

ARTICLE TYPE

An asynchronous variational integrator for the phase field approach to dynamic fracture

Zongwu Niu¹ | Vahid Ziaei-Rad^{1,3} | Zongyuan Wu¹ | Yongxing Shen^{*1,2}

¹University of Michigan – Shanghai Jiao Tong University Joint Institute, Shanghai Jiao Tong University, Shanghai, China

²Shanghai Key Laboratory for Digital Maintenance of Buildings and Infrastructure, Shanghai, China

³Department of Environmental Informatics, Helmholtz Centre for Environmental Research – UFZ, Leipzig, Germany

Correspondence

*Yongxing Shen, University of Michigan – Shanghai Jiao Tong University Joint Institute, Shanghai Jiao Tong University, Shanghai 200240, China. Email: yongxing.shen@sjtu.edu.cn

Summary

The phase field approach is widely used to model fracture behaviors due to the absence of the need to track the crack topology and the ability to predict crack nucleation and branching. In this work, the asynchronous variational integrator (AVI) is adapted for the phase field approach of dynamic brittle fracture. The AVI is derived from Hamilton's principle and allows each element in the mesh to have its own local time step that may be different from others'. While the displacement field is explicitly updated, the phase field is implicitly solved, with upper and lower bounds strictly and conveniently enforced. In particular, two important variants of the phase field approach, the AT1 and AT2 models, are equally easily implemented. Several benchmark problems are used to study the performances of both the AT1 and AT2 models, and the results show that the AVI for the phase field approach significantly speeds up the computational efficiency and successfully captures the complicated dynamic fracture behavior.

KEYWORDS:

phase field approach, asynchronous variational integrators, dynamic fracture, computational efficiency

1 | INTRODUCTION

Dynamic fracture refers to crack development processes accompanied by fast changes in applied loads and rapid crack propagation, where inertial forces play an important role during the evolution. Application examples of dynamic fracture include drop tests of electronic devices¹, oil recovery², and impact of automotive laminated glass³.

Dynamic fracture of solids has been extensively studied^{4,5,6,7}. Over the past decades, various numerical methods⁸ to simulate dynamic fracture have been proposed, which can be classified into two groups: discrete approaches and smeared-crack ones. A discrete approach explicitly describes the crack topology, such as the extended finite element method⁹, cohesive zone model¹⁰, element deletion method¹¹, cracking element method¹², and phantom nodes method¹³, just to name a few in the context of dynamic fracture. Conversely, a smeared-crack approach represents the crack by a smeared crack band, which includes the gradient damage model¹⁴, the thick level set approach¹⁵, and so on.

The regularized variational fracture method¹⁶, also called the phase field method for fracture, belongs to the group of smeared-crack approaches. It originates from Griffith's energetic theory and was developed based on the variational approach to brittle fracture by Francfort and Marigo¹⁷. The formulation solves crack problems by minimizing an energy functional that consists of the elastic energy, the external work, and the crack surface energy. This way, crack evolution is a natural outcome of the solution. The phase field method possesses the following advantages: (1) the crack evolves naturally and there is no need of a crack tracking algorithm; (2) there is no need of additional criterion for crack branching and merging; (3) the implementation

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is straightforward even for complicated crack problems in 3D. These advantages facilitate its application for various fracture problems, such as shell fracture¹⁸, beam fracture¹⁹, and carbon dioxide fracturing²⁰. For details about the implementation of this approach, we refer the reader to the work by Shen et al.²¹

For dynamic fracture, Borden et al.²² combined the phase field method with isogeometric analysis with local adaptive refinement to simulate dynamic brittle fracture. Nguyen and Wu²³ presented a phase field regularized cohesive zone model for dynamic brittle fracture. Hao et al.^{24,25} developed the formulations for high-speed impact problems for metals, accounting for volumetric and shear fracture.

However, the phase field model suffers from high computational cost partly because of the small critical time step, which, in turn, results from the necessary fine spatial discretization near the crack to resolve the regularization length scale. In order to overcome this challenge, various schemes to accelerate such computation have been proposed. Tian et al.²⁶ presented a multilevel hybrid adaptive finite element phase field method for quasi-static and dynamic brittle fracture, wherein the refinement is based on the crack tip identified with a certain scheme. Ziaei-Rad and Shen²⁷ developed a parallel algorithm on the graphical processing unit with a time adaptivity strategy to speed up the computation. Li et al.²⁸ proposed a variational h-adaption method with both a mesh refinement and a coarsening scheme based on an energy criterion. Engwer et al.²⁹ proposed a linearized staggered scheme with dynamic adjustments of the stabilization parameters throughout the iteration to reduce the computational cost.

In this work, we adapt the asynchronous variational integrator (AVI) to accelerate the computation for the phase field approach to dynamic fracture. The AVI is an instance of variational integrators. Variational integrators are a class of time integration algorithms derived from Hamilton's principle of stationary action and have the advantages of symplectic momentum conservation and remarkable energy (or Hamiltonian) behavior for long-time integration. In essence, they can be classified into synchronous variational integrators and asynchronous variational integrators. The former, such as central difference, requires all unknown variables to be solved with the same time step, taking into account the global requirement of stability and accuracy.

In contrast, the latter allows independent time steps for each term contributing to the action functional, effectively independent time steps for each element in the context of finite elements. This asynchrony allows the elements with smaller time steps to be more frequently updated. Moreover, the method may be made fully explicit and even in the implicit case, only assembly of the local reaction force vector and stiffness matrix instead of global ones is needed. For linear elastodynamics, the AVI was first introduced by Lew et al.^{30,31}, and the stability and convergence of AVI have been proved by Fong et al.³² and by Focardi and Mariano³³, respectively. In addition, the AVI has been extended to the contact problem³⁴, wave propagation³⁵ and computer graphics³⁶.

In the case of AVI for the phase field approach to fracture, a few adjustments need to be made. First and foremost, the overall Lagrangian is free of the time derivative of the phase field; hence solving the phase field is a local steady-state problem. More precisely, the coupled multi-field system is solved by employing a staggered scheme, in which the displacement and velocity fields are integrated with an explicit scheme while the phase field is the solution of an inequality-constrained optimization problem. In essence, the phase field of only one element is solved at a time, for which it is very convenient to enforce the inequality constraint compared to doing so for the entire domain. This feature allows implementing the AT1 variant³⁷ of the method with a similar cost to the more widely used AT2 variant¹⁶, the two variants differing in terms of the crack surface density function.

The AVI for the phase field approach has many advantages. Firstly, the formulation is derived from the (discrete) Hamilton's principle and possesses a variational structure, which conserves the local symplectic momentum and possesses good global energy behavior of the system. More importantly, this formulation alleviates the high computational cost for certain problems such as those involving bi-materials or functionally graded materials, by allowing each element to have an independent time step only restricted by the respective elemental critical time step for stability. In addition, the formulation only needs to assemble the elemental (or patchwise) residual vector and tangent stiffness matrix and enforces the irreversibility condition on only one element instead of over the entire domain. These features render the proposed formulation highly efficient.

There are other asynchronous methods for dynamic fracture with the phase field. For example, Ren et al.³⁸ proposed an explicit phase field formulation where the mechanical field is solved with a larger time step while the phase field is updated with smaller sub-steps. Suh and Sun³⁹ presented a subcycling method to capture the brittle fracture in porous media, where the heat transfer between the fluid and solid constituents is solved with different time steps as integer multiples of each other. Note that the formulations in Refs. 38,39 are not variational but obtained by directly discretizing the time-dependent mechanical and phase field differential equations, and hence may not enjoy the said advantages of variational integrators, i.e., they may not preserve the symplectic and variational structure of the system.

The paper is organized as follows. In Section 2, we briefly review the formulation of the phase field model for brittle fracture and introduce Hamilton's principle in the continuum Lagrangian framework. In Section 3, we present the asynchronous space-time discretization scheme and derive the discrete Euler-Lagrange equations by discrete variational principle, then we present how to solve the mechanical field and phase field by a staggered scheme. In addition, we summarize the overall implementation of AVI for the phase field approach to fracture. In Section 4, we showcase four benchmark examples under dynamic loading for verification and examining the performance. Finally, we conclude this work in Section 5.

2 | FORMULATIONS

This section devotes to the formulation of a dynamic fracture phase field model through Hamilton's principle for an elastic body with possible cracks represented by a phase field.

2.1 | Hamilton's principle

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be the domain occupied by the reference configuration of a body with possible cracks. Hamilton's principle states that the true trajectory of a body with prescribed initial and final conditions is the stationary point of the action functional with respect to arbitrary admissible variations. Here, we consider Ω with possible internal cracks during a specified time interval $t \in [t_0, t_f]$ with the action functional given by

$$S(\mathbf{u}, d) = \int_{t_0}^{t_f} L(\mathbf{u}, \dot{\mathbf{u}}, d) dt, \quad (1)$$

where $\mathbf{u}(\mathbf{X}, t)$, $\mathbf{X} \in \Omega$, denotes the displacement field of the body, and $\dot{\mathbf{u}} = d\mathbf{u}/dt$ is the velocity field. The scalar field $d : \Omega \times [t_0, t_f] \rightarrow [0, 1]$ is called the phase field, which approximates possible sharp cracks in a diffusive way. Herein, the Lagrangian function is in the form

$$L(\mathbf{u}, \dot{\mathbf{u}}, d) = T(\dot{\mathbf{u}}) - V(\mathbf{u}, d) - \Gamma(d), \quad (2)$$

where $V(\mathbf{u}, d)$ is the potential energy, $\Gamma(d)$ is the crack surface energy, and

$$T(\dot{\mathbf{u}}) = \int_{\Omega} \frac{1}{2} \rho \dot{\mathbf{u}} \cdot \dot{\mathbf{u}} d\Omega \quad (3)$$

is the kinetic energy, where ρ is the initial mass density.

2.2 | Phase field approximation

In this subsection, we revisit the two versions of the phase field model as a basis for subsequent development. In the phase field model of fracture, the sharp crack surface is approximated by a scalar phase field d as shown in Figure 1. The range of this field d has to be between 0 and 1. In particular, our convention is such that the region with $d = 1$ represents the fully cracked state and that with $d = 0$ represents the pristine state of the material. Following Ref. 16, the crack surface energy $\Gamma(d)$ in (2) is given by

$$\Gamma(d) = \int_{\Omega} g_c \gamma(d, \nabla d) d\Omega, \quad (4)$$

where $g_c > 0$ is the critical crack energy release rate, and $\gamma(d, \nabla d)$ is the crack surface density per unit volume,

$$\gamma(d, \nabla d) = \frac{1}{4c_w} \left(\frac{w(d)}{\ell} + \ell |\nabla d|^2 \right), \quad (5)$$

where $\ell > 0$ is the regularization length scale parameter, which controls the width of the transition region of the smoothed crack. Crack geometric function $w(d)$ and normalization constant $c_w = \int_0^1 \sqrt{w(d)} dd$ are model dependent. Specifically, for the brittle fracture, classical examples are $w(d) = d^2$ and $c_w = 1/2$ for the AT2 model; and $w(d) = d$ and $c_w = 2/3$ for the AT1 model³⁷. In addition, a notable difference between the AT2 and AT1 models is that the former gives rise to a more diffuse phase field profile while the latter generates a phase field profile with a narrow support near the crack.

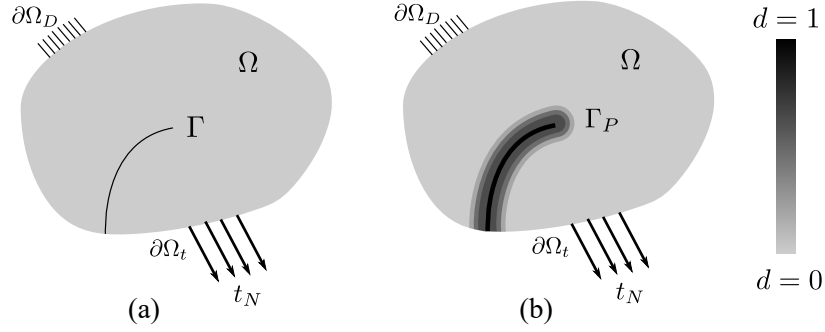


FIGURE 1 Body with an internal crack with: (a) a sharp crack; (b) a crack approximated by the phase field.

The potential energy $V(\mathbf{u}, d)$ in (2) is expressed as

$$V(\mathbf{u}, d) = \int_{\Omega} \psi(\boldsymbol{\varepsilon}(\mathbf{u}), d) \, d\Omega - \int_{\partial\Omega_t} \mathbf{t}_N \cdot \mathbf{u} \, dA - \int_{\Omega} \mathbf{b} \cdot \mathbf{u} \, d\Omega, \quad (6)$$

where ψ is the strain energy density, $\mathbf{t}_N : \partial\Omega_t \rightarrow \mathbb{R}^n$ is the prescribed traction boundary condition, and $\mathbf{b} : \Omega \rightarrow \mathbb{R}^n$ is the body force. The strain tensor is given by $\boldsymbol{\varepsilon} = (\nabla \mathbf{u} + \nabla \mathbf{u}^T)/2$, where $\nabla(\cdot)$ is the gradient operator with respect to \mathbf{X} .

Here we adopt a form for ψ that accounts for the unilateral constraint following Miehe et al.⁴⁰ which involves spectral decomposition of $\boldsymbol{\varepsilon}$. Other choices are, for example, the volumetric-deviatoric split by Amor et al.⁴¹, the micromechanics-informed model by Liu et al.⁴², and the model by Wu et al.⁴³ In the chosen formulation, the strain energy density takes the following form

$$\psi(\boldsymbol{\varepsilon}, d) = g(d)\psi_+ + \psi_-, \quad (7)$$

where $g(d) = (1 - d)^2$ is the degradation function, and ψ_+ and ψ_- are, respectively, the crack-driving and persistent portions of the strain energy density as

$$\psi_{\pm}(\boldsymbol{\varepsilon}) = \frac{\lambda}{2} \langle \text{tr}(\boldsymbol{\varepsilon}) \rangle_{\pm}^2 + \mu \, \text{tr}(\boldsymbol{\varepsilon}_{\pm}^2), \quad (8)$$

where λ and μ are Lamé constants such that $\mu > 0$ and $\lambda + 2\mu > 0$, the Macauley bracket is defined as $\langle \cdot \rangle_{\pm} = (\cdot \pm |\cdot|)/2$, and

$$\boldsymbol{\varepsilon}_{\pm} = \sum_{a=1}^3 \langle \varepsilon_a \rangle_{\pm} \mathbf{n}_a \otimes \mathbf{n}_a, \quad (9)$$

where $\{\varepsilon_a\}_{a=1}^3$ denote the principal strains, \mathbf{n}_a are the corresponding orthonormal principal directions, and the operator \otimes represents the dyadic product. Correspondingly, the Cauchy stress tensor is

$$\boldsymbol{\sigma}_{\pm}(\boldsymbol{\varepsilon}) = \frac{\partial \psi_{\pm}}{\partial \boldsymbol{\varepsilon}} = \lambda \langle \text{tr}(\boldsymbol{\varepsilon}) \rangle_{\pm} \mathbf{1} + 2\mu \sum_{a=1}^3 \langle \varepsilon_a \rangle_{\pm} \mathbf{n}_a \otimes \mathbf{n}_a, \quad (10)$$

where $\mathbf{1}$ is the second-order identity tensor.

2.3 | Spatial discretization

In this subsection, we obtain the semi-discrete Lagrangian by discretizing the displacement field and the phase field with a finite element mesh \mathcal{T}_h for Ω . Let η be the set of nodes of \mathcal{T}_h . The discretized fields take the following form

$$\mathbf{u}(\mathbf{X}) = \sum_{a \in \eta} N_a(\mathbf{X}) \mathbf{u}_a, \quad d(\mathbf{X}) = \sum_{a \in \eta} N_a(\mathbf{X}) d_a, \quad (11)$$

where $\mathbf{u}_a \in \mathbb{R}^n$ and $d_a \in \mathbb{R}$ are the displacement vector and phase field value at node $a \in \eta$, respectively, and N_a is the finite element shape function associated with node a .

The Lagrangian L may be decomposed as

$$L(\mathbf{u}, \dot{\mathbf{u}}, d) = \sum_{e \in \mathcal{T}_h} L_e(\mathbf{u}_e, \dot{\mathbf{u}}_e, \mathbf{d}_e), \quad (12)$$

113 where e is an element of the mesh \mathcal{T}_h , and \mathbf{u}_e , $\dot{\mathbf{u}}_e$, and \mathbf{d}_e are the vectors containing the displacements, velocities, and phase field
114 values of all the nodes of element e , respectively. The quantity L_e is given by

$$L_e(\mathbf{u}_e, \dot{\mathbf{u}}_e, \mathbf{d}_e) = T_e(\dot{\mathbf{u}}_e) - V_e(\mathbf{u}_e, \mathbf{d}_e) - \Gamma_e(\mathbf{d}_e), \quad (13)$$

115 where V_e and Γ_e are the elemental potential energy and elemental surface energy, respectively, and

$$T_e(\dot{\mathbf{u}}_e) = \frac{1}{2} \dot{\mathbf{u}}_e^T \mathbf{m}_e \dot{\mathbf{u}}_e \quad (14)$$

116 is the elemental kinetic energy, where \mathbf{m}_e is the diagonal element mass matrix. Hence, the space-discretized action is in the form

$$S(\{\mathbf{u}_e\}, \{\mathbf{d}_e\}) = \int_{t_0}^{t_f} \sum_{e \in \mathcal{T}_h} \left(\frac{1}{2} \dot{\mathbf{u}}_e^T \mathbf{m}_e \dot{\mathbf{u}}_e - V_e(\mathbf{u}_e, \mathbf{d}_e) - \Gamma_e(\mathbf{d}_e) \right) dt, \quad (15)$$

117 where the curly braces $\{\cdot\}$ represent the collection of all components of all $e \in \mathcal{T}_h$ over the entire time span.

118 3 | ASYNCHRONOUS VARIATIONAL INTEGRATOR WITH THE FRACTURE PHASE 119 FIELD

120 The main feature of the AVI is to assign different time steps to different elements of \mathcal{T}_h . The key idea is the stationarity of (15),
121 a functional over space and time, which allows to divide the total Lagrangian into contributions from elemental terms which
122 may possess independent time steps. For example, the smaller elements in the mesh may be updated a few times while the larger
123 elements are held, according to either a preset schedule or a schedule determined on the fly.

124 In the context of fracture phase field in this work, the phase field is implicitly solved at the element level instead of at the
125 global level, permitting more efficient solvers of inequality constraints.

126 In this section, we detail a phase field formulation and implementation for dynamic fracture through the AVI. The reader inter-
127 ested in the overall algorithmic implementation can directly go to Algorithm 2. In essence, we derive the proposed formulation
128 from the discrete Hamilton's principle of stationary action with the fracture phase field incorporated. In addition, we adapt the
129 reduced-space active set method to enforce the irreversibility constraint involved in the phase field problem.

130 3.1 | Asynchronous discretization

131 This subsection presents the discretization of the time domain through an asynchronous strategy. Such an asynchronous dis-
132 cretization allows each element to have an independent time step. As an example, Figure 2 shows the spacetime diagram of a
133 three-element mesh with asynchronous time steps.

134 Here, we assign to the element $e \in \mathcal{T}_h$ the update schedule

$$\Theta_e = \left\{ t_0 = t_e^0 < t_e^1 < t_e^2 \dots t_e^{N_e-1} < t_f \leq t_e^{N_e} \right\}. \quad (16)$$

135 At these instants, the displacements and velocities, and the phase field values of all nodes $a \in \eta(e)$ are updated, where $\eta(e)$ is
136 the set of nodes of e . In addition, we define the discrete elemental displacements $\mathbf{u}_e^j \equiv \mathbf{u}_e(t_e^j)$ and the discrete elemental phase
137 fields $\mathbf{d}_e^j \equiv \mathbf{d}_e(t_e^j)$ at $t_e^j \in \Theta_e$, and the entire update schedule of the mesh is

$$\Theta = \bigcup_{e \in \mathcal{T}_h} \Theta_e. \quad (17)$$

138 For simplicity, we assume that there are no coincident instants for any pair of elements except for the initial time, i.e., $\Theta_e \cap \Theta_{e'} =$
139 $\{t_0\}$ if $e \neq e'$. The general case with coincident update instants can be handled without much difficulty and will not change the
140 results as long as elements with coincident instants are far away enough from each other. More discussions on the case of update
141 coincidences for the adjacent elements are given in Appendix A.

142 Similarly, we also gather the schedule for node $a \in \eta$ as

$$\Theta_a = \bigcup_{e \ni a} \Theta_e = \left\{ t_0 = t_a^0 < t_a^1 < \dots < t_a^{n_a-1} < t_a^{n_a} \right\}. \quad (18)$$

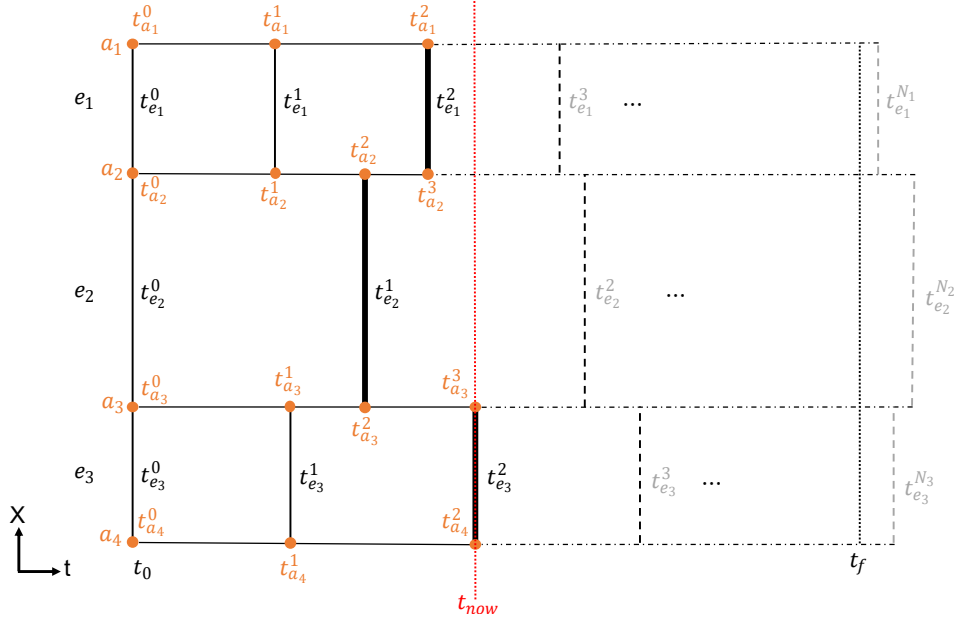


FIGURE 2 Asynchronous discretization of the time domain for a three-element mesh in the reference configuration. The entire update schedule follows the chronological order, i.e., $\Theta = \{t_{e_1}^0, t_{e_2}^0, t_{e_3}^0, t_{e_1}^1, t_{e_3}^1, t_{e_2}^1, t_{e_1}^2, t_{e_3}^2, t_{e_1}^3, t_{e_2}^3, t_{e_3}^3, \dots\}$, where $t_{now} = t_{e_3}^2$ is the current time and e_3 is the current active element.

143 We additionally define $\mathbf{u}_a^i \equiv \mathbf{u}_a(t_a^i)$, $d_a^i \equiv d_a(t_a^i)$, $t_a^i \in \Theta_a$, and the set of nodal displacements

$$\mathcal{U}_a = \{\mathbf{u}_a^i : i = 0, 1, \dots, n_a\}, \quad a \in \eta, \quad (19)$$

144 and the set of nodal phase fields

$$\mathcal{D}_a = \{d_a^i : i = 0, 1, \dots, n_a\}, \quad a \in \eta. \quad (20)$$

145 The triple $(\Theta, \mathcal{U}_a, \mathcal{D}_a)$ defines the discrete trajectory of the system. To solve for this triple, we write the discrete action sum as

$$S_{\text{dis}}(\Theta, \mathcal{U}_a, \mathcal{D}_a) = \sum_{e \in \mathcal{T}_h} \sum_{j=0}^{N_e-1} L_e^j, \quad (21)$$

146 where $L_e^j \approx \int_{t_e^j}^{t_e^{j+1}} L_e dt$.

147 This approximation can be realized by multiple schemes. In this paper, we adopt one such that each node $a \in \eta$ follows a linear
148 trajectory within the time interval $[t_a^i, t_a^{i+1}]$; consequently, the corresponding nodal velocities are constant in the said interval.
149 Moreover, the potential energy and the crack energy terms are approximated with the rectangular rule using their values at t_e^{j+1} .

150 Then the discrete Lagrangian is

$$\int_{t_e^j}^{t_e^{j+1}} L_e dt \approx L_e^j = \sum_{a \in e} \sum_{t_a^i \in [t_e^j, t_e^{j+1})} \frac{1}{2} m_{e,a} (t_a^{i+1} - t_a^i) \left\| \frac{\mathbf{u}_a^{i+1} - \mathbf{u}_a^i}{t_a^{i+1} - t_a^i} \right\|^2 - (t_e^{j+1} - t_e^j) (V_e(\mathbf{u}_e^{j+1}, \mathbf{d}_e^{j+1}) + \Gamma_e(\mathbf{d}_e^{j+1})), \quad (22)$$

151 where $m_{e,a}$ is the mass matrix entry of node a contributed by element e and $\mathbf{d}_e^{j+1} \equiv \mathbf{d}_e(t_e^{j+1})$ is the elemental phase field vector.

152 Finally, the discrete action sum (21) takes the following form

$$S_{\text{dis}} = \sum_{a \in \eta} \sum_{i=0}^{n_a-1} \frac{1}{2} M_a (t_a^{i+1} - t_a^i) \left\| \frac{\mathbf{u}_a^{i+1} - \mathbf{u}_a^i}{t_a^{i+1} - t_a^i} \right\|^2 - \sum_{e \in \mathcal{T}_h} \sum_{j=0}^{N_e-1} (t_e^{j+1} - t_e^j) (V_e(\mathbf{u}_e^{j+1}, \mathbf{d}_e^{j+1}) + \Gamma_e(\mathbf{d}_e^{j+1})), \quad (23)$$

153 where $M_a = \sum_{e, a \in \eta(e)} m_{e,a}$.

3.2 | Discrete variational principle

In this subsection, we derive the formulation of the AVI for the phase field approach to dynamic fracture using the discrete Hamilton's principle⁴⁴. Taking the partial derivative of the discrete action sum (23) with respect to \mathbf{u}_a^i follows

$$0 = \frac{\partial \mathcal{S}_{dis}}{\partial \mathbf{u}_a^i} = \frac{\partial}{\partial \mathbf{u}_a^i} \left(\frac{1}{2} M_a (t_a^i - t_a^{i-1}) \left\| \frac{\mathbf{u}_a^i - \mathbf{u}_a^{i-1}}{t_a^i - t_a^{i-1}} \right\|^2 + \frac{1}{2} M_a (t_a^{i+1} - t_a^i) \left\| \frac{\mathbf{u}_a^{i+1} - \mathbf{u}_a^i}{t_a^{i+1} - t_a^i} \right\|^2 - (t_e^j - t_e^{j-1}) V_e(\mathbf{u}_e^j, \mathbf{d}_e^j) \right), \quad (24)$$

where $a \in \eta(e)$ such at $t_a^i = t_e^j$, which yields the discrete Euler-Lagrange equations

$$\mathbf{p}_a^{i+1/2} - \mathbf{p}_a^{i-1/2} = I_{e,a}^i = - (t_e^j - t_e^{j-1}) \frac{\partial V_e(\mathbf{u}_e^j, \mathbf{d}_e^j)}{\partial \mathbf{u}_a^i}, \quad (25)$$

where $I_{e,a}^i$ may be regarded as the impulse component of node $a \in e$ at the time $t_a^i = t_e^j$, and the discrete linear momentum is defined as

$$\mathbf{p}_a^{i-1/2} = M_a \frac{\mathbf{u}_a^i - \mathbf{u}_a^{i-1}}{t_a^i - t_a^{i-1}} = M_a \mathbf{v}_a^{i-1/2}. \quad (26)$$

Similarly, we take the partial derivative of (23) with respect to \mathbf{d}_e^j as follows

$$0 = \frac{\partial \mathcal{S}_{dis}}{\partial \mathbf{d}_e^j} = \frac{\partial}{\partial \mathbf{d}_e^j} [V_e(\mathbf{u}_e^j, \mathbf{d}_e^j) + \Gamma_e(\mathbf{d}_e^j)], \quad (27)$$

for element e at time t_e^j , then the phase field of element e are updated by

$$\mathbf{d}_e^j = \arg \min_{\mathbf{d}_e^* \leq \mathbf{d}_e \leq 1} \{V_e(\mathbf{u}_e^j, \mathbf{d}_e) + \Gamma_e(\mathbf{d}_e)\}, \quad (28)$$

where \mathbf{d}_e^* represents the phase field value of $\eta(e)$ at their most recent time of update; namely, for node $a \in \eta(e)$, at step i , \mathbf{d}_e^* contains the phase field value d_a^{i-1} . Note that normally in the same element e , \mathbf{d}_e^* may contain phase field values at different times. Here the stationarity condition (27) becomes a minimization in (28) since Γ_e is elliptic. The irreversibility constraint in (28) may be enforced in many ways, for which we have chosen the reduced-space active set strategy, to be discussed in Section 3.4.

Now, we consider an element $e \in \mathcal{T}_h$ with \mathbf{u}_e^{j-1} and $\mathbf{p}_a^{i-1/2}$, $a \in \eta(e)$, known at time t_e^{j-1} , also known \mathbf{d}_e^* , the provisional solution procedure is thus:

- For all $a \in \eta(e)$, solve \mathbf{u}_a^i from (26): $\mathbf{u}_a^i = \mathbf{u}_a^{i-1} + (t_a^i - t_a^{i-1}) M_a^{-1} \mathbf{p}_a^{i-1/2}$.
- Solve \mathbf{d}_e^j from (28).
- For all $a \in \eta(e)$, solve $\mathbf{p}_a^{i+1/2}$ from (25): $\mathbf{p}_a^{i+1/2} = \mathbf{p}_a^{i-1/2} - (t_e^j - t_e^{j-1}) \partial V_e(\mathbf{u}_e^j, \mathbf{d}_e^j) / \partial \mathbf{u}_a^i$.

3.3 | Reformulation for solving the phase field with element patches

The solution procedure mentioned above is variational; however, the results obtained with (28) show an unreasonable crack pattern (see Appendix B), hence we reformulate the constrained optimization problem (28) as follows. Essentially we want to minimize $V + \Gamma$ with the newly obtained \mathbf{u}_e^j (same as before) while the field values of all nodes not belonging to e frozen to their most recent values. To this end, we define the patch for element e

$$\mathcal{T}_e = \{e' \in \mathcal{T}_h : \eta(e) \cap \eta(e') \neq \emptyset\}, \quad (29)$$

as shown in Figure 3. In this way, Eq. (28) is modified to take into account the contributions of its neighboring elements

$$\mathbf{d}_e^j = \arg \min_{\mathbf{d}_e^* \leq \mathbf{d}_e \leq 1} \sum_{e' \in \mathcal{T}_e} [V_{e'}(\{\mathbf{u}_{e'}^*, \mathbf{u}_e^j\}, \{\mathbf{d}_{e'}^*, \mathbf{d}_e\}) + \Gamma_e(\{\mathbf{d}_{e'}^*, \mathbf{d}_e\})], \quad (30)$$

where the superscript $*$ represents the nodal values of $\eta(e') \setminus \eta(e)$ (hollow nodes in Figure 3) at their most recent time of update.

Based on the spatial discretization, the minimization problem (30) leads to the phase field residual of the element

$$(\mathbf{r}_e)_a := \int_{\mathcal{T}_e} \left[g'(d) \psi_+(\epsilon) N_a + \frac{g_c}{4c_w} \left(\frac{w'(d) N_a}{\ell} + 2\ell \nabla d \cdot \nabla N_a \right) \right] d\Omega, \quad a \in \eta(e), \quad (31)$$

180 where $g' = dg/dd$ and the tangent stiffness matrix of the element

$$(\mathbf{k}_e)_{aa'} := \int_{\mathcal{T}_e} \left[g''(d) \psi_+(\varepsilon) N_a N_{a'} + \frac{g_c}{4c_w} \left(\frac{w''(d) N_a N_{a'}}{\ell} + 2\ell \nabla N_a \cdot \nabla N_{a'} \right) \right] d\Omega, \quad a, a' \in \eta(e). \quad (32)$$

181 For the detailed derivation of (31) and (32), the reader is referred to Shen et al.²¹

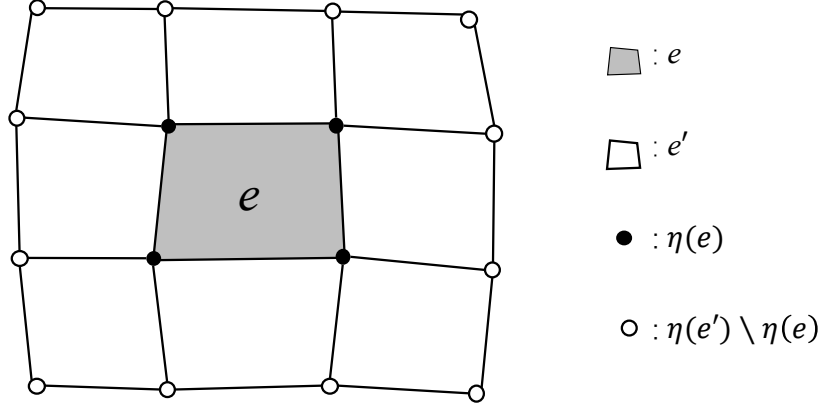


FIGURE 3 Diagram of a patch \mathcal{T}_e that consists of the element e (gray element) and its adjacent elements $e' \in \mathcal{T}_e$ (white elements). At the beginning of an iteration for element e at time t_e^j for solving the phase field \mathbf{d}_e^j , the displacements \mathbf{u}_e^j are known. Correspondingly, both the displacements \mathbf{u}_e^* , and the phase fields \mathbf{d}_e^* , of nodes in $\eta(e') \setminus \eta(e)$ (hollow nodes) assume their most recent values for the iteration.

182 3.4 | Reduced-space active set method for irreversibility constraint

183 There are several approaches to impose the inequality constraints of the phase field when solving (30), such as the local history
184 variable method⁴⁰, the penalty method⁴⁵, and the augmented Lagrangian method⁴⁶.

185 In this work, we employ the reduced-space active set strategy⁴⁷ to ensure the phase field bounds $d \in [0, 1]$ and the
186 irreversibility condition $\dot{d} > 0$. In the discrete setting, the phase field needs to satisfy the condition

$$0 \leq d_a^{i-1} \leq d_a^i \leq 1, \quad \forall i = 1, 2, \dots, n_a \text{ and } a \in \eta(e). \quad (33)$$

187 Note that we solve this inequality-constrained optimization problem efficiently for only one element instead of for the entire
188 domain. Then the solutions are determined by a mixed complementarity problem⁴⁸

$$\begin{cases} d_a^{i-1} = d_a^i, & r_a \geq 0, \\ d_a^{i-1} \leq d_a^i \leq 1, & r_a = 0, \\ d_a^i = 1, & r_a \leq 0, \end{cases} \quad (34)$$

189 where d_a^i is the nodal phase field at time $t_a^i = t_e^j$, and r_a is the phase field residual corresponding to node a . For each iteration,
190 with \mathbf{d}_e^* , \mathbf{u}_e^j , \mathbf{d}_e^* , and \mathbf{u}_e^* of the patch at time t_e^j known, the phase field value of element e can be updated through Algorithm 1.

191 An explanation of Algorithm 1 is as follows. Firstly, the phase field residual \mathbf{r}_e is computed from (31) and the new phase field
192 \mathbf{d}_e^j is initialized with the old phase field \mathbf{d}_e^* . Next, the set $\eta(e)$ is divided into an active set \mathcal{A} and its complementary inactive
193 set $\mathcal{I} = \eta(e) \setminus \mathcal{A}$ according to which case of (34) each node falls into, up to the given tolerance. If \mathcal{I} is an empty set, then the
194 procedure is returned with \mathbf{d}_e^j . Otherwise, the components of \mathbf{r}_e corresponding to nodes of \mathcal{I} need to be close enough to zero,
195 up to a certain tolerance. For this purpose, Newton iteration is performed until convergence. After that, the sets \mathcal{A} and \mathcal{I} are
196 adjusted based on (33). This process of Newton iteration and adjustment of node sets is iterated until all nodes of \mathcal{A} satisfy (34).

Algorithm 1 Reduced-space active set method for solving the phase field

Input: \mathbf{d}_e^* , \mathbf{u}_e^j , \mathbf{d}_e^* , \mathbf{u}_e^* , $\forall e' \in \mathcal{T}_e$ and tol ▷ tol may be chosen differently at each occurrence
Output: \mathbf{d}_e^j

- 1: Compute \mathbf{r}_e from (31) and initialize $\mathbf{d}_e^j = \mathbf{d}_e^*$
- 2: ▷ Let d_a^i and d_a^{i-1} denote the components of \mathbf{d}_e^j and \mathbf{d}_e^* corresponding to node a , respectively, where d_a^i is the nodal phase field at time $t_a^i = t_e^j$. Initially $d_a^i = d_a^{i-1}$.
- 3: $\mathcal{A} = \{a \in \eta(e) : r_a > \text{tol or } (d_a^i = 1, r_a < -\text{tol})\}$, $\mathcal{I} = \eta(e) \setminus \mathcal{A}$ ▷ All components of \mathbf{d}_e^j are in either \mathcal{A} or \mathcal{I} at any time
- 4: **while** $\mathcal{I} \neq \emptyset$ **do**
- 5: **while** $\|\mathbf{r}_{\mathcal{I}}\| > \text{tol}$ **do** ▷ $\{\}_{\mathcal{I}}$ is the component of $\{\}_{e}$ corresponding to nodes of \mathcal{I}
- 6: Compute $\mathbf{k}_{\mathcal{I}}$ from (32), solve $\mathbf{d}_{\mathcal{I}} \leftarrow \mathbf{d}_{\mathcal{I}} - \mathbf{k}_{\mathcal{I}}^{-1} \mathbf{r}_{\mathcal{I}}$, and update $\mathbf{r}_{\mathcal{I}}$
- 7: **end while**
- 8: **for** $a \in \mathcal{I}$ **do**
- 9: **if** $d_a^i > 1 + \text{tol}$ **then**
- 10: Set $d_a^i \leftarrow 1$ and $\mathcal{A} \leftarrow \mathcal{A} \cup \{a\}$, $\mathcal{I} \leftarrow \mathcal{I} \setminus \{a\}$
- 11: **else if** $d_a^i < d_a^{i-1} - \text{tol}$ **then**
- 12: Set $d_a^i \leftarrow d_a^{i-1}$ and $\mathcal{A} \leftarrow \mathcal{A} \cup \{a\}$, $\mathcal{I} \leftarrow \mathcal{I} \setminus \{a\}$
- 13: **end if**
- 14: **end for**
- 15: Compute $\mathbf{r}_{\mathcal{A}}$ from (31)
- 16: **if** $\forall a \in \mathcal{A}$ satisfy (34) **then**
- 17: **return** with \mathbf{d}_e^j
- 18: **else**
- 19: For each $a \in \mathcal{A}$ not satisfying (34), $\mathcal{I} \leftarrow \mathcal{I} \cup \{a\}$, $\mathcal{A} \leftarrow \mathcal{A} \setminus \{a\}$
- 20: **end if**
- 21: **end while**

3.5 | Algorithmic implementation

This section focuses on the algorithmic implementation of the AVI for the phase field fracture. The overall pseudo-code is provided in Algorithm 2. The time step of each element is taken as a fraction of their critical time step and is computed by

$$t_{crit} = C_{CFL} \frac{2}{\omega_e}, \quad (35)$$

where C_{CFL} is taken as 0.6 and ω_e is the maximum natural frequency of the element, which is the square root of the maximum eigenvalue of the generalized eigenvalue problem $\mathbf{k}_e \mathbf{U} = \omega^2 \mathbf{m}_e \mathbf{U}$. The time step of each element allows certain adaptivity, although we keep C_{CFL} constant in this work for simplicity.

Due to the asynchrony of the algorithm, we employ the priority queue⁴⁹ to keep track of the causality. The priority queue assigns each element a priority according to their next update time where the element to be updated at a sooner time has a higher priority. In other words, the priority queue ensures that all elements in the queue are ordered according to their next time to be updated, and the top element in the queue is always the one whose next update time is the closest to the current time in the future.

The implementation details are shown in Algorithm 2. First, the first time steps of all elements in the mesh are computed and pushed into the priority queue to establish the initial queue. Within each iteration, the priority queue pops an element (calls the active element) and its next update time. The nodal displacements, phase fields, and momenta of the active element are updated accordingly. Subsequently, the next update time of this element is computed and if this time is less than t_f , it is pushed into the priority queue. The algorithm continues until the priority queue is empty.

4 | NUMERICAL EXAMPLES

In this section, we showcase four benchmark examples to demonstrate the ability of the proposed formulation in capturing the key features of dynamic fracture. In addition, we compare the computational costs and the energy conservation behavior of

Algorithm 2 Algorithm of AVI for the phase field to fracture

Input: $\mathcal{T}_h, \eta, \Theta, Q = \emptyset$ and $\{\mathbf{u}_a^0, \mathbf{d}_e^0, \mathbf{p}_a^{1/2} \mid a \in \eta\}$
Output: $\mathbf{u}_a^i, \mathbf{d}_e^j, \mathbf{p}_a^{i+1/2}$, where $i, j = 1, 2, 3, \dots$, corresponding to Θ

- 1: Initialization: $\mathbf{u}_a \leftarrow \mathbf{u}_a^0, \mathbf{d}_e \leftarrow \mathbf{d}_e^0, \mathbf{p}_a \leftarrow \mathbf{p}_a^{1/2}, \tau_a \leftarrow t_0$ for all $a \in \eta$
- 2: **for** all $e \in \mathcal{T}_h$ **do**
- 3: $\tau_e \leftarrow t_0$
- 4: Compute t_e^1 and push (t_e^1, e) into priority queue Q
- 5: **end for**
- 6: **while** Q is not empty **do**
- 7: Extract next element: pop (t, e) from Q
- 8: Compute displacements \mathbf{u}_e with (26) and update node's time $\tau_a \leftarrow t$ for all $a \in \eta(e)$
- 9: Compute the phase fields \mathbf{d}_e with (30) following Algorithm 1
- 10: **if** $t < t_f$ **then**
- 11: Compute momentum \mathbf{p}_a with (25) for all $a \in \eta(e)$
- 12: Update element's time: $\tau_e \leftarrow t$
- 13: Compute t_e^{next} and schedule e for next iterate: push (t_e^{next}, e) into Q
- 14: **end if**
- 15: **end while**

TABLE 1 Material properties for the numerical examples of Section 4.1 through Section 4.3.

Parameter	Symbol	Section 4.1	Section 4.2	Section 4.3
Material	-	Silica glass	Soda-lime glass	Maraging steel 18Ni(300)
Young's modulus (GPa)	E	32	72	190
Poisson's ratio	ν	0.2	0.22	0.3
Density (kg/m ³)	ρ	2450	2440	8000
Critical energy release rate (J/m ²)	g_c	3	3.8	2.213×10^4
Rayleigh wave speed (m/s)	v_R	2119	3172	2803

our approach for the AT1 and AT2 models. While all examples conducted here are two-dimensional, the proposed formulation is easy to be generalized to three dimensions. In particular, we use an unstructured mesh with first-order quadrilateral finite elements, which is refined along the potential crack paths.

A note on the post-processing is as follows. We sample the solution at a frequency of every 500,000 elemental iterations. For example, if the current time $t_{now} = t_{e_3}^2$ in Figure 2 happens to be a sampling time, then post-processing results are obtained using the most recent nodal displacement, velocity, and phase field values prior to t_{now} , i.e., values at nodal times $t_{a_1}^2, t_{a_2}^3, t_{a_3}^3, t_{a_4}^2$ for the nodes shown. For example, the crack patterns to be plotted are obtained using the most recent phase field values prior to the sampling times. More accurate results can be obtained by interpolation using the values before and after the sampling time, which is not undertaken in this work for simplicity.

4.1 | Boundary tension test

In this section, a pre-notched rectangular plate loaded dynamically in tension is modeled. The geometry and boundary conditions are shown in Figure 4. A constant traction $\sigma^* = 1$ MPa is applied on the top and bottom edges throughout the simulation and the remaining boundary is traction free. This benchmark problem has been widely studied, for example by Song et al.⁵⁰ using the extended finite element method, by Nguyen¹⁰ with the cohesive zone method, and by Borden et al.²² with a synchronous phase field approach to fracture, as well as in experimental studies^{51,52}. As described in Ref. 51, a crack emerges at the notch tip and starts propagating to the right in a stable way. Over a certain distance, the main crack branches into two symmetrical sub-cracks and continue growing until it reaches the right surface.

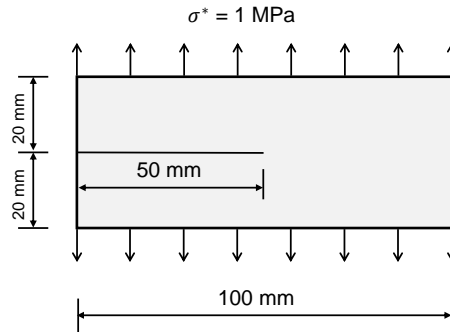


FIGURE 4 Geometry and boundary conditions of the boundary tension test, in which a pre-notched plate is under tension.

The material used in this test is silica glass and its properties are listed in Table 1. A plane strain state with a unit thickness is assumed. The length scale parameter takes $\ell = 1 \times 10^{-3}$ m, which is small enough with respect to the specimen dimensions. Two different mesh levels are used: Mesh 1 with $h = 5 \times 10^{-4}$ m = $\ell/2$, and Mesh 2 with $h = 2.5 \times 10^{-4}$ m = $\ell/4$ in the refined region.

The final phase field results are shown in Figure 5. As seen, there is no significant difference of in the crack pattern between the AT2 and AT1 models. The crack branches at between 34 and 36 μ s and reaches the right boundary at $t \approx 80 \mu$ s. The upper crack branching angle is around 27.5° , which agrees well with the results in Refs. 53,23. In addition, the bifurcation angle of the lower branch is slightly different from that of the upper one, which may be caused by the non-symmetric discretization of the mesh. This non-perfect symmetry was also observed by Ren et al.³⁸.

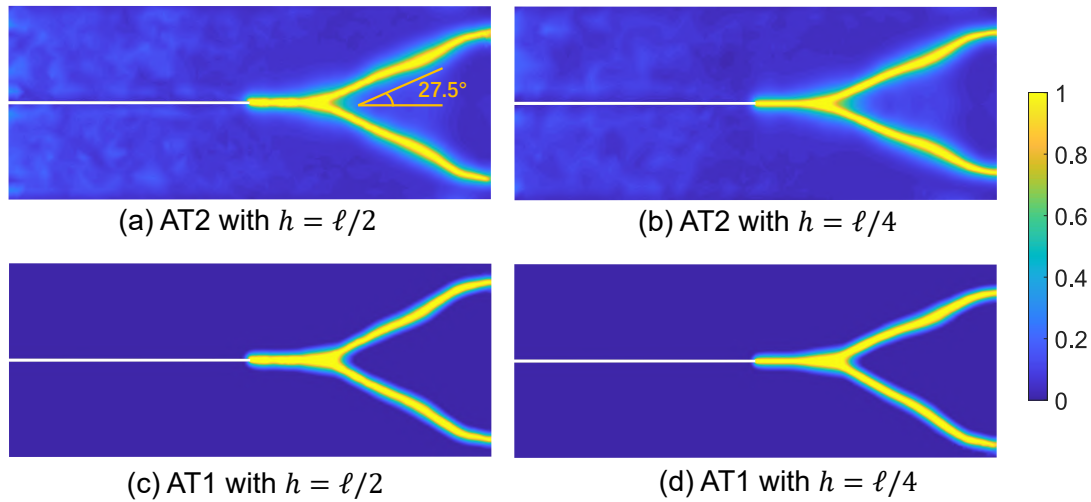


FIGURE 5 Phase field results of the test in Section 4.1.

Figure 6(a) shows the evolution of the total crack tip velocity calculated by

$$v_{tip} = \frac{1}{g_c} \frac{d\Gamma}{dt}, \quad (36)$$

and normalized by the Rayleigh wave speed. In particular, the derivative is obtained by comparing the values of Γ at consecutive sampling times. It is observed that at the beginning, a single main crack propagates to the right with a speed of less than 60% of the Rayleigh wave speed. Then, the main crack branches into two sub-cracks and in this respect the total crack tip velocity of both branches is plotted, which is still less than 60% of twice the Rayleigh wave speed. Therefore, whether before or after the branching emerges, the velocity is within a reasonable range. Moreover, the overall propagation speed during the evolution is

247 in good agreement with the results reported by Borden et al.²² Note that the velocity of Borden et al.²² is only for one of the
 248 branches while it is the total velocity of both branches in this work.

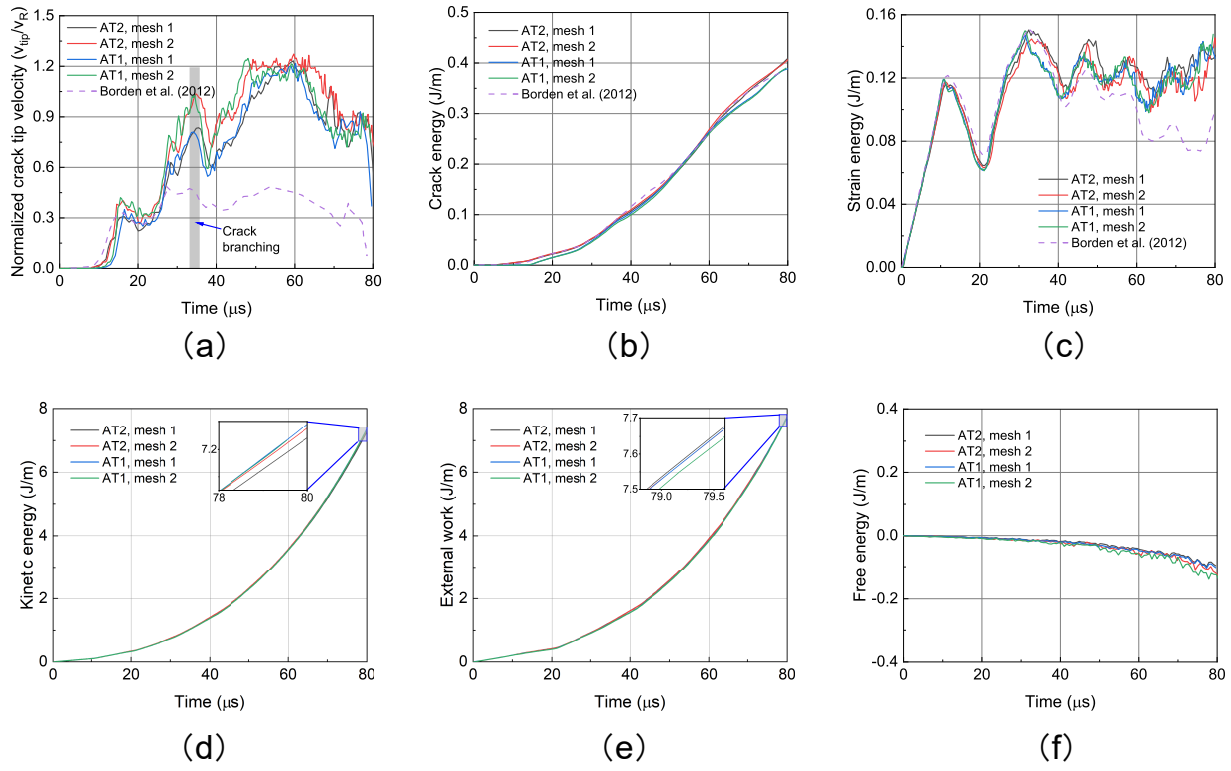


FIGURE 6 Results of the test in Section 4.1 compared with those of Borden et al.²² Evolution of: (a) normalized total crack tip velocity; (b) crack energy $\Gamma(d)$; (c) strain energy; (d) kinetic energy $T(\dot{\mathbf{u}})$; (e) external work; and (f) free energy $T(\dot{\mathbf{u}}) + V(\mathbf{u}, d) + \Gamma(d)$. Note that for (a), the velocity result in Borden et al.²² is only for one of the branches while it is the total velocity of both branches in this work. Here for (e) as well as for (b)(c)(d) the results are obtained by sampling, see the text for more details. It can be seen that the free energy is 1.32% of the external work at the end, indicating the conservation of energy.

249 Figure 6 (b)-(d) present the evolution of the crack surface energy, the strain energy, and the kinetic energy, respectively. The
 250 crack surface energy monotonically increases as expected due to the unilaterality of the phase field. In addition, the strain energy
 251 evidently shows the periodic oscillation at the beginning and this trend gradually weakens with crack evolution, because the
 252 stress wave is reflected at the boundaries and cracks, and interacts with itself.

253 Figure 6(e) shows the evolution of the external work, which is calculated from the second term on the right hand side of
 254 Eq. (6). It is clear that the kinetic energy accounts for most of the energy converted from external work.

255 Figure 6(f) shows the evolutions of the free energy $T(\dot{\mathbf{u}}) + V(\mathbf{u}, d) + \Gamma(d)$. The free energy is negative and its magnitude is
 256 only 1.32% of the external work at the end. This small negative numbers demonstrate that the method possesses remarkable
 257 energy conservation property and is energetically stable.

258 Figure 7 shows the maximum principal stress with Mesh 1 at $t = 70 \mu\text{s}$. Therein, stress concentration is clearly seen at the
 259 crack tips and the results are in good agreement with those in Ref. 54.

260 Figure 8(a) shows the number of updates per element for Mesh 1. By construction, the elements near the cracks are updated
 261 more frequently than those far away from the cracks. Figure 8(b) shows the wall time corresponding to $t_f = 80 \mu\text{s}$ for the AVI
 262 and a synchronous method (central difference). The simulations are conducted using an in-house MATLAB code on a personal
 263 computer with Intel(R) Core(TM) i7-6700 CPU @ 3.40GHz and 16.0 GB RAM. The results indicate that the AVI for both AT2
 264 and AT1 models take similar computational time. In addition, the computational efficiency of the AVI is approximately four
 265 times that of the synchronous method.

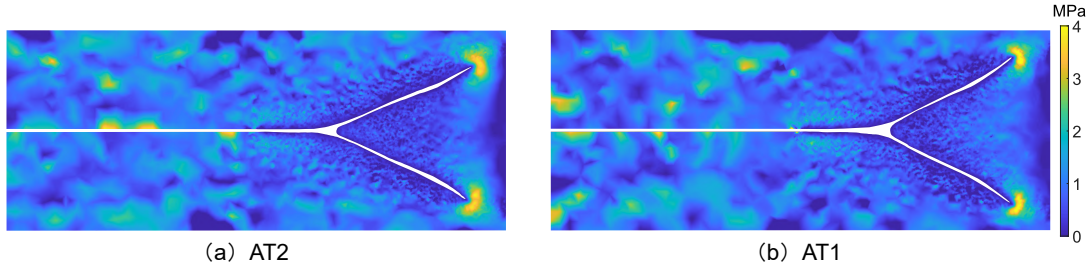


FIGURE 7 Maximum principal stress of the test in Section 4.1 with Mesh 1 at $t = 70 \mu s$, where the region $d > 0.9$ is removed.

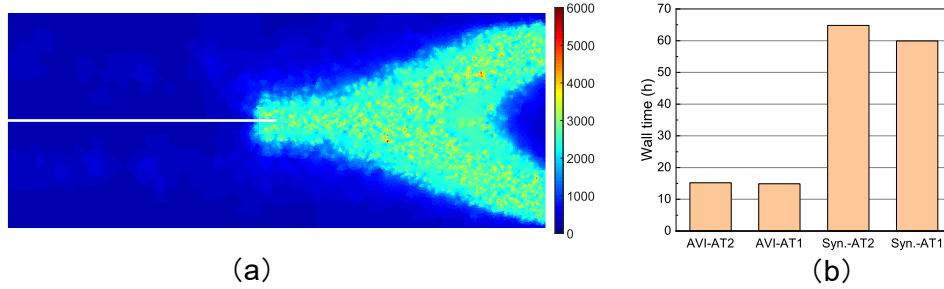


FIGURE 8 The performance indicators of the tests in Section 4.1 with Mesh 1: (a) number of updates per element; and (b) wall time corresponding to $t_f = 80 \mu s$, where 'Syn.' represents the results of a synchronous method (central difference).

Table 2 collects statistics of the computational cost for the example at hand. As a platform-independent indicator, the number of updates of each element throughout the simulation for each case is counted. The second, third, and fourth columns represent the minimum, maximum, and median numbers of updates among the elements, respectively. The fifth column is the total numbers of updates for all elements of AVI. The sixth column is the total numbers of updates of synchronous integration, where the data is estimated by assuming the global critical time step is used for the same time interval $[t_0, t_f]$. As shown, the total numbers of AVI updates is approximately 31% of those of a synchronous integration. Considering that it is even more costly to implicitly solve for the phase field with a synchronous method per time step, the data in Table 2 indicates that the proposed scheme effectively reduces the computational cost compared with a synchronous method.

TABLE 2 Numbers of elemental updates for the test in Section 4.1 during the entire simulation.

Mesh	Minimum	Maximum	Median	AVI total [†]	Synchronous integration (estimated) [‡]
Mesh 1	178	5,983	2,106	27,400,002	86,382,554
Mesh 2	160	10,150	3,440	62,285,189	201,051,200

[†]Total numbers of the elements involved in the update of the mechanical field and phase field.

[‡]This column of data is estimated by assuming the global critical time step is used throughout the computation for the same desired time interval $[t_0, t_f]$.

4.2 | Compact tension test

In this section, we investigate a series of dynamic loads applied on pre-crack surfaces as the compact tension (CT) test. The geometry and boundary conditions are shown in Figure 9. Three different constant normal tractions $\sigma^* = \{0.5, 3, 6\}$ MPa are applied on the pre-crack surfaces. This benchmark problem has been studied by Bobaru and Zhang⁵⁵ using peridynamics and Mandal et al.⁵⁶ with a synchronous phase field approach.

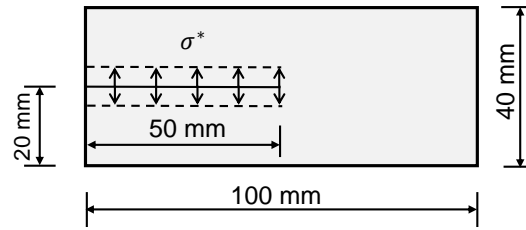


FIGURE 9 Geometry and boundary conditions for the dynamic CT test.

279 The material is assumed to be soda-lime glass, whose properties are given in Table 1. Plane strain state is assumed. The length
 280 scale parameter $\ell = 5 \times 10^{-4}$ m and the mesh size $h = 2.5 \times 10^{-4}$ m = $\ell/2$ in the refined region are used for all cases. Figure 10
 281 shows the phase field results for the CT test. For $\sigma^* = 0.5$ MPa, a straight crack without branching is obtained. For larger values
 282 of σ^* crack branching is observed and the branching location moves to the left with the increase of σ^* . The crack branching
 283 happens at around $17.3 \mu\text{s}$ and $9.2 \mu\text{s}$, and the branching angles are 52° and 46° for $\sigma^* = 3$ MPa and 6 MPa, respectively. Also,
 284 there is no significant difference of the crack patterns between the AT2 and AT1 models for the same load. Moreover, the crack
 285 patterns, branching instants, and branching angles are all in good agreement with the results reported in Ref. 56.

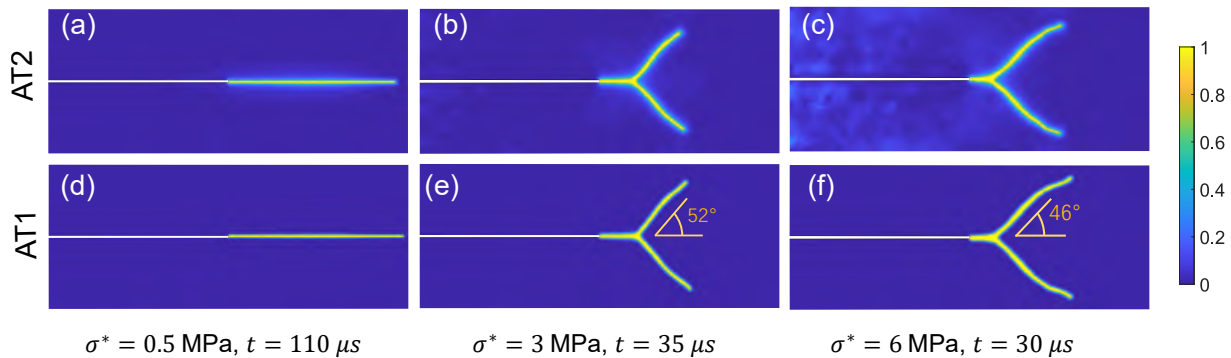


FIGURE 10 Phase field results of the test in Section 4.2 under different loads.

286 Figure 11(a) illustrates the evolution of the normalized total crack tip velocity of CT test for $\sigma^* = 3$ MPa. Like the case of
 287 Figure 6(a), a main crack propagates to the right with an increasing speed less than 60% of the Rayleigh wave speed. Then, then
 288 main crack branches into two sub-cracks and the total speed is still less than 60% of twice the Rayleigh wave speed. Note that
 289 the velocity of Mandal et al.⁵⁶ is only for one of the branches while our results are the total velocity of both branches. Figure
 290 11(b) shows the evolution of the crack energy, and the results of both models are slightly higher than the result reported by
 291 Mandal et al.⁵⁶. Figure 11(c) shows the evolution of the strain energy. An interesting observation is that the curve presents a
 292 periodic oscillation with a period of approximately $6.8 \mu\text{s}$, which can be explained as follows. During the process, the stress
 293 waves propagate from the crack to the top and bottom boundaries and then are reflected until they meet the crack again. The
 294 time it takes the stress wave to travel a round trip can be estimated by $l/v_D = 6.89 \mu\text{s}$, where $l = 40$ mm is twice the half-width
 295 of the specimen and $v_D = 5800$ m/s is the dilatational wave speed of soda-lime glass. This process is repeated, and hence the
 296 periodicity. Figure 11(d-e) show the evolution of the kinetic energy and the external work, both of which monotonically increase.
 297 Figure 11(f) shows the free energy of the AT2 and AT1 model during the evolution. As we can see, the magnitude of the free
 298 energy only accounts for 1.69% of the external work, which indicates the conservation of energy.

299 Figure 12 shows the maximum principal stress for $\sigma^* = 3$ MPa and 6 MPa, respectively. Therein, stress concentration is
 300 clearly seen at the crack tips.

301 Figure 13(a) shows the number of updates per element with the mesh of the case $\sigma^* = 3$ MPa. Like the previous example, the
 302 elements in the refined region are updated more frequently while those without refinement are updated less frequently. Figure

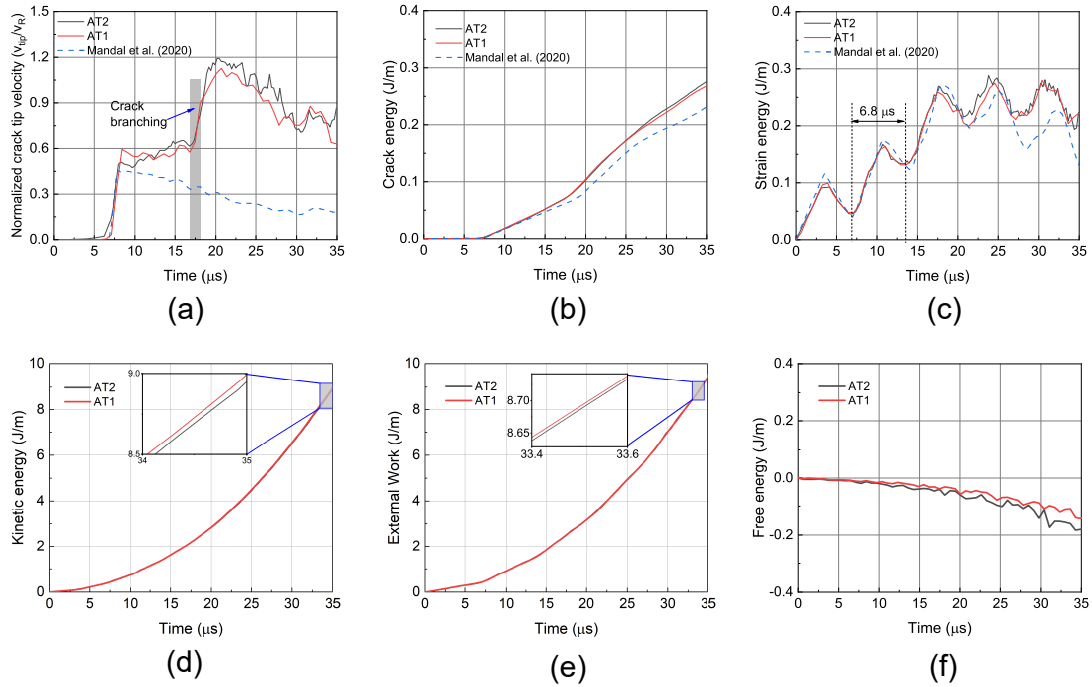


FIGURE 11 Results of the test in Section 4.2 with $\sigma^* = 3$ MPa compared with Mandal et al.⁵⁶ Evolution of: (a) normalized total crack propagation velocity. Note that the velocity of Mandal et al.⁵⁶ is only for one of the branches while our result is the total velocity of both branches.; (b) crack energy; (c) strain energy; (d) kinetic energy; (e) external work; and (f) free energy. The free energy is only 1.69% of the external work, which indicates the conservation of energy.

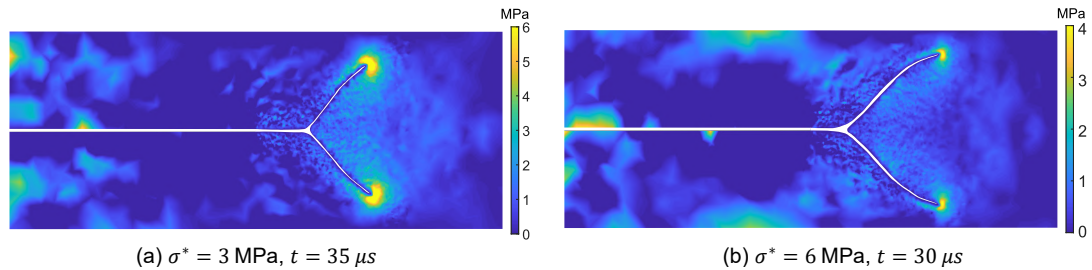


FIGURE 12 Maximum principal stress of the test in Section 4.2.

303 13(b) shows the wall time corresponding to $t_f = 35 \mu s$ of the AVI and a synchronous method for both models. As can be seen,
304 the AVI takes one third time of a synchronous method.

305 4.3 | The Kalthoff-Winkler test

306 This section studies the Kalthoff-Winkler experiment in which an edge-cracked plate is under impact velocity. Due to symmetry,
307 only half of the plate is considered. The geometry and boundary conditions are shown in Figure 14. In the experiment^{57,58}, the
308 brittle failure mode with a crack propagating at about 70° was observed at a certain impact speed, and the relevant numerical
309 results were reported by other researchers using the extended finite element method⁵⁰, peridynamics⁵⁹, and the gradient damage
310 method¹⁴.

311 The material is maraging steel 18Ni(300), whose properties are given in Table 1. A plane strain state is assumed. The length
312 scale parameter $\ell = 3.9 \times 10^{-4}$ m and two different meshes are used: Mesh 1 with size $h = 1.95 \times 10^{-4}$ m = $\ell/2$ and Mesh 2
313 with $h = 9.75 \times 10^{-5}$ m = $\ell/4$ in the refined region.

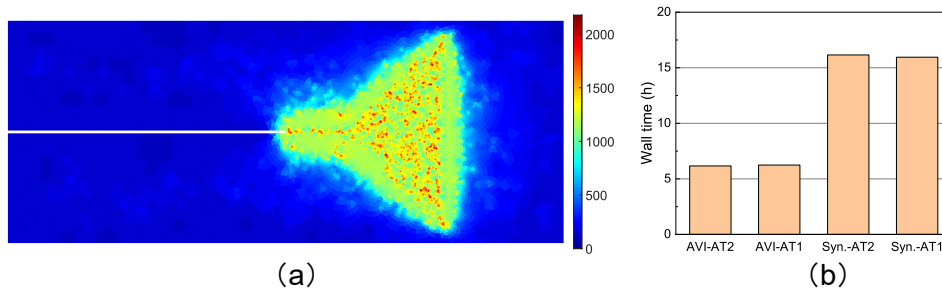


FIGURE 13 The performance indicators of the tests in Section 4.2 with the mesh of the case $\sigma^* = 3 \text{ MPa}$: (a) number of updates per element; and (b) wall time corresponding to $t_f = 35 \mu s$.

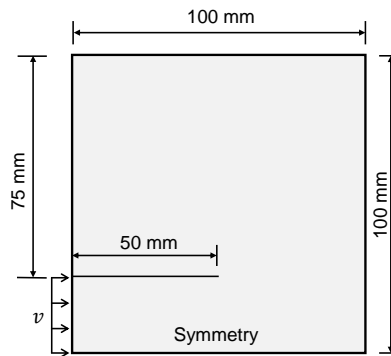


FIGURE 14 Geometry and boundary conditions of the Kalthoff test, where $v = 16.5 \text{ m/s}$.

314 Figure 15 shows the final phase field patterns at $t = 87 \mu s$ for different meshes and models. The crack propagates at $25.5 \mu s$
 315 and with an angle of about 67° with the horizontal line, which is in good agreement with the experimental results⁵⁷ and the
 316 numerical results using the phase field method^{60,61}.

317 Figure 16(a) presents the evolution of the normalized crack tip velocity of the Kalthoff test. The velocity results here are
 318 almost two times that reported by Liu et al.⁵⁴ The differences in crack tip velocity may be caused by the different post-processing
 319 methods, where Ref. 54 employed an alternative method that is different from ours by Eq. (36), to be discussed later. Figure
 320 16(b) shows the evolution of the crack energy calculated by (4), which agrees well with the numerical results in Ref. 46. In
 321 addition, the crack energy of the AT2 model is a little higher than the AT1 model for both meshes. Figure 16(c) and (d) show
 322 the evolution of the strain energy and kinetic energy, respectively, and the results are consistent with numerical results reported
 323 by Zhang et al.⁶² Figure 16(e) shows the evolution of the external work, to the best of our knowledge, there is no relevant report
 324 on external work of the Kalthoff test by the phase field method at present. However, our result is in good agreement with the
 325 result using the cohesive zone model by Park et al.⁶³ Figure 16(f) shows the evolution of the free energy. As we can see, the
 326 free energy gradually increases, reaching between 715.34 J and 1403.51 J at the end of the simulation, which seems to violate
 327 the law of conservation of energy. This phenomenon appears to be an open question.

328 Figure 17 shows the distribution of the maximum principal stress for Mesh 2. The stress concentration is clearly seen at the
 329 crack tip and the bottom right corner, in both AT2 and AT1 models. The result is in good agreement with those in Liu et al.⁵⁴

330 Figure 18(a) shows the number of updates per element of the Kalthoff test with Mesh 1. Similar to the previous examples,
 331 the elements near the crack are updated more frequently by design while those far away are updated less frequently. Figure 18(b)
 332 shows the wall time corresponding to $t_f = 87 \mu s$, indicating that the AVI takes only one-third of the wall time of a synchronous
 333 method.

334 Alternative method to calculate the crack tip velocity

335 As an attempt to reconcile the discrepancy, we employ the iso-curve strategy to calculate the crack tip velocity, as also done by
 336 Liu et al.⁵⁴ In this approach, the position of the crack tip is determined by the iso-curve with phase field $d = 0.9$. Therefore,
 337 the crack tip velocity is recalculated by $v_n = \|\mathbf{x}_n - \mathbf{x}_{n-1}\| / (t_n - t_{n-1})$, where \mathbf{x}_n represents the location of the crack tip at n th

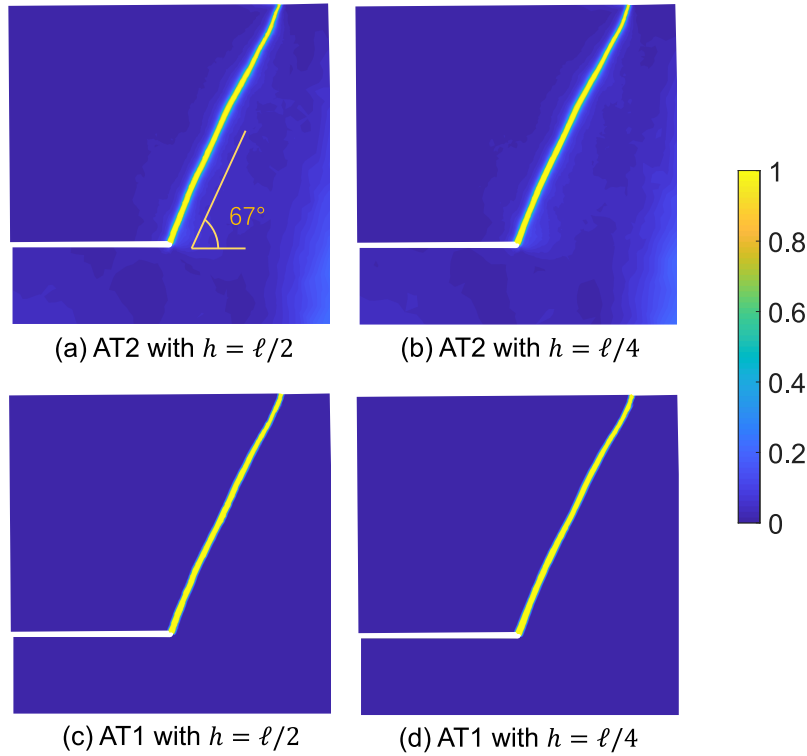


FIGURE 15 Phase field results of the test in Section 4.3 at $t = 87 \mu\text{s}$.

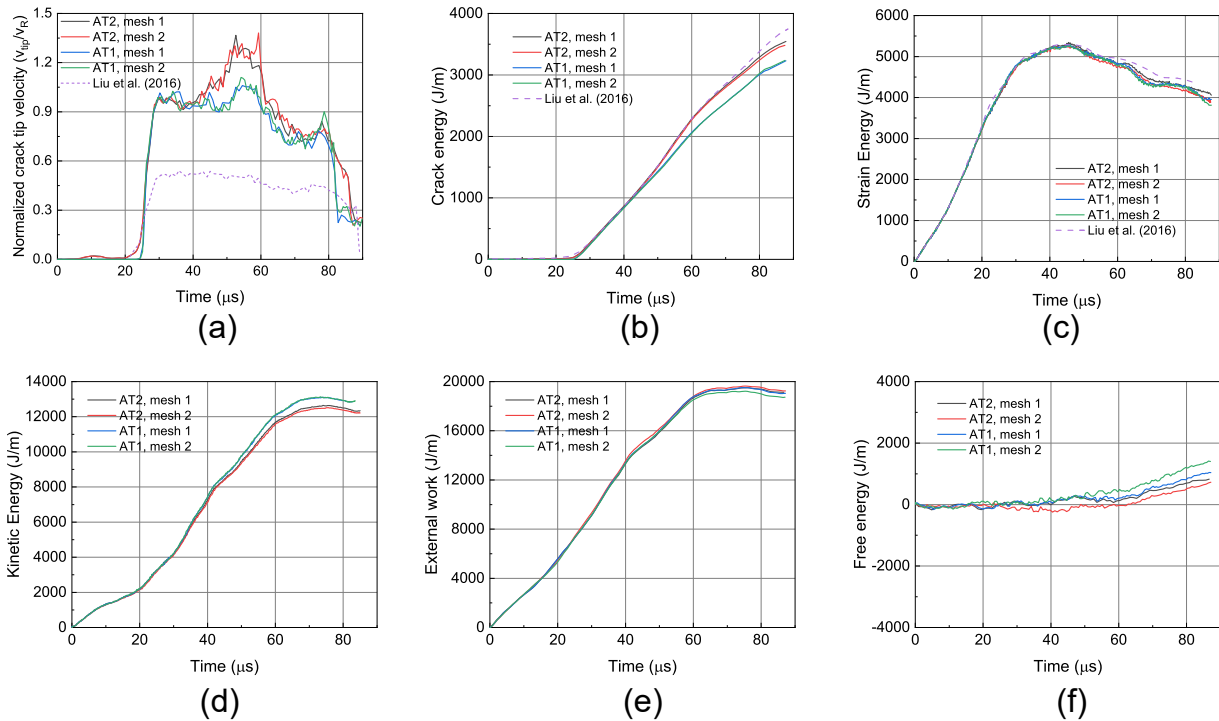


FIGURE 16 Results of the test in Section 4.3 compared with Liu et al.⁵⁴ Evolution of: (a) normalized crack tip velocity; (b) crack energy; (c) strain energy; (d) kinetic energy; (e) external work; and (f) free energy. The external work (e) is obtained by sampling the power of the reaction force, and then integrating this power with respect to time. The free energy gradually increases, reaching between 3.74% and 7.34% of the external work in the end, which seems to violate the law of conservation of energy. See Figure 19 for a comparison.

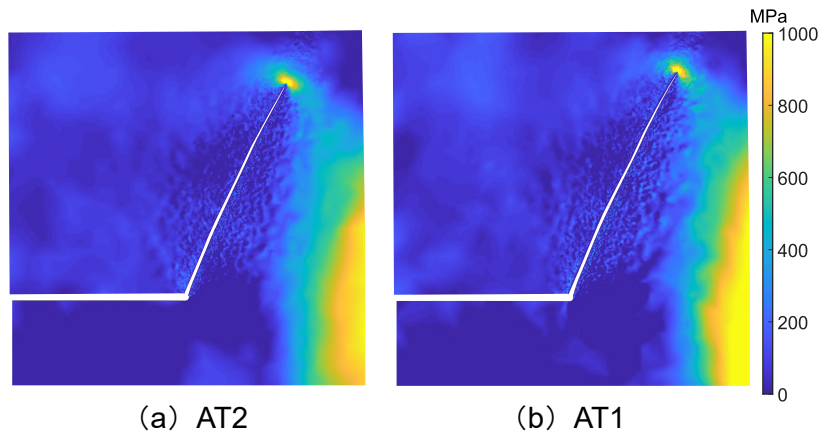


FIGURE 17 Maximum principal stress of the test in Section 4.3 with Mesh 2 at $t = 70 \mu s$.

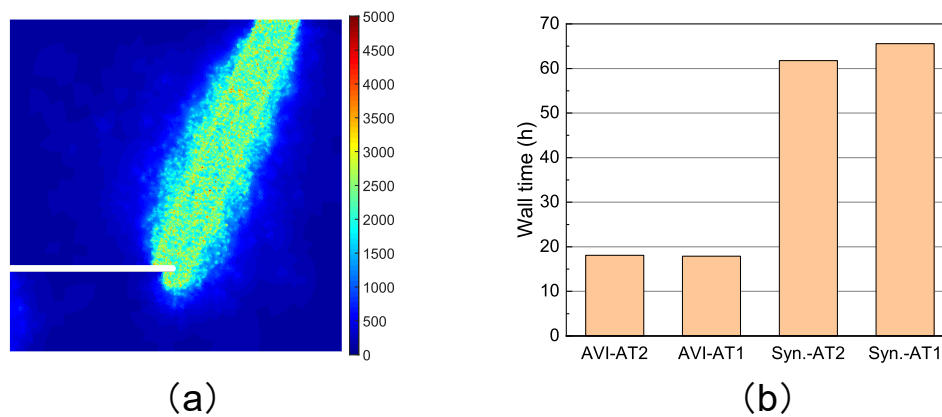


FIGURE 18 The performance indicators of the tests in Section 4.3 with Mesh 1: (a) number of updates per element; and (b) wall time corresponding to $t_f = 87 \mu s$.

sampling time t_n , and the result is shown in Figure 19(a). As can be seen, the crack accelerates to near $0.6v_R$ and then remains with this velocity during the propagation until it reaches the top boundary, which agrees well with the result reported in Ref. 54.

With this iso-curve scheme, the four cases of Kalthoff test show a similar final crack length of approximately $l_{crack} = 83$ mm, see Figure 19(b) with the right vertical axis. Correspondingly, the crack energy can be computed as

$$\hat{\Gamma} = g_c l_{crack}, \quad (37)$$

with the value of 1836.79 J for a sharp crack, see Figure 19(b) with the left vertical axis. A significant difference is that $\hat{\Gamma}$ is much smaller than Γ , and the ratios of $\Gamma/\hat{\Gamma}$ are 1.9 and 1.75 for the AT2 and AT1 models, respectively.

In addition, we recalculate the free energy by using (37) instead of (4), i.e., $T(\dot{\mathbf{u}}) + V(\mathbf{u}, d) + \hat{\Gamma}$, and the result is shown in Figure 19(c). As seen, with $\hat{\Gamma}$, the results are energetically stable and satisfy the conservation of energy.

Discussions

In the Kalthoff test, the crack energy calculated by (4) is higher than that by (37). This phenomenon is not unique to this work but also reported in Refs. 22,54,61,46,62,64,65, in which the ratio of $\Gamma/\hat{\Gamma}$ is between 1.90 and 2.45, equal to or even higher than our value. Meanwhile, in Ref. 23, this ratio is 1.37. In addition, this phenomenon was also reported in other dynamic phase field fracture by Ziaei-Rad and Shen²⁷, where the ratio is approximately 2.

Although the main reason why Γ is higher than $\hat{\Gamma}$ need to be further investigated, we suggest that the way of enforcing irreversibility constraint is not an ideal candidate. Borden et al.²² suggested that the strain-history field (alternative way to enforce the irreversibility) could play an important role, but the ratio of $\Gamma/\hat{\Gamma}$ is still 1.4 despite the strain-history field being

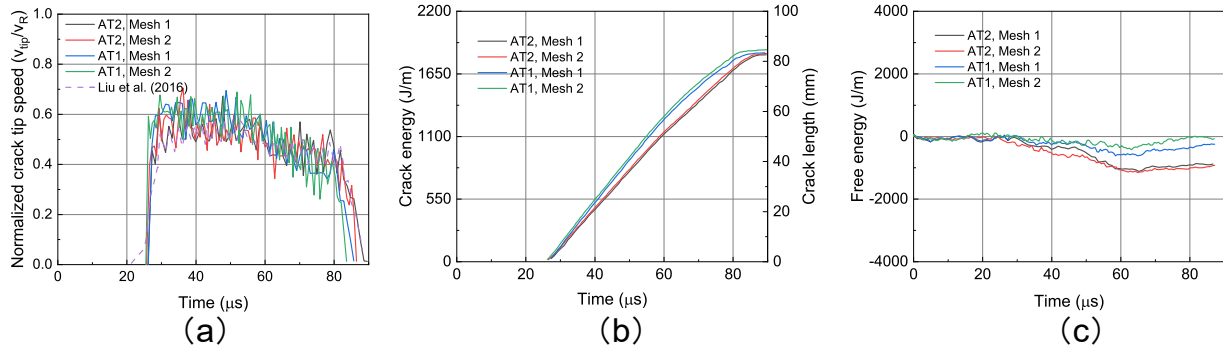


FIGURE 19 Results of the test in Section 4.3. Evolution of: (a) normalized crack tip velocity calculated by iso-curve strategy; (b) crack energy calculated by Eq. (37), i.e., $\hat{\Gamma}$ (left vertical axis), and crack length (right vertical axis); and (c) free energy $T(\dot{\mathbf{u}}) + V(\mathbf{u}, d) + \hat{\Gamma}(d)$, which satisfies the conservation of energy.

354 removed which allows the crack to heal. In addition, Geelen et al.⁴⁶ employed the augmented Lagrangian method to enforce the
 355 irreversibility and the resulting ratio is 2.

356 Moreover, Li et al.¹⁴ stated that the numerical phase field of the Kalthoff test is wider than the analytical one, and the wider
 357 damage profile will lead to an amplified effective fracture toughness, which had also been reported by Bourdin et al.⁶⁶ Further-
 358 more, Bleyer et al.⁶⁷ suggested that the mesh size has an influence on the result of both quasi-static and dynamic fracture and
 359 that will further lead to an overestimated crack energy (see Eqs. (16) and (17) in Ref. 67 for more details).

360 This issue appears to be an open question for the Kalthoff test.

361 4.4 | The fiber-reinforced composite test

362 This section studies the cracking behavior in a fiber-reinforced composite with two asymmetric initial explicit cracks. Figure
 363 20(a) illustrates the geometry and the boundary conditions, where the cracks are located at the end of the parallel fibers and with
 364 a length of 3 mm. Plane stress state is assumed. The material parameters for the matrix are selected as $E_m = 35$ GPa, $\nu_m = 0.42$,
 365 $\rho_m = 1450$ kg/m³, $g_{cm} = 5 \times 10^2$ J/m², $\ell_m = 2 \times 10^{-4}$ m while those for the fiber are $E_f = 208$ GPa, $\nu_f = 0.42$, $\rho_f = 8000$
 366 kg/m³, $g_{cf} = 1 \times 10^4$ J/m², $\ell_f = 1 \times 10^{-3}$ m. The mesh size in the refined region is $h = 5 \times 10^{-5}$ m = $\ell_m/4$.

367 Figure 20(b) and (c) show the phase field results of the AT2 and AT1 models, respectively. It is observed that the cracks
 368 propagate from the pre-crack tips and toward each other in a curved shape. This crack pattern was also observed in Refs. 68,69.

369 Figure 21(a) shows the evolution of the normalized crack tip velocity. The cracks propagate at 3 μs and rapidly increase
 370 to the Rayleigh wave speed v_R , then decrease to $v_R/2$ at the end of the simulation. Figure 21(b) shows the evolution of the
 371 crack energy, which increases monotonically during the simulation. Figure 21(c) shows the evolution of the strain energy, which
 372 increases at the beginning and then decreases after the cracks begin to propagation. Figure 21(d) shows the evolution of the
 373 kinetic energy. Interestingly the kinetic energy shows a plateau at the beginning of the crack propagation (about 3 μs) and then
 374 increases. Figure 21(e) and (f) show the evolution of the external work and the free energy respectively, indicating that the free
 375 energy is approximately 6% of the external work at the end of the simulation.

376 Figure 22(a) shows the number of updates per element. Again, by design, the elements near the cracks are updated more
 377 frequently. Furthermore, it is noted that the fiber elements are updated three times more frequently than the matrix elements far
 378 away from the cracks. Figure 22(b) shows the wall time corresponding to $t_f = 5.5$ μs for the AVI and a synchronous method,
 379 and the results indicate that the time the AVI takes is approximately a quarter times that of a synchronous method.

380 5 | CONCLUSIONS

381 In this paper, we have proposed an asynchronous variational formulation for the phase field approach to dynamic brittle fracture.
 382 The formulation is derived from Hamilton's principle of stationary action and to a great extent, retains the advantages of varia-
 383 tional integrators. A major characteristic of the formulation is that it allows elements to have independent time steps. The result
 384 indicates that the formulation is able to simulate dynamic fracture propagation and branching successfully. As a result of the

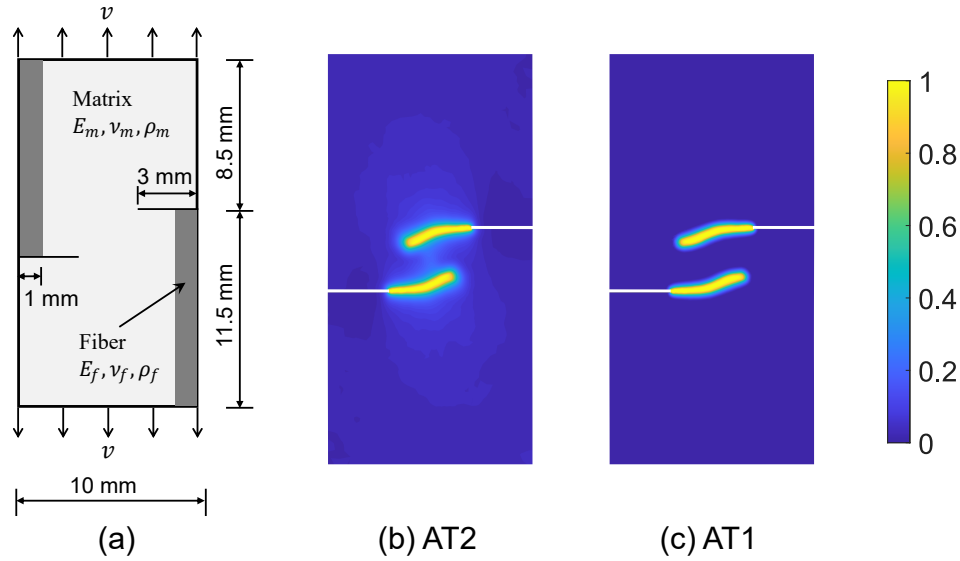


FIGURE 20 The fiber-reinforced composite test. (a) The geometry and boundary conditions, where $v = 5$ m/s. The phase field results of the (b) AT2 and (c) AT1.

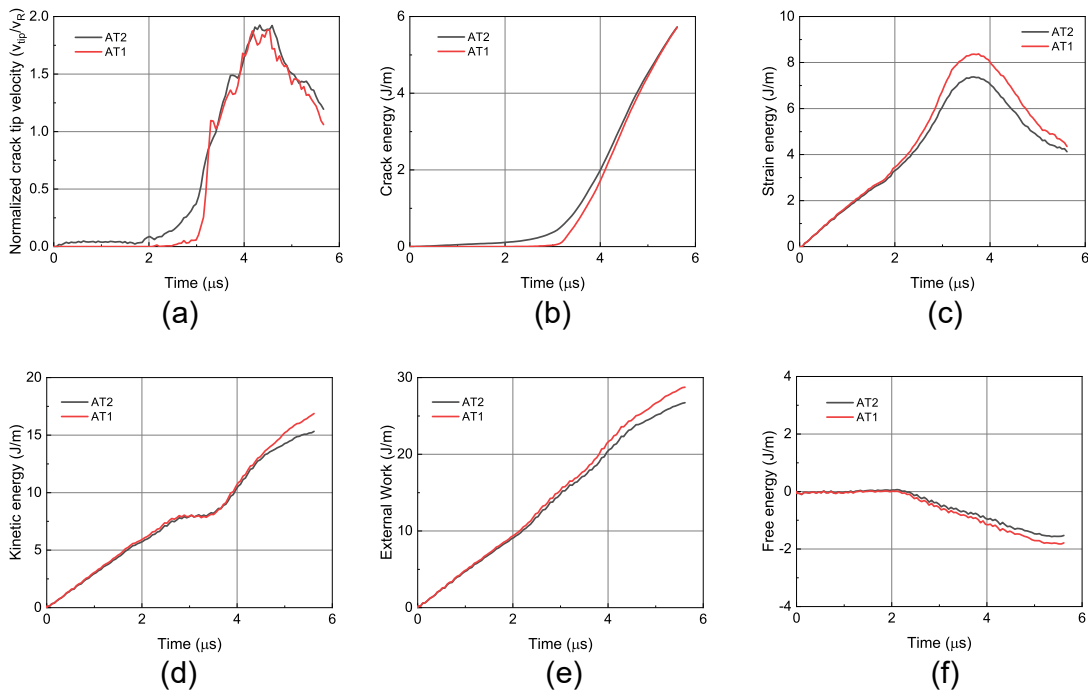


FIGURE 21 Results of the fiber-reinforced composite test in Section 4.4 as the evolution of: (a) normalized total crack tip velocity; (b) crack energy; (c) strain energy; (d) kinetic energy; (e) external work; and (f) free energy.

385 variational structure, the formulation performs a remarkable energy behavior during the simulation. Compared to synchronous
 386 method, the presented formulation is computationally more efficient for problems involving a high contrast in element sizes or
 387 material properties, such as bi-materials.

388 Another characteristic is that the phase field irreversibility condition is enforced by the reduced-space active set method at the
 389 level of element patches. As a result, the AT2 and AT1 variants of the phase field approach may be implemented with similar
 390 costs. The present study shows that these two variants lead to similar results at roughly the same computational cost.

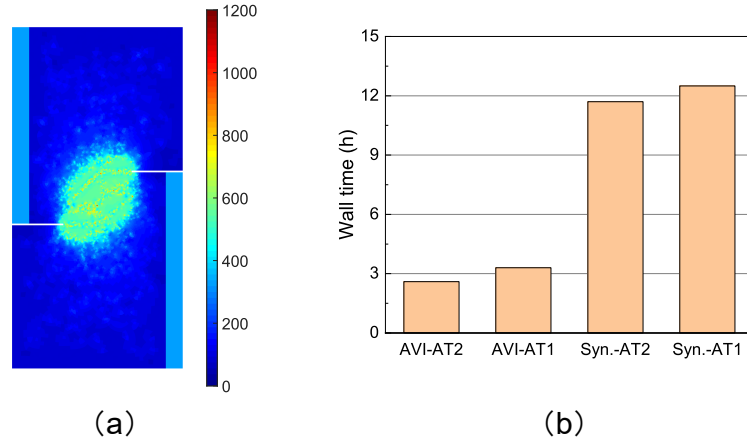


FIGURE 22 The performance indicators of the tests in Section 4.4: (a) number of updates per element; and (b) wall time corresponding to $t_f = 5.5 \mu\text{s}$.

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APPENDIX

A THE CASE WITH COINCIDENCE IN THE UPDATE SCHEDULE

The assumption of no coincident instants is not a restrictive assumption but for simplifying the derivation. Without this assumption, consider the case in which there are coincidences in the update instants for multiple adjacent elements sharing node a . Let the set of these nodes be denoted \mathcal{T}_a . It can be shown that the result of the simulation is equal for the pure elasticity case even if the update order of the elements in \mathcal{T}_a is exchanged.

However, when considering the phase field, the results are slightly dependent on the update order due to the irreversibility of the phase field. An alternative and the most orthodox treatment is to rewrite (24) as

$$0 = \frac{\partial S_{dis}}{\partial \mathbf{u}_a^i} = \frac{\partial}{\partial \mathbf{u}_a^i} \left(\frac{1}{2} M_a(t_a^i - t_a^{i-1}) \left\| \frac{\mathbf{u}_a^i - \mathbf{u}_a^{i-1}}{t_a^i - t_a^{i-1}} \right\|^2 + \frac{1}{2} M_a(t_a^{i+1} - t_a^i) \left\| \frac{\mathbf{u}_a^{i+1} - \mathbf{u}_a^i}{t_a^{i+1} - t_a^i} \right\|^2 - \sum_{e \in \mathcal{T}_a} (t_e^j - t_e^{j-1}) V_e(\mathbf{u}_e^j, \mathbf{d}_e^j) \right),$$

with the last term modified compared with (24). In this treatment, the elements in \mathcal{T}_a need to be simultaneously updated. It is believed that such more rigorous treatment will not alter too much the numerical results.

B PHASE FIELD RESULT WITHOUT USING ELEMENT PATCHES

Figure B1 shows the phase field result obtained with (28), i.e., without patches, the boundary conditions and material properties are the same as those of Section 4.1. As seen, the crack patterns are too diffused.

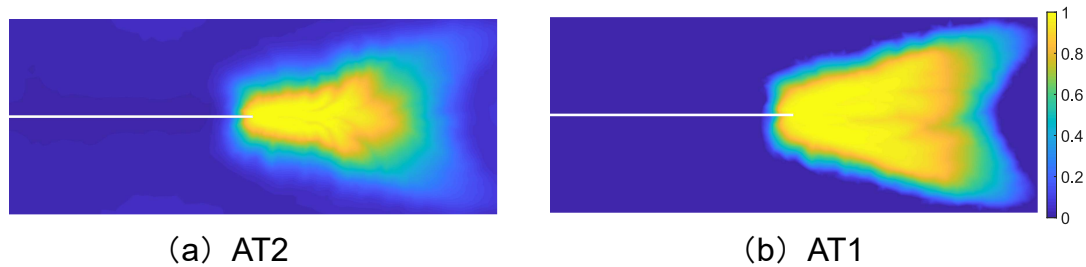


FIGURE B1 Phase field result with the formulation without using patches.

References

1. Yao Y, Keer LM. Cohesive fracture mechanics based numerical analysis to BGA packaging and lead free solders under drop impact. *Microelectronics Reliability* 2013; 53(4): 629-637. doi: <https://doi.org/10.1016/j.microrel.2012.12.007>
2. Zhang R, Zhang X, Kang T, He F. Dynamic fracture propagation model for oriented perforation steering fracturing in low permeability reservoir based on microelement method. *Journal of Natural Gas Science and Engineering* 2020; 74: 103105. doi: <https://doi.org/10.1016/j.jngse.2019.103105>
3. Chen S, Zang M, Wang D, Yoshimura S, Yamada T. Numerical analysis of impact failure of automotive laminated glass: A review. *Composites Part B: Engineering* 2017; 122: 47-60. doi: <https://doi.org/10.1016/j.compositesb.2017.04.007>
4. Freund LB. *Dynamic Fracture Mechanics*. Cambridge University Press . 1998.
5. Ravi-Chandar K. *Dynamic Fracture*. Elsevier . 2004.
6. Fineberg J, Bouchbinder E. Recent developments in dynamic fracture: Some perspectives. *International Journal of Fracture* 2015; 196(1-2): 33-57. doi: <https://doi.org/10.1007/s10704-015-0038-x>
7. Sun Y, Edwards MG, Chen B, Li C. A state-of-the-art review of crack branching. *Engineering Fracture Mechanics* 2021; 257: 108036. doi: <https://doi.org/10.1016/j.engfracmech.2021.108036>
8. Rabczuk T. Computational methods for fracture in brittle and quasi-brittle solids: State-of-the-art review and future perspectives. *International Scholarly Research Notices* 2013; 2013: 1-38. doi: <https://doi.org/10.1155/2013/849231>
9. Réthoré J, Gravouil A, Combescure A. An energy-conserving scheme for dynamic crack growth using the eXtended finite element method. *International Journal for Numerical Methods in Engineering* 2005; 63(5): 631-659. doi: <https://doi.org/10.1002/nme.1283>
10. Nguyen VP. Discontinuous Galerkin/extrinsic cohesive zone modeling: Implementation caveats and applications in computational fracture mechanics. *Engineering Fracture Mechanics* 2014; 128: 37-68. doi: <https://doi.org/10.1016/j.engfracmech.2014.07.003>
11. Liu Y, Filonova V, Hu N, et al. A regularized phenomenological multiscale damage model. *International Journal for Numerical Methods in Engineering* 2014; 99(12): 867-887. doi: <https://doi.org/10.1002/nme.4705>
12. Zhang Y, Zhuang X. Cracking elements method for dynamic brittle fracture. *Theoretical and Applied Fracture Mechanics* 2019; 102: 1-9. doi: <https://doi.org/10.1016/j.tafmec.2018.09.015>
13. Song JH, Areias PMA, Belytschko T. A method for dynamic crack and shear band propagation with phantom nodes. *International Journal for Numerical Methods in Engineering* 2006; 67(6): 868-893. doi: <https://doi.org/10.1002/nme.1652>
14. Li T, Marigo JJ, Guilbaud D, Potapov S. Gradient damage modeling of brittle fracture in an explicit dynamics context. *International Journal for Numerical Methods in Engineering* 2016; 108(11): 1381-1405. doi: <https://doi.org/10.1002/nme.5262>

- 438 15. Moreau K, Moës N, Picart D, Stainier L. Explicit dynamics with a non-local damage model using the thick
439 level set approach. *International Journal for Numerical Methods in Engineering* 2015; 102(3-4): 808-838. doi:
440 <https://doi.org/10.1002/nme.4824>
- 441 16. Bourdin B, Francfort GA, Marigo JJ. Numerical experiments in revisited brittle fracture. *Journal of the Mechanics and*
442 *Physics of Solids* 2000; 48(4): 797-826. doi: [https://doi.org/10.1016/S0022-5096\(99\)00028-9](https://doi.org/10.1016/S0022-5096(99)00028-9)
- 443 17. Francfort GA, Marigo JJ. Revisiting brittle fracture as an energy minimization problem. *Journal of the Mechanics and*
444 *Physics of Solids* 1998; 46(8): 1319-1342. doi: [https://doi.org/10.1016/S0022-5096\(98\)00034-9](https://doi.org/10.1016/S0022-5096(98)00034-9)
- 445 18. Amiri F, Millán D, Shen Y, Rabczuk T, Arroyo M. Phase-field modeling of fracture in linear thin shells. *Theoretical and*
446 *Applied Fracture Mechanics* 2014; 69: 102-109. doi: <https://doi.org/10.1016/j.tafmec.2013.12.002>
- 447 19. Lai W, Gao J, Li Y, Arroyo M, Shen Y. Phase field modeling of brittle fracture in an Euler-Bernoulli beam accounting
448 for transverse part-through cracks. *Computer Methods in Applied Mechanics and Engineering* 2020; 361: 112787. doi:
449 <https://doi.org/10.1016/j.cma.2019.112787>
- 450 20. Mollaali M, Ziaei-Rad V, Shen Y. Numerical modeling of CO2 fracturing by the phase field approach. *Journal of Natural*
451 *Gas Science and Engineering* 2019; 70: 102905. doi: <https://doi.org/10.1016/j.jngse.2019.102905>
- 452 21. Shen Y, Mollaali M, Li Y, Ma W, Jiang J. Implementation details for the phase field approaches to fracture. *Journal of*
453 *Shanghai Jiaotong University (Science)* 2018; 23(1): 166–174. doi: <https://doi.org/10.1007/s12204-018-1922-0>
- 454 22. Borden MJ, Verhoosel CV, Scott MA, Hughes TJ, Landis CM. A phase-field description of dynamic brittle
455 fracture. *Computer Methods in Applied Mechanics and Engineering* 2012; 217-220: 77-95. doi:
456 <https://doi.org/10.1016/j.cma.2012.01.008>
- 457 23. Nguyen VP, Wu JY. Modeling dynamic fracture of solids with a phase-field regularized cohesive zone model. *Computer*
458 *Methods in Applied Mechanics and Engineering* 2018; 340: 1000-1022. doi: <https://doi.org/10.1016/j.cma.2018.06.015>
- 459 24. Hao S, Shen Y, Cheng JB. Phase field formulation for the fracture of a metal under impact with a fluid formulation.
460 *Engineering Fracture Mechanics* 2022; 261: 108142. doi: <https://doi.org/10.1016/j.engfracmech.2021.108142>
- 461 25. Hao S, Chen Y, Cheng JB, Shen Y. A phase field model for high-speed impact based on the updated Lagrangian formulation.
462 *Finite Elements in Analysis and Design* 2022; 199: 103652. doi: <https://doi.org/10.1016/j.finel.2021.103652>
- 463 26. Tian F, Tang X, Xu T, Yang J, Li L. A hybrid adaptive finite element phase-field method for quasi-static and
464 dynamic brittle fracture. *International Journal for Numerical Methods in Engineering* 2019; 120(9): 1108-1125. doi:
465 <https://doi.org/10.1002/nme.6172>
- 466 27. Ziaei-Rad V, Shen Y. Massive parallelization of the phase field formulation for crack propagation with time adaptivity. *Com-*
467 *puter Methods in Applied Mechanics and Engineering* 2016; 312: 224-253. doi: <https://doi.org/10.1016/j.cma.2016.04.013>
- 468 28. Li Y, Lai W, Shen Y. Variational h-adaption method for the phase field approach to fracture. *International Journal of*
469 *Fracture* 2019; 217: 83-103. doi: <https://doi.org/10.1007/s10704-019-00372-y>
- 470 29. Engwer C, Pop IS, Wick T. Dynamic and weighted stabilizations of the L-scheme applied to a phase-field model for fracture
471 propagation. In: Springer International Publishing; 2021; Cham: 1177–1184
- 472 30. Lew A, Marsden JE, Ortiz M, West M. Asynchronous variational integrators. *Archive for Rational Mechanics and Analysis*
473 2003; 167(2): 85–146. doi: <https://doi.org/10.1007/s00205-002-0212-y>
- 474 31. Lew A, Marsden JE, Ortiz M, West M. Variational time integrators. *International Journal for Numerical Methods in*
475 *Engineering* 2004; 60(1): 153–212. doi: <https://doi.org/10.1002/nme.958>
- 476 32. Fong W, Darve E, Lew A. Stability of asynchronous variational integrators. *Journal of Computational Physics* 2008;
477 227(18): 8367-8394. doi: <https://doi.org/10.1016/j.jcp.2008.05.017>

- 478 33. Focardi M, Mariano PM. Convergence of asynchronous variational integrators in linear elastodynamics. *International*
479 *Journal for Numerical Methods in Engineering* 2008; 75(7): 755-769. doi: <https://doi.org/10.1002/nme.2271>
- 480 34. Ryckman RA, Lew AJ. An explicit asynchronous contact algorithm for elastic body-rigid wall interaction. *International*
481 *Journal for Numerical Methods in Engineering* 2012; 89(7): 869-896. doi: <https://doi.org/10.1002/nme.3266>
- 482 35. Liu P, Yang JZ, Yuan C. Extended synchronous variational integrators for wave propagations on non-uniform meshes.
483 *Communications in Computational Physics* 2020; 28(2): 691-722. doi: <https://doi.org/10.4208/cicp.OA-2019-0167>
- 484 36. Thomaszewski B, Pabst S, Straßer W. Asynchronous Cloth Simulation. In: The 2008 International Conference on Computer
485 Graphics and Virtual Reality; 2008.
- 486 37. Pham K, Amor H, Marigo JJ, Maurini C. Gradient damage models and their use to approximate brittle fracture. *International*
487 *Journal of Damage Mechanics* 2011; 20(4): 618-652. doi: [10.1177/1056789510386852](https://doi.org/10.1177/1056789510386852)
- 488 38. Ren HL, Zhuang X, Anitescu C, Rabczuk T. An explicit phase field method for brittle dynamic fracture. *Computers &*
489 *Structures* 2019; 217: 45-56. doi: <https://doi.org/10.1016/j.compstruc.2019.03.005>
- 490 39. Suh HS, Sun W. Asynchronous phase field fracture model for porous media with thermally non-equilibrated
491 constituents. *Computer Methods in Applied Mechanics and Engineering* 2021; 387: 114182. doi:
492 <https://doi.org/10.1016/j.cma.2021.114182>
- 493 40. Miehe C, Hofacker M, Welschinger F. A phase field model for rate-independent crack propagation: Robust algorithmic
494 implementation based on operator splits. *Computer Methods in Applied Mechanics and Engineering* 2010; 199(45): 2765-
495 2778. doi: [10.1016/j.cma.2010.04.011](https://doi.org/10.1016/j.cma.2010.04.011)
- 496 41. Amor H, Marigo JJ, Maurini C. Regularized formulation of the variational brittle fracture with unilateral con-
497 tact: Numerical experiments. *Journal of the Mechanics and Physics of Solids* 2009; 57(8): 1209-1229. doi:
498 <https://doi.org/10.1016/j.jmps.2009.04.011>
- 499 42. Liu Y, Cheng C, Ziaei-Rad V, Shen Y. A micromechanics-informed phase field model for brittle frac-
500 ture accounting for unilateral constraint. *Engineering Fracture Mechanics* 2021; 241: 107358. doi:
501 <https://doi.org/10.1016/j.engfracmech.2020.107358>
- 502 43. Wu JY, Nguyen VP, Zhou H, Huang Y. A variationally consistent phase-field anisotropic damage model for fracture. *Com-*
503 *puter Methods in Applied Mechanics and Engineering* 2020; 358: 112629. doi: <https://doi.org/10.1016/j.cma.2019.112629>
- 504 44. Marsden JE, West M. Discrete mechanics and variational integrators. *Acta Numerica* 2001; 10(10): 357-514. doi:
505 [10.1017/S096249290100006X](https://doi.org/10.1017/S096249290100006X)
- 506 45. Gerasimov T, De Lorenzis L. On penalization in variational phase-field models of brittle fracture. *Computer Methods in*
507 *Applied Mechanics and Engineering* 2019; 354: 990-1026. doi: <https://doi.org/10.1016/j.cma.2019.05.038>
- 508 46. Geelen RJM, Liu Y, Hu T, Tupek MR, Dolbow JE. A phase-field formulation for dynamic cohesive fracture. *Computer*
509 *Methods in Applied Mechanics and Engineering* 2019; 348: 680-711. doi: <https://doi.org/10.1016/j.cma.2019.01.026>
- 510 47. Wu JY. Robust numerical implementation of non-standard phase-field damage models for failure in solids. *Computer*
511 *Methods in Applied Mechanics and Engineering* 2018; 340: 767-797. doi: <https://doi.org/10.1016/j.cma.2018.06.007>
- 512 48. Farrell P, Maurini C. Linear and nonlinear solvers for variational phase-field models of brittle fracture. *International Journal*
513 *for Numerical Methods in Engineering* 2017; 109(5): 648-667. doi: <https://doi.org/10.1002/nme.5300>
- 514 49. Knuth DE. *The Art of Computer Programming*. Addison Wesley . 1998.
- 515 50. Song JH, Wang H, Belytschko T. A comparative study on finite element methods for dynamic fracture. *Computational*
516 *Mechanics* 2008; 42: 239-250. doi: [10.1007/s00466-007-0210-x](https://doi.org/10.1007/s00466-007-0210-x)
- 517 51. Ramulu M, Kobayashi AS. Mechanics of crack curving and branching – a dynamic fracture analysis. *International Journal*
518 *of Fracture* 1985; 27: 187-201. doi: <https://doi.org/10.1007/BF00017967>

- 519 52. Sharon E, Fineberg J. Microbranching instability and the dynamic fracture of brittle materials. *Physical Review B Condensed*
520 *Matter* 1996; 54(10): 7128-7139. doi: 10.1103/physrevb.54.7128
- 521 53. Schlüter A, Kuhn C, Müller R. Phase field approximation of dynamic brittle fracture. *Proceedings in Applied Mathematics*
522 *and Mechanics* 2014; 14(1): 143-144. doi: <https://doi.org/10.1002/pamm.201410059>
- 523 54. Liu G, Li Q, Msekh MA, Zuo Z. Abaqus implementation of monolithic and staggered schemes for quasi-
524 static and dynamic fracture phase-field model. *Computational Materials Science* 2016; 121: 35-47. doi:
525 <https://doi.org/10.1016/j.commatsci.2016.04.009>
- 526 55. Bobaru F, Zhang G. Why do cracks branch? A peridynamic investigation of dynamic brittle fracture. *International Journal*
527 *of Fracture* 2015; 196: 59-98. doi: 10.1007/s10704-015-0056-8
- 528 56. Mandal TK, Nguyen VP, Wu JY. Evaluation of variational phase-field models for dynamic brittle fracture. *Engineering*
529 *Fracture Mechanics* 2020; 235: 107169. doi: <https://doi.org/10.1016/j.engfracmech.2020.107169>
- 530 57. Kalthoff JF. Shadow optical analysis of dynamic shear fracture. *Optical Engineering* 1988; 27(10): 835 – 840. doi:
531 10.1117/12.7976772
- 532 58. Kalthoff JF. Modes of dynamic shear failure in solids. *International Journal of Fracture* 2000; 101(1): 1–31. doi:
533 <https://doi.org/10.1023/A:1007647800529>
- 534 59. Wang H, Xu Y, Huang D. A non-ordinary state-based peridynamic formulation for thermo-visco-plastic
535 deformation and impact fracture. *International Journal of Mechanical Sciences* 2019; 159: 336-344. doi:
536 <https://doi.org/10.1016/j.ijmecsci.2019.06.008>
- 537 60. Chu D, Li X, Liu Z. Study the dynamic crack path in brittle material under thermal shock loading by phase field modeling.
538 *International Journal of Fracture* 2017; 208: 115-130. doi: <https://doi.org/10.1007/s10704-017-0220-4>
- 539 61. Zhou S, Rabczuk T, Zhuang X. Phase field modeling of quasi-static and dynamic crack propagation:
540 COMSOL implementation and case studies. *Advances in Engineering Software* 2018; 122: 31-49. doi:
541 <https://doi.org/10.1016/j.advengsoft.2018.03.012>
- 542 62. Zhang Y, Ren H, Areias P, Zhuang X, Rabczuk T. Quasi-static and dynamic fracture modeling by the
543 nonlocal operator method. *Engineering Analysis with Boundary Elements* 2021; 133: 120-137. doi:
544 <https://doi.org/10.1016/j.enganabound.2021.08.020>
- 545 63. Park K, Paulino GH, Celes W, Espinha R. Adaptive mesh refinement and coarsening for cohesive zone model-
546 ing of dynamic fracture. *International Journal for Numerical Methods in Engineering* 2012; 92(1): 1-35. doi:
547 <https://doi.org/10.1002/nme.3163>
- 548 64. Tangella RG, Kumbhar P, Annabattula RK. Hybrid Phase Field Modelling of Dynamic Brittle Fracture and Implementation
549 in FEniCS. In: Krishnapillai S, R. V, Ha SK., eds. *Composite Materials for Extreme Loading* Springer Singapore; 2022;
550 Singapore: 15–24
- 551 65. Reddy SSK, Amirtham R, Reddy JN. Modeling fracture in brittle materials with inertia effects using the phase field method.
552 *Mechanics of Advanced Materials and Structures* 2021; 0(0): 1-16. doi: 10.1080/15376494.2021.2010289
- 553 66. Bourdin B, Francfort GA, Marigo JJ. The variational approach to fracture. *Journal of Elasticity* 2008; 91: 5-148. doi:
554 <https://doi.org/10.1007/s10659-007-9107-3>
- 555 67. Bleyer J, Roux-Langlois C, Molinari JF. Dynamic crack propagation with a variational phase-field model: Limiting
556 speed, crack branching and velocity-toughening mechanisms. *International Journal of Fracture* 2017; 204: 79-100. doi:
557 <https://doi.org/10.1007/s10704-016-0163-1>
- 558 68. Yin BB, Zhang LW. Phase field method for simulating the brittle fracture of fiber reinforced composites. *Engineering*
559 *Fracture Mechanics* 2019; 211: 321-340. doi: <https://doi.org/10.1016/j.engfracmech.2019.02.033>
- 560 69. Sumi Y, Wang ZN. A finite-element simulation method for a system of growing cracks in a heterogeneous material.
561 *Mechanics of Materials* 1998; 28(1): 197-206. doi: [https://doi.org/10.1016/S0167-6636\(97\)00048-3](https://doi.org/10.1016/S0167-6636(97)00048-3)