Colour Patterns for Polychromatic Four-colourings of Rectangular Subdivisions

H. Haverkort* M. Löffler† E. Mumford* M. O'Meara‡ J. Snoeyink§ B. Speckmann*

Abstract

A non-degenerate rectangular subdivision is a subdivision of a rectangle into a set of non-overlapping rectangles S, such that no four rectangles meet in a point. We consider a problem that Katz and colleagues call strong polychromatic four-colouring: Colouring the vertices of the subdivision with four colours, such that each rectangle of S has all colours among its four corners. By considering the possible colouring patterns, we can give short constructive proofs of colourability for subdivisions that are *sliceable* or *one-sided*. We also present techniques and observations for non-sliceable, two-sided subdivisions.

1 Introduction

A *rectangular subdivision* is a set S of rectangles with disjoint interiors whose union is a rectangle $r(S)$. The set of *vertices* of S is the union of the sets of vertices (corners) of the rectangles in S. If S contains four rectangles that meet in a single vertex, we say that S is *degenerate*. A *non-degenerate rectangular subdivision* is a rectangular subdivision in which each vertex is a corner of only one or two rectangles. Unless otherwise specified, a *subdivision* is a non-degenerate rectangular subdivision.

Dinitz et al. [1] showed that it is possible to colour the vertices of any subdivision S with three colours so that each rectangle in S is *polychromatic*—has at least one vertex of each colour. They conjectured that this is also possible with four colours. This conjecture is in fact a special case of a much older conjecture by Seymour [6] concerning the edge-colouring of a special class of planar graphs, so-called $4-$*graphs*. Seymour's conjecture was proven by Guenin [3]. We are thankful to Dimitrov et al. [2] for pointing out that Guenin's result implies that each non-degenerate subdivision has indeed a strong polychromatic four-colouring.

*Department of Mathematics and Computer Science, Eindhoven University of Technology, `cs.herman@haverkort.net, e.mumford@tue.nl, speckman@win.tue.nl`

†Department of Information and Computing Sciences, Utrecht University, `loffler@cs.uu.nl`

‡Department of Mathematics, University of Chicago, `mattjomeara@gmail.com`

§Department of Computer Science, University of North Carolina at Chapel Hill, `snoeyink@cs.unc.edu`

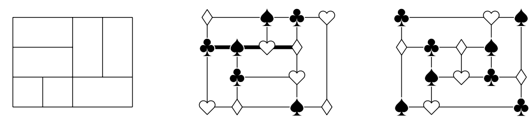

Figure 1: From left to right: a degenerate subdivision that is not colourable; a sliceable subdivision (but not one-sided, because of the fat segment); a one-sided (but not sliceable) subdivision.

Guenin's proof is non-constructive. In the remainder of this abstract we focus on constructive proofs for specialized subdivisions and report on observations concerning so-called colour patterns. For any subdivision S let a *colouring pattern* of S be an assignment of four colours to the vertices of S such that each rectangle is polychromatic. We say S is *colourable* if it admits at least one colour pattern. Let a *boundary colouring pattern* of S be the restriction of a colouring pattern of S to the corners of the boundary of S.

Known and new results. Not all degenerate rectangular subdivisions are colourable (see Figure 1).

Sliceable subdivisions. A subdivision is called *sliceable* if it can be obtained by recursively slicing a rectangle with horizontal and vertical lines. Horev et al. [5] call these *guillotine* subdivisions, and show that they are always colourable. We give a short proof using *boundary colour patterns*.

Canonical form. We can order the rectangles of a subdivision so that prefixes form monotone staircases; for some subdivisions this order is unique. We show that any colouring pattern of a subdivision can be realized by a subdivision with a unique order, and give a procedure to convert any given subdivision into such a *canonical form*.

One-sided subdivisions. A *maximal line segment of S* is a line segment that is completely covered by edges of the rectangles of S for which no extension is covered. A subdivision is *one-sided* if and only if, for every maximal line segment s, the vertices in the interior of s are T-junctions that all have the leg on the same side of s. We will prove that every one-sided subdivision is colourable.

Other subdivisions. Here we present some observations regarding possible algorithms or hypothetical counterexamples. We call a subdivision *atomic* or *semi-atomic* if every proper subset $S' \subset S$ such that S' is a subdivision, consists of only one rectangle, or at most two rectangles, respectively. We show that if there is a subdivision that is not colourable, the smallest such subdivision must be semi-atomic, or semi-atomic with one additional rectangle glued to an external edge. We also study sets of boundary colour patterns of different subdivisions of staircases which are surprisingly dissimilar.

2 Corner colour patterns

Consider any subdivision S, whose union is a rectangle $r(S)$. We say S is even if $|S|$ is even, and S is odd if $|S|$ odd.

Lemma 1 *If S is odd and colourable, then every boundary colouring pattern is polychromatic.*

If S is even and colourable, then in every boundary colouring pattern of S, either all four corners have the same colour or two corners have the same colour and two corners have another colour.

Proof. Number the colours to be used from 1 to 4. For a given colouring pattern of S, let C_c be the number of corners of $r(S)$ with colour c, and let I_c the number of other vertices with colour c. Note that each of the corners of $r(S)$ gives its colour to exactly one rectangle of S, while the remaining vertices of S give their colour to exactly two rectangles of S. No rectangle can have two corners of the same colour. Thus, for every colour c, we have $C_c + 2I_c = |S|$.

When S is odd, C_c must be odd, and therefore positive for each colour. Since $\sum_{c\in\{1,2,3,4\}} C_c = 4$, this implies that $C_c = 1$ for each $c \in \{1, 2, 3, 4\}$.

When S is even, C_c must be even, and therefore either 0, 2 or 4, for each colour. □

Subdivisions thus allow five boundary colouring patterns:

- ∷ all corners have different colours;
- = corners use two colours, paired horizontally;
- ‖ corners use two colours, paired vertically;
- × corners use two colours, paired diagonally;
- □ all corners have the same colour.

3 Sliceable subdivisions

Theorem 2 *Sliceable subdivisions are colourable.*

Proof. We prove by induction on the number of rectangles, that every odd sliceable subdivision with $|S| \leq n$ can be coloured and must have ∷ as its only boundary colouring pattern, and every even sliceable subdivision with $|S| \leq n$ can be coloured and has at least two of {=, ‖, ×} as boundary colouring patterns.

When $n = 1$, S has boundary colouring pattern ∷.

Now, consider a subdivision with $|S| = n + 1$ composed of two subdivisions, L and R, separated by a vertical line (the case of a horizontal separating line is symmetric). By induction, both L and R can be coloured separately. We distinguish four cases, depending on whether L and R are odd or even.

(i) If L and R are both odd, then, if necessary, we relabel the colours of R to match the two corners shared with L. The corners of $r(S)$ now use the remaining two colours. We may swap these colours in R so on the boundary they are paired either horizontally = or diagonally ×, satisfying the induction hypothesis.

(ii) If L is even and R is odd, then S is odd. By induction, L has as a boundary colouring pattern = or ×. We can recolour R to match L on shared vertices, resulting in a ∷ pattern for the boundary of S.

(iii) The case of L odd and R even is symmetric.

(iv) For the final case, in which L, R and S are all even, we use the following notation for composing boundary colouring patterns: $P_L P_R \to P_S$ means that joining boundary colouring pattern P_L for L with P_R for R gives boundary colouring pattern P_S for S.

If L and R are both even and admit the ‖ pattern, then S admits the pattern ‖, (‖‖ → ‖). Furthermore, by induction, both L and R admit at least one more pattern out of = and ×. Since == → =, ×× → =, =× → ×, and ×= → ×, we have that S has as a pattern at least one of = and ×.

If L and R are both even and L does not admit the ‖ pattern, then, by induction, L admits *both* the = and × patterns. R admits at least one of = and ×. Thus we can obtain the boundary colouring patterns == → = and ×= → ×, or =× → × and ×× → =; in both cases we obtain patterns = and × for S.

If L and R are both even and R does not admit the ‖ pattern, we apply the above arguments symmetrically and again S admits = and × as boundary colouring patterns.

Therefore the theorem holds for S by induction. □

4 Canonical form

One can build up a subdivision by listing rectangles $R_1, \ldots, R_n$ such that, for any i, the rectangles $R_1, \ldots, R_i$ cover the rectangle defined by the lower right corner of S and the upper left corner of R_i. That is, the rectangles with indices $\leq i$ are separated from those with index $> i$ by a *staircase* that is monotonically increasing in x and y. Such an ordering can be found for any subdivision by rotating it clockwise and extending the "aboveness" partial order to a total order [4]. We say that a subdivision has a *canonical ordering* if there is a unique extension. We can use aboveness to convert a subdivision into a *canonical form* – a subdivision with a unique extension.

Lemma 3 *One can in $O(n \log n)$ time convert a subdivision into canonical form – having unique ordering – without changing the colouring patterns.*

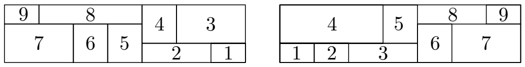

Figure 2: A subdivision in $\nwarrow$-order and $\nearrow$-order.

Proof. [Sketch] Consider the rotated subdivision S as a collection of open rectangles, and open *maximal line segments* (i.e. not containing their endpoints, but extending vertically or horizontally as far as possible – ending at T-junctions.) These are convex and disjoint; any set of disjoint convex objects can be given a total order consistent with aboveness for the direction from upper left to lower right corners of S.

Now, for the vertical and horizontal maximal segments, replace the x and y coordinates, respectively, by their ranks, redraw the subdivision and rotate it back, as in Figure 2. Including the segments in the ordering makes the ordering unique, and the transformation builds that uniqueness into the subdivision. This transformation does not affect the bipartite graph in which each rectangle connects to the vertices at its four corners. □

5 One-sided subdivisions

Theorem 4 *One-sided subdivisions are colourable.*

Proof. We may assume that $S = \{R_1, \ldots, R_n\}$ is a one-sided subdivision in canonical form ($\nwarrow$-order), since conversion to canonical form preserves one-sidedness. We claim that any two consecutive rectangles in $\nwarrow$-order share a corner: Assume inductively this claim holds after adding R_{i-1}. Up to reflection we may assume we are adding R_i above R_{i-1}; let h be the maximal line segment that contains the top edge of R_{i-1}. Since S is in $\nwarrow$-order, the left endpoint $l(h)$ of h must be the top left corner $tl(R_{i-1})$ of R_{i-1}, and the right endpoint $r(h)$ of h must be the bottom right corner of R_i. If $r(h)$ is also the top right corner $tr(R_{i-1})$ of R_{i-1}, then R_i and R_{i-1} share that corner. Otherwise $tr(R_{i-1})$ is a downward T-junction on h. Since S is one-sided, the bottom left corner $bl(R_i)$ of R_i cannot be an upward T-junction on h, so we have $bl(R_i) = l(h) = tl(R_{i-1})$, proving our claim.

We now define a path through S that we can use to colour S. Consider the n rectangles of S in $\nwarrow$-order, $R_1, \ldots, R_n$. They define a path $u_0, \ldots, u_n$ as follows: let u_0 be the lower right corner of R_1, let u_n be the upper left corner of R_n, and let u_i (for $0 < i < n$) be the corner shared by R_i and R_{i+1} (in case of a tie, the left- and bottommost corner common to R_i and R_{i+1} is chosen). Symmetrically, we define a path $v_0, \ldots, v_n$ from the lower left corner to the upper right corner of S, following the rectangles in $\nearrow$-order – the canonical order of their horizontal mirror image. Ties are now broken in favour of the right- and topmost corner shared by two rectangles.

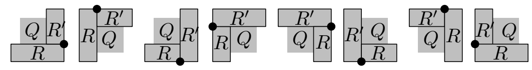

Figure 3: In all cases, Q appears between R and R' either in $\nwarrow$-order or in $\nearrow$-order.

We claim that these two paths are vertex-disjoint. Suppose, for the sake of contradiction, that there is a vertex w that appears on both paths. Then the two rectangles R and R' of which w is a corner must be adjacent in both $\nwarrow$- and $\nearrow$-order. Moreover, R and R' share only one corner, otherwise the tie-breaking mechanism would have put w in one path and the other shared corner in the other path. Now consider all ways in which R and R' can share exactly one corner. One can verify (see Figure 3) that in at least one of the two orderings, R and R' are *not* adjacent. This contradicts our assumption, proving our claim.

We now colour u_i black and v_i red for even i, and we colour u_i white and v_i green for odd i. Since each rectangle has two successive corners on each path, this ensures that each rectangle is polychromatic. □

6 On the hypothetical smallest counterexample

Lemma 5 *If S is even and colourable, it allows at least one boundary colouring pattern out of* ⊏⊐ *and* ╷╷, *and at least one pattern out of* × *and* □.

Proof. Consider a subdivision with boundary colour pattern □ and assume that all corners are coloured black. Consider the graph whose nodes are the vertices of S that are coloured black or white, and whose arcs are given by the pair of the black corner and the white corner of each rectangle in S. This graph consists of two paths whose four end nodes are the corners of S, and possibly a number of cycles; if on any of these paths or cycles we swap all black and white vertices, we maintain a valid colouring pattern.

Since each rectangle contains only one arc, the two paths cannot cross inside a rectangle; since each vertex has degree at most two, the two paths cannot cross in a vertex either. So the path that starts in the lower left corner of S ends in either the upper left or the lower right corner of S (never in the diagonally opposite corner). In the first case, we can change the □-pattern into a ╷╷-pattern by swapping the colours on that path; in the second case, we can change the □-colouring into a ⊏⊐-colouring by swapping the colours on that path—see Figure 4. Note that by swapping colours in this way, every rectangle is either left unchanged (if the arc defined by it is not on the swapping path) or it has its black corner turned white and its white corner turned black, so that each rectangle remains polychromatic. Hence every subdivision that

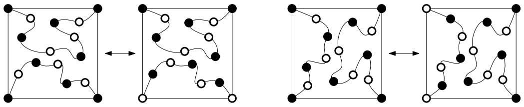

Figure 4: Changing colour patterns into others.

admits a $\square$-pattern also admits a $\sqsupset$ or a $\shortparallel$ pattern.

With similar arguments we can show that a $\times$ pattern can always be changed into a $\sqsupset$ or $\shortparallel$, and that $\sqsupset$ and $\shortparallel$ can always be changed into $\square$ or $\times$. $\square$

For any subdivision S let the set of boundary colouring patterns that S admits be $P(S)$. Then $P(S)$ is one of the eleven sets: $\emptyset$, $\{\sqsupset, \square\}$, $\{\shortparallel, \square\}$, $\{\shortparallel, \times\}$, $\{\sqsupset, \times\}$, $\{\shortparallel, \times, \square\}$, $\{\sqsupset, \times, \square\}$, $\{\shortparallel, \sqsupset, \times\}$, $\{\shortparallel, \sqsupset, \square\}$, $\{\shortparallel, \sqsupset, \times, \square\}$, and $\{::\}$. Figure 5 shows the smallest subdivisions for the last eight sets.

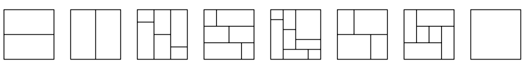

Figure 5: Subdivisions for each good pattern set.

Observe that if there exists a subdivision S such that $P(S) = \{\sqsupset, \square\}$, for every colouring pattern of S, the top corners of S will have the same colour. We can extend S to a subdivision S' by gluing an a rectangle R across the top of S. Then S' is not colourable as every colouring pattern of S forces R to be non-polychromatic. Hence, if any subdivision has $P(S) = \{\sqsupset, \square\}$ or (by symmetry) $P(S) = \{\shortparallel, \square\}$, then there are subdivisions that are not colourable. Therefore, we say that $\emptyset$, $\{\shortparallel, \square\}$ and $\{\sqsupset, \square\}$ are *bad* pattern sets and the other pattern sets are *good*.

Theorem 6 *Suppose there exist subdivisions with bad boundary colouring pattern sets. Let S be a smallest such subdivision. Then S does not contain any proper subset T whose union forms a rectangle with $|T| > 2$, that is, S is semi-atomic.*

Proof. [Sketch] We claim that if S contains a proper rectangular subset T with $|T| > 2$, then we can construct a subdivision S' with $|S'| < |S|$, such that each colouring pattern of S' can be transformed into a colouring pattern of S with the same boundary colouring pattern. Since S is the smallest subdivision with a bad pattern set, the smaller subdivision S' must have a good pattern set. Now, since $P(S') \subseteq P(S)$, and no superset of a good pattern set is bad, S must have a good boundary colouring pattern set. But, this contradicts our assumption that S is a smallest subdivision with a bad pattern set.

To prove our claim we show how to construct S' and transform a colouring pattern for S' into a colouring pattern for S. If $P(T) = \{::\}$, then let S' be the subdivision obtained from S by replacing T with a single rectangle T'. Consider a colouring pattern of S'. By definition of a valid colouring, T' must be polychromatic. Since $:: \in P(T)$, we can remove T' from S' again, colour T with the same colours on its corners as T', and insert T in S'. Thus we obtain a colouring pattern for S, such that $P(S) = P(S')$. If $P(T)$ differs from $\{::\}$, similar constructions are possible: if $P(T) \supseteq \{\sqsupset, \times\}$, we replace T by two rectangles separated by a vertical line; if $P(T) \supseteq \{\shortparallel, \times\}$, we replace T by two rectangles separated by a horizontal line; if $P(T) \supseteq \{\sqsupset, \shortparallel, \square\}$, we shrink T to a line segment. $\square$

7 Looking for algorithms and counterexamples

Trying to generalize the argument for colouring sliceable subdivisions, we consider cutting a subdivision S in $\nwarrow$-order into a prefix set of rectangles S_1 and a postfix set of rectangles S_2 and colouring each separately. Let $P(S_1)$ be the set of boundary colouring patterns along the cut where each pattern is relabeled to be in lex min order. Let S_2' be S_2, reflected in the line $x = y$. We say that $P(S_1)$ and $P(S_2')$ *couple* with pattern $p \in P(S_1)$ if S_2 can be relabeled so that S is colourable with pattern p along the cut.

We enumerated sets of rectangles that may arise as a prefix in a subdivision in $\nwarrow$-order. For stairs with four corners, 359 distinct stair colouring pattern sets were found. Of the possible 15 patterns the min set size was 3 while the max was 14 with average 8.6. For each pair of stair colouring pattern sets, the min size of the coupling set was 1 while the max was 14 with average 5.5. For each stair pattern we found a pair where that was the only pattern they coupled along. In the light of these results Guenin's intricate proof is particularly impressive.

References

[1] Y. Dinitz, M. J. Katz, and R. Krakovski. Guarding rectangular partitions. In *Abstr. 23rd Europ. Workshop Comp. Geom.*, pages 30-33, 2007.

[2] D. Dimitrov, E. Horev, and R. Krakovski. Polychromatic 4-colorings of rectangular partitions. In *Abstr. 24th Europ. Workshop Comp. Geom.*, 2007 (to appear).

[3] B. Guenin. Packing T-joins and edge colouring in planar graphs. Manuscript.

[4] L. J. Guibas and F. F. Yao. On translating a set of rectangles. *Computational Geometry* 1:61–77, 1983.

[5] E. Horev, M. J. Katz, M. Löffler, and R. Krakovski. Polychromatic 4-coloring of guillotine subdivisions. Manuscript.

[6] P. D. Seymour. On multi-colourings of cubic graphs, and conjectures of Fulkerson and Tutte. *Proc. London Math. Soc.* s3-38(3):423-460, 1979.