Closed Geodesics and Stability of Negatively Curved Metrics

by

Karen Aleksandra Butt

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
at The University of Michigan
2023

Doctoral Committee:

Professor Ralf Spatzier, Chair
Assistant Professor Pierre-Louis Blayac
Professor Victoria Booth
Professor Richard Canary
Associate Professor Alexander Wright

Karen Aleksandra Butt
kbutt@umich.edu
ORCID iD: 0000-0002-8691-8837

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ACKNOWLEDGEMENTS

First and foremost, it is a pleasure to thank my advisor, Ralf Spatzier, for his unparalleled generosity and support with all matters, both big and small. Of all the things I could have done better in graduate school, choosing an advisor is certainly not one of them.

I would like to thank my colleagues at the University of Michigan for valuable discussions about my research, particularly Thang Nguyen and Yuping Ruan. I would also like to thank Alex Wright for suggesting such an interesting thesis problem, and I thank my doctoral committee for their work reviewing this thesis. I thank Nicholas Wawrykow for help with making the figures as well as Bradley Zykoski for figuring out how to make the list of figures.

Over the past year, I am also very lucky to have met many wonderful people in the dynamical systems community at large. I am particularly grateful to Andrey Gogolev and Amie Wilkinson for their encouragement and for the interest they showed in my work. I also thank David Constantine, Vaughn Climenhaga, and Dima Dolgopyat for discussing various arguments in this thesis with me.

On a more personal note, I would like to thank my friends and classmates at the University of Michigan for making grad school a fun experience over the years, particularly Carsten Peterson, Nicholas Wawrykow, Yueqiao Wu, and Bradley Zykoski for their camaraderie during our sixth and final year. Finally, I thank my husband and the rest of my family, without whose unwavering support this thesis would certainly not have been possible.

This research was partially supported by the National Science Foundation under Grant Numbers DMS-2003712 and DMS-1607260.

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ABSTRACT

This dissertation explores the extent to which lengths of closed geodesics on a Riemannian manifold determine the underlying metric. In the setting of closed manifolds of negative sectional curvature, it is known in certain cases—and conjectured to be true in general—that in order to determine a Riemannian metric on a given manifold up to isometry, it suffices to measure the lengths of all closed geodesics (as a function of their free homotopy classes). This phenomenon is known as *marked length spectrum rigidity.* It was proved in dimension 2 (independently) by Otal and Croke [Ota90, Cro90], in dimension at least 3 if one of the metrics is locally symmetric by Hamenstädt [Ham99] (also using work of Besson–Courtois–Gallot [BCG95, BCG96]), and in any dimension by Guillarmou–Lefeuvre [GL19], provided the metrics are sufficiently close in a suitable C^k topology.

Still, even in the cases where rigidity is known to hold, there is more to be understood about the extent to which the marked length spectrum determines the metric. In this thesis, we prove *quantitative* versions of marked length spectrum rigidity in dimension 2, as well as in higher dimensions when one of the metrics is locally symmetric, thereby refining the previously known rigidity results of Otal [Ota90] and Hamenstädt [Ham99], respectively. In each of these settings, we consider pairs of Riemannian manifolds whose marked length spectra agree—only *approximately*—on a *finite* set of closed geodesics. We prove the two metrics are "approximately isometric", meaning bi-Lipschitz equivalent with constant close to 1. We obtain explicit estimates for this Lipschitz constant in terms of the measurement error and the length of the longest geodesic in the finite set. Our estimates depend only on concrete geometric information about the given metrics, such as the dimension, sectional curvature bounds, and injectivity radii.

CHAPTER I
Introduction

I.1: Statement of the problem

In broad terms, this dissertation contributes to the fields of differential geometry and dynamical systems, in particular the interplay between the two arising via the geodesic flow. More specifically, this work is concerned with closed geodesics on Riemannian manifolds (or, equivalently, periodic orbits of the geodesic flow), namely the extent to which they determine the underlying metric.

A fundamental question in Riemannian geometry is determining a set of parameters which describe a metric up to isometry. In negative curvature, a natural candidate is the set of lengths of closed geodesics, also known as the *length spectrum* due to its close connection with the Laplace spectrum. In fact, to what extent the Laplace spectrum determines the metric is a question which falls into a broad class of *inverse spectral problems*, famously known by the tagline "Can one hear the shape of a drum?" [Kac66]. It turns out that one cannot hear the shape of a negatively curved drum: the first examples of isospectral non-isometric surfaces of constant negative curvature were constructed by Vignéras [Vig80], and Sunada later provided a method to generate more general counterexamples [Sun85]. As a result, it is natural to consider lengths of closed geodesics together with the additional information of their associated free homotopy classes, which leads to the following definition.

Definition I.1. Given a closed, negatively curved Riemannian manifold (M, g), the *marked length spectrum* $\mathcal{L}_g$ is the function on free homotopy classes of closed curves in M which associates to each class the length of its unique geodesic representative.

Note that any point in the Teichmüller space of metrics of constant curvature -1 on a topological surface S is determined by $6\,\mathrm{genus}(S)-6$ *Fenchel–Nielsen coordinates*; moreover, it is known that these coordinates are in turn determined by the lengths of finitely many closed geodesics (as few as $6\,\mathrm{genus}(S)-5$) [FM11, Sch93]. In other words, the marked length spectrum (in fact, a finite part of it) uniquely determines a hyperbolic surface up to isometry.

In arbitrary dimension and variable negative curvature, on the other hand, this is still an open question.

Conjecture I.2 (Conjecture 3.1 in [BKB+85]). *The marked length spectrum of a closed Riemannian manifold of negative curvature determines the metric up to isometry.*

This question was resolved for (variably curved) surfaces independently by Otal and Croke [Ota90, Cro90]. For higher dimensions, this was solved by Hamenstädt in the case where one of the metrics is locally symmetric [Ham99], using the entropy rigidity theorem of Besson–Courtois–Gallot [BCG95, BCG96]. More recently, the conjecture was solved locally, that is, for two metrics which are sufficiently close in some suitable C^k topology, by Guillarmou–Lefeuvre using techniques from microlocal analysis [GL19].

Still, even in the cases where rigidity does hold, there is more to be understood about to what extent the marked length spectrum determines the metric. In this thesis, we consider two natural follow-up questions to Conjecture I.2.

Question I.3. *What if two metrics have marked length spectra which are not equal but close in some sense? Is there a sense in which the metrics are close?*

Question I.4. *Does the marked length spectrum on a sufficiently large finite set approximately determine the metric?*

Question I.4 has not been previously considered anywhere in the literature as far as we know (aside from the case of surfaces of constant curvature). All known proofs of marked length spectrum rigidity in the variable curvature setting rely on limiting procedures involving longer and longer closed geodesics. For example, $\mathcal{L}_g$ determines the topological entropy of the geodesic flow because the latter is the exponential growth rate of closed geodesics. However, it is not at all clear what information can be obtained about topological entropy from knowing $\mathcal{L}_g$ on a finite set. On a related note, Guillarmou–Knieper–Lefeuvre proved that in the local setting, it is enough for $\mathcal{L}_g$ and $\mathcal{L}_{g_0}$ to agree *asymptotically* in order to conclude g and g_0 are isometric [GKL22]. In particular, rigidity holds if $\mathcal{L}_g$ and $\mathcal{L}_{g_0}$ coincide *outside* of a finite set, and Question I.4 is a natural counterpart to this. Note that, in general, we can only hope to gain *approximate* information from finite data, since in variable curvature, the space of all possible metrics is infinite-dimensional.

Question I.3 was previously known for hyperbolic surfaces and in general for pairs of metrics g and g_0 on the same manifold M which are sufficiently close in some suitable C^k topology. The first case is due to Thurston [Thu98]. He showed that if (M, g) and (N, g_0) are both surfaces of constant negative curvature, and $f : M \to N$ is a fixed homeomorphism, then the best possible Lipschitz constant for a map $F : M \to N$ in the same homotopy class as f is precisely $\sup_{\gamma\in\Gamma} \frac{\mathcal{L}_{g_0}(f_*\gamma)}{\mathcal{L}_g(\gamma)}$. The second case is part of the previously mentioned work of Guillarmou–Knieper–Lefeuvre [GKL22]. Their techniques provide explicit estimates (in a suitable Sobolev norm) for how close the metrics are in terms of the ratio $\frac{\mathcal{L}_{g_0}}{\mathcal{L}_g}$, or more precisely the geodesic stretch; in fact, their results hold more generally for non-positively curved metrics with Anosov geodesic flow. However, this work requires g and g_0 to be sufficiently close metrics (in some C^k topology) on the same manifold. The results proved in this thesis do not require the metrics to be close, nor do they require the two metrics to be on the same manifold, but only on pairs of manifolds with isomorphic fundamental groups.

I.2: Statements of main results

This thesis provides new answers to Questions I.3 and I.4. We reduce Question I.4 to Question I.3 for closed negatively curved manifolds in general (see III). Moreover, we answer Question I.3 in dimension 2 (see I.2.2), and in higher dimensions when one of the metrics is locally symmetric (I.2.3). These are two of the main cases where marked length spectrum rigidity is known—due to Otal and Croke for surfaces [Ota90, Cro90], and Hamenstädt and Besson–Courtois–Gallot for higher dimensions [Ham99, BCG95].

While it is customary to compare $\mathcal{L}_g$ and $\mathcal{L}_{g_0}$ for metrics g and g_0 on the same manifold M, this also makes sense more generally for manifolds (M, g) and (N, g_0) with isomorphic fundamental groups. This is because the set of free homotopy classes of M can be identified with conjugacy classes in the fundamental group Γ of M, and as such we can view $\mathcal{L}_g$ as a function on Γ. In our setting (negative curvature), a standard result in algebraic topology states any isomorphism of fundamental groups is induced by a homotopy equivalence; however, M and N need not be diffeomorphic, as shown by Farrell–Jones [FJ89]. Our results cover this case as well.

I.2.1: Finiteness

Let (M, g) and (N, g_0) be closed negatively curved Riemannian manifolds with fundamental group Γ. Consider the situation where $\mathcal{L}_g$ and $\mathcal{L}_{g_0}$ coincide only on a *finite set* of closed geodesics. In fact, we can consider the more general situation where $\mathcal{L}_g$ and $\mathcal{L}_{g_0}$ are only *multiplicatively close* on this set.

Hypothesis I.5. *For $L > 0$, let $\Gamma_L := \{\gamma \in \Gamma \,|\, \mathcal{L}_g(\gamma) \leq L\}$. Now let $\varepsilon > 0$ small and suppose*

$$1 - \varepsilon \leq \frac{\mathcal{L}_{g_0}(\gamma)}{\mathcal{L}_g(\gamma)} \leq 1 + \varepsilon$$

for all $\gamma \in \Gamma_L$.

If L is sufficiently large, we obtain estimates for the ratio $\mathcal{L}_{g_0}/\mathcal{L}_g$ on all of Γ in terms of ε and L. Note that the estimates do not depend on the particular pair of metrics under consideration; they are uniform for all (M, g) and (N, g_0) with pinched sectional curvatures and injectivity radii bounded away from zero.

Theorem I.6 (Butt, [But22a] Theorem 1.2)**.** *Let (M, g) and (N, g_0) be closed Riemannian manifolds of dimension n with sectional curvatures contained in the interval $[-\Lambda^2, -\lambda^2]$. Let $\mathcal{L}_g$ and $\mathcal{L}_{g_0}$ denote their marked length spectra. Let Γ denote the fundamental group of M and let i_M denote its injectivity radius. Suppose there is a homotopy equivalence $f : M \to N$ and let f_* denote the induced map on fundamental groups.*

Then there is $L_0 = L_0(n, \Gamma, \lambda, \Lambda, i_N)$ so that the following holds: Suppose the marked length spectra $\mathcal{L}_g$ and $\mathcal{L}_{g_0}$ satisfy Hypothesis I.5 for some $\varepsilon > 0$ and $L \geq L_0$. Then there exist constants $C > 0$ and $0 < \alpha < 1$, depending only on n, Γ, λ, Λ, i_N, so that

$$1 - (\varepsilon + CL^{-\alpha}) \leq \frac{\mathcal{L}_{g_0}(f_*\gamma)}{\mathcal{L}_g(\gamma)} \leq 1 + (\varepsilon + CL^{-\alpha})$$

for all $\gamma \in \Gamma$.

This theorem says that the lengths of a sufficiently large finite set of closed geodesics determine the full marked length spectrum approximately, and the approximation improves as the size of the set of known closed geodesics increases. In other words, Question I.4 reduces to Question I.3. In light of this, we now suppose that $\mathcal{L}_g$ and $\mathcal{L}_{g_0}$ approximately agree on all of Γ:

Hypothesis I.7. *There is some small $\tilde{\varepsilon} > 0$ so that*

$$1 - \tilde{\varepsilon} \leq \frac{\mathcal{L}_{g_0}(\gamma)}{\mathcal{L}_g(\gamma)} \leq 1 + \tilde{\varepsilon}$$

for all $\gamma \in \Gamma$.

We prove (M, g) and (N, g_0) are bi-Lipschitz equivalent with constant close to 1 in the case of dimension 2, as well as in higher dimensions when (N, g_0) is locally symmetric. The Lipschitz constants depend only on concrete geometric and topological information about M and N such as the dimension, sectional curvature bounds, and injectivity radii. Moreover, we prove explicit estimates for the Lipschitz constant in terms of $\tilde{\varepsilon}$ in the case where (N, g_0) is a locally symmetric space of dimension at least 3. We only assume that $\mathcal{L}_g$ and $\mathcal{L}_{g_0}$ are close; we do not assume that g and g_0 are close.

I.2.2: Surfaces

To state our result precisely, let $\mathcal{C}(2, \lambda, \Lambda, v_0, D_0)$ consist of all closed C^∞ Riemannian manifolds of dimension 2 with sectional curvatures contained in the interval $[-\Lambda^2, -\lambda^2]$, volume bounded below by v_0, and diameter bounded above by D_0. We show pairs of such spaces become more isometric as their marked length spectra get closer to one another, refining the main result in [Ota90].

Theorem I.8 (Butt, [But22b] Theorem 1.1)**.** *Fix λ, Λ, v_0, $D_0 > 0$. Fix $L > 1$. Then there exists $\tilde{\varepsilon} = \tilde{\varepsilon}(L, \lambda, \Lambda, v_0, D_0) > 0$ small enough so that for any pair $(M, g), (M, h) \in \mathcal{C}(2, \lambda, \Lambda, v_0, D_0)$ satisfying*

$$1 - \tilde{\varepsilon} \leq \frac{\mathcal{L}_g}{\mathcal{L}_h} \leq 1 + \tilde{\varepsilon},$$

there exists an L-Lipschitz diffeomorphism $f : (M, g) \to (M, h)$.

I.2.3: Dimension at least 3, locally symmetric

Consider the case where (N, g_0) is a negatively curved locally symmetric space of dimension at least 3. We quantify how close g and g_0 are to being isometric by estimating the derivative of a map $F : M \to N$ in terms of $\tilde{\varepsilon}$. This is considerably stronger than Theorem I.8, since we are able to determine how the Lipschitz constant depends on $\tilde{\varepsilon}$. This refines the rigidity result in [Ham99, Corollary to Theorem A], which corresponds to the case $\tilde{\varepsilon} = 0$ in the theorem below. As in the previous theorem, we only assume the marked length spectra of the two metrics are close; we do not assume the metrics themselves are close in any C^k topology.

Theorem I.9 (Butt, [But22b] Theorem 1.2)**.** *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$ with fundamental group Γ and sectional curvatures contained in the interval $[-\Lambda^2, 0)$. Let (N, g_0) be a locally symmetric space. Assume there is a homotopy equivalence $f : M \to N$ and let f_* denote the induced map on fundamental groups. Then there exists small enough ε_0 (depending on Γ) so that whenever $\tilde{\varepsilon} \leq \varepsilon_0$ and*

$$1 - \tilde{\varepsilon} \leq \frac{\mathcal{L}_{g_0}(f_*\gamma)}{\mathcal{L}_g(\gamma)} \leq 1 + \tilde{\varepsilon}$$

for all $\gamma \in \Gamma$, there is a C^2 map $F : M \to N$ homotopic to f and constants $c_1(\tilde{\varepsilon}, n, \Gamma, \Lambda) < 1$, $C_2(\tilde{\varepsilon}, n, \Gamma, \Lambda) > 1$ such that for all $v \in TM$ we have

$$c_1 \|v\|_g \leq \|dF(v)\|_{g_0} \leq C_2 \|v\|_g.$$

More precisely, there is a constant $C = C(n, \Gamma, \Lambda)$ so that $c_1 = 1 - C\tilde{\varepsilon}^{1/8(n+1)}$ and $C_2 = 1 + C\tilde{\varepsilon}^{1/8(n+1)}$.

Remark I.10. The conclusion of Theorem I.9 can be restated as $\|g - F^*g_0\|_{C^0} \leq C\varepsilon^{1/8(n+1)}$. The author thanks Thibault Lefeuvre for this remark.

Remark I.11. If $\tilde{N}$ is a real, complex or quaternionic hyperbolic space, we can take $c_1 = 1 - C\varepsilon^{1/4(n+1)}$ and $C_2 = 1 + C\varepsilon^{1/4(n+1)}$. See Remark V.32.

In [Ham99, Theorem A], Hamenstädt proves that two negatively curved manifolds with the same marked length spectrum have the same volume, provided one of the manifolds has geodesic flow with C^1 Anosov splitting, a condition which holds in particular for locally symmetric spaces. (The *Anosov splitting* of the geodesic flow on the unit tangent bundle T^1N refers to the flow-invariant decomposition of TT^1N into the stable, unstable and flow directions; see the introduction to [Ham99].)

Thus, if M and N satisfy the assumptions of Theorem I.9 for $\varepsilon = 0$, they must have the same volume. Then, since the marked length spectrum determines the topological entropy of the geodesic flow, the fact that the two manifolds are isometric follows from the celebrated entropy rigidity theorem of Besson–Courtois–Gallot [BCG96, BCG95].

To prove Theorem I.9, we start by proving an analogue of [Ham99, Theorem A] under the assumption the marked length spectra satisfy equation (I.7), ie, we estimate the ratio $\mathrm{Vol}(M)/\mathrm{Vol}(N)$ in terms of ε. In order to obtain an explicit estimate, we assume the Anosov splitting is $C^{1+\alpha}$ instead of C^1. (For geodesic flows on manifolds with strictly $\frac{1}{4}$-pinched negative curvature, the Anosov splitting is $C^{1+\alpha}$ for some $\alpha > 0$. The splitting is C^1 by work of Hirsch–Pugh [HP75] and $C^{1+\alpha}$ by work of Hasselblatt [Has94, Theorem 5, Remark after Theorem 6].) Unlike in Theorem I.9, the constants here do not depend on (M, g) in any way.

Theorem I.12 (Butt, [But22b] Theorem 1.4)**.** *Let (M, g) be a closed negatively curved Riemannian manifold with fundamental group Γ. Let (N, g_0) be another closed negatively curved manifold with fundamental group Γ and assume the geodesic flow on T^1N has $C^{1+\alpha}$ Anosov splitting. Suppose the marked length spectra of M and N satisfy Hypothesis I.7. Then there is a constant C depending only on $\tilde{N}$ such that*

$$(1 - C\tilde{\varepsilon}^{\alpha})(1 - \tilde{\varepsilon})^n \mathrm{Vol}(M) \leq \mathrm{Vol}(N) \leq (1 + C\tilde{\varepsilon}^{\alpha})(1 + \tilde{\varepsilon})^n \mathrm{Vol}(M).$$

If, in addition, (N, g_0) is locally symmetric and $\tilde{\varepsilon}$ is sufficiently small (depending on $n = \dim N$), then α can be replaced with 2 in the above estimates and the constant C depends only n.

Remark I.13. If the Anosov splitting of T^1N is only C^1, then our proof shows the quantities $(1 \pm C\tilde{\varepsilon}^{\alpha})$ can be replaced with constants that converge to 1 as $\tilde{\varepsilon} \to 0$, but we are not able to determine the explicit dependence of these constants on $\tilde{\varepsilon}$; see the statement above Lemma IV.15.

Remark I.14. If N is locally symmetric, then $\mathrm{Vol}(N) \leq (1+\varepsilon)^n \mathrm{Vol}(M)$ follows from Lemma V.1 and the proof of the main theorem in [BCG96]. (See Remark V.6 for more details.) However, the lower bound for $\mathrm{Vol}(N)/\mathrm{Vol}(M)$ in Theorem I.12 is also crucial for the proof of Theorem I.9.

Remark I.15. If $\dim M = \dim N = 2$, then our proof of Theorem I.12 shows

$$(1-\varepsilon)^2 \mathrm{Vol}(M) \leq \mathrm{Vol}(N) \leq (1+\varepsilon)^2 \mathrm{Vol}(M),$$

which is the optimal estimate. This result also follows from [CD04, Theorem 1.1].

I.3: Structure of this dissertation

In Chapter II, we present background on marked length spectrum rigidity. The finiteness theorem (Theorem I.6) is then proved in Chapter III. In Chapter IV, we prove the volume estimate (Theorem I.12). In Chapter V, we use this volume estimate to prove a quantitative version of the Besson–Courtois–Gallot entropy rigidity theorem [BCG95], which gives our main quantitative rigidity result in the case of locally symmetric spaces of dimension at least 3 (Theorem I.9). Finally, we prove our approximate rigidity result for surfaces (Theorem I.8) in Chapter VI.

CHAPTER II
Preliminaries

In this dissertation, we prove quantitative versions of certain *marked length spectrum rigidity* results due to Otal [Ota90] and Hamenstädt [Ham99]. In this chapter, we present preliminary material on the marked length spectrum rigidity conjecture [BKB+85, Conjecture 3.1] as well as some aspects of Otal and Hamenstädt's proofs. We begin with background on the geometry of negatively curved manifolds and the dynamics of their associated geodesic flows.

II.1: The marked length spectrum

Throughout this thesis, (M, g) denotes a closed negatively curved Riemannian manifold; in other words, M is a compact manifold without boundary and g is a Riemannian metric of negative sectional curvature. In this setting, there is a well-defined way to measure the lengths of closed geodesics as a function of their free homotopy classes. This function is known as the *marked length spectrum* (Definition I.1), and its definition makes use of the following fact.

Lemma II.1. *Let (M, g) be a closed negatively curved Riemannian manifold. Then any closed curved in M is freely homotopic to a unique closed geodesic.*

There are two parts to this statement: existence of the geodesic representative as well as its uniqueness. Existence follows from compactness of M alone, whereas the negative curvature assumption is used to ensure uniqueness.

Proof of existence. We follow the argument in [dC92, Theorem 12.2.2]. Let c be a closed curve in M and let d be the infimum of the lengths of all curves freely homotopic to c. Either c is freely homotopic to a point or $d > 0$. In the latter case, consider a minimizing sequence of curves c_n so that $l_g(c_n) \to d$, where l_g is length measured with respect to the Riemannian metric g. Without loss of generality, we can assume each $c_n : [0, 1] \to M$ is piecewise geodesic, parametrized according to arclength. (To see this, we connect pairs of sufficiently nearby points on c_n by length-minimizing geodesics.) Moreover $L := \sup_n l_g(c_n) < \infty$. As such,

$$d(c_n(t_1), c_n(t_2)) \leq \int_{t_1}^{t_2} |c_n'(t)|\, dt \leq L(t_1 - t_2).$$

This shows the c_n are equicontinuous. Since M is compact, by Arzela–Ascoli there is a convergent subsequence $c_{n_k} \to c_0$. Let c be the piecewise geodesic obtained by connecting sufficiently nearby points on c_0 by locally length-minimizing geodesics. Then $l_g(c) = d$, and the fact that geodesics are locally length-minimizing implies that c is in fact geodesic at all of its points, i.e. has no corners. See [dC92, Theorem 12.2.2] for further details. □

Before proving uniqueness, we recall the required background on the geometry of negatively curved manifolds. First, Hadamard's theorem states that any complete simply connected Riemannian n-manifold of nonpositive sectional curvature is diffeomorphic to an open ball (see, for instance, [dC92, Theorem 7.3.1]). As such, if $\tilde{M}$ denotes the universal cover of the compact negatively curved n-manifold (M, g), then $\tilde{M}$ is diffeomorphic to an open ball of dimension n. Let Γ denote the fundamental group of M. The action of Γ on $\tilde{M}$ by deck transformations is also an action by isometries when $\tilde{M}$ is endowed with the natural Riemannian metric $\tilde{g}$ obtained by lifting g. Let $\gamma \in \Gamma$ and let γ also denote the corresponding isometry of $\tilde{M}$. Then γ leaves invariant a bi-infinite geodesic $\tilde{\gamma} : \mathbb{R} \to \tilde{M}$ (see, for instance, [dC92, Proposition 12.2.6]). These properties do not require the full strength of the Riemann curvature tensor, and can instead be deduced from more general "thin triangle" conditions, i.e. CAT(0). See [BH13].

In addition to the above setup, our proof of uniqueness of the geodesic representative in Lemma II.1 uses *strict convexity* of the distance function in negative curvature. Let $c_1, c_2 : [0,1] \to \tilde{M}$ be two distinct geodesics, and let $f(t)$ denote the function $t \mapsto d_g(c_1(t), c_2(t))$. Then for all $t \in [0,1]$, the following inequality holds:

$$f(t) < tf(0) + (1-t)f(1). \tag{II.1.1}$$

(See [BH13, Proposition II.2.2], which proves $f(t)$ is (not strictly) convex in the case of CAT(0) spaces. The strict inequality follows as soon as the CAT(0) inequality in [BH13, Definition II.1.1] can be replaced with a strict inequality, which is the case for Riemannian manifolds of sectional curvature strictly less than 0.)

Proof of uniqueness. Suppose for the sake of contradiction that there is a free homotopy $c_s(t)$, where $0 \leq s, t \leq 1$, between two distinct closed geodesics $c_0(t)$ and $c_1(t)$ in M. Since $c_0(t)$ and $c_1(t)$ are both closed curves, we have that $c_s(0)$ and $c_s(1)$ coincide for all s. Now let $\tilde{c}_s(t)$ be a lift of this homotopy to the universal cover $p : \tilde{M} \to M$. Since $p(\tilde{c}_s(0)) = p(\tilde{c}_s(1))$

for all s, this means $\tilde{c}_s(0)$ and $\tilde{c}_s(1)$ differ by some deck transformation $\gamma \in \Gamma$. In particular, $\gamma(c_0(0)) = c_0(1)$, and since c_0 is a closed geodesic, the action of γ also takes the tangent vector $c_0'(0)$ to the tangent vector $c_0'(1)$. This means the concatenation of the curves $\gamma^n(\tilde{c}_0(t))$ for $n \in \mathbb{Z}$ is a bi-infinite geodesic in $\tilde{M}$. The same is true for $\gamma^n(\tilde{c}_1(t))$. Since γ acts by isometries, the distance between these parametrized curves is bounded above by the distance between $c_0(t)$ and $c_1(t)$ in the compact quotient M. This boundedness contradicts the strict convexity of the distance function in (II.1.1). Therefore, $c_0(t)$ and $c_1(t)$ must coincide, which proves uniqueness. $\square$

Now that Lemma II.1 is proved, we can formally define the marked length spectrum.

Definition II.2. Given a closed, negatively curved Riemannian manifold (M, g), the *marked length spectrum* $\mathcal{L}_g$ is the function on free homotopy classes of closed curves in M which associates to each class the length of its unique geodesic representative.

Remark II.3. The function $\mathcal{L}_g$ can also be viewed as a function on conjugacy classes in the fundamental group of M, since these are identified with free homotopy classes of closed curves in M.

It is natural to ask to what extent $\mathcal{L}_g$ determines g. Conjecturally, $\mathcal{L}_g$ completely determines g up to isometry.

Conjecture II.4 (Conjecture 3.1 in [BKB+85])**.** *Let M be a closed manifold and g and g_0 be two Riemannian metrics of negative sectional curvature on M. Suppose $\mathcal{L}_g = \mathcal{L}_{g_0}$. Then g is isometric to g_0 (and the isometry preserves the marking, ie, is homotopic to the identity map on M).*

Remark II.5. One can ask the same question if g_0 is instead a negatively curved metric on some other manifold N whose fundamental group is isomorphic to that of M (but M and N are not assumed to be diffeomorphic). In light of Remark II.3, one can still make sense of the hypothesis $\mathcal{L}_g = \mathcal{L}_{g_0}$ in this setting. Then the desired conclusion is that g and g_0 are isometric via a diffeomorphism that induces the initial isomorphism between the fundamental groups of M and N.

II.1.1: Hyperbolic surfaces

We begin by discussing marked length spectrum rigidity (Conjecture II.4 above) in a particular setting, namely when $\dim M = 2$ and g and g_0 are both hyperbolic metrics, that is metrics of constant sectional curvature, say -1. In this case, Conjecture II.4 holds, and in fact, a stronger statement holds: it suffices to verify $\mathcal{L}_g = \mathcal{L}_{g_0}$ on a certain finite set of free homotopy classes to guarantee g and g_0 are isometric. This finiteness is closely related to the fact that the Teichmüller space of all possible hyperbolic metrics (up to isotopy) on a fixed topological surface M has finite dimension (equal to $6\,\mathrm{genus}(M) - 6$). One way to see that Teichmüller space is $(6\,\mathrm{genus}(M) - 6)$-dimensional is using Fenchel–Nielsen coordinates (see, for instance, [FM11, Chapter 10]). To summarize briefly, one can cut a closed surface up along certain closed geodesics so that each component is a *pair of pants* (topologically, a sphere with three punctures). A hyperbolic metric on each pair of pants is determined by the three "cuff lengths". There are $3\,\mathrm{genus}(M) - 3$ total cuff lengths in any pants decomposition of M, but these lengths alone do not suffice to determine the isometry type of M. (They account for exactly half of the $6\,\mathrm{genus}(M) - 6$ Fenchel–Nielsen coordinates.) In addition to these cuff lengths, one needs to keep track of "twist parameters", which dictate how the pants are glued back together to reconstruct the surface, since twisting a cuff before gluing it to another of the same length will change the isometry type of M. It turns out that these twist parameters can be recovered from the lengths of (finitely many) additional closed geodesics (see [FM11, Theorem 10.7]).

Now one can ask if marked length spectrum rigidity generalizes beyond this setting. The pants decomposition method above is not at all applicable in variable curvature. Indeed, one can simply perturb the metric in the interior of a pair of pants without changing any of the cuff lengths. Nevertheless, affirmative answers to Conjecture II.4 have been obtained in the case of (variably curved) surfaces by Otal [Ota90] and Croke [Cro90], in higher dimensions when one of the two metrics is locally symmetric by Hamenstädt [Ham99], and for pairs of metrics which are nearby (with respect to a suitable C^k topology) by Guillarmou–Lefeuvre [GL19]. Their proofs all make use of dynamical properties of the *geodesic flow*, which is the subject of the next section.

II.2: The geodesic flow on the unit tangent bundle of M

Aside from the fact that Conjecture II.4 has been known for quite some time in the case of hyperbolic surfaces, some additional intuition for why it should be true comes from considering the question from the perspective of dynamics. The underlying dynamical system in this context is the *geodesic flow*, which we denote by ϕ^t. This is a flow on the unit tangent bundle T^1M of (M, g). The flow ϕ^t is defined as follows: Given a unit tangent vector v, first consider the unique unit speed geodesic $c(t)$ with initial condition $c'(0) = v$. Now for any $t \in \mathbb{R}$, define $\phi^t v$ to be the unit tangent vector $c'(t)$. A simple but important observation is that *periodic orbits* of ϕ^t correspond precisely to closed geodesics in M.

In our setting, that is, when (M, g) is closed and negatively curved, the geodesic flow is *uniformly hyperbolic*, more commonly known as *Anosov* (Definition II.7 below). This hyperbolicity turns out to reveal a lot of information about the overall orbit structure of the flow (despite the fact that individual trajectories are highly sensitive to small changes in initial conditions). For instance, periodic orbits of Anosov flows are dense (Corollary II.15). In our geometric setting, this means vectors tangent to closed geodesics are dense in T^1M. At the very least, closed geodesics must be dense in M for marked length spectrum rigidity to hold; otherwise, one can simply perturb the metric in a neighborhood that does not intersect any closed geodesics, thereby producing a pair of non-isometric metrics with the same marked length spectrum. There are also stronger results about approximating certain trajectories of Anosov flows with periodic ones (some of which are discussed in detail below). That is to say, from the perspective of hyperbolic dynamics, it is natural to expect periodic orbits of the geodesic flow to provide significant information about the flow. This is some naive intuition behind Conjecture II.4.

For what follows it is convenient to fix a notion of distance on the unit tangent bundle T^1M. We do this using the natural Riemannian metric on T^1M arising from the Riemannian metric g on M.

Definition II.6. (See also [dC92, Exercise 3.2].) Let $v \in T^1M$ and let $V, W \in T_v(T^1M)$. Let $\alpha(t)$ and $\beta(t)$ be curves in T^1M such that $\alpha'(0) = V$ and $\beta'(0) = W$. Let $\pi : T^1M \to M$ denote the footpoint map. The curve $\alpha(t)$ is a curve $\pi(\alpha(t))$ in M together with a (unit) vector field along that curve, and analogously for $\beta(t)$. The *Sasaki metric* g^S is the following inner product on T_vT^1M:

$$g^S(V, W) := g(d\pi(V), d\pi(W))_{\pi(v)} + g\left(\frac{D\alpha}{dt}, \frac{D\beta}{dt}\right),$$

where $\frac{D}{dt}$ denotes covariant differentiation of a vector field along a curve (see [dC92, Proposition 2.2].)

This Riemannian metric gives rise to a distance function on T^1M which we will denote by d in the rest of this section.

II.2.1: Hyperbolicity

Definition II.7. (See [FH19, Definition 5.1.1].) A C^1 flow ϕ^t on a closed connected smooth manifold X is called *Anosov* if there is a continuous $D\phi^t$-invariant splitting of the tangent bundle $TX = E^c \oplus E^s \oplus E^u$ and constants $C \geq 1$, $\lambda \in (0, 1)$, $\mu \geq 1$ so that

- $E^c(x) := \mathbb{R}\frac{d}{dt}\phi^t(x) \neq 0$ for all $x \in X$,

- $\|D\phi^t(v)\| \leq C\lambda^t\|v\|$ for all $v \in E^s$,
- $\|D\phi^{-t}(v)\| \leq C\mu^{-t}\|v\|$ for all $v \in E^u$.

The subbundles E^s and E^u are referred to as the *stable* and *unstable distributions*, respectively.

Remark II.8. The Anosov property does not depend on the choice of norm on X. Since X is assumed to be compact, changing the norm will change the constants in the above definition, but not the fact that vectors in E^s and E^u are exponentially contracted by forward and backward iteration of the flow, respectively.

When X is the unit tangent bundle T^1M of a closed negatively curved manifold (M, g) and ϕ^t is the associated geodesic flow, then ϕ^t is, in fact, Anosov. See, for instance, [KH97, Section 17.6]. We omit the proof, but in order to provide some insight as to what the stable and unstable subbundles E^s and E^u are, we will describe the (strong) stable and (strong) unstable *foliations* $W^{ss}, W^{su} \subset T^1M$. The leaves of these foliations are tangent to the stable and unstable distributions E^s and E^u, respectively. For this we begin by defining the (visual) boundary at infinity of the universal cover $\tilde{M}$. See [BH13, Chapter II.8] for further details.

Definition II.9. (See [BH13, Definition II.8.1].) The *boundary at infinity of* $\tilde{M}$, denoted $\partial\tilde{M}$, is the set of asymptotic classes of geodesic rays, where two rays $c_0, c_1 : [0, \infty) \to \tilde{M}$ are said to be asymptotic (and in the same equivalence class) if the distance $d_g(c_0(t), c_1(t))$ is bounded for all $t \in [0, \infty)$.

Remark II.10. There is a topology on $\partial\tilde{M}$ arising from identifying it with the unit tangent space $T^1_p\tilde{M}$ for some fixed $p \in \tilde{M}$, and with respect to this topology, $\partial\tilde{M}$ is thus homeomorphic to the $(n-1)$-dimensional sphere, where $n = \dim M$. Indeed, any $v \in T^1_p\tilde{M}$ determines a geodesic ray $c(t)$ with $c'(0) = v$. It follows from [BH13, Proposition II.8.2] that this correspondence is a bijection. It follows from [BH13, Proposition II.8.8] that the induced topology on $\partial\tilde{M}$ is independent of the choice of basepoint p.

Remark II.11. In negative curvature, strict convexity of the distance function (see (II.1.1)) implies any two distinct points in $\partial\tilde{M}$ can be joined by a unique geodesic.

Let $W^s(v)$ denote the *weak stable* set of v, that is, $W^s(v) = \cup_{t\in\mathbb{R}} W^{ss}(\phi^t v)$. These are all points in T^1M which have the "same infinite future" as v. In other words, these are all vectors in T^1M whose associated geodesic rays (in the universal cover $\tilde{M}$) are asymptotic to that of v. Analogously, the *weak unstable* set of v is all unit tangent vectors whose associated geodesic rays are asymptotic to $-v$ in backward time. Before describing the (strong) stable and (strong) unstable manifolds $W^{ss}(v)$ and $W^{su}(v)$ for the geodesic flow on T^1M, we recall the notions of Busemann functions and horospheres.

Definition II.12. (See [BH13, Definition 8.17].) Let $p \in \tilde{M}$, $\xi \in \partial\tilde{M}$, and let $c(t)$ be the unique geodesic ray with $c(0) = p$ and $c(\infty) = \xi$. The function

$$B_{\xi,p}(q) = \lim_{t\to\infty}(d(q, c(t)) - t)$$

is called a *Busemann function*. Level sets of Busemann functions are called *horospheres*.

Remark II.13. The zero set of $\{B_{\xi,p} = 0\}$ can be thought of as a sphere tangent to the boundary $\partial\tilde{M}$ at ξ which passes through p. Other level sets $\{B_{\xi,p} = r\}$ are spheres tangent to the boundary at ξ whose (signed) distance from the zero set $\{B_{\xi,p} = 0\}$ is equal to r.

The stable and unstable manifolds W^{ss} and W^{su} for the geodesic flow have the following geometric description (see, for instance, [Bal95, p. 72]). Let $v \in T^1\tilde{M}$. Let $p \in \tilde{M}$ be the footpoint of v and let $\xi \in \partial\tilde{M}$ be the forward projection of $v \in T^1\tilde{M}$ to the boundary. Let $B_{\xi,p}$ denote the Busemann function on $\tilde{M}$ and let $H_{\xi,p}$ denote its zero set. Then the lift of $W^{ss}(v)$ to $T^1\tilde{M}$ is given by

$$\{-\mathrm{grad} B_{\xi,p}(q) \,|\, q \in H_{\xi,p}\}.$$

In other words, these are vectors normal to the horosphere $H_{\xi,p}$ which are pointing towards ξ. If η denotes the projection of $-v$ to the boundary $\partial\tilde{M}$, then the lift of $W^{su}(v)$ to $T^1\tilde{M}$ is analogously given by

$$\{\mathrm{grad} B_{\eta,p}(q) \,|\, q \in H_{\eta,p}\}.$$

These are vectors orthogonal to $H_{\eta,p}$ and whose negatives point towards η.

Such a family of vectors gives rise to a geodesic variation, and the verification that the geodesic flow on T^1M is Anosov boils down to considerations about the Jacobi fields associated to these variations. More precisely, let $c(s)$ denote a curve in the horosphere $H_{\eta,p}$ and let $V(s) := \mathrm{grad} B_{\eta,p}(c(s))$ be a vector field along this curve. Consider the geodesic variation $f(s,t) = \exp_{c(s)}(tV(s))$, where exp denotes the Riemannian exponential map. Then $J(t) := \frac{\partial}{\partial s}|_{s=0} f(s,t)$ is a *Jacobi field*, and verifying the exponential contraction property in Definition II.7 amounts to obtaining estimates of the form $\|J(t)\|, \|J'(t)\| \leq Ce^{-\lambda t}$. This is achieved by comparing with the constant curvature setting (using the Rauch comparison theorem), where the Jacobi equation can be solved explicitly. As such, one can find constants C, λ, μ as in Definition II.7 which depend on the sectional curvature bounds of M in an explicit way (in the case where the norm in Definition II.7 is taken to be the norm arising from the Sasaki metric defined in Definition II.6). For further details see [Bal95, Proposition IV.1.13 and Proposition IV.2.15].

II.2.2: Periodic orbits

Anosov flows are often described as *chaotic*, since a slight change of initial condition (in the unstable direction) causes exponential divergence of trajectories. Nevertheless, we have the following strong result about approximating certain trajectories with periodic ones. See, for instance, [FH19, Theorem 5.3.10].

Lemma II.14 (Anosov Closing Lemma)**.** *There is $\delta_0 > 0$ sufficiently small, T sufficiently large, and a constant $C > 0$ so that the following holds for all $\delta \leq \delta_0$ and all $t \geq T$. Suppose $v, \phi^t v \in T^1M$ so that $d(v, \phi^t v) < \delta$. Then either v and $\phi^t v$ are on the same local flow line or there is w with $d(v, w) < C\delta$ so that w is tangent to a closed geodesic of length $t' \in [t - C\delta, t + C\delta]$.*

Corollary II.15. *When M is closed and negatively curved, periodic orbits of the geodesic flow ϕ^t on T^1M are dense in T^1M.*

Proof. First note that ϕ^t preserves the Liouville measure μ, which is the measure arising from the Riemannian volume form on T^1M associated to the Sasaki metric g^S from Definition II.6 (see [Bur83], and Section II.3 below). By the Poincaré Recurrence Theorem [KH97, Theorem 4.1.19], μ-almost every v satisfies: for all $\delta > 0$ there exists sufficiently large $t = t(v, \delta)$ so that $d(v, \phi^t v) < \delta$. By the Anosov Closing Lemma, there is w so that $d(v, w) < C\delta$ and the orbit of w is periodic. □

The point w in the conclusion of the lemma has to be chosen carefully, in light of the fact that ϕ^t is chaotic. The mechanism which allows for this is *local product structure*, which we define below. Any Anosov flow has local product structure [FH19, Proposition 6.2.2], but our exposition will focus on our particular geometric setting of geodesic flows. First we explain the requisite notion of stable and unstable distances in this context. Let $v \in T^1M$ and $w \in W^{ss}(v)$. Let p and q denote the footpoints of v and w respectively. Define the stable distance $d_{ss}(v, w)$ to be the horospherical distance $h(p, q)$, i.e., the distance obtained from restricting the Riemannian metric g on $\tilde{M}$ to a given horosphere. The unstable distance is defined analogously.

Definition II.16. (See also [FH19, Proposition 6.2.2].) We say the flow ϕ^t on T^1M has *local product structure* if every point $v \in T^1M$ has a neighborhood V which satisfies: for all $\varepsilon > 0$, there is $\delta > 0$ so that whenever $x, y \in V$ with $d(x, y) \leq \delta$ there is a point $[x, y] \in V$ and a time $|\sigma(x, y)| < \varepsilon$ such that

$$[x, y] = W^{ss}(x) \cap W^{su}(\phi^{\sigma(x,y)} y).$$

Moreover, there is a constant $C_0 = C_0(\delta)$ so that $d(x, y) < \delta$ implies

$$d_{ss}(x, [x, y]), d_{su}(\phi^{\sigma(x,y)}[x, y], y) \leq C_0 d(x, y),$$

where d_{ss} and d_{su} denote the distances along the strong stable and strong unstable manifolds, respectively, and d denotes the distance coming from the Sasaki metric in Definition II.6.

The bracket $[x, y]$ can be thought of as a point with the same infinite future as x and the same infinite past as y. The proof of [FH19, Proposition 6.2.2] shows that all Anosov flows satisfy the above local product structure properties. However, for geodesic flows, the first part of the statement (about the existence of $[x, y]$ and $\sigma(x, y)$) can be deduced directly from the geometric description of W^{ss} and W^{su} in terms of normal fields to horospheres. In fact, the neighborhoods V in the above definition can be taken to be "very large" subsets of T^1M.

To see this, let $x, y \in T^1M$ and let $\tilde{x}$ and $\tilde{y}$ be lifts to $T^1\tilde{M}$. Let $\xi := \pi(\tilde{x})$ and $\eta := \pi(-\tilde{y})$ denote the forward projections to $\partial\tilde{M}$. The bracket $[x, y]$ is defined whenever ξ and η are distinct. If this is indeed the case, then the points η and ξ can be joined by a unique (bi-infinite) geodesic $c(t)$ (see Remark II.11). Note that $c(t)$ passes through the horosphere associated to $W^{ss}(x)$ as well as the one associated to $W^{su}(y)$. The bracket $[x, y]$ is simply the tangent vector to $c(t)$ based at the point at which $c(t)$ intersects the horosphere associated to $W^{ss}(x)$ (see, for instance, [Cou04]). The number $\sigma(x, y)$ is the distance between the horospheres $W^{ss}(x)$ and $W^{su}(x)$. Thus $|\sigma(x, y)| \leq d(x, y)$, which shows $\varepsilon = \varepsilon(\delta)$ can be taken to equal δ in the above definition.

We now explain how to deduce the Anosov Closing Lemma (Lemma II.14) from local product structure. We will use the fact that for $y \in W^{ss}(x)$, the stable distance $d_{ss}(\phi^t x, \phi^t y)$ is exponentially decaying in t, uniformly in x and y (and exponentially growing in t when $y \in W^{su}(x)$). This follows from the definition of an Anosov flow (Definition II.7). For the geodesic flow on a closed manifold M with sectional curvatures bounded above by $-a^2$, we have the more precise statement

$$d_{si}(\phi^t x, \phi^t y) \leq e^{-at} d_{si}(x, y) \tag{II.2.1}$$

for $i = s, u$ [HIH77, Proposition 4.1].

Proof of Lemma II.14 (Anosov Closing Lemma). We use the approach outlined in [Fra18, Figure 2] (see also [Bow75, 3.6, 3.8]). Fix δ_0 so that $d(x, y) < \delta_0$ implies $[x, y]$ is defined (see Definition II.16). Let C_0 be the constant in the second part of Definition II.16. Choose T large enough such that $\alpha := 2C_0 e^{-aT/2} < 1$, where a is as in (II.2.1).

Now let v_0 and $t \geq T$ such that $d_0 := d(v_0, \phi^t v_0) < \delta \leq \delta_0$. If $\phi^t v$ and v are not on the same local flow line, let $v_1 = [v_0, \phi^t v_0]$. Let $\sigma_0 = \sigma(v_0, \phi^t v_0) \leq d_0$. Let $t_0 = t + \sigma_0$ and note that $v_1 = W^{ss}(v_0) \cap W^{su}(\phi^{t_0} v_0)$. Let $d_1 := d(\phi^{-t_0/2} v_1, \phi^{t_0/2} v_1)$. We proceed to estimate d_1 in terms of d_0. By the second part of Definition II.16, we have $d_{ss}(\phi^{t_0} v_0, v_1) < C_0 d_0$ By (II.2.1), we have $d_{ss}(\phi^{t_0/2} v_0, \phi^{-t/2} v_1) < e^{-a t_0/2} C_0 d_0$. Similarly, $d_{su}(\phi^{t_0/2} v_0, \phi^{t_0/2} v_1) < e^{-\lambda t_0} C_0 d_0$. The triangle inequality, together with the fact that $d \leq d_{si}$ for $i = s, u$, gives $d(\phi^{t_0/2} v_1, \phi^{-t_0/2} v_1) \leq 2e^{-\lambda t_0} C_0 d_0 \leq \alpha d_0$, where $\alpha < 1$ by the choice of T in the first paragraph.

Now let $v_2 = [\phi^{-t_0/2} v_1, \phi^{t_0/2} v_1]$. Let t_1 such that $v_2 = W^{ss}(\phi^{-t_1/2} v_1) \cap W^{su}(\phi^{t_1/2} v_1)$. Then the above argument shows $d_2 := d(\phi^{-t_1/2} v_2, \phi^{t_1/2} v_2) \leq \alpha d_1$. Iterating the above procedure, define a sequence of points v_i, times t_i and distances $d_i := d(\phi^{-t_{i-1}/2} v_i, \phi^{t_{i-1}/2} v_i)$. Then by the same argument as before, $d_{i+1} \leq \alpha d_i$ and $|t_{i+1} - t_i| \leq d_i$. By compactness, the sequence v_i subconverges to some point w. Since $d_i \to 0$, the orbit of w is periodic. The length of the orbit of w differs from the length t of the original almost periodic orbit of v_0 by at most $\sum_{i=0}^{\infty} d_i \leq d_0 \alpha/(1-\alpha) \leq C\delta$, where $C = \alpha/(1-\alpha)$. □

As mentioned above, the Anosov Closing Lemma suggests that knowledge of the marked length spectrum should provide significant information about the underlying geodesic flow. This intuition can be formalized as follows.

Proposition II.17. *Let (M, g) and (N, g_0) be a pair of homotopy-equivalent closed negatively curved manifolds such that their marked length spectra $\mathcal{L}_g$ and $\mathcal{L}_{g_0}$ coincide. Let ϕ^t and ψ^t denote the associated geodesic flows on the unit tangent bundles T^1M and T^1N, respectively. Then the flows ϕ^t and ψ^t are* conjugate, *that is, there is a homeomorphism $\mathcal{F} : T^1M \to T^1N$ so that*

$$\mathcal{F}(\phi^t v) = \psi^t \mathcal{F}(v)$$

for all $v \in T^1M$.

Proof. Without assuming anything about $\mathcal{L}_g$ and $\mathcal{L}_{g_0}$, that is, whenever (M, g) and (N, g_0) are homotopy-equivalent closed negatively curved manifolds, the associated geodesic flows are *orbit-equivalent*. This means there is a homeomorphism $\mathcal{F} : T^1M \to T^1N$ such that

$$\mathcal{F}(\phi^t v) = \psi^{b(t,v)} \mathcal{F}(v)$$

for some function $b(t, v)$ on $\mathbb{R} \times T^1M$. See [Gro00]. When the additional condition $\mathcal{L}_g = \mathcal{L}_{g_0}$ holds, the Livsic theorem [KH97, Theorem 19.2.1] allows one to upgrade the orbit equivalence to a conjugacy. See [KH97, Section 2.2]. □

Remark II.18. The proof of the Livisic theorem, in turn, relies crucially on the Anosov Closing Lemma. See [KH97, Theorem 19.2.1].

Proposition II.17 shows a certain dynamical equivalence between the geodesic flows ϕ^t and ψ^t. Indeed, a conjugacy of flows preserves many dynamically defined invariants and structures associated to the flows (for instance, the topological entropy and the stable and unstable manifolds). However, the map $\mathcal{F} : T^1M \to T^1N$ does not provide any immediate information about the underlying Riemannian metrics g and g_0 on M and N. Showing g and g_0 are isometric entails finding a map from M to N, and it is not clear the conjugacy $\mathcal{F} : T^1M \to T^1N$ between unit tangent bundles descends to the base manifolds. (Another difficulty to note is that the the proof of Proposition II.17 only shows $\mathcal{F}$ is C^α, ie, $\mathcal{F}$ is not necessarily differentiable.)

Proving marked length spectrum rigidity (Conjecture II.4) thus requires more sophisticated considerations of the geometry and dynamics of the geodesic flow. In the remainder of this chapter, we explain some of the further tools used in Otal and Hamenstädt's partial solutions to this conjecture. We also discuss Gromov compactness, an additional tool used to prove our quantitative marked length spectrum rigidity results.

II.3: The Liouville measure and the Liouville current

One key idea in Otal and Hamenstädt's proofs of marked length spectrum rigidity is that the conjugacy of geodesic flows given in Proposition II.17 preserves a certain natural geometric measure on the unit tangent bundle, called the *Liouville measure*. Note that for the present discussion we do not require (M, g) to be negatively curved.

The Liouville measure has several equivalent descriptions. One way to define the Liouville measure is the measure associated to the Riemannian volume from on T^1M induced by the Sasaki metric g^S in Definition II.6. It is straightforward to verify that this measure is locally the product of the Riemannian volume measure on the base M and Lebesgue measure on the sphere S^{n-1} in the fiber. A less straightforward, though standard, fact is that this measure is ϕ^t-invariant, where ϕ^t is the geodesic flow.

To see this, we introduce another description of the Liouville measure, which comes from a contact structure on the unit tangent bundle. We begin with the tautological 1-form α on the cotangent bundle T^*M. The idea behind its construction is that a 1-form on T^*M is a way of associating to each $v^* \in T^*M$ a linear functional on $T_{v*}(T^*M)$, but each element of T^*M is already a linear functional on the tangent space of M. Let $\pi_{T^*M} : T^*M \to M$ denote the bundle projection. Then set $\alpha(v^*) := v^* \circ d\pi_{v^*}$. It is a standard fact that the 2-form $d\alpha$ is a symplectic form on T^*M, that is, $(d\alpha)^n$ is a volume form on T^*M, where $n = \dim M$.

Now a Riemannian metric g on M gives rise to an identification between the tangent bundle TM and the cotangent bundle T^*M. Let $G : TM \to T^*M$ be given by $v \mapsto g(v, \cdot)$.

Let ω denote the pullback $G^*\alpha$. Since pullback commutes with exterior differentiation, $d\omega$ is a symplectic form on TM, that is, $(d\omega)^n$ is non-degenerate on TM.

Lemma II.19. *Let ω be the 1-form on TM defined as above. That is, $\omega = G^*\alpha$, where α is the tautological one-form on T^*M and $G : TM \to T^*M$ is the identification arising from the Riemannian metric g on M. Then ω restricted to T^1M is a* contact form*, that is, the form $\omega \wedge (d\omega)^{n-1}$ is non-degenerate.*

Proof. Let X be the vector field on T^1M which generates the geodesic flow ϕ^t. We claim $\omega(X) \equiv 1$. To see this, we start by writing

$$\begin{aligned}
\omega_v(\xi) &:= (G^*\alpha)_v(\xi) \\
&= \alpha_{G(v)}(dG(\xi)) && \text{(definition of pullback)} \\
&= G(v) \circ d\pi_{T^*M}(dG(\xi)) && \text{(definition of } \alpha) \\
&= G(v) \circ d(\pi_{T^*M} \circ G)(\xi) && \text{(chain rule)} \\
&= g(v, d\pi_{TM}(\xi)).
\end{aligned}$$

Let $v \in T^1M$. Then $X(v) = \frac{d}{dt}|_{t=0}\phi^t v$. Then $d\pi_{TM}(X) = \frac{d}{dt}|_{t=0}\pi_{TM}(\phi^t v) = v$. Hence $\omega_v(X) = g(v, v) = 1$ in this case. This, together with the fact that $d\omega$ is symplectic on TM, shows that ω is contact. □

This lemma shows that $\omega \wedge (d\omega)^{n-1}$ is a volume form on T^1M. See [Bur83, 1.E] for a proof that this volume form coincides with the Sasaki volume form (up to a factor of $(n-1)!$). In other words the Liouville measure is (up to a constant multiple) the measure arising from $\omega \wedge (d\omega)^{n-1}$. This latter description of the Liouville measure is more readily seen to be invariant under the geodesic flow.

Lemma II.20. *The Liouville measure is ϕ^t-invariant.*

Proof. It suffices to check that $(\phi^t)^*\omega = \omega$, or that $\frac{d}{dt}(\phi^t)^*\omega = 0$. This latter expression is the Lie derivative $\mathcal{L}_X\omega$, where X is, as usual, the vector field generating ϕ^t. By Cartan's magic formula, we thus have

$$\frac{d}{dt}(\phi^t)^*\omega = d\iota_X\omega + \iota_X d\omega,$$

where ι_X denotes the operation of contracting a differential form along the vector field X.

The first term $d\iota_X$ is easily seen to be zero in light of the fact that $\iota_X = \alpha(X) \equiv 1$ (see the proof of Lemma II.19 above). To see that $\iota_X d\omega = 0$, we use that

$$d\omega(V, W) = g\left(d\pi(V), \frac{D\beta}{dt}\right) - g\left(d\pi(W), \frac{D\alpha}{dt}\right),$$

where $V, W \in T_v T^1 M$ are tangent to curves $\alpha(t)$ and $\beta(t)$, respectively, as in Definition II.6. (See, for instance, [Bur83, Proposition 1.3] for a proof).

Now set $V = X$, where X is the vector field generating the geodesic flow. Then $\frac{D\alpha}{dt} = 0$ because α is the tangent vector field along a geodesic, so the second term above is 0. The first term becomes $g(d\pi(X), \nabla_{b'(0)}\beta(0))$, where $b(t) = \pi(\beta(t))$. Since X is tangent to geodesic flow lines in $T^1 M$, we have that $d\pi(X) = \alpha(0)$. But $\alpha(0) = \beta(0) = v$, so

$$g\left(d\pi(X), \frac{D\beta}{dt}\right) = g(v, \nabla_{b'(0)} v) = \frac{1}{2}\frac{d}{dt}\Big|_{t=0} g_{b(t)}(v, v) = 0,$$

since $g(v, v) = 1$ for all $v \in T^1 M$. □

We now define the Liouville current on the space of geodesics of $\tilde{M}$.

Definition II.21. The *space of geodesics of* $\tilde{M}$, denoted $\mathcal{G}\tilde{M}$, is the quotient of $T^1\tilde{M}$ by the equivalence relation $v \sim \phi^t v$ for all $t \in \mathbb{R}$.

Remark II.22. When M has negative curvature, the space $\mathcal{G}\tilde{M}$ is identified with the space $\partial^2\tilde{M}$ of pairs of distinct points in the boundary $\partial\tilde{M}$.

Since the 2-form $d\omega$ on $T^1 M$ is ϕ^t-invariant, it descends to a 2-form on $\partial\tilde{M}$, where it is a symplectic form.

Definition II.23. The *Liouville current* is the measure on $\mathcal{G}\tilde{M}$ arising from the volume form $(d\omega)^{n-1}$.

A key step in both Otal and Hamenstädt's proofs of marked length spectrum rigidity is showing that the conjugacy in Proposition II.17 preserves the Liouville current. This is discussed in greater detail in Sections VI.2 and IV.2, respectively.

II.4: The BCG map

Hamenstädt's proof of marked length spectrum rigidity, in the case where M and N have dimension at least 3 and one the two metrics, say (N, g_0), is locally symmetric, crucially uses the entropy rigidity theorem of Besson–Courtois–Gallot [BCG95]. In fact, the following special case of the theorem is sufficient in this setting.

Theorem II.24. *(See [BCG96, Theorem 1.1].) Suppose (N, g_0) is negatively curved, locally symmetric, and of dimension at least 3 and (M, g) is negatively curved and homotopy-equivalent to N. Suppose the total volumes of M and N agree and the topological entropies $h(g)$ and $h(g_0)$ of their associated geodesic flows are equal as well. Then there is an isometry $F : M \to N$.*

Hamenstädt proves that if $\mathcal{L}_g = \mathcal{L}_{g_0}$ then $\mathrm{Vol}(M, g) = \mathrm{Vol}(N, g_0)$, essentially by showing the conjugacy in Proposition II.17 preserves the Liouville measure described in the previous section. The conjugacy of geodesic flows immediately implies $h(g) = h(g_0)$, from which it follows that M and N are isometric.

In this section, we explain the construction of the *BCG map*, that is, the isometry $F : M \to N$ constructed in [BCG96]. This is also the map in the conclusion of Theorem I.9, our quantitative version of marked length spectrum rigidity in this setting. In Chapter V, we show that if the volumes and entropies of (M, g) and (N, g_0) agree only approximately, then the BCG map is "almost an isometry", in the sense that it is bi-Lipschitz with Lipschitz constants close to 1.

The construction of the BCG map uses three main ingredients:

1. The family of Patterson–Sullivan measures $\{\mu_p\}_{p \in \tilde{M}}$ on the boundary at infinity $\partial\tilde{M}$ (the boundary at infinity was defined in Definition II.9),
2. A map $\overline{f} : \partial\tilde{M} \to \partial\tilde{N}$, induced by the assumed identification between the fundamental groups of M and N,
3. The barycenter of a measure on the boundary at infinity.

These are combined to obtain a Γ-equivariant map $F : \tilde{M} \to \tilde{N}$ as follows. Let $\mathcal{M}(\partial\tilde{M})$ denote the space of probability measures on $\partial\tilde{M}$. The *BCG map* is the Γ-equivariant map $F : \tilde{M} \to \tilde{N}$ given by

$$\begin{array}{ccccccc} \tilde{M} & \to & \mathcal{M}(\partial\tilde{M}) & \to & \mathcal{M}(\partial\tilde{N}) & \to & \tilde{N} \\ p & \mapsto & \mu_p & \mapsto & \overline{f}_*\mu_p & \mapsto & \mathrm{bar}(\overline{f}_*\mu_p). \end{array}$$

where $\mathrm{bar}(\overline{f}_*\mu_p) \in \tilde{N}$ denotes the barycenter of the measure $\overline{f}_*\mu_p$.

We now briefly recall what each of these ingredients are. The construction of the BCG map is more thoroughly explained in the survey article [BCG96]. See also [Fer96].

II.4.1: The Patterson–Sullivan measure

As usual, let Γ denote the fundamental group of the compact negatively curved manifold (M, g). We construct a family of measures on the boundary $\partial\tilde{M}$ indexed by points $p \in \tilde{M}$.

This is a standard construction known as the *Patterson–Sullivan measure* or a *conformal density* in the literature. See [Pat, Sul79, Rob03].

The construction of the Patterson–Sullivan measure begins with the *Poincaré series* for Γ, which is given by

$$P(s,x,y) = \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma.y)}$$

for $x, y \in \tilde{M}$ and $s \in \mathbb{R}$.

Definition II.25. The *critical exponent* of Γ is the number δ_Γ such that the Poincaré series $P(s,x,y)$ converges for $s > \delta_\Gamma$ and diverges for $s < \delta_\Gamma$. By the triangle inequality, it is clear that δ_Γ depends only on Γ, and not on x and y.

The following well-known lemma explains how the topological entropy of the geodesic flow of (M,g) is related to the BCG map. We follow the argument in [Fer96, Lemma 4.1].

Lemma II.26. *The critical exponent of Γ is the topological entropy $h(g)$.*

Proof. Let A_0 denote the closed ball of radius $1/2$ centered at p, and for any positive integer k, let A_k denote the annulus $B(p,k+1/2) \setminus B(p,k-1/2)$. Let $\Gamma_k = \{\gamma \in \Gamma \,|\, \gamma.p \in A_k\}$. Then we can write

$$\sum_{\gamma\in\Gamma} e^{-s(d(p,\gamma.p)} = \sum_{k=0}^{\infty}\sum_{\gamma\in\Gamma_k} e^{-sd(p,\gamma.p)},$$

and for each $\gamma \in \Gamma_k$, the quantity $d(p,\gamma.p)$ is within $1/2$ of k. Thus, the above Poincaré series is proportional to $\sum_{k=0}^{\infty} S_k e^{-sk}$, where S_k is the cardinality of Γ_k. Hence, the critical exponent equal to $\limsup_{k\to\infty} \frac{\log(S_k)}{k}$. Since Γ is cocompact, there are constants c_1 and c_2, depending on the diameter, injectivity radius, and sectional curvature bounds of the quotient $\tilde{M}/\Gamma$, so that $c_1\mathrm{vol}(A_k) \le S_k \le c_2\mathrm{vol}(A_k)$.

Next, we claim that the exponential growth rate of $\mathrm{vol}(A_k)$ is equal to the exponential growth rate of the volume of a ball of radius k. Indeed,

$$\mathrm{vol}(A_k) = \mathrm{vol}(B(p,k+1/2))\left(1 - \frac{\mathrm{vol}(B(p,k-1/2))}{\mathrm{vol}(B(p,k-1/2))}\right).$$

The proof in [Man79, p.568] shows that

$$\frac{\mathrm{vol}(B(p,k-1/2))}{\mathrm{vol}(B(p,k-1/2))} \ge \frac{1}{\mathrm{vol}(B(p,A))},$$

where A is a constant depending on (M,g), but independent of k. This shows that $\mathrm{vol}(A_k)$ and $\mathrm{vol}(B(p,k+1/2))$ have the same exponential growth rate in k. So the critical exponent of Γ is the volume growth entropy of $(\tilde{M}, g)$. By [Man79, Theorem 2], this is equal to the topological entropy of the geodesic flow. $\square$

We now outline the construction of the Patterson–Sullivan measure, following [Kni97]. Fix $x_0 \in \tilde{M}$ and let $\mathcal{D}_{x_0}$ denote the Dirac measure at x_0. For any $s > \delta_\Gamma$ and any $p \in \tilde{M}$, let

$$\mu_p^s = \mu_{p,y}^s = \frac{\sum_{\gamma \in \Gamma} e^{-sd(p,\gamma.x_0)} \mathcal{D}_{\gamma.x_0}}{P(s, x_0, x_0)}.$$

Since $|d(p, \gamma.x_0) - d(x_0, \gamma.x_0)| \leq d(p, x_0)$ for all γ, it follows that the total mass of μ_p^s is bounded between $e^{\pm sd(p,x_0)}$. Thus, we can take a weak limit of μ_p^s as $s \to \delta_\Gamma$. (For our purposes, it does not matter whether or not such a weak limit is unique, ie, independent of the choice of subsequence s_k converging to δ_Γ.) In our setting (Γ cocompact), the series $P(s, x_0, x_0)$ diverges for $s = \delta_\Gamma$ [Kni97, Corollary 5.2], which means the limiting measure μ_p is supported on the boundary $\partial\tilde{M}$. Thus we obtain a family of measures μ_p on $\partial\tilde{M}$ indexed by $p \in \tilde{M}$. One can check that this family satisfies the following properties (see, for instance, [Sul79]):

1. For all $p, q \in \tilde{M}$ the Radon–Nikodym derivatives satisfy $\frac{d\mu_p}{d\mu_q} = \exp(-\delta_\Gamma) B_{\xi,p}(q)$.

2. For all $p \in \tilde{M}$ and $\gamma \in \Gamma$ the pushforward measures satisfy $\gamma_* \mu_p = \mu_{\gamma.p}$.

II.4.2: The boundary map

In order to compare the marked length spectra $\mathcal{L}_g$ and $\mathcal{L}_{g_0}$ for Riemannian manifolds (M, g) and (N, g_0), an identification between the fundamental groups $\pi_1(M)$ and $\pi_1(N)$ is required, and as such, we always assume that we have one. Since M and N are both $K(\pi, 1)$ spaces (their universal covers are contractible by Hadamard's theorem), it follows that there is a homotopy equivalence $f : M \to N$, where f induces the starting identification of fundamental groups.

Let $\partial\tilde{M}$ denote the visual boundary of $\tilde{M}$ and Γ denote the fundamental group of M. We construct a map $\overline{f} : \partial\tilde{M} \to \partial\tilde{N}$ such that for all $\gamma \in \Gamma$ and all $\xi \in \partial\tilde{M}$ we have $\overline{f}(\gamma.\xi) = (f_*\gamma).\overline{f}(\xi)$. To do so, we first lift the homotopy equivalence $f : M \to N$ to a Γ-equivariant map $\tilde{f} : \tilde{M} \to \tilde{N}$. Since M and N are compact, it follows that $\tilde{f}$ is additionally a *quasi-isometry* (we explain this argument in detail in Section III.4). Hence $\tilde{f}$ induces a Γ-equivariant map (homeomoprhism) $\overline{f}$ between the boundaries $\partial\tilde{M}$ and $\partial\tilde{N}$ (see, for instance, [BH13, Theorem III.H.3.9].)

II.4.3: The barycenter

We follow the discussion in [BCG96, Section 3]. Fix $y_0 \in \tilde{N}$ and let λ be a measure on $\partial\tilde{N}$. Consider the function

$$\mathcal{B}(y) = \int_{\partial\tilde{N}} B_\xi(y, y_0) d\lambda(\xi).$$

This can be seen as measuring the average distance from y to the boundary. When $\mathcal{B}$ uniquely achieves a minimum at the point $y' \in \tilde{N}$, we call y' the *barycenter* of the measure λ. (Note that changing y_0 changes the function $\mathcal{B}$ by a constant, and does not affect the location of the barycenter.)

To define the BCG map, we want to show the barycenter is defined for the measure $\lambda = \overline{f}_*\mu_p$ on $\partial\tilde{N}$. Here, μ_p is the Patterson–Sullivan measure on $\partial\tilde{M}$, the boundary map $\overline{f} : \partial\tilde{M} \to \partial\tilde{N}$ is induced by the assumed homotopy equivalence between M and N, and finally, $\tilde{N}$ is a negatively curved symmetric space.

Put briefly, to see that $\mathcal{B}$ uniquely attains a minimum, one needs to check that $\mathcal{B}(x) \to \infty$ as x goes to infinity along a geodesic and that $\mathcal{B}(x)$ is strictly convex. The details can be found in [BCG95, Appendix A]. We explain the argument for strict convexity assuming N is locally symmetric. We start by differentiating under the integral to get

$$\mathrm{Hess}\mathcal{B}(\cdot) = \int_{\partial\tilde{N}} \mathrm{Hess}B_{\xi,y}(\cdot) d(\overline{f}_*\mu_p)(\xi).$$

Since N is locally symmetric, the Hessian of the Busemann function satisfies the formula

$$(\mathrm{Hess}B_\xi)_{F(p)}(\cdot) = \sqrt{-R(v_{F(p),\xi}, \cdot, v_{F(p),\xi}, \cdot)}, \tag{II.4.1}$$

where R is the Riemann curvature tensor (see [CF03, p. 16]). Let $v_{y,\xi}$ be the unit tangent vector based at y so that the geodesic with initial vector v has forward boundary point ξ, ie, $v_{y,\xi}$ is the gradient of $B_{\xi,y}$. Let θ_ξ denote the angle between $v_{y,\xi}$ and u. Then we can write $u = (\cos\theta_\xi)v_{y,\xi} + (\sin\theta_\xi)w$ for some unit vector w perpendicular to $v_{y,\xi}$. Since $(\mathrm{Hess}B_\xi)_{F(p)}(u) = \langle \nabla_u v_{y,\xi}, u\rangle$, we obtain $(\mathrm{Hess}B_\xi)_y(u) = \sin^2\theta_\xi(\mathrm{Hess}B_\xi)_y(w)$. Using the fact that the sectional curvatures of $\tilde{N}$ are at most -1, it follows that

$$(\mathrm{Hess}B_\xi)_y(u) \geq \sin^2\theta_\xi.$$

Hence, the integrand in the expression for $\mathrm{Hess}\mathcal{B}$ is 0 if and only if $\theta_\xi = 0, \pi$. This occurs precisely when $\xi = \pi(\pm u)$, where π is the projection of a unit tangent vector to its forward boundary point in $\partial\tilde{N}$. From the shadow lemma for Patterson–Sullivan measures (see, for instance, [Rob03, Lemma 1.3]), it follows that μ_p is non-atomic. This means

$$(\overline{f}_*\mu_p)(\partial\tilde{N} \setminus \{\pi(\pm u)\}) = 1 > 0.$$

Thus $(\mathrm{Hess} B_\xi)_y(u) > 0$ for a set of ξ of positive $\overline{f}_*\mu_p$-measure, which shows that $\mathcal{B}$ is strictly convex.

Definition II.27. Given $p \in \tilde{M}$, let μ_p be the Patterson-Sullivan measure on $\partial\tilde{M}$. Let $\overline{f} : \partial\tilde{M} \to \partial\tilde{N}$ as above. Define $F(p) = \mathrm{bar}(f_*\mu_p)$, where bar denotes the barycenter map (see [BCG96] for more details). We call F the *BCG map*.

II.5: Gromov compactness

In this final section of this chapter, we explain a type of convergence of Riemannian metrics that is used in the proofs of the main results of this thesis (Theorems I.6, I.8, and I.9), though it does not appear in the original proofs of marked length spectrum rigidity.

Let $\mathcal{C}(n, \Lambda, v_0, D_0)$ consist of all closed C^∞ Riemannian manifolds of dimension n with absolute sectional curvatures bounded by Λ^2, volume bounded below by v_0, and diameter bounded above by D_0. The space $\mathcal{C}(n, \Lambda, v_0, D_0)$ has the property that any sequence has a convergent subsequence in the Lipschitz topology; this is often called the Gromov compactness theorem [GKPS99]. In this thesis, we use refinements of Gromov's theorem due to Pugh and Greene–Wu [Pug87, GW88].

It follows from [GW88] that any sequence $(M, g_k) \in \mathcal{C}(2, \lambda, \Lambda, v_0, D_0)$ has a subsequence (M, g_{k_l}) converging in the following sense: there is a Riemannian metric g_0 on M such that in local coordinates we have $g_{k_l}^{ij} \to g_0^{ij}$ in the $C^{1,\alpha}$ norm, and the limiting g_0^{ij} have regularity $C^{1,\alpha}$. Moreover, the distance functions $d_{g_{k_l}}$ converge uniformly (with respect to the Lipschitz distance) to d_{g_0} on compact sets; see [GW88, p. 122].

We also recall some additional properties of the limit (M, g_0) due to Pugh. By [Pug87, Theorem 1], this limiting metric will have a Lipschitz geodesic flow, and the geodesics themselves are of $C^{1,1}$ regularity. Moreover, the exponential maps converge uniformly on compact sets [Pug87, Lemma 2].

CHAPTER III
Finiteness

In this chapter, we prove Theorem I.6 (reproduced below), that is, that short closed geodesics determine the full marked length spectrum approximately.

Theorem I.6 (Butt, [But22a] Theorem 1.2)**.** *Let (M, g) and (N, g_0) be closed Riemannian manifolds of dimension n with sectional curvatures contained in the interval $[-\Lambda^2, -\lambda^2]$. Let $\mathcal{L}_g$ and $\mathcal{L}_{g_0}$ denote their marked length spectra. Let Γ denote the fundamental group of M and let i_M denote its injectivity radius. Suppose there is a homotopy equivalence $f : M \to N$ and let f_* denote the induced map on fundamental groups.*

Then there is $L_0 = L_0(n, \Gamma, \lambda, \Lambda, i_N)$ so that the following holds: Suppose the marked length spectra $\mathcal{L}_g$ and $\mathcal{L}_{g_0}$ satisfy Hypothesis I.5 for some $\varepsilon > 0$ and $L \geq L_0$. Then there exist constants $C > 0$ and $0 < \alpha < 1$, depending only on n, Γ, λ, Λ, i_N, so that

$$1 - (\varepsilon + CL^{-\alpha}) \leq \frac{\mathcal{L}_{g_0}(f_*\gamma)}{\mathcal{L}_g(\gamma)} \leq 1 + (\varepsilon + CL^{-\alpha})$$

for all $\gamma \in \Gamma$.

In Section III.1, we start by stating the key dynamical facts used in our proof of Theorem I.9. Specifically, we use an estimate for the size of a covering of the unit tangent bundle T^1M by certain small "flow boxes" in addition to a Hölder estimate for a certain orbit equivalence between the geodesic flows of M and N. We then prove the theorem assuming these two facts. See the introduction to Section III.1 below for a rough sketch of the argument.

The rest, and vast majority, of this chapter is devoted to proving the above-mentioned covering lemma and Hölder estimate. The proofs rely on a few well-established consequences of the hyperbolicity of the geodesic flow. However, the standard results from the theory of Anosov flows (uniformly hyperbolic flows) are stated very generally and thus contain a multitude of constants which depend on the given flow in arguably mysterious ways. As a result, considerable technical difficulties arise in ensuring the constants depend only on select geometric and topological properties of (M, g) and (N, g_0).

The main components of this analysis are as follows. In Section III.2, we use geometric arguments involving horospheres to investigate the local product structure of the geodesic flow, a key mechanism responsible for many of the salient features of hyperbolic dynamical systems. Indeed, the results of this section are used to prove both the covering lemma and the Hölder estimate. The covering lemma is then quickly proved in Section III.3. Before proving the desired Hölder estimate, we show that the homotopy equivalence $f : M \to N$ (via which we are able to compare the marked length spectrum functions $\mathcal{L}_g$ and $\mathcal{L}_{g_0}$) can be taken to be a quasi-isometry with controlled quasi-isometry constants, i.e., depending only on n, Γ, λ, Λ, i_M, i_N. This is done in Section III.4. Finally, in Section III.5, we prove the orbit equivalence of geodesic flows in [Gro00] is Hölder continuous, also with controlled constants.

III.1: Proof of finiteness theorem

In this section, we will prove Theorem I.9 assuming two key statements: a covering lemma (Lemma III.1 below) and a Hölder estimate (Proposition III.4 below). These statements are proved in Sections III.3 and III.5, respectively.

The basic idea is to start by covering the unit tangent bundle T^1M with finitely many sufficiently small "flow boxes", that is, sets obtained by flowing local transversals for some small fixed time interval $(0, \delta)$. On the one hand, any periodic orbit of the flow that visits each of these boxes at most once is short, i.e., has period at most δ times the total number of boxes. On the other hand, any periodic orbit that is long, i.e., of length more than δ times the number of boxes, must return to at least one of the boxes more than once before it closes up. In other words, long periodic orbits contain shorter almost-periodic segments. By the Anosov closing lemma, these are in turn shadowed by periodic orbits. This allows us to approximate the lengths of long closed geodesics with sums of lengths of short ones. We then use a Hölder continuous orbit equivalence $\mathcal{F} : T^1M \to T^1N$ to argue that similar approximations hold for the corresponding closed geodesics in N. From this, we are able to estimate the ratio of $\mathcal{L}_g(\gamma)/\mathcal{L}_{g_0}(\gamma)$ for all long geodesics γ given our assumed estimate holds for short ones (Hypothesis I.5).

We now introduce the precise statements of the aforementioned covering lemma and Hölder estimate. Let W^{si} for $i = s, u$ denote the strong stable and strong unstable foliations for the geodesic flow ϕ^t on the unit tangent bundle T^1M. For $\delta > 0$, let

$$W^{si}_\delta(v) = W^{si}(v) \cap B(v, \delta),$$

where $B(v,\delta)$ denotes a ball of radius δ in T^1M with respect to the Sasaki metric. (See Section III.2 for some background on the stable/unstable foliations and the Sasaki metric. See also Section II.2.1.)

Let $P(v,\delta) = \cup_{v' \in W^{ss}_\delta(v)} W^{su}_\delta(v')$ and let $R(v,\delta) = \cup_{t\in(-\delta/2,\delta/2)} \phi^t P(v,\delta)$. We will call $R(v,\delta)$ a *δ-rectangle*. For our proof of Theorem I.9, we use the following estimate for the number of δ-rectangles needed to cover T^1M.

Lemma III.1. *Let i_M denote the injectivity radius of M. There is small enough $\delta_0 = \delta_0(n,\lambda,\Lambda,i_M)$ and a constant $C = C(n,\Gamma,\lambda,\Lambda,i_M)$ so that for any $\delta < \delta_0$, there is a covering of T^1M by at most C/δ^{2n+1} δ-rectangles.*

Remark III.2. The main difficulty is showing that the constant C does not depend on the metric g, but only on n, Γ, λ, Λ, $\operatorname{diam}(M)$.

Remark III.3. Rectangles of the form $R(v,\delta)$ are often used to construct Markov partitions, e.g. in [Rat73]. However, in Lemma III.1, we are not constructing a partition, meaning we do not require the rectangles to be measurably disjoint.

Now consider the geodesic flows ϕ^t and ψ^t on T^1M and T^1N, respectively. Recall that a homeomorphism $\mathcal{F} : T^1M \to T^1N$ is an *orbit equivalence* if there is some function (cocycle) $a(t,v)$ so that

$$\mathcal{F}(\phi^t v) = \psi^{a(t,v)}\mathcal{F}(v)$$

for all $v \in T^1M$ and for all $t \in \mathbb{R}$. Since M and N are homotopy-equivalent compact negatively curved manifolds, such an $\mathcal{F}$ exists by [Gro00]. Our proof of Theorem I.9 relies on the following estimates for the regularity of $\mathcal{F}$.

Proposition III.4. *Suppose (M,g) and (N,g_0) are a pair of homotopy-equivalent compact Riemannian manifolds with sectional curvatures contained in the interval $[-\Lambda^2,-\lambda^2]$. Let i_M and i_N denote their respective injectivity radii. Then there exists an orbit equivalence of geodesic flows $\mathcal{F} : T^1M \to T^1N$ which is C^1 along orbits and transversally Hölder continuous. More precisely, there is small enough $\delta_0 = \delta_0(\lambda,\Lambda,i_M)$ together with constants C and A, depending only on n, Γ, λ, Λ, i_M, i_N, so that the following hold:*

1. *$d(\mathcal{F}(v),\mathcal{F}(\phi^t v)) \leq At$ for all $v \in T^1\tilde{M}$ and $t \in \mathbb{R}$,*
2. *$d(\mathcal{F}(v),\mathcal{F}(w)) \leq Cd(v,w)^{A^{-1}\lambda/\Lambda}$ for all $v,w \in T^1\tilde{M}$ with $d(v,w) < \delta_0$.*

Remark III.5. It is a standard fact that any orbit equivalence of Anosov flows is C^0-close to a Hölder continuous one; in other words, there are constants C and α, depending on the given flows, i.e., on the metrics g and g_0, so that $d(\mathcal{F}(v),\mathcal{F}(w)) \leq Cd(v,w)^\alpha$ [FH19, Theorem

6.4.3]. However, we are claiming the stronger statement that for the orbit equivalence in [Gro00], there is a uniform choice of C and α for all (M, g) and (N, g_0) with pinched sectional curvatures and injectivity radii bounded away from 0.

To prove Theorem I.9, we start with a covering of T^1M by δ-rectangles (see Lemma III.1). Let δ_0 be as in Proposition III.4, then make δ_0 smaller if necessary so that Lemma III.1 holds as well. This choice of δ_0 depends only on n, λ, Λ, i_M. Now fix $\delta \leq \delta_0$, together with a covering $T^1M = \cup_{i=1}^m R(v_i, \delta)$. By Lemma III.1, we can take $m \leq C\delta^{2n+1}$. Since δ is now fixed, we use the notation R_i for the rectangle $R(v_i, \delta)$ and P_i for the transversal $P(v_i, \delta)$.

Let $v \in T^1M$. Then $v \in P_i$ if and only $\phi^t v \in R_i$ for all $t \in (-\delta/2, \delta/2)$. Moreover, if v is tangent to a closed geodesic of length τ, then for any rectangle R_i, the set

$$\{t \in (-\delta/2, \tau - \delta/2) | \, \phi^t v \cap R_i \neq \emptyset\}$$

is a (possibly empty) disjoint union of intervals of length δ.

Definition III.6. Fix a covering of T^1M by δ-rectangles $R_1, \ldots, R_m$ as above. Suppose η is a closed geodesic of length τ with $\eta'(0) = v$. Suppose that for each i, the set

$$\{t \in (-\delta/2, \tau - \delta/2) | \, \phi^t v \cap R_i \neq \emptyset\}$$

consists of at most a single interval. Then we say η is a *short geodesic* (with respect to the covering $R_1, \ldots R_m$).

Remark III.7. Let $L = L(\delta) = C\delta^{-2n}$, where C is the constant in the statement of Lemma III.1. If η is a short geodesic, then $l_g(\eta) \leq m\delta \leq C\delta^{-2n} = L$.

Proposition III.8. *Let γ be any closed geodesic in M. Then there is $k \in \mathbb{N}$ (depending on γ) and short geodesics $\eta_1, \ldots, \eta_{k+1}$ so that*

$$\left| l_g(\gamma) - \sum_{i=1}^{k+1} l_g(\eta_i) \right| < 2kC\delta$$

for some constant $C = C(\lambda, \Lambda, i_M)$.

Proof. If γ is already a short geodesic, then $k = 0$ and $\eta_1 = \gamma$. If not, then let i be the smallest index so that γ crosses through R_i in at least two time intervals. Let $v \in P_i$ tangent to γ and let $t_1 > 0$ be the first time so that $\phi^{t_1} v \in P_i$. By the Anosov Closing Lemma (Lemma II.14), there is w_1 tangent to a closed geodesic γ_1 of length t_1' with $|t_1 - t_1'| < C\delta$, where C depends only on the sectional curvature bounds λ and Λ and the injectivity radius i_M (see Lemma III.22). Similarly, applying the Anosov Closing Lemma to the orbit segment $\{\phi^t v \,|\, t \in [t_1, \tau]\}$ gives w_2 tangent to a closed geodesic γ_2 of length t_2' with $|(\tau - t_1) - t_2'| < C\delta$. This means $|l_g(\gamma) - l_g(\gamma_1) - l_g(\gamma_2)| < 2C\delta$.

Iterating the above process, we can "decompose" γ into short geodesics. More precisely, if γ_1 is not a short geodesic, then there is some other rectangle R_j through which γ_1 crosses twice. By the same argument as above, we get $|l_g(\gamma_1) - l_g(\gamma_{1,1}) - l_g(\gamma_{1,2})| < 2C\delta$ for some $\gamma_{1,1}, \gamma_{1,2} \in \Gamma$. Continuing in this manner, we get the desired conclusion. □

Next, we show that $l_{g_0}(\gamma)$ is still well-approximated by the sum of the g_0-lengths of the *same* free homotopy classes $\eta_1, \dots, \eta_{k+1}$ that were used to do the approximation with respect to g. For this, we use the estimates for the regularity of the orbit equivalence $\mathcal{F} : T^1M \to T^1N$ in Proposition III.4. Recall that $a(t, v)$ denotes the time-change cocycle, i.e. $\mathcal{F}(\phi^t v) = \psi^{a(t,v)}\mathcal{F}(v)$.

Lemma III.9. *Let γ and $\eta_1, \dots, \eta_{k+1}$ as in Proposition III.8. Then an analogous estimate holds in (N, g_0), namely,*

$$\left| l_{g_0}(\gamma) - \sum_{i=1}^{k+1} l_{g_0}(\eta_i) \right| < 2kC\delta^\alpha,$$

where C depends only on Γ, λ, Λ, i_M, i_N, and α is the Hölder exponent in the statement of Proposition III.4.

Proof. As in the proof of Proposition III.8, let $v \in T^1M$ tangent to γ. By the Anosov closing lemma, there is $w_1 \in T^1M$ tangent to a closed geodesic γ_1 of length t_1' such that $d(v, w_1) < C\delta$, for some $C = C(\lambda, \Lambda, i_M)$ (Lemma III.22). Additionally, $d(\phi^{t_1} v, \phi^{t_1'} w_1) < C\delta$.

By Proposition III.4, we know $d(\mathcal{F}(v), \mathcal{F}(w_1)) < C\delta^\alpha$. Moreover, since $\mathcal{F}(v)$ and $\mathcal{F}(w_1)$ remain $C\delta^\alpha$-close after being flowed by times $a(t_1, v)$ and $a(t_1', w)$, respectively, it follows that $|a(t_1, v) - a(t_1', w_1)| < 2C\delta^\alpha$. (We defer the short proof of this fact to Section III.2; see Lemma III.12.)

Similarly, the Anosov closing lemma applied to the orbit segment $\{\phi^t v \,|\, t \in [t_1, l(\gamma)]\}$ gives w_2 tangent to a closed geodesic γ_2 of length t_2'. By an analogous argument, $|a(l_g(\gamma) - t_1, v) - a(t_2', w_2)| < 2C\delta^\alpha$. Since $a(t, v)$ is a cocycle we get $|a(l_g(\gamma), v) - a(t_1', w_1) - a(t_2', w_2)| < 4C\delta^\alpha$.

Using that $\mathcal{F}$ is a Γ-equivariant orbit-equivalence, it follows that $a(l_g(\gamma), v) = l_{g_0}(\gamma)$ whenever $v \in T^1M$ is tangent to the closed geodesic γ. So the estimate in the previous paragraph can be rewritten as $|l_{g_0}(\gamma) - l_{g_0}(\gamma_2) - l_{g_0}(\gamma_2)| < 4C\delta^\alpha$. As such, we can iterate the process in Proposition III.8 and get an additive error of $4C\delta^\alpha$ at each stage. □

Proof of Theorem I.9. Recall from Remark III.7 that $L = L(\delta) = C\delta^{-2n}$ for some $C =$

$C(n, \Gamma, \lambda, \Lambda, i_M)$. Since we fixed $\delta \leq \delta_0 = \delta_0(n, \lambda, \Lambda, i_M)$, we see that $L \geq L_0 = L(\delta_0)$. By Lemma III.1, this choice of L_0 depends only on n, Γ, λ, Λ, i_M.

Recall as well that we are assuming

$$1 - \varepsilon \leq \frac{\mathcal{L}_g(\gamma)}{\mathcal{L}_{g_0}(\gamma)} \leq 1 + \varepsilon$$

for all $\gamma \in \Gamma_L := \{\gamma \in \Gamma \,|\, l_g(\gamma) \leq L\}$ (see Hypothesis I.5). We then have

$$\begin{aligned}
l_g(\gamma) &\leq \sum_{i=1}^{k+1} l_g(\gamma_i) + 2kC\delta && \text{(Proposition III.8)}\\
&\leq (1+\varepsilon)\sum_{i=1}^{k+1} l_{g_0}(\gamma_i) + 2kC\delta && \text{(Hypothesis I.5)}\\
&\leq (1+\varepsilon) l_{g_0}(\gamma) + (1+\varepsilon)2k(2C'\delta^\alpha + C\delta) && \text{(Proposition III.9)}\\
&\leq (1+\varepsilon) l_{g_0}(\gamma) + kC''\delta^\alpha.
\end{aligned}$$

Using this, we consider the ratio

$$\begin{aligned}
\frac{l_g(\gamma)}{l_{g_0}(\gamma)} &\leq (1+\varepsilon) + \frac{kC''\delta^\alpha}{l_{g_0}(\gamma)}\\
&\leq 1 + \varepsilon + \frac{kC''\delta^\alpha}{\sum_{i=1}^{k+1} l_{g_0}(\gamma_i) - 2k\delta} && \text{(Proposition III.9)}\\
&\leq 1 + \varepsilon + \frac{kC''\delta^\alpha}{2ki_N - 2k\delta}\\
&= 1 + \varepsilon + \frac{C''\delta^\alpha}{2i_N - 2\delta}.
\end{aligned}$$

In the last inequality, we used the fact that $l_{g_0}(\gamma) \geq 2i_N$ for all γ.

Finally, by the definition of L in Remark III.7, we have $\delta = CL^{-1/2n}$, where C is a constant depending only on n, Γ, a, b, i_M. So we can write that the ratio $l_g(\gamma)/l_{g_0}(\gamma)$ is between $1 \pm (\varepsilon + C'L^{-\alpha/2n})$, where α is the Hölder exponent in the statement of Proposition III.4. $\square$

Remark III.10. There is a way to obtain approximate control of the marked length spectrum from finitely many geodesics by combining Proposition III.4 with the finite Livsic theorem in [GL21], but our direct method above yields better estimates.

Let $a(t, v)$ denote the time change function for the orbit equivalence $\mathcal{F}$ in Proposition III.4. By the definition of $a(t, v)$ in (IV.3.1), (see also Lemma IV.24), this cocycle is differentiable in the t direction. Let $a(v) = \frac{d}{dt}|_{t=0} a(t, v)$. It follows from (IV.3.1) and Lemma

III.40 that $a(v)$ is of C^α regularity, where α is the same Hölder exponent as in the statement of Proposition III.4. It follows from Lemma III.34 and the proof of Lemma IV.24 that $\|a(v)\|_{C^0} \leq A$, where A is the constant in Lemma III.34. Hence, $\|a\|_{C^\alpha} \leq A + C$, where C is the constant in Proposition III.4.

Now let $\{\phi^t v\}_{0 \leq t \leq l_g(\gamma)}$ be the g-geodesic representative of the free homotopy class γ. Then $l_{g_0}(\gamma) = \int_0^{l_g(\gamma)} a(\phi^t v)\, dt$. Let $f(v) = (a(v) - 1)/\|a - 1\|_{C^\alpha}$. Then $\|f\|_{C^\alpha} \leq 1$ and Hypothesis I.5 implies

$$\frac{1}{l_g(\gamma)} \left| \int_0^{l_g(\gamma)} f(\phi^t v)\, dt \right| \leq \frac{\varepsilon}{A + C}$$

for all $\gamma \in \Gamma_L$. Setting $L = \left(\frac{\varepsilon}{C+A}\right)^{-1/2}$ means that f satisfies the hypotheses of Theorem 1.2 in [GL21]. This theorem implies that for all $\gamma \in \Gamma$, the ratio $\mathcal{L}_g/\mathcal{L}_{g_0}$ is between $1 \pm C' \left(\frac{\varepsilon}{C+A}\right)^\tau$, where C' and τ are constants depending on the given flow. Our direct method above yields an exponent of $\alpha/4n$ in place of τ.

III.2: Local product structure

We consider the distance d on T^1M induced by the *Sasaki metric* g^S on T^1M, which is in turn defined in terms of the Riemannian inner product g on M (see Definition II.6). Throughout the rest of this thesis, we will make use of the following standard facts relating the Sasaki distance d to the distance d_M on M coming from the Riemannian metric g and the distance $d_{T^1_qM}$ on $S^{n-1} \cong T^1_qM$. Let $v, w \in T^1M$ be unit tangent vectors with footpoints p and q respectively. Let $v' \in T^1_qM$ be the vector obtained by parallel transporting v along the geodesic joining p and q. Then we have

$$d_M(p,q), d_{T^1_qM}(v', w) \leq d(v, w) \leq d_M(p,q) + d_{T^1_qM}(v', w). \tag{III.2.1}$$

For convenience, we will often write d in place of d_M when it is clear from context that we are considering the distance between points as opposed to between unit tangent vectors.

Recall the geodesic flow on the unit tangent bundle of a negatively curved manifold is Anosov, and thus has *local product structure*. This means every point v has a neighborhood V which satisfies: for all $\varepsilon > 0$, there is $\delta > 0$ so that whenever $x, y \in V$ with $d(x,y) \leq \delta$ there is a point $[x,y] \in V$ and a time $|\sigma(x,y)| < \varepsilon$ such that

$$[x,y] = W^{ss}(x) \cap W^{su}(\phi^{\sigma(x,y)} y)$$

[FH19, Proposition 6.2.2]. Moreover, there is a constant $C_0 = C_0(\delta)$ so that $d(x,y) < \delta$

implies $d_{ss}(x,[x,y]), d_{su}(\phi^{\sigma(x,y)}[x,y],y) \leq C_0 d(x,y)$, where d_{ss} and d_{su} denote the distances along the strong stable and strong unstable manifolds, respectively.

To describe the stable and unstable distances d_{ss} and d_{su}, we first recall the stable and unstable manifolds W^{ss} and W^{su} for the geodesic flow have the following geometric description (see, for instance, [Bal95, p. 72]). Let $v \in T^1\tilde{M}$. Let $p \in \tilde{M}$ be the footpoint of v and let $\xi \in \partial\tilde{M}$ be the forward projection of $v \in T^1\tilde{M}$ to the boundary. Let $B_{\xi,p}$ denote the Busemann function on $\tilde{M}$ and let $H_{\xi,p}$ denote its zero set. Then the lift of $W^{ss}(v)$ to $T^1\tilde{M}$ is given by $\{-\mathrm{grad}B_{\xi,p}(q) \,|\, q \in H_{\xi,p}\}$. If η denotes the projection of $-v$ to the boundary $\partial\tilde{M}$, then the lift of $W^{su}(v)$ to $T^1\tilde{M}$ is analogously given by $\{\mathrm{grad}B_{\eta,p}(q) \,|\, q \in H_{\eta,p}\}$.

Now let $v \in T^1M$ and $w \in W^{ss}(v)$. Let p and q denote the footpoints of v and w respectively. Define the stable distance $d_{ss}(v,w)$ to be the horospherical distance $h(p,q)$, i.e., the distance obtained from restricting the Riemannian metric g on $\tilde{M}$ to a given horosphere. The unstable distance is defined analogously.

From the above description of W^{ss} and W^{su} in terms of normal fields to horospheres, it follows that the local product structure for the geodesic flow enjoys stronger properties than those for a general Anosov flow given in the first paragraph. First, the product structure is globally defined, meaning the neighborhood V in the first paragraph can be taken to be all of $T^1\tilde{M}$ (see, for instance, [Cou04]). Second, the bound on the temporal function σ can be strengthened:

Lemma III.11. *If $d(v,w) < \delta$, then $|\sigma(v,w)| < \delta$. for all δ*

Proof. Let p and q denote the footpoints of v and w respectively. Then by (III.2.1), we know $d(p,q) < \delta$. Let ξ denote the forward boundary point of v and let η denote the backward boundary point of w. Let $p' \in H_{\xi,p}$ and $q' \in H_{\eta,q}$ be points on the geodesic through η and ξ. Then $d(p',q') = |\sigma(v,w)|$. Moreover, since the geodesic segment through p' and q' is orthogonal to both $H_{\xi,p}$ and $H_{\eta,q}$, it minimizes the distance between these horospheres. In other words, $|\sigma(v,w)| = d(p',q') \leq d(p,q) < \varepsilon$. □

This allows us to deduce the following key lemma, which was used in the proof of Proposition III.9.

Lemma III.12. *Consider the geodesic flow ϕ^t on the universal cover $T^1\tilde{M}$. Suppose $d(v,w) < \delta_1$ and $d(\phi^s v, \phi^t w) < \delta_2$. Then $|s-t| < \delta_1 + \delta_2$.*

Proof. Since $[\phi^s v, \phi^s w] = [\phi^s v, \phi^t w]$, we have

$$\phi^{\sigma(\phi^s v, \phi^s w)}\phi^s w = \phi^{\sigma(\phi^s v, \phi^t w)}\phi^t w.$$

Thus $\sigma(\phi^s v, \phi^s w) + s = \sigma(\phi^s v, \phi^t w) + t$. Rearranging gives

$$s - t = \sigma(\phi^s v, \phi^t w) - \sigma(\phi^s v, \phi^s w) = \sigma(\phi^s v, \phi^t w) - \sigma(v, w).$$

By Lemma III.11, the absolute value of the right hand side is bounded above by $\delta_1 + \delta_2$, which completes the proof. □

Now assume (M, g) has sectional curvatures between $-\Lambda^2$ and $-\lambda^2$. We will show the constant C_0 in the definition of local product structure can be taken to depend only on λ, Λ and $\operatorname{diam}(M)$, whereas *a priori* it depends on the metric g. For our purposes, it will suffice to show the following proposition, which is formulated using the Sasaki distance d between vectors in T^1M instead of the stable/unstable distances d_{ss} and d_{su} between vectors on the same horosphere. In fact, we will show later (Lemma III.37) that the Sasaki distance d between vectors on the same stable/unstable manifold is comparable to d_{ss} and d_{su}, respectively.

Proposition III.13. *Suppose (M, g) has sectional curvatures between $-\Lambda^2$ and $-\lambda^2$. Then there is small enough $\delta_0 = \delta_0(\lambda, \Lambda, \operatorname{diam}(M))$ so that the following holds. Let $u \in T^1_p\tilde{M}$. Let $u_1 \in W^{ss}(u)$ and $u_2 \in W^{ss}(x)$ so that $d(u_1, u_2) \le \operatorname{diam}(T^1M)$, where d denotes the distance in the Sasaki metric. Then there exists a constant $C_0 = C_0(\lambda, \Lambda, \operatorname{diam}(M))$ so that whenever $d(u_1, u_2) < \delta_0$, we have $d(u, u_i) \le C_0 d(u_1, u_2)$ for $i = 1, 2$.*

Remark III.14. In our context, the dependence of the constant C_0 on the diameter of M can be replaced with a dependence on the injectivity radius i_M. Indeed, by [Gro82, Section 0.3], the volume of M is bounded above by a constant V_0 depending only on n, Γ, and λ. A standard argument (see, for instance, the proof of Lemma 3.9 in [But22b]) then shows the diameter is bounded above by $D_0 = D_0(i_M, V_0, \Lambda)$.

Our proof of Proposition III.13 relies on the geometry of horospheres, and we use many of the methods and results from the paper [HIH77] of the same title. However, we additionally consider the Sasaki distances between unit tangent vectors in $T^1\tilde{M}$ instead of just distances between points in $\tilde{M}$.

Let $\xi \in \partial\tilde{M}$ and let $B = B_\xi$ be the associated Busemann function. Suppose $p \in \tilde{M}$ is such that $B(p) = 0$. Let $v \in T^1_p\tilde{M}$ perpendicular to $\operatorname{grad}B(p)$ and consider the geodesic $\gamma(s) = \exp_p(sv)$. Define $f(s) = B(\gamma(s))$. This is the distance from $\gamma(s)$ to the zero set of B. Moreover, $f'(s) = \langle \operatorname{grad}B, \gamma' \rangle = \cos\theta$, where θ is the angle between $\gamma'(s)$ and $\operatorname{grad}B(\gamma(s))$. In particular, $f'(0) = 0$.

Lemma III.15. *For all $s \in \mathbb{R}$ we have $f(s) \le \frac{\Lambda}{2}s^2$ and $\cos\theta(s) = f'(s) \le \Lambda s$.*

Proof. We have $f''(s) = \langle \nabla_{\gamma'} \mathrm{grad} B, \gamma' \rangle = \langle \nabla_{\gamma'_T} \mathrm{grad} B, \gamma'_T \rangle$, where γ'_T denotes the component of γ' which is tangent to the horosphere through ξ and $\gamma(s)$. Note $\|\gamma'_T\| = \sin(\theta)$, where as before, θ is the angle between $\gamma'(s)$ and $\mathrm{grad} B(\gamma(s))$.

Thus $f''(s) = \langle J'(0), J(0) \rangle$, where J is the stable Jacobi field along the geodesic through $\gamma(s)$ and ξ with $J(0) = \gamma'_T(s)$. (See, for instance, [BCG95, p.750–751].) By [Bal95, Proposition IV.2.9 ii)], we have $\|J'(0)\| \leq \Lambda \|J(0)\|$, which shows $f''(s) \leq \Lambda \|J(0)\|^2 \leq \Lambda$. Since $f(0)$ and $f'(0)$ are both 0, Taylor's theorem implies that for any s, there is $\tilde{s} \in [0, s]$ so that $f(s) = \frac{f''(\tilde{s})}{2} s^2$. Thus, $f(s) \leq \frac{\Lambda}{2} s^2$ for all $s \geq 0$. Moreover, since $f'(0) = 0$, integrating $f''(s)$ shows $\cos\theta = f'(s) \leq \Lambda s$. $\square$

Lemma III.16. *Fix $S > 0$. Then there is a constant $c = c(\lambda, S)$ such that for all $s \in [0, S]$ we have $f(s) \geq \frac{c}{2} s^2$ and $\cos\theta = f'(s) \geq cs$.*

Proof. As in [HIH77, Section 4], we use $f_\lambda(s)$ to denote the analogue of the function $f(s)$, but defined in the space of constant curvature $-\lambda^2$. By considering the appropriate comparison triangles, it follows that $f(s) \geq f_\lambda(s)$ and $f'(s) \geq f'_\lambda(s)$ [HIH77, Lemma 4.2]. As in the proof of the previous lemma, we know $f''_\lambda(s) = \langle J(0), J'(0) \rangle$, where $\|J(0)\| = \sin\theta$. Solving the Jacobi equation explicitly in constant curvature gives $f''_\lambda(s) = \lambda \sin^2\theta$. For all $s \in [0, S]$, this is bounded below by $\lambda \sin^2\theta(S)$, which is a constant depending only on the value of S and the space of constant curvature $-\lambda^2$. In other words, there is a constant $c = c(\lambda, S)$ so that $f''_\lambda(s) \geq c$ for all $s \in [0, S]$. As in the proof of the previous lemma, Taylor's theorem then implies $f_\lambda(s) \geq \frac{c}{2} s^2$, and integrating f''_λ on the interval $[0, s]$ gives $f'_\lambda(s) \geq cs$. $\square$

Remark III.17. From the above proof it is evident that $f'_\lambda(s)/s \to 0$ as $s \to \infty$, and as such the only way to get a positive lower bound for $\cos\theta/s$ is to restrict to a compact interval $[0, S]$. This is reasonable for our purposes, since in the end, we will be applying the results of this section to the compact manifold M as opposed to its universal cover $\tilde{M}$. In Hypothesis III.18 below, we explain how we choose S based on $\mathrm{diam}(M)$.

For the proofs of the next several lemmas, we will consider the following setup (see Figure III.2 below). Let u be a unit tangent vector with footpoint p . Let $v \in T^1_p M$ perpendicular to u and let $\gamma(t) = \exp_p(tv)$. Fix $s > 0$ and let $u_1 \in W^{ss}(u)$ be such that such that the geodesic determined by u_1 passes through $\gamma(s)$. Let p_1 denote the footpoint of u_1. Let η denote the geodesic segment joining p and p_1 and let α denote the angle this segment makes with the vector u. Let q be the orthogonal projection of p_1 onto the geodesic γ. Consider the geodesic right triangle with vertices $p_1, q, \gamma(s)$. Let θ denote the angle at $\gamma(s)$ and let θ_1 denote the angle at p_1.

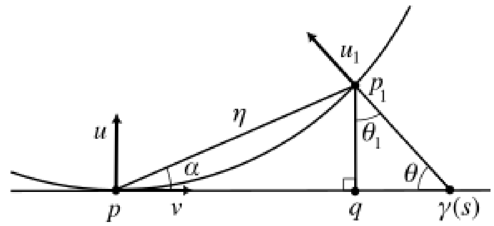

Figure 1: The setup for the proofs of Lemmas III.19, III.20, and III.21

Hypothesis III.18. *For our purposes, it is reasonable to assume* $d(p, p_1) \leq \operatorname{diam}(M)$, *where* $p, p_1 \in \tilde{M}$. *By [HIH77, Theorem 4.6, Proposition 4.7], this forces* $s \leq S$, *where* S *is a constant depending only on* $\operatorname{diam}(M)$ *and the lower sectional curvature bound* $-\Lambda^2$. *So we assume* $s \leq S$ *from now on.*

Lemma III.19. *Let* $u_1 \in W^{ss}(u)$ *as in Figure III.2, and assume* $s \leq S$ *(see Hypothesis III.18). Then there is a constant* $C = C(\Lambda, \operatorname{diam}(M))$ *so that* $d(u, u_1) \leq Cs$. *If* $u_2 \in W^{ss}(u)$, *then* $d(u, u_2) \leq Cs$ *as well.*

Proof. Consider the setup in Figure III.2. Let η denote the geodesic joining p and p_1 and let $P_\eta : T_pM \to T_{p_1}M$ denote parallel transport along this geodesic. Recall

$$d(u, u_1) \leq d_M(p, p_1) + d_{T^1_{p_1}M}(Pu, u_1).$$

To bound $d_M(p, p_1)$, we use the triangle inequality together, Lemma III.15, and Hypothesis III.18:

$$d(p, p_1) \leq d(p, q) + d(p_1, q) \leq s + d(p_1, \gamma(s)) \leq s + \Lambda s^2/2 \leq (1 + \Lambda S/2)s.$$

To bound $d_{T^1_{p_1}M}(Pu, u_1)$, we first find bounds for the angles θ and θ_1. We know from Lemma III.15 that $\sin(\pi/2 - \theta) = \cos\theta \leq \Lambda s$. Moreover, $\sin(\pi/2 - \theta) \geq (2/\pi)(\pi/2 - \theta)$ for $0 \leq \pi/2 \leq \theta$. Since the interior angles of geodesic triangles in M sum to less than π, we know $\theta + \theta_1 < \pi/2$. Thus, $\theta_1 < \pi/2 - \theta \leq (\pi/2)\Lambda s$.

Now let α denote the angle between u and η' at the point p. Then α is also the angle between Pu and η' at the point p_1, since parallel transport is an isometry and η' is a geodesic. Since the angle sum of the geodesic triangle with vertices p, p_1 and q is less than π, the angle in $T_{p_1}M$ between η' and $[p_1, q]$ is strictly less than α. Thus if we rotate η' towards Pu, we must pass through the tangent vector to $[p_1, q]$ along the way. Hence $d_{T_{p_1}M}(Pu, u_1) < \theta_1 \leq (\pi/2)\Lambda s$, which completes the proof of the upper bound for $d(u, u_1)$. The estimate for $d(u, u_2)$ follows by an analogous argument. $\square$

Lemma III.20. *Again, consider the setup in Figure III.2 and Hypothesis III.18. For all* $s \in [0, S]$ *we have* $\theta_1 \geq cs$ *for some* $c = c(\lambda, \Lambda, \mathrm{diam}(M))$.

Proof. Consider the following comparison triangle with vertices p_1', q', x in the space of constant curvature $-\Lambda^2$: suppose there is a right angle at the vertex q' and the lengths of the two legs are equal to $d_M(q, p_1)$ and $d_M(q, \gamma(s))$. Let θ' denote the angle at x and let θ_1' denote the angle at p_1'. Since triangles in M are thicker than in the space of constant curvature $-\Lambda^2$, we have $\theta_1 \geq \theta_1'$ and $\cos(\theta') \geq \cos(\theta)$. Now by [Bea12, Theorem 7.11.3] we have

$$\theta_1' \geq \sin(\theta_1') = \frac{\cos(\theta')}{\cosh(\Lambda\, d(q, p_1))} \geq \frac{\cos(\theta)}{\cosh(\Lambda\, d(\gamma(s), p_1))}.$$

By Lemma III.16, we can bound the numerator below by cs for some $c = c(\lambda, \Lambda, \mathrm{diam}(M))$. Using Lemma III.15 and Hypothesis III.18, we get $d(\gamma(s), p_1) = f(s) \leq \Lambda s^2/s \leq \Lambda S^2/2$. So the denominator is bounded above by some constant depending only $\lambda, \Lambda, \mathrm{diam}(M)$, which completes the proof. □

Lemma III.21. *Let* $u \in T_p^1 M$. *Let* $u_1 \in W^{ss}(u)$ *be such that the footpoints* p *and* p_1 *of* u *and* u_1 *are distance* t *apart. Then* $d(u, u_1) \leq (1 + \Lambda)t$.

Proof. Let η denote the geodesic joining p and p_1. Let $P_\eta : T_p M \to T_{p_1} M$ denote parallel transport along η. Let $v_0 \in T_p^1 M$ be the vector contained in the plane spanned by u and $\eta'(0)$ so that $\langle u, v_0 \rangle = 0$ and $\langle \eta'(0), v_0 \rangle > 0$. Let $V(s)$ denote the parallel vector field along $\eta(s)$ with initial value $V(0) = v_0$. Let $\theta(s)$ be the angle between $V(s)$ and $-\mathrm{grad}B(\eta(s))$. Then $\theta_1 = \pi/2 - \theta(t)$ is the angle between u_1 and $P_\eta u$. We have

$$\sin(\pi/2 - \theta) = \cos(\theta) = \langle V(t), -\mathrm{grad}B(\eta(t)) \rangle = \int_0^t \langle V(s), \nabla_V \mathrm{grad}B(\eta(s)) \rangle\, ds.$$

By the same argument as in the proof of Lemma III.15, this integral is bounded above by Λt. Hence, $d(u, u_1) \leq d_M(p, p_1) + d_{T_{p_1} M}(P_\eta u, u_1) \leq t + \Lambda t$. □

Proof of Proposition III.13. Consider the hypersurface formed by taking the exponential image of $\mathrm{grad}B(p)^\perp$. For $i = 1, 2$, let x_i denote the point on this hypersurface which is on the geodesic determined by u_i. Let $v_i \in T_p^1 M$ perpendicular to u for $i = 1, 2$ such that $\exp_p(s_i v_i) = x_i$, where $s_i = d(p, x_i)$. Let $\gamma_i(s) = \exp_p(s v_i)$.

Now suppose without loss of generality that $s_1 \geq s_2$ and let $s = s_1$. By Lemma III.19, we have $d(u, u_1) \leq Cs$ for some $C = C(\Lambda, \mathrm{diam}(M))$. So it suffices to bound $d(u_1, u_2)/s$ from below by some constant depending only on the desired parameters. Now let $c = c(\lambda, \Lambda, \mathrm{diam}(M))$ be the constant from the statement of Lemma III.20, and let β such that

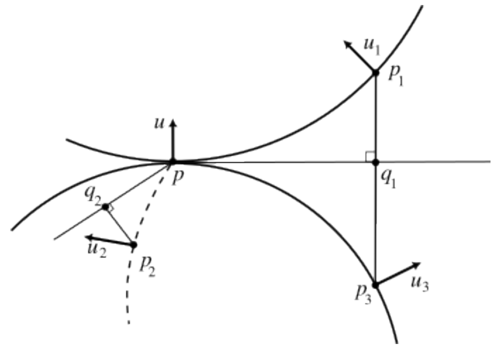

Figure 2: The horospherical configuration for the proof of Proposition III.13

$c - 2(1+\Lambda)\beta = c/2$. Let $s' = \beta s$. Let q_1 and q_2 denote the orthogonal projections of p_1 and p_2 onto the tangent plane $\text{grad}B(p)^{\perp}$. See Figure III.2 below.

We consider the cases $d(q_1, q_2) \geq s'$ and $d(q_1, q_2) \leq s'$ separately. In the first case, we obtain $d(u_1, u_2) \geq d_M(p_1, p_2) \geq d(q_1, q_2) \geq \beta s_1$, which shows $d(u, u_1) \leq \frac{C}{\beta} d(u_1, u_2)$.

We now consider the case $d(\gamma_1(s), \gamma_2(s)) \leq s'$. The geodesic determined by p_1 and q_1 intersects the unstable horosphere $W^{su}(u)$ at some point we will call p_3. Let $u_3 = \text{grad}B_{\eta,p}(p_3)$. As before, let θ_1 denote the angle between u_1 and the geodesic determined by the points p_1 and q_1. Then $d(u_1, u_3) \geq \theta_1 \geq cs$ by Lemma III.20.

We now bound $d(u_2, u_3)$ from above. Lemma III.21 gives $d(u_2, u_3) \leq (1+\Lambda) d(p_2, p_3)$. By the triangle inequality and Lemma III.15, $d(p_2, p_3) \leq s' + \Lambda s^2$. So if $s' \leq \beta s$, we obtain $d(u_2, u_3) \leq (1+\Lambda)(\beta + \Lambda s)s$. We now claim that there is δ_0 sufficiently small so whenever $d(u_1, u_2) < \delta_0$, we also have s is small enough to guarantee $\Lambda s \leq \beta$. To see this, first note that $d(u_1, u_2) \geq d(p_1, p_2) \geq d(p_1, q_1)$. Now consider a comparison right triangle in the space of constant curvature $-\lambda^2$ with hypotenuse equal to $d(\gamma(s_0), p_1) = f(s)$ and an angle θ equal to the angle between gradB and γ' at the point $\gamma(s_0)$. Let x denote the length of the side opposite to the angle θ. Then, using the fact that triangles in M are thinner than this comparison triangle, together with [Bea12, Theorem 7.11.2 ii)], Lemma III.16 and Hypothesis III.18, gives

$$\sinh(d(p_1, q_1)) \geq \sinh(x) = \sin(\theta(s)) \sinh(f(s)) \geq \sin(\theta(S)) \sinh(cs^2).$$

So if $d(u_1, u_2) \leq \delta$ we see that $\sinh(cs^2) \leq C\delta$, where C depends only on λ and S. In other words, there is small enough $\delta_0 = \delta_0(\lambda, \Lambda, \text{diam}(M))$ to ensure s is as small as desired, which in this case means small enough for $\Lambda s \leq \beta = \beta(\lambda, \Lambda, \text{diam}(M))$. Thus, we now have $d(u_2, u_3) \leq 2(1+\Lambda)\beta s$.

Finally, $d(u_1, u_2) \geq d(u_1, u_3) - d(u_2, u_3) \geq s_1(c - 2(1+\Lambda)\beta)$. By the choice of β, this is bounded below by $\frac{c}{2}s$. Hence $d(u, u_1) \leq Cs_1 \leq \frac{2C}{c} d(u_1, u_2)$. Reversing the roles of u_1 and u_2 and repeating the same argument gives the analogous upper bound for $d(u, u_2)$. $\square$

Proposition III.13 allows us to deduce the following refinement of the Anosov Closing Lemma, where we can say the constants involved depend only on concrete geometric information about (M, g), namely the diameter and the sectional curvature bounds. Note that now the setting is T^1M as opposed to the universal cover $T^1\tilde{M}$.

Lemma III.22. *There is $\delta_0 = \delta_0(\lambda, \Lambda, \mathrm{diam}(M))$ sufficiently small so that the following holds. Suppose $v, \phi^t v \in T^1M$ so that $d(v, \phi^t v) < \delta \leq \delta_0$. Then either v and $\phi^t v$ are on the same local flow line or there is w with $d(v, w) < C\delta$ so that w is tangent to a closed geodesic of length $t' \in [t - C\delta, t + C\delta]$, where C is a constant depending* only *on the diameter of M and the sectional curvature bounds λ and Λ.*

Proof. Let δ_0 be the constant in Proposition III.13. The proof of the usual Anosov Closing Lemma in [Fra18, Figure 2] (see also [Bow75, 3.6, 3.8]) shows the constant C depends only on the local product structure constant C_0. By Proposition III.13, we know this depends only on $\lambda, \Lambda, \mathrm{diam}(M)$. $\square$

III.3: Covering lemma

In this section, we prove the following covering lemma, which was one of the key statements we used in the proof of the main theorem.

Lemma III.1. There is small enough $\delta_0 = \delta_0(n, \lambda, \Lambda, \mathrm{diam}(M))$ together with a constant $C = C(n, \Gamma, \lambda, \Lambda, \mathrm{diam}(M))$ so that for any $\delta < \delta_0$, there is a covering of T^1M by at most C/δ^{2n+1} δ-rectangles.

We start with a preliminary lemma.

Lemma III.23. *Let $B(v, \delta)$ be a ball of radius δ in T^1M with respect to the Sasaki metric. There is small enough $\delta_0 = \delta_0(n)$, depending only on the dimension n, so that for all $\delta < \delta_0$ we have $\mathrm{vol}(B(v, \delta)) \geq c\delta^{2n+1}$ for some constant $c = c(n)$.*

Proof. First we claim $B(v, \delta) \supset B_M(p, \delta/2) \times B_{S^{n-1}}(v, \delta/2)$, where $B_M(p, \delta/2)$ is a ball of radius $\delta/2$ in M and $B_{S^{n-1}}(v, \delta/2)$ is a ball of radius $\delta/2$ in the unit tangent sphere T^1_pM. This follows immediately from (III.2.1). Since M is negatively curved, Theorem 3.101 ii) in [GHL90] implies $\mathrm{vol} B_M(p, \delta/2) \geq \beta_n \delta^n/2^n$, where β_n is the volume of the unit ball in $\mathbb{R}^n$. By Theorem 3.98 in [GHL90], we have $\mathrm{vol} B_{S^{n-1}}(v, \delta/2) = \frac{\beta_{n-1}\delta^{n-1}}{2^{n-1}}(1 - \frac{n-1}{6(n+1)}\delta^2 + o(\delta^4))$. Then for δ less than some small enough δ_0, we can write

$$B_{S^{n-1}}(v,\delta/2) \geq \frac{\beta_{n-1}\delta^{n-1}}{2^{n-1}}\left(1-2\frac{n-1}{6(n+1)}\delta^2\right) \geq c\delta^{n+1},$$

for some $c = c(n)$. The quantity δ_0 depends only on the coefficients of the Taylor expansion of $\mathrm{vol}B_{S^{n-1}}(v,\delta/2)$, which depend only on the geometry of S^{n-1}. So we can say δ_0 depends only on n. Therefore, the volume of the Sasaki ball $B(v,\delta)$ is bounded below by $c\delta^{2n+1}$ for some other constant $c = c(n)$ depending only on n. □

Proof of Lemma III.1. Let δ_0 and C as in Proposition III.13. Let $c = 1/C$ and let $\delta < \delta_0/2c$. Let $v_1,\ldots,v_m$ be a maximal $c\delta$-separated set in T^1M with respect to the Sasaki metric. We claim that the balls $B(v_1,c\delta),\ldots,B(v_m,c\delta)$ cover T^1M. If not, there is some v such that $d(v,v_i) \geq c\delta$ for all i. This contradicts the fact that $v_1,\ldots,v_m$ was chosen to be a *maximal* $c\delta$-separated set.

This implies that the rectangles $R(v_1,\delta)\ldots R(v_m,\delta)$ cover T^1M as well. Indeed, let $w \in B(v,c\delta)$. Then by Lemma III.11 there is a time $\sigma = \sigma(v,w) < c\delta$ and a point $[v,w] \in T^1M$ so that $[v,w] = W^{ss}(v) \cap W^{su}(\phi^\sigma w)$. Thus $d(v,\phi^\sigma w) \leq \delta_0$ and Proposition III.13 implies $d_ss(v,[v,w]), d_{su}([v,w],\phi^\sigma w) < Cc\delta = \delta$ as desired.

Now we estimate m. Since $v_1,\ldots,v_m$ if $c\delta$-separated, it follows that for $i \neq j$ we have $B(v_i,c\,\delta/2) \cap B(v_j,c\,\delta/2) = \emptyset$. Hence

$$m\inf_i \mathrm{vol}(B(v_i,c\delta/2)) \leq \mathrm{vol}(T^1M) = \mathrm{vol}(S^{n-1})\mathrm{vol}(M).$$

By [Gro82, 0.3 Thurston's Theorem], we have $\mathrm{vol}(M)$ is bounded above by a constant depending only on n, Γ and the upper sectional curvature bound $-\lambda^2$. This, together with Lemma III.23, gives $m \leq C/\delta^{2n+1}$ for some constant $C = C(n,\Gamma,\lambda,\Lambda,\mathrm{diam}(M))$. □

III.4: Pseudo-isometry estimates

Recall (M,g) and (N,g_0) are compact negatively curved manifolds with a given isomorphism between their fundamental groups. Since M and N are $K(\pi,1)$ spaces, there is a homotopy equivalence $M \to N$ inducing this isomorphism; moreover, we can assume it is of C^1 regularity, since every continuous map is homotopic to a differentiable one. Now lift this C^1 homotopy equivalence to a map $f : \tilde{M} \to \tilde{N}$, which is equivariant with respect to the actions of $\Gamma \cong \pi_1(M) \cong \pi_1(N)$ on $\tilde{M}$ and $\tilde{N}$. It is well-known that f is a *pseudo-isometry* (see, for instance, [BP92, Proposition C.1.2]), meaning there exist constants A and B so that for all

$x_1, x_2 \in \tilde{M}$ we have

$$A^{-1} d_g(x_1, x_2) - B \leq d_{g_0}(f(x_1), f(x_2)) \leq A d_g(x_1, x_2). \tag{III.4.1}$$

This is a special case of the perhaps more widely used concept of a *quasi-isometry*, which is when there is also an additive constant on the righthand side of (III.4.1). In our case, however, the absence of this additive constant is crucial for our remaining arguments.

In this section, we show the constants A and B depend only on the fundamental group Γ, the injectivity radius of (M, g) and the sectional curvature bounds for (M, g) and (N, g_0) (Proposition III.24 below).

Proposition III.24. *Suppose $f : \tilde{M} \to \tilde{N}$ is a Γ-equivariant C^1 map as above. Now let g and g_0 be Riemannian metrics on M and N with sectional curvatures contained in the interval $[-\Lambda^2, -\lambda^2]$, and suppose the injectivity radii of (M, g) and (N, g_0) are bounded below by i_M and i_N, respectively. Then there are constants A and B depending only n, Γ, a, b, i_M so that (III.4.1) holds for all $x_1, x_2 \in \tilde{M}$.*

We start by finding a uniform Lipschitz bound for f, in other words, proving the second inequality in the above proposition (Corollary III.27 below). A key tool we use is *Gromov compactness.* Let $\mathcal{M}(D_0, v_0, \Lambda)$ be the space of all Riemannian metrics on M with diameter at most D_0, volume at least v_0 and absolute sectional curvatures at most Λ^2. This space satisfies certain pre-compactness properties. We will use a refinement of Gromov's theorem due to Greene–Wu [GW88], namely that any sequence $(M, g_n) \in \mathcal{M}(D_0, v_0, \Lambda)$ has a subsequence (M, g_{n_k}) converging in the following sense: there is a Riemannian metric g_∞ on M such that in local coordinates we have $g^{ij}_{n_k} \to g^{ij}_\infty$ in the $C^{1,\alpha}$ norm and the limiting g^{ij}_∞ have regularity $C^{1,\alpha}$, for some $0 < \alpha < 1$.

Lemma III.25. *Suppose $f : M \to N$ is a C^1 map. Suppose g and g_0 are Riemannian metrics on M and N with $(M, g) \in \mathcal{M}(D_0, v_0, \Lambda)$ and $(N, g_0) \in \mathcal{M}(D_0', v_0', \Lambda')$. Then there exists a constant $A = A(f, D_0, D_0', v_0, v_0', \Lambda, \Lambda')$ so that*

$$\|df\|_{g,g_0} := \sup_{v \in TM} \frac{\|df_p(v)\|_{g_0}}{\|v\|_g} \leq A.$$

Proof. If $g^n \to g$ in the $C^{1,\alpha}$ topology, then, in particular, $\|v\|_{g_n} \to \|v\|_{g_\infty}$ uniformly on compact sets. This means if $g_n \to g$ and $g_0^n \to g_0$, then $df_{g^n, g_0^n} \to df_{g, g_0}$.

Now suppose for contradiction that the statement of the lemma is false. This means there are sequences $g^n \in \mathcal{M}(D_0, v_0, \Lambda)$ and $g_0^n \in \mathcal{M}(D_0', v_0', \Lambda')$ so that $\|df\|_{g^n, g_0^n} \to \infty$. After passing to convergent subsequences, we have $\|df\|_{g^n, g_0^n} \to \|df\|_{g^\infty, g_0^\infty}$ for some $C^{1,\alpha}$

Riemannian metrics g^∞ and g_0^∞. Since f is C^1 and the unit tangent bundle of M is compact, the derivative $df(v)$ is uniformly bounded in v. In other words, $\|df\|_{g^\infty, g_0^\infty} < \infty$, which is a contradiction. So the statement of the lemma must be true. $\square$

Lemma III.26. *Suppose that (M, g) has sectional curvatures in the interval $[-\Lambda^2, -\lambda^2]$ and injectivity radius at least i_M. Then there are constants $v_0 = v_0(i_M, n)$ and $D_0 = D_0(n, \lambda, \Gamma, i_M)$ so that $(M, g) \in \mathcal{M}(D_0, v_0, \Lambda)$.*

Proof. First, the desired absolute sectional curvature bound holds by assumption. Second, by Gromov's systolic inequality, we know $\mathrm{vol}(M, g) \geq v_0$, where v_0 is a constant depending only on n and i_M [Gro83, 0.1.A].

It now remains to bound the diameter from above. By [Gro82, Section 0.3], the volume is bounded above by a constant V_0 depending only on n, Γ, and λ. Thus it suffices to show diameter is bounded above by $D_0 = D_0(i_M, V_0, \Lambda)$. To see this, let p and q be such that $\mathrm{diam}(M) = d(p, q)$ and let $c(t)$ be the geodesic joining p and q. Let m be the unique positive integer such that $2(m-1)i_M \leq \mathrm{diam}(M) \leq 2mi_M$. Take balls of radius i_M centered at $c(0), c(2i_M), c(4i_M), \ldots, c(2(m-1)i_M)$. Since M is negatively curved, the volume of any such ball is bounded below by the volume of a ball of radius i_M in $\mathbb{R}^n$ [GHL90, Theorem 3.101 ii)], which we will denote by $v(i_M, n)$. Then $mv(r, n) \leq \mathrm{Vol}(M) \leq V_0$. This gives an upper bound for m, therefore

$$d(p, q) = \mathrm{diam}(M) \leq 2i_M m \leq i_M \frac{2V_0}{v(i_M, n)},$$

which completes the proof. $\square$

Corollary III.27. *Let (M, g) and (N, g_0) be closed Riemannian manifolds of dimension n with sectional curvatures in the interval $[-\Lambda^2, -\lambda^2]$, and assume there is an isomorphism between their fundamental groups. Then there is an A-Lipschitz map $f : M \to N$ inducing this isomorphism, where A depends only on $n, \Gamma, \lambda, \Lambda$, and the injectivity radii i_M and i_N.*

Given that f is A-Lipschitz, we now show the second estimate in the definition of pseudo-isometry. We follow the approach of [BP92, Proposition C.12], but we need to check the constants depend only on the desired parameters n, Γ, λ, Λ and i_N.

First, let $h : \tilde{N} \to \tilde{M}$ be a C^1 homotopy inverse of f. By Corollary III.27, h is also A-Lipschitz. Now consider the following fundamental domain D_M for the action of Γ on $\tilde{M}$ (see [BP92, Proposition C.1.3]). Fix $p \in \tilde{M}$. Let

$$D_M = \{x \in \tilde{M} \mid d(x, p) \leq d(x, \gamma.p) \, \forall \gamma \in \Gamma\}. \tag{III.4.2}$$

Claim III.28. *The diameter of* D_M *satisfies* $\operatorname{diam}(D_M) \leq 2\operatorname{diam}(M)$*.*

Proof. Let $x \in \tilde{M}$ so that $d(p,x) > \operatorname{diam}(M)$. This means there is some $\gamma \in \Gamma$ so that $d(x, \gamma.p) < d(x,p)$. In other words, any geodesic in $\tilde{M}$ starting at p stays in D_M for a time of at most $\operatorname{diam}(M)$. So if $x_1, x_2 \in D_M$ so that $d(x_1,x_2) = \operatorname{diam}(D_M)$, then $d(x_1,x_2) \leq d(x_1,p) + d(x_2,p) \leq 2\operatorname{diam}(M)$, which proves the claim. □

Claim III.29. *For all* $x \in \tilde{M}$*, we have* $d(h \circ f(x), x) \leq 2(1+A^2)\operatorname{diam}(M)$*.*

Proof. Since h and f are both continuous and Γ-equivariant, so is $h \circ f$, and thus it suffices to check the statement for x in a compact fundamental domain D_M. Since f and h are A-Lipschitz, it follows that the function $x \mapsto d(h \circ f(x), x)$ is $(1+A^2)$-Lipschitz:

$$|d(h \circ f(x), x) - d(h \circ f(y), y)| \leq d(h \circ f(x), h \circ f(y)) + d(x,y) \leq (1+A^2)d(x,y).$$

Noting $d(x,y) \leq D_M \leq 2\operatorname{diam}(M)$ completes the proof. □

Proof of Proposition III.24. We can now use the argument in [BP92] verbatim. By the previous claim, we obtain

$$d(h(f(x_1)), h(f(x_2))) \geq d(x_1,x_2) - 4(1+A^2)\operatorname{diam}(M).$$

Then, the Lipschitz bounds for f and h give

$$d(f(x_1), f(x_2)) \geq A^{-1} d(h \circ f(x_1), h \circ f(x_2)) \geq A^{-1}\big(d(x_1,x_2) - 4(1+A^2)\operatorname{diam}(M)\big),$$

which completes the proof. □

III.5: Hölder estimate

In this section, we show the following, which was one of the main ingredients in the proof of Theorem I.9.

Proposition III.4. Suppose (M,g) and (N,g_0) are a pair of homotopy-equivalent compact Riemannian manifolds with sectional curvatures contained in the interval $[-\Lambda^2, -\lambda^2]$. Let i_M and i_N denote their respective injectivity radii. Then there exists an orbit equivalence of geodesic flows $\mathcal{F} : T^1M \to T^1N$ which is C^1 along orbits and transversally Hölder continuous. More precisely, there is small enough $\delta_0 = \delta_0(\lambda, \Lambda, i_M)$ together with constants C and A, depending only on n, Γ, λ, Λ, i_M, i_N, so that the following hold:

1. $d(\mathcal{F}(v), \mathcal{F}(\phi^t v)) \leq At$ for all $v \in T^1\tilde{M}$ and $t \in \mathbb{R}$,

2. $d(\mathcal{F}(v), \mathcal{F}(w)) \leq Cd(v,w)^{A^{-1}\lambda/\Lambda}$ for all $v, w \in T^1\tilde{M}$ with $d(v,w) < \delta_0$.

We will take $\mathcal{F}$ to be the map in [Gro00], whose construction we now recall. The construction starts with a preliminary Γ-equivariant orbit map $\mathcal{F}_0 : T^1\tilde{M} \to T^1\tilde{N}$ which is not necessarily injective. As in Section III.4, consider a C^1 homotopy equivalence $M \to N$, which we lift to a Γ-equivariant map $f : \tilde{M} \to \tilde{N}$. By Proposition III.24, there are constants A and B, depending only on n, λ, Λ, Γ, i_M, so that

$$A^{-1}d(p,q) - B \leq d(f(p), f(q)) \leq Ad(p,q). \tag{III.5.1}$$

Let η be a bi-infinite geodesic in $\tilde{M}$ and let $\zeta = \overline{f}(\eta)$ be the corresponding geodesic in $\tilde{N}$, where $\overline{f} : \partial^2\tilde{M} \to \partial^2\tilde{N}$ is obtained from extending the quasi-isometry f to a map $\partial\tilde{M} \to \partial\tilde{N}$.

Let $P_\zeta : \tilde{N} \to \zeta$ denote the orthogonal projection. Note this projection is Γ-equivariant, i.e., $\gamma P_\zeta(x) = P_{\gamma\zeta}(\gamma x)$. If $(p,v) \in T^1\tilde{M}$ is tangent to η, define $\mathcal{F}_0(p,v)$ to be the tangent vector to ζ at the point $P_\zeta \circ \tilde{f}(p)$. Thus $\mathcal{F}_0 : T^1\tilde{M} \to T^1\tilde{N}$ is a Γ-equivariant map which sends geodesics to geodesics. As such, we can define a cocycle $b(t,v)$ to be the time which satisfies

$$\mathcal{F}_0(\phi^t v) = \psi^{b(t,v)}\mathcal{F}_0(v). \tag{III.5.2}$$

Remark III.30. Since $\tilde{f}$ is C^1 and the orthogonal projection is smooth in the t-direction, we have $t \mapsto b(t,v)$ is C^1.

It is possible for a fiber of the orthogonal projection map to intersect the quasi-geodesic $\tilde{f}(\eta)$ in more than one point; thus, $\mathcal{F}_0$ is not necessarily injective. In order to obtain an injective orbit equivalence, we follow the method in [Gro00] and average the function $b(t,v)$ along geodesics. We will repeatedly use the following standard fact about quasi-geodesics remaining bounded distance away from their corresponding geodesics:

Lemma III.31 (Theorem III.H.1.7 of [BH13]). *Let f be the quasi-isometry from Section III.4. Let $c(t)$ be any geodesic in $\tilde{M}$ and let η be its corresponding geodesic in $\tilde{N}$ obtained from the boundary map $\overline{f} : \partial\tilde{M} \to \partial\tilde{N}$. Then there is a constant R, depending only on the pseudo-isometry constants A and B of f and the upper sectional curvature bound $-\lambda^2$ for N, so that $d(f(c(t)), P_\eta(f(c(t))) \leq R$ for any $t \in \mathbb{R}$.*

Lemma III.32. *Let*

$$a_l(t,v) = \frac{1}{l}\int_t^{t+l} b(s,v)\, ds.$$

There is a large enough l so that $t \mapsto a_l(t,v)$ is injective for all v.

Proof. The fundamental theorem of calculus gives

$$\frac{d}{dt}a_l(t,v) = \frac{b(t+l,v)-b(t,v)}{l}. \tag{III.5.3}$$

We claim there is a large enough l so that this quantity is always positive. To this end, suppose $b(t+l,v)-b(t,v)=0$. This means $\mathcal{F}_0(\phi^t v)$ and $\mathcal{F}_0(\phi^{t+l}v)$ are in the same fiber of the normal projection onto the geodesic $\overline{f}(v)$. Since $s \mapsto \tilde{f}(\phi^s v)$ is a quasi-geodesic, there is a constant R, depending only on the quasi-isometry constants A and B of $\tilde{f}$, so that all points on $\tilde{f}(\phi^s v)$ are of distance at most R from the geodesic $\psi^t\mathcal{F}_0(v)$ [BH13, Theorem 3.H.1.7]. Thus two points on the same fiber of the normal projection are at most distance $2R$ apart, which gives

$$A^{-1}l - B \leq d(f(\phi^t v), f(\phi^{t+l}v)) \leq 2R.$$

Taking $l > A(2R+B)$ guarantees $\frac{d}{ds}a_l(s,v)$ is never 0, and hence $a_l(s,v)$ is injective. □

Proposition III.33. *For each $v \in T^1M$, let*

$$\mathcal{F}_l(v) = \psi^{a_l(0,v)}\mathcal{F}_0(v)$$

for a_l as in Lemma IV.24. Then $\mathcal{F}_l$ is an orbit equivalence of geodesic flows.

Proof. Since $\mathcal{F}_l$ sends geodesics to geodesics, there exists a cocycle $k_l(t,v)$ so that $\mathcal{F}_l(v) = \psi^{k_l(t,v)}\mathcal{F}_l(v)$. We need to check $t \mapsto k_l(t,v)$ is injective. Note that

$$\begin{aligned} a_l(0,\phi^t v) &= \frac{1}{l}\int_0^l b(s,\phi^t v)\,ds \\ &= \frac{1}{l}\int_0^l b(s+t,v) - b(t,v)\,ds \\ &= a_l(t,v) - b(t,v). \end{aligned}$$

This means

$$\begin{aligned} \mathcal{F}_l(\phi^t v) &= \psi^{a_l(0,\phi^t v)}\mathcal{F}_0(\phi^t v) \\ &= \psi^{a_l(0,\phi^t v)+b(t,v)}\mathcal{F}_0(v) \\ &= \psi^{a_l(t,v)}\mathcal{F}_0(v). \end{aligned}$$

Therefore, $\mathcal{F}_l(\phi^t v) = \psi^{k_l(t,v)}\mathcal{F}_l(v) = \psi^{a_l(t,v)}\mathcal{F}_0(v)$, and hence

$$\frac{d}{dt}|_{t=0}k_l(t,v) = \frac{d}{dt}|_{t=0}a_l(t,v) = \frac{b(l,v)}{l}. \tag{III.5.4}$$

The proof of Lemma IV.24 shows the above quantity is positive. So $\mathcal{F}_l$ is injective along geodesics, as desired. $\square$

We now proceed to find a Hölder estimate for $\mathcal{F}_l$, which we will denote by $\mathcal{F}$ from now on for simplicity. Most of the work is finding estimates for the map $\mathcal{F}_0$ from (III.5.2) (Proposition III.38).

Lemma III.34. *Let $b(t,v)$ as in (III.5.2). Let A, B as in (III.5.1). Then $b(t,v)$ satisfies*

$$At - B' \leq b(t,v) \leq At.$$

for all t, where B' is a constant depending only on λ, A, B.

Proof. Recall $b(t,v) = d(P_\eta f(p), P_\eta f(q))$, which is bounded above by $d(f(p), f(q))$ because orthogonal projection is a contraction in negative curvature. This quantity is in turn bounded above by At, using the Lipschitz bound for f in (III.5.1).

Next, let R be the constant in Lemma III.31. Then $d(f(p), P_\eta(f(p)) \leq R$, which implies $b(t,v) \geq d(f(p), f(q)) - 2R$. The desired estimate then follows from the lower bound for $d(f(p), f(q))$ in (III.5.1). $\square$

Lemma III.35. *There is small enough $\delta_0 = \delta_0(\Lambda)$ so that for any $\delta \leq \delta_0$ the following holds. Fix $v \in T^1\tilde{M}$ and let $x \in \tilde{M}$ be a point such that the orthogonal projection $P_v(x)$ of x onto the bi-infinite geodesic determined by v is the footpoint of v. Let $w \in W^{su}(v)$ and suppose further that $d_{su}(v,w) < \delta$. Then there is a constant $C = C(n, \Gamma, \lambda, \Lambda, i_M)$ so that $d(P_v(x), P_w(x)) < C\delta$.*

Proof. Let p and q denote the footpoints of v and w, respectively. Let $u \in T^1_pM$ be the vector tangent to the curve in the horosphere connecting p and q. Let $\gamma(s) = \exp_p(su)$. Let s_0 be such that $\gamma(s_0)$ intersects the geodesic determined by w. We claim there are positive constants $\delta_0 = \delta_0(\Lambda)$ and $C = C(\Lambda)$ so that if $d(v,w) \leq \delta \leq \delta_0$ then $s_0 \leq C\delta$. By [HIH77, Proposition 4.7] we know $\tanh(\Lambda s_0) \leq Cd_{su}(v,w) \leq C\delta$, where C is some constant depending only on Λ, which proves the claim.

Now let θ denote the angle between the geodesic segment $[x, \gamma(s_0)]$ and the geodesic determined by w. We start by showing θ is close to $\pi/2$. In the case where x and p coincide, the above angle θ is the same as the angle θ in Lemma III.15. Thus, $\cos\theta \leq \Lambda s_0$.

Otherwise, let $t_0 = d(x,p) \neq 0$. We consider two further cases: $d(x, \gamma(s_0)) \leq \delta$ and $d(x, \gamma(s_0)) \geq \delta$. For the proof in the first case, we start by noting that

$$\begin{aligned} d(p, P_w(x)) &\leq d(p, \gamma(s_0)) + d(P_w(x), \gamma(s_0)) \\ &\leq d(p,q) + d(q, \gamma(s_0) + d(P_w(x), \gamma(s_0)) + d(x, \gamma(s_0)). \end{aligned}$$

Since $d(v, w) < \delta$ (by assumption), so is $d(p, q)$. By Lemma III.15, $d(q, \gamma(s_0)) \leq \Lambda s_0^2 \leq C\delta^2$. Finally, note $d(x, P_w(x)) \leq d(x, \gamma(s_0))$. So applying the hypothesis $d(x, \gamma(s_0)) \leq \delta$ completes the proof in this case.

Now we consider the case $d(x, \gamma(s_0)) \geq \delta$. Let $v_0 \in T_x^1 M$ such that $\exp_x(t_0 v_0) = p$. For $0 < s \leq s_0$, let $v(s) \in T_{x_0} M$ such that $\exp_x(t_0 v(s)) = \gamma(s)$. Then $X(s) := \frac{d}{dt}|_{t=t_0} \exp_x(tv(s))$ is a vector field along $\gamma(s)$. The hypothesis $d(x, \gamma(s_0)) \geq \delta$ allows us to bound

$$\frac{\|X(s)\|}{\|X(s_0)\|} = \frac{d(x, \gamma(s))}{d(x, \gamma(s_0)} \leq 1 + \frac{s_0}{d(x, \gamma(s_0))} \leq 1 + \frac{s_0}{\delta} \leq 1 + C, \tag{III.5.5}$$

where C is a constant depending only on Λ.

We now claim there is a constant $C = C(n, \Gamma, \lambda, \Lambda, i_M)$ so that

$$\cos\theta = \frac{\langle \text{grad} B_\xi(\gamma(s_0)), X(s_0)\rangle}{\|X(s_0)\|} \leq C s_0. \tag{III.5.6}$$

Since $\langle \text{grad} B_\xi(\gamma(0)), X(0)\rangle = 0$, the fundamental theorem of calculus gives

$$\langle \text{grad} B_\xi(\gamma(s_0)), X(s_0)\rangle = \int_0^{s_0} \frac{d}{ds} \langle \text{grad} B_\xi(\gamma(s)), X(s)\rangle \, ds.$$

So the desired bound for $\cos(\theta)$ follows from bounding the integrand from above by $C\|X(s_0)\|$ for all $s \in [0, s_0]$. In light of (III.5.5), it suffices to find an upper bound of the form $C\|X(s)\|$. To this end, we rewrite integrand using the product rule:

$$\frac{d}{ds}\langle \text{grad} B(\gamma(s)), X(s)\rangle = \langle \nabla_{\gamma'} \text{grad} B(\gamma(s)), X(s)\rangle + \langle \text{grad} B(\gamma(s)), \nabla_{\gamma'} X(s)\rangle. \tag{III.5.7}$$

The first term on the righthand side is bounded above by

$$\|X(s)\| |\langle \nabla \text{grad} B_{\gamma'}(\gamma(s)), u\rangle| = \|X(s)\| \text{Hess} B_\xi(\gamma'(s), u)$$

for some unit vector u. Next, using that the Hessian is symmetric bilinear form, together with Lemma III.15, we have

$$\text{Hess} B_\xi(\gamma'(s), u) \leq \frac{1}{4} \text{Hess} B_\xi(\gamma'(s) + u, \gamma'(s) + u) \leq \frac{\Lambda}{4} \|\gamma'(s) + u\| \leq \frac{\Lambda}{2}.$$

Now we consider the second term in (III.5.7). First note that $\nabla_{\gamma'} X(s) = J_s'(t_0)$, where $J_s(t)$ is the Jacobi field along the geodesic $\eta_s(t) = \exp_x(tv(s))$ with initial conditions $J_s(0) = 0$ and $J_s'(0) = v(s)$. In order to bound $\|J_s'(t_0)\|$, we let $e_1(t) = \eta'(t), e_2(t), \ldots, e_n(t)$ be a parallel orthonormal frame along $\eta(t)$. Let $f_1(t), \ldots, f_n(t)$ such that $J_s(t) = \sum_{i=1}^n f_i(t) e_i(t)$. The fact that J_s satisfies the Jacobi equation means $f_1''(t) = 0$, so $f_1(t) = \langle v(s), \eta'(0)\rangle t$ and $|f_1'(t)| \leq \|v(s)\| = \|X(s)\|$. Now let $J_s^\perp$ denote the component of J_s which is perpendicular to η_s. By [Bal95, Proposition IV.2.5], we have

$$\|(J_s^\perp)'(t_0)\| \leq \cosh(\Lambda t_0)\|(J_s^\perp)'(0)\| \leq \cosh(\Lambda R)\|X(s)\|,$$

where R is the constant in Lemma III.31. This completes the verification of (III.5.6).

Now let q' be the orthogonal projection of x onto the geodesic determined by w. We use our bound for $\cos\theta$ to show $d(p, q')$ is small. Consider the geodesic triangle with vertices x, q' and $\gamma(s_0)$. The angle at q' is $\pi/2$ by definition of orthogonal projection, and we have just shown the angle θ at $\gamma(s_0)$ satisfies $\cos\theta \leq Cs_0$, where $s_0 < C\delta$. Then by [Bea12, Theorem 7.11.2 iii)] $\tanh(d(q', \gamma(s_0)) \leq C\delta \tanh(d(x, q')) \leq C\delta$, where C is the constant in (III.5.6). Thus, for δ_0 sufficiently small in terms of C, we see that $d(q', \gamma(s_0) \leq 2C\delta$ whenever $\delta < \delta_0$. Now recall from the first paragraph that $d(p, \gamma(s_0)) = s_0 \leq C\delta$. Noting that $d(p, q') \leq d(p, \gamma(s_0)) + d(q', \gamma(s_0))$ completes the proof. □

Proposition III.36. *Let $\delta_0 = \delta_0(\Lambda)$ be as small as in the previous lemma. Suppose $w \in W^{su}(v)$ and $d_{su}(v, w) < \delta_0$. Then there is a constant $C = C(n, \Gamma, \lambda, \Lambda, i_M)$ so that $d(\mathcal{F}_0(v), \mathcal{F}_0(w)) \leq C d_{su}(v, w)^{A\lambda/\Lambda}$, where A is the constant in Proposition III.24. The analogous statement holds if $w \in W^{ss}(v)$ instead.*

Proof. Let p and q denote the footpoints of v and w, respectively. By definition, $\mathcal{F}_0(v) = P_{\eta_1}(f(p))$ and $\mathcal{F}_0(w) = P_{\eta_2}(f(q))$ for the appropriate bi-infinite geodesics η_1 and η_2 in $\tilde{N}$. By the triangle inequality,

$$d(P_{\eta_1}(f(p)), P_{\eta_2}(f(q)) \leq d(P_{\eta_1}(f(p)), P_{\eta_1}(f(q))) + d(P_{\eta_1}(f(q)), P_{\eta_2} f(q)). \tag{III.5.8}$$

We start by estimating the first term. Let $d_{su}(v, w) = \delta$. Then $d(p, q) < \delta$. By (III.5.1), we have $d(f(p), f(q)) < A\delta$. Since orthogonal projection is a contraction in negative curvature, the second term is bounded above by $d(f(p), f(q)) \leq A\, d(p, q)$.

Thus it remains to bound $d(P_{\eta_1}(f(q)), P_{\eta_2}(f(q)))$, which we do by applying Lemma III.35. Since $\mathcal{F}(v)$ and $\mathcal{F}(w)$ are on the same weak unstable leaf, there is w' on the orbit of w so that $\mathcal{F}(v)$ and $\mathcal{F}(w')$ are on the same strong unstable leaf. In light of Lemma III.35, it suffices to find a Hölder estimate for $d_{su}(\mathcal{F}(v), \mathcal{F}(w'))$.

Again, let $\delta = d_{su}(v, w)$ for simplicity. Since the unstable distance exponentially expands under the geodesic flow, there is some positive time t so that $d_{su}(\phi^t v, \phi^t w) = 1$. More precisely, [HIH77, Proposition 4.1] implies $\Lambda t \geq \log(1/\delta)$.

Next, note that $d(\mathcal{F}_0(\phi^t v), \mathcal{F}_0(\phi^t w')) \leq d(\mathcal{F}_0(\phi^t v), \mathcal{F}_0(\phi^t w)) \leq 2R + A$, where R is as in

Lemma III.31 and A is the Lipschitz constant for f. Indeed, since f is A-Lipschitz, we have $d(f(\phi^t v), f(\phi^t w)) \leq A$, and $d(f(\phi^t v), P_{\eta_1} f(\phi^t v)) \leq R$. By [HIH77, Theorem 4.6], we also have the bound $d_{su}(\mathcal{F}_0(\phi^t v), \mathcal{F}_0(\phi^t w')) \leq \frac{2}{\Lambda} \sinh(\Lambda(2R + A)/2)$.

By [HIH77, Proposition 4.1], we have the following estimate for how the unstable distance gets contracted under the geodesic flow:

$$d_{su}(\mathcal{F}_0(v), \mathcal{F}_0(w')) \leq e^{-\lambda b(t,v)} d_{su}(\psi^{b(t,v)} \mathcal{F}_0(v), \psi^{b(t,v)} \mathcal{F}_0(w')).$$

Now recall $b(t, v) \geq A^{-1} t - B'$ from Lemma III.34. This, together with the previous paragraph, gives

$$d_{su}(\mathcal{F}(v), \mathcal{F}(w')) \leq e^{-\lambda(A^{-1}t - B')} \frac{2}{\Lambda} \sinh(\Lambda(2R + A)/2) = Ce^{-\lambda A^{-1} t}$$

for some constant $C = C(\lambda, \Lambda, A, B)$. Finally, we use $t \geq \frac{log(1/\delta)}{\Lambda}$ to obtain $d_{su}(\mathcal{F}_0(v), \mathcal{F}_0(w')) \leq C\delta^{A^{-1}\lambda/\Lambda}$ for some other constant $C = C(\lambda, A, B)$. By Lemma III.35, the second term in (III.5.8) is thus bounded above by $C\delta^{A^{-1}\lambda/\Lambda}$ for some other constant $C = C(\lambda, \Lambda, A, B)$, which completes the proof. $\square$

Lemma III.37. *There is small enough δ_0, depending only on the curvature bounds λ and Λ, so that if $w \in W^{ss}(v)$ and $d(v, w) < \delta_0$, then*

$$c_1 d(v, w) \leq d_{ss}(v, w) \leq c_2 d(v, w),$$

where c_1 and c_2 are constants depending only on λ and Λ. The analogous statement holds for d_{su}.

Proof. By [HIH77, Theorem 4.6], we have $d_{ss}(v, w) \leq \frac{\Lambda}{2} \sinh(2/\Lambda\, d(p, q))$. Thus, if $d(p, q)$ is small enough (depending on Λ), we have $h(p, q) \leq \frac{4}{\Lambda} d(p, q) \leq \frac{4}{\Lambda} d(v, w)$.

By Lemma III.21, $d(v, w) \leq (1 + \Lambda) d(p, q)$. By the other estimate in [HIH77, Theorem 4.6], there is a constant C, depending only on λ, so that $d(p, q) \leq Ch(p, q)$ for all p, q with $d(p, q)$ sufficiently small in terms of λ. $\square$

Proposition III.38. *There exists small enough δ_0, depending only on λ, Λ, $\mathrm{diam}(M)$, so that for any $v, w \in T^1\tilde{M}$ satisfying $d(v, w) < \delta_0$ we have $d(\mathcal{F}_0(v), \mathcal{F}_0(w)) \leq Cd(v, w)^{A^{-1}\lambda/\Lambda}$ for some constant $C = C(n, \Gamma, \lambda, \Lambda, i_M)$.*

Proof. By Lemma III.11, we know that for any $v, w \in T^1 M$ with $d(v, w) = \delta$, there is a time $\sigma = \sigma(v, w) \in [-\delta, \delta]$ and a point $[v, w] \in T^1\tilde{M}$ so that

$$[v, w] = W^{ss}(v) \cap W^{su}(\phi^\sigma w).$$

Let $\alpha = A^{-1}\lambda/\Lambda$ be the exponent from Proposition III.36. Applying Proposition III.36, followed by Lemma III.37 and Proposition III.13, and finally Lemma III.11, we have

$$d(\mathcal{F}_0(v), \mathcal{F}_0([v,w])) \leq C' d_{ss}(v,[v,w])^\alpha \leq Cd(v,\phi^\sigma w)^\alpha \leq C(2d(v,w))^\alpha$$

for some constants C and C' depending only on $n, \Gamma, \lambda, \Lambda, i_M, \mathrm{diam}(M)$. By a similar argument,

$$d(\mathcal{F}_0([v,w], \phi^\sigma w) \leq Cd(v,w)^\alpha.$$

Finally, as in the beginning of the proof of Proposition III.36, we have

$$d(\mathcal{F}(w), \mathcal{F}(\phi^\sigma w)) \leq A\delta.$$

Now, $d(\mathcal{F}_0(v), \mathcal{F}_0(w)) \leq Cd(v,w)^\alpha$ follows from the triangle inequality. □

Lemma III.39. *There is a constant $C = C(\lambda, \Lambda, t)$ so that $d(\phi^t v, \phi^t w) < Cd(v,w)$ for all $d(v,w) \leq \delta_0$, where δ_0 depends only on λ, Λ,* diam(M)*.*

Proof. As before, consider $[v,w] = W^{ss}(v) \cap W^{su}(\phi^{\sigma(v,w)}w)$. The distance between w and $\phi^{\sigma(v,w)}$ remains constant under application of ϕ^t, and since v and $[v,w]$ are on the same stable leaf, their distance contracts under application of ϕ^t. Finally, since $[v,w]$ and $\phi^{\sigma(v,w)}w$ are on the same strong unstable leaf, [HIH77, Proposition 4.1], Lemma III.37 and Proposition III.13 imply

$$d(\phi^t[v,w], \phi^t\phi^{\sigma(v,w)}w) \leq e^{\Lambda t} d_{su}([v,w], \phi^{\sigma(v,w)}w) \leq e^{\Lambda t} Cd(v,w)$$

for some constant C depending only on λ, Λ, diam(M). □

Lemma III.40. *Let C denote the constant in Proposition III.38, and let $\alpha = A^{-1}\lambda/\Lambda$ denote the Hölder exponent. Then there is a constant $C_1 = C_1(C,t)$ so that*

$$|b(t,v) - b(t,w)| \leq C_1 d(v,w)^\alpha.$$

Proof. By Proposition III.38, we have $d(\mathcal{F}_0(v), \mathcal{F}_0(w)) \leq Cd(v,w)^\alpha$ and $d(\mathcal{F}_0(\phi^t v), \mathcal{F}_0(\phi^t w)) \leq Cd(\phi^t v, \phi^t w)^\alpha$. Applying Lemma III.39 shows $d(\mathcal{F}_0(\phi^t v), \mathcal{F}_0(\phi^t w)) \leq C_1 d(v,w)^\alpha$, where C_1 depends on C and t. The desired result now follows from Lemma III.12. □

Proof of Proposition III.4. We want to find a Hölder estimate for $\mathcal{F}_l(v) = \psi^{a_l(0,v)}\mathcal{F}_0(v)$, where $a_l(0,v) = \frac{1}{l}\int_0^l b(t,v)\,dt$. By the triangle inequality,

$$d(\mathcal{F}_l(v), \mathcal{F}_l(w)) \leq d(\psi^{a_l(0,v)}\mathcal{F}_0(v), \psi^{a_l(0,v)}\mathcal{F}_0(w)) + d(\psi^{a_l(0,v)}\mathcal{F}_0(w), \psi^{a_l(0,w)}\mathcal{F}_0(w)).$$

To bound the first term, note that for all $t \in [0, l]$ we have $b(t, v) \leq At \leq Al$ by Lemma III.34. Hence the average $a_l(0, v)$ is bounded above by A. By Lemma III.39 and Proposition III.38, we have

$$d(\psi^{a_l(0,v)}\mathcal{F}_0(v), \psi^{a_l(0,v)}\mathcal{F}_0(w)) \leq Cd(\mathcal{F}_0(v), \mathcal{F}_0(w)) \leq Cd(v, w)^\alpha,$$

where C depends only on l and the constant from Proposition III.38. As such, C depends only on n, Γ, λ, Λ, i_M. By Lemma III.40, the second term is bounded above by

$$|a_l(0, v) - a_l(0, w)| \leq \frac{1}{l}\int_0^l |b(t, v) - b(t, w)|\, dt, \leq Cd(v, w)^\alpha,$$

where C again depends only on n, Γ, λ, Λ, i_M. $\square$

CHAPTER IV
Volume Estimate

This chapter is devoted to proving Theorem I.12. We will repeatedly use the following standard construction, part of which can be found in [BCG96, Section 4]:

Construction IV.1. Let $f : M \to N$ be a homotopy equivalence of negatively curved manifolds. Let $\partial\tilde{M}$ denote the visual boundary of $\tilde{M}$. We can construct a map $\overline{f} : \partial\tilde{M} \to \partial\tilde{N}$ such that for all $\gamma \in \Gamma$ and all $\xi \in \partial\tilde{M}$ we have $\overline{f}(\gamma.\xi) = (f_*\gamma).\overline{f}(\xi)$. Indeed, the homotopy equivalence $f : M \to N$ can be lifted to a Γ-equivariant map $\tilde{f} : \tilde{M} \to \tilde{N}$ such that $\tilde{f}$ is additionally a quasi-isometry (details in [BCG96, Section 4]). Hence $\tilde{f}$ induces a Γ-equivariant map $\overline{f}$ between the boundaries $\partial\tilde{M}$ and $\partial\tilde{N}$.

Now recall the space of geodesics of $\tilde{M}$ is the quotient of $T^1\tilde{M}$ obtained by identifying any two unit tangent vectors on the same orbit of the geodesic flow. This space can be identified with the set $\partial^2\tilde{M}$ of pairs of distinct points in $\partial\tilde{M}$ by associating the equivalence class of the unit tangent vector v with the pair $(\pi(v), \pi(-v))$ of its forward and backward endpoints. Thus, the product $\overline{f} \times \overline{f}$ gives a map between the spaces of geodesics of $\tilde{M}$ and $\tilde{N}$. For notational simplicity, we will write this map as $\overline{f} : \partial^2\tilde{M} \to \partial^2\tilde{N}$.

Note the case $\varepsilon = 0$ of Theorem I.12 is Theorem A in [Ham99]. We follow the same overall approach as in [Ham99], which we now summarize. It follows from arguments in [Ota90] that the marked length spectrum of M determines the so-called *cross-ratio* of four points on the boundary $\partial\tilde{M}$. We start by generalizing these arguments to analyze how perturbing the marked length spectrum as in (I.7) affects the cross-ratio (Proposition IV.3).

In [Ham99], Hamenstädt proves the cross-ratio determines the so-called *Liouville current*, a measure on $\partial^2\tilde{M}$ which can be used to reconstruct the Liouville measure on T^1M. In the $\varepsilon = 0$ case, that is, equality of the marked length spectra, the geodesic flows on T^1N and T^1M are conjugate [Ham92], so one can use equality of Liouville currents to obtain equality of Liouville measures and hence volumes.

In Theorem IV.6, we generalize the arguments in [Ham99] to analyze how perturbing the cross-ratio – due to perturbing the marked length spectrum – affects the Liouville current.

However, an estimate of the Liouville currents does not immediately imply a volume estimate since when $\varepsilon > 0$, the geodesic flows of M and N need not be conjugate. We instead obtain controlled orbit equivalences between the geodesic flows on T^1M and T^1N by delicately implementing the construction in [Gro00].

IV.1: The cross-ratio

We now define the cross-ratio associated to any negatively curved Riemannian manifold (M, g). Let $p : \tilde{M} \to M$ be the universal cover of M and let $\partial\tilde{M}$ be the visual boundary of $\tilde{M}$. Let $\pi : T^1\tilde{M} \to \partial\tilde{M}$ denote the map which sends v to the forward boundary point of the geodesic determined by v. Let $\partial^4\tilde{M}$ denote pairwise distinct quadruples of points in $\partial\tilde{M}$.

Definition IV.2. [Ota92, Lemma 2.1] Let $a, b, c, d \in \partial^4\tilde{M}$. Let $a_i, b_i, c_i, d_i \in \tilde{M}$ be sequences converging to a, b, c, d respectively. Define

$$[a, b, c, d] = \lim_{i\to\infty} d(a_i, c_i) + d(b_i, d_i) - d(a_i, d_i) - d(b_i, c_i), \tag{IV.1.1}$$

where d is the Riemannian distance function. By [Ota92, Lemma 2.1], this limit exists and is independent of the chosen sequences a_i, b_i, c_i, d_i. We call $[\cdot\,,\cdot\,,\cdot\,,\cdot]$ the *cross-ratio*.

Theorem 2.2 in [Ota92] shows the cross-ratio is completely determined by the marked length spectrum, and the argument is not specific to dimension 2. In this section, we prove the following result which shows how perturbing the marked length spectrum affects the cross-ratio.

Proposition IV.3. *Let (M, g) and (N, g_0) be negatively curved manifolds with ε-close marked length spectra as in (I.7). Let $\overline{f} : \partial\tilde{M} \to \partial\tilde{N}$ be the map constructed from the homotopy equivalence $f : M \to N$ as in Construction IV.1. We then have*

$$(1 - \varepsilon)[a, b, c, d] \leq [\overline{f}(a), \overline{f}(b), \overline{f}(c), \overline{f}(d)] \leq (1 + \varepsilon)[a, b, c, d].$$

Over the course of the proof of [Ota92, Theorem 2.2], the following lemma is proved, giving more precise information about how the marked length spectrum determines the cross-ratio. We include a careful proof, since the setup will be needed to prove Proposition IV.3.

Lemma IV.4. *Given $(a, b, c, d) \in \partial^4\tilde{M}$, there exist sequences a_i, b_i, c_i, d_i converging to a, b, c, d, respectively, so that the terms $d(a_i, c_i), d(b_i, d_i), d(a_i, d_i), d(b_i, c_i)$ can be approximated arbitrarily well by lengths of closed geodesics.*

Proof. Since M is negatively curved, the geodesic flow ϕ^t on T^1M is Anosov; hence there exists $v \in T^1M$ with dense forward and backward orbit. Let $v_1, v_2 \in T^1\tilde{M}$ vary over all lifts of v which determine two distinct geodesics in $\tilde{M}$. Then quadruples of the form $(\pi(v_1), \pi(-v_1), \pi(-v_2), \pi(v_2))$ are dense in $\partial^4\tilde{M}$. Since the cross-ratio is continuous [Ota92], it suffices to check the proposition on this dense set of quadruples.

For $i = 1, 2$, let $T_i^+, T_i^- > 0$ be large enough such that the expression

$$d(\phi^{T_1^+}v_1, \phi^{-T_2^-}v_2) + d(\phi^{T_2^+}v_2, \phi^{-T_1^-}v_1) - d(\phi^{T_1^+}v_1, \phi^{-T_1^-}v_1) - d(\phi^{T_2^+}v_2, \phi^{-T_2^-}v_2) \qquad \text{(IV.1.2)}$$

is arbitrarily close to $[\pi(v_1), \pi(-v_1), \pi(-v_2), \pi(v_2)]$. In (IV.1.2), expressions of the form $d(v, w)$ for $v, w \in T^1\tilde{M}$ should be understood as the distances in $\tilde{M}$ between their footpoints. Now fix $w \in T^1\tilde{M}$. Since the geodesic tangent to v_1 projects to a geodesic with dense forward orbit in T^1M, we can make T_1^+ larger if necessary so that $Dp(\phi^{T_1^+}v_1)$ is arbitrarily close to $Dp(w)$ in T^1M. Hence there is some $\gamma_1^+ \in \Gamma$ such that $\gamma_1^+.w$ is arbitrarily close to $\phi^{T_1^+}v_1$ in $\tilde{M}$. By the same argument, there exist $\gamma_i^\pm \in \Gamma$ such that $\gamma_i^\pm.w$ is close to $\phi^{T_i^\pm}v_i$ for $i = 1, 2$.

We now use this setup to show terms in (IV.1.2) can be approximated arbitrarily well by lengths of closed geodesics. Consider the geodesic c in $\tilde{M}$ joining the basepoints of $\gamma_1^-.w$ and $\gamma_2^+.w$. Since the endpoints of c can be made arbitrarily close to $\pi(-v_1)$ and $\pi(v_2)$, the tangent vectors to c are arbitrarily close to the geodesic $(\pi(-v_1), \pi(v_2))$. Also, $\phi^{-T_1^-}v_1$ gets arbitrarily close to the tangent vector to $(\pi(-v_1), \pi(v_2))$ as T_1^- gets larger. So the tangent vector to c at the footpoint $\gamma_1^-.w$ is arbitrarily close to $\phi^{-T_1^-}v_1$, and hence to $\gamma_1^-.w$ as well. Similarly, the tangent vector to c at the footpoint of $\gamma_2^+.w$ is arbitrarily close to the vector $\gamma_2^+.w$.

Now consider the projection $p(c)$ in M. This is a closed curve which is freely homotopic to $\gamma_2^+ \circ (\gamma_1^-)^{-1}$, and is a geodesic away from the basepoint of $Dp(w)$. In the previous paragraph we showed the two tangent vectors to $p(c)$ at that point are both arbitrarily close to the vector $Dp(w)$. The Anosov closing lemma then implies $p(c)$ is shadowed by a closed geodesic; see [Fra18, p. 105] and [KH97, Theorem 6.4.15]. In particular, this closed geodesic is in the same free homotopy class as $p(c)$. So $d(\phi^{T_1^+}v_1, \phi^{T_1^-}v_1)$ is approximately $\mathcal{L}_g(\gamma_2^+ \circ (\gamma_1^-)^{-1})$. Using an analogous argument, the other three terms in equation (IV.1.2) can also be approximated by terms of the form $\mathcal{L}_g(\gamma_i^+ \circ (\gamma_j^-)^{-1})$. $\square$

Proof of Proposition IV.3. Let $(a, b, c, d) \in \partial^4\tilde{M}$. By the previous lemma, there are sequences $a_i, b_i, c_i, d_i \in \tilde{M}$ converging to a, b, c, d along with sequences $\gamma_{a_i}, \gamma_{b_i}, \gamma_{c_i}, \gamma_{d_i}$ such that $d(a_i, b_i)$ is approximately $\mathcal{L}_g(\gamma_{b_i} \circ \gamma_{a_i}^{-1})$ and analogously for the other three terms in the defining equation for $[a, b, c, d]$. Let v_{a_i}, v_{b_i} be tangent vectors to the geodesic through a and b based at a_i and b_i respectively. Let v_{c_i} and v_{d_i} be defined analogously. Recall the γ_i were chosen such that there is $w \in T^1\tilde{M}$ satisfying the condition that the vectors $\gamma_{a_i}.w, \gamma_{b_i}.w, \gamma_{c_i}.w, \gamma_{d_i}.w$ are arbitrarily close to the vectors $v_{a_i}, v_{b_i}, v_{c_i}, v_{d_i}$.

By [Gro00], there exists a Γ-equivariant homeomorphism $\mathcal{F} : T^1\tilde{M} \to T^1\tilde{N}$, which is an orbit equivalence of geodesic flows. Moreover, $\mathcal{F}$ sends the geodesic through $a, b \in \partial\tilde{M}$ to the geodesic through $\overline{f}(a), \overline{f}(b) \in \partial\tilde{N}$. Consider the distance between the footpoints of $\mathcal{F}(v_{a_i})$ and $\mathcal{F}(v_{b_i})$. Since $\mathcal{F}$ is continuous we know $\mathcal{F}(v_{a_i})$ is close to $\gamma_{a_i}.\mathcal{F}(w)$ and $\mathcal{F}(v_{b_i})$ is close to $\gamma_{b_i}.\mathcal{F}(w)$. By the Anosov closing lemma, (the projection of) the geodesic through $\gamma_{a_i}.\mathcal{F}(w)$ and $\gamma_{b_i}.\mathcal{F}(w)$ can be approximated with a closed geodesic of length $\mathcal{L}_g(f_*(\gamma_{b_i} \circ \gamma_{a_i}^{-1}))$. The same argument can be used to approximate the other three distances in the limit definition of $[\overline{f}(a), \overline{f}(b), \overline{f}(c), \overline{f}(d)]$. The desired result then follows from the assumption $1 - \varepsilon \leq \frac{\mathcal{L}_{g_0}}{\mathcal{L}_g} \leq 1 + \varepsilon$. ☐

Remark IV.5. This proof does not use that $1 \pm \varepsilon$ is close to 1, so this generalizes [Ota92, Proposition 4.2].

IV.2: The Liouville current

Let ω be the 1-form on T^1M obtained by pulling back the canonical 1-form on T^*M to TM via the identification induced by the Riemannian metric and then restricting to T^1M. (See Section II.3 for more details.) Then ω and $d\omega$ are both flow-invariant, and ω is a contact form, meaning $\omega \wedge (d\omega)^{n-1}$ is a volume form on T^1M. The associated measure on T^1M is called the *Liouville measure*. The total Liouville volume of T^1M is the product of the Riemannian volume of M and the volume of the unit sphere in dimension $n-1$; thus the ratio of the volumes of M and N is the same as the ratio of the Liouville volumes of their respective unit tangent bundles.

Recall the space of geodesics is the quotient of $T^1\tilde{M}$ by the action of the geodesic flow, and can also be identified with the set $\partial^2\tilde{M}$ of pairs of distinct points in the boundary (see Construction IV.1). Since $d\omega$ is flow-invariant, it descends to a 2-form on the space of geodesics $\partial^2\tilde{M}$. This form is also symplectic, meaning $(d\omega)^{n-1}$ is a volume form on $\partial^2\tilde{M}$. The associated measure is called the *Liouville current*. In this section, we establish the following relation between the marked length spectra and the Liouville currents:

Theorem IV.6. *Let (M, g) be a closed negatively curved Riemannian manifold of dimension at least 3 with fundamental group Γ. Let (N, g_0) be another closed negatively curved manifold with fundamental group Γ and assume the geodesic flow on T^1N has Anosov splitting of $C^{1+\alpha}$ regularity. Suppose that the marked length spectra of M and N are ε-close as in (I.7). Let λ^M and λ^N denote the Liouville currents on $\partial^2\tilde{M}$ and $\partial^2\tilde{N}$ respectively, and let $\overline{f} : \partial^2\tilde{M} \to \partial^2\tilde{M}$*

as in Construction IV.1. Then there is a constant $C = C(\tilde{N}, \tilde{g}_0)$ such that

$$(1 - C\varepsilon^\alpha)(1-\varepsilon)^{n-1}\overline{f}_*\lambda^M \leq \lambda^N \leq (1 + C\varepsilon^\alpha)(1+\varepsilon)^{n-1}\overline{f}_*\lambda^M. \tag{IV.2.1}$$

If, in addition, (N, g_0) is locally symmetric and ε is sufficiently small (depending on $n = \dim N$), then α can be replaced with 2 in the above estimates and the constant C depends only n.

Remark IV.7. If the Anosov splitting of T^1N is only C^1, then our proof shows the quantities $(1 \pm C\varepsilon^\alpha)$ can be replaced with constants that converge to 1 as $\varepsilon \to 0$, but we are not able to determine the explicit dependence of these constants on ε; see the statement above Lemma IV.15.

The proof of this theorem relies on relating the Liouville current to the cross-ratio, in order to then apply Proposition IV.3. We begin by explaining the explicit relation between the Liouville current and the cross-ratio in the case where $\dim(M) = 2$. Let $a, b, c, d \in \partial\tilde{M}$ be four distinct points. Since $\partial\tilde{M}$ is a circle, the pair of points (a, b) determines an interval in the boundary (after fixing an orientation). Let $(a, b) \times (c, d) \in \partial^2\tilde{M}$ denote the geodesics starting in the interval (a, b) and ending in the interval (c, d). Then

$$\lambda((a, b) \times (c, d)) = \frac{1}{2}[a, b, c, d]. \tag{IV.2.2}$$

(See [Ota90, Proof of Theorem 2] and [HP97, Theorem 4.4].)

In [Ham99], Hamenstädt relates the Liouville current and the cross-ratio for manifolds of any dimension. If, in addition, the manifold N is such that TT^1N has C^1 Anosov splitting, then the Liouville current is completely determined by the cross-ratio, and hence by the marked length spectrum, as is the case for surfaces. Hamenstädt's argument shows more specifically that if N satisfies the C^1 Anosov splitting condition and M is another manifold with the same marked length spectrum, and hence cross-ratio, as N, then the Liouville currents of M and N agree. In particular, this shows Theorem IV.6 when $\varepsilon = 0$.

Before proving Theorem IV.6, we recall notation, terminology and select arguments from [Ham99]: Hamenstädt constructs measures $\mathcal{S}$ and $\mathcal{P}$ (to be defined in Constructions IV.9 and IV.21 respectively) on the space of geodesics, both completely determined by the cross-ratio, such that

$$\mathcal{S} \leq \lambda \leq \mathcal{P} \tag{IV.2.3}$$

[Ham99, Propositions 3.8 and 3.13 a)]. If the underlying manifold has C^1 Anosov splitting, which is the case for the locally symmetric space N, the stronger conclusion

$$\mathcal{S} = \lambda = \mathcal{P} \tag{IV.2.4}$$

holds by [Ham99, Proposition 3.13 b)]. If M is such that its cross-ratio agrees with the cross-ratio of a locally symmetric space N, then $\mathcal{S}^N = \overline{f}_*\mathcal{S}^M$ and $\mathcal{P}^N = \overline{f}_*\mathcal{P}^M$. Combining this with (IV.2.3) and (IV.2.4) gives

$$\overline{f}_*\lambda^M \leq \overline{f}_*\mathcal{P}^M = \lambda^N = \overline{f}_*\mathcal{S}^M \leq \overline{f}_*\lambda^M, \tag{IV.2.5}$$

which forces $\lambda^N = \overline{f}_*\lambda^M$. Thus, in order to see the effects of the cross-ratio on the Liouville current, we need to use the exact constructions of $\mathcal{S}$ and $\mathcal{P}$ from [Ham99]. We start with preliminary definitions.

IV.2.1: Definition of $\mathcal{S}$

Definition IV.8. [Ham99, p. 123] Fix $\eta > 0$. Let $B(r) \subset \mathbb{R}^n$ be the ball of radius r centered at the origin and let $\phi_0(x, y)$ denote the dot product of $x, y \in \mathbb{R}^n$. Let $\beta_1, \beta_2 : B(r) \to \partial\tilde{M}$ be continuous embeddings so that

$$|[\beta_1(x), \beta_1(0), \beta_2(y), \beta_2(0)] - \phi_0(x, y)| \leq \eta r^2 \tag{IV.2.6}$$

for all $x, y \in B(r)$. We say the image $\beta_1(B(r)) \times \beta_2(B(r)) \subset \partial\tilde{M} \times \partial\tilde{M} \setminus \Delta$ is a $(1+\eta)$ *quasi-symplectic r-ball*. We let $\mathcal{Q}(\eta)$ denote the collection of all $(1+\eta)$-quasisymplectic r-balls for arbitrary r.

Fix any distance d on $\partial\tilde{M}$ that induces the visual topology. For $Q \in Q(\eta)$, we let $\mathrm{diam}(Q)$ be the $d \times d$ diameter of $Q \subset \partial^2\tilde{M}$.

For $Q \in \mathcal{Q}(\eta)$, define a quantity $\delta(Q)$ as follows. Write $Q = A \times B$, ie, $A = \beta_1(B(r))$, $B = \beta_2(B(r))$. First let $\delta(A \times B; a, b) = \sup_{\xi \in A, \zeta \in B}[a, \xi, b, \zeta]$. Now define

$$\delta(A \times B) = \inf_{a \in A, b \in B} \delta(A \times B; a, b). \tag{IV.2.7}$$

(See [Ham99, p. 124]).

Construction IV.9. [Ham99, p. 124] Let $C \subset \partial\tilde{M} \times \partial\tilde{M}$ a Borel set and let a_{n-1} denote the volume of the unit ball in $\mathbb{R}^{n-1}$. Define

$$\mathcal{S}_\eta(C) = \inf \left\{ a_{n-1}^2 \sum_{i=1}^{\infty} \delta(Q_i)^{n-1} \mid Q_i \in Q(\eta), \mathrm{diam}(Q_i) \leq \eta, C \subset \cup_{i=1}^{\infty} Q_i \right\}.$$

Finally, let $\mathcal{S}(C) = \limsup_{\eta \to 0} S_\eta(C)$.

This concludes our summary of [Ham99].

Hypothesis IV.10. *For the remainder of this section, we assume*

$$1 - \varepsilon \leq \frac{\mathcal{L}_{g_0}(f_*\gamma)}{\mathcal{L}_g(\gamma)} \leq 1 + \varepsilon.$$

(See (I.7) in the statement of Theorem I.9.)

By Proposition IV.3, the cross-ratios of M and N satisfy

$$(1-\varepsilon)[a,b,c,d] \leq [\overline{f}(a), \overline{f}(b), \overline{f}(c), \overline{f}(d)] \leq (1+\varepsilon)[a,b,c,d].$$

One might hope that this cross-ratio estimate implies $\mathcal{S}^N \leq (1+\varepsilon)^{n-1}\overline{f}_* S^M$, since this would immediately yield $\lambda^N \leq (1+\varepsilon)^{n-1}\overline{f}_*\lambda^M$ by (IV.2.3) and (IV.2.4). However, we are only able to conclude $\mathcal{S}^N_\varepsilon \leq (1+\varepsilon)^{n-1}\overline{f}_*\mathcal{S}^M$, so (IV.2.4) does not apply (Proposition IV.14). As such, we proceed to estimate the rate at which $\mathcal{S}^N_\varepsilon$ converges to λ^N as $\varepsilon \to 0$ (Proposition IV.15), and we also show how this estimate can be improved in the case where N is locally symmetric (Lemma IV.18 and Remark IV.20). The analysis for the measure $\mathcal{P}$ is similar, but the directions of the inequalities are reversed.

IV.2.2: Comparing $\overline{f}_*\mathcal{S}^M$ and $\mathcal{S}^N$

Changing the cross-ratio used to define $\mathcal{S}$ will change the the set $\mathcal{Q}(\eta)$ and the quantity $\delta(Q)$. We investigate this precisely below.

Lemma IV.11. *Let $Q \subset \partial^2\tilde{M}$. If $Q \in \mathcal{Q}(\eta)$ then $\overline{f}(Q) \in \mathcal{Q}(\eta + (1+\eta)\varepsilon)$.*

Proof. If $Q \in \mathcal{Q}(\eta)$, there are maps $\beta_i : B(r) \to \partial\tilde{M}$ for $i = 1, 2$ with $Q = \beta_1(B(r)) \times \beta_2(B(r))$ such that

$$|[\beta_1(x), \beta_1(0), \beta_2(y), \beta_2(0)]_M - \phi_0(x,y)| \leq \eta r^2. \tag{IV.2.8}$$

Using the triangle inequality, Proposition IV.3, and (IV.2.8) gives

$$\begin{aligned}
&|[\overline{f}\circ\beta_1(x), \overline{f}\circ\beta_1(0), \overline{f}\circ\beta_2(y), \overline{f}\circ\beta_2(0)]_N - \phi_0(x,y)| \\
&\leq \varepsilon[\beta_1(x), \beta_1(0), \beta_2(y), \beta_2(0)]_M + |[\beta_1(x), \beta_1(0), \beta_2(y), \beta_2(0)]_M - \phi_0(x,y)| \\
&\leq \varepsilon[\beta_1(x), \beta_1(0), \beta_2(y), \beta_2(0)]_M + \eta r^2 \\
&\leq \varepsilon(1+\eta)r^2 + \eta r^2,
\end{aligned}$$

which shows $\overline{f}(Q) \in \mathcal{Q}(\eta + (1+\eta)\varepsilon)$. ☐

Lemma IV.12. *For any $\eta > 0$, let $Q \in \mathcal{Q}(\eta)$. We then have*

$$(1-\varepsilon)\delta^M(Q) \le \delta^N(\overline{f}(Q)) \le (1+\varepsilon)\delta^M(Q).$$

Proof. This follows immediately from the definition of δ in Equation IV.2.7 together with Proposition IV.3. □

Corollary IV.13. *Let*

$$\chi_\eta^C = \left\{(A_i \times B_i)_{i\in\mathbb{N}} \,|\, C \subset \cup_{i=1}^\infty A_i \times B_i,\ \mathrm{diam}(A_i \times B_i) \le \eta,\ A_i \times B_i \in \mathcal{Q}(\eta)\right\}.$$

Then

$$\overline{f}\left(\chi_\eta^{\overline{f}^{-1}(C)}\right) \subset \chi^C_{\eta+(1+\eta)\varepsilon}$$

for sufficiently small η.

Proof. If $(A_i \times B_i)_{i\in\mathbb{N}} \in \chi_\eta^{\overline{f}^{-1}(C)}$ then $\overline{f}(A_i \times B_i)_{i\in\mathbb{N}}$ clearly satisfies the first condition in the definition of $\chi^C_{\eta+(1+\eta)\varepsilon}$. To check the second condition, note that since $\overline{f}$ is continuous, for any $\varepsilon > 0$ there exists $\eta > 0$ so that $\mathrm{diam}(A_i \times B_i) \le \eta$ implies $\mathrm{diam}(\overline{f}(A_i) \times \overline{f}(B_i)) \le \varepsilon \le \eta + (1+\eta)\varepsilon$. The third condition follows from the previous lemma. □

Proposition IV.14. *The following inequality of measures holds:*

$$\mathcal{S}^N_\varepsilon \le (1+\varepsilon)^{n-1}\overline{f}_*\mathcal{S}^M.$$

Proof. For any $C \subset \partial^2\tilde{N}$, Corollary IV.13 and Lemma IV.12 give

$$\begin{aligned}
\overline{f}_*\mathcal{S}^M_\eta(C) &= \mathcal{S}^M_\eta(\overline{f}^{-1}(C)) \\
&= \inf\left\{ a^2_{n-1}\sum_{i=1}^\infty \delta(A_i\times B_i)^{n-1} \,\middle|\, (A_i \times B_i)_{i\in\mathbb{N}} \in \chi_\eta^{\overline{f}^{-1}(C)}\right\} \\
&= \inf\left\{ a^2_{n-1}\sum_{i=1}^\infty \delta(A_i\times B_i)^{n-1} \,\middle|\, (\overline{f}(A_i)\times\overline{f}(B_i))_{i\in\mathbb{N}} \in \overline{f}(\chi_\eta^{\overline{f}^{-1}(C)})\right\} \\
&\ge \inf\left\{ a^2_{n-1}\sum_{i=1}^\infty \delta(A_i\times B_i)^{n-1} \,\middle|\, (\overline{f}(A_i)\times\overline{f}(B_i))_{i\in\mathbb{N}} \in \chi^C_{\varepsilon(1+\eta)+\eta}\right\} \\
&\ge \inf\left\{ \frac{a^2_{n-1}}{(1+\varepsilon)^{n-1}}\sum_{i=1}^\infty \delta(\overline{f}(A_i)\times\overline{f}(B_i))^{n-1} \,\middle|\, (\overline{f}(A_i)\times\overline{f}(B_i))_{i\in\mathbb{N}} \in \chi^C_{\varepsilon(1+\eta)+\eta}\right\} \\
&\ge \mathcal{S}^N_{\varepsilon(1+\eta)+\eta}(C)/(1+\varepsilon)^{n-1}
\end{aligned}$$

Taking $\eta \to 0$ completes the proof. □

IV.2.3: Comparing $\mathcal{S}_\varepsilon^N$ and λ^N

If TT^1N has C^1 Anosov splitting, then $\mathcal{S}_\varepsilon^N \to \lambda^N$ as $\varepsilon \to 0$ (see the proof of [Ham99, Corollary 3.12]). If we assume instead the splitting is $C^{1+\alpha}$, we obtain a more precise convergence statement:

Lemma IV.15. *Suppose the Anosov splitting of N is of class $C^{1+\alpha}$. Then there is a constant C, depending only on $\tilde{N}$, so that for all $\varepsilon > 0$ (sufficiently small in terms of n) we have*

$$S_\varepsilon^N \geq \frac{1}{1 + C\varepsilon^\alpha}\lambda^N.$$

Recall from the proof of [Ham99, Corollary 3.12] that if $\delta > 0$ and $\chi(\delta)$ is chosen as in [Ham99, Lemma 3.11] then $S_\chi \geq (1+\delta)^{-1}\lambda$. (This requires δ to be sufficiently small in terms of $n = \dim N$.) As such, we prove Lemma IV.15 by explicitly determining the dependence of $\chi(\delta)$ on δ. Note it follows from the proof of [Ham99, Lemma 3.11] that $\chi(\delta)$ is in turn equal to the quantity $\kappa(\delta)$ from [Ham99, Property 4), p. 130]. We now recall all the relevant definitions in the statement of [Ham99, Property 4)]:

First we recall the definition of the function ϕ at the beginning of [Ham99, Section 3]. Let ρ be a symplectic form on $\mathbb{R}^n \times \mathbb{R}^n$ so that for all $x, y \in \mathbb{R}^n$, the submanifolds $\{x\} \times \mathbb{R}^n$ and $\mathbb{R}^n \times \{y\}$ are Lagrangian. For $x \in \mathbb{R}^n$, let c_x be a curve in $\mathbb{R}^n$ such that $c_x(0) = 0$ and $c_x(1) = x$. Similarly define a curve c_y. Then define a surface $\Psi_{x,y}(s,t) = (c_x(s), c_y(t))$. Let $\phi(x,y) = \int_{\Psi_{x,y}} \rho$. By [Ham99, Lemma 3.1], the function ϕ is well-defined, ie, does not depend on the choice of curves c_x and c_y. Note that if ρ_0 is the standard symplectic form $\sum_i dx_i \wedge dy_i$, then the associated function $\phi_0(x,y)$ is the dot product of x and y in $\mathbb{R}^n$.

Hamenstädt also defines such a function ϕ associated to the symplectic form $d\omega$ on the space of geodesics using special coordinates $\Psi : \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \to \partial\tilde{M} \times \partial\tilde{M} \setminus \Delta$ to view $d\omega$ as a symplectic form on $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$. We recall the construction of Ψ, which can be found above the statement of [Ham99, Lemma 3.9]: There exists a geodesic flow invariant connection ∇ on T^1N called the Kanai connection. This connection is flat when restricted to the leaves of the strong stable and strong unstable foliations W^{ss} and W^{su}, respectively (see the discussion in [Ham99] for more details). Fix $v \in T^1\tilde{M}$ and let $L^{su} : T_vW^{su} \to W^{su}$ and $L^{ss} : T_vW^{ss} \to W^{ss}$ be exponential maps with respect to the restriction of this connection to W^{ss} and W^{su} respectively. Let $\{X_i\}$ and $\{Y_j\}$ be orthonormal bases for T_vW^{su} and T_vW^{ss} respectively so that $d\omega(X_i, Y_j) = \delta_{ij}$. For $w \in W^{su}(v)$ and $z \in W^{ss}(v)$ both sufficiently close to v, define $[w,z]$ to be the unique point in $W^{ss}(w) \cap W^u(z)$. The regularity of the function $[\cdot,\cdot]$ is the same as that of the Anosov splitting. Define $\Psi(x_1 \ldots, x_{n-1}, y_1, \ldots, y_{n-1}) = [L^{su}(\sum_i x_iX_i), L^{ss}(\sum_j y_jY_j)]$. Let ρ be the symplectic form on $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ given by $\rho(x, y) = d\omega(d\Psi x, d\Psi y)$, which is continuous when Ψ is C^1. Use this form ρ to define a function ϕ as above. Recall ϕ_0 was defined similarly, but in place of the symplectic form ρ, the standard symplectic form ρ_0 was used. Then the function ϕ has the following property:

Lemma IV.16. *[Ham99, Property 4, p. 130] Suppose the Anosov splitting of N is C^1. Then for any $\delta > 0$ there is $\kappa(\delta)$ so that whenever $\|x\|, \|y\| < \kappa(\delta)$ we have*

$$\frac{|\phi(x, y) - \phi_0(x, y)|}{\|x\|\|y\|} < \delta.$$

We now show how $\kappa(\delta)$ depends on δ in the case where the Anosov splitting is $C^{1+\alpha}$.

Lemma IV.17. *Suppose the Anosov splitting of N is $C^{1+\alpha}$. Then there is a constant $C = C(\tilde{N})$ so that $\kappa(\delta) = \left(\frac{\delta}{C}\right)^{1/\alpha}$ in the above lemma.*

Proof. Fix $x, y \in \mathbb{R}^{n-1}$ and consider the parametrized surface $\Psi_{x,y}(s, t) = (sx, ty)$. Then, definitionally, we have

$$\frac{\phi(x, y) - \phi_0(x, y)}{\|x\|\|y\|} = \frac{1}{\|x\|\|y\|} \int_{\Psi_{x,y}} \rho - \rho_0.$$

Write $\rho - \rho_0 = \sum_{ij} a_{ij} dx_i \wedge dy_j$. Since Ψ is $C^{1+\alpha}$, the a_{ij} are C^α. Moreover, $a_{ij}(0, 0) = 0$ [Ham99, Property 1), p. 128]. Thus $|a_{ij}(sx, ty)| \leq C\|(sx, ty)\|^\alpha \leq C\|(x, y)\|^\alpha$ for some constant C depending on $\tilde{N}$.

We now have

$$\begin{aligned}
\frac{|\phi(x, y) - \phi_0(x, y)|}{\|x\|\|y\|} &= \left| \frac{1}{\|x\|\|y\|} \int_0^1 \int_0^1 (\rho - \rho_0) \left(\frac{\partial \Psi}{\partial s}, \frac{\partial \Psi}{\partial t} \right) dsdt \right| \\
&= \left| \int_0^1 \int_0^1 \sum_{i,j=1}^n a_{ij}(sx, ty) dx_i \wedge dy_j \left(\frac{x}{\|x\|}, \frac{y}{\|y\|} \right) dsdt \right| \\
&\leq C\|(x, y)\|^\alpha \int_0^1 \int_0^1 \sum_{i,j=1}^n dx_i \wedge dy_j \left(\frac{x}{\|x\|}, \frac{y}{\|y\|} \right) dsdt \\
&\leq n^2 C\|(x, y)\|^\alpha.
\end{aligned}$$

If $\|x\|, \|y\| < \kappa$, we get $|\phi(x, y) - \phi_0(x, y)| \leq C\kappa^\alpha \|x\|\|y\|$ for some constant $C = C(\tilde{N})$. So we can take $\kappa(\delta) = (\delta/C)^{1/\alpha}$ for some other $C = C(\tilde{N})$ and the conclusion of Lemma IV.16 will hold. $\square$

Next we show how to improve the value of $\kappa(\delta)$ when N is a locally symmetric space.

Lemma IV.18. *If N is a locally symmetric space, and δ is sufficiently small (depending on n) then we can take $\kappa(\delta) = C\delta^{1/2}$ for some constant C depending only on the dimension of N.*

Proof. Let $g(x, y) = \phi(x, y) - \phi_0(x, y)$. Since N is locally symmetric, the stable and unstable foliations are real-analytic, and so is g. We now compute the first nonzero term of the power series expansion of g centered at $(0, 0)$.

Since $\phi(0, 0) = \phi_0(0, 0) = 0$, we get $g(0, 0) = 0$. Now fix y and let $g_y(x)$ denote the function $x \mapsto g(x, y)$. Let $g_x(y)$ be defined analogously. We know $g_0(x) = 0$ for all x and $g_0(y) = 0$ for all y [Ham99, Property 1)]. Hence the k-th derivative $D^k g_0(x) = 0$ for all x and $D^k g_0(y) = 0$ for all y.

Additionally, the function $g_x(y)$ satisfies $Dg_x(0) = 0$ [Ham99, Property 3)]. Analogously we have $Dg_y(0) = 0$. This, together with the previous paragraph, means if α and β are both n-dimensional multi-indices, then we have $\frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^\beta}{\partial y^\beta} g(0, 0) = 0$ whenever $|\alpha| = 0, 1$ or $|\beta| = 0, 1$. Hence the first nonzero term of the power series expansion of g centered at $(0, 0)$ is of the form $\sum_{i,j,k,l} a_{ijkl} x_i x_j y_j y_l$. This means there is a constant C depending only on $\tilde{N}$ such that

$$\frac{|\phi(x, y) - \phi_0(x, y)|}{\|x\|\|y\|} \leq \frac{1}{\|x\|\|y\|} C\|x\|^2\|y\|^2$$

so long as $\|x\|, \|y\|$ are small enough for the power series expansion of g centered at the origin to converge at (x, y). Set $\kappa(\delta) = (\delta/C)^{1/2}$. Then for small enough δ (depending on $\tilde{N}$), we obtain $|\phi(x, y) - \phi_0(x, y)| \leq \delta\|x\|\|y\|$ whenever $\|x\|, \|y\| \leq \kappa$, as desired. □

IV.2.4: Comparing λ_N and $\overline{f}_*\lambda_M$

Proposition IV.19. *Let λ^M and λ^N denote the Liouville currents on $\partial^2\tilde{M}$ and $\partial^2\tilde{N}$ respectively. There is a constant $C = C(\tilde{N})$ so that*

$$\lambda^N \leq (1 + \varepsilon)^{n-1}(1 + C\varepsilon^\alpha)\overline{f}_*\lambda^M.$$

Proof. Combining Proposition IV.14 together with [Ham99, Proposition 3.8] (see also (IV.2.3)), we obtain

$$\mathcal{S}^N_\varepsilon \leq (1 + \varepsilon)^{n-1}\overline{f}_*\mathcal{S}^M \leq (1 + \varepsilon)^{n-1}\overline{f}_*\lambda^M.$$

Lemma IV.15 together with the proof of [Ham99, Corollary 3.12] gives

$$\mathcal{S}^N_\varepsilon \geq (1 + C\varepsilon^\alpha)^{-1}\lambda^N$$

for some C depending only on $\tilde{N}$, which completes the proof. □

Remark IV.20. If $\tilde{N}$ is a symmetric space, Lemma IV.18 shows we can take $\alpha = 2$ in the statement of the above proposition. Additionally, since there are only finitely many negatively curved symmetric spaces of a given dimension, we can say C depends only on $n = \dim \tilde{N}$.

To complete the proof of Theorem IV.6, we need a lower estimate for λ_N analogous to the upper estimate in Proposition IV.19. We obtain this by mimicking the above analysis for the measure $\mathcal{P}$ instead of $\mathcal{S}$, see (IV.2.3). We first recall the construction of $\mathcal{P}$:

Construction IV.21. [Ham99, Proposition 3.13] Let $\eta > 0$ and U be an open subset of $\partial^2 \tilde{M}$. Define

$$\mathcal{P}_\eta(U) = \sup \left\{ a_{n-1}^2 \sum_{i=1}^{\infty} \delta(Q_i)^{n-1} \mid Q_i \in Q(\eta), \operatorname{diam}(Q_i) \leq \eta, Q_i \subset U,\, Q_i \cap Q_j = \emptyset \right\}.$$

Let $\mathcal{P}(U) = \liminf_{\eta \to 0} \mathcal{P}_\eta(U)$. For $C \subset \partial^2 \tilde{M}$ a Borel set, define $\mathcal{P}(C) = \inf\{P(U) \,|\, U \supset C\}$.

Proposition IV.22. *Let λ^M and λ^N denote the Liouville currents on $\partial^2 \tilde{M}$ and $\partial^2 \tilde{N}$ respectively. There is a constant $C = C(\tilde{N})$ so that*

$$\lambda_N \geq (1-\varepsilon)^{n-1}(1 - C\varepsilon^\alpha)\overline{f}_* \lambda_M.$$

If $\tilde{N}$ is a symmetric space, we can take $\alpha = 2$, and the constant C depends only on $n = \dim \tilde{N}$.

Proof. Let

$$\chi_\eta^U = \{(Q_i)_{i \in \mathbb{N}} \,|\, Q_i \cap Q_j = \emptyset,\, Q_i \subset U,\, \operatorname{diam}(Q_i) \leq \eta,\, Q_i \in \mathcal{Q}(\eta)\}.$$

Then $\overline{f}(\chi_\eta^U) \subset \chi_{(1+\eta)\varepsilon+\eta}^{\overline{f}(U)}$ by Corollary IV.13. Using Lemma IV.12 gives

$$\begin{aligned}
\overline{f}_* \mathcal{P}_\eta^M(U) &= \mathcal{P}_\eta^M(\overline{f}^{-1}(U)) \\
&= \sup \left\{ a_{n-1}^2 \sum_{i=1}^{\infty} \delta(Q_i)^{n-1} \,\middle|\, (Q_i)_{i \in \mathbb{N}} \in \chi_\eta^{\overline{f}^{-1}(U)} \right\} \\
&\leq \sup \left\{ \frac{a_{n-1}^2}{(1-\varepsilon)^{n-1}} \sum_{i=1}^{\infty} \delta(\overline{f}(Q_i))^{n-1} \,\middle|\, (\overline{f}(Q_i))_{i \in \mathbb{N}} \in \overline{f}(\chi_\eta^{\overline{f}^{-1}(U)}) \right\} \\
&\leq \sup \left\{ \frac{a_{n-1}^2}{(1-\varepsilon)^{n-1}} \sum_{i=1}^{\infty} \delta(\overline{f}(Q_i))^{n-1} \,\middle|\, (Q_j)_{j \in \mathbb{N}} \in \chi_{(1+\eta)\varepsilon+\eta}^{U} \right\} \\
&\leq \mathcal{P}_N^{(1+\eta)\varepsilon+\eta}(U)/(1-\varepsilon)^{n-1}.
\end{aligned}$$

Taking $\eta \to 0$ gives

$$(1-\varepsilon)^{n-1}\overline{f}_*\mathcal{P}^M(U) \le \mathcal{P}^N_\varepsilon(U).$$

This, together with [Ham99, Proposition 3.13 a)] (see also (IV.2.3)), gives

$$(1-\varepsilon)^{n-1}\overline{f}_*\lambda^M(U) \le (1-\varepsilon)^{n-1}\overline{f}_*\mathcal{P}^M(U) \le \mathcal{P}^N_\varepsilon(U).$$

It follows from Lemma IV.17 together with the proof of [Ham99, Proposition 3.13 b)] that

$$\mathcal{P}^N_\varepsilon \le (1-C\varepsilon^\alpha)^{-1}\lambda^N$$

for some constant C depending only on $\tilde{N}$.

Hence $\lambda^N(U) \ge (1-\varepsilon)^{n-1}(1-C\varepsilon^\alpha)\overline{f}_*\mathcal{P}^M(U)$ for any open set $U \subset \partial^2\tilde{M}$. To obtain this inequality for any Borel set $A \subset \partial^2\tilde{M}$ we take the infimum over all open sets $U \supset A$. Finally, noting that $\mathcal{P}^M \ge \lambda^M$ (see (IV.2.3)) completes proof. □

Proof of Theorem IV.6. The first part of the statement follows immediately from Propositions IV.19 and IV.22. The refinement of the statement in the case where $\tilde{N}$ is a symmetric space follows from Remark IV.20. □

IV.3: A controlled orbit equivalence

In this section, we will use the estimate for the ratio $\overline{f}_*\lambda^M/\lambda^N$ of the Liouville currents in Theorem IV.6 to compare $\mathrm{Vol}(M)$ and $\mathrm{Vol}(N)$. Note the Riemannian volumes of M and N are determined by the Liouville volumes of T^1M and T^1N. To obtain the Liouville measure from the Liouville current, we integrate the Liouville current in the geodesic flow direction. Let ϕ^t denote the geodesic flow of M and let ψ^t denote the geodesic flow of N. If the marked length spectra of M and N are equal, then the flows ϕ^t and ψ^t are *conjugate* [Ham92], ie, there is a homeomorphism $\mathcal{F} : T^1M \to T^1N$ such that

$$\mathcal{F}(\phi^t v) = \psi^t \mathcal{F}(v)$$

for all $t \in \mathbb{R}, v \in T^1M$. If, in addition to this, M and N have the same Liouville current, then T^1M and T^1N have the same Liouville measure, so $\mathrm{Vol}(M) = \mathrm{Vol}(N)$.

If the lengths of closed geodesics of M and N are instead ε-close as in (I.7), the geodesic flows may not be conjugate. However, so long as M and N are negatively curved and have isomorphic fundamental groups, their geodesic flows are *orbit-equivalent* [Gro00]. This means there exists a function $a(t,v)$ such that

$$\mathcal{F}(\phi^t v) = \psi^{a(t,v)}\mathcal{F}(v)$$

for all $t \in \mathbb{R}, v \in T^1M$.

In this section, we will use the assumption of approximately equal lengths (I.7) to show the time change $a(t,v)$ is close to t on sets of large measure, thereby allowing us to show the total Liouville measures of T^1M and T^1N are close.

We begin by recalling the setup from [Gro00]. The construction starts with a preliminary Γ-equivariant orbit map $\mathcal{F}_0 : T^1\tilde{M} \to T^1\tilde{N}$ which is not necessarily injective. Recall there is a homotopy equivalence $f : M \to N$ by assumption. We can assume f is C^1 by using that every continuous map is homotopic to a differentiable map; see [BP92, p. 86] and [MW97]. Let $\tilde{f} : \tilde{M} \to \tilde{N}$ be a lift of f.

Let η be a bi-infinite geodesic in $\tilde{M}$ and let $\zeta = \overline{f}(\eta)$ be the corresponding geodesic in $\tilde{N}$, where $\overline{f} : \partial^2\tilde{M} \to \partial^2\tilde{N}$ is obtained from extending the quasi-isometry $\tilde{f}$ to a map $\partial\tilde{M} \to \partial\tilde{N}$; see Construction IV.1. Let $P_\zeta : \tilde{N} \to \zeta$ denote the orthogonal projection. Note this projection is Γ-equivariant, ie, $\gamma P_\zeta(x) = P_{\gamma\zeta}(\gamma x)$. If $(p,v) \in T^1\tilde{M}$ is tangent to η, then we can define $\mathcal{F}_0(p,v)$ to be the tangent vector to ζ at the point $P_\zeta \circ \tilde{f}(p)$. Thus $\mathcal{F}_0 : T^1\tilde{M} \to T^1\tilde{N}$ is a Γ-equivariant map which sends geodesics to geodesics. As such, we can define a cocycle $b(t,v)$ to be the time which satisfies

$$\mathcal{F}_0(\phi^t v) = \psi^{b(t,v)}\mathcal{F}_0(v).$$

Remark IV.23. Since $\tilde{f}$ is C^1 and the orthogonal projection is smooth in the t-direction, we have $t \mapsto b(t,v)$ is C^1.

It is possible for a fiber of the orthogonal projection map to intersect the quasi-geodesic $\tilde{f}(\eta)$ in more than one point; thus, $\mathcal{F}_0$ is not necessarily injective. In order to obtain an injective orbit equivalence, we follow the method in [Gro00] and average the function $b(t,v)$ along geodesics. We include a proof below, since the setup will be used throughout this section.

Lemma IV.24. *Let*

$$a_l(t,v) = \frac{1}{l}\int_t^{t+l} b(s,v)\,ds.$$

There is a large enough l so that $t \mapsto a_l(t,v)$ is injective for all v.

Proof. The fundamental theorem of calculus gives

$$\frac{d}{dt}a_l(t,v) = \frac{b(t+l,v) - b(t,v)}{l}. \tag{IV.3.1}$$

We claim there is a large enough l so that this quantity is always positive. To this end, suppose $b(t+l,v) - b(t,v) = 0$. This means $\mathcal{F}_0(\phi^t v)$ and $\mathcal{F}_0(\phi^{t+l}v)$ are in the same fiber of the normal projection onto the geodesic $\overline{f}(v)$. Since $s \mapsto \tilde{f}(\phi^s v)$ is a quasi-geodesic, there is a constant R, depending only on the quasi-isometry constants A and B of $\tilde{f}$, so that all points on $\tilde{f}(\phi^s v)$ are of distance at most R from the geodesic $\psi^t\mathcal{F}_0(v)$ [BH13, Theorem 3.H.1.7]. Thus two points on the same fiber of the normal projection are at most distance $2R$ apart, which gives

$$A^{-1}l - B \leq d(f(\phi^t v), f(\phi^{t+l}v)) \leq 2R.$$

Taking $l > A(2R+B)$ guarantees $\frac{d}{ds}a_l(s,v)$ is never 0, and hence $a_l(s,v)$ is injective. $\square$

Proposition IV.25. *For each $v \in T^1M$, let*

$$\mathcal{F}_l(v) = \psi^{a_l(0,v)}\mathcal{F}_0(v)$$

for a_l as in Lemma IV.24. Then $\mathcal{F}_l$ is an orbit equivalence of geodesic flows.

Proof. Since $\mathcal{F}_l$ sends geodesics to geodesics, there exists a cocycle $k_l(t,v)$ so that $\mathcal{F}_l(v) = \psi^{k_l(t,v)}\mathcal{F}_l(v)$. We need to check $t \mapsto k_l(t,v)$ is injective. Note that

$$\begin{aligned} a_l(0,\phi^t v) &= \frac{1}{l}\int_0^l b(s,\phi^t v)\,ds \\ &= \frac{1}{l}\int_0^l b(s+t,v) - b(t,v)\,ds \\ &= a_l(t,v) - b(t,v). \end{aligned}$$

This means

$$\begin{aligned} \mathcal{F}_l(\phi^t v) &= \psi^{a_l(0,\phi^t v)}\mathcal{F}_0(\phi^t v) \\ &= \psi^{a_l(0,\phi^t v)+b(t,v)}\mathcal{F}_0(v) \\ &= \psi^{a_l(t,v)}\mathcal{F}_0(v). \end{aligned}$$

Therefore, $\mathcal{F}_l(\phi^t v) = \psi^{k_l(t,v)}\mathcal{F}_l(v) = \psi^{a_l(t,v)}\mathcal{F}_0(v)$, and hence

$$\frac{d}{dt}|_{t=0}k_l(t,v) = \frac{d}{dt}|_{t=0}a_l(t,v) = \frac{b(l,v)}{l}. \tag{IV.3.2}$$

The proof of Lemma IV.24 shows the above quantity is positive. So $\mathcal{F}_l$ is injective along geodesics, as desired. $\square$

Now we will use the assumption $1 - \varepsilon \leq \frac{\mathcal{L}_{g_0}}{\mathcal{L}_g} \leq 1 + \varepsilon$ (Hypothesis IV.10) to say more about this orbit equivalence.

Lemma IV.26. *Let $v \in T^1M$ be tangent to the axis of γ and let $\tau = l(\gamma)$. Then*

$$1 - \varepsilon \leq \frac{b(\tau, v)}{\tau} \leq 1 + \varepsilon.$$

Proof. By definition, $b(\tau, v)$ is the distance from $\mathcal{F}_0(v)$ to $\mathcal{F}_0(\gamma v) = \overline{f}(\gamma)F_0(v)$. In addition, if v is on the axis of γ, then $\mathcal{F}(v)$ is on the axis of $\overline{f}(\gamma)$, which means $b(\tau, v)$ is equal to the translation length of $\overline{f}(\gamma)$. The hypothesis $1-\varepsilon \leq \frac{\mathcal{L}_{g_0}}{\mathcal{L}_g} \leq 1+\varepsilon$ implies the translation length of $\overline{f}(\gamma)$ is between $(1-\varepsilon)\tau$ and $(1+\varepsilon)\tau$, which completes the proof. $\square$

Lemma IV.27. *There is a number L with $1+\varepsilon \leq L \leq 1-\varepsilon$ such that for almost every $v \in T^1\tilde{M}$, we have*

$$\frac{b(t, v)}{t} \to L$$

as $t \to \infty$.

Proof. Let $\beta(v) = \frac{d}{dt}|_{t=0}\, b(t, v)$ (see Remark IV.23). Then the fundamental theorem of calculus implies

$$b(T, v) = \int_0^T \beta(\phi^t v)\, dt.$$

Indeed,

$$\begin{aligned}
\int_0^T \beta(\phi^t v)\, dt &= \int_0^T \frac{d}{ds}|_{s=0}\, b(s, \phi^t v)\, dt \\
&= \int_0^T \frac{d}{ds}|_{s=0}\, [b(s+t, v) - b(t, v)]\, dt \qquad \text{(cocycle condition)} \\
&= \int_0^T \frac{d}{ds}|_{s=0}\, b(s+t, v)\, dt \\
&= b(T, v) - b(0, v).
\end{aligned}$$

The ergodic theorem then implies

$$\lim_{T\to\infty} \frac{b(T, v)}{T} = \int_{T^1M} \beta(v) d\mu(v)$$

for μ-almost every v, where μ is normalized Liouville measure on T^1M. The integral of β on the right-hand side can be approximated by averaging β along closed geodesics (see [Sig72]). Lemma IV.26 then implies the value of this integral is between $1-\varepsilon$ and $1+\varepsilon$. $\square$

Now we will explicitly relate the Liouville current λ on $\partial^2\tilde{M}$ and the Liouville measure μ on $T^1\tilde{M}$. (This is a special case of a more general correspondence between geodesic-flow-invariant measures on $T^1\tilde{M}$ and finite measures on $\partial^2\tilde{M}$ due to Kaimanovich [Kai90, Theorem 2.1].)

Let X denote the vector field on T^1M which generates the geodesic flow. For every $v \in T^1M$, we can choose local coordinates $(t, x_1, \ldots, x_m)$ near v so that $\partial/\partial t = X$. Then $(0, x_1, \ldots, x_m)$ defines a local smooth hypersurface $K_0 \subset T^1M$ which is transverse to X. Let $K = \pi(K_0) \subset \partial^2\tilde{M}$. Then $\int_{K_0}(d\omega)^{n-1} = \lambda(K)$.

For $T > 0$ define

$$K_T = \{\phi^t v \mid v \in K_0,\ t \in [0,T]\}. \tag{IV.3.3}$$

If T is sufficiently small, then with respect to our choice of local coordinates, we have $K_T = \{(t, x_1, \ldots, x_m) \mid 0 \le t \le T\}$ and $\omega = dt$. We thus obtain

$$\mu(K_T) = \int_{K_T} \omega \wedge (d\omega)^{n-1} = T\int_{K_0}(d\omega)^{n-1} = T\lambda(K). \tag{IV.3.4}$$

Lemma IV.28. *Suppose*

$$\overline{f}_*\lambda^M \ge C'\lambda^N \tag{IV.3.5}$$

for some constant C'. For $T > 0$ (sufficiently small as above) and $K_0 \subset T^1M$ a local transversal to the geodesic flow, define K_T as in (IV.3.3) above. For all $\delta > 0$, there is a large enough l (depending on K_T and δ) so that

$$\mu^N(\mathcal{F}_l(K_T)) \ge C'(1-\varepsilon-\delta)(1-\delta)\mu^M(K_T).$$

Proof. For almost every $v \in K_T$, Lemma IV.27 gives $\lim_{l\to\infty}\frac{b(l,v)}{l} = L$, where $1-\varepsilon \le L \le 1+\varepsilon$. By Egorov's theorem, there is a large subset of vectors v (meaning of measure at least $(1-\delta)\lambda^M(K_T)$) for which $\frac{b(l,v)}{l} \to L$ uniformly in v. In fact, this subset can be taken to be of the form $E_T := K_T \cap \pi^{-1}(E)$ for some $E \subset K \subset \partial^2\tilde{M}$. To see this, we compare the convergence of $\frac{b(l,v)}{l}$ with that of $\frac{b(l,\phi^t v)}{l}$ for $t \in [0,T]$. The cocycle condition implies

$$\frac{b(l,\phi^t v) - b(l,v)}{l} = \frac{b(t,\phi^l v) - b(t,v)}{l}.$$

The numerator of the right hand side is bounded on the compact set $[0,T] \times T^1M$ independent of l, so the left hand side goes to zero uniformly in v as $l \to \infty$.

Thus we can choose large enough l (depending on δ) so that $L - \delta \leq \frac{b(l,v)}{l} \leq L + \delta$ for all $v \in E_T$. Using Lemma IV.27 and (IV.3.2) we get

$$1 - \varepsilon - \delta \leq \frac{d}{dt}|_{t=0}\, k_l(t, v) \leq 1 + \varepsilon + \delta$$

for all $v \in E_T$. The cocycle condition implies

$$\int_0^t \frac{d}{dt}|_{t=0} k_l(t, \phi^s v)\, ds = \int_0^t \frac{d}{dt}|_{t=0} k_l(s + t, v)\, ds = k_l(t, v) - k_l(0, v).$$

This, together with the previous inequalities, gives

$$1 - \varepsilon - \delta \leq k_l(t, v)/t \leq 1 + \varepsilon + \delta.$$

This means

$$\mathcal{F}_l(E_T) = \{\psi^{k_l(t,v)}\mathcal{F}_l(v) \,|\, v \in E_0, t \in [0, T]\} \supset \{\psi^s \mathcal{F}_l(v) \,|\, v \in E_0, s \in [0, (1 - \varepsilon - \delta)T]\}. \tag{IV.3.6}$$

The Liouville measure of the rightmost set is $(1 - \varepsilon - \delta)T\lambda^N(\overline{f}(E))$ by (IV.3.4). Moreover,

$$T\lambda^M(E) = \mu^M(E_T) \geq (1 - \delta)\mu^M(K_T) = (1 - \delta)T\lambda^M(K)$$

shows $\lambda^M(E) \geq (1 - \delta)\lambda^M(K)$. Then we have

$$\begin{aligned}
\mu^N(\mathcal{F}_l(K_T)) &\geq \mu^N(\mathcal{F}_l(E_T)) \\
&\geq (1 - \varepsilon - \delta)T\lambda^N(\overline{f}(E)) && \text{(equation IV.3.6)} \\
&\geq (1 - \varepsilon - \delta)TC'\lambda^M(E) && \text{(equation IV.3.5)} \\
&\geq (1 - \varepsilon - \delta)TC'(1 - \delta)\lambda^M(K) \\
&= C'(1 - \varepsilon - \delta)(1 - \delta)\mu^M(K_T),
\end{aligned}$$

which is the desired result. □

Now we are ready to prove the main theorem of this section, which relates the volumes of M and N.

Proof of Theorem I.12. Let $\delta > 0$. Choose finitely many disjoint sets of the form $K^i_{T^i} \subset T^1M$

(as defined in (IV.3.3)) so that

$$\sum_{i=1}^{k} \mu(K^i_{T^i}) \geq \mu(T^1M) - \delta.$$

Now choose large enough l (depending on δ) so that the conclusion of Lemma IV.28 holds for $K^1_{T^1}, \ldots, K^k_{T^k}$ simultaneously. By Theorem IV.6, the hypothesis of Lemma IV.28 holds with $C' = (1 - C\varepsilon^\alpha)(1 - \varepsilon)^{n-1}$. We then have

$$\begin{aligned}
\mu^N(T^1N) &\geq \Sigma_i \mu^N(\mathcal{F}_l(K^i_{T_i})) \\
&\geq C'(1 - \varepsilon - \delta)(1 - \delta)\Sigma_i \mu^M(K^i_{T_i}) \\
&\geq C'(1 - \varepsilon - \delta)(1 - \delta)(\mu^M(T^1M) - \delta).
\end{aligned}$$

Taking $\delta \to 0$ implies $\mathrm{Vol}(N) \geq (1 - C\varepsilon^\alpha)(1 - \varepsilon)^{n-1}\mathrm{Vol}(M)$. Switching the roles of M and N in all the arguments in this section gives the estimate in the other direction. $\square$

CHAPTER V
Estimates for the BCG Map

If $\mathcal{L}_g = \mathcal{L}_{g_0}$, then it follows from [Ham99, Theorem A] that $\mathrm{Vol}(M, g) = \mathrm{Vol}(N, g_0)$. Since $\mathcal{L}_g$ determines the topological entropy of the geodesic flow, the entropy rigidity theorem of Besson-Courtois-Gallot [BCG96] states there is an isometry $F : M \to N$.

In the case where $1 - \varepsilon \leq \frac{\mathcal{L}_{g_0}}{\mathcal{L}_g} \leq 1 + \varepsilon$ (Hypothesis IV.10), Theorem I.12 states the volumes of M and N satisfy $(1 - C\varepsilon^2)(1 - \varepsilon)^n \leq \frac{\mathrm{Vol}(N)}{\mathrm{Vol}(M)} \leq (1 + C\varepsilon^2)(1 + \varepsilon)^n$, where C is a constant depending only on n. Moreover, the entropies are related as follows.

Lemma V.1. *Let h denote the topological entropy of the geodesic flow. Then with the above marked length spectrum assumptions we have*

$$\frac{1}{1+\varepsilon} h(g) \leq h(g_0) \leq \frac{1}{1-\varepsilon} h(g). \tag{V.0.1}$$

Proof. This follows from the following description of the topological entropy in terms of periodic orbits due to Margulis [Mar69]:

$$h(g) = \lim_{t \to \infty} \frac{1}{t} \log P_g(t), \tag{V.0.2}$$

where $P_g(t) = \#\{\gamma \,|\, l_g(\gamma) \leq t\}$. □

We use the results of Theorem I.12 and Lemma V.1 to modify the proof in [BCG96] that there is an isometry $F : M \to N$. More specifically, we use the same construction for the map F as in [BCG96] and show the matrix of dF_p with respect to suitable orthonormal bases is close to the identity matrix.

V.1: Construction of the BCG map

From now on, we will assume N is a locally symmetric space. This means $\tilde{N}$ is either a real, complex or quaternionic hyperbolic space or the Cayley hyperbolic space of real dimension 16; let $d = 1, 2, 4$ or 8 respectively.

We now normalize the metric g_0 so the sectional curvatures are all -1 in the case $d = 1$ and contained in the interval $[-4, -1]$ otherwise. Since $\dim N \geq 3$, Mostow rigidity implies (N, g_0) is determined up to isometry by its fundamental group Γ [Mos73]. Thus, from now on, any constants arising from the geometry of N, such as the diameter and the injectivity radius, can be thought of as depending only on Γ. We also rescale the metric g by the same factor as g_0 in order to preserve the assumed marked length spectrum ratio in (I.7) as well as the established volume and entropy ratios. From now on, we will also assume the sectional curvatures of (M, g) are in the interval $[-\Lambda^2, -0)$ for some constant Λ. Such a constant always exists since M is assumed to be compact; however some of our estimates will depend on its particular value.

We first recall the construction of the map $F : M \to N$ in [BCG96]. We then summarize the proof that F is an isometry in the case of equal entropies and volumes, before explaining how to modify it for approximately equal entropies and volumes.

Given $p \in M$, let μ_p be the Patterson-Sullivan measure on $\partial\tilde{M}$. Let $\overline{f} : \partial\tilde{M} \to \partial\tilde{N}$ as before (see Construction IV.1). Define $F(p) = \mathrm{bar}(f_*\mu_p)$, where bar denotes the barycenter map (see [BCG96] for more details). We call F the *BCG map*. By the definition of the barycenter, the BCG map has the implicit description

$$\int_{\partial\tilde{N}} dB_{F(p),\xi}(\cdot) d(\overline{f}_*\mu_p)(\xi) = 0, \tag{V.1.1}$$

where $\xi \in \partial\tilde{N}$ and $B_{F(p),\xi}$ is the Busemann function on $(\tilde{N}, g_0)$. By the implicit function theorem, the BCG map F is C^1 (actually, C^2 since Busemann functions on $\tilde{M}$ are C^2 [Bal95, Proposition IV.3.2]), and its derivative dF_p satisfies

$$\int_{\partial\tilde{N}} \mathrm{Hess}B^N_{F(p),\xi}(dF_p(v), u)\, d(\overline{f}_*\mu_p)(\xi) = h(g) \int_{\partial\tilde{N}} dB^N_{F(p),\xi}(u) dB^M_{p,\overline{f}^{-1}(\xi)}(v) d(\overline{f}_*\mu_p)(\xi) \tag{V.1.2}$$

for all $v \in T_pM$ and $u \in T_{F(p)}N$ [BCG96, (5.2)]. In light of this, it is natural to define the following quadratic forms H and K:

$$\langle K_{F(p)}u, u\rangle := \int_{\partial\tilde{N}} (\mathrm{Hess}B_{F(p),\xi})(u)\, d(\overline{f}_*\mu_p)(\xi), \tag{V.1.3}$$

$$\langle H_{F(p)}u, u\rangle := \int_{\partial\tilde{N}} (dB_{F(p),\xi}(u))^2\, d(\overline{f}_*\mu_p)(\xi), \tag{V.1.4}$$

where $\langle\cdot,\cdot\rangle$ denotes the Riemannian inner product coming from g_0 [BCG96, p. 636].

Without any assumptions about the volumes or entropies, the following three inequalities hold; see [Rua22] for the Cayley case.

Lemma V.2. *[BCG96, Lemma 5.4]*

$$|\mathrm{Jac}F(p)| \leq \frac{h^n(g)}{n^{n/2}} \frac{\det(H)^{1/2}}{\det(K)}.$$

Lemma V.3. *[BCG95, Lemma B3] Let $n \geq 3$ and let H and K be the $n \times n$ positive definite symmetric matrices coming from the operators in (V.1.3) and (V.1.4), respectively. Then*

$$\frac{\det H}{\det(K)^2} \leq \frac{(n-1)^{\frac{2n(n-1)}{n+d-2}}}{(n+d-2)^{2n}} \frac{\det(H)^{\frac{n-d}{n+d-2}}}{\det(I-H)^{\frac{2(n-1)}{n+d-2}}},$$

with equality if and only if $H = \frac{1}{n}I$.

Lemma V.4. *[BCG95, Lemma B4] Let H be an $n \times n$ positive definite symmetric matrix with trace 1, where $n \geq 3$. Let $1 < \alpha \leq n-1$. Then*

$$\frac{\det H}{\det(I-H)^\alpha} \leq \left(\frac{n^\alpha}{n(n-1)^\alpha}\right)^n.$$

Moreover, equality holds if and only if $H = \frac{1}{n}I$.

Combining the above three inequalities (setting $\alpha = \frac{2(n-1)}{n-d}$) together with the fact that $h(g_0) = n + d - 2$, we obtain:

Lemma V.5. *[BCG96, Proposition 5.2 i)]*

$$|\mathrm{Jac}F(p)| \leq \left(\frac{h(g)}{h(g_0)}\right)^n.$$

As in the proof of [BCG96, Theorem 5.1], the above lemma relates the volumes of M and N as follows:

$$\mathrm{Vol}(N, g_0) \leq \int_M |F^* d\mathrm{Vol}| = \int_M |(\mathrm{Jac}F) d\mathrm{Vol}| \leq \left(\frac{h(g)}{h(g_0)}\right)^n \mathrm{Vol}(M, g). \tag{V.1.5}$$

Remark V.6. This, together with Lemma V.1, improves one of the inequalities in Theorem I.12 in the special case where N is a locally symmetric space.

With this setup in mind, the argument in [BCG96] showing that F is an isometry consists of the following components:

1. If the volumes and entropies are equal, then the inequalities in (V.1.5) are all equalities, which gives equality in Lemma V.5.

2. Thus, equality also holds in Lemmas V.2 and V.3, from which it follows that $H = \frac{1}{n}I$ and $K = \frac{n+d-2}{n}I = \frac{h(g_0)}{n}I$. See [BCG96, p. 639].

3. With H and K as above, the end of the proof of Proposition 5.2 ii) in [BCG96] shows that $dF_p = \left(\frac{h(g_0)}{h(g)}\right) I$, which means F is an isometry in the case where the entropies are equal. This concludes the proof of Theorem 1 in [BCG96].

Assuming instead that $1 - \varepsilon \leq \frac{\mathcal{L}_{g_0}}{\mathcal{L}_g} \leq 1 + \varepsilon$, the equalities of volumes and entropies are replaced with the conclusions of Theorem I.12 and Lemma V.1 respectively. Proceeding as in the above outline, we can instead obtain estimates for $\|dF_p\|$ in terms of ε:

1. We show equality almost holds in (V.1.5); that is, we find a lower bound for $\mathrm{Jac}F(p)$ of the form $\beta(h(g)/h(g_0))^n$ for suitable β (Proposition V.27).

2. This implies the eigenvalues of H are all close to $1/n$ and the eigenvalues of K are all close to $h(g_0)/n$ (Proposition V.31).

3. With H and K as above, we mimic the proof of [BCG96, Proposition 5.2 ii)] to obtain bounds for $\|dF_p\|$, which completes the proof of Theorem I.9 (Proposition V.35).

The main difficulty is step (1), where we cannot simply mimic the arguments in [BCG96]. Indeed, with the above assumptions about the entropies (Lemma V.1) and the volumes (Theorem I.12), the inequalities in (V.1.5) become

$$(1 - C\varepsilon^2)(1-\varepsilon)^n \frac{1}{(1+\varepsilon)^n} \left(\frac{h(g)}{h(g_0)}\right)^n \mathrm{Vol}(M) \leq \int_M |\mathrm{Jac}F| \leq \left(\frac{h(g)}{h(g_0)}\right)^n \mathrm{Vol}(M),$$

which does not give a lower bound for the integrand. In order to obtain a lower bound for $|\mathrm{Jac}F|$, we use the above lower bound for its integral together with a Lipschitz bound for the function $p \mapsto |\mathrm{Jac}F(p)|$ (Proposition V.25). The fact that this function is Lipschitz is immediate from the fact that F is C^2; however, it is not clear a priori how the Lipschitz bound depends on (M, g). Assuming $1 - \varepsilon \leq \frac{\mathcal{L}_{g_0}}{\mathcal{L}_g} \leq 1 + \varepsilon$ holds (Hypothesis IV.10) for ε sufficiently small (depending on n and Γ), we will show there is a Lipschitz bound for $\mathrm{Jac}F(p)$ depending only on the dimension n, the fundamental group Γ and the lower bound $-\Lambda^2$ for the sectional curvatures of M.

V.2: Lower bound for K

Recall the BCG map F is defined implicitly (see (V.1.1)), and its derivative dF_p satisfies the following equation

$$\langle K dF_p(v), u\rangle = h(g) \int_{\partial\tilde{N}} dB^N_{F(p),\xi}(u) dB^M_{p,\overline{f}^{-1}\xi}(v) d(\overline{f}_*\mu_p)(\xi).$$

(See (V.1.2) and (V.1.3).) In order to use this equation to find a Lipschitz bound for Jac$F(p)$, we start by bounding the quadratic form K away from zero (Proposition V.17). Recall

$$\langle K_{F(p)}u, u\rangle := \int_{\partial\tilde{N}} (\mathrm{Hess}B_\xi)_{F(p)}(u)\, d(\overline{f}_*\mu_p)(\xi). \tag{V.2.1}$$

Note that K depends not only on the symmetric space (N, g_0), but also on (M, g), since μ_p is the Patterson-Sullivan measure on $\partial\tilde{M}$ defined with respect to the metric g. We start by recalling that K is positive-definite for any given (negatively curved) metric g on M (see [BCG96, Definition 3.2]). We include a detailed proof as we will refer to the arguments later.

Lemma V.7. *There is $\kappa_g > 0$ so that $\langle K_{F(p)}u, u\rangle \geq \kappa_g$ for all $p \in \tilde{M}$, $u \in T^1_{F(p)}\tilde{N}$.*

Proof. First we examine the integrand in (V.2.1). Fix $p \in \tilde{M}$ and $u \in T^1_{F(p)}\tilde{N}$ and consider $(\mathrm{Hess}B_\xi)_{F(p)}(u)$. Let $v_{F(p),\xi}$ be the unit tangent vector based at $F(p)$ so that the geodesic with initial vector v has forward boundary point ξ, ie, $v_{F(p),\xi}$ is the gradient of $B_{\xi,F(p)}$. Let θ_ξ denote the angle between $v_{F(p),\xi}$ and u. Then we can write $u = (\cos\theta_\xi)v_{F(p),\xi} + (\sin\theta_\xi)w$ for some unit vector w perpendicular to $v_{F(p),\xi}$. Since $(\mathrm{Hess}B_\xi)_{F(p)}(u) = \langle \nabla_u v_{F(p),\xi}, u\rangle$, we obtain $(\mathrm{Hess}B_\xi)_{F(p)}(u) = \sin^2\theta_\xi(\mathrm{Hess}B_\xi)_{F(p)}(w)$. Let R denote the curvature tensor of $(\tilde{N}, \tilde{g}_0)$. Using the formula

$$(\mathrm{Hess}B_\xi)_{F(p)}(\cdot) = \sqrt{-R(v_{F(p),\xi}, \cdot, v_{F(p),\xi}, \cdot)} \tag{V.2.2}$$

(see [CF03, p. 16]), together with the fact the sectional curvatures of $\tilde{N}$ are at most -1, it follows that

$$(\mathrm{Hess}B_\xi)_{F(p)}(u) \geq \sin^2\theta_\xi.$$

Hence, the integrand in the definition of $K_{F(p)}$ is 0 if and only if $\theta_\xi = 0, \pi$. This occurs precisely when $\xi = \pi(\pm u)$, where π is the projection of a unit tangent vector to its forward boundary point in $\partial\tilde{N}$. Since μ_p is non-atomic, we have $(\overline{f}_*\mu_p)(\partial\tilde{N}\setminus\{\pi(\pm u)\}) = 1 > 0$. Thus $(\mathrm{Hess}B_\xi)_{F(p)}(u) > 0$ for a set of ξ of positive $\overline{f}_*\mu_p$-measure, which means $K_{F(p)}(u, u) > 0$ for all $(F(p), u) \in T^1\tilde{N}$.

Moreover, there is $\kappa > 0$ so that $\langle K_{F(p)}u, u\rangle \geq \kappa$ for all $p \in \tilde{M}$ and $u \in T^1_{F(p)}\tilde{N}$. To see this, first note $\langle K_{\gamma F(p)}\gamma u, \gamma u\rangle = \langle K_{F(p)}u, u\rangle$ for all $\gamma \in \Gamma$, since the action of Γ is by isometries. Thus it suffices to bound $\langle K_{F(p)}u, u\rangle$from below as $(F(p), u) \in T^1\tilde{N}$ varies over a compact fundamental domain for T^1N. This follows from the fact that $\langle K_{F(p)}u, u\rangle$ varies continuously with respect to $(F(p), u) \in T^1\tilde{N}$. □

While K is positive-definite for any given negatively curved metric g on M, it is not clear from the above analysis that there is a lower bound which is uniform in g. To this end, we establish a type of compactness of the space of all metrics g on M with sectional curvatures in the interval $[-\Lambda^2, 0)$ and marked length spectrum satisfying $1 - \varepsilon \leq \mathcal{L}_g/\mathcal{L}_{g_0} \leq 1 + \varepsilon$ (Corollary V.10). We start with some preliminary lemmas.

Lemma V.8. *The injectivity radii of (M, g) and (N, g_0) satisfy*

$$(1 - \varepsilon)\mathrm{inj}(M, g) \leq \mathrm{inj}(N, g_0) \leq (1 + \varepsilon)\mathrm{inj}(M, g).$$

Proof. This follows from the fact that in negative curvature, the injectivity radius is half the length of the shortest closed geodesic [Pet06, p.178] together with the marked length spectrum assumption.

Indeed, let γ be the shortest closed geodesic in (M, g) and let γ_0 be the shortest closed geodesic in (N, g_0). Then the marked length spectrum assumption gives

$$2(1 + \varepsilon)\mathrm{inj}(M, g) = (1 + \varepsilon)\mathcal{L}_g(\gamma) \geq \mathcal{L}_{g_0}(f_*\gamma) \geq \mathcal{L}_{g_0}(\gamma_0) = 2\,\mathrm{inj}(N, g_0).$$

An analogous argument gives the other estimate. □

Lemma V.9. *There is an upper bound for diam(M) depending only on ε, the dimension n, and the fundamental group Γ.*

Proof. Let p and q be such that $\mathrm{diam}(M) = d(p, q)$ and let $c(t)$ be the geodesic joining p and q. Let r be the injectivity radius of (M, g). Let m be the unique positive integer such that $2(m-1)r \leq \mathrm{diam}(M) \leq 2mr$. Take balls of radius r centered at $c(0), c(2r), c(4r), \ldots, c(2(m-1)r)$. Since M is negatively curved, the volume of any such ball is bounded below by the volume of a ball of radius r in $\mathbb{R}^n$ [GHL90, Theorem 3.101 ii)], which we will denote by $v(r, n)$.

Then $mv(r, n) \leq \mathrm{Vol}(M) \leq C\mathrm{Vol}(N)$, for some $C = C(\varepsilon, n)$ (see Theorem I.12). This gives an upper bound for m, therefore

$$d(p, q) = \mathrm{diam}(M) \leq 2rm \leq r\frac{2C\mathrm{Vol}(N)}{v(r, n)}.$$

Combining with the previous lemma gives

$$\operatorname{diam}(M) \leq \frac{\operatorname{inj}(N)}{1-\varepsilon}(1+\varepsilon)^n \frac{2C\operatorname{Vol}(N)}{v(\operatorname{inj}(N), n)}.$$

Finally, since N is locally symmetric, it follows from Mostow Rigidity that $\operatorname{inj}(N)$ and $\operatorname{Vol}(N)$ depend only on Γ. □

Corollary V.10. *Fix (N, g_0) a rank 1 locally symmetric space of dimension at least 3, and let M be another manifold with the same fundamental group as N. Fix $\varepsilon, \Lambda > 0$. Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of Riemannian metrics on M with sectional curvatures in the interval $[-\Lambda^2, 0)$ and marked length spectra satisfying $1-\varepsilon \leq \mathcal{L}_{g_0}/\mathcal{L}_g \leq 1+\varepsilon$. Then there is a $C^{1,\alpha}$ Riemannian metric g_∞ on M and a subsequence $\{g_{n_k}\}_{k\in\mathbb{N}}$ so that the distance functions d_{g_k} converge to d_{g_∞} uniformly on compact sets.*

Proof. Let $\mathcal{M} = \mathcal{M}(M, D_0, v_0, \Lambda)$ be the space of all Riemannian metrics on M with diameter bounded above by D_0, volume bounded below by v_0, and absolute sectional curvatures bounded above by Λ^2. Then, by [GW88, Theorem 1], the space of all such metrics is pre-compact in the following sense: every sequence in $\mathcal{M}$ has a subsequence which converges in the Lipschitz topology to a limiting metric g_∞ whose coordinate functions g_∞^{ij} are of $C^{1,\alpha}$ regularity for some $0 < \alpha < 1$ (see [GW88] for more details). Moreover, the associated distance functions converge uniformly on compact sets [GW88, p. 122].

Thus, it suffices to show any g_n as in the statement of the Corollary is contained in $\mathcal{M}(M, D_0, v_0, \Lambda)$. First, by Lemma V.9, these metrics all satisfy $\operatorname{diam}(M, g) \leq D_0$ for some $D_0 = D_0(n, \varepsilon, \Gamma)$. Second, we know $\operatorname{Vol}_g(M) \geq (h(g)/h(g_0))^n \operatorname{Vol}_{g_0}(N) \geq (1-\varepsilon)^n \operatorname{Vol}_{g_0}(N)$ by [BCG96, Theorem 5.1 i)] and Lemma V.1. Finally, the desired sectional curvature bound holds by assumption. This completes the proof. □

Remark V.11. The metric space (M, g_∞) is CAT(0) as it is a suitable limit of such spaces; see [BH13, Theorem II.3.9].

Lemma V.12. *Suppose g_n is a sequence of Riemannian metrics on M (as in the statement of Corollary V.10) so that the distance functions d_{g_n} converge uniformly to d_{g_∞} on compact sets for some $C^{1,\alpha}$ Riemannian metric g_∞. Lift the g_n and g_∞ to metrics on $\tilde{M}$. Then for any $A > 1$ there is sufficiently large k so that for all $n \geq k$ we have*

$$A^{-1}\, d_{g_\infty}(p, q) \leq d_{g_n}(p, q) \leq A\, d_{g_\infty}(p, q)$$

for all $p, q \in \tilde{M}$. In other words, for sufficiently large n, the distance d_{g_n} on $\tilde{M}$ is A-bi-Lipschitz equivalent to d_{g_∞}.

Proof. Let $D \subset \tilde{M}$ be a fundamental domain for M. Since $d_{g_n} \to d_{g_\infty}$ uniformly on M (with respect to the Lipschitz topology defined in [GW88]), given any constant $A > 1$, there is large enough k so that $k \geq n$ implies

$$A^{-1}\, d_{g_\infty}(p,q) \leq d_{g_n}(p,q) \leq A\, d_{g_\infty}(p,q)$$

for all $p, q \in D$. We can extend these inequalities to all $p, q \in \tilde{M}$ as follows. Consider the g_∞-geodesic from p to q in $\tilde{M}$, and let $p = p_1, \ldots, p_l = q$ be points on this geodesic such that each g_∞-geodesic segment joining p_i to p_{i+1} is contained in a single fundamental domain of the form γD for some $\gamma \in \Gamma$. By the triangle inequality,

$$d_{g_n}(p,q) \leq \sum_{i=1}^{l-1} d_{g_n}(p_i, p_{i+1}) \leq A \sum_{i=1}^{l-1} d_{g_\infty}(p_i, p_{i+1}) = A d_{g_\infty}(p,q).$$

An analogous argument gives the estimate in the other direction. □

Recall the CAT(0) boundary (visual boundary) $\partial\tilde{M}$ of (M, g) is defined as asymptotic classes of geodesic rays [BH13, Definition II.8.1]. If $p \in \tilde{M}$ is fixed, then for any $\xi \in \partial\tilde{M}$, there is a unique geodesic ray connecting x and ξ [BH13, Proposition II.8.2]. Thus, there is a natural identification between $\partial\tilde{M}$ and the unit tangent space $T_p^1 M$. In light of this, we can can make sense of the visual boundaries with respect to all our metrics g_n and g_∞ simultaneously, and we will denote this boundary by $\partial\tilde{M}$.

Lemma V.13. *As above, let g_n be a sequence of Riemannian metrics so that the distance functions d_{g_n} converge uniformly on compact sets to the distance function of some limiting $C^{1,\alpha}$ metric g_∞. Fix $p \in \tilde{M}$ and let $\xi \in \partial\tilde{M}$. For $x \in \tilde{M}$ let $b^n(x) := B_\xi^{g_n}(p,x)$ be the associated Busemann function with respect to the g_n metric, and let $b^\infty(x)$ be defined analogously. Then there is a subsequence b^{n_k} converging to b^∞ uniformly on compact sets.*

Proof. Since b^n is a Busemann function, we have $d_{g_n}(b^n(x), b^n(y)) \leq d_{g_n}(x,y)$. For any $A > 1$, there is large enough k so that $d_{g_n}(x,y) \leq A d_{g_\infty}(x,y)$ for all $n \geq k$. So the b^n form an equicontinuous family and thus converge uniformly on compact sets to some function b (after passing to a subsequence).

We claim b is in fact the Busemann function $b^\infty(x)$ on $\tilde{M}$ with respect to the distance induced by g_∞. Since (M, g_∞) is a CAT(0) space, we use the characterization of Busemann functions in [Bal95, Proposition IV.3.1]. First, $b^\infty(p) = 0$ since this holds for all b^n by assumption. Second, we claim b^∞ is convex. To see this, fix $(q,w) \in T^1\tilde{M}$. Let $\exp_q^n$ denote the exponential map with respect to the metric g_n. Since each b^n is convex, we have

$$b^n(\exp_q^n(tw)) \leq (1-t)b^n(q) + tb^n(\exp_q^n(w))$$

for all $t \in [0,1]$. By [Pug87, Lemma 2], we have $\exp_q^n(tw) \to \exp_q^\infty(tw)$ for all $t \in [0,1]$. Since the b^n converge uniformly on compact sets, taking $n \to \infty$ gives

$$b^\infty(\exp_q^\infty(tw)) \leq (1-t)b^\infty(q) + tb^n(\exp_q^\infty(w))$$

for each n and all $t \in [0,1]$, which shows convexity. Third, $|b^n(p) - b^n(q)| \leq d_{g_n}(p,q)$ for all n; taking $n \to \infty$ shows b^∞ has Lipschitz constant 1. Finally, we need to verify that for any $q \in \tilde{M}$, there is $q_1 \in \tilde{M}$ with $b^\infty(q) - b^\infty(q_1) = 1$. For any n, we know there is q_1^n with $b^\infty(q) - b^n(q_1^n) = 1$, and we can choose q_1^n to also satisfy $d_{g_n}(q, q_1^n) = 1$. By Lemma V.12, the q_1^n are all contained in a bounded set for sufficiently large n, and hence we can pass to a convergent subsequence. The limit of this subsequence is the desired q_1. □

We now consider the Patterson–Sullivan measures $\mu_p^{g_n}$ on $\partial\tilde{M}$. For any negatively curved metric g on M, define $P_t^g = \{\gamma \in \Gamma \,|\, d_g(x, \gamma x) \leq t\}$ and let

$$\delta(g) = \limsup_{t\to\infty} \frac{\log(\# P_t^g)}{t}.$$

Then $\delta(g)$ is independent of the choice of x (in the definition of P_t^g), and $\delta(g) = h(g)$, the critical exponent of μ_p^g. (See [Qui06, Lemma 4.5].)

Now suppose we have a sequence of metrics g_n converging to a CAT(0) metric g_∞ of $C^{1,\alpha}$ regularity, as in the conclusion of Corollary V.10. Define $\delta(g_\infty)$ as above.

Lemma V.14. *If $d_{g_n} \to d_{g_\infty}$ on compact sets, then $\delta(g_n) \to \delta(g_\infty) < \infty$.*

Proof. Fix $A > 1$. By Lemma V.12, there is large enough k so that for any $n \geq k$ the distances d_{g_n} and d_{g_∞} are A-bi-Lipschitz equivalent on all of $\tilde{M}$. Then $P_t^{g_n} \subset P_{At}^{g_\infty}$ which implies $\delta(g_n) \leq A\,\delta(g_\infty)$. Analogously, $\delta(g_\infty) \leq A\,\delta(g_n)$. Thus,

$$|\delta(g_\infty) - \delta(g_n)| \leq \max(A-1, 1-A^{-1})\delta(g_n).$$

Since g_n satisfies $1 - \varepsilon \leq \frac{\mathcal{L}_{g_0}}{\mathcal{L}_{g_n}} \leq 1 + \varepsilon$, Lemma V.1 shows $\delta(g_n) = h(g_n) \leq (1-\varepsilon)^{-1}h(g_0)$ for all n, where $h(g_0)$ is the topological entropy of the geodesic flow of the symmetric space (N, g_0). Thus $|\delta(g_\infty) - \delta(g_n)| \to 0$ as $n \to \infty$. □

Fix $p \in \tilde{M}$ and consider the sequence $\{\mu_p^{g_n}\}$ of probability measures on $\partial\tilde{M}$. By the Banach–Aologlu theorem, this sequence must have a weakly convergent subsequence $\{\mu_p^{g_{n_k}}\}$, ie, there exists a probability measure ν_p such that for any continuous function ϕ on $\partial\tilde{M}$ we have

$$\int_{\partial\tilde{M}} \phi(\xi)\, d(\mu_p^{g_{n_k}})(\xi) \to \int_{\partial\tilde{M}} \phi(\xi)\, d(\nu_p)(\xi).$$

Lemma V.15. *Suppose $d_{g_n} \to d_{g_\infty}$ on compact sets. Consider any family of measures $\{\nu_p\}_{p\in\tilde{M}}$ on $\partial\tilde{M}$ obtained by the above limiting procedure. Then the family $\{\nu_p\}_{p\in\tilde{M}}$ satisfies the following properties.*

1. *For all $p, q \in \tilde{M}$ the Radon–Nikodym derivatives satisfy $\frac{d\nu_p}{d\nu_q} = \exp(-\delta(g_\infty)B^\xi_{g_\infty}(p,q))$.*

2. *For all $p \in \tilde{M}$ and $\gamma \in \Gamma$ the pushforward measures satisfy $\gamma_*\nu_p = \nu_{\gamma.p}$.*

Proof. To show 1), fix p and q and take a subsequence $\{g_n\}$ so that both $\mu_p^{g_n} \to \nu_p$ and $\mu_q^{g_n} \to \nu_q$ as $n \to \infty$. For any continuous function ϕ on $\partial\tilde{M}$ we then have

$$\begin{aligned}
\int_{\partial\tilde{M}} \phi(\xi)\, d(\nu_q)(\xi) &= \lim_{n\to\infty} \int_{\partial\tilde{M}} \phi(\xi)\, d(\mu_q^{g_n})(\xi) \\
&= \lim_{n\to\infty} \int_{\partial\tilde{M}} \phi(\xi) \exp(-\delta(g_n)B^n_\xi(p,q))\, d(\mu_p^{g_n})(\xi) \\
&= \int_{\partial\tilde{M}} \phi(\xi) \exp(-\delta(g_\infty)B^\infty_\xi(p,q))\, d(\nu_p)(\xi).
\end{aligned}$$

In the last equality, we use Lemmas V.13 and V.14. By an analogous argument, we also see $\gamma_*\nu_p = \nu_{\gamma p}$ for all $\gamma \in \Gamma$. □

Corollary V.16. *(See [Rob03, Lemma 1.3].) Let ν as in the previous lemma. Let $x \in \tilde{M}$ and let $\xi \in \partial\tilde{M}$. Let $c_{x,\xi}$ be the unique g_∞-geodesic through x and ξ. Let*

$$\mathcal{O}_x(y,R) = \{\xi \in \partial\tilde{M} \mid c_{x,\xi} \cap B(y,R) \neq \emptyset\}.$$

Then

$$\nu_x(\mathcal{O}_x(\gamma.x, R)) \leq \exp(-h(g)(d_{g_\infty}(x,\gamma.x) - 2R)).$$

Proof. By the previous lemma, one can use the proof of [Rob03, Lemma 1.3] verbatim. □

Proposition V.17. *There is $\kappa > 0$, depending only on n, ε, Γ, Λ, so that for all $p \in \tilde{M}$ and all $u \in T^1_{F(p)}\tilde{N}$, we have $\langle K_{F(p)}u, u\rangle \geq \kappa$.*

Proof. By Lemma V.7, for any fixed metric g, there is $\kappa_g > 0$ so that $\langle K_{F(p)}u, u\rangle \geq \kappa_g$ for all $p \in \tilde{M}$, $u \in T^1_{F(p)}\tilde{N}$. Now let $\mathcal{M}$ as in the proof of Corollary V.10 and suppose for contradiction there is a sequence $\{g_n\} \in \mathcal{M}$ so that $\kappa_{g_n} \to 0$. This means there are $p_n \in M$, together with $u_n \in T^1_{F(p_n)}N$, so that $\langle K_{F(p_n)}u_n, u_n\rangle \to 0$. By compactness of T^1N, we can assume $p_n \to p$ for some $p \in M$ and also $u_n \to u$ for some $u \in T^1_{F(p}\tilde{N}$ (after passing to a subsequence). Thus, $(\mathrm{Hess}B^\xi)_{F(p_n)}(u_n) \to (\mathrm{Hess}B^\xi)_{F(p)}(u)$ uniformly in ξ. After passing to a further subsequence, we can assume $\mu_p^{g_n} \to \mu_p^{g_\infty}$ (using Corollary V.10 and Lemma V.15). Thus, as $n \to \infty$, we have

$$\begin{aligned}\kappa_{g_n} = \langle K_{F(p_n)}u_n, u_n\rangle &= \int_{\partial\tilde{M}} (\mathrm{Hess}B^\xi)_{F(p_n)}(u_n)\exp(-h(g_n)B_\xi(p_n,p)(\overline{f}_*\mu_p^{g_n})\\ &\to \int_{\partial\tilde{M}} (\mathrm{Hess}B^\xi)_{F(p)}(u)(\overline{f}_*\nu_p).\end{aligned}$$

Since we assumed $\kappa_{g_n} \to 0$, the above limit is zero. However, the same argument as in the proof of Lemma V.7 shows this expression is positive. Indeed, the only fact used about g was that the Patterson–Sullivan measure μ_p^g of the complement of two points in the boundary is positive. This still holds for ν_p by Corollary V.16. Thus, we have arrived at a contradiction, and we conclude that κ_g is bounded away from 0 uniformly for g in $\mathcal{M}$. □

V.3: Lipschitz constant for $\mathrm{Jac}F(p)$

To find such a Lipschitz constant, we start by finding a preliminary Lipschitz estimate for F. This uses the lower bound κ for K established in Proposition V.17. While the fact that F is Lipschitz follows from the fact that F is C^2, it is not clear a priori which properties of (M,g) this Lipschitz constant depends on. In the end, this Lipschitz constant will turn out to be close to 1 in a way that depends only on $\varepsilon, n, \Gamma, \Lambda$ by Theorem I.9.

Lemma V.18. *Let F be the BCG map. Then $\|dF_p\| \leq \frac{h(g)}{\kappa}$ for all $p \in \tilde{M}$.*

Proof. Using (V.1.2), we get the following inequality by applying Cauchy–Schwarz (see [BCG96, (5.3)]) together with the fact that $\|dB(w)\| \leq \|w\|$ for any Busemann function:

$$\langle K_{F(p)}dF_p v, u\rangle \leq h(g)\|v\|\|u\|.$$

Now let $\|v\| = 1$ and let $u = dF_p(v)$. Then the above inequality and Proposition V.17 give

$$\kappa\|dF_p v\|^2 \leq \langle K_{F(p)}dF_p(v), dF_p(v)\rangle \leq h(g)\|dF_p(v)\|.$$

Thus

$$\|dF_p(v)\| \leq \frac{h(g)}{\kappa},$$

which completes the proof. □

Let $p, q \in \tilde{M}$ and let $c(t)$ be unit speed the geodesic joining p and q such that $c(0) = p$. Let $P_{c(t)}$ denote parallel transport along the curve $c(t)$. For $i = 1, 2$, let $u_i \in T^1_{F(p)}\tilde{N}$ and let $U_i(t) = P_{F(c(t))}u_i$.

We begin by finding a bound for the derivative of the function $t \mapsto \langle K_{F(c(t))}U_1(t), U_2(t)\rangle$ for $0 \le t \le T_0$. This bound will depend only on ε, n, Γ, Λ and T_0.

Lemma V.19. *Let* $K^\xi_{F(p)}(u_1, u_2) = (\mathrm{Hess} B_\xi)_{F(p)}(u_1, u_2)$. *Let* $U_i(t) = P_{F(c(t))}u_i$ *as above. Then the function* $t \mapsto K^\xi_{F(c(t))}(U_1(t), U_2(t))$ *has derivative bounded by a constant depending only on* ε, n, Γ, Λ.

Proof. Let $X = \frac{d}{dt}|_{t=0}F(c(t))$. Then it suffices to find a uniform bound for $\|X(K^\xi(U_1, U_2))\|$ on $\tilde{N}$. Since the U_i are parallel along X, we have $X(K^\xi(U_1, U_2)) = \nabla K^\xi(U_1, U_2, X)$ (see [dC92, Definition 4.5.7]). So $\|X(K^\xi(U_1, U_2))\| \le \|\nabla K^\xi\|\|U_1\|\|U_2\|\|X\|$. Since $\|X\| \le h(g)/\kappa$ by the previous lemma and $\|U_1\| = \|U_2\| = 1$, it remains to control $\|\nabla K^\xi\|$. We claim this quantity is uniformly bounded on $\tilde{N}$.

First note that if a is an isometry fixing ξ, then

$$K^\xi_x(v, w) = K^\xi_{a(x)}(a_*v, a_*w).$$

Now fix $x_0 \in \tilde{N}$ and let $e_1, \cdots e_n \in T_{x_0}\tilde{N}$ orthonormal frame. For any other $x \in \tilde{N}$, there exists an isometry a taking x to x_0 fixing ξ (since $\tilde{N}$ is a symmetric space). As such, we can extend the e_i to vector fields E_i on all of $\tilde{N}$. Then the quantity

$$\nabla K^\xi(E_i, E_j, E_k) = E_k(K^\xi(E_i, E_j)) - K^\xi(\nabla_{E_k}E_i, E_j) - K^\xi(\nabla_{E_k}E_i, E_j)$$

is invariant by isometries a fixing ξ, and is thus constant on $\tilde{N}$. This shows the desired claim that $\|\nabla K^\xi\|$ is uniformly bounded on $\tilde{N}$. The bound depends only on the symmetric space $\tilde{N}$ and hence only on the dimension n. $\square$

Lemma V.20. *Consider the function*

$$t \mapsto \langle K_{F(c(t))}U_1(t), U_2(t)\rangle$$

for $0 \le t \le T_0$. *Its derivative is bounded by a constant depending only on* $\varepsilon, n, \Gamma, \Lambda, T_0$.

Proof. Note that $\overline{f}_*\mu_{c(t)}(\xi) = \exp\left[-h(g)B^M_{\overline{f}^{-1}(\xi)}(p, c(t))\right]\overline{f}_*\mu_p(\xi)$. Then

$$\langle K_{F(c(t))}U_1(t), U_2(t)\rangle = \int_{\partial\tilde{N}} K^\xi(U_1(t), U_2(t))\,\exp\left[-h(g)B^M_{\overline{f}^{-1}(\xi)}(p, c(t))\right]\overline{f}_*\mu_p(\xi).$$

The first term in the integrand is bounded above as a consequence of (V.2.2), and this bound depends only on the dimension n. By the previous lemma, the derivative of this function is bounded by a constant depending only on $n, \varepsilon, \Gamma, \Lambda$. Since $|B_{\overline{f}^{-1}(\xi)}(p, c(t))| \leq d(p, c(t) \leq T_0$, the second term is bounded by a constant depending only on n, ε, T_0. The same is true of its derivative, since Busemann functions have gradient 1. Hence the derivative of $\langle K_{F(c(t))} U_1(t), U_2(t) \rangle$ is bounded by a constant depending only on the desired parameters. $\square$

Corollary V.21. *The function $t \mapsto \det K_{F(c(t))}$ on the interval $0 \leq t \leq T_0$ is L_1-Lipschitz for some $L_1 = L_1(\varepsilon, n, \Gamma, \Lambda, T_0)$.*

Proof. By Lemma V.20, the entries of the matrix $K_{F(c(t))}$ (with respect to a g_0-orthonormal basis) vary in a Lipschitz way. Using (V.2.2), we see that for u_1 and u_2 unit vectors, the expression $\mathrm{Hess} B^N_{F(p),\xi}(u_1, u_2)$ is uniformly bounded above by some constant depending only on $(\tilde{N}, \tilde{g}_0)$. Since the entries of the matrix $K_{F(c(t))}$ are Lipschitz and bounded, it follows the determinant of this matrix is Lipschitz. $\square$

Recall (V.1.2) implies

$$\langle K_{F(p)} dF_p(v), u \rangle = h(g) \int_{\partial \tilde{M}} dB^N_{F(p), \overline{f}(\xi)}(u) dB^M_{p,\xi}(v) d\mu_p(\xi).$$

This formula, together with the Lipschitz bound for $p \mapsto \det K_{F(p)}$ established in Corollary V.21, will allow us to find a Lipschitz bound for $p \mapsto \det(dF_p) = \mathrm{JacF}(p)$.

Lemma V.22. *Let p, q and $c(t)$ be as above. Then the function*

$$t \mapsto dB^M_{c(t),\xi}(P_{c(t)} v)$$

is $\Lambda/2$-Lipschitz for all $v \in T^1_p \tilde{M}$.

Proof. We have

$$\frac{d}{dt}|_{t=0} dB_{c(t),\xi}(P_{c(t)} v) = \mathrm{Hess} B_{p,\xi}(c'(0), v) = \mathrm{Hess} B_{p,\xi}(c'(0)^T, v^T),$$

where $c'(0)^T$ and v^T are the components of $c'(0)$ and v in the direction tangent to the horosphere through p and ξ. Using that $\mathrm{Hess} B_{p,\xi}$ is bilinear and positive definite on $\mathrm{grad} B^\perp_{p,\xi}$, we obtain

$$4 \mathrm{Hess} B_{p,\xi}(c'(0)^T, v^T) \leq \mathrm{Hess} B_{p,\xi}(c'(0)^T + v^T, c'(0)^T + v^T).$$

Let $v' = c'(0)^T + v^T$ and note $\|v'\| \leq 2$. Let $\beta(s)$ be a curve in the horosphere such that $\beta'(0) = v'$. Consider the geodesic variation $j(s,t) = \exp_{\beta(s)}(t \mathrm{grad} B_{\beta(s),\xi})$ and let $J(t) =$

$\frac{d}{ds}|_{s=0} j(s,t)$ be the associated Jacobi field. Then $J(0) = v'$ and $J'(0) = \nabla_{v'} \mathrm{grad} B_{p,\xi}$. This means

$$\mathrm{Hess} B_{p,\xi}(c'(0)^T + v^T, c'(0)^T + v^T) = \langle J'(0), J(0) \rangle.$$

Let $\chi = \frac{1}{2}\sqrt{\lambda^2 + \Lambda^2}$. According to [BK84, 4.2],

$$\langle J'(0), J(0) \rangle \leq \langle J'(0) + \chi J(0), J(0) \rangle \leq |J(0)|(\chi - \lambda).$$

Since $|J(0)| = |v'| \leq 2$, we get $4\frac{d}{dt}|_{t=0} dB_{c(t),\xi}(P_{c(t)}v) \leq \langle J'(0), J(0) \rangle \leq 2\Lambda$. □

Lemma V.23. *The function $t \mapsto dB^N_{F(c(t)),\xi}(P_{F(c(t))}u)$ is $(\frac{h(g)}{\kappa} + 1)$-Lipschitz for all $u \in T^1_{F(p)}\tilde{N}$.*

Proof. We repeat the same proof as in the previous lemma, but replacing λ^2 and Λ^2 with 1 and 4, respectively. In this case, $\chi - \lambda < 1$. This gives

$$\frac{d}{dt}|_{t=0}\, dB^N_{F(c(t)),\xi}(P_{F(c(t))}u) = \mathrm{Hess} B^N_{F(p),\xi}(dF_p(c'(0)), u) < |dF_p(c'(0)) + u|.$$

Since $c'(0)$ has norm 1, the Lipschitz bound from Lemma V.18 gives $|dF_p(c'(0)) + u| \leq \frac{h(g)}{\kappa} + 1$, which completes the proof. □

Lemma V.24. *The function $t \mapsto \det K_{F(c(t))} \mathrm{Jac} F(c(t))$ on the interval $0 \leq t \leq T_0$ is L_2-Lipschitz, where L_2 depends only on $\varepsilon, n, \Gamma, \Lambda$, T_0.*

Proof. Consider the function

$$t \mapsto h(g) \int_{\partial \tilde{M}} dB^N_{F(c(t)),f(\xi)}(P_{F(c(t))}u) dB^M_{c(t),\xi}(P_{c(t)}v) e^{-h(g)B_\xi(p,c(t))} d\mu_p(\xi).$$

The first two terms in the integrand are bounded by 1 in absolute value. The third term is bounded above by a constant depending only on ε, n, T_0 as in the proof of Lemma V.20 and $h(g) \leq (1+\varepsilon)h(g_0)$ by Lemma V.1. Moreover, the three terms in the integrand are each Lipschitz – the first two by Lemmas V.23 and V.22, respectively, and the last one as in the proof of Lemma V.20. Since the entries of the matrix $K_{F(c(t))}(dF_{c(t)})$ are bounded and Lipschitz, the determinant of this matrix is also Lipschitz. □

Proposition V.25. *The function $p \mapsto |\mathrm{Jac} F(p)|$ is L-Lipschitz, where the constant L depends only on $\varepsilon, n, \Gamma, \Lambda$.*

Proof. Since $K_{F(p)}$ is a symmetric matrix, it has an orthonormal basis of eigenvectors u_i. Moreover, $\langle K_{F(p)}u_i, u_i \rangle \geq \kappa \langle u_i, u_i \rangle$ by Proposition V.17. It follows that $\det K_{F(p)} \geq \kappa^n$.

Using this, we obtain

$$\begin{aligned}\kappa^n|\mathrm{Jac}F(p)-\mathrm{Jac}F(q)| &\le |\det K_{F(p)}\mathrm{Jac}F(p)-\det K_{F(p)}\mathrm{Jac}F(q)|\\ &\le L_2 d(p,q)+|\mathrm{Jac}F(q)||\det K_{F(p)}-\det K_{F(q)}| \qquad \text{(Lemma V.24)}\\ &\le L_2 d(p,q)+(1+\varepsilon)^n L_1 d(p,q),\end{aligned}$$

where the last inequality follows from Corollary V.21 and Lemmas V.5 and V.1. Moreover, Corollary V.21 and Lemma V.24 imply L_1 and L_2 depend only on $\varepsilon, n, \Gamma, \Lambda$. Proposition V.17 states κ depends only on ε, n, Γ. □

Let $c_\varepsilon := (1-C\varepsilon^2)(1-\varepsilon)^n$ be the constant from Theorem I.12 satisfying $c_\varepsilon \mathrm{Vol}(M) \le \mathrm{Vol}(N)$. (Recall C depends only on n since N is locally symmetric.) Let $c(n)$ denote the volume of the unit ball in $\mathbb{R}^n$. Choose ε_0 small enough so that

$$\frac{1-c_{\varepsilon_0}/(1+\varepsilon_0)^n}{\varepsilon_0^{1/(n+1)}}c_{\varepsilon_0}^{-1}(1+\varepsilon_0)^n \le \frac{c(n)\mathrm{inj}(N,g_0)^n}{\mathrm{Vol}(N)}.$$

This is possible since the first term on the lefthand side approaches 0 as $\varepsilon_0 \to 0$, while the other two approach 1. Indeed, the numerator of the first term can be written as $2n\varepsilon_0+O(\varepsilon_0^2)$. The righthand side depends only on n and Γ, so the choice of ε_0 depends only on n and Γ.

Hypothesis V.26. *From now on, we assume $\varepsilon \le \varepsilon_0$. (The reason for this will become apparent in the proof of the next proposition, see (V.4.1).) Then for L as in Proposition V.25, we have $L(\varepsilon, n, \Gamma, \Lambda) \le L(\varepsilon_0, n, \Gamma, \Lambda)$ for all $\varepsilon \le \varepsilon_0$. From now on, we will use $L = L(n, \Gamma, \Lambda)$ to denote $L(\varepsilon_0, n, \Gamma, \Lambda)$.*

V.4: Lower bound for $|\mathrm{Jac}F(p)|$

Now that we have a Lipschitz bound for $\mathrm{Jac}F(p)$, we can use the fact that (M,g) and (N,g_0) have approximately equal volumes (Theorem I.12) and approximately equal entropies (Lemma V.1) to show equality almost holds in the inequality $\mathrm{Jac}F(p) \le (h(g)/h(g_0))^n$ (Lemma V.5).

Proposition V.27. *There is a constant $\beta < 1$, depending only on $\varepsilon, n, \Gamma, \Lambda$, such that*

$$\beta\left(\frac{h(g)}{h(g_0)}\right)^n \le |\mathrm{Jac}F(p)|$$

for all $p \in \tilde{M}$. In particular, there is a constant $C = C(n,\Gamma,\Lambda)$ so that $\beta = 1 - C\varepsilon^{1/(n+1)} + O(\varepsilon^{2/(n+1)})$.

We need two preliminary lemmas. Let ν denote the measure on M coming from the Riemannian volume.

Lemma V.28. *Let $\phi : M \to \mathbb{R}$ be a ν-measurable function such that $\phi \geq 0$. Suppose the integral of ϕ satisfies $0 \leq \int_M \phi \leq \delta$. Let $B = \{x \in M \mid \phi > \omega\}$ where ω is some constant. Then $\nu(B) \leq \delta/\omega$.*

Proof. Note that $\omega\, \nu(B) \leq \int_B \phi \leq \int_X \phi \leq \delta$, which gives the desired bound. □

Lemma V.29. *Let i_M denote the injectivity radius of M and let $c(n)$ denote the volume of the unit ball in $\mathbb{R}^n$. Fix $\delta < c(n)(i_M)^n$. Let $B \subset M$ be an open set with $\nu(B) < \delta$. Then there is $r = r(\delta)$ such that for any $p \in B$ there is $q \in M \setminus B$ with $d(p,q) \leq r$. Moreover, $r \leq c(n)^{-1/n}\delta^{1/n}$.*

Proof. Let $p \in B$. Let $q \in M \setminus B$ be the point such that $d(p,q) = \min_{x \in M\setminus B} d(p,x)$. Let $r = d(p,q)$. Then the open ball $B(p,r)$ is contained in the set B. We consider the cases $r \leq i_M$ and $r > i_M$ separately:

In the case $r \leq i_M$, we can apply Theorem 3.101 ii) in [GHL90] to obtain the inequality $\mathrm{Vol}B(p,r) \geq c(n)r^n$, where $c(n)$ is the volume of the unit ball in $\mathbb{R}^n$. Since $B(p,r) \subset B$, this gives $r^n \leq \frac{\delta}{c(n)}$.

In the case $r > i_M$, we do not have the above volume estimate for the ball $B(p,r)$. However, $B(p,i_M) \subset B(p,r) \subset B$ so the same argument as in the first case gives a bound $(i_M)^n \leq \frac{\delta}{c(n)}$. This is a contradiction for small enough δ, so we must be in the first case. □

Remark V.30. We have $i_M \geq i_0$ where i_0 is a constant depending only on Γ (and on $\varepsilon_0 = \varepsilon_0(\Gamma)$). Indeed, Lemma V.8 gives $i_M \geq \frac{1}{1+\varepsilon_0} i_N$, and i_N depends only on Γ by Mostow rigidity and our choice of normalization for the metric g_0.

Proof of Proposition V.27. Let $c_\varepsilon := (1 - C\varepsilon^2)(1-\varepsilon)^n$ be the constant from Theorem I.12 satisfying $c_\varepsilon \mathrm{Vol}(M) \leq \mathrm{Vol}(N)$. (Recall C depends only on n since N is locally symmetric.) Using this theorem together with the bound $\frac{h(g)}{h(g_0)} \leq 1+\varepsilon$ from Lemma V.1, we get

$$c_\varepsilon \frac{1}{(1+\varepsilon)^n} \left(\frac{h(g)}{h(g_0)}\right)^n \mathrm{Vol}(M) \leq c_\varepsilon \mathrm{Vol}(M) \leq \mathrm{Vol}(N).$$

Combining with (V.1.5) gives

$$c_\varepsilon \frac{1}{(1+\varepsilon)^n} \left(\frac{h(g)}{h(g_0)}\right)^n \mathrm{Vol}(M) \leq \int_M |(\mathrm{Jac}F)|\, d\,\mathrm{Vol} \leq \left(\frac{h(g)}{h(g_0)}\right)^n \mathrm{Vol}(M).$$

Next, we apply Lemma V.28 to $\phi(p) = (h(g)/h(g_0))^n - |\mathrm{Jac}F(p)| \geq 0$. In this case, we indeed have $0 \leq \int_M \phi \leq \delta$ with $\delta = (1 - c_\varepsilon/(1+\varepsilon)^n)\,(h(g)/h(g_0))^n\,\mathrm{Vol}(M)$. Let $\alpha < 1$ and write

$$M_\alpha = \left\{ |\mathrm{Jac}F(p)| \geq \alpha \left(\frac{h(g)}{h(g_0)}\right)^n \right\}.$$

Then $M_\alpha = \{\phi \leq (1-\alpha)(h(g)/h(g_0))^n\}$. So Lemma V.28 gives

$$\nu\left(M \setminus M_\alpha\right) \leq \frac{1 - c_\varepsilon/(1+\varepsilon)^n}{1-\alpha}\mathrm{Vol}(M).$$

Let $1 - \alpha = \varepsilon^{1/(n+1)}$. Let $\varepsilon_0 = \varepsilon_0(n, \Gamma)$ as in Hypothesis V.26. Then

$$\nu(M \setminus M_\alpha) \leq c(n)(\mathrm{inj}(M, g))^n, \tag{V.4.1}$$

so the hypotheses of Lemma V.29 are satisfied. The lemma gives $r(\varepsilon) = c(n)\nu(M \setminus M_\alpha)^{1/n}$ so that for all $p \in M \setminus M_\alpha$ there is $q \in M_\alpha$ satisfying $d(p, q) < r(\varepsilon)$. Applying Proposition V.25 with $T_0 = r(\varepsilon_0)$, we then have

$$\alpha \left(\frac{h(g)}{h(g_0)}\right)^n \leq |\mathrm{Jac}F(q)| \leq Lr(\varepsilon) + |\mathrm{Jac}F(p)|$$

for some $L = L(n, \Gamma, \Lambda)$. Rearranging and applying the entropy estimate in Lemma V.1 gives

$$\left(\alpha - (1-\varepsilon)^{-n}Lr(\varepsilon)\right)\left(\frac{h(g)}{h(g_0)}\right)^n \leq \mathrm{Jac}|F(p)|.$$

Let $\beta = \alpha - (1-\varepsilon)^{-n}Lr(\varepsilon)$. Using $\alpha = 1 - \varepsilon^{1/(n+1)}$ gives $\mu(M \setminus M_\alpha) \leq C\varepsilon^{1-1/(n+1)} + O(\varepsilon^{2-1/(n+1)})$ and $r(\varepsilon) \leq C\varepsilon^{1/(n+1)} + O(\varepsilon^{2/(n+1)})$, where the constants C depend only on n, Γ, Λ. So $\beta = 1 - C\varepsilon^{1/(n+1)} + O(\varepsilon^{2/(n+1)})$ for some $C = C(n, \Gamma, \Lambda)$. □

V.5: Estimates for $\|dF_p\|$

Recall $H_{F(p)}$ and $K_{F(p)}$ are symmetric bilinear forms on $T_{F(p)}\tilde{N}$ (see (V.1.3) and (V.1.4)). We will use the lower bound we just established for $\mathrm{Jac}F(p)$ in Proposition V.27 to show H and K are close to scalar matrices. This will then allow us to mimic the proof of [BCG96, Proposition 5.2 ii)] to find bounds for the derivative of the BCG map that are close to 1.

Proposition V.31. *Let $F : \tilde{M} \to \tilde{N}$ be the BCG map and assume there is a constant $\beta < 1$*

as in the conclusion of Proposition V.27 so that the Jacobian of F satisfies

$$\beta\left(\frac{h(g)}{h(g_0)}\right)^n \leq |\mathrm{Jac}F(p)|$$

for all $p \in \tilde{M}$. Let $H_{F(p)}$ and $K_{F(p)}$ be the symmetric bilinear forms on $T_{F(p)}\tilde{N}$ defined in (V.1.3) and (V.1.4). Then there are constants $a, a' < 1$ and $A, A' > 1$, depending only on β and n, such that

$$a\frac{1}{n}\langle v, v\rangle \leq \langle H_{F(p)}v, v\rangle \leq A\frac{1}{n}\langle v, v\rangle,$$
$$a'\frac{h(g_0)}{n}\langle v, v\rangle \leq \langle K_{F(p)}v, v\rangle \leq A'\frac{h(g_0)}{n}\langle v, v\rangle$$

for all $p \in M$ and all $v \in T_{F(p)}\tilde{N}$. In particular, there is a constant $C = C(n, \Gamma, \Lambda)$ so that $a = 1 - C\varepsilon^{1/2(n+1)} + O(\varepsilon^{1/(n+1)})$, $A = 1 + C\varepsilon^{1/2(n+1)} + O(\varepsilon^{1/2(n+1)})$, $a' = 1 - C\varepsilon^{1/4(n+1)} + O(\varepsilon^{1/2(n+1)})$, $A' = 1 - C\varepsilon^{1/4(n+1)} + O(\varepsilon^{1/2(n+1)})$.

Remark V.32. If $\tilde{N}$ is not Cayley hyperbolic space, we can take $a = a'$ and $A' = A$. This is explained right after the proof of the proposition.

The lower bound on $\mathrm{Jac}F(p)$ can be thought of as equality almost holding in Lemma V.5. This lower bound, together with the inequalities in Lemmas V.2 and V.3, implies equality almost holds in Lemma V.4, that is,

$$\beta^{\frac{2(n+d-2)}{n-d}}\left(\frac{n^{\alpha-1}}{(n-1)^\alpha}\right)^n \leq \frac{\det H}{\det(I-H)^\alpha} \leq \left(\frac{n^{\alpha-1}}{(n-1)^\alpha}\right)^n, \tag{V.5.1}$$

where $\alpha = \frac{2(n-1)}{n-d}$.

In order to prove Proposition V.31, we will first show that since β is close to 1, the matrix H is almost $\frac{1}{n}I$.

Lemma V.33. *Let H be a symmetric positive definite $n \times n$ matrix with trace 1 for $n \geq 3$. Let $1 < \alpha \leq n-1$ and let $m = \left(\frac{n^{\alpha-1}}{(n-1)^\alpha}\right)^n$. Suppose*

$$\frac{\det H}{\det(I-H)^\alpha} \geq \beta' m,$$

where $\beta' = \beta^{\frac{2(n+d-2)}{n-d}}$ and β is as in Proposition V.27. (Note $0 < \beta' < 1$.) Let λ_i denote the eigenvalues of H. Then there are constants $a < 1$ and $A > 1$, depending on β and n, such that

$$a\frac{1}{n} \leq \lambda_i \leq A\frac{1}{n}$$

for $i = 1, \dots, n$. In particular, there is a constant $C = C(n, \Gamma, \Lambda)$ so that $a = 1 - C\varepsilon^{1/4n} + O(\varepsilon^{1/2n})$, $A = 1 + C\varepsilon^{1/4n} + O(\varepsilon^{1/2n})$.

Proof. It follows from [BCG95, Proposition B.5] (see [Rua22] for the Cayley case), Lemma V.2 and Proposition V.27 that there is a constant $B(n) > 0$ so that

$$\sum_{i=1}^{n} \left(\lambda_i - \frac{1}{n}\right)^2 \leq \frac{1-\beta'}{B}.$$

Write $\delta = \sqrt{(1-\beta')/B}$. Then $|\lambda_i - 1/n| < \delta$ implies we can take $a = 1 - n\delta$, $A = 1 + n\delta$. Recall $\beta = 1 - C\varepsilon^{1/(n+1)} + O(\varepsilon^{2/(n+1)})$ and $\beta' = \beta^p$ for some $p(n,d) > 1$. Then $\beta' = 1 - C'\varepsilon^{1/(n+1)} + O(\varepsilon^{2/(n+1)})$, where C' is a possibly different constant still depending only on n, Γ, Λ. Thus there is a constant $C = C(n, \Gamma, \Lambda)$ so that $\delta = C\varepsilon^{1/2(n+1)} + O(\varepsilon^{1/(n+1)})$. So we can take $a = 1 - C\varepsilon^{1/2(n+1)} + O(\varepsilon^{1/(n+1)})$ and $A = 1 + C\varepsilon^{1/2(n+1)} + O(\varepsilon^{1/(n+1)})$, where C is another constant depending on the same parameters. □

Next, we need an analogue of Lemma V.33 for the arithmetic-geometric mean inequality.

Lemma V.34. *Let L be a symmetric positive-definite $n \times n$ matrix with $b \leq trace(L) \leq b'$ for positive constants b, b' depending only on $\varepsilon, n, \Gamma, \Lambda$. Suppose*

$$\det L \geq \alpha \left(\frac{1}{n}\mathrm{trace} L\right)^n$$

for some $0 < \alpha < 1$. Let $\mu_1, \dots, \mu_n$ denote the eigenvalues of L. Then there are constants $a' < 1$ and $A' > 1$ such that

$$a'\frac{\mathrm{trace}(L)}{n} \leq \mu_i \leq A'\frac{\mathrm{trace}(L)}{n}$$

for $i = 1, \dots, n$. In particular, there is a constant $C = C(n, \Gamma, \Lambda)$ so that $a' = 1 - C\sqrt{1-\alpha}$, $A' = 1 + C\sqrt{1-\alpha}$.

Proof. We will use the approach of the proof of [BCG95, Proposition B5]. Let $\phi(\mu_1, \dots, \mu_n) = \log(\mu_1 \cdot \dots \cdot \mu_n)$. Since ϕ is concave, there is a constant $B > 0$ so that the inequality

$$\log(\mu_1 \cdot \dots \cdot \mu_n) \leq \log\left(\frac{\mathrm{trace}(L)}{n}\right)^n - B\sum_{i=1}^{n}\left(\mu_i - \frac{\mathrm{trace}(L)}{n}\right)^2$$

holds on the set of all $\mu_i \geq 0$ satisfying $\mu_1 + \dots + \mu_n = \mathrm{trace}(L)$. The constant B depends only on the function ϕ. In other words, it does not depend on any topological or geometric properties of the manifolds M and N other than the number $n = \dim M = \dim N$.

Since L is positive definite, we know $0 < \mu_i < \operatorname{trace}(L)$ for all i. So there exists $T = T(\varepsilon, n, \Gamma, \Lambda)$ such that $B\sum_{i=1}^n \left(\mu_i - \frac{\operatorname{trace}(L)}{n}\right)^2 \leq T$. Following the same steps as in the proof of [BCG95, Proposition B.5], we then obtain

$$\sum_{i=1}^n \left(\mu_i - \frac{\operatorname{trace}(L)}{n}\right)^2 \leq \frac{1-\alpha}{B^{\frac{1-e^{-T}}{T}}}.$$

Let $\delta^2 = (1-\alpha)/(B^{\frac{1-e^{-T}}{T}})$. Then we can write $\delta = C\sqrt{1-\alpha}$ for some $C = C(n, \Gamma, \Lambda)$. Using the boundedness assumption $b \leq \operatorname{trace}(L) \leq b'$, we conclude $a' = 1 - C\sqrt{1-\alpha}$, $A' = 1 + C\sqrt{1-\alpha}$ for some $C = C(n, \Gamma, \Lambda)$. □

Proof of Proposition V.31. First, note that $\det K \geq a^{n/2}(h(g_0)/n)^n$ follows from [BCG95, Proposition B5] and Lemma V.33. So equality almost holds in the arithmetic-geometric mean inequality. By Lemma V.34, the eigenvalues of K are between $a'h(g_0)/n$ and $A'h(g_0)/n$, where $a' = 1 - C\sqrt{1-a^{n/2}}$ and $A' = 1 + C\sqrt{1-a^{n/2}}$. In terms of ε, we have $a' = 1 - C\varepsilon^{1/4(n+1)} + O(\varepsilon^{1/2(n+1)})$ and $A' = 1 - C\varepsilon^{1/4(n+1)} + O(\varepsilon^{1/2(n+1)})$. □

Proof of Remark V.32. When N is a real, complex or quaternionic hyperbolic space, we can write

$$K = I - H - \sum_{k=1}^{d-1} J_k H J_k, \tag{V.5.2}$$

for $d = 1, 2, 4$, respectively. Here, $J_1, \ldots J_{d-1}$ are the orthogonal endomorphisms at each point defining the complex or quaternionic structure. They are parallel and satisfy $J_i^2 = -Id$; see [BCG96, p. 638]. Now recall that Lemma V.33 gives

$$a\frac{1}{n}\langle v, v\rangle \leq \langle Hv, v\rangle \leq A\frac{1}{n}\langle v, v\rangle$$

for all v. To prove the corresponding statement for K, first note $\langle J_k H J_k u, u\rangle = \langle -HJ_k u, J_k u\rangle$. Since $\langle J_k u, J_k u\rangle = \langle u, u\rangle$, we have

$$a\frac{1}{n}\langle u, u\rangle \leq \langle HJ_k u, J_k u\rangle \leq A\frac{1}{n}\langle u, u\rangle.$$

We can use equation (V.5.2) to write

$$\begin{aligned}\langle Ku,u\rangle &= \langle u,u\rangle - \langle Hu,u\rangle + \sum_{k=1}^{d-1}\langle HJ_k u, J_k u\rangle \\ &\le (1 - a\frac{1}{n} + A\frac{d-1}{n})\langle u,u\rangle \\ &= \left(\frac{n+d-2}{n} + n\delta\frac{1}{n} + n\delta\frac{d-1}{n}\right)\langle u,u\rangle \qquad (\text{using } a = 1-n\delta,\ A = 1+n\delta) \\ &\le A\frac{n+d-2}{n}\langle u,u\rangle.\end{aligned}$$

By a similar argument, $\langle Ku,u\rangle \ge a\frac{n+d-2}{n}\langle u,u\rangle$. □

Proposition V.35. *Let F denote the BCG map, and suppose H and K satisfy the conclusion of Proposition V.31. Then there are constants $c_1 = c_1(\varepsilon,n,\Gamma,\Lambda) < 1$, $C_2 = C_2(\varepsilon,n,\Gamma,\Lambda) > 1$ such that for all $v \in TM$ we have*

$$c_1\|v\|_g \le \|dF(v)\|_{g_0} \le C_2\|v\|_g. \tag{V.5.3}$$

Moreover, there is a constant $C = C(n,\Gamma,\Lambda)$ so that $c_1 = 1 - C\varepsilon^{1/8(n+1)} + O(\varepsilon^{1/4(n+1)})$ and $c_2 = 1 + C\varepsilon^{1/8(n+1)} + O(\varepsilon^{1/4(n+1)})$.

Proof. We closely follow the proof of [BCG96, Proposition 5.2 ii)]. First note it suffices to prove the claim for v a unit vector. Using the definitions of H and K together with the Cauchy-Schwarz inequality, we obtain

$$\langle KdF_p v, u\rangle \le h(g)\langle Hu,u\rangle^{1/2}\left(\int_{X(\infty)}(dB_{p,\xi}(v))^2 d\mu_p(\xi)\right)^{1/2}.$$

(See [BCG96, (5.3)].) Using the upper bound for H in Proposition V.31, the above inequality implies

$$\langle KdF_p v, u\rangle \le \sqrt{A}\frac{h(g)}{\sqrt{n}}\|u\|\left(\int_{X(\infty)}(dB_{p,\xi}(v))^2 d\mu_p(\xi)\right)^{1/2}.$$

Now let $u = dF_p(v)/\|dF_p(v)\|$. Using the lower bound for K in Proposition V.31 gives

$$\|dF_p\| \le \frac{\sqrt{A}}{a'}\frac{h(g)}{h(g_0)}\sqrt{n}\left(\int_{X(\infty)}(dB_{p,\xi}(v))^2 d\mu_p(\xi)\right)^{1/2}.$$

Now let $L = dF_p \circ dF_p^T$ and let v_i be an orthonormal basis for $T_p\tilde{M}$. Then, since $\|dB_{p,\xi}(v)\| \le$

$\|v\| = 1$, we get

$$\mathrm{trace}(L) = \sum_{i=1}^{n} \langle Lv_i, v_i \rangle = \langle dF_p(v_i), dF_p(v_i) \rangle \leq \left(\frac{\sqrt{A}}{a'} \frac{h(g)}{h(g_0)} \right)^2 n.$$

Combining this with Proposition V.27 and the arithmetic-geometric mean inequality gives

$$\beta^2 \left(\frac{h(g)}{h(g_0)} \right)^{2n} \leq |\mathrm{Jac}F(p)|^2 = \det L \leq \left(\frac{1}{n} \mathrm{trace} L \right)^n \leq \left(\frac{\sqrt{A}}{a'} \frac{h(g)}{h(g_0)} \right)^{2n}. \tag{V.5.4}$$

Hence the hypotheses of Lemma V.34 hold with $\alpha = \beta^2 (a')^{2n}/A^n$. Using the expressions for β, a', A in Propositions V.27 and V.31, we can write $\alpha = 1 - C\varepsilon^{1/4(n+1)} + O(\varepsilon^{1/2(n+1)})$ for some $C = C(n, \Gamma, \Lambda)$. Lemma V.34 thus implies

$$a_1 \frac{1}{n} \mathrm{trace} L \langle v, v \rangle \leq \langle Lv, v \rangle = \langle dF_p v, dF_p v \rangle \leq A_2 \frac{1}{n} \mathrm{trace} L \langle v, v \rangle,$$

where $a_1 = 1 - C\varepsilon^{1/8(n+1)} + O(\varepsilon^{1/4(n+1)})$ and $A_2 = 1 + C\varepsilon^{1/8(n+1)} + O(\varepsilon^{1/4(n+1)})$. Using (V.5.4) gives

$$a_1 \beta^{2/n} \left(\frac{h(g)}{h(g_0)} \right)^2 \leq \frac{\langle dF_p v, dF_p v \rangle}{\langle v, v \rangle} \leq A_2 \left(\frac{\sqrt{A}}{a} \frac{h(g)}{h(g_0)} \right)^2.$$

Hence, there is a constant $C = C(n, \Gamma, \Lambda)$ so that the lower bound can be written as $1 - C\varepsilon^{1/8(n+1)} + O(\varepsilon^{1/4(n+1)})$ and the upper bound as $1 + C\varepsilon^{1/8(n+1)} + O(\varepsilon^{1/4(n+1)})$. $\square$

CHAPTER VI
Surfaces

In this chapter, we prove a generalization of Otal's marked length spectrum rigidity result for negatively curved surfaces [Ota90]. We show that pairs of negatively curved metrics on a surface become more isometric as the ratio of their marked length spectrum functions gets closer to 1. Aside from some background on the Liouville measure and Liouville current from Chapter IV, this section does not rely on earlier parts of this paper.

Let $\mathcal{C}(2, \lambda, \Lambda, v_0, D_0)$ consist of all closed C^∞ Riemannian manifolds of dimension 2 with sectional curvatures contained in the interval $[-\Lambda^2, -\lambda^2]$, volume bounded below by v_0, and diameter bounded above by D_0. In this section we will prove the following theorem about surfaces whose marked length spectra are close:

Theorem I.8. *Fix $\lambda, \Lambda, v_0, D_0 > 0$. Fix $L > 1$. Then there exists $\varepsilon = \varepsilon(L, \lambda, \Lambda, v_0, D_0) > 0$ small enough so that for any pair $(M, g), (M, h) \in \mathcal{C}(2, \lambda, \Lambda, v_0, D_0)$ satisfying*

$$1 - \varepsilon \leq \frac{\mathcal{L}_g}{\mathcal{L}_h} \leq 1 + \varepsilon, \tag{VI.0.1}$$

there exists an L-Lipschitz diffeomorphism $f : (M, g) \to (M, h)$.

The space $\mathcal{C}(2, \lambda, \Lambda, v_0, D_0)$ has the property that any sequence has a convergent subsequence in the Lipschitz topology; this is often called the Gromov compactness theorem [GKPS99]. In this paper, we use refinements of Gromov's theorem due to Pugh and Greene–Wu [Pug87, GW88].

It follows from [GW88] that any sequence $(M, g_n) \in \mathcal{C}(2, \lambda, \Lambda, v_0, D_0)$ has a subsequence (M, g_{n_k}) converging in the following sense: there is a Riemannian metric g_0 on M such that in local coordinates we have $g^{ij}_{n_k} \to g^{ij}_0$ in the $C^{1,\alpha}$ norm, and the limiting g^{ij}_0 have regularity $C^{1,\alpha}$. Additionally, the distance functions $d_{g_{n_k}}$ converge uniformly (with respect to the Lipschitz distance) to d_{g_0} on compact sets; see [GW88, p. 122]. In particular, this implies the following (see also Lemma V.12):

Lemma VI.1. *Given any $A > 1$, there is a sufficiently large k so that for all $p, q \in M$ we have $A^{-1} d_{g_0}(p,q) \leq d_{g_{n_k}}(p,q) \leq A\, d_{g_0}(p,q)$.*

We will use Gromov compactness to prove Theorem I.8 by contradiction. Indeed, suppose the statement is false. Then for every $\varepsilon > 0$, there are $(M, g_\varepsilon), (M, h_\varepsilon) \in \mathcal{C}(2, \lambda, \Lambda, v_0, D_0)$ so that there is no L-Lipschitz map $f : (M, g_\varepsilon) \to (M, h_\varepsilon)$. By [GW88], there is a subsequence $\varepsilon_n \to 0$ so that $(M, g_{\varepsilon_n}) \to (M, g_0)$ and $(M, h_{\varepsilon_n}) \to (M, h_0)$ in the sense described above. From now on we will relabel g_{ε_n} as g_n and h_{ε_n} as h_n. To prove the main theorem, it suffices to prove the following statement:

Proposition VI.2. *Let (M, g_0) and (M, h_0) be the Greene–Wu limits of the counterexamples above. Then there is a map $f : M \to M$ such that for all $p, q \in M$ we have $d_{g_0}(p,q) = d_{h_0}(f(p), f(q))$.*

Proof of Theorem I.8. Fix $L > 1$ and suppose the theorem is false. Let $(M, g_n), (M, h_n)$ be the convergent sequences of counter-examples defined above. Since $(M, g_n) \to (M, g_0)$, Lemma VI.1 gives large enough n so that $\sqrt{L}^{-1} d_{g_0}(p,q) \leq d_{g_n}(p,q) \leq \sqrt{L}\, d_{g_0}(p,q)$ for all $p, q \in M$, and similarly for d_{h_n}. Then Proposition VI.2 gives

$$d_{g_n}(p,q) \leq \sqrt{L}\, d_{h_0}(f(p), f(q)) \leq L\, d_{h_n}(f(p), f(q)).$$

So $f : (M, g_n) \to (M, h_n)$ is an L-Lipschitz map, which is a contradiction. □

VI.1: The marked length spectra of (M, g_0) and (M, h_0)

To prove Proposition VI.2, we will first show (M, g_0) and (M, h_0) have the same marked length spectrum. Then we will construct an isometry $f : (M, g_0) \to (M, h_0)$. We use the same main steps as in [Ota90]; however, since g_0 and h_0 are only of $C^{1,\alpha}$ regularity, there are additional technicalities that arise when verifying the requisite properties of the Liouville measure and Liouville current in this context.

We first recall some additional properties of the limit (M, g_0). By a theorem of Pugh [Pug87, Theorem 1], this limiting metric will have a Lipschitz geodesic flow, and the geodesics themselves are of $C^{1,1}$ regularity. Moreover, the exponential maps converge uniformly on compact sets [Pug87, Lemma 2], which is equivalent to the following:

Lemma VI.3. *Let ϕ_n and ϕ_0 denote the geodesic flows on (T^1M, g_n) and (T^1M, g_0) respectively. Fix $T > 0$ and let $K \subset T^1M$ compact. Then $\phi_n^t v \to \phi_0^t v$ uniformly for $(t, v) \in [0, T] \times K$.*

In addition, the space (M, g_0) is CAT$(-\lambda^2)$ because it is a suitable limit of such spaces; see [BH13, Theorem II.3.9]. Thus, even though the curvature tensor is not defined for the $C^{1,\alpha}$ metric g_0, this limiting space still exhibits many key properties of negatively curved manifolds. One such property, heavily used in Otal's proof of marked length spectrum rigidity [Ota90], is the fact that the angle sum of a non-degenerate geodesic triangle is strictly less than π [dC92, Lemma 12.3.1 ii)]. This still holds for CAT$(-\lambda^2)$ spaces, essentially by definition [BH13, Proposition II.1.7 4].

Moreover, we can define the marked length spectrum of (M, g_0) the same way as for negatively curved manifolds. The fact that there exists a geodesic representative for each homotopy class is a general application of the Arzelà-Ascoli theorem; see [BH13, Proposition I.3.16]. The proof that this geodesic representative is unique in the negatively curved case immediately generalizes to the CAT$(-\lambda^2)$ case; see [dC92, Lemma 12.3.3].

We will now show (M, g_0) and (M, h_0) have the same marked length spectrum. We start with a preliminary lemma.

Lemma VI.4. *Let $\langle\gamma\rangle$ be a free homotopy class. Let γ_0 and γ_n denote the geodesic representatives with respect to g_0 and g_n respectively. Write $\gamma_0(t) = \phi_0^t v_0$ and $\gamma_n(t) = \phi_n^t v_n$. Then for all $0 \leq t \leq l_{g_0}(\gamma_0)$, we have $\phi_n^t v_n \to \phi_0^t v_0$ in T^1M as $n \to \infty$.*

Proof. Let $T = l_{g_0}(\gamma_0)$. By Lemma VI.3, choose n large enough so that $d(\phi_n^t v_0, \phi_0^t v_0) < \varepsilon$ for all $t \in [0, T]$. In particular, $\phi_n^T v_0$ is close to $\phi_0^T v_0 = v_0$. The Anosov closing lemma applied to the geodesic flow on (T^1M, g_n) gives $\phi_n^t v_0$ is shadowed by a closed orbit. By construction, this closed orbit is close to γ_0 and is also homotopic to it, which completes the proof. □

Proposition VI.5. *The Riemannian surfaces (M, g_0) and (M, h_0) have the same marked length spectrum.*

Proof. The previous lemma, together with Lemma VI.1, implies $l_{g_n}(\gamma_n) \to l_{g_0}(\gamma_0)$ as $n \to \infty$. Let $\tilde{\gamma}_n$ be the geodesic representatives of $\langle\gamma\rangle$ with respect to the h_n metrics. Then we also have $l_{h_n}(\tilde{\gamma}_n) \to l_{h_0}(\tilde{\gamma}_0)$ as $n \to \infty$. Since $\mathcal{L}_{g_n}/\mathcal{L}_{h_n} \to 1$, we obtain $l_{g_0}(\gamma_0)/l_{h_0}(\tilde{\gamma}_0) = 1$, which completes the proof. □

VI.2: Liouville current

Now that we have two surfaces with the same marked length spectrum, we will follow the method of [Ota90] to show they are isometric. Two key tools used in Otal's proof are the Liouville current and the Liouville measure (both defined at the beginning of Section IV.2). In this section and the next, we will construct analogous measures for the limit (M, g_0) and show they still satisfy the properties required for Otal's proof.

Recall the Liouville current is a Γ-invariant measure on the space of geodesics of $\tilde{M}$; see Section IV.2. Recall as well the following relation between the cross-ratio and Liouville current for surfaces. Let $a, b, c, d \in \partial\tilde{M}$ be four distinct points. Since $\partial\tilde{M}$ is a circle, the pair of points (a, b) determines an interval in the boundary (after fixing an orientation). Let $(a, b) \times (c, d) \in \partial^2\tilde{M}$ denote the geodesics starting in the interval (a, b) and ending in the interval (c, d). Then

$$\lambda((a, b) \times (c, d)) = \frac{1}{2}[a, b, c, d]. \tag{VI.2.1}$$

(See (IV.2.2), also [Ota90, Proof of Theorem 2] and [HP97, Theorem 4.4].)

We can use the above equation to define the Liouville current λ_0 on (M, g_0). Let λ_n denote the Liouville current with respect to the smooth metric g_n. It is then clear from Lemma IV.3 that $\lambda_n(A) \to \lambda_0(A)$ for any Borel set $A \subset \partial^2\tilde{M}$.

We now recall a key property of the Liouville current used in Otal's proof. We begin by defining coordinates on the space of geodesics: Fix $v \in T^1M$ and $T > 0$, and let $t \mapsto \eta(t)$ be the geodesic segment of length T with $\eta'(0) = v$. Let $\mathcal{G}_v^T$ denote the (bi-infinite) geodesics which intersect the geodesic segment η transversally. Let $b : [0, T] \times (0, \pi) \to T^1M$ be the map defined by sending (t, θ) to the unit tangent vector with footpoint $\eta(t)$ obtained by rotating $\eta'(t)$ by angle θ. We can then identify each vector $b(t, \theta)$ with a unique geodesic in $\mathcal{G}_v^T$ (see [Ota90, p. 155]).

When g is a smooth Riemannian metric on M, the Liouville current with respect to the above coordinates is of the form $\frac{1}{2}\sin\theta\, d\theta\, dt$. The same proof works for the measure λ_0 defined in terms of the $C^{1,\alpha}$ Riemannian metric g_0. To see this, we begin by describing the space T_vT^1M. If $\xi \in T_vTM$, then ξ is tangent to a curve $\beta(t) \in TM$, which is in turn a vector field along a curve $b(t) \in M$. Let ∇ be the Levi-Civita connection for the metric g_0 and let $\kappa_v(\xi) := \nabla_{b'(t)}\beta(0)$ denote the connector map, which is of C^α regularity. Let $\pi_{TM} : TM \to M$ be the natural projection; then $d\pi(\xi) = b'(0)$. The map $T_vTM \to T_pM \oplus T_pM$ given by $\xi \mapsto (d\pi(\xi), \kappa_v(\xi))$ is an isomorphism [Bur83, 1.D].

Now for $v \in T^1M$ and $\xi_1, \xi_2 \in T_vT^1M$, define the C^α 2-form

$$\tau_v(\xi_1, \xi_2) = \langle d\pi_{TM}\xi_2, \kappa_v\xi_1\rangle - \langle d\pi_{TM}\xi_1, \kappa_v\xi_2\rangle.$$

In the case of a smooth Riemannian metric, the above formula is the coordinate expression for the symplectic form $d\omega$ defined at the beginning of Section IV.2 [Bur83, 1.D]. Since the g_{ij}^n and their derivatives converge to those of g_0, this means τ is the limit of the $d\omega^n$ for the metrics g_n. Since each $d\omega^n$ is invariant under the geodesic flow ϕ_n, Lemma VI.3 implies τ is invariant under the geodesic flow g_0. Therefore, we can think of τ as a C^α 2-form on the space of geodesics, which in turn gives rise to a measure.

Lemma VI.6. *Let $b : [0, T] \times (0, \pi) \to T^1M$ as above. Then $b^*\tau = \sin\theta \, d\theta \, dt$.*

Proof. Fix (t, θ) and let $u = b(t, \theta)$. Let $\beta_1(t)$ denote the coordinate curve $t \mapsto b(t, \theta)$. This gives a parallel vector field along η making fixed angle θ with η'. Thus if ξ_1 is the vector tangent to β_1 at u, we get $\kappa_v \xi_1 = 0$ and $d\pi\xi_1 = \eta'(t)$. This latter vector is obtained by rotating u by angle θ, which we will denote by $\theta \cdot u$.

Next, let $\beta_2(\theta)$ denote the coordinate curve $\theta \mapsto b(t, \theta)$. This is a curve in the fiber over $\eta(t)$, which means $d\pi(\xi_2) = 0$. This curve traces out a circle in the unit tangent space, and its tangent vector is thus perpendicular to the circle. This means $\kappa_v(\xi_2) = (\pi/2) \cdot u$.

Hence

$$\tau_{b(t,\theta)}(\xi_2, \xi_1) = \langle \pi/2 \cdot u, \theta \cdot u \rangle - \langle 0, 0 \rangle = \sin\theta,$$

as claimed. □

We now claim the measure on the space of geodesics coming from the symplectic form $\frac{1}{2}\tau$ is equal to the Liouville current. Indeed, this follows from [Ota90, Theorem 2]. To show this theorem is still true for (M, g_0), it suffices to verify the geodesic flow ϕ_0 satisfies the Anosov closing lemma (see the proof of Proposition IV.4).

Lemma VI.7. *The Anosov closing lemma holds for the g_0-geodesic flow, ie, given $\delta > 0$, there exist $T_0 > 0$, $\delta_0 > 0$ with the following property: for any v so that $d(\phi^t v, v) < \delta_0$ for $t \geq T_0$, there exists w tangent to a periodic orbit of length t_0 where $|t - t_0| < \delta$ and $d(\phi^s v, \phi^s w) < \delta$ for $s \in [0, \min(t, t_0)]$.*

Proof. We can choose T_0 and δ_0 so that the conclusion of Anosov closing lemma holds for all g_n with n sufficiently large. Indeed, this follows from the fact that the stable/unstable distributions of the g_n geodesic flows converge uniformly on compact sets to those of the g_0 geodesic flow as $n \to \infty$; see Lemma V.13 and [Fra18, p. 105].

Now take v and $t \geq T_0$ so that $d_{g_0}(\phi_0^t v, v) < \delta_0/2$. Choose n large enough such that $\phi_n^t v$ is within $\delta_0/2$ of $\phi_0^t v$. Applying the Anosov closing lemma to g_n gives w and t_0 with $|t - t_0| < \delta$, $\phi_n^{t_0} w = w$ and $d(\phi_n^s v, \phi_n^s w)$ for $s \in [0, \min(t, t_0)]$. By Lemma VI.4, this g_n-closed orbit is δ-close to a g_0-closed orbit, which completes the proof. □

Since (M, g_0) and (M, h_0) are CAT$(-\lambda^2)$ spaces, we can define a correspondence of geodesics $\phi : (\partial^2\tilde{M}, g_0) \to (\partial^2\tilde{M}, h_0)$ as in Construction IV.1. The following fact is still true in this context; see [Ota90, p. 156].

Proposition VI.8. *Let $\mathcal{G}_v \subset \partial^2\tilde{M}$ be a coordinate chart with coordinates (t, θ) and let $\phi(\mathcal{G}_v) = \mathcal{G}_{\phi(v)}$ have coordinates (t, θ'). Then ϕ takes the measure $\sin\theta \, d\theta \, dt$ to $\sin\theta' \, d\theta' \, dt'$.*

VI.3: Liouville measure

Let μ_n denote the Liouville measure on T^1M with respect to the metric g_n on M. Let g_n^S denote the associated Sasaki metric on T^1M. Then μ_n is a constant multiple of the measure arising from the Riemannian volume form of g_n^S. In local coordinates, the measure μ_n can be written in terms of the g_n^{ij} and their first derivatives. Since $g_n^{ij} \to g_0^{ij}$ in the $C^{1,\alpha}$ norm, we see the measures μ_n converge to a measure μ_0, which is the Riemannian volume associated to the C^α Sasaki metric g_0^S. Hence, the measure μ_0 can be written locally as the product $dm \times d\theta$, where dm is the Riemannian volume on M coming from g_0, and $d\theta$ is Lebesgue measure on the circle T_p^1M.

We now recall the average change in angle function $\Theta' : [0,\pi] \to [0,\pi]$ from [Ota90, Section 2]. First Otal considers the function $\theta' : T^1M \times [0,\pi] \to \mathbb{R}$ defined as follows. Given a unit tangent vector v and an angle θ, let $\theta \cdot v$ denote the vector obtained by rotating v by θ. Consider lifts of the geodesics determined by v and $\theta \cdot v$ passing through the same point in $\tilde{M}$. The correspondence of geodesics ϕ (see above Proposition VI.8 and Construction IV.1) takes intersecting geodesics to intersecting geodesics (since $\dim M = 2$). Let $\theta'(\theta, v)$ denote the angle between the image geodesics in $(\tilde{M}, h_0)$ at their point of intersection. Finally, let $\Theta'(\theta) = \int_{T^1M} \theta'(\theta, v) d\mu_0(v)$.

The function Θ' satisfies symmetry and subadditivity properties [Ota90, Proposition 6]. Indeed, the proof of [Ota90, Proposition 6] uses the above local product structure of the Liouville measure along with the fact that in negative curvature, the angle sum of a non-degenerate geodesic triangle is strictly less than π. As mentioned before, this latter fact holds for CAT($-\lambda^2$) spaces as well [BH13, Proposition II.1.7.4].

To deduce the third key property of Θ' (see [Ota90, Proposition 7] for the exact statement), we require the following fact about μ_0, which holds by [Sig72] in the original smooth case. Since ϕ_0 is a geodesic flow on a CAT(–1) space, it satisfies a sufficiently strong specification property such that the proof of [Sig72] works verbatim in this context; see [CLT20, Theorem 3.2, Lemma 4.5].

Proposition VI.9. *Let $f : T^1M \to \mathbb{R}$ be a continuous function. Let $\varepsilon > 0$. Then there is a closed geodesic γ_0 so that*

$$\left| \int_{T^1M} f \, d\mu_0 - \frac{1}{l_{g_0}(\gamma_0)} \int_{\gamma_0} f \, dt \right| < \varepsilon.$$

VI.4: Constructing a distance-preserving map $f : (M, g_0) \to (M, h_0)$

Using Propositions VI.8 and VI.9, the proof of [Ota90, Proposition 7] shows the hypotheses of [Ota90, Lemma 8] are satisfied. Thus, the function Θ' defined at the beginning of Section VI.3 is the identity. From this, it follows that ϕ takes triples of geodesics intersecting in a single point to triples of geodesics intersecting in a single point; see the proof of [Ota90, Theorem 1]. We then define $f : (M, g_0) \to (M, h_0)$ exactly as in [Ota90]: given $p \in \tilde{M}$, take any two geodesics through p. Then their images under ϕ must also intersect in a single point, which we call $\tilde{f}(p)$. Then $\tilde{f}$ is distance-preserving and Γ-equivariant by the same argument as in [Ota90].

This proves Proposition VI.2, and hence Theorem I.8 is proved.

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