F_v-schemes, so the map $U'(F_v) \to U(F_v)$ is proper as a morphism of topological spaces; this implies that the preimage of the compact subgroup $U(O_v)$ under f'' is a compact subgroup of $U'(F_v)$, and so lies in $U'(O_v)$, the maximal compact subgroup. Thus, if $t \in T(F_v)$ is such that $f(t) \in U(O_v)$, then $f'(t) \in U'(O_v)$, so that $f'(t) = f'(x)$ for some $x \in T(O_v)$, and now $f(t) = f(x)$, as desired. □

We now give a restricted product structure to the groups $H^i(\overline{\mathbb{A}}/\mathbb{A}, T \xrightarrow{f} U)$ with respect to the subgroups of the above lemma:

Proposition A.3.6 *We have a canonical isomorphism*

$$H^i(\overline{\mathbb{A}}/\mathbb{A}, T \xrightarrow{f} U) \xrightarrow{\sim} \prod_{v \in V_F}' H^i(\overline{F_v}/F_v, T \xrightarrow{f} U),$$

where the product is restricted with respect to the subgroups $H^i(O_v^{nr}/O_v, T \xrightarrow{f} U)$ for $v \notin S$ (which are indeed subgroups by Lemma A.3.5). When $i \geq 2$, this restricted product is a direct sum.

Proof. The first step is to use Lemma A.3.1 to replace $H^i(\overline{\mathbb{A}}/F, T \xrightarrow{f} U)$ by $H^i(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U)$, and Lemma A.1.1 to replace $H^i(\overline{F_v}/F_v, T \xrightarrow{f} U)$ by $H^i(F_v^{\text{sep}}/F_v, T \xrightarrow{f} U)$. Consider a finite Galois extension K/F, and let $S_{(K)}$ denote a large finite set of places containing S such that K is unramified outside $S_{(K)}$. For any place $w \in V_K$ lying over $v \notin S_{(K)}$, the natural map $H^i(O_w/O_v, T \xrightarrow{f} U) \to H^i(O_v^{\text{nr}}/O_v, T \xrightarrow{f} U)$ is an isomorphism (replace O_v^{nr} by O_w in the proof of Lemma A.3.5). From here, we may work with group cohomology and use the identical argument of [KS99, Lemma C.1.B] to deduce the result. □

Continuing to follow [KS99, §C], we topologize our adelic cohomology groups. We work with the Galois versions $H^i(F^{\text{sep}}/F, T \xrightarrow{f} U)$, $H^i(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U)$, and $\bar{H}^i(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U)$. We give $H^i(F^{\text{sep}}/F, T \xrightarrow{f} U)$ the discrete topology for all i. We give $H^0(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U)$ the topology it inherits as a closed subgroup of $T(\mathbb{A})$, and $H^1(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U)$ the topology determined by declaring that the map $U(\mathbb{A})/f[T(\mathbb{A})] \to H^1(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U)$ is an open immersion; note that $f[T(\mathbb{A})]$ is closed in $U(\mathbb{A})$, since $f(T(F_v)) \cap U(O_v) = f(T(O_v))$ for $v \notin S$ and $\prod_{v \notin S} f(T(O_v))$ is compact, and $f(T(F_v))$ is closed in $U(F_v)$ for $v \in S$ (by an argument that we made earlier in this subsection). In the above discussion, we are using [Česl6, Theorem 2.20] to decompose $T(\mathbb{A})$ and $U(\mathbb{A})$ as restricted products. We give the groups $H^i(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U)$ the discrete topology for $i \geq 2$.

We now turn to topologizing the groups $\bar{H}^i(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U)$, which is the most complicated of the three cases. Note that for any F-torus S, the group $S(\mathbb{A}^{\text{sep}})$ carries a natural topology, given by the direct limit topology of the topological groups $S(\mathbb{A}_K)$, where K/F ranges over all finite Galois extensions. These topologies coincide with the topologies induced by giving $\mathbb{A}^{\text{sep}}$ the structure of a topological ring via the direct limit topology. Note that the ring $\mathbb{A}^{\text{sep}}$ is Hausdorff; to see, this, note that each $\mathbb{A}_K$ is a metrizable topological space (by [KS20, Proposition 1.1]), and is thus normal; now the direct limit of normal spaces with transition maps that are closed immersions (as is the case with $\mathbb{A}_K \to \mathbb{A}_L$) is a normal topological space, and hence a fortiori Hausdorff.

It follows that $S(\mathbb{A}^{\text{sep}})$ is Hausdorff (by [Con11, Proposition 2.1]). Since $S(K)$ is closed in $S(\mathbb{A}_K)$ for all K, it follows that $S(F^{\text{sep}})$ is a closed subgroup of $S(\mathbb{A}^{\text{sep}})$ (using that $S(F^{\text{sep}}) \cap S(\mathbb{A}_K) = S(K)$), so the topological group $S(\mathbb{A}^{\text{sep}})/S(F^{\text{sep}})$ makes sense. Moreover, the subgroup

$[S(\mathbb{A}^{\text{sep}})/S(F^{\text{sep}})]^{\Gamma}$ is closed, since it's the intersection over all elements $\sigma \in \Gamma$ of the subsets $[S(\mathbb{A}^{\text{sep}})/S(F^{\text{sep}})]^{\sigma}$, which are the preimages of the (closed) diagonal $\Delta(S(\mathbb{A}^{\text{sep}})/S(F^{\text{sep}}))$ under the continuous map

$$\text{id} \times (-)^{\sigma} \colon S(\mathbb{A}^{\text{sep}})/S(F^{\text{sep}}) \to S(\mathbb{A}^{\text{sep}})/S(F^{\text{sep}}) \times S(\mathbb{A}^{\text{sep}})/S(F^{\text{sep}}).$$

Moreover, using these topologies, the natural map

$$[T(\mathbb{A}^{\text{sep}})/T(F^{\text{sep}})]^{\Gamma} \to [U(\mathbb{A}^{\text{sep}})/U(F^{\text{sep}})]^{\Gamma}$$

is continuous, and hence the closed kernel (our group $\bar{H}^0(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U)$) has the natural structure of a topological group, settling the $i = 0$ case.

We claim that the image of the map

$$[T(\mathbb{A}^{\text{sep}})/T(F^{\text{sep}})]^{\Gamma} \to [U(\mathbb{A}^{\text{sep}})/U(F^{\text{sep}})]^{\Gamma}$$

is in fact a closed subgroup (with topologies given as above). First, observe that for K/F finite, the map $T(\mathbb{A}_K) \xrightarrow{f} U(\mathbb{A}_K)$ is closed; this follows from the closedness of f as a map from $T(K_v)$ to $U(K_v)$ for all v, the observation that $f(T(K_v)) \cap U(O_{K_v}) = f(T(O_{K_v}))$, and the structure of the adelic topology on $U(\mathbb{A}_K)$ (using the restricted-product decomposition of $U(\mathbb{A}_K)$ from [Čes16, Theorem 2.20]). Now note that the image of the map in question is the direct limit of the images of the maps of topological groups $[T(\mathbb{A}_K)/T(K)]^{\Gamma_{K/F}} \to [U(\mathbb{A}_K)/U(K)]^{\Gamma_{K/F}}$, and so it's enough to show that all of these images are closed. This follows immediately from the closedness of $T(\mathbb{A}_K) \xrightarrow{f} U(\mathbb{A}_K)$ and the fact that $[T(\mathbb{A}_K)/T(K)]^{\Gamma_{K/F}}$ is closed in $T(\mathbb{A}_K)/T(K)$ (implied by our above arguments).

We give $\bar{H}^1(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U)$ the topology determined by declaring that the map

$$\text{cok}([T(\mathbb{A}^{\text{sep}})/T(F^{\text{sep}})]^{\Gamma} \to [U(\mathbb{A}^{\text{sep}})/U(F^{\text{sep}})]^{\Gamma}) \to \bar{H}^1(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U)$$

is an open immersion (where the left-hand side has the natural quotient topology). For any $i \geq 2$, we give $\bar{H}^i(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U)$ the discrete topology.

A.4 Complexes of tori over global function fields—duality

We now discuss duality for the groups $\bar{H}^i(\overline{\mathbb{A}}/\mathbb{A}, T \xrightarrow{f} U)$; it will be more convenient to replace these groups by (the canonically-isomorphic) $\bar{H}^i(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U)$. As in the local case, we have a Tate-Nakayama pairing

$$\bar{H}^r(\mathbb{A}^{\text{sep}}/F, T \xrightarrow{f} U) \times H^{3-r}(\Gamma, X^*(U) \xrightarrow{f^*} X^*(T)) \to \mathbb{Q}/\mathbb{Z}, \tag{A.7}$$

where the $\mathbb{Q}/\mathbb{Z}$ comes from identifying the 2nd cohomology group of the complex with degree-n (≥ 0) term

$$\mathbb{G}_m((\mathbb{A}^{\text{sep}})^{\otimes_{\mathbb{A}}(n+1)})/\mathbb{G}_m((F^{\text{sep}})^{\otimes_F (n+1)})$$

with $H^2(\Gamma, C)$ (where $C = \varinjlim_{K/F} C_K$ is the universal idéle class group) and then identifying this last group with $\mathbb{Q}/\mathbb{Z}$ via the global invariant map. For an F-torus S, denote by $\bar{H}^i(\mathbb{A}^{\text{sep}}/\mathbb{A}, S)$ the ith cohomology of the complex with degree-n term

$$S((\mathbb{A}^{\text{sep}})^{\otimes_{\mathbb{A}}(n+1)})/S((F^{\text{sep}})^{\otimes_F (n+1)})$$

(we can define an analogue for $\overline{\mathbb{A}}$, but we won't use that here).

According to [KS99, Lemma D.2.A],(which relies on the results of [Mil06, §4], which are stated for arbitrary nonarchimedean local fields) the groups $\bar{H}^r(\mathbb{A}^{\text{sep}}/\mathbb{A}, T)$ vanish for $r \geq 3$, and for $r = 1, 2$ we having a pairing

$$\bar{H}^r(\mathbb{A}^{\text{sep}}/\mathbb{A}, T) \times H^{2-r}(\Gamma, X^*(T)) \to \mathbb{Q}/\mathbb{Z}$$

which induces isomorphisms $\bar{H}^r(\mathbb{A}^{\text{sep}}/\mathbb{A}, T) \xrightarrow{\sim} \text{Hom}_{\mathbb{Z}}(H^{2-r}(\Gamma, X^*(T)), \mathbb{Q}/\mathbb{Z})$, and the group $\bar{H}^1(\mathbb{A}^{\text{sep}}/\mathbb{A}, T)$ is finite.

We now extend this to our complexes:

Lemma A.4.1 *For $r \geq 4$, the groups $\bar{H}^r(\mathbb{A}^{sep}/\mathbb{A}, T \xrightarrow{f} U)$ vanish. For $r = 2, 3$, the pairing* (A.7) *induces an isomorphism*

$$\bar{H}^r(\mathbb{A}^{sep}/\mathbb{A}, T \xrightarrow{f} U) \xrightarrow{\sim} \mathit{Hom}_{\mathbb{Z}}(H^{3-r}(\Gamma, X^*(U) \xrightarrow{f^*} X^*(T)), \mathbb{Q}/Z).$$

For $r = 2, 3$, the group $\bar{H}^r(\mathbb{A}^{sep}/\mathbb{A}, T \xrightarrow{f} U)$ is finitely-generated, and for $r = 3$ it is free.

Proof. See the explanation following [KS99, Lemma C.2.A]. □

We now give a duality theorem for $r = 1$, which will use the absolute Weil group W_F of F (corresponding to the inverse limit of extensions of $\mathbb{A}_K/K^*$ by $\Gamma_{K/F}$ corresponding to the canonical H^2-class, as K/F ranges over all finite Galois extensions) as in the local case. We define the cochain groups $C^m(W_F, \widehat{T})$ and cohomology groups $H^m(W_F, \widehat{T})$ in the same way as in the local case. Note that $H^m(W_F, \widehat{T})$ vanishes for $m \geq 2$, and $H^1(W_F, \widehat{T})$ is canonically isomorphic to

$\mathrm{Hom}_{\mathrm{cts}}(\bar{H}^0(\mathbb{A}^{\mathrm{sep}}/\mathbb{A}, T), \mathbb{C}^\times)$, by [Lan97]. We define the hypercochain groups $C^m(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})$ the same way as in the local case, and take $H^m(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})$ to be the cohomology of the corresponding complex. Note that $H^m(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T}) = 0$ for $m \geq 3$. We have the following global analogue of Proposition A.2.2:

Proposition A.4.2 *When $T \xrightarrow{f} U$ is an isogeny, the canonical inclusion $H^1(\Gamma, \widehat{U} \xrightarrow{\hat{f}} \widehat{T}) \to H^1(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})$ is an isomorphism.*

Proof. As in the proof of Proposition A.2.2, the inflation-restriction sequence shows that it's enough to show that the image of any element in $\mathrm{Hom}_{\mathrm{cts}}(\mathbb{A}_K/K^*, \ker(\hat{f}))$ is zero in some large finite Galois extension L/F containing K, which follows from the fact that the kernel of any such homomorphism is open and finite-index and the universal norm group of (the idele class groups of) a global function field is trivial (see [NSW08, Proposition 8.1.26]). □

We may define a pairing

$$\bar{H}^1(\mathbb{A}^{\mathrm{sep}}/\mathbb{A}, T \xrightarrow{f} U) \times H^1(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T}) \to \mathbb{C}^\times \tag{A.8}$$

exactly as in the local case, and, like in the local case, it induces a surjective homomorphism

$$H^1(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T}) \to \mathrm{Hom}_{\mathrm{cts}}(\bar{H}^1(\mathbb{A}^{\mathrm{sep}}/\mathbb{A}, T \xrightarrow{f} U), \mathbb{C}^\times)$$

with kernel the image of $(T^\Gamma)^\circ \subseteq H^0(W_F, \widehat{T})$ in $H^1(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})$, the quotient by which we will denote by $H^1(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})_{\mathrm{red}}$.

We now define a compact subgroup $\bar{H}^i(\mathbb{A}^{\mathrm{sep}}/\mathbb{A}, T \xrightarrow{f} U)_1$ of $\bar{H}^i(\mathbb{A}^{\mathrm{sep}}/\mathbb{A}, T \xrightarrow{f} U)$ for $i = 0, 1$. We first set $\bar{H}^0(\mathbb{A}^{\mathrm{sep}}/\mathbb{A}, T)_1$ to be the kernel of the group homomorphism

$$H\colon [T(\mathbb{A}^{\mathrm{sep}})/T(F^{\mathrm{sep}})]^\Gamma \to X_*(T)^\Gamma$$

determined by, for all $\lambda \in X^*(T)^\Gamma$, the equality

$$\langle \lambda, H(\bar{t}) \rangle = \deg(\lambda(\bar{t})),$$

where we are using the fact that $[(\mathbb{A}^{\mathrm{sep}})^\times/(F^{\mathrm{sep}})^\times]^\Gamma = \mathbb{A}^\times/F^\times$, and $\deg\colon \mathbb{A}^\times/F^\times \to \mathbb{Z}$ is the homomorphism defined by $\deg(\bar{\alpha}) = \sum_{v \in V} v(\alpha_v)[k_v\colon k]$, where k denotes the constant field of the global function field F.

Lemma A.4.3 *The kernel of the above homomorphism is a compact subgroup of $[T(\mathbb{A}^{sep})/T(F^{sep})]^\Gamma$ (topologized as in the previous subsection).*

Proof. This follows from elementary results concerning the structure of tori over global fields. We have a canonical isogeny $T \to T_a \times T_s$, where T_a is the maximal F-anisotropic subtorus of T and T_s is the maximal F-split subtorus of T. Note that this induces an injective group homomorphism $X_*(T)^\Gamma \to X_*(T_a \times T_s)^\Gamma = X_*(T_s)$. Then we have a commutative diagram

$$\begin{array}{ccc} (T(\mathbb{A}^{\text{sep}})/T(F^{\text{sep}}))^\Gamma & \longrightarrow & ((T_a \times T_s)(\mathbb{A}^{\text{sep}})/(T_a \times T_s)(F^{\text{sep}}))^\Gamma \\ \downarrow & & \downarrow \\ X_*(T)^\Gamma & \longrightarrow & X_*(T_s), \end{array}$$

and since the lower horizontal map is injective, the kernel of the left-hand vertical map (the group we're analyzing) is the kernel of the right-down composition, i.e, the preimage of the kernel of the right-hand vertical map. Since the top horizontal map is induced by the isogeny $T \to T_a \times T_s$ (which is proper), if we can show that the kernel of the right-hand vertical map is compact, then its preimage in $[T(\mathbb{A}^{\text{sep}})/T(F^{\text{sep}})]^\Gamma$ is also compact (since the properness of f implies that the map of topological groups $T(\mathbb{A}^{\text{sep}}) \xrightarrow{f_K} U(\mathbb{A}^{\text{sep}})$ is proper, by [Con11, Proposition 5.8]). Rewriting the group $[(T_a \times T_s)(\mathbb{A}^{\text{sep}})/(T_a \times T_s)(F^{\text{sep}})]^\Gamma$ as

$$[T_a(\mathbb{A}^{\text{sep}})/T_a(F^{\text{sep}})]^\Gamma \times [T_s(\mathbb{A}^{\text{sep}})/T_s(F^{\text{sep}})]^\Gamma,$$

it's clear that the kernel in question equals $[T_a(\mathbb{A}^{\text{sep}})/T_a(F^{\text{sep}})]^\Gamma \times K_s$, where K_s denotes the kernel of the map $H_s\colon [T_s(\mathbb{A}^{\text{sep}})/T_s(F^{\text{sep}})]^\Gamma \to X_*(T_s)$. First, note that the group $[T_a(\mathbb{A}^{\text{sep}})/T_a(F^{\text{sep}})]^\Gamma$ is already compact; this follows from the fact that it contains $T_a(\mathbb{A})/T_a(F)$ as a finite-index closed subgroup, and this latter group is compact (by [Con20, Theorem 8.1.3], using that T_a is F-anisotropic).

We have thus reduced the lemma to the case in which $T = T_s$ is F-split. Pick a $\mathbb{Z}$-basis $\lambda_1, \dots \lambda_n$ of $X^*(T) = X^*(T)^\Gamma$. Then $\bar{t}$ lies in the kernel of H if and only if $\deg[\lambda_i(\bar{t})] = 0$ for all i. In fact, we have an F-isomorphism

$$T \xrightarrow{(\lambda_i)} \mathbb{G}_m^n,$$

and now the kernel of H is the preimage under the above isomorphism of the kernel of the map $(\mathbb{A}^\times/F^\times)^n \xrightarrow{\deg^n} \mathbb{Z}^n$, which is the n-fold product of the compact subgroups C_F^0 of $\mathbb{A}^\times/F^\times$ (by [NSW08, Proposition 8.1.25]). □

We then define $\bar{H}^0(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U)_1$ to be the intersection of the group $\bar{H}^0(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U) \subseteq [T(\mathbb{A}^{\text{sep}})/T(F^{\text{sep}})]^\Gamma$ with the above kernel. It is easy to check that when f is an isogeny this intersection is all of $\bar{H}^0(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U)$. We now proceed to the $i = 1$ construction.

For any $\lambda \in X^*(U)^\Gamma$, we have a map of complexes from $[T \xrightarrow{f} U]$ to $[1 \to \mathbb{G}_m]$ given by

$$\begin{array}{ccc} T & \xrightarrow{f} & U \\ \downarrow & & \downarrow \lambda \\ 1 & \longrightarrow & \mathbb{G}_m, \end{array}$$

which induces a map $\bar{H}^1(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U) \to \bar{H}^1(\mathbb{A}^{\text{sep}}/\mathbb{A}, 1 \to \mathbb{G}_m) = \bar{H}^0(\mathbb{A}^{\text{sep}}/\mathbb{A}, \mathbb{G}_m) = \mathbb{A}^\times/F^\times$, which we may then map to $\mathbb{Z}$ via deg, as above. This determines a homomorphism $\bar{H}^1(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U) \to X_*(U)^\Gamma$, and we declare $\bar{H}^1(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U)_1$ to be the kernel of the composition

$$H^{(1)} \colon \bar{H}^1(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U) \to X_*(U)^\Gamma \to \frac{X_*(U)^\Gamma}{f_*(X_*(T)^\Gamma)}.$$

Note that we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \bar{H}^0(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U) & \longrightarrow & \bar{H}^0(\mathbb{A}^{\text{sep}}/\mathbb{A}, T) & \xrightarrow{f} & \bar{H}^0(\mathbb{A}^{\text{sep}}/\mathbb{A}, U) & \xrightarrow{\delta} & \bar{H}^1(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U) \\ & & \downarrow & & \downarrow H_T & & \downarrow H_U & & \downarrow H^{(1)} \\ 0 & \longrightarrow & \text{Ker}(f_*|_{X_*(T)^\Gamma}) & \longrightarrow & X_*(T)^\Gamma & \xrightarrow{f_*} & X_*(U)^\Gamma & \longrightarrow & X_*(U)^\Gamma/f_*(X_*(T)^\Gamma). \end{array} \tag{A.9}$$

We claim now that the map $H_T \colon \bar{H}^0(\mathbb{A}^{\text{sep}}/\mathbb{A}, T) \to X_*(T)^\Gamma$ is split; indeed, this time using the isogeny $T_a \times T_s \to T$, we get the commutative diagram

$$\begin{array}{ccc} ((T_a \times T_s)(\mathbb{A}^{\text{sep}})/(T_a \times T_s)(F^{\text{sep}}))^\Gamma & \longrightarrow & (T(\mathbb{A}^{\text{sep}})/T(F^{\text{sep}}))^\Gamma \\ \downarrow H_{T_a \times T_s} & & \downarrow H_T \\ X_*(T_s) & \xrightarrow{\sim} & X_*(T)^\Gamma, \end{array}$$

where, as we have indicated, the bottom horizontal map is an isomorphism. As in the proof of the Lemma A.4.3, to split $H_{T_a \times T_s}$, it's enough to split H_{T_s}. As before, we have characters $\lambda_i \in X^*(T_s)$ such that $T_s \xrightarrow{(\lambda_i)} \mathbb{G}_m^n$ is an isomorphism, and so it's enough to split the map $(\mathbb{A}^\times/F^\times)^n \xrightarrow{\deg^n} \mathbb{Z}^n$, which is clearly possible. Our splitting of $H_{T_a \times T_s}$ gives a splitting of H_T by applying the inverse isomorphism $X_*(T)^\Gamma \to X_*(T_s)$, giving the main claim. Of course the same argument works with T replaced by U. Along with the obvious product decompositions of $\bar{H}^0(\mathbb{A}^{\text{sep}}/\mathbb{A}, T)$ and $\bar{H}^0(\mathbb{A}^{\text{sep}}/\mathbb{A}, U)$, we get an induced splitting $X_*(U)^\Gamma/f_*(X_*(T)^\Gamma) \to \bar{H}^1(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U)$ of $H^{(1)}$, realizing $\bar{H}^1(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U)$ as the product $\bar{H}^1(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U)_1 \times [X_*(U)^\Gamma/f_*(X_*(T)^\Gamma)]$.

Lemma A.4.4 *The group $\bar{H}^1(\mathbb{A}^{sep}/\mathbb{A}, T \xrightarrow{f} U)_1$ is compact (as a subgroup of the topological group $\bar{H}^1(\mathbb{A}^{sep}/\mathbb{A}, T \xrightarrow{f} U)$).*

Proof. We have a natural injection $\bar{H}^0(\mathbb{A}^{\text{sep}}/\mathbb{A}, U)_1/f(\bar{H}^0(\mathbb{A}^{\text{sep}}/\mathbb{A}, T)_1) \hookrightarrow \bar{H}^1(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U)_1$, which, by the definition of our topologies, is a closed immersion. We claim that, in fact, this is a subgroup of a finite index in the target. By the commutative diagram (A.9), we have

$$\delta^{-1}[\bar{H}^1(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U)_1] = \bar{H}^0(\mathbb{A}^{\text{sep}}/\mathbb{A}, U)_1 \cdot f[\bar{H}^0(\mathbb{A}^{\text{sep}}/\mathbb{A}, T)] \subseteq \bar{H}^0(\mathbb{A}^{\text{sep}}/\mathbb{A}, U),$$

and hence the image of the above natural injection equals

$$\delta[\bar{H}^0(\mathbb{A}^{\text{sep}}/\mathbb{A}, U)] \cap \bar{H}^1(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U)_1,$$

and hence is of finite index, since $\delta[\bar{H}^0(\mathbb{A}^{\text{sep}}/\mathbb{A}, U)]$ is of finite index in $\bar{H}^1(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U)$, by the finiteness of $\bar{H}^1(\mathbb{A}^{\text{sep}}/\mathbb{A}, T)$. Since $\bar{H}^0(\mathbb{A}^{\text{sep}}/\mathbb{A}, U)_1/f(\bar{H}^0(\mathbb{A}^{\text{sep}}/\mathbb{A}, T)_1)$ is itself compact (by Lemma A.4.3), the result follows. □

Corollary A.4.5 *When f is an isogeny, the group $\bar{H}^1(\mathbb{A}^{sep}/\mathbb{A}, T \xrightarrow{f} U)$ is compact.*

Proof. This follows immediately from the above lemma and the fact that $X_*(U)^\Gamma/X_*(T)^\Gamma$ is finite, due to the fact that $X_*(T) \subseteq X_*(U)$ is finite-index and $X_*(U)^\Gamma \cap X_*(T) = X_*(T)^\Gamma$. □

We conclude this section by giving new global duality results that involve the above cohomology groups. We have a natural map

$$H^i(F^{\text{sep}}/F, T \xrightarrow{f} U) \to H^i(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U),$$

and we will denote its kernel by $\ker^i(F^{\text{sep}}/F, T \xrightarrow{f} U)$ and its cokernel by $\text{cok}^i(F^{\text{sep}}/F, T \xrightarrow{f} U)$; our primary case of interest in this paper is when $i = 1$; Using Proposition A.3.6, we may also describe $\ker^i(F^{\text{sep}}/F, T \xrightarrow{f} U)$ as the kernel of the map

$$H^i(F^{\text{sep}}/F, T \xrightarrow{f} U) \to \prod_{v \in V} H^i(F_v^{\text{sep}}/F_v, T \xrightarrow{f} U).$$

We have, from the long exact sequence (A.6), the short exact sequences

$$1 \to \text{cok}^i(F^{\text{sep}}/F, T \xrightarrow{f} U) \to \bar{H}^i(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U) \to \ker^{i+1}(F^{\text{sep}}/F, T \xrightarrow{f} U) \to 1. \quad \text{(A.10)}$$

The following is an analogue of [KS99, Lemma C.3.A]:

Lemma A.4.6 *For all i, the image of $H^i(F^{sep}/F, T \xrightarrow{f} U)$ is discrete in $H^i(\mathbb{A}^{sep}/\mathbb{A}, T \xrightarrow{f} U)$. Moreover, the map*

$$\text{cok}^i(F^{sep}/F, T \xrightarrow{f} U) \to \bar{H}^i(\mathbb{A}^{sep}/\mathbb{A}, T \xrightarrow{f} U)$$

induces an isomorphism of topological groups from $\text{cok}^i(F^{sep}/F, T \xrightarrow{f} U)$ *to an open subgroup of* $\bar{H}^i(\mathbb{A}^{sep}/\mathbb{A}, T \xrightarrow{f} U)$ *for* $i = 0, 1$.

Proof. The first statement is clear for $i \neq 1$ (cf. the analogous argument in [KS99]), so we only need to prove both statements for $i = 1$. For the first statement, it's enough to show that the intersection of $f[H^1(F^{\text{sep}}/F, T \xrightarrow{f} U)]$ with the open subgroup $U(\mathbb{A})/f(T(\mathbb{A}))$ is discrete. Since the image of $U(F)/f(T(F))$ is of finite index in $[U(\mathbb{A})/f(T(\mathbb{A}))] \cap f[H^1(F^{\text{sep}}/F, T \xrightarrow{f} U)]$ (because the kernel of $H^1(F,T) \to \prod_v' H^1(F_v, T)$ is finite), it's enough to show that the image of $U(F)/f(T(F))$ is discrete in $U(\mathbb{A})/f(T(\mathbb{A}))$.

Similarly to what we've done before, we have a split surjective homomorphism $T(\mathbb{A}) \to X_*(T)^\Gamma$ with closed (not necessarily compact) kernel $T(\mathbb{A})_1$, similarly for U, and the induced product structures are compatible with the homomorphism f, allowing us to rewrite f as

$$T(\mathbb{A})_1 \times X_*(T)^\Gamma \xrightarrow{f \times f_*} U(\mathbb{A})_1 \times X_*(U)^\Gamma,$$

leading to a decomposition

$$U(\mathbb{A})/f(T(\mathbb{A})) = U(\mathbb{A})_1/f(T(\mathbb{A})_1) \times X_*(U)^\Gamma / f_*(X_*(T)^\Gamma),$$

and image of $U(F)/f(T(F))$ in $U(\mathbb{A})/f(T(\mathbb{A}))$ lands in the factor $U(\mathbb{A})_1/f(T(\mathbb{A})_1)$.

The subgroup $f(T(F))$ is evidently discrete in $U(\mathbb{A})_1$, since the subgroup $U(F)$ is discrete in $U(\mathbb{A})$ (by [Con11], Example 2.2, using that F is discrete in $\mathbb{A}$). Thus, $U(\mathbb{A})_1/f(T(F))$ contains the discrete subgroup $U(F)/f(T(F))$ and the compact subgroup $f(T(\mathbb{A})_1)/f(T(F))$ (the compactness follows from Lemma A.4.3). The desired discreteness then follows by the analogous argument in the proof of [KS99, Lemma C.3.A].

As in [KS99], to prove the second statement for $i = 1$ it suffices to show that the map

$$U(\mathbb{A}) \to [U(\mathbb{A}^{\text{sep}})/U(F^{\text{sep}})]^\Gamma / f[T(\mathbb{A}^{\text{sep}})/T(F^{\text{sep}})]^\Gamma$$

is open. Note that the image $U(\mathbb{A})/U(F) \hookrightarrow [U(\mathbb{A}^{\text{sep}})/U(F^{\text{sep}})]^\Gamma$ is closed (a straightforward exercise in the topology of adelic points), and is also finite index (by the finiteness of the kernel of $H^1(F,U) \to H^1(\mathbb{A}, U)$), and is hence open. Since quotient maps are open, the composition

$$U(\mathbb{A}) \to U(\mathbb{A})/U(F) \to [U(\mathbb{A}^{\text{sep}})/U(F^{\text{sep}})]^\Gamma \to [U(\mathbb{A}^{\text{sep}})/U(F^{\text{sep}})]^\Gamma / f[T(\mathbb{A}^{\text{sep}})/T(F^{\text{sep}})]^\Gamma$$

is open, as desired.

It remains to show that the injection $\text{cok}^0(F^{\text{sep}}/F, T \xrightarrow{f} U) \to \bar{H}^0(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U)$ has open image. As in [KS99], it's enough to show that the map $H^0(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U) \to \ker[T(\mathbb{A})/T(F) \to$

$U(\mathbb{A})/U(F)]$ is open (because, as in [KS99], $T(\mathbb{A})/T(F)$ is open in $[T(\mathbb{A}^{\text{sep}})/T(F^{\text{sep}})]^{\Gamma}$). Define the closed subgroup $B := \{t \in T(\mathbb{A}) \mid f(t) \in U(F)\}$ of $T(\mathbb{A})$. Note that $H^0(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U)$ is a closed subgroup of B, and we thus have a closed immersion $B/[H^0(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U)] \hookrightarrow U(F) \hookrightarrow U(\mathbb{A})$, where the last closed immersion has discrete image. It follows that, since $H^0(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U)$ is a closed subgroup of B with discrete quotient, it's open, and then the result follows from the fact that $B/T(F) = \ker[T(\mathbb{A})/T(F) \to U(\mathbb{A})/T(F)]$. □

We immediately obtain:

Corollary A.4.7 *The group*

$$\mathit{cok}^1(F^{sep}/F, T \xrightarrow{f} U)_1 := \mathit{cok}^1(F^{sep}/F, T \xrightarrow{f} U) \cap \bar{H}^1(\mathbb{A}^{sep}/\mathbb{A}, T \xrightarrow{f} U)_1$$

is compact. Moreover, when f is an isogeny, the group $\mathit{cok}^1(F^{sep}/F, T \xrightarrow{f} U)$ is compact.

Define the group

$$\ker^1(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})_{\text{red}} := \ker[H^1(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})_{\text{red}} \to \prod_{v \in V} H^1(W_{F_v}, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})_{\text{red}}].$$

We have the following useful result:

Proposition A.4.8 *We have a duality isomorphism*

$$\mathit{Hom}_{cts}(\mathit{cok}^1(F^{sep}/F, T \xrightarrow{f} U), \mathbb{C}^{\times}) \xrightarrow{\sim} H^1(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})_{red}/\mathit{ker}^1(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})_{red}.$$

Moreover, the group $\mathit{ker}^1(F^{sep}/F, T \xrightarrow{f} U)$ is finite.

Proof. Using that $\text{cok}^1(F^{\text{sep}}/F, T \xrightarrow{f} U)$ is an open subgroup of $\bar{H}^1(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U)$, applying the functor $\text{Hom}_{\text{cts}}(-, \mathbb{C}^{\times})$ to the short exact sequence (A.10) with $i = 1$ gives that the group $\text{Hom}_{\text{cts}}(\text{cok}^1(F^{\text{sep}}/F, T \xrightarrow{f} U), \mathbb{C}^{\times})$ is canonically isomorphic to the quotient

$$\text{Hom}_{\text{cts}}(\bar{H}^1(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U), \mathbb{C}^{\times})/\text{Hom}_{\text{cts}}(\ker^2(F^{\text{sep}}/F, T \xrightarrow{f} U), \mathbb{C}^{\times}).$$

Moreover, the same short exact sequence tells us that $\text{Hom}_{\text{cts}}(\ker^2(F^{\text{sep}}/F, T \xrightarrow{f} U), \mathbb{C}^{\times})$ is canonically isomorphic to the subgroup

$$\ker[\text{Hom}_{\text{cts}}(\bar{H}^1(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U), \mathbb{C}^{\times}) \to \text{Hom}_{\text{cts}}(H^1(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U), \mathbb{C}^{\times})].$$

But now we know that $\text{Hom}_{\text{cts}}(\bar{H}^1(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U), \mathbb{C}^{\times})$ is canonically isomorphic to the group $H^1(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})_{\text{red}}$ via the pairing (A.8), that $H^1(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U)$ is canonically isomorphic to $\prod'_v H^1(F_v^{\text{sep}}/F_v, T \xrightarrow{f} U)$ (by Proposition A.3.6), and that each $H^1(F_v^{\text{sep}}/F_v, T \xrightarrow{f} U)$ has continuous dual canonically isomorphic to $H^1(W_{F_v}, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})_{\text{red}}$, which gives the result.

For the finiteness of $\ker^1(F^{\text{sep}}/F, T \xrightarrow{f} U)$, one checks that the map

$$\bar{H}^0(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U) \to \text{Ker}(f_*|_{X_*(T)^\Gamma})$$

from the diagram (A.9) remains surjective when restricted to the subgroup $\text{cok}^0(F^{\text{sep}}/F, T \xrightarrow{f} U)$, which means that $\bar{H}^0(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U)_1$ surjects onto $\ker^1(F^{\text{sep}}/F, T \xrightarrow{f} U)$ with open kernel (this openness follows from Lemma A.4.6). Since $\bar{H}^0(\mathbb{A}^{\text{sep}}/\mathbb{A}, T \xrightarrow{f} U)_1$ is compact, its quotient by an open subgroup is finite. □

In fact, we have the following exact analogue of [KS99, Lemma C.3.B], whose adaptation we leave here (for completeness) as an exercise (Proposition A.4.8 is the only part of this result used in the above paper):

Proposition A.4.9 *The groups* $\ker^i(F^{sep}/F, T \xrightarrow{f} U)$ *are finite for all* i *and vanish unless* $i = 1, 2, 3$. *For* $i = 1, 2, 3$*, we have dual finite abelian groups*

$$\mathit{Hom}(\ker^1(F^{sep}/F, T \xrightarrow{f} U), \mathbb{C}^\times) = \ker^2(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T}),$$

$$\mathit{Hom}(\ker^1(F^{sep}/F, T \xrightarrow{f} U), \mathbb{C}^\times) = \ker^1(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})_{red},$$

$$\mathit{Hom}(\ker^3(F^{sep}/F, T \xrightarrow{f} U), \mathbb{Q}/\mathbb{Z}) = \ker^1(\Gamma, X^*(U) \xrightarrow{f^*} X^*(T)).$$

The groups $\text{cok}^i(F^{sep}/F, T \xrightarrow{f} U)$ *vanish for* $i \geq 4$*, and for* $i \leq 3$ *we have duality isomorphisms*

$$\mathit{Hom}_{cts}(\text{cok}^0(F^{sep}/F, T \xrightarrow{f} U), \mathbb{C}^\times) = H^2(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})/\ker^2(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T}),$$

$$\mathit{Hom}_{cts}(\text{cok}^1(F^{sep}/F, T \xrightarrow{f} U), \mathbb{C}^\times) = H^1(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})_{red}/\ker^1(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})_{red},$$

$$\text{cok}^2(F^{sep}/F, T \xrightarrow{f} U) = \mathit{Hom}(H^1(\Gamma, X^*(U) \xrightarrow{f^*} X^*(T))/\ker^1(\Gamma, X^*(U) \xrightarrow{f^*} X^*(T)), \mathbb{Q}/\mathbb{Z}),$$

$$\text{cok}^3(F^{sep}/F, T \xrightarrow{f} U) = \bar{H}^3(\mathbb{A}^{sep}/\mathbb{A}, T \xrightarrow{f} U) = \mathit{Hom}(H^0(\Gamma, X^*(U) \xrightarrow{f^*} X^*(T)), \mathbb{Q}/\mathbb{Z}),$$

where all groups not already defined above are defined in analogy to the corresponding objects in [KS99].

We conclude this section with a few results involving the group $H^1(\Gamma, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})$. First, we define $H^1(\Gamma, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})_{\text{red}}$ to be the quotient of $H^1(\Gamma, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})$ by the image of $(\widehat{T}^\Gamma)^\circ \subseteq H^0(\Gamma, \widehat{T})$. For any $v \in V$, we define the quotient $H^1(\Gamma_v, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})_{\text{red}}$ of $H^1(\Gamma_v, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})$ analogously, with Γ

replaced by Γ_v. Finally, we set

$$\ker^1(\Gamma, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})_{\rm red} := \ker[H^1(\Gamma, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})_{\rm red} \to \prod_{v \in V} H^1(\Gamma_v, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})_{\rm red}].$$

We have the following analogue of [KS99, Lemma C.3.C]:

Lemma A.4.10 *The natural map from $H^1(\Gamma, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})_{red}$ to $H^1(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})_{red}$ maps the group ker$^1(\Gamma, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})_{red}$ isomorphically onto ker$^1(W_F, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})_{red}$. Moreover, we have natural isomorphisms*

$$H^1(\Gamma, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})_{red} \xrightarrow{\sim} H^2(\Gamma, X^*(U) \xrightarrow{f^*} X^*(T))$$

and

$$ker^1(\Gamma, \widehat{U} \xrightarrow{\hat{f}} \widehat{T})_{red} \xrightarrow{\sim} ker^2(\Gamma, X^*(U) \xrightarrow{f^*} X^*(T)).$$

Proof. These second two maps are induced by the boundary map coming from the commutative diagram of short exact sequences of Γ-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X^*(U) & \longrightarrow & \mathrm{Lie}(\widehat{U}) & \longrightarrow & \widehat{U} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow \hat{f} & & \\ 0 & \longrightarrow & X^*(T) & \longrightarrow & \mathrm{Lie}(\widehat{T}) & \longrightarrow & \widehat{T} & \longrightarrow & 1, \end{array}$$

viewed as a short exact sequence of length-2 complexes. The proof of this result is identical to that of [KS99, Lemma C.3.C]. □

BIBLIOGRAPHY

[ABV92] Jeffrey Adams, Dan Barbasch, and David A. Vogan, Jr., *The Langlands classification and irreducible characters for real reductive groups.* Progress in Mathematics, 104, Birkhäuser Boston, Inc., Boston, MA,1992.

[Bor79] Armand Borel. *Automorphic L-functions*. Proc. of Symposia in Pure Mathematics, vol. 33, part 2, 1979, pp. 27-61.

[Bor91] Armand Borel. *Linear algebraic groups* (2nd expanded ed.). Graduate Texts in Mathematics, Springer-Verlag, Heidelberg, 1991.

[BS68] A. Borel and T.A. Springer. *Rationality properties of linear algebraic groups II*. Tohoku Math. Journal, vol. 20, no. 4, pp.443-497, 1968.

[Bro82] Kenneth Brown. *Cohomology of groups*. Graduate Texts in Mathematics, Springer-Verlag, Heidelberg, 1982.

[Căl00] Andrei Căldăraru. *Derived category of twisted sheaves on Calabi-Yau manifolds*. PhD Thesis, Cornell University, 2000.

[Čes16] Kęstutis Česnavičius, *Poitou-Tate without restrictions on the order*. Mathematical Research Letters, vol 22., no. 6, 2016.

[Con18] Brian Conrad, “Discriminants and étale algebras," from Math 248A at Stanford University, http://virtualmath1.stanford.edu/ conrad/248APage/handouts/etaledisc.pdf, 2018.

[Con20] Brian Conrad, “Reductive groups over fields," from Math 249B course notes typed by Tony Feng, Stanford University, URL http://virtualmath1.stanford.edu/ conrad/249BW16Page/handouts/249B_2016.pdf, 2020.

[Con11] Brian Conrad, “Weil and Grothendieck approaches to adelic points." Available online at http://math.stanford.edu/ conrad/papers/adelictop.pdf, 2011.

[CS80] W. Casselman and J. Shalika. *The unramified principal series of p-adicgroups. II. The Whittaker function*. Compositio Math. 41, no. 2, 207–231,1980.

[Deb06] Stephen Debacker. *Parametrizing conjugacy classes of maximal unramified tori via Bruhat-Tits theory*. Michigan Math Journal, vol. 4, no. 1, pp. 157-178, 2006.

[DP08] Ron Donagi and Tony Pantev. *Torus fibrations, gerbes, and duality*. Memoirs of the American Mathematical Society, vol. 193, no. 901, 2008.

[FV02] I.B. Fesenko and S.V. Vostokov. *Local fields and their extensions* (2nd ed.). Translations of Mathematical Monographs, American Mathematical Society, vol. 121, 2nd ed., Providence, RI, 2002.

[Gil21] Philippe Gille, "Introduction to reductive group schemes over rings." Available online at http://math.univ-lyon1.fr/homes-www/gille/prenotes/reductive.pdf, 2021.

[Gir71] Jean Giraud. *Cohomologie non-abélienne*. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Springer-Verlag, Heidelberg, vol. 179, 1971.

[Hoc10] Mel Hochster, "Math 615 lecture notes." Available online at http://www.math.lsa.umich.edu/ hochster/615W10/615.pdf, 2010.

[Kal16] Tasho Kaletha. *Rigid inner forms of real and p-adic groups*. Annals of Mathematics, vol. 184, no. 2, pp. 559-632, 2016.

[Kal18] Tasho Kaletha. *Global rigid inner forms and multiplicities of discrete automorphic representations*. Invent. Math. 213, no. 1, 271–369, 2018.

[Kal19] Tasho Kaletha. *Regular supercuspidal representations*. Journal of the AMS, vol. 32, pp. 1071-1170, 2019.

[Kal20] Tasho Kaletha. Errata webpage. http://www-personal.umich.edu/ kaletha/errata.pdf, 2020.

[KS20] J. Kelly and C. Samuels. *Direct limits of adele rings and their completions*. Rocky Mountain Journal of Mathematics. vol, 50. Issue, 3., 1021-1043, 2020.

[Kot82] R.E. Kottwitz. *Rational conjugacy classes in reductive groups*. Duke Mathematical Journal, vol. 49, no. 4, pp. 785-806, 1982.

[Kot83] R. E. Kottwitz, *Sign changes in harmonic analysis on reductive groups*. Trans. Amer. Math. Soc., vol. 278, no.1, pp. 289-297, 1983.

[Kot84] R.E. Kottwitz. *Stable trace formula: cuspidal tempered terms*. Duke Mathematical Journal, vol. 51, no. 3, pp. 611-650, 1984.

[Kot86] R. E. Kottwitz. *Stable trace formula: elliptic singular terms*. Mathematische Annalen, vol. 275, pp. 365-399, 1986.

[KS99] R. E. Kottwitz and D. Shelstad. *Foundations of twisted endoscopy*. Astérisque No. 255, 1999.

[KS12] R.E. Kottwitz and D. Shelstad. *On Splitting Invariants and Sign Conventions in Endoscopic Transfer.* arXiv:1201.5658, 2012.

[Lan79] R. P. Langlands. *Stable conjugacy: definitions and lemmas.* Canadian Journal of Mathematics, vol. 31, pp. 700-725, 1979.

[Lan83] R.P. Langlands. *Les débuts d'une formules des traces stable.* Publ. Math. de l'Univ. Paris VII, vol. 13, Paris, 1983.

[Lan97] R.P. Langlands. *Representations of abelian algebraic groups.* Pacific Journal of Mathematics, vol. 181, no. 3, pp. 231-250, 1997.

[LS87] R. P. Langlands and D. Shelstad, *On the definition of transfer factors*, Mathematische Annalen, vol. 278, pp. 219-271, 1987.

[LR87] R.P. Langlands and M. Rapoport, *Shimuravarietäten und Gerben.* Reine Angew. Math. vol. 378, pp. 113–220, 1987.

[Lie04] Max Lieblich. *Moduli of twisted sheaves and generalized Azumaya algebras*. PhD Thesis, Massachusetts Institute of Technology, 2004.

[Ols16] Martin Olsson. *Algebraic spaces and stacks*. American Mathematical Society Colloquium Publications, American Mathematical Society, vol. 62, Providence, RI, 2016.

[Mil06] James Milne. *Arithmetic duality theorems* (2nd ed.). Kea Books, Booksurge, LLC, 2006.

[MS89] J.S. Milne and K.Y. Shih. "Conjugates of Shimura varieties," *Hodge cycles, motives, and Shimura varieties*, edited by P. Deligne, J.S. Milne, A. Ogus, and K.Y. Shih, Lecture Notes in Mathematics, Springer-Verlag, vol. 900, pp. 280-356, 1989.

[Moe02] Ieke Moerdijk, "Introduction to the language of stacks and gerbes," from Third Summer Lisbon Lectures in Geometry, Instituto Tecnico Superior, Lisbon, 2002.

[Mor72] R. A. Morris, *The inflation-restriction theorem for Amitsur cohomology*. Pacific Journal of Mathematics, vol. 41, no. 3, pp. 791-797, 1972.

[NSW08] J. Neukirch, A. Schmidt, and K. Wingberg, *Cohomology of number fields*, second ed., Grundlehren der Mathematischen Wissenschaften, vol. 323, Springer-Verlag, Berlin, 2008.

[Poo17] Bjorn Poonen. *Rational points on varieties*. Graduate Studies in Mathematics, American Mathematical Society, vol. 186, Providence, RI, 2017.

[PR94] Vladimir Platonov and Andrei Rapinchuk. *Algebraic Groups and Number Theory*. Academic Press, 1994.

[RZ00] Luis Ribes and Pavel Zelesskii. *Profinite groups*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, vol. 40, Heidelberg, 2000.

[Ros19] Zev Rosengarten. *Tate duality in positive dimension over function fields*. arXiv:1805.00522, 2019.

[Ser92] J.P. Serre. *Lie Algebras and Lie Groups*, second ed ., Lecture Notes in Mathematics, vol. 1500, Springer-Verlag, Berlin, 1992.

[Ser94] J.P. Serre. *Galois cohomology*. Springer Monographs in Mathematics, Spring-Verlag, Heidelberg, 1994.

[Ser95] J.P. Serre. *Cohomologie Galoisienne: progrés et problémes*. Astérisque no. 227, pp. 229-257, 1995.

[Sh90] Freydoon Shahidi, *A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups*. Annals of Mathematics, vol. 131, no. 2., pp. 273-330, 1990.

[Sha64] Stephen Shatz. *Cohomology of Artinian group schemes over local fields*. Annals of Mathematics, vol. 79, no. 3, pp. 411-449, 1964.

[Shi19] Minseon Shin. *The cohomological Brauer group of a torsion* $\mathbb{G}_m$*-gerbe*. to appear in Int. Mathematics Research Notices, 2019.

[Stacks] The Stacks project authors. The Stacks Project. https://stacks.math.columbia.edu, 2020.

[Ste65] Robert Steinberg. *Regular elements of semi-simple algebraic groups.* Publications Mathématiques de l'IHÉS., vol. 25, pp. 49-80, 1965.

[Taï18] Olivier Taïbi. *Akizuki-Witt maps and Kaletha's global rigid inner forms*. Algebra Number Theory 12, no. 4, 833–884, 2018.

[Tat66] John Tate. *The cohomology groups of tori in finite Galois extensions of number fields*. Nagoya Math Journal, vol. 27, no.2, pp. 706-719, 1966.

[Tat79] John Tate *Number theoretic background*, Automorphic Forms, Representations and L-functions, Proc. Sympos. Pure Math, vol. 33, part 2, , pp. 165-184, 1979.

[Tha08] Nguyen Quoc Thang. *On Galois cohomology of semisimple groups over local and global fields of positive characteristic*. Mathematische Zeitschrift, vol. 259, pp. 457-467, 2008.

[Tha11] Nguyen Quoc Thang. *On Galois cohomology and weak approximation of connected reductive groups over fields of positive characteristic*. Proc. Japan Acad, vol. 87, no. 10, pp. 203-208, 2011.

[Tha13] Nguyen Quoc Thang. *On Galois cohomology of semisimple groups over local and global fields of positive characteristic, III*. Mathematische Zeitschrift, vol. 275, pp. 1287-1315, 2013.

[Tha19] Nguyen Quoc Thang. *On Galois cohomology of connected reductive groups and Kottwitz exact sequence.* Bulletin des Sciences Mathématiques, vol. 151, pp. 66-138, 2019.

[Tit79] Jacques Tits. "Reductive groups over local fields," in *Proc. of AMS Summer Institute on L-functions and automorphic representations* by A. Borel, W. Casselman, AMS Proc. of Symposia in Pure Mathematics, 1979.

[Vis15] Angelo Vistoli."Gerbes," from summer school: Algebraic stacks and related topics, University of Mainz, typed by D. Holmes, edited by R. van Bommel, 2015.

[Vog93] David A. Vogan Jr. "The Local Langlands Conjecture," *Representation Theory of Groups and Algebras*, edited by J. Adams, R. Herb, S. Kudla, J.S. Li, R. Lipsman, and J. Rosenberg, Contemp. Math., Amer. Math. Soc., vol. 145, pp. 305-379, Providence, RI, 1993.