Gromov-Witten Theory of Non-Convex Complete Intersections

by

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To my family

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ABSTRACT

This thesis provides a technique to compute the Gromov-Witten invariants of complete intersections in GIT quotient stacks, regardless of any convexity assumptions. In particular, our technique addresses the failure of the Quantum Lefshetz Hyperplane theorem for such targets, and recovers all the invariants one would expect from such a target, if not more.

Our technique revolves around modifying the GIT presentation of the target based on a chosen set of Chen-Ruan cohomology classes. We first do this for toric stacks, where we provide explicit formulas for these modifications through geometric motivations. Then, using the orbifold quasimap theory of Cheong, Ciocan-Fontanine, and Kim, we compute a series associated to this presentation known as an I-function, analogous to the I-functions of Givental. After a mirror transformation, we show that this series lies on Givental's Lagrangian cone, as well as proving that this mirror transformation is invertible. More concretely, we show we are able to obtain explicit values of Gromov-Witten invariants with insertions coming from the classes we extend by, which we illustrate through examples. Notably, these examples recover previously known results, uncover interesting numerical phenomena, and provide cases where invariants with primitive insertions can be computed.

We also extend the above results to non-abelian quotients via Webb's Abelian/Non-abelian Correspondence. We study how the above ideas interact with this correspondence, and prove analogous results for Weyl-invariant Chen-Ruan classes. We then apply these techniques to the example of a stacky del Pezzo to recover its full quantum period, proving a conjecture of Oneto and Petracci.

CHAPTER I
Introduction

Inspired by the physical calculations of Candelas, de la Ossa, Green, and Parkes [9], Givental proved the celebrated genus zero Gromov-Witten theory formulas for the quintic threefold [34], and later extended these proofs to complete intersections in toric varieties [32, 33]. The arguments of these proofs rely on formulating the genus zero invariants of the complete intersection as twisted invariants of the ambient scheme, a process whose modern formulation takes the form of the so-called Quantum Lefshetz Hyperplane Theorem [21, 46], and has become the de facto standard method to computing genus zero Gromov-Witten invariants of complete intersections.

However, this method falls apart when the target is an orbifold, or, in algebraic terms, a Deligne-Mumford stack. As shown in [20], the Quantum Lefshetz Hyperplane Theorem doesn't hold for almost all complete intersections in Deligne-Mumford stacks, including the most simple examples. The culprit is linked to a common requirement for Quantum Lefshetz known as convexity, which is a very mild ask for scheme targets, yet turns into a highly restrictive condition for stack targets. As a result, the most valuable tool for computing the Gromov-Witten invariants of such targets is rendered ineffective for most of them.

The purpose of this thesis is to provide a new method to computing the genus zero Gromov-Witten theory of a complete intersection in a Deligne-Mumford quotient stack. This method is independent of the convexity condition required of Quantum Lefshetz, and provides a computation of all the invariants one would expect from a Quantum Lefshetz type theorem, if not more. We will work in the context of quasimap theory, which we review along with other required topics. We will then explain the motivation behind the methodology and prove that one can recover all of the desired Gromov-Witten invariants. Examples of computations will be provided to illustrate the method at work in more concrete settings.

I.1: Motivation

The idea of counting is as rudimentary as one can get in mathematics, but the questions that are born of it can be surprisingly complex. A simple question might be how many lines go through two points, with an equally simple answer of one line. However, what if one replaces the line with a conic? It might not take much convincing to see that there are infinitely many conics through two points, but then the natural question creeps in of how many points do we need to have finitely many conics. Eventually curiosity leads to even more questions; for instance, one can ask about cubics, or maybe change the condition of going through points to instead be tangency to other fixed shapes. As the variations increase in complexity, the solutions grow wilder and more difficult, but the simplicity of the questions continue to make them tantalizing to mathematicians.

These questions gave rise to the field of enumerative geometry, which seeks to answer all such geometric counting questions. This ranges from the more ancient Apollonius's problem in Euclidean geometry to more modern questions involving intersections in projective space. A simple example of the latter is asking how many degree d rational curves pass through $3d+1$ points in the projective plane. However, it turns out this simple problem was surprisingly difficult; despite being tackled since the late 1800's, it was only solved for $d \leq 5$ by 1990! The breakthrough that led to a full solution of this problem is what is now known as Gromov-Witten theory.

Gromov-Witten theory is an enumerative theory that gives "virtual counts" of curves with some incidence relations inside of a scheme X. Following motivations in physics, the idea of Gromov-Witten theory is to count these curves through intersection theory on what is known as the *moduli space of stable maps*, denoted $\overline{\mathcal{M}}_{g,n}(X, \beta)$. This moduli space parameterizes morphisms f from a nodal curve C of genus g with n-marked points to the space X such that $f_*[C] = \beta$, where the degree class β is a chosen Chow or homology class in $A_2(X)$ or $H_2(X, \mathbb{Z})$. Stability conditions are placed on the morphisms f so that they have finite automorphisms, making $\overline{\mathcal{M}}_{g,n}(X, \beta)$ a Deligne-Mumford stack.

The moduli stack $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is often highly singular, and may have irreducible components of varying dimension, a byproduct of the compactification required to make the intersection theory well-defined. However, given something called a perfect obstruction theory, one can construct what is known as a *virtual fundamental class* $\left[\overline{\mathcal{M}}_{g,n}(X, \beta)\right]^{\text{vir}}$, which is a Chow or homology class that is of the expected dimension of the moduli space [6].

To get the desired curve count, we utilize the natural *evaluation morphisms* $\text{ev}_i : \overline{\mathcal{M}}_{g,n}(X, \beta) \to X$, which are defined by taking a morphism f to the image of the i-th marked point. A *Gromov-Witten invariant* is then given by pulling back Chow classes γ_i on

X by the ev_i and integrating them against the virtual fundamental class,

$$\text{(I.1.1)} \qquad \langle \gamma_1, \ldots, \gamma_n \rangle_{g,\beta}^X := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_i \mathrm{ev}_i^* \gamma_i$$

The above invariant is meant to count curves in of genus g in X which are incident to the corresponding Chow cycles γ_i at the n marked points, and in some cases they do, such as in the case of Kontsevich's formula for rational plane curves [45]. However, as alluded by the phrase "virtual count", this is not always the case; the counts themselves can be rational or even negative. Still, the invariants have some very nice properties, such as being deformation invariant, and satisfy many rich relations and recursion structures that provide the framework for a highly interesting enumerative theory.

It is often useful to include other classes in the integrand of (I.1.1), producing invariants that are a spin on the one above. The most common inclusion is that of the ψ-classes. The class ψ_i is defined to be the first Chern class of the tautological line bundles L_i on $\overline{\mathcal{M}}_{g,n}(X,\beta)$, where L_i is the bundle whose fiber over a point f is the fiber of the cotangent line bundle of the source curve at the i-th marked point. Integrals with ψ-classes are known as *descendant invariants*, and are often considered when studying structural properties of Gromov-Witten invariants.

Another useful variation is to include characteristic classes of obstruction bundles on $\overline{\mathcal{M}}_{g,n}(X,\beta)$ into the integrand. The resulting invariants are known as *twisted invariants*, and often show up when trying to relate Gromov-Witten theories of related spaces, such as the case of local Gromov-Witten invariants. A notable appearance of twisted invariants is in the Quantum Lefshetz Hyperplane Theorem, which we now turn to.

Let Y be a smooth projective variety, and let $E \to Y$ be a vector bundle that is a direct sum of line bundles, $E = \bigoplus_j E_j$. Let $X \subset Y$ be a complete intersection that is cut out by the vanishing of a generic section of E. Furthermore, assume that E is *convex*, which is to say that for any genus zero stable map $f : C \to Y$, we have $H^1(C, f^*E) = 0$. Then consider the diagram

$$\begin{tikzcd} C_{0,n,\beta} \arrow[r, "f"] \arrow[d, "\pi"] & Y \\ \overline{\mathcal{M}}_{0,n}(Y,\beta) & \end{tikzcd}$$

where $C_{0,n,\beta}$ is the universal curve and f is the universal morphism. The convexity condition on E implies that $E_{0,n,\beta} := R\pi_* f^* E$ is a vector bundle on $\overline{\mathcal{M}}_{g,n}(Y,\beta)$.

For a degree class δ on Y, let $\iota : \overline{\mathcal{M}}_{g,n}(X,\delta) \hookrightarrow \overline{\mathcal{M}}_{g,n}(Y, i_*\delta)$ be the map on moduli stacks induced by the inclusion $i : X \to Y$. Then the Quantum Lefshetz Hyperplane Theorem states that we have the following equality of virtual fundamental classes

$$\sum_{\delta : i_* \delta = \beta} [\overline{\mathcal{M}}_{0,n}(X, \delta)]^{\mathrm{vir}} = [\overline{\mathcal{M}}_{0,n}(Y, \beta)]^{\mathrm{vir}} \cap e(E_{0,n,\beta}) \tag{I.1.2}$$

In other words, the theorem states that the Gromov-Witten invariants of X can be expressed as twisted Gromov-Witten invariants of Y, where we twist by the Euler class of $E_{0,n,\beta}$.

This theorem is incredibly useful when the invariants of the ambient space Y are easier to compute, which is often the case. For example, the quintic threefold $Q_5 \subset \mathbb{P}^4$ does not carry a nice non-trivial torus action, while the ambient space has a very simple scaling action. As a result, the twisted Gromov-Witten invariants of the ambient space can be computed using techniques such as Atiyah-Bott localization, greatly simplifying the computation for Q_5. Consequently, this theorem lies at the heart of the computation for many complete intersections, especially if the ambient space carries a nice group action or is combinatorially rich such as in the case of toric varieties [32, 33].

The above story can be replicated when the target is a Deligne-Mumford stack instead of a scheme. However, the convexity condition imposed on E is much more restrictive. For smooth schemes, the convexity of E is implied by a positivity condition; if $\int_\beta c_1(E_j) \geq 0$ for all curve classes β, then the bundle E is convex. But for Deligne-Mumford stacks, this positivity condition no longer implies convexity [20]. One way to see this is to look at the Riemann-Roch formula for a vector bundle $\mathcal{E}$ on an orbifold curve $\mathcal{C}$, [2]

$$\chi(\mathcal{E}) = \mathrm{rank}(\mathcal{E})\chi(\mathcal{O}_{\mathcal{C}}) + \deg \mathcal{E} - \sum_{i=1}^{n} \mathrm{age}_{p_i}(\mathcal{E}). \tag{I.1.3}$$

Here, the p_i denote the stacky points of the curve $\mathcal{C}$. The quantity $\mathrm{age}_{p_i}(\mathcal{E})$ is always non-negative, and is based on how the isotropy groups at p_i act on the fibers of $\mathcal{E}$. The important point is that one can show that these numbers are zero for all stacky points on curves involved in the Gromov-Witten theory of X only when the bundle $\mathcal{E}$ is pulled back from the coarse moduli space of Y. When this isn't the case, which is for most bundles, we can always consider a stable map from a curve with high enough amounts of stacky points so that the right side of (I.1.3) is negative, hence the convexity condition fails.

As a result, it is unlikely that an equation like I.1.2 holds in the orbifold setting, and explicit counterexamples are detailed in [20]. The Gromov-Witten invariants of orbifold complete intersections are thus rarely obtained as a twisted theory of an ambient stack, which has severely limited the development of the genus zero orbifold theory of targets such as complete intersections in toric stacks compared to their non-stacky counterparts.

I.2: Overview of Present Work

In this thesis, we address the failure of Quantum Lefshetz for orbifold complete intersections and develop techniques to compute the genus zero Gromov-Witten theory for such targets. The ambient space for our targets will be GIT quotient stacks

$$Y = [V /\!\!/_{\theta} G] = [V^{\mathrm{ss}}/G]$$

where V is a vector space, G is an algebraic reductive group acting on W, and θ is a character of G. Our actual targets will be complete intersections inside these stacks, $X \subset Y$, which is cut out by some split vector bundle $E = \bigoplus_j E_j$. In particular, X can be written as a GIT quotient stack

$$X = [W /\!\!/_{\theta} G]$$

for some affine scheme W.

These quotient stacks are precisely the spaces for which quasimap theory has been developed, first done in [15] for schemes and later extended to orbifolds in [12]. In quasimap theory, one instead looks at morphisms from (orbifold) curves $\mathcal{C}$ to the non-GIT stack quotient $[W/G]$ where almost all the curve lands in the GIT locus X. From this viewpoint, one can form a ϵ-family of moduli spaces $\mathcal{Q}^{\epsilon}_{g,n}(X,\beta)$, for $\epsilon \in (0,\infty] \cap \mathbb{Q}$. Each of these moduli spaces are different compactifications of the open locus $\mathcal{M}_{g,n}(X,\beta)$ in the moduli space of stable maps where the source curve is smooth. Notably, we also have that $\mathcal{Q}^{\infty}_{g,n}(X,\beta)$ is precisely the usual compactification $\overline{\mathcal{M}}_{g,n}(X,\beta)$.

The main purpose of using quasimap theory is that it provides a setting for us to do a Givental-style argument, where we compute the invariants of X through what is known as a mirror theorem. Classically, instead of computing invariants, one usually tries to find a formula for a generating series of Gromov-Witten invariants known as the *J-function*. This was done by explicitly solving for a hypergeometric series known as an *I-function*, which arises as the solution to the Picard-Fuchs equation on a mirror space $\tilde{X}$. By the general ideas of mirror symmetry, one can show that the I-function equals the J-function up to a change of variables, thus giving an explicit formula for J and allows one to explicitly unwrap the individual invariants.

In quasimap theory, the I-function is instead obtained as a generating function of similarly defined invariants on an $\epsilon = 0^+$ compactification, $\mathcal{Q}^{0^+}_{0,1}(X,\beta)$. This moduli space is in many ways simpler to work with than the moduli space of stable maps, including the fact that the I-function can be computed via a localization procedure. Meanwhile, we have that J-function as the generating series of Gromov-Witten invariants on the $\epsilon = \infty$ side, which is what we truly want. The goal is to then compare the two functions on both sides of the ϵ spectrum; by taking $\epsilon \to \infty$ and seeing how the invariants change, we can use the formula for I in order to get an explicit formula for J, a process known as *wall-crossing*.

For now, let us restrict to the case of an abelian quotient, which is when the group G is a complex torus. This case captures all the main ideas, while also being necessary for the case of a non-abelian quotient.

Our first step is to compute the I-function. However, it is in this step that we already run into a troubling problem. The curves involved in the I-function are relatively simple, being irreducible genus zero curves with at most one stacky point. One can check that given a map f from such a curve $\mathcal{C}$, we have that $H^1(\mathcal{C}, f^*E) = 0$ as long as E is a positive bundle, just as in the scheme case. This means that a Quantum Lefshetz theorem would apply to the computation for the I-function, even when this is not the case for the Gromov-Witten theory of X. This suggests that the naïve I-function does not have the ability to recover all the Gromov-Witten invariants, and it indeed turns out to be the case that one cannot capture those invariants responsible for the failure of convexity.

Another related problem with these I-functions is that they may not have enough parameters to account for the full scope of invariants desired in the J-function. Admittedly, this is somewhat by our choice of set-up; we use what is known as a small I-function, which classically only computes invariants where the cohomological insertions are of degree ≤ 2 (i.e. divisor classes). Meanwhile, the J-function we want is sometimes referred to as a big J-function in the literature, a generating series where the invariants have any number of insertions and the insertions are any Chen-Ruan cohomology class from X. However, we point out that even when one only cares about insertions of degree ≤ 2, such as in the case of Calabi-Yau threefolds, the naïve I-function still lacks enough parameters. This can be traced to the fact that the divisor equation for invariants is limited to Chen-Ruan classes coming from the twisted sectors [2, Theorem 8.3.1].

Our solution to this problem is to extend the GIT presentation of X in a particular way. Suppose $X = [W /\!\!/_\theta (\mathbb{C}^*)^r]$ where the weight matrix of the $(\mathbb{C}^*)^r$ action is a matrix $r \times n$ matrix A. Then given cohomology classes $\phi_1, \ldots, \phi_m \in H_{\mathrm{CR}}(X)$ associated to Chow cycles in the inertia stack of a certain form, we construct a new GIT presentation

$$X = [W_e /\!\!/_{\theta_e} (\mathbb{C}^*)^{r+m}] \subset [\mathbb{C}^{n+m} /\!\!/_{\theta_e} (\mathbb{C}^*)^{r+m}]$$

where the weight matrix of the new action is given by

$$\left(\begin{array}{c|c} A & 0_{r\times m} \\ \hline B & \mathrm{Id}_{m\times m} \end{array}\right)$$

for some $m \times n$ matrix B. The matrix B here is chosen in a specific way to correspond to the choice of the ϕ_i.

The key idea behind this new presentation is that it allows us to repackage quasimaps from orbifold curves into quasimaps from their coarse curves. This uses the fact that quasimaps are morphisms into the honest stack quotient $[W/G]$ which does change under the new GIT presentation, even though the GIT quotient stack itself does not. The extra torus actions means that a morphism into the extended stack quotient requires the data of additional line bundles, which can be used to transform the stack data into base points while simultaneously retaining the stacky information, thus allowing for the process to be reversed if needed.

We illustrate this idea in the following example of a degree 7 hypersurface in $\mathbb{P}(1,1,1,1,3)$.

Example I.1. Let C be a smooth curve with a marking p, and let $\mathcal{C}$ be the 3rd root stack of C along p. There is an associated inclusion $B\mu_3 \to \mathcal{C}$, and a 3rd root $\mathcal{O}_{\mathcal{C}}(p/3)$ of $\mathcal{O}_C(p)|_{\mathcal{C}}$ together with an inclusion $\mathcal{O}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{C}}(p/3)$.

Consider a now a map $f\colon \mathcal{C} \to \mathbb{P}(1,1,1,1,3)$ mapping the orbifold point to the orbifold point, and such that the restriction $B\mu_3 \to B\mu_3$ is the identity. Recall that $\mathbb{P}(1,1,1,1,3)$ is a GIT quotient $V /\!\!/_\theta \mathbb{C}^*$ where $V = \mathbb{C}^5$ with $\mathbb{C}^*$-action specified by the charge matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 3 \end{pmatrix},$$

and with character $\theta\colon \mathbb{C}^* \to \mathbb{C}^*, \theta(\lambda) = \lambda$. The map f is thus equivalent to a line bundle $\mathcal{L}$ on $\mathcal{C}$ together with a section $s \in H^0(\mathcal{L}^{\oplus 4} \oplus \mathcal{L}^{\otimes 3})$. The conditions on the map imply that $\mathcal{L}$ can be written as $\mathcal{L} = L|_{\mathcal{C}} \otimes \mathcal{O}_{\mathcal{C}}(p/3)$ where L is a line bundle on C. We have the identification

$$H^0(\mathcal{L}^{\oplus 4} \oplus \mathcal{L}^{\otimes 3}) = H^0(L^{\oplus 4}|_{\mathcal{C}} \oplus (L^{\otimes 3} \otimes \mathcal{O}_C(p))|_{\mathcal{C}}),$$

and hence the line bundles L and $\mathcal{O}_C(p)$, the section s, and the inclusion $\mathcal{O}_C \to \mathcal{O}_C(p)$ define a map $C \to [V'/T']$ for $V' = \mathbb{C}^6$ and $T' = (\mathbb{C}^*)^2$ with T'-action on V' specified by the charge matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

With the character $\theta'\colon T' \to \mathbb{C}^*$ given by $\theta'(\lambda, \mu) = \lambda\mu$, the set of semi-stable points is

$$(V')^{\mathrm{ss}} = V' \setminus (\{x_0 = \cdots = x_4 = 0\} \cup \{x_5 = 0\}) \subset \{(x_0, \ldots, x_5) \in \mathbb{C}^6\} = V',$$

and hence the GIT quotient $V' /\!\!/_{\theta'} T'$ recovers $\mathbb{P}(1,1,1,1,3)$. Therefore, the map f gives rise to a map $f' \colon C \to [V'/T']$ that falls into X_7 outside of p, and that falls with order one into

$$[(\{x_5 = 0\} \setminus (\{x_0 = \cdots = x_4 = 0\})/T']$$

at p. Conversely, given such an f', we may reconstruct the orbifold map f.

We can do an analogous construction for the hypersurface $X_7 \subset \mathbb{P}(1,1,1,1,3)$. The equation of X_7 is of the form

$$F_7(x_0,x_1,x_2,x_3) + x_4F_4(x_0,x_1,x_2,x_3) + x_4^2F_1(x_0,x_1,x_2,x_3), \tag{I.2.1}$$

where F_7, F_4 and F_1 are homogeneous polynomials of degree 7, 4 and 1, respectively, and F_1 is necessarily non-zero. We may then write $X_7 = W/\!\!/_\theta\mathbb{C}^*$, where W is the zero locus of (I.2.1). Then, $\mathfrak{X}_7 = [W/\mathbb{C}^*]$ is an extension of X_7 to $[V/\mathbb{C}^*]$. In order to extend X_7 to $\mathfrak{X}_7' = [W'/T']$, we use the equation

$$x_5^2F_7(x_0,x_1,x_2,x_3) + x_4x_5F_4(x_0,x_1,x_2,x_3) + x_4^2F_1(x_0,x_1,x_2,x_3),$$

which has weights $(7,2)$ with respect to T'. With these definitions, there is a correspondence between maps $f \colon \mathcal{C} \to X_7$ mapping p to the twisted sector $\mathbf{1}_{1/3}$, and maps $f \colon C \to \mathfrak{X}_7'$ that map p into part of the unstable locus.

We could extend the GIT further to incorporate the twisted sector $\mathbf{1}_{2/3}$ to be a hypersurface in $[\mathbb{C}^7/(\mathbb{C}^*)^3]$ for the charge matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 2 & 0 & 1 \end{pmatrix},$$

although this is not needed as $\mathbf{1}_{2/3}$ is a degree 4 class and X_7 is a Calabi-Yau threefold.

As evidenced in the example, the stacky points in the original source curve $\mathcal{C}$ become base points in the morphism to the extended presentation. The upshot of this extended presentation is that the simplicity of the source curves for our I-function is no longer a barrier, as we can now hide additional stack data in the base points. This allows us to create a more expressive I-function which should be able to recover invariants involving the cohomology classes ϕ_i.

Theorem A (Theorem V.28, Corollary V.31)**.** *Consider $X \subset \mathfrak{X}_e$, where $\mathfrak{X}_e$ denotes the* *extended stack quotient. Then extended I-function for X is given by*

$$I^X(q,z) = \sum_{\beta \in \mathrm{Eff}^I} \frac{q^\beta}{(\prod_{a=1}^m (d_{r+a}!) z^{d_{r+a}})} \prod_{i=1}^n \prod_{j=1}^s \times$$

$$\frac{\prod\limits_{\substack{\langle k \rangle = \beta\cdot\psi_{\bullet i} \\ \beta\cdot\psi_{\bullet i} < k < 0}} \left(c_1\left(\mathcal{O}_{\mathfrak{X}_e}(\beta\cdot\psi_{\bullet i})\right)|_{F_\beta} + kz\right)}{\prod\limits_{\substack{\langle k \rangle = \beta\cdot\psi_{\bullet i} \\ 0 < k \leq \beta\cdot\psi_{\bullet i}}} \left(c_1\left(\mathcal{O}_{\mathfrak{X}_e}(\beta\cdot\psi_{\bullet i})\right)|_{F_\beta} + kz\right)} \times \frac{\prod\limits_{\substack{\langle k \rangle = \beta\cdot\xi_{\bullet j} \\ 0 < k \leq \beta\cdot\xi_{\bullet j}}} \left(c_1\left(\mathcal{O}_{\mathfrak{X}_e}(\beta\cdot\xi_{\bullet j})\right)|_{F_\beta} + kz\right)}{\prod\limits_{\substack{\langle k \rangle = \beta\cdot\xi_{\bullet j} \\ \beta\cdot\xi_{\bullet j} < k < 0}} \left(c_1\left(\mathcal{O}_{\mathfrak{X}_e}(\beta\cdot\xi_{\bullet j})\right)|_{F_\beta} + kz\right)} \cdot \iota\left([F_\beta]^{\mathrm{vir}}\right)$$

where $[F_\beta]^{\mathrm{vir}}$ is associated with its image under the evaluation map, as in (V.3.5), ι is the involution map on the inertia stack, and $q^\beta := \prod_{i=1}^{r+m} q_i^{d_i}$ for $\beta = (d_1, \ldots, d_{r+m})$. Moreover, when $[F_\beta]^{\mathrm{vir}}$ is obtained via a regular sequence, we have that we can identify $\iota([F_\beta]^{\mathrm{vir}})$ with the cohomology class $t_{\alpha_{-\beta}}^{I_\beta^{<0}}$.

There is a lot of notation in the above formula that we refrain from defining now and will be clarified in Chapter V. The takeaway one can take, however, is that the I-function has an explicit formula with easily computable pieces. A reader familiar with I-functions of complete intersections may also notice that the factor coming from the vector bundle defining the intersection has possible negative indexing and may contribute to the denominator of the above expression, which indicates the possible non-convexity of the bundle.

The computation of the I-function follows the recipe set-out by Ciocan-Fontanine, Cheong, and Kim [12] with some extra details in order to account for the enlargened presentation. Some notable features we point out is that the above I-function is valued in the cohomology of X rather than the ambient space Y, as a twisted I-function might be, and that there are additional Novikov variables arising from the extra torus actions, which are crucial to showing that this I-function is indeed capable of recovering the full-range of invariants we desire.

With our I-function in hand, we proceed to show that we can wall-cross to the $\epsilon = \infty$ moduli space to obtain the J-function. Define

$$\mu(q,z) = [zI(q,z) - z]_+$$

where $[\cdot]_+$ takes the terms of the series with non-negative powers of z. Then we make use of a quasimap wall-crossing formula proved by Zhou [59, Theorem 1.12.2] which states

$$J(\mu(q,-z), q, z) = I(q,z).$$

Therefore, we can obtain a formula for a J-function where the generic insertion in our invariants is $\mu(q,-z)$. Alternatively, this can be phrased as saying that our I-function lies on the Lagrangian cone.

However, if one wants to recover individual Gromov-Witten invariants, they would need to show that the change of variables given by $\mu(q,z)$ is invertible. This is made possible by the additional Novikov parameters incorporated into our I-function from the extension of the torus, which allows us to circumvent the disparity in parameters between the I and J functions.

Theorem B (Theorem VI.6, Lemma VI.7,)**.** *Consider a GIT extension given by the choice of cohomology classes $\{\phi_1,\ldots,\phi_m\}$. Let $\mu(q,z)$ be the series associated to the extended I-function as above. Then $\mu(q,z)$ can be written in the form*

$$\mu(q,z) = \sum_{k=1}^{m} q^{\beta_k}\phi_k + \text{ other terms}$$

for some explicit degrees β_k.

Moreover, if all the cohomology classes appearing in $\mu(q,z)$ are contained in the extension set $\{\phi_i\}_{i=1}^m$, then the mirror transformation given by $\mu(q,z)$ is invertible.

The first part of the statement indicates that the Gromov-Witten invariants in the J-function obtained by the mirror transformation can have arbitrary amounts of insertions of the form ϕ_i. The second part of the statement can be interpreted as saying that one can extract values for each of these individual invariants from the I-function formula, hence can compute all the Gromov-Witten invariants with ϕ_i insertions. The hypothesis of this statement can be easily checked via other sufficient conditions, and is often managed for the targets one sees in practice. This theorem is also independent of how complicated $\mu(q,z)$; when $\mu(q,z)$ has positive powers of z, reminiscent of a general type intersection in the classical case, we use a process known as *Birkhoff Factorization* in order to show that the mirror map is still invertible.

By combining the above results, we have essentially proven a generalization of the Quantum Lefshetz theorem for toric stacks. In particular, we remove any need for the convexity hypothesis, and, by virtue of the amount of cohomology classes we can keep track of, we are able to recover more invariants than what a Quantum Lefshetz type argument would normally compute.

Finally, we show that one can employ similar techniques to deal with non-abelian quotients as well. Here, the key tool is the use of Webb's abelian non-abelian correspondence [58], which allows us to write the I-functions of the non-abelian quotient in terms of an I-function for a corresponding quotient by a torus. When the latter is a complete intersection in a toric stack, we can use the extension techniques above to obtain an extended I-function for the non-abelian quotient.

We note that there are some subtleties when doing this, as we need the corresponding extension for the non-abelian group to retain properties necessary for quasimap theory, e.g. having the semi-stable and stable locus agree. These issues are addressed in Proposition VIII.4 and Lemma VIII.8, where we provide sufficient conditions for the extension that can be easily checked. There is also some clarification needed in how to choose an appropriate extension for a given class, which we address by discussing the correspondence between Weyl group-invariant cohomology classes in the abelian quotient and cohomology classes of the non-abelian quotient.

We apply these non-abelian techniques to the more involved example of a $\frac{1}{3}(1,1)$ del Pezzo surface $X_{1,7/3}$, which is a complete intersection inside the weighted Grassmannian $wGr(2,5)$. By using an appropriate extension, we are able to obtain a J-function for this example that contains a more diverse set of invariants than any previous computation. We then extract what is known as the quantum period from this J-function, obtaining the following formula.

Theorem C (Theorem VIII.21)**.** *The quantum period of $X_{1,7/3}$ is given by*

$$G(x,t) = e^{-t(x+5)}\times$$
$$\sum_{\tilde{\beta}_i\in\mathbb{Z}_{\geq 0}} A_{\tilde{\beta}}(t,t(x-3),1)\left(1+\frac{\tilde{\beta}_1-\tilde{\beta}_2}{2}(-3B_{\tilde{\beta}_1}+3B_{\tilde{\beta}_2}-2B_{2\tilde{\beta}_1+\tilde{\beta}_2+\tilde{\beta}_3}+2B_{\tilde{\beta}_1+2\tilde{\beta}_2+\tilde{\beta}_3})\right),$$

where $A_{\tilde{\beta}}$ and B_n are explicit formulas defined in Section VIII.3.4. Setting $x=3$, we recover the conjectured specialized formula given in [51, Section 6.2].

The above formula contains more information than the formula conjectured in [51, Section 6.2], which is a testament to the usefulness of our techniques, and is a necessary step towards approaching other conjectures about quantum periods, such as [4, Conj. B]. While the above formula is an example, the techniques are much more general, and we hope that we can find other interesting non-abelian situations to apply them.

I.3: Relation to Past Work

The work in this thesis is based on two upcoming works [39, 54] by the author and his collaborators. The work involving extensions for toric stacks is based upon and generalizes work with collaborators Felix Janda and Yang Zhou, where similar statements were proved in the case of a Calabi-Yau threefold in a weighted projective stack. The work with non-abelian quotients is done in collaboration with Rachel Webb, where we take the extent of this technique to its perceivable limits in the non-abelian case.

We also make mention that the idea of extensions for toric stacks has been seen before in the notion of an extended stacky fan [40]. The relation between the extended stacky fan and the GIT extensions we consider can be given by translating between the stacky fan picture to the GIT picture, as in [22, Section 4], albeit up to some row operations. One can view the reasonings presented for the extensions in this thesis as providing geometric intuition to the more combinatorial picture in [40].

The stacky fan was utilized by [18, 19] to prove a mirror theorem for toric stacks and convex complete intersections. The results on toric complete intersections in this thesis can be seen as a generalization of those results. The most notable generalization is that we remove the convexity requirement, which they require in order to use a Quantum Lefshetz argument. However, we also note that the extensions in this thesis comprise of a larger subset of cohomology classes than just fundamental classes of twisted sectors, and that our I-function lives directly in the cohomology of the complete intersection rather than being a twisted I-function. As a result, the breadth of invariants we can recover is greater, such as being able to recover invariants with some primitive insertions. Additionally, we provide some examples of invariants where condition $S - \sharp$ of [19] does not hold, which may prove of interest.

We also make mention of some other work that has appeared in recent years regarding non-convex complete intersections:

In a closely related approach [56], Wang gives an I-function for non-convex complete intersections, and has an independent proof to that of [59] of the wall-crossing formula for quasimaps to a toric stack. We expect that applying his approach to the extended GIT presentations in this thesis will recover the same results.

In [37], Guéré develops a new technique called "Hodge Gromov-Witten theory" which allows computing genus-zero and certain higher genus invariants of possibly non-convex hypersurfaces in weighted projective space. It would be interesting to verify if his computations lead to the same results as ours.

In [38], Heath and Shoemaker develop a general Quantum Lefshetz and Serre duality statement for 2-pointed quasimaps to possibly non-convex orbifold complete intersections in a stacky GIT quotient. Combined with quasimap wall-crossing, this allows computing the Gromov-Witten invariants with ambient insertions in terms of the invariants of the ambient space.

In [58], Webb proves an abelian/non-abelian correspondence for orbifolds, and as a consequence, constructs an I-function for the Gromov-Witten theory of complete intersections.

These results are used in a collaborative paper with the author [54] as indicated above.

I.4: Detailed Outline

In Chapter II, we provide an overview of Deligne-Mumfod stacks and their Gromov-Witten theory. The information provided here is intended to give inexperienced readers an intuitive understanding of the ideas, while also serving as a reference for more experienced readers.

In Chapter III, we briefly cover the main ideas and properties of quasimap theory used in this thesis. Notably, we introduce the definition I-function as a localization residue on the stacky loop space.

In Chapter IV, we establish the definition of a toric stack, as well as highlighting the main properties, conventions, and assumptions we make about these targets. We also describe complete intersections in such spaces, including their Chen-Ruan cohomology and other necessary properties.

In Chapter V, we focus on the computation of the extended I-function. We introduce the notion of the extended GIT quotient, and detail its construction in relation to a chosen set of Chen-Ruan cohomology classes. Afterwards, we give an explicit description of the fixed locus corresponding to the I-function computation, and then proceed to computing the fixed and moving parts of the perfect obstruction theory to obtain an I-function formula.

In Chapter VI, we describe the mirror theorem and prove invertibility of the mirror map. We show that the I-function recovers all the invariants with insertions corresponding to the chosen cohomology classes in the extension. We also discuss the complexity of the mirror map in relation to the chosen extension.

In Chapter VII, we give examples of the extended I-function in various scenarios. We compute explicit invariants for certain Calabi-Yau threefolds in weighted projective stacks and show that the invariants receive match known ones in the literature. We also highlight an example where invariants with primitive insertions can be recovered, as well as an example of a complete intersection in a toric stack.

In Chapter VIII, we explain how one can use the abelian/non-abelian correspondence in order to extend the results to that of non-abelian quotients. We discuss how the extension interacts with the above correspondence, and what can be recovered in different scenarios. We also provide an example of a stacky Del Pezzo surface surface inside of a weighted Grassmannian, and show that with the proper extension, we can provide a full formula for the quantum period of the Del Pezzo, proving a conjecture of Oneto and Petracci [51].

I.5: Conventions and Notation

We will assume that all the spaces we work with are defined over $\mathbb{C}$, and we assume that all the Deligne-Mumford stacks we work with are separated. Points of a stack will refer to $\mathbb{C}$ points, unless otherwise specified. We will also assume all our Chow and cohomology groups have $\mathbb{Q}$ coefficients unless otherwise stated. We remark that one can replace $\mathbb{C}$ with other algebraically closed fields of characteristic zero if desired.

We will also use the terms "orbifold" and "Deligne-Mumford stack" interchangeably, but will always use the algebro-geometric definition in our proofs. All our stacky curves will be assumed to have cyclic isotropy groups.

We also use the following notation throughout the paper:

- For a matrix $M = (m_{ij})$, we use the notation $m_{\bullet j}$ to refer to the j-column vector.
- The stacky GIT quotient is defined as the stack quotient of the semi-stable locus by the group, i.e.
$$[W /\!\!/_\theta G] := [W^{\mathrm{ss}}/G]$$
- The weighted projective space $\mathbb{P}(w_0, \dots, w_n)$ refers to the stack quotient
$$[\mathbb{C}^{n+1} - \{0\}/\mathbb{C}^*], \quad \lambda \cdot (x_0, \dots, x_n) = (\lambda^{w_0} x_0, \dots, \lambda^{w_n} x_n) \text{ for } \lambda \in \mathbb{C}^*$$
- For a character χ of G, we define the line bundle $\mathcal{O}(\chi)$ over the stack quotient $[W/G]$ to be the line bundle induced by giving the trivial line bundle $W \times \mathbb{C}$ the G-equivariant structure given by G acting on the fiber by χ.
- For $n \in \mathbb{Q}$, we define $\lfloor n \rfloor \in \mathbb{Z}$ to be the smallest integer such that $n - \lfloor n \rfloor \geq 0$. Similarly, $\lceil n \rceil$ is the smallest integer such that $\lceil n \rceil - n \geq 0$. We define $\langle n \rangle = n - \lfloor n \rfloor$ as the fractional part of n.

CHAPTER II
Orbifold Gromov-Witten Theory

II.1: Preliminaries on Orbifolds

Throughout this paper, the primary spaces of interest will be orbifolds. To be more precise, we will actually be working with the algebraic analog of an orbifold, which is a Deligne-Mumford stack, and will also see the appearance of Artin stacks. The notion of a stack is notorious in its difficulty to fully understand, but many new references in recent years have made the subject much more approachable (see e.g. [29, 50]). Because of this, we will focus on providing a working knowledge of stacks through intuition via orbifolds, as well as specific details about the types of stacks that can appear. Hopefully, this intuition provides enough confidence for those unfamiliar with the material to start working with stacks, which will ultimately lead to more understanding when learning the more precise definitions later.

For those already familiar with stacks, we remark that all the stacks we work in this paper are separated and are of finite type over a characteristic zero ground field $k = \bar{k}$.

An *orbifold* is a singular space or variety that locally looks like the quotient of a manifold by a finite group. The data of an n-dimensional complex orbifold consists of a topological space X along with an atlas of compatible triples $\{(U_i, G_i, \phi_i)\}_{i\in I}$, where $U_i \subset \mathbb{C}^n$, G_i acts on U_i smoothly, $\phi_i : U_i \to X$ is a G_i-invariant morphism that induces a homemorphism of U_i/G onto its image, and $\bigcup_{i\in I} \operatorname{im}\phi_i$ is an open cover of X. We will ignore issues of chart compatibility here, simply noting that it can be done, and will refer to this collection of data as $\mathcal{X}$.

Importantly, the data of an orbifold consists of more than just the singular topological space, and remembers the group action that forms the quotient singularities. Given a point $x \in X$, there exists a chart (U, G, ϕ) such that $x = \phi(y)$, for some $y \in U$. Then there is a subgroup $H \subset G$ that fixes the point y under the G-action. One can show that this group is independent of the chart chosen, and that a neighborhood of $x \in X$ is isomorphic to $\mathbb{C}^n/H$, where the origin maps to x. This group, subsequently referred to as G_x is known as the *isotropy group* at x and is implicit in the data of an orbifold.

There is a forgetful morphism π that takes $\mathcal{X}$ to the underlying topological space X, forgetting the data of the atlas. X here is often referred to as the *coarse moduli space* of $\mathcal{X}$, and $\pi : \mathcal{X} \to X$ is the *coarse moduli morphism.* While $\mathcal{X}$ and X may look the same from a topological point of view, X does not remember the isotropy data of its points.

Example II.1. A classic example of an orbifold is that of a weighted projective stack $\mathbb{P}(w_0, \ldots, w_n)$. The underlying topological space is given by altering the weights of the $\mathbb{C}^*$ action in the usual quotient construction of projective space

$$\left(\mathbb{C}^{n+1} - \{0\}\right)/\mathbb{C}^*, \qquad \lambda \cdot (x_0, \ldots, x_n) = (\lambda^{w_0} x_0, \ldots, \lambda^{w_n} x_n)$$

One can also phrase this as the GIT quotient $\mathbb{C}^{n+1} /\!\!/_\theta \mathbb{C}^*$ for a positive character θ, or as Proj of the ring $\mathbb{C}[x_0, \ldots, x_n]$ where weight of the variable x_i is w_i. It's not hard to see that every point has a finite abelian isotropy group; in particular, the i-th coordinate point will have isotropy group isomorphic to μ_{w_i}. We can find a local chart around each point that is a quotient of an affine space by the isotropy of the point, and hence endow the space with an orbifold structure.

An important example for us is the weighted projective line $\mathbb{P}(a, b)$. Note that as a scheme, constructed by the GIT quotient as above, we have that $\mathbb{P}(a, b) \cong \mathbb{P}^1$ [26, Proposition 1.3.1]. However, as an orbifold or stack, we have that $\mathbb{P}(a, b) \neq \mathbb{P}^1$ since the isotropy data does not match!

In a step towards the algebraic notion of a stack, we can rephrase the definition of orbifold in terms of what is known as a *Lie groupoid.* This encompasses the notion of an orbifold in terms of charts, and historically is a better way to view the subject (see [3]).

To define a Lie groupoid, we first take a groupoid object in topological spaces, $\mathcal{G} = (G_0, G_1, s, t, m, u, i)$. Here, G_0 is a set of objects and G_1 is a set of arrows between said objects. Saying that $\mathcal{G}$ is a groupoid means that all the arrows in G_1 have an inverse. The maps $s, t : G_1 \to G_0$ are source and target maps, $m : G_1 \times_{s,t} G_1 \to G_1$ is a composition map, $u : G_0 \to G_1$ is a unit (identity) map, and $i : G_1 \to G_1$ is an inverse map. Now the set of data $\mathcal{G}$ is a Lie groupoid when the spaces G_0, G_1 are smooth manifolds, all the maps are smooth, and s, t are both submersions.

The above seems like a lot of data, but we can relate it to the first, more concrete description. First, assume that furthermore the maps s, t are both étale and proper. Then the topological space associated to $\mathcal{G}$ is the quotient space G_0/G_1 under the obvious action. This will be the coarse moduli space of our orbifold. On the other hand, the isotropy data associated to a point $x \in G_0/G_1$ is given by taking a preimage $\tilde{x} \in G_0$, and defining G_x to be the group of all arrows in G_1 whose source and target are $\tilde{x}$. Thus, we see that $\mathcal{G}$ is a presentation of an orbifold with coarse modui space G_0/G_1. There are different presentations possible for a given orbifold and one might want to identify them all; this is the notion of Morita equivalence, and we refer the reader to see details in [3]

The above view of an orbifold as a groupoid is very much inspired by the idea of a stack, and hopefully bridges some intuition on why a stack is defined in the way that it is. We will now give a flavor of what goes into a stack.

On the more formal and general side, a stack is a fibered category over a category with a Grothendieck topology for which one has descent. More informally, and more particular to our usage, we will think of a stack as a "sheaf of groupoids" over the category of schemes equip with a Grothendieck topology (typically the étale or fpqc topology).

Disregarding formality and embracing vagueness, we explain the above. By a fibered category, we mean a category $\mathcal{F}$ with a map to another category $\rho : \mathcal{F} \to \mathcal{C}$ with some technical conditions about pulling back objects in $\mathcal{F}$ along morphisms in $\mathcal{C}$. Being fibered in groupoids means that given any object $c \in \mathcal{C}$, the category "fibered" over c, consisting of objects and morphisms in $\mathcal{F}$ that map to c and id_c, is a groupoid. Finally, by a sheaf or descent, it means that given an open cover of an object in $c \in \mathcal{C}$, and objects in $\mathcal{F}$ lying over that open cover that satisfy some compatibility conditions, we can "glue" to obtain an object over c. The last condition should be though of something akin to saying that given an étale or fpqc cover of a scheme, we can locally glue objects on these covers as one would expect in a sheaf.

We can relate this admittedly vague and abstract definition to the groupoid definition of an orbifold as so. Suppose we have two objects X_0, X_1 of a category $\mathcal{C}$, and that there are morphisms as in the orbifold definition above that make this a groupoid. For another object $U \in \mathcal{C}$, let $X_i(U)$ denote set of arrows $U \to X_i$ in $\mathcal{C}$. Then we can define a fibered category $\rho : [X_0/X_1] \to \mathcal{C}$ where the fiber over U is the groupoid $[X_0(U)/X_1(U)]$ whose objects are objects in $X_0(U)$, and morphisms are objects in $X_1(U)$. When $\mathcal{C}$ is the category of schemes with an appropriate topology, this is a stack. When $\mathcal{C}$ is the category of topological spaces, this recovers the notion of an orbifold as before (the geometric points of X_i are recovered by taking U to be a point).

The final note we make is regarding the usage of the terminology around stacks, all over the category of schemes.

- A *representable* morphism of stacks is one in which the base-changing along a morphism from a scheme to the target results in a morphism of schemes. Intuitively, one can think of this as saying given a morphism of orbifolds, the "induced" group homorphism on the level of isotropy groups is injective.

- A *Deligne-Mumford stack* is a stack over schemes for which the diagonal morphism is representable and separated, and is the target of an étale surjective morphism from a scheme. This is the algebraic notion of an orbifold as we have defined it, and is informally identified by the condition of all the isotropy groups being finite.

- An *algebraic* or *Artin* stack is a stack over schemes for which the diagonal morphism is representable and separated, and is the target of an smooth surjective morphism from a scheme. One can informally think of this as the case where we allow our isotropy groups to be non-finite

If it wasn't clear, we also note that all schemes can be though of as stacks with trivial isotropy groups, and that there are obvious inclusions from the categories of schemes to Deligne-Mumford stacks to Artin stacks.

The upshot of this all is that one can wrestle with the abstractness of stacks by first working with the notion of an orbifold in either setting, or thinking of schemes were one keeps track of the data of isotropy groups. Combined with the specifics of the stacks in the next section (Section II.2), this should be enough for the paper.

Convention. We will use the terms "orbifold" to refer to "Deligne-Mumford stack" throughout this paper.

Remark II.2. Due to many of the details being brushed aside, one should be careful when moving between the different pictures presented. For instance, the definition of a coarse moduli space is more specific for stacks, and it is not true that all stacks have one, e.g. the stack $[\mathbb{A}^1/\mathbb{C}^*]$. However, the relevant stacks for our purposes will admit a coarse moduli, and in general for separated Deligne-Mumford stacks one can show the existence of a coarse moduli space via the Keel-Mori Theorem [41]. Thinking of the underlying topological space as k-points of the stack should be sufficient for getting the general picture in this paper.

II.2: Quotient and Root Stacks

The most important examples of stacks that we will work with are quotient stacks, root stacks, and gerbes. We will list out basic, yet important, properties of each, as well as some useful examples.

Given a smooth group scheme G acting on a scheme X, one can form the *quotient stack* $[X/G]$. Given a scheme T, the quotient stack parameterizes G-torsors on T that come with an equivariant map to X, i.e.

$$[X/G](T) = \left\{ \begin{array}{c|c} \begin{tikzcd} \mathcal{P} \arrow[r, "\pi"] \arrow[d] & X_T \\ T & \end{tikzcd} & \mathcal{P} \text{ is a } G_T \text{ torsor and } \pi \text{ is } G_T \text{ equivariant} \end{array} \right\}$$

where G_T and X_T are base-changes of the schemes over T.

From a differential geometry perspective, one should be thinking of the case of principal G-bundles over T. As it turns out, this is the picture when G is also an affine scheme group scheme [50, Proposition 4.5.6], so we have that

$$[X/G](T) = \left\{ \begin{array}{c|c} \begin{tikzcd} P \arrow[r, "\pi"] \arrow[d] & X \\ T & \end{tikzcd} & P \text{ is a principal } G\text{-bundle and } \pi \text{ is } G \text{ equivariant} \end{array} \right\}$$

for affine G. Quotient stacks parameterize the G-equivariant data of X, and provide some of the nicest examples of stacks due to their explicit description. From an intuitive point of view, one can understand the geometric points of the underlying topological space as the orbit space, and the isotropy groups of a geometric point in the quotient to be the stabilizer of a lift of that point in X under the G-action.

Example II.3. Given a smooth group scheme G over $\mathbb{C}$, we can consider the quotient stack $BG = [\bullet/G]$, where G acts on $\operatorname{Spec}\mathbb{C}$ by the trivial group action. This is known as the *classying stack of* G. From the above, a map from a scheme $T \to BG$ is equivalent to a G-torsor over T.

A particularly important example is the case when $G = \mathbb{C}^*$. Then by the equivalence of categories between $\mathbb{C}^*$-torsors and line bundles, we a map from $T \to B\mathbb{C}^*$ is equivalent to a choice of line bundle on T.

Example II.4. Consider the quotient stack $[\mathbb{A}^n/\mathbb{C}^*]$, where $\mathbb{C}^*$ acts by scaling with weight $(1, \ldots, 1)$. A map from a test scheme T to $[\mathbb{A}^n/\mathbb{C}^*]$ is given by a $\mathbb{C}^*$ torsor on T and a $\mathbb{C}^*$ equivariant map from said torsor to $\mathbb{A}^n$. We may consider the torsor as a line bundle $\mathcal{L}$ on T, from which we require a $\mathbb{C}^*$-equivariant map to $\mathbb{A}^n$. One can think of the morphism as a map to trivial rank n bundle, hence show that that data is equivalent to n cosections of $\mathcal{L}$. Taking duals, we see that the one can represent the category $[\mathbb{A}^n/\mathbb{C}^*](T)$ as the category whose objects are a choice of line bundle on T, along with n sections of that line bundle.

One can generalize this example to different $\mathbb{C}^*$ weights $(w_0, \ldots, w_n)$ to see that the sections are of corresponding powers of the line bundle, or increase the dimension of the torus to $(\mathbb{C}^*)^m$ to see that the data involves the choice of m line bundles.

We can now discuss line bundles on the quotient stack $[X/G]$. We refer to [48, Page 64], or [50] for the definition of the Picard group of an algebraic stack.

Lemma II.5. *A line bundle on $[X/G]$ is equivalent to a G-linearized line bundle on X, i.e.*

$$\mathrm{Pic}([X/G]) = \mathrm{Pic}^G(X)$$

Proof. The data of G-linearized line bundle on X is equivalent to the descent data of a line bundle associated to the cover $\{X \to [X/G]\}$, hence the result is immediate. $\square$

The next type of stack we will often use is that of a *root stack* [2, 8]. Let T be a scheme, and let $(\mathcal{L}, s)$ be the data of a line bundle and global section on T. Then we construct the r-th root stack $T_{\mathcal{L},s,r}$ as the fiber product of the following diagram in the category of algebraic stacks

$$\begin{array}{ccc} T_{\mathcal{L},s,r} & \longrightarrow & [\mathbb{A}^1/\mathbb{C}^*] \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \cdot r} \\ T & \xrightarrow{(\mathcal{L},s)} & [\mathbb{A}^1/\mathbb{C}^*] \end{array}$$

where the right vertical map is the map induced by the r-th power map on $\mathbb{A}^1$ and $\mathbb{C}^*$. Fiber-wise, for a scheme S, we have that the category $T_{\mathcal{L},s,r}(S)$ consist of quadruples (f, M, t, φ) where $f : S \to T$ is a morphism, M is an invertible sheaf on S, t is a section of M, and $\varphi : M^{\otimes r} \to f^*\mathcal{L}$ is an isomorphism such that $\varphi(t^r) = f^*s$.

From the orbifold perspective, one should think of the root stack as the scheme T but with "stackiness" added to the vanishing of the section s. Letting $D := V(s)$, we have that the pre-image of D in $T_{\mathcal{L},s,r}$ topologically looks the same, but has isotropy groups μ_r at every k-point. Outside of D, π is an isomorphism. Thus, this construction allows us to add stacky data to a divisor in any scheme or stack.

Example II.6. Consider $\mathbb{P}^1$ with a special point ∞. Then doing the r-th root construction above gives us $\mathbb{P}(1, r)$. One can re-iterate the root construction with a different values r and different points as many times as one wants. The result will be a $\mathbb{P}^1$ with stackiness along any divisor D, and with any choice of finite, cyclic isotropy group attached to each point in D.

In general, one can use this construction to form any smooth, complex orbifold curve $\mathcal{C}$ that has finitely many stacky points, all with finite, cyclic isotropy groups [8, Ex 2.4.6].

One thing to note is that the top horizontal map of the fiber diagram induces a line bundle $\mathcal{T}$ and section $t \in \Gamma(\mathcal{T}, T_{\mathcal{L},s,r})$. Tracing the diagram, we have that $\mathcal{T}^{\otimes r} \cong \pi^*\mathcal{L}$ and $t^r \cong \pi^* s$.

Lemma II.7. *[8, Corollary 3.1.2, 3.1.3] Every line bundle $\mathcal{F}$ on $T_{\mathcal{L},s,r}$ can be written in the form*

$$\mathcal{F} \cong \pi^*\mathcal{M} \otimes \mathcal{T}^k$$

for some unique $0 \leq k < r$ and for M a line bundle on T unique up to isomorphism. Moreover, we have that every global section of $\mathcal{F}$ is of the form $\pi^* m \otimes t^k$ for a unique global section m of M.

Applying the above lemma recursively, we obtain a good understanding of the Picard group of any root stack. In the above decomposition, we will call $\mathcal{M}$ the *round-down* of $\mathcal{F}$.

Lastly, we give a brief discussion of gerbes. We skip the formal definition and instead refer the reader to a reference like [50, Chapter 12]. Loosely speaking, a gerbe is akin to fiber bundle whose fiber is BG for some group G. One can think of a gerbe over a scheme or stack as a stack that adds an extra copy of a group G to all the isotropy groups, but carries the same topological data. Notably, under this description, all the k-points of a gerbe have non-trivial isotropy group. We say that a gerbe is a G-gerbe if locally the fiber looks likes BG.

Example II.8. The trivial example is that BG is a G-gerbe over a point. More generally, given a scheme X, one has the trivial G-gerbe $X \times BG$ over X.

Another example is the weighted projective line $\mathbb{P}(2,2)$. Note that it has μ_2 isotropy at every point. One can show that $\mathbb{P}(2,2)$ is a gerbe with μ_2 banding over $\mathbb{P}^1$, and that is not the trivial gerbe, i.e. $\mathbb{P}(2,2) \ncong \mathbb{P}^1 \times B\mu_2$.

One way to construct gerbes is via root construction as before. Given a scheme T and a line bundle $\mathcal{L}$, one can again construct a fiber product $T_{\mathcal{L},r}$ as

$$\begin{tikzcd}
T_{\mathcal{L},r} \arrow[r] \arrow[d, "\pi"] & B\mathbb{C}^* \arrow[d, "\cdot r"] \\
T \arrow[r, "(\mathcal{L})"] & B\mathbb{C}^*
\end{tikzcd}$$

Then $T_{\mathcal{L},r}$ is a μ_r-gerbe over T, and carries a universal line bundle $\mathcal{T}$ such that $\mathcal{T}^{\otimes r} \cong \pi^*\mathcal{L}$. Different choices of $\mathcal{L}$ produce different gerbes (see [2, Appendix B] for details). We also comment that this construction is not the same as taking $s = 0$ in the construction of the root stack with a section (see [8, Section 2.2])!

Finally, there is a process to remove the gerbe structure known as *rigidification*. Informally, given a stack with non-trivial isotropy everywhere, the process produces a stack where the common isotropy among the points is quotiented out. For instance, given a G-gerbe over a stack $\mathcal{X}$, the rigidication process applied to the gerbe returns the underlying stack $\mathcal{X}$. More details of this can be found in [2, Appendix C].

II.3: Cohomology of DM Stacks

In this section, we collect facts about the Chow and cohomology theories of a Deligne-Mumford stack. For this section, we will always have $\pi : \mathcal{X} \to X$ denote a separated Deligne-Mumford stack over $\mathbb{C}$ with coarse moduli space X. Additionally, all the theories considered will have $\mathbb{Q}$ coefficients.

First, we note that the theory of Chow rings extends nicely to these situations. One can define the rational Chow group of $\mathcal{X}$ as the free abelian group on integral closed substacks up to rational equivalence. There is a pushforward morphism $\pi_* : A_*(\mathcal{X}) \to A_*(X)$ that behaves as follows: let $\pi_{\mathcal{V}} : \mathcal{V} \to V$ be a closed substack of $\mathcal{X}$ with coarse moduli space V. Then $[\mathcal{V}]$ and $[V]$ are cycles in $A_*(\mathcal{X})$ and $A_*(X)$ respectively. If we let r be the order of the isotropy group of a generic geometric point in $\mathcal{V}$, then we have

$$\pi_*[\mathcal{V}] = \frac{1}{r}[V]$$

As for the ring structure, we simply remark that it is constructed in a similar way to the scheme case, and that we have a well-defined notion of characteristic classes of vector bundles on orbifolds.

The cohomology of an orbifold is a little more complex, as we want a cohomology theory that sees more than the underlying topological space. In order to capture the isotropy data, we will instead consider the cohomology of a related space known as the *inertia stack* of $\mathcal{X}$, denoted by $\mathcal{IX}$. The quickest way to define this stack is as the fiber product of the below diagram

$$\begin{tikzcd} \mathcal{IX} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\Delta"] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X} \end{tikzcd}$$

where Δ denotes the diagonal morphism. The inertia stack has a useful decomposition into multiple components

$$\mathcal{IX} = \sqcup_r \mathcal{IX}_{\mu_r}$$

where the stacks $\mathcal{IX}_{\mu_r}$ have a nice description. Given a scheme T, the objects of the category

$\mathcal{IX}_{\mu_r}(T)$ are representable morphisms $(B\mu_r)_T \to \mathcal{X}$, while arrows over $f : T' \to T$ are 2-morphisms of the commutative diagram formed by the objects and the map $f_* : (B\mu_r)_{T'} \to (B\mu_r)_T$. Note that since the objects are representable morphisms, we see that the disjoint product in $\mathcal{IX}$ has only finitely many components when $\mathcal{X}$ is a Deligne-Mumford stack, as r is bounded by the maximal order of the isotropy groups of $\mathcal{X}$.

The purpose of these stacks is to keep track of the automorphism data in the stack $\mathcal{X}$. As we will see in the next example, many of the inertia stacks we work with have $\mathcal{IX}_{\mu_r}$ isomorphic to the closed substack of $\mathcal{X}$ consisting of points whose isotropy groups have an order r cyclic subgroup, and so those unfamiliar with the concept can think of them as such. In particular, we have that $\mathcal{IX}_{\mu_1} \cong \mathcal{X}$. Moving forward, we will refer to $\mathcal{IX}_{\mu_1}$ as the *untwisted sector*, whereas the connected components of the $\mathcal{IX}_{\mu_r}$ for $r \neq 1$ will be known as the *twisted sectors*

We can also rigidify the inertia stack to produce the *rigidified inertia stack*, denoted $\overline{\mathcal{I}}\mathcal{X}$. As a stack, the objects of $\overline{\mathcal{I}}\mathcal{X}_{\mu_r}(T)$ are given by a morphisms $f : \mathcal{G} \to \mathcal{X}$, where $\mathcal{G}$ is a μ_r-gerbe over T. In other words, we remove the data of an identification $\mathcal{G} \cong (B\mu_r)_T$, which is equivalent to the data of a section of the gerbe. From an orbifold perspective, the rigidification $\overline{\mathcal{I}}\mathcal{X}_{\mu_r}$ removes the common μ_r isotropy that all the points of $\mathcal{IX}_{\mu_r}$ have.

Example II.9. Consider a Deligne-Mumford quotient stack $\mathcal{X} = [W/G]$, where W is an affine scheme and G is an abelian group. Then the inertia stack has the following decomposition

$$\mathcal{IX} = \bigsqcup_{g \in G} [W^g/G]$$

where W^g denotes the fixed locus under the action by g. The rigidified inertia stack has a similar decomposition as $\overline{\mathcal{I}}\mathcal{X} = \sqcup_{g \in G}[W^g/(G/Z(g))]$, where $Z(g)$ is the centralizer of the group G.

For non-abelian G, we have a similar decomposition where the index runs over the conjugacy classes in G. We write this as

$$\mathcal{IX} = \bigsqcup_{(g) \subset G} [W^g/Z(g)]$$

where (g) is a conjugacy class with representative g and $Z(g)$ is the centralizer.

We can now talk about the Chen-Ruan cohomology. As a vector space, we have that

$$H^*_{\mathrm{CR}}(\mathcal{X}, \mathbb{C}) = H^*(\mathcal{IX}, \mathbb{C})$$

However, the grading and the ring structure are not the usual ones.

To describe the grading, we first talk about the *age* of a locally free sheaf. Given a locally free sheaf $\mathcal{F}$ of rank n, one can look at its fiber over a point $x \in \mathcal{X}$. The isotropy group G_x lifts to an action on the fiber. Since the isotropy group is finite, we can write the action of an element $g \in G_x$ in the form of a diagonal matrix

$$\operatorname{diag}(e^{\frac{2\pi i k_1}{r}}, \ldots, e^{\frac{2\pi i k_n}{r}})$$

for some $0 \leq k_i < r-1$, where r is the order of G_x. Then we define the age $\iota_{(x,g)}(\mathcal{F})$ as the quantity

$$\iota_{(x,g)}(\mathcal{F}) = \sum_{i=1}^{n} k_i$$

Now, for ease, suppose we have a decomposition of the inertia stack as in Example II.9. We can consider the tangent bundle $T\mathcal{X}$ as living on a connected component $\Omega \subset \mathcal{I}\mathcal{X}$ by restricting to the closed substack in $\mathcal{X}$ isomorphic to that component. Then we can consider the age of $T\mathcal{X}$ with respect to the pair (x, g), where x is a point in the closed substack corresponding to Ω and g is the indexing element of Ω. It turns out that the age is independent of conjugacy class, and is locally constant on Ω, so that this is a well-defined number associated to Ω itself. We then define

$$\iota(\Omega) = \iota_{(x,g)}(T\mathcal{X}|_\Omega).$$

This definition is more closely related to the differential picture [11], although there is an analogous algebraic definition involving gerbes [2, Section 7].

The grading on the Chen-Ruan cohomology is then given as follows

$$H^n_{\mathrm{CR}}(\mathcal{X}, \mathbb{C}) = \oplus_{\Omega \subset \mathcal{I}\mathcal{X}} H^{n-\iota(\Omega)}(\Omega, \mathbb{C})$$

where we note that we are using the complex, rather than real, grading for our cohomologies.

As for the ring structure, we will not go into detail as it is somewhat complicated, and not completely necessary for understanding the main ideas in this paper. We will simply note that it is derived as the classical limit of the quantum cohomology ring, and refer to [11, 3] for details.

II.4: Gromov-Witten Theory of DM Stacks

The Gromov-Witten theory of a Deligne-Mumford stack models closely after the Gromov-Witten theory of a scheme, with some slight technical changes.

We first discuss the notion of an *balanced twisted curve*, which will be the source of our morphisms. A *twisted curve* is a connected, one-dimensional Deligne-Mumford stack that is étale locally a nodal curve, and such that the only stack structure appears at markings and nodes. In particular, the coarse moduli space is the usual notion of a marked nodal curve, and the markings and nodes in the twisted curve are gerbes over their image in the coarse curve. A twisted curve being *balanced* means that étally locally around the nodes, the picture is isomorphic to the quotient stack

$$[k[x, y]/\mu_r], \quad \zeta \cdot (x, y) = (\zeta \cdot x, \zeta^{-1} \cdot y) \text{ for } \zeta \in \mu_r$$

Now let $\pi : \mathcal{X} \to X$ be a Deligne-Mumford stack with coarse moduli space X. We can construct a Deligne-Mumford stack

$$\mathcal{K}_{g,n}(\mathcal{X}, \beta)$$

that parametrizes stable representable morphisms of degree $\beta \in A_1(X)$ from a genus g twisted curve $\mathcal{C}$ with n markings. By stable, we mean that the induced morphism from the coarse moduli curve C to X is stable in the usual sense. We point out the requirement of the morphisms being representable, as this is relevant only to the stack case, and also make note that $A_1(X, \mathbb{Q}) \cong A_1(\mathcal{X}, \mathbb{Q})$ so one can think of β as a class in either.

As in the scheme case, this stack contains *evaluation morphisms* ev_i. However, the target of these morphisms is not $\mathcal{X}$, but rather $\overline{\mathcal{I}}\mathcal{X}$. Indeed, given a twisted stable map $f : \mathcal{C} \to \mathcal{X}$ over a base scheme T, and letting $\Sigma_i \subset \mathcal{C}$ denote the gerbe at the i-th marking, we define the i-th evaluation map $\mathrm{ev}_i : \mathcal{K}_{g,n}(\mathcal{X}, \beta) \to \overline{\mathcal{I}}\mathcal{X}$ as the following on objects over T

$$\mathrm{ev}_i(f) = \left\{ \begin{array}{ccc} \Sigma_i & \xrightarrow{f|_{\Sigma_i}} & \mathcal{X} \\ \big\downarrow & & \\ T & & \end{array} \right\} \in \overline{\mathcal{I}}\mathcal{X}(T)$$

Sometimes, it also makes sense to make use of a variation of the evaluation map where we compose with the involution $\iota : \overline{\mathcal{I}}\mathcal{X} \to \overline{\mathcal{I}}\mathcal{X}$. The involution on the inertia stack is defined by changing the banding of the gerbe via the inverse map $G \to G$ that takes $g \to g^{-1}$. For quotient stacks, as in Example II.9, it takes the component corresponding to g to the component corresponding to g^{-1}. We define $\check{\mathrm{ev}}_i := \iota \circ \mathrm{ev}_i$.

The last piece before we define Gromov-Witten invariants is the existence of a virtual cycle on the stack $\mathcal{K}_{g,n}(\mathcal{X}, \beta)$. As it turns out, the usual candidate $R\pi_*(f^*T\mathcal{X})$ does the job [2, Section 4.5], where *pi* and f are the universal maps associated to the universal curve of $\mathcal{K}_{g,n}(\mathcal{X}, \beta)$ in the obvious way. Thus, we have a virtual cycle $[\mathcal{K}_{g,n}(\mathcal{X}, \beta)]^{\mathrm{vir}}$.

Before we discuss the virtual dimension, we remark that the stack $\mathcal{K}_{g,n}(\mathcal{X},\beta)$ has a decomposition into connected components determined by which connected components of the rigidified inertia stack the marked points of the source curves land in. If we label the connected components of the inertia stack by some indexing set $\{h_i\}$, then we can write the decomposition

$$\mathcal{K}_{g,n}(\mathcal{X},\beta) = \sqcup_{\vec{h}=(h_1,\ldots,h_n)} \mathcal{K}_{g,\vec{h}}(\mathcal{X},\beta)$$

where h_i corresponds to the connected component that is the image of ev_i. Now by the usual calculation, using the orbifold version of Riemann-Roch, one can show that

$$\mathrm{vdim}[\mathcal{K}_{g,\vec{h}}(\mathcal{X},\beta)]^{\mathrm{vir}} = \int_\beta c_1(T\mathcal{X}) + (1-g)(\dim \mathcal{X} - 3) + n - \sum_i \mathrm{age}(f^*T\mathcal{X})|_{p_i}$$

where p_i denote the n marked points, and $\int_\beta c_1(T\mathcal{X})$ is defined as in [2, Section 7.2].

We can now formally define a Gromov-Witten invariant.

Definition II.10. Let $\mathcal{X}$ be a Deligne-Mumford stack. Fix g, n, β as before, and let $\{\gamma_i\}_{i=1}^n \in H^*_{\mathrm{CR}}(\mathcal{X})$ be a set of n Chen-Ruan cohomology classes. A *Gromov-Witten invariant* associated to this data is the number

$$\langle \gamma_1, \ldots, \gamma_n \rangle^{\mathcal{X}}_{g,n,\beta} := \int_{[\mathcal{K}_{g,n}(\mathcal{X},\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^*\gamma_1 \cup \cdots \cup \mathrm{ev}_n^*\gamma_n \quad \in \mathbb{Q}$$

Similarly, one can do all the usual variations seen in the scheme theory. One important one is that of descendant invariants, where we incorporate the ψ_i classes. As a quick reminder, we can construct the tautological line bundles $\mathcal{L}_i$ on $\mathcal{K}_{g,n}(\mathcal{X},\beta))$ as the bundles whose fiber at a point is the cotangent space to the associated coarse curve at the i-th marking. Note that we use the coarse curve here rather than the twisted curve so that $\mathcal{L}_i$ is truly a line bundle. Then we have that

$$\psi_i = c_1(\mathcal{L}_i)$$

and we can define the descendant invariants as

$$\langle \gamma_1\psi_1^{a_1}, \ldots, \gamma_n\psi_n^{a_n} \rangle^{\mathcal{X}}_{g,n,\beta} := \int_{[\mathcal{K}_{g,n}(\mathcal{X},\beta)]^{\mathrm{vir}}} \left(\prod_i \psi_i^{a_i}\right) \mathrm{ev}_1^*\gamma_1 \cup \cdots \cup \mathrm{ev}_n^*\gamma_n$$

Other variations are also well-defined, e.g. twisting invariants by a characteristic class, and are defined as one would expect (although the definition of a Gromov-Witten Chow class requires a bit more nuance [2, Section 6]).

Remark II.11. Note that our evaluation maps have the rigidifed inertia stack as the target, since we do not keep track of sections of our gerbes. This is different from the early days of orbifold Gromov-Witten theory, where the data of a section was part of the moduli data and so the evaluation maps landed in the inertia stack itself. As a result, one must be careful when traversing the orbifold Gromov-Witten theory literature, as the moduli stack and virtual cycles they use have slight differences e.g. [55, 1]. These differences are outlined in [2, Section 6.1] and [54].

II.5: The Lagrangian Cone

In the proof of the invertibility of the mirror theorem VI.6, we will make use of Givental's Lagrangian cone [35]. We collect some of the main ideas here, following the algebro-geometric framework outlined in [17, Appendix B] and referring to the same paper for the technical details.

Let $\mathrm{NE}(X)$ denote the Mori cone of curves, and define the Novikov ring Λ to be the completion of $\mathbb{C}[\mathrm{NE}(X)]$ with respect to the maximal ideal generated by q^β for $\beta in\, \mathrm{NE}(X)$. In other words, the elements of Λ are given by

$$\Lambda = \left\{ \sum_{\beta \in \mathrm{NE}(X)} a_\beta q^\beta | \, a_\beta \in \mathbb{C} \right\}.$$

Note that this is a topological ring whose topology is linear, complete, and Hausdorff. We will also assume this assumption for all of our rings R in this section. Under this assumption, limits are well defined, and from a ring R we can form

$$R\{z, z^{-1}\} := \{\sum_{n \in \mathbb{Z}} r_n z^n | \; r_n \in R, \, r_n \to 0 \text{ as } |n| \to \infty\}$$

which is the space of convergent Laurent polynomials, and

$$R^{\mathrm{nilp}} := \{r \in R | \; \lim_{n \to \infty} r^n = 0\}$$

which is the ideal of topologically nilpotent elements. Then we define Givental's symplectic vector space as an infinite-dimensional space over Λ

$$\mathcal{H} = H^*_{\mathrm{CR}}(X, \mathbb{C}) \otimes_{\mathbb{C}} \Lambda\{z, z^{-1}\}.$$

Let $\{\phi_i\}_{i=1}^N$ denote a basis of $H^*_{\mathrm{CR}}(X)$ and let $\{\phi^i\}_{i=1}^N$ denote its dual basis. For reasons we will see later, we will write a general element $f \in \mathcal{H}$ in the form

$$f = -z + \mathbf{t}(z) + \mathbf{p}(z), \quad \mathbf{t}(z) = \sum_{k \geq 0} t_k^\alpha \phi_\alpha z^k, \quad \mathbf{p}(z) = \sum_{l \geq 0} p_{l\beta} \frac{\phi^\beta}{(-z)^{l+1}}$$

where we use Einstein's summation convention and suppress the sums over the Greek indices.

We now want to take a formal germ at $-z$ in Givental's space. Algebraically this is given as the affine formal scheme of some complete ring S over Λ , which can be found in [17, Appendix B]. Instead, we will give the functor of points presentation of this scheme. From this viewpoint, we have that the formal germ $(\mathcal{H}, -z)$ is a functor from the category of topological Λ-algebras to Sets, where we have

$$(\mathcal{H}, -z)(R) \cong \left\{ -z + \sum_{n \in \mathbb{Z}} r_n^\alpha \phi_\alpha z^n |\ r_n^\alpha \in R^{\mathrm{nilp}},\ r_n^\alpha \to 0 \text{ as } |n| \to \infty \right\}$$

As before, we will write elements of this space in the form $f = -z + \mathbf{t}(z) + \mathbf{p}(z)$.

Now for a power series $\mathbf{t}(z)$ as above, we set the following conventional notation for genus zero invariants,

$$\langle\!\langle \gamma_1 \psi^{k_1}, \ldots, \gamma_m \psi^{k_m} \rangle\!\rangle_{\mathbf{t}} := \sum_{\beta \in \mathrm{NE}(X)} \sum_{n \geq 0} \frac{q^\beta}{n!} \langle \gamma_1 \psi^{k_1}, \ldots, \gamma_m \psi^{k_m}, \mathbf{t}(\psi), \ldots, \mathbf{t}(\psi) \rangle_{0,m+n,\beta}^X$$

for $\gamma_i \in H^*_{\mathrm{CR}}(X)$. Then we define a formal subscheme $\mathcal{L}_X \subset (\mathcal{H}, -z)$ whose R-valued points are given as

$$\mathcal{L}_X(R) = \left\{ -z + \mathbf{t}(z) + \mathbf{p}(z) \in (\mathcal{H}, -z)(R) |\ p_{k\alpha} = \langle\!\langle \phi_\alpha \psi^k \rangle\!\rangle_{\mathbf{t}},\ \forall\, k, \alpha \right\} \tag{II.5.1}$$

This is known as Givental's Lagrangian cone. It also as an explicit description as an affine formal scheme defined by the closure of the ideal of relations $p_{k\alpha} - \langle\!\langle \phi_\alpha \psi^k \rangle\!\rangle_{\mathbf{t}}$ in S.

We will also make use of the tangent spaces to the Lagrangian cone. We define the tangent functor $T\mathcal{L}_X$ to be

$$T\mathcal{L}_X(R) = \mathcal{L}_X(R[\epsilon]/(\epsilon^2))$$

and define the tangent space $T_f \mathcal{L}_X(R)$ at a point $f \in \mathcal{L}_X(R)$ to be the pre-image of f under the natural map $\mathcal{L}_X(R[\epsilon]/(\epsilon^2)) \to \mathcal{L}_X(R)$.

As seen above, points on the Lagrangian cone look like generating functions of Gromov-Witten invariants, and are related to what we refer to as a J-function.

Definition II.12. Let $\tau = \sum_{i=1}^N \tau^i \phi_i$ be a formal element of $H^*_{\mathrm{CR}}(X)$. Then the *J-function* is defined as

$$J(\tau, z) = z + \tau + \sum_{k \geq 0} \langle\!\langle \phi_\alpha \psi^k \rangle\!\rangle_{\mathbf{t}=\tau} \frac{\phi^\alpha}{(-z)^{k+1}}$$
$$= z + \tau + \sum_{n \geq 0} \sum_{\beta \in \mathrm{NE}(X)} \sum_{i=1}^{N} \frac{q^\beta}{n!} \langle \frac{\phi_\alpha}{z - \psi}, \tau, \ldots, \tau \rangle^X_{0,n+1,\beta} \phi^\alpha$$

It's not hard to see from the above definition that $J(\tau, -z)$ defines a $\Lambda[\tau^1, \ldots, \tau^N]$ point of the Lagrangian cone $\mathcal{L}_X$. Note that if one were to replace τ with the more general $\mathbf{t}(z)$ in the definition of the J-function, one would get a description of the points of the Lagrangian cone. From this viewpoint, one can intuitively view the Lagrangian cone as a family of J-function like generating series; in other words, saying a function is a point on the Lagrangian cone is to say that the function is some form of generalized J-function.

In Gromov-Witten theory, it is natural to consider generating functions of invariants rather than individual invariants, hence studying functions like the J-function. The subspace $\mathcal{L}_X$ is a geometric formulation of these considerations. A stunning and useful feature is that relations in Gromov-Witten theory can now be interpreted as properties of $\mathcal{L}_X$. For instance, a priori $\mathcal{L}_X$ is merely a subspace of $(\mathcal{H}, -z)$. The fact that it is a Lagrangian subspace, and moreover a cone as in [17, Proposition B.2], can be interpreted as a consequence of genus zero orbifold Gromov-Witten theory satisfying the Dilaton Equation, the String Equation, and the Topological Recursion Relations (see [55, Section 3.1]). More concretely, these three properties imply the following:

Theorem II.13 ([35])**.** *The subscheme $\mathcal{L}_X \subset (\mathcal{H}, -z)$ is a Lagrangian cone with vertex at the origin. Moreover, the tangent spaces $L_f := T_f\mathcal{L}_X$ are tangent to $\mathcal{L}_X$ exactly along zL_f.*

In particular, the above theorem tells us that $zL_f \subset L_f$ and $zL_f \subset \mathcal{L}_X$ for all $f \in \mathcal{L}_X$, and that L_f is the tangent space to all the smooth points in zL_f. This theorem will prove useful to us when we apply the technique of Birkhoff factorization in Section VI.3.

Another way to view the above theorem is that the Lagrangian cone is ruled by these subspaces zL_f. Hence, understanding the elements of the tangent spaces of $\mathcal{L}_X$ is useful to understanding $\mathcal{L}_X$ itself. Remarkably, the tangent space at $f = -z + \mathbf{t}(z) + \mathbf{p}(z)$ has a complete description in terms of the derivatives of the J-function for $\tau = \tau(\mathbf{t})$ [17, Proposition B.4], and even has a $\mathcal{D}$-module structure [17, Corollary B.7]. We refer to the references for the details on these properties, but we do write out one lemma about tangent vectors which will prove useful to us later.

Lemma II.14. *[17, Lemma B.1] If $I(t) \in \mathcal{L}_X(R[\![t]\!])$, then the derivative $\frac{dI}{dt}(t)$ lies in $T_{I(t)}\mathcal{L}_X(R[\![t]\!])$.*

CHAPTER III
Quasimap Theory

We compute our I-functions via quasimaps to GIT targets, first constructed in [47] (under the name of "stable quotients") and [15] for schemes and later extended to Deligne–Mumford (DM) stacks in [12]. We review some of the definitions for the convenience of the reader, and refer the reader to [12] for more details.

III.1: Quasimap Definitions

Suppose we have the following data:

- W an affine scheme with at worst lci singularities,
- G an algebraic reductive group acting on W,
- $\theta\colon G \to \mathbb{C}^*$ a character of G,

such that the θ-semistable locus $W^{\mathrm{ss}}_\theta(G)$ is equal to the θ-stable locus $W^{\mathrm{s}}_\theta(G)$, and is smooth. When understood, we will drop the θ and G from the notation.

Then we construct the stacky GIT quotient

$$X := [W^{\mathrm{ss}}/G],$$

a DM stack whose coarse moduli $\underline{X}$ is the standard GIT quotient $W /\!\!/_\theta G$. We also have the Artin stack quotient

$$\mathfrak{X} := [W/G].$$

Note that there is an open embedding $X \hookrightarrow \mathfrak{X}$ induced by the open embedding $W^{\mathrm{ss}} \hookrightarrow W$. The character θ gives the trivial bundle on W a G-equivariant structure, and thus defines a line bundle L_θ on $\mathfrak{X}$. We use the same notation for its restriction to X and refer to it as the *polarization*.

Let $(\mathcal{C}, x_1, \ldots, x_n)$ denote an n-pointed twisted curve with balanced nodes ([2, Section 4]).

Definition III.1. An n-pointed quasimap to X is a representable morphism

$$f\colon (\mathcal{C}, x_1, \ldots, x_n) \to \mathfrak{X}$$

where $f^{-1}(\mathfrak{X} \setminus X)$ is a purely zero-dimensional substack.

The substack $f^{-1}(\mathfrak{X} \setminus X)$ is called the *base locus* of the quasimap. For this thesis, we will additionally require that the base locus is disjoint from the nodes and markings (known as a pre-stable quasimap in [12]), so that the base locus is represented by a subscheme of the coarse curve C.

The *curve class* (or *degree*) β of the quasimap is defined to be the homomorphism

$$\beta \in \operatorname{Hom}(\operatorname{Pic}(\mathfrak{X}), \mathbb{Q}), \quad \beta(L) = \deg(f^*L), \ \forall L \in \operatorname{Pic}(\mathfrak{X}), \tag{III.1.1}$$

where the degree of a line bundle on a twisted curve is defined as in [2].

Notice that the notion of quasimaps to X, as well as their degree, actually depends on the ambient $\mathfrak{X}$. This dependence is key to our extended GIT construction in Chapter V.

For any $\epsilon \in \mathbb{Q}_{>0}$, a quasimap is said to be ϵ-stable if

(i) the $\mathbb{Q}$-line bundle $(f^*L_\theta)^{\otimes\epsilon} \otimes \omega_{C,\log}$ is positive, i.e. it has positive degree on each irreducible component of C, where $\omega_{C,\log} = \omega_C(\sum_{i=1}^n x_i)$ is the log-dualizing sheaf of C

(ii) $\epsilon \cdot \ell(x) \leq 1$ for all base points x, where $\ell(x)$ is the length of the base locus scheme at the point x.

The two stability conditions when $\epsilon \to +\infty$ and $\epsilon \to 0^+$ are well-defined and will be denoted by $\epsilon = \infty$ and $\epsilon = 0^+$, respectively. These two special cases will be the most important in this paper:

- When $\epsilon = \infty$, ϵ-stable quasimaps are the same as twisted stable maps, in the sense of [2].
- For $\epsilon = 0^+$, the length condition is trivially satisfied, and the positivity condition disallows, in particular, any rational tails. This will be the stability condition most relevant to the I-functions.

The moduli space $\mathcal{Q}^\epsilon_{g,n}(X,\beta)$ of genus-g, n-pointed ϵ-stable quasimaps to X with degree β is a DM stack, proper over the affine quotient $W/\!\!/_0 G$, and equipped with a perfect obstruction theory

$$\phi_{\mathcal{Q}^\epsilon_{g,n}(X,\beta)/\mathfrak{M}^{\mathrm{tw}}_{g,n}} \colon (R^\bullet\pi_* f^*\mathbb{T}_{\mathfrak{X}})^\vee \to \mathbb{L}_{\mathcal{Q}^\epsilon_{g,n}(X,\beta)/\mathfrak{M}^{\mathrm{tw}}_{g,n}}$$

relative to the Artin stack $\mathfrak{M}^{\mathrm{tw}}_{g,n}$ of prestable twisted curves, where $\pi\colon \mathcal{C} \to \mathcal{Q}^{\epsilon}_{g,n}(X,\beta)$ is the universal curve and $f\colon \mathcal{C} \to \mathfrak{X}$ is the universal quasimap. This induces a virtual fundamental class $[\mathcal{Q}^{\epsilon}_{g,n}(X,\beta)]^{\mathrm{vir}}$. (See [10, Section A.2.2] for more details on the perfect obstruction theory.)

III.2: Stacky Loop Space

The I-function is usually defined via $\mathbb{C}^*$-localization on a variant of the quasimap moduli space known as the graph-quasimap moduli space [12, Section 2.5]. However, since we are only interested in a special case of the graph moduli space, we can instead work with an alternative moduli space, known as the "stacky loop space" in the sense of [12, Section 4.2]. This moduli space has a more explicit description, which will make our later computations clearer.

For any positive integer r, let

$$\mathcal{Q}_{\mathbb{P}(1,r)}(X,\beta) \subset \mathrm{Hom}^{\mathrm{rep}}_{\beta}(\mathbb{P}(1,r),\mathfrak{X})$$

be the substack of representable morphisms $\mathbb{P}(1,r) \to \mathfrak{X}$ which are quasimaps of curve class β, i.e. the generic point of $\mathbb{P}(1,r)$ falls into X. Here, the domain curve is a fixed weighted projective line $\mathbb{P}(1,r)$, with one marking at the possibly stacky point $\infty := [0:1]$. The *stacky loop space* is then constructed as

$$\mathcal{Q}_{\mathbb{P}(1,\star)}(X,\beta) := \coprod_{r=1}^{\infty} \mathcal{Q}_{\mathbb{P}(1,r)}(X,\beta).$$

Remark III.2. We note that this is technically an open locus inside the stacky loop space of [12, Section 4.2], since our quasimaps are assumed to be pre-stable. However, this does not affect the definition of the I-function.

Because X is a DM-stack and because of representability, only finitely many terms of the coproduct are non-empty. This moduli space has a universal curve $\pi\colon \mathcal{C}_\beta \to \mathcal{Q}_{\mathbb{P}(1,\star)}(X,\beta)$, which is trivial over each component of $\mathcal{Q}_{\mathbb{P}(1,\star)}(X,\beta)$, together with a universal morphism $f\colon \mathcal{C}_\beta \to \mathfrak{X}$. There is an absolute perfect obstruction theory

$$\phi_{\mathcal{Q}_{\mathbb{P}(1,\star)}(X,\beta)}\colon (R^\bullet \pi_* f^* \mathbb{T}_{\mathfrak{X}})^\vee \to \mathbb{L}_{\mathcal{Q}_{\mathbb{P}(1,\star)}(X,\beta)}$$

with an associated virtual class $[\mathcal{Q}_{\mathbb{P}(1,\star)}(X,\beta)]^{\mathrm{vir}}$.

There is a $\mathbb{C}^*$-action on $\mathbb{P}(1,r)$ given by

$$\lambda \cdot [x : y] = [x : \lambda y] \tag{III.2.1}$$

and by precomposition, this induces a $\mathbb{C}^*$-action on our moduli space. Let F_β be the component of the fixed locus where the only basepoint is located at $0 := [1 : 0]$, and the entire class β is supported over 0, i.e. it is a basepoint is of length $\deg(\beta)$. The perfect obstruction theory is equivariant under this $\mathbb{C}^*$-action, so that F_β has a virtual fundamental class $[F_\beta]^{\mathrm{vir}}$, and a virtual normal bundle $N_{F_\beta/\mathcal{Q}_{\mathbb{P}(1,\star)}(X,\beta)}$ as in [36].

By [12, Lemma 4.8], the fixed locus F_β of the stacky loop space is isomorphic to the corresponding fixed locus in the graph-quasimap space in a way that preserves the perfect obstruction theories. This justifies our use of the stacky loop space to define the I-function.

Let $ev_\star \colon \mathcal{Q}_{\mathbb{P}(1\star)}(X,\beta) \to \overline{\mathcal{I}}X$ be the evaluation map to the rigidified inertia stack, as defined in [2]. We define

$$\widetilde{ev}_\star := \iota \circ \mathbf{r}_\star(ev_\star).$$

where ι is the inversion of band automorphism, and $\mathbf{r}_\star$ is the order of the band of the gerbe structure at $\star$.

Definition III.3. The I-function is defined as

$$I(q,z) = \sum_{\beta \in \mathrm{Eff}(W,G,\theta)} q^\beta (\widetilde{ev}_\star)_* \left(\frac{[F_\beta]^{\mathrm{vir}}}{e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_\beta/\mathcal{Q}_{\mathbb{P}(1,\star)}(X,\beta)})} \right). \tag{III.2.2}$$

The I-function is a homogenous function of degree 0 when we apply the following degree conventions

$$\deg(z^k) = k, \quad \deg(q^\beta) = \beta(\omega_{\mathfrak{X}}^\vee), \quad \deg(\phi) = i \text{ for } \phi \in H^{2i}_{\mathrm{CR}}(X). \tag{III.2.3}$$

Moreover, we remark that the I-function as defined above agrees with the I-function of Givental [32], which is a hypergeometric series obtained as the solution of the Picard-Fuchs equation on the B-model of the target space, up to a possible multiplication by z.

CHAPTER IV
Toric Stacks

The definition will we use of a toric Deligne-Mumford stack was first introduced in [7], where they are constructed via a combinatorial object known as a *stacky fan.*These stacky fans contain the data of an underlying rational simplicial fan, from which one can show that the coarse moduli space of a toric Deligne-Mumford stack is the toric variety associated to this underlying fan.

By further imposing that the coarse moduli space is semi-projective, meaning it is projective over its affinization $\mathrm{Spec}(H^0(X, \mathcal{O}_X))$, and that the coarse moduli space has a non-empty torus fixed set, we can express these stacks as the stacky GIT quotient of a vector space V by a torus T. We will work in this setting, giving a brief outline of the basics of toric stacks from the GIT perspective rather than the stacky fan approach. The relation between the two, as well as more details, can be found in [22, Section 4].

IV.1: GIT Structure

Let V be an n-dimensional vector space with the action of a complex torus T, and θ a character of T. As mentioned before, our toric stack will ultimately be of the form $[V /\!\!/_\theta T]$, but we will try to make all the pieces of this quotient more explicit.

We will choose a basis $x_1, \ldots, x_n$ for $V = \mathbb{C}^n$ that diagonalizes the T action, so that for all $t \in T$, the torus action can be described by

$$t \cdot (x_1, \ldots, x_n) = (\chi_1(t)x_1, \ldots, \chi_n(t)x_n)$$

for some characters χ_i of T. Picking an isomorphism $T \cong (\mathbb{C}^*)^r$, we will write $t = (t_1, \ldots, t_r)$. The above action can then be described by an $r \times n$ matrix $\mathcal{W}$ known as the *weight matrix*, defined by

$$\mathcal{W} = (w_{ij}), \quad \text{such that } \chi_i(t_j) = t_j^{w_{ij}} \text{for all } i, j.$$

Following our set conventions, we will use $w_{\bullet j}$ to denote the column vectors of $\mathcal{W}$. Note, by definition, that $w_{\bullet j}$ is equal to the image of χ_j under the obvious isomorphism of the character space $\mathrm{Hom}((\mathbb{C}^*)^r, \mathbb{C}^*) \cong \mathbb{Z}^r$ that sends a character $\chi(t_1, \dots, t_r) = \prod_i t_i^{c_i}$ to the r-tuple $(c_1, \dots, c_r)$.

Now given the above, we define the set

$$\mathcal{A}_\theta = \{I \subset \{1, \dots, n\} \,|\, \theta = \sum_{i \in I} a_i w_{\bullet i}, \text{ for some } a_i \in \mathbb{R}_{>0}\} \tag{IV.1.1}$$

where we consider $\theta \in \mathbb{Z}^r$ via the earlier isomorphism. The set $\mathcal{A}_\theta$ is known as the set of *anticones* in [22], hence we will similarly refer to elements of this set as anticones. We impose the following assumptions for our set $\mathcal{A}_\theta$ for the rest of this paper:

Assumption IV.1. Given θ, we have

- $\{1, \dots, n\} \in \mathcal{A}_\theta$
- For all $I \in \mathcal{A}_\theta$, we have that $\{w_{\bullet i}\}_{i \in I}$ spans $\mathrm{Hom}((\mathbb{C}^*)^r, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^r$.

A consequence of these assumptions is that anticones are closed under enlargement.

Lemma IV.2. *Assume θ satisfies Assumption IV.1. Then if $I \in A_\theta$ and $I \subset J$, we have $J \subset A_\theta$.*

Proof. This is immediate from the spanning property of the assumption □

For each anticone $I \in \mathcal{A}_\theta$, we also define the open set

$$U_I = \{(x_1, \dots, x_n) \,|\, x_i \neq 0 \text{ for } i \in I\} \subset \mathbb{C}^n\}.$$

Then the semi-stable locus with respect to θ, denoted by $V_\theta^{\mathrm{ss}}(T)$, can be described as follows:

Lemma IV.3. *The θ semi-stable locus is given by*

$$V_\theta^{\mathrm{ss}}(T) = \bigcup_{I \in \mathcal{A}_\theta} U_I$$

Proof. This can be checked via the numerical criterion [42, Prop. 2.5], which states that a point x is semi-stable if and only if for every one parameter subgroup λ of T for which $\lim_{t \to 0} \lambda(t) \cdot x$ exists, we have that the character $\theta \circ \lambda : \mathbb{C}^* \to \mathbb{C}^*$ is defined by taking a non-negative power.

Let $\lambda : \mathbb{C}^* \to (\mathbb{C}^*)^r$ be a one parameter subgroup given by weights $s = (s_1, \dots, s_r)$, so that

$$\lambda(t) \cdot x = (t^{s \cdot w_{\bullet 1}} x_1, \dots, t^{s \cdot w_{\bullet n}} x_n).$$

Now suppose we have a point $x \in U_I$ for some $I \in \mathcal{A}_\theta$. Since $x_i \neq 0$ for $i \in I$, we have that $\lim_{t\to 0} \lambda(t) \cdot x$ exists only when $s \cdot w_{\bullet i} \geq 0$ for $i \in I$. But then by the definition of an anticone, the character $\theta \circ \lambda : \mathbb{C}^* \to \mathbb{C}^*$ is defined by taking the positive power $\sum_{i \in I} (a_i w_{\bullet i} \cdot s) \geq 0$, hence the point is semi-stable.

Now suppose we have a point $x \notin \bigcup_{I \in \mathcal{A}_\theta} U_I$. To show that x is not semi-stable, it suffices to show that there is no θ^k-invariant monomial that is non-zero at x. Let $F = \prod_{i=1}^n x_i^{c_i}$ be such a monomial, and suppose $F(x) \neq 0$. This implies that $k\theta = \sum c_i w_{\bullet i}$ where the sum ranges over the i for which the coordinate x_i of x is not zero. However, since c_i and k are positive, this implies that the set of i for which x_i is non-zero forms an anticone, hence a contradiction.

□

Note that $V_\theta^{\mathrm{ss}}(T) \neq \emptyset$ by the first assumption in Assumption IV.1. Moreover, these assumptions allow us to satisfy a necessary requirement for quasimap theory, as seen in the next lemma.

Lemma IV.4. *Suppose that θ satisfies Assumption IV.1. Then we have that $V^{\mathrm{ss}} = V^s$.*

Proof. We again apply the stability criterion of [42, Prop. 2.5], which states that a point is stable if and only if in addition to the criterion for being semi-stable, we have that $\theta \circ \lambda$ is the identity map only if λ acts trivially.

Suppose we have a semi-stable point $x \in U_I$ for some $I \in \mathcal{A}_\theta$, and suppose we have a one parameter subgroup λ with weights $s = (s_1, \ldots, s_r)$ such that $s \cdot w_{\bullet i} = 0$ for all $i \in I$. Then $s \cdot w_{\bullet j} = 0$ for any j by the spanning assumption, so λ acts trivially hence x is stable. □

With this, we now define what we mean by a toric stack for the rest of this paper.

Definition IV.5. A *toric stack* is a Deligne-Mumford stack Y such that

$$Y = [V /\!\!/_\theta T]$$

for some complex torus T, a T-representation V, and character θ, where θ satisfies Assumption IV.1. We call the triple (V, T, θ) a *GIT representation* for the toric stack Y.

Remark IV.6. Note that there is not a unique GIT representation for a toric stack. For example, we can represent $\mathbb{P}^1$ with the usual representation $[\mathbb{C}^2 /\!\!/_\theta \mathbb{C}^*]$, where $\mathbb{C}^*$ has weight matrix $\begin{pmatrix} 1 & 1 \end{pmatrix}$ and θ is a positive character. However, we can also write it as the GIT representation $[\mathbb{C}^3 /\!\!/_\vartheta (\mathbb{C}^*)^2]$, where $(\mathbb{C})^*$ acts by weights $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and the character ϑ takes $(\lambda, \mu) \to \lambda\mu$. This flexibility in GIT representation will be key in our main results.

Remark IV.7. While not needed for the main results in the paper, we briefly describe the wall and chamber structure associated to varying θ, as seen in [22]. For each $I \subset \{1, \dots, n\}$, we let $\angle_I = \{\sum_i a_i w_{\bullet i} | \, a_i \in \mathbb{R}_{>0}, \, i \in I\}$. Then the $\mathbb{R}$-character space $\mathrm{Hom}((\mathbb{C}^*)^r, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^r$ is divided into chambers given by

$$C_\theta = \bigcap_{I \in \mathcal{A}_\theta} \angle_I$$

and the walls are the codimension 1 boundaries of the C_θ. By Assumption IV.1, we have that the C_θ for varying θ intersect along these codimension 1 boundaries, and $C_\theta = C_{\theta'}$ if and only if $\theta' \in C_\theta$. The main gist of this is that our GIT quotient only varies when U_θ varies, and this is precisely determined by the sets I in $\mathcal{A}_\theta$. This wall and chamber structure reflects when the sets $\mathcal{A}_\theta$ change based on θ.

IV.2: Chen-Ruan Cohomology

Since a toric stack $Y := [V /\!\!/_\theta T]$ is a global quotient by an abelian group, the inertia stack, and the rigidified inertia stack, can be easily described as in II.9

$$\mathcal{I}Y = \bigsqcup_{t \in T} [V^t /\!\!/_\theta T], \qquad \overline{\mathcal{I}}Y = \bigsqcup_{t \in T} [V^t /\!\!/_\theta (T/\langle t \rangle)]$$

where V^t is the fixed locus of V under the action by t, and $T/\langle t \rangle$ is the group obtained by quotienting out the subgroup generated by t.

To see which $t \in T$ have a non-zero fixed locus, we can use the combinatorial data provided in the previous section. For $\beta \in \mathrm{Hom}(\chi(T), \mathbb{Q}) \cong \mathbb{Q}^r$, define the set

$$S_\beta = \{i \in \{1, \dots, n\} | \, \beta(w_{\bullet i}) \in \mathbb{Z}\} \tag{IV.2.1}$$

and the group element

$$g_\beta = (e^{2\pi i d_1}, \dots, e^{2\pi i d_r}) \in (\mathbb{C}^*)^r \tag{IV.2.2}$$

where $\beta = (d_1, \dots, d_r)$ under the natural identification $\mathrm{Hom}(\chi(T), \mathbb{Q}) \cong \mathbb{Q}^r$ given by identifying $\chi(T) \cong \mathbb{Z}^r$ with basis the projection characters. Since X is Deligne-Mumford, every possible $t \in T$ with non-empty fixed set has finite order, hence can be written as g_β for some $\beta \in \mathbb{Q}^r$.

Lemma IV.8. *For $\beta \in \mathrm{Hom}(\chi(T), \mathbb{Q})$, $V^{g_\beta} \neq \emptyset$ if and only if $S_\beta \in \mathcal{A}_\theta$.*

Proof. Suppose $V^{g_\beta} \neq \emptyset$. Then there is a fixed point $x = (x_1, \dots, x_n) \in U_I$ for some $I \subset \mathcal{A}_\theta$.

Since x is g_β fixed and $x_i \neq 0$ for $i \in I$, we have $e^{2\pi i \beta(w_{\bullet i})} = 1$ for all such i, where we note that $\beta(w_{\bullet i})$ is the same as $\beta \cdot w_{\bullet i}$, depending on if we think of $\beta \in \operatorname{Hom}(\chi(T), \mathbb{Q})$ versus $\beta \in \mathbb{Q}^r$. Thus, we have $I \subset S_\beta$, hence $S_\beta \in \mathcal{A}_\theta$ by Lemma IV.2

On the other hand, suppose $S_\beta \in \mathcal{A}_\theta$. Then the points $x \in U_{S_\beta}$ such that $x_j = 0$ for $j \notin S_\beta$ are fixed by g_β.

□

Note that both S_β and g_β only depend on β up to translation by $\mathbb{Z}^r \subset \mathbb{Q}^r$, and the denominators of the d_i are bounded [2, Proposition 2.1.1], hence we only need to consider finitely many such β.

Let $\alpha = (\alpha_1, \ldots, \alpha_r) \in ([0,1) \cap \mathbb{Q})^r$. Then we set the notation

$$Y_\alpha := [V^{g_\alpha} /\!\!/_\theta T], \quad g_\alpha = (e^{2\pi i \alpha_1}, \ldots, e^{2\pi i \alpha_r}) \tag{IV.2.3}$$

for description of all the sectors of our inertia stack. If any values $n \geq 1$ are used, then take their fractional part $\langle n \rangle$.

The age of the inertia sectors X_α can be deduced from the *generalized Euler triangle* associated to stack quotients of the form $[V/G]$ [15, Section 5.1]

$$V \times_G \mathfrak{g} \to V \times_G V \to \mathbb{T}_{[V/G]} \to \tag{IV.2.4}$$

where G acts by the adjoint action on $\mathfrak{g}$, and $\mathbb{T}$ is the tangent complex. We can restrict this to the GIT locus to obtain

$$V^{\mathrm{ss}} \times_G \mathfrak{g} \to V^{\mathrm{ss}} \times_G V \to \mathbb{T}_{[V /\!\!/_\theta G]} \to \tag{IV.2.5}$$

which, for a toric stack, further simplies to the following exact triangle

$$\mathcal{O}^{\oplus r} \to \bigoplus_i \mathcal{O}(w_{\bullet i}) \to \mathbb{T}_{[V /\!\!/_\theta T]} \to \tag{IV.2.6}$$

Note that the first two terms in IV.2.6 are quasi-coherent sheaves. The age of a twisted sector Y_α is given by computing the age of the tangent bundle $T_{[V /\!\!/_\theta T]}$ with respect to the g_α action. Since the age is additive among exact triangles, we can simply this to studying the age of sheaves $\mathcal{O}(\xi)$ for some r-tuple ξ. From direct computation, we have

$$\operatorname{age}_{g_\alpha}(\mathcal{O}(\xi)) = \langle \xi \cdot \alpha \rangle$$

where we recall that $\langle n\rangle = n - \lfloor n \rfloor$. From this, one obtains that

$$\text{age}\, Y_\alpha = \sum_{i=1}^{n} \langle w_{\bullet i} \cdot \alpha \rangle \tag{IV.2.7}$$

We now set the following notation for the most relevant Chen-Ruan cohomology classes:

- $\mathbf{1} \in H^0_{\text{CR}}(Y)$ is the unit class of the untwisted sector and multiplicative unit of the ring.
- $\mathbf{1}_\alpha \in H^{2(\text{age}\, Y_\alpha)}_{\text{CR}}(Y)$ is the unit class of the twisted sector Y_α
- $H_i = c_1(\mathcal{O}(\text{pr}_i)) \in H^2_{\text{CR}}(Y)$, where pr_i is the i-th projection character of T.

Understanding the relations requires further analysis of the cohomology rings of the underlying toric varieties associated to the Y_α, e.g. as in [28], but will not be needed for our purposes.

IV.3: Complete Intersections

Let (V, T, θ) be the GIT presentation of a toric stack as before. In addition to that data, we consider a s-dimensional T-representation E that splits into one-dimensional representations $E = \oplus_{j=1}^s E_j$, where the torus action T on E_j is given by $(t_1, \dots, t_r) \to \prod_{i=1}^r t_i^{b_{ij}}$. Then we get a split T-linearized bundle

$$E \times V = (\oplus_{j=1}^s E_j) \times V \to V$$

given by the data of the representation. Let s_E be a T-equivariant section of this bundle, and let $W = V(s_E)$. Consider the stack s

$$X := [W /\!\!/_\theta T] \subset [V /\!\!/_\theta T] =: Y.$$

Note that $E \times V$ descends to a split vector bundle

$$\oplus_{i=1}^s \mathcal{O}(b_{\bullet j}) \to [V/T]$$

where $b_{\bullet j} = (b_{1j}, \dots, b_{rj})$, and the section s_E induces a section of this bundle. We will consolidate notation and simply refer to $\oplus_{i=1}^s \mathcal{O}(b_{\bullet j})$ and the induced section as E and s_E as well. Then

$$X = Z(s_E) \subset Y$$

from this viewpoint. We will want to choose s_E so that X satisfies our requirements of being a complete intersection.

Definition IV.9. A smooth Deligne-Mumford stack X is a *complete intersection in a toric stack* $[V /\!\!/_\theta T]$ if it can be realized as a quotient stack

$$X = [W /\!\!/_\theta T] \subset [V /\!\!/_\theta T]$$

where X is the vanishing of a regular section s_E of some split vector bundle $E = \oplus_{j=1}^{s}$ on $[V /\!\!/_\theta T]$.

We will refer to the data (V, T, θ, E, s_E) as a GIT representation for a complete intersection, where the components are defined as above. Equivalently, we may also use (W, T, θ), where the data of E and s_E is implicit in $W = Z(s_E)$.

Remark IV.10. The key thing to note about our definition is that we are looking at regular sections of split vector bundles which can be identified by the data of a T-linearization of a trivial bundle on V. In general, the T-linearized bundles on the semi-stable locus $\mathrm{Pic}^T((V_\theta^{\mathrm{ss}}(T)))$ may include more than T-linearizations on trivial bundles; for instance, it's possible there is a non-trivial bundle after removing the unstable locus, and for which a T-linearization exists. We will not work with those.

Equivalently, one can say that the vector bundles we work with are restrictions from vector bundles over the ambient stack quotient $[V/T]$, since, by descent, those are exactly T-linearizations of the trivial bundle over V.

The inertia stack for these complete intersections has a similar decomposition as the inertia stack of the ambient toric variety. Since they are global quotients, we still have

$$\mathcal{I}X = \bigsqcup_{t \in T} [W^t /\!\!/_\theta T], \qquad \overline{\mathcal{I}}X = \bigsqcup_{t \in T} [W^t /\!\!/_\theta (T/\langle t \rangle)].$$

We will denote X_α to refer to sector as before, where we have that

$$X_\alpha := Y_\alpha \cap X$$

where the latter intersection is viewed as substacks in Y. All of the cohomology classes from before on Y can be pulled back to X, and generate the subring of ambient Chen-Ruan cohomology classes $H^*_{\mathrm{CR,amb}}(X) \subset H^*_{\mathrm{CR}}(X)$.

The big change we mention is that the age of the twisted sectors X_α may differ from those of Y_α, since we want to understand the age with respect to the tangent bundle of X rather than Y. These two are related by what we refer to as the *conormal* or *adjunction* triangle, which can be found in [49]

$$\mathbb{T}_{[W/G]} \to \mathbb{T}_{[V/G]} \to E \to \tag{IV.3.1}$$

where E is the bundle on $[V/G]$ whose section cuts out $[W/G]$, or in other words the normal bundle. Letting $G = T$, and restricting this sequence to the GIT stacks, we easily see, via a similar computation as before, that

$$\operatorname{age} X_\alpha = \sum_{i=1}^{n} \langle w_{\bullet i} \cdot \alpha \rangle - \sum_{j=1}^{s} \langle b_{\bullet j} \cdot \alpha \rangle \tag{IV.3.2}$$

Remark IV.11. It should be warned that when we say pullback of Chen-Ruan classes, we mean the pullback on the singular cohomology of the sectors. Because of the age discrepancy, the pulled back Chen-Ruan class may not have the same degree as the class being pulled back.

CHAPTER V
Extended I-function

Let Y be a toric stack, E a split vector bundle on Y, and X a smooth complete intersection given by the vanishing of a generic section of E. Choosing a GIT presentation for Y, we express our set-up in the following diagram:

$$\begin{tikzcd} & E = \oplus_{j=1}^{s}\mathcal{O}(b_{1j},\dots,b_{rj}) \arrow[d] \\ {[W/\!\!/_\theta(\mathbb{C}^*)^r] = X} \arrow[r, hook] \arrow[d, hook] & Y = [\mathbb{C}^n/\!\!/_\theta(\mathbb{C}^*)^r] \arrow[d, hook] \\ {[W/(\mathbb{C}^*)^r] = \mathfrak{X}} \arrow[r, hook] & \mathfrak{Y} = [\mathbb{C}^n/(\mathbb{C}^*)^r] \end{tikzcd}$$

where W is the affine cone of X, and $\mathfrak{X}$, $\mathfrak{Y}$ are the ambient stack quotients associated to the presentation of X and Y respectively.

In this section, we will modify the GIT presentation of our toric stack in a specific way and then compute the corresponding quasimap I-function. As mentioned in the introduction, this change will allow us to encode the data of more complicated quasimaps than normally seen by the I-function, and will ultimately produce an extended I-function that is capable of capturing a much greater variety of Gromov-Witten invariants.

For convenience, we impose a few more assumptions onto the above setting

Assumption V.1.

- The generic isotropy group of X and Y are trivial
- The GIT presentation of Y is *minimal*, i.e. the dimension of the torus in the presentation is the smallest possible dimension over all GIT presentations of Y.

Neither of these assumptions are strictly necessary. Stacks that don't satisfy the first assumption can be viewed as gerbes over their rigidification, and can be dealt with via the work of [5] or by directly modifying the following arguments. Removing the second assumption increases book-keeping and slightly changes the proof of invertibility VI.8. However, imposing these conditions allows us to avoid additional divisions in the cohomology ring or degree parameters in the I-function that may obfuscate the main ideas.

Remark V.2. We allow for the case where E is the zero bundle, in which case you recover the Gromov-Witten theory of the toric stack itself.

V.1: GIT Data Extension

We start by explaining the matrix extension that allows for our more expressive I-function. For each cohomology class type that we want to appear as insertions in our Gromov-Witten invariants, we will add an additional $\mathbb{C}^*$ action to our GIT presentation. While the fundamental idea behind the extensions are the same, we will separate the extensions into two cases, dependent on the cohomology class they seek to parameterize, as this provides a cleaner narrative.

V.1.1: Type I: Fundamental Classes

Let $\mathbf{1}_\alpha$ denote the fundamental class of the sector X_α. In order to capture these types of insertions, we want to rewrite quasimaps from twisted curves that have points evaluating into X_α as quasimaps from curves with no orbifold structure. The geometric motivation for why we extend our GIT structure is given in the following example, generalizing the earlier example I.1 we have seen.

Example V.3. For a given α, let G_α be the subgroup generated by g_α. Let $f : \mathcal{C} \to Y$ be a quasimap with one orbifold marking whose isotropy group is $B\mu_{|G_\alpha|}$, and whose evaluation lands in X_α. Let $\rho : \mathcal{C} \to \underline{C}$ be the coarse moduli morphism, and let p be the underlying point to the orbifold marking. Then f is determined by the data of r line bundles on $\mathcal{C}$ with the form

$$\mathcal{L}_i = \rho^* L_i \otimes \mathcal{O}(\alpha_i p)$$

where L_i is the round-down line bundle on $\underline{C}$, and sections

$$s_k \in H^0(\mathcal{C}, \mathcal{L}^{w_{\bullet k}}), \quad 1 \leq k \leq n$$

where we define

$$\mathcal{L}^{w_{\bullet k}} := \otimes_{i=1}^{r} \mathcal{L}_i^{w_{ik}}, \tag{V.1.1}$$

with w_{ij} the weight matrix entries.

We ideally would like to define a quasimap from $\underline{C}$ that recovers this data, and the ideal choice for our bundles would be our round-downs L_i. However, the choice of sections are not so clear. Indeed, we have

$$\rho_*(\mathcal{L}^{w_{\bullet k}}) = \rho_*(\otimes_{i=1}^r(\rho^* L_i^{w_{ik}} \otimes \mathcal{O}(w_{ik} \cdot \alpha_i p))) \cong (\otimes_{i=1}^r L_i^{w_{ik}}) \otimes \mathcal{O}_{\underline{C}}\left(\left\lfloor \sum_i w_{ik} \cdot \alpha_i \right\rfloor p\right) \neq L_i^{w_{\bullet k}}$$

hence our sections for the corresponding line bundles involving the round-downs are not one-to-one with those of the originals as seen via Lemma II.7.

In order to rectify the above issue, we extend our GIT quotient by an additional $\mathbb{C}^*$ factor, where we choose the weights appropriately so that the extra line bundle coming from this factor can accommodate the loss of data associated with the missing root bundles. For orbifold points landing in X_α as above, we extend the GIT quotient as

$$Y = [\mathbb{C}^n \times \mathbb{C} /\!\!/_{\theta_e} (\mathbb{C}^*)^r \times \mathbb{C}^*]$$

where $\theta_e = \theta \times id$, and our $(r+1) \times (n+1)$ weight matrix is given by

$$\left(\begin{array}{ccc|c} & \mathcal{W} & & 0_{r\times 1} \\ \hline m_1 & \cdots & m_n & 1 \end{array}\right)$$

where

$$m_j = \lfloor w_{\bullet j} \cdot \alpha \rfloor = \left\lfloor \sum_{i=1}^r w_{ij}\alpha_i \right\rfloor .$$

Now we can construct a quasimap $g : \underline{C} \to Y$ using this new GIT presentation. If we choose our line bundles L_i as above, with the additional line bundle being $\mathcal{O}(p)$, then we see that we can choose our sections as those that determine f, with the last section $s_{n+1} \in \mathcal{O}(p)$ being the tautological section with vanishing at p.

Conversely, given a quasimap from $\underline{C}$ to this extended GIT presentation, we can construct a quasimap from an orbifold curve $\mathcal{C}$ to the original GIT presentation, where the amount of $B\mu_{|G_\alpha|}$ stacky points corresponds to the degree of the "extra" bundle. Imposing that all these stacky points evaluate into the X_α sector gives us unique inverse to the above process.

It is worth mentioning here that the orbifold markings of the original quasimap f have now become base-points when rewritten as the quasimap g (see Lemma V.8). Geometrically, this is what allows us to capture the data of these orbifold markings in the very restrictive curves of the I-function, as we can hide the orbifold data in the allowed base-point of such curves. An intuitive way of thinking about this geometric fix may be that there isn't a good way to "collide" orbifold points into one another, but base-points can be collided and combined into one.

Remark V.4. While this type of extension is normally used for a twisted sector X_α, one could use the untwisted sector and keep track of the unit class $\mathbf{1}$. While one can always rewrite any correlator with $\mathbf{1}$ insertions as one without any via the string equation, keeping track of $\mathbf{1}$ can still prove useful, such as when we prove invertibility of the mirror map in Theorem VI.6.

V.1.2: Type 2: Hyperplane Intersections

The second type of extension is a modification on the first, and captures cohomology classes related to hyperplane intersections. Let $x_1, \ldots, x_n$ denote the standard coordinates on $\mathbb{C}^n$. Since Y_α is given by a vector subspace, we can let $I_\alpha \subset \{1, \ldots, n\}$ denote the coordinates which span Y_α, and subsequently call the coordinates on the sector x_i as well.

Now let $J \subset I_\alpha$, and consider the intersection $\left(\bigcap_{j\in J} V(x_j) \cap X_\alpha\right) \subset Y_\alpha$. We can think of this as a cycle in X_α, and will denote such a cycle as

$$t_\alpha^J := \left[\bigcap_J V(x_j) \cap X_\alpha\right] \in A^*(X_\alpha).$$

We will treat these cycles as Chen-Ruan cohomology classes via the cycle map and Poincaré duality, and will use t_α^J to also refer to the corresponding cohomology class. The goal of this type of extension is account for Gromov-Witten invariants with insertions of this form.

Geometrically, we want to take a quasimap with marked points landing in these cycles, and turn them into quasimaps with base-points at those marks instead. This inherently doesn't require the marked points to be stacky, and can be understood quite nicely in the following example:

Example V.5. Let $f : C \to \mathbb{P}^2$ be a morphism such that $f(p) = [0 : 0 : 1]$ for some $p \in C$. This morphism is given by a choice of bundle L on C, as well as three sections s_i. However, we see that the first two sections vanish at p, hence both those sections are in the image of the map on sections induced by the inclusion $L(-p) \hookrightarrow L$.

Now we introduce a GIT extension $\mathbb{P}^2 = [\mathbb{C}^4 /\!\!/_\theta (\mathbb{C}^*)^2]$, where the weight matrix is given by

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

A quasimap $g : C \to [\mathbb{C}^4 /\!\!/_\theta (\mathbb{C}^*)^2]$ is composed of two line bundles. We can choose the bundles $L(-p)$ and $\mathcal{O}(p)$, and then choose the sections $s_0, s_1 \in L(-p)$, $s_3 \in L$, and $s_4 \in \mathcal{O}(p)$ the tautological section associated to $\mathcal{O} \hookrightarrow \mathcal{O}(p)$. This new quasimap g now has a basepoint at p, but otherwise agrees with f. However, the information about the point p is retained in the extra bundle, hence we can reconstruct f from g.

The above manipulation is easily generalized to orbifold quasimaps to toric stacks; essentially you want to shift the line bundle of the j-th section down by the divisor p whenever you want to land in the closed locus $V(x_j)$. By combining this extension idea with the extensions of Type I, we are able to keep track of a these hyperplane intersections in twisted sectors as well, such as t_J^α.

More explicitly, to keep track of t_α^J we extend by an additional $\mathbb{C}^*$ action whose weights m_i are given by

$$m_j = \begin{cases} \lfloor w_{\bullet j} \cdot \alpha \rfloor & j \notin J \\ \lfloor w_{\bullet j} \cdot \alpha \rfloor - 1 & j \in J \end{cases} \quad \text{for } 1 \leq j \leq n$$

Note that this differs from the weights of the extension for $\mathbf{1}_\alpha$ by subtracting 1 from the entries whose indexes lie in J.

One thing to point out is that the cycles t_α^J may end up being primitive classes. This can occur when the expected dimension of the cycle doesn't match the actual dimension, an example of which can be seen in Example VII.4. When the codimension of t_α^J in X_α is $|J|$, we have that the classes are ambient and can be expressed as cohomology products $t_\alpha^J = \prod_{j \in J} c_1(\mathcal{O}(w_{\bullet j})) \cdot \mathbf{1}_\alpha$

For the rest of the thesis, we assume the following assumptions for our sets J.

Assumption V.6. For a cohomology class t_α^J, we require

- $J \subset I_\alpha$.
- $\bigcap_J V(x_j) \cap X_\alpha \neq \emptyset$.
- J is minimal in the sense that there is no $J' \subset J$ such that $t_\alpha^{J'} = t_\alpha^J$ (this includes $J' = \emptyset$).

The first condition ensures that the intersection makes sense, the second ensures that t_α^J is an actual cycle, and the third is to avoid redundancies in the defining set J.

Remark V.7. Similar to Remark V.4, you can use this type of extension to keep track of divisor classes in the untwisted sector. One can reduce Gromov-Witten invariants with such insertions to those without using the divisor equation [2, Theorem 8.3.1], but as in the case of $\mathbf{1}$, it can sometimes be useful for theoretical reasons to track such classes.

V.1.3: General Extension Data

We now proceed with extending our GIT presentation by all the cohomology classes we can track of, and then ensuring that this presentation satisfies all the properties and assumptions we expect for quasimap theory.

To combine the notation from the previous two sections, we will let J possibly be the empty set, so that $t_\alpha^\emptyset = \mathbf{1}_\alpha$. Now choose some collection of cohomology classes of the above form to extend by, say $t_{\alpha_1}^{J_1}, \ldots, t_{\alpha_m}^{J_m}$ for sets J_i and r-tuples $\alpha_i \in ([0,1)\cap\mathbb{Q})^r$. We then present our toric stack Y with the GIT presentation

$$Y = [\mathbb{C}^n \times \mathbb{C}^m /\!\!/_{\theta_e} (\mathbb{C}^*)^{r+m}] \tag{V.1.2}$$

where our polarization is given by

$$\theta_e = \theta \times (C_\theta \cdot id)^m : (\mathbb{C}^*)^r \times (\mathbb{C}^*)^m \to \mathbb{C}^*$$

for where we define the constant $C_\theta := \theta \cdot (1, \ldots, 1)$, and our $(r+m)\times(n+m)$ weight matrix is of the form

$$\Psi = \left(\begin{array}{c|c} \mathcal{W} & 0_{r\times m} \\ \hline A & \mathrm{Id}_{m\times m} \end{array}\right). \tag{V.1.3}$$

Here, $A = (a_{ij})$ is the $m \times n$ matrix whose entries are given by

$$a_{ij} = \lfloor w_{\bullet j} \cdot \alpha_i \rfloor - \delta_{J_i}(j) = \left\lfloor \sum_{k=1}^{r} w_{kj} (\alpha_i)_k \right\rfloor - \delta_{J_i}(j), \tag{V.1.4}$$

where δ_{J_i} is the indicator function on the set J_i, i.e. $\delta_{J_i}(k) = \begin{cases} 1 & \text{for } k \in J_i \\ 0 & \text{for } k \notin J_i \end{cases}$. As usual, we will denote the columns of our weight matrix Ψ with the notation $\psi_{\bullet j}$.

To show that (V.1.2) holds and that this is a valid GIT presentation, we verify the following

Lemma V.8. *The θ_e-semistable locus $(\mathbb{C}^{n+m})^{\mathrm{ss}}_{\theta_e}$ is given by*

$$(\mathbb{C}^{n+m})^{\mathrm{ss}}_{\theta_e} = (\mathbb{C}^n)^{\mathrm{ss}}_\theta \times (\mathbb{C} - \{0\})^m.$$

Furthermore, we have that it agrees with the θ_e-stable locus, i.e. $(\mathbb{C}^{n+m})^{\mathrm{ss}}_{\theta_e} = (\mathbb{C}^{n+m})^{s}_{\theta_e}$.

Proof. By Lemma IV.3, it is enough to show that

$$\mathcal{A}_{\theta_e} = \{I \cup \{n+1, \dots, n+m\} \,|\, I \in \mathcal{A}_\theta\}. \tag{V.1.5}$$

Let $R \in \mathcal{A}_{\theta_e}$, and let $R_\theta := R \cap \{1, \dots, n\}$. It is immediate from the description of θ_e and the weights (V.1.3) that $R \in \mathcal{A}_{\theta_e}$ if and only if $R_\theta \in \mathcal{A}_\theta$. Thus, it suffices to show that $R = R_\theta \cup \{n+1, \dots, n+m\}$.

Suppose $\theta = \sum_{r \in R_\theta} c_r w_{\bullet r}$ for some $c_r \in \mathbb{R}_{>0}$. Then showing $n+i \in R$ is equivalent to showing that $\sum_{r \in R_\theta} c_r a_{ir} < C_\theta$, where a_{ir} is as in (V.1.4). However, we have that

$$\begin{aligned}
\sum_{r \in R_\theta} c_r a_{ir} &= \sum_{r \in R_\theta} c_r \left(\left\lfloor \sum_{k=1}^{n} w_{kr} (\alpha_i)_k \right\rfloor - \delta_{J_i}(r) \right) \\
&\leq \sum_{r \in R_\theta} c_r \sum_{k=1}^{n} w_{kr} (\alpha_i)_k \\
&< \sum_{r \in R_\theta} c_r w_{\bullet r} \cdot (1, \dots, 1) = C_\theta
\end{aligned}$$

where we use that $(\alpha_i)_k < 1$ for all i, k. Since this holds for all i, we have $\{n+1, \dots, n+m\} \subset R$ and the claim is proven.

For the claim about the stable locus, we note that $\mathcal{A}_{\theta_e}$ satisfies Assumption IV.1, hence the result follows from Lemma IV.4

□

Remark V.9. Parallel to remark IV.7, we have that the chamber and wall structure of our extended GIT is given in the exact same way as before, and our choice of θ_e is one of many possible extensions. However, as evidenced in the proof above, crossing any of the codimension one boundaries associated to $\{n+i\}$ results in an Artin stack, i.e. to be Deligne-Mumford, all $I \in \mathcal{A}_{\theta'_e}$ must contain the subset $\{n+1, \dots, n+m\}$.

Now we turn our attention to the complete intersection $X \subset Y$. We want to extend the affine cone W to a $(\mathbb{C}^*)^{r+m}$ invariant closed subscheme $W_e \subset \mathbb{C}^{n+m}$ such that $W_e \cap (\mathbb{C}^n \times \{1, \dots, 1\}) = W$. To find W_e, let $x_1, \dots, x_n, y_1, \dots, y_m$ be the standard coordinates of $\mathbb{C}^{n+m}$. Suppose that

$$W = V(F_1, \dots, F_s)$$

where $F_k \in \mathbb{C}[x_1, \dots, x_n]$ is a quasi-homogenous polynomial of multi-degree $b_{\bullet j}$, where we recall that $b_{\bullet j} = (b_{1j}, \dots, b_{rj})$. We can now regard F_k as an element of $\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_m]$ via the obvious inclusion and consider its multi-degree under the weights of (V.1.3).

To define W_e, we need to homogenize the F_k with respect to this new grading. We address this in the following lemmas.

Lemma V.10. *The twisted sector X_α is a complete intersection in Y_α, and is given by the vanishing of the polynomials F_k such that $\alpha \cdot b_{\bullet k} \in \mathbb{Z}$.*

Proof. We will consider $Y_\alpha \subset Y$ via the inclusion of the fixed locus.

First, we note that for F_k such that $\alpha \cdot b_{\bullet k} \notin \mathbb{Z}$, we have that $F_k|_{Y_\alpha} = 0$. This is because Y_α is the linear span of those x_i such that $w_{\bullet i} \cdot \alpha \in \mathbb{Z}$.

Now consider a closed point $p \in X_\alpha$. Because X is a complete intersection, we have that $\{dF_k\}_{k=1}^s$ are linearly indepenedent in T_pY. Since p is fixed under the action of g_α, this action lifts to the tangent space and we get a decomposition $T_pY = T_pY_\alpha \oplus T'$, where T' is the non-fixed part under the action. We have that F_k with $b_{\bullet k} \cdot \alpha \in \mathbb{Z}$ are invariant under the action, hence $dF_k = 0$ on T'. Thus, $\{dF_k\}_{b_{\bullet k} \cdot \alpha \in \mathbb{Z}}$ are linear linearly independent in T_pY_α, hence X_α is a complete intersection. $\square$

Define the set $\mathcal{F}_\alpha \subset \{1, \dots, s\}$ to be the set of indices such that $V(\{F_i\}_{i \in \mathcal{F}_\alpha}) = X_\alpha \subset Y_\alpha$, or equivalently

$$\mathcal{F}_\alpha = \{k | \ b_{\bullet k} \cdot \alpha \in \mathbb{Z}\}$$

For each class $t_{\alpha_i}^{J_i}$, we define a corresponding set $\Phi_{\alpha_i}^{J_i} \subseteq \mathcal{F}_{\alpha_i}$

$$\Phi_{\alpha_i}^{J_i} = \{k \in \mathcal{F}_{\alpha_i} | \ F_k \in (\{x_j\}_{j \in J_i})\}.$$

Note the following size restriction on these new sets:

Lemma V.11. *We have that $|\Phi_{\alpha_i}^{J_i}| \leq |J_i|$.*

Proof. We have that $i_* t_{\alpha_i}^{J_i} = V(\{F_k\}_{k \in \mathcal{F}_\alpha}, \{x_j\}_{j \in J_i})$ by definition, and for dimension reasons we require this ideal to be generated by at least $|\mathcal{F}_\alpha|$ elements.. However, by definition of $\Phi_{\alpha_i}^{J_i}$, we see that this ideal has $|\mathcal{F}_\alpha| - |\Phi_{\alpha_i}^{J_i}| + |J_i|$ generators, so the result follows. $\square$

With these definitions made, we can now explain how to homogenize the F_k in the extension.

Lemma V.12. *For $j = 1, \dots, s$, there exists a unique $\tilde{F}_k \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_m]$ such that*

(1) $\tilde{F}_k$ is quasi-homogenous of multi-degree $(b_{1k}, \dots, b_{rk}, \lfloor b_{\bullet k} \cdot \alpha_1 \rfloor - \delta_{\Phi_{\alpha_1}^{J_1}}(k), \dots, \lfloor b_{\bullet k} \cdot \alpha_m \rfloor - \delta_{\Phi_{\alpha_m}^{J_m}}(k))$.

(2) $\tilde{F}_k(x_1, \dots, x_n, 1 \dots, 1) = F_k$.

where $\delta_{\Phi^{J_i}_{\alpha_i}}$ is the indicator function on the set $\Phi^{J_i}_{\alpha_i}$.

Proof. It suffices to prove this for the case where F_k is a monomial. Subsequently, we will set $F_k = x_1^{c_1} \cdots x_n^{c_n}$ to be a monomial of degree $b_\bullet = (b_1, \dots, b_r)$. Then it's clear that the unique $\tilde{F}$ must be given by $\tilde{F} = x_1^{c_1} \cdots x_n^{c_n} y_1^{d_1} \cdots y_m^{d_m}$ where

$$d_\ell = \left(\lfloor b_\bullet \cdot \alpha_\ell \rfloor - \delta_{\Phi^{J_\ell}_\alpha}(k) \right) - \sum_{j=1}^{n} c_j \left(\lfloor w_{\bullet j} \cdot \alpha_\ell \rfloor - \delta_{J_\ell}(j) \right)$$

Thus it suffices to show that $d_i \geq 0$. Noting that $b_i = \sum_j c_j w_{ij}$, we have that

$$\begin{aligned} \lfloor b_\bullet \cdot \alpha_i \rfloor &= \left\lfloor \sum_{i=1}^{r} \sum_{j=1}^{n} c_j w_{ij} (\alpha_\ell)_i \right\rfloor \\ &= \left\lfloor \sum_{j=1}^{n} c_j (w_{\bullet j} \cdot \alpha_\ell) \right\rfloor \\ &\geq \sum_{j=1}^{n} c_j \lfloor w_{\bullet j} \cdot \alpha_\ell \rfloor \end{aligned}$$

Combined with the fact that $\delta_{\Phi^{J_\ell}_\alpha}(k)$ is nonzero if and only if $c_j \delta_{J_\ell}(j)$ is nonzero for some j, the result follows.

□

Lettting $\tilde{F}_1, \dots, \tilde{F}_s$ be as in the above lemma, we have set

$$W_e = V(\tilde{F}_1, \dots, \tilde{F}_s) \subset \mathbb{C}^{n+m}$$

Then our extended GIT presentation for X is given by

$$X = [W_e /\!\!/_{\theta_e} (\mathbb{C}^*)^{r+m}]$$

Lemmas (V.8) and (V.12) ensure that this equality holds, and that the stability assumptions for quasimap theory are satisfied.

Similar to the extension for the affine cone, we also extend the vector bundle E to a bundle $\mathcal{E}$ on the extended stack quotient $[W_e/(\mathbb{C}^*)^{r+m}]$, defined as

$$\mathcal{E} = \bigoplus_{j=1}^{s} \mathcal{O} \left(b_{1j}, \dots, b_{rj}, \lfloor b_{\bullet j} \cdot \alpha_1 \rfloor - \delta_{\Phi^{J_1}_\alpha}(j), \dots, \lfloor b_{\bullet j} \cdot \alpha_m \rfloor - \delta_{\Phi^{J_m}_\alpha}(j) \right)$$

The extended degrees are chosen to match with that of the $\tilde{F}_j$, hence we have that $[W_e/(\mathbb{C}^*)^{r+m}]$

is the vanishing locus of a section of $\mathcal{E}$ in the ambient stack quotient $[\mathbb{C}^{n+m}/(\mathbb{C}^*)^{r+m}]$. For future convenience, we set the notation $\xi_{\bullet k}$ for the multi-degree of $\tilde{F}_k$, and set the notation

$$\mathcal{E} = \bigoplus_{j=1}^{s} \mathcal{O}(\xi_{\bullet j})$$

for our extended bundle.

Despite our original sections defining a regular sequence, it is possible that this property disappears after extension. The resulting affine scheme may then not be a complete intersection and could have worse that lci singularities, which is problematic for quasimap theory and the perfect obstruction theory calculation. An example of this is as follows:

Example V.13. Consider the complete intersection $X := X_{4,4,4} \subset \mathbb{P}(1,1,1,1,1,3)$ defined by the vanishing of three generic quasi-homogenous polynomials of degree 4. Such a polynomial looks like

$$F_4(x_0, \dots, x_4) + F_1(x_0, \dots, x_4)x_5$$

where F_i is of degree i, and hence we see that the $B\mu_3$ point is always contained in X. If we extend by the class $\mathbf{1}_{1/3}$, which corresponds to the fundamental class of the twisted sector $X_{1/3}$, we obtain a weight matrix of the form

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Letting y be the extra variable obtained post-extension, we have after the homogenization in Lemma V.12 that the extended polynomials are of the form

$$F_4(x_0, \dots, x_4)y + F_1(x_0, \dots, x_4)x_5.$$

However, we then see that the codimension 2 locus $V(x_5, y)$ is contained in the vanishing of all the polynomials, whereas the expected dimension is codimension 3. Since polynomial rings over $\mathbb{C}$ are Cohen-Macaulay, the polynomials cannot define a regular sequence, hence W_e will not be a complete intersection.

As a result, we will assume the following for the rest of the thesis

Assumption V.14. We will assume that the extended polynomials $\tilde{F}_k$ from Lemma V.12 define a regular sequence.

Remark V.15. We note that the above condition is automatically satisfied when $s \leq 2$. It suffices to consider the case where we extend by one class, i.e. $m = 1$, as the rest follows by induction. For $s = 1$, we only have a hypersurface, hence the result holds trivially.

For $s = 2$, we note that by the second property of V.12, the locus where W_e possibly fails to be a complete intersection is in the unstable locus with respect to the GIT data. We also note that our choice of homogenization in Lemma V.12 is minimal in the sense that the $\tilde{F}_k$ are not divisible by extra variable obtained via extension. The latter implies that the intersection of the vanishing locus of the $\tilde{F}_k$ with the unstable locus is of codimension at least 2. However, since we are working with a Cohen-Macaulay affine scheme, we only need to check that the vanishing of the $\tilde{F}_k$ has the expected dimension in order to ensure that they form a regular sequence, hence it will always be regular for $s = 2$.

Moving forward, we will work with the extended presentation above, which can be summarized in the following diagram.

$$\begin{tikzcd} & \mathcal{E} = \oplus_j \mathcal{O}(\xi_{\bullet j}) \arrow[d] \\ \left[W_e /\!\!/_{\theta_e} (\mathbb{C}^*)^{r+m}\right] = X \arrow[r, hook] \arrow[d, hook] & Y = \left[\mathbb{C}^{n+m} /\!\!/_{\theta_e} (\mathbb{C}^*)^{r+m}\right] \arrow[d, hook] \\ \left[W_e/(\mathbb{C}^*)^{r+m}\right] = \mathfrak{X}_e \arrow[r, hook] & \mathfrak{Y}_e = \left[\mathbb{C}^{n+m}/(\mathbb{C}^*)^{r+m}\right] \end{tikzcd} \tag{V.1.6}$$

We end this section with a description of the anticanonical bundle of the complete intersection in this extended presentation, which will be relevant later when we discuss the asymptotics of the mirror map in Lemma VI.9.

Lemma V.16. *Let $\omega^\vee_{\mathfrak{X}_e}$ be the anticanonical bundle of $\mathfrak{X}_e$. Then we have that*

$$\omega^\vee_{\mathfrak{X}_e} = \mathcal{O}(\zeta_1, \ldots, \zeta_r, \eta_1, \ldots . \eta_{r+m})$$

where

$$\zeta_k = \sum_{i=1}^n w_{ki} - \sum_{j=1}^s b_{kj}$$

$$\eta_k = \sum_{i=1}^n \lfloor w_{\bullet i} \cdot \alpha_k \rfloor - \sum_{j=1}^s \lfloor b_{\bullet j} \cdot \alpha_k \rfloor - (|J_k| - |\Phi^{J_k}_{\alpha_k}| - 1)$$

Furthermore, if the following condition is satisfied

$$V\left(\{F_j\}_{j \notin \Phi^{J_k}_{\alpha_k}}, \{x_i\}_{i \in J_k}\}\right) \subset Y_{\alpha_k} \text{ is a complete intersection} \tag{$*$}$$

then we have that

$$\eta_k = \zeta \cdot \alpha_k + 1 - \frac{\deg_{CR}(t_{\alpha_k}^{J_k})}{2}$$

where $\zeta = (\zeta_1, \ldots, \zeta_r)$ *and* $\deg_{\mathrm{CR}}(\cdot)$ *is the Chen-Ruan degree of the cohomology class.*

Proof. This follows from taking the determinant of the exact triangles (IV.2.4) and (IV.3.1).

For the second statement, condition $(*)$ ensures that the cycle $t_{\alpha_k}^{J_k}$ is of codimension $|J_k| - |\phi_{\alpha_k}^{J_k}|$ inside of X_{α_k} by virtue of being a complete intersection. We therefore have that

$$\deg_{\mathrm{CR}}(t_{\alpha_k}^{J_k}) = 2(\mathrm{age}(X_{\alpha_k}) + |J_k| - |\phi_{\alpha_k}^{J_k}|)$$

where $\mathrm{age}(X_{\alpha_k}) = \sum_{i=1}^{n} \langle w_{\bullet i} \cdot \alpha_k \rangle - \sum_{j=1}^{s} \langle b_{\bullet j} \cdot \alpha_k \rangle$. The result then immediately follows. $\square$

V.2: Geometry of Quasimap Space

Here we will collect the relevant geometric information about the quasimap moduli space and the fixed locus necessary for the I-function computation.

When computing the I-function, we only need to focus on quasimaps in the fixed loci F_β, i.e. those whose source curves $\mathcal{C}$ are isomorphic to $\mathbb{P}(1, \star)$ for $\star \in \mathbb{Z}_{>0}$, and are invariant under the scaling $\mathbb{C}^*$ action (III.2.1). Under this isomorphism, we denote the two points ∞ and 0 as

$$0 := [1 : 0], \quad \infty := [0 : 1]$$

We also note that such curves are root stacks, and we have that $\mathrm{Pic}(\mathbb{P}(1, \star)) \cong \mathbb{Z}$, where the generator $\mathcal{O}(1)$ is the $\star$-th root bundle associated to the stacky point [8, Section 3.1].

While the target of these quasimaps are $\mathfrak{X}_e$, we will consider them as quasimaps to $\mathfrak{Y}_e$ via post-composition with the inclusion. This is because, by descent, $\mathrm{Pic}(\mathfrak{X}_e)$ corresponds to torus-linearized bundles of $\mathrm{Pic}(W_e)$, which are hard to describe in generality. On the other hand, the torus-linearized bundles on a vector space are precisely given by a choice of character, so we have

$$\mathrm{Pic}(\mathfrak{Y}_e) = \mathbb{C}^{r+m}$$

Thus, the quasimap degrees to $\mathfrak{Y}_e$ are much easier to describe. As a result, we will write a quasimap degree as

$$\beta = (d_1, \ldots, d_{r+m}) \text{ for } \beta \in \mathrm{Hom}(\mathrm{Pic}(\mathfrak{Y}_e, \mathbb{Q}) \cong \mathbb{Q}^{r+m}$$

and will often treat β as a vector through the above isomorphism. The possible quasimap degrees β from curves as above constitute a subset of the entire effective cone, which is described in the following proposition.

Proposition V.17. *For $\beta = (d_1, \ldots, d_{r+m})$, define the set $S_\beta^{\geq 0} \subset \{1, \ldots, n+m\}$ as*

$$S_\beta^{\geq 0} = \{i | \ \beta \cdot \psi_{\bullet i} \in \mathbb{Z}_{\geq 0}\}$$

Then the fixed locus F_β is non-empty if and only if $S_\beta^{\geq 0} \in \mathcal{A}_{\theta_e}$, as in (V.1.5).

Proof. Recall that a quasimap of degree β is equivalent to choosing $r+m$ line bundles $\mathcal{L}_i$ such that $\deg \mathcal{L}_i = d_i$, and $n+m$ sections $s_j \in \mathcal{L}^{\psi_{\bullet i}}$. Since our curves are isomorphic to $\mathbb{P}(1, \star)$ and have at most one orbifold point, we have that any line bundle with integer degree is pulled back from the coarse curve. On the other hand, any line bundle with non-integer degree only has sections that vanish at the orbifold point.

From this, the claim is evident. Given a quasimap from $\mathbb{P}(1, \star)$, of degree β, we must have that the image of ∞ lands in the semi-stable locus, hence lands in U_I for some $I \subset \mathcal{A}_{\theta_e}$. By the above, this means that $\deg \mathcal{L}^{\psi_{\bullet i}} \in \mathbb{Z}$ for $i \in I$, hence $I \subset S_\beta^{\geq 0} \in \mathcal{A}_{\theta_e}$ from IV.2.

On the other hand, suppose $S_\beta^{\geq 0} \in \mathcal{A}_{\theta_e}$. Then we can choose line bundles with degrees corresponding to β, and set sections $s_i = 0$ for $i \notin S_\beta^{\geq 0}$. For $i \in S_\beta^{\geq 0}$, we have that $\mathcal{L}^{\psi_{\bullet i}} \cong \pi^* \mathcal{O}(\beta \cdot \psi_{\bullet i})$; we then choose s_i to be the unique section, up that scaling, that only vanishes at zero, i.e. if $[x : y]$ are the local coordinates on the coarse curve $\mathbb{P}^1$, then we choose s_i such that it is identified through the isomorphism with a multiple of $\pi^*(y^n)$. One can easily check that this defines a quasimap that satisfies all the required conditions of being in the fixed locus.

$\square$

Following the proposition, we define the set

$$\mathrm{Eff}^I(W_e, (\mathbb{C}^*)^{r+m}, \theta_e) = \{\beta | \ F_\beta \neq \emptyset\} \tag{V.2.1}$$

as the subset of the effective cone that is most relevant to the I-function computation. Note that this depends on the GIT presentation of our target. However, when the argument is understood, we will often shorten this to Eff^I.

Now for $\beta \in \mathrm{Eff}^I$, let

$$\begin{tikzcd} \mathbb{P}(1, \star) \times F_\beta \arrow[r, "f"] \arrow[d, "\pi"] & \mathfrak{X}_e \\ F_\beta & \end{tikzcd} \tag{V.2.2}$$

be the universal quasimap. There is a $\mathbb{C}^*$ action on $\mathbb{P}(1,\star) \times F_\beta$ given by the scaling action on the first factor and the trivial action on the second. By definition of the fixed locus, the map f is a $\mathbb{C}^*$ invariant, i.e. it is $\mathbb{C}^*$ equivariant where $\mathfrak{X}_e$ has the trivial $\mathbb{C}^*$ action. Given a vector bundle $\mathcal{E}$ on fX_e, we can regard it as a $\mathbb{C}^*$ linearized bundle with the trivial linearization. Then the pullback $f^*\mathcal{E}$ inherits a canonical equivariant structure.

Lemma V.18. *For any line bundle $\mathcal{L}$ on $\mathfrak{X}_e$, up to the n-th power map $\mathbb{C}^* \xrightarrow{n} \mathbb{C}^*$ for some n, the equivariant structure on $f^*\mathcal{L}$ is the unique one whose action on $f^*\mathcal{L}|_{\Sigma_\infty \times F_\beta}$ is trivial, where $\Sigma_\infty \subset \mathbb{P}(1,\star)$ is the stacky locus.*

Proof. After composition with the n-th power map for some n, we can treat the $\mathbb{C}^*$ action on $\Sigma_\infty \times F_\beta$ as the trivial one. We then note that the action map $\sigma : \mathbb{C}^* \times \Sigma_\infty \times F_\beta \to \Sigma_\infty \times F_\beta$ coincides with that of the projection map π, hence a linearization of $f^*\mathcal{L}|_{\Sigma_\infty \times F_\beta}$ is equivalent to an automorphism of the bundle $\pi^* f^*\mathcal{L}$ over $\mathbb{C}^* \times \Sigma_\infty \times F_\beta$ satisfying the cocycle condition. Under the identification $\sigma = \pi$, one can check that this cocycle condition is equivalent to descent data in the smooth topology with regards to the cover composing only of π, and hence we have that a linearization is equivalent to a $\Sigma_\infty \times F_\beta$ morphism

$$\mathbb{C}^* \times \Sigma_\infty \times F_\beta \to \underline{\mathrm{Aut}}_{\Sigma_\infty \times F_\beta}(f^*\mathcal{L}|_{\Sigma_\infty \times F_\beta}) = \mathbb{C}^* \times \Sigma_\infty \times F_\beta$$

Now notice that the $\mathbb{C}^*$ action on both $\Sigma_\infty \times F_\beta$ and $\mathfrak{X}_e$ are both trivial. Restricting the morphism f to $\Sigma_\infty \times F_\beta$, we have that making the map f equivariant is equivalent to choosing a two-morphism in the usual action diagram, and, by the triviality of the actions, this is given by an automorphism of f that satisfies the required cocyle conditions. Since the linearization of the pullback is determined by the structure of the equivariant map, we have that the linearization morphism above must factor through

$$\mathrm{Aut}_{\Sigma_\infty \times F_\beta}(f|_{\Sigma_\infty \times F_\beta})$$

However, since $f|_{\Sigma_\infty \times F_\beta}$ factors through the Deligne-Mumford stack X, we have that this is a finite group scheme. As a result, the group homomorphism that determines our linearization must be trivial, in which case the two-morphism determining our equivariant map must be trivial. Since $\mathcal{L}$ had trivial linearization, we then have that $f^*\mathcal{L}|_{\Sigma_\infty \times F_\beta}$ has trivial linearzation.

For uniqueness, we note that any difference in linearization is equivalent to a difference in linearization of the trivial bundle, so it suffices to show that the linearization on the trivial bundle is the trivial one. A linearization on the trivial bundle $\mathcal{O}_{\mathbb{P}(1,\star) \times F_\beta}$ is an automorphism of $\mathcal{O}_{\mathbb{C}^* \times \mathbb{P}(1,\star) \times F_\beta}$ that satisfies the cocyle condition, which is given by multiplication by a non-vanishing section. A non-vanishing section that is equal to 1 on $\mathbb{C}^* \times \Sigma_\infty \times F_\beta$ must be identically 1 on $\mathbb{C}^* \times \mathbb{P}(1,\star) \times F_\beta$ since such sections are constant along $\mathbb{P}(1,\star)$. Hence any linearization of $\mathcal{O}_{\mathbb{P}(1,\star)\times F_\beta}$ trivial at $\Sigma_\infty \times F_\beta$ must be trivial everywhere.

□

We can now focus our attention on line bundles on $\mathbb{P}(1,\star)$ that have trivial equivariant structure at ∞. The weights of the sections of such line bundles are easily determined, and will allow us to fully describe the fixed locus, as well as the perfect obstruction theory.

Lemma V.19. *Let $\mathcal{L}$ be an equivariant line bundle of degree $\frac{r}{\star}$ over $\mathbb{P}(1,\star)$ such that, up to n-th power map on $\mathbb{C}^*$, the action on $\mathcal{L}_{\Sigma_\infty}$ is trivial. Then we have that*

- *$H^0(\mathbb{P}(1,\star),\mathcal{L})$ is the $\mathbb{C}$-vector space with basis given by $x^i y^j$, where $i+\star j = r$, $i,j \in \mathbb{Z}_{\geq 0}$*
- *$H^1(\mathbb{P}(1,\star),\mathcal{L})$ is the $\mathbb{C}$-vector space with basis given by $x^i y^j$, where $i+\star j = r$, $i,j \in \mathbb{Z}_{<0}$*

Moreover, the weight of $x^i y^j$ under the $\mathbb{C}^*$ action is given by $\frac{i}{\star}$.

Proof. The description of the cohomology follows immediately from a standard Čech cohomology computation (see [12, Section 5.2]). For the weights of the sections, we note that the weight of y^n is the weight of $\mathcal{L}|_{\Sigma\times F_\beta}$ after the n-th power map, hence is zero by assumption. Then we note that the meromorphic section $\frac{x^\star}{y}$ has weight 1 by definition of the $\mathbb{C}^*$ action, from which we deduce that the weight of x is $\frac{1}{\star}$.

□

From now on, we will also fix the choice of a section σ_∞ of $\mathbb{P}(1,\star) \to \operatorname{Spec}\mathbb{C}$ that maps into Σ_∞, which is unique up to non-canonical isomorphisms. This choice gives a canonical isomorphism of Σ_∞ with $B\mu_\star$. We will also abuse notation and use σ_∞ to denote $\sigma_\infty \times id_S : S \to \Sigma_\infty \times S$ for any S.

Lemma V.20. *Let $\mathcal{L}$ be a line bundle on $\mathfrak{X}_e$ such that f^*L has fiberwise degree $\frac{r}{\star}$. Then*

- *if $r \geq 0$,*
$$R\pi_* f^*\mathcal{L} \cong \bigoplus_{\substack{i+\star j = r\\ i,j\in\mathbb{Z}_{\geq 0}}} \mathbb{C}x^i y^j \otimes \sigma_\infty^*\mathcal{L},$$
- *if $r < 0$,*
$$R\pi_* f^*\mathcal{L} \cong \bigoplus_{\substack{i+\star j = r\\ i,j\in\mathbb{Z}_{<0}}} \mathbb{C}x^i y^j \otimes \sigma_\infty^*\mathcal{L}[-1].$$

Moreover, this commutes with base change along any $S \to F_\beta$.

Proof. We can base change to any scheme S and prove the result there. Using the fact that $H^1(\mathbb{P}(1,\star), \mathcal{O}_{\mathbb{P}(1,\star)}) = 0$, a standard argument using cohomology and base change shows that

$$f^*\mathcal{L}_S \cong \mathcal{O}_{\mathbb{P}(1,\star)}(r) \boxtimes \mathcal{M}$$

for some $M \in \mathrm{Pic}(S)$. By Lemma V.18, we have that, up to some n-th power map on $\mathbb{C}^*$, that this is an isomorphism of $\mathbb{C}^*$-equivariant line bundles where $\mathcal{M}$ has the trivial linearization, and $\mathcal{O}_{\mathbb{P}(1,\star)}(r)$ is an equivariant bundle whose fiber at ∞ has weight zero. After taking $R\pi_*$, the result follows from the cohomology description in Lemma V.19.

□

Corollary V.21. *Let $\mathcal{L}$ be a line bundle on $\mathfrak{X}_e$ such that f^*L has fiberwise degree $\frac{r}{\star}$. Then if $\frac{r}{\star} \geq 0$, we have a canonical isomorphism*

$$(R\pi_* f^*\mathcal{L})^{\mathbb{C}^*} \cong \sigma^*\mathcal{L}$$

Otherwise, we have $(R\pi_* f^*\mathcal{L})^{\mathbb{C}^*} = 0$.

Proof. This is immediate from Lemma V.20 and noticing that there is at most one weight zero section, which only appears when the bundle has positive degree. □

We end this section with an explicit description of the stack F_β. For a fixed β, we define the set $I_\beta^{\geq 0} \subset \{1, \dots, n+m\}$ via

$$I_\beta^{\geq 0} = \{i \mid \beta \cdot \psi_{\bullet i} \in \mathbb{Z}_{\geq 0}\}, \tag{V.2.3}$$

and define the subset $W_e^\beta \subset W_e$ as the linear subspace cut out by all the x_i for $i \notin I_\beta^{\geq 0}$. Similarly, let $(W_e^\beta)^{\mathrm{ss}} = W_e^\beta \cap W_e^{\mathrm{ss}}$, $W^\beta = W_e \cap W$, and $(W^\beta)^{\mathrm{ss}} = W^\beta \cap W^{\mathrm{ss}}$. In particular, we note that

$$[(W_e^\beta)^{\mathrm{ss}}/(\mathbb{C}^*)^{r+m}] = [(W^\beta)^{\mathrm{ss}}/(\mathbb{C}^*)^r] \subset X$$

Proposition V.22. *The composition $f \circ \sigma_\infty : F_\beta \to X$ induces an isomorphism*

$$F_\beta \cong [(W_e^\beta)^{\mathrm{ss}}/(\mathbb{C}^*)^{r+m}] = [(W^\beta)^{\mathrm{ss}}/(\mathbb{C}^*)^r]$$

Proof. We show that there is a fiberwise equivalence of categories between the two stacks, hence the two stacks are isomorphic. Given a scheme S, we have that a map from $S \to \mathfrak{Y}_e$ is given by $r+m$ line bundles $\mathcal{M}_1, \dots, \mathcal{M}_{r+m}$ on S, with a choice of sections $a_i \in H^0(S, \mathcal{M}^{\psi_{\bullet i}})$ for all i. Further requiring that the corresponding morphism factors through $[(W_e^\beta)^{\mathrm{ss}}/(\mathbb{C}^*)^{r+m}]$ gives us an object of $[(W_e^\beta)^{\mathrm{ss}}/(\mathbb{C}^*)^{r+m}](S)$.

Now let $\beta = (d_1, \dots, d_{r+m})$. Given the datum above, we can then set

$$\mathcal{L}_i = \mathcal{O}_{\mathbb{P}(1,\star)}(\star \cdot d_i) \boxtimes \mathcal{M}_i \quad \text{and} \quad s_j = y^{\beta \cdot \psi_{\bullet j}} \boxtimes a_j \tag{V.2.4}$$

where we set $s_j = 0$ whenever $\beta \cdot \psi_{\bullet j} \notin \mathbb{Z}_{\geq 0}$, i.e. whenever $j \notin I_\beta^{\geq 0}$. Then the datum $\{L_i, s_j\}$ defines map from $\mathbb{P}(1,\star) \times F_\beta \to \mathfrak{X}_e$, and by Lemma V.19 this is a $\mathbb{C}^*$-invariant map, hence defines an object of $F_\beta(S)$. Thus, we have defined a functor

$$[(W_e^\beta)^{\text{ss}}/(\mathbb{C}^*)^{r+m}](S) \to F_\beta(S)$$

It's clear that this functor is fully faithful. On the other hand, given an object of $F_\beta(S)$, the proof of Lemma V.20 as well as Corollary V.21 shows that it must be of the form (V.2.4), hence the functor is essentially surjective. Thus, the functor is an equivalence of categories. It's also clear that the inverse to this functor is $f \circ \sigma_\infty$, as restricting the data $\{L_i, s_j\}$ to $\Sigma_\infty \times S$ recovers the data $\{\mathcal{M}_i, a_j\}$.

$\square$

In light of the above proposition, we will often consider F_β as a substack of X through the isomorphism described.

Remark V.23. The proof of Proposition V.22 also describes the universal map (V.2.2) explicitly in terms of line bundles and sections. Indeed, one can take $\mathcal{M}_i$ and a_j to be the universal line bundles and sections over the stack $[(W_e^\beta)^{\text{ss}}/(\mathbb{C}^*)^{r+m}]$, from which the line bundles and sections as in (V.2.4) gives the universal map.

V.3: Perfect Obstruction Theory

The perfect obstruction theory $\phi_{Q_{\mathbb{P}(1,\star)}(X,\beta)}\colon (R^\bullet\pi_* f^*\mathbb{T}_{\mathfrak{X}_e})^\vee \to \mathbb{L}_{Q_{\mathbb{P}(1\star)}(X,\beta)}$ is $\mathbb{C}^*$-equivariant [12]. Let

$$\mathbb{E} := (R^\bullet\pi_* f^*\mathbb{T}_{\mathfrak{X}_e})|_{F_\beta}.$$

Then by [36, Proposition 1], the fixed part $\mathbb{T}_{F_\beta} \to \mathbb{E}^{\text{fix}}$ is a perfect obstruction theory, defining the virtual cycle $[F_\beta]^{\text{vir}}$, and the virtual normal bundle is the moving part of $\mathbb{E}$

$$N^{\text{vir}}_{F_\beta/Q_{\mathbb{P}(1\star)}(X,\beta)} := \mathbb{E}^{\text{mov}}.$$

The goal of this section is to compute $[F_\beta]^{\text{vir}}$ and the equivariant Euler class $e^{\mathbb{C}^*}(\mathbb{E}^{\text{mov}})$. We recall that given an explicit presentation of the complex $\mathbb{E}^{\text{mov}} = [E^\bullet]$, we define $e^{\mathbb{C}^*}(\mathbb{E}^{\text{mov}})$ as the product $\prod_i (e^{\mathbb{C}^*} E^i)^{(-1)^i}$. Putting these together will give us a closed formula for the

I-function.

We can understand $\mathbb{E}$ through a pair of exact triangles. The first one is the generalized Euler sequence (IV.2.6) (c.f. [13, Section 5.2]) for $\mathfrak{Y}_e = [\mathbb{C}^{n+m}/(\mathbb{C}^*)^{r+m}]$

$$\mathcal{O}_{\mathfrak{Y}_e}^{r+m} \to \bigoplus_{i=1}^{n+m} \mathcal{O}_{\mathfrak{Y}_e}(\psi_{\bullet i}) \to \mathbb{T}_{\mathfrak{Y}_e} \xrightarrow{+1} . \tag{V.3.1}$$

The second triangle is obtained by dualizing the conormal triangle for the closed embedding $i : \mathfrak{X}_e \hookrightarrow \mathfrak{Y}_e$ (IV.3.1)

$$\mathcal{E}[-1] \to \mathbb{T}_{\mathfrak{X}_e} \to i^*\mathbb{T}_{\mathfrak{Y}_e} \xrightarrow{+1} . \tag{V.3.2}$$

Now consider the universal family (V.2.2). Pulling back and pushing forward the exact triangles (V.3.1) and (V.3.2) to F_β gives the exact triangles

$$R^\bullet\pi_*\left(f^*\mathcal{O}_{\mathfrak{X}_e}^{\oplus r+m}\right) \to R^\bullet\pi_*\left(f^*\bigoplus_i \mathcal{O}_{\mathfrak{X}_e}(\psi_{\bullet i})\right) \to R^\bullet\pi_*\left(f^*\mathbb{T}_{\mathfrak{Y}_e}\right) \xrightarrow{+1}, \tag{V.3.3}$$

and

$$\mathbb{E} \to R^\bullet\pi_*(f^*\mathbb{T}_{\mathfrak{Y}_e}) \to R^\bullet\pi_*(f^*\mathcal{E}) \xrightarrow{+1} . \tag{V.3.4}$$

Let Y_β (resp. $\mathfrak{Y}_{e,\beta}$) be the substack of Y (resp. $\mathfrak{Y}_e$) defined by the vanishing of all the x_i for $i \notin I_\beta^{\geq 0}$ as in (V.2.3). Set

$$\mathcal{E}_\beta = \bigoplus_{\beta\cdot\xi_{\bullet j}\in\mathbb{Z}_{\geq 0}} \mathcal{O}_Y(\xi_{\bullet j})$$

and let

$$\tilde{F} = (\tilde{F}_1, \ldots, \tilde{F}_s)$$

denote the section $\mathcal{E}$ that defines $\mathfrak{X}_e$, as in Lemma V.12.

Lemma V.24. *The section $\tilde{F}|_{Y_\beta}$ factors through $\mathcal{E}_\beta|_{Y_\beta}$.*

Proof. Suppose that $\beta \cdot \xi_j \notin \mathbb{Z}_{\geq 0}$. Then all the terms of $\tilde{F}_j$ must involve some x_i for which $\beta \cdot \psi_{\bullet j} \notin \mathbb{Z}_{\geq 0}$, hence $\tilde{F}_j$ vanishes on Y_β.

□

Proposition V.25. *The virtual cycle of F_β is the localized Euler class of $\mathcal{E}_\beta|_{Y_\beta}$ with respect to the section $\tilde{F}|_{Y_\beta}$. That is, we have*

$$[F_\beta]^{\text{vir}} = e_{\text{loc},\tilde{F}|_{Y_\beta}}\left(\mathcal{E}_\beta|_{Y_\beta}\right)$$

Proof. We can take the fixed part of the exact triangle (V.3.3). By Corollary V.21, we have that this becomes the exact sequence

$$0 \to \mathcal{O}_{F_\beta}^{r+m} \to \bigoplus_{\beta\cdot\psi_{\bullet i}\in\mathbb{Z}_{\geq 0}} \mathcal{O}_{\mathfrak{X}_e}(\psi_{\bullet i})|_{F_\beta} \to \left(\pi_*(f^*\mathbb{T}_{\mathfrak{Y}_e})|_{F_\beta}\right)^{\text{fix}} \to 0$$

where we recall that by Proposition V.22 we are viewing F_β as a substack of $\mathfrak{X}_e$. However, the map between the first two terms in the above exact sequence is that of the generalized Euler sequence for Y^β, hence we have that $\left(\pi_*(f^*\mathbb{T}_{\mathfrak{Y}_e})|_{F_\beta}\right)^{\text{fix}} \cong T_{Y_\beta}|_{F_\beta}$.

Now we can take the fixed part of the exact triangle (V.3.4), and, by Corollary V.21 and the above isomorphism, we have that this becomes the exact triangle

$$\mathbb{E}^{\text{fix}} \to T_{Y_\beta}|_{F_\beta} \to \mathcal{E}_\beta|_{F_\beta} \xrightarrow{+1}$$

Since this sequence comes from the conormal sequence, one can see that the map $T_{Y_\beta}|_{F_\beta} \to \mathcal{E}_\beta|_{F_\beta}$ comes from the differentiation of the section $\tilde{F}$. It is then evident that the virtual fundamental class is given by the localized Euler class corresponding to the section $\tilde{F}$ (see the basic example in [6]).

□

On the other hand, we can take the moving parts of the triangles (V.3.3) and (V.3.4). This allows us to compute the equivariant Euler class of the virtual normal bundle.

Lemma V.26. *Let $\beta = (d_1, \ldots, d_{r+m})$. We have*

$$e^{\mathbb{C}^*}\left(N^{\text{vir}}_{F_\beta/Q_{\mathbb{P}(1\star)}(X,\beta)}\right) = \left(\prod_{a=1}^{m} \frac{1}{(d_{r+a}!)z^{d_{r+a}}}\right)\prod_{i=1}^{n}\prod_{j=1}^{s} \times \frac{\prod_{\substack{\langle k\rangle=\beta\cdot\psi_{\bullet i}\\ \beta\cdot\psi_{\bullet i}<k<0}}\left(c_1\left(\mathcal{O}_{\mathfrak{X}_e}(\beta\cdot\psi_{\bullet i})\right)|_{F_\beta}+kz\right)}{\prod_{\substack{\langle k\rangle=\beta\cdot\psi_{\bullet i}\\ 0<k\leq\beta\cdot\psi_{\bullet i}}}\left(c_1\left(\mathcal{O}_{\mathfrak{X}_e}(\beta\cdot\psi_{\bullet i})\right)|_{F_\beta}+kz\right)} \times \frac{\prod_{\substack{\langle k\rangle=\beta\cdot\xi_{\bullet j}\\ 0<k\leq\beta\cdot\xi_{\bullet j}}}\left(c_1\left(\mathcal{O}_{\mathfrak{X}_e}(\beta\cdot\xi_{\bullet j})\right)|_{F_\beta}+kz\right)}{\prod_{\substack{\langle k\rangle=\beta\cdot\xi_{\bullet j}\\ \beta\cdot\xi_{\bullet j}<k<0}}\left(c_1\left(\mathcal{O}_{\mathfrak{X}_e}(\beta\cdot\xi_{\bullet j})\right)|_{F_\beta}+kz\right)}$$

Proof. This follows from taking $\mathbb{C}^*$-equivariant Euler classes of the moving parts of the triangles (V.3.3) and (V.3.4), as well as Lemma V.20. □

The last piece we need to consider in our I-function is the evaluation map $ev_\star$. Recall that given a quasimap degree $\beta = (d_1, \ldots, d_{r+m})$, there is an associated group element $g_\beta = (e^{2\pi i d_1}, \ldots, e^{2\pi i d_r}) \in (\mathbb{C}^*)^r$ (IV.2.2), and this group element then gives rise to a twisted sector X_α, where $\alpha = (d_1, \ldots, d_r)$. Moving forward, we will denote the twisted sector corresponding to β as X_{α_β}.

Then for F_β, the image of $ev_\star$ lies in X_{α_β}. Furthermore, we can consider X_{α_β} as a subset of X through inclusion of the fixed locus. This is an embedding, and together forms the following commutative diagram

$$\begin{tikzcd} F_\beta \arrow[r, "ev_\star"] \arrow[dr, "f\circ\sigma_\infty"'] & X_{\alpha_\beta} \arrow[d] \\ & X \end{tikzcd} \tag{V.3.5}$$

Since the I-function is valued in the cohomology of the inertia stack, we can view F_β as a subset of X_{α_β} via the $ev_\star$. By the above diagram, this is compatible with our previous discussion.

Remark V.27. Note that in general, $Y_\beta \neq Y_{\alpha_\beta}$ as stacks, where Y_β is defined as in Proposition V.25. For example, consider the quotient stack $\mathfrak{Y} = [\mathbb{C}^6/(\mathbb{C}^*)^2]$ where the weight matrix is given by

$$\begin{pmatrix} 1 & 4 & 4 & 6 & 9 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

It's straightforward to check that the corresponding toric stack is $Y = \mathbb{P}(1,4,4,6,9)$, and that $\beta = (-1/3, 3)$ is an I-effective class, i.e. $\beta \in \mathrm{Eff}^I$. One can check that $Y_\beta = B\mu_9$ here, while the twisted sector $Y_{\alpha_\beta} = \mathbb{P}(6,9)$. It is true, however, that Y_β is a substack of Y_{α_β} when viewing both as substacks of Y. Similarly, one can check that $F_\beta \neq X_{\alpha_\beta}$ in general, as can be seen in the related example of a degree 24 hypersurface in Y VII.4.

Finally, we note that the rigidification map $\mathcal{I}_r X \to \overline{\mathcal{I}}_r X$ has degree r^{-1} in the sense that pushing forward is equivalent to multiplying by r^{-1} in the Chow ring. This factor will cancel with the factor of r that is implicit in the definition of the I-function from $\widetilde{ev}_\star$. Combining this discussion with Proposition V.25 and Lemma V.26 gives us the following formula for the I-function.

Theorem V.28. *The extended I-function for X is given by*

$$I^X(q,z) = \sum_{\beta \in \mathrm{Eff}^I} \frac{q^\beta}{\left(\prod_{a=1}^m (d_{r+a}!) z^{d_{r+a}}\right)} \prod_{i=1}^n \prod_{j=1}^s \times$$
$$\frac{\prod_{\substack{\langle k\rangle = \beta\cdot\psi_{\bullet i} \\ \beta\cdot\psi_{\bullet i} < k < 0}} \left(c_1\left(\mathcal{O}_{\mathfrak{X}_e}(\beta\cdot\psi_{\bullet i})\right)|_{F_\beta} + kz\right)}{\prod_{\substack{\langle k\rangle = \beta\cdot\psi_{\bullet i} \\ 0 < k \leq \beta\cdot\psi_{\bullet i}}} \left(c_1\left(\mathcal{O}_{\mathfrak{X}_e}(\beta\cdot\psi_{\bullet i})\right)|_{F_\beta} + kz\right)} \times \frac{\prod_{\substack{\langle k\rangle = \beta\cdot\xi_{\bullet j} \\ 0 < k \leq \beta\cdot\xi_{\bullet j}}} \left(c_1\left(\mathcal{O}_{\mathfrak{X}_e}(\beta\cdot\xi_{\bullet j})\right)|_{F_\beta} + kz\right)}{\prod_{\substack{\langle k\rangle = \beta\cdot\xi_{\bullet j} \\ \beta\cdot\xi_{\bullet j} < k < 0}} \left(c_1\left(\mathcal{O}_{\mathfrak{X}_e}(\beta\cdot\xi_{\bullet j})\right)|_{F_\beta} + kz\right)} \cdot \iota\left([F_\beta]^{\mathrm{vir}}\right)$$

where $[F_\beta]^{\mathrm{vir}}$ *is associated with its image under the evaluation map, as in* (V.3.5)*,* ι *is the involution map on the inertia stack, and* $q^\beta := \prod_{i=1}^{r+m} q_i^{d_i}$ *for* $\beta = (d_1, \ldots, d_{r+m})$*.*

In general, the cohomology classes obtained from $[F_\beta]^{\mathrm{vir}}$ from Poincaré duality can be complicated to describe further than what is given in Proposition V.25. However, the cohomology classes obtained from $[F_\beta]$ can be described more explicitly, especially since Proposition V.22 gives an explicit description of the cycles. In the situation where $[F_\beta]^{\mathrm{vir}} = [F_\beta]$ for all β, the I-function takes a much nicer form. Hence, we will make the following assumption for the rest of this section.

Assumption V.29. For all $\beta \in \mathrm{Eff}^I$, we have that the section $\tilde{F}|_{Y_\beta}$ defines a regular sequence.

By definition of the localized Euler class (see [31, Proposition 14.1]), this ensures that $[F_\beta]^{\mathrm{vir}} = [F_\beta]$.

Now for each β, consider the set $I_{\alpha_\beta} = \{i | \ \alpha_\beta \cdot w_{\bullet i} \in \mathbb{Z}\} \subset \{1, \ldots, n\}$ associated to the span of the twisted sector X_{α_β}. We note that the set $I_\beta^{\geq 0} \cap \{1, \ldots, n\}$ from Proposition V.22 is a subset of I_{α_β}, given by those i such that $\alpha_\beta \cdot w_{\bullet i} \in \mathbb{Z}_{\geq 0}$. Define $I_\beta^{<0}$ to be the complement of $I_\beta^{\geq 0}$ in I_{α_β}, i.e.

$$I_\beta^{<0} = \{i | \ \alpha_\beta \cdot w_{\bullet i} \in \mathbb{Z}_{<0}\} \subset I_{\alpha_\beta}$$

Lemma V.30. *We have an equality of cycles*

$$[F_\beta] = t_{\alpha_\beta}^{I_\beta^{<0}}$$

Proof. We note that $Y_\beta = V\left(\{x_i\}_{i \in I_\beta^{<0}}\right) \subset Y_{\alpha_\beta}$. Viewing F_β as a substack of X_{α_β}, we have from Proposition V.22 that $F_\beta \cong X_{\alpha_\beta} \cap Y_\beta \subset Y_{\alpha_\beta}$. As a cycle in X_{α_β}, the latter is precisely $t_{\alpha_\beta}^{I_\beta^{<0}}$, hence the equality follows. □

In light of this, we can write the extended I-function of Theorem V.28 as a series only involving the more familiar cohomology classes t_α^J.

Corollary V.31. *If Assumption V.29, then the extended I-function of Theorem V.28 can be written as*

$$I^X(q,z) = \sum_{\beta \in \mathrm{Eff}^I} \frac{q^\beta}{\left(\prod_{a=1}^m (d_{r+a}!) z^{d_{r+a}}\right)} \prod_{i=1}^n \prod_{j=1}^s \times$$
$$\frac{\prod_{\substack{\langle k \rangle = \beta \cdot \psi_{\bullet i} \\ \beta \cdot \psi_{\bullet i} < k < 0}} \left(c_1\left(\mathcal{O}_{\mathfrak{X}_e}(\beta \cdot \psi_{\bullet i})\right)|_{F_\beta} + kz\right)}{\prod_{\substack{\langle k \rangle = \beta \cdot \psi_{\bullet i} \\ 0 < k \leq \beta \cdot \psi_{\bullet i}}} \left(c_1\left(\mathcal{O}_{\mathfrak{X}_e}(\beta \cdot \psi_{\bullet i})\right)|_{F_\beta} + kz\right)} \times \frac{\prod_{\substack{\langle k \rangle = \beta \cdot \xi_{\bullet j} \\ 0 < k \leq \beta \cdot \xi_{\bullet j}}} \left(c_1\left(\mathcal{O}_{\mathfrak{X}_e}(\beta \cdot \xi_{\bullet j})\right)|_{F_\beta} + kz\right)}{\prod_{\substack{\langle k \rangle = \beta \cdot \xi_{\bullet j} \\ \beta \cdot \xi_{\bullet j} < k < 0}} \left(c_1\left(\mathcal{O}_{\mathfrak{X}_e}(\beta \cdot \xi_{\bullet j})\right)|_{F_\beta} + kz\right)} \cdot \iota^* t_{\alpha_\beta}^{I_\beta^{<0}}$$

where ι is the involution map on the inertia stack and $q^\beta := \prod_{i=1}^{r+m} q_i^{d_i}$ for $\beta = (d_1, \ldots, d_{r+m})$.

Note that

$$\iota^* t_{\alpha_{(-\beta)}}^{I_\beta^{<0}}$$

since $g_{(-\beta)} = g_\beta^{-1}$.

As we mentioned before, the cohomology classes t_α^J can possibly be primitive classes of X. However, in the nicest situations, it's possible that one only has t_α^J which are ambient classes, hence one might be able to get even nicer forms of the I-function. An example of this phenomenon can be seen in some of the examples, e.g. Example VII.1.

Remark V.32. One can weaken Assumption V.29 to instead require that $[F_\beta]^{\mathrm{vir}}$ is a constant multiple of F_β, e.g. it has some non-reduced structure. As long as one can identify the constant, then a similar equation to that of Corollary V.31 is obtained.

CHAPTER VI
Mirror Theorem and Invertibility

In this section, we apply quasimap wall-crossing in order to recover the Gromov-Witten invariants from the extended I-function computed in the previous section.

VI.1: The $\epsilon = \infty$ stability condition

Recall that the notion of $\epsilon = \infty$ stability coincides with the usual notion of stability in Gromov-Witten theory. In particular, this implies that our maps have no basepoints, hence the image lies in entirely in X. As a result, we note that the GIT presentation of our quotients do not affect the invariants on this side, and since our extended GIT presentation does not change our quotient by Lemma V.8, we have that the $\epsilon = \infty$ quasimap invariants of our extended presentation agree with the Gromov-Witten invariants of X.

To be more precise, we address some small subtleties in our notion of degree. Let $\mathrm{NE}(X)$ denote the Mori cone of curves of X. Typically, the degree of a Gromov-Witten invariant is defined in terms of a curve class in $H_2(X)$, but there is a natural map $\mathrm{NE}(X) \to H_2(X)$ which is injective for Deligne-Mumford stacks with projective coarse moduli space (see [43, Proposition 5.15] and [44, Proposition 14]), and which is an isomorphism for toric targets [25, Proposition 6.2.15]. As such, we will use $\mathrm{NE}(X)$ for our curve classes.

On the other hand, the degree for quasimaps is defined as an element of $\mathrm{Hom}(\mathrm{Pic}(\mathfrak{X}_e), \mathbb{Q})$, which a priori depends on the GIT presentation corresponding to $\mathfrak{X}_e$. However, there is a natural morphism

$$\iota : \mathrm{NE}(X) \to \mathrm{Hom}(\mathrm{Pic}(\mathfrak{X}_e), \mathbb{Q})$$

defined by

$$\gamma \to \left(\mathcal{L} \to \int_\gamma c_1(\mathcal{L})\right) \text{ for all } \mathcal{L} \in \mathrm{Pic}(\mathfrak{X}_e, \mathbb{Q})$$

Since our curve classes lie in X, we naturally have that ι factors as

$$\mathrm{NE}(X) \to \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{Q}) \to \mathrm{Hom}(\mathrm{Pic}(\mathfrak{X}), \mathbb{Q}) \to \mathrm{Hom}(\mathrm{Pic}(\mathfrak{X}_e), \mathbb{Q})$$

with the maps corresponding to the restriction maps $\mathrm{Pic}(\mathfrak{X}_e) \to \mathrm{Pic}(\mathfrak{X}) \to \mathrm{Pic}(X)$. The first and last map in the above presentation of ι are both injective; the first is from the definition of $\mathrm{NE}(X)$, which identifies curves based on numerical equivalence, while the last comes from surjectivity of the corresponding Picard groups. The second map requires some more care, as indicated in Remark VI.2; for ease, we will impose the rather mild assumption that it has finite fibers.

Lastly, we note that since we often consider β as an element of $\mathrm{Hom}(\mathrm{Pic}(\mathfrak{Y}_e), \mathbb{Q})$, it makes sense to post-compose ι with the natural map $\mathrm{Hom}(\mathrm{Pic}(\mathfrak{X}_e), \mathbb{Q}) \to \mathrm{Hom}(\mathrm{Pic}(\mathfrak{Y}_e), \mathbb{Q})$. By abuse of notation, we will continue to call this composition ι, as all the relevant results hold with or without the post-composition.

Then by tracing definitions, one arrives at the following lemma

Lemma VI.1. *Assume that for $\beta \in \mathrm{Hom}(\mathrm{Pic}(\mathfrak{Y}_e), \mathbb{Q})$, we have $\iota^{-1}(\beta)$ is finite. Then*

$$\langle \mathbf{t}_1(\psi), \ldots, \mathbf{t}_n(\psi) \rangle^{\infty}_{g,\beta} = \sum_{\iota(\gamma)=\beta} \langle \mathbf{t}_1(\psi), \ldots, \mathbf{t}_n(\psi) \rangle_{g,\gamma}$$

for $\mathbf{t}_i \in H^*_{\mathrm{CR}}(X, \mathbb{Q})[\![z]\!]$.

Note that in particular that due to how ι factors, we have that $\langle \mathbf{t}_1(\psi), \ldots, \mathbf{t}_n(\psi) \rangle^{\infty}_{g,\beta} = 0$ for any $\beta \in \mathrm{Hom}(\mathrm{Pic}(\mathfrak{Y}_e), \mathbb{Q})$ whose extended degrees are non-zero. This is expected, and can also be proven directly by the fact that we allow no basepoints in our $\epsilon = \infty$ stability condition.

Moving forward, we will consolidate the notation and refer to the image of ι as $\mathrm{NE}(X)$ as well.

Remark VI.2. Ideally, we would like ι to be injective, which is true if we have that both $\mathrm{Pic}(\mathfrak{X}) \to \mathrm{Pic}(X)$ and $\mathrm{Pic}(\mathfrak{Y}_e) \to \mathrm{Pic}(\mathfrak{X}_e)$ are surjective. If so, we would have that

$$\langle \mathbf{t}_1(\psi), \ldots, \mathbf{t}_n(\psi) \rangle^{\infty}_{g,\iota(\gamma)} = \langle \mathbf{t}_1(\psi), \ldots, \mathbf{t}_n(\psi) \rangle_{g,\gamma}$$

so that we can unravel individual invariants.

The Picard groups $\mathrm{Pic}(\mathfrak{X})$ and $\mathrm{Pic}(X)$ parameterize G-equivariant line bundles on W and W^{ss} respectively, so surjectivity of the first map becomes a question of whether one can extend G-equivariant line bundles from W^{ss} to W. A sufficient, but not necessary, condition for this would be if W was locally factorial and the unstable locus, $W \setminus W^{\mathrm{ss}}$, had codimension 2. In this case, one can extend line bundles and the G-equivariant data by Hartog's lemma.

For the second morphism, we have that $\mathrm{Pic}(\mathfrak{Y}_e)$ is equivalent to a linearization of the trivial bundle over V, which is given by a character of the extended torus. The restriction to

$\mathrm{Pic}(\mathfrak{X}_e)$ is injective, and is surjective exactly when the only line bundle on W_e is the trivial bundle.

Alternatively, one can push ι completely to the ambient toric stack and show injectivity there, which is reminiscent of the more classical quantum Lefschetz hyperplane arguments. More explicitly, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{NE}(X) & \xrightarrow{\iota_X} & \mathrm{Hom}(\mathrm{Pic}\,\mathfrak{X}_e, \mathbb{Q}) \\ \downarrow & & \downarrow \\ \mathrm{NE}(Y) & \xrightarrow{\iota_Y} & \mathrm{Hom}(\mathrm{Pic}\,\mathfrak{Y}_e, \mathbb{Q}) \end{array}$$

where ι_X references ι as above, and the left-side vertical map is the pushforward on curve classes by the inclusion $X \hookrightarrow Y$. Since Y is a quotient of a vector space, one has by Hartog's lemma that ι_Y is injective when the unstable locus in the non-extended GIT presentation $V \setminus V^{\mathrm{ss}}$ is codimension at least 2, which is a simple check in practice (e.g. this holds for weighted projective spaces).

VI.2: Wall-Crossing Formula

Quasimap wall-crossing is an explicit relationship between ϵ-stable quasi-map invariants for different values of ϵ. It has been established for complete intersections in ordinary projective space in [14, 16], and has recently been extended by Yang Zhou to all (including orbifold) GIT quotients in [59]. In particular, the wall-crossing formula for the latter applies to our target spaces. We will present the relevant results of [59] without proof in the following discussion.

Let $\phi_1, \ldots, \phi_N$ denote a basis for $H^*_{\mathrm{CR}}(X, \mathbb{Q})$, and let $\phi^1, \ldots, \phi^N$ denote the dual basis under the Poincaré pairing. For generic $\mathbf{t}(z) \in H^*_{\mathrm{CR}}(X, \mathbb{Q})[\![z]\!]$, we define the big $J^{+\infty}$-function as

$$J^{+\infty}(\mathbf{t}(z), q, z) = 1 + \frac{\mathbf{t}(-z)}{z} + \sum_{\beta \in \mathrm{NE}(X), k \geq 0} \frac{q^\beta}{k!} \sum_{i=1}^{N} \phi_i \langle \frac{\phi^i}{z(z-\psi)}, \mathbf{t}(\psi), \ldots, \mathbf{t}(\psi) \rangle^{X,\infty}_{0,1+k,\beta} \tag{VI.2.1}$$

where the unstable terms (corresponding to non-existent moduli spaces) are interpreted as zero.

Define

$$\mu(q, z) = [zI(q, z) - z]_+, \tag{VI.2.2}$$

where $[\cdot]_+$ refers to truncating the terms with negative powers of z. By taking $\epsilon \to 0^+$ and $\mathbf{t}(z) = 0$ in [59, Theorem 1.12.2], or by [56, Theorem 6.7] in the toric hypersurface case, we get the following elegant wall-crossing formula

Theorem VI.3 ([56, 59]). *We have*

$$J^{+\infty}(\mu(q,-z), q, z) = I(q,z), \tag{VI.2.3}$$

for any quasimap I-function.

It's important to note that this statement is independent of the GIT presentation chosen, so it applies to the I-function of Theorem V.28.

Remark VI.4. Note that $J^{+\infty}$ is related to the usual J-function in Definition II.12 by Lemma VI.1 and by multiplying by z. When ι is injective, we can interpret this statement as saying that $-zI(q,-z)$ is on the Lagrangian cone $\mathcal{L}_X$ (II.5.1).

When ι is not injective, we can modify the definition of the Lagrangian cone by changing the Novikov ring to be the completion with respect to $\epsilon = \infty$ quasimap degrees (the image of ι). It's not hard to see that one can prove all the same statements of [17, Appendix B] for this version of the Lagrangian cone as well; essentially, one needs to check that the dilaton, string, and TRR equations all hold for quasimap invariants. This can be done by writing these invariants in terms of the usual Gromov-Witten ones via VI.1, using the relations in that setting, and translating back. Consequently, this theorem can still be interpreted as a statement about I lying on the Lagrangian cone.

In light of the above remark, we will drop the $+\infty$ superscript moving forward and will not make a distinction between the versions of the Lagrangian cone.

VI.3: Invertibility

In this section, we will show that with the change of variables defined by $\mu(q,z)$ in (VI.2.3) is an "invertible" transformation for an appropriately extended I-function. More specifically, we will show that we can extract all the individual Gromov-Witten invariants from the J-function by writing J in terms of an explicit change of coordinates of I.

Moving forward, we will extend our GIT presentation as in Section V.1.3 by cohomology elements $\{t^{J_i}_{\alpha_i}\}_{i=1}^m$ such that the following hold

- The unit class $\mathbf{1}$ and a basis of $H^2(X,\mathbb{Q})$, e.g. the untwisted divisor classes $\{H_i\}_{i=1}^r$, are contained in $\{t^{J_i}_{\alpha_i}\}_{i=1}^m$.

- The cohomology classes $\{t_{\alpha_i}^{J_i}\}_{i=1}^m$ are a maximal, linearly independent collection of classes of the form t_α^J in $H^*_{\mathrm{CR}}(X, \mathbb{Q})$
- The resulting presentation satisfies Assumption V.29.

Technically the second condition subsumes the first, but it is worth mentioning for clarity. We also can relax the other two conditions to something slightly weaker, as touched upon in Remark VI.8. However, these current conditions are effectively the conditions one would want to work under, and showcase the complete proof with minimal extra bookkeeping. To reduce notation strain, we will also set

$$t_i := t_{\alpha_i}^{J_i}$$

for the remainder of this section.

Finally, we remark that the proof of invertibility relies heavily on the language of Givental's Lagrangian cone. To match the conventions set in [17, Appendix B] and [35], we need to have the $\beta = 0$ term of our I and J functions to be z, in which case we would need to multiply by z in the I-function of Theorem V.28 and the definition of the J-function as given in (VI.2.1) so that it matches the definition in II.12. This is an easily fixable matter of convention, so we will consider it done:

Convention. For the remainder of this section, we have

- $I(q, z)$ refers to z times the function of Theorem V.28.
- The function $\mu(q, z)$ is now given by $\mu(q, z) = [I(q, z) - z]_+$ with the new convention for $I(q, z)$ above.

With our conventions set, we now we move to proving invertibility of the mirror map. To make the computations easier, we will choose a more convenient basis for $\mathrm{Hom}(\mathrm{Pic}(\mathfrak{X}), \mathbb{Q})$ so that the effective curve classes Eff^I can be represented by a tuple of natural numbers.

Lemma VI.5. *For $0 \neq \beta \in \mathrm{Eff}^I$, we have $-\beta \notin \mathrm{Eff}^I$.*

Proof. From the definition of S_β, we have that $\beta \in \mathrm{Eff}^I$ means that $\beta \cdot \theta_e \geq 0$. If both β and $-\beta$ are effective, then $\beta \cdot \theta = 0$, and hence $\beta \cdot \psi_{\bullet i} = 0$ for all $i \in S_\beta$. However, by Assumption IV.1, this means that $\beta = 0$. $\square$

The above lemma implies that the convex cone generated by $\mathrm{Eff}^I \subset \mathbb{Q}^{r+m}$ contains no lines, and hence there exists a change of variables so that the entire cone can be represented by positive integers. If we identify $\mathrm{Hom}(\mathrm{Pic}(\mathfrak{X}), \mathbb{Q}) \cong \mathbb{Q}^{r+m}$ via the identification

$$\beta \to (\beta(\mathcal{O}(1,0,\dots 0),\dots,\beta(0,\dots,0,1)) = (d_1,\dots,d_{r+m}),$$

then we will change to a new basis $\beta = (e_1, \dots, e_{r+m})$ via a transformation

$$\begin{pmatrix} d_1 \\ \vdots \\ d_{r+m} \end{pmatrix} = \left(\begin{array}{c|c} A & -\alpha_{r\times m} \\ \hline 0_{m\times r} & \mathrm{Id}_{r\times m} \end{array}\right) \begin{pmatrix} e_1 \\ \vdots \\ e_{r+m} \end{pmatrix} \tag{VI.3.1}$$

Here, the matrix $-\alpha_{r\times m}$ has (i,j) entry given by $-(\alpha_i)_j$. Note that the last m columns give linearly independent effective classes in the d_i coordinates. We choose the matrix A so that the first r columns complete a basis for $\mathbb{Q}^{r+m}$, and such that the positive convex cone generated by the columns of the change of base matrix contain the convex cone generated by Eff^I, possible by Lemma VI.5.

From this description, it is clear now that in the e_i basis of $\mathrm{Hom}(\mathrm{Pic}(\mathfrak{X}), \mathbb{Q})$, we can represent the effective classes Eff^I by vectors in $\mathbb{Q}^{r+m}_{\geq 0}$. By appropriately scaling the matrix A, we can further make it so that $\mathrm{Eff}^I \subset \mathbb{Z}^{r+m}_{\geq 0}$, as desired. This change of basis is equivalent to changing the basis of $\mathrm{Pic}(\mathfrak{X})$ to be given by line bundles corresponding to the columns of the change of basis matrix (VI.3.1). For future reference, we will refer to this new basis as the *positive basis*, and refer to the old one as the *standard basis*.

Under this new basis, for $\beta = (e_1, \dots, e_{r+m}) \in \mathbb{Z}^{r+m}_{\geq 0}$, the following changes are made to our extended I-function in Theorem V.28

- The sum becomes a sum over $\mathbb{Z}^{r+m}_{\geq 0}$
- The indices β are to be replaced with the right-hand side of (VI.3.1)
- The Novikov variable is replaced with $q^\beta := q_1^{e_1} \cdots q_r^{e_r} p_1^{e_{r+1}} \cdots p_m^{e_{r+m}}$

These are mostly indexing changes. However, we make a point of the Novikov variable change, as keeping track of the Novikov variables is crucial to understanding why this extended I-function is necessary for invertibility. In particular, we point out that we are using p_i to differentiate the Novikov variables that arise from the extra torus factors in the extension.

Regarding the Novikov variables, we also point out that for $\beta \in \mathrm{NE}(X)$, we necessarily have that $e_i = 0$ for $i > r$ by Lemma VI.1. This means that the p_i variables do not naturally show up in the J-function, and we can identify the Novikov ring as

$$\Lambda = \mathbb{C}[\![q_1, \dots, q_r]\!]$$

Letting $\tau := \sum_{i=1}^m \tau_i t_i$, we have that

$$J(\tau, q, -z) \in \mathcal{L}_X(\Lambda[\![\tau_1, \ldots, \tau_m]\!]). \tag{VI.3.2}$$

On the other hand, the "extended" variables p_i do show up in our extended I-function, so that $I(q, z)$ is an element in $H^*_{\mathrm{CR}}(X, \mathbb{Q}) \otimes \Lambda[\![p_1, \ldots, p_m]\!]\{z, z^{-1}\}$. By (VI.2.3), we have that I defines a $\Lambda[\![p_1, \ldots, p_m]\!]$ point of the Lagrangian cone

$$I(p, q, z) \in \mathcal{L}_X(\Lambda[\![p_1, \ldots, p_m]\!]). \tag{VI.3.3}$$

By Assumption V.29, we know that our fixed loci are of the form t^J_α, and so all the cohomology classes in $\mu(q, z)$ are contained in our extension set $\{t_i\}_{i=1}^m$. We can then write out $\mu(q, p, -z)$ as a sum of the form

$$\mu(q, p, -z) = \sum C_i(q, p, z) t_i \tag{VI.3.4}$$

where $C_i(q, p, z) \in \Lambda[\![p_1, \ldots, p_m, z]\!]$.

Before proving invertibility, let us first discuss what we mean by invertibility, as well as a sketch of the proof idea. Let us first assume that $\mu(q, p, -z)$ has no terms with positive powers of z. If that is the case, then all the coefficients C_i are actually elements of $\Lambda[\![p_1, \ldots, p_m]\!]$. Then we can interpret μ to be an explicit change of coordinates

$$\Lambda[\![\tau_1, \ldots, \tau_m]\!] \to \Lambda[\![p_1, \ldots, p_m]\!], \quad \tau_i \to C_i(q, p) \tag{VI.3.5}$$

and that (VI.2.3) says that the J-function, regarded as an element of $\mathcal{L}_X(\Lambda[\![p_1, \ldots, p_m]\!])$ via the above change of coordinates, agrees with the extended I-function. Now invertibility of the mirror map simply means that (VI.3.5) is an invertible morphism.

However, in general we may have positive powers of z in $\mu(q, p, -z)$, so that the change of coordinates is much more complicated. In this case, we will employ a method that has come to be known as Birkhoff Factorization. The key to Birkhoff Factorization is that one can deform the I-function within the Lagrangian cone using Theorem II.13. Through an appropriate choice of tangent vectors, we can get a new function on the cone where, up to a certain degree in the Novikov variables, we have that μ has no positive powers of z. Then, up to higher order terms, we can show that μ invokes a change of coordinates similar to (VI.3.5), and that this is invertible. By showing we can do this up to arbitrary order in the Novikov variables, we can effectively recover individual Gromov-Witten invariants, hence we have shown the "invertibility" of the mirror map. Following the idea above, we will now prove this invertibility statement in a rigorous fashion.

Theorem VI.6. *The mirror map defined in (VI.2.3) is an invertible transformation. In other words, for $\tau = \sum_{i=1}^m \tau_i t_i$, the Gromov-Witten invariants of $J(\tau, q, z)$ can be explicitly computed from the extended I-function of Theorem V.28.*

Proof. First, we will show that we can deform our I-function so that $\mu(q, p, -z)$ has no terms involving z up to any choice of degree. When discussing the asymptotics of $\mu(q, p, -z)$, we will consider the degree of the Novikov variables q_i, p_j to all be one and we will consider the degree of z to be zero. In other words, for $\beta = (e_1, \ldots, e_{r+m})$, we have

$$\deg(q^\beta) := \sum_{i=1}^{r+m} e_i, \quad \deg(z^k) = 0.$$

As a warning, this is not the degree that one usually associates to the Novikov variables in the literature, i.e. it is not the degree associated to the valuation $q^\beta \to \beta(\omega^\vee)$, nor is it the same degree conventions which we use to say the I-function is homogenous.

Note that the constant term of $\mu(q, p, -z)$ is zero, since the term of the I-function corresponding to $\beta = 0$ is simply z.

Now we proceed by induction on the degree of the terms in μ. Assume that all terms of degree $< k$ do not have a factor of z, i.e. we have that

$$\mu(q, p, -z) = \mu^{<k}(q, p) + O(\text{degree } k \text{ terms})$$

where $\mu^{<k}(q, p, -z)$ refers to the degree $< k$ terms of μ, and more generally the superscript refers to the allowed degrees in the truncation. Then we want to show that we can adjust our I-function so that $\mu^{\leq k}(q, p, -z)$ does not depend on z.

To do this, we will make use of the following asymptotical description of the derivatives of the I-function.

Lemma VI.7.
$$\frac{\partial I}{\partial p_i}(q, p, z) = t_i + O(q, p, z)$$

Proof. The linear term in p_i of the extended I-function in Theorem V.31 is given by the degree β corresponding to the $r + i$-th unit vector in the positive basis, or the $r + i$-th column vector of the matrix in (VI.3.1) in the old basis. By direct calculation, one can check that this is precisely the term $p_i t_i$ for all i. □

Since $I(q,p,-z) \in \mathcal{L}_X(\Lambda[\![p_1,\ldots,p_m]\!])$, we have by Lemma II.14 ([17, Lemma B.1]) that

$$\frac{\partial I}{\partial p_i}(q,p,-z) \in \mathcal{T}_{I(q,p,-z)}\mathcal{L}_X(\Lambda[\![p_1,\ldots,p_m]\!])$$

Then, by the properties of the Lagrangian cone, we have that

$$I(q,p,-z) + \sum_{i=1}^{m} A_i(q,p,z) z \frac{\partial I}{\partial p_i}(q,p,-z) \in \mathcal{L}_X(\Lambda[\![p_1,\ldots,p_m]\!]) \tag{VI.3.6}$$

for any choice of functions $A_i(Q,p,z) \in \Lambda[\![p_1,\ldots,p_m]\!][z]$. Now expand $\mu^k(q,p,-z)$ into terms based on cohomological factors t_i, so that we have

$$\mu^k(q,p,-z) = \sum_{i=1}^{m} B_i(q,p,z) t_i$$

where the B_i are polynomial in z. We can then set $A_i(q,p,z) = -[z^{-1}B_i]_+$ in (VI.3.6), and let $I'(q,p,-z)$ be the resulting function. Since I' is still on the Lagrangian cone, the wall-crossing statement (VI.2.3) holds for I'.

Let $\mu'(q,p,-z) = [I' - z]_+$. By the choice of A_i as well as the description of the derivatives in Lemma VI.7, we have that $(\mu')^k$ does not depend on z. Furthermore, since $\deg B_i = k$, we have that $(\mu')^{<k} = \mu^{<k}$. Hence, I' satisfies the requirements of $(\mu')^{\le k}$ not involving z.

Finally, we note that terms involving the derivatives in (VI.3.6) all have a factor of z, so that the derivatives of I' also satisfy the description given in Lemma VI.7. Thus, letting I' be our new I-function, we can inductively repeat the above to show that we can obtain an I-function where $\mu^{\le k}$ does not involve z for any choice of k.

Now fix a choice of k. We can rewrite (VI.2.3) in the following form

$$J(\mu^{\le k}, q, z) + O(\text{ degree } k+1 \text{ terms }) = I(q,p,z) + O(\text{degree } k+1 \text{ terms}).$$

By the above, we can assume that $\mu^{\le k}$ does not involve z. Then mirror map, up to degree k is given by the following ring homomorphism over Λ

$$\Lambda[\![\tau_1,\ldots,\tau_m]\!] \to \Lambda[\![p_1,\ldots,p_m]\!], \quad \tau_i \to C_i^{\le k}(q,p)$$

where $C_i^{\le k}$ is defined in terms of $\mu^{\le k}$ as in (VI.3.4).

By the inverse function theorem, we have that the above ring homomorphism is invertible if the corresponding Jacobian matrix is of full rank at $(q,p) = 0$. This is immediate from Lemma VI.7, hence the mirror map is invertible up to degree k. Thus, letting k be as large as needed, we can compute any individual Gromov-Witten invariant with insertions of the form t_i. $\square$

Remark VI.8. The assumption that the set $\{t_i\}_{i=1}^m$ be a maximally linear set is not strictly necessary in the proof of invertibility. Indeed, to show that the mirror map is invertible, one really only needs that the set $\{t_i\}_{i=1}^m$ contains all the possible cohomology classes that $[F_\beta]^{\text{vir}}$ can be. Linear independence of the elements of the set is also not necessary; for instance, one can extend by the same class twice in the GIT presentation, and the resulting I-function should still contain enough information to recover the invariants. If one has extra, redundant variables, or if linear independence is not clear, then one can replace the inverse function theorem with the more general constant rank theorem in the proof of invertibility and obtain the same result.

One thing that we want to make sure to clarify is that our I-function indeed captures all the invariants with insertions t_i. The invariants of the J-function have insertions coming from $\mu(q, p, -z)$, and we have, from Lemma VI.7 that

$$\mu(q, p, z) = \sum_{i=1}^{m} t_i p_i + O(q, z, p_i p_j)$$

hence all the possible invariants with t_i insertions will appear in our J-function. Recalling that all the ambient cohomology classes can be represented as classes of the form t_α^J, the above theorem thus recovers all the invariants a Quantum Lefshetz-type theorem would, as well as possibly more.

Lastly, we want to mention the asymptotical behavior one expects of the extended I-function and how it might lead to more complicated mirror maps. Recall that in the case of complete intersections in projective space, the complexity of the mirror map increased as the degree of the canonical bundle grew. In particular, the mirror theorem was trivial for Fano cases, yet required Birkhoff factorization for cases of general type. The same principal holds for the orbifold case, but we note that complexity of the mirror map also depends on the GIT presentation.

One way to see this is that the degree of the canonical bundle depends on the GIT presentation, as seen in Lemma V.16. In particular, we have the following:

Lemma VI.9. *Suppose that we extend the GIT presentation by the data corresponding to a Chen-Ruan cohomology class ϕ such that $\deg(\phi) > 2$. Then $\mu(q, z)$ of the corresponding extended I-function has terms with positive powers of z.*

Proof. Suppose ϕ is a cohomology class supported on the twisted sector X_α. For our purposes, we can assume that we only extend by this one class.

Since we assume that Assumption V.29 holds, we have that condition $(*)$ of Lemma V.16 holds true, and so we have that

$$\omega_{\mathfrak{X}_e}^{\vee} = \mathcal{O}(\zeta_1, \ldots, \zeta_r, \eta)$$

where ζ_i and η are defined as in Lemma V.16. Now consider the effective quasimap term $\beta = (-\alpha_1, \ldots, -\alpha_r, 1)$, where we are using the standard coordinates for β, and where $\alpha = (\alpha_1, \ldots, \alpha_r)$. Then in our I-function, we have

$$\begin{aligned}\deg(q^\beta) &= \beta(\omega_{\mathfrak{X}_e}^{\vee}) \\ &= (-\alpha_1, \ldots, -\alpha_r, 1) \cdot (\zeta_1, \ldots, \zeta_r, \eta) \\ &= 1 - \frac{\deg_{\mathrm{CR}}(\phi)}{2}\end{aligned}$$

where in the last line we use the explicit degree of η given by the assumption in Lemma V.16. From this. we see that $\deg(q^\beta) < 0$ whenever $\deg(\phi) > 2$. Moreover, we know that the fixed locus $[F_\beta]^{\mathrm{vir}} = \phi$ for such a β by Lemma VI.7. By homogeneity of the I-function, we thus see that the coefficient of $q^\beta \phi$ must have a term with a positive power of z. $\square$

As a consequence of the lemma and its proof, we see that the mirror map gets much worse when we extend by cohomology classes of higher degree. In particular, we could have a mirror map with positive powers of z even if the GIT stack quotient is Fano. When the resulting mirror map has no positive powers of z even after extension, e.g. the case of extending a Calabi-Yau threefold by classes of degree 2 only, then the Birkhoff Factorization arguments in Theorem VI.6 become unnecessary.

CHAPTER VII

Examples

In this section, we compute some examples of extensions for complete intersections in toric stack. We use these examples to illustrate how the extension works, including nuances that may not be immediately apparent, and give examples of invariants when possible.

The first three examples we give will be of Calabi-Yau threefolds inside of weighted projective spaces. As a reminder, we construct the weighted projective space via the usual GIT quotient

$$\mathbb{P}(w_0, \ldots, w_n) = [\mathbb{C}^{n+1} /\!\!/_\theta \mathbb{C}^*], \quad t \cdot (x_0, \ldots, x_n) = (t^{a_0} x_0, \ldots, t^{a_n} x_n) \text{ for } t \in \mathbb{C}^*$$

where θ is the identity morphism. Note the indexing change on the coordinates; we start with 0 as that is more conventional when working with projective spaces. We work with Calabi-Yau threefolds since they are often of more numerical interest, and the dimensions of the corresponding moduli spaces are nice. These three examples are, in order:

- $X_7 \subset \mathbb{P}(1, 1, 1, 1, 3)$
- $X_{17} \subset \mathbb{P}(2, 2, 3, 3, 7)$
- $X_{4,4} \subset \mathbb{P}(1, 1, 1, 1, 1, 3)$

The first example is the simplest, and done most in depth. The second is a hypersurface with a more complicated inertia stack. The third is a simple example of a complete intersection.

Since these are Calabi-Yau threefolds, we only need to extend by degree 2 classes. Moreover, we focus on those coming from twisted sectors, as we can use the divisor equation and string [2, Theorem 8.3.1] to deal with the untwisted classes. With these examples, we will discuss the geometry of each space, compute the extension data and the corresponding extended I-function, and then show some invariants recovered by the mirror theorem. The invariants are unravelled via a program by Yang Zhou written in Sage.

We do remark that all three of these examples will make use of a specific change of coordinates: namely, we use the positive basis obtained from the change of coordinates

(VI.3.1) where we set

$$A = \frac{1}{w}, \quad w = \mathrm{lcm}_i(w_i) \tag{VII.0.1}$$

. where the w_i are the weights of the ambient weighted projective space. This is so that we can sum over positive integers in our I-function for comfort.

Another thing we remark about these three examples is that the I-function is written in a different form compared to how it is in Theorem V.28. In the formulation of the I-function for these examples, it is possible to see divisions by the hyperplane class H show up. However, it turns out that if one formally defines

$$\frac{H^i}{H^j} = \begin{cases} H^{i-j} & i > j \\ 0 & \text{else} \end{cases} \tag{VII.0.2}$$

then the cohomology classes in the denominators will always cancel with some coming from the numerators. Moreover, the resulting cohomology class will correspond to $\widetilde{ev}_\star[F_\beta]^{\mathrm{vir}}$, hence recovers the usual formula. This form is chosen in order to make the I-function more compact and pleasing to look at.

The fourth example we will consider is the Calabi-Yau threefold $X_{24} \subset \mathbb{P}(1, 4, 4, 6, 9)$. This example is listed in order to show when one can extend by a non-ambient class. Moreover, it turns out that in order to have invertibility, it is also necessary to extend by a non-ambient class, even if one is only interested in Gromov-Witten classes where the insertions are all ambient. We explain the extension data for this example, but leave out the I-function since its computation is similar to the other three.

The fifth example is that of a Fano complete intersection in a more complicated toric stack. This example is more general than the others in terms of complexity of the ambient space, and also explores a non-Calabi-Yau complete intersection. This example is also relevant to a future non-abelian example seen in Section VIII.3, and we will focus on computations with this non-abelian example in mind.

For the first four examples, we note that Assumption V.14 is automatically satisfied by Remark V.15. For the last one, we defer the discussion to Section VIII.3 (see Lemma VIII.17).

VII.1: X_7

VII.1.1: Geometry

Let X_7 be a smooth hypersurface of degree 7 in $\mathbb{P}(1,1,1,1,3)$. Possible equations for X_7 include

$$x_0^7 + x_1^7 + x_2^7 + x_3^7 + x_3 x_4^2 = 0,$$
$$x_0^7 + x_0 x_1^6 + x_1 x_2^6 + x_2 x_3^6 + x_3 x_4^2 = 0.$$

The second equation is called an invertible polynomial of chain type as considered in [37]; the first equation is a sum of three Fermat polynomials and one of chain-type. Note that any equation for X_7 has a term $F_1(x_0, x_1, x_2, x_3)x_4^2$, where F_1 is a linear form. Furthermore, any X_7 will pass through the single orbifold point $B\mu_3 \subset \mathbb{P}(1,1,1,1,3)$ at $x_0 = x_1 = x_2 = x_3 = 0$. By the adjunction sequence,

$$0 \to T_{X_7} \to T_{\mathbb{P}(1,1,1,1,3)}|_{X_7} \to \mathcal{O}(7)|_{X_7} \to 0$$

we have $\omega_{X_7} \cong \mathcal{O}_{X_7}$, and hence X_7 is a Calabi–Yau 3-orbifold.

The inertia stack of X_7 is

$$\mathcal{I}X_7 = X_7 \sqcup B\mu_3 \sqcup B\mu_3,$$

and the rigidified inertia stack [2, Section 3] is

$$\overline{\mathcal{I}}X_7 = X_7 \sqcup \mathrm{pt} \sqcup \mathrm{pt}.$$

The following is a homogeneous basis for the ambient Chen–Ruan cohomology of X_7:

class	$\mathbf{1}$	$H \cdot \mathbf{1}$	$H^2 \cdot \mathbf{1}$	$H^3 \cdot \mathbf{1}$	$\mathbf{1}_{1/3}$	$\mathbf{1}_{2/3}$
degree	0	2	4	6	2	4

VII.1.2: I-function

The equation of X_7 is of the form

(VII.1.1) $$F_7(x_0, x_1, x_2, x_3) + x_4 F_4(x_0, x_1, x_2, x_3) + x_4^2 F_1(x_0, x_1, x_2, x_3),$$

where F_7, F_4 and F_1 are homogeneous polynomials of degree 7, 4 and 1, respectively, and F_1 is necessarily non-zero. We may then write $X_7 = [W /\!\!/_\theta \mathbb{C}^*] \subset [W/\mathbb{C}^*] = \mathfrak{X}_7$, where W is the zero locus of (VII.1.1).

The only twisted degree 2 class of X_7 is $\mathbf{1}_{1/3}$, so the extended ambient quotient will be $[\mathbb{C}^6/(\mathbb{C}^*)^2]$ with charge matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

In order to extend X_7 to $\mathfrak{X}'_7 = [W'/(\mathbb{C}^*)^2] \subset [\mathbb{C}^6/(\mathbb{C}^*)^2]$, we use the equation

$$x_5^2 F_7(x_0, x_1, x_2, x_3) + x_4 x_5 F_4(x_0, x_1, x_2, x_3) + x_4^2 F_1(x_0, x_1, x_2, x_3),$$

which has weights $(7, 2)$ with respect to $(\mathbb{C}^*)^2$.

The effective degrees $\beta = (e_0, e_1) \in \mathbb{Q}^2$ are given by $(e_0, e_1) \in (1/3)\mathbb{Z} \times \mathbb{Z}_{\geq 0}$ such that $3e_0 + e_1 \geq 0$ when using the standard basis, but under the coordinate change (VII.0.1), we can write all our degrees as $\beta = (d_0, d_1) \in \mathbb{N}^2$. The extended I-function then takes the form (VII.1.2)

$$I_{X_7} = \sum_{d_0, d_1 \geq 0} \frac{q_0^{d_0} q_1^{d_1}}{(d_1!) z^{d_1}} \frac{\prod\limits_{\substack{\langle i \rangle = \langle \frac{d_0 - d_1}{3} \rangle \\ i \leq 0}} (H + iz)^4 \prod\limits_{\substack{\langle j \rangle = \langle \frac{d_0 - d_1}{3} \rangle \\ j \leq 2d_0 + \frac{d_0 - d_1}{3}}} (7H + jz)}{\prod\limits_{\substack{\langle i \rangle = \langle \frac{d_0 - d_1}{3} \rangle \\ i \leq \frac{d_0 - d_1}{3}}} (H + iz)^4 \prod\limits_{\substack{\langle j \rangle = \langle \frac{d_0 - d_1}{3} \rangle \\ j \leq 0}} (7H + jz) \prod\limits_{k=1}^{d_0} (3H + kz)} \mathbf{1}_{\langle \frac{d_1 - d_0}{3} \rangle}$$

where we point out that we define any division $\frac{H^i}{H^j}$ as in (VII.0.2). An example of this division occurs when we consider the term correpsonding to $\beta = (0, 3)$, which is written out as

$$\frac{H^4}{7H} \mathbf{1}.$$

Under our conventions, this becomes $\frac{H^3}{7}$. On the other hand, if one were to compute $[F_\beta]^{\text{vir}}$, it would be the cycle $[B\mu_3] \subset X_7$, seen as a cycle in the untwisted sector.

It turns out that $H^3 7$ is Poincaré dula to $[B\mu_3]$.To see this, it suffices to show that $\int_{X_7} \frac{H^3}{7} = \frac{1}{3}$. Consider the map $f : \mathbb{P}(1, 1, 1, 1, 3) \to \mathbb{P}^4$ given by $(x_0, \ldots, x_4) \to (x_0^3, \ldots, x_3^3, x_4)$, which is a finite cover of degree 27. If we let h denote the hyperplane class on $\mathbb{P}^4$, then we have that $f^*(h) = 3H$. Then, by the projection formula, we have

$$\int_{X_7} \frac{H^3}{7} = \int_{\mathbb{P}(1,1,1,1,3)} H^4 = 27 \int_{\mathbb{P}^4} (h/3)^4 = \frac{1}{3}$$

hence we see that we get the correct term in the above I-function as in V.28

VII.1.3: Wall-Crossing and Invariants

We now apply (VI.2.3) to our I-function. In the Calabi–Yau threefold case, where we only extend by the degree 2 twisted classes, we can use the string, divisor, and dilaton equations in order to present Theorem VI.3 in a more explicit form:

Lemma VII.1. *If X is a Calabi–Yau threefold, we have*

$$\mu(q,z) = z(I_0(q)-1) + \mathbf{I}_1(q) + \mathbf{I}_1'(q), \tag{VII.1.3}$$

where

$$I_0 \in \mathbb{Q}[\![q]\!], \quad \mathbf{I}_1 \in H^2(X,\mathbb{Q})[\![q]\!], \quad \mathbf{I}_1' \in H^0(\mathrm{age}^{-1}(1),\mathbb{Q})[\![q]\!].$$

and equation (VI.2.3) can be rewritten in the form

$$\exp\left(-\frac{\mathbf{I}_1}{zI_0}\right)\frac{I}{I_0} = 1 + \frac{\mathbf{t}}{z} + \sum_{\substack{\gamma\in \mathrm{NE}(X), k\geq 0\\ (k,\gamma)\neq(1,0),(0,0)}} \frac{Q^\gamma}{k!}\sum_p T_p\left\langle \frac{T^p}{z(z-\psi)}, \mathbf{t},\ldots,\mathbf{t}\right\rangle^X_{0,1+k,\gamma} \quad , \tag{VII.1.4}$$

where

$$\mathbf{t} = \frac{\mathbf{I}_1'}{I_0}, \quad Q^\gamma = q^{\iota(\gamma)}\exp\left(\int_\gamma \mathbf{I}_1/I_0\right), \quad \forall \gamma\in \mathrm{NE}(X). \tag{VII.1.5}$$

This form of the wall-crossing statement is more akin to what one sees in classical I to J mirror theorems for Calabi-Yau threefolds, e.g. as in the case of the quintic threefold. The proof of this is a straightforward application of the string, divisor, and dilaton equations, hence we leave it to the reader.

Returning to our example X_7, we have that (VII.1.4) in this case looks like

(VII.1.6)
$$\frac{I}{I_0}\exp\left(-\frac{\mathbf{I}_1}{I_0}\frac{H}{z}\right) = 1 + \frac{t}{z}\phi_{1/3} + \sum_{(d_0,d_1)\neq(0,1),(0,0)} \frac{Q^{d_0}t^{d_1}}{d_1!}\sum_p T_p\left\langle \frac{T^p}{z(z-\psi)}, \phi_{1/3}^{\boxtimes d_1}\right\rangle_{0,1+d_1,d_0/3},$$

under the mirror map

$$\begin{cases} Q = q_0\exp\left(\frac{\mathbf{I}_1}{3I_0}\right) \\ t = \frac{\mathbf{I}_1'}{I_0} \end{cases} . \tag{VII.1.7}$$

Define

$$N_{d_0,d_1} = \frac{1}{d_1!}\left\langle \phi_{1/3},\ldots,\phi_{1/3}\right\rangle_{0,d_1,d_0/3}$$

and set

$$F(Q,t)=\sum_{d_0>0}\sum_{d_1\geq 0}N_{d_0,d_1}Q^{d_0}t^{d_1}+\sum_{d_1\geq 3}N_{0,d_1}t^{d_1}.$$

If we look at the $\frac{H^2}{z^2}$ coefficient of the right hand side (VII.1.6), then we have

$$\begin{aligned}
&\sum_{(d_0,d_1)\neq(0,1),(0,0)}\frac{Q^{d_0}t^{d_1}}{d_1!}\langle\frac{3}{7}H,\phi_{1/3}^{\boxtimes d_1}\rangle_{0,1+d_1,d_0/3}\\
&=\frac{1}{7}\sum_{d_0>0}\sum_{d_1\geq 0}d_0Q^{d_0}t^{d_1}N_{d_0,d_1}\\
&=\frac{1}{7}Q\frac{\partial F}{\partial Q}(Q,t).
\end{aligned}\tag{VII.1.8}$$

where we use that $\int_{X_7}H^3=\frac{7}{3}$ and that $\langle H,\phi_{1/3},\phi_{1/3}\rangle=0$. Similarly, we can look at the coefficient of $\phi_{2/3}$, and we would get

$$\begin{aligned}
&\sum_{(d_0,d_1)\neq(0,1),(0,0)}\frac{Q^{d_0}t^{d_1}}{d_1!}\langle 3\phi_{1/3},\phi_{1/3}^{\boxtimes d_1}\rangle_{0,1+d_1,d_0/3}\\
&=3\sum_{(d_0,d_1)\neq(0,1),(0,0)}(d_1+1)Q^{d_0}t^{d_1}N_{d_0,d_1+1}\\
&=3\frac{\partial F}{\partial t}(Q,t).
\end{aligned}\tag{VII.1.9}$$

We can then look at the corresponding terms of the left-hand side of (VII.1.6). From (VII.1.2), we have the following formulas

$$I_0=1+2q_0q_1+840q_0^3+6q_0^2q_1^2+15120q_0^4+O(q_0^6,q_1^6)$$

$$\mathbf{I}_1=15q_0q_1+7266q_0^3+\frac{121}{2}q_0^2q_1^2+144438q_0^4q_1+O(q_0^6,q_1^6).$$

$$\begin{aligned}
\mathbf{I}_1'=&q_1+\frac{385}{3}q_0^2+\frac{5}{9}q_0q_1^2+\frac{130900}{81}q_0^3q_1-\frac{1}{648}q_1^4+\frac{5084951872}{6075}q_0^5\\
&+\frac{220}{243}q_0^2q_1^3+O(q_0,q_1)^6
\end{aligned}$$

The mirror map gives us the change of variables

$$\begin{cases} Q = & q_0 + 5q_0^2q_1 + 2422q_0^4 + \frac{68}{3}q_0^3q_1^2 + O(q_0,q_1)^6 \\ t = & q_1 + \frac{385}{3}q_0^2 - \frac{13}{9}q_0q_2^2 + \frac{42070}{81}q_0^3q_1 - \frac{1}{648}q_1^4 + \frac{4430066872}{6075}q_0^5 - \frac{536}{243}q_0^2q_1^3 + O(q_0,q_1)^6 \end{cases}$$

After applying this change of variables, we have that the coefficients of $\frac{H^2}{z^2}$ and $\phi_{2/3}$ on the left-hand side are given by

$$\begin{aligned} \mathrm{Coeff}_{H^2/z^2} =& 4Qt + \frac{19873}{3}Q^3 - \frac{47}{9}Q^2t^2 + \frac{617288}{81}Q^4t \\ &+ \frac{1}{162}Qt^4 + O(Q,t)^6 \end{aligned}$$

$$\begin{aligned} \mathrm{Coeff}_{\phi_{2/3}} =& 84Q + \frac{1}{2}t^2 - \frac{329}{3}Q^2t + \frac{1080254}{27}Q^4 \\ &+ \frac{14}{27}Qt^3 + \frac{3094}{27}Q^3t^2 - \frac{1}{1080}t^5 + O(Q,t)^6 \end{aligned}$$

Comparing with the right-hand side of (VII.1.6) gives us the partial derivatives $\frac{\partial F}{\partial Q}(Q,t)$ and $\frac{\partial F}{\partial t}(Q,t)$, which we can then use to compute that

$$\begin{aligned} F(Q,t) =& 28Qt + \frac{139111}{9}Q^3 + \frac{1}{18}t^3 - \frac{329}{18}Q^2t^2 \\ &+ \frac{1080254}{81}Q^4t + \frac{7}{162}Qt^4 + O(Q,t)^6 \end{aligned} \tag{VII.1.10}$$

For convenience, a table of some low degree invariants is provided below:

d \ k	0	1	2	3	4	5	6
0				$\frac{1}{3}$			$-\frac{1}{27}$
1		28			$\frac{28}{27}$		
2			$-\frac{329}{9}$			$\frac{707}{243}$	
3	$\frac{139111}{9}$			$\frac{6188}{81}$			$\frac{10052}{243}$
4		$\frac{1080254}{81}$			$\frac{534751}{4374}$		
5			$-\frac{726355322}{18225}$			$\frac{1672112666}{492075}$	
6	$\frac{1533417713597}{48600}$			$\frac{5386105627}{36450}$			$\frac{12986899639}{328050}$

VII.1.4: Invariants of $[\mathbb{C}^3/\mathbb{Z}_3]$

The above computation recovers the non-equivariant Gromov–Witten theory of $[\mathbb{C}^3/\mathbb{Z}_3]$, first computed in [17]. Taking the I-function (VII.1.2), we can set $d_0 = 0$, so that we are only considering degree 0 invariants, and obtain:

$$I_{X_7}|_{d_0=0} = \sum_{d_1 \geq 0} \frac{q_1^{d_1}}{(d_1!)z^{d_1}} \frac{\prod\limits_{\substack{\langle i\rangle = \langle -d_1/3\rangle \\ -d_1/3 < i \leq 0}} (H+iz)^4}{\prod\limits_{\substack{\langle i\rangle = \langle -d_1/3\rangle \\ -d_1/3 < i \leq 0}} (7H+iz)} \mathbf{1}_{\langle \frac{d_1}{3}\rangle}$$

We can then identify the factor $\frac{H^3}{7}$ with the class of the orbifold point, allowing us to write the function in terms of the cohomology of $[\mathbb{C}^3/\mathbb{Z}_3]$, resulting in

$$\sum_{d_1 \geq 0} \frac{q_1^{d_1}}{(d_1!)z^{d_1}} \prod_{\substack{\langle i\rangle = \langle -d_1/3\rangle \\ -d_1/3 < i < 0}} (iz)^3 \mathbf{1}_{\langle \frac{d_1}{3}\rangle} \tag{VII.1.11}$$

which agrees with the twisted I-function in [17, Section 6.3] after setting the equivariant parameters, and the variables x_0 and x_2 in their I-function to zero. Note that the parameters x_0 and x_2 in their I-function are due to their extension including the identity class and the class $\mathbf{1}_{2/3}$. Via a similar extension, we could recover these variables as well.

Setting $k = \lfloor d_1 \rfloor$, the z^{-1} coefficient of (VII.1.11), agrees with their τ^1 defined via

$$\tau^1 = \sum_{k \geq 0} \frac{(-1)^{3k}(x_1)^{3k+1}}{(3k+1)!} \left(\frac{\Gamma(k+\frac{1}{3})}{\Gamma(\frac{1}{3})}\right)^3,$$

and hence the mirror maps agree. It is then straightforward to check by comparing the coefficients of $\frac{\mathbf{1}_{2/3}}{z^2}$ on both the I and J side that we recover [17, Proposition 6.4] as well, hence the invariants agree.

VII.1.5: An enumerative invariant

By (VII.1.10), the one-pointed degree $\frac{1}{3}$ invariant of X_7 with one insertion of $\phi_{1/3}$ is equal to 28. It turns out that this invariant is enumerative in the sense that it agrees with the number of orbifold lines $\mathbb{P}(1,3)$ inside a generic septic hypersurface in $\mathbb{P}(1,1,1,1,3)$.

To see this, notice that $X_7 \subset \mathbb{P}(1,1,1,1,3)$ is in general given by an equation of the form

$$F_7(x_0,x_1,x_2,x_3) + x_4 \cdot F_4(x_0,x_1,x_2,x_3) + x_4^2 F_1(x_0,x_1,x_2,x_3) = 0,$$

where F_1, F_4 and F_7 are polynomials homogeneous of degrees 1, 4 and 7, respectively. A line $\ell \cong \mathbb{P}(1,3)$ on X_7 must pass through the unique orbifold point, and is uniquely determined by its intersection $p = (p_0, p_1, p_2, p_3, 0)$ with the hypersurface $\{x_4 = 0\} \cong \mathbb{P}^3$. In order for p to lie on X_7, we need that $F_7(p_0, p_1, p_2, p_3) = 0$. In addition, for all of ℓ to lie on X_7, we also need to require that $F_4(p_0, p_1, p_2, p_3) = F_1(p_0, p_1, p_2, p_3) = 0$. This defines a set of $7 \cdot 4 \cdot 1$ points on $\{x_4 = 0\}$, and hence there are 28 possible choices for ℓ.

This appears to be the only enumerative or even integral invariant of X_7. In the case of Calabi–Yau 3-manifold, the BPS invariants are linear combinations of Gromov–Witten invariants that are conjectured to be integers.

Question VII.2. Is there an analog of BPS invariants for Calabi–Yau 3-orbifolds like X_7?

VII.2: X_{17} in $\mathbb{P}(2,2,3,3,7)$

VII.2.1: Geometry

Let X_{17} be a smooth hypersurface of degree 17 in $\mathbb{P}(2,2,3,3,7)$. A possible equation for X_{17} is

$$x_0 x_2^5 + x_2 x_4^2 + x_4 x_1^5 + x_1 x_3^5 + x_3 x_0^7.$$

This is an equation of loop-type. In fact, there are no equations for X_{17} of Fermat- or chain-type (or combinations thereof) since 17 is divisible by neither 2, 3 nor 7.

From the adjunction sequence, we again observe that X_{17} is a Calabi–Yau 3-orbifold. The stacky loci of the ambient $\mathbb{P}(2,2,3,3,7)$ are a $B\mu_7$-point at $x_0 = x_1 = x_2 = x_3 = 0$, a gerby projective line $\mathbb{P}(2,2)$ at $x_2 = x_3 = x_4 = 0$, and a gerby projective line $\mathbb{P}(3,3)$ at $x_0 = x_1 = x_4 = 0$. We may check that any X_{17} must pass through all of these stacky loci, and in order to be nonsingular at each of these, the equation of X_{17} must contain a term linear in x_2, x_3 and quadratic in x_4, a term quintic in x_0, x_1 and linear in x_4, and a term linear in x_0, x_1 and quintic in x_2, x_3.

The inertia stack of X_{17} is

$$\mathcal{I}X_{17} = X_{17} \sqcup \bigsqcup_{i=1}^{6} B\mu_7 \sqcup \mathbb{P}(2,2) \sqcup \bigsqcup_{i=1}^{2} \mathbb{P}(3,3),$$

and the rigidified inertia stack is

$$\overline{\mathcal{I}}X_{17} = X_{17} \sqcup \bigsqcup_{i=1}^{6} \mathrm{pt} \sqcup \mathbb{P}^1 \sqcup \bigsqcup_{i=1}^{2} \mathbb{P}^1.$$

The ambient Chen–Ruan cohomology of X_{17} has homogeneous basis:

class	$\mathbf{1}$	$H \cdot \mathbf{1}$	$H^2 \cdot \mathbf{1}$	$H^3 \cdot \mathbf{1}$	$\mathbf{1}_{1/7}$	$\mathbf{1}_{2/7}$	$\mathbf{1}_{3/7}$	$\mathbf{1}_{4/7}$	$\mathbf{1}_{5/7}$
degree	0	2	4	6	2	4	4	2	2

class	$\mathbf{1}_{6/7}$	$\mathbf{1}_{1/2}$	$H \cdot \mathbf{1}_{1/2}$	$\mathbf{1}_{1/3}$	$H \cdot \mathbf{1}_{1/3}$	$\mathbf{1}_{2/3}$	$H \cdot \mathbf{1}_{2/3}$
degree	4	2	4	2	4	2	4

VII.2.2: I-function

We extend the ambient quotient stack to be of the form $[\mathbb{C}^{11}/(\mathbb{C}^*)^7]$ with charge matrix given by

$$\begin{pmatrix} 2 & 2 & 3 & 3 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 4 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 2 & 5 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 2 & 4 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The 6 extra rows correspond (in order) to the twisted divisors $\mathbf{1}_{1/7}$, $\mathbf{1}_{4/7}$, $\mathbf{1}_{5/7}$, $\mathbf{1}_{1/2}$, $\mathbf{1}_{1/3}$, $\mathbf{1}_{2/3}$. Furthermore, we can extend the equation for X_{17} to an equation of weight $(17, 2, 9, 12, 8, 5, 11)$.

Let $\beta = (d_0, d_1, \ldots, d_6) \in \mathbb{Z}^7_{\geq 0}$, where we use the basis change (VII.0.1). We make the following definitions

- $\rho_1 = \frac{d_0 - 6d_1 - 3d_2 - 9d_3 - 14d_5 - 7d_6}{21}$
- $\rho_2 = \frac{d_0 - 6d_1 - 10d_2 - 2d_3 - 7d_4}{14}$
- $\rho_3 = \frac{d_0 - 3d_4 - 2d_5 - 4d_6}{6}$
- $\rho_4 = \frac{17d_0}{42} - \frac{3d_1}{7} - \frac{5d_2}{7} - \frac{d_3}{7} - \frac{d_4}{2} - \frac{2d_5}{3} - \frac{d_6}{3}$
- $\sigma = \frac{d_0 - 6d_1 - 24d_2 - 30d_3 - 21d_4 - 14d_5 - 28d_6}{42}$

Then we have the extended I-function is given by

$$\text{(VII.2.1)} \quad I_{X_{17}} = \sum_{d_i \in \mathbb{Z}_{\geq 0}} \frac{\prod_{i=0}^{6} q_i^{d_i}}{\prod_{i=1}^{6}(d_i)! z^{d_i}} \times$$
$$\frac{\prod_{\substack{\langle j \rangle = \langle \rho_1 \rangle \\ j \leq 0}} (2H+jz)^2 \prod_{\substack{\langle j \rangle = \langle \rho_2 \rangle \\ j \leq 0}} (3H+jz)^2 \prod_{\substack{\langle j \rangle = \langle \rho_3 \rangle \\ j \leq 0}} (7H+jz) \prod_{\substack{\langle j \rangle = \langle \rho_4 \rangle \\ j \leq \rho_4}} (17H+jz)}{\prod_{\substack{\langle j \rangle = \langle \rho_1 \rangle \\ j \leq \rho_1}} (2H+jz)^2 \prod_{\substack{\langle j \rangle = \langle \rho_2 \rangle \\ j \leq \rho_2}} (3H+jz)^2 \prod_{\substack{\langle j \rangle = \langle \rho_3 \rangle \\ j \leq \rho_3}} (7H+jz) \prod_{\substack{\langle j \rangle = \langle \rho_4 \rangle \\ j \leq 0}} (17H+jz)} \cdot \mathbf{1}_{\langle -\sigma \rangle}$$

where again we use the conventions of (VII.0.2).

VII.2.3: Wall-Crossing and Invariants

We have that (VII.1.4) in this case looks like

(VII.2.2)

$$\frac{I}{I_0}\exp\left(-\frac{H}{z}\frac{\mathbf{I}_1}{I_0}\right) = 1 + \frac{\sum t_i\phi_i}{z} + \sum_{(r,d)\neq(1,0),(0,0)} \frac{Q^d}{r!}\sum_p T_p \langle \frac{T^p}{z(z-\psi)}, (\sum t_i\phi_i)^{\boxtimes r}\rangle_{0,1+r,d/42}$$

where we set

$$(\phi_1, \ldots, \phi_6) = (\mathbf{1}_{1/7}, \mathbf{1}_{4/7}, \mathbf{1}_{5/7}, \mathbf{1}_{1/2}, \mathbf{1}_{1/3}, \mathbf{1}_{2/3})$$

The mirror map is given by

$$Q = q_0 \exp(\mathbf{I}_1/42 I_0) \quad t_i = \frac{I'_{1i}}{I_0}$$

where

$$\mathbf{I}'_1 = \sum_{i=1}^{6} I'_{1i\phi_i}$$

We can define the generating function

$$F(Q, t_1, \ldots, t_6) = \sum_{d>0}\sum_{r=0}^{\infty} \frac{Q^d}{r!}\langle(\sum t_i\phi_i)^{\boxtimes r}\rangle_{0,r,d/42}$$
$$+ \sum_{r=3}^{\infty} \frac{1}{r!}\langle(\sum t_i\phi_i)^{\boxtimes r}\rangle_{0,r,d/42}.$$

and we can again look at coefficients of cohomology classes on both sides to obtain explicit formulas for the partial derivatives of F. The following table lists the coefficients for each cohomology class on the right hand side

base element	coefficient
H^2	$\frac{6}{17}Q\frac{\partial}{\partial Q}F + \frac{63}{34}t_4^2 + \frac{28}{17}t_5t_6$
$H_{1/2}$	$2\frac{\partial}{\partial t_4}F$
$H_{1/3}$	$3\frac{\partial}{\partial t_6}F$
$H_{2/3}$	$3\frac{\partial}{\partial t_5}F$
$\mathbf{1}_{2/7}$	$7\frac{\partial}{\partial t_3}F$
$\mathbf{1}_{3/7}$	$7\frac{\partial}{\partial t_2}F$
$\mathbf{1}_{6/7}$	$7\frac{\partial}{\partial t_1}F$
H^3	$\frac{252}{17}(\sum_i t_i\frac{\partial}{\partial t_i} - 2)F$

Comparing with the left-hand side of (VII.2.2), we obtain the following explicit computation of F

$$
\begin{aligned}
&F(Q, t_1, \ldots, t_6) \\
=&\frac{1}{14}t_1^2t_3 + \frac{1}{14}t_2t_3^2 - \frac{13}{54}t_5^3 + \frac{7}{54}t_6^3 + Q^2t_3t_5 - \frac{1}{147}t_1^3t_2 - \frac{1}{98}t_1t_2^2t_3 + \frac{1}{48}t_4^4 \\
&+ \frac{1}{18}t_5^2t_6^2 + 5Q^3t_2t_4 + \frac{1}{7}Q^2t_1t_2t_5 + \frac{1}{6}Q^2t_3t_6^2 + \frac{1}{686}t_1^2t_2^3 + \frac{3}{2744}t_1t_3^4 \\
&+ \frac{1}{8232}t_2^4t_3 - \frac{1}{324}t_5^4t_6 - \frac{1}{324}t_5t_6^4 + \frac{1}{42}Q^2t_1t_2t_6^2 + \frac{1}{294}Q^2t_2^3t_5 \\
&- \frac{1}{18}Q^2t_3t_5^2t_6 - \frac{43}{115248}t_1^4t_3^2 - \frac{31}{28812}t_1^2t_2t_3^3 - \frac{11}{144060}t_1t_2^5 - \frac{5}{28812}t_2^2t_3^4 \\
&+ \frac{1}{2880}t_4^6 + \frac{1}{9720}t_5^6 + \frac{1}{486}t_5^3t_6^3 + \frac{1}{9720}t_6^6 + \frac{85}{6}Q^6t_1 - \frac{1}{4}Q^4t_3^2t_6 - \frac{5}{98}Q^3t_1t_3^2t_4 \\
&- \frac{5}{24}Q^3t_2t_4^3 + \frac{1}{196}Q^2t_1^2t_3^2t_5 - \frac{1}{126}Q^2t_1t_2t_5^2t_6 + \frac{1}{1764}Q^2t_2^3t_6^2 + \frac{5}{2058}Q^2t_2t_3^3t_5 \\
&+ \frac{1}{648}Q^2t_3t_5^4 - \frac{1}{162}Q^2t_3t_5t_6^3 + \frac{37}{6050520}t_1^7 + \frac{311}{2016840}t_1^5t_2t_3 + \frac{69}{134456}t_1^3t_2^2t_3^2 \\
&+ \frac{3}{16807}t_1t_2^3t_3^3 + \frac{11}{6050520}t_2^7 + \frac{2}{252105}t_3^7 - \frac{1}{3240}t_5^5t_6^2 - \frac{1}{3240}t_5^2t_6^5 \\
&+ O(Q, t_1, t_2, t_3, t_4, t_5, t_6)^8
\end{aligned}
$$

Unfortunately, the computational power to compute even the first invariant without markings, the coefficient of Q^{42}, is too high.

Similarly to the discussion in Section VII.1.5, the integer invariants $\langle \phi_3, \phi_5 \rangle_{0,2,1/21} = 1$ and $\langle \phi_2, \phi_4 \rangle_{0,2,1/14} = 5$ are enumerative.

More explicitly, the first invariant is the number of weighted projective lines $\mathbb{P}(3,7)$ between the gerby line $\mathbb{P}(3,3)$ and the $B\mu_7$-point. Such lines are contained in the weighted projective plane $\mathbb{P}(3,3,7)$ defined via $x_0 = x_1 = 0$. In this plane, the equation for X_{17} becomes of the form $x_4^2 \cdot F_1(x_2, x_3)$, where F_1 is a linear form. Hence, the unique line lying inside X_{17} is the one passing through the point in $\mathbb{P}(3,3)$ where F_1 vanishes.

The second invariant is the number of weighted projective lines $\mathbb{P}(2,7)$ between the gerby line $\mathbb{P}(2,2)$ and the $B\mu_7$-point. In the case, the number 5 appears because in the weighted projective plane $\mathbb{P}(2,2,7)$ given by $x_2 = x_3 = 0$, the equation for X_{17} becomes of the form $x_4 \cdot F_5(x_0, x_1)$, where F_5 is a homogeneous quintic polynomial.

VII.3: $X_{4,4}$ in $\mathbb{P}(1,1,1,1,1,3)$

VII.3.1: Geometry

Let $X_{4,4}$ be a smooth complete intersection of two degree 4 hypersurfaces in $\mathbb{P}(1,1,1,1,1,3)$. It is not straightforward to write down concrete equations for such a smooth complete intersection, though by a Bertini argument, a generic complete intersection of that type is smooth. The complete intersection $X_{4,4}$ must pass through the unique orbifold point $B\mu_3$ of $\mathbb{P}(1,1,1,1,1,3)$ at $x_0 = x_1 = x_2 = x_3 = x_4 = x_5 = 0$. To ensure smoothness at the orbifold point, the equations for $X_{4,4}$ take the form

$$\begin{aligned} F_{4a}(x_0, \ldots, x_4) + x_5 F_{1a}(x_0, \ldots, x_4) &= 0 \\ F_{4b}(x_0, \ldots, x_4) + x_5 F_{1b}(x_0, \ldots, x_4) &= 0 \end{aligned}$$

for quartic polynomials F_{4a} and F_{4b}, and for linear forms F_{1a} and F_{1b} whose matrix of coefficients has full rank 2. By the adjunction sequence, we see that $X_{4,4}$ is another example of a Calabi–Yau 3-orbifold.

The inertia stack of $X_{4,4}$ is

$$\mathcal{I}X_{4,4} = X_{4,4} \sqcup B\mu_3 \sqcup B\mu_3,$$

and the rigidified inertia stack is

$$\overline{\mathcal{I}}X_{4,4} = X_{4,4} \sqcup \mathrm{pt} \sqcup \mathrm{pt}.$$

The following is a homogeneous basis for the ambient Chen–Ruan cohomology of $X_{4,4}$:

class	$\mathbf{1}$	$H \cdot \mathbf{1}$	$H^2 \cdot \mathbf{1}$	$H^3 \cdot \mathbf{1}$	$\mathbf{1}_{1/3}$	$\mathbf{1}_{2/3}$
degree	0	2	4	6	2	4

VII.3.2: I-function

For $X_{4,4}$, we extend the quotient in the same way as in X_7, that is as $[\mathbb{C}^6/(\mathbb{C}^*)^2]$ with the same charge matrix as above. The equations of the complete intersection are extended to two polynomials of weight $(4,1)$.

The effective cone also agrees with that of X_7, so we will use $\beta = (d_0, d_1) \in \mathbb{Z}^2_{\geq 0}$. The extended I-function is as follows

(VII.3.1)

$$I_{X_{4,4}} = \sum_{d_0,d_1\geq 0} \frac{q_0^{d_0} q_1^{d_1}}{z^{d_1} d_1!} \frac{\prod\limits_{\substack{\langle i\rangle = \langle \frac{d_0-d_1}{3}\rangle \\ i\leq \frac{4d_0-d_1}{3}}} (4H+iz)^2 \prod\limits_{\substack{\langle j\rangle = \langle \frac{d_0-d_1}{3}\rangle \\ j\leq 0}} (H+jz)^5}{\prod\limits_{\substack{\langle i\rangle = \langle \frac{d_0-d_1}{3}\rangle \\ i\leq 0}} (4H+iz)^2 \prod\limits_{\substack{\langle j\rangle = \langle \frac{d_0-d_1}{3}\rangle \\ j\leq d}} (H+jz)^5 \prod_{k=1}^{d_0}(3H+kz)} \cdot \mathbf{1}_{\langle -\frac{d_0-d_1}{3}\rangle}$$

VII.3.3: Wall-Crossing and Invariants

The mirror formula reads the exact same as it does in the X_7 case, with the same mirror map as before. The only change here is the I-function leading to different invariants. Since the computation is otherwise the same, we will just give a partial formula for $F(Q,t)$ and a table of some invariants for this case

$$\begin{aligned} F(Q,t) =& 16Qt + \frac{20800}{9}Q^3 + \frac{1}{18}t^3 - \frac{46}{9}Q^2t^2 + \frac{46490}{81}Q^4t + \frac{2}{81}Qt^4 + \frac{2329313056}{6075}Q^6 \\ &+ \frac{304}{243}Q^3t^3 - \frac{1}{19440}t^6 - \frac{9256192}{18225}Q^5t^2 + \frac{77}{7290}Q^2t^5 + \frac{1704994246016}{8037225}Q^7t \\ &+ \frac{1391}{13122}Q^4t^4 - \frac{29}{229635}Qt^7 + \frac{1690784332712}{10935}Q^9 + \frac{17945392}{54675}Q^6t^3 \\ &+ \frac{122}{10935}Q^3t^6 + \frac{1}{3265920}t^9 + O(Q,t)^{10} \end{aligned}$$

Note that when setting $Q = 0$, we get

$$\begin{aligned} F(0,t) =& \frac{1}{18}t^3 - \frac{1}{19440}t^6 + \frac{1}{3265920}t^9 - \frac{1093}{349192166400}t^{12} + \frac{119401}{2859883842816000}t^{15} \\ &- \frac{27428707}{420059738832814080000}t^{18} + O(t^{20}). \end{aligned}$$

which matches the part of X_7 not involving Q, and hence can also be used to derive the invariants of $[\mathbb{C}^3/\mathbb{Z}^3]$.

The invariant $\langle \mathbf{1}_{1/3}\rangle_{0,1,1/3} = 16$ enumerates the number of orbifold lines $\mathbb{P}(1,3)$ in a generic $X_{4,4}$. This can be derived in the same way as in the X_7 case, and 16 arises as $(4\cdot 1)^2$.

VII.4: X_{24}

This example is meant to highlight how it is possible to keep track of more than just the ambient classes of the target, as well as how this may be necessary in order to have invertibility of the mirror theorem.

VII.4.1: Geometry

Let X_{24} be a smooth degree 24 hypersurface in $\mathbb{P}(1,4,4,6,9)$. One can check as before that this is also an example of a Calabi-Yau threefold. We also note that any degree 24 polynomial $F(x_0,\ldots,x_4)$ cannot have a term solely composed of the degree 9 variable x_4, hence we have that X_{24} must contain the $B\mu_9$ stacky point in $\mathbb{P}(1,4,4,6,9)$.

The inertia stack of $\mathbb{P}(1,4,4,6,9)$ can be described as having sectors $\mathbb{P}_\alpha$, indexed by $\alpha \in (0,1] \in \mathbb{Q}$ such that the denominator of α is either 4, 6, or 9. We can describe the $\mathbb{P}_\alpha$ as isomorphic to linear substacks of the weighted projective space

$$\mathbb{P}_\alpha \cong V(\{x_i | \ i\cdot\alpha \notin \mathbb{Z}) \subset \mathbb{P}(1,4,4,6,9).$$

Similarly, we have a similar decomposition for the inertia stack of X_{24} into sectors X_α. As stacks, we have that

$$X_\alpha \cong X_{24} \cap \mathbb{P}_\alpha$$

where we view $\mathbb{P}_\alpha$ as a substack of $\mathbb{P}(1,4,4,6,9)$.

However, we note that the inertia stack $\mathcal{I}X_{24}$ has more connected components than that of the inertia stack of the ambient projective space. Indeed, we can look at the component

$$X_{1/3} \subset \mathbb{P}_{1/3} \cong \mathbb{P}(6,9)$$

where we have that $X_{1/3}$ is isomorphic to the vanishing of the defining polynomial $F(x_0,\ldots,x_4)$ when restricted to $\mathbb{P}(6,9)$. However, since we know that $B\mu_9 \subset V(F)$, we see that $X_{1/3} \neq \emptyset$. Moreover, the generic degree 24 polynomial F on $\mathbb{P}(6,9)$ looks like

$$F|_{\mathbb{P}(6,9)}(x,y) = a_1x^4 + a_2xy^2, \quad a_1,a_2 \in \mathbb{C}$$

where x and y are of degree 6 and 9 respectively. As a result, we expect $X_{1/3}$ to look like disjoint union of a $B\mu_9$ point and five $B\mu_3$ points.

From the above, we see that the Chen-Ruan cohomology of X_{24} is not generated by the ambient cohomology coming from the weighted projective space. Indeed, the cohomology of $X_{1/3}$ is already 7-dimensional, and the pullback of $H \cdot \mathbf{1}_{1/3}$ from $\mathbb{P}_{1/3}$ vanishes due to dimension reasons.

Denote by H the pullback of the hyperplane class to X_{24}, $\mathbf{1}_\alpha$ to be the unit class of X_α, and $\phi_{1/3}$ the Poincaré dual of the $[B\mu_9]$ point in $X_{1/3}$. From the above discussion, we know that $\phi_{1/3}$ is a non-ambient class. However, we have that

$$[B\mu_9] = V(x_3) \cap X_{1/3}$$

where we view the right-hand side as substacks of X_{24}. In particular, this implies that $\phi_{1/3}$ is a cohomology class that we can extend by, despite being primitive.

VII.4.2: Extension Data

Since X is a Calabi-Yau threefold, we will restrict ourselves to extending by the degree twisted 2 classes. Note that we do not need to worry about the degree 2 classes coming from the untwisted sector, as we can use the divisor equation [2, Theorem 8.3.1].

A basis of the degree 2 ambient twisted classes consist of the following list

$$\mathbf{1}_{1/4},\ \mathbf{1}_{2/4},\ \mathbf{1}_{1/9},\ \mathbf{1}_{1/3},\ \mathbf{1}_{5/9},\ \mathbf{1}_{7/9}. \tag{VII.4.1}$$

However, since the twisted sector $X_{1/3}$ has age 2, all the cohomology classes in $H^0(X_{1/3})$ correspond to degree 2 Chen-Ruan classes. In particular, this implies that $\phi_{1/3}$ is a degree 2 class, in addition to the Poincaré dual of the other connected components of $X_{1/3}$.

Now suppose we wanted to just focus on computing the invariants whose insertions are the ambient cohomology classes. Then we can attempt to extend by only those classes and ignore the non-ambient ones coming from $X_{1/3}$. The resulting weight matrix will have the submatrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 2 \\ 0 & 2 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 2 & 2 & 3 & 5 \\ 0 & 3 & 3 & 4 & 7 \end{pmatrix}$$

where A is as in V.1.3. The rows of A correspond, in order, to the extensions by the cohomology classes of (VII.4.1). The bundle $\mathcal{O}(24)$ on the ambient space is extended to become the line bundle $\mathcal{O}(24, 6, 12, 2, 8, 13, 18)$, and the section F is homogenized appropriately.

A quasimap degree is then given by $\beta \in \mathrm{Hom}(\chi((\mathbb{C}^*)^7), \mathbb{Q}) \cong \mathbb{Q}^7$. Consider the quasimap degree

$$\beta = (-1/3, 0, 0, 3, 0, 0, 0).$$

It is straightfoward to check that this effective, since β pairs as an integer with the columns related to the weights 6 and 9 with at least one of them being non-negative. Since $\langle -1/3 \rangle = 2/3$, we have that $[F_\beta]$ is a cycle in $X_{2/3}$. However, note that β pairs as a negative integer with the column corresponding to the weight 6, while it is non-negative when paired with the column coming from 9. Therefore, by Proposition V.25, we have that

$$[F_\beta]^{\mathrm{vir}} = (V(x_3) \cap X_{2/3}$$

hence

$$\widetilde{ev}_\star[F_\beta]^{\mathrm{vir}} = \phi_{1/3}.$$

By explicit computation of the I-function and the mirror map μ associated to it, we see that there is a term involving $\phi_{1/3}$. Consequently, even though we originally only cared about the ambient insertions, we see that we still need to track a non-ambient class, namely $\phi_{1/3}$, in order to hope for any invertibility statement.

Therefore, if one wants an invertible mirror theorem, one should use the extension

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 2 \\ 0 & 2 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 2 & 2 & 3 & 5 \\ 0 & 3 & 3 & 4 & 7 \\ 0 & 1 & 1 & 1 & 3 \end{pmatrix} \tag{VII.4.2}$$

where the last row corresponds to the extension given by $\phi_{1/3}$. The line bundle extends to the bundle $\mathcal{O}(24, 6, 12, 2, 8, 13, 18, 7)$ in this case, and the polynomial F is homogenized accordingly. From here, one can follow as in the previous examples to compute the I-function, and as indicated in the above discussion, one can check that invertibility holds with this appropriate extension.

Remark VII.3. We remark that we cannot recover all the primitive classes with our methods. Indeed, the other six primitive classes corresponding to the duals of the $B\mu_3$ points in $X_{1/3}$ are not of the form t^J_α. Intuitively, there is nothing special that distinguishes these points from one another, as opposed to the $B\mu_9$ point, hence there is some added difficulty in trying to parameterize each individually. Further techniques likely need to be developed in order to find a way to distinguish each of these classes, which we hope to explore in the future.

VII.5: Fano Complete Intersection in Toric Stack

VII.5.1: Geometry

We now consider an example of a complete intersection inside of a more complicated toric stack. This example will be related to the weighted Grassmannian of Section VIII.3. We will consider the following GIT data:

- $V := \mathbb{C}^{10}$ is the space of 2×5 matrices M over $\mathbb{C}$ with coordinates $\{m_{ij}\}$.
- $T = (\mathbb{C}^*)^2$ is a torus action on V with weight matrix

$$\text{(VII.5.1)} \qquad \begin{array}{cccccccccc} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} & m_{21} & m_{22} & m_{23} & m_{24} & m_{25} \end{array}$$
$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \end{pmatrix}$$

- $\theta : T \to \mathbb{C}^*$ is the character $\theta(t_1, t_2) = t_1 t_2$.

The θ-invariant polynomials in this case are generated by $\{m_{1i}m_{2j}\}_{1\leq i,j\leq 5}$. The union of the non-vanishing locus of these polynomials gives the semi-stable locus, hence we have

$$\text{(VII.5.2)} \qquad V^{\text{ss}}_\theta(T) = \{M \in V \mid M \text{ does not have a zero row.}\}$$

One can check that the stable locus also agrees with this. From this data, we can form the resulting stacks

$$Y = [V/\!\!/_\theta T], \quad \mathfrak{Y} = [V/T]$$

Now consider the split vector bundle $E = \mathcal{O}(2,2)^{\oplus 4} \to Y$. We want to consider a smooth complete intersection cut out by a generic section s of E. We note that the sections of E are generated by the following list of polynomials on V

$$\text{(VII.5.3)} \qquad \{m_{1i}m_{24}, m_{1i}m_{25}, m_{14}m_{2i}, m_{15}m_{2i}, (m_{1i}m_{2j})^2\}_{1\leq i,j\leq 3}$$

Picking four polynomials $F_1, \dots, F_4$ of degree $(2n, 2n)$ such that the Jacobian matrix of the F_j is full rank at every point of $V^{\text{ss}}_\theta(T)$, we define $W := V(F_1, \dots, F_4)$. This gives us the following smooth complete intersection stacks,

$$X := [W/\!\!/_\theta T], \quad \mathfrak{X} = [W/T]$$

where X is a Fano complete intersection in Y with $\omega_X^\vee = \mathcal{O}(1,1)$.

The inertia stacks of Y and X can be understood by finding the elements of T which have non-trivial fixed loci, which are $(-1,1)$, $(1,-1)$, and (ξ^i, ξ^i), where $\xi = e^{2\pi i/3}$ is a third root of unity. We thus have the decompositions

$$IY = Y \sqcup Y_{0,\frac{1}{2}} \sqcup Y_{\frac{1}{2},0} \sqcup Y_{\frac{1}{3},\frac{1}{3}} \sqcup Y_{\frac{2}{3},\frac{2}{3}}$$

$$IX = X \sqcup X_{0,\frac{1}{2}} \sqcup X_{\frac{1}{2},0} \sqcup X_{\frac{1}{3},\frac{1}{3}} \sqcup X_{\frac{2}{3},\frac{2}{3}}$$

where

- $Y_{\frac{1}{2},0}$ is the substack given by the vanishing locus $V(m_{11}, m_{12}, m_{13}, m_{24}, m_{25})$
- $Y_{0,\frac{1}{2}}$ is the substack given by the vanishing locus $V(m_{14}, m_{15}, m_{21}, m_{22}, m_{23})$
- $Y_{\frac{1}{3},\frac{1}{3}} = Y_{\frac{2}{3},\frac{2}{3}}$ is the substack given by the vanishing locus $V(m_{11}, m_{12}, m_{13}, m_{21}, m_{22}, m_{23})$
- $X_\alpha = Y_\alpha \cap X$ as a substack in Y

In particular, we note that the fixed loci $Y^{\frac{1}{3},\frac{1}{3}} = Y^{\frac{2}{3},\frac{2}{3}}$ is entirely contained in the base locus of E, as can be seen by (VII.5.3) and the vanishing locus given above. Therefore, we have

$$X_{\frac{i}{3},\frac{i}{3}} \cong Y_{\frac{i}{3},\frac{i}{3}} \qquad \text{for } i = 1, 2$$

Let $H_1 = c_1(\mathcal{O}(1,0))$ and $H_2 = c_1(\mathcal{O}(0,1))$. The following classes generate the ambient Chen–Ruan cohomology ring of X:

class	$\mathbf{1}$	H_1	H_2	$\mathbf{1}_{1/2,0}$	$\mathbf{1}_{0,1/2}$	$\mathbf{1}_{1/3,1/3}$	$\mathbf{1}_{2/3,2/3}$
degree	0	2	2	5	5	4/3	8/3

where the ages can be computed via the usual combination of the Euler sequence and conormal sequence.

VII.5.2: Extended I-function

We now extend the GIT presentation to keep track of the class $\mathbf{1}_{1/3,1/3}$. While we can also extend to keep track of the other twisted classes, we will only need to extend by this one class when we consider the quantum period of a non-abelian example VIII.3.

The extended GIT presentation is given by

$$X = [W_e /\!\!/_{\theta_e} (\mathbb{C}^*)^3] \subset [W_e/(\mathbb{C}^*)^3] = \mathfrak{X}_e$$

where $(\mathbb{C}^*)^3$ acts with weight matrix

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \tag{VII.5.4}$$

the extended character θ_e is given by

$$\theta_e : (\mathbb{C}^*)^3 \to \mathbb{C}^*, \qquad \theta_e(t_1, t_2, s) = t_1 t_2 s$$

and the affine scheme W_e is cut out by a lift of a generic section s_e of the bundle

$$\mathcal{E} := \mathcal{O}(2,2,1)^{\oplus 4} \to \mathfrak{X}_e$$

such that W_e is smooth.

Let $\psi_{\bullet i}$ refer to the i-th column of the weight matrix (VII.5.4). From the description of a quasimap as sections of line bundles (see Proposition V.17), we can compute the effective cone of the extended GIT presentation by looking at what positivity conditions we require of $\beta \cdot \psi_{\bullet i}$ so that the image is in $(W_e)^{\mathrm{ss}}_{\theta_e} = W^{\mathrm{ss}} \times \mathbb{C}^*$.

Let $\beta = (d_1, d_2, d_3) \in \mathrm{Hom}(\mathrm{Pic}([V \times \mathbb{C}/(\mathbb{C}^*)^3]), \mathbb{Q}) \cong \mathbb{Q}^3$, where we use the usual basis corresponding to the projections $\mathrm{pr}_i : (\mathbb{C}^*)^3 \to \mathbb{C}^*$. From the description of the semi-stable locus (VII.5.2), or by looking at which S_β form an anticone, we can directly compute that the I-effective classes β are all contained in the set given by the following conditions

- $(\langle d_1 \rangle, \langle d_2 \rangle)$ is either $(0,0)$, $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$, or $(\frac{i}{3}, \frac{i}{3})$ for $i = 1, 2$.
- $d_3 \in \mathbb{Z}_{\geq 0}$
- $\begin{cases} d_1 \geq 0 \text{ and } d_1 + 2d_2 + d_3 \geq 0 & \text{if } (\langle d_1 \rangle, \langle d_2 \rangle) = (0, \frac{1}{2}) \\ d_2 \geq 0 \text{ and } 2d_1 + d_2 + d_3 \geq 0 & \text{if } (\langle d_1 \rangle, \langle d_2 \rangle) = (\frac{1}{2}, 0) \\ 2d_1 + d_2 + d_3 \geq 0 \text{ and } d_1 + 2d_2 + d_3 \geq 0 & \text{otherwise} \end{cases}$

We will denote this set by Eff^I, and this will be what we sum over in our I-function. We remark that this is the I-effective set for $\mathfrak{Y}_e$, which is possibly larger than the I-effective set for $\mathfrak{X}_e$, but any extra terms will contribute zero so it doesn't change the I-function.

With this, we have that the extended I-function of X is given by

$$I^X = \sum_{\beta=(d_1,d_2,d_3)\in \mathrm{Eff}^I} \frac{q^\beta}{d_3! z^{d_3}} \frac{\prod\limits_{\substack{\langle k\rangle=\langle d_1\rangle \\ d_1<k<0}} (H_1+kz)^3}{\prod\limits_{\substack{\langle k\rangle=\langle d_1\rangle \\ 0<k\leq d_1}} (H_1+kz)^3} \frac{\prod\limits_{\substack{\langle k\rangle=\langle d_2\rangle \\ d_2<k<0}} (H_2+kz)^3}{\prod\limits_{\substack{\langle k\rangle=\langle d_2\rangle \\ 0<k\leq d_2}} (H_2+kz)^3}$$
$$\times \frac{1}{\prod\limits_{\substack{\langle k\rangle=\langle 2d_1+d_2+d_3\rangle \\ 0<k\leq 2d_1+d_2+d_3}} (2H_1+H_2+kz)^2} \frac{1}{\prod\limits_{\substack{\langle k\rangle=\langle d_1+2d_2+d_3\rangle \\ 0<k\leq d_1+2d_2+d_3}} (H_1+2H_2+kz)^2}$$
$$\times \frac{\prod\limits_{\substack{\langle k\rangle=\langle 2d_1+2d_2+d_3\rangle \\ 0<k\leq 2d_1+2d_2+d_3}} (2H_1+2H_2+kz)^4}{\prod\limits_{\substack{\langle k\rangle=\langle 2d_1+2d_2+d_3\rangle \\ 2d_1+2d_2+d_3<k<0}} (2H_1+2H_2+kz)^4} \cdot \iota\left([F_\beta]^{\mathrm{vir}}\right) \tag{VII.5.5}$$

where the different $\iota\left([F_\beta]^{\mathrm{vir}}\right)$ are written out in the following list:

- $\iota\left([F_\beta]^{\mathrm{vir}}\right) = \mathbf{1}$ for $\beta \in \mathbb{Z}^3_{\geq 0}$
- $\iota\left([F_\beta]^{\mathrm{vir}}\right) = H_1^3$ for $\beta \in \mathbb{Z}_{<0} \times \mathbb{Z}^2_{\geq 0}$
- $\iota\left([F_\beta]^{\mathrm{vir}}\right) = H_2^3$ for $\beta \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{<0} \times \mathbb{Z}_{\geq 0}$
- $\iota\left([F_\beta]^{\mathrm{vir}}\right) = [X^{(\frac{1}{3},\frac{1}{3})}]$, treated as a cycle in X, for $\beta \in \mathbb{Z}^2_{<0} \times \mathbb{Z}_{\geq 0}$
- $\iota\left([F_\beta]^{\mathrm{vir}}\right) = \mathbf{1}_{(\langle -d_1\rangle, \langle -d_2\rangle)}$ whenever $(\langle d_1\rangle, \langle d_2\rangle) \neq (0,0)$.

CHAPTER VIII
Non-Abelian Targets

In this section, we will discuss how one can apply the above techniques to the case of stack quotients by non-abelian groups. We will then explore the example of a del Pezzo surface in a weighted Grassmannian, where we compute its quantum period and satisfy a conjecture of [51].

VIII.1: Abelian/Non-Abelian Correspondence

The main tool to work with non-abelian quotients is Webb's Abelian/Non-Abelian Correspondence [58]. In essence, this theorem allows us to relate the I-function of a non-abelian quotient with the I-function of the corresponding abelian quotient given by a maximal torus of the group. Here, we will set the scene for this correspondence and present the exact relationship between the two I-functions.

Let (W, G, θ) be the data of a quasimap target, and let $T \subset G$ be a fixed maximal torus. By the inclusion $T \hookrightarrow G$ and by letting θ also denote its restriction to T, we obtain another triple (W, T, θ). We will denote $W^{\mathrm{ss}}(G)$ to be the semi-stable locus with respect to the action by G, and $W^{\mathrm{ss}})(T)$ to be the semi-stable locus with respect to the action by T.

For the rest of this section, we will impose the following assumptions:

Assumption VIII.1.

- (W, T, θ) satisfies the data of a quasimap target.
- The group G is connected.
- Suppose the fixed locus $(W^{\mathrm{ss}}(G))^g$ is non-empty for some $g \in G$. Then we must have that g is contained in a maximal torus of G, and that the centralizer $Z_G(g)$ is connected.

By the first assumption, we can form two different GIT stack quotients, $[W /\!\!/_\theta G]$ and $[W /\!\!/_\theta T]$. However, because of the potential change in the stable locus, there is no clear morphism between the two stacks. In light of this, we define a new intermediate stack

$$[W /\!\!/_G T] := [W^{\mathrm{ss}}(G) /\!\!/_\theta T]$$

where we take the quotient of the G-semi-stable points by the action of T. In order to lessen confusion, we will subsequently suppress the θ in the notation for the GIT stack quotients. We have the following commutative diagram relating the three stacks:

$$\begin{tikzcd} {[W /\!\!/_G T]} \arrow[r, "j"] \arrow[d, "\varphi"] & {[W /\!\!/ T]} \\ {[W /\!\!/ G]} & \end{tikzcd} \tag{VIII.1.1}$$

where j is an open immersion. An analogous diagram holds for the inertia stacks of the three spaces. On the other hand, the there is an obvious map of the non-GIT stack quotients

$$[W/T] \to [W/G]$$

which in turn induces a natural map

$$\mathrm{Hom}(\mathrm{Pic}([W/T]), \mathbb{Q}) \xrightarrow{r^W_{G,T}} \mathrm{Hom}(\mathrm{Pic}([W/G]), \mathbb{Q}). \tag{VIII.1.2}$$

Now given a degree $\tilde{\beta} \in \mathrm{Hom}(\mathrm{Pic}([W/T]), \mathbb{Q})$ and a character $\xi \in \chi(T)$, we define operational Chow Classes

$$C^\circ(\tilde{\beta}, \xi) := \begin{cases} \displaystyle\prod_{\substack{\tilde{\beta}(\xi) < k < 0 \\ k - \tilde{\beta}(\xi) \in \mathbb{Z}}} (c_1(\mathcal{O}(\xi)) + kz) & \text{if } \tilde{\beta}(\xi) \le 0 \\ \left(\displaystyle\prod_{\substack{0 < k \le \tilde{\beta}(\xi) \\ k - \tilde{\beta}(\xi) \in \mathbb{Z}}} (c_1(\mathcal{O}(\xi)) + kz) \right)^{-1} & \text{if } \tilde{\beta}(\xi) > 0 \end{cases}$$

$$C(\tilde{\beta}, \xi) := \begin{cases} c_1(\mathcal{O}(\xi)) C^\circ(\tilde{\beta}, \xi) & \text{it } \tilde{\beta}(\xi) \in \mathbb{Z}_{<0} \\ C^\circ(\tilde{\beta}, \xi) & \text{else} \end{cases}$$

Now we can state the Abelian/Non-Abelian Correspondence. We will use $I^{[W/\!\!/G]}(z)$ and $I^{[W/\!\!/T]}(z)$ to denote the I-functions of the corresponding GIT stack quotients.

Theorem VIII.2 ([58]). *Let $\rho_1, \ldots, \rho_m \in \chi(T)$ denote the roots of G with respect to T. For* *every degree $\beta \in \mathrm{Hom}(\mathrm{Pic}([W/G]), \mathbb{Q})$, the I-functions of $[W/\!\!/T]$ and $[W/\!\!/G]$ satisfy*

$$\varphi^* I_\beta^{[W/\!\!/G]}(z) = \sum_{\tilde{\beta}\to\beta} \left(\prod_{i=1}^m C(\tilde{\beta}, \rho_i)^{-1} \right) j^* I_{\tilde{\beta}}^{[W/\!\!/T]}(z)$$

where $I_\beta^{[W/\!\!/G]}$ (resp. $I_{\tilde{\beta}}^{[W/\!\!/T]}$) refers to the degree β (resp. $\tilde{\beta}$) term of the corresponding I-functions, and the sum is over all $\tilde{\beta}$ mapping to β under the map $r^W_{G,T}$. Moreover, these formulas uniquely determine $I^{[W/\!\!/G]}(z)$.

Remark VIII.3. One thing to note is that when $\beta(\rho_i) \in \mathbb{Z}_{<0}$, then we get a division by $c_1(\mathcal{O}(\rho_i))$. While this may seem nonsensical, it turns out that the numerator of any such term will contain a cohomology class that has $c_1(\mathcal{O}(\rho_i))$ as a factor. Moreover, this factorization is unique, hence one can formally divide to rid the denominators of these classes and make sense of these expressions. The details of this can be found in [57, Lemma 5.3.1].

VIII.2: Non-Abelian Extensions

We now address how the GIT extensions work in the non-abelian setting. In somewhat of a reverse order to the toric setting, we will first discuss what type of extensions are allowed, independent of a choice of cohomology class. We will then discuss a strategy to how to associate an extension to the Chen-Ruan classes we wish to parameterize. Lastly, we discuss the effective cone for the corresponding I-function.

VIII.2.1: Extension Data

For now, assume that our quotients are of the form $X = [V/\!\!/_\theta G]$, where V is an n-dimensional vector space. The goal is to construct a new GIT presentation for X, namely $X = [\mathbb{C}^n \times \mathbb{C}/\!\!/_\vartheta G \times \mathbb{C}^*]$, where ϑ is a $G \times \mathbb{C}^*$ character. Note that being a valid quasimap target is included as part of our definition of GIT presentation, hence one of the conditions we require is that the semi-stable and stable loci of the new presentation agree. The following proposition gives sufficient conditions for such an extension:

Proposition VIII.4. *Let ν be a 1-parameter subgroup of $\mathrm{GL}(V)$ that commutes with the image of G in $\mathrm{GL}(V)$. Assume moreover that there is an integer $r \geq 1$ and a 1-parameter subgroup $\nu' : \mathbb{C}^* \to G$ such that*

$$\lim_{t\to 0} \nu'(t^s)\nu(t^{-rs}) \cdot x \quad \textit{exists} \tag{VIII.2.1}$$

for every $v \in V$ and $s \geq 0$. Then there is a GIT presentation $X = [V \times \mathbb{C} /\!\!/_{\vartheta} G \times \mathbb{C}^*]$, where $G \times \mathbb{C}^*$ acts on $V \times \mathbb{C}$ by

$$(g,t) \cdot (x,y) = (g\nu(t)x, ty). \tag{VIII.2.2}$$

for $(g,t) \in G \times \mathbb{C}^*$ and $(x,y) \in V \times \mathbb{C}$, and ϑ is equal to the character

$$\vartheta : G \times \mathbb{C}^* \to \mathbb{C}^* \qquad \vartheta(g,t) = \theta(g)t^N. \tag{VIII.2.3}$$

for any $N > \langle \theta, \nu' \rangle / r$.

Note that the pairing $\langle \theta, \nu' \rangle \in \mathbb{Z}$ is defined to be the power associated to the map $\theta \circ \nu' : \mathbb{C}^* \to \mathbb{C}^*$.

Remark VIII.5. Since G is reductive, V can be written as a direct sum of irreducible representations $V_1, \ldots, V_k$. Requiring $\nu(t)$ to commute with the image of G in $\mathrm{GL}(V)$ is equivalent to requiring each V_i to be contained in a weight space of ν.

The proof of Proposition VIII.4 consists of two easy lemmas.

Lemma VIII.6. *Let G be an algebraic group with a representation $\rho : G \to \mathrm{GL}(V)$. Let $\nu : \mathbb{C}^* \to \mathrm{GL}(V)$ be a 1-parameter subgroup that commutes with $\rho(G) \subset \mathrm{GL}(V)$. Let $G \times \mathbb{C}^*$ act on $V \times \mathbb{C}$ as in (VIII.2.2). There is a morphism $j : [V/G] \to [(V \times \mathbb{C})/(G \times \mathbb{C}^*)]$ with a retraction, such that if $U \subset V$ is any G-invariant subscheme, j restricts to an isomorphism of $[U/G]$ and $[(U \times \mathbb{C}^*)/(G \times \mathbb{C}^*)]$.*

Proof. Recall from e.g. [52, Prop 2.6, Thm 4.1] the prestack $[V/G]^{pre}$: the objects over a scheme S are $V(S)$ and an arrow from $x \in V(S)$ to $y \in V(S)$ is an element $g \in G(S)$ such that $g \cdot x = y$. The map j is induced by the map of prestacks $[V/G]^{pre} \to [(V \times \mathbb{C})/(G \times \mathbb{C}^*)]^{pre}$ given by

$$(x; g) \mapsto (x, 1; g, 1) \quad (x; g) \in V \times G, \tag{VIII.2.4}$$

and its retraction p is induced by the map of prestacks $[(V \times \mathbb{C})/(G \times \mathbb{C}^*)]^{pre} \to [V/G]^{pre}$ given by

$$(x, y; g, t) \mapsto (\nu(y)^{-1}x, g) \quad (x, y; g, t) \in (V \times \mathbb{C}^*) \times (G \times \mathbb{C}^*).$$

One checks that $p \circ j$ is the identity, that j factors through $[(V \times \mathbb{C}^*)/(G \times \mathbb{C}^*)]^{pre}$, and that after restricting the codomain accordingly j is essentially surjective and fully faithful on every groupoid fiber. $\square$

For $N \in \mathbb{Z}$, define a character

$$\vartheta_N : G \times \mathbb{C}^* \to \mathbb{C}^* \qquad \vartheta_N(g,t) = \theta(g)t^N.$$

Recall that $V_\theta^{\mathrm{ss}}(G)$ is defined to be the set of semi-stable points of V with respect to θ and G (we will drop the group G from the notation when it is clear). We also define the notation

$$\mathrm{Stab}_G(V) := \{g \in G | \, V^g \neq \emptyset\}$$

where we recall that V^g is the fixed point locus under the action by g.

Lemma VIII.7. *Let ν be as in Lemma VIII.6. Moreover, assume there is an integer $r \geq 1$ and a 1-parameter subgroup ν' of G satisfying (VIII.2.1) for every $x \in V$ and $s \geq 0$. Choose an integer N such that $N > \langle \theta, \nu' \rangle / r$.*

1. *There are equalities*

$$(V \times \mathbb{C})^{ss}_{\vartheta_N}(G \times \mathbb{C}^*) = (V \times \mathbb{C})^{s}_{\vartheta_N}(G \times \mathbb{C}^*) = V_\theta^{ss}(G) \times \mathbb{C}^*.$$

2. *The natural map $\mathrm{Stab}_G(V_\theta^{ss}) \to \mathrm{Stab}_{G\times\mathbb{C}^*}((V \times \mathbb{C})^{ss}_{\vartheta_N})$ given by $g \mapsto (g,1)$ is a bijection.*

Proof. We use the numerical criterion of [42, Prop 2.5]. To prove (1), we first show $(V \times \mathbb{C})^{ss}_{\vartheta_N} \subset V_\theta^{ss} \times \mathbb{C}^*$. Let $(x,y) \in (V \times \mathbb{C})^{ss}_{\vartheta_N}$. If x is not in V_θ^{ss}, then there is a 1-parameter subgroup $\lambda : \mathbb{C}^* \to G$ such that $\lim_{t\to 0} \lambda(t) \cdot x$ exists but $\langle \theta, \lambda \rangle < 0$. The composition $\mathbb{C}^* \xrightarrow{\lambda} G \to G \times \mathbb{C}^*$ is a 1-parameter subgroup of $G \times \mathbb{C}^*$, where the second arrow is $g \mapsto (g,1)$, and this subgroup witnesses the unstability of (x,y) for any $y \in \mathbb{C}$. Similarly, if $y = 0$, then $t \mapsto (\nu'(t), t^{-r})$ defines a 1-parameter subgroup of $G \times \mathbb{C}^*$ witnessing the unstability of (x,y) for any $x \in V$.

Next we show $V^{ss} \times \mathbb{C}^* \subset (V \times \mathbb{C})^s$. Let $(x,y) \in V_\theta^{ss} \times \mathbb{C}^*$. An arbitrary 1-parameter subgroup of $G \times \mathbb{C}^*$ has the form $(\lambda(t), t^s)$. If $\lim_{t\to 0}(\lambda(t)\nu(t^s)x, t^s y)$ exists then $s \geq 0$ and $\lim_{t\to 0} \lambda(t)\nu(t^s)x$ exists. This means that $\lim_{t\to 0} \lambda(t^r)\nu(t^{rs})x$ exists. Since $\lim_{t\to 0} \nu'(t^s)\nu(t^{-rs})x'$ exists for every x', we assert that

$$\lim_{t\to 0} \nu'(t^s)\nu(t^{-rs})\lambda(t^r)\nu(t^{rs})x = \lim_{t\to 0} \lambda(t^r)\nu'(t^s)x$$

exists. This can be proved using the triangle inequality and the fact that the 1-parameter subgroup $\nu'(t^s)\nu(t^{-rs})$ of $\mathrm{GL}(V)$ can be written as diagonal matrices $\mathrm{diag}(t^{a_1}, \ldots, t^{a_n})$ in some basis for V, for some integers $a_i \geq 0$, so when $|t| < 1$ the functions $V \to V$ defined by $\nu'(t^s)\nu(t^{-rs})$ are contractions.

But $x \in V_\theta^{ss}$ and $\lambda(t^r)\nu'(t^s)$ is a 1-parameter subgroup of G, so we must have

$$\langle \theta, r\lambda + s\nu' \rangle = r\langle \theta, \lambda \rangle + s\langle \theta, \nu' \rangle \geq 0.$$

Since $\langle \theta, \nu' \rangle < rN$ and $s \geq 0$ this implies

$$r\langle \theta, \lambda \rangle + rsN \geq 0 \tag{VIII.2.5}$$

or equivalently, since $r \geq 1$,

$$\langle \vartheta, (\lambda(t), t^s) \rangle = \langle \theta, \lambda \rangle + sN \geq 0. \tag{VIII.2.6}$$

If equality holds in (VIII.2.6), then equality also holds in (VIII.2.5), which implies that both $r\langle \theta, \lambda \rangle + s\langle \theta, \nu' \rangle$ and s are zero. Since $x \in V_\theta^{ss} = V_\theta^s$, this implies $\lambda(t^r)\nu'(t^s)$ acts trivially on V, but since $s = 0$ we have $\lambda(t^r)\nu'(t^s) = \lambda(t^r)$. So $\lambda(t)$ also acts trivially on V. Since $s = 0$ we have that $(\lambda(t), t^s)$ acts trivially on $V \times \mathbb{C}$. Assertion (2) is immediate using part (1).

□

Proof of Proposition VIII.4. We let $G \times \mathbb{C}^*$ act on $V \times \mathbb{C}$ as in (VIII.2.2) and we let $\vartheta = \vartheta_N$ as defined in Lemma VIII.7. By Lemma VIII.7 we know that $(V \times \mathbb{C}, G \times \mathbb{C}^*, \vartheta_N)$ satisfies the assumptions of VIII.1 and of being a quasimap target. Combined with Lemma VIII.6 we get an isomorphism

$$[V/\!\!/_\theta G] = ([V \times \mathbb{C})/\!\!/_{\vartheta_N} G \times \mathbb{C}^*].$$

□

Now suppose we have a complete intersection $[W/\!\!/_\theta G] \subset [V/\!\!/_\theta G]$, where $W = Z(s)$ is the zero locus of a G-equivariant section s of a G-linearized vector bundle $E \times V$ on V. We will assume that $[W/\!\!/_\theta G]$ satisfies all the hypothesis of being a quasimap target. We want to find a GIT extension

$$[W/\!\!/_\theta G] = [W_e/\!\!/_\vartheta G \times \mathbb{C}^*] \subset [V \times \mathbb{C}/\!\!/_\vartheta G \times \mathbb{C}^*]$$

as before.

The first step to doing so is to find an extension of the ambient space; for example, we can find some $\nu : \mathbb{C}^* \to \mathrm{GL}(V)$ satisfying the hypothesis of Proposition VIII.4. Afterwards, in order to obtain W_e, we need to find a $G \times \mathbb{C}^*$ representation of E and a section $\tilde{s}$ that is $G \times \mathbb{C}^*$-equivariant such that both the representation and section extend the original as in the toric case. Then we can set $W_e = Z(\tilde{s})$.

Let $\rho : G \to \mathrm{GL}(E)$ be the G-action on E corresponding to the original G-linearization, and let $\mu : \mathbb{C}^* \to \mathrm{GL}(E)$ be a 1-parameter subgroup commuting with the image of ρ as in Remark VIII.5. Denote by $E(\mu)$ the $G \times \mathbb{C}^*$-representation that is the same underlying vector space as E equip with the homomorphism

$$G \times \mathbb{C}^* \to \mathrm{GL}(E) \qquad (g,t) \mapsto \rho(g)\mu(t). \tag{VIII.2.7}$$

Lemma VIII.8. *Let $\mu : \mathbb{C}^* \to GL(E)$ be a homomorphism commuting with the image of ρ such that the morphism*

$$\tilde{s} : V \times \mathbb{C}^* \to E \qquad \tilde{s}(x,y) = \mu(y)s(\nu(y)^{-1}x)$$

extends to all of $X \times \mathbb{C}$. Then $\tilde{s}$ induces a $G \times \mathbb{C}^*$-equivariant section of $E(\mu) \times V \to V$ such that $[Z(\tilde{s})/\!\!/_{\vartheta}(G \times \mathbb{C}^*)] = [Z(s)/\!\!/_{\theta}G]$ and $Z(\tilde{s}) \cap (V \times \mathbb{C})^{\mathrm{ss}}$ is smooth.

Proof. One checks using the definitions in (VIII.2.2) and (VIII.2.7) that $\tilde{s}$ is $G \times \mathbb{C}^*$-equivariant. Next, the isomorphism (VIII.2.4) induces an isomorphism

$$[Z(s)/\!\!/_{\theta}G] \simeq [Z(\tilde{s})/\!\!/_{\vartheta}(G \times \mathbb{C}^*)].$$

This implies moreover that $Z(\tilde{s}) \cap (V \times \mathbb{C})^{\mathrm{ss}}$ is a $G \times \mathbb{C}^*$-torsor over $[Z(s)/\!\!/_{\theta}G]$. Since $[Z(s)/\!\!/_{\theta}G]$ is known to be smooth, so is $Z(\tilde{s}) \cap (V \times \mathbb{C})^{\mathrm{ss}}$. □

VIII.2.2: Extension Choice

With the above results established, we now want a way to pick extension data that relates to tracking Chen-Ruan classes as insertions. The practical strategy that we will explore is relating the desired Chen-Ruan classes in terms of those of the maximal torus, finding an extension in the abelian setting, and then using Theorem VIII.2 to obtain an extended non-abelian I-function.

Recall that given a maximal torus $T \subset G$, we have the Weyl group

$$W_G := N(T)/Z(T)$$

where $N(T)$ and $Z(T)$ are, respectively, the normalizer and centralizer groups of T in G. In order to avoid confusion with the affine scheme W used before, we will set the notation

$$X_G = [W/\!\!/G], \quad X_T = [W/\!\!/T], \quad X_{G,T} = [W/\!\!/_G T]$$

so that (VIII.1.1) becomes

$$\begin{tikzcd} X_{G,T} \arrow[r, "j"] \arrow[d, "\varphi"] & X_T \\ X_G \end{tikzcd}$$

The above diagram induces a similar diagram for inertia stacks, for which we will denote the morphisms by the same names. We will also use the decomposition

$$\mathcal{I}X_G = \bigsqcup_{(g)\in \mathrm{Conj}(G)} X_G^g$$

as in Example I.1, where $X_G^g := [W^g /\!\!/ Z(g)]$ is our notation for the corresponding component of the inertia stack in the decomposition. We will extend this decomposition and notation to the other two stacks X_T and $X_{G,T}$ in the obvious way.

The Weyl group W_G acts on T by conjugation in G, i.e. for $w \in W_G$, we have $w\cdot t = \tilde{w}t\tilde{w}^{-1}$ for some lift $\tilde{w} \in G$. This induces an action on the inertia stack $\mathcal{I}X_T$ by sending $X_T^t \to X_T^{w\cdot t}$ through the isomorphism induced by the action of $\tilde{w}$. Note that we can similarly define an action of W_G on the inertia stack $\mathcal{I}X_{G,T}$.

As seen in [58], the morphism φ on the level of inertia stacks is flat, hence we can consider the pullback morphsim φ^* on the level of Chow groups. Using the above Weyl group action, we know state the Abelian/Non-Abelian Correspondence for Chow groups.

Lemma VIII.9. *[58, Lemma 2.2.1] The pullback φ^* induces an isomorphism*

$$A_*(\mathcal{I}X_G) \xrightarrow{\sim} \left(A_*(\mathcal{I}X_{G,T})\right)^{W_G}.$$

This statement also holds for the rigidified inertia stacks.

Since j is an open immersion, (VIII.1.1) gives us the corresponding map of Chow groups

$$\begin{tikzcd} \left(A_*(\mathcal{I}X_{G,T})\right)^{W_G} & \left(A_*(\mathcal{I}X_T)\right)^{W_G} \arrow[l, "j^*"'] \\ A_*(\mathcal{I}X_G) \arrow[u, "\varphi^*", "\wr"'] \end{tikzcd} \tag{VIII.2.8}$$

Now let $t_\alpha^J \in A_*(\mathcal{I}X_T)$ be a Chow class as in the toric case, and further assume that t_α^J is Weyl invariant. Through the above diagram, we obtain a Chow class

$$\phi_\alpha^J := (\varphi^*)^{-1} j^* t_\alpha^J \in A_*(\mathcal{I}X_G).$$

As usual, we will use the same notation to denote its associated Chen-Ruan cohomology class, and will assume that $\phi_\alpha^J \neq 0$. Now choose the characters ν, μ to be as in the extension for t_α^J in the toric case. As long as these choices satisfy Proposition VIII.4 and Lemma VIII.8, then this extension data will result in extended presentation of X_G,

$$X_G = [W_e /\!\!/_\vartheta G \times \mathbb{C}^*] \subset [W_e/G \times \mathbb{C}^*]$$

whose associated toric stack is the extended presentation of X_T. By Theorem VIII.2, we have an extended I-function for this presentation of X_G. We claim that after wall-crossing, this I-function results in a J-function that captures invariants with ϕ_α^J as insertions.

Proposition VIII.10. *Let $X_G = [W_e /\!\!/_\vartheta G \times \mathbb{C}^*]$ be the extended GIT presentation as above, and let $\mu(q,z)$ be the mirror map associated to the extended I-function, as in (VI.2.2). Then, up to a negative, $\mu(q,z)$ contains a term of the form $q^\beta \phi_\alpha^J$, for some quasimap degree β.*

Proof. We know that the I-function for the extended toric stack has a term of the form $z^{-1}q^{\tilde{\beta}}t_\alpha^J$, where $\tilde{\beta} = (-\alpha_1, \ldots, -\alpha_n, 1) \in \mathrm{Hom}(\mathrm{Pic}([W/T], \mathbb{Q}))$. Setting $\beta = r_{G,T}^W(\tilde{\beta})$, we have by Theorem VIII.2 that the extended I-function for X_G has a term of the form

$$(\varphi^{-1})^* \left(\prod_{i=1}^m C(\tilde{\beta}, \rho_i) \right) \cdot q^\beta z^{-1} \phi_\alpha^J \tag{VIII.2.9}$$

Note that the roots ρ_i come in pairs, i.e. for every i, there is some j such that $\rho_i = -\rho_j$. By explicit computation, one has that

$$C(\tilde{\beta}, \rho)C(\tilde{\beta}, -\rho) = (-1)^b \frac{c_1(\mathcal{O}(\rho)) + \tilde{\beta}(\rho)z}{c_1(\mathcal{O}(\rho))} = (-1)^b \left(1 + \frac{\tilde{\beta}(\rho)z}{c_1(\mathcal{O}(\rho))} \right)$$

where b is some constant depending on $\tilde{\beta}$ and ρ. We note that the last term on the right-hand side doesn't make sense alone, due to the division by a cohomology class, but does make sense in the total I-function (see Remark VIII.3). It is then immediate from expanding the product that the (VIII.2.9) has a term of the desired form. □

As a final remark, we note that Weyl-invariance of the cohomology class on the abelian side is a crucial assumption. However, if one restricts the insertions in the J-function of X_G to only involve those cohomology classes ϕ_α^J as above, then it is easy to modify the invertibility theorem VI.6 to obtain similar results to the abelian case.

VIII.2.3: Effective Cone

We now give a brief description of the I-effective cone for $X_G = [W/\!\!/_\theta G] \subset [V/\!\!/_\theta G]$. By I-effective cone, we mean the set

$$\mathrm{Eff}^I(W, G, \theta) = \{\beta \in \mathrm{Hom}(\mathrm{Pic}([V/G]), \mathbb{Q}) |\ F_\beta \neq 0\}$$

which is the set of β that we sum over in the I-function (see (V.2.1)).

Recall that for a toric stack $[V/\!\!/_\theta T]$, we have a set of anticones $\mathcal{A}_\theta$ (IV.1.1) and for each $I \in \mathcal{A}_\theta$, we have an open set U_I. We now define a locally closed subset associated to each $I \in \mathcal{A}_\theta$,

$$V_I = \{(x_1, \ldots, x_n) \in U_I |\ x_i = 0 \text{ for } i \notin I\}.$$

Let $V^{\mathrm{ss}}(G)$ denote the semi-stable with respect to the G-action and character θ.

Definition VIII.11. An anticone $I \in \mathcal{A}_\theta$ is G-effective if $V_I \cap V^{\mathrm{ss}}(G) \neq \emptyset$.

Lemma VIII.12. *Let I, J be subsets of $\{1, \ldots, n\}$ with $I \subset J$. If I is a G-effective anticone, then so is J.*

Proof. The fact that J is an anticone follows from Lemma IV.2.

Since I is G-effective, we have that $V_I \cap V^{\mathrm{ss}}(G) \neq \emptyset$. We also have

$$V_I \cap V^{\mathrm{ss}}(G) \subset \{x_i = 0 |\ i \notin I\} \cap V^{\mathrm{ss}}(G) \subset \{x_i = 0 |\ i \notin J\} \cap V^{\mathrm{ss}}(G)$$

hence the rightmost set is non-empty open subset of the subspace $\{x_i = 0 |\ i \notin J\}$. But V_J is also a non-empty open subset of $\{x_i = 0 |\ i \notin J\}$, hence

$$V_J \cap \{x_i = 0 |\ i \notin J\} \cap V^{\mathrm{ss}}(G) = V_J \cap V^{\mathrm{ss}}(G)$$

is non-empty. □

Remark VIII.13. Suppose we have an extended GIT presentation of $(V \times \mathbb{C}, G \times \mathbb{C}^*, \vartheta)$ satisfying the assumptions of Lemma VIII.7. Then, as was the case for anticones, the G-effective anticones for $(V \times \mathbb{C}, T \times \mathbb{C}^*, \vartheta)$ are $\{I \cup \{n+1\}\}$, where $I \in \mathcal{A}_\theta$ is an anticone for the original presentation (V, T, θ). This follows immediately from the definition and Lemma VIII.7.

For $\tilde{\beta} \in \mathrm{Hom}(\chi(T), \mathbb{Q}) = \mathrm{Hom}(\mathrm{Pic}([V/T]), \mathbb{Q})$, recall the set $S^{\geq 0}_{\tilde{\beta}}$ defined in Proposition V.17. Define a subset

$$\mathbb{K} := \{r^V_{X,G}(\tilde{\beta}) |\ S^{\geq 0}_{\tilde{\beta}} \text{ is a } G\text{-effective anticone}\} \subset \mathrm{Hom}(\chi(G), \mathbb{Q}).$$

Then we claim that this is the set of I-effective classes for $[V/\!\!/_\theta G]$.

Lemma VIII.14. *Let $\beta \in \mathrm{Hom}(\chi(G), \mathbb{Q})$. Then the following are equivalent:*

1. *The fixed locus $F_\beta \subset \mathcal{Q}_{\mathbb{P}(1,\star)}([V/\!\!/_\theta G], \beta)$ is non-empty.*
2. *There exists some $\tilde{\beta} \in \mathrm{Hom}(\chi(T), \mathbb{Q})$ with $r^X_{G,T}(\tilde{\beta}) = \beta$ such that $S_{\tilde{\beta}}$ is a G-effective anticone.*

Before we prove this lemma, we recall some background. A quasimap $f : \mathbb{P}(1, r) \to [V/G]$ is equivalent to a choice of principal G bundle on $\mathbb{P}(1, r)$ and an equivariant morphism from the principal bundle to V satisfying the quasimap conditions. Equivalently, this can be phrased as taking the associated vector bundle to the principal G-bundle and a section on that bundle. Since the curve is a weighted projective line, we have that such an associated vector bundle splits into a direct sum of line bundles. Because every principal G-bundle is induced from a principal T-bundle, the morphism f lifts to a morphism $\tilde{f} : \mathbb{P}(1, r) \to [X/T]$ where the data is given by the same vector bundle and section. Moreover, the degree of $\tilde{f}$ maps to the degree of f under $r^X_{G,T}$.

proof of Lemma VIII.14. Assume that $F_\beta \neq \emptyset$. Then there exists a $\mathbb{C}^*$-fixed quasimap $f : \mathbb{P}(1, r) \to [V/G]$, which by the above, lifts to a quasimap $\tilde{f} : \mathbb{P}(1, r) \to [V/T]$ of degree $\tilde{\beta}$ mapping to β.

Since $\tilde{f}$ is a quasimap, we have that the orbifold point ∞ must land in $[V/\!\!/_\theta T]$, and by the proof of Proposition V.17, we have that there is an anticone $I \subset S^{\geq 0}_{\tilde{\beta}}$ such that $\tilde{f}(\infty) \in U_I$. Let $p := \tilde{f}(\infty)$, and write $p = (p_1, \ldots, p_n)$ in coordinates coming from V. Define the set $J \subset \{1, \ldots n\}$ as

$$J = \{j \in \{1, \ldots, n\} |\ p_j \neq 0\}.$$

Then $I \subset J$, and from the description of the section associated to the quasimap in the proof of Proposition V.17, we have that $J \subset S^{\geq 0}_{\tilde{\beta}}$ still. By Lemma IV.2, we have that J is an anticone. Morever, we have by definition that $p \in V_J$, hence J is G-effective. By Lemma VIII.12, we have $S_{\tilde{\beta}}$ is G-effective.

Now assume $S^{\geq 0}_{\tilde{\beta}}$ is G-effective for some $\tilde{\beta}$ mapping to β. In particular, $S^{\geq 0}_{\tilde{\beta}}$ is an anticone, so by Proposition V.17 we have a $\mathbb{C}^*$-fixed quasimap to (V, T, θ). Then using the same vector bundle and section, one can construct a $\mathbb{C}^*$-fixed quasimap of degree β to (V, G, θ), hence $F_\beta \neq \emptyset$. $\square$

As in Proposition V.17, this condition can be extended to describe the I-effective classes of complete intersections $X_G \subset [V/\!\!/_\theta G]$ by requiring that the image of the quasimap lies in

X_G. As before, let

$$V^{\tilde{\beta}} := \{(x_1, \ldots, x_n) \in V |\ x_i = 0 \text{ for } i \notin S_{\tilde{\beta}}\}.$$

Then it is straightforward to adjust the above lemma to complete intersections by further requiring that $V^{\tilde{\beta}} \cap W^{\mathrm{ss}}(G) \neq \emptyset$ in addition to $S_{\tilde{\beta}}^{\geq 0}$ being G-effective. This leads to the following corollary:

Corollary VIII.15. *Let X_G be given by the GIT presentation (W, G, θ). Then $\mathrm{Eff}^I(X, G, \theta)$ is equivalent to the set of $\beta \in \mathrm{Hom}(\chi(G), \mathbb{Q})$ such that*

1. $\beta \in \mathbb{K}$
2. $V^{\tilde{\beta}} \cap W^{\mathrm{ss}}(G) \neq \emptyset$ *for some $\tilde{\beta}$ mapping to β*

In practice, one first computes the minimal G-effective cones, which are those that contain no proper subset that is also G-effective. For each G-effective anticone J, we get a set

$$C_J := \{\tilde{\beta} \in \mathrm{Hom}(\chi(T), \mathbb{Q})|\ J \subset S_{\tilde{\beta}}\}.$$

The set of I-effective classes is equal to the union of the images of the sets C_J under the natural map $\mathrm{Hom}(\chi(T), \mathbb{Q}) \to \mathrm{Hom}(\chi(G), \mathbb{Q})$.

VIII.3: Example: Orbifold Del-Pezzo $X_{1,7/3}$

In this section, we will apply the techniques of the non-abelian extension to compute invariants of a stacky del Pezzo surface inside of a weighted Grassmannian. In particular, we are able to recover the quantum period and verify that it satisfies a conjectured formula.

VIII.3.1: Geometry

Weighted Grassmannians were introduced in [24]. Donagi and Sharpe [27] give the following GIT data for the weighted Grassmannian $wGr(2, 5)$:

- $V = \mathbb{C}^{10}$ is the space of 2×5 matrices M over $\mathbb{C}$
- $G := (\mathrm{SL}_2 \times \mathbb{C}^*)/\mu_2$, where $(\Lambda, \gamma) \in \mathrm{SL}_2 \times \mathbb{C}^*$ acts on V by

  $$(\Lambda, \gamma) \cdot M = \Lambda M \operatorname{diag}(\gamma, \gamma, \gamma, \gamma^3, \gamma^3)$$

  and μ_2 is the subgroup $(\operatorname{diag}(-1, -1), -1) \subset \mathrm{SL}_2 \times \mathbb{C}^*$.
- $\theta \in \chi(G)$ is induced by the function $\mathrm{SL}_2 \times \mathbb{C}^* \to \mathbb{C}^*$ sending $(\Lambda, \gamma) \to \gamma^2$.

The semistable locus $V^{\mathrm{ss}}(G)_\theta$ is easily checked to be equal to the set of full rank matrices. The resulting stack $Y_G := [V /\!\!/_\theta G]$ is the weighted Grassmannian $wGr(2,5)$. We note that as a group, we have $G \cong \mathrm{GL}_2$, and that the change in how the determinant factor $\mathbb{C}^*$ acts M is what makes this a weighted version of the original Grassmannian construction.

Now we can also consider the data of a G-linearized vector bundle on V

$$V \times E := V \times (\mathbb{C}_\chi)^{\oplus 4}$$

where $\mathbb{C}_\chi$ is the G-representation given by the character

$$\chi : G \to \mathbb{C}^*, \quad \chi(\Lambda, \gamma) \to \gamma^4$$

We will also use E to denote the induced split vector bundle on $\mathfrak{Y}_G$. The complete intersection we will consider is given by

$$X_G := [W /\!\!/_\theta G],$$

where $W = Z(s)$ is the vanishing locus of a generic section s. Via a Bertini style argument, as will be seen in the next section, one can choose this section so that the resulting stack X_G is smooth. Moreover, as seen in [23], this stack is isomorphic to the $1/3(1,1)$ stacky del Pezzo surface $X_{1,7/3}$, which was also studied in [51, Section 6.2].

Since we will be using the Abelian/Non-Abelian Correspondence to study this stack, we will also take a look at the stacks corresponding to a maximal torus $T \subset G$. We can choose the maximal torus $T' \subset G$ to be the quotient of the subgroup $(\mathrm{diag}(\lambda, \lambda^{-1}), \gamma)$. Under the isomorphism $G \cong \mathrm{GL}_2$, we have that the image of T' is the diagonal matrices $T \subset \mathrm{GL}_2$. More explicitly, this isomorphism is induced by the morphism

$$T' \to (\mathbb{C}^*)^2 =: T, \quad (\mathrm{diag}(\lambda, \lambda^{-1}), \gamma) \to (\lambda\gamma, \lambda^{-1}\gamma)$$

The weights of T on V is precisely the weight matrix (VII.5.1), and the character θ descends to the same character on T as in Section VII.5. Moreover, the induced T representation on E gives the exact same bundle as in Section VII.5. Thus, we refer to our earlier example for the geometry of the toric stacks associated to the maximal torus. Moving forward, we will refer to the stacks X and Y in Section VII.5 as X_T and Y_T.

Lastly, we address the inertia stack of X_G. First, we note that the the elements of G with non-trivial fixed loci in $V^{\mathrm{ss}}_\theta(G)$ are precisely those that are conjugate to an element in T' that maps to the set $\{(1,1), (1,-1), (-1,1), (\xi,\xi), (\xi^2,\xi^2)\} \subset T$, where $\xi = e^{2\pi i/3}$. Thus, the weighted Grassmannian Y_G has a decomposition similar to that of Y_T.

However, the inertia stack for X_G slightly differs from that of X_T. Let E again denote the G-equivariant vector bundle on $\mathfrak{Y}_G$, and let $\Gamma(\mathfrak{Y}, E_G)$ denote the global sections of E, which are G-sections of the corresponding G-linearized bundle. Let $\Delta_{ij} := m_{1i}m_{2j} - m_{1j}m_{2i}$. Then we have that $\Gamma(\mathfrak{Y}_G, E)$ is spanned by 4-tuples from the set

$$S = \left\{ \begin{array}{c} \Delta_{12}^2,\ \Delta_{13}^2,\ \Delta_{23}^2,\ \Delta_{12}\Delta_{13},\ \Delta_{12}\Delta_{23},\ \Delta_{13}\Delta_{23}, \\ \Delta_{14},\ \Delta_{15},\ \Delta_{24},\ \Delta_{25},\ \Delta_{34},\ \Delta_{35} \end{array} \right\}.$$

Lemma VIII.16. *There is an open subset $U \subset \Gamma(\mathfrak{Y}_G, E)$ such that if $s \in U$, then $Z(s) \cap V_\theta^{\mathrm{ss}}(G)$ is smooth and connected with no μ_2-stabilizers.*

Proof. Let $Z(S)$ denote the vanishing set of all the sections in S. One can check that every component of $Z(S) \subset V$ has codimension at least 4. Morevoer, the intersection $Z(S) \cap V^{\mathrm{ss}}(G)$ is the locus where the first three columns of $M = (m_{ij})$ are zero and $\Delta_{45}(M) \neq 0$, hence has codimension 6.

Let $K := \mathrm{span}\{S\} = \Gamma(V, \mathcal{O}_V^{\oplus 4})$. By Bertini's Theorem [30, Thm 3.4.10] applied to the restriction of the linear system K to the variety $V^{ss}(G)$, there is an open subset $U_1 \subset K^{\oplus 4}$ consisting of sections s such that $Z(s) \cap V^{ss}(G)$ is irreducible. Moreover, by [30, Thm 3.4.8] we can choose U_1 to consist of sections whose vanishing locus is smooth outside $Z(S)$.

On the other hand, if we set $s = (\Delta_{14}, \Delta_{24}, \Delta_{34}, \Delta_{15}) \in K^{\oplus 4}$ then one can check directly that the Jacobian matrix of s has full rank when $m_{14} = m_{25} = 1$ and all other coordinates are zero. Since this point is in the unique orbit of G on $Z(S) \cap V^{ss}(G)$, we conclude that $Z(s)$ is smooth along $Z(S) \cap V^{ss}(G)$. Hence there is an open subset $U_2 \subset U_1$ consisting of sections whose vanishing locus is smooth on V^{ss}.

Let $D \subset V$ be the locus of points with μ_2 isotropy; it has dimension 7. Let $Z(S)^c$ be the complement of $Z(S) \subset V$. By [30, 1.5.4(1)] and [53, Tag 05F7], there is an open subset $U_3 \subset U_2$ such that if $s \in U_3$, then $Z(s) \cap D \cap Z(S)^c \cap V^{ss}$ has the expected dimension $7-4 = 3$, or it is empty. Since $Z(S) \cap V^{ss}$ has μ_3 isotropy (see the first paragraph), we know that for $s \in U_3$ the locus $Z(s) \cap D \cap V^{ss}$ has dimension 3 or is empty. Finally, $G = \mathrm{GL}(2)$ has dimension 4 and the orbits of semi-stable points also have dimension 4, so we have that for $s \in U_3$ the locus $Z(s) \cap V^{ss}$ does not contain any point with μ_2 stabilizer. □

By the lemma, we see that the inertia stack of X_G decomposes as

$$\mathcal{I}X_G = X_G \sqcup (X_G)_{(\frac{1}{3},\frac{1}{3})} \sqcup (X_G)_{(\frac{2}{3},\frac{2}{3})}$$

One can check that as a stack, we have that

$$(X_G)_{(\frac{i}{3},\frac{i}{3})} \cong B\mu_e$$

by looking at the fixed loci of the corresponding group element. Moreover, using the generalized Euler sequence (IV.2.4) and conormal sequence (IV.3.1), we can compute that

$$\operatorname{age}(X_G)_{(\frac{i}{3},\frac{i}{3})} = \frac{2i}{3}.$$

As for the Chen-Ruan cohomology, we suffice with stating the three classes $\mathbf{1}$ and $\mathbf{1}_{i/3}$, which are the unit classes of the untwisted sector and twisted sector $(X_G)_{(\frac{i}{3},\frac{i}{3})}$ respectively.

VIII.3.2: Extension Data

We now consider extensions to keep track of the Chen-Ruan class $\mathbf{1}_{1/3}$, the unit class of the twisted sector $(X_G)_{(\frac{i}{3},\frac{i}{3})}$.

Note that the Weyl group $W_G \cong \mathbb{Z}/2\mathbb{Z}$ by direct calculation, and the action on $T = (\mathbb{C}^*)^2$ is given by permuting the two factors. Moreover, the twisted sector $(X_T)_{(\frac{1}{3},\frac{1}{3})}$ is invariant under the Weyl group action, and the degree $\tilde{\beta} = (-1/3, -1/3) \in \operatorname{Hom}(\chi(T), \mathbb{Q})$ is also invariant under the Weyl group action. Therefore, as suggested by the strategies outlined in Section VIII.2.2, we consider the GIT extension given by the character

$$\nu : \mathbb{C}^* \to \mathrm{GL}(V), \quad \nu(s) \cdot M = M \operatorname{diag}(1,1,1,s,s) \qquad M \in V,\ s \in \mathbb{C}^*.$$

The hypotheses of Proposition VIII.4 are satisfied with $r = 1$, $N = 3$, and ν' equal to the quotient of $s \to (Id, s) \in \mathrm{SL}_2 \times \mathbb{C}^*$. As a result, we get an extended GIT presentation $(V \times \mathbb{C}, G \times \mathbb{C}^*, \vartheta)$ of Y_G where the action of $G \times \mathbb{C}^*$ is given by

$$(\Lambda, \gamma, s) \cdot (M, y) = (\Lambda M \operatorname{diag}(\gamma, \gamma, \gamma, \gamma^3 s, \gamma^3 s), \gamma s)$$

for $(\Lambda, \gamma, s) \in \mathrm{SL}_2 \times \mathbb{C}^* \times \mathbb{C}^*$ and $(M, y) \in V \times \mathbb{C}$.

To extend the complete intersection we extend the bundle E to $\mathcal{E} \to (\mathfrak{Y}_G)_e = [V \times \mathbb{C}/G \times \mathbb{C}^*]$. We define $\mathcal{E}$ to be the bundle obtained from the $G \times \mathbb{C}$ linearized bundle

$$(V \times \mathbb{C}) \times (\mathbb{C}_{\chi_e})^{\oplus 4}$$

where $\chi_e : G \times \mathbb{C}^* \to \mathbb{C}^*$ is induced by $(\Lambda, \gamma, s) \to \gamma^4 s$. As before, let $\Gamma((\mathfrak{Y}_G)_e, \mathcal{E})$ denote the global sections of $\mathcal{E}$, which we write as $G \times \mathbb{C}^*$-equivariant sections of the corresponding linearized bundle over $V \times \mathbb{C}$. Then $\Gamma((\mathfrak{Y}_G)_e, \mathcal{E})$ is spanned by 4-tuples from the set

$$S' = \left\{ \begin{array}{c} y\Delta_{12}^2,\ y\Delta_{13}^2,\ y\Delta_{23}^2,\ y\Delta_{12}\Delta_{13},\ y\Delta_{12}\Delta_{23},\ y\Delta_{13}\Delta_{23}, \\ \Delta_{14},\ \Delta_{15},\ \Delta_{24},\ \Delta_{25},\ \Delta_{34},\ \Delta_{35} \end{array} \right\}.$$

Note that restricting the elements of S' to $y = 1$ recovers the set S from before.

Lemma VIII.17. *There is a dense set of $G \times \mathbb{C}^*$-equivariant sections s of $\mathcal{E}$ such that*

1. *$Z(s) \cap (V^{ss}(G) \times \mathbb{C}^*)$ is smooth, connected, and does not contain any points with stabilizer isomorphic to μ_2, and*
2. *s is regular.*

Proof. Let $Z(S')$ denote the common vanishing locus of all elements of S'. A computation shows that $Z(S') \cap (V \times \mathbb{C})^{\mathrm{ss}}_{\vartheta}(G \times \mathbb{C}^*)$ is equal to the locus where the first three columns of $M = (m_{ij})$ are zero and $\Delta_{45}(M) = 0$ (and $y \neq 0$). This is precisely the locus of points wtih $B\mu_3$ isotropy and it has has codimension 6 in $(V \times \mathbb{C})^{\mathrm{ss}}$. Let $\mathcal{S}$ be the space of sections equal to the span of S'.

Consider the variety

$$\mathcal{Z} = \{(x, s) \in (V \times \mathbb{C})^{\mathrm{ss}} \times \mathcal{S}^{\oplus 4} \mid s(x) = 0\}$$

together with its projection $p : \mathcal{Z} \to \mathcal{S}^{\oplus 4}$. By Bertini's Theorem [30, Thm 3.4.10] and [53, Tag 055A], there is an open subset $U_1 \subset \mathcal{S}^{\oplus 4}$ consisting of sections s such that the fiber $\mathcal{Z}_s$ is irreducible.

The map p is equal to the composition $(V \times \mathbb{C})^{\mathrm{ss}} \times \mathcal{S}^{\oplus 4} \to Y_G \times \mathcal{S}^{\oplus 4} \to \mathcal{S}^{\oplus 4}$ where the first map is a $G \times \mathbb{C}^*$-torsor and the second map is proper, so p is closed. Let $\Sigma \subset \mathcal{Z}$ denote the singular locus of p. Since p is closed we know that the set of $s \in \mathcal{S}^{\oplus 4}$ where $\Sigma_s \cap Z(S') \neq \emptyset$ is closed; in fact it has positive codimension, so that there is an open subset $U_2' \subset \mathcal{S}^{\oplus 4}$ where $\mathcal{Z}_s$ is smooth along $Z(S')$. To show that some s has $\Sigma_s \cap Z(S') = \emptyset$ we use $s = (\Delta_{14}, \Delta_{24}, \Delta_{34}, \Delta_{15}) \in \mathcal{S}^{\oplus 4}$. One can check directly that the Jacobian matrix of s has full rank when $m_{14} = m_{25} = 1$ and all other coordinates are zero. Finally, by [30, Thm 3.4.8] we can in fact choose $s \in U_2'$ so that Σ_s is empty. Since p is closed we get an open subset $U_2 \subset \mathcal{S}^{\oplus 4}$ where fibers of p are smooth.

Let $D \subset V$ be the locus of points with $B\mu_2$ isotropy; it has dimension 8. Let $Z(S')^c$ be the complement of $Z(S')$. By [30, 1.5.4(1)] and [53, Tag 05F7], there is an open subset $U_3 \subset \mathcal{S}^{\oplus 4}$ such that if $s \in U_3$, then

$$Z(s) \cap (D \cap Z(S')^c) \tag{VIII.3.1}$$

has the expected dimension $8 - 4 = 4$, or it is empty. Since $Z(S')$ has $B\mu_3$ isotropy, we can replace $D \cap Z(S')^c$ with D in (VIII.3.1). Finally, $\mathrm{GL}_2 \times \mathbb{C}^*$ has dimension 5 and the orbits of semi-stable points also have dimension 5, so for $s \in U_3$ the locus (VIII.3.1) must be empty.

We have constructed an open subset $U_1 \cap U_2 \cap U_3$ of $\mathcal{S}^{\oplus 4}$ where fibers of p are smooth irreducible and have no $B\mu_2$-stabilizers. To finish the proof we apply the following general lemma to show that we can find a dense subset of regular sections:

Lemma VIII.18. *Let $V = \mathbb{C}^n$ let $H \subset \Gamma(V, \mathcal{O}_V)$ be a finite-dimensional vector subspace with basis $\{s_i\}_{i=1}^m$. If the codimension of every component of $Z(s_1, \ldots, s_m)$ is at least k, the set of regular sequences of length k form a dense subset of H^k.*

Proof. For each $i = 1, \ldots, k$, set

$$S_i = \{(h_1, \ldots, h_k) \in H^k \mid (h_1, \ldots, h_i) \text{ is regular}\}.$$

Clearly $S_1 = (H \setminus \{0\}) \times H^{k-1}$. For $i > 1$ we will show that S_i is a dense subset of S_{i-1}.

Let $U \subset S_{i-1}$ be open and let $\mu = (h_1, \ldots, h_k) \in U$, so in particular $(h_1, \ldots, h_{i-1})$ is regular. We claim that there is an open subset $H' \subset H$ such that for $h \in H'$, the sequence $(h_1, \ldots, h_{i-1}, v)$ is regular. Granting this, since H' is dense in H, the intersection $U \cap (h_1 \times \ldots \times h_{i-1} \times H' \times H^{k-i})$ is not empty and the lemma statement follows.

Now we prove the claim. The complement of H' is the set of $h \in V$ such that h is in an associated prime of the ideal $(h_1, \ldots, h_{i-1}) \subset \Gamma(V, \mathcal{O}_V)$. Since each associated prime is a vector subspace of $\Gamma(V, \mathcal{O}_V)$, it is in particular a closed subset, and we see that the complement of H' is closed. (This uses the fact that there are only finitely many associated primes). To see that H' is nonempty, suppose for contradiction that every $h \in H$ is contained in some associated prime of $(h_1, \ldots, h_{i-1})$. Since H is not a union of proper subspaces, we have $H \subset \mathfrak{p}$ for some associated prime $\mathfrak{p}$. Since $\Gamma(V, \mathcal{O}_V)$ is Cohen-Macaulay and $(h_1, \ldots, h_{i-1})$ is regular, the height of $\mathfrak{p}$ is equal to $i - 1$. Now $\mathfrak{p}$ defines a point of V that is contained in the base locus of H but has codimension $i - 1 < k$, a contradiction. □

To use VIII.18, we need to check that $Z(S')$ has codimension at least 4 in $V \times \mathbb{C}$; this is straight-forward.

□

With this lemma proved, we can take W_e to be the vanishing of a section as above, hence we successfully obtain an extended GIT presentation for X_G.

Lastly, we note that the maximal torus of the extended GIT presentation, and the corresponding extended presentation for the abelian side, is precisely that given in Section VII.5.2.

We will use the following notation moving forward for all the different stacks

$$\begin{tikzcd}
X_G = [W/\!\!/_\theta G] \arrow[r] \arrow[d, "\cong"] & {[W/G] =: \mathfrak{X}_G} \\
X_G = [W_e/\!\!/_\vartheta G \times \mathbb{C}^*] \arrow[r] & {[W_e/G \times \mathbb{C}^*] =: (\mathfrak{X}_G)_e,}
\end{tikzcd}$$

$$\begin{tikzcd}
X_T = [W/\!\!/_\theta T] \arrow[r] \arrow[d, "\cong"] & {[W/T] =: \mathfrak{X}_T} \\
X_T = [W_e/\!\!/_\vartheta T \times \mathbb{C}^*] \arrow[r] & {[W_e/T \times \mathbb{C}^*] =: (\mathfrak{X}_T)_e}
\end{tikzcd}$$

We also have analogous diagrams where $X, \mathfrak{X}$ are replaced by $Y, \mathfrak{Y}$ to denote the ambient weighted Grassmannian and ambient toric stack.

VIII.3.3: Extended I-function

We will now compute the I-function for X_G.

First, we will discuss the I-effective cone. Note that $\chi(G) \cong \mathbb{Z}$, where we choose the basis to be the character θ that defines the weighted Grassmannian. This can be seen by identifying $G \cong \mathrm{GL}_2$, and noting that θ maps to the determinant character under this identification. Therefore, we have that

$$\mathrm{Hom}(\mathrm{Pic}((\mathfrak{Y}_G)_e), \mathbb{Q}) = \mathrm{Hom}(\chi(G \times \mathbb{C}^*), \mathbb{Q}) \cong \mathbb{Q}^2$$

where the last isomorphism is determined by the image of θ and the identity character on $\mathbb{C}^*$. Recall the morphism $r := r^V_{G,T}$ as in (VIII.1.2),

$$\mathrm{Hom}(\mathrm{Pic}((\mathfrak{Y}_T)_e), \mathbb{Q}) \cong \mathbb{Q}^3 \xrightarrow{r} \mathbb{Q}^2 \cong \mathrm{Hom}(\mathrm{Pic}((\mathfrak{Y}_G)_e), \mathbb{Q})$$

Let $\tilde{\beta} = (\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3) \in \mathrm{Hom}(\mathrm{Pic}((\mathfrak{Y}_T)_e), \mathbb{Q}) \cong \mathbb{Q}^3$. Then, tracing through the definitions, the map r above is identified with the linear map $r : \mathbb{Q}^3 \to \mathbb{Q}^2$ given by

$$r(\tilde{\beta}) = (\tilde{\beta}_1 + \tilde{\beta}_2, \tilde{\beta}_3).$$

Now we want to identify the quasimap degrees $\beta \in \mathbb{Q}^2 \cong \mathrm{Hom}(\mathrm{Pic}((\mathfrak{Y}_G)_e), \mathbb{Q})$ which are I-effective. Using the I-effective classes computed for $\mathfrak{Y}_T$ in Section VII.5.2, we can see which of those satisfy the conditions of Corollary VIII.15. If $(\langle\tilde{\beta}_1\rangle, \langle\tilde{\beta}_2\rangle)$ is equal to $(1/2, 0)$ or $(0, 1/2)$, it is easy to check that $V^{(\tilde{\beta}_1, \tilde{\beta}_2)}$ corresponds to the locus of points with μ_2 isotropy, hence does not intersect with $W_e^{\mathrm{ss}}(G)$. As a result, we only need to work with the $\tilde{\beta}$ such that

$$\{\tilde{\beta} \in (1/3, 1/3)\mathbb{Z}^2 \times \mathbb{Z}_{\geq 0} |\ 2\tilde{\beta}_1 + \tilde{\beta}_2 + \tilde{\beta}_3 \geq 0, \text{ and } \tilde{\beta}_1 + 2\tilde{\beta}_2 + \tilde{\beta}_3 \geq 0\} \tag{VIII.3.2}$$

For any such $\tilde{\beta}$ above, we have that $V^{(\tilde{\beta}_1, \tilde{\beta}_2)}$ contains the locus where the first three columns of M are zero. This locus is also contained in $W^{\mathrm{ss}}(G)$, hence we see that all the β above satisfy condition 2 of Corollary VIII.15. Similarly, it is easy to see that $S_{\tilde{\beta}}$ is G-effective for all the above $\tilde{\beta}$ by virtue of the corresponding locally closed subset containing the μ_3 fixed locus. Therefore, the I-effective classes are the image of the above set under r, which we write as

$$\mathrm{Eff}^I := \{\beta = (d_1, d_2) \in (1/3)\mathbb{Z} \times \mathbb{Z}_{\geq 0} |\ 3d_1 + 2d_2 \geq 0\}. \tag{VIII.3.3}$$

Note that the image of the set (VIII.3.2) under r is clearly contained in Eff^I. On the other hand, every $\beta \in \mathrm{Eff}^I$ has some $\tilde{\beta}$ in (VIII.3.2) in its pre-image under r.

The second piece we need to use Theorem VIII.2 is an explicit formula for the factors $C(\tilde{\beta}, \rho)$, where ρ is a root of G with respect to T. Let $e_i \in \chi(T)$ be the i-th projection character, $e_i : T \cong (\mathbb{C}^*)^2 \to \mathbb{C}^*$. Then a standard computation shows that G has two roots, namely

$$\rho_1 = e_1 - e_2, \quad \rho_2 = e_2 - e_1$$

Note that $\tilde{\beta}(\rho_i) \in \mathbb{Z}$ for all $\tilde{\beta}$ in (VIII.3.2). By direct computation, we have

$$\begin{aligned} \prod_{i=1}^{2} C(\tilde{\beta}, \rho_i)^{-1} &= (-1)^{\tilde{\beta}(\rho)} \frac{c_1(\mathcal{O}(\rho)) + \tilde{\beta}(\rho)z}{c_1(\mathcal{O}(\rho))} \\ &= (-1)^{\tilde{\beta}_1 - \tilde{\beta}_2} \frac{H_1 - H_2 + (\tilde{\beta}_1 - \tilde{\beta}_2)z}{H_1 - H_2} \end{aligned} \tag{VIII.3.4}$$

where ρ can be either ρ_i, and $H_i := c_1(\mathcal{O}(e_i))$. We remark that this expression is not sensible on its own due to the division by $H_1 - H_2$, but that this $H_1 - H_2$ will cancel out with numerators appearing elsewhere in the total I-function.

Now we can apply Theorem VIII.2, from which we obtain the formula

$$\varphi^* I(q, z)^{X_G} = \sum_{\beta = (d_1, d_2) \in \mathrm{Eff}^I} q_1^{d_1} q_2^{d_2} \sum_{\tilde{\beta} \to \beta} j^* \left((-1)^{\tilde{\beta}_1 - \tilde{\beta}_2} \frac{H_1 - H_2 + (\tilde{\beta}_1 - \tilde{\beta}_2)z}{H_1 - H_2} I_{\tilde{\beta}}(z) \right) \tag{VIII.3.5}$$

where $I_{\tilde{\beta}}$ is the coefficient of $q^{\tilde{\beta}}$ in (VII.5.5). For convenience, we list out $j^*\iota([F_{\tilde{\beta}}^{\mathrm{vir}})$:

- $j^*\iota\left([F_{\tilde{\beta}}]^{\mathrm{vir}}\right) = \mathbf{1}$ for $\tilde{\beta} \in \mathbb{Z}^3_{\geq 0}$

- $j^*\iota\left([F_{\tilde{\beta}}]^{\mathrm{vir}}\right) = j^*H_1^3$ for $\tilde{\beta} \in \mathbb{Z}_{<0} \times \mathbb{Z}_{\geq 0}^2$
- $j^*\iota\left([F_{\tilde{\beta}}]^{\mathrm{vir}}\right) = j^*H_2^3$ for $\tilde{\beta} \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{<0} \times \mathbb{Z}_{\geq 0}$
- $j^*\iota\left([F_{\tilde{\beta}}]^{\mathrm{vir}}\right) = [B\mu_3]$, treated as a cycle in the untwisted sector, for $\tilde{\beta} \in \mathbb{Z}_{<0}^2 \times \mathbb{Z}_{\geq 0}$
- $j^*\iota\left([F_{\tilde{\beta}}]^{\mathrm{vir}}\right) = \mathbf{1}_{(\langle d_1 \rangle)}$ whenever $\langle d_1 \rangle \neq 0$.

where we remind that the μ_3 locus in the untwisted sector of X_G is isomorphic to $B\mu_3$.

We remark the Weyl action of $\mathbb{Z}_2$ acts on the cohomology ring of X_T by taking $H_1 \to H_2$, but preserves $\mathbf{1}_{i/3}$. Applying this action to φ^*I, we see that it merely switches the term corresponding to $\tilde{\beta} = (\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3)$ with that of $\tilde{\beta}' = (\tilde{\beta}_2, \tilde{\beta}_1, \tilde{\beta}_3)$. Therefore, φ^*I is invariant under the Weyl action, and so, by treating the right-hand side of (VIII.3.5) as having cohomology in X_G, we can regard it as a formula for the extended I-function, $I^{X_G}(q, z)$.

Lemma VIII.19.

$$I^{X_G}(q, z) = \mathbf{1} + z^{-1}(8q_1 + q_2)\mathbf{1} + z^{-1}(q_1^{-2/3}q_2 + 3q_1^{1/3})\mathbf{1}_{1/3} + O(z^{-2})$$

Proof. Recall that the I-function is homogenous of degree 0, where we have $\deg z = 1$, $\deg q^\beta = \beta(\omega_{X_G}^\vee)$, and the degree of a cohomology class is it's complex Chen-Ruan degree, e.g. $\deg H_i = 1$ rather than 2. Taking determinants of the conormal and Euler sequences associated to X_G, and noting that the determinant of the sheaf $\mathfrak{g} \times_G V$ is trivial since the roots come in opposing pairs, we see

$$\omega_{X_G}^\vee = \mathcal{O}(1, 1)$$

where we use the identification $\chi(G \times \mathbb{C}^*) \cong \mathbb{Q}^2$ with basis given by θ and $id_{\mathbb{C}^*}$. Note that the set (VIII.3.2) is generated over $\mathbb{Z}_{\geq 0}$ by $\frac{1}{3}(2, -1, 0)$, $\frac{1}{3}(-1, 2, 0)$, and $\frac{1}{3}(-1, -1, 3)$. Under r, these map to $\beta = (1/3, 0)$ and $\beta = (-2/3, 1)$. Applying these to $\omega_{X_G}^\vee$, we see that

$$\deg q^\beta \geq \frac{1}{3}$$

for all $\beta \in \mathrm{Eff}^I$. Since the degrees of the cohomology classes are also always positive, we see that the power of z in each term of the I-function has to be non-positive, and that the only term with z^0 comes from $\beta = (0, 0)$, which is easily seen to contribute the term $\mathbf{1}$ in the I-function.

Now we want to find which terms have z factor z^{-1}. Note that given a toric quasimap degree $\tilde{\beta} = (\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3)$, the power of z in each term is bounded above by

$$-\left(\sum_{i=1}^{3} \tilde{\beta}_i + \deg j^*\iota\left([F_{\tilde{\beta}}]^{\mathrm{vir}}\right)\right),$$

where we use that $\deg \omega_{X_T}^{\vee} = (1,1,1)$. From the description of the fixed loci above, we have that the only possible cases that result in $\sum_{i=1}^{3} \tilde{\beta}_i + \deg j^*\iota\left([F_{\tilde{\beta}}]^{\mathrm{vir}}\right) = 1$ come from $\tilde{\beta} \cdot (1,1,1) = 1$ and $\deg j^*\iota\left([F_{\tilde{\beta}}]^{\mathrm{vir}}\right) = \mathbf{1}$, or $\tilde{\beta} \cdot (1,1,1) = 1/3$ and $\deg j^*\iota\left([F_{\tilde{\beta}}]^{\mathrm{vir}}\right) = \mathbf{1}_{1/3}$.

By explicitly checking which $\tilde{\beta}$ satisfy one of the above two conditions, we see that we are left with the $\tilde{\beta}$ degrees:

$$(1,0,0) \quad (0,1,0) \quad (0,0,1) \quad (-1/3,-1/3,1) \quad (2/3,-1/3,0) \quad (-1/3,2/3,0)$$

and after applying r, we get the degrees β equal to one of the following

$$(1,0) \quad (0,1) \quad (-2/3,1) \quad (1/3,0).$$

The coefficients of the z^{-1} term for each of these degrees follows from direct computation.

$\square$

Remark VIII.20. In the proof, we found the $\tilde{\beta} = (\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3)$ for which $I_{\tilde{\beta}}$ has a term with the power of z at least -1. But, in the root factor (VIII.3.4), there is a term $\frac{Cz}{H_1 - H_2}$ which we multiply $I_{\tilde{\beta}}$ by, hence one might think that we should look for all $\tilde{\beta}$ for which $I_{\tilde{\beta}}$ has a term with power z^{-2} instead. However, one can check that after multiplying by $\frac{Cz}{H_1 - H_2}$, the resulting z^{-1} term cancels with the corresponding z^{-1} term coming from $\tilde{\beta}' = (\tilde{\beta}_2, \tilde{\beta}_1, \tilde{\beta}_3)$, hence we do not need to consider them. It's recommended to try a computation with $\tilde{\beta} = (2,0,0)$ to see this in effect.

The above lemma tells us that

$$\mu(q,z) = [zI - z]_+ = (8q_1 + q_2)\mathbf{1} + (q_1^{-2/3} q_2 + 3q_1^{1/3})\mathbf{1}_{1/3}$$

and hence by the mirror theorem (VI.2.3), we get

$$J^{\infty}(\mu(q_1,q_2), q, z) = I(q,z). \tag{VIII.3.6}$$

VIII.3.4: Quantum Period

We compute the quantum period of $Y /\!\!/ G$, following the definition in [4]. The quantum period for X_G is given by

$$G(x,t) := [J\left(\gamma(t,x), t^{-\omega_{X_G}}, 1\right)]_{\mathbf{1}} \qquad \text{where} \qquad \gamma(t,x) := t^{1/3}x\mathbf{1}_{1/3}.$$

Here, $[\cdot]_{\mathbf{1}}$ denotes the coefficient of the class $\mathbf{1}$, and the argument $t^{-\omega_{X_G}}$ means that we are replacing the Novikov variable q^β with $t^{\beta(-\omega_{X_G})}$. Note that we are setting $z = 1$, so that we do not have to worry about any difference in convention between J-functions, i.e. whether one multiplies by z to match the style of Givental, or not.

To compute $G(x,t)$, we set $q_1 = t$ and $q_2 = t(x-3)$ in (VIII.3.6). On the left-hand side, we have that

$$\mu(t, t(x-3)) = t(x+5)\mathbf{1} + t^{1/3}x\mathbf{1}_{1/3} = t(x+5)\mathbf{1} + \gamma(t,x).$$

Moreover, we recall that the degrees $\beta = (d_1, d_2)$ for J^∞ require that $d_2 = 0$. With the description of the anticanonical bundle, we have after the change of coordinates that q^β becomes t^{d_1}, which is equal to $t^{\beta(-\omega_{X_G})}$ for all effective β in the J-function.

On the right-hand side of (VIII.3.6), after writing $I(q,z) = I(q_1, q_2, z)$, we get $I(t, t(x-3), z)$. Thus, after setting $q_1 = t$, $q_2 = t(x-3)$, and $z = 1$, equation (VIII.3.6) becomes

$$[J\left(t(x+5)\mathbf{1} + \gamma(t,x), t^{-\omega_{X_G}}, 1\right)]_{\mathbf{1}} = [I(t, t(x-3), q)]_{\mathbf{1}}$$

and hence by the string equation, we get

$$G(x,t) = e^{-t(x+5)}[I(t, t(x-3), q)]_{\mathbf{1}} \tag{VIII.3.7}$$

From the explicit description of $j^*\iota\left([F_{\tilde{\beta}}]^{\mathrm{vir}}\right)$, it's easy to see that an I-effective $\tilde{\beta} = (\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3)$ term can only contribute to the coefficient of $\mathbf{1}$ in the I-function if $\tilde{\beta}_1, \tilde{\beta}_2 \in \mathbb{Z}_{\geq 0}$.

Let

$$A_{\tilde{\beta}}(q_1, q_2, z) := \frac{q_1^{\tilde{\beta}_1+\tilde{\beta}_2} q_2^{\tilde{\beta}_3}(-1)^{\tilde{\beta}_1-\tilde{\beta}_2}(2\tilde{\beta}_1 + 2\tilde{\beta}_2 + \tilde{\beta}_3)!^4}{\tilde{\beta}_3!\tilde{\beta}_1!^3\tilde{\beta}_2!^3(2\tilde{\beta}_1 + \tilde{\beta}_2 + \tilde{\beta}_3)!^2(\tilde{\beta}_1 + 2\tilde{\beta}_2 + \tilde{\beta}_3)!^2 z^{\tilde{\beta}_1+\tilde{\beta}_2+\tilde{\beta}_3}}$$

Then from (VII.5.5) and (VIII.3.5), one can check that $\tilde{\beta}$ with $\tilde{\beta}_1, \tilde{\beta}_2 \in \mathbb{Z}_{\geq 0}$ contributes the following to the I-function: is equal to

$$A_{\tilde{\beta}}\left(1+\frac{(\tilde{\beta}_1-\tilde{\beta}_2)z}{H_1-H_2}\right)\left(-\sum_{k=1}^{\tilde{\beta}_1}\frac{3H_1}{kz}-\sum_{k=1}^{\tilde{\beta}_2}\frac{3H_2}{kz}-\sum_{k=1}^{2\tilde{\beta}_1+\tilde{\beta}_2+\tilde{\beta}_3}\frac{2(2H_1+H_2)}{kz}\right.$$
$$\left.-\sum_{k=1}^{\tilde{\beta}_1+2\tilde{\beta}_2+\tilde{\beta}_3}\frac{2(H_1+2H_2)}{kz}+\sum_{k=1}^{2\tilde{\beta}_1+2\tilde{\beta}_2+\tilde{\beta}_3}\frac{4(2H_1+2H_2)}{kz}\right)+O(H_i)$$

Define

$$B_\ell := \sum_{k=1}^{\ell}\frac{1}{k}.$$

After adding the $(\tilde{\beta}_1,\tilde{\beta}_2,\tilde{\beta}_3)$ and $(\tilde{\beta}_2,\tilde{\beta}_1,\tilde{\beta}_3)$ terms together and setting $q_1=t$, $q_2=t(x-3)$ and $z=1$, we get

$$A_{\tilde{\beta}}(t,t(x-3),1)\left(2+(\tilde{\beta}_1-\tilde{\beta}_2)(-3B_{\tilde{\beta}_1}+3B_{\tilde{\beta}_2}-2B_{2\tilde{\beta}_1+\tilde{\beta}_2+\tilde{\beta}_3}+2B_{\tilde{\beta}_1+2\tilde{\beta}_2+\tilde{\beta}_3})\right)+O(H_i).$$

Therefore, from (VIII.3.7), we get the following formula:

Theorem VIII.21.

$$G(x,t)=e^{-t(x+5)}\times$$
$$\sum_{\tilde{\beta}_i\in\mathbb{Z}_{\geq 0}}A_{\tilde{\beta}}(t,t(x-3),1)\left(1+\frac{\tilde{\beta}_1-\tilde{\beta}_2}{2}(-3B_{\tilde{\beta}_1}+3B_{\tilde{\beta}_2}-2B_{2\tilde{\beta}_1+\tilde{\beta}_2+\tilde{\beta}_3}+2B_{\tilde{\beta}_1+2\tilde{\beta}_2+\tilde{\beta}_3})\right).$$

Observe that if we set $x=3$, we recover the specialization at the end of [51, Section 6.2].

We can also regularize this quantum period in hopes that the regularization is mirror dual to a maximally mutable Laurent polynomial associated to this Del Pezzo surface, which was the aim of the computation in [51] (see [4, Conj. B] for details on this duality). The first few terms of the regularization are provided below:

$$1+(14x+70)t^2+(6x^2+210x+966)t^3+(546x^2+6888x+22470)t^4+\dots$$

One can check that this agrees with the first few terms of the classical period of the Laurent polynomial $ay+\frac{x}{y^2}(1+y)^3+\frac{1}{xy^2}(1+y)^4+\frac{7}{y}+\frac{2}{y^2}$. Note that after specialization at $a=3$, this is the same Laurent polynomial as in [51, Section 7, Ex. 9].

BIBLIOGRAPHY

[1] D. Abramovich, T. Graber, and A. Vistoli. Algebraic orbifold quantum products. In *Orbifolds in mathematics and physics (Madison, WI, 2001)*, volume 310 of *Contemp. Math.*, pages 1–24. Amer. Math. Soc., Providence, RI, 2002.

[2] D. Abramovich, T. Graber, and A. Vistoli. Gromov-Witten theory of Deligne-Mumford stacks. *Amer. J. Math.*, 130(5):1337–1398, 2008. arXiv:math/0603151.

[3] A. Adem, J. Leida, and Y. Ruan. *Orbifolds and stringy topology*, volume 171 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2007.

[4] M. Akhtar, T. Coates, A. Corti, L. Heuberger, A. Kasprzyk, A. Oneto, A. Petracci, T. Prince, and K. Tveiten. Mirror symmetry and the classification of orbifold del Pezzo surfaces. *Proc. Amer. Math. Soc.*, 144(2):513–527, 2016.

[5] E. Andreini, Y. Jiang, and H.-H. Tseng. Gromov-Witten theory of root gerbes I: Structure of genus 0 moduli spaces. *J. Differential Geom.*, 99(1):1–45, 2015.

[6] K. Behrend and B. Fantechi. The intrinsic normal cone. *Invent. Math.*, 128(1):45–88, 1997. arXiv:alg-geom/9601010.

[7] L. A. Borisov, L. Chen, and G. G. Smith. The orbifold Chow ring of toric Deligne-Mumford stacks. *J. Amer. Math. Soc.*, 18(1):193–215, 2005. arXiv:math/0309229.

[8] C. Cadman. Using stacks to impose tangency conditions on curves. *American Journal of Mathematics*, 129(2):405–427, 2007.

[9] P. Candelas, X. De la Ossa, P. Green, and L. Parkes. A pair of calabi-yau manifolds as an exactly soluble superconformal theory. *Nuclear Physics B*, 359:21–74, 07 1991.

[10] Q. Chen, F. Janda, and R. Webb. Virtual cycles of stable (quasi)-maps with fields. arXiv:1911.09825, 2019.

[11] W. Chen and Y. Ruan. A new cohomology theory of orbifold. *Comm. Math. Phys.*, 248(1):1–31, 2004.

[12] D. Cheong, I. Ciocan-Fontanine, and B. Kim. Orbifold quasimap theory. *Math. Ann.*, 363(3-4):777–816, 2015. arXiv:1405.7160.

[13] I. Ciocan-Fontanine and B. Kim. Moduli stacks of stable toric quasimaps. *Adv. Math.*, 225(6):3022–3051, 2010. arXiv:0908.4446.

[14] I. Ciocan-Fontanine and B. Kim. Quasimap wall-crossings and mirror symmetry. *Publ. Math. Inst. Hautes Études Sci.*, 2020. arXiv:1611.05023.

[15] I. Ciocan-Fontanine, B. Kim, and D. Maulik. Stable quasimaps to GIT quotients. *J. Geom. Phys.*, 75:17–47, 2014. arXiv:1106.0804.

[16] E. Clader, F. Janda, and Y. Ruan. Higher-genus quasimap wall-crossing via localization. arXiv:1706.03427, 2017.

[17] T. Coates, A. Corti, H. Iritani, and H.-H. Tseng. Computing genus-zero twisted Gromov-Witten invariants. *Duke Math. J.*, 147(3):377–438, 2009. arXiv:math/0702234.

[18] T. Coates, A. Corti, H. Iritani, and H.-H. Tseng. A mirror theorem for toric stacks. *Compositio Mathematica*, 151(10):1878?1912, 2015.

[19] T. Coates, A. Corti, H. Iritani, and H.-H. Tseng. Some applications of the mirror theorem for toric stacks. *Adv. Theor. Math. Phys.*, 23(3):767–802, 2019.

[20] T. Coates, A. Gholampour, H. Iritani, Y. Jiang, P. Johnson, and C. Manolache. The quantum Lefschetz hyperplane principle can fail for positive orbifold hypersurfaces. *Math. Res. Lett.*, 19(5):997–1005, 2012. arXiv:1202.2754.

[21] T. Coates and A. Givental. Quantum Riemann-Roch, Lefschetz and Serre. *Ann. of Math. (2)*, 165(1):15–53, 2007.

[22] T. Coates, H. Iritani, and Y. Jiang. The crepant transformation conjecture for toric complete intersections. *Adv. Math.*, 329:1002–1087, 2018. arXiv:1410.0024.

[23] A. Corti and L. Heuberger. Del Pezzo surfaces with $\frac{1}{3}(1,1)$ points. *Manuscripta Math.*, 153(1-2):71–118, 2017.

[24] A. Corti and M. Reid. Weighted Grassmannians. In *Algebraic geometry*, pages 141–163. de Gruyter, Berlin, 2002.

[25] D. A. Cox, J. B. Little, and H. K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.

[26] I. Dolgachev. Weighted projective varieties. In *Group actions and vector fields (Vancouver, B.C., 1981)*, volume 956 of *Lecture Notes in Math.*, pages 34–71. Springer, Berlin, 1982.

[27] R. Donagi and E. Sharpe. GLSMs for partial flag manifolds. *J. Geom. Phys.*, 58(12):1662–1692, 2008.

[28] G. Ellingsrud and S. A. Strø mme. On the Chow ring of a geometric quotient. *Ann. of Math. (2)*, 130(1):159–187, 1989.

[29] B. Fantechi, L. Göttsche, L. Illusie, S. L. Kleiman, N. Nitsure, and A. Vistoli. *Fundamental algebraic geometry*, volume 123 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005. Grothendieck's FGA explained.

[30] H. Flenner, L. O'Carroll, and W. Vogel. *Joins and intersections*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1999.

[31] W. Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

[32] A. Givental. A Mirror Theorem for Toric Complete Intersections. In M. Kashiwara, A. Matsuo, K. Saito, and I. Satake, editors, *Topological Field Theory, Primitive Forms and Related Topics*, page 141, Jan. 1998.

[33] A. Givental. Elliptic Gromov-Witten invariants and the generalized mirror conjecture. In *Integrable systems and algebraic geometry (Kobe/Kyoto, 1997)*, pages 107–155. World Sci. Publ., River Edge, NJ, 1998. arXiv:math/9803053.

[34] A. B. Givental. Equivariant Gromov-Witten invariants. *Internat. Math. Res. Notices*, (13):613–663, 1996.

[35] A. B. Givental. Symplectic geometry of Frobenius structures. In *Frobenius manifolds*, Aspects Math., E36, pages 91–112. Friedr. Vieweg, Wiesbaden, 2004.

[36] T. Graber and R. Pandharipande. Localization of virtual classes. *Invent. Math.*, 135(2):487–518, 1999. arXiv:alg-geom/9708001.

[37] J. Guéré. Hodge–Gromov–Witten theory, 2019. arXiv:1908.11409.

[38] L. Heath and M. Shoemaker. Quantum Serre duality for quasimaps, 2021. arXiv:2107.05751.

[39] F. Janda, N. Sultani, and Y. Zhou. Computing gromov-witten invariants of non-convex complete intersections in weighted projective space. (in preparation).

[40] Y. Jiang. The orbifold cohomology ring of simplicial toric stack bundles. *Illinois J. Math.*, 52(2):493–514, 2008.

[41] S. Keel and S. Mori. Quotients by groupoids. *Ann. of Math. (2)*, 145(1):193–213, 1997.

[42] A. D. King. Moduli of representations of finite-dimensional algebras. *Quart. J. Math. Oxford Ser. (2)*, 45(180):515–530, 1994.

[43] J. Kollár. *Singularities of the minimal model program*, volume 200 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2013. With a collaboration of Sándor Kovács.

[44] J. Kollár. Seshadri's criterion and openness of projectivity. *Proc. Indian Acad. Sci. Math. Sci.*, 132(2):Paper No. 40, 2022.

[45] M. Kontsevich and Y. Manin. Gromov-Witten classes, quantum cohomology, and enumerative geometry [MR1291244 (95i:14049)]. In *Mirror symmetry, II*, volume 1 of *AMS/IP Stud. Adv. Math.*, pages 607–653. Amer. Math. Soc., Providence, RI, 1997.

[46] Y.-P. Lee. Quantum Lefschetz hyperplane theorem. *Invent. Math.*, 145(1):121–149, 2001. arXiv:math/0304140.

[47] A. Marian, D. Oprea, and R. Pandharipande. The moduli space of stable quotients. *Geom. Topol.*, 15(3):1651–1706, 2011. arXiv:0904.2992.

[48] D. Mumford. Picard groups of moduli problems. In *Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963)*, pages 33–81. Harper & Row, New York, 1965.

[49] M. Olsson. Sheaves on Artin stacks. *J. Reine Angew. Math.*, 603:55–112, 2007.

[50] M. Olsson. *Algebraic spaces and stacks*, volume 62 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2016.

[51] A. Oneto and A. Petracci. On the quantum periods of del Pezzo surfaces with $\frac{1}{3}(1,1)$ singularities. *Adv. Geom.*, 18(3):303–336, 2018.

[52] M. Romagny. Group actions on stacks and applications. *Michigan Math. J.*, 53(1):209–236, 2005.

[53] T. Stacks Project Authors. *Stacks Project*. https://stacks.math.columbia.edu, 2019.

[54] N. Sultani and R. Webb. Some applications of nonabelian Quantum Lefschetz for orbifold quasimap I-functions. (in preparation).

[55] H.-H. Tseng. Orbifold quantum Riemann-Roch, Lefschetz and Serre. *Geom. Topol.*, 14(1):1–81, 2010.

[56] J. Wang. A mirror theorem for Gromov-Witten theory without convexity. arXiv:1910.14440, 2019.

[57] R. Webb. The abelian-nonabelian correspondence for i-functions, 2018.

[58] R. Webb. Abelianization and Quantum Lefschetz for orbifold quasimap I-functions. arXiv:2109.12223, 2021.

[59] Y. Zhou. Quasimap wall-crossing for GIT quotients. *Invent. Math.*, 227(2):581–660, 2022.