Hodge Theory of Twisted Derived Categories and the Period-index Problem

by

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To Mom and Dad

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ABSTRACT

We study the Hodge theory of twisted derived categories and its relation to the period-index problem. Our main contribution is the development of a theory of twisted Mukai structures for topologically trivial Brauer classes on arbitrary smooth proper varieties and in families. As applications, we construct Hodge classes whose algebraicity would imply period-index bounds; construct new counterexamples to the integral Hodge conjecture on Severi-Brauer varieties; and prove the integral Hodge conjecture for derived categories of Deligne-Mumford surfaces. Finally, we solve the period-index problem for the complex-analytic Brauer group of a general complex torus of dimension at least three.

CHAPTER I
Introduction

I.1: The period-index problem

Let K be a field. A finite-dimensional (noncommutative) algebra A/K is a *central simple algebra* if its center is K, and the only two-sided ideals of A are 0 and A itself. The simplest examples of central simple algebras are the matrix algebras $\mathrm{M}_n(K)$, for each $n > 0$. In fact, for any central simple algebra A/K, there is an isomorphism $A \otimes K^{\mathrm{sep}} \simeq \mathrm{M}_n(K^{\mathrm{sep}})$ where K^{sep} is a separable closure of K; the integer n is called the *degree* of A.

Two central simple algebras A and B over K are *Morita-equivalent* if there exist integers $r, s > 0$ and an isomorphism

$$A \otimes \mathrm{M}_r(K) \simeq B \otimes \mathrm{M}_s(K).$$

The set of Morita-equivalence classes of central simple algebras with tensor product forms a group $\mathrm{Br}(K)$, the *Brauer group* of K.

A fundamental problem, and a precursor to the period-index problem described below, is to determine which fields have a trivial Brauer group. Some classical results are as follows:

- If K is separably closed, then the Brauer group of K is trivial.

- If K is a finite field, then the Brauer group of K is trivial, by Wedderburn's little theorem.

- If K has transcendence degree 1 over an algebraically closed field, then the Brauer group of K is trivial, by Tsen's theorem.

On the other hand, many fields of interest (such as local or global fields, or the real numbers $\mathbf{R}$, or suitable transcendental extensions of all of the above) have nonzero Brauer groups. For these fields, one tries instead to get a handle on the complexity of the Brauer classes themselves.

It is a general fact that $\mathrm{Br}(K)$ is a torsion group, and the order of a class $\alpha \in \mathrm{Br}(K)$ is called its *period*, $\mathrm{per}(\alpha)$. One may also consider the *index* $\mathrm{ind}(\alpha)$, which is the greatest common divisor (in fact, minimum) of the degrees of central simple algebras of class α.

(1) The period of α divides its index.

(2) The period and the index of α have the same prime factors.

The period-index problem is the issue of determining a bound on $\mathrm{ind}(\alpha)$ in terms of $\mathrm{per}(\alpha)$:

Period-Index Problem. *Determine $\epsilon > 0$ such that* $\mathrm{ind}(\alpha)$ *divides* $\mathrm{per}(\alpha)^\epsilon$.

Generally speaking, one seeks an ϵ which works for *any* $\alpha \in \mathrm{Br}(K)$. We mention a few notable examples. According to the Albert–Hasse–Noether–Brauer theorem [Roq05], period and index coincide in the Brauer group of a local field. (Indeed, any Brauer class is represented by a cyclic algebra of degree equal to the period). Artin, in the context of his work on Severi–Brauer varieties over C_2 fields, conjectured that period and index coincide for Brauer classes over C_2 fields [Art82]. Finally, Saltman showed that $\mathrm{ind}(\alpha)$ divides $\mathrm{per}(\alpha)^2$ for a Brauer class α on a p-adic curve, assuming that $\mathrm{per}(\alpha)$ is prime to p. These results contribute to the general picture that Brauer groups of fields with bounded "dimension" (for instance, cohomological dimension or Tsen rank) should have a uniform period-index bound ϵ.

We now turn to the conjectural picture, which seems to have been folklore for some time before its appearance in unpublished notes of Colliot–Thélène [CT01]:

Period-Index Conjecture. *Let k be an algebraically closed field, and let K/k be a finitely generated extension of transcendence degree d. Then for any $\alpha \in \mathrm{Br}(K)$,*

$$\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^{d-1}.$$

One could formulate the conjecture more generally for C_d fields. We note, however, that the analogous conjecture for fields of cohomological dimension d is false. In fact, Merkurjev has constructed a family of fields K_i of cohomological dimension 2 and Brauer classes $\alpha_i \in \mathrm{Br}(K_i)$ of period 2 such that $\mathrm{ind}(\alpha_i)$ is unbounded [Mer91]. There are also versions when k is a finite or p-adic field:

- With $k = \mathbf{F}_q$, one conjectures that for $\alpha \in \mathrm{Br}(K)$, $\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^d$.

- With $k = \mathbf{Q}_p$, one conjectures that for $\alpha \in \mathrm{Br}(K)$, $\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^{d+1}$.

It is not known if there is a clean formulation of the conjecture which covers all known and expected cases.

The period-index conjecture is trivial for $d = 0$ and $d = 1$. A crucial result is the theorem of de Jong [dJ04], which proves the conjecture when $d = 2$ and $\mathrm{per}(\alpha)$ is prime to the characteristic of k. De Jong's result was later extended by Lieblich [Lie08] to the general case:

Theorem I.1 (de Jong, Lieblich). *Let k be an algebraically closed field, and let K/k be a finitely generated extension of transcendence degree 2. Then for any $\alpha \in \mathrm{Br}(K)$,*

$$\mathrm{ind}(\alpha) = \mathrm{per}(\alpha).$$

On the other hand, the picture for an extension K/k of transcendence degree at least 3 is obscure. In fact, there is no known example of a finitely generated extension K/k of transcendence degree at least 3 over an algebraically closed, and an integer $\epsilon > 0$, such that

$$\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^\epsilon$$

for *all* $\alpha \in \mathrm{Br}(K)$.

I.2: Hodge theory of twisted derived categories

We now introduce the main point of view of this thesis. We will adopt the viewpoint of *twisted sheaves*, which were introduced by Giraud [Gir71], but became widespread after the appearance of Căldăraru's thesis [Căl00]. The importance of twisted sheaves to the study of all aspects of the Brauer group was underscored by their role in de Jong's proof of Gabber's result that $\mathrm{Br}(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_m)$ for a scheme with an ample line bundle [dJ03].

First, given a scheme X and a class $\alpha \in \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_m)$, one may consider the abelian category $\mathrm{Coh}(X, \alpha)$ of α-twisted sheaves in the sense of Căldăraru [Căl00].

The Grothendieck group $\mathrm{K}_0(X, \alpha)$ of $\mathrm{Coh}(X, \alpha)$ admits a rank homomorphism

$$\mathrm{rk} : \mathrm{K}_0(X, \alpha) \to \mathbf{Z},$$

and a key fact is that the index of α coincides with the positive generator of $\mathrm{rk}(\mathrm{K}_0(X, \alpha))$.

Remark I.2. We make a simple observation in the case $\alpha = 0$. There is a forgetful homomorphism from the algebraic K-group $\mathrm{K}_0(X)$ to the topological K-group $\mathrm{K}_0^{\mathrm{top}}(X)$, which is the Grothendieck group of complex topological vector bundles on X. Moreover, the forgetful

homomorphism admits a factorization

$$
\begin{array}{ccc}
\mathrm{K}_0(X) & \longrightarrow & \mathrm{K}_0^{\mathrm{top}}(X) \\
& \searrow {\scriptstyle(\star)} \quad \nearrow & \\
& \mathrm{Hdg}(X) &
\end{array}
$$

where $\mathrm{Hdg}(X) \subseteq \mathrm{K}_0^{\mathrm{top}}(X)$ is the subgroup of classes in $\mathrm{K}_0^{\mathrm{top}}(X)$ whose Chern characters are of Hodge type in each degree. The question of whether $(\star)$ is surjective is a variant of the integral Hodge conjecture on X.

We now explain an analogous picture for $\mathrm{K}_0(X, \alpha)$, relying on a framework recently introduced by Perry [Per22]. Begin with a $\mathbf{C}$-linear triangulated category $\mathcal{C}$ (suitably enhanced), which admits an admissible embedding into a category of the form $\mathrm{D}_{\mathrm{perf}}(Y)$, for Y a smooth, projective variety over $\mathbf{C}$; such a category $\mathcal{C}$ will be called *geometric*. To a geometric category $\mathcal{C}$, Perry associates a natural weight-0 Hodge structure $\mathrm{K}_0^{\mathrm{top}}(\mathcal{C})$, satisfying several properties:

First, in the case when $\mathcal{C} = \mathrm{D}_{\mathrm{perf}}(X)$ for a smooth, projective variety X, $\mathrm{K}_0^{\mathrm{top}}(\mathcal{C})$ is naturally isomorphic to $\mathrm{K}_0^{\mathrm{top}}(X)$, with the Hodge structure induced by the rational Chern character isomorphism

$$
\mathrm{K}_0^{\mathrm{top}}(X) \otimes \mathbf{Q} \simeq \bigoplus_i \mathrm{H}^{2i}(X, \mathbf{Q})(i).
$$

Second, there is a natural homomorphism from the Grothendieck group $\mathrm{K}_0(\mathcal{C})$ to $\mathrm{K}_0^{\mathrm{top}}(\mathcal{C})$, admitting a factorization

$$
\begin{array}{ccc}
\mathrm{K}_0(\mathcal{C}) & \longrightarrow & \mathrm{K}_0^{\mathrm{top}}(\mathcal{C}) \\
& \searrow {\scriptstyle(\star)} \quad \nearrow & \\
& \mathrm{Hdg}(\mathcal{C}) &
\end{array}
$$

where $\mathrm{Hdg}(\mathcal{C}) \subseteq \mathrm{K}_0^{\mathrm{top}}(\mathcal{C})$ is the subgroup of integral Hodge classes.

Integral Hodge Conjecture (Perry)**.** *The homomorphism $(\star)$ is surjective.*

Remark I.3. The integral Hodge conjecture for $\mathcal{C}$ is an analogue of the classical integral Hodge conjecture [Hod52], which asserts that for a smooth projective variety X over $\mathbf{C}$, the cycle class map

$$
\mathrm{CH}^k(X) \to \mathrm{Hdg}^{2k}(X, \mathbf{Z}(k)) \subseteq \mathrm{H}^{2k}(X, \mathbf{Z}(k))
$$

is surjective in each degree.

Returning to the case of twisted sheaves, one may take $\mathcal{C} = \mathrm{D}_{\mathrm{perf}}(X, \alpha)$, the derived category of α-twisted sheaves. A result of Bernardara [Ber09] implies that $\mathrm{D}_{\mathrm{perf}}(X, \alpha)$ is geometric, so Perry's formalism applies. In particular, we obtain a diagram

$$
\begin{array}{ccc}
\mathrm{K}_0(X, \alpha) & \xrightarrow{\hspace{3cm}} & \mathrm{K}_0^{\mathrm{top}}(X, \alpha) \\
& \searrow{\scriptstyle(\star)} \quad \nearrow & \\
& \mathrm{Hdg}(X, \alpha) &
\end{array}
$$

where we have written $\mathrm{K}_0^{\mathrm{top}}(X, \alpha) = \mathrm{K}_0^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(X, \alpha))$, and likewise with $\mathrm{Hdg}(X, \alpha)$.

It is natural, then, to divide the period-index problem for $\alpha \in \mathrm{Br}(X)$ into two parts:

(1) Construct a Hodge class $v \in \mathrm{Hdg}(X, \alpha)$ of rank $\mathrm{per}(\alpha)^{\dim X - 1}$.

(2) Show that v is algebraic, i.e., lies in the image of an element of $\mathrm{K}_0(X, \alpha)$.

In order to carry out step (1), it is necessary for one to be able to compute the Hodge structure $\mathrm{K}_0^{\mathrm{top}}(X, \alpha)$.

Let X be a smooth, projective variety over $\mathbf{C}$. From the exponential sequence, there is an exact sequence

$$
0 \longrightarrow \frac{\mathrm{H}^2(X, \mathbf{Q}(1))}{\mathrm{H}^2(X, \mathbf{Z}(1)) + \mathrm{NS}(X)_{\mathbf{Q}}} \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{H}^3(X, \mathbf{Z}(1))^{\mathrm{tors}} \longrightarrow 0.
$$

A Brauer class $\alpha \in \mathrm{Br}(X)$ is *topologically trivial* if it lies in the kernel of the right-hand map. Equivalently, there is an element $B \in \mathrm{H}^2(X, \mathbf{Q}(1))$ mapping to α; such a B is called a *rational B-field*.

Theorem I.4. *Let X be a smooth, projective variety over $\mathbf{C}$, and let $\alpha \in \mathrm{Br}(X)$ be topologically trivial.*

(1) *There is an isomorphism of abelian groups between $\mathrm{K}_0^{\mathrm{top}}(X, \alpha)$ and $\mathrm{K}_0^{\mathrm{top}}(X)$.*

(2) *Under the isomorphism from (1), the Hodge structure on $\mathrm{K}_0^{\mathrm{top}}(X, \alpha)$ may be identified with the Hodge structure on $\mathrm{K}_0^{\mathrm{top}}(X)$ induced by the isomorphism*

$$
\mathrm{K}_0^{\mathrm{top}}(X) \otimes \mathbf{Q} \simeq \bigoplus_i \mathrm{H}^{2i}(X, \mathbf{Q})(i), \quad v \mapsto \exp(B) \cdot \mathrm{ch}(v),
$$

where B is any rational B-field for α.

Theorem I.4 says that the Hodge structure $\mathrm{K}_0^{\mathrm{top}}(X, \alpha)$ is a form of *twisted Mukai structure*, a notion which has been used extensively in the context of twisted derived categories of K3 surfaces since the work of Huybrechts and Stellari [HS05].

I.3: Applications to the period-index problem

There is a natural rank homomorphism from $\mathrm{K}_0^{\mathrm{top}}(X, \alpha)$, which is compatible with the rank morphism from $\mathrm{K}_0(X, \alpha)$:

$$\begin{array}{ccc} \mathrm{K}_0(X, \alpha) & \longrightarrow & \mathrm{K}_0^{\mathrm{top}}(X, \alpha) \\ & \searrow \quad \swarrow & \\ & \mathbf{Z}. & \end{array}$$

We define *Hodge-theoretic index* $\mathrm{ind}_{\mathrm{H}}(\alpha)$ as the positive generator of the image of $\mathrm{Hdg}(X, \alpha)$ under the rank homomorphism. We will show (among other things) that

$$\mathrm{per}(\alpha) \mid \mathrm{ind}_{\mathrm{H}}(\alpha) \mid \mathrm{ind}(\alpha).$$

In particular, if one assumes the period-index conjecture, then it must be the case that $\mathrm{ind}_{\mathrm{H}}(\alpha)$ is bounded by $\mathrm{per}(\alpha)^{\dim X - 1}$. We show this up to a constant factor:

Theorem I.5. *Let α be a topologically trivial Brauer class. Then*

$$\mathrm{ind}_{\mathrm{H}}(\alpha) \mid \mathrm{per}(\alpha)^{\dim X - 1} \cdot ((\dim X - 1)!)^{\dim X - 2}.$$

While the period-index problem is concerned with upper bounds for $\mathrm{ind}(\alpha)$, it is worth noting that there are few methods available for producing *lower* bounds, with perhaps the most notable (in the global setting) being Gabber's method [CT02], which works on products of curves. In many cases where α is topologically trivial and the cohomology ring of X is well understood, one may compute $\mathrm{ind}_{\mathrm{H}}(\alpha)$ concretely using Theorem I.4, which results in a lower bound for α. In this way, one can recover (and generalize) a result of Kresch:

Theorem I.6 (Kresch)**.** *Let X be a smooth, projective variety over $\mathbf{C}$. For $b \in \mathrm{H}^2(X, \mathbf{Z}(1))$, let $\alpha \in \mathrm{Br}(X)[2]$ be the 2-torsion Brauer class associated to b. If $\mathrm{ind}(\alpha) = 2$, then there exists $\tau \in \mathrm{Hdg}^4(X, \mathbf{Z}(1))$ such that*

$$b^2 + \tau \equiv 0 \mod 2$$

The Brauer class α associated to b is given by taking $b/2$ as the rational B-field. The importance of Kresch's result is in its converse implication; that is, it allows one to construct examples of Brauer classes α with $\mathrm{per}(\alpha) = 2$ and $\mathrm{ind}(\alpha) = 4$. When X is a threefold, these examples are sharp with respect to the period-index conjecture. As an extension of Kresch's result, we obtain a Hodge-theoretic obstruction for α to have index 4:

Theorem I.7. *Let X be a smooth, projective variety over $\mathbf{C}$. For $b \in \mathrm{H}^2(X, \mathbf{Z}(1))$, let $\alpha \in \mathrm{Br}(X)[2]$ be the 2-torsion Brauer class associated to b. If $\mathrm{ind}(\alpha) = 4$, then there exist $\sigma \in \mathrm{NS}(X)$, $\tau \in \mathrm{Hdg}^4(X, \mathbf{Z}(1))$ such that*

$$(\mathrm{I.3.1}) \qquad\qquad b^2 + \sigma \cdot b + \tau \equiv 0 \mod 2.$$

The interesting case is when $\dim X = 3$. On the one hand, the period-index conjecture predicts that $\mathrm{ind}(\alpha) = 4$. On the other hand, we do not know if the condition (I.3.1) is satisfied by all degree 2 cohomology classes b on threefolds in general, although it is satisfied in all examples known to us.

I.4: Applications to the integral Hodge conjecture

In his 1950 ICM address [Hod52], Hodge conjectured that if X is a smooth, projective variety, then for each $k \geq 0$, the cycle class map

$$\mathrm{cl} : \mathrm{CH}^k(X) \to \mathrm{H}^{2k}(X, \mathbf{Z}(k))$$

surjects onto the subspace of *integral Hodge classes*

$$\mathrm{Hdg}^{2k}(X, \mathbf{Z}(k)) = \{v \in \mathrm{H}^{2k}(X, \mathbf{Z}(k)) : v_{\mathbf{C}} \in \mathrm{F}^0 \mathrm{H}^{2k}(X, \mathbf{C}(k))\}.$$

Atiyah and Hirzebruch observed Hodge's formulation would imply that each *torsion* class $v \in \mathrm{H}^{2k}(X, \mathbf{Z}(k))$ lies in the image of cycle class map, and constructed counterexamples with topological methods. As a result, Hodge's formulation (which, we emphasize, is known to be false) is called the *integral Hodge conjecture*, whereas the *rational Hodge conjecture* is the analogous statement with rational coefficients.

We note that the method of Atiyah and Hirzebruch (and subsequent authors, notably Totaro [Tot97]) is based on approximating the Deligne–Mumford stack $\mathrm{B}G$ for G a finite group. On the other hand, it is not difficult to see from basic representation theory that the integral Hodge conjecture holds for the category $\mathrm{D}_{\mathrm{perf}}(\mathrm{B}G)$, illustrating the principle that Perry's integral Hodge conjecture for categories is more robust than the usual integral Hodge conjecture for cohomology.

Theorem I.8. *Let X be a smooth, proper Deligne–Mumford stack of dimension at most 2. Then the integral Hodge conjecture holds for $\mathrm{D}_{\mathrm{perf}}(X)$.*

If X is a smooth, proper variety, then Theorem I.8 follows from Lefschetz's (1,1) theorem, along with the degeneration of the Atiyah–Hirzebruch spectral sequence relating singular

cohomology and topological K-theory. In general case, Theorem I.8 reduces to the case of $D(X, \alpha)$ when X is a smooth, proper *orbifold* of dimension 2, and the proof boils down to an application of de Jong's result that $\mathrm{per}(\alpha) = \mathrm{ind}(\alpha)$.

The Hodge-theoretic index introduced above also provides a method for producing new families of counterexamples to the integral Hodge conjecture, both in its categorical and cohomological forms. The idea is to begin with a Brauer class α with a previously known nontrivial lower bound on the index, so that $\mathrm{per}(\alpha) < \mathrm{ind}(\alpha)$. Then one constructs a Hodge class v in $\mathrm{K}_0^{\mathrm{top}}(X, \alpha)$ of rank $\mathrm{per}(\alpha)$; the algebraicity of v would imply that $\mathrm{per}(\alpha) = \mathrm{ind}(\alpha)$. For instance, using lower bounds of Gabber for certain Brauer classes on products of curves [CT02], we show the following:

Theorem I.9. *Let C be a curve of genus ≥ 2, and let $E_1, \ldots, E_k$ be elliptic curves for $2 \leq k \leq g$. Suppose that $C, E_1, \ldots, E_k$ are very general, and let $X = C \times \prod E_i$.*

  (1) For each prime ℓ, there is a Brauer class α_ℓ such that the integral Hodge conjecture fails for $\mathrm{D}_{\mathrm{perf}}(X, \alpha_\ell)$.

  (2) For each Severi–Brauer variety of class $\mathbf{P}$ of class α_ℓ, the integral Hodge conjecture fails for $\mathbf{P}$.

We note that the counterexamples to the integral Hodge conjecture from (2) are topologically rather simple: Each Severi–Brauer variety over a product of curves is (topologically) the projectivization of a complex (topological) vector bundle. We prove a similar result for Severi–Brauer varieties over abelian threefolds:

Theorem I.10. *Let A be an abelian threefold, with $\alpha \in \mathrm{Br}(X)[2]$ such that $\mathrm{ind}(\alpha) > 2$. For any Severi–Brauer variety $\mathbf{P} \to X$ of class α, the integral Hodge conjecture fails for $\mathbf{P}$*

In contrast to Theorem I.9, we do not assert that the integral Hodge conjecture fails for $D(X, \alpha)$. In fact, we expect it to be true.

I.5: The period-index problem for complex tori

Let X be a connected complex manifold. An *Azumaya algebra* $\mathcal{A}$ over X is a sheaf of $\mathcal{O}_X$-algebras which is locally isomorphic to a matrix $\mathcal{O}_X$-algebra $\mathrm{M}_n(\mathcal{O}_X)$. Since the rank of an Azumaya algebra $\mathcal{A}$ is a square, one defines the *degree* of $\mathcal{A}$ to be $\sqrt{\mathrm{rk}\,\mathcal{A}}$. Two Azumaya algebras $\mathcal{A}$ and $\mathcal{B}$ are *Morita equivalent* if there exist vector bundles E and F, and an isomorphism of $\mathcal{O}_X$-algebras

$$\mathcal{A} \otimes \mathcal{E}nd(E) \simeq \mathcal{B} \otimes \mathcal{E}nd(F).$$

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The Brauer group $\operatorname{Br}(X)$ is the group of Morita-equivalence classes of Azumaya algebras with tensor product.

Given a Brauer class α, the *period* $\operatorname{per}(\alpha)$ of α is its order in $\operatorname{Br}(X)$, which is a torsion group. The *index* $\operatorname{ind}(\alpha)$ of α is the greatest common divisor of the set of degrees of Azumaya algebras of class α. It follows from a general result of Antieau and Williams that $\operatorname{per}(\alpha)$ divides $\operatorname{ind}(\alpha)$, and that they share prime factors [AW15]. In analogy with the algebraic case considered above, the *period-index problem* asks for an integer $\epsilon > 0$ such that

$$\operatorname{ind}(\alpha) \mid \operatorname{per}(\alpha)^{\dim X - 1}.$$

Little is known about the period-index problem for Brauer groups of non-algebraic complex manifolds beyond the case of surfaces. When X is an analytic K3 surface, an analysis of the argument given in [HS03] shows that period and index coincide, in accordance with de Jong's theorem. Moreover, the same argument applies to the case when X is a 2-dimensional complex torus. When X is a Stein manifold, the period-index problem is equivalent to the topological period-index problem by the Grauert–Oka principle, and one may obtain bounds from the work of Antieau and Williams [AW21] (Remark VIII.2).

If X is a complex torus of dimension g with $\alpha \in \operatorname{Br}(X)$, then there is a naïve upper bound on $\operatorname{ind}(\alpha)$ given by the *annihilator* $\operatorname{Ann}(\alpha)$, which is the least degree of a finite isogeny $f : X' \to X$ such that $f^* \alpha = 0$. When $\operatorname{NS}(X) = 0$, one may compute $\operatorname{Ann}(\alpha)$ explicitly (Lemma VIII.15), and as α ranges over $\operatorname{Br}(X)[n]$, $\operatorname{Ann}(\alpha)$ attains any positive integer value which divides n^g. Our main result is that for X general, $\operatorname{Ann}(\alpha)$ is also a lower bound for $\operatorname{ind}(\alpha)$:

Theorem I.11. *Let X be a general complex torus of dimension $g \geq 3$, and let $\alpha \in \operatorname{Br}(X)$. Then*

$$\operatorname{ind}(\alpha) = \operatorname{Ann}(\alpha)$$

More precisely, Theorem I.11 holds for a complex torus X with $\operatorname{NS}(X) = \operatorname{Hdg}^4(X) = 0$. The proof of Theorem I.11 is based on an argument of Voisin [Voi02] showing that a general complex torus of dimension at least three does not satisfy the resolution property, along with an analysis of the Hodge-theoretic properties of α-twisted sheaves. As an immediate consequence of Theorem I.11, we obtain the following result:

Corollary I.12. *Let X be a general complex torus of dimension $g \geq 3$. For each $n > 0$, there exists a Brauer class α with $\operatorname{per}(\alpha) = n$, $\operatorname{ind}(\alpha) = n^g$.*

In particular, the extension of the period-index conjecture to even the most familiar compact Kähler manifolds is false. By contrast, we prove the period-index conjecture for

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abelian threefolds in forthcoming work with Perry [HP22].

One may interpret Theorem I.11 as the statement that the rank of any locally free α-twisted sheaf is divisible by $\mathrm{Ann}(\alpha)$. In fact, we show that the rank of an arbitrary α-twisted coherent sheaf on X is divisible by $\mathrm{Ann}(\alpha)$ (Theorem VIII.17), so that using a *coherent* index (Definition VIII.4) does not correct for the failure of the expected period-index bounds.

CHAPTER II
Twisted Derived Categories

II.1: Linear categories

The content of §II.2 is rather formal, so we work in a greater level of generality than is necessary for the rest of the paper. Note, however, that for the main results of the paper, it is sufficient to consider smooth, separated Deligne–Mumford stacks over a field of characteristic 0.

In general, we follow the conventions and terminology of [Per19], with the exception that our base S is permitted to be a *perfect algebraic stack*, as in [BZFN10]. In contrast to [BZFN10], we avoid for simplicity the language of derived algebraic geometry.

Definition II.1. An algebraic stack S is *perfect* if the following conditions hold:

(1) $D_{qc}(S)$ is compactly generated.

(2) The compact and perfect objects of $D_{qc}(S)$ coincide.

(3) The diagonal of S is affine.

Quasi-compact tame Deligne–Mumford stacks with affine diagonal are perfect [HR17]. For example, separated Deligne–Mumford stacks of finite type over a field of characteristic 0 are perfect.

Let S be a perfect algebraic stack. Then $D_{perf}(S)$, with the tensor product of complexes, may be regarded as a commutative algebra object of the category Cat_{st} of small, idempotent-complete stable ∞-categories.

An *S-linear category* is a $D_{perf}(S)$-module object of Cat_{st}. The collection of S-linear categories forms an ∞-category $Cat_S = Mod_{D_{perf}(S)}(Cat_{st})$, with a symmetric monoidal structure given by the tensor product of $D_{perf}(S)$-modules $\mathcal{C} \otimes_S \mathcal{C}'$. Moreover, given a $\mathcal{C}, \mathcal{C}' \in Cat_S$, there is a mapping object $Fun_S(\mathcal{C}, \mathcal{C}') \in Cat_S$, satisfying the property that

$$\operatorname{Map}_{Cat_S}(\mathcal{C}, \mathcal{C}') = Fun_S(\mathcal{C}, \mathcal{C}')^{\simeq}$$

where the left-hand side denotes the morphism space in the category Cat_S, and the right-hand side denotes the maximal ∞-subgroupoid of $\mathrm{Fun}_S(\mathcal{C}, \mathcal{C}')$, obtained from $\mathrm{Fun}_S(\mathcal{C}, \mathcal{C}')$ by discarding non-invertible 1-morphisms.

Example II.2. Let $f : X \to S$ be a morphism. Then $\mathrm{D}_{\mathrm{perf}}(X)$ is an S-linear category, with the action of $\mathrm{D}_{\mathrm{perf}}(S)$ given by

$$E \mapsto E \otimes f^*(F), \quad E \in \mathrm{D}_{\mathrm{perf}}(X), F \in \mathrm{D}_{\mathrm{perf}}(S).$$

Let $T \to S$ be a morphism, and let $\mathcal{C}$ be an S-linear category. The *base change* $\mathcal{C}_T$ is the tensor product

$$\mathcal{C} \otimes_{\mathrm{D}_{\mathrm{perf}}(S)} \mathrm{D}_{\mathrm{perf}}(T)$$

regarded as an T-linear category.

Example II.3. Let $T \to S$ be a morphism between perfect algebraic stacks, and let $\mathcal{C} = \mathrm{D}_{\mathrm{perf}}(S)$. Then

$$\mathcal{C}_T \simeq \mathrm{D}_{\mathrm{perf}}(T)$$

by [BZFN10, Theorem 1.2]

II.2: Twisted derived categories

In this section, we establish some basic notation and results about twisted derived categories. We begin by briefly introducing the abstract theory, due to Toën and extended by Antieau–Gepner [AG14] and Antieau [Ant17].

Let X be a perfect stack. An X-linear category is a *twisted derived category* over X if there exists an fppf cover $U \to X$ such that $\mathcal{C}_U$ is equivalent to $\mathrm{D}_{\mathrm{perf}}(U)$. According to a result of Toën [Toë12], any twisted derived category $\mathcal{C}$ is determined up to equivalence by a class

$$[\mathcal{C}] \in \mathrm{dBr}(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_m) \times \mathrm{H}^1_{\text{ét}}(X, \mathbf{Z}),$$

where $\mathrm{dBr}(X)$ is the *derived Brauer group* of X. Briefly, one may construct $[\mathcal{C}]$ as follows: the higher stack $\mathcal{I}so_X(\mathcal{C}, \mathrm{D}_{\mathrm{perf}}(X))$ of X-linear equivalences is a torsor under $G = \mathcal{A}ut_X(\mathrm{D}_{\mathrm{perf}}(X)) = \mathrm{B}\mathbf{G}_m \times \mathbf{Z}$, and is determined up to isomorphism by the cohomology class $[\mathcal{C}] \in \mathrm{H}^1_{\text{ét}}(X, G)$. Conversely, for any $\alpha \in \mathrm{dBr}(X)$, there exists an X-linear category $\mathcal{C}$, unique up to equivalence, with $\alpha = [\mathcal{C}]$.

Notation II.4. For $\alpha \in \mathrm{dBr}(X)$, we write $\mathrm{D}_{\mathrm{perf}}(X, \alpha)$ for the choice of a category $\mathcal{C}$ with $[\mathcal{C}] = \alpha$.

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We warn the reader that $\mathrm{D}_{\mathrm{perf}}(X, \alpha)$ is unique up to potentially non-unique X-linear equivalence. More precisely, if $\mathcal{C}$ and $\mathcal{C}'$ are twisted derived categories categories with $[\mathcal{C}] = [\mathcal{C}']$, then the set of isomorphism classes of X-linear equivalences $\mathcal{C} \simeq \mathcal{C}'$ is a torsor under $\mathrm{Pic}(X) \times \mathrm{H}^0_{\mathrm{\acute{e}t}}(X, \mathbf{Z})$.

Example II.5. Let $\mathcal{X} \to X$ be a $\mathbf{G}_m$-gerbe, and let $\mathrm{D}^1_{\mathrm{perf}}(\mathcal{X}/X)$ be the derived category of perfect 1-twisted complexes on X. When the gerbe structure on $\mathcal{X}$ is understood, we simply write $\mathrm{D}^1_{\mathrm{perf}}(\mathcal{X})$. Then $\mathrm{D}^1_{\mathrm{perf}}(\mathcal{X})$ is a twisted derived category over X, with

$$[\mathrm{D}^1_{\mathrm{perf}}(\mathcal{X})] = [\mathcal{X}] \in \mathrm{H}^2_{\mathrm{\acute{e}t}}(X, \mathbf{G}_m) \subseteq \mathrm{dBr}(X).$$

To prove this, observe that $\mathcal{I}so_X(\mathrm{D}^1_{\mathrm{perf}}(\mathcal{X}), \mathrm{D}_{\mathrm{perf}}(X))$ may be identified with $\mathcal{P}\mathrm{ic}^{-1}(\mathcal{X}/X) \times \mathrm{H}^0_{\mathrm{\acute{e}t}}(X, \mathbf{Z})$, where the first factor is the stack of -1-twisted line bundles on $\mathcal{X}/X$. Then $\mathcal{P}\mathrm{ic}^{-1}(\mathcal{X}/X) \to X$ is a $\mathbf{G}_m$-gerbe of class $[\mathcal{X}]$ (cf. [Shi21, Prop. 2.11]).

Example II.6. For $n > 0$, let $\mathcal{X} \to X$ be a $\boldsymbol{\mu}_n$-gerbe. As in the previous example, the X-linear category $\mathrm{D}^1_{\mathrm{perf}}(\mathcal{X})$ of perfect 1-twisted complexes on $\mathcal{X}$ is a twisted derived category of class $\alpha \in \mathrm{H}^2_{\mathrm{\acute{e}t}}(X, \mathbf{G}_m)$, where α is the image of $[\mathcal{X}]$ under the homomorphism

$$\mathrm{H}^2_{\mathrm{\acute{e}t}}(X, \boldsymbol{\mu}_n) \to \mathrm{H}^2_{\mathrm{\acute{e}t}}(X, \mathbf{G}_m).$$

In fact, there is an orthogonal decomposition

$$(\mathrm{II}.2.1) \qquad \mathrm{D}_{\mathrm{perf}}(\mathcal{X}) = \langle \mathrm{D}^0_{\mathrm{perf}}(\mathcal{X}), \mathrm{D}^1_{\mathrm{perf}}(\mathcal{X}), \ldots, \mathrm{D}^{n-1}_{\mathrm{perf}}(\mathcal{X}) \rangle,$$

where $\mathrm{D}^k_{\mathrm{perf}}(\mathcal{X})$ is the category of perfect k-twisted complexes on $\mathcal{X}$, and $\mathrm{D}^0_{\mathrm{perf}}(\mathcal{X})$ is the pullback of $\mathrm{D}_{\mathrm{perf}}(X)$.

Example II.7. Let $\pi : \mathbf{P} \to X$ be a Severi–Brauer variety of relative dimension $n - 1$. According to a result of Bernardara [Ber09] (and [BS21] in the case of an algebraic stack X), there is an X-linear semiorthogonal decomposition

$$(\mathrm{II}.2.2) \qquad \mathrm{D}_{\mathrm{perf}}(\mathbf{P}) = \langle \mathrm{D}_{\mathrm{perf}}(\mathbf{P})_0, \mathrm{D}_{\mathrm{perf}}(\mathbf{P})_1, \ldots, \mathrm{D}_{\mathrm{perf}}(\mathbf{P})_{n-1} \rangle,$$

where $\mathrm{D}_{\mathrm{perf}}(\mathbf{P})_0$ is the pullback of $\mathrm{D}_{\mathrm{perf}}(X)$, and for any k, $\mathrm{D}_{\mathrm{perf}}(\mathbf{P})_k$ is a twisted derived category over X of class $[\mathbf{P}]^k$, where $[\mathbf{P}]$ is the class of $\mathbf{P}$ in the cohomological Brauer group of X.

Warning II.8. We warn the reader that we follow Giraud's convention [Gir71, Example V.4.8] regarding the Brauer class of a Severi–Brauer variety, which differs from Bernardara's

convention by a sign.

Remark II.9. When $[\mathbf{P}] = 0$, (II.2.2) recovers Beilinson's semiorthogonal decomposition for a projective bundle, with $\mathrm{D}_{\mathrm{perf}}(\mathbf{P})_k = \pi^* \mathrm{D}_{\mathrm{perf}}(X) \otimes \mathcal{O}_{\mathbf{P}}(1)$ for any tautological line bundle $\mathcal{O}_{\mathbf{P}}(1)$.

Remark II.10. We explain the compatibility between Example II.7 and Example II.6. Let X be a separated scheme of finite type over $\mathbf{C}$. Let $\mathcal{X} \to X$ be a $\boldsymbol{\mu}_n$-gerbe, and suppose that there exists a Severi–Brauer variety $\mathbf{P} \to X$ which represents the image of $[\mathcal{X}]$ in $\mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_m)$.

We observe that $\mathbf{P}_{\mathcal{X}} \to \mathbf{P}$ is a projective bundle. Consider the pullback diagram

$$
\begin{array}{ccc}
\mathbf{P}_{\mathcal{X}} & \xrightarrow{\ f'\ } & \mathbf{P} \\
\downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\
\mathcal{X} & \xrightarrow{\ f\ } & X
\end{array}
$$

Let $\mathrm{Pic}^{-1}(\mathbf{P}_{\mathcal{X}})_1$ be the set of relative hyperplane bundles on $\mathbf{P}_{\mathcal{X}}/\mathcal{X}$ which are -1-twisted for the gerbe structure $\mathbf{P}_{\mathcal{X}} \to \mathbf{P}$. Then $\mathrm{Pic}^{-1}(\mathcal{X})_1$ is nonempty (cf. the proof of [BR19, Theorem 6.2], for instance), and moreover it is a torsor under $\mathrm{Pic}(X)$.

Lemma II.11. *For any* $\mathcal{O}_{\mathbf{P}_{\mathcal{X}}}(1) \in \mathrm{Pic}^{-1}(\mathbf{P}_{\mathcal{X}})_1$, *the Fourier–Mukai functor*

$$
\Phi_{\mathcal{O}_{\mathbf{P}_{\mathcal{X}}(1)}} : \mathrm{D}^1_{\mathrm{perf}}(\mathcal{X}) \to \mathrm{D}_{\mathrm{perf}}(\mathbf{P})_1
$$

is an X*-linear equivalence of categories.*

Proof. From the semiorthogonal decompositions above, tensoring with $\mathcal{O}_{\mathbf{P}_{\mathcal{X}}}(1)$ induces an equivalence
$$
\mathrm{D}^1_{\mathrm{perf}}(\mathcal{X}) \otimes_X \mathrm{D}_{\mathrm{perf}}(\mathbf{P}) \to \mathrm{D}^0_{\mathrm{perf}}(\mathcal{X}) \otimes_X \mathrm{D}_{\mathrm{perf}}(\mathbf{P}),
$$
and an equivalence

$$
\mathrm{D}_{\mathrm{perf}}(\mathcal{X}) \otimes_X \mathrm{D}_{\mathrm{perf}}(\mathbf{P})_0 \to \mathrm{D}_{\mathrm{perf}}(\mathcal{X}) \otimes_X \mathrm{D}_{\mathrm{perf}}(\mathbf{P})_1,
$$

where each category is regarded as an admissible subcategory of $\mathbf{P}_{\mathcal{X}}$ by pullback on both factors. Comparing the two, we see that tensoring with $\mathcal{O}_{\mathbf{P}_{\mathcal{X}}}(1)$ induces an equivalence

$$
\mathrm{D}^1_{\mathrm{perf}}(\mathcal{X}) \otimes_X \mathrm{D}_{\mathrm{perf}}(\mathbf{P})_0 \to \mathrm{D}^0_{\mathrm{perf}}(\mathcal{X}) \otimes_X \mathrm{D}_{\mathrm{perf}}(\mathbf{P})_1,
$$

which may be identified with the Fourier–Mukai transform in the statement of the lemma. □

II.3: Derived categories of non-abelian gerbes

Let X be a perfect algebraic stack, and let $\mathcal{X} \to X$ be a gerbe with finite inertia. Let $\hat{\mathcal{X}} \to X$ be the moduli stack of simple coherent sheaves on $\mathcal{X} \to X$, and let $\hat{\mathcal{X}}^{\mathrm{sh}} \to X$ be the sheafification of $\hat{\mathcal{X}}$ over X. We note that $\hat{\mathcal{X}} \to \hat{\mathcal{X}}^{\mathrm{sh}}$ is a $\mathbf{G}_m$-gerbe.

Let $\mathcal{E} \in \mathrm{D}_{\mathrm{perf}}(\mathcal{X} \times_X \hat{\mathcal{X}})$ be the universal simple, perfect sheaf, and let

$$\Phi_{\mathcal{E}} : \mathrm{D}_{\mathrm{perf}}(\mathcal{X}) \longrightarrow \mathrm{D}^1_{\mathrm{perf}}(\hat{\mathcal{X}}/\hat{\mathcal{X}}^{\mathrm{sh}})$$

be the Fourier-Mukai transform associated to $\mathcal{E}$.

Proposition II.12. *In the context above, if X is a $\mathbf{Q}$-stack, then $\Phi_{\mathcal{E}}$ is an equivalence.*

Proof. The question is fppf-local on X [BZFN12, Prop. 2.1, Prop. 2.2]. Since the statement is compatible with fppf base change, we ultimately reduce to the case when $X = \operatorname{Spec} k$, for k an algebraically closed field with $\mathrm{char}(k) = 0$, and $\mathcal{X} = BG$ for a finite constant group scheme G. Then $\hat{\mathcal{X}}^{\mathrm{sh}}$ may be identified with the set $\{[V_1], \ldots, [V_\ell]\}$ of isomorphism classes of irreducible G-representations.

In terms of G-representations, the component of $\Phi_{\mathcal{E}}$ which maps to $\mathrm{D}_{\mathrm{perf}}(\{[V_i]\})$ is given by

$$V \mapsto (V \otimes V_i)^G \simeq \operatorname{Hom}_G(V^\vee, V_i),$$

and it follows from Maschke's theorem and Schur's lemma that $\Phi_{\mathcal{E}}$ gives an equivalence at the level of categories of coherent sheaves. $\qquad\square$

Corollary II.13. *Let X be a perfect algebraic stack over $\mathbf{Q}$, and let $\mathcal{X} \to X$ be a gerbe with finite inertia. There exists a finite étale cover $Y \to X$ which is representable by algebraic spaces, a class $\alpha \in \mathrm{H}^2_{\mathrm{\acute{e}t}}(Y, \mathbf{G}_m)$, and an X-linear equivalence $\mathrm{D}_{\mathrm{perf}}(\mathcal{X}) \simeq \mathrm{D}_{\mathrm{perf}}(Y, \alpha)$.*

Proof. We may take $Y = \hat{\mathcal{X}}^{\mathrm{sh}}$ and $\alpha = [\hat{\mathcal{X}}]$ in the context of Proposition II.12. $\qquad\square$

Remark II.14. When $\mathcal{X} \to X$ is a $\boldsymbol{\mu}_n$-gerbe, $\hat{\mathcal{X}}^{\mathrm{sh}}$ is a disjoint union of n copies of X, labeled by the characters of $\boldsymbol{\mu}_n$, and the equivalence of Proposition II.12 recovers the orthogonal decomposition (II.2.1).

Example II.15 (Total rigidification)**.** Let X be a smooth Deligne–Mumford stack over a field k of characteristic 0. Then X admits a *total rigidification* [AOV08, Appendix A], which is a gerbe $X \to X^{\mathrm{rig}}$ with finite inertia such that X^{rig} is an orbifold over k. (We recall our convention that an orbifold over k is a Deligne–Mumford stack of finite type with trivial generic stabilizers). By Corollary II.13, $\mathrm{D}_{\mathrm{perf}}(X)$ is equivalent to $\mathrm{D}_{\mathrm{perf}}(Y, \alpha)$, where Y is an orbifold and $\alpha \in \mathrm{H}^2_{\mathrm{\acute{e}t}}(Y, \mathbf{G}_m)$.

II.4: Root stacks

In what follows, we have followed the notation and conventions of [BLS16, §3], to which we refer for details.

Definition II.16. Let X be an algebraic stack, and let $E \subseteq X$ be an effective Cartier divisor. For an integer $r > 0$, we recall that the *rth root stack* $X_{r^{-1}E} \to X$ of X along E is given by the pullback

$$\begin{array}{ccc} X_{r^{-1}E} & \longrightarrow & [\mathbf{A}^1/\mathbf{G}_m] \\ \downarrow & & \downarrow{\scriptstyle \pi_r} \\ X & \xrightarrow{\ f\ } & [\mathbf{A}^1/\mathbf{G}_m], \end{array}$$

where f is the morphism induced by E, and π_r is induced by rth power map on both $\mathbf{A}^1$ and $\mathbf{G}_m$.

Let $\pi : X_{f^{-1}E} \to X$ be the rth stack of X along E. According to [BLS16, Theorem 4.7], $D_{\mathrm{perf}}(X_{r^{-1}E})$ admits an X-linear semiorthogonal decomposition of the form

(II.4.1) $$D_{\mathrm{perf}}(X_{r^{-1}E}) = \langle \mathcal{C}_{r-1}, \mathcal{C}_{r-2}, \dots, \mathcal{C}_1, \pi^* D_{\mathrm{perf}}(X) \rangle,$$

where each $\mathcal{C}_i$ is equivalent to $D_{\mathrm{perf}}(E)$. In fact, the base change of (II.4.1) to the preimage $r^{-1}E$ of E in $X_{r^{-1}E}$ recovers the semiorthogonal decomposition (II.2.1) associated to the $\boldsymbol{\mu}_n$-gerbe $r^{-1}E \to E$, which has a trivial cohomological Brauer class.

Example II.17 (Weak factorization). According to a result of Harper [Har17] (see also [BR19]), if X and Y are smooth, separated Deligne–Mumford stacks over a field of characteristic 0, which are isomorphic over an open substack U, then X and Y may be connected by a chain

$$X = X_0 \xdashrightarrow{\ f_0\ } X_1 \xdashrightarrow{\ f_1\ } \cdots \xdashrightarrow{\ f_{n-1}\ } X_n = Y,$$

where, for each i, either f_i or f_i^{-1} is a root stack over a smooth divisor in the complement of U or a blowup along a smooth closed substack in the complement of U.

If $X_i \to X_i^{\mathrm{rig}}$ denotes the total rigidification of X_i (Example II.15), then there is a diagram

$$\begin{array}{ccccccc} X = X_0 & \xdashrightarrow{\ f_0\ } & X_1 & \xdashrightarrow{\ f_1\ } & \cdots & \xdashrightarrow{\ f_{n-1}\ } & X_n = Y \\ \downarrow & & \downarrow & & & & \downarrow \\ X^{\mathrm{rig}} = X_0^{\mathrm{rig}} & \xdashrightarrow{\ f_0^{\mathrm{rig}}\ } & X_1^{\mathrm{rig}} & \xdashrightarrow{\ f_1^{\mathrm{rig}}\ } & \cdots & \xdashrightarrow{\ f_{n-1}^{\mathrm{rig}}\ } & X_n^{\mathrm{rig}} = Y^{\mathrm{rig}}, \end{array}$$

where the vertical morphisms are gerbes with finite inertia.

Variant II.18. Suppose that X is smooth over a field k, and let E be a simple normal crossing divisor of X, with ordered components $E_1, \ldots, E_n$. For a multi-index $r \in \mathbf{Z}_{>0}^n$, the *iterated rth root stack* $X_{r^{-1}E}$ is given by the fiber product

$$X_{r_1^{-1}E_1} \times_X X_{r_2^{-1}E_2} \times_X \cdots \times_X X_{r_n^{-1}E_n}.$$

The stack $X_{r^{-1}E}$ is smooth over k, and the preimage of E is a simple normal crossing divisor.

The following lemma is standard, but we include it for completeness.

Lemma II.19. *Let X be a smooth variety over a field k, and let $U \subseteq X$ be an open subvariety such that $X - U$ has simple normal crossings with components E_i. If $X_{n^{-1}E} \to X$ is an iterated $\bar{n}$th root stack over $E = (E_i)$, where $\bar{n} = (n, n, \ldots, n)$, then the restriction*

$$\mathrm{Br}(X_{n^{-1}D})[n] \to \mathrm{Br}(U)[n]$$

is an isomorphism.

Proof. By purity, one may reduce to the case of a DVR, which is treated in [Lie11, §3.2]. □

Lemma II.20. *Let U be a smooth, quasi-projective variety over a field k of characteristic 0, and let $\alpha \in \mathrm{Br}(U)$. There exists a smooth, proper Deligne–Mumford stack X, and an open immersion $j : U \to X$, such that the following conditions hold:*

(1) There exists a class $\alpha' \in \mathrm{Br}(X)$ whose restriction to U is α.

(2) Any $\boldsymbol{\mu}_n$-gerbe $\mathcal{U} \to U$ with Brauer class α extends to a $\boldsymbol{\mu}_n$-gerbe $\mathcal{X} \to X$ of Brauer class α'.

Proof. Let X_0 be a smooth, projective variety over k which compactifies U such that $X_0 - U$ has simple normal crossings. We observe that $\mathrm{Pic}(X_0) \to \mathrm{Pic}(U)$ is surjective. By Lemma II.19, there is an iterated root stack $X \to X_0$ such that α extends to a cohomological Brauer class $\alpha' \in \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_m)[n]$.

Let $\mathcal{U} \to U$ be a $\boldsymbol{\mu}_n$-gerbe of Brauer class α. Consider the commutative diagram

$$
\begin{array}{ccccc}
\mathrm{Pic}(X) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(X, \boldsymbol{\mu}_n) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_m)[n] \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Pic}(U) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(U, \boldsymbol{\mu}_n) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(U, \mathbf{G}_m)[n].
\end{array}
$$

The outer vertical morphisms are surjective, so $[\mathcal{U}]$ lifts to a class $[\mathcal{X}] \in \mathrm{H}^2_{\text{ét}}(X, \boldsymbol{\mu}_n)$. Finally, since X is a global quotient stack with quasi-projective coarse space ([BLS16, Lemma

3.4]), the covering theorem of Kresch–Vistoli [KV04] implies that $H^2_{\text{ét}}(X, \mathbf{G}_m)$ coincides with $\text{Br}(X)$ $\qquad\qquad\square$

CHAPTER III
Hodge Theory of Categories

III.1: Topological K-theory

In [Bla16], Blanc constructs a functor

$$K^{\mathrm{top}} : \mathrm{Cat}_{\mathbf{C}} \longrightarrow \mathrm{Sp}$$

from the ∞-category of $\mathbf{C}$-linear categories to the stable ∞-category of spectra, satisfying the following properties:

  (i) K^{top} commutes with filtered colimits.

  (ii) Given an exact sequence[1]

$$\mathcal{C} \longrightarrow \mathcal{C}' \longrightarrow \mathcal{C}''$$

   of $\mathbf{C}$-linear categories, the resulting sequence

$$K^{\mathrm{top}}(\mathcal{C}) \longrightarrow K^{\mathrm{top}}(\mathcal{C}') \longrightarrow K^{\mathrm{top}}(\mathcal{C}'')$$

   is an exact triangle in Sp.

  (iii) There is a commutative diagram

(III.1.1)
$$
\begin{array}{ccc}
K(\mathcal{C}) & \xrightarrow{\ \mathrm{ch}\ } & \mathrm{HN}(\mathcal{C}) \\
\downarrow & & \downarrow \\
K^{\mathrm{top}}(\mathcal{C}) & \xrightarrow{\ \mathrm{ch}\ } & \mathrm{HP}(\mathcal{C})
\end{array}
$$

   in Sp, functorial in $\mathcal{C}$.

---

[1] A sequence of $\mathbf{C}$-linear categories is exact if it is both a fiber and a cofiber sequence.

(iv) If X is a separated scheme of finite type over $\mathbf{C}$, then $\mathrm{K}^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(X))$ may be identified with the complex topological K-theory $\mathrm{K}^{\mathrm{top}}_0(X)$, and (iii) may be identified (functorially in X) with the corresponding diagram

$$
\begin{array}{ccc}
\mathrm{K}(X) & \xrightarrow{\ \mathrm{ch}\ } & \mathrm{HN}(X) \\
\downarrow & & \downarrow \\
\mathrm{K}^{\mathrm{top}}(X^{\mathrm{an}}) & \xrightarrow{\ \mathrm{ch}\ } & \mathrm{HP}(X)
\end{array}
$$

of Chern characters for X.

For our purposes, it is enough to understand $\mathrm{K}^{\mathrm{top}}(\mathcal{C})$ in the case when $\mathcal{C}$ occurs as a semiorthogonal component of a category of the form $\mathrm{D}_{\mathrm{perf}}(X)$, when X is a scheme or a Deligne–Mumford stack. From (ii), the functor $\mathrm{K}^{\mathrm{top}}$ satisfies the following additivity property: Given a semiorthogonal decomposition

$$
\mathcal{C} = \langle \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n \rangle
$$

of a $\mathbf{C}$-linear category $\mathcal{C}$, there is an equivalence

$$
\mathrm{K}^{\mathrm{top}}(\mathcal{C}) \simeq \mathrm{K}^{\mathrm{top}}(\mathcal{C}_1) \oplus \mathrm{K}^{\mathrm{top}}(\mathcal{C}_2) \oplus \cdots \oplus \mathrm{K}^{\mathrm{top}}(\mathcal{C}_n)
$$

induced by the projections $\mathcal{C} \to \mathcal{C}_i$.

Remark III.1. Let $\mathcal{C}$ be a *proper* $\mathbf{C}$-linear category. Perry [Per22, Lemma 5.2] constructs an *Euler pairing*

$$
\chi^{\mathrm{top}}(-,-) : \mathrm{K}^{\mathrm{top}}_i(\mathcal{C}) \otimes \mathrm{K}^{\mathrm{top}}_i(\mathcal{C}) \to \mathbf{Z},
$$

which satisfies the following properties:

(1) If $\mathcal{C}$ admits a semiorthogonal decomposition, then the resulting splitting of $\mathrm{K}^{\mathrm{top}}_i(\mathcal{C})$ is semiorthogonal for the Euler pairing.

(2) If $\chi(-,-)$ is the Euler pairing on $\mathrm{K}_0(\mathcal{C})$ given by

$$
\chi(v, w) = \sum_i (-1)^i \dim \mathrm{Ext}^i_{\mathbf{C}}(v, w),
$$

then the homomorphism $\mathrm{K}_0(\mathcal{C}) \to \mathrm{K}^{\mathrm{top}}_0(\mathcal{C})$ preserves Euler pairings.

(3) If $\mathcal{C} = \mathrm{D}_{\mathrm{perf}}(X)$ for a proper scheme X over $\mathbf{C}$, then for $v, w \in \mathrm{K}^{\mathrm{top}}_i(X)$,

$$
\chi^{\mathrm{top}}(v, w) = s_*(v^{\vee} \otimes w) \in \mathrm{K}^{\mathrm{top}}_{2i}(\mathrm{Spec}\,\mathbf{C}) \simeq \mathbf{Z},
$$

where $s : X \to \mathrm{Spec}(\mathbf{C})$ is the structure morphism.

Blanc's topological K-theory satisfies a localization sequence:

Lemma III.2. *Let $i : Z \to X$ be a closed immersion of algebraic stacks over $\mathbf{C}$, with complement $j : U \to X$. There is an exact triangle*

$$\mathrm{K}^{\mathrm{top}}(\mathrm{D}^b_{\mathrm{coh}}(Z)) \xrightarrow{\ i_*\ } \mathrm{K}^{\mathrm{top}}(\mathrm{D}^b_{\mathrm{coh}}(X)) \xrightarrow{\ j^*\ } \mathrm{K}^{\mathrm{top}}(\mathrm{D}^b_{\mathrm{coh}}(X - Z)).$$

The following argument appeared in a preprint version of [HLP20], and we include it here for completeness. Note that it exploits details of Blanc's construction (cf. [Bla16, Def. 1.2]), which we have not described.

Proof. For a $\mathbf{C}$-linear category $\mathcal{C}$, consider the presheaf of spectra

$$\underline{\mathrm{K}}(\mathcal{C})(U) = \underline{\mathrm{K}}(\mathcal{C} \otimes_{\mathbf{C}} \mathrm{D}_{\mathrm{perf}}(U))$$

on the category of smooth, affine schemes over $\mathbf{C}$.

For any smooth, affine scheme U, and any algebraic stack X, there is an equivalence

$$\mathrm{D}^b_{\mathrm{coh}}(X) \otimes_{\mathbf{C}} \mathrm{D}_{\mathrm{perf}}(U) \simeq \mathrm{D}^b_{\mathrm{coh}}(X \times_{\mathbf{C}} U)$$

by [DG13, Corollary 4.2.3]. From the localization sequence for G-theory of algebraic stacks, it follows that there is a fiber sequence of presheaves of spectra

(III.1.2) $$\underline{\mathrm{K}}(\mathrm{D}^b_{\mathrm{coh}}(Z)) \longrightarrow \underline{\mathrm{K}}(\mathrm{D}^b_{\mathrm{coh}}(X)) \longrightarrow \underline{\mathrm{K}}(\mathrm{D}^b_{\mathrm{coh}}(U))$$

on the category of smooth, affine schemes over $\mathbf{C}$. Finally, (III.1.2) remains a fiber sequence after geometric realization and inverting the Bott element. $\qquad\square$

Example III.3. Let X be a smooth, projective Deligne–Mumford stack over $\mathbf{C}$, and suppose that $X = [Y/G]$, where Y is quasi-projective and G is reductive. According to a result of Halpern-Leistner and Pomerleano, one may identify $\mathrm{K}^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(X))$ with the equivariant topological K-theory $\mathrm{K}^{\mathrm{top}}_G(Y^{\mathrm{an}})$ [HLP20, Theorem 2.10].

Example III.4. Let X be a separated scheme of finite type over $\mathbf{C}$, and $\alpha \in \mathrm{Br}(X)$. Moulinos [Mou19] constructs an equivalence

$$\mathrm{K}^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(X, \alpha)) \simeq \mathrm{KU}^{\bar{\alpha}}(X^{\mathrm{an}}),$$

where $\bar{\alpha} \in \mathrm{H}^3(X^{\mathrm{an}}, \mathbf{Z})^{\mathrm{tors}}$ is the topological Brauer class associated to α, and $\mathrm{KU}^{\bar{\alpha}}(X^{\mathrm{an}})$ is the $\bar{\alpha}$-twisted topological K-theory spectrum of X in the sense of Atiyah–Segal [AS06].

Remark III.5. Let $\mathcal{C}$ be a $\mathbf{C}$-linear category. According to a result of Antieau–Heller [AH18], the functor

$$T \mapsto \mathrm{K}^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(T) \otimes_{\mathbf{C}} \mathcal{C})$$

satisfies étale hyperdescent on the site of smooth, separated schemes over $\mathbf{C}$. In the case $\mathcal{C} = \mathrm{D}_{\mathrm{perf}}(\operatorname{Spec} \mathbf{C})$, one recovers the fact that topological K-theory satisfies étale hyperdescent.

Variant III.6. There is a relative version of Blanc's construction, due to Moulinos [Mou19]. Let S be a separated scheme of finite type over $\mathbf{C}$. There is a functor

$$(\mathrm{III}.1.3) \qquad\qquad \mathrm{K}^{\mathrm{top}}(-/S) : \mathrm{Cat}_S \longrightarrow \mathrm{Shv}_{S^{\mathrm{an}}}(\mathrm{Sp}),$$

valued in sheaves of spectra on S^{an}, satisfying the following properties:

(i) When $S = \operatorname{Spec} \mathbf{C}$, $\mathrm{K}^{\mathrm{top}}(\mathcal{C}/S)$ is the constant sheaf associated to $\mathrm{K}^{\mathrm{top}}(\mathcal{C})$.

(ii) If $\mathcal{C}$ admits an S-linear semiorthogonal decomposition

$$\mathcal{C} = \langle \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n \rangle,$$

then there is a decomposition

$$\mathrm{K}^{\mathrm{top}}(\mathcal{C}/S) = \mathrm{K}^{\mathrm{top}}(\mathcal{C}_1/S) \oplus \mathrm{K}^{\mathrm{top}}(\mathcal{C}_2/S) \oplus \cdots \oplus \mathrm{K}^{\mathrm{top}}(\mathcal{C}_n/S).$$

(iii) If $f : X \to S$ is a proper morphism of separated schemes over $\mathbf{C}$, then $\mathrm{K}^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(X)/S)$ may be identified (functorially in S) with the sheaf of spectra $U \mapsto \mathrm{K}^{\mathrm{top}}((f^{\mathrm{an}})^{-1}(U))$.

III.2: The Hodge filtration

Let $\mathcal{C}$ be a $\mathbf{C}$-linear category. In the terminology of Orlov [Orl16], $\mathcal{C}$ is *geometric* if it arises as a semiorthogonal component of $\mathrm{D}_{\mathrm{perf}}(X)$, for a smooth, proper scheme X over $\mathbf{C}$. Examples of $\mathcal{C}$-linear categories include $\mathrm{D}_{\mathrm{perf}}(X)$ for smooth, proper Deligne–Mumford stacks [BLS16], and, consequently, $\mathrm{D}_{\mathrm{perf}}(X, \alpha)$ where X is a smooth, proper Deligne–Mumford stack, and $\alpha \in \mathrm{H}^2(X, \mathbf{G}_m)$.

Construction III.7. Let $\mathcal{C}$ be a geometric $\mathbf{C}$-linear category. For each $i \in \mathbf{Z}$, the morphism from (iii)

$$\mathrm{K}_i^{\mathrm{top}}(\mathcal{C}) \otimes \mathbf{C} \longrightarrow \mathrm{HP}_i(\mathcal{C})$$

is an isomorphism, the noncommutative Hodge-to-de Rham spectral sequence

$$(\text{III.2.1}) \qquad\qquad \text{HH}_*(\mathcal{C})[u^{\pm 1}] \implies \text{HP}_*(\mathcal{C})$$

degenerates [Kal08], and the resulting filtration endows $\text{K}_i^{\text{top}}(\mathcal{C})$ with a pure Hodge structure of weight $-i$.

Example III.8. Let X be a smooth, proper scheme over $\mathbf{C}$, and let $i \in \mathbf{Z}$. Identifying $\text{K}_i^{\text{top}}(X)$ with $\text{K}_i^{\text{top}}(\text{D}_{\text{perf}}(X))$, the result of Construction III.7 is the unique integral Hodge structure on $\text{K}_i^{\text{top}}(X)$ such that the Chern character homomorphism

$$\text{K}_i^{\text{top}}(X) \otimes \mathbf{Q} \to \bigoplus_k \text{H}^{2k-i}(X^{\text{an}}, \mathbf{Q})(k)$$

is an isomorphism of rational Hodge structures.

Remark III.9. If X is a smooth, proper Deligne–Mumford stack over $\mathbf{C}$, then the Hodge structure on $\text{K}_i^{\text{top}}(\text{D}_{\text{perf}}(X))$ is not necessarily pulled back from the Hodge structures on the rational cohomology of X through a Chern character. For example, when $X = \text{B}G$ for a finite cyclic group G, then $\text{K}_0^{\text{top}}(\text{D}_{\text{perf}}(\text{B}G))$ has rank n, whereas $\text{H}^{2*}(X^{\text{an}}, \mathbf{Q})$ has rank 1.

For a geometric category $\mathcal{C}$, we write $\text{Hdg}(\mathcal{C}, \mathbf{Z})$ for the group of integral Hodge classes in $\text{K}_0^{\text{top}}(\mathcal{C})$. From considering the case of $\text{D}_{\text{perf}}(X)$, it follows that there is a factorization

$$\text{K}_0(\mathcal{C}) \longrightarrow \text{Hdg}(\mathcal{C}, \mathbf{Z}) \subseteq \text{K}_0^{\text{top}}(\mathcal{C}).$$

Perry formulates a noncommutative integral Hodge conjecture [Per22, Conjecture 5.13]:

Conjecture III.10 (Perry)**.** *Let $\mathcal{C}$ be a geometric $\mathbf{C}$-linear category. The homomorphism*

$$\text{K}_0(\mathcal{C}) \to \text{Hdg}(\mathcal{C}, \mathbf{Z})$$

is surjective.

As with the integral Hodge conjecture for the integral cohomology of a smooth, proper variety, Conjecture III.10 is known to fail in certian examples. As Perry notes, if X is a hypersurface in $\mathbf{P}^4$ for which the integral Hodge conjecture fails [BCC92], then Conjecture III.10 fails for $\text{D}_{\text{perf}}(X)$.

Remark III.11. If the Atiyah–Hirzebruch spectral sequence relating $\text{K}_*^{\text{top}}(X)$ with $\text{H}^*(X^{\text{an}}, \mathbf{Z})$ degenerates (for example, if the cohomology ring $\text{H}^*(X^{\text{an}}, \mathbf{Z})$ is torsion-free) then the integral Hodge conjecture in all degrees for X implies the integral Hodge conjecture for $\text{D}_{\text{perf}}(X)$, as

one may see from the proof of [Per22, Prop. 5.16]. In general, however, there do not appear to be implications in either direction between the integral Hodge conjecture for X and the integral Hodge conjecture for $D_{\mathrm{perf}}(X)$.

It is often useful to measure the failure of the integral Hodge conjecture using the *Voisin group*,

$$V(\mathcal{C}) = \mathrm{Coker}\left(K_0(\mathcal{C}) \to \mathrm{Hdg}(\mathcal{C}, \mathbf{Z})\right).$$

The integral Hodge conjecture holds for $\mathcal{C}$ if and only if $V(\mathcal{C}) = 0$.

The most important property of the integral Hodge conjecture for categories is its compatibility with semiorthogonal decompositions: If $\mathcal{C}$ admits a semiorthogonal decomposition

$$\mathcal{C} = \langle \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n \rangle,$$

then there is a decomposition

$$V(\mathcal{C}) = V(\mathcal{C}_1) \oplus V(\mathcal{C}_2) \oplus \cdots \oplus V(\mathcal{C}_n).$$

In particular, the integral Hodge conjecture holds for $\mathcal{C}$ if and only if it holds for each $\mathcal{C}_i$.

III.3: Variations of Hodge structure

Let S be a separated scheme of finite type over $\mathbf{C}$, and $f : X \to S$ be a smooth, proper morphism. Suppose that there is an S-linear semiorthogonal decomposition

$$D_{\mathrm{perf}}(X) = \langle \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n \rangle.$$

For each $i \in \mathbf{Z}$, there is a splitting

$$(\mathrm{III.3.1}) \qquad K_i^{\mathrm{top}}(X/S) = K_i^{\mathrm{top}}(\mathcal{C}_1/S) \oplus K_i^{\mathrm{top}}(\mathcal{C}_2/S) \oplus \cdots \oplus K_i^{\mathrm{top}}(\mathcal{C}_n/S),$$

and combining Construction III.7 with Variant III.6, one may show that (III.3.1) is a splitting of pure variations of Hodge structure.

CHAPTER IV
Twisted Derived Categories

IV.1: Topologically trivial Brauer classes

For an integer n, we write $\mathbf{Z}(n) = (2\pi i)^n \mathbf{Z}$.

Definition IV.1. Let X be a separated scheme of finite type over $\mathbf{C}$, and let $\alpha \in \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_m)^{\text{tors}}$ be a cohomological Brauer class. Then α is *topologically trivial* if it lies in the kernel of the composition

$$\mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_m) \to \mathrm{H}^2(X^{\text{an}}, \mathbf{G}_m^{\text{cont}}) \to \mathrm{H}^3(X^{\text{an}}, \mathbf{Z}(1)),$$

where $\mathbf{G}_m^{\text{cont}}$ is the sheaf of invertible continuous functions on X^{an}, and the right-hand morphism arises from the continuous exponential sequence.

Remark IV.2. If X is proper, then $\mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_m)^{\text{tors}}$ is isomorphic to $\mathrm{H}^2(X^{\text{an}}, \mathcal{O}^{\times}_{X^{\text{an}}})^{\text{tors}}$. From the exponential sequence for the complex-analytic space X^{an}, there is an exact sequence

$$\mathrm{H}^2(X^{\text{an}}, \mathbf{Q}(1)) \longrightarrow \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_m)^{\text{tors}} \longrightarrow \mathrm{H}^3(X^{\text{an}}, \mathbf{Z}(1))^{\text{tors}},$$

so $\alpha \in \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_m)^{\text{tors}}$ is topologically trivial if and only if α lies in the image of $\mathrm{H}^2(X^{\text{an}}, \mathbf{Q}(1))$.

It is often convenient to phrase topological triviality at the level of $\boldsymbol{\mu}_n$-gerbes:

Definition IV.3. Let X be a separated scheme of finite type over $\mathbf{C}$, and let $\mathcal{X} \to X$ be a $\boldsymbol{\mu}_n$-gerbe, for $n > 0$.

(1) If the $\mathbf{G}_m$-gerbe associated to $\mathcal{X}$ is trivial, then $\mathcal{X}$ is *essentially trivial*.

(2) If the topological $\mathbf{G}_m$-gerbe associated to $\mathcal{X}^{\text{an}}$ is trivial, then $\mathcal{X}$ is *essentially topologically trivial*.

We observe that $\mathcal{X}$ is essentially trivial if and only if $[\mathcal{X}]$ lies in the kernel of the homomorphism

$$\mathrm{H}^2_{\text{ét}}(X, \boldsymbol{\mu}_n) \to \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_m).$$

Similarly, $\mathcal{X}$ is essentially topologically trivial if and only if $[\mathcal{X}]$ lies in the kernel of the connecting homomorphism

$$\mathrm{H}^2(X^{\mathrm{an}}, \boldsymbol{\mu}_n) \to \mathrm{H}^3(X^{\mathrm{an}}, \mathbf{Z}(1))$$

arising from the sequence

$$\mathbf{Z}(1) \xrightarrow{\quad n \quad} \mathbf{Z}(1) \xrightarrow{\exp(-/n)} \boldsymbol{\mu}_n.$$

Equivalently, $[\mathcal{X}]$ lies in the image of an element $v \in \mathrm{H}^2(X^{\mathrm{an}}, \mathbf{Z}(1))$.

Remark IV.4. Let $\mathcal{X} \to X$ be a $\boldsymbol{\mu}_n$-gerbe. By a *topological line bundle* on $\mathcal{X}$, we mean a line bundle on the associated gerbe on the topological site of X. If L is a topological line bundle on $\mathcal{X}$, then $L^{\otimes n}$ descends to a topological line bundle on X, which we simply call $L^{\otimes n}$.

Similarly, we will consider 1-twisted topological line bundles on $\mathcal{X}$. If $\mathcal{X} \to X$ is topologically essentially trivial, the set of 1-twisted topological line bundles on $\mathcal{X}$ is a torsor under the group of topological line bundles on X

Definition IV.5. Let X be a separated scheme of finite type over $\mathbf{C}$, and let $\mathcal{X} \to X$ be a $\boldsymbol{\mu}_n$-gerbe. Let L be a topological line bundle on $\mathcal{X}$.

(1) The K-theory class $[L]$ of L in $\mathrm{K}_0^{\mathrm{top}}(X) \otimes \mathbf{Q}$ is the nth root $([L^{\otimes}])^{1/n}$ of the class of $L^{\otimes n}$.

(2) The *first Chern class* of L is $c_1(L) = c_1(L^{\otimes n})/n$, regarded as an element of $\mathrm{H}^2(X^{\mathrm{an}}, \mathbf{Q}(1))$.

(3) The *Chern character* of L is the image of $[L]$ under the Chern character:

$$\mathrm{ch}(L) = \exp(c_1(L)) \in \mathrm{H}^{2*}(X^{\mathrm{an}}, \mathbf{Q}(*)).$$

Lemma IV.6. *Let L be a 1-twisted topological line bundle on $\mathcal{X}$. The image of $c_1(L^{\otimes n})$ under*

$$\exp(-/n) : \mathrm{H}^2(X^{\mathrm{an}}, \mathbf{Z}(1)) \to \mathrm{H}^2(X^{\mathrm{an}}, \boldsymbol{\mu}_n)$$

is $[\mathcal{X}]$.

Proof. This is an analogue of [Lie07, Prop. 2.3.4.4] for the topological site, and the proof is identical. Briefly, if L is a 1-twisted topological line bundle on $\mathcal{X}$, then the topological gerbe associated to $\mathcal{X}$ is equivalent to the gerbe of nth roots of $L^{\otimes n}$. $\qquad\square$

IV.2: Severi–Brauer varieties

Let X be a separated scheme of finite type over $\mathbf{C}$. Let $\mathfrak{X} \to X$ be a $\boldsymbol{\mu}_n$-gerbe, and suppose that there exists a Severi–Brauer variety $\mathbf{P} \to X$ which represents the image of $[\mathfrak{X}]$ in $\mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_m)$. In what follows, we frequently refer to the notation of Example II.7.

Consider the pullback diagram

$$
\begin{array}{ccc}
\mathbf{P}_{\mathfrak{X}} & \xrightarrow{\;f'\;} & \mathbf{P} \\
\downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\
\mathfrak{X} & \xrightarrow{\;f\;} & X.
\end{array}
$$

Let L be a 1-twisted topological line bundle on $\mathfrak{X}$. Then $\pi'^* L \otimes \mathcal{O}_{\mathbf{P}_{\mathfrak{X}}}(1)$ is a 0-twisted topological line bundle on $\mathbf{P}_{\mathfrak{X}}$, so it descends to a line bundle on $\mathbf{P}$, which we simply call $L(1)$.

Proposition IV.7. *In the setting above, multiplication by $L(1)$ on $\mathrm{K}^{\text{top}}_0(\mathbf{P})$ induces an equivalence of spectra*

$$
\mathrm{K}^{\text{top}}(\mathrm{D}_{\text{perf}}(\mathbf{P})_0) \to \mathrm{K}^{\text{top}}(\mathrm{D}_{\text{perf}}(\mathbf{P})_1).
$$

Proof. First, assume that $\mathfrak{X} \to X$ is essentially trivial. We may write $L = L^{\text{alg}} \otimes f^* L_0$, where L^{alg} is a 1-twisted algebraic line bundle on $\mathfrak{X}$ and L_0 is a topological line bundle on X. Then $L(1) = L^{\text{alg}}(1) \otimes \pi^* L_0$, and $L^{\text{alg}}(1)$ is a tautological bundle on the projective bundle $\mathbf{P} \to X$, so the desired result is clear from Beilinson's semiorthogonal decomposition.

In the general case, consider the pullback square

$$
\begin{array}{ccc}
\mathbf{P}^{(2)} = \mathbf{P} \times_X \mathbf{P} & \xrightarrow{\;\pi_2\;} & \mathbf{P} \\
\downarrow{\scriptstyle \pi_1} & & \downarrow \\
\mathbf{P} & \xrightarrow{\hspace{2cm}} & X.
\end{array}
$$

Let $\mathrm{D}_{\text{perf}}(\mathbf{P}^{(2)})^{\pi_1}_k$ be the kth piece of the semiorthogonal decomposition of $\mathrm{D}_{\text{perf}}(\mathbf{P}^{(2)})$ for the projective bundle structure from π_1. The bottom horizontal morphism of the commutative diagram

$$
\begin{array}{ccc}
\mathrm{K}^{\text{top}}(\mathrm{D}_{\text{perf}}(\mathbf{P})_0) & \xrightarrow{\;L(1)\;} & \mathrm{K}^{\text{top}}(\mathrm{D}_{\text{perf}}(\mathbf{P})_1) \\
\downarrow{\scriptstyle \pi_2^*} & & \downarrow{\scriptstyle \pi_2^*} \\
\mathrm{K}^{\text{top}}(\mathrm{D}_{\text{perf}}(\mathbf{P}^{(2)})^{\pi_1}_0) & \xrightarrow{\;\pi_2^* L(1)\;} & \mathrm{K}^{\text{top}}(\mathrm{D}_{\text{perf}}(\mathbf{P}^{(2)})^{\pi_1}_1).
\end{array}
$$

is an equivalence by the previous case, and moreover the vertical morphisms are inclusions of summands. Comparing with the analogous diagram for multiplication by $L(1)^{-1}$, we see that

multiplication by $L(1)$ is an equivalence. □

Remark IV.8. The assumption in Proposition IV.7 that the Brauer class of $\mathcal{X}$ is represented by a Severi–Brauer variety is, we expect, inessential. However, we have imposed it in the absence of a comparison theorem between the topological K-theory of the gerbe $\mathcal{X}$ and Blanc's K-theory in the case when $\mathcal{X}$ is not a global quotient.

IV.3: Twisted Mukai structures

The goal of this section is to prove that the Hodge structures of twisted derived categories are given by twisted Mukai structures. This is stated as Corollary IV.14 below.

Definition IV.9. Let X be a smooth, proper variety over $\mathbf{C}$. Given a class $B \in \mathrm{H}^2(X^{\mathrm{an}}, \mathbf{Q}(1))$, the *$B$-twisted Mukai structure* $\mathrm{K}_i^{\mathrm{top}}(X)^B$ is the unique integral Hodge structure of weight $-i$ on $\mathrm{K}_i^{\mathrm{top}}(X)$ such that the homomorphism

$$\mathrm{K}_i^{\mathrm{top}}(X)^B \to \bigoplus_k \mathrm{H}^{2k-i}(X^{\mathrm{an}}, \mathbf{Q}(k)), \quad v \mapsto \exp(B) \cdot \mathrm{ch}(v)$$

induces, after extension of scalars, an isomorphism between rational Hodge structures. We endow $\mathrm{K}_i^{\mathrm{top}}(X)^B$ with an *Euler pairing*, which by definition is the Euler pairing of Remark III.1 on the underlying group $\mathrm{K}_i^{\mathrm{top}}(X)$.

Theorem IV.10. *Let X be a smooth, proper variety over $\mathbf{C}$, and let $\mathcal{X} \to X$ be an essentially topologically trivial $\boldsymbol{\mu}_n$-gerbe. Suppose that the cohomological Brauer class of $\mathcal{X}$ lies in $\mathrm{Br}(X)$. Then for each i and each 1-twisted topological line bundle L on $\mathcal{X}$, there is an isomorphism of Hodge structures*
$$\mathrm{K}_i^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}^1(\mathcal{X})) \to \mathrm{K}_i^{\mathrm{top}}(X)^{c_1(L)},$$
which is compatible with Euler pairings.

Proof. Let $\mathbf{P}$ be a Severi–Brauer variety which represents the image of $[\mathcal{X}]$ in $\mathrm{Br}(X)$. In the notation of Proposition IV.7, consider the composition

$$\mathrm{K}_i^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(\mathbf{P})_0) \xrightarrow{\;[L(1)]\;} \mathrm{K}_i^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(\mathbf{P})_1) \xrightarrow{\;[\mathcal{O}_{\mathbf{P}_{\mathcal{X}}}(-1)]\;} \mathrm{K}_i^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(\mathbf{P})_0) \otimes \mathbf{Q},$$

where, in the right-hand morphism, we use multiplication by the K-theory class of the twisted line bundle (Definition IV.5).

Since $\mathcal{O}_{\mathbf{P}_{\mathcal{X}}}(-1)$ is an algebraic twisted line bundle, the right-hand morphism induces an isomorphism of rational Hodge structures. On the other hand, by Proposition IV.7, the

left-hand morphism is an isomorphism of abelian groups. It follows that the Hodge structure on $\mathrm{K}_i^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(\mathbf{P})_1))$ is identified with $\mathrm{K}_i^{\mathrm{top}}(X)^{c_1(L)}$ by $[L(1)]$, and we conclude the theorem by Lemma II.11, which implies that $\mathrm{K}_i^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(\mathbf{P})_1)$ is isomorphic to $\mathrm{K}_i^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}^1(\mathfrak{X}))$ as a Hodge structure. The compatibility with Euler pairings follows from the observation that multiplication by $[L(1)]$ preserves the Euler pairing on $\mathrm{K}_i^{\mathrm{top}}(\mathbf{P})$. $\qquad\square$

Remark IV.11. We frequently use the observation that, in the case $i = 0$, the isomorphism constructed in the course of the proof preserves the rank homomorphisms

$$\mathrm{K}_0^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}^1(\mathfrak{X})) \to \mathbf{Z}, \quad \mathrm{K}_0^{\mathrm{top}}(X)^{c_1(L)} \to \mathbf{Z}$$

on both sides.

Definition IV.12. Let X be a smooth, proper variety over $\mathbf{C}$, and let $\alpha \in \mathrm{Br}(X)$. A *rational B-field* for α is a class $B \in \mathrm{H}^2(X^{\mathrm{an}}, \mathbf{Q}(1))$ which maps to α under the exponential

$$\exp : \mathrm{H}^2(X^{\mathrm{an}}, \mathbf{Q}(1)) \to \mathrm{H}^2(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}^\times)^{\mathrm{tors}} \simeq \mathrm{H}_{\mathrm{\acute{e}t}}^2(X, \mathbf{G}_m).$$

Remark IV.13. Let $\mathfrak{X} \to X$ be a $\boldsymbol{\mu}_n$-gerbe with cohomological Brauer class α. If L is a 1-twisted topological line bundle on $\mathfrak{X}$, then $c_1(L^{\otimes n})/n$ is a rational B-field for α. To prove this, observe that there is a commmutative diagram

$$\begin{array}{ccc}
\mathrm{H}^2(X^{\mathrm{an}}, \mathbf{Z}(1)) & \xrightarrow{\exp(-/n)} & \mathrm{H}^2(X^{\mathrm{an}}, \boldsymbol{\mu}_n) \\
\Big\downarrow{-/n} & & \Big\downarrow \\
\mathrm{H}^2(X^{\mathrm{an}}, \mathbf{Q}(1)) & \xrightarrow{\exp} & \mathrm{H}^2(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}^\times),
\end{array}$$

and apply Lemma IV.6.

Corollary IV.14. *Let X be a smooth, proper variety over $\mathbf{C}$, and let $\alpha \in \mathrm{Br}(X)$ be a topologically trivial Brauer class. If B is a rational B-field for α, then for each i there is an isomorphism of Hodge structures*

$$\mathrm{K}_i^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(X, \alpha)) \simeq \mathrm{K}_i^{\mathrm{top}}(X)^B,$$

which is compatible with Euler pairings.

Proof. By Remark IV.13 and Theorem IV.10, it is enough to show that the isomorphism class of the integral Hodge structure $\mathrm{K}_i^{\mathrm{top}}(X)^B$ does not depend on the choice of the rational B-field B. If B and B' are two rational B-fields for α, then from the exponential sequence, the difference $B - B'$ lies in $\mathrm{H}^2(X^{\mathrm{an}}, \mathbf{Z}(1)) + \mathrm{NS}(X)_{\mathbf{Q}}$.

If $B - B'$ lies in $\mathrm{NS}(X)_{\mathbf{Q}}$, then the identity map on $\mathrm{K}_i^{\mathrm{top}}(X)$ induces an isomorphism of Hodge structures

$$\mathrm{K}_i^{\mathrm{top}}(X)^B \to \mathrm{K}_i^{\mathrm{top}}(X)^{B'}.$$

Therefore, we may suppose that $x = B - B'$ lies in $\mathrm{H}^2(X^{\mathrm{an}}, \mathbf{Z}(1))$. In that case, the desired isomorphism of Hodge structures

$$\mathrm{K}_i^{\mathrm{top}}(X)^B \to \mathrm{K}_i^{\mathrm{top}}(X)^{B'}$$

is given by multiplication by $[L_x]$, where L_x is a topological line bundle on X with $c_1(L_x) = x$. $\qquad\square$

IV.4: Variation of twisted Mukai structure

Let $f : X \to S$ be a smooth, proper morphism, and let $\mathcal{X} \to X$ be a $\boldsymbol{\mu}_n$-gerbe. Let P^0 be the local system of topological line bundles on the fibers X_s of f, and let P^1 be the local system of 1-twisted topological line bundles on the fibers of $\mathcal{X} \to S$. We observe that P^1 is a torsor under P^0.

Remark IV.15. The category of P^0-sheaves is symmetric monoidal, with the monoidal structure given by the *contracted product* of P^0-sheaves $A \wedge B$, which is the sheafification of the quotient presheaf

$$A \times B/(a,b) \sim (ga, g^{-1}b).$$

If A is a P^0-module, and B is a P^0-torsor, then $A \wedge B$ is a P^0-module, with addition given locally by

$$a \wedge b + a' \wedge b' = a \wedge b + a' \wedge gb$$
$$= (a + ga') \wedge b,$$

where $b' = gb$ for a unique $g \in \mathrm{P}^0$.

For $i \in \mathbf{Z}$, the local system $\mathrm{K}_i^{\mathrm{top}}(X/S)$ is a P^0-module, and we may define a variation of Hodge structure on the P^0-module $\mathrm{K}_i^{\mathrm{top}}(X/S) \wedge \mathrm{P}^1$ as follows:

Definition IV.16. The *variation of twisted Mukai structure* on $\mathrm{K}_i^{\mathrm{top}}(X/S) \wedge \mathrm{P}^1$ is the unique integral variation of Hodge structure such that the Chern character

$$\mathrm{K}_i^{\mathrm{top}}(X/S) \wedge \mathrm{P}^1 \to \bigoplus_k \mathrm{R}^{2k-i} f_* \mathbf{Q}(k), \quad v \wedge L \mapsto \mathrm{ch}(L) \cdot \mathrm{ch}(v)$$

induces an isomorphism between $\mathbf{Q}$-variations of Hodge structure.

Theorem IV.17. *Let $f : X \to S$ be a smooth, proper morphism, where S is a separated scheme of finite type over $\mathbf{C}$, and let $\mathcal{X} \to X$ be a $\boldsymbol{\mu}_n$-gerbe such that the fiber $\mathcal{X}_s \to X_s$ is essentially topologically trivial for each $s \in S(\mathbf{C})$. Assume that the cohomological Brauer class of $\mathcal{X}$ lies in $\mathrm{Br}(X)$. Then there is an isomorphism of variations of Hodge structure*

$$\mathrm{K}_i^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}^1(\mathcal{X})/S) \to \mathrm{K}_i^{\mathrm{top}}(X/S) \wedge \mathrm{P}^1.$$

Proof. Let $\mathbf{P} \to X$ be a Severi–Brauer variety which represents the image of $[\mathcal{X}]$ in $\mathrm{Br}(X)$. Following the proof of Theorem IV.10, we may consider a composition of morphisms of P^0-modules

(IV.4.1)
$$\mathrm{K}_i^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(\mathbf{P})_0/S) \wedge \mathrm{P}^1 \xrightarrow{\ \Phi\ } \mathrm{K}_i^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(\mathbf{P})_1/S) \xrightarrow{\ \Psi\ } \mathrm{K}_i^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(\mathbf{P})_0/S) \otimes \mathbf{Q},$$

where:

$$\Phi(v \wedge L) = \left[L(1)\right] \cdot v, \qquad \Psi(v) = \left[\mathcal{O}_{\mathbf{P}_{\mathcal{X}}}(-1)\right] \cdot v$$

On stalks, (IV.4.1) recovers (after choosing a 1-twisted topological line bundle) the analogous sequence in the proof of Theorem IV.10. In particular, Φ is an isomorphism of local systems, and Ψ induces an isomorphism of $\mathbf{Q}$-variations of Hodge structure. $\qquad\qquad\square$

Remark IV.18. In the context of Theorem IV.17, let $\mathbf{V} = \mathrm{K}_0^{\mathrm{top}}(X/S) \wedge \mathrm{P}^1$. Then $\mathbf{V}$ admits a rational section $\tau \in \Gamma(S^{\mathrm{an}}, \mathbf{V}) \otimes \mathbf{Q}$, whose restriction to the stalk at $s \in S(\mathbf{C})$ is given by $[L^\vee] \wedge L$ for any 1-twisted topological line bundle L on $\mathcal{X}_s$. Then τ is Hodge, since $\Psi(\Phi(\tau)) = [L] \cdot [L^\vee] = 1$.

By the theorem of the fixed part [Del71, 4.1.1], $\Gamma(\mathbf{V}) = \Gamma(S^{\mathrm{an}}, \mathbf{V})$ carries a natural Hodge structure, determined by the property that for each $s \in S(\mathbf{C})$, the inclusion $\Gamma(\mathbf{V}) \to \mathbf{V}_s$ is a morphism of Hodge structures. Regarding $\Gamma(\mathbf{V})$ as a $\mathrm{K}_0^{\mathrm{top}}(X/S)$-module, the section $\tau \in \Gamma(\mathbf{V}) \otimes \mathbf{Q}$ determines an isomorphism of $\mathbf{Q}$-variations of Hodge structure

$$\mathrm{K}_0^{\mathrm{top}}(X/S) \otimes \mathbf{Q} \to \mathbf{V} \otimes \mathbf{Q},$$

and hence an isomorphism of rational Hodge structures

$$\Gamma(\mathrm{K}_0^{\mathrm{top}}(X/S)) \otimes \mathbf{Q} \simeq \Gamma(\mathbf{V}) \otimes \mathbf{Q}.$$

In particular, if $\mathbf{v} \in \Gamma(\mathbf{V})$ is everywhere Hodge, then the image of $\mathbf{v}$ in $\Gamma(\mathbf{V}) \otimes \mathbf{Q}$ may be written $\mathbf{w} \cdot \tau$, where $\mathbf{w}$ is a global section of $\mathrm{K}_0^{\mathrm{top}}(X/S) \otimes \mathbf{Q}$ which is everywhere Hodge.

IV.5: Quasi-projective varieties

We now develop a theory of twisted Mukai structures on smooth, quasi-projective varieties. On the one hand, our treatment is rather ad hoc, but on the other, it is amenable to computation. The usefulness of the quasi-projective case comes from the following well-known lemma:

Lemma IV.19. *Let X be a smooth scheme over $\mathbf{C}$, and let $\alpha \in \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_m)$ be a cohomological Brauer class. There is a dense open subscheme U of X such that α_U is topologically trivial.*

Proof. Let $\bar{\alpha} \in \mathrm{H}^3(X^{\text{an}}, \mathbf{Z}(1))^{\text{tors}}$ be the topological Brauer class of α. According to [CTV12, Théorème 3.1], $\mathcal{H}^3(\mathbf{Z}(1))$ is torsion-free, where $\mathcal{H}^3(\mathbf{Z}(1))$ is the sheafification of the Zariski presheaf $U \mapsto \mathrm{H}^3(U^{\text{an}}, \mathbf{Z}(1))$. In particular, $\bar{\alpha}$ lies in the kernel of the homomorphism

$$\mathrm{H}^3(X^{\text{an}}, \mathbf{Z}(1)) \to \mathrm{H}^0(X, \mathcal{H}^3(\mathbf{Z}(1))).$$

The right-hand side is the third unramified cohomology group $\mathrm{H}^3_{\text{nr}}(X, \mathbf{Z}(1))$ of X, and the kernel consists of classes with coniveau ≤ 1. $\qquad\square$

Let U be a smooth, separated variety over $\mathbf{C}$. Each singular cohomology group $\mathrm{H}^k(U^{\text{an}}, \mathbf{Q})$ carries a functorial mixed Hodge structure, with weights in the interval $[k, 2k]$. The lowest-weight part $\mathrm{W}_k\mathrm{H}^k(U^{\text{an}}, \mathbf{Q})$ is a pure Hodge structure of weight k, and may be described concretely as the image of the restriction map

$$j^* : \mathrm{H}^k(X^{\text{an}}, \mathbf{Q}) \to \mathrm{H}^k(U^{\text{an}}, \mathbf{Q})$$

for any smooth compactification $j : U \to X$ [Del71, 3.2.16].

Definition IV.20. Let U be a smooth, separated scheme over $\mathbf{C}$, and let $B \in \mathrm{H}^2(U^{\text{an}}, \mathbf{Q}(1))$. The *$B$-twisted Mukai structure* $\mathrm{W}^{\text{top}}_i(U)^B$ is the Hodge structure of weight $-i$, given by the preimage of

$$\text{(IV.5.1)} \qquad\qquad \bigoplus_k \mathrm{W}_{-i}(\mathrm{H}^{2k-i}(U^{\text{an}}, \mathbf{Q}(k))$$

under the B-twisted Chern character

$$\mathrm{K}^{\text{top}}_i(U) \longrightarrow \bigoplus_k \mathrm{H}^{2k-i}(U^{\text{an}}, \mathbf{Q}(k)), \quad v \mapsto \exp(B)\mathrm{ch}(v).$$

The Hodge filtration on $\mathrm{W}^{\text{top}}_i(U)^B$ is pulled back along the Chern character from (IV.5.1).

When $B = 0$, we write $\mathrm{W}_i^{\mathrm{top}}(U)^B = \mathrm{W}_i^{\mathrm{top}}(U)$, and call $\mathrm{W}_i^{\mathrm{top}}(U)$ the *lowest-weight part* of $\mathrm{K}_i^{\mathrm{top}}(U)$.

Lemma IV.21. *Let $U \to X$ be an open immersion from a smooth, separated scheme U to a smooth, proper Deligne–Mumford stack X over $\mathbf{C}$. For each i, the homomorphism $\mathrm{K}_i^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(X))$ to $\mathrm{K}^{\mathrm{top}}(U)$ factors through $\mathrm{W}_i^{\mathrm{top}}(U)$, and the induced homomorphism*

$$\mathrm{K}_i^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(X)) \to \mathrm{W}_i^{\mathrm{top}}(U)$$

is a morphism of Hodge structures.

Proof. First, suppose that X is a smooth, proper variety. Then the lemma follows from the fact that for each k, the induced map $\mathrm{H}^k(X^{\mathrm{an}}, \mathbf{Q}) \to \mathrm{H}^k(U^{\mathrm{an}}, \mathbf{Q})$ preserves the weight filtration.

Next, we argue that if the lemma holds for a single compactification X of U, then it holds for all compactifications. By weak factorization (Example II.17), it is enough to prove the following claim: If X is a compactification of U, and $X' \to X$ is either a root stack along a smooth divisor in the complement of U, or a blowup along a smooth closed substack in the complement of U, then the conclusion of the lemma holds for X if and only if it holds for X'. The claim follows from considering the semiorthogonal decomposition associated to either a blowup or a root stack. $\qquad\square$

Definition IV.22. Let U be a smooth, separated scheme over $\mathbf{C}$, and let $\mathbf{P} \to U$ be a Severi–Brauer variety. The *lowest-weight part* of $\mathrm{K}_i^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(\mathbf{P})_1)$ is the subgroup

$$\mathrm{W}_i^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(\mathbf{P})_1) \subseteq \mathrm{K}_i^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(\mathbf{P})_1)$$

which is the intersection of $\mathrm{K}_i^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(\mathbf{P})_1))$ with $\mathrm{W}_i^{\mathrm{top}}(\mathbf{P})$.

Theorem IV.23. *Let U be a smooth, quasi-projective variety. Let $\mathcal{U} \to U$ be an essentially topologically trivial $\boldsymbol{\mu}_n$-gerbe, and let $\mathbf{P} \to U$ be a Severi–Brauer variety which represents the Brauer class of $\mathcal{U}$. Let i be an integer.*

(1) $\mathrm{W}_i^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(\mathbf{P})_1)$ is a Hodge substructure of $\mathrm{W}_i^{\mathrm{top}}(\mathbf{P})$.

(2) For each 1-twisted topological line bundle L on $\mathcal{U}$, there is an isomorphism of Hodge structures

$$\mathrm{W}_i^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(\mathbf{P})_1) \to \mathrm{W}_i^{\mathrm{top}}(U)^{c_1(L)}.$$

Proof. Following the proof of Theorem IV.10, we consider the composition

$$\mathrm{K}_i^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(\mathbf{P})_0) \xrightarrow{\ [L(1)]\ } \mathrm{K}_i^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(\mathbf{P})_1) \xrightarrow{\ [\mathcal{O}_{\mathbf{P}_\mathcal{U}}(-1)]\ } \mathrm{K}_i^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(\mathbf{P})_0) \otimes \mathbf{Q}.$$

Multiplication by $[\mathcal{O}_{\mathbf{P}_\mathcal{U}}(-1)]$ gives an automorphism of the rational Hodge structure $\mathrm{W}_i^{\mathrm{top}}(\mathbf{P}) \otimes \mathbf{Q}$, so (1) follows since the right-hand map is an isomorphism after extending scalars to $\mathbf{Q}$. Then (2) follows, since the left-hand map identifies the Hodge structure on $\mathrm{W}_i^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(\mathbf{P})_1))$ with the twisted Mukai structure $\mathrm{W}_i^{\mathrm{top}}(U)^{c_1(L)}$. $\qquad\square$

IV.6: Computations on quasi-projective varieties

In this section, we prove several technical results which will be used later. In Lemma IV.26, we give a lower bound for the ranks of Hodge classes for twisted Mukai structures on smooth, quasi-projective varieties, and in Lemma IV.27, we compute the set of Hodge classes for twisted Mukai structures on affine surfaces.

For a smooth, separated scheme U over $\mathbf{C}$, we define

$$\mathrm{W}_0\mathrm{H}^2(U^{\mathrm{an}}, \mathbf{Z}(1)) \subseteq \mathrm{H}^2(U^{\mathrm{an}}, \mathbf{Z}(1))$$

to be the set of class v whose image in $\mathrm{H}^2(U^{\mathrm{an}}, \mathbf{Q}(1))$ lies in $\mathrm{W}_0\mathrm{H}^2(X^{\mathrm{an}}, \mathbf{Q}(1))$. The proof of Lemma IV.24 below implies that $\mathrm{W}_0\mathrm{H}^2(U^{\mathrm{an}}, \mathbf{Z}(1))$ coincides with the image of $\mathrm{H}^2(X^{\mathrm{an}}, \mathbf{Z}(1))$ for any smooth compactification of U, although we do not need this fact.

Lemma IV.24. *Let U be a smooth, separated scheme over $\mathbf{C}$. Then*

$$\mathrm{NS}(U) = \mathrm{Hdg}(\mathrm{W}_0\mathrm{H}^2(U^{\mathrm{an}}, \mathbf{Z}(1))).$$

Proof. Let X be a smooth compactification of U so that $D = X - U$ has simple normal crossings. Consider the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{NS}(X) & \xrightarrow{\ f\ } & \mathrm{H}^2(X^{\mathrm{an}}, \mathbf{Z}(1)) & \longrightarrow & \mathrm{Coker}\, f & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\
0 & \longrightarrow & \mathrm{Hdg}(\mathrm{W}_0\mathrm{H}^2(U^{\mathrm{an}}, \mathbf{Z}(1))) & \xrightarrow{\ g\ } & \mathrm{H}^2(U^{\mathrm{an}}, \mathbf{Z}(1)) & \longrightarrow & \mathrm{Coker}\, g & \longrightarrow & 0.
\end{array}
$$

Since $a \otimes \mathbf{Q}$ is an isomorphism, we need to show that $\mathrm{Coker}\, a$ is torsion-free. It is well-known that $\mathrm{Ker}\, b$ is generated by the cycle classes of the components of D (see for instance [BO21, Prop. 2.2]). In particular, $\mathrm{Ker}\, a$ is isomorphic to $\mathrm{Ker}\, b$, and the connecting homomorphism $\mathrm{Ker}\, c \to \mathrm{Coker}\, a$ is injective. Since f is the inclusion of a saturated sublattice, $\mathrm{Coker}\, f$ is

torsion-free, hence so is $\operatorname{Ker} c$. In particular, to show that $\operatorname{Coker} a$ is torsion-free, it is enough to show that $\operatorname{Coker} b$ is torsion-free.

Finally, the torsion-freeness of $\operatorname{Coker} b$ follows from the Leray spectral sequence for $j : U \to S$, and the observations that $\mathrm{H}^0((D^{(2)})^{\mathrm{an}}, \mathbf{Z})$ and $\mathrm{H}^1((D^{(1)})^{\mathrm{an}}, \mathbf{Z})$ are torsion-free, where $D^{(1)}$ is the disjoint union of the components of D and $D^{(2)}$ is the disjoint union of the pairwise intersections of the components of D. $\qquad\square$

Let U be a smooth, quasi-projective variety over $\mathbf{C}$. Given a class $v \in \mathrm{H}^2(U^{\mathrm{an}}, \mathbf{Z}(1))$ and an integer $n > 0$, let $\alpha(v, n) \in \operatorname{Br}(U)$ be the image of v under the composition

$$\mathrm{H}^2(U^{\mathrm{an}}, \mathbf{Z}(1)) \xrightarrow{\exp(-/n)} \mathrm{H}^2_{\mathrm{\acute{e}t}}(U, \boldsymbol{\mu}_n) \longrightarrow \mathrm{H}^2_{\mathrm{\acute{e}t}}(U, \mathbf{G}_m).$$

The following lemma shows that $\mathrm{W}^{\mathrm{top}}_i(U)^{v/n}$ depends only on $\alpha(v, n)$.

Lemma IV.25. *In the situation above, let $v_1, v_2 \in \mathrm{H}^2(U^{\mathrm{an}}, \mathbf{Z}(1))$, and let $n_1, n_2 > 0$ be integers. If $\alpha(v_1, n_1) = \alpha(v_2, n_2)$, then there is an isomorphism*

$$\mathrm{W}^{\mathrm{top}}_i(U)^{v_1/n_1} \simeq \mathrm{W}^{\mathrm{top}}_i(U)^{v_2/n_2}.$$

Proof. For simplicity, we write $\mathrm{W}(v/n) = \mathrm{W}^{\mathrm{top}}_i(\mathrm{D}_{\mathrm{perf}}(U))^{v/n}$. We make three observations:

(1) For $t > 0$, $\mathrm{W}(tv/tn) \simeq \mathrm{W}(v/n)$.

(2) For $w \in \mathrm{H}^2(U^{\mathrm{an}}, \mathbf{Z}(1))$, $\mathrm{W}((v + nw)/n) \simeq \mathrm{W}(v/n)$.

(3) For $w \in \mathrm{NS}(U)$, $\mathrm{W}((v + w)/n) \simeq \mathrm{W}(v/n)$.

Note that (1) is from the definition, and for (2) and (3) one follows the proof of Corollary IV.14.

For the lemma, by (1) we may assume that $n_1 = n_2$. Then from the Kummer sequence, $v_1 - v_2$ lies in $n_1 \cdot \mathrm{H}^2(U^{\mathrm{an}}, \mathbf{Z}(1)) + \mathrm{NS}(U)$, and we apply (2) and (3). $\qquad\square$

Lemma IV.26. *Let U be a smooth, quasi-projective variety over $\mathbf{C}$, with a class $v \in \mathrm{H}^2(U^{\mathrm{an}}, \mathbf{Z}(1))$ and an integer $n > 0$. Then the rank of any Hodge class in $\mathrm{W}_0(U)^{v/n}$ is divisible by n_0, where n_0 is the order of $\alpha(v, n)$ in $\operatorname{Br}(U)$.*

Proof. By Lemma IV.25, we may suppose that $n = n_0$. Let $w \in \mathrm{W}^{\mathrm{top}}_0(U)^{v/n}$ be a Hodge class, and write $w = (w_0, w_2, \dots)$, so that

$$\exp(B) \cdot \mathrm{ch}(w) = (w_0, w_2 + w_0 \cdot B, \dots).$$

Then $w_2 + w_0 \cdot (v/n)$ is Hodge, so $nw_2 + w_0 v$ is an integral Hodge class in $W_0(H^2(U^{an}, \mathbf{Z}(1)))$, and hence is the Chern class of an algebraic line bundle by Lemma IV.24. In particular, the image of $w_0 v + n w_2$ under the composition

$$H^2(U^{an}, \mathbf{Z}(1)) \to H^2(U^{an}, \boldsymbol{\mu}_n) \to Br(U)[n]$$

is trivial. On the other hand, the image coincides with $w_0 \cdot \alpha(v/n)$. Therefore, n divides w_0. $\qquad\square$

Lemma IV.27. *Let U be a smooth, affine surface over $\mathbf{C}$, with a class $v \in H^2(U^{an}, \mathbf{Z}(1))$ and an integer $n > 0$. There is a short exact sequence*

$$0 \longrightarrow NS(U) \longrightarrow Hdg(W_0^{top}(U)^{v/n}) \xrightarrow{\ rk\ } n_0 \cdot \mathbf{Z} \longrightarrow 0,$$

where n_0 is the order of $\alpha(v/n)$ in $Br(U)$.

Proof. From the Atiyah–Hirzebruch spectral sequence, the map

$$(rk, c_1): K^{top}(U) \to H^0(U^{an}, \mathbf{Z}) \oplus H^2(U^{an}, \mathbf{Z}(1))$$

is an isomorphism. By Lemma IV.25, we may suppose that $n = n_0$. By Lemma IV.24, (w_0, w_2) is a Hodge class in $W_0^{top}(U)^{v/n}$ if and only if $w_0 \cdot (v/n) + w_2$ lies in $NS(U)_{\mathbf{Q}}$. If $w_0 = 0$, then $w_2 \in NS(U)$ by Lemma IV.24. The morphism

$$NS(U) \to Hdg(W_0^{top}(U)^{v/n})$$

in the statement of the lemma is given by sending w_2 to $(0, w_2)$. Since $(n, -v)$ is a Hodge class of rank n, it remains to show that the rank of any Hodge class is divisible by n, which is Lemma IV.26. $\qquad\square$

CHAPTER V
The Hodge-theoretic Index

V.1: The period-index problem

We refer to [CT06] for an account of the classical theory of Brauer groups and the period-index conjecture.

Definition V.1. Let K be a field, and let $\alpha \in \mathrm{Br}(K)$. The *period* of α is its order in the torsion group $\mathrm{Br}(K)$. The *index* of α is equal to the following positive integers:

(1) The minimum rank of an α-twisted vector space.

(2) The minimum degree of a separable extension K'/K such that $\alpha_{K'} = 0$.

(3) The integer $d' + 1$, where d' is the minimum dimension of a Severi–Brauer variety of class α.

(4) The degree $\sqrt{\dim D}$ of the unique division algebra D of class α.

One may show that $\mathrm{ind}(\alpha)$ divides $\mathrm{per}(\alpha)$, and that $\mathrm{per}(\alpha)$ and $\mathrm{ind}(\alpha)$ share the same prime factors. In particular, for each $\alpha \in \mathrm{Br}(K)$, there is an integer $\epsilon(\alpha)$ such that

$$\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^{\epsilon(\alpha)}.$$

The *period-index problem* for K is the problem of determining a bound for $\epsilon(\alpha)$, preferably one which is uniform over the Brauer group. The main conjecture is the following:

Conjecture V.2 (The period-index conjecture). *Let K be a field of transcendence degree d over an algebraically closed field k. For any Brauer class $\alpha \in \mathrm{Br}(K)$,*

$$\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^{d-1}.$$

There are variants of Conjecture V.2, for instance when k is a finite field or a p-adic field.

The status of Conjecture V.2 is as follows:

$d \le 1$ The Brauer group $\operatorname{Br}(K)$ is trivial.

$d = 2$ Conjecture V.2 was proved by de Jong [dJ04] when $\operatorname{per}(\alpha)$ is prime to the characteristic of k. When the characteristic of k divides $\operatorname{per}(\alpha)$, the conjecture was proved through different methods by de Jong–Starr [SdJ10] and Lieblich [Lie08].

$d \ge 3$ Conjecture V.2 is not known for any given field K.

When $d \ge 3$, it is not known in general that there exists an exponent ϵ such that for each $\alpha \in \operatorname{Br}(K)$,

$$\operatorname{ind}(\alpha) \mid \operatorname{per}(\alpha)^{\epsilon}.$$

However, Matzri [Mat16] has obtained non-uniform exponents $\epsilon(\alpha)$, which depend only on $\operatorname{tr.deg}(K/k)$ and the prime factors of $\operatorname{per}(\alpha)$, based on estimates for the symbol length of a central simple algebra of given period. Aside from the cases described above, Matrzi's bounds are the best available.

V.2: The global period-index problem

We now explain a version of the period-index conjecture for orbifolds over fields.

Definition V.3. Let X be a connected Deligne–Mumford stack of finite type over a field k, and let $\alpha \in \operatorname{H}^2_{\text{ét}}(X, \mathbf{G}_m)^{\text{tors}}$. The *period* $\operatorname{per}(\alpha)$ of α is its order in $\operatorname{H}^2_{\text{ét}}(X, \mathbf{G}_m)$. The *index* $\operatorname{ind}(\alpha)$ of α is the positive generator of the image of a rank homomorphism,

$$\operatorname{K}_0(\operatorname{D}_{\text{perf}}(X, \alpha)) \to \mathbf{Z}$$

which arises by viewing $\operatorname{D}_{\text{perf}}(X, \alpha)$ as $\operatorname{D}^1_{\text{perf}}(\mathfrak{X})$ for $\mathfrak{X} \to X$ a $\boldsymbol{\mu}_n$-gerbe of class α.

We recall our convention that an *orbifold* over a field k is a Deligne–Mumford stack of finite type over k with trivial generic stabilizers.

Lemma V.4. *Let X be a smooth, connected orbifold over k. If $\alpha \in \operatorname{H}^2_{\text{ét}}(X, \mathbf{G}_m)$, then*

$$\operatorname{ind}(\alpha) = \operatorname{ind}(\alpha_K),$$

where α_K is the image of α in $\operatorname{Br}(K)$.

Proof. We first show that $\operatorname{ind}(\alpha_K)$ divides $\operatorname{ind}(\alpha)$. Let $\mathfrak{X} \to X$ be a $\mathbf{G}_m$-gerbe of class α, and let $\mathfrak{X}_K \to \operatorname{Spec} K$ be the fiber over the generic point. Choose a 1-twisted vector bundle V_K on $\mathfrak{X}_K$ of rank $\operatorname{ind}(\alpha_K)$. By [Lie08, Prop. 3.1.1.9], there exists a coherent extension of V_K to

a 1-twisted coherent sheaf V on $\mathfrak{X}$, and since $\mathfrak{X}$ is smooth, V is quasi-isomorphic to a perfect complex E, with $\operatorname{rk} E = \operatorname{ind}(\alpha_K)$. We next show that $\operatorname{ind}(\alpha)$ divides $\operatorname{ind}(\alpha_K)$. Let E be a 1-twisted perfect complex on $\mathfrak{X}$ of rank r. Then the restriction of E to $\mathfrak{X}_K$ is perfect, and the rank of any 1-twisted locally free sheaf on $\mathfrak{X}_K$ is divisible by $\operatorname{ind}(\alpha)$. $\qquad\square$

Conjecture V.5 (Global period-index)**.** *Let X be a smooth, proper, connected orbifold of dimension d over an algebraically closed field k. For any $\alpha \in \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_m)$,*

$$\operatorname{ind}(\alpha) \mid \operatorname{per}(\alpha)^{d-1}.$$

Remark V.6. The global period-index conjecture over $\mathbf{C}$ (Conjecture V.5) is equivalent to the period-index conjecture for complex function fields (Conjecture V.2). Lemma V.4 shows that the conjecture for function fields implies the global conjecture. In the other direction, let K be a function field over $\mathbf{C}$. Then any Brauer class extends to a quasi-projective model of K, and one may apply Lemma II.19 to pass to a smooth, proper orbifold.

Alternatively, the discriminant-avoidance theorem of de Jong and Starr [SdJ10] implies that, in order to prove the period-index conjecture for complex function fields of transcendence degree $\leq d$, it is enough to verify the global period index conjecture for smooth, complex, projective varieties of dimension $\leq d$. However, the reduction is inexplicit.

V.3: The Hodge-theoretic index

Let X be a smooth, proper, connected orbifold over $\mathbf{C}$, and let $\alpha \in \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_m)$. Given a closed point $\operatorname{Spec} \mathbf{C} \to X$, the pullback

$$\mathrm{D}_{\text{perf}}(X, \alpha) \to \mathrm{D}_{\text{perf}}(\operatorname{Spec} \mathbf{C})$$

defines a rank homomorphism $\mathrm{K}_0^{\text{top}}(X, \alpha) \to \mathrm{K}_0^{\text{top}}(\operatorname{Spec} \mathbf{C}) = \mathbf{Z}$, compatible with the rank homomorphism $\mathrm{K}_0(\mathrm{D}_{\text{perf}}(X, \alpha)) \to \mathbf{Z}$ on algebraic K-theory.

Definition V.7. The *Hodge-theoretic index* $\operatorname{ind}_{\mathrm{H}}(\alpha)$ is the positive generator of the image of the rank homomorphism

$$\mathrm{Hdg}(\mathrm{D}_{\text{perf}}(X, \alpha), \mathbf{Z}) \to \mathbf{Z}.$$

Observe that $\operatorname{ind}_{\mathrm{H}}(\alpha)$ divides $\operatorname{ind}(\alpha)$, and if they differ, then the integral Hodge conjecture fails for $\mathrm{D}_{\text{perf}}(X, \alpha)$. In general, the Hodge-theoretic index may differ from both the period and the index, but it enjoys a number of the same formal properties, as indicated in Lemma V.8 below.

We first establish some basic properties of $\mathrm{ind}_\mathrm{H}(\alpha)$. Since $\mathrm{Br}(X)$ is a torsion group, any nonzero element α admits a prime decomposition

$$\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n,$$

characterized by the property that $\mathrm{per}(\alpha_i) = \ell_i^{r_i}$, for distinct primes $\ell_1, \ldots, \ell_n$ and $r_i > 0$.

Lemma V.8. *Let X be a smooth, proper, connected orbifold over $\mathbf{C}$ with function field K, and let $\alpha \in \mathrm{H}^2_{\mathrm{\acute{e}t}}(X, \mathbf{G}_m)$.*

(1) *If $f : X' \to X$ is generically finite, then $\mathrm{ind}_\mathrm{H}(\alpha) \mid (\deg f) \cdot \mathrm{ind}_\mathrm{H}(f^*\alpha)$.*

(2) *If X' is smooth, proper connected orbifold over $\mathbf{C}$ which is birational to X, and $\alpha_K \in \mathrm{Br}(K)$ extends to a class $\alpha' \in \mathrm{H}^2_{\mathrm{\acute{e}t}}(X', \mathbf{G}_m)$, then $\mathrm{ind}_\mathrm{H}(\alpha) = \mathrm{ind}_\mathrm{H}(\alpha')$.*

(3) *If $\alpha = \alpha_1 + \cdots + \alpha_n$ is the prime decomposition of α, then $\mathrm{ind}_\mathrm{H}(\alpha) = \prod_i \mathrm{ind}_\mathrm{H}(\alpha_i)$.*

(4) $\mathrm{per}(\alpha) \mid \mathrm{ind}_\mathrm{H}(\alpha) \mid \mathrm{ind}(\alpha)$.

Proof. For (1), observe that

$$\mathrm{rk}\, f_* f^* v = (\deg f)\, \mathrm{rk}\, v$$

for any $v \in \mathrm{K}_0^{\mathrm{top}}(S, \alpha)$. Then (2) follows from (1) and weak factorization for a $\boldsymbol{\mu}_n$-gerbe of Brauer class α, which (as noted in Example II.17) is compatible with the gerbe structure.

For (3), it is not hard to see that $\mathrm{ind}_\mathrm{H}(\alpha)$ divides $\prod \mathrm{ind}_\mathrm{H}(\alpha_i)$, so we show that for each i, $\mathrm{ind}_\mathrm{H}(\alpha_i)$ divides $\mathrm{ind}_\mathrm{H}(\alpha)$. Let E be an object in $\mathrm{D}_{\mathrm{perf}}(X, \alpha_i - \alpha)$ of rank $\mathrm{ind}(\alpha_i - \alpha)$. Since

$$\mathrm{Fun}_X(\mathrm{D}_{\mathrm{perf}}(X, \alpha), \mathrm{D}_{\mathrm{perf}}(X, \alpha_i)) \simeq \mathrm{D}_{\mathrm{perf}}(X, \alpha_i - \alpha),$$

the object E gives a functor from $\mathrm{D}_{\mathrm{perf}}(X, \alpha)$ to $\mathrm{D}_{\mathrm{perf}}(X, \alpha_i)$, and the induced map on topological K-theory sends a Hodge class of rank $\mathrm{ind}_\mathrm{H}(\alpha)$ to a Hodge class of rank $\mathrm{ind}_\mathrm{H}(\alpha) \cdot \mathrm{ind}(\alpha - \alpha_i)$. Since it is clear that $\mathrm{ind}_\mathrm{H}(\alpha_i)$ divides $\mathrm{ind}(\alpha_i)$, $\mathrm{ind}_\mathrm{H}(\alpha_i)$ is prime to $\mathrm{ind}(\alpha - \alpha_i)$, and so $\mathrm{ind}_\mathrm{H}(\alpha_i)$ divides $\mathrm{ind}_\mathrm{H}(\alpha)$.

For (4), it is clear that $\mathrm{ind}_\mathrm{H}(\alpha)$ divides $\mathrm{ind}(\alpha)$, so we show that $\mathrm{per}(\alpha)$ divides $\mathrm{ind}_\mathrm{H}(\alpha)$. Let $U \subseteq X$ be a substack which is equivalent to a smooth quasi-projective variety. By Lemma IV.19, we may suppose (after perhaps shrinking U) that the restriction of α to U is topologically trivial. By Lemma II.20 and (2), we may assume that there is a Severi–Brauer variety $\mathbf{P} \to X$ whose restriction over U has class α_U.

It follows from Lemma IV.21 that the induced map

$$\mathrm{K}_0^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(\mathbf{P})_1) \to \mathrm{W}_0^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(\mathbf{P}_U)_1).$$

is a morphism of Hodge structures. We conclude by observing that the rank of any Hodge class in the right-hand side is divisible by $\mathrm{per}(\alpha)$, by Theorem IV.23 and Lemma IV.26. $\qquad\square$

Example V.9. Let X be a smooth, proper variety, and let $v \in \mathrm{H}^2(X^{\mathrm{an}}, \mathbf{Z}(1))$ be a cohomology class. In [Kre03], Kresch shows that if the image $\exp(v/2)$ of v in $\mathrm{Br}(X)[2]$ has index 2, then there exists a Hodge class $h \in \mathrm{Hdg}^4(X^{\mathrm{an}}, \mathbf{Z}(2))$ such that

$$(\mathrm{V.3.1}) \qquad\qquad v^2 \equiv h \mod 2.$$

Kresch observes that (V.3.1) imposes a nontrivial condition on v, and so constructs examples of threefolds with Brauer classes of period 2 and index 4.

From our calculation of the twisted Mukai structure, one may give a quick proof of Kresch's result. Let $B = v/2$. If the index of α is 2, then let

$$\mathrm{ch}(w) = (2, w_2, w_4, \dots)$$

be the Chern character of a Hodge class w of rank 2 in $\mathrm{K}_0^{\mathrm{top}}(X)^B$. Since w is Hodge,

$$\exp(B) \cdot \mathrm{ch}(w) = (2, 2B + w_2, B^2 + w_2 \cdot B + w_4, \dots)$$

is Hodge in each degree. Since $2w_4$ is integral,

$$4(B^2 + w_2 \cdot B + w_4) = v^2 + 2v \cdot w_2 + 4 \cdot w_4$$
$$\equiv v^2 \mod 2,$$

which recovers Kresch's obstruction (V.3.1).

Remark V.10. We briefly compare the Hodge-theoretic index with the étale index $\mathrm{ind}_{\mathrm{\acute{e}t}}(\alpha)$ studied by Antieau [Ant11a], [Ant11b], and Antieau–Williams [AW13]. Let X be a smooth, proper scheme over $\mathbf{C}$, and let $\mathrm{K}_0^{\mathrm{\acute{e}t}}(X)$ be the étale K-theory of X. There is a sequence of morphisms

$$(\mathrm{V.3.2}) \qquad \mathrm{K}_0(X) \xrightarrow{\ s\ } \mathrm{K}_0^{\mathrm{\acute{e}t}}(X) \longrightarrow \mathrm{K}_0^{\mathrm{top}}(X) \xrightarrow{\ \mathrm{rk}\ } \mathbf{Z}.$$

From [Tho85, Theorem 2.15], s becomes an isomorphism after tensoring with $\mathbf{Q}$. It follows that the image of $\mathrm{K}_0^{\mathrm{\acute{e}t}}(X)$ in $\mathrm{K}_0^{\mathrm{top}}(X)$ is contained in $\mathrm{Hdg}(\mathrm{D}_{\mathrm{perf}}(X), \mathbf{Z})$.

Let $\alpha \in \mathrm{Br}(X)$ be a Brauer class, and let $\mathbf{P} \to X$ be a Severi–Brauer variety of class α.

As a summand of the sequence (V.3.2) on $\mathbf{P}$, we obtain a sequence

$$(\text{V.3.3}) \qquad \mathrm{K}_0(X,\alpha) \longrightarrow \mathrm{K}_0^{\text{ét}}(X,\alpha) \longrightarrow \mathrm{K}_0^{\text{top}}(\mathrm{D}_{\text{perf}}(X,\alpha)) \xrightarrow{\text{rk}} \mathbf{Z}.$$

From above, $\mathrm{K}^{\text{ét}}(X,\alpha)$ maps into $\mathrm{Hdg}(\mathrm{D}_{\text{perf}}(X,\alpha),\mathbf{Z})$.

In [Ant11b], Antieau defines the *étale index* $\mathrm{ind}_{\text{ét}}(\alpha)$ of α to be the positive generator of $\mathrm{rk}(\mathrm{K}_0^{\text{ét}}(S,\alpha))$. In [AW13], Antieau and Williams construct Brauer classes on Serre-Godeaux varieties with $\mathrm{per}(\alpha)$ strictly less than $\mathrm{ind}_{\text{ét}}(\alpha)$.

The Hodge-theoretic index provides a straightforward method for finding such examples. From the discussion above,

$$\mathrm{ind}_{\mathrm{H}}(\alpha) \mid \mathrm{ind}_{\text{ét}}(\alpha),$$

so it suffices to construct examples where $\mathrm{ind}_{\mathrm{H}}(\alpha)$ exceeds $\mathrm{per}(\alpha)$. For instance, any pair (X,α) such that $\alpha \in \mathrm{Br}(X)[2]$ is obstructed by Kresch's method (Example V.9) furnishes an example. Alternatively, it is straightforward to construct Brauer classes on abelian varieties of dimension $d \geq 3$ with $\mathrm{ind}_{\mathrm{H}}(\alpha) > \mathrm{per}(\alpha)$, using Corollary IV.14.

V.4: Bounding the Hodge-theoretic index

In this section, we obtain an upper bound for the Hodge-theoretic index of a topologically trivial Brauer class.

Theorem V.11. *Let X be a smooth, proper variety of dimension d over $\mathbf{C}$. For any topologically trivial Brauer class $\alpha \in \mathrm{Br}(X)$ with $\mathrm{per}(\alpha) = n$,*

$$\mathrm{ind}_{\mathrm{H}}(\alpha) \mid n^{d-1} \cdot ((d-1)!)^{d-2}.$$

In particular, if n is prime to $(d-1)!$, then $\mathrm{ind}_{\mathrm{H}}(\alpha) \mid n^{d-1}$.

Remark V.12. Theorem V.11 is a Hodge-theoretic analogue of a recent result of Antieau–Williams on the topological period-index conjecture [AW21].

We begin with a simple result on the image of topological K-theory in integral cohomology.

Lemma V.13. *Let M be a finite CW complex of dimension $2d$. For any $v \in \mathrm{H}^2(M,\mathbf{Z})$ and any polynomial $f(t) \in \mathbf{Z}[t]$, there exists a constant $c \in \mathbf{Q}$ so that*

$$((d-1)!)^{d-v_0(f)-1}(f(v) + c \cdot v^d) \in \mathrm{H}^{\text{ev}}(M,\mathbf{Q})$$

lies in the image of $\mathrm{K}_0^{\text{top}}(M)$ under the Chern character, where $v_0(f)$ is the order of vanishing of $f(t)$ at 0.

Proof. Since $\mathbf{CP}^\infty = \mathrm{K}(\mathbf{Z}, 2)$, it suffices to treat the case of its $2d$-skeleton $M = \mathbf{CP}^d$, $v = c_1(\mathcal{O}_{\mathbf{CP}^d}(1))$. We use the following fact: If $x \in \mathrm{H}^{2k}(M, \mathbf{Z})$ is an integral class, then there exists $z \in \mathrm{K}_0^{\mathrm{top}}(M)$ such that

$$\mathrm{ch}(z) = x + \mathrm{ch}_{k+1}(z) + \mathrm{ch}_{k+2}(z) + \cdots$$

The fact is a consequence of the degeneration of the Atiyah–Hirzebruch spectral sequence for M at E_2 [AH62].

We proceed by ascending induction on $k = v_0(f)$. If $k = d - 1$, then the result is clear, since (from the fact) we may find $z \in \mathrm{K}_0^{\mathrm{top}}(M)$ so that $f(v)$ and $\mathrm{ch}(z)$ differ by $c \cdot v^d$. If $k < d - 1$, then we may apply the fact to obtain a class $z \in \mathrm{K}_0^{\mathrm{top}}(M)$ so that the polynomial $f(v) - \mathrm{ch}(z) \in \mathbf{Q}[v]$ vanishes to order $k + 1$ at 0. Since $(d - 1)! \cdot \mathrm{ch}(z)$ is integral except in the top degree, there exists a constant $c_0 \in \mathbf{Q}$ so that

$$g(v) = (d - 1)! \cdot (f(v) - \mathrm{ch}(z)) + c_0 v^d$$

is integral, and $v_0(g) = k + 1$. From the inductive hypothesis, there exists $c_1 \in \mathbf{Q}$ so that

$$((d-1)!)^{d-k-2} \cdot (g(v) + c_1 v^d) = ((d-1)!)^{d-k-1}\left(f(v) - \mathrm{ch}(z) + \frac{1}{(d-1)!}(c_0 + c_1)v^d\right)$$

lies in the image of $\mathrm{K}_0^{\mathrm{top}}(M)$. $\qquad\square$

Proof of Theorem V.11. Let $v \in \mathrm{H}^2(X^{\mathrm{an}}, \mathbf{Z}(1))$ be an integral class so that v/n is a rational B-field for α. Let

$$f(v) = n^{d-1} \exp(-v/n) - \left(\frac{(-1)^d}{n \cdot d!} + \frac{(n^{d-2})^d}{d!}\right) v^d$$

$$= n^{d-1} - n^{d-2}v + \frac{n^{d-3}}{2}v^2 - \cdots + \frac{(-1)^{d-1}}{(d-1)!}v^{d-1} - \frac{(n^{d-2})^d}{d!}v^d$$

For each $c \in \mathbf{Q}$, the class $((d-1)!)^{d-2} \cdot f(v) + cv^d$ is a rational Hodge class in $\mathrm{K}_0^{\mathrm{top}}(X)^B \otimes \mathbf{Q}$. We claim that it lies in the image of the Chern character for an appropriate choice of $c \in \mathbf{Q}$.

The polynomial

$$g(v) = (d-1)! \cdot \left(f(v) - n^{d-1} + (\exp(n^{d-2}v) - 1)\right)$$

$$= (d-1)! \cdot \sum_{k=2}^{d-1} \frac{(n^{d-2})^k + (-1)^k n^{d-k-1}}{k!} \cdot v^k$$

is integral, and $v_0(g) = 2$. By Lemma V.13, there exists a constant $c' \in \mathbf{Q}$ such that

$$((d-1)!)^{d-3} \left(g(v) + c'v^d \right)$$

lies in the image of the Chern character, which implies the claim. $\qquad\qquad\square$

Remark V.14. The factor of $((d-1)!)^{d-2}$ appearing in Theorem V.11 is not optimal. One could improve it by incorporating a more careful analysis of the image of $\mathrm{K}_0^{\mathrm{top}}(\mathbf{CP}^d)$ in $\mathrm{H}^{\mathrm{ev}}(\mathbf{CP}^{d,\mathrm{an}}, \mathbf{Q})$ into the proof of Lemma V.13, or, alternatively, with the integrality theorem of Adams [Ada61]. Both routes lead to better but more complicated bounds, which unfortunately still contain a factor of $(d-1)!$.

V.5: Potential obstructions to period-index bounds

Let X be a smooth, proper threefold, and let $\alpha \in \mathrm{Br}(X)$ be a topologically trivial Brauer class. Then Theorem V.11 provides the bound

$$\mathrm{ind}_\mathrm{H}(\alpha) \mid 8,$$

whereas the period-index conjecture would imply that $\mathrm{ind}_\mathrm{H}(\alpha)$ divides 4. The goal of this section is to analyze the situation in detail.

Theorem V.15. *Let X be a smooth, proper variety over $\mathbf{C}$ with $v \in \mathrm{H}^2(X^{\mathrm{an}}, \mathbf{Z}(1))$, and let α be the image of $v/2$ in $\mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_m)[2]$. If $\mathrm{ind}_\mathrm{H}(\alpha)$ divides 4, then there exists $H \in \mathrm{NS}(X)$ such that*

$$(\mathrm{V.5.1}) \qquad\qquad v^2 + v \cdot H \in \mathrm{H}^4(X^{\mathrm{an}}, \mathbf{Z}(2))$$

is congruent to a Hodge class modulo 2.

Proof. If $\pi : X' \to X$ is a blowup along a smooth subvariety, then v satisfies the conclusion of the theorem if and only if $\pi^* v$ does. Therefore, we may suppose that X is projective, so that $\mathrm{Br}(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_m)$ and Corollary IV.14 applies.

Suppose that $w \in \mathrm{K}_0^{\mathrm{top}}(X)^B$ is a Hodge class of rank 4. Then

$$\exp(B) \cdot (4, w_2, w_4, \dots) = (4, 4B + w_2, 2B^2 + B \cdot w_2 + w_4, \dots)$$

is Hodge in each degree, where $\mathrm{ch}(w) = (4, w_2, w_4, \dots)$. We may write:

- $H = 4B + w_2$, for $H \in \mathrm{NS}(X)$.

- $w_4 = \frac{1}{2}w_2^2 + \epsilon$, where ϵ is an integral class.

The second point comes from the condition that $(4, w_2, w_4, \dots)$ lies in the image of K-theory, and the leading terms of Chern characters are integral.

Let $Z = 2B^2 + B \cdot w_2 + w_4$. Then Z is a rational Hodge class, and

$$2Z = v^2 + v(H - 2v) + (H - 2v)^2 + 2\epsilon$$
$$\equiv v^2 + v \cdot H - H^2 \mod 2.$$

In particular, $v^2 + v \cdot H$ is congruent to a Hodge class modulo 2. $\qquad \square$

Question V.16. *Let X be a smooth, proper threefold over $\mathbf{C}$, and let $v \in \mathrm{H}^2(X^{\mathrm{an}}, \mathbf{Z}(1))$. Does there exist a class $H \in \mathrm{NS}(X)$ such that $v^2 + H \cdot v$ is congruent modulo 2 to a Hodge class in $\mathrm{H}^4(X^{\mathrm{an}}, \mathbf{Z}(2))$?*

A negative answer to Question V.16 would imply that the period-index conjecture fails for Brauer classes of period 2 on complex threefolds.

Remark V.17. The potential obstruction from Theorem V.15 is reminiscent of a topological obstruction for the period-conjecture for period 2 Brauer classes on threefolds proposed by Antieau and Williams [AW14], which was later shown to be ineffective [CG20].

CHAPTER VI

Counterexamples to the Integral Hodge Conjecture

VI.1: Abelian threefolds

By Gabber's method [CT02, Appendice], one may construct Brauer classes on very general abelian threefolds of period 2 and index 4. Alternatively, one may use twisted Mukai structures to produce examples of Brauer classes on abelian threefolds with

$$2 = \operatorname{per}(\alpha) < \operatorname{ind}_{\mathrm{H}}(\alpha) = 4$$

The following theorem shows that all such examples give rise to counterexamples to the integral Hodge conjecture:

Theorem VI.1. *Let X be an abelian threefold over $\mathbf{C}$, and let $\alpha \in \operatorname{Br}(X)$ be a Brauer class with $\operatorname{per}(\alpha) = 2$ and $\operatorname{ind}(\alpha) > 2$. For any Severi–Brauer variety $\mathbf{P} \to X$ of class α and relative dimension d, the integral Hodge conjecture fails in $\mathrm{H}^{2d}(\mathbf{P}^{\mathrm{an}}, \mathbf{Z}(d))$.*

The integral Hodge conjecture for abelian threefolds is a result of Grabowski [Gra04], so Theorem VI.1 shows that the integral Hodge conjecture for a Severi–Brauer variety $\mathbf{P} \to X$ may fail even if it holds for the base.

Proof. Let $\pi : \mathbf{P} \to X$ be a Severi–Brauer variety of class α and relative dimension $d \geq 3$. The relative Picard group of $\mathbf{P}/X$ is generated by a line bundle $\mathcal{O}_{\mathbf{P}}(2)$, whose restriction to the fiber over each $s \in X(\mathbf{C})$ is $\mathcal{O}_{\mathbf{P}_s^n}(2)$. Let $Q \in \mathrm{H}^2(\mathbf{P}^{\mathrm{an}}, \mathbf{Z}(1))$ be the first Chern class of $\mathcal{O}_{\mathbf{P}}(2)$.

Since α is topologically trivial, $\mathbf{P}^{\mathrm{an}} \to X^{\mathrm{an}}$ is a topological projective bundle, and there exists a class $H \in \mathrm{H}^2(\mathbf{P}^{\mathrm{an}}, \mathbf{Z}(1))$ whose restriction to each closed fiber is the hyperplane class. We may write

$$H = \frac{1}{2}Q + \frac{1}{2}\pi^* v,$$

for a class $v \in \mathrm{H}^2(X^{\mathrm{an}}, \mathbf{Z}(1))$. Since Q is algebraic, the class

$$\exp(-\pi^* v/2) \cdot \exp(H) = (1, w_2, w_4, \dots) \in \bigoplus_k \mathrm{H}^{2k}(X^{\mathrm{an}}, \mathbf{Q}(k))$$

is Hodge in each degree. In particular, the component

$$w_d = \frac{1}{d!}H^d - \frac{1}{2(d-1)!}(\pi^*v)H^{d-1} + \frac{1}{2^2 \cdot 2(d-2)!}(\pi^*v^2)H^{d-2} - \frac{1}{2^3 \cdot 3!(d-3)!}(\pi^*v^3)H^{d-3}$$

is a rational Hodge class. Consider the class

$$Z = 2 \cdot d! \cdot \left(w_d + \frac{1}{2^3 \cdot 3!(d-3)!}(\pi^*v^3)H^{d-3} \right)$$
$$= 2H^d - d(\pi^*v)H^{d-1} + \frac{(d)(d-1)}{4}(\pi^*v^2)H^{d-2}.$$

First, Z is a rational Hodge class, since w_d and $(\pi^*v^3)H^{d-3}$ are Hodge classes, as v^3 is Hodge. Second, Z is integral, since v^2 is divisible by 2, as is any degree 2 cohomology class on an abelian variety.

It remains to argue that Z is not algebraic. The Gysin homomorphism

$$\mathrm{H}^{2d}(\mathbf{P}^{\mathrm{an}}, \mathbf{Z}(d)) \to \mathrm{H}^6(X^{\mathrm{an}}, \mathbf{Z}(6))$$

sends Z to $2 \cdot [X]$, where $[X]$ is the fundamental class. If Z is algebraic, then there exists a zero-cycle of degree 2 on the generic fiber $\mathbf{P}_\eta$ of π. However, $\mathrm{ind}(\alpha)$ coincides with the minimum degree of a zero-cycle on $\mathbf{P}_\eta$. $\qquad\square$

Remark VI.2. The assumption that X is an abelian threefold, as opposed to an arbitrary threefold, enters in only two places: First, so that one may arrange that $\mathrm{ind}(\alpha) > 2$, and second, to ensure that v^2 is divisible by 2.

Remark VI.3. If X is a smooth, proper variety over $\mathbf{C}$, the truth of the integral Hodge conjecture for a Severi–Brauer variety $\mathbf{P} \to X$ depends only on the subgroup of $\mathrm{Br}(X)$ generated by $[\mathbf{P}]$.

Indeed, in the trivial case when $\mathbf{P} \to X$ is a projective bundle, then the integral Hodge conjecture for $\mathbf{P}$ is equivalent to the integral Hodge conjecture for X. In the general case, if $\mathbf{P}$ and $\mathbf{P}'$ are Severi–Brauer varieties which generate the same subgroup of $\mathrm{Br}(X)$, then we may consider the diagram

$$\begin{array}{ccc} \mathbf{P} \times_X \mathbf{P}' & \xrightarrow{\pi_2} & \mathbf{P}' \\ \downarrow{\scriptstyle \pi_1} & & \downarrow \\ \mathbf{P} & \longrightarrow & X. \end{array}$$

We observe that π_1 and π_2 are projective bundles.

VI.2: Products of curves

Let C be a smooth, projective curve of genus $g \geq 2$ over $\mathbf{C}$, and let $E_1, \ldots, E_k$ be elliptic curves over $\mathbf{C}$, with $2 \leq k \leq g$. Consider a symplectic basis

$$s_1, \ldots, s_g, t_1, \ldots, t_g \in \mathrm{H}^1(C^{\mathrm{an}}, \mathbf{Z}),$$

and for each i, let $u_i \in \mathrm{H}^1(E_i^{\mathrm{an}}, \mathbf{Z})$ be a nonzero class. Let $X = C \times \prod_{i=1}^k E_i$, and define the class

$$v = 2\pi i \cdot \sum_{i=1}^k s_i \cup u_i \in \mathrm{H}^2(X^{\mathrm{an}}, \mathbf{Z}(1)).$$

For each prime ℓ, let $\alpha_\ell \in \mathrm{Br}(X)[\ell]$ be the Brauer class given by the rational B-field v/ℓ.

Theorem VI.4. *For a prime ℓ, let $X = C \times \prod_{i=1}^k E_i$ and $\alpha_\ell \in \mathrm{Br}(X)$ be as above. If $C, E_1, \ldots, E_k$ are very general, then the integral Hodge conjecture fails for $\mathrm{D}_{\mathrm{perf}}(X, \alpha)$.*

Proof. According to Gabber's result [CT02, Appendice], $\mathrm{ind}(\alpha_\ell) = \ell^k$. Therefore, it suffices to show that $\mathrm{ind}_{\mathrm{H}}(\alpha) = \ell$.

We observe that $v^2 = 0$. If L be a topological line bundle on X with $c_1(L) = v$, then consider the class

$$w = \ell + (1 - [L]) \in \mathrm{K}_0^{\mathrm{top}}(X), \quad \mathrm{ch}(w) = (\ell, -v, 0, \ldots, 0).$$

Since $\exp(v/\ell)\mathrm{ch}(w)$ is Hodge, w is a Hodge class for the twisted Mukai structure $\mathrm{K}_0^{\mathrm{top}}(X)^{v/\ell}$.
$\square$

Corollary VI.5. *Let ℓ be a prime. In the context of Theorem VI.4, if $\mathbf{P} \to X$ is a Severi–Brauer variety of class α_ℓ, then the integral Hodge conjecture fails for $\mathbf{P}$.*

Proof. The cohomology ring of X is torsion-free, so the integral Hodge conjecture for $\mathbf{P}$ in all degrees implies the integral Hodge conjecture for $\mathrm{D}_{\mathrm{perf}}(\mathbf{P})$ (Remark III.11).
$\square$

CHAPTER VII

The Integral Hodge Conjecture for DM Surfaces

VII.1: Preliminaries on DM curves

Let C be a smooth, proper Deligne–Mumford stack of pure dimension 1 over $\mathbf{C}$. Then C is a gerbe over its total rigidification C^{rig} (Example II.15), which a smooth, proper orbifold. By Corollary II.13 and Tsen's theorem, there exists a representable finite étale cover $C' \to C^{\mathrm{rig}}$ and an equivalence

$$(\text{VII.1.1}) \qquad\qquad \mathrm{D}_{\mathrm{perf}}(C) \simeq \mathrm{D}_{\mathrm{perf}}(C').$$

As in the classical setting, $\mathrm{K}_0^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(C))$ is spanned by algebraic classes:

Lemma VII.1. *Let C be a smooth, proper Deligne–Mumford curve over $\mathbf{C}$. Then*

$$\mathrm{K}_0(\mathrm{D}_{\mathrm{perf}}(C)) \longrightarrow \mathrm{K}_0^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(C))$$

is surjective.

Proof. From (VII.1.1), it suffices to consider the case when C is a smooth, proper orbifold. Let $p_1, \ldots, p_r$ be the orbifold points of C. The coarse space C^{cs} is smooth, and from the semiorthogonal decomposition of a root stack (§II.4),

$$\mathrm{D}_{\mathrm{perf}}(C) = \langle \mathrm{D}_{\mathrm{perf}}(C^{\mathrm{cs}}), \mathcal{C} \rangle,$$

where $\mathcal{C}$ admits a semiorthogonal decomposition into copies of $\mathrm{D}_{\mathrm{perf}}(p_i)$. Since the desired result holds for $\mathrm{D}_{\mathrm{perf}}(\operatorname{Spec} \mathbf{C})$ and $\mathrm{D}_{\mathrm{perf}}(C^{\mathrm{cs}})$, it holds for $\mathrm{D}_{\mathrm{perf}}(C)$. $\qquad\square$

Lemma VII.2. *Let D be a simple normal crossing Deligne–Mumford curve over $\mathbf{C}$. Then*

$$\mathrm{K}_0(\mathrm{D}_{\mathrm{coh}}^b(D)) \to \mathrm{K}_0^{\mathrm{top}}(\mathrm{D}_{\mathrm{coh}}^b(D))$$

is surjective.

Proof. Let $D^\circ \subseteq D$ be the smooth locus of D, and let $Z = D - D^\circ$ be the complement. Note that Z is smooth. From Lemma III.2, there is a diagram with exact rows

$$
\begin{array}{ccccccc}
\mathrm{K}_0(\mathrm{D}_{\mathrm{perf}}(Z)) & \longrightarrow & \mathrm{K}_0(\mathrm{D}^b_{\mathrm{coh}}(D)) & \longrightarrow & \mathrm{K}_0(\mathrm{D}_{\mathrm{perf}}(D^\circ)) & \longrightarrow & 0 \\
{\scriptstyle\sim}\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\
\mathrm{K}_0^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(Z)) & \longrightarrow & \mathrm{K}_0^{\mathrm{top}}(\mathrm{D}^b_{\mathrm{coh}}(D)) & \longrightarrow & \mathrm{K}_0^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(D^\circ)) & \longrightarrow & 0.
\end{array}
$$

The surjectivity on the right and the fact that a is an isomorphism follow from the observation that $\mathrm{D}_{\mathrm{perf}}(Z)$ is equivalent to a product of copies of $\mathrm{D}_{\mathrm{perf}}(\mathrm{Spec}\,\mathbf{C})$.

Our goal is to show that b is surjective. Since a is an isomorphism, it suffices to show that c is surjective. But D° is open inside of $D' = \coprod_i D_i$, where each D_i is a gerbe over an irreducible component D_i^{cs} of D^{cs}. We apply Lemma III.2 again to obtain an exact sequence

$$
\mathrm{K}_0^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(Z')) \longrightarrow \mathrm{K}_0^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(D')) \longrightarrow \mathrm{K}_0^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(D^\circ)) \longrightarrow 0,
$$

where Z' is a finite set of stacky points. By Lemma VII.1, $\mathrm{K}_0^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(D'))$ is spanned by algebraic classes, so the same holds for $\mathrm{K}_0^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(D^\circ))$. $\qquad\square$

VII.2: DM surfaces

Our goal is to prove the following theorem:

Theorem VII.3. *Let X be a smooth, proper Deligne–Mumford surface over $\mathbf{C}$. Then the integral Hodge conjecture holds for $\mathrm{D}_{\mathrm{perf}}(X)$.*

The strategy of the proof is to reduce to the case of $\mathrm{D}_{\mathrm{perf}}(X, \alpha)$, for a smooth, proper orbifold X, and then to analyze the localization sequence arising from an affine surface contained in X. At the key step, we produce a Hodge class of rank $\mathrm{per}(\alpha)$ using de Jong's theorem [dJ04] that $\mathrm{per}(\alpha) = \mathrm{ind}(\alpha)$. We mention a corollary:

Corollary VII.4. *The truth of the integral Hodge conjecture for $\mathrm{D}_{\mathrm{perf}}(X)$ is a birational invariant for smooth, proper Deligne–Mumford threefolds.*

Proof. Combine Theorem VII.3 with weak factorization (Example II.17). $\qquad\square$

We now begin the proof of Theorem VII.3.

Step 1. By Example II.15, it suffices to prove the integral Hodge conjecture for $\mathrm{D}^1_{\mathrm{perf}}(\mathfrak{X})$, where $\mathfrak{X} \to X$ is a $\boldsymbol{\mu}_n$-gerbe over a smooth, proper orbifold.

Step 2. Since the integral Hodge conjecture holds for smooth Deligne–Mumford stacks of dimension at most 1 by Lemma VII.1, weak factorization (Example II.17) implies that the integral Hodge conjecture for $\mathrm{D}^1_{\mathrm{perf}}(\mathfrak{X})$ is a birational invariant of $\mathfrak{X}$.

Step 3. Let $U \subseteq X$ be an open substack equivalent to an affine surface. Possibly shrinking U, we may assume that $\mathrm{NS}(U) = 0$. We observe that since U is an affine surface, the restriction $\mathcal{U} \to U$ of $\mathfrak{X}$ over U is essentially topologically trivial.

From the previous step, we may replace $\mathfrak{X}$ by any other smooth compactification of $\mathcal{U}$. In particular, following the proof of Lemma II.20, we adopt the following setup: $U \to X_0$ is the inclusion of U into a smooth, projective variety such that $D = X_0 - U$ has simple normal crossings; $X \to X_0$ is an iterated root stack over the components of D; $\mathfrak{X} \to X$ is a $\boldsymbol{\mu}_n$-gerbe extending $\mathcal{U}$; and $\mathbf{P} \to X$ is a Severi–Brauer variety representing the Brauer class of $\mathfrak{X}$.

Step 4. There is a commutative diagram

$$
\begin{array}{ccccc}
\mathrm{Hdg}(\mathrm{D}_{\mathrm{perf}}(\mathbf{P})_1, \mathbf{Z}) & \xrightarrow{\ f\ } & \mathrm{Hdg}(\mathrm{W}_0^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(\mathbf{P}_U)_1)) & \longrightarrow & \mathrm{K}_0^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(\mathbf{P}_U)_1) \\
\Big\downarrow{\scriptstyle\sim} & & & & \Big\downarrow{\scriptstyle\sim} \\
\mathrm{Hdg}(\mathrm{D}^1_{\mathrm{perf}}(\mathfrak{X}), \mathbf{Z}) & & \xrightarrow{\hspace{4cm} g \hspace{4cm}} & & \mathrm{K}_0^{\mathrm{top}}(\mathrm{D}^1_{\mathrm{perf}}(\mathcal{U})),
\end{array}
$$

where the vertical isomorphisms come from Lemma II.11, after choosing $\mathcal{O}_{\mathbf{P}_{\mathfrak{X}}}(1) \in \mathrm{Pic}^{-1}(\mathbf{P}_{\mathfrak{X}})_1$.

The key claim is that any fiber of g contains an algebraic class. To prove the claim, it suffices to prove the corresponding statement for f. Observe that by Lemma IV.27 and the assumption that $\mathrm{NS}(U) = 0$, the rank map gives an isomorphism

$$
\mathrm{Hdg}(\mathrm{W}_0^{\mathrm{top}}(\mathrm{D}_{\mathrm{perf}}(\mathbf{P}_U)_1) \simeq \mathrm{per}(\alpha) \cdot \mathbf{Z},
$$

where α is the Brauer class of $\mathfrak{X}$. Therefore, it suffices to show that there is an algebraic class of rank $\mathrm{per}(\alpha)$ in $\mathrm{Hdg}(\mathrm{D}^1_{\mathrm{perf}}(\mathfrak{X}), \mathbf{Z})$. This follows from de Jong's theorem [dJ04] that $\mathrm{per}(\alpha) = \mathrm{ind}(\alpha)$.

Step 5. Let E be the preimage of D under the map $\mathfrak{X} \to X_0$. From the localization sequence (Lemma III.2), we may consider a diagram with exact rows

$$
\begin{array}{ccccc}
K & \longrightarrow & \mathrm{Hdg}(\mathrm{D}^1_{\mathrm{perf}}(\mathfrak{X}), \mathbf{Z}) & \longrightarrow & \mathrm{K}_0^{\mathrm{top}}(\mathrm{D}^1_{\mathrm{perf}}(\mathcal{U})) \\
\Big\downarrow & & \Big\downarrow & & \Big\| \\
\mathrm{K}_0^{\mathrm{top}}(\mathrm{D}^b_{\mathrm{coh}}(E)) & \longrightarrow & \mathrm{K}_0^{\mathrm{top}}(\mathrm{D}^1_{\mathrm{perf}}(\mathfrak{X})) & \longrightarrow & \mathrm{K}_0^{\mathrm{top}}(\mathrm{D}^1_{\mathrm{perf}}(\mathcal{U})),
\end{array}
$$

where the vertical maps are inclusions. Our goal is to show that any $v \in \mathrm{Hdg}(\mathrm{D}^1_{\mathrm{perf}}(\mathfrak{X}), \mathbf{Z})$ is

algebraic. From the result of the previous step, we may suppose that v lies in the image of K. Then we apply Lemma VII.2, which asserts that the map $\mathrm{K}_0(\mathrm{D}^b_{\mathrm{coh}}(E)) \to \mathrm{K}^{\mathrm{top}}_0(\mathrm{D}^b_{\mathrm{coh}}(E))$ is surjective. $\qquad\square$

CHAPTER VIII
The Period-index Problem for Complex Tori

VIII.1: Preliminaries

For the definition of the Brauer group of a complex manifold and its basic properties, we refer to [Sch05]. If X is a connected complex manifold and $\alpha \in \mathrm{Br}(X)$, we recall that the *period* $\mathrm{per}(\alpha)$ is the order of α in $\mathrm{Br}(X)$, and the *index* of α is given by

$$\mathrm{ind}(\alpha) = \gcd(\deg \mathcal{A} : [\mathcal{A}] = \alpha),$$

where $\mathcal{A}$ runs over the Azumaya algebras of class α. (We recall that the *degree* of an Azumaya algebra is the square root of its rank.)

Lemma VIII.1. *Let X be a connected complex manifold, and $\alpha \in \mathrm{Br}(X)$. Then $\mathrm{per}(\alpha)$ divides $\mathrm{ind}(\alpha)$, and $\mathrm{per}(\alpha)$ and $\mathrm{ind}(\alpha)$ share prime factors.*

Proof. See [AW15]. □

Remark VIII.2. Suppose that X is a connected Stein manifold, and consider the homomorphism

$$\mathrm{Br}(X) \to \mathrm{H}^3(X, \mathbf{Z})^{\mathrm{tors}}.$$

By the Grauert–Oka principle [For17, §8.2], the sets of isomorphism classes of holomorphic and continuous torsors under PGL_{n+1} coincide, for each n. It follows that the map above is an isomorphism, and that the period-index problem for X is equivalent to the topological period-index problem for the CW complex underlying X. Hence, one may obtain bounds from [AW21]. For instance,

$$\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^{\dim X - 1}$$

if $\mathrm{per}(\alpha)$ is prime to $(\dim X - 1)!$.

It is frequently useful to work with α-twisted sheaves in the sense of Căldăraru [Căl00]. We write $\mathrm{Coh}^\alpha(X)$ for the abelian category of α-twisted sheaves (where, as is customary, we

often suppress the choice of a cocycle representative for α). One has the following well-known result:

Lemma VIII.3. *Let X be a complex manifold, and let $\alpha \in \mathrm{Br}(X)$. Then*

$$F \mapsto \mathcal{E}nd(F)$$

gives a bijection between the set of isomorphism classes of locally free α-twisted sheaves, and the set of isomorphism classes of Azumaya algebras on X of class α.

Proof. We refer to [Căl00, Theorem 1.3.5], which holds without change in the analytic case. $\square$

From Lemma VIII.3, the index of α coincides with the greatest common divisor of the ranks of locally free α-twisted sheaves. It is also natural to consider the ranks of arbitrary α-twisted coherent sheaves, which leads to the following variant of the index:

Definition VIII.4. Let X be a complex manifold, with $\alpha \in \mathrm{Br}(X)$. The *coherent index* $\mathrm{ind}_{\mathrm{Coh}}(\alpha)$ is the greatest common divisor of the ranks of α-twisted coherent sheaves on X.

We note that $\mathrm{ind}_{\mathrm{Coh}}(\alpha)$ divides $\mathrm{ind}(\alpha)$. It follows from Theorem VIII.17 below that $\mathrm{ind}_{\mathrm{Coh}}(\alpha)$ and $\mathrm{ind}(\alpha)$ coincide when X is a general complex torus, but we do not know if they coincide for Brauer classes on arbitrary complex manifolds.

Remark VIII.5. A famous question of Grothendieck [Gro68] asks when the inclusion

$$\mathrm{Br}(X) \subseteq \mathrm{H}^2(X, \mathcal{O}_X^\times)^{\mathrm{tors}}$$

is an equality. While the answer is positive in the projective case [dJ03], the question remains widely open in the compact Kähler case. It is known, however, for compact Kähler surfaces (cf. [HS03] and [Sch05]), as well as complex tori (cf. [EN83], or one may deduce it from the proof of Lemma VIII.6 below).

Lemma VIII.6. *Let $f : X' \to X$ be a finite cover of complex manifolds. Given $\alpha \in \mathrm{Br}(X)$ such that $f^*\alpha = 0$,*

$$\mathrm{ind}(\alpha) \mid \deg f.$$

Proof. Let $\tilde{\alpha}$ be a cocycle representative for α, and let L be a $f^*\tilde{\alpha}$-twisted line bundle on X'. Then f_*L is a locally free $\tilde{\alpha}$-twisted coherent sheaf of rank $\deg f$ on X, and we conclude by Lemma VIII.3. $\square$

In order to apply an analytic result in the proof of Theorem VIII.17 below, it will be convenient to treat α-twisted sheaves on X as genuine sheaves on a Severi–Brauer variety. Recall that a Severi–Brauer variety over a connected complex manifold X is a smooth, proper PGL_{n+1}-equivariant morphism $\mathbf{P} \to X$ (where the action on X is trivial), which is locally on X of the form $\mathbf{P}^n \times U \to U$, for $U \subseteq X$. A Severi–Brauer variety $\mathbf{P} \to X$ has a Brauer class $[\mathbf{P}] \in \mathrm{Br}(X)$, which obstructs the existence of a holomorphic vector bundle E on X such that $\mathbf{P} \simeq \mathbf{P}(E)$.

We warn the reader that we follow Giraud's convention [Gir71, Example V.4.8] for the Brauer class of a Severi–Brauer variety. If one adopts the opposite convention, then the category of weight-k sheaves defined below is equivalent to the category of α^{-k}-twisted sheaves.

Definition VIII.7. Let X be a complex manifold, and let $\mathbf{P} \to X$ be a Severi–Brauer variety. A coherent sheaf E on $\mathbf{P}$ has *weight k* if, for any $x \in X$, the restriction of E to the fiber $\mathbf{P}_x$ is isomorphic to $V \otimes \mathcal{O}_{\mathbf{P}_x}(k)$ for a $\mathbf{C}$-vector space V.

Let $\mathrm{Coh}^k(\mathbf{P}/X)$ be the abelian category of weight-k coherent sheaves on $\mathbf{P}$. For simplicity, we often write $\mathrm{Coh}^k(\mathbf{P})$, with the morphism to X implicit.

Remark VIII.8 (Descent). In the setting above, let E be a weight-0 coherent sheaf on $\mathbf{P}$. Then the natural morphism

$$\pi^* \pi_* E \to E$$

is an isomorphism.

Lemma VIII.9. *Let X be a complex manifold, and let $\alpha \in \mathrm{Br}(X)$, and let $\pi : \mathbf{P} \to X$ be a Severi–Brauer variety of class α. There exists a rank-preserving equivalence*

$$\mathrm{Coh}^\alpha(X) \simeq \mathrm{Coh}^1(\mathbf{P}).$$

Proof. Let $\tilde{\alpha}$ be a cocycle representative for α. Since $\pi^* \alpha$ is trivial, there exists an $\pi^* \tilde{\alpha}^{-1}$-twisted line bundle $\mathcal{O}_{\mathbf{P}}^{\mathrm{tw}}(1)$ on $\mathbf{P}$. The equivalence is given by $E \mapsto \pi^* E \otimes \mathcal{O}_{\mathbf{P}}^{\mathrm{tw}}(1)$, with inverse equivalence $F \mapsto \pi_*(F \otimes \mathcal{O}_{\mathbf{P}}^{\mathrm{tw}}(-1))$. $\square$

Remark VIII.10 (Topologically trivial Brauer classes). Let $\mathbf{P} \to X$ be a Severi–Brauer variety, and suppose that the Brauer class α of $\mathbf{P}$ is *topologically trivial*, i.e., lies in the kernel of the homomorphism

$$\mathrm{Br}(X) \subseteq \mathrm{H}^2(X, \mathcal{O}_X^\times)^{\mathrm{tors}} \to \mathrm{H}^3(X, \mathbf{Z})^{\mathrm{tors}}$$

arising from the exponential sequence. Then $\mathbf{P} \to X$ is the projectivization of a topological complex vector bundle.

In particular, there exists a *weight–1 topological line bundle* $\mathcal{O}_{\mathbf{P}}^{\mathrm{top}}(1)$ on $\mathbf{P}$, whose restriction to the fiber $\mathbf{P}_x$ over $x \in X$ is isomorphic to $\mathcal{O}_{\mathbf{P}_x}(1)$. Moreover, by the Leray–Hirsch theorem, there is a decomposition

$$\mathrm{K}_0^{\mathrm{top}}(\mathbf{P}) = \mathrm{K}_0^{\mathrm{top}}(X) \oplus \mathrm{K}_0^{\mathrm{top}}(X) \cdot [\mathcal{O}_{\mathbf{P}}^{\mathrm{top}}(1)] \oplus \cdots \oplus \mathrm{K}_0^{\mathrm{top}}(X) \cdot [\mathcal{O}_{\mathbf{P}}^{\mathrm{top}}(1)^{\otimes d}],$$

where d is the relative dimension of $\mathbf{P} \to X$.

Definition VIII.11. Let X be a complex manifold, and let $\alpha \in \mathrm{Br}(X)$. A *rational B-field* for α is a class $B \in \mathrm{H}^2(X, \mathbf{Q})$ whose image under the homomorphism

$$\exp(2\pi i \cdot -) : \mathrm{H}^2(X, \mathbf{Q}) \to \mathrm{H}^2(X, \mathcal{O}_X^\times)^{\mathrm{tors}}$$

is α. We note that a rational B-field for α exists if and only if α is topologically trivial.

Remark VIII.12. Let X be a compact Kähler manifold, and let $\pi : \mathbf{P} \to X$ be a Severi–Brauer variety of class α. Suppose that α is topologically trivial. For any weight-1 topological line bundle $\mathcal{O}_{\mathbf{P}}^{\mathrm{top}}(1)$ on $\mathbf{P}$,

$$c_1(\mathcal{O}_{\mathbf{P}}^{\mathrm{top}}(1)) = H + \pi^* B,$$

where $H \in \mathrm{NS}(\mathbf{P})_{\mathbf{Q}}$ is a class whose restriction to any fiber $\mathbf{P}_x$ is $\mathcal{O}_{\mathbf{P}_x}(1)$, and $B \in \mathrm{H}^2(X^{\mathrm{an}}, \mathbf{Q})$ is a rational B-field for α. We refer to [dP22, Lemma 5.9], which however uses the opposite convention for the class of a Severi–Brauer variety (hence the difference in sign), and which is stated for a smooth, proper variety X but holds in the compact Kähler case by an identical argument.

VIII.2: The annihilator of a Brauer class

Definition VIII.13. Let X be a complex torus.

1. The *annihilator* of a class $\omega \in \mathrm{H}^2(X, \mathbf{Z}/n)$ is the least degree of a finite isogeny $f : X' \to X$ such that $f^* \omega = 0$.

2. The *annihilator* of a Brauer class $\alpha \in \mathrm{Br}(X)$ is the least degree of a finite isogeny $f : X' \to X$ such that $f^* \alpha = 0$.

By Lemma VIII.6, $\mathrm{ind}(\alpha)$ divides $\mathrm{Ann}(\alpha)$.

Remark VIII.14. Suppose that X is a complex torus with $\mathrm{NS}(X) = 0$. From the Kummer sequence, the exponential map

$$\mathrm{H}^2(X, \mathbf{Z}/n) \to \mathrm{Br}(X)[n]$$

is an isomorphism. If $\alpha \in \operatorname{Br}(X)[n]$, then α lies in the image of a unique class $\omega \in \operatorname{H}^2(X, \mathbf{Z}/n)$, and $\operatorname{Ann}(\alpha) = \operatorname{Ann}(\omega)$.

The following lemma computes $\operatorname{Ann}(\omega)$ in terms of linear algebra. For an element $a \in \mathbf{Z}/n$, we write $\operatorname{Ann}_{\mathbf{Z}}(a) \in \mathbf{Z}$ for the positive generator of the annihilator ideal of a.

Lemma VIII.15. *Let X be a complex torus of dimension g, with $\omega \in \operatorname{H}^2(X, \mathbf{Z}/n)$. Suppose that $e_1, \ldots, e_{2g}$ is a basis for $\operatorname{H}^1(X, \mathbf{Z})$ such that*

$$\omega = \sum_{i=1}^{r} a_i e_i \wedge e_{i+g},$$

for $0 \neq a_i \in \mathbf{Z}/n$ and $0 \leq r \leq g$. Then

$$\operatorname{Ann}(\omega) = \prod_{i=1}^{r} \operatorname{Ann}_{\mathbf{Z}}(a_i).$$

Proof. First, consider an isogeny $f : X' \to X$ of degree $\prod \operatorname{Ann}_{\mathbf{Z}}(a_i)$ such that

$$f^* e_i = \operatorname{Ann}(a_i) \cdot e_i', \quad 1 \leq i \leq r$$

where $e_1', \ldots, e_{2g}'$ is a basis for $\operatorname{H}^1(X', \mathbf{Z})$. Then $f^*\omega = 0$, so $\operatorname{Ann}(\omega) \leq \prod \operatorname{Ann}_{\mathbf{Z}}(a_i)$.

In the other direction, we may suppose that $n = p^e$ is a prime power. Let

$$\eta = \sum_{i=1}^{r} a_i' e_i \wedge e_{i+g} \in \operatorname{H}^2(X, \mathbf{Z})$$

be a lift of ω, and let $f : X' \to X$ be a finite isogeny such that $f^*\omega = 0$. Then $f^*\eta$ is divisible by n. In particular,

$$f_* f^* \eta^r / r! = \deg f \cdot \prod_{i=1}^{r} a_i'$$

$$\equiv 0 \mod n^r.$$

Since n is a prime power, it follows that $\prod \operatorname{Ann}_{\mathbf{Z}}(a_i)$ divides $\deg f$. $\square$

If X is a complex torus, then $\operatorname{H}^3(X, \mathbf{Z})^{\operatorname{tors}} = 0$, so any Brauer class is topologically trivial and admits a rational B-field.

Lemma VIII.16. *Let X be a complex torus with $\operatorname{NS}(X) = 0$, and let $\alpha \in \operatorname{Br}(X)$. For any rational B-field B for α, let N be the least positive integer such that*

$$N \cdot \exp(B) \in \operatorname{H}^{\operatorname{ev}}(X, \mathbf{Q})$$

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lies in $H^{ev}(X, \mathbf{Z})$. *Then* $N = \mathrm{Ann}(\alpha)$.

Proof. Let $n = \mathrm{per}(\alpha)$. We write

$$B = \sum_{i=1}^{r} \frac{m_i}{n} e_i \wedge e_{i+g}, \quad m_i \neq 0 \in \mathbf{Z}, \quad 0 \leq r \leq g.$$

Then the image ω of $n \cdot B$ in $H^2(X, \mathbf{Z}/n)$ maps to α under the exponential. By Remark VIII.14, $\mathrm{Ann}(\alpha)$ is equal to $\mathrm{Ann}(\omega)$. By Lemma VIII.15, $\mathrm{Ann}(\omega) = \prod \mathrm{Ann}_{\mathbf{Z}}(m_i \bmod n)$.

With N as in the statement of the lemma, we observe that $N \cdot B^r/r!$ is integral. For each prime factor p of n, with $v_p(m) = e$, it follows that

$$N \cdot \prod m_i \equiv 0 \mod p^{er},$$

so $\prod \mathrm{Ann}_{\mathbf{Z}}(m_i \bmod p^e)$ divides N for all such p, which implies that $\prod \mathrm{Ann}_{\mathbf{Z}}(m_i \bmod n)$ divides N. In the other direction, it is straightforward to show that

$$\prod \mathrm{Ann}_{\mathbf{Z}}(m_i \bmod n) \cdot \exp(B)$$

is integral, so that N divides $\prod \mathrm{Ann}_{\mathbf{Z}}(m_i \bmod n)$. $\qquad\square$

VIII.3: The main result

In this section, we prove our main result, phrased in terms of the coherent index of Definition VIII.4.

Theorem VIII.17. *Let X be a complex torus such that* $\mathrm{NS}(X) = \mathrm{Hdg}^4(X) = 0$, *with* $\alpha \in \mathrm{Br}(X)$. *Then*
$$\mathrm{ind}_{\mathrm{Coh}}(\alpha) = \mathrm{Ann}(\alpha)$$

Theorem VIII.17 implies Theorem I.11, and also implies Corollary I.12 by the formula given for $\mathrm{Ann}(\alpha)$ through Remark VIII.14 and Lemma VIII.15. We note that if a complex torus X satisfies the condition that $\mathrm{NS}(X) = \mathrm{Hdg}^4(X) = 0$, and E is a locally free coherent sheaf on X, then $c_i(E) = 0$ for each $i > 0$ [Voi02]. In particular, X does not have the resolution property.

Proof. First, from the definition of $\mathrm{ind}_{\mathrm{Coh}}(\alpha)$ and Lemma VIII.6,

$$\mathrm{ind}_{\mathrm{Coh}}(\alpha) \mid \mathrm{ind}(\alpha) \mid \mathrm{Ann}(\alpha),$$

so it remains to show that $\mathrm{Ann}(\alpha)$ divides $\mathrm{ind}_{\mathrm{Coh}}(\alpha)$, i.e., that the rank of any α-twisted coherent sheaf is divisible by $\mathrm{Ann}(\alpha)$.

Let $\mathbf{P} \to X$ be a Severi–Brauer variety of class α, and let E be a coherent sheaf of weight 1 on $\mathbf{P}$. By Lemma VIII.9, it is enough to show the rank of E is divisible by $\mathrm{Ann}(\alpha)$. We may suppose that E is torsion-free, and by induction on the rank, we may suppose that E contains no nonzero subsheaves of smaller rank. Then $E^{\vee\vee}$ contains no nonzero subsheaves of smaller rank as well, and is reflexive. We replace E with $E^{\vee\vee}$.

Let $f : X' \to X$ be a finite isogeny such that $f^*\alpha = 0$, and consider a pullback diagram

$$
\begin{array}{ccc}
\mathbf{P}' & \xrightarrow{\ g\ } & \mathbf{P} \\
\downarrow & & \downarrow \\
X' & \xrightarrow{\ f\ } & X.
\end{array}
$$

We note that E is μ-stable with respect to a Kähler form ω on $\mathbf{P}$, so by [BS09, Prop. 2.3], g^*E is μ-polystable with respect to $g^*\omega$. Since $\mathbf{P}' \to X'$ is a projective bundle, there is a holomorphic relative hyperplane bundle $\mathcal{O}_{\mathbf{P}'}(1)$ so that $E' = g^*E \otimes \mathcal{O}_{\mathbf{P}'}(-1)$ has weight 0, and in particular descends to a coherent sheaf E'_0 on X' (Remark VIII.8).

Our assumption on X implies that $\mathrm{NS}(X') = \mathrm{Hdg}^4(X') = 0$, so that $c_1(E') = c_2(E') = 0$. Since E' is reflexive and μ-polystable, the theorem of Bando and Siu [BS94, Corollary 3] implies that E' is a flat holomorphic vector bundle, and in particular, $c_i(E') = 0$ for all $i > 0$.

Next, we compute Chern characters. Let $\mathcal{O}_{\mathbf{P}}^{\mathrm{top}}(1)$ be a 1-twisted topological line bundle on $\mathbf{P}$. From the Leray–Hirsch theorem (Remark VIII.10), we may write

$$
\mathrm{ch}(E) = \mathrm{ch}(\mathcal{O}_{\mathbf{P}}^{\mathrm{top}}(1)) \cdot \mathrm{ch}(\xi),
$$

where $\xi \in \mathrm{K}_0^{\mathrm{top}}(X)$. From Remark VIII.12, we may write $c_1(\mathcal{O}_{\mathbf{P}}^{\mathrm{top}}(1)) = H + B$, where H is a rational Hodge class and B is a rational B-field for α. Since H is Hodge,

$$
\zeta = \exp(B) \cdot \mathrm{ch}(\xi) \in \mathrm{H}^{\mathrm{ev}}(X, \mathbf{Q})
$$

is Hodge in each degree.

We claim that $f^*\zeta = \mathrm{ch}(E'_0)$. To prove the claim, first observe that f^*B is a rational B-field for $f^*\alpha = 0$, hence must be integral because $\mathrm{NS}(X') = 0$. Therefore, g^*h is an integral Hodge class, and generates $\mathrm{NS}(\mathbf{P}')$. It follows that $c_1(\mathcal{O}_{\mathbf{P}'}(1)) = g^*h$, which implies the claim.

From the first part of the proof,

$$
\mathrm{ch}(E'_0) = N \cdot \mathbf{1} \in \mathrm{H}^{\mathrm{ev}}(X', \mathbf{Z}),
$$

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for an integer $N \geq 0$, where $\mathbf{1}$ is the multiplicative unit. It follows that $\zeta = N \cdot \mathbf{1}$.

It remains to show that $\mathrm{Ann}(\alpha)$ divides N. But

$$\mathrm{ch}(\xi) = N \cdot \exp(-B)$$

lies in $\mathrm{H}^{\mathrm{ev}}(X, \mathbf{Z})$, so we conclude by Lemma VIII.16. $\qquad \square$

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