SYMMETRIC NASH EQUILIBRIUM IN THE CONDORCET CYCLE

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ABSTRACT. Symmetric Nash equilibrium in strategical voting may reveal voters' cardinal information, and voting rules that encourage strategical voting may still be desirable. This paper provides such an example. It shows that the symmetric Nash equilibrium of the extension of the majority voting rule on the preference domain with only a Condorcet cycle depends on the cardinal information of voters' utility functions. By comparing the result of this voting rule with that of the random selection and the best total utility, the paper shows that this voting rule will always be better than a random selection. Voters' manipulation reveals information about their utility function. This information suggests a range for the best possible utility. The utility resulting from the extension of the majority voting rule is always within this range.

SECTION 1. INTRODUCTION

This paper provides an example of a voting rule which Symmetric Nash equilibrium in strategical voting reveals voters' cardinal information. Though this voting rule encourages strategical voting, it is still consequentially desirable.

A voting rule may be evaluated either procedurally or consequentially. In other words, we may evaluate a voting rule by assessing whether its *process* is fair (the proceduralist approach) or whether it *results* in good outcomes (the consequentialist approach).

Both the consequentialist and proceduralist approaches have been well studied in the game-theoretic literature. Under the proceduralist approach, whether a voting rule is fair or not is determined by whether it suffices "a set of ideals with which any collective decision-making procedure ought to comply"[1] or not. One of such widely discussed ideals is strategyproofness. The strategyproofness axiom requires voting rules to induce voters to vote according to their true preferences, and to prevent them from voting strategically. In the paper by Dasgupta and Maskin (2020),[2] they prove that for

1. Jules Coleman and John Ferejohn, "Democracy and Social Choice," *Ethics* 97, no. 1 (1986): 6–25, ISSN: 00141704, 1539297X, http://www.jstor.org/stable/2381403.

2. Partha Dasgupta and Eric Maskin, "Strategy-Proofness, Independence of Irrelevant Alternatives, and Majority Rule," 2020, eprint: AmericanEconomicReview:Insights.Forthcoming;2(4):1-16.

any set of voters whose domain of true preferences is without Condorcet cycles, the extension of the majority voting rule is strategyproof. In Gibbard and Satterthwaite's paper, they prove that in any non-dictatory deterministic ordinal system with more than two candidates, a given voting rule will always be susceptible to strategic voting.[34]

Under the consequentialist approach, ideal good outcomes may be the objectively better outcomes, or if there are no objectively better outcomes, some good, compromised outcomes. If the objectively better outcomes exist, the purpose of a voting procedure will be to collect information with the goal of achieving a common goal. If there aren't such outcomes, the purpose of the voting rule will be to reconcile voters' conflicting interests with the goal of achieving some good compromises. The Condorcet Jury Theorem is one of the crucial claims under the consequentialist approach that focuses on information aggregation. The theorem assumes a set of voters (jury), every one of whom chooses between a correct and an incorrect candidate. The winner under this voting rule are they who are preferred by a majority. The Condorcet Jury Theorem states that if the probability of each voter choosing the correct outcome is greater than 50%, adding more voters increases the probability that the majority will choose correctly. Also, as the number of voters increases, the probability of a correct decision approaches 1. On the other hand, if the probability of an individual choosing correctly is less than 50%, adding more voters reduces the probability of a correct choice by the majority, and the probability of a correct decision is maximized for a jury of size one.[5]

One of the interesting discussions that link the proceduralist and the consequentialist approaches is the trade-off between strategyproofness and Pareto efficiency. In this trade-off, it is assumed that only the better outcomes under the consequentialist approach are considered Pareto efficient. In Lin Zhou's paper, they prove that symmetric mechanisms can't have both

3. Allan Gibbard, "Manipulation of Voting Schemes: A General Result," *Econometrica* 41, no. 4 (1973): 587–601, ISSN: 00129682, 14680262, http://www.jstor.org/stable/1914083.

4. Mark Allen Satterthwaite, "Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions," *Journal of Economic Theory* 10, no. 2 (1975): 187–217, ISSN: 0022-0531, https://doi.org/https://doi.org/10.1016/0022-0531(75)90050-2, https://www.sciencedirect.com/science/article/pii/0022053175900502.

5. Marquis de Condorcet, "Essay on the Application of Analysis to the Probability of Majority Decisions," *Paris: Imprimerie Royale*, 1785,

strategy-proofness and Pareto efficiency.[6] One such example is given by Abdulkadiroğlu, Che, and Yasuda in their paper on school choice.[7] Their paper compares the non-strategy-proof mechanism—the Boston mechanism—with the strategy-proof mechanism—the student-proposing deferred acceptance (henceforth DA) mechanism. The paper points out that there is a clear welfare loss associated with the DA relative to the Boston mechanism, which loss can be seen as the 'price' paid for achieving strategy-proofness.

The trade-off between Pareto efficiency and strategy-proofness questions our obsession with strategy-proofness. This paper takes the consequentialist approach and studies the extension of the majority voting rule proposed in Dasgupta and Maskin's paper. Yet, instead of looking at the voting rule on a preference domain without the Condorcet cycle, this paper focuses on voters' behavior in a domain where the mechanism fails strategyproofness—the domain with only the Condorcet cycle.

The following paper will first formalize the model of the extension of the model of the majority voting rule on the preference domain with only a Condorcet cycle. After quantifying and standardizing voters' utility, the paper will present the symmetric Nash equilibrium of the game. By applying a utilitarian approach, the paper will evaluate the voting rule consequentially, and compare the Nash equilibria with the outcome of random selections and that of the best possible utility.

Section 2. Model

Let $X = \{x, y, z\}$ be the set of three candidates for a given office, and $I = \{1, 2, 3\}$ the set of voters voting for the candidates. Each voter $i \in I$ is described by its utility function $u_i : X \to [0, 1]$ such that

$$\max\{u_i(x') : x' \in X\} = 1, \text{ and } \min\{u_i(x') : x' \in X\} = 0. \tag{1}$$

Let the cumulative distribution function of voters' utility of their second-preference candidate be $F(t_i)$, where $t_i \in (0, 1)$ is the voter's utility of their second-preferred candidate. Note that since this paper discusses only the symmetric Nash equilibrium in not weakly dominated strategy, the distribution of t_i is the same for all $i \in I$.

Let a voting rule be a function that takes a profile of voters' utility function to a winner, which is one of the candidates in X. The winner of the extension of the majority rule is the candidate who is preferred by more than fifty percent of voters in each head-to-head contest with other candidates. If

6. Lin Zhou, "On a conjecture by gale about one-sided matching problems," *Journal of Economic Theory* 52, no. 1 (1990): 123–135, ISSN: 0022-0531, https://doi.org/https://doi.org/10.1016/0022-0531(90)90070-Z, https://www.sciencedirect.com/science/article/pii/002205319090070Z.

7. Atila Abdulkadiroğlu, Yeon-Koo Che, and Yosuke Yasuda, "Resolving Conflicting Preferences in School Choice: The "Boston Mechanism" Reconsidered," *American Economic Review* 101, no. 1 (February 2011): 399–410, https://doi.org/10.1257/aer.101.1.399, https://www.aeaweb.org/articles?id=10.1257/aer.101.1.399.

there isn't such a candidate, then the winner is randomly selected from the smallest set of candidates, in which set each candidate beats any candidate in the complement set by a majority.

This paper analyzes the simplest Condorcet Cycle, in which three voters vote for three candidates following the extension of majority rule. It assumes that all voters are aware that if they vote according to their true preferences, their votes will lead to a Condorcet Cycle.

Let $\succ$ denote an ordering of the candidates. Without loss of generality, assume voters' preferences to be the ones shown in table 1.

Table 1. Voters' Preferences of Candidates x,y, and z

	Preference
Voter 1	$x \succ y \succ z$
Voter 2	$y \succ z \succ x$
Voter 3	$z \succ x \succ y$

Section 3. Symmetric Nash equilibrium

Theorem 3.1. *In not weakly dominated strategies, if we rule out weakly dominated strategies and only search for symmetric Nash equilibria, then there is and only is a symmetric Nash equilibrium. In the symmetric Nash equilibrium, voters share a threshold* $v_0 \in (0, 1)$*. Fix a voter whose true preference as* $x_1 \succ x_2 \succ x_3$ *and let the utility of its second preferred candidate be* t*. If* $t \leq v_0$*, the voter will report their true preferences* $x_1 \succ x_2 \succ x_3$*; if* $v_0 \leq t$*, voters will manipulate their preferences and report* $x_2 \succ x_1 \succ x_3$*.*

Proof. To prove the theorem, without loss of generality, the following will first find the best response for voter 1, whose true preference is $x \succ y \succ z$. The paper will then find the Nash equilibrium of the game by solving the system of best response for all voters.

Voter 1's best response is some combination of the preferences in the set of *Reported Preference* (henceforth RP), where

(2)
$$RP = \{x \succ y \succ z, x \succ z \succ y, y \succ x \succ z, y \succ z \succ x, z \succ x \succ y, z \succ y \succ x\},$$

depending on other voters' strategies, the probability of other voter playing each strategy, and voter 1's utility function.

Taking voter 2 and 3 into consideration, in table 5 of the appendix, the paper list the outcome for the $|RP|^3 = 6^3 = 216$ possible combinations. The six columns $(RP_1, RP_2, RP_3, RP_4, RP_5, RP_6)$ in table 5 corresponds to the six possible preferences Voter 1 may report. The 36 rows $(Case1, ..., Case36)$ correspond to the $|RP|^2 = 36$ possible combinations of preferences Voter 2 and 3 may report. If the combination of preferences do not constitute a Condorcet Cycle, the outcome will be denoted as one of x, y, z. Otherwise, it will be written as *Condorcet Cycle*.

As we can see from the table 5, for voter 1, reporting preference 1 ($x \succ y \succ z$, the true preference) weakly dominates both reporting preference 4 ($z \succ y \succ x$) and preference 6 ($z \succ x \succ y$); reporting preference 3 ($y \succ x \succ z$) weakly dominates reporting preference 5 ($y \succ z \succ x$). Therefore, in a symmetric Nash equilibrium in not weakly dominated strategy, voter 1 will report one of the preferences in the set

$$\{x \succ y \succ z, x \succ z \succ y, y \succ x \succ z\}. \tag{3}$$

In a symmetric Nash equilibrium, all voters will adopt the same strategy. Thus, for an arbitrary voter i whose true preference is $x_1 \succ x_2 \succ x_3$, the voter will report one of the preferences in the set

$$\{x_1 \succ x_2 \succ x_3, x_1 \succ x_3 \succ x_2, x_2 \succ x_1 \succ x_3\}, \tag{4}$$

with a utility function $u_i : \{x_1, x_2, x_3\} \to [0, 1]$, where $u_i(x_1) = 1, u_i(x_2) = t_i, u_i(x_3) = 0$ for $i \in I$. To simplify the discussion of other voters' probability of reporting each preference, the following assumes that voter 2 and 3 behave identically. Though adding this assumption may change voter 1's best response, it will not influence the symmetric Nash equilibria. With this assumption, the paper denotes p_1 the probability voter i will report preference 1 ($x_1 \succ x_2 \succ x_3$), p_2 the probability it will report preference 2 ($x_1 \succ x_3 \succ x_2$), and p_3 ($p_3 = 1 - p_1 - p_2$) the probability it will report preference 3 ($x_2 \succ x_1 \succ x_3$). The paper lists the preferences voters report as well as their corresponding probabilities in table 2.

TABLE 2. Voters' Reported Preference and Probabilities

	Reported Preference 1 (RP1)	Reported Preference 2 (RP2)	Reported Preference 3 (RP3)
Voter 2	y $\succ$ z $\succ$ x	y $\succ$ x $\succ$ z	x $\succ$ y $\succ$ z
Voter 3	z $\succ$ x $\succ$ y	z $\succ$ y $\succ$ x	y $\succ$ x $\succ$ z
Probability	p_1	p_2	p_3

Note: The RP of voters 2 and 3 and their probabilities.

After eliminating the weakly dominated strategies from table 5, and adding the probability and payoff of each combination of voters 2 and 3's reported preference, we obtain table 6 in the appendix. The column *Probability* refers to the probability each combination will occur in a symmetric Nash equilibrium. The column *payoff* refers to voter 1's payoff in each combination.

Fix p_1, p_2, t_1, we can calculate voter 1's expected utility when it reports preference RP_1, RP_2, RP_3, respectively:

$$\begin{cases} \mathbb{E}(u_1|RP_1, p_1, p_2, t_1) = & p_1^2\frac{t_1+1}{3} + p_1p_2t_1 + p_1p_3 + p_1p_2 + p_2^2t_1 + p_2p_3 + p_3^2 \\ \mathbb{E}(u_1|RP_2, p_1, p_2, t_1) = & p_1p_3 + p_1p_2 + p_2^2\frac{t_1+1}{3} + p_2p_3 + p_3^2 \\ \mathbb{E}(u_1|RP_3, p_1, p_2, t_1) = & p_1^2t_1 + p_1p_2t_1 + p_1p_3t_1 + p_1p_2t_1 + p_2^2t_1 + p_2p_3t_1 \\ & +p_3^2\frac{t_1+1}{3}. \end{cases} \tag{5}$$

Fix p_1, p_2, the equations above are linear functions of t. Thus, if the best response of voter 1 exists, then the best response can be represented by either no threshold, or one threshold $v_0 \in (0, 1)$, or two thresholds $v_1, v_2 \in (0, 1)$, where $v_1 < v_2$, such that:

- Case 1: voter 1 reports one preference independent of the value of t_1.
- Case 2: voter 1 reports one preference if $t_1 \leq v_0$, and another preference if $t_1 \geq v_0$.
- Case 3: voter 1 reports one preference if $t_1 \leq v_1$, another preference if $v_1 \leq t_1 \leq v_2$, and another preference if $t_1 \geq v_2$.

Fix reported preferences RP, p_1, p_2, and consider the functions in system 10 as functions of t_1. The slopes of the functions of RP_1, RP_2, RP_3 are

$$\begin{aligned} k_1 &= \frac{1}{3}p_1^2 + p_1p_2 + p_2^2, \\ k_2 &= \frac{1}{3}p_2^2, \\ k_3 &= p_1^2 + 2p_1p_2 + p_1p_3 + p_2^2 + p_2p_3 + \frac{1}{3}p_3^2, \end{aligned} \tag{6}$$

respectively. Note that $k_2 < k_1 < k_3$.

When $t_1 \to 0$, the expected utilities are

$$\begin{cases} lim_{t_1\to 0}\mathbb{E}(u_1|RP_1, p_1, p_2, t_1) = & p_1^2\frac{1}{3} + p_1p_3 + p_1p_2 + p_2p_3 + p_3^2 \\ lim_{t_1\to 0}\mathbb{E}(u_1|RP_2, p_1, p_2, t_1) = & p_1p_3 + p_1p_2 + p_2^2\frac{1}{3} + p_2p_3 + p_3^2 \\ lim_{t_1\to 0}\mathbb{E}(u_1|RP_3, p_1, p_2, t_1) = & p_3^2\frac{1}{3}. \end{cases} \tag{7}$$

Note that

$$lim_{t_1\to 0}\mathbb{E}(u_1|RP_1, p_1, p_2, t_1) > lim_{t_1\to 0}\mathbb{E}(u_1|RP_3, p_1, p_2, t_1), \tag{8}$$

$$lim_{t_1\to 0}\mathbb{E}(u_1|RP_2, p_1, p_2, t_1) > lim_{t_1\to 0}\mathbb{E}(u_1|RP_3, p_1, p_2, t_1). \tag{9}$$

When $t_1 \to 1$, the expected utilities are

$$\begin{cases} lim_{t_1\to 1}\mathbb{E}(u_1|RP_1, p_1, p_2, t_1) = & p_1^2\frac{2}{3} + p_1p_2 + p_1p_3 + p_1p_2 + p_2^2 + p_2p_3 + p_3^2 \\ lim_{t_1\to 1}\mathbb{E}(u_1|RP_2, p_1, p_2, t_1) = & p_1p_3 + p_1p_2 + p_2^2\frac{2}{3} + p_2p_3 + p_3^2 \\ lim_{t_1\to 1}\mathbb{E}(u_1|RP_3, p_1, p_2, t_1) = & p_1^2 + p_1p_2 + p_1p_3 + p_1p_2 + p_2^2 + p_2p_3 \\ & +p_3^2\frac{2}{3}. \end{cases} \tag{10}$$

Note that

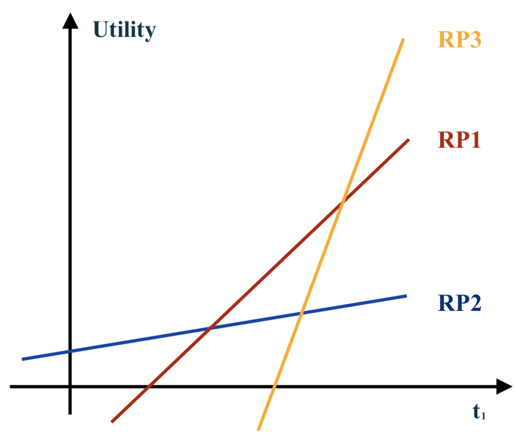
$$lim_{t_1 \to 1}\mathbb{E}(u_1|RP_1, p_1, p_2, t_1) > lim_{t_1 \to 1}\mathbb{E}(u_1|RP_2, p_1, p_2, t_1). \quad (11)$$

According to the relationships between the slopes of the functions in system 10, the possible best response functions for voter 1 are:

- Case 1: Report RP_1 regardless of the value of t_1
- Case 2:
  (1) Reports RP_2 if $t_1 \le v_0$, RP_1 if $t_1 \ge v_0$;
  (2) Reports RP_1 if $t_1 \le v_0$, RP_3 if $t_1 \ge v_0$;
  (3) Reports RP_2 if $t_1 \le v_0$, RP_3 if $t_1 \ge v_0$;
- Case 3: Reports RP_2 if $t_1 \le v_1$, RP_1 if $v_1 \le t_1 \le v_2$, RP_3 if $t_1 \ge v_2$.

As an example, figure 1 visualizes the expected utility of different reported preferences in Case 3.

Figure 1. Visualization of Expected Utility



Note: The expected utility of different reported preferences with respect to voter 1's utility to its second preferred candidate.

Note that, fix p_1, p_2, the voter will play one of the strategies listed in the cases above, but some of the listed strategies may not appear in voter 1's best response function. It is rather hard to write out the specific conditions of p_1, p_2 for each strategy explicitly, the following will find the symmetric Nash equilibrium by exclusion.

If voter 2 and 3 plays case 1, then $p_2 = p_3 = 0$, which contradicts with the necessary condition of voter 1 plays case 1:

$$lim_{t_1 \to 1}\mathbb{E}(u_1|RP_1, p_1, p_2, t_1) > lim_{t_1 \to 1}\mathbb{E}(u_1|RP_3, p_1, p_2, t_1). \quad (12)$$

If voter 2 and 3 plays case 2 – (1), then $p_3 = 0$, which contradicts the necessary condition of voter 2 plays case 2 – (1):

$$lim_{t_1 \to 1}\mathbb{E}(u_1|RP_1, p_1, p_2, t_1) > lim_{t_1 \to 1}\mathbb{E}(u_1|RP_3, p_1, p_2, t_1). \tag{13}$$

If voter 2 and 3 plays case 2 – (3), then $p_1 = 0$, which contradicts with the necessary consition of voter 1 plays case 2 – (3):

$$lim_{t_1 \to 1}\mathbb{E}(u_1|RP_1, p_1, p_2, t_1) < lim_{t_1 \to 1}\mathbb{E}(u_1|RP_3, p_1, p_2, t_1). \tag{14}$$

Thus, two of the cases in the best responses may appear in symmetric Nash equilibria. The one-threshold case is voter 1 reports RP_1 if $t_1 \leq v_0$, reports RP_3 if $t_1 \geq v_0$ with conditions:

$$\begin{gathered} lim_{t_1 \to 0}\mathbb{E}(u_1|RP_1, p_1, p_2, t_1) \geq lim_{t_1 \to 0}\mathbb{E}(u_1|RP_2, p_1, p_2, t_1) \\ lim_{t_1 \to 1}\mathbb{E}(u_1|RP_1, p_1, p_2, t_1) < lim_{t_1 \to 1}\mathbb{E}(u_1|RP_3, p_1, p_2, t_1) \\ \mathbb{E}(u_1|RP_1, p_1, p_2, t^*) \leq \mathbb{E}(u_1|RP_2, p_1, p_2, t^*), \\ \text{where } \mathbb{E}(u_1|RP_1, p_1, p_2, t^*) = \mathbb{E}(u_1|RP_3, p_1, p_2, t^*). \end{gathered} \tag{15}$$

Note that $\mathbb{E}(u_1|RP_1, p_1, p_2, t^*) \leq \mathbb{E}(u_1|RP_2, p_1, p_2, t^*)$ is implied by the other three functions, and the conditions can be simplified into

$$p_2 < p_1, p_1 > p_3. \tag{16}$$

The two-threshold case is voter 1 reports RP_2 if $t_1 \leq v_1$, reports RP_1 if $v_1 \leq t_1 \leq v_2$, and reports RP_3 if $t_1 \geq v_2$, with conditions

$$\begin{gathered} lim_{t_1 \to 0}\mathbb{E}(u_1|RP_1, p_1, p_2, t_1) < lim_{t_1 \to 0}\mathbb{E}(u_1|RP_2, p_1, p_2, t_1) \\ lim_{t_1 \to 1}\mathbb{E}(u_1|RP_1, p_1, p_2, t_1) < lim_{t_1 \to 1}\mathbb{E}(u_1|RP_3, p_1, p_2, t_1) \\ \mathbb{E}(u_1|RP_1, p_1, p_2, t^*) \geq \mathbb{E}(u_1|RP_2, p_1, p_2, t^{**}), \\ \text{where } \mathbb{E}(u_1|RP_2, p_1, p_2, t^*) = \mathbb{E}(u_1|RP_3, p_1, p_2, t^{**}). \end{gathered} \tag{17}$$

Note that $\mathbb{E}(u_1|RP_1, p_1, p_2, t^*) \geq \mathbb{E}(u_1|RP_2, p_1, p_2, t^{**})$ is implied by the other three functions, and the conditions can be simplified into

$$p_2 > p_1, p_1 > p_3. \tag{18}$$

If the best response exists, then the one-threshold case and the two-threshold case are the only two remaining possibilities. Next, the paper will rule out the two-threshold case and prove that the one-threshold case is the only possible symmetric Nash equilibrium for this game.

To find the symmetric Nash equilibrium, this paper considers the probability of voter 2 and 3 reporting RP1 and RP2 the same as that of voter 1. The exact value of v_0 and v_1, v_2 can be found by solving the following systems.

For the one-threshold case:

$$\begin{cases} p_1 = F(v_0) \\ p_2 = 0 \\ p_3 = 1 - p_1 - p_2 \\ \mathbb{E}(u_1|RP_1, p_1, p_2, v_0) = \mathbb{E}(u_1|RP_3, p_1, p_2, v_0) \\ p_2 \leq p_1, p_1 > p_3 \\ p_1, p_2, p_3, v_0 \in (0, 1); \end{cases} \tag{19}$$

and for the two-threshold case:

$$\begin{cases} p_1 = F(v_2) - F(v_1) \\ p_2 = F(v_1) \\ p_3 = 1 - p_1 - p_2 \\ \mathbb{E}(u_1|RP_2, p_1, p_2, v_1) = \mathbb{E}(u_1|RP_1, p_1, p_2, v_1) \\ \mathbb{E}(u_1|RP_1, p_1, p_2, v_2) = \mathbb{E}(u_1|RP_3, p_1, p_2, v_2) \\ p_2 < p_1, p_1 < p_3 \\ p_1, p_2, p_3, v_1, v_2 \in (0, 1) \\ v_1 < v_2; \end{cases} \tag{20}$$

where F is the distribution function of voters' utility for their second preference.

Let us consider first the system of the one-threshold case. System 19 can be rewritten as

$$\begin{cases} F(v_0) = \frac{3}{v_0+1} - 1 \\ F(v_0) \in [\frac{1}{2}, 1] \\ v_0 \in (0, 1). \end{cases} \tag{21}$$

On the domain of $(0, 1)$, since $F(v_0)$ is an increasing function with range $(0, 1)$, and $\frac{3}{v_0+1} - 1$ is a decreasing function with range $(\frac{1}{2}, 1)$, there is and only is one solution to the first equation of the system, which solution suffices the latter two inequalities in system 21. Therefore, system 19 has and only has one solution.

Next, we will prove that the system doesn't have any solution to the two-threshold case. System 20 can be rewritten as

$$\begin{cases} F(v_2) = F(v_1) \cdot \frac{2-v_1}{v_1+1} \\ F(v_2) \cdot \frac{v_2+1}{2-v_2} = F(v_1) + 1 \\ 2F(v_1) > F(v_2) > \frac{1+F(v_1)}{2} \\ F(v_1), F(v_2) \in (0, 1). \end{cases} \tag{22}$$

From the first two equations, we have

$$F(v_2)\cdot\frac{v_2+1}{2-v_2}-F(v_2)\cdot\frac{v_2+1}{2-v_2}$$
$$=F(v_1)\cdot\frac{2-v_1}{v_1+1}\cdot\frac{v_2+1}{2-v_2}-F(v_1)-1 \tag{23}$$
$$=0$$

Define function g such that for $v_1\in(0,1)$ and constant $v_2\in(0,1)$,

$$g(v_1)=F(v_1)\cdot\frac{2-v_1}{v_1+1}\cdot\frac{v_2+1}{2-v_2}-F(v_1)-1. \tag{24}$$

Since $g(0),g(1)<0$, if $g(v_1)=0$ has any solution on $(0,1)$, then there exists a local extremum $v_1'\in(0,1)$ such that

$$\left.\frac{\mathrm{d}g(v_1)}{\mathrm{d}v_1}\right|_{v_1=v_1'}=0, g(v_1')>0. \tag{25}$$

The former gives us

$$\frac{\mathrm{d}F(v_1)}{\mathrm{d}v_1}=F(v_1)\cdot\frac{\frac{2v_1-1}{(v_1+1)^2}}{\frac{2-v_1}{v_1+1}-\frac{2-v_2}{v_2+1}}. \tag{26}$$

Since $F(v_1)$ is a increasing function of v_1, $\frac{\mathrm{d}F(v_1)}{\mathrm{d}v_1}>0$, which gives us $v_1'>\frac{1}{2}$. Thus

$$g(v_1')=F(v_1')\cdot\frac{2-v_1'}{v_1'+1}\cdot\frac{v_2+1}{2-v_2}-F(v_1')-1$$
$$<F(v_1')\cdot(\frac{v_2+1}{2-v_2}-1)-1 \tag{27}$$
$$<F(v_1')-1$$
$$<0.$$

As a result, $g(v_1)\leq g(v_1')<0$ for all $v_1\in(0,1)$. Thus, system 22 has no solution.

Therefore, there is and only is a symmetric Nash equilibrium. □

Example 3.2 (Uniform Distribution)**.** Let t_i distributed uniformly over $(0,1)$, id est $F(t_i)=t_i$. Plugging the distribution function to system 19, we have

$$\begin{cases} p_1=v_0=0.7321\\ p_2=0\\ p_3=0.2679, \end{cases} \tag{28}$$

Therefore, the symmetric Nash equilibrium in not weakly dominated strategy for this scenario is: for $i\in\{1,2,3\}$, if $u_i(x_2)<0.7321$, voter i reports its preference as $x_1\succ x_2\succ x_3$; if $u_i(x_2)>0.7321$, voter i reports $x_2\succ x_1\succ x_3$; otherwise, voter i is indifferent between reporting $x_1\succ x_2\succ x_3$ and $x_2\succ x_1\succ x_3$.

Example 3.3 (Truncated Normal Distribution)**.** To generalize our distribution function, let voters' utility of their second preference follows a truncated normal distribution. Here, the truncated normal distribution is derived from that of a normally distributed random variable by bounding the random variable from both below and above. The cumulative distribution function of the distribution is

$$F(x, \mu, \sigma, a, b) = \frac{\Phi(\xi) - \Phi(\alpha)}{z}, \tag{29}$$

where $\xi = \frac{x-\mu}{\sigma}$, $\alpha = \frac{0-\mu}{\sigma}$, $\beta = \frac{1-\mu}{\sigma}$, $z = \Phi(\beta) - \Phi(\alpha)$, and Φ is the cumulative distribution function of the standard normal distribution. Note that μ and σ here are the mean and variance of Φ, but not necessarily the mean and variance of the truncated normal distribution. By varying the μ and σ of the distribution, we have the result in table 3.

Table 3. More General Symmetric Nash equilibrium

	μ	σ	v_0
1	0.5	0.5	0.7177
2	0.5	0.25	0.6838
3	0.2	0.25	0.5690
4	0.8	0.25	0.8111

Note: The mean μ and the variance σ of the cumulative distribution function Φ, and their corresponding threshold v_0.

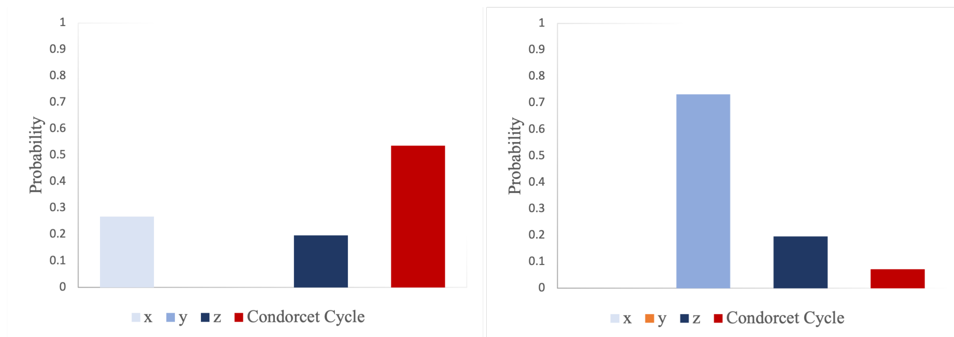
As shown in previous proofs and examples, the symmetric Nash equilibrium depends on the distribution function of voters' utility. The intuition behind this is that voters are motivated to manipulate their preferences if they disfavor the Condorcet Cycle. The following will elaborate on this intuition with the example in which voters' utility distributed uniformly over $(0, 1)$.

Let voter 1's true preference be $x \succ y \succ z$ as before. Fix other voters' reported preference, figure 2 shows the probability of each voting outcome when the voter 1 reports RP1 ($x \succ y \succ y$) and RP3 ($y \succ x \succ z$), respectively.

As we can see from the figure, by switching from RP1 to RP3, voter 1 increases the probability of its second-preferred candidate winning and decreases the probability of the occurrence of the Condorcet Cycle. The rationale behind this is that voters' utilities to their second preference are not the same as their expected utility resulting from the Condorcet Cycle. This motivates voters to take risks and encourages them to vote according to whether they want to avoid the Condorcet cycle or not.

Figure 2. Probability of Outcome for the Symmetric Nash Equilibrium.

RP1. RP3.



Note: Probability of each candidate wining when the voter reports RP1 and RP3, respectively.

Section 4. Outcome Evaluation

In this section, this paper applies a utilitarian approach to evaluate the voting outcome. Given the voters' utility, the paper assumes that the greater their expected total utility is, the better the outcome is. The following will compare the outcome of the majority voting rule with that of the random selection and the best total utility. It is through this comparison that we study the mapping from the *distribution of voters' utility to their second-preferred candidate* to the *expected value of voters' total utility* under different rules.

Next, the following will derive the general formula of the expected total utility for the symmetric Nash equilibrium in the previous section. Denote the *probability density function* for voters' utility of their second-preferred candidate as $f(t)$, where $t \in (0,1)$ is the value of their utility to their second-preferred candidate. Given a symmetric Nash equilibrium, there are $2 \times 2 \times 2 = 8$ possible combinations of voters' reported preferences. Let the set $\{No.j : j \in J\}$, where $J = \{1, 2, ..., 8\}$, denote the set of combinations of voters' reported preferences. Let $p_{No.j}$ be the probability that combination $No.j$ will happen in the symmetric Nash equilibrium. Let the reported preference $\overline{RP}$ be a function of t such that

(30)
$$\overline{RP} : (0,1) \to \{x \succ y \succ z, x \succ z \succ y, y \succ z \succ x, y \succ x \succ z, z \succ x \succ y, z \succ y \succ z\}.$$

Let $u_{i,j}$, where $i \in I = \{1, 2, 3\}$ be voter i's utility in combination $No.j$. Let

$$U_{sum} : (0,1) \times (0,1) \times (0,1) \to [0,3] \tag{31}$$

be the expected value of voters' total utility.

With all the notations above, voters' expected total utility can be represented as:

$$U_{sum}(t_1,t_2,t_3) = \sum_{j\in J} p_{No.j} \cdot (u_{1,j} + u_{2,j} + u_{3,j}). \tag{32}$$

Therefore, the expectation of the random variable U_{sum} is:

$$\begin{aligned} &\mathbb{E}[U_{sum}(t_1,t_2,t_3)] \\ &= \sum_{j\in J} p_{No.j} \cdot \mathbb{E}[u_{1,j} + u_{2,j} + u_{3,j}|\text{combination } j] \\ &= \sum_{j\in J} p_{No.j} \cdot \mathbb{E}[u_{1,j} + u_{2,j} + u_{3,j}|\overline{RP}(t_1), \overline{RP}(t_2), \overline{RP}(t_3)]. \end{aligned} \tag{33}$$

Table 7 of the appendix lists all possible combinations of reported preferences and their corresponding total utility. In the table, each row represents a combination of the reported preferences. The columns refer to the probability that the corresponding case will occur in the symmetric Nash equilibrium; the preferences reported by voters; the voting outcome; the sum of voters' utility, respectively. By plugging in the result in table 7 to equation 33, we have:

$$\begin{aligned} &\mathbb{E}[U_{sum}(t_1,t_2,t_3)] \\ &= p_1^2 \cdot \int_0^{v_1} t\mathrm{d}F(t) + (3\cdot p_1^2 + 3\cdot p_1 \cdot p_3 + p_3^2) \cdot \int_{v_1}^1 t\mathrm{d}F(t) + 1. \end{aligned} \tag{34}$$

calculation is included in appendix B.1.

Subsection 4.1. **Expected Total Utility and Comparison with Random Selection.**

Theorem 4.1. *Voters will be the same, if not better off, under the symmetric Nash equilibrium than under the rule of random selection.*

Proof. The expected value for the random selection is

$$\mathbb{E}_r[U_{sum}(t_1,t_2,t_3)] = 1 + \int_0^1 f(t)\cdot t\mathrm{d}t. \tag{35}$$

Thus,

$$\begin{aligned} &\mathbb{E}[U_{sum}(t_1,t_2,t_3)] - \mathbb{E}_r[U_{sum}(t_1,t_2,t_3)] \\ &= (F(v_1)+1)[\mathbb{E}_r[U_{sum}(t_1,t_2,t_3)]\cdot(F(v_1)-1)+1] \\ &\geq 0 \end{aligned} \tag{36}$$

Therefore, voters will always be better off under the symmetric Nash equilibrium than under the rule of random selection. □

Example 4.2. By plugging in the values in both example 3.2 and 3.3, we obtain the expected utility for the uniform and the truncated normal distribution, which is shown in table 4 in the appendix. In the table, the column *Uniform Distribution*, *Truncated Distribution*, and the parameters indicate which specific category of distribution that column refers to. The row *Random Selection* corresponds to the expected value of the sum of voters' utility if the winner is selected randomly. The row *Symmetric Nash Equilibrium* corresponds to the expected value of the sum of voters' utility in symmetric Nash equilibrium. As shown in table 4, compared to random selection, voters are in general better off in the symmetric Nash equilibrium.

TABLE 4. Examples of Expected Sum

Expected Total Utility	Uniform Distribution	Truncated Distribution			
	mean $= 0.5$	$\mu = 0.5$ $\sigma = 0.5$	$\mu = 0.5$ $\sigma = 0.25$	$\mu = 0.2$ $\sigma = 0.25$	$\mu = 0.8$ $\sigma = 0.25$
Random Selection	1.5	1.5	1.5	1.2912	1.7088
Symmetric Nash Equilibrium	1.6699	1.6547	1.6169	1.3559	1.8179

Note: The expected total utility of random selection and symmetric Nash equilibrium for different distribution functions of t.

Subsection 4.2. **Case Study and Comparison with the Best Possible Uitlity.** With the approach of utilitarianism, we want to pick a candidate that maximizes voters' total utility. Ideally, we would want voters to expose their utility functions so that we can pick a winner that maximizes their total utility. This exposure is rarely realistic. However, in the game this paper sets up, given voters' true preferences and the Nash equilibrium they were in, we can obtain information about their utility functions according to the way they manipulate their preferences.

For example, in the symmetric Nash equilibrium, for a voter whose true preference is $x_1 \succ x_2 \succ x_3$, their utility to the second-preferred candidate, which this paper denotes as t, is less or equal to v_0; if they report $x_1 \succ x_3 \succ x_2$, then their utility to the second-preferred candidate is larger or equal to v_0.

We then revisit tables 7. Instead of presenting voters' preferences explicitly, the paper denotes their reported preferences as $RP1, RP2, RP3$ as indicated in table 2. The paper also adds the column *Implication*, which records the information voters' reported preferences indicate, and add the column *Best Utility*, which keeps track of the best possible utility, which gives us table 8. As we can see from the tables, voters' manipulation reveals information about their utility function. This information provides a range for the best possible utility, and the utility resulted from the extension of the majority voting rule is always within this range.

Though ordinal voting rules don't request voters' utility functions, in some contexts, voters nevertheless reveal information concerning the cardinal information about their utility functions. In the game this paper set up, it is exactly through voters' manipulation that voters reveal information about their strength of preferences.

Section 5. Conclusion

This paper serves as a footnote for the discussion of the tradeoff between strategyproofness and Pareto efficiency.

The studies of the extension of the majority voting rule oftentimes exclude the discussion about the domain that induces the Condorcet cycle, for including this domain will fail the strategyproofness axiom. This paper, however, focuses only on this excluded domain and compares the same (yet now non-strategyproof) majority voting rule with the strategyproof voting rule–random selection. It shows that despite its strategyproofness, the outcome of the rule of random selection is less, or at most equally as efficient as the extension of the majority voting rule.

In this game setting, the Pareto efficient outcome is the best possible outcome under the utilitarian approach. Though the information a voting profile provides is not enough to determine the exact best possible utility, it oftentimes nevertheless excludes some absolutely not best utility. While the outcome of random selection may be excluded from this additional information, such information never precludes the outcome of the extended majority voting rule. Thus, in the scenario this paper studies, the non-strategyproofness voting rule is equally efficient as, if not more than, the strategyproof voting rule.

Admittedly, while evaluating the outcome of the extension of the majority voting rule, this paper uses only the symmetric Nash equilibrium and doesn't discuss the asymmetric Nash equilibrium. The dynamics between asymmetric Nash equilibrium and the result of the random selection will be an interesting topic for future studies.

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Appendix A. Tables

Note for table 5: The winner of the game in different voting profiles. The six columns (RP1,RP2,RP3,RP4,RP5,RP6) in the table corresponds to the six possible preferences Voter 1 may report. The 36 rows (Case1, ..., Case36) correspond to the $|RP|^2 = 36$ possible combinations of preferences voter 2 and 3 may report. If the combination of preferences does not constitute a Condorcet Cycle, the outcome will be denoted as one of x, y, z. Otherwise, it will be written as Condorcet Cycle.

Note for table 6: The winner and the payoff for voter 1 in different voting profiles. After eliminating the weakly dominated strategies from table 5, and adding the probability and payoff of each combination of voters 2 and 3's reported preference, we obtain table 6.

Note for table 7: The probability, outcome, and the sum of voters' utility for each voting profile. In the table, each row represents a combination of the reported preferences. The columns refer to the probability that the corresponding case will occur in the symmetric Nash equilibrium; the preferences reported by voters; the voting outcome; the sum of voters' utility, respectively.

Note for table 8: The implication of t, the sum of voters' utility, and the best utility for each voting profile. The column *Implication* records the information voters' reported preferences indicate. the column *Best Utility* keeps track of the best possible utility.

TABLE 5. Possible Outcomes

# Case	Voter 2 & 3		Voter 1					
	# Voter	Reported Preference	RP1	RP2	RP3	RP4	RP5	RP6
			$x \succ y \succ z$	$x \succ z \succ y$	$y \succ x \succ z$	$z \succ y \succ x$	$y \succ z \succ x$	$z \succ x \succ y$
1	2	$y \succ z \succ x$	Condorcet Cycle	z	y	z	y	z
	3	$z \succ x \succ y$						
2	2	$y \succ z \succ x$	x	x	y	z	y	z
	3	$x \succ z \succ y$						
3	2	$y \succ z \succ x$	x	x	y	y	y	Condorcet Cycle
	3	$x \succ y \succ z$						
4	2	$y \succ z \succ x$	y	y	y	y	y	y
	3	$y \succ x \succ z$						
5	2	$y \succ z \succ x$	y	y	y	y	y	y
	3	$y \succ z \succ x$						
6	2	$y \succ z \succ x$	y	z	y	z	y	z
	3	$z \succ y \succ x$						
7	2	$y \succ x \succ z$	x	x	y	z	y	z
	3	$z \succ x \succ y$						
8	2	$y \succ x \succ z$	x	x	y	Condorcet Cycle	y	x
	3	$x \succ z \succ y$						
9	2	$y \succ x \succ z$	x	x	y	y	y	x
	3	$x \succ y \succ z$						
10	2	$y \succ x \succ z$	y	y	y	y	y	y
	3	$y \succ x \succ z$						
11	2	$y \succ x \succ z$	y	y	y	y	y	y
	3	$y \succ z \succ x$						
12	2	$y \succ x \succ z$	y	Condorcet Cycle	y	z	y	z
	3	$z \succ y \succ x$						

Table 5. Possible Outcomes (continued)

# Case	Voter 2 & 3		Voter 1					
	# Voter	Reported Preference	RP1	RP2	RP3	RP4	RP5	RP6
			$x \succ y \succ z$	$x \succ z \succ y$	$y \succ x \succ z$	$z \succ y \succ x$	$y \succ z \succ x$	$z \succ x \succ y$
13	2	$z \succ x \succ y$	z	z	z	z	z	z
	3	$z \succ x \succ y$						
14	2	$z \succ x \succ y$	x	x	x	z	z	z
	3	$x \succ z \succ y$						
15	2	$z \succ x \succ y$	x	x	x	z	Condorcet Cycle	z
	3	$x \succ y \succ z$						
16	2	$z \succ x \succ y$	x	x	y	z	y	z
	3	$y \succ x \succ z$						
17	2	$z \succ x \succ y$	Condorcet Cycle	z	y	z	y	z
	3	$y \succ z \succ x$						
18	2	$z \succ x \succ y$	z	z	z	z	z	z
	3	$z \succ y \succ x$						
19	2	$z \succ y \succ x$	z	z	z	z	z	z
	3	$z \succ x \succ y$						
20	2	$z \succ y \succ x$	x	x	Condorcet Cycle	z	z	z
	3	$x \succ z \succ y$						
21	2	$z \succ y \succ x$	x	x	y	z	y	z
	3	$x \succ y \succ z$						
22	2	$z \succ y \succ x$	y	Condorcet Cycle	y	z	y	z
	3	$y \succ x \succ z$						
23	2	$z \succ y \succ x$	y	z	y	z	y	z
	3	$y \succ z \succ x$						
24	2	$z \succ y \succ x$	z	z	z	z	z	z
	3	$z \succ y \succ x$						

TABLE 5. Possible Outcomes (continued)

# Case	Voter 2 & 3		Voter 1					
	# Voter	Reported Preference	RP1	RP2	RP3	RP4	RP5	RP6
			$x \succ y \succ z$	$x \succ z \succ y$	$y \succ x \succ z$	$z \succ y \succ x$	$y \succ z \succ x$	$z \succ x \succ y$
25	2	$x \succ y \succ z$	x	x	x	z	Condorcet Cycle	z
	3	$z \succ x \succ y$						
26	2	$x \succ y \succ z$	x	x	x	x	x	x
	3	$x \succ z \succ y$						
27	2	$x \succ y \succ z$	x	x	x	x	x	x
	3	$x \succ y \succ z$						
28	2	$x \succ y \succ z$	x	x	y	y	y	x
	3	$y \succ x \succ z$						
29	2	$x \succ y \succ z$	x	x	y	y	y	Condorcet Cycle
	3	$y \succ z \succ x$						
30	2	$x \succ y \succ z$	x	x	y	z	y	z
	3	$z \succ y \succ x$						
31	2	$x \succ z \succ y$	x	x	x	z	z	z
	3	$z \succ x \succ y$						
32	2	$x \succ z \succ y$	x	x	x	x	x	x
	3	$x \succ z \succ y$						
33	2	$x \succ z \succ y$	x	x	x	x	x	x
	3	$x \succ y \succ z$						
34	2	$x \succ z \succ y$	x	x	y	Condorcet Cycle	y	x
	3	$y \succ x \succ z$						
35	2	$x \succ z \succ y$	x	x	y	z	y	z
	3	$y \succ z \succ x$						
36	2	$y \succ z \succ x$	y	z	y	z	y	z
	3	$z \succ y \succ x$						

TABLE 6. Payoff Function of Voter 1

				Voter 1					
				RP1 x $\succ$ y $\succ$ z p_1		RP2 x $\succ$ z $\succ$ y p_2		RP3 y $\succ$ x $\succ$ z p_3	
#Case	Probability	#Voter	RP	winner	payoff	winner	payoff	winner	payoff
1	$p_1 \cdot p_1$	2	y $\succ$ z $\succ$ x	Condorcet Cycle	$\frac{t_1+1}{3}$	z	0	y	t_1
		3	z $\succ$ x $\succ$ y						
6	$p_1 \cdot p_2$	2	y $\succ$ z $\succ$ x	y	t_1	z	0	y	t_1
		3	z $\succ$ y $\succ$ x						
2	$p_1 \cdot p_3$	2	y $\succ$ z $\succ$ x	x	1	x	1	y	t_1
		3	x $\succ$ z $\succ$ y						
7	$p_2 \cdot p_1$	2	y $\succ$ x $\succ$ z	x	1	x	1	y	t_1
		3	z $\succ$ x $\succ$ y						
12	$p_2 \cdot p_2$	2	y $\succ$ x $\succ$ z	y	t_1	Condorcet Cycle	$\frac{t_1+1}{3}$	y	t_1
		3	z $\succ$ y $\succ$ x						
8	$p_2 \cdot p_3$	2	y $\succ$ x $\succ$ z	x	1	x	1	y	t_1
		3	x $\succ$ z $\succ$ y						
19	$p_3 \cdot p_1$	2	z $\succ$ y $\succ$ x	z	0	z	0	z	0
		3	z $\succ$ x $\succ$ y						
24	$p_3 \cdot p_2$	2	z $\succ$ y $\succ$ x	z	0	z	0	z	0
		3	z $\succ$ y $\succ$ x						
21	$p_3 \cdot p_3$	2	z $\succ$ y $\succ$ x	x	1	x	1	Condorcet Cycle	$\frac{t_1+1}{3}$
		3	x $\succ$ z $\succ$ y						

Table 7. Outcome Evaluation

#No.	Probability p_j	Reported Preference			Voting Outcome	Sum of Utility $u_1(t_1)+u_2(t_2)+u_3(t_3)$
		Voter 1	Voter 2	Voter 3		
1	p1·p1·p1	x $\succ y \succ z$	y $\succ z \succ x$	z $\succ x \succ y$	Condorcet	$1+\frac{1}{3}(t_1+t_2+t_3)$
2	p1·p1·p3	y $\succ x \succ z$	y $\succ z \succ x$	z $\succ x \succ y$	y	t_1+1
3	p1·p1·p3	x $\succ y \succ z$	y $\succ z \succ x$	x $\succ z \succ y$	x	t_3+1
4	p1·p3·p3	y $\succ x \succ z$	y $\succ z \succ x$	x $\succ z \succ y$	y	t_1+1
5	p1·p1·p3	x $\succ y \succ z$	z $\succ y \succ x$	z $\succ x \succ y$	z	t_2+1
6	p1·p3·p3	y $\succ x \succ z$	z $\succ y \succ x$	z $\succ x \succ y$	z	t_2+1
7	p1·p3·p3	x $\succ y \succ z$	z $\succ y \succ x$	x $\succ z \succ y$	x	t_3+1
8	p3·p3·p3	y $\succ x \succ z$	z $\succ y \succ x$	x $\succ z \succ y$	Condorcet	$1+\frac{1}{3}(t_1+t_2+t_3)$

Table 8. Implication from Nash Equilibrium

#No.	Reported Preference			Implication	Sum of Utility $u_1(t_1)+u_2(t_2)+u_3(t_3)$	Best Utility
	Voter 1	Voter 2	Voter 3			
1	RP1	RP1	RP1	-	$1+\frac{1}{3}(t_1+t_2+t_3)$	$1+\max\{t_1,t_2,t_3\}$
2	RP3	RP1	RP1	$t_1>t_2,t_3$	t_1+1	t_1+1
3	RP1	RP1	RP3	$t_3>t_1,t_2$	t_3+1	t_3+1
4	RP3	RP1	RP3	$t_1,t_3>t_2$	t_1+1	$1+\max\{t_1,t_3\}$
5	RP1	RP3	RP1	$t_2>t_1,t_3$	t_2+1	t_2+1
6	RP3	RP3	RP1	$t_1,t_2>t_3$	t_2+1	$1+\max\{t_1,t_2\}$
7	RP1	RP3	RP3	$t_2,t_3>t_1$	t_3+1	$1+\max\{t_2,t_3\}$
8	RP3	RP3	RP3	-	$1+\frac{1}{3}(t_1+t_2+t_3)$	$1+\max\{t_1,t_2,t_3\}$

Appendix B. Proofs and Calculations

Subsection B.1. **Additional Calculation for Section .** For the expected utility of the Nash equilibrium,

(37)

$$
\begin{aligned}
&\mathbb{E}[U_{sum}(t_1,t_2,t_3)]\\
&= p_1 \cdot p_1 \cdot p_1 \cdot \mathbb{E}[1+\frac{1}{3}\cdot(t_1+t_2+t_3)|t_1 \le v_0, t_2 \le v_0, t_3 \le v_0]\\
&+ p_1 \cdot p_1 \cdot p_3 \cdot \mathbb{E}[t_1+1|t_1 \ge v_0, t_2 \le v_0, t_3 \le v_0]\\
&+ p_1 \cdot p_1 \cdot p_3 \cdot \mathbb{E}[t_3+1|t_1 \le v_0, t_2 \le v_0, t_3 \ge v_0]\\
&+ p_1 \cdot p_3 \cdot p_3 \cdot \mathbb{E}[t_1+1|t_1 \ge v_0, t_2 \le v_0, t_3 \ge v_0]\\
&+ p_1 \cdot p_1 \cdot p_3 \cdot \mathbb{E}[t_2+1|t_1 \le v_0, t_2 \ge v_0, t_3 \le v_0]\\
&+ p_1 \cdot p_3 \cdot p_3 \cdot \mathbb{E}[t_2+1|t_1 \ge v_0, t_2 \ge v_0, t_3 \le v_0]\\
&+ p_1 \cdot p_3 \cdot p_3 \cdot \mathbb{E}[t_3+1|t_1 \le v_0, t_2 \ge v_0, t_3 \ge v_0]\\
&+ p_3 \cdot p_3 \cdot p_3 \cdot \mathbb{E}[1+\frac{1}{3}\cdot(t_1+t_2+t_3)|t_1 \ge v_0, t_2 \ge v_0, t_3 \ge v_0]\\
&= p_1^3 \cdot \{1+\mathbb{E}[t|t \le v_0]\} + p_1^2 \cdot p_3 \cdot \{1+\mathbb{E}[t|t \ge v_0]\} + p_1^2 \cdot p_3 \cdot \{1+\mathbb{E}[t|t \ge v_0]\}\\
&+ p_3^2 \cdot p_1 \cdot \{1+\mathbb{E}[t|t \ge v_0]\} + p_1^2 \cdot p_3 \cdot \{1+\mathbb{E}[t|t \ge v_0]\}\\
&+ p_1 \cdot p_3^2 \cdot \{1+\mathbb{E}[t|t \ge v_0]\} + p_1 \cdot p_3^2 \cdot \{1+\mathbb{E}[t|t \ge v_0]\} + \cdot p_3^3 \cdot \{1+\mathbb{E}[v|t \ge v_0]\}\\
&= p_1^3 \cdot \mathbb{E}[t|t \le v_0] + (3 \cdot p_1^2 \cdot p_3 + 3 \cdot p_1 \cdot p_3^2 + p_3^3) \cdot \mathbb{E} \cdot [t|t \ge v_0] + 1\\
&= p_1^3 \cdot \int_0^{v_0} f(t|t \le v_0) \cdot t\mathrm{d}t + (3 \cdot p_1^2 \cdot p_3 + 3 \cdot p_1 \cdot p_3^2 + p_3^3) \cdot \int_{v_0}^1 f(t|t \ge v_0) \cdot t\mathrm{d}t + 1.
\end{aligned}
$$

Since

$$f(t|t \le v_0) = \frac{f(t)}{\int_0^{v_0} f(t)\mathrm{d}t} = \frac{f(t)}{p_1}, \tag{38}$$

we have

$$\int_0^{v_0} f(t|t \le v_0) \cdot \mathrm{d}t = {p_1}^{-1} \int_0^{v_0} t \cdot \mathrm{d}F(t). \tag{39}$$

Similarly,

$$\int_{v_0}^1 f(t|t \ge v_0) \cdot \mathrm{d}t = {p_3}^{-1} \int_{v_0}^1 t \cdot \mathrm{d}F(t). \tag{40}$$

Thus,

$$
\begin{aligned}
&\mathbb{E}[U_{sum}(t_1,t_2,t_3)]\\
&= p_1^2 \cdot \int_0^{v_1} t\mathrm{d}F(t) + (3 \cdot p_1^2 + 3 \cdot p_1 \cdot p_3 + p_3^2) \cdot \int_{v_1}^1 t\mathrm{d}F(t) + 1
\end{aligned} \tag{41}
$$