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TECHNICAL NOTE

Computation of Maximal Output Admissible Sets for Linear Systems with Polynomial Constraints

Andres Cotorruelo*[1] | Emanuele Garone[1] | Ilya V. Kolmanovsky[2] | Daniel R. Ramirez[3] | Daniel Limon[3]

[1]Department of Automation and System Analysis, Université Libre de Bruxelles, 1050 Brussels, Belgium
[2]College of engineering, University of Michigan, Ann Arbor 48109MI, USA
[3]Department of Systems Engineering and Automation, Universidad de Sevilla, Seville, Spain

Correspondence
*Corresponding author, Email: Andres.Cotorruelo.Jimenez@ulb.be

Present Address
Service d'Automatique et d'Analyse des Systèmes École polytechnique de Bruxelles Université libre de Bruxelles 50, av. F.D. Roosevelt, CP 165/55 1050 Bruxelles – Belgique

Summary

In this technical note we study the computation of the Maximal Output Admissible Set for linear systems subject to polynomial constraints. The computation of an inner approximation of the Maximal Output Admissible Sets requires the determination of constraint redundancy. We use a procedure to determine polynomial constraint redundancy based on a consequence of Putinar's Positivstellensatz. Further, we present a modification of the algorithm to compute the Maximal Output Admissible Set with improved performance. Lastly, demonstrate the potential for practical applications in two case studies of spacecraft rendezvous and control of an electromagnetic actuator.

KEYWORDS:
Constrained control, Maximal Output Admissible Set, Nonlinear systems, Sum of Squares, Redundant constraints

1 | INTRODUCTION

With applications ranging from estimating regions of attraction[1,2], to the analysis of uncertain systems[3], and to constrained control[4,5], set-theoretic methods keep proving themselves very useful to characterize and control dynamic systems[6].

This note focuses on a particularly useful family of sets: the Maximal Output Admissible Sets (MOASs). These sets were considered in[7], where a MOAS was defined as *"the set of initial states such that the output will fulfill the constraints in the future"*. More recently, this definition was extended from considering only outputs to considering every arbitrary combination of states and constant input signals[8,9]. The MOAS is very important for the design of many constrained control schemes. In particular, it is the key ingredient for the formulation of Reference/Command Governor schemes[9], of multimode control schemes[10], and is also a commonly used terminal set in Model Predictive Control (MPC) schemes[11]. The use of these schemes is quite common, for instance, in aerospace control applications, see *e.g.*[12,13,14,15,16,17,18].

However, at the current stage, the availability of algorithms to compute efficiently this MOASs is mainly restricted to linear systems subject to linear constraints[9]. This note develops a method to compute an efficient representation of the MOAS for linear systems subject to general polynomial constraints. This method significantly enlarges the classes of applications where MOAS is needed, e.g., to systems which are feedback linearized in which case the dynamics become linear while constraints become nonlinear after the transformations and can be approximated as polynomial.

The note is organized as follows. We first give a brief introduction to the algorithmic determination of the MOAS based on certificates of constraint redundancy. We then provide a systematic way to generate these certificates in the case of polynomial constraints. Further, we modify the algorithm to determine the MOAS to improve its performance based on several observations.

Finally, we demonstrate the methodology on two relevant examples of spacecraft rendezvous and electromagnetic actuator control that suggest significant potential for its practical impact.

2 | PROBLEM STATEMENT

Consider a pre-stabilized[1] closed loop discrete-time linear system,

$$x(t+1) = Ax(t) + Bv(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $v(t) \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$ is Schur, $B \in \mathbb{R}^{n\times p}$, and $t \in \mathbb{Z}_{\geq 0}$ designates the discrete-time instant. This system is subject to constraints in the form $(x, v) \in \mathcal{C}$, where

$$\mathcal{C} = \{(x, v) : c_i(x, v) \geq 0,\ i = 1, \dots, n_c\}, \tag{2}$$

with $c_i \in \mathbb{R}[x, v]$, where by $\mathbb{R}[x, v]$ we denote the set of all polynomials in the variables $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$. Since system (1) describes a closed-loop system, the input $v(t)$ can be seen as a set-point reference for the closed loop system. Furthermore, the set $\mathcal{C}$ is determined by constraints in $x(t)$ and $v(t)$ that characterize state and input constraints on the original open-loop system.

The objective of this paper is to present a procedure to compute the *Maximal Output Admissible Set* (MOAS) for this class of systems and constraints. For a pre-stabilized system, the MOAS can be defined as the set of states x and references v such that, if x is the initial state and the set-point v is kept constant, constraints will not be violated. More formally the MOAS can be defined as

Definition 1. (Maximal Output Admissible Set.) Consider a pre-stabilized system (1) subject to $(x, v) \in \mathcal{C}$. The MOAS is defined as $\mathcal{O}_\infty = \{(x, v) : (\hat{x}(k|x, v), v) \in \mathcal{C},\ k \in \mathbb{Z}_{\geq 0}\}$, where $\hat{x}(k|x, v) = A^k x + \sum_{j=0}^{k-1} A^j Bv$ is the prediction of the state at time k given the initial state x and constant applied reference v.

3 | GENERAL CONSIDERATIONS ON THE COMPUTATION OF $\tilde{\mathcal{O}}_\infty$

Since $\mathcal{O}_\infty$ is, in general, described by an infinite number of constraints, it is a common practice to use $\tilde{\mathcal{O}}_\infty$, which is a slightly tightened version of $\mathcal{O}_\infty$, defined as $\tilde{\mathcal{O}}_\infty = \mathcal{O}_\infty \cap \mathcal{O}^\epsilon$ with $\mathcal{O}^\epsilon = \{(x, v) : (\overline{x}_v, v) \in (1-\epsilon)\mathcal{C}\}$, where $\overline{x}_v = (I_n - A)^{-1}Bv$ is the steady-state associated to the constant applied reference v and $\epsilon > 0$ is a small constant. The set $\tilde{\mathcal{O}}_\infty$ has a number of very desirable properties[7]: it is positively invariant, it can approximate arbitrary well $\mathcal{O}_\infty$ by decreasing ϵ, and, most notably, if $\tilde{\mathcal{O}}_\infty$ is compact, then it is also *finitely determined*, *i.e.* and there exists a k^* such that

$$\tilde{\mathcal{O}}_\infty = \tilde{\mathcal{O}}_{k^*} = \{(x, v) : (\hat{x}(k|x, v), v) \in \mathcal{C}, \forall k = 1, \dots, k^*\} \cap \mathcal{O}^\epsilon.$$

In general, using Lyapunov arguments[20] and possibly solving two (in general non-convex) optimization problems it is possible to compute an upper bound on k^*, $\bar{k} \geq k^*$ for general class of systems and constraints[9]. However, this upper bound is typically very conservative. As a consequence, to describe the MOAS as $\tilde{\mathcal{O}}_\infty = \tilde{\mathcal{O}}_{\bar{k}}$ results in an unnecessarily large number of constraints which might sensibly slow down the control algorithms making use of such a set.

In[7] it was proven that k^* can be defined as the smallest k such that $\tilde{\mathcal{O}}^*_k = \tilde{\mathcal{O}}_{k^*+1}$. Accordingly, an iterative algorithm able to determine $\tilde{\mathcal{O}}_\infty$ (Algorithm 1) and k^* was presented, which, in principle, allows to find an optimal[2] representation of $\tilde{\mathcal{O}}_\infty$.

The main difficulty in the implementation of Algorithm 1 is to check if the constraint $c_i(\hat{x}(k|x, v), v) \geq 0$ is redundant or not with respect to to the set $\tilde{\mathcal{O}}$. By definition, given a general constraint $c(z) \geq 0$ with $z \in \mathbb{R}^{n_z}$, $c(z) > 0$ is redundant with respect to a set $\mathcal{K} \subset \mathbb{R}^{n_z}$ if and only if $c(z) \geq 0,\ \forall z \in \mathcal{K}$. Using this fact, a common way to build a redundancy certificate is to compute

$$\gamma^* = \min_{z\in\mathcal{K}} c(z) \tag{3}$$

and to note that $\gamma^* \geq 0$ if and only if the constraint is redundant.

[1] It is important to remark that this is not a limiting assumption; in fact, the RG literature is based on this assumption as well as a significant part of the MPC literature, *e.g.*, semi-feedback MPC[19].

[2] For the sake of precision it is worth remarking that, in principle, the representation output by Algorithm 1 may not be minimal: once a constraint is added to $\tilde{\mathcal{O}}_k$ it is not checked for redundancy at a later stage and it might have become redundant in later iterations of the algorithm. Consequently, to obtain a minimal representation of $\tilde{\mathcal{O}}_\infty$ it is necessary to assess the redundancy of every constraint that defines this set, eliminating those which are determined to be so.

Algorithm 1 Computation of $\tilde{\mathcal{O}}_\infty$[7].

$\tilde{\mathcal{O}} \leftarrow \mathcal{O}^\epsilon$
$k \leftarrow 0$
$\rho \leftarrow 1$
while $\rho \neq 0 \wedge k \leq \bar{k}$ **do**
  $\rho \leftarrow 0$
  for $i = 1, .., n_c$ **do**
    if $c_i(\hat{x}(k|x, v), v) \geq 0$ is not redundant with respect to $\tilde{\mathcal{O}}$ **then**
      $\rho \leftarrow 1$
      $\tilde{\mathcal{O}} \leftarrow \tilde{\mathcal{O}} \cap \{(x, v)|c_i(\hat{x}(k|x, v), v) \geq 0\}$
    end if
  end for
  $k \leftarrow k + 1$
end while
$\tilde{\mathcal{O}}_\infty = \tilde{\mathcal{O}}$

The main problem with this formulation is that it provides a redundancy certificate only in the case one can compute the exact optimal solution of (3). If instead one can compute only a sub-optimal solution $\gamma \leq \gamma^*$, the method only provides a non-redundancy certificate in the case $\gamma < 0$, otherwise nothing can be said. Accordingly, the above approach can be used to remove redundant constraints only when it is reasonable to solve exactly problem (3), *e.g.* when (3) is convex. Note that for (3) to be convex the constraint to be checked must be concave and the set $\mathcal{K}$ needs to be convex. This means that this method can be used for the computation of $\tilde{\mathcal{O}}_\infty$ only in the case of linear systems subject to linear constraints. A number of efficient implementations of this idea of certifying the redundancy of linear constraints have been proposed in the literature, including Linear Programming approaches[21], various heuristics[22] and deterministic methods[23]. For an extensive survey on the subject, the reader is referred to[24]. In the Reference Governor literature, the Linear Programming approach is customarily used in conjunction with Algorithm 1 to compute $\tilde{\mathcal{O}}_\infty$ for systems subject to linear constraints[9].

For what concerns the case of nonlinear constraints, to the best of the authors' knowledge, the existing literature does not provide systematic methods for the certification of redundancy, and as a consequence for the computation of $\tilde{\mathcal{O}}_\infty$. Indeed, so far most of the literature dealing with nonlinear systems and/or constraints has used either the upper bound $\bar{k}$ derived by Lyapunov arguments that can be very conservative or, more often, empirically estimated "sufficiently long" prediction horizons[20,25]. In this paper we propose a novel procedure that allows to ascertain whether a polynomial constraint is redundant with respect to a set defined by polynomial constraints, thus allowing to compute for efficient representations of $\tilde{\mathcal{O}}_\infty$. The proposed solution makes use of the Sum of Squares framework.

4 | REDUNDANCY CERTIFICATE FOR POLYNOMIAL CONSTRAINTS

In order to build a redundancy certificate for nonlinear constraints we recall the Putinar's formulation of the Positivstellensatz (P-satz)[26]

Theorem 1 (Putinar, 1993)**.** Consider a polynomial $p(z) \in \mathbb{R}[z]$ and a set $\mathcal{K} = \{z : q_i(z) \geq 0, i = 1, \dots, N\}$, where $q_i(z) \in \mathbb{R}[z], i = 1, ..., N$. Then every $p(z) > 0 \; \forall z \in \mathcal{K}$ can be written as

$$p(z) - \sum_{i=1}^{N} s_i(z)q_i(z) \in \Sigma[z], \tag{4}$$

for some $s_i \in \Sigma[z]$, $i = 1, \dots, N$ in which $\Sigma[z]$ represents the set of all Sum of Squares polynomials[27].

It is important to remark that this theorem is nonconstructive and that the implication it states is

$$p(z) > 0 \; \forall z \in \mathcal{K} \Rightarrow \text{(4) holds}.$$

Instead, we will use the converse result, which is summarized in the following corollary:

Corollary 1. Let $p(x) \in \mathbb{R}[x]$ and $\mathcal{K} = \left\{x \in \mathbb{R}^n : \{f_i(x)\}_{i=1}^m \geq 0\right\}$, then $p(x)$ is *nonnegative* over $\mathcal{K}$ if there exist $\{s_i(x)\}_{i=0}^m \in \Sigma[x]$ such that

$$p(x) = s_0(x) + \sum_{i=1}^{m} s_i(x) f_i(x).$$

Note that, by construction, the polynomial $p(x)$ is positive over $\mathcal{K}$ since it is the sum of strictly positive terms over $\mathcal{K}$. This corollary can be used to obtain a *redundancy certificate* of a polynomial constraint with respect to a set consisting of polynomial inequalities. In fact, if there exist $s_1, \dots, s_n \in \Sigma[z]$ such that $p(z) - \sum_{i=1}^n s_i(z) q_i(z)$ is a sum of squares polynomial, $p(z) \geq 0$ is certified to be redundant with respect to $\mathcal{K}$. This condition can be checked using the following Sum of Squares Programming (SOSP) feasibility test

$$\begin{aligned} &\text{find } s_i(z),\ i = 1, \dots, n_c \\ &\text{s.t.} \\ &\qquad p(z) - \textstyle\sum_{i=1}^{n_c} s_i(z) q_i(z) \in \Sigma[z] \\ &\qquad\qquad s_i \in \Sigma[z],\ i = 1, \dots, n_c. \end{aligned} \tag{5}$$

In many cases it is also of interest to quantify "how much" a constraint is redundant. A possible way to do so is to maximize on a slack variable γ as follows

$$\begin{aligned} \gamma^* = &\max \gamma \\ &\text{s.t} \\ &\quad p(z) - \textstyle\sum_{i=1}^{n_c} s_i(z) q_i(z) - \gamma \in \Sigma[z] \\ &\qquad s_i(z) \in \Sigma[z],\ i = 1, \dots, n_c. \end{aligned} \tag{6}$$

In this formulation, if γ^* is positive, the constraint $p(z) \geq 0$ is redundant with respect to $\mathcal{K}$. This slack variable approach can also be used to assess which constraints are "almost redundant". Almost redundant constraints are those whose associated γ^* is negative and small in absolute value, and can be potentially eliminated using inner approximations. This concept is explored in[28] where it was shown that the elimination of almost redundant constraints and a pull-in transformation can be used to compute simple inner approximations of $\tilde{\mathcal{O}}_\infty$ and implement reference governors with a significant reduction in online computational time and effort.

As proven in[29], it is possible to determine whether a polynomial is SOS through a Semi-Definite Programming (SDP) optimization problem, which means that optimization problem (6) can be efficiently solved by off-the-shelf solvers *e.g.* [30]. Additionally, there exist software suites that provide an interface between SOSP and SDP *e.g.* [31], thus transforming the optimization problem (6) into an LMI problem.

5 | EFFICIENT COMPUTATION OF $\tilde{\mathcal{O}}_\infty$

Note that using Algorithm 1 and the proposed redundancy certificate to compute $\tilde{\mathcal{O}}_\infty$ requires performing $(k^* + 1) \cdot n_c$ LMI feasibility tests as in (5). Accordingly, every iteration will take longer than the previous one, since the number of inequalities describing $\tilde{\mathcal{O}}_k$ grows by up of n_c every iteration. Furthermore, the number of variables of the optimization problem increases as well, since we need to declare a new SOS multiplier s_i for every new inequality. Interestingly, it is possible to reduce the number of redundancy checks and consequently the computational time by using the following proposition.

Proposition 1. If a constraint $c_j(x, v) \geq 0$ is redundant at iteration k' with respect to $\tilde{\mathcal{O}}_{k'}$, then it will be redundant for any iterations $k > k'$.

PROOF - Let $\mathcal{C}_i = \{(x, v) : c_i(x, v) \geq 0\}$, $i = 1, \dots, n_c$ and let us assume that constraint j becomes redundant at iteration k'. By definition

$$(\hat{x}(k|x, v), v) \in \bigcap_{i=1}^{n_c} \mathcal{C}_i,\ k = \{1, \dots, k' - 1\}, \tag{7}$$

and since the j-th constraint is redundant at iteration k'

$$(\hat{x}(k'|x, v), v) \in \mathcal{C}_j.$$

Then for any $(x, v) \in \tilde{\mathcal{O}}_{k'}$, $(\hat{x}(k'|x, v), v) \in \mathcal{C}_j$. Finally, since $\tilde{\mathcal{O}}_{k'+1} \subseteq \tilde{\mathcal{O}}_{k'}$, $(x, v) \in \mathcal{O}_{k'}$ implies $(f(x, v), v) \in \mathcal{C}_j$, therefore, the j-th constraint is redundant at iteration number $k' + 1$. ■

Using Proposition 1 we can refine Algorithm 1 into the more computationally efficient Algorithm 2 which only checks the redundancy of constraints that have not been redundant so far.

Algorithm 2 More efficient computation of $\tilde{\mathcal{O}}_\infty$

$\tilde{\mathcal{O}} \leftarrow \mathcal{O}^\varepsilon$
$k \leftarrow 0$
$\rho \leftarrow 1$
$\rho_i \leftarrow 1,\ i = 1, ..., n_c$
while $\rho \neq 0 \wedge k \leq \bar{k}$ **do**
  $\rho \leftarrow 0$
  for $i = 1, .., n_c$ **do**
    if $\rho_i \neq 0$ **then**
      if $c_i(\hat{x}(k|x, v), v) \geq 0$ is redundant with respect to $\tilde{\mathcal{O}}$ **then**
        $\rho_i \leftarrow 0$
      else
        $\rho \leftarrow 1$
        $\tilde{\mathcal{O}} \leftarrow \tilde{\mathcal{O}} \cap \{(x, v) | c_i(\hat{x}(k|x, v), v) \geq 0\}$
      end if
    end if
  end for
  $k \leftarrow k + 1$
end while
$\tilde{\mathcal{O}}_\infty \leftarrow \tilde{\mathcal{O}}$

6 | ILLUSTRATIVE EXAMPLES

6.1 | Application: Satellite Rendezvous

In this example we apply the proposed methodology to compute $\tilde{\mathcal{O}}_\infty$ for a deputy-chief satellite rendezvous. In this setting, the control objective is for the deputy satellite to approach the chief while staying in line of sight[32]. A non-inertial Hill frame is attached to the chief spacecraft where the three axes are the radial direction towards earth, the along-track direction towards the chief spacecraft, and the cross-track direction along the chief's angular momentum vector, respectively.

The orbit can be represented by the following linearized model,

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3\nu^2 & 0 & 0 & 0 & 2\nu & 0 \\ 0 & 0 & 0 & -2\nu & 0 & 0 \\ 0 & 0 & -\nu^2 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u(t), \tag{8}$$

where $\nu = 0.0011$ rad s^{-1}, $x = [x_1\ x_2\ x_3\ \dot{x}_1\ \dot{x}_2\ \dot{x}_3]^{\mathrm{T}}$ are the relative positions in km and velocities in km s^{-1} along the three axes of the Hill frame, and $u = [u_1\ u_2\ u_3]^{\mathrm{T}}$ are the three components of the thrust in the aforementioned axes. System (8) is subject to the following constraints:

$$\begin{aligned} (\tan^2\gamma)(x_2 + 0.01)^2 - x_1^2 - x_3^2 \geq 0 \\ u_{max}^2 - u_1^2 - u_2^2 - u_3^2 \geq 0 \\ x_2 \geq 0 \end{aligned} \tag{9}$$

where $\gamma = 15\,^\circ$ is the Line of Sight cone half angle, and $u_{max} = 0.001$ km s^{-2} is the maximum thrust acceleration. Discretizing (8) with $T_s = 30$ s and controlling it with the control law $u(t) = Fx(t) + Gv(t)$, with LQR gain F and feedforward gain G such

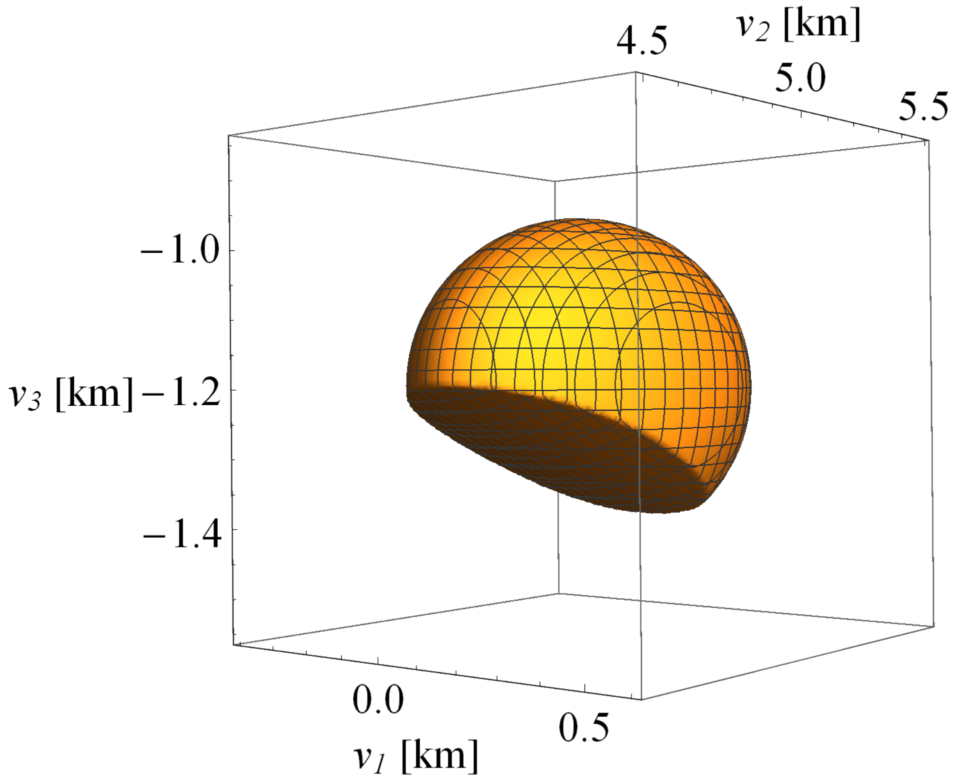

FIGURE 1 3D slice of the 9-dimensional $\tilde{\mathcal{O}}_\infty$ of system (10) corresponding to $x_1 = 0.1$, $x_2 = 5$, $x_3 = -1.2$, $x_4 = x_5 = x_6 = 0$.

that v becomes the reference position of the deputy spacecraft yields

$$x(t+1) = Ax + Bv, \tag{10}$$

where

$$A = \begin{bmatrix} 0.0111 & 0.0007 & 0 & 0.1676 & 0.0129 & 0 \\ 0.0611 & 0.1247 & 0 & 0.8913 & 1.9383 & 0 \\ 0 & 0 & 0.0138 & 0 & 0 & 0.2079 \\ -0.0659 & -0.0006 & 0 & -0.9879 & -0.0205 & 0 \\ 0.0048 & -0.0583 & 0 & 0.0822 & -0.8705 & 0 \\ 0 & 0 & -0.0657 & 0 & 0 & -0.9861 \end{bmatrix}, \quad B = \begin{bmatrix} 0.9889 & -0.0007 & 0 \\ -0.0611 & 0.8753 & 0 \\ 0 & 0 & 0.9862 \\ 0.0659 & 0.0006 & 0 \\ -0.0048 & 0.0583 & 0 \\ 0 & 0 & 0.0657 \end{bmatrix}.$$

A 3D slice of the $\tilde{\mathcal{O}}_\infty$ computed applying Algorithm 2 is depicted in **FIGURE** 1. For this system, k^* was determined to be $k^* = 19$ and $\tilde{\mathcal{O}}_\infty$ is defined by 31 inequalities. This represents a very significant decrease compared to the 30 step prediction horizon (and the resulting 90 constraints) which were empirically estimated in [32]. The computation of $\tilde{\mathcal{O}}_\infty$ took 2.07 s on an Intel Core i7-7500 at 2.7 GHz with 16 GB of RAM. All optimization problems were solved using MOSEK [33] interfaced in Julia 1.5.3.

6.2 | Electromagnetically Actuated Mass-Spring Damper

In this example we apply the proposed methodology to an electromagnetically actuated mass-spring damper system [25], depicted in **FIGURE** 2. This system has been used to describe many electromagnetic actuators, *e.g.* injectors, valves, [34,35] and is modeled by the following equations:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{\alpha}{m}\frac{u}{(d_0 - x_1)^\gamma}, \end{aligned} \tag{11}$$

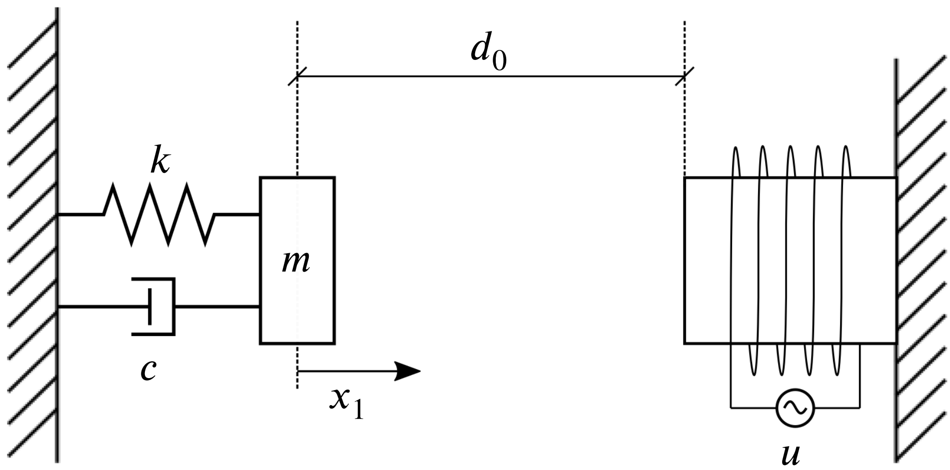

FIGURE 2 Electromagnetically actuated mass-spring damper system.

where x_1 and x_2 are the position in m and velocity in $\mathrm{m\,s^{-1}}$ of the armature, respectively, $k = 38.94\ \mathrm{N\,m^{-1}}$, $m = 1.54$ kg, $c = 0.65\ \mathrm{N\,s\,m^{-1}}$, $\alpha = 4.5 \cdot 10^{-5}\ \mathrm{C^2\,kg^{-1}\,m^{-3}}$, $d_0 = 0.0102$ m, and $\gamma = 2$. This system can be feedback linearized using the control law

$$u = \frac{1}{\alpha}(d_0 - x_1)^{\gamma}(kv - c_d x_2), \tag{12}$$

with $c_d = 4$. The closed loop system becomes

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{m}x_1 - \frac{c + c_d}{m}x_2 + \frac{k}{m}v. \end{aligned} \tag{13}$$

System (13) is subject to the following constraints:

$$x_1 \leq 0.008, \quad kv - c_d x_2 \geq 0, \quad u \leq 0.3,$$

where u is given by (12). Finally, the system is discretized with a sampling time of $T_s = 0.05$ s, yielding

$$x(t+1) = \begin{bmatrix} 0.9701 & 0.0459 \\ -1.1610 & 0.8312 \end{bmatrix} x(t) + \begin{bmatrix} 0.0299 \\ 1.1610 \end{bmatrix} v(t). \tag{14}$$

The resulting $\tilde{\mathcal{O}}_\infty$ is depicted in **FIGURE** 3. For its computation we considered $\varepsilon = 10^{-2}$. $\tilde{\mathcal{O}}_\infty$ was finitely determined after $k^* = 35$ iterations and it is described by 87 inequalities. The elapsed time to compute $\tilde{\mathcal{O}}_\infty$ was 0.58 s. Out of the total 105 inequalities resulting from this horizon, 18 were determined to be redundant. It must be remarked that the upper bound $\bar{k}$ computed using the Lyapunov approach as in[20] would give an unreasonably long horizon of $\bar{k} = 786$ and that, in absence of a sound methodology, in previous publications reporting this example the horizon k was estimated empirically.

7 | CONCLUSION

In this note we presented a systematic procedure to compute Maximal Output Admissible Sets for linear systems subject to polynomial constraints. To do so, we introduced a redundancy certificate for polynomial constraints based on Sum of Squares programming. Further, we provided a modification of the traditional algorithm for the computation of the Maximal Output Admissible Set for increased computational efficiency. Finally, we demonstrated the effectiveness of the proposed methodology by computing the Maximal Output Admissible Sets for two relevant examples of spacecraft rendezvous and an electromagnetic actuator control, indicating a significant potential for practical impact of the proposed methodology.

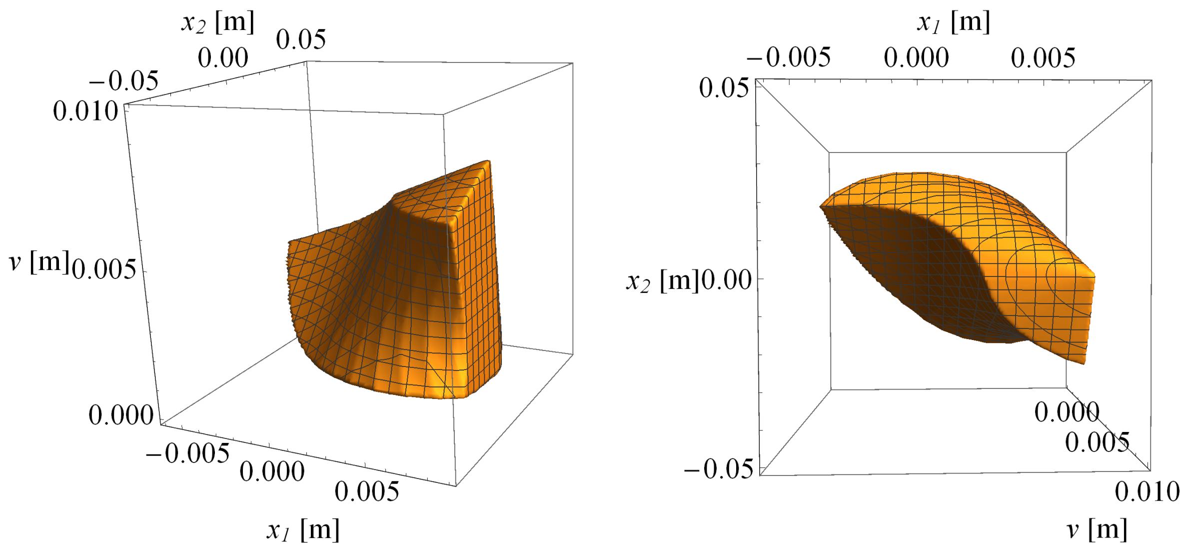

FIGURE 3 $\tilde{\mathcal{O}}_\infty$ for system (14) viewed from two different angles.

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Author contributions

Every author has contributed substantially to the development of both the research presented in this manuscript and the manuscript itself.

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Conflict of interest

The authors declare no potential conflict of interests.

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