Left Inversion, System Zeros, and Input Estimation

by

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To Amma and Pappa for their love and support

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ABSTRACT

This dissertation focuses on input estimation, that is, estimation of the input to a linear system using knowledge of the output measurements and the system model for tall or square systems with full column rank. First, finite-time input estimation for discrete-time linear time-invariant (LTI) systems with zero nonzero zeros and unknown initial conditions is considered. Necessary and sufficient conditions for finite-time input estimation are derived. For systems with zero nonzero zeros, a specific construction of finite-impulse-response (FIR) delayed left inverse with minimal delay using the Smith-McMillan form at infinity is given.

Since zeros play a vital role in input estimation, further research on system zeros is considered. Expressions for the number of transmission zeros and the number of infinite zeros in terms of the defect of a block-Toeplitz matrix of Markov parameters and the observability matrix are obtained. For counting zeros, these results serve as duals to the counting of poles using the block-Hankel matrix. Furthermore, the zero dynamics of input-output models are explored, and their properties are elucidated. Output zeroing in input-output models is considered and its equivalence to output zeroing in state space models is discussed.

Next, retrospective cost input estimation (RCIE), which is an adaptive input estimation technique for discrete-time linear time-varying (LTV) systems that depends on a target model based on the closed-loop dynamics, is considered. In particular, the decomposition of the retrospective performance variable into the sum of a performance term and a model-matching term, which provides insight into the achievable performance of RCIE, is presented. Since the system dynamics and target model are LTV, the construction of LTV state space realizations for LTV input-output models as well as the construction of LTV input-output models for LTV state space models are given in this dissertation. Using the same technique used for RCIE, the decomposition of the retrospective performance variable in retrospective cost adaptive control (RCAC) is also derived.

Finally, as an application of input estimation, causal numerical differentiation is considered. When the dynamics of the system consist of a cascade of one or more integrators, the estimates of the input provide estimates of one or more derivatives of the output signal. The performance of RCIE as a causal differentiator is analyzed through numerical simulations. RCIE as a causal differentiator is then applied to the position data of a small rover to estimate its velocity and acceleration.

CHAPTER 1

Introduction

1.1 System Inversion and Input Estimation

Left inverses and right inverses of functions and systems play an important role in many fields of engineering. The need to invert dynamical systems arises in many control-system related applications such as feedforward control, output tracking, and input estimation.

This dissertation focuses on input estimation using left inversion; however, the role of right inverses is briefly reviewed in order to clarify the distinction. The theory of right inverses for continuous-time systems was studied in [1–5] and the theory of right inverses for discrete-time systems were studied in [5–8]. Right inverses are used for feedforward control and output tracking. For illustration, consider the command $y_{\rm d}$ shown in Figure 1.1, and suppose that the input u to the transfer function G is given by the output of the delayed right inverse H of G. Letting $y_{\rm d}$ be the input to H, it follows that $y = Gu = GHy_{\rm d} = y_{\rm d}$, which implies that the output y follows the command $y_{\rm d}$. However, this calculation ignores the effect of pole-zero cancellation and initial conditions. The construction of right inverses is challenging in the case where G has a nonminimum-phase transmission zero, which may entail a hidden instability. For this case, approximate right inverses have been developed for feedforward control and output tracking [9–24].

Unlike right inverses, left inverses estimate unknown inputs. For example, let u be an unknown input, as shown in Figure 1.2, and suppose that the output y of the plant G is the input to the delayed left inverse H. In the case where the initial conditions of G and H are zero and G has

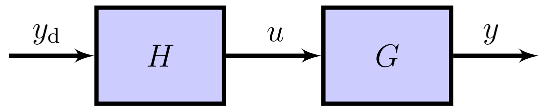

Figure 1.1: Cascade of a right invertible system and a delayed right inverse H.

no zeros, it follows that the output of H is $z = Hy = HGu = u$. The theory of left inverses for linear systems was developed in [25–27]. Necessary and sufficient conditions for the existence of delayed left inverses in terms of the ranks of successive block-Toeplitz matrices were given in [25]. An algorithm for constructing the inverse of an invertible square system was given in [27]. Delayed left inverses were constructed in [26] for both continuous-time and discrete-time linear systems. Additional work on the theory of left inverses includes [2–4, 6–8, 28–31].

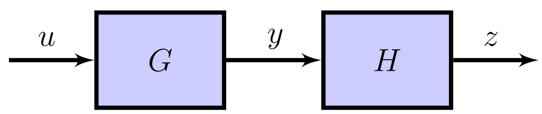

Figure 1.2: Cascade of a left invertible system and a delayed left inverse H. Although $z = Hy = HGu = u$, it does not necessarily follow that the output z matches a delayed version of u due to initial conditions and zeros.

1.2 Role of Zeros in Input Estimation

The zeros of a dynamical system present an impediment to input estimation. If a system has a transmission zero, then there exist an initial condition and nonzero input such that the response of the system is identically zero. For example, consider the discrete-time transfer function $G(\mathrm{z}) = \dfrac{\mathrm{z}+1}{\mathrm{z}+2}$, whose inverse is $H(\mathrm{z}) = \dfrac{\mathrm{z}+2}{\mathrm{z}+1}$. Now consider the minimal realization of G given by

$$x_{k+1} = -2x_k + u_k, \tag{1.1}$$

$$y_k = -x_k + u_k. \tag{1.2}$$

Letting $x_0 = 1$ and $u_k = (-1)^k$, it follows that $y \equiv 0$. On the other hand, letting $x_0 = 0$ and $u \equiv 0$, it also follows that $y \equiv 0$. Hence, it is impossible to estimate the input from the output. Although G is invertible, the zero at -1 prevents unambiguous estimation of the input. Therefore, the study of system zeros is extremely important within the context of input estimation.

1.3 Literature Review on Input Estimation

Within a deterministic, discrete-time setting, finite-time input estimation provides the exact values of the input. For finite-time input estimation, there are three key issues. The first issue concerns the delay under which the input can be estimated. The minimal delay was determined in [25], which showed that the minimal delay is the smallest index for which the difference of the ranks of two successive block-Toeplitz matrices is equal to the number of inputs. The second issue concerns the presence of zeros. Since zeros block inputs, the presence of zeros prevents unambiguous estimation of the system input. The third issue concerns the effect of unknown, nonzero initial conditions. In particular, the free response of the system contributes to its output, thus making it difficult to determine the input by inverting the system.

Results on finite-time input estimation, that is, exact input estimation after a finite number of steps were given in [32–36]. In particular, for systems with no zeros, a state-estimation and input-estimation algorithm was given in [36] and an alternative input-estimation algorithm based on the generalized inverse of a partitioned matrix was given in [34].

If a system has at least one zero that is not zero, then finite-time input estimation cannot be achieved for any delay. However, in this case, input estimation is possible asymptotically; a more appropriate name for this problem is *asymptotic input estimation*. Input estimation was considered for systems with no zeros in [37, 38], for minimum-phase systems in [39] and for nonminimum phase systems in [11, 40–48].

Input estimation is distinct from the design of unknown input observers [49–58]. In particular, unknown input observers are constructed to estimate states despite the presence of an unknown input that is not captured by zero-mean white noise as in the case of the Kalman filter. Unknown input observers are thus state estimators that are robust to the presence of non-stochastic unknown inputs. It is important to emphasize, however, that, unlike input estimation, unknown input observers do not estimate unknown inputs.

1.4 Applications of Input Estimation

One of the main applications of input estimation is target tracking [59–62]. Target tracking is the determination of the present and often future position and velocity of a moving target from noisy measurements of its present states. In order to perform target tracking, the acceleration of the target is treated as an input and then estimated using input estimation techniques. The estimated acceleration is then used in conjunction with Kalman filter to estimate the states.

Input estimation is also used in fault detection and diagnosis [63–65]. In sensor fault detection and diagnosis, a causal, delayed left inverse of a dynamical system that represents the relationship between two sets of sensors, namely, input sensors, which are suspect, and output sensors, which are assumed to be healthy, is constructed. Measurements from the healthy sensors are used to drive the delayed left inverse, whose output provides estimates of the expected measurements from the suspect input sensors. By comparing the estimates of the measurements of the suspect sensors with the actual measurements, it is possible to detect and diagnose faults in these sensors.

Additional applications of input estimation include determination of features of disturbances [40, 66], and automotive control [67–69].

Another application of input estimation is causal numerical differentiation, which has not been considered previously in the literature and is part of the work in this dissertation. When the dynamics of the system consist of a cascade of one or more integrators, the estimates of the input provide estimates of one or more derivatives of the output signal. Since, like state estimation, input estimation is an online technique, this approach is suitable for causal numerical differentiation.

1.5 Adaptive Input Estimation

Adaptive input estimation is considered in [70], where the goal is to estimate the velocity and acceleration of a maneuvering vehicle. In this technique known as *retrospective cost input estima-*

tion (RCIE), input estimation is combined with state estimation based on the discrete-time Kalman filter. Additional prior work on RCIE includes [64, 71–75].

RCIE is applicable to discrete-time linear time-varying (LTV) multi-input, multi-output (MIMO) systems. In RCIE, the error metric for adaptation is given by the estimation residual, that is, the innovations. A retrospective performance variable based on the innovations is defined. The retrospective performance variable depends on a target model that is based on the closed-loop system dynamics. A cost function involving the retrospective performance variable is minimized using retrospective cost optimization [76] to update the coefficients of the input estimator. Retrospective cost optimization is based on recursive least squares (RLS). RCIE then replicates the estimated input in the Kalman filter to estimate the states.

1.6 Contributions

This section summarizes the contributions of the work presented in this dissertation relative to the prior literature.

Finite-Time Input Estimation

Finite-time input estimation for discrete-time LTI systems is considered in [32–36], where the input is reconstructed based on a state space approach for systems with no zeros. These works do not give an explicit construction of left inverses.

The work in this dissertation has three key contributions relative to prior work. First, a specific construction of a finite-impulse-response (FIR) delayed left inverse with minimal delay for systems with zero nonzero zeros is presented. Next, it is shown that, in the presence of an arbitrary unknown initial condition, finite-time input estimation is possible using a delayed left inverse H if and only if H is FIR. Finally, it is shown that a transfer function with full column normal rank has an FIR delayed left inverse with the minimal delay if and only if the system has zero nonzero zeros. The presented works on finite-time input estimation were published in [77, 78].

System Zeros

The number of poles of a transfer function can be obtained from knowledge of the Markov parameters since the rank of a block Hankel matrix of Markov parameters is equal to the McMillan degree [79]. This is useful for estimating the number of poles in the case where the system model is unknown. The number of transmission zeros of a transfer function can be counted by forming the Smith-McMillan form, and the number of infinite zeros of a transfer function can be counted by forming the Smith-McMillan form at infinity [80]. Another approach is to compute the number of transmission and infinite zeros by using pole and zero modules [81]. However, the above approaches for counting zeros are not feasible if the system model is unknown.

Within the context of discrete-time LTI systems, the present dissertation describes alternative characterizations of the number of transmission zeros and the number of infinite zeros. In particular, the number of zeros is related to the defect of a block-Toeplitz matrix of Markov parameters. For counting zeros, these results serve as duals to the counting of poles using the block-Hankel matrix and provide a method to estimate the number of zeros from the Markov parameters when the system model is unknown.

Next, this dissertation presents several novel results on the zero dynamics of input-output models. The main motivation for this work is the fact that RCIE uses input-output models whose zeros directly impact the ability to perform input estimation. Zeros of state space models have been extensively studied over the years [82–87]. However, zeros of input-output models have not been studied in the literature.

The presented works on system zeros were published in [88, 89].

Decomposition of the Retrospective Performance Variable

The purpose of the decomposition of retrospective performance variable is to investigate the underlying mechanism and performance of RCIE. In this direction, this paper provides a detailed analysis of the decomposition of the retrospective performance variable, which provides insight into the achievable performance of RCIE. In particular, the retrospective performance variable is decomposed into the sum of a performance term and a model-matching term. The performance term consists of a closed-loop time-domain transfer function, whereas the model-matching term involves a closed-loop time-domain transfer function and the target model, both driven by the virtual external input perturbation. This work is motivated by the decomposition of the retrospective performance variable given in [90] within the context of retrospective cost adaptive control (RCAC) [76, 91–93]. However, unlike [90], the system dynamics and target model in the present paper are linear time-varying (LTV), and hence the approach given in [90] is not applicable here.

The main contribution of the present work is thus the development of an alternative approach to the decomposition of the retrospective performance variable that is applicable to LTV models. This approach depends on the construction of discrete-time LTV state space realizations for LTV input-output models as well as the construction of LTV input-output models for LTV state space models. The existing results on LTV input-output models in [94–99] are presented in terms of abstract input-output maps and infinite power series, are not directly implementable and thus not applicable to the problems considered here. Consequently, the present paper gives simple and easily implementable algebraic results on LTV input-output dynamics needed to derive the decomposition of the retrospective performance variable in RCIE.

The presented works on the decomposition of retrospective performance variable in RCIE were published in [100].

This dissertation also presents the decomposition of the retrospective performance variable in RCAC for the case where the system dynamics and target model are LTV. The same approach that was used for the decomposition of retrospective performance variable in RCIE is used in RCAC.

Causal Numerical Differentiation

Many applications of estimation and control benefit from the ability to perform causal differentiation, that is, numerical differentiation that provides estimates of the derivative of a signal based on current and past data [101–105]. Numerous techniques have been developed for numerical differentiation, including integration-based methods [106, 107], observer-based methods [108, 109], and sliding-mode techniques [110–113]. However, many of these works are either noncausal implementations or they are difficult to tune in practice.

The work presented in this dissertation formulates causal numerical differentiation as an input estimation problem. The accuracy of retrospective cost input estimation (RCIE) and the high-gain observer (HGO) given in [108] for causal numerical differentiation in the presence of noisy measurements are compared through numerical simulations. For simplicity and clarity, the numerical study in this dissertation considers harmonic signals corrupted by Gaussian white noise. For each input signal, backward-difference numerical differentiation provides a baseline for performance comparison. The different methods are then applied to the position data of a small rover to estimate its velocity and acceleration.

The work on causal numerical differentiation was done in collaboration with Shashank Verma and was published in [114].

1.7 Dissertation Outline

This dissertation is organized as follows.

Chapter 2 Summary

Chapter 2 lists all the notations and definitions used in this dissertation.

Chapter 3 Summary

Chapter 3 presents results on finite-time input estimation for discrete-time linear time-invariant systems using FIR inverses. Numerical examples are provided for illustrating the results.

Chapter 4 Summary

Chapter 4 gives expressions for the number of transmission zeros and he number of infinite zeros of a MIMO transfer function in terms of the defect of an augmented matrix involving an observability matrix and a block-Toeplitz matrix. These results are illustrated with a numerical example.

Chapter 5 Summary

Chapter 5 elucidates the properties of the zero dynamics within the context of input-output models. In addition, output zeroing in input-output models is considered, and its equivalence to output zeroing in state space models is discussed. Finally, a numerical example is presented to illustrate the results.

Chapter 6 Summary

Chapter 6 gives a concise description of the RCIE algorithm.

Chapter 7 Summary

Chapter 7 gives the construction of LTV state space realizations for LTV input-output models as well as the construction of LTV input-output models for LTV state space models.

Chapter 8 Summary

Chapter 8 presents the decomposition of the retrospective performance variable in RCIE into the sum of a performance term and a model-matching term. A numerical example is used to illustrate the derived results and observations.

Chapter 9 Summary

Chapter 9 presents and compares causal numerical differentiation using RCIE and HGO. The velocity and acceleration of a small rover are estimated by numerical differentiation of experimental position data of the rover.

Chapter 10 Summary

Chapter 10 gives a concise description of the RCAC algorithm and presents the decomposition of the retrospective performance variable in RCAC into the sum of a performance term and a model-matching term.

Finally, chapter 11 gives the conclusions and future work.

CHAPTER 2

Preliminaries

This chapter lists all the notations and definitions used in this dissertation.

2.1 Notations

$\mathbb{R}[\mathrm{z}]^{p\times m}$	the set of $p \times m$ matrices each of whose entries is a polynomial with real coefficients
$\mathbb{R}(\mathrm{z})^{p\times m}$	the set of $p \times m$ matrices each of whose entries is a rational function with real coefficients
$\mathbb{R}(\mathrm{z})^{p\times m}_{\mathrm{prop}}$	the proper transfer functions in $\mathbb{R}(\mathrm{z})^{p\times m}$
$\overset{\min}{\sim}$	a minimal realization of a transfer function
$\dim V$	the dimension of a vector space V
$\mathcal{R}(A)$	the range of A
$\operatorname{def} A$	the defect of A
$\operatorname{ind} A$	the index of A
0^0	1
$\operatorname{McDeg} G$	the McMillan degree of G
$\mathbf{q}$	the forward shift operator
$\mathbf{q}^{-1}$	the backward shift operator

2.2 Definitions

Definition 2.1. Let $G \in \mathbb{R}(\mathrm{z})^{p\times m}_{\mathrm{prop}}$, and, for each $i \geq 0$, let H_i be the ith Markov parameter of G. Then, for all $i \geq 0$, the ith *Markov block-Toeplitz matrix* associated with G is defined by

$$\mathcal{T}_i \triangleq \begin{bmatrix} H_0 & 0 & 0 & \cdots & 0 \\ H_1 & H_0 & 0 & \cdots & 0 \\ H_2 & H_1 & H_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ H_i & H_{i-1} & \cdots & H_1 & H_0 \end{bmatrix} \in \mathbb{R}^{(i+1)p\times(i+1)m}. \tag{2.1}$$

In the case where i is a negative integer, $\mathcal{T}_i$ is an empty matrix.

Definition 2.2. Let $G \in \mathbb{R}(\mathrm{z})^{m\times p}_{\mathrm{prop}}$, and let d be a nonnegative integer. Then, G is *delayed left invertible with delay* d if there exists $H \in \mathbb{R}(\mathrm{z})^{m\times p}_{\mathrm{prop}}$ such that $H(\mathrm{z})G(\mathrm{z}) = \mathrm{z}^{-d}I_m$. In this case, H is a *delayed left inverse of* G *with delay* d. Furthermore, G is *delayed left invertible* if there exists $d \geq 0$ such that G is delayed left invertible with delay d, and H is a *delayed left inverse of* G if there exists $d \geq 0$ such that H is a delayed left inverse of G with delay d. Finally, H is a *left inverse of* G if H is a delayed left inverse of G with delay $d = 0$.

Definition 2.3. Let $A \in \mathbb{R}^{n\times n}$. Then, the *index* of A, denoted by $\operatorname{ind} A$, is the smallest nonnegative integer ν such that rank A^ν = rank $A^{\nu+1}$.

Note that, if A is nilpotent, then $\operatorname{ind} A$ is the smallest positive integer ν such that $A^\nu = 0$.

Definition 2.4. Let $G \in \mathbb{R}(\mathrm{z})^{p\times m}_{\mathrm{prop}}$, where $G \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and $A \in \mathbb{R}^{n\times n}$. Then, the *index* of G, denoted by $\operatorname{ind} G$, is $\operatorname{ind} A$.

Definition 2.5. Let $U \in \mathbb{R}[\mathrm{z}]^{n\times n}$. Then U is *unimodular* if $\det U$ is a nonzero constant.

Definition 2.6. Let $W \in \mathbb{R}(\mathrm{z})^{m\times m}_{\mathrm{prop}}$. Then W is *biproper* if $W_\infty \triangleq \lim_{\mathrm{z}\to\infty} W(\mathrm{z})$ is nonsingular.

Definition 2.7. Let $G \in \mathbb{R}(\mathrm{z})^{p\times m}_{\mathrm{prop}}$ and $i \geq 0$. Then $\beta_i(G) \triangleq \operatorname{rank} \mathcal{T}_i - \operatorname{rank} \mathcal{T}_{i-1}$, where $\mathcal{T}_i$ is ith Markov block-Toeplitz matrix associated with G.

Definition 2.8. Let $G \in \mathbb{R}(\mathrm{z})^{p\times m}_{\mathrm{prop}}$. The *rank* of G is the maximum value of $\operatorname{rank} G(\mathrm{z})$ taken over the set of complex numbers z such that, for all $i = 1, \ldots, p$ and $j = 1, \ldots, m$, z is not a pole of the (i, j) entry of G.

Definition 2.9. Let $G \in \mathbb{R}(\mathrm{z})^{p\times m}_{\mathrm{prop}}$, and assume that G has full column normal rank. Then η_G denotes the smallest nonnegative integer d for which there exists a delayed left inverse of G with delay d.

Definition 2.10. Let $G \in \mathbb{R}(\mathrm{z})^{p\times m}_{\mathrm{prop}}$. Then the roots of the polynomial $p_1 p_2 \ldots p_\rho$ are the *transmission zeros* of G, where the polynomials $p_1, p_2, \ldots, p_\rho$ are the numerators of the nonzero diagonal entries of the Smith-McMillan form given by Theorem 3.2.

Definition 2.11. Let $A \in \mathbb{R}^{m\times n}$. Then, the *defect* of A, denoted by $\operatorname{def} A$, is the rank of the nullspace of A.

Definition 2.12. Let $V \subseteq \mathbb{R}^n$ and let

$$\begin{bmatrix} A \\ C \end{bmatrix} V \subseteq \begin{bmatrix} I \\ 0 \end{bmatrix} V + \mathcal{R}\left(\begin{bmatrix} B \\ D \end{bmatrix}\right). \tag{2.2}$$

Then V is an *output-nulling invariant subspace* of (A, B, C, D). The sum of all output-nulling invariant subspaces of (A, B, C, D) is the *maximal output-nulling invariant subspace* of (A, B, C, D).

Definition 2.13. Let $G \in \mathbb{R}(\mathrm{z})^{p\times m}_{\mathrm{prop}}$, where $G \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and $A \in \mathbb{R}^{n\times n}$. Then, the *Rosenbrock system matrix* is defined as

$$\mathcal{Z}(\mathrm{z}) \triangleq \begin{bmatrix} \mathrm{z}I - A & -B \\ C & D \end{bmatrix}. \tag{2.3}$$

Furthermore, $\mathrm{z} \in \mathbb{C}$ is an invariant zero of the realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ if $\operatorname{rank} \mathcal{Z}(\mathrm{z}) < \operatorname{rank} \mathcal{Z}$.

Definition 2.14. Let $P \in \mathbb{R}[\mathrm{z}]^{p\times m}$ and $R \in \mathbb{R}[\mathrm{z}]^{p\times p}$. Then R *left divides* P if there exists $\hat{P} \in \mathbb{R}[\mathrm{z}]^{p\times m}$ such that $P = R\hat{P}$.

Definition 2.15. Let $P \in \mathbb{R}[\mathrm{z}]^{p\times n}$ and $Q \in \mathbb{R}[\mathrm{z}]^{p\times m}$. Then P and Q are *coprime* if every $R \in \mathbb{R}[\mathrm{z}]^{p\times p}$ that left divides both P and Q is unimodular.

Definition 2.16. Let $P \in \mathbb{R}[\mathrm{z}]^{p\times m}$. Then $\deg P$ is the maximum degree of the entries of P. Furthermore, P is *monic* if $p = m$ and $P(\mathrm{z}) = \mathrm{z}^{\deg P} I_m + P_0(\mathrm{z})$, where $P_0 \in \mathbb{R}[\mathrm{z}]^{m\times m}$ and $\deg P_0 < \deg P$.

Definition 2.17. Let $D \in \mathbb{R}[\mathrm{z}]^{p\times p}$, and $N \in \mathbb{R}[\mathrm{z}]^{p\times m}$, assume that D is nonsingular, and assume that $G = D^{-1}N$. Then (D, N) is a *left polynomial fraction description (LPFD)* of G. Furthermore, if D and N are coprime, then (D, N) is a *coprime left polynomial fraction description (CLPFD)* of G. In addition, if D is monic, then (D, N) is a *monic left polynomial fraction description (MLPFD)* of G. Finally, if D and N are coprime and D is monic, then (D, N) is a *monic coprime left polynomial fraction description (MCLPFD)* of G.

Note that the terms ‘matrix fraction description’ and ‘polynomial matrix fraction description’ are used as alternatives to the term ‘polynomial fraction description’ in the literature.

Definition 2.18. Let $P \in \mathbb{R}[\mathrm{z}]^{p\times n}$, $Q \in \mathbb{R}[\mathrm{z}]^{p\times m}$ and $R \in \mathbb{R}[\mathrm{z}]^{p\times p}$. Then R is a *greatest common left divisor* of P and Q if there exists $\hat{P} \in \mathbb{R}[\mathrm{z}]^{p\times n}$, $\hat{Q} \in \mathbb{R}[\mathrm{z}]^{p\times m}$ such that $P = R\hat{P}$, $Q = R\hat{Q}$ and $\hat{P}$ and $\hat{Q}$ are coprime.

Definition 2.19. Let $G \in \mathbb{R}(\mathrm{z})^{p\times m}_{\mathrm{prop}}$, let (D, N) be an LPFD of G, and, for all $k \geq 0$, let $u_k \in \mathbb{C}^m$ satisfy

$$N(\mathbf{q})u_k = 0. \tag{2.4}$$

Then, (2.4) is the *zero dynamics of* (D, N). If, in addition, (D, N) is a CLPFD of G, then (2.4) is the *zero dynamics of* G.

Definition 2.20. Let $y_{-n}, y_{-n+1}, \ldots, y_{-1} \in \mathbb{R}^p$. Consider the input-output model of a linear time-varying system given by, for all $k \geq 0$,

$$y_k + D_{1,k} y_{k-1} + \cdots + D_{n,k} y_{k-n} = N_{0,k} u_k + \cdots + N_{n,k} u_{k-n}, \tag{2.5}$$

where, $u_k \in \mathbb{R}^m$ is the input, $y_k \in \mathbb{R}^p$ is the output, $D_{1,k}, \ldots, D_{n,k} \in \mathbb{R}^{p\times p}$, and $N_{0,k}, \ldots, N_{n,k} \in \mathbb{R}^{p\times m}$. Define

$$D_k(\mathbf{q}^{-1}) \stackrel{\triangle}{=} I_p + D_{1,k}\mathbf{q}^{-1} + \cdots + D_{n,k}\mathbf{q}^{-n}, \tag{2.6}$$

$$N_k(\mathbf{q}^{-1}) \stackrel{\triangle}{=} N_{0,k} + N_{1,k}\mathbf{q}^{-1} + \cdots + N_{n,k}\mathbf{q}^{-n}. \tag{2.7}$$

Then, $G_k \stackrel{\triangle}{=} D_k^{-1} N_k$ is the *time-domain transfer function at step* k of the system represented by (2.5). In terms of G_k, (2.5) is written as

$$y_k = G_k(\mathbf{q}^{-1}) u_k, \tag{2.8}$$

and in terms of N_k and D_k, (2.5) is written as

$$D_k(\mathbf{q}^{-1}) y_k = N_k(\mathbf{q}^{-1}) u_k, \tag{2.9}$$

Definition 2.21. Consider the LTV state space model

$$x_{k+1} = A_k x_k + B_k u_k, \tag{2.10}$$

$$y_k = C_k x_k + E_k u_k, \tag{2.11}$$

where, for all $k \geq 0$, $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^m$ is the input, and $y_k \in \mathbb{R}^p$ is the output. Define the *observability matrix* at step k as

$$\mathcal{O}_k \triangleq \begin{bmatrix} C_k \\ C_{k+1}A_k \\ C_{k+2}A_{k+1}A_k \\ \vdots \\ C_{k+n-1}A_{k+n-2}\ldots A_{k+1}A_k \end{bmatrix}, \tag{2.12}$$

and the *controllability matrix* at step k as

$$\mathcal{C}_k \triangleq \begin{bmatrix} B_{k-1} & A_{k-1}B_{k-2} & A_{k-1}A_{k-2}B_{k-3} & \cdots & A_{k-1}\ldots A_{k-n+1}B_{k-n} \end{bmatrix}. \tag{2.13}$$

If, for all $k \geq 0$, $\operatorname{rank} \mathcal{O}_k = n$, then (A, C) is *completely observable*. If, for all $k \geq n$, $\operatorname{rank} \mathcal{C}_k = n$, then (A, B) is *completely controllable*. Furthermore, if (A, B) is completely controllable and (A, C) is completely observable, then (A, B, C, E) is *minimal*.

Definition 2.22. Let $D_{1,k}, \ldots, D_{n,k} \in \mathbb{R}^{p\times p}$, let $N_{0,k}, \ldots, N_{n,k} \in \mathbb{R}^{p\times m}$, let $y_{-n}, \ldots, y_{-1} \in \mathbb{R}^p$ be initial output data, let $(\theta_k)_{k=-n}^{\infty} \in \mathbb{R}^r$, and, for all $k \geq -n$, let $u_k : \mathbb{R}^r \to \mathbb{R}^m$. Then, the *FIA sequence* $(y_k(\theta_k))_{k=0}^{\infty}$ is given by the *fixed-input-argument (FIA) filter*

$$y_k(\theta_k) + D_{1,k}y_{k-1}(\theta_{k-1}) + \cdots + D_{n,k}y_{k-n}(\theta_{k-n}) = N_{0,k}u_k(\theta_k) + \cdots + N_{n,k}u_{k-n}(\theta_k), \tag{2.14}$$

where, for all $k \in [-n, -1]$, $y_k(\theta_k) \triangleq y_k$.

Note that, in (2.14), at each step k, the arguments of $u_{k-n}, \ldots, u_k$ are fixed at the current value θ_k. In contrast, the left hand side defines the current output $y_k(\theta_k)$ which depends on the past output values $y_{k-n}(\theta_{k-n}), \ldots, y_{k-1}(\theta_{k-1})$. In terms of $\mathbf{q}^{-1}$, (2.14) is written as either

$$D_k(\mathbf{q}^{-1})y_k(\theta_k) = N_k(\mathbf{q}^{-1})u_k(\theta_{\overline{k}}), \tag{2.15}$$

or

$$y_k(\theta_k) = G_k(\mathbf{q}^{-1})u_k(\theta_{\overline{k}}), \tag{2.16}$$

where $G_k \stackrel{\triangle}{=} D_k^{-1} N_k$.

CHAPTER 3

Finite-Time Input Estimation

This chapter considers finite-time input estimation for discrete-time linear time-invariant systems in the case where the initial condition is unknown. Finite-time input estimation is the exact reconstruction of input after a finite number of steps. First, two specific constructions of finite-impulse-response (FIR) delayed left inverse for systems with zero nonzero zeros are presented; one using the Smith-McMillan form does not necessarily provide the minimal delay possible and the second using the Smith-McMillan form at infinity gives an FIR delayed left inverse with the minimal delay. Next, it is shown that, in the presence of an arbitrary unknown initial condition, finite-time input estimation is possible using a delayed left inverse H if and only if H is FIR. Finally, it is shown that a transfer function with full column normal rank has an FIR delayed left inverse with the minimal delay if and only if the system has zero nonzero zeros.

Let $G \in \mathbb{R}(\mathrm{z})^{p\times m}_{\mathrm{prop}}$. If H is a delayed left inverse of G with delay d, then the output of HG is equal to the d-step-delayed input of HG. However, HG does not account for the free response of the state space model formed by cascading state space models of G and H. The missing free response can be accounted for by specifying initial conditions of realizations of G and H. Let

$$G \overset{\min}{\sim} \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array}\right], \quad H \overset{\min}{\sim} \left[\begin{array}{c|c} A_H & B_H \\ \hline C_H & D_H \end{array}\right], \tag{3.1}$$

and, for all $k \geq 0$, consider the state space equations

$$x_{G,k+1} = A_G x_{G,k} + B_G u_k, \tag{3.2}$$

$$y_k = C_G x_{G,k} + D_G u_k, \tag{3.3}$$

and

$$x_{H,k+1} = A_H x_{H,k} + B_H y_k, \tag{3.4}$$

$$z_k = C_H x_{H,k} + D_H y_k. \tag{3.5}$$

Then, the state space realization of the cascade HG is given by

$$x_{k+1} = A x_k + B u_k, \tag{3.6}$$

$$z_k = C x_k + D u_k, \tag{3.7}$$

where

$$x \triangleq \begin{bmatrix} x_G \\ x_H \end{bmatrix}, \ A \triangleq \begin{bmatrix} A_G & 0 \\ B_H C_G & A_H \end{bmatrix}, \quad B \triangleq \begin{bmatrix} B_G \\ B_H D_G \end{bmatrix}, \tag{3.8}$$

$$C \triangleq \begin{bmatrix} D_H C_G & C_H \end{bmatrix}, \quad D \triangleq D_H D_G. \tag{3.9}$$

Note that the realization (3.6), (3.7) of HG is not necessarily minimal.

3.1 Effect of Zeros on Input Estimation

If the continuous-time system G has a transmission zero, then it follows from [115, p. 398] that there exist an initial condition and nonzero input such that the response of a minimal state space realization of G is identically zero. The following result is the discrete-time analogue and is partially given by Lemma 2.7 in [86, p. 25].

Proposition 3.1. Let $G \in \mathbb{R}(\mathrm{z})^{p\times m}_{\mathrm{prop}}$, where $G \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and $A \in \mathbb{R}^{n\times n}$, and, for all $k \geq 0$, consider

$$x_{k+1} = Ax_k + Bu_k, \tag{3.10}$$

$$y_k = Cx_k + Du_k. \tag{3.11}$$

Assume that $\mathrm{z}_0 \in \mathbb{C}$ is a transmission zero of G, and let $\begin{bmatrix} \overline{x} \\ \overline{u} \end{bmatrix} \in \mathcal{N}(\mathcal{Z}(\mathrm{z}_0))$ have nonzero real part, where $\mathcal{Z}$ is the Rosenbrock system matrix. Define the initial state $x_0 \triangleq \mathrm{Re}(\overline{x})$, and, for all $k \geq 0$, define the input sequence $u_k \triangleq \mathrm{Re}(\mathrm{z}_0^k \overline{u})$. Then, for all $k \geq 0$, $y_k = 0$. Furthermore, $\overline{u} \neq 0$.

Proof. By assumption,

$$\begin{bmatrix} \mathrm{z}_0 I - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} \overline{x} \\ \overline{u} \end{bmatrix} = 0,$$

and thus

$$(\mathrm{z}_0 I - A)\overline{x} = B\overline{u}, \tag{3.12}$$

$$C\overline{x} + D\overline{u} = 0. \tag{3.13}$$

Using (3.12) and the fact that $\mathrm{z}_0^0 = 1$, it follows from (3.10) that $x_1 = A\,\mathrm{Re}(\overline{x}) + B\,\mathrm{Re}(\overline{u}) = A\,\mathrm{Re}(\overline{x}) + \mathrm{Re}(\mathrm{z}_0\overline{x}) - A\,\mathrm{Re}(\overline{x}) = \mathrm{Re}(\mathrm{z}_0\overline{x})$. Proceeding similarly, it follows that, for all $k \geq 0$, $x_k = \mathrm{Re}(\mathrm{z}_0^k\overline{x})$. Thus (3.11) and (3.13) together imply that, for all $k \geq 0$, $y_k = C\,\mathrm{Re}(\mathrm{z}_0^k\overline{x}) + D\mathrm{Re}(\mathrm{z}_0^k\overline{u}) = \mathrm{Re}(\mathrm{z}_0^k(C\overline{x} + D\overline{u})) = 0$.

Next, suppose that $\overline{u} = 0$. Hence (3.13) implies that $C\overline{x} = 0$. Then it follows from (3.10) and

(3.11) that

$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix} = \mathcal{O}\mathrm{Re}(\overline{x}),$$

where $\mathcal{O}$ is the observability matrix obtained from (A, C). Since $\mathcal{O}$ has full column rank and, for all $k \geq 0$, $y_k = 0$, it follows that $\mathrm{Re}(\overline{x}) = 0$, which is a contradiction. Thus $\overline{u} \neq 0$. □

Note that, in the case where $\mathrm{z}_0 \neq 0$, the input u that produces the zero output has the property that, for all $k \geq 0$, $u_k \neq 0$. Since the zero input also produces the zero output, finite-time input estimation is impossible. However, in the case where $\mathrm{z}_0 = 0$, the input u that produces the zero output is $\{\mathrm{Re}(\overline{u}), 0, 0, \ldots.\}$. The fact that u_k is nonzero only at the initial time step suggests that finite-time input estimation may be possible in this case as long as G has zero nonzero zeros. In fact, delayed left inverses for systems with this property is constructed in the next two sections.

3.2 Smith-McMillan Construction of a Delayed Left Inverse

In this section, we use the *Smith-Mcmillan form* to construct an FIR delayed left inverse for systems with zero nonzero zeros. The following result given by Theorem 6.7.5 in [116, p. 514] presents the Smith-McMillan form.

Theorem 3.2. Let $G \in \mathbb{R}(\mathrm{z})^{p\times m}_{\mathrm{prop}}$, and let $\rho \stackrel{\triangle}{=} \mathrm{rank}\, G$. Then there exist unimodular matrices $S_1 \in \mathbb{R}(\mathrm{z})^{p\times p}$ and $S_2 \in \mathbb{R}(\mathrm{z})^{m\times m}$ and unique monic polynomials $p_1, \ldots, p_\rho, q_1, \ldots, q_\rho \in \mathbb{R}(\mathrm{z})$ such that p_i and q_i are coprime for all $i \in \{1, \ldots, \rho\}$, p_i divides p_{i+1} for all $i \in \{1, \ldots, \rho - 1\}$,

q_{i+1} divides q_i for all $i \in \{1, \ldots, \rho - 1\}$, and $G = S_1 S S_2$, where

$$S = \begin{bmatrix} p_1/q_1 & & & 0_{\rho\times(m-\rho)} \\ & \ddots & & \\ & & p_\rho/q_\rho & \\ 0_{(p-\rho)\times\rho} & & & 0_{(p-\rho)\times(m-\rho)} \end{bmatrix}. \tag{3.14}$$

S is the *Smith-McMillan form* of G. The roots of the polynomial $q_1 q_2 \ldots q_\rho$ are the *poles* of G, and the roots of the polynomial $p_1 p_2 \ldots p_\rho$ are the *transmission zeros* of G.

Proposition 3.3. Let $G \in \mathbb{R}(\mathrm{z})^{p\times m}_{\mathrm{prop}}$, assume that G has full column normal rank, and assume that G has zero nonzero zeros. Then, $H_s \triangleq S_2^{-1} S^{+} S_1^{-1}$ has zero nonzero poles, where S, S_1, and S_2 are defined in Theorem 3.2 and $S^{+} \triangleq (S^{\mathrm{T}} S)^{-1} S^{\mathrm{T}}$.

Proof. Since G has zero nonzero zeros, each polynomial $p_1, p_2, \ldots, p_\rho$, defined in Theorem 3.2, is a power of z. Hence S^{+} has zero nonzero poles. Note that since S_1 and S_2 are unimodular matrices, the entries of S_1^{-1} and S_2^{-1} are polynomials. Thus H_s has zero nonzero poles.

Corollary 3.4. Let G and H_s be as defined in Proposition 3.3, and let d_0 be the smallest nonnegative integer such that $H(\mathrm{z}) = \mathrm{z}^{-d_0} H_s(\mathrm{z})$ is a proper transfer function. Then, H is an FIR delayed left inverse of G.

Example 3.5. Let $G(\mathrm{z}) = \begin{bmatrix} \frac{1}{\mathrm{z}} \\ \frac{1}{\mathrm{z}^2} \end{bmatrix}$. Then

$$S(\mathrm{z}) = \begin{bmatrix} \frac{1}{\mathrm{z}^2} \\ 0 \end{bmatrix}, \quad S_1(\mathrm{z}) = \begin{bmatrix} \mathrm{z} & 1 \\ 1 & 0 \end{bmatrix}, \quad S_2(\mathrm{z}) = 1,$$

such that S is the Smith-Mcmillan form of G and $G = S_1 S S_2$. Evaluating the expression for H given in Corollary 3.4 yields $H = \begin{bmatrix} 0 & 1 \end{bmatrix}$ such that $H(\mathrm{z})G(\mathrm{z}) = \mathrm{z}^{-2}$. Hence, H is an FIR delayed left inverse of G with delay 2. However, note that $H = \begin{bmatrix} 1 & 0 \end{bmatrix}$ is a delayed left inverse of G such that $H(\mathrm{z})G(\mathrm{z}) = \mathrm{z}^{-1}$. Hence this method does not necessarily give an FIR delayed left inverse with the minimal possible delay. $\diamond$

3.3 Construction of an FIR Delayed Left Inverse with the Minimal Delay

In this section, we use the Smith-McMillan form at infinity [80] to construct an FIR delayed left inverse with the minimal delay for systems with zero nonzero zeros. The main result is Theorem 3.12, which presents the expression for the constructed FIR inverse.

Lemma 3.6. Let $U \in \mathbb{R}[\mathrm{z}]^{n\times n}$, assume that U is unimodular, and, for all $\mathrm{z} \neq 0$, define $V(\mathrm{z}) \triangleq U(1/\mathrm{z})$. Then V is biproper and FIR.

Proof. Since U is a polynomial matrix, each entry of U is of the form $\alpha_k \mathrm{z}^k + \cdots + \alpha_1 \mathrm{z} + \alpha_0$, where k is a nonnegative integer and $\alpha_0, \ldots, \alpha_k$ are real numbers. Then the corresponding entry of V has the form $\alpha_k \mathrm{z}^{-k} + \cdots + \alpha_1 \mathrm{z}^{-1} + \alpha_0$, which is proper and FIR. Hence V is proper and FIR. Next, define the nonzero constant $\beta \triangleq \det U(\mathrm{z})$, and note that $\lim_{\mathrm{z}\to\infty} \det V(\mathrm{z}) = \lim_{\mathrm{z}\to\infty} \det U(1/\mathrm{z}) = \lim_{\mathrm{z}\to 0} \det U(\mathrm{z}) = \beta \neq 0$. Hence V is biproper. $\square$

Lemma 3.7. Let $G \in \mathbb{R}(\mathrm{z})^{p\times m}_{\mathrm{prop}}$ and, for all $\mathrm{z} \neq 0$, define $\hat{G}(\mathrm{z}) \triangleq G(1/\mathrm{z})$. Then the following statements hold:

i) $\hat{G}$ has no poles at zero.

ii) If G has zero nonzero zeros, then $\hat{G}$ has zero nonzero zeros.

Proof. To prove *i*), suppose that $\hat{G}$ has at least one pole at zero. Then at least one entry of $\hat{G}$ is of the form $\dfrac{N(\mathrm{z})}{\mathrm{z}^k D(\mathrm{z})}$, where k is a positive integer, N and D are polynomials such that $N(0) \neq 0$, and

$D(0) \neq 0$. Then the corresponding entry of G is $\dfrac{\mathrm{z}^k N(1/\mathrm{z})}{D(1/\mathrm{z})}$. Since $N(0) \neq 0$ and $D(0) \neq 0$, it follows that $\dfrac{N(1/\mathrm{z})}{D(1/\mathrm{z})}$ is exactly proper and hence $\dfrac{\mathrm{z}^k N(1/\mathrm{z})}{D(1/\mathrm{z})}$ is improper, which is a contradiction. Hence $\hat{G}$ has no poles at zero.

To prove *ii*), define $\rho \triangleq \operatorname{rank} G = \operatorname{rank} \hat{G}$. Suppose that z_0 is a nonzero zero of $\hat{G}$. Then $\operatorname{rank} G(1/\mathrm{z}_0) = \operatorname{rank} \hat{G}(\mathrm{z}_0) < \rho$. Thus $1/\mathrm{z}_0$ is a nonzero zero of G, which is a contradiction. Hence $\hat{G}$ has zero nonzero zeros. □

The following result presents the *Smith-McMillan form at infinity* S_∞ of G. The proof presented here, which is different from the proof in [80], is constructive; this construction is also used to prove Theorem 3.12.

Theorem 3.8. Let $G \in \mathbb{R}(\mathrm{z})^{p\times m}_{\mathrm{prop}}$, define $\rho \triangleq \operatorname{rank} G$, and define $\rho_0 \triangleq \rho - \operatorname{rank} G(\infty)$. Then there exist biproper transfer functions $W \in \mathbb{R}(\mathrm{z})^{p\times p}_{\mathrm{prop}}$ and $V \in \mathbb{R}(\mathrm{z})^{m\times m}_{\mathrm{prop}}$ and integers $\iota_1 \geq \iota_2 \geq \cdots \geq \iota_{\rho_0} > 0$ such that $G = W S_\infty V$, where

$$S_\infty(\mathrm{z}) \triangleq \begin{bmatrix} \mathrm{z}^{-\iota_1} & & & & & & 0_{\rho\times(m-\rho)} \\ & \ddots & & & & & \\ & & \mathrm{z}^{-\iota_{\rho_0}} & & & & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & \\ 0_{(p-\rho)\times\rho} & & & & & & 0_{(p-\rho)\times(m-\rho)} \end{bmatrix}. \tag{3.15}$$

Proof. For all $\mathrm{z} \neq 0$, define $\hat{G}(\mathrm{z}) \triangleq G(1/\mathrm{z})$. Note that $\operatorname{rank} \hat{G} = \operatorname{rank} G = \rho$. Let $\hat{G} = \hat{S}_1 \hat{S} \hat{S}_2$, where $\hat{S} \in \mathbb{R}(\mathrm{z})^{p\times m}$ is the Smith-McMillan form of $\hat{G}$, and $\hat{S}_1 \in \mathbb{R}(\mathrm{z})^{p\times p}$ and $\hat{S}_2 \in \mathbb{R}(\mathrm{z})^{m\times m}$ are unimodular matrices. Define $S_1(\mathrm{z}) \triangleq \hat{S}_1(1/\mathrm{z})$, $S(\mathrm{z}) \triangleq \hat{S}(1/\mathrm{z})$, and $S_2(\mathrm{z}) \triangleq \hat{S}_2(1/\mathrm{z})$. It follows from Lemma 3.7 that $\hat{G}$ has no poles at zero and thus $\hat{S}$ has no poles at zero. Hence $\hat{S}$ is of the form

$$\hat{S}(\mathrm{z}) = \begin{bmatrix} \frac{\mathrm{z}^{\iota_1} N_1(\mathrm{z})}{D_1(\mathrm{z})} & & & & & & 0_{\rho\times(m-\rho)} \\ & \ddots & & & & & \\ & & \frac{\mathrm{z}^{\iota_\kappa} N_\kappa(\mathrm{z})}{D_\kappa(\mathrm{z})} & & & & \\ & & & \frac{N_{\kappa+1}(\mathrm{z})}{D_{\kappa+1}(\mathrm{z})} & & & \\ & & & & \ddots & & \\ & & & & & \frac{N_\rho(\mathrm{z})}{D_\rho(\mathrm{z})} & \\ 0_{(p-\rho)\times\rho} & & & & & & 0_{(p-\rho)\times(m-\rho)} \end{bmatrix},$$

where $\iota_1 \geq \cdots \geq \iota_\kappa > 0$ and $\kappa \triangleq \rho - \operatorname{rank} \hat{G}(0)$. N_i and D_i, for $i = 1, \ldots, \rho$, are polynomials such that $N_i(0) \neq 0$, and $D_i(0) \neq 0$. Then

$$S(\mathrm{z}) = \begin{bmatrix} \frac{\mathrm{z}^{-\iota_1} N_1(1/\mathrm{z})}{D_1(1/\mathrm{z})} & & & & & & 0_{\rho\times(m-\rho)} \\ & \ddots & & & & & \\ & & \frac{\mathrm{z}^{-\iota_\kappa} N_\kappa(1/\mathrm{z})}{D_\kappa(1/\mathrm{z})} & & & & \\ & & & \frac{N_{\kappa+1}(1/\mathrm{z})}{D_{\kappa+1}(1/\mathrm{z})} & & & \\ & & & & \ddots & & \\ & & & & & \frac{N_\rho(1/\mathrm{z})}{D_\rho(1/\mathrm{z})} & \\ 0_{(p-\rho)\times\rho} & & & & & & 0_{(p-\rho)\times(m-\rho)} \end{bmatrix}.$$

Since, for all $i = 1, \ldots, \rho$, $N_i(0) \neq 0$, and $D_i(0) \neq 0$, it follows that $N_i(1/\mathrm{z})$ and $D_i(1/\mathrm{z})$ are exactly proper and hence S is proper. Therefore, S can be factored as $S = S_0 D_v$, where

$$S_0(\mathrm{z}) \triangleq \begin{bmatrix} \mathrm{z}^{-\iota_1} & & & & & & 0_{\rho\times(m-\rho)} \\ & \ddots & & & & & \\ & & \mathrm{z}^{-\iota_\kappa} & & & & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & \\ 0_{(p-\rho)\times\rho} & & & & & & 0_{(p-\rho)\times(m-\rho)} \end{bmatrix}.$$

and

$$D_v(\mathrm{z}) \triangleq \begin{bmatrix} \frac{N_1(1/\mathrm{z})}{D_1(1/\mathrm{z})} & & & & & \\ & \ddots & & & & \\ & & \frac{N_\rho(1/\mathrm{z})}{D_\rho(1/\mathrm{z})} & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix}. \tag{3.16}$$

Note that $D_v \in \mathbb{R}(\mathrm{z})^{m\times m}_{\mathrm{prop}}$ is a biproper diagonal matrix. Since $\hat{G}(0) = G(\infty)$, it follows that $\kappa = \rho_0$ and thus $S_0 = S_\infty$. Now, for all $\mathrm{z} \neq 0$, $G(\mathrm{z}) = \hat{G}(1/\mathrm{z}) = \hat{S}_1(1/\mathrm{z})\hat{S}(1/\mathrm{z})\hat{S}_2(1/\mathrm{z}) = S_1(\mathrm{z})S(\mathrm{z})S_2(\mathrm{z})$. Since $\hat{S}_1$ and $\hat{S}_2$ are unimodular, it follows from Lemma 3.6 that S_1 and S_2 are biproper. Defining $W \triangleq S_1$ and $V \triangleq D_v S_2$, it follows that $G = S_1 S S_2 = S_1 S_0 D_v S_2 = S_1 S_\infty D_v S_2 = W S_\infty V$. □

Note that ρ_0 is the *number of infinite zero directions*, for all $i = 1, \ldots, \rho_0$, ι_i is the *number of infinite zeros in the ith direction*, and $\iota \triangleq \sum_{j=1}^{\rho_0} \iota_j$ is the *number of infinite zeros* of G.

The following result is given by Theorem 1 in [117].

Lemma 3.9. Let $G_1 \in \mathbb{R}(\mathrm{z})^{p\times m}_{\mathrm{prop}}$ and $G_2 \in \mathbb{R}(\mathrm{z})^{p\times m}_{\mathrm{prop}}$ be such that $G_2 = W G_1 V$, where $W \in$

$\mathbb{R}(z)_{\text{prop}}^{p\times p}$ and $V \in \mathbb{R}(z)_{\text{prop}}^{m\times m}$ are biproper. Then, for all $i \geq 0$, $\beta_i(G_1) = \beta_i(G_2)$.

The following result is given by Theorem 4 in [25].

Proposition 3.10. Let $G \in \mathbb{R}(z)_{\text{prop}}^{p\times m}$ and $d \geq 0$. Then G is delayed left invertible with delay d if and only if $\operatorname{rank} \mathcal{T}_d - \operatorname{rank} \mathcal{T}_{d-1} = m$.

The following result is based on the discussion of the pole/zero structure at infinity given in [117].

Proposition 3.11. Let $G \in \mathbb{R}(z)_{\text{prop}}^{p\times m}$, assume that G has full column normal rank, and define ι_1 as in Theorem 3.8. Then $\eta_G = \iota_1$.

Proof. Let $H_{\infty,i}$ be the ith Markov parameter of S_∞ and $\mathcal{T}_{\infty,i}$ be the ith Markov block-Toeplitz matrix associated with S_∞, where S_∞ is the Smith-McMillan form at infinity of G. Define the multiset $F \triangleq \{\iota_1, \ldots, \iota_{\rho_0}, 0, \ldots, 0\}$ with ρ elements, where $\iota_1, \ldots, \iota_{\rho_0}$ and ρ are defined in Theorem 3.8. For all $i \geq 0$, let F_i be the multiset consisting of all elements of F that are less than or equal to i, and let $|F_i|$ denote the cardinality of F_i.

Note that, for all $i \geq 0$, each row of $\mathcal{T}_{\infty,i}$ is either zero or has exactly one nonzero entry that is equal to one, and the nonzero rows of $\mathcal{T}_{\infty,i}$ are linearly independent. It thus follows that $\beta_i(S_\infty) = \operatorname{rank} \mathcal{T}_{\infty,i} - \operatorname{rank} \mathcal{T}_{\infty,i-1} = \operatorname{rank} \begin{bmatrix} H_{\infty,0} & \cdots & H_{\infty,i} \end{bmatrix} = |F_i|$. Hence, Theorem 3.8 and Lemma 3.9 imply that, for all $i \geq 0$, $\beta_i(G) = \beta_i(S_\infty) = |F_i|$.

Note that $\max_{i\geq 0} \beta_i(S_\infty) = \max_{i\geq 0} |F_i| = |F| = \rho$. Since ι_1 is the largest element in F, it follows that the smallest i such that $|F_i| = \rho$ is ι_1. Thus $\rho = |F_{\iota_1}| = \beta_{\iota_1}(G) = \operatorname{rank} \mathcal{T}_{\iota_1} - \operatorname{rank} \mathcal{T}_{\iota_1 - 1}$, where $\mathcal{T}_i$ is the ith Markov block-Toeplitz matrix associated with G. Since G has full column rank, it follows that $\rho = m$, and thus ι_1 is the smallest i such that $\operatorname{rank} \mathcal{T}_i - \operatorname{rank} \mathcal{T}_{i-1} = m$. Hence Proposition 3.10 implies that $\eta_G = \iota_1$. $\square$

The following result constructs an FIR delayed left inverse of G with the minimal delay.

Theorem 3.12. Let $G \in \mathbb{R}(z)_{\text{prop}}^{p\times m}$, assume that G has full column rank, and assume that G has zero nonzero zeros. Then there exist biproper transfer functions $W \in \mathbb{R}(z)_{\text{prop}}^{p\times p}$ and $V \in \mathbb{R}(z)_{\text{prop}}^{m\times m}$

such that

$$H_\infty(\mathrm{z}) \triangleq \mathrm{z}^{-\eta_G} V^{-1}(\mathrm{z}) S_\infty^{\mathrm{T}}(1/\mathrm{z}) W^{-1}(\mathrm{z}) \tag{3.17}$$

is an FIR delayed left inverse of G with delay η_G, where

$$S_\infty(\mathrm{z}) \triangleq \begin{bmatrix} \mathrm{z}^{-\iota_1} & & & & & \\ & \ddots & & & & \\ & & \mathrm{z}^{-\iota_{\rho_0}} & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \\ & & & 0_{(p-m)\times m} & & \end{bmatrix} \tag{3.18}$$

is the Smith-McMillan form at infinity of G, $\iota_1 \geq \iota_2 \geq \cdots \geq \iota_{\rho_0} > 0$ are integers, and $\rho_0 \triangleq m - \operatorname{rank} G(\infty)$.

Proof. Define, for all $\mathrm{z} \neq 0$, $\hat{G}(\mathrm{z}) \triangleq G(1/\mathrm{z})$. Note that $\operatorname{rank}\hat{G} = \operatorname{rank} G = m$. Let $\hat{G} = \hat{S}_1\hat{S}\hat{S}_2$, where $\hat{S}$ is the Smith-McMillan form of $\hat{G}$, and $\hat{S}_1$ and $\hat{S}_2$ are unimodular matrices. Define $S_1(\mathrm{z}) \triangleq \hat{S}_1(1/\mathrm{z})$, $S(\mathrm{z}) \triangleq \hat{S}(1/\mathrm{z})$, and $S_2(\mathrm{z}) \triangleq \hat{S}_2(1/\mathrm{z})$. Following the same steps given in the proof of Theorem 3.8 yields $G = WS_\infty V$, where $W \triangleq S_1$, $V \triangleq D_v S_2$, S_∞ is given by (3.18), and D_v is given by (3.16) with ρ replaced by m. Since $\hat{S}_1$ and $\hat{S}_2$ are unimodular, it follows that $\hat{S}_1^{-1}$ and $\hat{S}_2^{-1}$ are unimodular and thus Lemma 3.6 implies that $W^{-1} = S_1^{-1}$ and S_2^{-1} are FIR. Since G has zero nonzero zeros, it follows from Lemma 3.7 that $\hat{G}$ has zero nonzero zeros. Hence, $\hat{S}$ has zero nonzero zeros. Hence, for all $i = 1, \ldots, m$, $N_i = 1$ in (3.16). Hence D_v^{-1} is FIR. Thus $V^{-1} = S_2^{-1}D_v^{-1}$ is FIR, and hence H_∞ is FIR. Next, it follows from Theorem 3.11 that $\eta_G = \iota_1$. Hence, $\mathrm{z}^{-\eta_G} S_\infty^{\mathrm{T}}(1/\mathrm{z})$ is proper. Note that W^{-1} and V^{-1} are biproper and thus H_∞ is proper. Since $H_\infty(\mathrm{z})G(\mathrm{z}) = \mathrm{z}^{-\eta_G} V^{-1}(\mathrm{z}) S_\infty^{\mathrm{T}}(1/\mathrm{z}) W^{-1}(\mathrm{z}) W(\mathrm{z}) S_\infty(\mathrm{z}) V(\mathrm{z}) = \mathrm{z}^{-\eta_G} I_m$, it follows that H_∞ is an FIR delayed left inverse of G with delay η_G. $\square$

3.4 Input Estimation using FIR Delayed Left Inverse

The main result in this section shows that, in the presence of an arbitrary unknown initial condition, finite-time input estimation is possible using a delayed left inverse H if and only if H is FIR. The following lemma will be needed.

Lemma 3.13. Let $G \in \mathbb{R}(\mathrm{z})_{\mathrm{prop}}^{p\times m}$ and $H \in \mathbb{R}(\mathrm{z})_{\mathrm{prop}}^{m\times p}$, with minimal state space realizations (3.1)-(3.5). Assume that H is FIR and that H is a delayed left inverse of G with delay d. Define $K(\mathrm{z}) \triangleq H(\mathrm{z})C_G(\mathrm{z}I - A_G)^{-1}$. Then K is FIR.

Proof. For the state space realization of HG given by (3.6)–(3.9), note that $\mathrm{spec}(A) = \mathrm{spec}(A_G) \cup \mathrm{spec}(A_H)$. Since H is FIR, it follows that $\mathrm{spec}(A_H) = \{0\}$. Therefore, each nonzero eigenvalue of A is an eigenvalue of A_G. Since HG is FIR, it follows that each nonzero eigenvalue of A (including multiplicity) is either an uncontrollable eigenvalue of (A, B) or an unobservable eigenvalue of (A, C). However, since (A_G, B_G) is controllable, each nonzero eigenvalue of A is contained in $\mathrm{spec}(A_G)$, and A is lower triangular, it follows from the PBH test that each nonzero eigenvalue of A is a controllable eigenvalue of (A, B) and thus an unobservable eigenvalue of (A, C). Defining

$$B_0 \triangleq \begin{bmatrix} I_{n_G} \\ 0 \end{bmatrix}, \quad D_0 \triangleq 0,$$

where $n_G \triangleq \mathrm{Mcdeg}\ G$, note that (A, B_0, C, D_0) is a state space realization of K. Since each nonzero eigenvalue of A is an unobservable eigenvalue of (A, C), it follows that none of the nonzero eigenvalues of A are poles of K. Hence, every pole of K is zero, and thus K is FIR. $\square$

Theorem 3.14. Let $G \in \mathbb{R}(\mathrm{z})_{\mathrm{prop}}^{p\times m}$ and $H \in \mathbb{R}(\mathrm{z})_{\mathrm{prop}}^{m\times p}$ with minimal state space realizations (3.1)-(3.5), assume that H is a delayed left inverse of G with delay d, and define $K(\mathrm{z}) \triangleq H(\mathrm{z})C_G(\mathrm{z}I - A_G)^{-1}$. Then the following statements hold:

$i)$ If there exists a nonnegative integer ν such that, for all $k \geq \nu$ and all initial conditions $x_{G,0}$

and $x_{H,0}$, $z_k = u_{k-d}$, then H is FIR.

$ii)$ If H is FIR, then for all $k \geq \nu = \max\{\operatorname{ind} H, \operatorname{ind} K, d\}$ and all initial conditions $x_{G,0}$ and $x_{H,0}$, $z_k = u_{k-d}$. If, in addition, $x_{H,0} = 0$, then $\nu = \max\{\operatorname{ind} K, d\}$.

Proof. Note that, for all $k \geq 0$, $z_k = z_{\mathrm{free,k}} + z_{\mathrm{forced,k}}$, where z_{free} and z_{forced} denote the free response and forced response, respectively, of (3.6)–(3.9). Since $H(\mathrm{z})G(\mathrm{z}) = \mathrm{z}^{-d}I_m$, it follows that, for all $k \geq d$, $z_{\mathrm{forced,k}} = u_{k-d}$. Next, note that, for all $k \geq 0$,

$$
\begin{aligned}
z_{\mathrm{free,k}} &= CA^k x_0 \\
&= \begin{bmatrix} D_H C_G & C_H \end{bmatrix} \begin{bmatrix} A_G^k & 0 \\ \sum_{i=0}^{k-1} A_H^i B_H C_G A_G^{k-i-1} & A_H^k \end{bmatrix} \begin{bmatrix} x_{G,0} \\ \\ x_{H,0} \end{bmatrix} \\
&= z_{G,k} + z_{H,k},
\end{aligned}
\tag{3.19}
$$

where

$$
\begin{aligned}
z_{G,k} &\triangleq \left(D_H C_G A_G^k + C_H \sum_{i=0}^{k-1} A_H^i B_H C_G A_G^{k-i-1} \right) x_{G,0}, \\
z_{H,k} &\triangleq C_H A_H^k x_{H,0}.
\end{aligned}
$$

To prove *i)*, note that there exists a nonnegative integer ν such that, for all $k \geq \nu$ and all $x_{G,0}$, $x_{H,0}$, $z_{\mathrm{free,k}} = 0$. Hence it follows from (3.19) that, for all $k \geq \nu$,

$$
\begin{bmatrix} D_H C_G & C_H \end{bmatrix} \begin{bmatrix} A_G^k & 0 \\ \sum_{i=0}^{k-1} A_H^i B_H C_G A_G^{k-i-1} & A_H^k \end{bmatrix} = 0,
$$

and thus, for all $k \geq \nu$, $C_H A_H^k = 0$. Hence H is FIR.

To prove *ii)*, note that since H is FIR and thus A_H is nilpotent, it follows that, for all $k \geq \operatorname{ind} H$, $z_{H,k} = 0$. Noting that z_G is the output of (3.6), (3.7) in the case where $u \equiv 0$ and $x_{H,0} = 0$, it follows from (3.4) and (3.5) that the Z transform of z_G is given by

$$\begin{aligned}
\hat{z}_G(\mathrm{z}) &= C_H\hat{x}_H(\mathrm{z}) + D_H\hat{y}(\mathrm{z}) \\
&= \big(C_H(\mathrm{z}I - A_H)^{-1}B_H + D_H\big)\hat{y}(\mathrm{z}) \\
&= \big(C_H(\mathrm{z}I - A_H)^{-1}B_H + D_H\big)C_G\hat{x}_G(\mathrm{z}) \\
&= z\big(C_H(\mathrm{z}I - A_H)^{-1}B_H + D_H\big)C_G(\mathrm{z}I - A_G)^{-1}x_{G,0} \\
&= \mathrm{z}K(\mathrm{z})x_{G,0} = \mathrm{z}\hat{w}_G(\mathrm{z}),
\end{aligned}$$

where $\hat{w}_G(\mathrm{z}) \stackrel{\triangle}{=} K(\mathrm{z})x_{G,0}$. Note that the inverse Z transform w_G of $\hat{w}_G$ is a linear combination of the n_G single-channel impulse responses of K. Lemma 3.13 implies that K is FIR and thus, for all $k \geq \operatorname{ind} K + 1$, $w_{G,k} = 0$. Since $z_{G,k} = w_{G,k+1}$, it follows that, for all $k \geq \operatorname{ind} K$, $z_{G,k} = 0$. Hence, for all $k \geq \nu = \max\{\operatorname{ind} H, \operatorname{ind} K, d\}$, $z_k = u_{k-d}$.

Finally, consider the case where $x_{H,0} = 0$. In this case, it follows that, for all $k \geq 0$, $z_{H,k} = 0$, and thus, for all $k \geq 0$, $z_k = z_{\mathrm{free,k}} + z_{\mathrm{forced,k}} = z_{G,k} + z_{H,k} + z_{\mathrm{forced,k}} = z_{G,k} + z_{\mathrm{forced,k}}$. Therefore, for all $k \geq \max\{\operatorname{ind} K, d\}$, $z_k = u_{k-d}$. □

Theorem 3.14 shows that, for all $k \geq \max\{\operatorname{ind} H, \operatorname{ind} K, d\}$, the output z is equal to the input u delayed by d steps. However, if $\max\{\operatorname{ind} H, \operatorname{ind} K\} > d$, then, for all $k = 0, \ldots, \max\{\operatorname{ind} H, \operatorname{ind} K\} - d - 1$, the input u_k is not reconstructed. Note that Theorem 3.14 does not assume any stability condition, and thus the result holds even in the case where both G and H are unstable.

3.5 Existence of FIR Delayed Left Inverse

The following result restates part of Theorem 3.12 and provides its converse. In particular, Theorem 3.12 shows that a transfer function with full column normal rank has an FIR delayed left inverse with the minimal delay if and only if it has zero nonzero zeros. It follows from this fact and Theorem 3.14 that finite-time input estimation is possible if and only if the system has zero nonzero zeros.

Theorem 3.15. Let $G \in \mathbb{R}(\mathrm{z})_{\mathrm{prop}}^{p\times m}$, and assume that G has full column normal rank. Then, for all $d \geq \eta_G$, there exists an FIR $H \in \mathbb{R}(\mathrm{z})_{\mathrm{prop}}^{m\times p}$ such that H is a delayed left inverse of G with delay d if and only if G has zero nonzero zeros.

Proof. Sufficiency follows from Theorem 3.12. To prove necessity, suppose that z_0 is a nonzero zero of G. Since H is FIR, it follows that z_0 is not a pole of H. Note that $\operatorname{rank} H(\mathrm{z}_0)G(\mathrm{z}_0) = \operatorname{rank} \mathrm{z}_0^{-d} I_m = m$. Since z_0 is a nonzero zero of G, it follows that $\operatorname{rank} G(\mathrm{z}_0) < m$. Hence $\operatorname{rank} H(\mathrm{z}_0)G(\mathrm{z}_0) \leq \min\{\operatorname{rank} H(\mathrm{z}_0), \operatorname{rank} G(\mathrm{z}_0)\} < m$, which is a contradiction. Hence G has zero nonzero zeros. $\square$

Consider the case where G has at least one zero zero and zero nonzero zeros. With $\mathrm{z}_0 = 0$, it follows from Proposition 3.1 that, if $y \equiv 0$, then either u is an impulse or $u \equiv 0$. Hence, the initial input u_0 cannot be reconstructed. However, the inability to reconstruct the initial input cannot be inferred from Theorem 3.14. As discussed at the end of this section, the following result strengthens Theorem 3.14 by implying that u_0 cannot be reconstructed in the case where $d = 0$.

Proposition 3.16. Let $G \in \mathbb{R}(\mathrm{z})_{\mathrm{prop}}^{p\times m}$ and $H \in \mathbb{R}(\mathrm{z})_{\mathrm{prop}}^{m\times p}$ with minimal state space realizations (3.1)-(3.5). Assume that H is an FIR left inverse of G, define $K(\mathrm{z}) \triangleq H(\mathrm{z})C_G(\mathrm{z}I - A_G)^{-1}$, and assume that G has at least one zero zero. Then $K \neq 0$.

Proof. Since $HG = I_m$, it follows that $D_H D_G = I_m$ and hence $\operatorname{rank} D_H = \operatorname{rank} D_G = m$. Thus there exists a nonsingular matrix $S \in \mathbb{R}^{p\times p}$ such that $\hat{D}_H \triangleq D_H S = \begin{bmatrix} I_m & 0 \end{bmatrix}$. Define $n_G \triangleq \operatorname{Mcdeg} G$, and define $\hat{C}_G \triangleq S^{-1} C_G = \begin{bmatrix} \hat{C}_1 \\ \hat{C}_2 \end{bmatrix}$, where $\hat{C}_1 \in \mathbb{R}^{m\times n_G}$ and $\hat{C}_G \in \mathbb{R}^{(p-m)\times n_G}$. Similarly, define $\hat{D}_G \triangleq S^{-1} D_G = \begin{bmatrix} \hat{D}_1 \\ \hat{D}_2 \end{bmatrix}$, where $\hat{D}_1 \in \mathbb{R}^{m\times m}$ and $\hat{D}_2 \in \mathbb{R}^{(p-m)\times m}$. Let $\hat{G} \in \mathbb{R}(\mathrm{z})^{p\times m}$, where $\hat{G} \sim \left[\begin{array}{c|c} A_G & B_G \\ \hline \hat{C}_G & \hat{D}_G \end{array}\right]$. Let $\mathcal{O}$ and $\hat{\mathcal{O}}$ denote the observability matrices corresponding to (A_G, C_G) and $(A_G, \hat{C}_G)$, respectively. Note that

$$\operatorname{rank}\hat{\mathcal{O}} = \operatorname{rank}\begin{bmatrix} S^{-1}C_G \\ S^{-1}C_G A_G \\ \vdots \\ S^{-1}C_G A_G^{n_G-1} \end{bmatrix} = \operatorname{rank}(I_{n_G} \otimes S^{-1})\mathcal{O} = \operatorname{rank}\mathcal{O} = n_G.$$

Thus $\hat{G} \overset{\min}{\sim} \left[\begin{array}{c|c} A_G & B_G \\ \hline \hat{C}_G & \hat{D}_G \end{array}\right]$. Note that

$$\hat{D}_1 = \begin{bmatrix} I_m & 0 \end{bmatrix}\begin{bmatrix} \hat{D}_1 \\ \hat{D}_2 \end{bmatrix} = \hat{D}_H\hat{D}_G = D_H S S^{-1} D_G = I_m. \tag{3.20}$$

Now, suppose that $K = 0$. Since $K(\mathrm{z}) = H(\mathrm{z})C_G(\mathrm{z}I - A_G)^{-1} = 0$, it follows that $(C_H(\mathrm{z}I - A_H)^{-1}B_H + D_H)C_G = H(\mathrm{z})C_G = 0$. Letting $\mathrm{z} \to \infty$ implies that $D_H C_G = 0$. Then

$$\hat{C}_1 = \begin{bmatrix} I_m & 0 \end{bmatrix}\begin{bmatrix} \hat{C}_1 \\ \hat{C}_2 \end{bmatrix} = \hat{D}_H\hat{C}_G = D_H S S^{-1} C_G = 0. \tag{3.21}$$

Let $\mathcal{Z}$ denote the Rosenbrock system matrix of the minimal realization (3.1) of G. Since G has at least one zero zero, it follows that

$$\begin{aligned} n_G + m > \operatorname{rank}\mathcal{Z}(0) &= \operatorname{rank}\begin{bmatrix} -A_G & B_G \\ C_G & -D_G \end{bmatrix} \\ &= \operatorname{rank}\begin{bmatrix} I & 0 \\ 0 & S^{-1} \end{bmatrix}\begin{bmatrix} -A_G & B_G \\ C_G & -D_G \end{bmatrix} = \operatorname{rank}\begin{bmatrix} -A_G & B_G \\ \hat{C}_G & -\hat{D}_G \end{bmatrix}. \end{aligned} \tag{3.22}$$

It follows from (3.20)-(3.22) that

$$n_G + m > \operatorname{rank}\begin{bmatrix} -A_G & B_G \\ \hat{C}_G & -\hat{D}_G \end{bmatrix} = \operatorname{rank}\begin{bmatrix} -A_G & B_G \\ 0 & -I_m \\ \hat{C}_2 & -\hat{D}_2 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} -A_G & 0 \\ 0 & I_m \\ \hat{C}_2 & 0 \end{bmatrix}. \tag{3.23}$$

Since $(A_G, \hat{C}_G)$ is observable, it follows from the PBH test that $\operatorname{rank}\begin{bmatrix} -A_G \\ \hat{C}_G \end{bmatrix} = n_G$. Hence $\operatorname{rank}\begin{bmatrix} -A_G & 0 \\ 0 & I_m \\ \hat{C}_2 & 0 \end{bmatrix} = n_G + m$, which contradicts (3.23). Therefore, $K \neq 0$. $\square$

In the case where $d = 0$ and G has at least one zero zero, Proposition 3.16 implies that $\operatorname{ind} K \geq 1$ and thus it follows from Theroem 3.14 that u_0 cannot be reconstructed.

3.6 Numerical Examples

Example 3.17. Let

$$G(\mathrm{z}) = \begin{bmatrix} \dfrac{1}{\mathrm{z}^2} \\ \dfrac{1}{\mathrm{z}+1} \end{bmatrix}, \quad H(\mathrm{z}) = \begin{bmatrix} \dfrac{\mathrm{z}}{\mathrm{z}+1} & \dfrac{1}{\mathrm{z}^2} \end{bmatrix}, \tag{3.24}$$

so that $H(\mathrm{z})G(\mathrm{z}) = \mathrm{z}^{-2}$ and thus H is a delayed left inverse of G with delay 2. Figure 3.1 shows the input and output of (3.6), (3.7) with zero initial conditions and with nonzero initial conditions. Note that H, which is an IIR transfer function, fails to reconstruct the input in the case where the initial conditions are nonzero. Now, let

$$H(\mathrm{z}) = \begin{bmatrix} 0 & \dfrac{\mathrm{z}+1}{\mathrm{z}} \end{bmatrix}, \tag{3.25}$$

so that $H(\mathrm{z})G(\mathrm{z}) = \mathrm{z}^{-1}$, and thus H is a delayed left inverse of G with delay 1. Figure 3.2 shows the input and output of (3.6), (3.7) with zero and nonzero initial conditions. Note that H, which is an FIR transfer function, correctly reconstructs the input in the case where the initial conditions are nonzero. $\diamond$

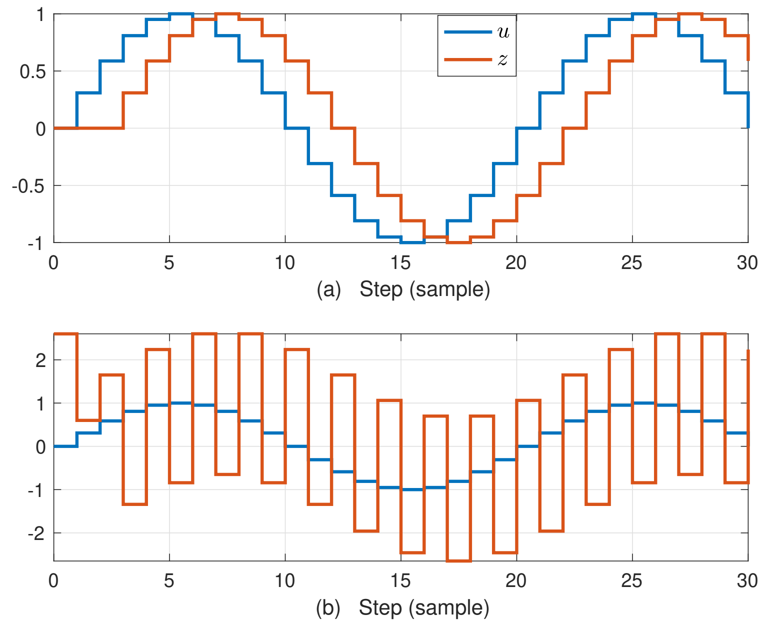

Figure 3.1: (a) shows the input and output of (3.6), (3.7) with zero initial conditions. (b) shows the input and output of (3.6), (3.7) with nonzero initial conditions.

Example 3.18. Let $G \in \mathbb{R}(\mathrm{z})^{3\times 2}_{\mathrm{prop}}$, where

$$G(\mathrm{z}) = \begin{bmatrix} \dfrac{\mathrm{z}}{\mathrm{z}+1} & \dfrac{1}{\mathrm{z}} \\ \dfrac{\mathrm{z}}{\mathrm{z}+2} & 0 \\ \dfrac{\mathrm{z}}{\mathrm{z}+1} & 1 \end{bmatrix} \tag{3.26}$$

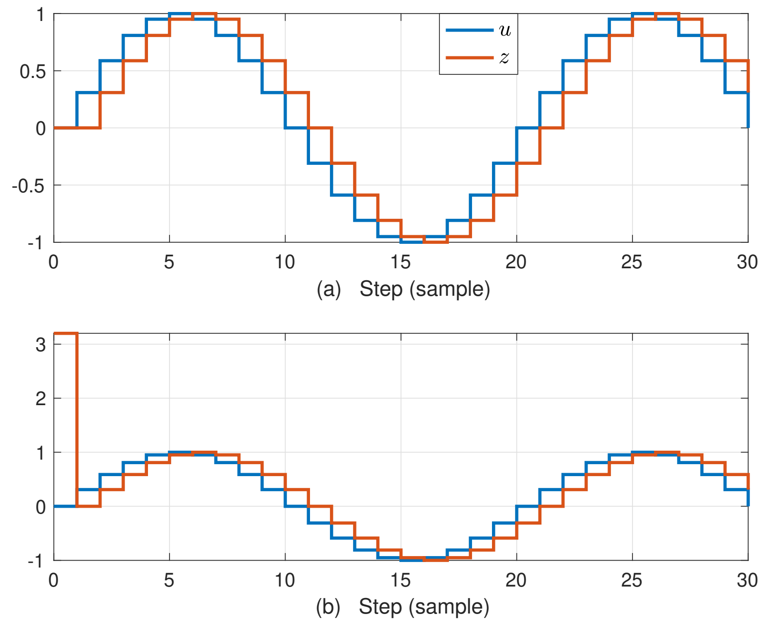

Figure 3.2: (a) shows the input and output of (3.6), (3.7) with zero initial conditions. (b) shows the input and output of (3.6), (3.7) with nonzero initial conditions.

Then

$$S_\infty(\mathrm{z}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, W(\mathrm{z}) = \begin{bmatrix} \frac{\mathrm{z}+2}{2\mathrm{z}} & \frac{2(\mathrm{z}+1)}{\mathrm{z}^2} & \frac{2(2\mathrm{z}+1)}{\mathrm{z}} \\ \frac{\mathrm{z}+1}{2\mathrm{z}} & \frac{\mathrm{z}+1}{\mathrm{z}^2} & \frac{2\mathrm{z}+1}{\mathrm{z}} \\ \frac{\mathrm{z}+2}{2\mathrm{z}} & \frac{\mathrm{z}^2+\mathrm{z}+2}{\mathrm{z}^2} & \frac{3\mathrm{z}+2}{\mathrm{z}} \end{bmatrix}, V(\mathrm{z}) = \begin{bmatrix} 1 & -\frac{\mathrm{z}^2+3\mathrm{z}+2}{\mathrm{z}^3} \\ 0 & 1 \end{bmatrix}, \tag{3.27}$$

where S_∞ is the Smith-McMillan form at infinity of G and $G = WS_\infty V$. It follows from Propostion 3.10 that $\eta_G = 0$. Evaluating the expression for H_∞ given in Theorem 3.12 yields

$$H_\infty(\mathrm{z}) = \begin{bmatrix} -\frac{2\mathrm{z}^3+7\mathrm{z}^2+7\mathrm{z}+2}{2\mathrm{z}^3} & \frac{4\mathrm{z}^4+11\mathrm{z}^3+3\mathrm{z}^2-8\mathrm{z}-4}{2\mathrm{z}^4} & \frac{2\mathrm{z}^3+7\mathrm{z}^2+7\mathrm{z}+2}{2\mathrm{z}^4} \\ -\frac{1}{2} & -\frac{\mathrm{z}+2}{2\mathrm{z}} & \frac{2\mathrm{z}+1}{2\mathrm{z}} \end{bmatrix}, \tag{3.28}$$

which satisfies $H_\infty(\mathrm{z})G(\mathrm{z}) = I_2$. Hence, H_∞ is an FIR left inverse of G. Constructing minimal realizations of H_∞ and K shows that $\mathrm{ind}\, H_\infty = \mathrm{ind}\, K = 4$, where K is defined in Theorem 3.14 with H replaced by H_∞. Theorem 3.14 thus implies that $\nu = \max\{\mathrm{ind}\, H_\infty, \mathrm{ind}\, K, d\} = 4$ and hence, for all $k \geq 4$, $z_k = u_k$, where u and z are defined in (3.6), (3.7). Figure 3.3 shows the input and output of (3.6), (3.7) with nonzero initial conditions. Note that, in this example, G is unstable and has a zero at zero. $\diamond$

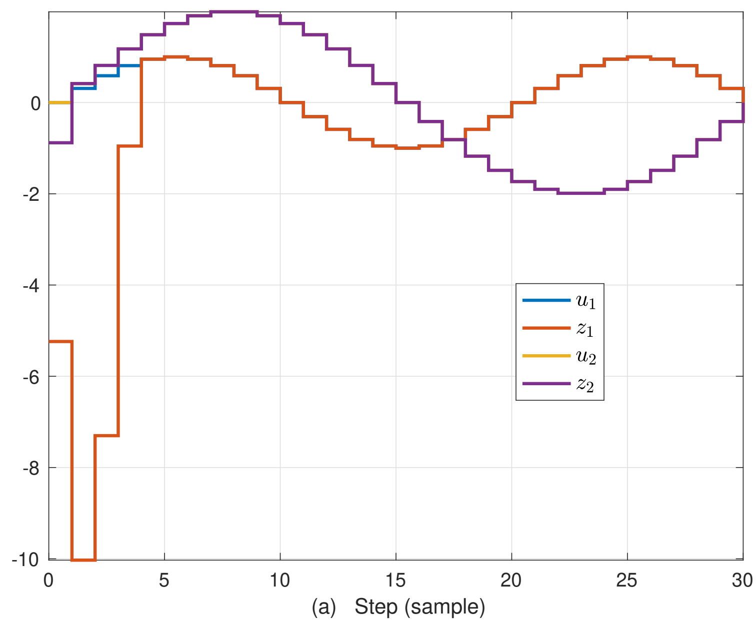

Figure 3.3: Input and output of (3.6), (3.7), where $u = [u_1\ u_2]^{\mathrm{T}}$ and $z = [z_1\ z_2]^{\mathrm{T}}$. Note that, for all $k \geq 4$, $z_{1,k} = u_{1,k}$, and, for all $k \geq 1$, $z_{2,k} = u_{2,k}$. Hence, for all $k \geq 4$, $z_k = u_k$.

3.7 An Application of Finite-Time Input Estimation

Many tall systems, that is, systems with number of outputs greater than the number of inputs have no transmission zeros. Exact input estimation can be done for such systems. Consider the mass-spring system shown in Figure 3.4. The dynamics of this system are given by

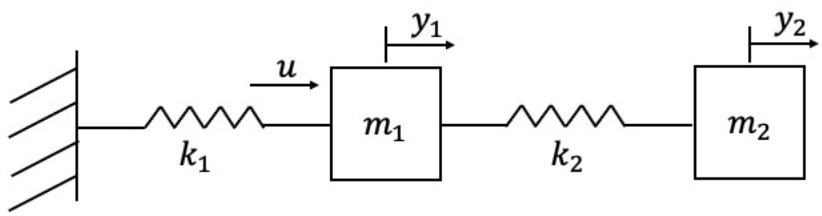

Figure 3.4: Mass-spring system

$$m_1\ddot{y}_1 = -k_1 y_1 - k_2(y_1 - y_2) + u, \tag{3.29}$$

$$m_2\ddot{y}_2 = -k_2(y_2 - y_1), \tag{3.30}$$

where m_1 and m_2 are the masses in kilograms, k_1 and k_2 are spring constants in Newton/meters, y_1 and y_2 are position of the masses m_1 and m_2 in meters, and u is the force in Newtons. Assume that u is unknown and let $m_1 = m_2 = 1$ kg and $k_1 = k_2 = 1$ N/m. Assume that the output measurement is $y = \begin{bmatrix} \dot{y}_1 \\ y_2 \end{bmatrix}$. Then the transfer function from u to y is given by

$$G(s) = \begin{bmatrix} \dfrac{s^3 + s}{s^4 + 3s^2 + 1} \\ \dfrac{1}{s^4 + 3s^2 + 1} \end{bmatrix}. \tag{3.31}$$

Discretization of (3.31) using zero-order hold yields the discrete-time transfer function given by

$$G_{\mathrm{d}}(z) = \begin{bmatrix} \dfrac{0.09967\mathrm{z}^3 - 0.298\mathrm{z}^2 + 0.298\mathrm{z} - 0.09967}{\mathrm{z}^4 - 3.97\mathrm{z}^3 + 5.94\mathrm{z}^2 - 3.97\mathrm{z} + 1} \\ \\ \dfrac{4.163 \times 10^{-6}\mathrm{z}^3 + 4.571 \times 10^{-5}\mathrm{z}^2 + 4.571 \times 10^{-5}\mathrm{z} + 4.163 \times 10^{-6}}{\mathrm{z}^4 - 3.97\mathrm{z}^3 + 5.94\mathrm{z}^2 - 3.97\mathrm{z} + 1} \end{bmatrix}. \tag{3.32}$$

Note that G_d has no transmission zeros. The minimal delay for which a delayed left inverse exists for G_d is $\eta_{G_\mathrm{d}} = 1$. Using the construction of FIR delayed inverse given in Section 3.3 yields

$$H(z) = \begin{bmatrix} \dfrac{8.295\mathrm{z}^6 - 25.2\mathrm{z}^5 + 19.26\mathrm{z}^4 + 10.25\mathrm{z}^3 - 18.27\mathrm{z}^2 + 4.964\mathrm{z} + 0.6982}{\mathrm{z}^6} \\ \dfrac{4.161\times10^4\mathrm{z}^6 - 1.7\times10^5\mathrm{z}^5 + 3.079\times10^5\mathrm{z}^4 - 3.591\times10^5\mathrm{z}^3 + 3.079\times10^5\mathrm{z}^2 - 1.7\times10^5\mathrm{z} + 4.161\times10^4}{\mathrm{z}^6} \end{bmatrix}^\mathrm{T}. \tag{3.33}$$

Note that $HG = 1/\mathrm{z}$ and hence H is a delayed left inverse of G with the minimal delay 1. Figure 3.5 compares the actual input u applied to the system with the output of the cascade of G and H. The input is exactly reconstructed after a finite number of steps. Figure 3.6 is a zoomed view of 3.5 and shows that the input is estimated with a delay of 1.

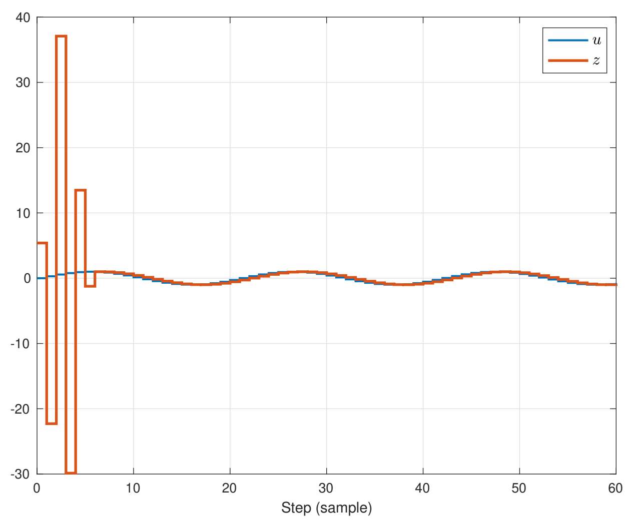

Figure 3.5: Input estimation for the mass-spring system shown in Figure 3.4. u is the actual input and z is the estimated input.

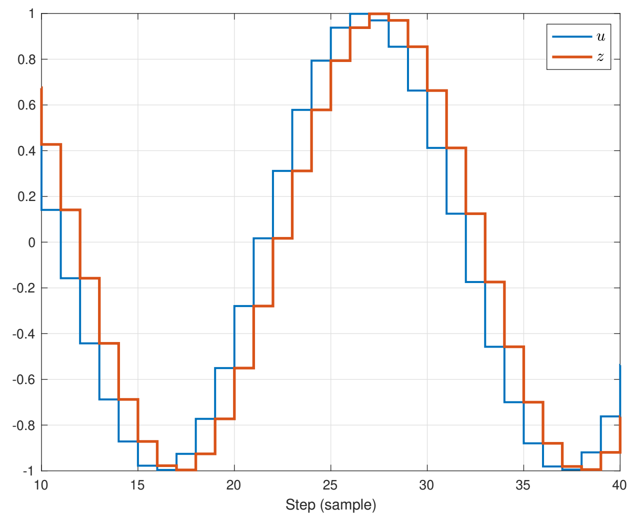

Figure 3.6: Zoomed view of Figure 3.5

CHAPTER 4

Transmission Zeros and Infinite Zeros

Transmission zeros can be counted by using the Smith-McMillan form, pole/zero modules, or the dimension of the largest output nulling invariant subspace. In this chapter, an alternative approach is provided by showing that the number of transmission zeros of a MIMO transfer function is given in terms of the defect of an augmented matrix involving an observability matrix and the Markov block-Toeplitz matrix. It is also shown that the number of infinite zeros is related to the defect of the Markov block-Toeplitz matrix. These results are illustrated with a numerical example.

Let $G \in \mathbb{R}(\mathrm{z})^{p\times m}_{\mathrm{prop}}$, where $p \geq m$, $G \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, and $A \in \mathbb{R}^{n\times n}$. Consider (3.10) and (3.11). For all $l \geq 0$, define the l^{th} Markov parameter

$$H_l \triangleq \begin{cases} D, & l = 0, \\ CA^{l-1}B, & l \geq 1. \end{cases}$$

For all $l \geq 0$, define

$$\mathcal{Y}_l \triangleq \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_l \end{bmatrix} \in \mathbb{R}^{(l+1)p}, \quad \mathcal{U}_l \triangleq \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_l \end{bmatrix} \in \mathbb{R}^{(l+1)m}, \quad \Gamma_l \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^l \end{bmatrix} \mathbb{R}^{(l+1)p\times n}.$$

Γ_l is the lth *observability matrix*. It follows from (3.10), (3.11) that, for all $l \geq 0$,

$$\mathcal{Y}_l = \Gamma_l x_0 + \mathcal{T}_l \mathcal{U}_l = \Psi_l \begin{bmatrix} x_0 \\ \mathcal{U}_l \end{bmatrix}, \tag{4.1}$$

where

$$\Psi_l \triangleq \begin{bmatrix} \Gamma_l & \mathcal{T}_l \end{bmatrix} \in \mathbb{R}^{(l+1)p \times [n+(l+1)m]},$$

and $\mathcal{T}_l$ is the lth Markov block-Toeplitz matrix associated with G. For all $l \geq 0$, define

$$Q_l \triangleq \begin{bmatrix} H_0 \\ H_1 \\ \vdots \\ H_l \end{bmatrix} \in \mathbb{R}^{(l+1)p \times m}, \quad P_l \triangleq \begin{bmatrix} 0 \\ \mathcal{T}_{l-1} \end{bmatrix} \in \mathbb{R}^{(l+1)p \times lm},$$

so that $\mathcal{T}_l = \begin{bmatrix} Q_l & P_l \end{bmatrix}$. Let ζ denote the number of transmission zeros of G counting multiplicity.

We assume for the rest of the chapter that G has full column normal rank, that is, $\operatorname{rank} G = m$. This assumption implies that G is square or tall, that is, $p \geq m$. However, since G and G^{T} have the same poles and zeros, the results in this chapter can be used in the case where G has full row rank, that is, $\operatorname{rank} G = p$. In this case, G is square or wide, that is, $m \geq p$.

4.1 Counting Transmission Zeros

In this section, we relate the number of transmission zeros of G to the defect of an augmented matrix involving an observability matrix and the Markov block-Toeplitz matrix. The concept of output nulling invariant subspaces [118] acts as a bridge in establishing this relationship. The main result is Theorem 4.3, which provides an expression for the number of transmission zeros.

The following result is given by Theorem 11 in [119].

Proposition 4.1. Let V^* be the maximal output-nulling invariant subspace of a minimal realization of G. Then, $\dim\ V^* = \zeta$.

Lemma 4.2. Let V^* be the maximal output-nulling invariant subspace of (3.10), (3.11), and let $x_0 \in V^*$. Then there exists an input sequence $(u_k)_{k\geq 0}$ such that, for all $k \geq 0$, $y_k = 0$.

Proof. Since $x_0 \in V^*$, it follows from (2.2) that there exists $u_0 \in \mathbb{R}^m$ such that

$$x_1 = Ax_0 + Bu_0,$$
$$0 = Cx_0 + Du_0,$$

where $x_1 \in V^*$. Since $x_1 \in V^*$, it follows from (2.2) that there exists $u_1 \in \mathbb{R}^m$ such that

$$x_2 = Ax_1 + Bu_1,$$
$$0 = Cx_1 + Du_1,$$

where $x_2 \in V^*$. By induction, it follows that there exists an input sequence $(u_k)_{k\geq 0}$ such that, for all $k \geq 0$, $y_k = 0$. □

The following result characterizes the number of transmission zeros in terms of the defect of the Markov block-Toeplitz matrix given by Definition 2.1 and the defect of a matrix consisting of an observability matrix and the Markov block-Toeplitz matrix.

Theorem 4.3. For all $l \geq n-1$,

$$\operatorname{def}\Psi_l - \operatorname{def}\mathcal{T}_l = \dim(\mathcal{R}(\Gamma_l) \cap \mathcal{R}(\mathcal{T}_l)) = \zeta. \tag{4.2}$$

Proof. It follows from Fact 3.14.15 in [116] that, for all $l \geq 0$,

$$\operatorname{def}\Psi_l = \operatorname{def}\Gamma_l + \operatorname{def}\mathcal{T}_l + \dim(\mathcal{R}(\Gamma_l) \cap \mathcal{R}(\mathcal{T}_l)). \tag{4.3}$$

Note that, for all $l \geq n-1$, $\operatorname{def} \Gamma_l = 0$. Hence (4.3) implies that, for all $l \geq n-1$,

$$\operatorname{def} \Psi_l - \operatorname{def} \mathcal{T}_l = \dim(\mathcal{R}(\Gamma_l) \cap \mathcal{R}(\mathcal{T}_l)).$$

Next, let V^* be the maximal output-nulling invariant subspace of (3.10), (3.11). Then Proposition 4.1 implies that $\dim\ V^* = \zeta$. Let $x_{1,0}, x_{2,0}, \ldots, x_{\zeta,0}$ be a basis for V^*. It follows from Lemma 4.2 that, for all $l \geq n-1$ and $i = 1, \ldots, \zeta$, there exists $\mathcal{U}_{l,i} \in \mathbb{R}^{(l+1)m}$ such that, when substituted for $\mathcal{U}_l$ in (4.1), yields $\mathcal{Y}_l = 0$. Thus, for all $l \geq n-1$ and $i = 1, \ldots, \zeta$, it follows that

$$\Gamma_l x_{i,0} + \mathcal{T}_l \mathcal{U}_{l,i} = 0.$$

For all $l \geq n-1$ and $i = 1, \ldots, \zeta$, define $z_{l,i} \triangleq \Gamma_l x_{i,0} = -\mathcal{T}_l \mathcal{U}_{l,i}$. For all $l \geq n-1$, let $\alpha_{l,1}, \ldots, \alpha_{l,\zeta}$ be real numbers such that $\sum_{i=1}^{\zeta} \alpha_{l,i} z_{l,i} = 0$. Then, for all $l \geq n-1$,

$$0 = \sum_{i=1}^{\zeta} \alpha_{l,i} z_{l,i} = \sum_{i=1}^{\zeta} \alpha_{l,i} \Gamma_l x_{i,0} = \Gamma_l \sum_{i=1}^{\zeta} \alpha_{l,i} x_{i,0}.$$

Since, for all $l \geq n-1$, Γ_l has full column rank, it follows that $\sum_{i=1}^{\zeta} \alpha_{l,i} x_{i,0} = 0$ and thus $\alpha_{l,i} = 0$. Hence, for all $l \geq n-1$, $z_{l,1}, \ldots, z_{l,\zeta}$ are linearly independent. Now, for all $l \geq n-1$, define $z_l \triangleq \Gamma_l x_0$, where $x_0 \triangleq \sum_{i=1}^{\zeta} \beta_i x_{i,0}$. It follows that, for all $l \geq n-1$,

$$z_l = \Gamma_l \sum_{i=1}^{\zeta} \beta_i x_{i,0} = \sum_{i=1}^{\zeta} \beta_i \Gamma_l x_{i,0} = \sum_{i=1}^{\zeta} \beta_i z_{l,i}.$$

Thus, for all $l \geq n-1$, $\operatorname{span}\{z_{l,1}, \ldots, z_{l,\zeta}\} = \mathcal{R}(\Gamma_l) \cap \mathcal{R}(\mathcal{T}_l)$, and hence $\dim(\mathcal{R}(\Gamma_l) \cap \mathcal{R}(\mathcal{T}_l)) = \zeta$.
$\square$

4.2 Counting Infinite Zeros

Infinite zeros extend the notion of relative degree to MIMO systems; in fact, for a SISO system, the number of infinite zeros is the relative degree of the transfer function. The main result in this section, Theorem 4.6, establishes a relationship between the number of infinite zeros and the defect of the Markov block-Toeplitz matrix. All of the definitions and results given below support the main result.

Proposition 4.4. η_G is finite.

Proof. Note that, since G has full column rank, $[G(\mathrm{z})^{\mathrm{T}}G(\mathrm{z})]^{-1}G(\mathrm{z})^{\mathrm{T}}$ is a left inverse of G and thus there exists $d \geq 0$ such that $H(\mathrm{z}) = \mathrm{z}^{-d}[G(\mathrm{z})^{\mathrm{T}}G(\mathrm{z})]^{-1}G(\mathrm{z})^{\mathrm{T}}$ is a delayed left inverse of G with delay d. Hence G is delayed left invertible with delay d. Then Proposition 3.10 implies that $\operatorname{rank} \mathcal{T}_d - \operatorname{rank} \mathcal{T}_{d-1} = m$ and hence η_G is finite. $\square$

Lemma 4.5. Let $l_0 \geq 0$. The following statements are equivalent:

$i)$ $\operatorname{rank} \mathcal{T}_{l_0} - \operatorname{rank} \mathcal{T}_{l_0-1} = m$.

$ii)$ $\operatorname{rank} Q_{l_0} = m$ and $\dim(\mathcal{R}(Q_{l_0}) \cap \mathcal{R}(P_{l_0})) = 0$.

$iii)$ For all $l \geq l_0$, $\operatorname{rank} \mathcal{T}_l - \operatorname{rank} \mathcal{T}_{l-1} = m$.

Proof. To prove *i*) $\Longrightarrow$ *ii*), note that it follows from Fact 3.14.15 in [116, p. 322] that $m = \operatorname{rank} \mathcal{T}_{l_0} - \operatorname{rank} \mathcal{T}_{l_0-1} = \operatorname{rank} Q_{l_0} - \dim(\mathcal{R}(Q_{l_0}) \cap \mathcal{R}(P_{l_0}))$. Thus, $m + \dim(\mathcal{R}(Q_{l_0}) \cap \mathcal{R}(P_{l_0})) = \operatorname{rank} Q_{l_0} \leq m$. Hence, $\operatorname{rank} Q_{l_0} = m$, and $\dim(\mathcal{R}(Q_{l_0}) \cap \mathcal{R}(P_{l_0})) = 0$.

To prove *ii*) $\Longrightarrow$ *iii*), note that, for all $l \geq 0$,

$$Q_{l+1} = \begin{bmatrix} Q_l \\ H_{l+1} \end{bmatrix}, \quad P_{l+1} = \left[\begin{array}{ccc|c} & P_l & & 0 \\ \hline H_l & \cdots & H_1 & H_0 \end{array}\right].$$

Furthermore, for all $l \geq l_0$, $\operatorname{rank} Q_{l+1} = \operatorname{rank} Q_{l_0} = m$. Since $\operatorname{rank} Q_{l_0} = m$ and $\dim(\mathcal{R}(Q_{l_0}) \cap \mathcal{R}(P_{l_0})) = 0$, it follows from Lemma A in [34] that $\dim(\mathcal{R}(Q_{l_0+1}) \cap \mathcal{R}(P_{l_0+1})) = 0$. By induction, it thus follows that, for all $l \geq l_0$, $\dim(\mathcal{R}(Q_l) \cap \mathcal{R}(P_l)) = 0$. Thus, for all $l \geq l_0$, Fact 3.14.15 in [116, p. 322] implies that $\operatorname{rank} \mathcal{T}_l - \operatorname{rank} \mathcal{T}_{l-1} = \operatorname{rank} Q_l - \dim(\mathcal{R}(Q_l) \cap \mathcal{R}(P_l)) = m$.

The proof of *iii*) $\Longrightarrow$ *i*) is immediate. $\square$

The following result characterizes the number of infinite zeros in terms of the defect of the Markov block-Toeplitz matrix.

Theorem 4.6. For all $l \geq \eta_G - 1$, $\operatorname{def} \mathcal{T}_l = \iota$.

Proof. Note that it follows from Proposition 4.4 that η_G is finite. Next, Fact 3.14.15 in [116, p. 322] implies that, for all $l \geq 0$,

$$\operatorname{def} \mathcal{T}_l = \operatorname{def} Q_l + \operatorname{def} P_l + \dim(\mathcal{R}(Q_l) \cap \mathcal{R}(P_l)). \tag{4.4}$$

For all $l \geq \eta_G$, Lemma 4.5 implies that $\operatorname{rank} Q_l = m$, and $\dim(\mathcal{R}(Q_l) \cap \mathcal{R}(P_l)) = 0$. Therefore, it follows from (4.4) that, for all $l \geq \eta_G$, $\operatorname{def} \mathcal{T}_l = \operatorname{def} P_l = \operatorname{def} \begin{bmatrix} 0 \\ \mathcal{T}_{l-1} \end{bmatrix} = \operatorname{def} \mathcal{T}_{l-1}$. Hence, for all $l \geq \eta_G$, $\operatorname{def} \mathcal{T}_l = \operatorname{def} \mathcal{T}_{\eta_G - 1}$.

Next, let S_∞ be the Smith-McMillan form at infinity of G, and let $\iota_1, \ldots, \iota_{\rho_0}$ be as defined in Theorem 3.8. Let $H_{\infty,j}$ be the jth Markov parameter of S_∞. It follows from Proposition 3.11 that $\eta_G = \iota_1$, and hence

$$\iota = \sum_{j=1}^{\rho_0} \iota_j = \sum_{j=1}^{\iota_1} j \operatorname{rank} H_{\infty,j} = \sum_{j=1}^{\eta_G} j \operatorname{rank} H_{\infty,j}. \tag{4.5}$$

Since G has full column normal rank, it follows that

$$m = \sum_{j=0}^{\eta_G} \operatorname{rank} H_{\infty,j}. \tag{4.6}$$

Let $\mathcal{T}_{\infty,i}$ be the ith Markov block-Toeplitz matrix associated with S_∞. Note that, for all $i \geq 0$, each row of $\mathcal{T}_{\infty,i}$ is either zero or has exactly one nonzero entry that is equal to one, and the nonzero rows of $\mathcal{T}_{\infty,i}$ are linearly independent. It thus follows that, for all $i \geq 0$,

$$\operatorname{rank} \mathcal{T}_{\infty,i} = \sum_{j=0}^{i} (i - j + 1) \operatorname{rank} H_{\infty,j}. \tag{4.7}$$

Hence (4.5), (4.6), and (4.7) imply that

$$\begin{aligned}
\operatorname{def} \mathcal{T}_{\infty,\eta_G - 1} &= \eta_G m - \sum_{j=0}^{\eta_G - 1} (\eta_G - j) \operatorname{rank} H_{\infty,j} \\
&= \eta_G (m - \operatorname{rank} H_{\infty,0}) - \eta_G \sum_{j=1}^{\eta_G - 1} \operatorname{rank} H_{\infty,j} + \sum_{j=1}^{\eta_G - 1} j \operatorname{rank} H_{\infty,j} \\
&= \eta_G (m - \operatorname{rank} H_{\infty,0}) - \eta_G (m - \operatorname{rank} H_{\infty,0} - \operatorname{rank} H_{\infty,\eta_G}) + \sum_{j=1}^{\eta_G - 1} j \operatorname{rank} H_{\infty,j} \\
&= \sum_{j=1}^{\eta_G} j \operatorname{rank} H_{\infty,j} = \iota.
\end{aligned}$$

Next, since $\mathcal{T}_{-1}$ is an empty matrix, it follows from Lemma 3.9 and Theorem 3.8 that, for all $l \geq 0$, $\operatorname{rank} \mathcal{T}_l = \operatorname{rank} \mathcal{T}_{\infty,l}$. Hence, $\operatorname{def} \mathcal{T}_{\eta_G - 1} = \operatorname{def} \mathcal{T}_{\infty,\eta_G - 1} = \iota$. $\square$

4.3 Numerical Example

Example 4.7. Let

$$G = \begin{bmatrix} \dfrac{1}{\mathrm{z}+1} & 1 \\ \dfrac{1}{\mathrm{z}+3} & \dfrac{1}{2\mathrm{z}} \\ \dfrac{1}{2\mathrm{z}} & 1 \end{bmatrix}. \tag{4.8}$$

Numerical computation using Matlab yields

i) $n = 4$, $\eta_G = 1$.

$ii)$ $\operatorname{def} \mathcal{T}_0 = 1$.

$iii)$ $\operatorname{def} \Psi_0 = 3$, and $\operatorname{def} \Psi_l = 2$, for $l = 1, 2, 3$.

Theorem 4.6 thus implies that, for all $l \geq 0$, $\operatorname{def} \mathcal{T}_l = 1$, and thus $\iota = 1$. Similarly, Theorem 4.3 implies that, for all $l \geq 3$, $\operatorname{def} \Psi_l = 2$, and thus $\zeta = 1$.

As a check, the numbers of infinite and transmission zeros are calculated from the Smith-McMillan form at infinity and the Smith-McMillan form, respectively, as follows. Note that $G = W S_\infty V$, where

$$S_\infty(\mathrm{z}) = \begin{bmatrix} \frac{1}{\mathrm{z}} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad V(\mathrm{z}) = \begin{bmatrix} \dfrac{1-\mathrm{z}}{\mathrm{z}} & \dfrac{(\mathrm{z}-1)(5\mathrm{z}^2-6\mathrm{z}-9)}{4\mathrm{z}^3} \\ -\dfrac{6\mathrm{z}}{\mathrm{z}^2+4\mathrm{z}+3} & \dfrac{6\mathrm{z}-9}{2\mathrm{z}} \end{bmatrix}, \tag{4.9}$$

$$W(\mathrm{z}) = \begin{bmatrix} -\dfrac{6\mathrm{z}+9}{2\mathrm{z}} & \dfrac{4\mathrm{z}^4+21\mathrm{z}^3+21\mathrm{z}^2-27\mathrm{z}-27}{12\mathrm{z}^4} & 1 \\ -\dfrac{2\mathrm{z}+3}{2\mathrm{z}} & \dfrac{7\mathrm{z}^3+7\mathrm{z}^2-9\mathrm{z}-9}{12\mathrm{z}^4} & \dfrac{1}{3} \\ -\dfrac{10\mathrm{z}^2+15\mathrm{z}+9}{4\mathrm{z}^2} & \dfrac{8\mathrm{z}^5+37\mathrm{z}^4+38\mathrm{z}^3-18\mathrm{z}^2-54\mathrm{z}-27}{24\mathrm{z}^5} & \dfrac{\mathrm{z}+1}{2\mathrm{z}} \end{bmatrix}. \tag{4.10}$$

It can be seen from S_∞ that $\iota = 1$. Next, note that $G = S_1 S S_2$, where

$$S(\mathrm{z}) = \begin{bmatrix} \dfrac{1}{\mathrm{z}(\mathrm{z}+1)(\mathrm{z}+3)} & 0 \\ 0 & \dfrac{\mathrm{z}-1}{\mathrm{z}} \\ 0 & 0 \end{bmatrix}, \quad S_2(\mathrm{z}) = \begin{bmatrix} 1 & \dfrac{2\mathrm{z}^3+9\mathrm{z}^2+10\mathrm{z}+3}{12} \\ 0 & 1 \end{bmatrix}, \tag{4.11}$$

$$S_1(\mathrm{z}) = \begin{bmatrix} \mathrm{z}(\mathrm{z}+3) & -\dfrac{\mathrm{z}(2\mathrm{z}+9)}{12} & -\dfrac{\mathrm{z}+6}{6} \\ \mathrm{z}(\mathrm{z}+1) & -\dfrac{2\mathrm{z}^2+5\mathrm{z}+6}{12} & -\dfrac{\mathrm{z}+4}{6} \\ \dfrac{\mathrm{z}^2+4\mathrm{z}+3}{2} & \dfrac{-2\mathrm{z}^2-11\mathrm{z}+3}{24} & -\dfrac{\mathrm{z}+7}{12} \end{bmatrix}. \tag{4.12}$$

It can be seen from S that $\zeta = 1$. $\diamond$

CHAPTER 5

Zero Dynamics of Input-Output Models

Zeros are of extreme importance in linear systems theory, especially unstable zeros, which degrade achievable performance. Furthermore, the presence of a zero in a state space model implies the existence of an initial condition and a nonzero input signal such that the output is identically zero; this property is called *output zeroing*. The purpose of this chapter is to elucidate the properties of the zero dynamics within the context of input-output models, which, like state space models, are time-domain models, but, unlike state space models, have no internal state. In particular, the focus is on the zero dynamics of left polynomial fraction description (LPFD) input-output models whose denominator polynomial is not necessarily monic. In addition, output zeroing in input-output models is considered, and its equivalence to output zeroing in state space models is discussed. Finally, a numerical example is presented to illustrate the results.

5.1 Preliminary Results

In this and all subsequent sections, let $G \in \mathbb{R}(\mathrm{z})^{p\times m}_{\mathrm{prop}}$. In the notation of Theorem 3.2, define

$$D_{\mathrm{S}} \triangleq \begin{bmatrix} q_1 & & & 0 \\ & \ddots & & \\ & & q_\rho & \\ 0 & & & I_{p-\rho} \end{bmatrix} S_1^{-1}, \quad N_{\mathrm{S}} \triangleq \begin{bmatrix} p_1 & & & 0 \\ & \ddots & & \\ & & p_\rho & \\ 0 & & & 0_{(p-\rho)\times(m-\rho)} \end{bmatrix} S_2. \tag{5.1}$$

Proposition 5.1. $(D_{\mathrm{S}}, N_{\mathrm{S}})$ is a CLPFD of G.

Proof. Note that, for all $\mathrm{z} \in \mathbb{C}$, $\mathrm{rank}[D_{\mathrm{S}}(\mathrm{z})\ N_{\mathrm{S}}(\mathrm{z})] = p$, and hence it follows from Theorem 16.16 in [120, p. 300] that D_{S} and N_{S} are coprime. □

The following result is given by Theorem 16.17 in [120, p. 301].

Proposition 5.2. Let (D, N) be a CLPFD of G and $(\hat{D}, \hat{N})$ be an LPFD of G. Then $(\hat{D}, \hat{N})$ is a CLPFD of G if and only if there exists a unimodular matrix $U \in \mathbb{R}[\mathrm{z}]^{p\times p}$ such that $\hat{D} = UD$ and $\hat{N} = UN$.

Corollary 5.3. Let (D, N) be an LPFD of G. Then (D, N) is a CLPFD of G if and only if there exists a unimodular matrix $U \in \mathbb{R}[\mathrm{z}]^{p\times p}$ such that $D = UD_{\mathrm{S}}$ and $N = UN_{\mathrm{S}}$.

Note that it follows from [121, p. 35] that, for all $P \in \mathbb{R}[\mathrm{z}]^{p\times n}$ and $Q \in \mathbb{R}[\mathrm{z}]^{p\times m}$, there exists a greatest common left divisor of P and Q.

Lemma 5.4. Let (D, N) be a CLPFD of G, and let $(\hat{D}, \hat{N})$ be an LPFD of G. Then there exists a nonsingular $L \in \mathbb{R}[\mathrm{z}]^{p\times p}$ such that $\hat{D} = LD$ and $\hat{N} = LN$.

Proof. Let $R \in \mathbb{R}[\mathrm{z}]^{p\times p}$ be a greatest common left divisor of $\hat{D}$ and $\hat{N}$. Then there exist $\overline{D} \in \mathbb{R}[\mathrm{z}]^{p\times p}$ and $\overline{N} \in \mathbb{R}[\mathrm{z}]^{p\times m}$ such that $\hat{D} = R\overline{D}$, $\hat{N} = R\overline{N}$, and $\overline{D}$ and $\overline{N}$ are coprime. Next, it follows from Proposition 5.2 that there exists a unimodular matrix $U \in \mathbb{R}[\mathrm{z}]^{p\times p}$ such that $\overline{D} = UD$ and $\overline{N} = UN$. Hence, $\hat{D} = LD$ and $\hat{N} = LN$, where $L \triangleq RU$. Since $\hat{D}$ is nonsingular, it follows that R is nonsingular and thus L is nonsingular. □

Proposition 5.5. Let (D, N) be an LPFD of G. Then $\deg\det D = \mathrm{McDeg}\ G$ if and only if (D, N) is a CLPFD of G.

Proof. To prove sufficiency, note that Corollary 5.3 implies that there exists a unimodular matrix $U \in \mathbb{R}[\mathrm{z}]^{p\times p}$ such that $D = UD_{\mathrm{S}}$. Hence, $\deg\det D = \deg\det U + \deg\det D_{\mathrm{S}} = \deg\det D_{\mathrm{S}} = \mathrm{McDeg}\ G$. To prove necessity, note that it follows from Lemma 5.4 and Proposition 5.1 that there exists a nonsingular $L \in \mathbb{R}[\mathrm{z}]^{p\times p}$ such that $D = LD_{\mathrm{S}}$ and $N = LN_{\mathrm{S}}$. Hence $\deg\det L = \deg\det D - \deg\det D_{\mathrm{S}} = \mathrm{McDeg}\ G - \mathrm{McDeg}\ G = 0$. Thus, L is unimodular and therefore Corollary 5.3 implies that (D, N) is a CLPFD of G. □

5.2 Results on Zero Dynamics of Input-Output Models

This section discusses various aspects of the zero dynamics of input-output models. In particular, Proposition 5.6 characterizes transmission zeros of G using an LPFD of G and a CLPFD of G. Next, Proposition 5.7 gives an expression for counting the number of transmission zeros of G using a CLPFD of G. Necessary and sufficient conditions for the existence of nonzero solutions to the zero dynamics of G are given in Proposition 5.8, and solutions of the zero dynamics are characterized by Proposition 5.9. Next, Theorem 5.10 relates nonzero solutions of the zero dynamics to the transmission zeros of G.

Proposition 5.6. Let (D, N) be an LPFD of G, and let z_0 be a transmission zero of G. Then $\operatorname{rank} N(\mathrm{z}_0) < \operatorname{rank} N$. Now assume that (D, N) is a CLPFD of G. Then z_0 is a transmission zero of G if and only if $\operatorname{rank} N(\mathrm{z}_0) < \operatorname{rank} N$.

Proof. To prove the first statement, note that Proposition 5.1 and Lemma 5.4 imply that there exists a nonsingular $L \in \mathbb{R}[\mathrm{z}]^{p\times p}$ such that $D = LD_{\mathrm{S}}$ and $N = LN_{\mathrm{S}}$, where D_{S} and N_{S} are defined in (5.1). Since z_0 is a transmission zero of G, it follows from Theorem 3.2 that $\operatorname{rank} N_{\mathrm{S}}(\mathrm{z}_0) < \operatorname{rank} N_{\mathrm{S}}$. Hence, $\operatorname{rank} N(\mathrm{z}_0) \le \operatorname{rank} N_{\mathrm{S}}(\mathrm{z}_0) < \operatorname{rank} N_{\mathrm{S}} = \operatorname{rank} N$. To prove sufficiency in the second statement, note that Corollary 5.3 implies that, for all $\mathrm{z} \in \mathbb{C}$, $\operatorname{rank} N(\mathrm{z}) = \operatorname{rank} N_{\mathrm{S}}(\mathrm{z})$. Hence, $\operatorname{rank} N_{\mathrm{S}}(\mathrm{z}_0) = \operatorname{rank} N(\mathrm{z}_0) < \operatorname{rank} N = \operatorname{rank} N_{\mathrm{S}}$, and thus it follows from Theorem 3.2 that z_0 is a transmission zero of G. $\square$

Proposition 5.7. Let (D, N) be a CLPFD of G, let $(\hat{D}, \hat{N})$ be a CLPFD of G^{T}, and let ζ be the number of transmission zeros of G counting multiplicity.

$i)$ If $\operatorname{rank} G = p$, then $\zeta = \frac{1}{2} \deg\det NN^{\mathrm{T}}$.

$ii)$ If $\operatorname{rank} G = m$, then $\zeta = \frac{1}{2} \deg\det \hat{N}\hat{N}^{\mathrm{T}}$.

Proof. To prove i), note that it follows from Corollary 5.3 that $NN^{\mathrm{T}} = UN_{\mathrm{S}}N_{\mathrm{S}}^{\mathrm{T}}U^{\mathrm{T}}$, where $U \in \mathbb{R}[\mathrm{z}]^{p\times p}$ is a unimodular matrix. Since $\operatorname{rank} G = p$, it follows that $N_{\mathrm{S}}N_{\mathrm{S}}^{\mathrm{T}}$ is nonsingular and thus NN^{T} is nonsingular. Thus $\deg\det NN^{\mathrm{T}} = \deg\det N_{\mathrm{S}}N_{\mathrm{S}}^{\mathrm{T}} = 2\zeta$.

To prove *ii*), note that the number of transmission zeros of G^{T} is equal to the number of transmission zeros of G. Since $\operatorname{rank} G^{\mathrm{T}} = \operatorname{rank} G = m$, applying *i*) to G^{T} yields *ii*). $\square$

The zero dynamics of an LPFD of G and the zero dynamics of G are defined in 2.19. The following result gives necessary and sufficient conditions for the existence of nonzero solutions of (2.4).

Proposition 5.8. Let $N \in \mathbb{R}[\mathrm{z}]^{p\times m}$. Then (2.4) has a nonzero solution if and only if there exists $\mathrm{z}_0 \in \mathbb{C}$ such that $\operatorname{rank} N(\mathrm{z}_0) < m$.

Proof. To prove sufficiency, let $N(\mathbf{q}) = \mathbf{q}^{\ell} B_0 + \mathbf{q}^{\ell-1} B_1 + \cdots + B_\ell$. Then

$$\begin{aligned} N(\mathbf{q})\mathrm{z}_0^k &= \left(\mathbf{q}^{\ell} B_0 + \mathbf{q}^{\ell-1} B_1 + \cdots + B_\ell\right)\mathrm{z}_0^k = \mathrm{z}_0^{k+\ell} B_0 + \mathrm{z}_0^{k+\ell-1} B_1 + \cdots + \mathrm{z}_0^k B_\ell \\ &= \left(\mathrm{z}_0^{\ell} B_0 + \mathrm{z}_0^{\ell-1} B_1 + \cdots + B_\ell\right)\mathrm{z}_0^k = N(\mathrm{z}_0)\mathrm{z}_0^k. \end{aligned}$$

Note that there exists $\overline{u} \neq 0$ such that $N(\mathrm{z}_0)\overline{u} = 0$. For all $k \geq 0$, define $u_k \triangleq \mathrm{z}_0^k \overline{u}$. Hence, for all $k \geq 0$, $N(\mathbf{q})u_k = N(\mathbf{q})\mathrm{z}_0^k\overline{u} = N(\mathrm{z}_0)\mathrm{z}_0^k\overline{u} = \mathrm{z}_0^k N(\mathrm{z}_0)\overline{u} = 0$. Since $\overline{u} \neq 0$, it follows that u is a nonzero solution of (2.4).

To prove necessity, note that, in the case where $\operatorname{rank} N < m$, it follows that, for all $\mathrm{z}_0 \in \mathbb{C}$, $\operatorname{rank} N(\mathrm{z}_0) < m$. In the case where $\operatorname{rank} N = m$, there exists a unimodular matrix $U \in \mathbb{R}[\mathbf{q}]^{p\times p}$ such that $\overline{N} \triangleq UN = \begin{bmatrix} N_0 \\ 0_{(p-m)\times m} \end{bmatrix}$, where $N_0 \in \mathbb{R}[\mathbf{q}]^{m\times m}$ is nonsingular. Then (2.4) implies that $\overline{N}(\mathbf{q})u_k = U(\mathbf{q})N(\mathbf{q})u_k = 0$, and thus $N_0(\mathbf{q})u_k = 0$. Now, suppose that N_0 is unimodular. Then $N_0^{-1}(\mathbf{q})$ is a polynomial matrix, and hence (2.4) is equivalent to $N_0^{-1}(\mathbf{q})N_0(\mathbf{q})u_k = 0$, and thus, for all $k \geq 0$, $u_k = 0$, which is a contradiction. It thus follows that N_0 is not unimodular, that is, $\det N_0$ is a nonconstant polynomial in $\mathbf{q}$. Let z_0 be a root of $\det N_0$. Hence, $\operatorname{rank} N_0(\mathrm{z}_0) < m$. Since, for all $\mathrm{z} \in \mathbb{C}$, $\operatorname{rank} N(\mathrm{z}) = \operatorname{rank} \overline{N}(\mathrm{z}) = \operatorname{rank} N_0(\mathrm{z})$, it follows that $\operatorname{rank} N(\mathrm{z}_0) = \operatorname{rank} N_0(\mathrm{z}_0) < m$. $\square$

The following result characterizes the possibly complex solutions of (2.4).

Proposition 5.9. Let $\mathrm{z}_0 \in \mathbb{C}$, and let $\overline{u} \in \mathbb{C}^m$. Then, for all $k \geq 0$, $u_k \triangleq \mathrm{z}_0^k \overline{u}$ satisfies (2.4) if and only if $N(\mathrm{z}_0)\overline{u} = 0$.

Proof. Let $N(\mathbf{q}) = \mathbf{q}^\ell B_0 + \mathbf{q}^{\ell-1} B_1 + \cdots + B_\ell$. Then

$$
\begin{aligned}
N(\mathbf{q})\mathrm{z}_0^k &= \left(\mathbf{q}^\ell B_0 + \mathbf{q}^{\ell-1} B_1 + \cdots + B_\ell\right) \mathrm{z}_0^k = \mathrm{z}_0^{k+\ell} B_0 + \mathrm{z}_0^{k+\ell-1} B_1 + \cdots + \mathrm{z}_0^k B_\ell \\
&= \left(\mathrm{z}_0^\ell B_0 + \mathrm{z}_0^{\ell-1} B_1 + \cdots + B_\ell\right) \mathrm{z}_0^k = N(\mathrm{z}_0)\mathrm{z}_0^k.
\end{aligned}
$$

To prove sufficiency, note that, for all $k \geq 0$, $N(\mathbf{q})u_k = N(\mathbf{q})\mathrm{z}_0^k\overline{u} = N(\mathrm{z}_0)\mathrm{z}_0^k\overline{u} = \mathrm{z}_0^k N(\mathrm{z}_0)\overline{u} = 0$. To prove necessity, note that, for all $k \geq 0$, $0 = N(\mathbf{q})u_k = \mathrm{z}_0^k N(\mathrm{z}_0)\overline{u}$. Letting $k = 0$ yields $N(\mathrm{z}_0)\overline{u} = 0$. $\square$

Proposition 5.8 and Proposition 5.9 discuss solutions of (2.4) in relation to an arbitrary complex number z_0. Since the focus of this chapter is on transmission zeros, we now give a result on the relationship between the solutions of (2.4) and a transmission zero z_0 of G.

Theorem 5.10. Let (D, N) be an LPFD of G. The following statements hold:

i) If $\operatorname{rank} N < m$, then, for all $\mathrm{z}_0 \in \mathbb{C}$, there exists a nonzero $\overline{u} \in \mathcal{N}(N(\mathrm{z}_0))$, and, for all $k \geq 0$, $u_k \triangleq \mathrm{z}_0^k\overline{u}$, is a nonzero solution of (2.4).

ii) If $\operatorname{rank} N = m$, and $\mathrm{z}_0 \in \mathbb{C}$ is a transmission zero of G, then there exists a nonzero $\overline{u} \in \mathcal{N}(N(\mathrm{z}_0))$, and, for all $k \geq 0$, $u_k \triangleq \mathrm{z}_0^k\overline{u}$, is a nonzero solution of (2.4).

iii) If $\operatorname{rank} N = m$, (D, N) is a CLPFD of G, and $\mathrm{z}_0 \in \mathbb{C}$, then the following statements are equivalent.

(a) z_0 is a transmission zero of G.

(b) There exists a nonzero $\overline{u} \in \mathcal{N}(N(\mathrm{z}_0))$.

(c) There exists $\overline{u} \neq 0$ such that, for all $k \geq 0$, $u_k \triangleq \mathrm{z}_0^k\overline{u}$ is a nonzero solution of (2.4).

Proof. *i)* follows from Proposition 5.9, and *ii)* follows from Proposition 5.6 and Proposition 5.9. a) $\Longrightarrow$ b) in *iii)* follows from Proposition 5.6, b) $\Longrightarrow$ c) in *iii)* follows from Proposition 5.9, and c) $\Longrightarrow$ a) in *iii)* follows from Proposition 5.8 and Proposition 5.6. $\square$

5.3 Equivalence of Output Zeroing in Input-Output Models and State Space Models

If G has a transmission zero, then it follows from [86, p. 25] that there exist an initial condition and a nonzero input such that the response of a minimal state space realization of G is identically zero. This is called *output zeroing* in state space models. Proposition 5.12 and Corollary 5.13 deal with output zeroing in state space models, and Theorem 5.14 relates output zeroing in state space models to the transmission zeros of G. In contrast, Theorem 5.16 and Corollary 5.17 discuss output zeroing in input-output models. Next, Theorem 5.18 relates output zeroing in input-output models to the transmission zeros of G, where it is shown that, if G has a transmission zero, then there exists a nonzero input such that the response of a time-domain input-output representation of G is identically zero. Furthermore, this section connects output zeroing in input-output models to output zeroing in state space models. In particular, Theorem 5.20 and Theorem 5.24 establish the equivalence between output zeroing in input-output models and output zeroing in state space models.

The following result is an immediate consequence of the definition of invariant zeros.

Proposition 5.11. Let (A, B, C, E) be a realization of G, where $A \in \mathbb{R}^{n\times n}$, and and let $\mathcal{Z}$ be the Rosenbrock system matrix. Then, the following statements hold:

$i)$ Assume that $\operatorname{rank} \mathcal{Z} < n + m$. Then, for all $\mathrm{z}_0 \in \mathbb{C}$, there exists nonzero $\begin{bmatrix} \overline{x} \\ \overline{u} \end{bmatrix} \in \mathcal{N}(\mathcal{Z}(\mathrm{z}_0))$.

$ii)$ Assume that $\operatorname{rank} \mathcal{Z} = n + m$. Then, $\mathrm{z}_0 \in \mathbb{C}$ is an invariant zero of (A, B, C, E) if and only if there exists nonzero $\begin{bmatrix} \overline{x} \\ \overline{u} \end{bmatrix} \in \mathcal{N}(\mathcal{Z}(\mathrm{z}_0))$.

$iii)$ Assume that $\operatorname{rank} \mathcal{Z} = n + m$ and (A, B, C, E) is minimal. Then, $\mathrm{z}_0 \in \mathbb{C}$ is a transmission zero of G if and only if there exists nonzero $\begin{bmatrix} \overline{x} \\ \overline{u} \end{bmatrix} \in \mathcal{N}(\mathcal{Z}(\mathrm{z}_0))$.

The following result on output zeroing in state space models is given by Lemma 2.7 and Lemma 2.9 in [86, p. 25, 31].

Proposition 5.12. Let (A, B, C, E) be a realization of G, where $A \in \mathbb{R}^{n\times n}$, and let $\mathrm{z}_0 \in \mathbb{C}$, $\overline{x} \in \mathbb{C}^n$, and $\overline{u} \in \mathbb{C}^m$. Furthermore, define $x_0 \triangleq \overline{x}$, and, for all $k \geq 0$, define $u_k \triangleq \mathrm{z}_0^k \overline{u}$ and consider

$$x_{k+1} = Ax_k + Bu_k, \tag{5.2}$$

$$y_k = Cx_k + Eu_k. \tag{5.3}$$

Then, the following statements hold:

$i)$ If $\begin{bmatrix} \overline{x} \\ \overline{u} \end{bmatrix} \in \mathcal{N}(\mathcal{Z}(\mathrm{z}_0))$, then, for all $k \geq 0$, $y_k = 0$.

$ii)$ If (A, C) is observable and, for all $k \geq 0$, $y_k = 0$, then $\begin{bmatrix} \overline{x} \\ \overline{u} \end{bmatrix} \in \mathcal{N}(\mathcal{Z}(\mathrm{z}_0))$.

In Proposition 5.12, the signal u and initial state x_0 are not necessarily real. In practice, however, it is desirable to consider real input signals and real states. For this case, the following result is a consequence of statement *i*) of Proposition 5.12.

Corollary 5.13. Let (A, B, C, E) be a realization of G, where $A \in \mathbb{R}^{n\times n}$, and let $\mathrm{z}_0 \in \mathbb{C}$ and $\begin{bmatrix} \overline{x} \\ \overline{u} \end{bmatrix} \in \mathcal{N}(\mathcal{Z}(\mathrm{z}_0))$. Define $x_0 \triangleq \operatorname{Re}(\overline{x})$, and, for all $k \geq 0$, define $u_k \triangleq \operatorname{Re}(\mathrm{z}_0^k \overline{u})$ and consider (5.2), (5.3). Then, for all $k \geq 0$, $y_k = 0$.

The following result relates output zeroing in state space models to transmission zeros of G.

Theorem 5.14. Let (A,B,C,E) be a realization of G, where $A \in \mathbb{R}^{n\times n}$, and let z_0 be a transmission zero of G. Then there exists $\begin{bmatrix} \overline{x} \\ \overline{u} \end{bmatrix} \in \mathcal{N}(\mathcal{Z}(\mathrm{z}_0))$, where $\overline{x} \neq 0$ and $\overline{u} \neq 0$. Furthermore, there exist $x_0 \neq 0$ and $u \neq 0$ such that $y \equiv 0$, where x_0, u, and y satisfy (5.2), (5.3), and where x_0 and u are real.

Proof. It follows from the Kalman decomposition (see Proposition 16.9.12 in [116, p. 1273] or Chapters 2 and 6 in [122]) that there exists a nonsingular matrix $S \in \mathbb{R}^{n\times n}$ such that

$$A_{\mathrm{d}} \triangleq SA_{\mathrm{ocf}}S^{-1} = \begin{bmatrix} A_1 & 0 & A_{13} & 0 \\ A_{21} & A_2 & A_{23} & A_{24} \\ 0 & 0 & A_3 & 0 \\ 0 & 0 & A_{43} & A_4 \end{bmatrix},$$

$$B_{\mathrm{d}} \triangleq SB_{\mathrm{ocf}} = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix},$$

$$C_{\mathrm{d}} \triangleq C_{\mathrm{ocf}}S^{-1} = \begin{bmatrix} C_1 & 0 & C_3 & 0 \end{bmatrix},$$

where, for all $i = 1, \ldots, 4$, $A_i \in \mathbb{R}^{n_i \times n_i}$, and (A_1, B_1, C_1, E) is a minimal realization of G. Let $\mathcal{Z}_1$ be the Rosenbrock system matrix of the realization (A_1, B_1, C_1, E). Since z_0 is a transmission zero of G, it follows that $\operatorname{rank} \mathcal{Z}_1(\mathrm{z}_0) < \operatorname{rank} \mathcal{Z}_1$. Let $\mathrm{z}_1 \in \mathbb{C}$ be such that $\operatorname{rank} \mathcal{Z}_1(\mathrm{z}_1) = \operatorname{rank} \mathcal{Z}_1$.

Hence $\operatorname{rank} \mathcal{Z}_1(\mathrm{z}_0) < \operatorname{rank} \mathcal{Z}_1(\mathrm{z}_1)$, and thus Fact 3.14.15 in [116, p. 322] implies that

$$\begin{aligned}
&\operatorname{rank} \begin{bmatrix} \mathrm{z}_0 I - A_1 \\ C_1 \end{bmatrix} + \operatorname{rank} \begin{bmatrix} B_1 \\ E \end{bmatrix} - \dim \left(\mathcal{R}\left(\begin{bmatrix} \mathrm{z}_0 I - A_1 \\ C_1 \end{bmatrix} \right) \cap \mathcal{R}\left(\begin{bmatrix} B_1 \\ E \end{bmatrix} \right) \right) \\
&\quad < \operatorname{rank} \begin{bmatrix} \mathrm{z}_1 I - A_1 \\ C_1 \end{bmatrix} + \operatorname{rank} \begin{bmatrix} B_1 \\ E \end{bmatrix} - \dim \left(\mathcal{R}\left(\begin{bmatrix} \mathrm{z}_1 I - A_1 \\ C_1 \end{bmatrix} \right) \cap \mathcal{R}\left(\begin{bmatrix} B_1 \\ E \end{bmatrix} \right) \right).
\end{aligned}$$

Since (A_1, C_1) is observable, it follows that $\operatorname{rank} \begin{bmatrix} \mathrm{z}_0 I - A_1 \\ C_1 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} \mathrm{z}_1 I - A_1 \\ C_1 \end{bmatrix} = n_1$. Hence,

$$\dim \left(\mathcal{R}\left(\begin{bmatrix} \mathrm{z}_0 I - A_1 \\ C_1 \end{bmatrix} \right) \cap \mathcal{R}\left(\begin{bmatrix} B_1 \\ E \end{bmatrix} \right) \right) > \dim \left(\mathcal{R}\left(\begin{bmatrix} \mathrm{z}_1 I - A_1 \\ C_1 \end{bmatrix} \right) \cap \mathcal{R}\left(\begin{bmatrix} B_1 \\ E \end{bmatrix} \right) \right) \geq 0.$$

Thus, there exists $\begin{bmatrix} \overline{x}_1 \\ \overline{u}_1 \end{bmatrix} \in \mathcal{N}(\mathcal{Z}_1(\mathrm{z}_0))$, where $\overline{x}_1 \neq 0$ and $\overline{u}_1 \neq 0$. Define $\overline{x} \triangleq \begin{bmatrix} \overline{x}_1 \\ 0 \end{bmatrix} \in \mathbb{C}^n$ and $\overline{u} \triangleq \overline{u}_1$. Then, $\overline{x} \neq 0$, $\overline{u} \neq 0$, and $\begin{bmatrix} \overline{x} \\ \overline{u} \end{bmatrix} \in \mathcal{N}(\mathcal{Z}(\mathrm{z}_0))$.

Without loss of generality, let $\begin{bmatrix} \overline{x} \\ \overline{u} \end{bmatrix} \in \mathcal{N}(\mathcal{Z}(\mathrm{z}_0))$, where $\mathrm{Re}(\overline{x}) \neq 0$ and $\overline{u} \neq 0$. Define $x_0 \triangleq \mathrm{Re}(\overline{x})$ and, for all $k \geq 0$, $u_k \triangleq \mathrm{Re}(\mathrm{z}_0^k \overline{u})$. In the case where $\mathrm{Re}(\overline{u}) \neq 0$, note that $u_0 = \mathrm{Re}(\overline{u}) \neq 0$, and thus $u \neq 0$. In the case where $\mathrm{Re}(\overline{u}) = 0$, suppose that $\mathrm{Im}(\mathrm{z}_0) = 0$. Then, it follows from $\mathrm{Re}\left(\mathcal{Z}(\mathrm{z}_0) \begin{bmatrix} \overline{x} \\ \overline{u} \end{bmatrix} \right) = 0$ that $\begin{bmatrix} \mathrm{z}_0 I - A \\ C \end{bmatrix} \mathrm{Re}(\overline{x}) = 0$, and thus $\mathrm{Re}(\overline{x}) = 0$, which is a contradiction. Hence, $\mathrm{Im}(\mathrm{z}_0) \neq 0$, which implies that $u_1 = \mathrm{Re}(\mathrm{z}_0 \overline{u}) = \mathrm{Im}(\mathrm{z}_0)\mathrm{Im}(\overline{u}) \neq 0$, and thus $u \neq 0$. Finally, Corollary 5.13 implies that, for all $k \geq 0$, $y_k = 0$. □

Theorem 5.16 concerns output zeroing in input-output models. The proof of this result takes advantage of the following lemma.

Lemma 5.15. Let $D_0, D_1, \ldots, D_\ell \in \mathbb{R}^{p\times p}$, assume that $D_\ell \neq 0$, and, for all $k \geq 0$, consider the difference equation

$$D_\ell y_{k+\ell} + \cdots + D_1 y_{k+1} + D_0 y_k = 0, \tag{5.4}$$

with the initial condition $y_0 = y_1 = \cdots = y_{\ell-1} = 0$. If $\det(\mathrm{z}^\ell D_\ell + \cdots + \mathrm{z}D_1 + D_0) \neq 0$, then, for all $k \geq \ell$, $y_k = 0$.

Proof. For all $i < 0$, define $D_i \triangleq 0$, and, for all $k \geq 0$, define

$$\overline{\mathcal{T}}_k \triangleq \begin{bmatrix} D_\ell & 0 & \cdots & 0 \\ D_{\ell-1} & D_\ell & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ D_{\ell-k} & D_{\ell-k+1} & \cdots & D_\ell \end{bmatrix} = \begin{bmatrix} Q_k & P_k \end{bmatrix},$$

where

$$Q_k \triangleq \begin{bmatrix} D_\ell \\ D_{\ell-1} \\ \vdots \\ D_{\ell-k} \end{bmatrix}, \quad P_k \triangleq \begin{bmatrix} 0 \\ \overline{\mathcal{T}}_{k-1} \end{bmatrix},$$

and $\overline{\mathcal{T}}_{-1}$ is the empty matrix. Next, define $D(\mathrm{z}) \triangleq \mathrm{z}^\ell D_\ell + \cdots + \mathrm{z}D_1 + D_0$ and $G(\mathrm{z}) \triangleq D(1/\mathrm{z})$. Note that G is a proper finite-impulse-response (FIR) transfer function, and $D_\ell, \ldots, D_0$ are the Markov parameters of G. Since D is nonsingular, it follows that G is invertible, and hence it follows from Proposition 2 in [34] that there exists $d \geq 0$ such that $\operatorname{rank} \overline{\mathcal{T}}_d - \operatorname{rank} \overline{\mathcal{T}}_{d-1} = p$. Next, it follows from Fact 3.14.15 in [116, p. 322] that $p = \operatorname{rank} \overline{\mathcal{T}}_d - \operatorname{rank} \overline{\mathcal{T}}_{d-1} = \operatorname{rank} Q_d - \dim(\mathcal{R}(Q_d) \cap \mathcal{R}(P_d))$. Thus, $p + \dim(\mathcal{R}(Q_d) \cap \mathcal{R}(P_d)) = \operatorname{rank} Q_d \leq p$. Hence, $\operatorname{rank} Q_d = p$, and $\dim(\mathcal{R}(Q_d) \cap \mathcal{R}(P_d)) = 0$.

Next, (5.4) implies that

$$\overline{\mathcal{T}}_d \begin{bmatrix} y_\ell \\ y_{\ell+1} \\ \vdots \\ y_{\ell+d+1} \end{bmatrix} = Q_d y_\ell + P_d \begin{bmatrix} y_{\ell+1} \\ y_{\ell+2} \\ \vdots \\ y_{\ell+d+1} \end{bmatrix} = 0.$$

Since $\dim(\mathcal{R}(Q_d) \cap \mathcal{R}(P_d)) = 0$, it follows that

$$Q_d y_\ell = P_d \begin{bmatrix} y_{\ell+1} \\ y_{\ell+2} \\ \vdots \\ y_{\ell+d+1} \end{bmatrix} = 0.$$

Since $\operatorname{rank} Q_d = p$, it follows that $y_\ell = 0$. Since $y_1 = \cdots = y_\ell = 0$, repeating the previous argument with ℓ replaced by $\ell + 1$ implies that $y_{\ell+1} = 0$. By induction, it follows that, for all $k \geq \ell$, $y_k = 0$. $\square$

Theorem 5.16. Let (D, N) be an LPFD of G, let $\mathrm{z}_0 \in \mathbb{C}$, and let $\overline{u} \in \mathbb{C}^m$. Let $y_0 = \cdots = y_{\ell-1} = 0$, where $\ell \triangleq \deg D$, and, for all $k \geq 0$, define $u_k \triangleq \mathrm{z}_0^k \overline{u}$ and consider

$$D(\mathbf{q}) y_k = N(\mathbf{q}) u_k. \tag{5.5}$$

Then, for all $k \geq \ell$, $y_k = 0$ if and only if $N(\mathrm{z}_0)\overline{u} = 0$.

Proof. To prove sufficiency, note that Proposition 5.9 implies that, for all $k \geq 0$, $N(\mathbf{q})u_k = 0$. Hence, for all $k \geq 0$, $D(\mathbf{q})y_k = 0$. Since D is nonsingular, Lemma 5.15 implies that, for all $k \geq \ell$, $y_k = 0$. To prove necessity, note that, for all $k \geq 0$, $N(\mathbf{q})u_k = D(\mathbf{q})y_k = 0$. Therefore, Proposition 5.9 implies that $N(\mathrm{z}_0)\overline{u} = 0$. $\square$

For the case of real input signals, the following result is a consequence of the sufficiency part of Theorem 5.16.

Corollary 5.17. Let (D, N) be an LPFD of G, let $\mathrm{z}_0 \in \mathbb{C}$, and let $\overline{u} \in \mathcal{N}(N(\mathrm{z}_0))$. Let $y_0 = \cdots = y_{\ell-1} = 0$, where $\ell \triangleq \deg D$, and, for all $k \geq 0$, define $u_k \triangleq \mathrm{Re}(\mathrm{z}_0^k \overline{u})$ and consider (5.5). Then, for all $k \geq \ell$, $y_k = 0$.

The following result relates output zeroing in input-output models to transmission zeros of G.

Theorem 5.18. Let (D, N) be an LPFD of G, and let z_0 be a transmission zero of G. Then $\mathcal{N}(N(\mathrm{z}_0)) \neq \{0\}$. Furthermore, let $y_0 = \cdots = y_{\ell-1} = 0$, where $\ell \triangleq \deg D$. Then there exists real $u \neq 0$ such that u and $y \equiv 0$ satisfy (5.5).

Proof. It follows from Proposition 5.6 that $\mathcal{N}(N(\mathrm{z}_0)) \neq \{0\}$. Let $\overline{u}$ be a nonzero vector in $\mathcal{N}(N(\mathrm{z}_0))$ such that $\mathrm{Re}(\overline{u}) \neq 0$, and define, for all $k \geq 0$, $u_k \triangleq \mathrm{Re}(\mathrm{z}_0^k \overline{u})$. Note that $u_0 = \mathrm{Re}(\overline{u}) \neq 0$, and thus $u \neq 0$. Then it follows from Corollary 5.17 that, for all $k \geq \ell$, $y_k = 0$ in (5.5). $\square$

Next, we consider the equivalence between output zeroing in input-output models and output zeroing in state space models. In particular, Theorem 5.20 shows the equivalence between output zeroing using an MLPFD of G and output zeroing using the observable canonical form realization of G corresponding to the MLPFD of G. The observable canonical form realization of G obtained from an MLPFD of G is given in Proposition 5.19.

The following result provides a MIMO extension of the observable canonical form realization given in [123].

Proposition 5.19. Let $(D_{\mathrm{M}}, N_{\mathrm{M}})$ be an MLPFD of G, and let

$$D_{\mathrm{M}}(\mathrm{z}) = \mathrm{z}^\ell I + \mathrm{z}^{\ell-1} A_1 + \cdots + A_\ell, \tag{5.6}$$

$$N_{\mathrm{M}}(\mathrm{z}) = \mathrm{z}^\ell B_0 + \mathrm{z}^{\ell-1} B_1 + \cdots + B_\ell, \tag{5.7}$$

where, for all $i = 1, \ldots, \ell$, $A_i \in \mathbb{R}^{p\times p}$, and, for all $i = 0, \ldots, \ell$, $B_i \in \mathbb{R}^{p\times m}$. Then, for all $k \geq 0$,

$$\hat{x}_{k+1} = A_{\mathrm{ocf}}\hat{x}_k + B_{\mathrm{ocf}}u_k, \tag{5.8}$$

$$y_k = C_{\mathrm{ocf}}\hat{x}_k + Eu_k, \tag{5.9}$$

where

$$A_{\mathrm{ocf}} \triangleq \begin{bmatrix} 0 & \cdots & 0 & -A_\ell \\ I & \cdots & 0 & -A_{\ell-1} \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & I & -A_1 \end{bmatrix}, \tag{5.10}$$

$$B_{\mathrm{ocf}} \triangleq \begin{bmatrix} B_\ell - A_\ell B_0 \\ B_{\ell-1} - A_{\ell-1}B_0 \\ \vdots \\ B_1 - A_1 B_0 \end{bmatrix}, \tag{5.11}$$

$$C_{\mathrm{ocf}} \triangleq \begin{bmatrix} 0 & \cdots & 0 & I \end{bmatrix}, \quad E \triangleq B_0, \tag{5.12}$$

$\hat{x}_k \triangleq \begin{bmatrix} \hat{x}_{1,k} & \cdots & \hat{x}_{l,k} \end{bmatrix}^{\mathrm{T}}$, and, for all $i = 0, 1, \ldots, \ell - 1$,

$$\hat{x}_{\ell-i,k} \triangleq y_{k+i} + \sum_{j=1}^{i} A_j y_{k+i-j} - \sum_{j=0}^{i} B_j u_{k+i-j}, \tag{5.13}$$

is an observable state space model of G.

$(A_{\mathrm{ocf}}, B_{\mathrm{ocf}}, C_{\mathrm{ocf}}, E)$ is the *observable canonical form* state space realization of G corresponding to $(D_{\mathrm{M}}, N_{\mathrm{M}})$.

Theorem 5.20. Let $(D_{\mathrm{M}}, N_{\mathrm{M}})$ be an MLPFD of G, let D_{M} and N_{M} be given by (5.6), (5.7), and let $\mathcal{Z}_{\mathrm{ocf}}$ be the Rosenbrock system matrix of the realization $(A_{\mathrm{ocf}}, B_{\mathrm{ocf}}, C_{\mathrm{ocf}}, E)$, where $(A_{\mathrm{ocf}}, B_{\mathrm{ocf}}, C_{\mathrm{ocf}}, E)$ is the observable canonical form realization of G corresponding to $(D_{\mathrm{M}}, N_{\mathrm{M}})$

given by (5.10)–(5.12). Furthermore, let $\overline{u} \in \mathbb{C}^m$ and $\mathrm{z}_0 \in \mathbb{C}$. Then, there exists $\overline{x} \in \mathbb{C}^{p\ell}$ such that $\begin{bmatrix} \overline{x} \\ \overline{u} \end{bmatrix} \in \mathcal{N}(\mathcal{Z}_{\mathrm{ocf}}(\mathrm{z}_0))$ if and only if $N_{\mathrm{M}}(\mathrm{z}_0)\overline{u} = 0$. If these conditions hold, then

$$\overline{x} = - \begin{bmatrix} \sum_{i=0}^{\ell-1} \mathrm{z}_0^{\ell-i-1} B_i \overline{u} \\ \sum_{i=0}^{\ell-2} \mathrm{z}_0^{\ell-i-2} B_i \overline{u} \\ \vdots \\ B_0 \overline{u} \end{bmatrix}. \tag{5.14}$$

Proof. To prove sufficiency, let

$$\overline{x} = - \begin{bmatrix} \sum_{i=0}^{\ell-1} \mathrm{z}_0^{\ell-i-1} B_i \overline{u} \\ \sum_{i=0}^{\ell-2} \mathrm{z}_0^{\ell-i-2} B_i \overline{u} \\ \vdots \\ B_0 \overline{u} \end{bmatrix}.$$

Then,

$$C_{\mathrm{ocf}}\overline{x} + E\overline{u} = -B_0\overline{u} + B_0\overline{u} = 0. \tag{5.15}$$

Next, note that

$$(\mathrm{z}_0 I - A_{\mathrm{ocf}})\overline{x} = -\begin{bmatrix} \mathrm{z}_0 I & 0 & \cdots & 0 & A_\ell \\ -I & \mathrm{z}_0 I & \cdots & 0 & A_{\ell-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & -I & \mathrm{z}_0 I + A_1 \end{bmatrix} \begin{bmatrix} \sum_{i=0}^{\ell-1} \mathrm{z}_0^{\ell-i-1} B_i \overline{u} \\ \sum_{i=0}^{\ell-2} \mathrm{z}_0^{\ell-i-2} B_i \overline{u} \\ \vdots \\ B_0 \overline{u} \end{bmatrix}$$

$$= \begin{bmatrix} -\mathrm{z}_0 \sum_{i=0}^{\ell-1} \mathrm{z}_0^{\ell-i-1} B_i \overline{u} - A_\ell B_0 \overline{u} \\ \sum_{i=0}^{\ell-1} \mathrm{z}_0^{\ell-i-1} B_i \overline{u} - \mathrm{z}_0 \sum_{i=0}^{\ell-2} \mathrm{z}_0^{\ell-i-2} B_i \overline{u} - A_{\ell-1} B_0 \overline{u} \\ \vdots \\ \sum_{i=0}^{1} \mathrm{z}_0^{1-i} B_i \overline{u} - \mathrm{z}_0 B_0 \overline{u} - A_1 B_0 \overline{u} \end{bmatrix}, \tag{5.16}$$

and

$$B_{\mathrm{ocf}} \overline{u} = \begin{bmatrix} B_\ell - A_\ell B_0 \\ B_{\ell-1} - A_{\ell-1} B_0 \\ \vdots \\ B_1 - A_1 B_0 \end{bmatrix} \overline{u} = \begin{bmatrix} B_\ell \overline{u} - A_\ell B_0 \overline{u} \\ B_{\ell-1} \overline{u} - A_{\ell-1} B_0 \overline{u} \\ \vdots \\ B_1 \overline{u} - A_1 B_0 \overline{u} \end{bmatrix}. \tag{5.17}$$

Subtracting (5.17) from (5.16) yields

$$(\mathrm{z}_0 I - A_{\mathrm{ocf}})\overline{x} - B_{\mathrm{ocf}}\overline{u} = \begin{bmatrix} -\sum_{i=0}^{\ell-1} \mathrm{z}_0^{\ell-i} B_i \overline{u} - B_\ell \overline{u} \\ \sum_{i=0}^{\ell-1} \mathrm{z}_0^{\ell-i-1} B_i \overline{u} - \sum_{i=0}^{\ell-2} \mathrm{z}_0^{\ell-i-1} B_i \overline{u} - B_{\ell-1}\overline{u} \\ \vdots \\ \sum_{i=0}^{1} \mathrm{z}_0^{1-i} B_i \overline{u} - \mathrm{z}_0 B_0 \overline{u} - B_1 \overline{u} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=0}^{\ell} \mathrm{z}_0^{\ell-i} B_i \overline{u} \\ \sum_{i=0}^{\ell-1} \mathrm{z}_0^{\ell-i-1} B_i \overline{u} - \sum_{i=0}^{\ell-1} \mathrm{z}_0^{\ell-i-1} B_i \overline{u} \\ \vdots \\ \mathrm{z}_0 B_0 \overline{u} + B_1 \overline{u} - \mathrm{z}_0 B_0 \overline{u} - B_1 \overline{u} \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{\ell} \mathrm{z}_0^{\ell-i} B_i \overline{u} \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{5.18}$$

Since $0 = N_{\mathrm{M}}(\mathrm{z}_0)\overline{u} = \sum_{i=0}^{\ell} \mathrm{z}_0^{\ell-i} B_i \overline{u}$, (5.18) implies that

$$(\mathrm{z}_0 I - A_{\mathrm{ocf}})\overline{x} - B_{\mathrm{ocf}}\overline{u} = 0. \tag{5.19}$$

It thus follows from (5.15) and (5.19) that $\begin{bmatrix} \overline{x} \\ \overline{u} \end{bmatrix} \in \mathcal{N}(\mathcal{Z}_{\mathrm{ocf}}(\mathrm{z}_0))$.

To prove necessity, let $\overline{x} = \begin{bmatrix} \overline{x}_1^{\mathrm{T}} & \cdots & \overline{x}_\ell^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, where $\overline{x}_1, \ldots, \overline{x}_\ell \in \mathbb{C}^p$. Then,

$$\begin{aligned} 0 &= (\mathrm{z}_0 I - A_{\mathrm{ocf}})\overline{x} - B_{\mathrm{ocf}}\overline{u} \\ &= \begin{bmatrix} \mathrm{z}_0 I & 0 & \cdots & 0 & A_\ell \\ -I & \mathrm{z}_0 I & \cdots & 0 & A_{\ell-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & -I & \mathrm{z}_0 I + A_1 \end{bmatrix} \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \vdots \\ \overline{x}_l \end{bmatrix} - \begin{bmatrix} B_\ell - A_\ell B_0 \\ B_{\ell-1} - A_{\ell-1} B_0 \\ \vdots \\ B_1 - A_1 B_0 \end{bmatrix} \overline{u}. \end{aligned} \tag{5.20}$$

Note that $0 = C\overline{x} + E\overline{u} = \overline{x}_\ell + B_0\overline{u}$ implies that

$$\overline{x}_\ell = -B_0\overline{u}. \tag{5.21}$$

Hence, (5.20) and (5.21) imply that

$$\begin{aligned}\overline{x}_{\ell-1} &= (\mathrm{z}_0 I + A_1)\overline{x}_\ell - B_1\overline{u} + A_1 B_0\overline{u} = -(\mathrm{z}_0 I + A_1)B_0\overline{u} - B_1\overline{u} + A_1 B_0\overline{u} \\ &= -\mathrm{z}_0 B_0\overline{u} - B_1\overline{u}. \end{aligned} \tag{5.22}$$

Next, (5.20)-(5.22) imply that

$$\begin{aligned}\overline{x}_{\ell-2} &= \mathrm{z}_0\overline{x}_{\ell-1} + A_2\overline{x}_\ell - B_2\overline{u} + A_2 B_0\overline{u} = -\mathrm{z}_0^2 B_0\overline{u} - \mathrm{z}_0 B_1\overline{u} - A_2 B_0\overline{u} - B_2\overline{u} + A_2 B_0\overline{u} \\ &= -\mathrm{z}_0^2 B_0\overline{u} - \mathrm{z}_0 B_1\overline{u} - B_2\overline{u}. \end{aligned} \tag{5.23}$$

Proceeding similarly, it follows that, for all $j = 1, \ldots, \ell$, $\overline{x}_j = -\sum_{i=0}^{\ell-j} \mathrm{z}_0^{\ell-i-j} B_i\overline{u}$, and thus (5.14) holds. Finally, it follows from (5.14) and (5.20) that

$$\begin{aligned}0 &= \mathrm{z}_0\overline{x}_1 + A_\ell\overline{x}_\ell - B_\ell\overline{u} + A_\ell B_0\overline{u} = -\sum_{i=0}^{\ell-1} \mathrm{z}_0^{\ell-i} B_i\overline{u} - A_\ell B_0\overline{u} - B_\ell\overline{u} + A_\ell B_0\overline{u} \\ &= -\sum_{i=0}^{\ell} \mathrm{z}_0^{\ell-i} B_i\overline{u} = -N(\mathrm{z}_0)\overline{u}. \qquad \square \end{aligned}$$

Note that Theorem 5.20 relates output zeroing using an MLPFD of G to output zeroing using a specific realization of G, which is obtained from the given MLPFD and is not necessarily minimal. In order to obtain a more general result, we next consider the equivalence between output zeroing using an arbitrary CLPFD of G and output zeroing using an arbitrary minimal realization of G. Given an arbitrary CLPFD of a continuous-time transfer function G, [124] describes an algorithm for obtaining a minimal realization of G. Since the algorithm in [124] is algebraic, the result holds true for discrete-time transfer functions by replacing the differentiation operator with the forward-shift operator. For illustration, the example given in [124] is reworked in terms of $\mathbf{q}$ as Example 5.21 below. Proposition 5.22 and Proposition 5.23 are consequences of the application of the algorithm in [124] to discrete-time transfer functions. Using Proposition 5.23, the equivalence between output zeroing using an arbitrary CLPFD of G and output zeroing using an arbitrary minimal realization of G is proved in Theorem 5.24.

Example 5.21. Let

$$G(\mathrm{z}) = \begin{bmatrix} \dfrac{\mathrm{z}^2 - 2\mathrm{z} + 3}{\mathrm{z}^4 + 3\mathrm{z}^3 + 7\mathrm{z}^2 + 18\mathrm{z} + 6} & \dfrac{-(2\mathrm{z}+3)(\mathrm{z}+3)}{\mathrm{z}^4 + 3\mathrm{z}^3 + 7\mathrm{z}^2 + 18\mathrm{z} + 6} \\ \dfrac{1}{\mathrm{z}^2 + 6} & \dfrac{\mathrm{z}+3}{\mathrm{z}^2+6} \end{bmatrix}, \tag{5.24}$$

$$\hat{D}(\mathrm{z}) = \begin{bmatrix} \mathrm{z}^2 + 3\mathrm{z} + 1 & 2\mathrm{z} + 3 \\ \mathrm{z}^3 + 3\mathrm{z}^2 + \mathrm{z} & 3\mathrm{z}^2 + 3\mathrm{z} + 6 \end{bmatrix}, \quad \hat{N}(\mathrm{z}) = \begin{bmatrix} 1 & 0 \\ \mathrm{z}+1 & \mathrm{z}+3 \end{bmatrix}. \tag{5.25}$$

Note that $G = \hat{D}^{-1}\hat{N}$, and $\deg\det\hat{D} = \mathrm{McDeg}\, G = 4$. Hence Proposition 5.5 implies that $(\hat{D}, \hat{N})$ is a CLPFD of G. Let $U(\mathrm{z}) = \begin{bmatrix} 1 & 0 \\ -\mathrm{z} & 1 \end{bmatrix}$. Since U is unimodular, it follows from Proposition 5.2 that (D, N) is a CLPFD of G, where

$$D(\mathrm{z}) \triangleq U(\mathrm{z})\hat{D}(\mathrm{z}) = \begin{bmatrix} \mathrm{z}^2 + 3\mathrm{z} + 1 & 2\mathrm{z} + 3 \\ 0 & \mathrm{z}^2 + 6 \end{bmatrix}, \quad N(\mathrm{z}) \triangleq U(\mathrm{z})\hat{N}(\mathrm{z}) = \begin{bmatrix} 1 & 0 \\ 1 & \mathrm{z}+3 \end{bmatrix}. \tag{5.26}$$

In terms of the forward-shift operator, (5.26) has the form

$$D(\mathbf{q}) = \begin{bmatrix} \mathbf{q}^2 + 3\mathbf{q} + 1 & 2\mathbf{q} + 3 \\ 0 & \mathbf{q}^2 + 6 \end{bmatrix}, \quad N(\mathbf{q}) = \begin{bmatrix} 1 & 0 \\ 1 & \mathbf{q}+3 \end{bmatrix}. \tag{5.27}$$

Now, for all $k \geq 0$, let u_k and y_k satisfy (5.5), let $u_k = \begin{bmatrix} u_{1,k} & u_{2,k} \end{bmatrix}^{\mathrm{T}}$, and let $y_k = \begin{bmatrix} y_{1,k} & y_{2,k} \end{bmatrix}^{\mathrm{T}}$.

Define $x_k \triangleq \begin{bmatrix} x_{1,k} & x_{2,k} & x_{3,k} & x_{4,k} \end{bmatrix}^{\mathrm{T}}$, where

$$x_{1,k} \triangleq y_{1,k}, \quad x_{2,k} \triangleq y_{1,k+1} + 3y_{1,k}, \tag{5.28}$$

$$x_{3,k} \triangleq y_{2,k}, \quad x_{4,k} \triangleq y_{2,k+1} - u_{2,k}. \tag{5.29}$$

Then, for all $k \geq 0$, u_k, y_k, and x_k satisfy (5.2) and (5.3) with

$$A \triangleq \begin{bmatrix} -3 & 1 & 0 & 0 \\ -1 & 0 & -3 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -6 & 0 \end{bmatrix}, B \triangleq \begin{bmatrix} 0 & 0 \\ 1 & -2 \\ 0 & 1 \\ 1 & 3 \end{bmatrix}, C \triangleq \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, E \triangleq 0. \tag{5.30}$$

It can be verified numerically that (A, B, C) is a minimal realization of G. ◇

The following result is a consequence of the application of the algorithm in [124] to discrete-time transfer functions.

Proposition 5.22. Let (D, N) be a CLPFD of G and, for all $k \geq 0$, let u_k and y_k satisfy (5.5). Then there exist $L_u \in \mathbb{R}[\mathbf{q}]^{n\times m}$, $L_y \in \mathbb{R}[\mathbf{q}]^{n\times p}$, and a minimal realization (A, B, C, E) of G such that, for all $k \geq 0$, (5.2) and (5.3) hold with $x_k \triangleq L_u(\mathbf{q})u_k + L_y(\mathbf{q})y_k$, where n is the McMillan degree of G.

The following result is needed in the proof of Theorem 5.24.

Proposition 5.23. Let (A, B, C, E) be a minimal realization of G, let (D, N) be a CLPFD of G and, for all $k \geq 0$, let u_k and y_k satisfy (5.5). Then, for all $k \geq 0$, there exist $L_u \in \mathbb{R}[\mathbf{q}]^{n\times m}$, $L_y \in \mathbb{R}[\mathbf{q}]^{n\times p}$ such that, for all $k \geq 0$, (5.2) and (5.3) hold with $x_k \triangleq L_u(\mathbf{q})u_k + L_y(\mathbf{q})y_k$, where n is the McMillan degree of G.

Proof. Note that Proposition 5.22 implies that there exist $\overline{L}_u \in \mathbb{R}[\mathbf{q}]^{n\times m}$, $\overline{L}_y \in \mathbb{R}[\mathbf{q}]^{n\times p}$, and a minimal realization $(\overline{A}, \overline{B}, \overline{C}, \overline{E})$ of G such that, for all $k \geq 0$, (5.2) and (5.3) hold with A, B, C, E, and x_k replaced with $\overline{A}, \overline{B}, \overline{C}, \overline{E}$, and $\overline{x}_k$ respectively, and $\overline{x}_k \triangleq \overline{L}_u(\mathbf{q})u_k + \overline{L}_y(\mathbf{q})y_k$. Next, Proposition 16.9.8 in [116, p. 1272] implies that there exists a unique nonsingular $S \in \mathbb{R}^{n\times n}$ such that $A = S\overline{A}S^{-1}$, $B = S\overline{B}$, and $C = \overline{C}S^{-1}$. Define $L_u \triangleq S\overline{L}_u$, and $L_y \triangleq S\overline{L}_y$. Hence, for all $k \geq 0$, (5.2) and (5.3) hold with $x_k \triangleq L_u(\mathbf{q})u_k + L_y(\mathbf{q})y_k$. $\square$

The following result establishes the equivalence between output zeroing using an arbitrary CLPFD of G and output zeroing using an arbitrary minimal realization of G. Note that, unlike Theorem 5.20, the LPFD in the following is coprime but not necessarily monic.

Theorem 5.24. Let (D, N) be a CLPFD of G, let (A, B, C, E) be an nth-order minimal realization of G, let $\mathrm{z}_0 \in \mathbb{C}$, and let $\overline{u} \in \mathbb{C}^m$. Then, there exists $\overline{x} \in \mathbb{C}^n$ such that $\begin{bmatrix} \overline{x} \\ \overline{u} \end{bmatrix} \in \mathcal{N}(\mathcal{Z}(\mathrm{z}_0))$ if and only if $N(\mathrm{z}_0)\overline{u} = 0$.

Proof. Suppose that, for all $k \geq 0$, u_k and y_k satisfy (5.5). Then Proposition 5.23 implies that, for all $k \geq 0$, there exist $L_u \in \mathbb{R}[\mathbf{q}]^{n\times m}$ and $L_y \in \mathbb{R}[\mathbf{q}]^{n\times p}$ such that, for all $k \geq 0$, (5.2) and (5.3) hold with $x_k \triangleq L_u(\mathbf{q})u_k + L_y(\mathbf{q})y_k$. For all $k \geq 0$, define $u_k \triangleq \mathrm{z}_0^k\overline{u}$. To prove necessity, define $x_0 \triangleq \overline{x}$. Then, statement *i)* in Proposition 5.12 implies that, for all $k \geq 0$, $y_k = 0$. Hence it follows from Theorem 5.16 that $N(\mathrm{z}_0)\overline{u} = 0$. To prove sufficiency, define $\overline{x} \triangleq L_u(\mathrm{z}_0)\overline{u}$, write $L_y(\mathbf{q}) = q^r P_r + \cdots + qP_1 + P_0$, define $\ell \triangleq \deg D$, and suppose that $y_0 = y_1 = \cdots = y_c = 0$, where $c \triangleq \max\{r, \ell-1\}$. Note that $x_k = L_u(\mathbf{q})u_k + L_y(\mathbf{q})y_k = L_u(\mathbf{q})\mathrm{z}_0^k\overline{u} + L_y(\mathbf{q})y_k = L_u(\mathrm{z}_0)\mathrm{z}_0^k\overline{u} + L_y(\mathbf{q})y_k$. Hence $x_0 = L_u(\mathrm{z}_0)\overline{u} + P_r y_r + \cdots + P_1 y_1 + P_0 y_0 = L_u(\mathrm{z}_0)\overline{u} = \overline{x}$. Next, it follows from Theorem 5.16 that, for all $k \geq 0$, $y_k = 0$. Hence statement *ii)* in Proposition 5.12 implies that $\begin{bmatrix} \overline{x} \\ \overline{u} \end{bmatrix} \in \mathcal{N}(\mathcal{Z}(\mathrm{z}_0))$. $\square$

In the case where z_0 is a transmission zero of G, Theorem 5.14 implies that there exists nonzero $\begin{bmatrix} \overline{x} \\ \overline{u} \end{bmatrix} \in \mathcal{N}(\mathcal{Z}(\mathrm{z}_0))$, where $\overline{x} \neq 0$ and $\overline{u} \neq 0$, and thus Theorem 5.24 implies that $N(\mathrm{z}_0)\overline{u} = 0$. Therefore, there exist $\overline{x} \neq 0$ and $\overline{u} \neq 0$ such that $\begin{bmatrix} \overline{x} \\ \overline{u} \end{bmatrix} \in \mathcal{N}(\mathcal{Z}(\mathrm{z}_0))$ and $N(\mathrm{z}_0)\overline{u} = 0$.

The following example illustrates the equivalence between output zeroing in input-output models and output zeroing in state space models due to transmission zeros.

Example 5.25. Consider the discrete-time transfer function

$$G(\mathrm{z}) = \begin{bmatrix} \frac{\mathrm{z}-3}{\mathrm{z}+2} & 0 \\ \frac{1}{\mathrm{z}+1} & \frac{\mathrm{z}}{\mathrm{z}+1} \\ 1 & \frac{9}{\mathrm{z}} \end{bmatrix}, \tag{5.31}$$

and consider the minimal realization of G given by

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}, \ C = \begin{bmatrix} -5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix}, \ E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{5.32}$$

Since $\operatorname{rank} \mathcal{Z}(3) = 4 < 5 = \operatorname{rank} \mathcal{Z}$, it follows that $\mathrm{z} = 3$ is a transmission zero of G. For all $k \geq 0$, let u_k, y_k, and x_k satisfy (5.2) and (5.3). Then, it follows from Proposition 5.12 that, if

$$\begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \in \mathcal{N}(\mathcal{Z}(3)), \tag{5.33}$$

and, for all $k \geq 1$, $u_k = 3^k u_0$, then $y \equiv 0$. For example, noting

$$\begin{bmatrix} 3/5 \\ 1 \\ -1/3 \\ 3 \\ -1 \end{bmatrix} \in \mathcal{N}(\mathcal{Z}(3)), \tag{5.34}$$

it follows that $y \equiv 0$ with $x_0 = \begin{bmatrix} 3/5 \\ 1 \\ -1/3 \end{bmatrix}$ and, for all $k \geq 0$, $u_k = 3^k \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

Next, taking the z-transform of (5.2), (5.3) yields

$$\hat{y}(\mathrm{z}) = G(\mathrm{z})\hat{u}(\mathrm{z}) + \mathrm{z}C(\mathrm{z}I - A)^{-1}x_0, \tag{5.35}$$

where $\hat{u}$ and $\hat{y}$ denote the z-transforms of u and y, respectively. Note that (5.35) includes separate terms for the free response and the forced response of (5.2), (5.3). An alternative time-domain representation of (5.2), (5.3) can be obtained by replacing z by the forward-shift operator $\mathbf{q}$. To do this, we first factor $G(\mathrm{z}) = D(\mathrm{z})^{-1}N(\mathrm{z})$, where

$$D(\mathrm{z}) = \mathrm{z}I_3 + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad N(\mathrm{z}) = \mathrm{z} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} -3 & 0 \\ 1 & 0 \\ 0 & 9 \end{bmatrix}. \tag{5.36}$$

Note that (D, N) is an MLPFD of G. Then, for all $k \geq 0$, (5.2), (5.3) has the equivalent time-domain input-output model (5.5), where

$$D(\mathbf{q}) = \begin{bmatrix} \mathbf{q}+2 & 0 & 0 \\ 0 & \mathbf{q}+1 & 0 \\ 0 & 0 & \mathbf{q} \end{bmatrix}, \; N(\mathbf{q}) = \begin{bmatrix} \mathbf{q}-3 & 0 \\ 1 & \mathbf{q} \\ \mathbf{q} & 9 \end{bmatrix}. \tag{5.37}$$

Note that the free response $\mathrm{z}C(\mathrm{z}I - A)^{-1}x_0$ in (5.35) has no counterpart in (5.5). In fact, the response of (5.5) includes both the free response and the forced response [125]. Now, in (5.5), letting $y_0 = 0$ and, for all $k \geq 0$, $u_k = 3^k \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ yields $y \equiv 0$. Furthermore, note that $u_0 \in \mathcal{N}(N(3))$. Hence, the input that produces identically zero output is obtained from the Rosenbrock matrix $\mathcal{Z}$ for a state space model as well as the numerator polynomial matrix N for an input-output model.

To further illustrate the connection between output zeroing in input-output models and output zeroing in state space models, we consider the observable canonical form realization $(A_{\rm ocf}, B_{\rm ocf}, C_{\rm ocf}, E)$ of G corresponding to (D, N), where

$$A_{\rm ocf} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B_{\rm ocf} = \begin{bmatrix} -5 & 0 \\ 1 & -1 \\ 0 & -9 \end{bmatrix}, C_{\rm ocf} = I_3. \tag{5.38}$$

By using (5.13), the state of the observable canonical form realization (5.8), (5.9) can be constructed in terms of input and output data. For this example, the initial condition obtained from (5.13) is given by

$$\hat{x}_0 = y_0 - \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u_0. \tag{5.39}$$

Now, setting $y_0 = 0$ and $u_0 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ in (5.39) yields $\hat{x}_0 = \begin{bmatrix} -3 \\ 1 \\ -3 \end{bmatrix}$, which is the initial condition that, along with the output-zeroing input, produces the identically zero output. It can be verified numerically that $\begin{bmatrix} \hat{x}_0 \\ u_0 \end{bmatrix} \in \mathcal{N}(\mathcal{Z}_{\rm ocf}(3))$, where $\mathcal{Z}_{\rm ocf}$ is the Rosenbrock system matrix of the realization $(A_{\rm ocf}, B_{\rm ocf}, C_{\rm ocf}, E)$. Hence, the vector consisting of this initial condition and the initial input lies in the null space of the Rosenbrock system matrix. $\diamond$

CHAPTER 6

Retrospective Cost Input Estimation

This chapter introduces retrospective cost input estimation [70] which is based on retrospective cost optimization [76]. In RCIE, input estimation is combined with state estimation based on the discrete-time Kalman filter. RCIE can be applied to any multi-input multi-output (MIMO) discrete-time linear-time varying system.

Consider the linear time-varying (LTV) discrete-time system

$$x_{k+1} = A_k x_k + B_k u_k + G_k d_k + D_{1,k} w_k, \tag{6.1}$$

$$y_k = C_k x_k + v_k, \tag{6.2}$$

where $x_k \in \mathbb{R}^{l_x}$ is the unknown state, $u_k \in \mathbb{R}^{l_u}$ is the known input, $d_k \in \mathbb{R}^{l_d}$ is the unknown deterministic input, $w_k \in \mathbb{R}^{l_w}$ is is unknown white process noise with zero mean and unit variance, $y_k \in \mathbb{R}^{l_y}$ is the the measured output, and $v_k \in \mathbb{R}^{l_y}$ is unknown white measurement noise with zero mean and variance $V_{2,k}$. This model may represent a sampled-data version of a continuous-time plant with sample time T_s, in which case x_k denotes the state at time $t = kT_\mathrm{s}$. The matrices $A_k \in \mathbb{R}^{l_x \times l_x}$, $B_k \in \mathbb{R}^{l_x \times l_u}$, $G_k \in \mathbb{R}^{l_x \times l_d}$, $D_{1,k} \in \mathbb{R}^{l_x \times l_w}$, $C_k \in \mathbb{R}^{l_y \times l_x}$, and $V_{2,k} \in \mathbb{R}^{l_y \times l_y}$ are assumed to be known. Define $V_{1,k} \triangleq D_{1,k} D_{1,k}^\mathrm{T} \in \mathbb{R}^{l_x \times l_x}$. The goal is to estimate the unknown input d_k and the unknown state x_k.

6.1 Input Estimation

Consider the Kalman filter forecast step

$$x_{\mathrm{fc},k+1} = A_k x_{\mathrm{da},k} + B_k u_k + G_k \hat{d}_k, \tag{6.3}$$

$$y_{\mathrm{fc},k} = C_k x_{\mathrm{fc},k}, \tag{6.4}$$

$$z_k = y_{\mathrm{fc},k} - y_k, \tag{6.5}$$

where $\hat{d}_k \in \mathbb{R}^{l_d}$ is the input estimate, $x_{\mathrm{da,k}} \in \mathbb{R}^{l_x}$ is the data-assimilation state, $x_{\mathrm{fc},k} \in \mathbb{R}^{l_x}$ is the forecast state, $z_k \in \mathbb{R}^{l_y}$ is the innovations. The input estimate $\hat{d}_k$ is obtained as the output of the *input-estimation subsystem* of order n_c given by

$$\hat{d}_k = \sum_{i=1}^{n_\mathrm{c}} P_{i,k} \hat{d}_{k-i} + \sum_{i=0}^{n_\mathrm{c}} Q_{i,k} z_{k-i}, \tag{6.6}$$

where $P_{i,k} \in \mathbb{R}^{l_d \times l_d}$ and $Q_{i,k} \in \mathbb{R}^{l_d \times l_y}$. RCIE minimizes z_{k+1} by updating $P_{i,k}$ and $Q_{i,k}$. The subsystem (6.6) can be reformulated as

$$\hat{d}_k = \Phi_k \theta_k, \tag{6.7}$$

where the regressor matrix Φ_k is defined by

$$\Phi_k \triangleq \begin{bmatrix} \hat{d}_{k-1} \\ \vdots \\ \hat{d}_{k-n_\mathrm{c}} \\ z_k \\ \vdots \\ z_{k-n_\mathrm{c}} \end{bmatrix}^\mathrm{T} \otimes I_{l_d} \in \mathbb{R}^{l_d \times l_\theta} \tag{6.8}$$

and the coefficient vector θ_k is defined by

$$\theta_k \triangleq \operatorname{vec}\begin{bmatrix} P_{1,k} & \cdots & P_{n_{\rm c},k} & Q_{0,k} & \cdots & Q_{n_{\rm c},k}\end{bmatrix} \in \mathbb{R}^{l_\theta}, \tag{6.9}$$

where $l_\theta \triangleq l_d^2 n_{\rm c} + l_d l_y(n_{\rm c}{+}1)$, "$\otimes$" is the Kronecker product, and "vec" is the column-stacking operator. The order $n_{\rm c}$ of the input-estimation subsystem must be chosen large enough to accommodate an internal model of the unknown input. In terms of the backward shift operator $\mathbf{q}^{-1}$, (6.6) can be written as

$$\hat{d}_k = G_{\hat{d}z,k}(\mathbf{q}^{-1})z_k, \tag{6.10}$$

where

$$G_{\hat{d}z,k} \triangleq D_{\hat{d}z,k}^{-1}N_{\hat{d}z,k}, \tag{6.11}$$

$$D_{\hat{d}z,k}(\mathbf{q}^{-1}) \triangleq I_{l_d} - P_{1,k}\mathbf{q}^{-1} - \cdots - P_{n_{\rm c},k}\mathbf{q}^{-n_{\rm c}}, \tag{6.12}$$

$$N_{\hat{d}z,k}(\mathbf{q}^{-1}) \triangleq Q_{0,k} + Q_{1,k}\mathbf{q}^{-1} + \cdots + Q_{n_{\rm c},k}\mathbf{q}^{-n_{\rm c}}. \tag{6.13}$$

Next, define the filtered signals

$$\Phi_{{\rm f},k} \triangleq G_{{\rm f},k}(\mathbf{q}^{-1})\Phi_k, \tag{6.14}$$

$$\hat{d}_{{\rm f},k} \triangleq G_{{\rm f},k}(\mathbf{q}^{-1})\hat{d}_k, \tag{6.15}$$

where $G_{{\rm f},k}$ is an $l_y \times l_d$ filter of window length $n_{\rm f} \geq 1$. Define the *retrospective performance variable*

$$z_{{\rm rc},k}(\hat{\theta}) \triangleq z_k - \left(\hat{d}_{{\rm f},k} - \Phi_{{\rm f},k}\hat{\theta}\right), \tag{6.16}$$

where the coefficient vector $\hat{\theta} \in \mathbb{R}^{l_\theta}$ denotes a variable for optimization. The retrospective performance variable $z_{\mathrm{rc},k}(\hat{\theta})$ is used to determine the updated coefficient vector θ_{k+1} by minimizing a function of $z_{\mathrm{rc},k}(\hat{\theta})$. The optimized value of $z_{\mathrm{rc},k}$ is thus given by

$$z_{\mathrm{rc},k}(\theta_{k+1}) = z_k - \left(\hat{d}_{\mathrm{f},k} - \Phi_{\mathrm{f},k}\theta_{k+1}\right), \tag{6.17}$$

which shows that the updated coefficient vector θ_{k+1} is "applied" retrospectively with the filtered regressor $\Phi_{\mathrm{f},k}$. Furthermore, note that the filter $G_{\mathrm{f},k}$ is used to obtain $\Phi_{\mathrm{f},k}$ from Φ_k by means of (6.14) but ignores past changes in the coefficient vector, as can be seen by the product $\Phi_{\mathrm{f},k}\theta_{k+1}$ in (6.17). Consequently, the filtering used to construct (6.17) ignores changes in the coefficient vector over the window $[k - n_{\mathrm{f}}, k]$.

Next, define the retrospective cost function

$$J_k(\hat{\theta}) \triangleq \sum_{i=0}^{k} \lambda^{k-i}\left(z_{\mathrm{rc},i}^{\mathrm{T}}(\hat{\theta})R_z z_{\mathrm{rc},i}(\hat{\theta}) + \hat{\theta}^{\mathrm{T}}\Phi_i^{\mathrm{T}}R_d\Phi_i\hat{\theta}\right) + \lambda^{k+1}(\hat{\theta}-\theta_0)^{\mathrm{T}}R_\theta(\hat{\theta}-\theta_0), \tag{6.18}$$

where $R_z \in \mathbb{R}^{l_y\times l_y}$, $R_d \in \mathbb{R}^{l_d\times l_d}$, and $R_\theta \in \mathbb{R}^{l_\theta\times l_\theta}$ are positive definite and $\lambda \in (0,1]$ is the forgetting factor. Define $P_0 \triangleq R_\theta^{-1}$, and define

$$\Gamma_k \triangleq (\lambda\widetilde{R}^{-1} + \widetilde{\Phi}_k P_k \widetilde{\Phi}_k^{\mathrm{T}})^{-1}, \quad \widetilde{\Phi}_k \triangleq \begin{bmatrix}\Phi_{\mathrm{f},k}\\ \Phi_k\end{bmatrix} \in \mathbb{R}^{(l_y+l_d)\times l_\theta}, \tag{6.19}$$

$$\widetilde{R} \triangleq \begin{bmatrix}R_z & 0\\ 0 & R_d\end{bmatrix} \in \mathbb{R}^{(l_y+l_d)\times(l_y+l_d)}, \quad \widetilde{z}_k \triangleq \begin{bmatrix}z_k - \hat{d}_{\mathrm{f},k}\\ 0\end{bmatrix} \in \mathbb{R}^{l_y+l_d}. \tag{6.20}$$

Furthermore, for all $k \geq 0$, denote the minimizer of the function given by (6.18) as

$$\theta_{k+1} \triangleq \underset{\hat{\theta}}{\operatorname{argmin}}\ J_k(\hat{\theta}). \tag{6.21}$$

Substituting (6.16) in (6.18) yields

$$J_k(\hat{\theta}) = \sum_{i=0}^{k} \lambda^{k-i} \left(\widetilde{z}_i + \widetilde{\Phi}_i\hat{\theta}\right)^{\mathrm{T}} \widetilde{R} \left(\widetilde{z}_i + \widetilde{\Phi}_i\hat{\theta}\right) + \lambda^{k+1}(\hat{\theta} - \theta_0)^{\mathrm{T}} P_0^{-1}(\hat{\theta} - \theta_0)$$
$$= \sum_{i=0}^{k} \lambda^{k-i} \left(\widetilde{R}^{1/2}\widetilde{z}_i + \widetilde{R}^{1/2}\widetilde{\Phi}_i\hat{\theta}\right)^{\mathrm{T}} \left(\widetilde{R}^{1/2}\widetilde{z}_i + \widetilde{R}^{1/2}\widetilde{\Phi}_i\hat{\theta}\right) + \lambda^{k+1}(\hat{\theta} - \theta_0)^{\mathrm{T}} P_0^{-1}(\hat{\theta} - \theta_0).$$

Then, it follows from Theorem 2 in [126] that, for all $k \geq 0$,

$$P_{k+1} = \frac{1}{\lambda}P_k - \frac{1}{\lambda}P_k\widetilde{\Phi}_k^{\mathrm{T}}\widetilde{R}^{1/2}(\lambda I + \widetilde{R}^{1/2}\widetilde{\Phi}_k P_k \widetilde{\Phi}_k^{\mathrm{T}}\widetilde{R}^{1/2})^{-1}\widetilde{R}^{1/2}\widetilde{\Phi}_k P_k,$$
$$\theta_{k+1} = \theta_k - P_k\widetilde{\Phi}_k^{\mathrm{T}}\widetilde{R}^{1/2}(\lambda I + \widetilde{R}^{1/2}\widetilde{\Phi}_k P_k \widetilde{\Phi}_k^{\mathrm{T}}\widetilde{R}^{1/2})^{-1}\widetilde{R}^{1/2}(\widetilde{z}_k + \widetilde{\Phi}_k\theta_k),$$

which imply that

$$P_{k+1} = \frac{1}{\lambda}(P_k - P_k\widetilde{\Phi}_k^{\mathrm{T}}\Gamma_k\widetilde{\Phi}_k P_k), \tag{6.22}$$

$$\theta_{k+1} = \theta_k - P_k\widetilde{\Phi}_k^{\mathrm{T}}\Gamma_k(\widetilde{z}_k + \widetilde{\Phi}_k\theta_k). \tag{6.23}$$

The RLS update equations (6.22) and (6.23) give the unique global minimizer θ_{k+1} of (6.18). Using the updated coefficient vector given by (6.23), the estimated input at step $k+1$ is given by replacing k by $k+1$ in (6.7). We choose $\theta_0 = 0$, and thus, $\hat{d}_0 = 0$.

6.2 State Estimation

In order to estimate the state x_k, $x_{\mathrm{fc},k}$ given by (6.3) is used to obtain the estimate $x_{\mathrm{da},k}$ of x_k given by the Kalman filter data-assimilation step

$$x_{\mathrm{da},k} = x_{\mathrm{fc},k} + K_{\mathrm{da},k} z_k, \tag{6.24}$$

$K_{\mathrm{da},k} \in \mathbb{R}^{l_x \times l_y}$ is the state estimator gain or the Kalman gain. The expression for the optimal state estimator gain that produces the minimum mean square error is derived in section 6.4.

6.3 Error Systems

Define the data-assimilation error $e_{\mathrm{da},k} \triangleq x_k - x_{\mathrm{da},k}$, the forecast error $e_{\mathrm{fc},k} \triangleq x_k - x_{\mathrm{fc},k}$, and the input-estimation error $e_{d,k} \triangleq d_k - \hat{d}_k$. Furthermore, define

$$\widetilde{K}_k \triangleq I + K_{\mathrm{da},k}C_k, \quad \overline{A}_k \triangleq A_k(I + K_{\mathrm{da},k}C_k), \quad \overline{B}_k \triangleq -A_kK_{\mathrm{da},k}. \tag{6.25}$$

Note that it follows from (6.1)–(6.5) and (6.24) that

$$\begin{aligned} e_{\mathrm{fc},k+1} &= x_{k+1} - x_{\mathrm{fc},k+1} \\ &= A_kx_k + D_{1,k}w_k + G_kd_k - A_kx_{\mathrm{da},k} - G_k\hat{d}_k \\ &= A_kx_k + G_ke_{d,k} + D_{1,k}w_k - A_k(x_{\mathrm{fc},k} + K_{\mathrm{da},k}z_k) \\ &= A_k(x_k - x_{\mathrm{fc},k}) + G_ke_{d,k} + D_{1,k}w_k - A_kK_{\mathrm{da},k}(C_kx_{\mathrm{fc},k} - C_kx_k - v_k) \\ &= \overline{A}_ke_{\mathrm{fc},k} + G_ke_{d,k} + D_{1,k}w_k - \overline{B}_kv_k, \end{aligned} \tag{6.26}$$

and

$$\begin{aligned} e_{\mathrm{da},k} &= x_k - x_{\mathrm{da},k} = x_k - x_{\mathrm{fc},k} - K_{\mathrm{da},k}(C_kx_{\mathrm{fc},k} - C_kx_k - v_k) \\ &= (I + K_{\mathrm{da},k}C_k)e_{\mathrm{fc},k} + K_{\mathrm{da},k}v_k. \end{aligned} \tag{6.27}$$

Replacing k by $k+1$ in (6.27), and using (6.1) and (6.3) yields

$$\begin{aligned} e_{\mathrm{da},k+1} &= \widetilde{K}_{k+1}(A_kx_k + D_{1,k}w_k + G_kd_k - A_kx_{\mathrm{da},k} - G_k\hat{d}_k) + K_{\mathrm{da},k+1}v_{k+1} \\ &= \widetilde{K}_{k+1}A_ke_{\mathrm{da},k} + \widetilde{K}_{k+1}G_ke_{d,k} + \widetilde{K}_{k+1}D_{1,k}w_k + K_{\mathrm{da},k+1}v_{k+1}. \end{aligned} \tag{6.28}$$

6.4 Kalman Gain

The following result gives the expression for the *Kalman gain* which is the optimal state estimator gain that produces the minimal mean-square error.

Theorem 6.1. Consider the system given by (6.1) and (6.2), and the Kalman filter given by (6.3)–(6.5), and (6.24). Let $P_{\mathrm{da},k} \in \mathbb{R}^{l_x \times l_x}$ be the variance of $e_{\mathrm{da},k}$ and let $P_{\mathrm{fc},k} \in \mathbb{R}^{l_x \times l_x}$ be the variance of $e_{\mathrm{fc},k}$. Assume that, for all $k \geq 0$, $C_k P_{\mathrm{fc},k} C_k^{\mathrm{T}} + V_{2,k}$ is nonsingular. Then, for all $k \geq 0$, the optimal state estimator gain that gives the minimal mean-square data-assimilation error is given by

$$K_{\mathrm{da},k} = -P_{\mathrm{fc},k} C_k^{\mathrm{T}} (C_k P_{\mathrm{fc},k} C_k^{\mathrm{T}} + V_{2,k})^{-1}, \tag{6.29}$$

and the variance update equations are given by

$$P_{\mathrm{fc},k+1} = A_k P_{\mathrm{da},k} A_k^{\mathrm{T}} + V_{1,k} + \widetilde{V}_{\mathrm{da},k}, \tag{6.30}$$

$$P_{\mathrm{da},k} = (I + K_{\mathrm{da},k} C_k) P_{\mathrm{fc},k}, \tag{6.31}$$

where

$$\widetilde{V}_{\mathrm{da},k} \triangleq G_k \mathrm{var}(e_{d,k}) G_k^{\mathrm{T}} + A_k \mathrm{cov}(e_{\mathrm{da},k}, e_{d,k}) G_k^{\mathrm{T}} + G_k \mathrm{cov}(e_{d,k}, e_{\mathrm{da},k}) A_k^{\mathrm{T}}. \tag{6.32}$$

Proof. Note that (6.1) and (6.3) imply

$$\begin{aligned}
P_{\mathrm{fc},k+1} &= \mathrm{var}(e_{\mathrm{fc},k+1}) = \mathrm{var}(A_k(x_k - x_{\mathrm{da},k}) + D_{1,k} w_k + G_k e_{d,k}) \\
&= A_k \mathrm{var}(e_{\mathrm{da},k}) A_k^{\mathrm{T}} + D_{1,k} \mathrm{var}(w_k) D_{1,k}^{\mathrm{T}} + G_k \mathrm{var}(e_{d,k}) G_k^{\mathrm{T}} + A_k \mathrm{cov}(e_{\mathrm{da},k}, w_k) D_{1,k}^{\mathrm{T}} \\
&\quad + D_{1,k} \mathrm{cov}(w_k, e_{\mathrm{da},k}) A_k^{\mathrm{T}} + G_k \mathrm{cov}(e_{d,k}, w_k) D_{1,k}^{\mathrm{T}} + D_{1,k} \mathrm{cov}(w_k, e_{d,k}) G_k^{\mathrm{T}} \\
&\quad + A_k \mathrm{cov}(e_{\mathrm{da},k}, e_{d,k}) G_k^{\mathrm{T}} + G_k \mathrm{cov}(e_{d,k}, e_{\mathrm{da},k}) A_k^{\mathrm{T}} \\
&= A_k P_{\mathrm{da},k} A_k^{\mathrm{T}} + V_{1,k} + \widetilde{V}_{\mathrm{da},k}.
\end{aligned}$$

Similarly, (6.2), (6.4), (6.5), and (6.24) imply that

$$\begin{aligned}
P_{\mathrm{da},k} &= \mathrm{var}(x_k - x_{\mathrm{fc},k} - K_{\mathrm{da},k}C_k x_{\mathrm{fc},k} + K_{\mathrm{da},k}(C_k x_k + v_k)) \\
&= \mathrm{var}((I + K_{\mathrm{da},k}C_k)(x_k - x_{\mathrm{fc},k}) + K_{\mathrm{da},k}v_k) \\
&= (I + K_{\mathrm{da},k}C_k)\mathrm{var}(e_{\mathrm{fc},k})(I + K_{\mathrm{da},k}C_k)^{\mathrm{T}} + K_{\mathrm{da},k}\mathrm{var}(v_k)K_{\mathrm{da},k}^{\mathrm{T}} \\
&\quad + (I + K_{\mathrm{da},k}C_k)\mathrm{cov}(e_{\mathrm{fc},k}, v_k)K_{\mathrm{da},k}^{\mathrm{T}} + K_{\mathrm{da},k}\mathrm{cov}(v_k, e_{\mathrm{fc},k})(I + K_{\mathrm{da},k}C_k)^{\mathrm{T}} \\
&= (I + K_{\mathrm{da},k}C_k)P_{\mathrm{fc},k}(I + K_{\mathrm{da},k}C_k)^{\mathrm{T}} + K_{\mathrm{da},k}V_{2,k}K_{\mathrm{da},k}^{\mathrm{T}} \\
&= P_{\mathrm{fc},k} + K_{\mathrm{da},k}C_kP_{\mathrm{fc},k} + P_{\mathrm{fc},k}C_k^{\mathrm{T}}K_{\mathrm{da},k}^{\mathrm{T}} + K_{\mathrm{da},k}(C_kP_{\mathrm{fc},k}C_k^{\mathrm{T}} + V_{2,k})K_{\mathrm{da},k}^{\mathrm{T}}. \qquad (6.33)
\end{aligned}$$

Next, note that

$$\arg\min_{K_{\mathrm{da},k}} \mathrm{E}\,(e_{\mathrm{da},k}^{\mathrm{T}}e_{\mathrm{da},k}) = \arg\min_{K_{\mathrm{da},k}} \mathrm{tr}(P_{\mathrm{da},k}).$$

Since the derivative of $P_{\mathrm{da},k}$ with respect to $K_{\mathrm{da},k}$ is zero at the minimum value of $\mathrm{tr}(P_{\mathrm{da},k})$ and since $P_{\mathrm{da},k}$ is a quadratic function of $K_{\mathrm{da},k}$, it follows that the the optimal state estimator gain that gives the minimal mean square error between the state x_k and estimate $x_{\mathrm{da},k}$ is given by

$$0 = \frac{\partial P_{\mathrm{da},k}}{\partial K_{\mathrm{da},k}} = 2(C_kP_{\mathrm{fc},k})^{\mathrm{T}} + 2K_{\mathrm{da},k}(C_kP_{\mathrm{fc},k}C_k^{\mathrm{T}} + V_{2,k}),$$

which implies (6.29). Multiplying both sides of (6.29) by $(C_kP_{\mathrm{fc},k}C_k^{\mathrm{T}} + V_{2,k})K_{\mathrm{da},k}^{\mathrm{T}}$ on the right yields

$$K_{\mathrm{da},k}(C_kP_{\mathrm{fc},k}C_k^{\mathrm{T}} + V_{2,k})K_{\mathrm{da},k}^{\mathrm{T}} = -P_{\mathrm{f},k}C_k^{\mathrm{T}}K_{\mathrm{da},k}^{\mathrm{T}},$$

which along with (6.33) implies (6.31). □

Note that (6.26) implies that

$$\begin{aligned}P_{\mathrm{fc},k+1} &= \mathrm{cov}(e_{\mathrm{fc},k+1}) = \mathrm{cov}(\overline{A}_k e_{\mathrm{fc},k} + G_k e_{d,k} + D_{1,k} w_k - \overline{B}_k v_k)\\ &= \overline{A}_k P_{\mathrm{fc},k} \overline{A}_k^{\mathrm{T}} + \overline{B}_k V_{2,k} \overline{B}_k^{\mathrm{T}} + V_{1,k} + \widetilde{V}_{\mathrm{fc},k},\end{aligned} \tag{6.34}$$

where

$$\widetilde{V}_{\mathrm{fc},k} \triangleq G_k \mathrm{var}(e_{d,k}) G_k^{\mathrm{T}} + \overline{A}_k \mathrm{cov}(e_{\mathrm{fc},k}, e_{d,k}) G_k^{\mathrm{T}} + G_k \mathrm{cov}(e_{d,k}, e_{\mathrm{fc},k}) \overline{A}_k^{\mathrm{T}}. \tag{6.35}$$

Similarly, (6.28) implies that

$$\begin{aligned}P_{\mathrm{da},k+1} &= \mathrm{cov}(e_{\mathrm{da},k+1}) = \mathrm{cov}(\widetilde{K}_{k+1} A_k e_{\mathrm{da},k} + \widetilde{K}_{k+1} G_k e_{d,k} + \widetilde{K}_{k+1} D_{1,k} w_k + K_{\mathrm{da},k+1} v_{k+1})\\ &= \widetilde{K}_{k+1} A_k P_{\mathrm{da},k} A_k^{\mathrm{T}} \widetilde{K}_{k+1}^{\mathrm{T}} + \widetilde{K}_{k+1} V_{1,k} \widetilde{K}_{k+1}^{\mathrm{T}} + K_{\mathrm{da},k+1} V_{2,k+1} K_{\mathrm{da},k+1}^{\mathrm{T}} + \widetilde{K}_{k+1} \widetilde{V}_{\mathrm{da},k} \widetilde{K}_{k+1}^{\mathrm{T}}.\end{aligned} \tag{6.36}$$

Here (6.34) and (6.36) are alternative update equations for $P_{\mathrm{fc},k}$ and $P_{\mathrm{da},k}$.

6.5 The Filter $G_{\mathrm{f},k}$

We choose $G_{\mathrm{f},k}$ to be the FIR filter

$$G_{\mathrm{f},k}(\mathbf{q}^{-1}) = \sum_{i=1}^{n_{\mathrm{f}}} \mathbf{q}^{-i} H_{i,k}, \tag{6.37}$$

where,

$$H_{i,k} \triangleq \begin{cases} C_k G_{k-1}, & k \geq i = 1,\\ C_k \left(\prod_{j=1}^{i-1} \overline{A}_{k-j} \right) G_{k-i}, & k \geq i \geq 2,\\ 0, & i > k. \end{cases} \tag{6.38}$$

This particular choice for the filter was given in [70] and is observed to be effective in the successful implementation of the RCIE algorithm.

CHAPTER 7

Conversion Between LTV State Space Models and LTV Input-Output Models

This chapter gives the construction of LTV state space realizations for LTV input-output models as well as the construction of LTV input-output models for LTV state space models. The decomposition of retrospective performance variable presented in the next chapter use the results in this chapter.

Consider the LTV state space model given by (2.10) and (2.11). Assume that, for all $k \le 0$, $A_k = 0, B_k = 0, C_k = 0$, and $E_k = 0$. The solution to (2.10) and (2.11), for all $k \ge 0$, is

$$y_k = C_k \left(\prod_{i=1}^{k} A_{k-i} \right) x_0 + C_k \sum_{i=0}^{k-2} \left(\prod_{j=1}^{k-i-1} A_{k-j} \right) B_i u_i + C_k B_{k-1} u_{k-1} + E_k u_k, \tag{7.1}$$

where $\prod_{i=0}^{r} X_i \triangleq X_0 X_1 \ldots X_r$.

In this and all subsequent sections, let $G \in \mathbb{R}(\mathbf{q}^{-1})^{p\times m}_{\mathrm{prop}}$ denote the time-domain transfer function of an LTV system, and let G_k denote the transfer function at step k. Define, for all $k \ge 0$, $n \triangleq \mathrm{McDeg}\ G_k$, and thus it is assumed that the order of a minimal state space model of G is constant. Let, for all $k \ge 0$, $G_k = D_k^{-1} N_k$, where D_k and N_k are defined in (2.6) and (2.7). Note that, for all $k < 0$, $D_{1,k} = \cdots = D_{n,k} = N_{0,k} = \cdots = N_{n,k} = 0$. The *Markov parameters* of G, for all $k \geq 0$, are defined as

$$H_{i,k} \triangleq \begin{cases} E_k, & i = 0, \\ C_k B_{k-1}, & k \geq i = 1, \\ C_k \prod_{j=1}^{i-1} A_{k-j} B_{k-i}, & k \geq i \geq 2, \\ 0, & i > k. \end{cases} \tag{7.2}$$

Proposition 7.1. Assume that (A, C) is completely observable. Then, for all $k \geq n$, an input-output model corresponding to (2.10) and (2.11) is given by (2.5) where, for all $k \geq n$,

$$N_{i,k} = \begin{cases} H_{0,k}, & i = 0, \\ H_{i,k} + \sum_{j=1}^{i} D_{j,k} H_{i-j,k-j}, & 1 \leq i \leq n, \end{cases} \tag{7.3}$$

$$\begin{bmatrix} D_{n,k} & \cdots & D_{1,k} \end{bmatrix} = -C_k \left(\prod_{i=1}^{n} A_{k-i} \right) \mathcal{O}_{k-n}^{\mathrm{L}}, \tag{7.4}$$

and $\mathcal{O}_k^{\mathrm{L}}$ is a left inverse of $\mathcal{O}_k$.

Proof. Assume that (7.3) and (7.4) hold. Post-multiplying (7.4) by $\mathcal{O}_{k-n}$ yields

$$\begin{aligned} 0 &= \begin{bmatrix} D_{n,k} & \cdots & D_{1,k} \end{bmatrix} \mathcal{O}_{k-n} + C_k \left(\prod_{i=1}^{n} A_{k-i} \right) \\ &= C_k \left(\prod_{i=1}^{n} A_{k-i} \right) + D_{1,k} C_{k-1} \left(\prod_{i=2}^{n} A_{k-i} \right) + \cdots + D_{n-1,k} C_{k-n+1} A_{k-n} + D_{n,k} C_{k-n}. \end{aligned} \tag{7.5}$$

Next, it follows from (2.10) and (2.11) that

$$y_k = C_k \left(\prod_{i=1}^{n} A_{k-i} \right) x_{k-n} + C_k \sum_{i=0}^{n-2} \left(\prod_{j=1}^{n-i-1} A_{k-j} \right) B_{k-n+i} u_{k-n+i} + C_k B_{k-1} u_{k-1} + E_k u_k.$$

Hence

$$
\begin{aligned}
&y_k + D_{1,k}y_{k-1} + \cdots + D_{n-1,k}y_{k-n+1} + D_{n,k}y_{k-n} \\
&= C_k\left(\prod_{i=1}^{n} A_{k-i}\right)x_{k-n} + C_k\sum_{i=0}^{n-2}\left(\prod_{j=1}^{n-i-1} A_{k-j}\right)B_{k-n+i}u_{k-n+i} + C_kB_{k-1}u_{k-1} + E_ku_k \\
&+ D_{1,k}\left(C_{k-1}\left(\prod_{i=2}^{n} A_{k-i}\right)x_{k-n} + C_{k-1}\sum_{i=0}^{n-3}\left(\prod_{j=2}^{n-i-1} A_{k-j}\right)B_{k-n+i}u_{k-n+i} + C_{k-1}B_{k-2}u_{k-2}\right. \\
&\left. + E_{k-1}u_{k-1}\right) + \cdots + D_{n-1,k}(C_{k-n+1}A_{k-n}x_{k-n} + C_{k-n+1}B_{k-n}u_{k-n} + E_{k-n+1}u_{k-n+1}) \\
&+ D_{n,k}(C_{k-n}x_{k-n} + E_{k-n}u_{k-n}).
\end{aligned} \tag{7.6}
$$

Regrouping the terms in (7.6) and using (7.2) yields

$$
\begin{aligned}
&y_k + D_{1,k}y_{k-1} + \cdots + D_{n-1,k}y_{k-n+1} + D_{n,k}y_{k-n} \\
&= \left(C_k\left(\prod_{i=1}^{n} A_{k-i}\right) + D_{1,k}C_{k-1}\left(\prod_{i=2}^{n} A_{k-i}\right) + \cdots + D_{n-1,k}C_{k-n+1}A_{k-n} + D_{n,k}C_{k-n}\right)x_{k-n} \\
&+ \left(H_{n,k} + \sum_{j=1}^{n} D_{j,k}H_{n-j,k-j}\right)u_{k-n} + \left(H_{n-1,k} + \sum_{j=1}^{n-1} D_{j,k}H_{n-1-j,k-j}\right)u_{k-n+1} + \ldots\ldots \\
&+ (H_{1,k} + D_{1,k}H_{0,k-1})\,u_{k-1} + H_{0,k}u_k.
\end{aligned} \tag{7.7}
$$

Then, substituting (7.5) and (7.3) in (7.7) gives (2.5). □

Proposition 7.2. A completely observable state space model corresponding to the input-output model in (2.5) is given by (2.10) and (2.11), where, for all $k \geq 0$,

$$A_k = \begin{bmatrix} 0 & \cdots & 0 & -D_{n,k+n} \\ I & \cdots & 0 & -D_{n-1,k+n-1} \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & I & -D_{1,k+1} \end{bmatrix}, \quad B_k = \begin{bmatrix} N_{n,k+n} - D_{n,k+n}N_{0,k} \\ N_{n-1,k+n-1} - D_{n-1,k+n-1}N_{0,k} \\ \vdots \\ N_{1,k+1} - D_{1,k+1}N_{0,k} \end{bmatrix}, \tag{7.8}$$

$$C_k = \begin{bmatrix} 0_{p\times p(n-1)} & I_p \end{bmatrix}, \quad E_k = N_{0,k}. \tag{7.9}$$

Furthermore, $x_k = \begin{bmatrix} x_{1,k}^{\mathrm{T}} & x_{2,k}^{\mathrm{T}} & \cdots & x_{n,k}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, where, for all $i = 0, \ldots, n-1$,

$$x_{n-i,k} = \sum_{j=i+1}^{n} N_{j,k+i}u_{k-j+i} - \sum_{j=i+1}^{n} D_{j,k+i}y_{k-j+i}. \tag{7.10}$$

Proof. Assume that (7.8), (7.9), and (7.10) hold. Rearranging terms in (2.5) yields

$$y_k = \sum_{j=1}^{n} N_{j,k}u_{k-j} - \sum_{j=1}^{n} D_{j,k}y_{k-j} + N_{0,k}u_k. \tag{7.11}$$

Substituting $i = 0$ in (7.10) yields

$$x_{n,k} = \sum_{j=1}^{n} N_{j,k}u_{k-j} - \sum_{j=1}^{n} D_{j,k}y_{k-j},$$

which along with (7.11) implies that

$$x_{n,k} = y_k - N_{0,k}u_k. \tag{7.12}$$

Next, substituting $i = n-1$ in (7.10) yields

$$x_{1,k} = N_{n,k+n-1}u_{k-1} - D_{n,k+n-1}y_{k-1}. \tag{7.13}$$

Hence it follows from (7.12) and (7.13) that

$$\begin{aligned} x_{1,k+1} &= N_{n,k+n}u_k - D_{n,k+n}y_k = -D_{n,k+n}(y_k - N_{0,k}u_k) + (N_{n,k+n} - D_{n,k+n}N_{0,k})u_k \\ &= -D_{n,k+n}x_{n,k} + (N_{n,k+n} - D_{n,k+n}N_{0,k})u_k. \end{aligned} \tag{7.14}$$

Furthermore, for all $i = 0, \ldots, n-2$, (7.10) and (7.12) imply that

$$\begin{aligned} x_{n-i,k+1} &= \sum_{j=i+1}^{n} N_{j,k+i+1}u_{k-j+i+1} - \sum_{j=i+1}^{n} D_{j,k+i+1}y_{k-j+i+1} \\ &= \sum_{j=i+2}^{n} N_{j,k+i+1}u_{k-j+i+1} - \sum_{j=i+2}^{n} D_{j,k+i+1}y_{k-j+i+1} + N_{i+1,k+i+1}u_k - D_{i+1,k+i+1}y_k \\ &= x_{n-i-1,k} + N_{i+1,k+i+1}u_k - D_{i+1,k+i+1}y_k \\ &= x_{n-i-1,k} - D_{i+1,k+i+1}(y_k - N_{0,k}u_k) + (N_{i+1,k+i+1} - D_{i+1,k+i+1}N_{0,k})u_k \\ &= x_{n-i-1,k} - D_{i+1,k+i+1}x_{n,k} + (N_{i+1,k+i+1} - D_{i+1,k+i+1}N_{0,k})u_k. \end{aligned} \tag{7.15}$$

Hence, it follows from (7.8), (7.14) and (7.15) that

$$\begin{aligned} &A_k x_k + B_k u_k \\ &= \begin{bmatrix} -D_{n,k+n}x_{n,k} \\ x_{1,k} - D_{n-1,k+n-1}x_{n,k} \\ \vdots \\ x_{n-1,k} - D_{1,k+1}x_{n,k} \end{bmatrix} + \begin{bmatrix} N_{n,k+n} - D_{n,k+n}N_{0,k} \\ N_{n-1,k+n-1} - D_{n-1,k+n-1}N_{0,k} \\ \vdots \\ N_{1,k+1} - D_{1,k+1}N_{0,k} \end{bmatrix} u_k = \begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \\ \vdots \\ x_{n,k+1} \end{bmatrix} = x_{k+1}. \end{aligned}$$

Finally, it follows from (7.9) and (7.12) that

$$C_k x_k + E_k u_k = x_{n,k} + N_{0,k}u_k = y_k. \tag{7.16}$$

Since, for all $k \geq 0$, $\operatorname{rank} \mathcal{O}_k = n$ for A_k and C_k as defined in (7.8) and (7.9), it follows that (A, C) is completely observable. □

CHAPTER 8

Decomposition of Retrospective Performance Variable in RCIE

In this chapter, in order to obtain a better understanding of the underlying mechanism and the performance of RCIE, a decomposition of the retrospective performance variable into the sum of a performance term and a model-matching term is presented. Since this decomposition involves time-varying input-output models, the results on conversion between LTV state space models and LTV input-output models given in the previous chapter are used here. Analysis of the decomposition shows how RCIE avoids convergence to an estimator that is destabilizing or has poor performance. A numerical example is used to illustrate the derived results and observations.

Define the *virtual external input perturbation* for RCIE as

$$\widetilde{d}_k(\theta_{k+1}) \overset{\triangle}{=} \hat{d}_k - \Phi_k\theta_{k+1}. \tag{8.1}$$

Let $\widetilde{d}_{\mathrm{f},k}(\theta_{k+1})$ be given by the FIA filter

$$\widetilde{d}_{\mathrm{f},k}(\theta_{k+1}) = G_{\mathrm{f},k}(\mathbf{q}^{-1})\widetilde{d}_k(\theta_{\overline{k+1}}). \tag{8.2}$$

Note that $\widetilde{d}_{\mathrm{f},k}(\theta_{k+1})$ ignores the changes in the argument θ_{k+1} of $\widetilde{d}_k$ over the interval $[k-n_{\mathrm{f}}, k]$ in accordance with retrospective optimization. Using (8.2), (6.17) can be written as

$$z_{\mathrm{rc},k}(\theta_{k+1}) = z_k - \widetilde{d}_{\mathrm{f},k}(\theta_{k+1}). \tag{8.3}$$

The following matrices are used in Theorem 3.1.

$$\hat{A}_k \triangleq \begin{bmatrix} 0 & \cdots & 0 & P_{n_\mathrm{c},k+n_\mathrm{c}+1} \\ I & \cdots & 0 & P_{n_\mathrm{c}-1,k+n_\mathrm{c}} \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & I & P_{1,k+2} \end{bmatrix}, \quad \hat{G} \triangleq \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{l_d n_\mathrm{c} \times l_d}, \tag{8.4}$$

$$\hat{B}_k \triangleq \begin{bmatrix} Q_{n_\mathrm{c},k+n_\mathrm{c}+1} + P_{n_\mathrm{c},k+n_\mathrm{c}+1} Q_{0,k+1} \\ Q_{n_\mathrm{c}-1,k+n_\mathrm{c}} + P_{n_\mathrm{c}-1,k+n_\mathrm{c}} Q_{0,k+1} \\ \vdots \\ Q_{1,k+2} + P_{1,k+2} Q_{0,k+1} \end{bmatrix}, \tag{8.5}$$

$$\hat{C} \triangleq \begin{bmatrix} 0 & \cdots & 0 & I \end{bmatrix} \in \mathbb{R}^{l_d \times l_d n_\mathrm{c}}, \quad \hat{D}_k \triangleq Q_{0,k+1}, \tag{8.6}$$

$$B_{\mathrm{a},k} \triangleq \begin{bmatrix} G_k & D_{1,k} & -\overline{B}_k \end{bmatrix}, \quad D_\mathrm{a} \triangleq \begin{bmatrix} 0_{l_y \times l_d} & 0_{l_y \times l_w} & -I_{l_y} \end{bmatrix}, \tag{8.7}$$

$$\widetilde{A}_k \triangleq \begin{bmatrix} \hat{A}_k & \hat{B}_k C_k \\ -G_k \hat{C} & \overline{A}_k - G_k \hat{D}_k C_k \end{bmatrix}, \quad \widetilde{B}_k \triangleq \begin{bmatrix} \hat{G} & \hat{B}_k D_\mathrm{a} \\ 0 & B_{\mathrm{a},k} - G_k \hat{D}_k D_\mathrm{a} \end{bmatrix}, \tag{8.8}$$

$$\widetilde{C}_k \triangleq \begin{bmatrix} 0 & C_k \end{bmatrix}, \quad \widetilde{D} \triangleq \begin{bmatrix} 0 & D_\mathrm{a} \end{bmatrix}. \tag{8.9}$$

The following result presents the *retrospective performance variable decomposition*, which shows that the retrospective performance variable is a combination of the closed-loop performance and the extent to which the updated closed-loop transfer function from $\widetilde{d}_k(\theta_{k+1})$ to z_k matches the filter $G_{\mathrm{f},k}$. Henceforth, $G_{\mathrm{f},k}$ is called the *target model* since it serves as the target for the closed-loop transfer function from $\widetilde{d}_k(\theta_{k+1})$ to z_k.

Theorem 8.1. For all $k \geq 0$,

$$z_{\mathrm{rc},k}(\theta_{k+1}) = z_{\mathrm{pp},k}(\theta_{k+1}) + z_{\mathrm{mm},k}(\theta_{k+1}), \tag{8.10}$$

where the performance term $z_{\mathrm{pp},k}(\theta_{k+1})$ and the model-matching term $z_{\mathrm{mm},k}(\theta_{k+1})$ are defined as

$$z_{\mathrm{pp},k}(\theta_{k+1}) \triangleq G_{z\overline{u},k}(\mathbf{q}^{-1})\overline{u}_k, \tag{8.11}$$

$$z_{\mathrm{mm},k}(\theta_{k+1}) \triangleq G_{z\widetilde{d},k}(\mathbf{q}^{-1})\widetilde{d}_k(\theta_{k+1}) - G_{\mathrm{f},k}(\mathbf{q}^{-1})\widetilde{d}_k(\theta_{\overline{k+1}}), \tag{8.12}$$

and $\overline{u}_k \triangleq \begin{bmatrix} d_k^{\mathrm{T}} & w_k^{\mathrm{T}} & v_k^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$. The time-domain transfer functions $G_{z\widetilde{d},k} \in \mathbb{R}(\mathbf{q}^{-1})_{\mathrm{prop}}^{l_y \times l_d}$ and $G_{z\overline{u},k} \in \mathbb{R}(\mathbf{q}^{-1})_{\mathrm{prop}}^{l_y \times (l_d + l_w + l_y)}$ are given by

$$\begin{bmatrix} G_{z\widetilde{d},k} & G_{z\overline{u},k} \end{bmatrix} \triangleq G_{z\widetilde{u},k}, \tag{8.13}$$

where $G_{z\widetilde{u},k}$ is the time-domain transfer function of the system represented by the state space model

$$\widetilde{x}_{k+1} = \widetilde{A}_k\widetilde{x}_k + \widetilde{B}_k\widetilde{u}_k, \tag{8.14}$$

$$z_k = \widetilde{C}_k\widetilde{x}_k + \widetilde{D}\widetilde{u}_k, \tag{8.15}$$

$\widetilde{u}_k \triangleq \begin{bmatrix} \widetilde{d}_k^{\mathrm{T}}(\theta_{k+1}) & \overline{u}_k^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, $\widetilde{x}_0 \triangleq \begin{bmatrix} 0_{1\times l_d n_{\mathrm{c}}} & (x_0 - x_{\mathrm{fc},0})^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, and $\widetilde{A}_k$, $\widetilde{B}_k$, $\widetilde{C}_k$, and $\widetilde{D}$ are defined in (8.8) and (8.9).

Proof. Note that (6.2), (6.4), and (6.5) imply that

$$z_k = C_k e_{\mathrm{fc},k} - v_k. \tag{8.16}$$

Then, (6.26) and (8.16) can be written as

$$e_{\mathrm{fc},k+1} = \overline{A}_k e_{\mathrm{fc},k} + B_{\mathrm{a},k}\overline{u}_k - G_k\hat{d}_k, \tag{8.17}$$

$$z_k = C_k e_{\mathrm{fc},k} + D_{\mathrm{a}}\overline{u}_k, \tag{8.18}$$

where $B_{\mathrm{a},k}$ and D_{a} are defined in (8.7). Next, it follows from (6.6) that

$$\Phi_k\theta_{k+1} = \sum_{i=1}^{n_\mathrm{c}} P_{i,k+1}\hat{d}_{k-i} + \sum_{i=0}^{n_\mathrm{c}} Q_{i,k+1}z_{k-i}. \tag{8.19}$$

Substituting (8.19) in (8.1) yields

$$\hat{d}_k = \widetilde{d}_k(\theta_{k+1}) + \sum_{i=1}^{n_\mathrm{c}} P_{i,k+1}\hat{d}_{k-i} + \sum_{i=0}^{n_\mathrm{c}} Q_{i,k+1}z_{k-i}. \tag{8.20}$$

Using (6.12) and (6.13), it follows from (8.20) that

$$\hat{d}_k = \widetilde{d}_k(\theta_{k+1}) + \hat{d}_k - D_{\hat{d}z,k+1}(\mathbf{q}^{-1})\hat{d}_k + N_{\hat{d}z,k+1}(\mathbf{q}^{-1})z_k,$$

which, using (6.11), can be rewritten as

$$\hat{d}_k = D^{-1}_{\hat{d}z,k+1}(\mathbf{q}^{-1})\widetilde{d}_k(\theta_{k+1}) + G_{\hat{d}z,k+1}(\mathbf{q}^{-1})z_k. \tag{8.21}$$

Note that (6.12), (6.13), and Proposition 7.2 imply that a state space model corresponding to (8.21) is given by

$$\hat{x}_{k+1} = \hat{A}_k\hat{x}_k + \hat{G}\widetilde{d}_k(\theta_{k+1}) + \hat{B}_k z_k, \tag{8.22}$$

$$\hat{d}_k = \hat{C}\hat{x}_k + \hat{D}_k z_k, \tag{8.23}$$

where $\hat{A}_k$, $\hat{G}$, $\hat{B}_k$, $\hat{C}$, and $\hat{D}_k$ are defined in (8.4), (8.5), and (8.6), and $\hat{x}_0 \triangleq 0_{l_d n_\mathrm{c}\times 1}$. Substituting

(8.23) and (8.18) in (8.17) yields

$$e_{\mathrm{fc},k+1} = (\overline{A}_k - G_k\hat{D}_kC_k)e_{\mathrm{fc},k} - G_k\hat{C}\hat{x}_k + (B_{\mathrm{a},k} - G_k\hat{D}_kD_{\mathrm{a}})\overline{u}_k, \tag{8.24}$$

Similarly, substituting (8.18) in (8.22) yields

$$\hat{x}_{k+1} = \hat{A}_k\hat{x}_k + \hat{B}_kC_ke_{\mathrm{fc},k} + \hat{G}\widetilde{d}_k(\theta_{k+1}) + \hat{B}_kD_{\mathrm{a}}\overline{u}_k. \tag{8.25}$$

Define $\widetilde{x}_k \triangleq \begin{bmatrix}\hat{x}_k^{\mathrm{T}} & e_{\mathrm{fc},k}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}$. Thus, (8.14) and (8.15) follow from (8.24), (8.25), and (8.18). Since $G_{z\widetilde{u},k}$ is the time-domain transfer function of the system represented by (8.14) and (8.15), it follows from (8.13) that

$$z_k = G_{z\overline{u},k}(\mathbf{q}^{-1})\overline{u}_k + G_{z\widetilde{d},k}(\mathbf{q}^{-1})\widetilde{d}_k(\theta_{k+1}). \tag{8.26}$$

Finally, substituting (8.26) in (8.3) yields (8.10). $\square$

Note that the expression for $G_{z\widetilde{u},k}$ is obtained using (8.8) and (8.9) in accordance with Definition 2.20 and Proposition 7.1. In order to use Proposition 7.1, (8.14) and (8.15) must be converted to a completely observable state space model. The time-varying Eigensystem Realization Algorithm explained in Section IV of [127] provides a method to reduce any given LTV state space model to a minimal state space model.

Proposition 8.2. Assume that $\lim_{k\to\infty} \theta_k$ exists and Φ_k is bounded. Then $\lim_{k\to\infty} \widetilde{d}_k(\theta_{k+1}) = 0$.

Proof. Equations (6.7) and (8.1) imply that

$$\widetilde{d}_k(\theta_{k+1}) = \hat{d}_k - \Phi_k\theta_{k+1} = \Phi_k(\theta_k - \theta_{k+1}).$$

Defining $\alpha \stackrel{\triangle}{=} \sup_{k\geq 0} \sigma_{\max}(\Phi_k)$, where $\sigma_{\max}$ denotes the maximum singular value, it follows that

$$\|\widetilde{d}_k(\theta_{k+1})\| \leq \sigma_{\max}(\Phi_k)\|\theta_k - \theta_{k+1}\| = \alpha\|\theta_k - \theta_{k+1}\|.$$

Hence,

$$\lim_{k\to\infty} \|\widetilde{d}_k(\theta_{k+1})\| \leq \alpha \lim_{k\to\infty} \|\theta_k - \theta_{k+1}\| = 0,$$

and thus $\lim_{k\to\infty} \widetilde{d}_k(\theta_{k+1}) = 0$. $\square$

In order to analyze the retrospective performance variable decomposition, assume that $R_z = I$, and $\lambda = 1$. Then, it follows from (6.18) and (8.10) that

$$J_k(\theta_{k+1}) = \sum_{i=0}^{k} \Big(z_{\mathrm{pp},i}^{\mathrm{T}}(\theta_{i+1}) z_{\mathrm{pp},i}(\theta_{i+1}) + z_{\mathrm{mm},i}^{\mathrm{T}}(\theta_{i+1}) z_{\mathrm{mm},i}(\theta_{i+1}) + 2 z_{\mathrm{pp},i}^{\mathrm{T}}(\theta_{i+1}) z_{\mathrm{mm},i}(\theta_{i+1})$$
$$+ \theta_{i+1}^{\mathrm{T}} \Phi_i^{\mathrm{T}} R_d \Phi_i \theta_{i+1} \Big) + (\theta_{k+1} - \theta_0)^{\mathrm{T}} R_\theta (\theta_{k+1} - \theta_0). \tag{8.27}$$

Note that the first two terms in the sum are nonnegative, whereas the third term can have arbitrary sign. This suggests that RLS can minimize $J_k(\theta_{k+1})$ by making the third term negative while the nonnegative terms remain large. In the case where R_θ and R_d are small, using RLS to minimize (8.27) yields, for $k \geq k_0 \in \mathbb{R}$,

$$z_{\mathrm{rc},k}(\theta_{k+1}) \approx 0, \tag{8.28}$$

which, using (8.10), implies that

$$z_{\mathrm{pp},k}(\theta_{k+1}) \approx -z_{\mathrm{mm},k}(\theta_{k+1}). \tag{8.29}$$

Example 8.3. Consider the state space model given by (6.1), (6.2), where, for all $k \geq 0$,

$$A_k \triangleq \begin{bmatrix} 0 & 1 \\ (0.9)^{k+1} & (0.5)^{k+1} \end{bmatrix}, \quad G_k = G \triangleq \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{8.30}$$

$$C_k = C \triangleq \begin{bmatrix} 1 & 1.1 \end{bmatrix}, \quad D_{2,k} = D_2 \triangleq 0.01, \tag{8.31}$$

$u_k = w_k = 0$, v_k is standard Gaussian white noise, and $x_0 = \begin{bmatrix} 0.2 & 0.2 \end{bmatrix}^{\mathrm{T}}$. Let $n_{\mathrm{c}} = 6, n_{\mathrm{f}} = 2, \lambda = 1, R_\theta = 10^{-4} I_{13}, R_d = 10^{-6}, R_z = 1, \widetilde{V} = 10^{-2} I_2$, and let the unknown input be $d_k = 1 + \sin(0.3k)$.

Plots (a) and (b) in Figure 8.1 show that, after an initial finite number of steps, (8.28) and (8.29) hold true. Plot (c) in Figure 8.1 shows that the difference between z_{rc} and $z_{\mathrm{pp}} + z_{\mathrm{mm}}$ is negligible, and thus confirms (8.10). The convergence of $\hat{d}$, θ, and $\widetilde{d}$ is depicted in Figure 8.2. Note that, in these plots, the time step at which the RCIE algorithm is started is assumed as the 0-th step. In order to observe the steady-state behavior of the time-domain transfer functions $G_{z\overline{u}}$ and $G_{z\widetilde{d}}$ after the estimator coefficient θ converges, the magnitude plots of $G_{zd,200}$, $G_{zw,200}$, and $G_{zv,200}$ are shown in Figure 8.3, where $\begin{bmatrix} G_{zd,200} & G_{zw,200} & G_{zv,200} \end{bmatrix} = G_{z\overline{u},200}$, and the extent to which the frequency response of $G_{z\widetilde{d},200}$ matches with that of $G_{\mathrm{f},200}$ is shown in Figure 8.4.

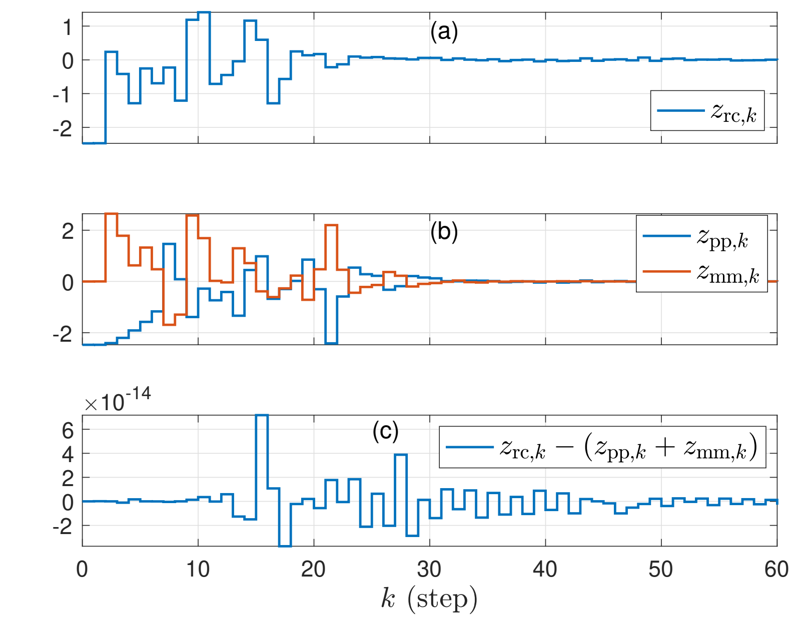

Figure 8.1: (a) For all $k \geq 21$, $z_{\mathrm{rc},k} \approx 0$, which confirms (8.28). (b) For all $k \geq 21$, $z_{\mathrm{pp},k} \approx z_{\mathrm{mm},k}$, which confirms (8.29). Furthermore, for all $k \geq 35$, $z_{\mathrm{pp},k} \approx z_{\mathrm{mm},k} \approx 0$. (c) For all $k \geq 0$, $|z_{\mathrm{rc},k} - (z_{\mathrm{pp},k} + z_{\mathrm{mm},k})| \leq 3 \times 10^{-14}$, which confirms (8.10).

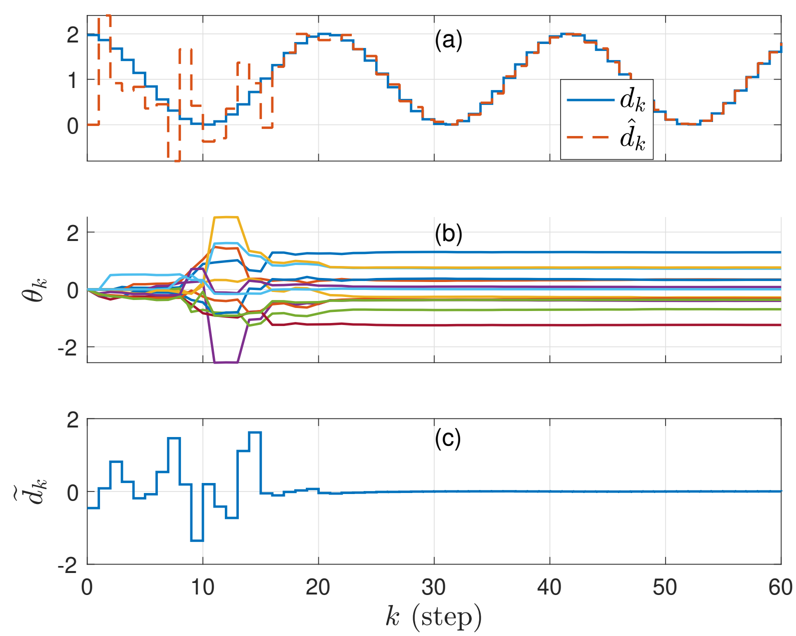

Figure 8.2: (a) After the initial transient period of about 25 steps, $\hat{d}$ follows d. (b) The estimator coefficients θ converges after about 25 steps. (c) The virtual external input perturbation $\widetilde{d}$ converges to zero after about 25 steps, in accordance with Proposition 8.2.

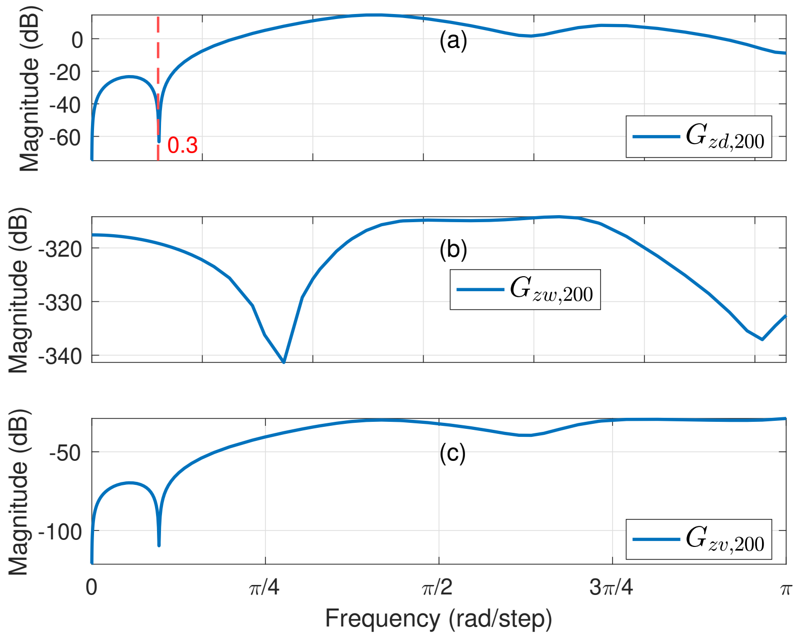

Figure 8.3: The magnitudes of $G_{zw,200}$ and $G_{zv,200}$ are close to zero at all frequencies. The magnitude of $G_{zd,200}$ at the frequencies 0 and 0.3 rad/step contained in the spectrum of the unknown input signal d is close to zero. These observations confirm that, for large values of k, $z_{\mathrm{pp},k} \approx 0$.

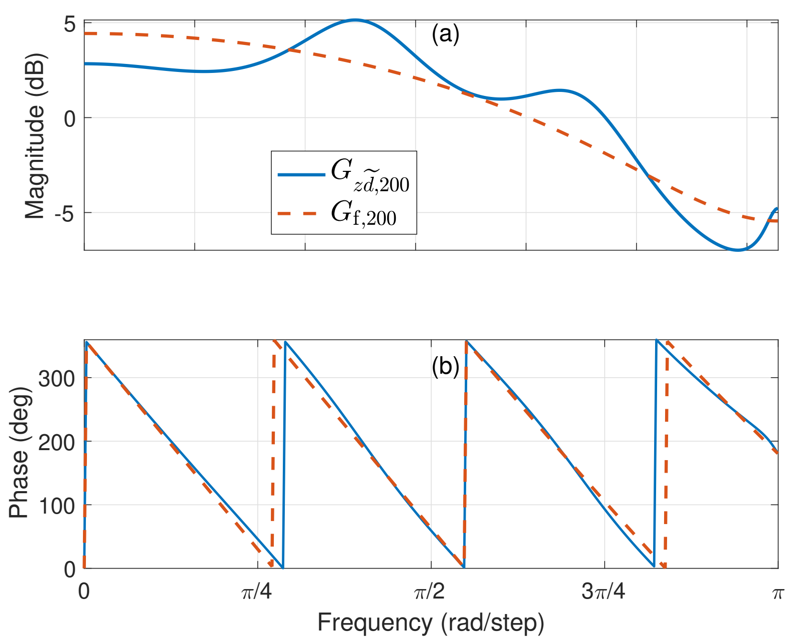

Figure 8.4: Comparison of the frequency response of $G_{z\widetilde{d},200}$ with that of $G_{\mathrm{f},200}$. The magnitude plots and the phase plots match approximately.

CHAPTER 9

Causal Numerical Differentiation

The ability to control a system is often enhanced by feeding back the derivatives of sensor signals, such as estimates of velocity and acceleration when only position is measured. Within this context, signal differentiation must be performed causally, that is, using only current and past data and with minimal computational latency. This chapter formulates causal[1] numerical differentiation as a sampled-data input-estimation problem, where the plant is a cascade of integrators. Using backward-difference differentiation (BDD) as a baseline comparison, high-gain observers (HGO) with bilinear discretization and retrospective cost input estimation are applied to harmonic signals under various noise levels for single and double differentiation. These methods are then applied to experimental position data of a small rover for estimating its velocity and acceleration. Neither method uses information about the noise statistics, and no analog or digital filtering is used for noise suppression.

9.1 Differentiation using RCIE

Since the objective is to use RCIE as a differentiator, the system given by (6.1) and (6.2) is modeled as the discrete-time equivalent of an integrator. Thus, the measured output $y(t)$ is an integral of the unknown input $d(t)$ or, in other words, the unknown input $d(t)$ is the derivative of the measured output $y(t)$. Hence, by applying RCIE and reconstructing $\hat{d}$ from the estimates $\hat{d}_k$,

[1]The estimation of the derivative of y_k uses the data y_k and hence the estimation of the derivative of y_k starts at step k. This implies that the estimate of derivative of y_k is available only at step $k+1$ and thus there is a delay of one step in the estimation.

we are estimating the derivative of the measured output y. Note that the concept of process noise is not applicable when the system is modeled as an integrator. Hence, for the rest of this chapter, it is assumed that $w = 0$, and thus $D_1 = 0$.

Consider the n-th order integrator dynamics

$$\dot{x} = A_{\mathrm{I}}x + B_{\mathrm{I}}y^{(n)}, \quad y = C_{\mathrm{I}}x, \tag{9.1}$$

$$A_{\mathrm{I}} \triangleq \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ 0 & 0_{1\times(n-1)} \end{bmatrix}, \quad B_{\mathrm{I}} \triangleq \begin{bmatrix} 0_{(n-1)\times 1} \\ 1 \end{bmatrix}, \tag{9.2}$$

$$C_{\mathrm{I}} \triangleq \begin{bmatrix} 1 & 0_{1\times(n-1)} \end{bmatrix}, \tag{9.3}$$

where $x, y \in \mathbb{R}$, and $y^{(n)}$ is the n-th derivative of y. The discretization of (9.1) using zero-hold results in the discrete-time state space model given by

$$x_{k+1} = A_{\mathrm{d}}x_k + B_{\mathrm{d}}y_k^{(n)}, \quad y_k = C_{\mathrm{I}}x_k, \tag{9.4}$$

$$A_{\mathrm{d}} \triangleq e^{A_{\mathrm{I}}T_{\mathrm{s}}}, \quad B_{\mathrm{d}} \triangleq \int_0^{T_{\mathrm{s}}} e^{A_{\mathrm{I}}(t-\tau)}B_{\mathrm{I}}d\tau, \tag{9.5}$$

where $x_k \triangleq x(kT_{\mathrm{s}})$, $y_k \triangleq y(kT_{\mathrm{s}})$, $y_k^{(n)} \triangleq y^{(n)}(kT_{\mathrm{s}})$, and T_{s} is the sampling time. Setting $A = A_{\mathrm{d}}$, $B = B_{\mathrm{d}}$, and $C = C_{\mathrm{I}}$ in (6.1) and (6.2), and applying RCIE gives an estimate ($\hat{y}^{(n)} = \hat{d}$) of $y^{(n)}$. Note that $A_{\mathrm{d}} = 1, B_{\mathrm{d}} = T_{\mathrm{s}}$, and $C_{\mathrm{I}} = 1$ in the case of single differentiation, and

$$A_{\mathrm{d}} = \begin{bmatrix} 1 & T_{\mathrm{s}} \\ 0 & 1 \end{bmatrix}, \quad B_{\mathrm{d}} = \begin{bmatrix} \frac{1}{2}T_{\mathrm{s}}^2 \\ T_{\mathrm{s}} \end{bmatrix}, \quad C_{\mathrm{I}} = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{9.6}$$

in the case of double differentiation.

9.2 Differentiation using HGO

The discrete-time implementation of casual differentiation using HGO was done in [108] and the same is explained here. A state space model for a high-gain observer designed for the system represented by (9.1) is given by

$$\dot{\hat{x}} = A_{\rm co}\hat{x} + B_{\rm co}y, \quad \hat{y} = C_{\rm o}\hat{x}, \tag{9.7}$$

$$A_{\rm co} \triangleq A_{\rm I} - HC_{\rm I}, \quad C_{\rm o} \triangleq \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} \end{bmatrix}, \tag{9.8}$$

$$B_{\rm co} = H \triangleq \begin{bmatrix} \frac{\alpha_1}{\varepsilon} & \frac{\alpha_2}{\varepsilon^2} & \cdots & \frac{\alpha_n}{\varepsilon^n} \end{bmatrix}^{\rm T}, \tag{9.9}$$

where ε is a small positive parameter, and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are constants chosen such that the polynomial

$$p(s) \triangleq s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n \tag{9.10}$$

is Hurwitz. The transfer function from y to $\hat{y}$ is given by

$$G(s) = C_{\rm o}(sI - A_{\rm I} + HC_{\rm I})^{-1}H = D_G^{-1}(s)N_G(s), \tag{9.11}$$

where

$$D_G(s) \triangleq \varepsilon^n s^n + \alpha_1\varepsilon^{n-1}s^{n-1} + \cdots + \alpha_{n-1}\varepsilon s + \alpha_n, \tag{9.12}$$

$$N_G(s) \triangleq \begin{bmatrix} \alpha_2\varepsilon^{n-2}s^{n-1} + \cdots + \alpha_{n-1}\varepsilon s^2 + \alpha_n s \\ \alpha_3\varepsilon^{n-3}s^{n-1} + \cdots + \alpha_{n-1}\varepsilon s^3 + \alpha_n s^2 \\ \vdots \\ \alpha_{n-1}\varepsilon s^{n-1} + \alpha_n s^{n-2} \\ \alpha_n s^{n-1} \end{bmatrix}. \tag{9.13}$$

Let

$$\hat{x} = \begin{bmatrix} \hat{x}_1 & \hat{x}_2 & \dots & \hat{x}_n \end{bmatrix}^{\mathrm{T}}, \tag{9.14}$$

$$\hat{y} = \begin{bmatrix} \hat{y}^{(1)} & \hat{y}^{(2)} & \dots & \hat{y}^{(n-1)} \end{bmatrix}^{\mathrm{T}}. \tag{9.15}$$

Since

$$\lim_{\varepsilon \to 0} G(s) = \begin{bmatrix} s & \cdots & s^{n-2} & s^{n-1} \end{bmatrix}^{\mathrm{T}}, \tag{9.16}$$

it follows that, for all $i = 1, \dots, n-1$, $\hat{y}^{(i)}$ is an approximation of $y^{(i)}$. Let the discrete-time observer state space model obtained by using bilinear transformation on (9.7) be

$$\hat{x}_{k+1} = A_{\mathrm{do}}\hat{x}_k + B_{\mathrm{do}}y_k, \quad \hat{y}_k = C_{\mathrm{o}}\hat{x}_k. \tag{9.17}$$

Thus, the implementation of (9.17) gives the estimates $(\hat{y}^{(1)}, \hat{y}^{(2)}, \dots, \hat{y}^{(n-1)})$ of $y^{(1)}, y^{(2)}, \dots, y^{(n-1)}$.

9.3 Differentiation using BDD

The *single derivative backward-difference differentiator* is given by

$$G_{\mathrm{sd}}(\mathrm{z}) \triangleq \frac{\mathrm{z} - 1}{T_{\mathrm{s}}\mathrm{z}}. \tag{9.18}$$

The *double derivative backward-difference differentiator* is given by

$$G_{\rm dd}({\rm z}) \triangleq \frac{({\rm z}-1)^2}{(T_{\rm s}{\rm z})^2}. \tag{9.19}$$

9.4 Numerical Examples

In this section, numerical examples are given to illustrate the accuracy of RCIE and HGO as differentiators. BDD will be used as a baseline for comparison. Note that the examples will deal with discrete-time signals only.

Example 9.1. *Differentiation in the Absence of Noise.*

In this example, it is assumed that there is no output noise, and hence $v \equiv 0$. Let the measured output be $y_k = \sin(0.2k)$.

Single Differentiation (SD)

In the case of RCIE, let $n_{\rm c} = 1$, $n_{\rm f} = 6$, $R_\theta = 10^{-3} I_3, R_d = 10^{-5}, R_z = 1$, $\widetilde{V} = 10^{-4}$. In the case of HGO, let $n = 3$ in (9.1), (9.2), and (9.3), let $\alpha_1 = 3, \alpha_2 = 3$, $\alpha_3 = 1$. Note that choosing $n = 3$ gave slightly better estimate of the first derivative as compared to choosing $n = 2$. The parameter ε is chosen as the value between 0.01 and 2 that gives the lowest root mean square error (RMSE) between the estimated values and the true values. Figure 9.1 compares the signals estimated by SD/RCIE, SD/HGO, and SD/BDD with the true first derivative.

Double Differentiation (DD)

In the case of RCIE, let $n_{\rm c} = 18$, $n_{\rm f} = 4$, $R_\theta = 10^{-1} I_{37}, R_d = 10^{-6}, R_z = 1$, $\widetilde{V} = 10^{-5}$. In the case of HGO, let $n = 4$ in (9.1), (9.2), and (9.3), let $\alpha_1 = 8, \alpha_2 = 24$, $\alpha_3 = 32$, $\alpha_4 = 16$. Note that choosing $n = 4$ gave slightly better estimate for the second derivative as compared to choosing $n = 3$. The parameter ε is chosen in the same way as chosen for single differentiation. Figure

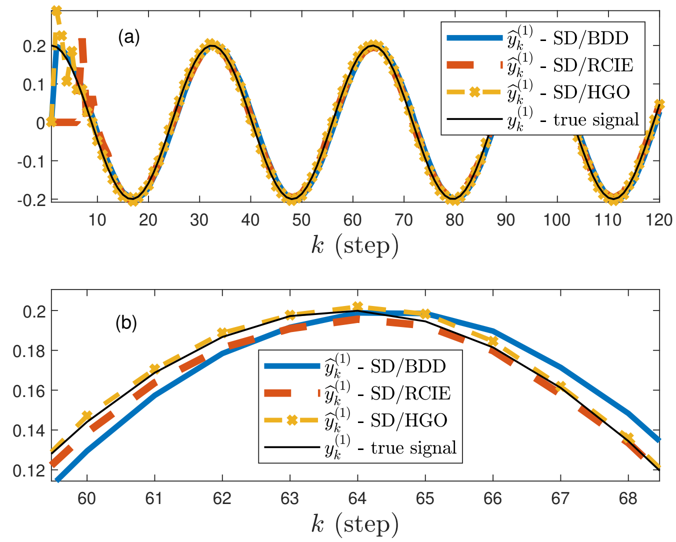

Figure 9.1: *Example 9.1 Single Differentiation.* (a) The signals estimated by SD/RCIE and SD/HGO follow the true first derivative $y^{(1)}$ after about 20 steps, whereas the signal estimated by SD/BDD follows $y^{(1)}$ without a transient period. (b) A zoomed view of plot (a). At steady state, SD/HGO is more accurate than SD/RCIE and SD/BDD.

9.2 compares the signals estimated by DD/RCIE, DD/HGO, and DD/BDD with the true second derivative.

Example 9.2. *Differentiation in the Presence of Noise.*

This example considers differentiation in the presence of output noise. Let the measured output be $y_k = \sin(0.2k) + D_2 v_k$, where v is standard Gaussian white noise.

Single Differentiation

In the case of RCIE, let $n_c = 1$, $n_f = 6$, $R_\theta = 10^{-6} I_3$, $R_d = 10^{-5}$, $R_z = 1$, $\widetilde{V} = 10^{-2}$. In the case of HGO, the parameters values are the same as they are for single differentiation in Example 9.1. For a signal-to-noise ratio (SNR) of 40 dB ($D_2 = 0.00699945$), Figure 9.3 compares the signals estimated by SD/RCIE, SD/HGO, and SD/BDD with the true first derivative.

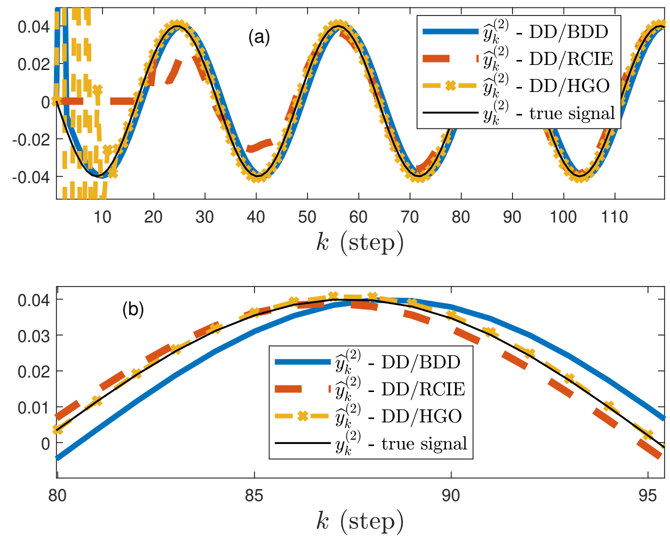

Figure 9.2: *Example 9.1 Double Differentiation.* (a) The signal estimated by DD/HGO follows the true second derivative $y^{(2)}$ after about 20 steps, the signal estimated by DD/RCIE follows $y^{(2)}$ after about 50 steps, and the signal estimated by DD/BDD follows $y^{(2)}$ without a transient period. The signal estimated by DD/HGO has large oscillations in the transient period. (b) A zoomed view of plot (a). At steady state, DD/HGO is more accurate than DD/RCIE and DD/BDD.

Double Differentiation

In the case of RCIE, let $n_{\rm c} = 18$, $n_{\rm f} = 4$, $R_\theta = 10^{-1} I_{37}$, $R_d = 10^{-6}$, $R_z = 1$, $\widetilde{V} = 10^{-5}$. In the case of HGO, the parameters values are the same as they are for double differentiation in Example 9.1. For an SNR of 40 dB, Figure 9.4 compares the signals estimated by DD/RCIE, DD/HGO, and DD/BDD with the true second derivative.

In order to do quantitative comparison among the different methods, the normalized RMSE in the estimation of the single derivative and the double derivative is plotted in Figures 9.5 and 9.6, respectively, for SNRs in the range of 40 dB to 60 dB.

Inference

Both HGO and RCIE give better estimates of the derivative than BDD. Though the performance of HGO is better than that of RCIE in the case of double differentiation, it is difficult to tune

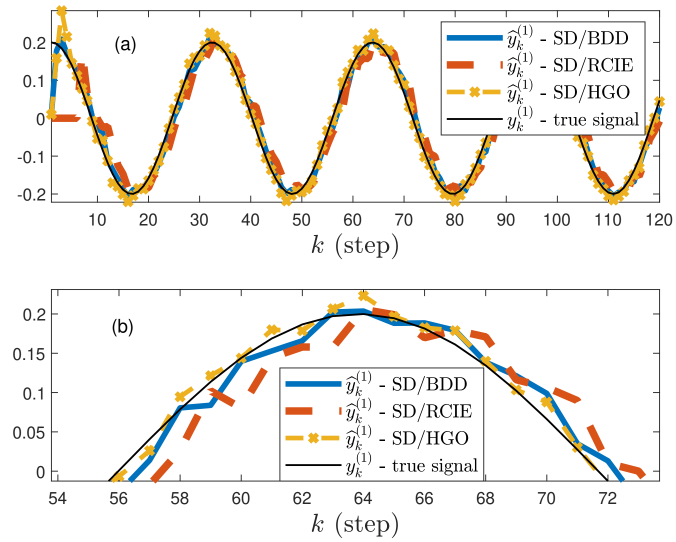

Figure 9.3: *Example 9.2 Single Differentiation.* (a) The signals estimated by SD/RCIE, SD/HGO, and SD/BDD follow the true first derivative $y^{(1)}$ after an initial transient period. SD/HGO exhibits a longer transient period as compared to SD/RCIE. (b) A zoomed view of plot (a). At steady state, SD/HGO is more accurate than SD/RCIE and SD/BDD.

the parameters of HGO and hence it is more practical to use RCIE for both single and double differentiation.

9.5 Differentiation of Experimental Data

RCIE, HGO, and BDD are applied to experimental position data of a small rover for estimating its velocity and acceleration. An OptiTrack camera sensor is used to collect the position data of the rover at a sample rate of 50 Hz. Figure 9.7 depicts the trajectory of the rover on the x-y plane and the position data along the x-axis. Differentiation of the position data along the x-axis is done to obtain the velocity and the acceleration along the x-axis. Since the true velocity and the true acceleration of the rover are not known, it is not possible to evaluate the accuracy of the estimated signals.

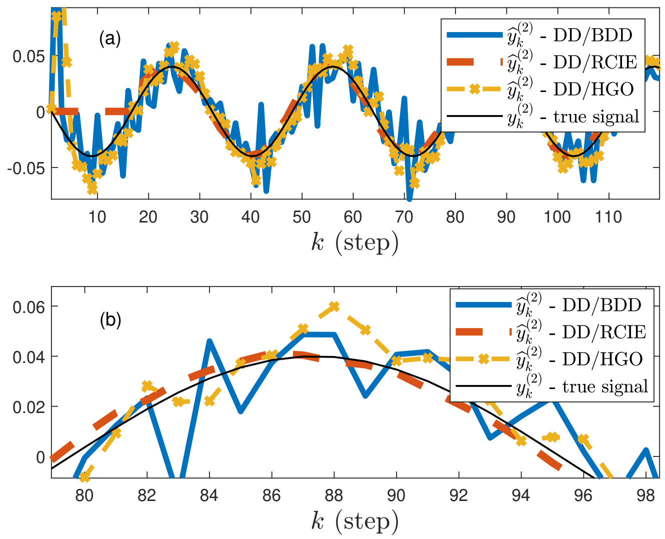

Figure 9.4: *Example 9.2 Double Differentiation.* (a) The signal estimated by DD/RCIE follows the true second derivative $y^{(2)}$ after an initial transient period. Though the signals estimated by DD/HGO and DD/BDD follow the general trend of $y^{(2)}$, they are noisy. (b) A zoomed view of plot (a). At steady state, DD/RCIE is more accurate than DD/HGO and DD/BDD.

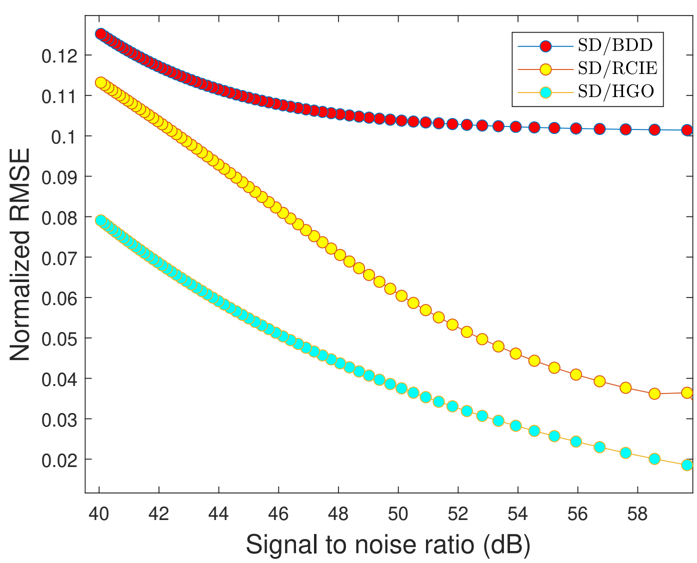

Figure 9.5: *Example 9.2. Normalized RMSE in the estimation of the first derivative.* SD/HGO performs better than SD/RCIE and SD/BDD.

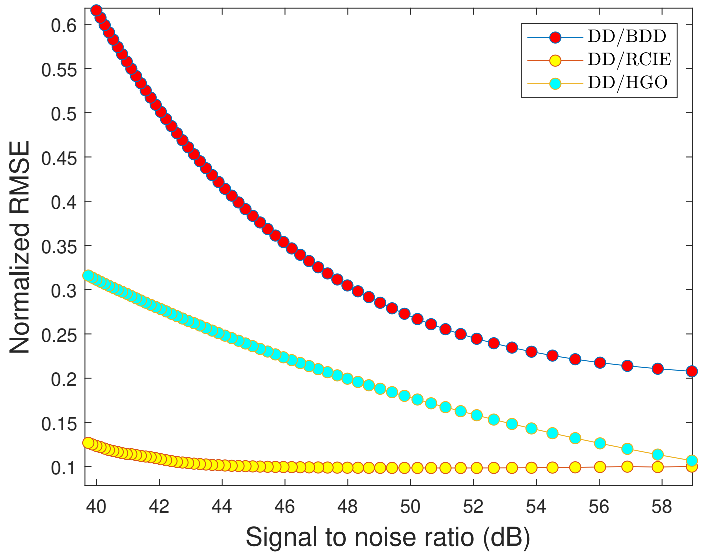

Figure 9.6: *Example 9.2. Normalized RMSE in the estimation of the second derivative.* DD/RCIE performs better than DD/HGO and DD/BDD.

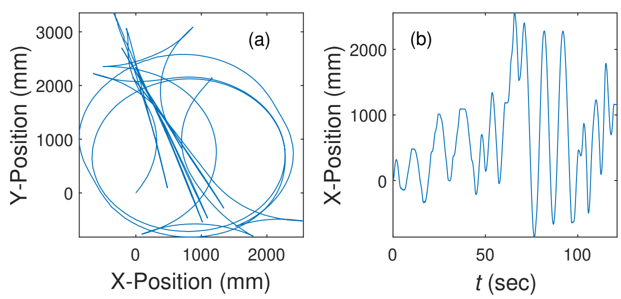

Figure 9.7: *Experimental Data.* (a) The trajectory of the rover on the x-y plane. (b) Position of the rover along x-axis versus time.

Single Differentiation

In the case of RCIE, let $n_{\rm c} = 1$, $n_{\rm f} = 6$, $R_\theta = 10^{-3} I_3$, $R_d = 10^{-5}$, $R_z = 1$, $\widetilde{V} = 10^{-4}$. In the case of HGO, let $n = 3$ in (9.1), (9.2), and (9.3), let $\alpha_1 = 3$, $\alpha_2 = 3$, $\alpha_3 = 1$. The parameter ε is given an optimum value that renders the estimated signal smooth and follow the general trend of the signals estimated by RCIE and BDD. Figure 9.8 compares the signals estimated by SD/RCIE, SD/HGO, and SD/BDD.

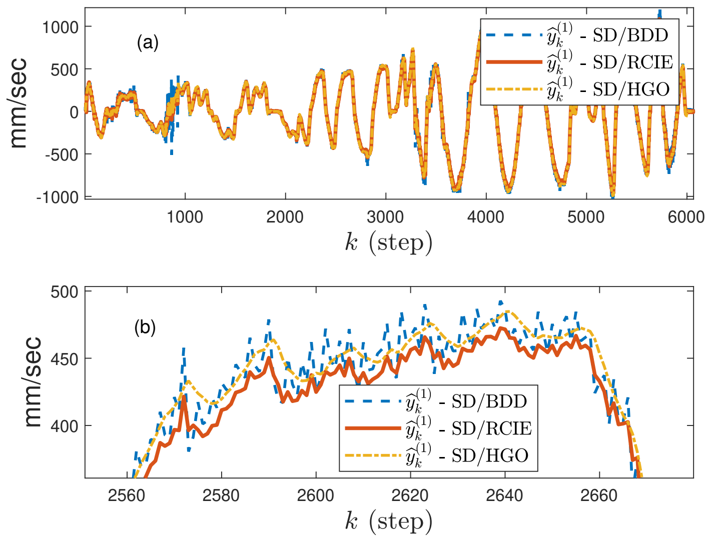

Figure 9.8: *Single Differentiation of Experimental Data.* (a) shows the signals estimated by SD/RCIE, SD/HGO, and SD/BDD. (b) A zoomed view of plot (a). The signal estimated by SD/BDD is noisy, whereas the signals estimated by SD/RCIE and SD/HGO are reasonably smooth.

Double Differentiation

In the case of RCIE, let $n_{\rm c} = 18$, $n_{\rm f} = 4$, $R_\theta = 10^{-1} I_{37}$, $R_d = 10^{-6}$, $R_z = 1$, $\widetilde{V} = 10^{-5}$. In the case of HGO, let $n = 4$ in (9.1), (9.2), and (9.3), let $\alpha_1 = 8$, $\alpha_2 = 24$, $\alpha_3 = 32$, $\alpha_4 = 16$. The parameter ε is chosen in the same way as chosen for single differentiation. Figure 9.9 compares the signals estimated by DD/RCIE, DD/HGO, and DD/BDD.

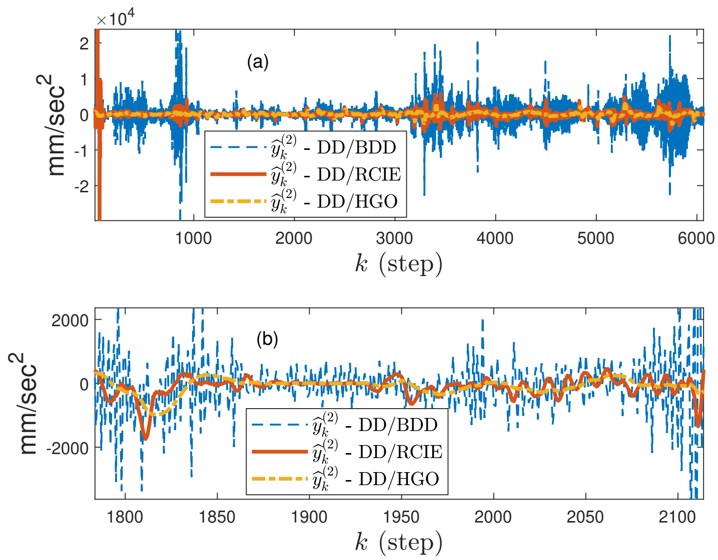

Figure 9.9: *Double Differentiation of Experimental Data.* (a) shows the signals estimated by DD/RCIE, DD/HGO, and DD/BDD. (b) A zoomed view of plot (a). The signal estimated by DD/BDD is noisy, whereas the signals estimated by DD/RCIE and DD/HGO are reasonably smooth.

CHAPTER 10

Decomposition of Retrospective Performance Variable in RCAC

This chapter presents the decomposition of retrospective performance variable within the context of RCAC. The same approach that was used for the decomposition of retrospective performance variable in RCIE in chapter 8 is used here.

10.1 Retrospective Cost Adaptive Control

Retrospective Cost Adaptive Control (RCAC) [76] is a direct, discrete-time adaptive control technique for stabilization, command following, and disturbance rejection. As a discrete-time approach, RCAC is motivated by the desire to implement control algorithms that operate at a fixed measurement sampling rate without the need for controller discretization. This discretization also means that the required modeling information can be estimated based on data sampled at the same rate as the control update. RCAC was motivated by the concept of retrospectively optimized control, where past controller coefficients used to generate past control inputs are reoptimized in the sense that, if the reoptimized coefficients had been used over a previous window of operation, then the performance would have been better. However, unlike signal processing applications such as estimation and identification, it is impossible to change past control inputs, and thus the reoptimized controller coefficients are used only to generate the next control input.

Consider the LTV discrete-time system

$$x_{k+1} = A_k x_k + B_k u_k + B_{w,k} w_k, \tag{10.1}$$

$$y_k = C_k x_k + v_k, \tag{10.2}$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^m$ is the control, $w_k \in \mathbb{R}^l$ is the disturbance, $y_k \in \mathbb{R}^p$ is the measured output, and $v_k \in \mathbb{R}^p$ is the sensor noise. Define the command-following error

$$z_k \stackrel{\triangle}{=} r_k - y_k, \tag{10.3}$$

where $r_k \in \mathbb{R}^p$ is the command signal. Consider the strictly proper, discrete-time dynamic compensator

$$u_k = \sum_{i=1}^{n_\mathrm{c}} P_{i,k} u_{k-i} + \sum_{i=1}^{n_\mathrm{c}} Q_{i,k} z_{k-i}, \tag{10.4}$$

where $k \geq 0$, $u_k \in \mathbb{R}^m$ is the requested control, n_c is the controller window length, and $Q_{1,k}, \ldots, Q_{n_\mathrm{c},k} \in \mathbb{R}^{m\times p}$ and $P_{1,k}, \ldots, P_{n_\mathrm{c},k} \in \mathbb{R}^{m\times m}$ are the numerator and denominator controller coefficient matrices, respectively. For convenience, a "cold" startup is assumed, where $Q_{1,0}, \ldots, Q_{n_\mathrm{c},0}$, $P_{1,0}, \ldots, P_{n_\mathrm{c},0}$, $u_{-n_\mathrm{c}}, \ldots, u_{-1}$, and $z_{-n_\mathrm{c}}, \ldots, z_{-1}$ are defined to be zero, and thus $u_0 = 0$. The controller (10.4) can be written as

$$u_k = \phi_{\mathrm{c},k} \theta_{\mathrm{c},k}, \tag{10.5}$$

where

$$\phi_{\mathrm{c},k} \triangleq \begin{bmatrix} u_{k-1} \\ \vdots \\ u_{k-n_\mathrm{c}} \\ z_{k-1} \\ \vdots \\ z_{k-n_\mathrm{c}} \end{bmatrix}^{\mathrm{T}} \otimes I_m \in \mathbb{R}^{m \times l_{\theta_\mathrm{c}}}, \tag{10.6}$$

is the *controller regressor*, $l_{\theta_\mathrm{c}} \triangleq n_\mathrm{c} m(m+p)$, and the *controller coefficient vector* is defined by

$$\theta_{\mathrm{c},k} \triangleq \mathrm{vec} \begin{bmatrix} P_{1,k} & \cdots & P_{n_\mathrm{c},k} & Q_{1,k} & \cdots & Q_{n_\mathrm{c},k} \end{bmatrix} \in \mathbb{R}^{l_{\theta_\mathrm{c}}}. \tag{10.7}$$

In terms of $\mathbf{q}^{-1}$, the controller (10.4) can be expressed as

$$u_k = G_{\mathrm{c},k}(\mathbf{q}^{-1}) z_k, \tag{10.8}$$

where

$$G_{\mathrm{c},k} \triangleq D_{\mathrm{c},k}^{-1} N_{\mathrm{c},k}, \tag{10.9}$$

$$N_{\mathrm{c},k}(\mathbf{q}^{-1}) \triangleq Q_{1,k}\mathbf{q}^{-1} + \cdots + Q_{n_\mathrm{c},k}\mathbf{q}^{-n_\mathrm{c}}, \tag{10.10}$$

$$D_{\mathrm{c},k}(\mathbf{q}^{-1}) \triangleq I_m - P_{1,k}\mathbf{q}^{-1} - \cdots - P_{n_\mathrm{c},k}\mathbf{q}^{-n_\mathrm{c}}. \tag{10.11}$$

Next, define the filtered signals

$$u_{\mathrm{f},k} \triangleq G_{\mathrm{f},k}(\mathbf{q}^{-1}) u_k, \tag{10.12}$$

$$\phi_{\mathrm{f},k} \triangleq G_{\mathrm{f},k}(\mathbf{q}^{-1}) \phi_{\mathrm{c},k}, \tag{10.13}$$

where, for startup, $u_{\mathrm{f},k}$ and $\phi_{\mathrm{f},k}$ are initialized at zero. The $p \times m$ filter $G_{\mathrm{f},k}$ has the form

$$G_{\mathrm{f},k} \stackrel{\triangle}{=} D_{\mathrm{f},k}^{-1} N_{\mathrm{f},k}, \tag{10.14}$$

where

$$N_{\mathrm{f},k}(\mathbf{q}^{-1}) \stackrel{\triangle}{=} N_{0,k} + N_{1,k}\mathbf{q}^{-1} + \cdots + N_{n_\mathrm{f},k}\mathbf{q}^{-n_\mathrm{f}}, \tag{10.15}$$

$$D_{\mathrm{f},k}(\mathbf{q}^{-1}) \stackrel{\triangle}{=} I_q + D_{1,k}\mathbf{q}^{-1} + \cdots + D_{n_\mathrm{f},k}\mathbf{q}^{-n_\mathrm{f}}, \tag{10.16}$$

n_f is the filter window length, and $N_{0,k}, \ldots, N_{n_\mathrm{f},k} \in \mathbb{R}^{p\times m}$ and $D_{1,k}, \ldots, D_{n_\mathrm{f},k} \in \mathbb{R}^{p\times p}$ are the numerator and denominator coefficients of $G_{\mathrm{f},k}$, respectively.

Next, in order to update the controller coefficient vector (10.7), define the retrospective performance variable

$$z_{\mathrm{rp},k}(\theta_\mathrm{c}) \stackrel{\triangle}{=} z_k - (u_{\mathrm{f},k} - \phi_{\mathrm{f},k}\theta_\mathrm{c}), \tag{10.17}$$

where θ_c is a generic variable for optimization. Note that $u_{\mathrm{f},k}$ depends on u_k and thus on the current controller coefficient vector $\theta_{\mathrm{c},k}$. The retrospective performance variable $z_{\mathrm{rp},k}(\theta_\mathrm{c})$ is used to determine the updated controller coefficient vector $\theta_{\mathrm{c},k+1}$ by minimizing a function of $z_{\mathrm{rp},k}$. The optimized value of $z_{\mathrm{rp},k}$ is thus given by

$$z_{\mathrm{rp},k}(\theta_{\mathrm{c},k+1}) = z_k - (u_{\mathrm{f},k} - \phi_{\mathrm{f},k}\theta_{\mathrm{c},k+1}), \tag{10.18}$$

which shows that the updated controller coefficient vector $\theta_{\mathrm{c},k+1}$ is "applied" retrospectively with the filtered controller regressor $\phi_{\mathrm{f},k}$. Furthermore, note that the filter $G_{\mathrm{f},k}$ is used to obtain $\phi_{\mathrm{f},k}$ from ϕ_k by means of (10.13) but ignores past changes in the controller coefficient vector, as can be seen by the product $\phi_{\mathrm{f},k}\theta_{\mathrm{c},k+1}$ in (10.18). Consequently, the filtering used to construct (10.18) ignores changes in the controller coefficient vector over the window $[k - n_\mathrm{f}, k]$.

Define the retrospective cost function

$$J_k(\theta_{\rm c}) \stackrel{\triangle}{=} \sum_{i=0}^{k} z_{{\rm rp},i}(\theta_{\rm c})^{\rm T} z_{{\rm rp},i}(\theta_{\rm c}) + (\theta_{\rm c} - \theta_{{\rm c},0})^{\rm T} P_{{\rm c},0}^{-1}(\theta_{\rm c} - \theta_{{\rm c},0}), \tag{10.19}$$

where $P_{{\rm c},0} \in \mathbb{R}^{l_{\theta_{\rm c}} \times l_{\theta_{\rm c}}}$ is positive definite. For all $k \geq 0$, the minimizer $\theta_{{\rm c},k+1}$ of (10.19) is given by the recursive least squares (RLS) solution [126]

$$P_{{\rm c},k+1} = P_{{\rm c},k} - P_{{\rm c},k}\phi_{{\rm f},k}^{\rm T}(I_p + \phi_{{\rm f},k}P_{{\rm c},k}\phi_{{\rm f},k}^{\rm T})^{-1}\phi_{{\rm f},k}P_{{\rm c},k}, \tag{10.20}$$

$$\theta_{{\rm c},k+1} = \theta_{{\rm c},k} + P_{{\rm c},k+1}\phi_{{\rm f},k}^{\rm T}(z_k - u_{{\rm f},k} - \phi_{{\rm f},k}\theta_{{\rm c},k}). \tag{10.21}$$

Using the updated controller coefficient vector given by (10.21), the requested control at step $k+1$ is given by replacing k by $k+1$ in (10.5). Although $\theta_{{\rm c},0}$ can be chosen arbitrarily, we chose $\theta_{{\rm c},0} = 0$ in order to reflect the absence of additional modeling information. Note that $P_{{\rm c},0}$ is a tuning parameter.

10.2 Decomposition of the Retrospective Performance Variable

Define the *virtual external input perturbation* for RCAC as

$$\widetilde{u}_k(\theta_{\rm c}) \stackrel{\triangle}{=} u_k - \phi_{{\rm c},k}\theta_{\rm c}. \tag{10.22}$$

Let $\widetilde{u}_{{\rm f},k}(\theta_{{\rm c},k+1})$ be given by the FIA filter

$$\widetilde{u}_{{\rm f},k}(\theta_{{\rm c},k+1}) \stackrel{\triangle}{=} G_{{\rm f},k}(\mathbf{q}^{-1})\widetilde{u}_k(\theta_{{\rm c},\overline{k+1}}) \tag{10.23}$$

Note that $\widetilde{u}_{\mathrm{f},k}(\theta_{\mathrm{c},k+1})$ ignores the change in the argument $\theta_{\mathrm{c},k+1}$ of $\widetilde{u}_k$ over the interval $[k-n_\mathrm{f},k]$ in accordance with retrospective optimization. Using (10.23), (10.18) can be written as

$$\hat{z}_{\mathrm{rp},k}(\theta_{\mathrm{c},k+1}) \triangleq z_k - \widetilde{u}_{\mathrm{f},k}(\theta_{\mathrm{c},k+1}). \tag{10.24}$$

The following matrices are used in Theorem 10.1.

$$\widetilde{A}_k \triangleq \begin{bmatrix} 0 & \cdots & 0 & P_{n_\mathrm{c},k+n_\mathrm{c}+1} \\ I & \cdots & 0 & P_{n_\mathrm{c}-1,k+n_\mathrm{c}} \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & I & P_{1,k+2} \end{bmatrix}, \quad B_{\widetilde{u}} \triangleq \begin{bmatrix} I_m \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{mn_\mathrm{c}\times m}, \tag{10.25}$$

$$\widetilde{B}_k \triangleq \begin{bmatrix} Q_{n_\mathrm{c},k+n_\mathrm{c}+1} \\ Q_{n_\mathrm{c}-1,k+n_\mathrm{c}} \\ \vdots \\ Q_{1,k+2} \end{bmatrix}, \quad \widetilde{C} \triangleq \begin{bmatrix} 0 & \cdots & 0 & I_{n_\mathrm{c}} \end{bmatrix} \in \mathbb{R}^{m\times mn_\mathrm{c}}, \tag{10.26}$$

$$\hat{A}_k \triangleq \begin{bmatrix} A_k & B_k\widetilde{C} \\ -\widetilde{B}_kC_k & \widetilde{A}_k \end{bmatrix}, \quad \hat{B}_k \triangleq \begin{bmatrix} 0 & B_{w,k} & 0 \\ B_{\widetilde{u}} & 0 & \widetilde{B}_k \end{bmatrix}, \tag{10.27}$$

$$\hat{C}_k \triangleq \begin{bmatrix} -C_k & 0_{p\times mn_\mathrm{c}} \end{bmatrix}, \quad \hat{D} \triangleq \begin{bmatrix} 0 & 0 & I_{l_y} \end{bmatrix}. \tag{10.28}$$

The following result presents the *retrospective performance variable decomposition*, which shows that the retrospective performance variable is a combination of the closed-loop performance and the extent to which the updated closed-loop transfer function from $\widetilde{u}_k(\theta_{k+1})$ to z_k matches the filter $G_{\mathrm{f},k}$. Henceforth, $G_{\mathrm{f},k}$ is called the *target model* since it serves as the target for the closed-loop transfer function from $\widetilde{u}_k(\theta_{k+1})$ to z_k.

Theorem 10.1. For all $k \geq 0$,

$$z_{\mathrm{rp},k}(\theta_{\mathrm{c},k+1}) = z_{\mathrm{pp},k}(\theta_{\mathrm{c},k+1}) + z_{\mathrm{mm},k}(\theta_{\mathrm{c},k+1}), \tag{10.29}$$

where the performance term $z_{\mathrm{pp},k}(\theta_{\mathrm{c},k+1})$ and the model-matching term $z_{\mathrm{mm},k}(\theta_{\mathrm{c},k+1})$ are defined as

$$z_{\mathrm{pp},k}(\theta_{\mathrm{c},k+1}) \triangleq G_{z\overline{u},k}(\mathbf{q}^{-1})\overline{u}_k, \tag{10.30}$$

$$z_{\mathrm{mm},k}(\theta_{\mathrm{c},k+1}) \triangleq G_{z\widetilde{u},k}(\mathbf{q}^{-1})\widetilde{u}_k(\theta_{\mathrm{c},k+1}) - G_{\mathrm{f},k}(\mathbf{q}^{-1})\widetilde{u}_k(\theta_{\mathrm{c},\overline{k+1}}), \tag{10.31}$$

and $\overline{u}_k \triangleq \begin{bmatrix} w_k^{\mathrm{T}} & (r_k - v_k)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$. The time-domain transfer functions $G_{z\overline{u},k} \in \mathbb{R}^{p\times(l+p)}$ and $G_{z\tilde{u},k} \in \mathbb{R}^{p\times m}$ are given by

$$\begin{bmatrix} G_{z\widetilde{u},k} & G_{z\overline{u},k} \end{bmatrix} \triangleq G_{z\hat{u},k}, \tag{10.32}$$

where $G_{z\hat{u},k}$ is the time-domain transfer function of the system represented by the state space model

$$\hat{x}_{k+1} = \hat{A}_k\hat{x}_k + \hat{B}_k\hat{u}_k, \tag{10.33}$$

$$z_k = \hat{C}_k\hat{x}_k + \hat{D}\hat{u}_k, \tag{10.34}$$

$\hat{u}_k \triangleq \begin{bmatrix} \widetilde{u}_k^{\mathrm{T}}(\theta_{\mathrm{c},k+1}) & \overline{u}_k^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, $\hat{x}_0 \triangleq \begin{bmatrix} x_0^{\mathrm{T}} & 0_{1\times n_{\mathrm{c}}m} \end{bmatrix}^{\mathrm{T}}$, and $\hat{A}_k$, $\hat{B}_k$, $\hat{C}_k$, and $\hat{D}$ are defined in (10.27) and (10.28).

Proof. Note that (10.4) implies that

$$\phi_k\theta_{\mathrm{c},k+1} = \sum_{i=1}^{n_{\mathrm{c}}} P_{i,k+1}u_{k-i} + \sum_{i=1}^{n_{\mathrm{c}}} Q_{i,k+1}z_{k-i}. \tag{10.35}$$

Substituting $\theta_{\mathrm{c}} = \theta_{\mathrm{c},k+1}$ and (10.35) in (10.22) yields

$$u_k = \widetilde{u}_k(\theta_{\mathrm{c},k+1}) + \sum_{i=1}^{n_{\mathrm{c}}} P_{i,k+1}u_{k-i} + \sum_{i=1}^{n_{\mathrm{c}}} Q_{i,k+1}z_{k-i}. \tag{10.36}$$

Using (10.10) and (10.11), it follows from (10.36) that

$$u_k = \widetilde{u}_k(\theta_{\mathrm{c},k+1}) + u_k - D_{c,k+1}(\mathbf{q}^{-1})u_k + N_{c,k+1}(\mathbf{q}^{-1})z_k,$$

which, using (10.9), can be rewritten as

$$u_k = D_{c,k+1}^{-1}(\mathbf{q}^{-1})\widetilde{u}_k(\theta_{\mathrm{c},k+1}) + G_{c,k+1}(\mathbf{q}^{-1})z_k. \tag{10.37}$$

Note that (10.10), (10.11), and Proposition 7.2 imply that a state space model corresponding to (10.37) is given by

$$\widetilde{x}_{k+1} = \widetilde{A}_k\widetilde{x}_k + B_{\widetilde{u}}\widetilde{u}_k(\theta_{\mathrm{c},k+1}) + \widetilde{B}_k z_k, \tag{10.38}$$

$$u_k = \widetilde{C}\widetilde{x}_k, \tag{10.39}$$

where $\widetilde{A}_k$, $B_{\widetilde{u}}$, $\widetilde{B}_k$, and $\widetilde{C}$ are defined in (10.25) and (10.26), and $\widetilde{x}_0 \triangleq 0_{n_c m\times 1}$. Next, substituting (10.2) in (10.3) yields

$$z_k = -C_k x_k - v_k + r_k = -C_k x_k + \begin{bmatrix} 0 & I_{l_y} \end{bmatrix} \overline{u}_k. \tag{10.40}$$

Furthermore, substituting (10.39) in (10.1) yields

$$x_{k+1} = A_k x_k + B_k\widetilde{C}\widetilde{x}_k + B_{w,k}w_k = A_k x_k + B_k\widetilde{C}\widetilde{x}_k + \begin{bmatrix} B_{w,k} & 0 \end{bmatrix} \overline{u}_k. \tag{10.41}$$

and substituting (10.40) in (10.38) yields

$$\begin{aligned}\widetilde{x}_{k+1} &= -\widetilde{B}_k C_k x_k + \widetilde{A}_k\widetilde{x}_k + B_{\widetilde{u}}\widetilde{u}_k(\theta_{\mathrm{c},k+1}) - \widetilde{B}_k v_k + \widetilde{B}_k r_k \\ &= -\widetilde{B}_k C_k x_k + \widetilde{A}_k\widetilde{x}_k + B_{\widetilde{u}}\widetilde{u}_k(\theta_{\mathrm{c},k+1}) + \begin{bmatrix} 0 & \widetilde{B}_k \end{bmatrix} \overline{u}_k. \end{aligned} \tag{10.42}$$

Define $\hat{x}_k \overset{\triangle}{=} \begin{bmatrix} x_k^{\mathrm{T}} & \widetilde{x}_k^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$. Then, (10.33) and (10.34) follow from (10.40), (10.41), and (10.42). Since $G_{z\widetilde{u},k}$ is the time-domain transfer function of the system represented by (10.33) and (10.34), it follows from (10.32) that

$$z_k = G_{z\overline{u},k}(\mathbf{q}^{-1})\overline{u}_k + G_{z\widetilde{u},k}(\mathbf{q}^{-1})\widetilde{u}_k(\theta_{\mathrm{c},k+1}). \tag{10.43}$$

Finally, substituting (10.43) in (10.24) yields (10.29). $\square$

Note that the expression for $G_{z\hat{u},k}$ is obtained using (10.27) and (10.28) in accordance with Definition 2.20 and Proposition 7.1. In order to apply Proposition 7.1, (10.33) and (10.34) must be converted to a completely observable state space model. The time-varying Eigensystem Realization Algorithm explained in Section IV of [127] provides a method to reduce any given LTV state space model to a minimal state space model.

Proposition 10.2. Assume that $\lim_{k\to\infty} \theta_{\mathrm{c},k}$ exists and $\phi_{\mathrm{c},k}$ is bounded. Then $\lim_{k\to\infty} \widetilde{u}_k(\theta_{\mathrm{c},k+1}) = 0.$

Proof. Equations (6.6) and (10.22) imply that

$$\widetilde{u}_k(\theta_{\mathrm{c},k+1}) = u_k - \phi_{\mathrm{c},k}\theta_{\mathrm{c},k+1} = \phi_{\mathrm{c},k}(\theta_{\mathrm{c},k} - \theta_{\mathrm{c},k+1}).$$

Defining $\alpha \overset{\triangle}{=} \sup_{k\geq 0} \sigma_{\max}(\phi_{\mathrm{c},k})$, where $\sigma_{\max}$ denotes the maximum singular value, it follows that

$$\|\widetilde{u}_k(\theta_{k+1})\| \leq \sigma_{\max}(\phi_{\mathrm{c},k})\|\theta_{\mathrm{c},k} - \theta_{\mathrm{c},k+1}\| = \alpha\|\theta_{\mathrm{c},k} - \theta_{\mathrm{c},k+1}\|.$$

Hence,

$$\lim_{k\to\infty} \|\widetilde{u}_k(\theta_{\mathrm{c},k+1})\| \leq \alpha \lim_{k\to\infty} \|\theta_{\mathrm{c},k} - \theta_{\mathrm{c},k+1}\| = 0,$$

and thus $\lim_{k\to\infty} \widetilde{u}_k(\theta_{\mathrm{c},k+1}) = 0.$ $\square$

CHAPTER 11

Conclusions and Future Work

11.1 Conclusions

Finite-time input estimation for discrete-time linear time-invariant (LTI) systems with zero nonzero zeros and unknown initial conditions was considered. Necessary and sufficient conditions for finite-time input estimation were derived. For systems with zero nonzero zeros, a specific construction of a finite-impulse-response (FIR) delayed left inverse with minimal delay using the Smith-McMillan form at infinity was given. Expressions for the number of transmission zeros and the number of infinite zeros in terms of the defect of a block-Toeplitz matrix of Markov parameters and the observability matrix were obtained. Furthermore, several results on the zero dynamics of input-output models were derived. Output zeroing in input-output models was considered, and its equivalence to output zeroing in state space models was established.

The decomposition of the retrospective performance variable in RCIE into the sum of a performance term and a model-matching term was presented. Construction of LTV state space realizations for LTV input-output models as well as the construction of LTV input-output models for LTV state space models were given and used to derive a decomposition of the retrospective performance variable. Similarly, the decomposition of the retrospective performance variable in RCAC was also presented.

Finally RCIE was applied to causal numerical differentiation. The performance of RCIE and HGO as causal differentiators was analyzed through numerical simulations. Both methods were then applied to the position data of a small rover to estimate its velocity and acceleration.

11.2 Future Work

Finite-time input estimation

Future work includes analysis of the robustness of the finite-time input estimation results to parameter uncertainties and noise. Furthermore, the work on zero dynamics of input-output models might help extend FIR delayed left inverses to nonlinear systems.

System Zeros

Future research will focus on numerically estimating the numbers of infinite zeros and transmission zeros in the presence of noisy data. In particular, by applying the singular value decomposition and nuclear norm minimization [128, 129] to the matrices Ψ and $\mathcal{T}$ obtained from subspace identification [130], it may be possible to estimate the number of zeros. The application of these results to improving the accuracy of the computation of zeros using standard methods [131] is another promising topic for future work.

Decomposition of the Retrospective Performance Variable

Observations from the analysis of the decomposition of the retrospective performance variable suggest that the closed-loop performance and model-matching terms might be helpful in determining the convergence of RCIE. A complete convergence analysis of RCIE, both deterministic and stochastic, is an interesting future research work.

Causal Numerical Differentiation

Future research will focus on the development of metrics that can be used to determine the relative accuracy of the RCIE estimates. Extension to include adaptation of bias and variance of sensor noise to address the situation where the sensor noise is unknown and possibly time-dependent is another research direction to be considered.

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