

Filling Volume Minimality and Boundary Rigidity of Metrics Close to a Negatively Curved Symmetric Metric

by

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To my family, and to Lin Chen, who planted the first seeds of mathematics in my young mind.

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ABSTRACT

We investigate the relation between boundary data of a compact manifold and its interior geometry. A compact Riemannian manifold D with smooth boundary ∂D is boundary rigid if its interior geometry is uniquely determined by ∂D and distances between points on ∂D . D is a minimal filling if for any D' with $\partial D' = \partial D$, having larger distances between points on ∂D implies $\text{Vol}(D') \geq \text{Vol}(D)$.

In this thesis, we generalize D. Burago and S. Ivanov's work [BI13] on filling volume minimality and boundary rigidity of almost real hyperbolic metrics. We show that regions with metrics close to a negatively curved symmetric metric are strict minimal fillings and hence boundary rigid. This includes perturbations of real, complex, quaternionic and Cayley hyperbolic metrics.

CHAPTER I

Introduction

I.1: Boundary rigidity and filling minimality

Let $M = (M^n, g)$ be a compact Riemannian manifold with boundary ∂M . For example, when M is the planet Earth, ∂M is its surface. Its *boundary distance function*, denoted by bd_M , is the restriction of the Riemannian distance d_M to $\partial M \times \partial M$. In the above example, the distances between boundary points must be realized by “digging into the Earth”. For example, the distance minimizing curve between the north pole and the south pole will pass through the very center part of the Earth.

Both filling minimality and boundary rigidity focus on the following question:

How much do the distances between boundary points determine the interior geometry?

Apart from its intrinsic interest in geometry, this question has major real-life implications. We continue with the example of the Earth. Most knowledge of the interior of the Earth comes from the study of seismic waves. The travel speed of seismic waves at a particular location is related to several factors like composition of rocks. Understanding travel times of seismic waves can help us understand such properties of the Earth. Mathematically, we can use a Riemannian metric to model how fast seismic waves can travel at a particular location. The travel times will then become “distances” under this particular Riemannian metric. Hence, information regarding the speed of seismic waves is converted to geometric information. Therefore understanding the above question is very interesting from this perspective.

We say a Riemannian manifold is *boundary rigid* if its metric is uniquely determined by its boundary distance function. More precisely, we have the following definition.

Definition I.1 (Boundary rigidity). A compact Riemannian manifold M (with boundary) is *boundary rigid* if every Riemannian manifold M' with $\partial M' = \partial M$ and $\text{bd}_{M'} = \text{bd}_M$ is isometric to M via a boundary-preserving isometry.

In the previous example, if the Earth, equipped with travel speed data of seismic waves, is “boundary rigid”, then the travel times of seismic waves determine the travel speed of seismic

waves at any location in the interior of the Earth.

It is easy to construct manifolds which are *not* boundary rigid. For example, if there exists some proper open subset which does not intersect any shortest path connecting boundary points, then the metric on this open subset does not affect the boundary distance function and hence such manifolds are not boundary rigid. In particular, “large” spherical caps, i.e. proper open subsets of a sphere $S^n \subset \mathbb{R}^{n+1}$ bounded by a hyperplane in \mathbb{R}^{n+1} which properly contain a hemisphere are not boundary rigid. An example of large spherical cap is the region on the surface of the Earth which are south of the $N^\circ 40$ latitude line. In this case, the boundary is the $N^\circ 40$ latitude line. Here, the boundary distances are realized by distances along the surface. We observe that flying from Beijing to New York (both of which are almost located at the boundary) along the $N^\circ 40$ latitude line takes much less time than flying from Beijing to the south pole and then to New York (if we assume flying speed are the same). Such manifolds must be avoided if one seeks boundary rigidity. Therefore it is reasonable to first consider simple manifolds, i.e. manifolds with strictly convex boundary such that every two points are connected by a unique geodesic segment and geodesics do not have conjugate points. In particular, we have the following conjecture by Michel.

Conjecture I.2 (Michel, [Mich81]). *All simple manifolds are boundary rigid.*

Here and below, by *region* we mean a connected open set with a smooth boundary.

A lot of progress has been made toward boundary rigidity. Pestov and Uhlmann [PU05] proved the above conjecture in dimension 2. In higher dimensions, regions in \mathbb{R}^n (Besikovich [Besi52]; Gromov [Grom83]), in the open hemisphere S_+^n (Michel [Mich81]) and in rank-1 symmetric spaces of non-compact type (following the volume entropy rigidity theorem by Besson, Courtois and Gallot [BCG95]) are known to be boundary rigid. Burago and Ivanov proved boundary rigidity for almost Euclidean ([BI10]) and almost real hyperbolic ([BI13]) regions. Recently a very general result by Stefanov, Uhlmann and Vasy in [SUV21, Corollary 1.2] showed that a simple manifold (M, g) is boundary rigid if it satisfies any of the following conditions:

- (1). (M, g) has non-positive sectional curvature;
- (2). (M, g) has non-negative sectional curvature;
- (3). (M, g) has no focal points.

We refer the readers to Croke [Crok04], Ivanov [Ivan10] and Stefanov-Uhlmann [SU08] for a survey on boundary rigidity.

Definition I.3 (Filling minimality). A compact Riemannian manifold M with boundary ∂M is a *minimal filling* if, for every compact Riemannian manifold M' with $\partial M' = \partial M$, the inequality

$$d_{M'}(x, y) \geq d_M(x, y) \quad \forall x, y \in \partial M$$

implies that

$$\text{Vol}(M') \geq \text{Vol}(M).$$

We say that M is a *strict minimal filling* if, in addition, the equality

$$\text{Vol}(M') = \text{Vol}(M)$$

holds only when M and M' are isometric via an isometry which fixes all boundary points.

Remark. The idea of filling Riemannian manifolds was introduced by Gromov in [Grom83]. If $M = (M, g)$ has a connected boundary with dimension ≥ 2 and is a minimal filling, then by [Grom83, 2.2A Proposition] $\text{Vol}(M)$ is called the *filling volume* of the boundary $(\partial M, \text{bd}_M)$ equipped with boundary distance function denoted by $\text{FillVol}(\partial M, \text{bd}_{(M,g)})$. See [Grom83] for a detailed discussion.

Similar to boundary rigidity, not all manifolds are minimal fillings. For example, “large spherical caps” fail to be minimal fillings because they have larger volume than the hemisphere with the same boundary and boundary distance function. In [BI10] and [BI13], Burago and Ivanov made the following conjecture.

Conjecture I.4 (Burago-Ivanov, [BI10, BI13]). *Every simple manifold is a strict minimal filling.*

When M is simple, its volume is uniquely determined by its boundary distance function due to Santaló’s formula [Sant04]. Moreover, if M' shares the same boundary distance function with M , M' also has to be simple due to strict triangular inequality and smoothness of bd_M . Therefore boundary rigidity is a direct corollary of filling minimality when the manifold is simple. A similar argument also works for *strong geodesically minimizing* (SGM) manifolds (See [Crok91, 1. Preliminaries] for the definition of the SGM condition and [Crok91, Lemma 5.1] for details of this argument.) which allows non-convex boundaries. In particular, compact regions with a smooth boundary inside a simply connected negatively curved manifold satisfies the SGM condition.

Unlike boundary rigidity, little is understood about filling minimality. For example, Gromov’s sphere filling conjecture ([Grom83, page 13]) asks whether a hemisphere of dimension $n + 1$ is a minimal filling for its boundary. This is still open with partial results proved by Gromov [Grom83, page 59] and Bangert-Croke-Ivanov-Katz [BCIK05, Corollary 1.8]. Croke, Dairbekov and Sharafutdinov proved “local filling minimality” in [CDS00, Proposition 1.2] for simple manifolds with “limited positive curvature along geodesics”. Local filling minimality here refers to the case when M' and M (See Definition I.3) have the same underlying manifold and very close metrics. Burago and Ivanov in [BI10] and [BI13] proved strict filling minimality for almost Euclidean and almost hyperbolic regions (when M is almost Euclidean and almost hyperbolic and M' is arbitrary).

The filling minimality problem can be more difficult than the boundary rigidity problem. This is because having the same boundary distance function provides more information than having a larger boundary distance function due to [Crok91, Lemma 5.1]. In the case when M is a simple manifold, we denote by $\partial_- T^1 M$ the collection of unit vectors on ∂M pointing inside M and $\partial_+ T^1 M$ the collection of unit vectors on ∂M pointing outside M . Since any maximally extended geodesic in a simple manifold M intersects the boundary transversely and has finite length, there is a one-one correspondence between maximally extended geodesics in (M, g) and the corresponding triples $(v, w, l) \in \partial_- T^1 M \times \partial_+ T^1 M \times \mathbb{R}_+$ recording their initial vectors, exit vectors and lengths. Recall that geodesics are length minimizing and do not admit conjugate points when M is simple, we have the following one-one correspondence.

$$\{(p, q) | p \neq q \in \partial M\} \leftrightarrow \text{Lens}(M),$$

where

$$\text{Lens}(M) := \left\{ \begin{array}{l} (\pi_-(\dot{\gamma}(0)), \pi_+(\dot{\gamma}(l)), l) \\ \in (T\partial M)^2 \times \mathbb{R}_+ \end{array} \left| \begin{array}{l} \gamma : [0, l] \rightarrow M \text{ a maximally} \\ \text{extended unit speed geodesic;} \\ \pi_{\pm} : \partial_{\pm} T^1 M \rightarrow T\partial M \\ \text{orthogonal projection.} \end{array} \right. \right\}.$$

We call $\text{Lens}(M)$ the *lens data* of M . Similar to the boundary rigidity problem, we have the *lens rigidity* problem which asks whether lens data can determine the manifold up to an isometry. If M' and a simple manifold M have the same boundary and the same boundary distance function, [Crok91, Lemma 5.1] implies that $\text{Lens}(M)$ and $\text{Lens}(M')$ are canonically identified. Therefore proving boundary rigidity in this case is the same as proving lens rigidity. This observation has been used in many results on boundary rigidity (for example the aforementioned result in [SUV21] by Stefanov-Uhlmann-Vasy.). Unfortunately, having a larger boundary distance function can mess up the lens data, especially the part of data recording initial and exit vectors. (In fact, if we neglect the length part from the lens data, we can still study the corresponding *scattering rigidity* problem. See for example [BGJ22] by Bonthonneau, Guillarmou and Jézéquel.) Hence methods using lens data cannot directly apply in a similar way when working on the filling minimality problem.

I.2: Statements of main results

In this paper, we will generalize Burago and Ivanov's work in [BI10] and [BI13] to metrics close to a negatively curved symmetric metric. Notice that negatively curved symmetric metrics come from rank-1 symmetric spaces of non-compact type, we only need to consider metric perturbations

of regions in real, complex, quaternionic and Cayley hyperbolic spaces. Let \mathbb{K} be one of the following:

- (i). \mathbb{R} , the field of real numbers;
- (ii). \mathbb{C} , the field of complex numbers;
- (iii). \mathbb{H} , the algebra of all quaternions;
- (iv). \mathbb{O} , the algebra of all octonions.

We denote by $\mathbb{R}\mathbb{H}^n, \mathbb{C}\mathbb{H}^n, \mathbb{H}\mathbb{H}^n, \mathbb{O}\mathbb{H}^2$ the real, complex, quaternionic and Cayley hyperbolic spaces respectively.

In this paper, we prove the following main theorem.

Theorem I.5 ([Ruan22, Theorem 1.5], [BI13, Theorem 1.6]). *Let M be a negatively curved symmetric space, i.e. $M = \mathbb{R}\mathbb{H}^n, \mathbb{C}\mathbb{H}^n, \mathbb{H}\mathbb{H}^n$ or $\mathbb{O}\mathbb{H}^2$. For any compact region $D \subset M$ (not necessarily simple) with a smooth boundary, there is a C^r -neighborhood (for a suitable r) of the symmetric metric on D such that, for every metric g from this neighborhood, the Riemannian manifold (D, g) is a strict minimal filling.*

Remark.

- (1). The real hyperbolic case was proved in [BI13, Theorem 1.6]. Other cases were proved in [Ruan22, Theorem 1.5];
- (2). The condition on g being sufficiently close to the symmetric metric appears in several statements throughout this paper. This is given by several complicated constraints on g related to the diameter of the region (with respect to the symmetric metric) and the dimension of the manifold. In particular we assume that g is negatively curved so that (M, g) satisfies the SGM condition. We do not track the number of derivatives required for our arguments to work but we will summarize all constraints on g at the end of this paper.

Since the SGM condition and strict filling minimality imply boundary rigidity as explained right after Conjecture I.4, we have the following direct corollary of Theorem I.5.

Corollary I.6. *Under the same assumption as in Theorem I.5, (D, g) is boundary rigid.*

I.3: Plan of the proof

While we employ the same general constructions as in [BI13], various proofs become much more complicated for general rank-1 symmetric spaces other than the special case of real hyperbolic

spaces. Therefore we need to improve some technical facts to prove filling minimality and boundary rigidity of almost rank-1 symmetric metrics.

In Chapter II, we introduce the classical models and some basic geometric properties for rank-1 symmetric spaces. The emphasis is on Section II.2, the introduction of the Cayley hyperbolic space. This is because there are more technical difficulties in the rest of this paper regarding the Cayley hyperbolic space, due to the non-associativity of octonionic multiplication.

Chapter III starts with a “projection argument” (Section III.1) and its application in proving filling minimality of regions in a Euclidean space (Section III.2). In Sections III.3, III.4, we want to use this “projection argument” to prove Theorem I.5 while leaving the technical constructions and verifications of necessary properties to Chapter IV. These two sections are essentially the same as [BI13, Sections 2 and 3]: Let (D, g) be as in Theorem I.5. We can then extend g smoothly to a metric on $\mathbb{K}\mathbf{H}^n$ which coincides with the symmetric metric on $\mathbb{K}\mathbf{H}^n$ (also denoted by g for simplicity) outside a compact neighborhood of D .

- (1). In Section III.3, we give the proof for Theorem I.5 by assuming Proposition III.4, which is a key technical result. This proposition claims the existence of the following tools for us to prove Theorem I.5 using the “projection argument” in Section III.1: two maps and a notion of Riemannian structure on an open subset of some L^∞ space. The first map is a distance-preserving map $\Phi : M \rightarrow \mathcal{L} := L^\infty(S)$, where $M = (\mathbb{K}\mathbf{H}^n, g)$ and S is a suitable measure space. (Following the traditions of geometric measure theory, we refer to Lipschitz maps from manifolds of any dimension to a normed space as *surfaces*). Later in Section IV.1, we choose S to be the visual boundary of M with a visual measure and Φ to be the Busemann function with respect to a fixed point in the interior as in [BI13]. The other map is a “projection” map P_σ from a suitable neighborhood of $\Phi(M) \subset \mathcal{L}$ to M in the sense that $P_\sigma \circ \Phi = \text{Id}_M$. We also assume that P_σ precomposing any 1-Lipschitz map f from a dn -dimensional manifold does not increase volumes. Moreover, we require that $P_\sigma \circ f$ preserves volumes if and only if the image of $P_\sigma \circ f$ is contained in $\Phi(M)$. If (D', g') shares the same boundary with (D, g) with larger boundary distance function, then we can extend $\Phi|_{\partial D}$ to a 1-Lipschitz map $\Phi' : (D', g') \rightarrow \mathcal{L}$. Therefore strict filling minimality of (D, g) follows from the above properties of P_σ .
- (2). In Section III.4, we introduce the aforementioned “Riemannian structure” on a suitable open neighborhood \mathcal{U} of $\Phi(M) \subset \mathcal{L}$ so that we can define dn -dimensional ($d = 1, 2, 4, 8$ when $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ respectively) Riemannian volumes on \mathcal{U} and Jacobians for maps involving \mathcal{U} . Under this “Riemannian structure”, Φ is volume preserving and any 1-Lipschitz map from a dn -dimensional manifold to \mathcal{U} is volume non-increasing. This construction is given by introducing a Riemannian metric on \mathcal{U} satisfying some natural conditions. With the help

of this “Riemannian structure”, it remains for us to find a P_σ whose dn -dimensional Jacobian is smaller than 1 on $\mathcal{U} \setminus \Phi(M)$.

In Chapter IV, we construct the aforementioned “embedding” Φ , “projection” P_σ and “Riemannian structure” on an open neighborhood \mathcal{U} of $\Phi(M) \subset \mathcal{L}$. Then it remains for us to verify that our constructions indeed satisfy the requirements in Proposition III.4. Most of the constructions can be found in Section IV.1 and most of the verification details are in IV.2-IV.4.

- (1). Section IV.1 introduces the construction of Φ , the “Riemannian structure” on \mathcal{U} and a “locally orthogonal projection” $P : \mathcal{U} \rightarrow M$ defined via a barycenter construction. The “projection” map P_σ will eventually be a small perturbation of this P and therefore we expect P to be almost volume non-increasing. For any ϕ in the domain of P , a direct computation decomposes the derivative map $d_\phi P$ into two different linear operators $A_\phi^{-1} : T_{P(\phi)}M \rightarrow T_{P(\phi)}M$ and $E_\phi : T_\phi \mathcal{L} \rightarrow T_{P(\phi)}M$. Then it remains for us to study the Jacobians of the above two maps.

In the cases of real hyperbolic spaces discussed in [BI13], the operator $A_\phi = \text{Id}$. This follows from the formula for the Hessian of the Busemann functions and sectional curvatures being constantly -1 (up to scaling). The main technical difficulty in our paper comes from the fact that other non-compact rank-1 symmetric spaces have sectional curvatures ranging from -4 to -1 (see [BH99]). This leaves A_ϕ much more complicated even in the model case when g is symmetric. We resolve this difficulty from the observation that A_ϕ and E_ϕ have closely related matrix expressions under a suitable choice of basis. The main difference of our proof compared to [BI13] comes from the remaining technical part of the paper.

- (2). In Section IV.2, we introduce some notion of “almost rank-1 structure” (depending smoothly on g) to construct suitable bases convenient for further computations. Then we construct an operator \widehat{A}_ϕ to approximate A_ϕ by using the data from the Hessian of the Busemann functions in symmetric spaces under these bases. (The construction of P mentioned in Section IV.1 is not unique. In the case of real hyperbolic spaces, a suitable choice of P will lead to $\widehat{A}_\phi = \text{Id}$. See [BI13] for more details.) Moreover, computation shows that \widehat{A}_ϕ and E_ϕ have matrix representations closely related to a positive definite matrix Q_ϕ (see (IV.2.7)). Hence the study of eigenvalues of Q_ϕ leads to the desired Jacobian and norm estimates. In particular, we proved that the dn -dimensional Jacobian of $\widehat{A}_\phi^{-1} \circ E_\phi$ is bounded above by 1. This implies that the dn -dimensional Jacobian of P is bounded above by 1 plus some error term. In other words, P is almost volume non-increasing.
- (3). Section IV.3 is the most technical part of this paper. The main goal of this section is to provide a detailed estimate on the error terms $A_\phi - \widehat{A}_\phi$ and $1 - \text{Jac}(\widehat{A}_\phi^{-1} \circ E_\phi)$ so that

we can construct P_σ as a perturbation of P which decreases dn -dimensional volumes. A similar version of the statements and proofs introduced in this section also applies to the real hyperbolic cases.

- (4). Section [IV.4](#) is similar to [[BI13](#), Section 7, a compression trick] which constructs the aforementioned P_σ and verifies the required properties introduced in Section [III.3](#) (see Proposition [III.4](#)).

CHAPTER II

Rank-1 Symmetric Spaces of Non-compact Type

In this chapter, we review some basic facts about rank-1 symmetric spaces of non-compact type. As is mentioned in Chapter I, real hyperbolic spaces $\mathbb{R}\mathbb{H}^n$, complex hyperbolic spaces $\mathbb{C}\mathbb{H}^n$, quaternionic hyperbolic spaces $\mathbb{H}\mathbb{H}^n$ and the Cayley hyperbolic space $\mathbb{O}\mathbb{H}^2$ are all possible rank-1 symmetric spaces. See [Helg78, Table II, page 354].

II.1: Real, complex and quaternionic hyperbolic spaces

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or \mathbb{H} , where $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ is the associative, non-commutative \mathbb{R} -algebra of quaternions with the standard multiplication laws $i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j$. For any element $a \in \mathbb{K}$, we denote by \bar{a} its conjugate. When $\mathbb{K} = \mathbb{H}$ and $a = x + yi + zj + wk$ for some $x, y, z, w \in \mathbb{R}$, similar to the complex conjugate, we have $\bar{a} = 2x - a$ and $\text{Re}(a) := x = (a + \bar{a})/2$. In particular, $|a| := \sqrt{a\bar{a}} = \sqrt{\bar{a}a}$ gives the Euclidean norm of a .

Denoted by $M_0 = (\mathbb{K}\mathbb{H}^n, g_0)$ the corresponding symmetric space with a symmetric metric. We recall following the model for M_0 in [BH99, Ch II.10]. Let $\mathbb{K}\mathbb{P}^n := (\mathbb{K}^{n+1} \setminus \{0\})/\{v \sim v\lambda, \lambda \in \mathbb{K} \setminus \{0\}\}$ and $[v] \in \mathbb{K}\mathbb{P}^n$ be the equivalence class of v . Denoted by q a quadratic form on \mathbb{K}^{n+1} defined as

$$q(v, w) = \bar{v}_0 w_0 - \sum_{j=1}^n \bar{v}_j w_j$$

for any $v = (v_0, \dots, v_n)$ and $w = (w_0, \dots, w_n)$ in \mathbb{K}^{n+1} . One can verify that q satisfies the following properties.

- (1). $q(v, v) \in \mathbb{R}$ for any $v \in \mathbb{K}^{n+1}$;
- (2). $q(v, w_1\lambda_1 + w_2\lambda_2) = q(v, w_1)\lambda_1 + q(v, w_2)\lambda_2$ for any $v, w_1, w_2 \in \mathbb{K}^{n+1}$ and $\lambda_1, \lambda_2 \in \mathbb{K}$;
- (3). $q(v_1\lambda_1 + v_2\lambda_2, w) = \bar{\lambda}_1 q(v_1, w) + \bar{\lambda}_2 q(v_2, w)$ for any $w, v_1, v_2 \in \mathbb{K}^{n+1}$ and $\lambda_1, \lambda_2 \in \mathbb{K}$.

Let

$$\mathbb{K}\mathbb{H}^n = \{[v] \in \mathbb{K}\mathbb{P}^n \mid v \in \mathbb{K}^{n+1} \setminus \{0\}, q(v, v) > 0\}.$$

For any $u \in \mathbb{K}^{n+1}$ with $q(u, u) = 1$, the tangent space of \mathbb{KH}^n at $[u]$ can be identified as $\{v \in \mathbb{K}^{n+1} | q(u, v) = 0\}$. The symmetric metric is defined as $g_0(v, w) = -\text{Re}(q(v, w))$. In particular, angles with respect to g_0 is given by

$$\cos(\angle_{g_0}(v, w)) = -\frac{\text{Re}(q(v, w))}{\sqrt{q(v, v)q(w, w)}}, \quad \forall v, w \in T_{[u]}M_0 \setminus \{0\}.$$

For any $[u] \in M_0$ with $q(u, u) = 1$ and any linearly independent $v, w \in T_{[u]}M_0$, we have the following classical result on curvatures in M_0 .

Proposition II.1 ([BH99, 10.12 Proposition]). *Under previous assumptions, for any $p \in M_0$ and any pair of non-zero vectors $v, w \in T_pM_0$ such that v is not parallel to w , the following holds.*

- (1). *If $v = w\lambda$ for some $\lambda \in \mathbb{K} \setminus \mathbb{R}$, then the sectional curvature $K_{M_0}(v, w) = -4$, which is the smallest possible sectional curvature;*
- (2). *If $q(v, w) \in \mathbb{R}$, then the sectional curvature $K_{M_0}(v, w) = -1$, which is the largest possible sectional curvature.*

As a consequence, we have the following linear algebra result.

Corollary II.2. *Let $M_0 = (\mathbb{KH}^n, g_0)$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or \mathbb{H} and g_0 is the symmetric metric mentioned above. Denoted by $d = \dim_{\mathbb{R}} \mathbb{K}$. Then there exists unit vector fields $\xi_{i,t}$ on M_0 and fiberwise orthogonal linear maps $\mathcal{J}_t : TM_0 \rightarrow TM_0$, $1 \leq i \leq n$ and $0 \leq t \leq d - 1$ such that the following holds*

- (1). $\mathcal{J}_0 = \text{Id}$;
- (2). $\{\xi_{i,t}\}_{1 \leq i \leq n, 0 \leq t \leq d-1}$ form an orthonormal basis at every point in M_0 ;
- (3). $\mathcal{J}_t(\xi_{l,0}) = \xi_{l,t}$ for any $1 \leq l \leq n$ and $0 \leq t \leq d - 1$.
- (4). For any non-zero vector $v \in TM_0$, the sectional curvature of between v and $\mathcal{J}_t(v)$ is -4 ;
- (5). For any $1 \leq l_1 \neq l_2 \leq n$ and $0 \leq t_1, t_2 \leq d - 1$, the sectional curvature of between ξ_{l_1, t_1} and ξ_{l_2, t_2} is -1 .

Proof. For any $[u] \in \mathbb{KH}^n$ with $u = (u_0, \dots, u_n)$, we have $u_0 > 0$. We define smooth vector fields η_l on M_0 by $\eta_l([u]) = (-u_l/u_0, 0, \dots, 0, 1, 0, \dots, 0)$ for any $1 \leq l \leq n$, where 1 is in the l -th component. Denoted by $|\cdot|$ the norm on TM_0 induced by the metric g_0 . Let $\mathcal{J}_1(v) = vi$ when

$\mathbb{K} = \mathbb{C}, \mathbb{H}$, $\mathcal{J}_2(v) = vj$ and $\mathcal{J}_3(v) = vk$ when $\mathbb{K} = \mathbb{H}$. The fourth property easily follows from these definitions. We inductively construct $\xi_{l,t}$ via a Gram-Schmidt process as follows:

$$\begin{aligned} \xi_{1,0} &= \eta_1 / |\eta_1| = \eta_1 / \sqrt{-q(\eta_1, \eta_1)}; & \xi_{1,t} &= \mathcal{J}_t(\xi_1); \\ \xi_{l,0} &= \frac{\eta_l + \sum_{m=1}^{l-1} \xi_{m,0} \overline{q(\eta_l, \xi_{m,0})}}{\left| \eta_l + \sum_{m=1}^{l-1} \xi_{m,0} \overline{q(\eta_l, \xi_{m,0})} \right|}; & \xi_{l,t} &= \mathcal{J}_t(\xi_l), \quad 2 \leq l \leq n. \end{aligned}$$

$\xi_{l,0}$ are well-defined because the l -th component of $\eta_l + \sum_{m=1}^{l-1} \xi_{m,0} \overline{q(\eta_l, \xi_{m,0})}$ is non-zero. Then $q(\xi_{l,0}, \xi_{l,0}) = -1$ for any $1 \leq l \leq n$. We prove inductively on l that $q(\xi_{l,0}, \xi_{l_0,0}) = 0$ for any $1 \leq l_0 < l \leq n$. When $l = 1$ the above holds automatically. Suppose the above holds true for $l \leq l_1 - 1$, then when $l = l_1$,

$$\begin{aligned} q(\xi_{l,0}, \xi_{l_0,0}) &= \frac{q(\eta_l, \xi_{l_0,0}) + \sum_{m=1}^{l-1} q(\xi_{m,0} \overline{q(\eta_l, \xi_{m,0})}, \xi_{l_0,0})}{\left| \eta_l + \sum_{m=1}^{l-1} \xi_{m,0} \overline{q(\eta_l, \xi_{m,0})} \right|} \\ &= \frac{q(\eta_l, \xi_{l_0,0}) + \sum_{m=1}^{l-1} q(\eta_l, \xi_{m,0}) q(\xi_{m,0}, \xi_{l_0,0})}{\left| \eta_l + \sum_{m=1}^{l-1} \xi_{m,0} \overline{q(\eta_l, \xi_{m,0})} \right|} \\ &= \frac{q(\eta_l, \xi_{l_0,0}) + q(\eta_l, \xi_{l_0,0}) q(\xi_{l_0,0}, \xi_{l_0,0})}{\left| \eta_l + \sum_{m=1}^{l-1} \xi_{m,0} \overline{q(\eta_l, \xi_{m,0})} \right|} = \frac{q(\eta_l, \xi_{l_0,0}) - q(\eta_l, \xi_{l_0,0})}{\left| \eta_l + \sum_{m=1}^{l-1} \xi_{m,0} \overline{q(\eta_l, \xi_{m,0})} \right|} = 0. \end{aligned}$$

Hence $q(\xi_{l,0}, \xi_{l_0,0}) = 0$ for any $1 \leq l_0 < l \leq n$. Since the maps $\mathcal{J}_t(\cdot)$ are \mathbb{K} -scalar multiplications on the right, by the second and the third properties of $q(\cdot, \cdot)$, $q(\xi_{l,t}, \xi_{l_0,t_0}) = 0$ for any $1 \leq l_0 < l \leq n$ and any $0 \leq t, t_0 \leq d - 1$. This verifies the second, the third and the fifth properties. \square

II.2: The Cayley hyperbolic space

The set of octonions \mathbb{O} is an 8-dimensional non-associative, non-commutative division algebra over \mathbb{R} . Let $\langle \cdot, \cdot \rangle$ be the Euclidean inner product on \mathbb{O} and $|\cdot|$ the induced norm. Then we have the following properties. (See [SV00, 1. Composition algebra])

- (1). $|ab| = |a||b|$ for any $a, b \in \mathbb{O}$;
- (2). $\langle ab, ac \rangle = \langle ba, ca \rangle = |a|^2 \langle b, c \rangle$ for any $a, b, c \in \mathbb{O}$;
- (3). $\langle ac, bd \rangle + \langle ad, bc \rangle = 2\langle a, b \rangle \langle c, d \rangle$ for any $a, b, c, d \in \mathbb{O}$;
- (4). $\bar{a} = 2\langle a, 1 \rangle - a$ for any $a \in \mathbb{O}$;
- (5). $2\langle a, b \rangle = 2\langle \bar{a}, \bar{b} \rangle = \bar{a}b + \bar{b}a = a\bar{b} + b\bar{a}$ for any $a, b \in \mathbb{O}$;

- (6). $(ba)\bar{a} = \bar{a}(ab) = |a|^2b$ for any $a, b \in \mathbb{O}$;
- (7). $a(\bar{b}c) + b(\bar{a}c) = (c\bar{a})b + (c\bar{b})a = \langle a, b \rangle c$ for any $a, b, c \in \mathbb{O}$;
- (8). (Moufang Identities)
- (i). $(ab)(ca) = a((bc)a)$ for any $a, b, c \in \mathbb{O}$;
 - (ii). $a(b(ac)) = (a(ba))c$ for any $a, b, c \in \mathbb{O}$;
 - (iii). $b(a(ca)) = ((ba)c)a$ for any $a, b, c \in \mathbb{O}$;
- (9). Multiplications involving only two octonions are associative.

Let

$$I_{1,2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and

$$\mathfrak{J}(1, 2, \mathbb{O}) = \{\mathfrak{X} \in \text{Mat}_{3 \times 3}(\mathbb{O}) : I_{1,2}\mathfrak{X}^*I_{1,2} = \mathfrak{X}\}.$$

Any element \mathfrak{X} can be written in the following form

$$\mathfrak{X}(\theta, a) = \begin{pmatrix} \theta_1 & a_3 & \bar{a}_2 \\ -\bar{a}_3 & -\theta_2 & -a_1 \\ -a_2 & -\bar{a}_1 & -\theta_3 \end{pmatrix}, \quad \theta_j \in \mathbb{R}, a_j \in \mathbb{O}, j = 1, 2, 3,$$

where $\theta = (\theta_1, \theta_2, \theta_3)$ and $x = (x_1, x_2, x_3)$.

Definition II.3 (“Matrix model”). We define the Cayley hyperbolic space $\mathbb{O}\mathbf{H}^2$ as

$$\mathbb{O}\mathbf{H}^2 = \{\mathfrak{X} \in \mathfrak{J}(1, 2, \mathbb{O}) : \mathfrak{X}^2 = \mathfrak{X}, \text{tr}(\mathfrak{X}) = 1, \mathfrak{X}_{11} > 0\}.$$

Proposition II.4. *For any trace 1 idempotent $\mathfrak{X} \in \mathfrak{J}(1, 2, \mathbb{O})$ with $\mathfrak{X}_{11} \neq 0$, there exists a unique vector $(\theta, a, b) \in \mathbb{R}_+ \times \mathbb{O}^2$ such that*

$$\mathfrak{X} = \text{sgn}(\mathfrak{X}_{11})I_{1,2}(\theta, b, c)^*(\theta, b, c),$$

where $\text{sgn}(t) = t/|t|$ when $t \neq 0$. The set

$$\mathfrak{J}_{1,0} := \{\mathfrak{X} \in \mathfrak{J}(1, 2, \mathbb{O}) : \mathfrak{X}^2 = \mathfrak{X}, \text{tr}(\mathfrak{X}) = 1, \mathfrak{X}_{11} = 0\}$$

is isomorphic to $\mathbb{O}\mathbf{P}^1 \cong S^8$.

Proof. Write

$$\mathfrak{X} = \begin{pmatrix} \theta_1 & a_3 & \bar{a}_2 \\ -\bar{a}_3 & -\theta_2 & -a_1 \\ -a_2 & -\bar{a}_1 & -\theta_3 \end{pmatrix}.$$

It suffices to show that

$$\begin{pmatrix} \theta_1 & a_3 & \bar{a}_2 \\ \bar{a}_3 & \theta_2 & a_1 \\ a_2 & \bar{a}_1 & \theta_3 \end{pmatrix} = \text{sgn}(\theta_1)(\theta, b, c)^*(\theta, b, c) \quad (\text{II.2.1})$$

for some $(\theta, b, c) \in \mathbb{R} \times \mathbb{O}^2$. Notice that

$$\mathfrak{X}^2 = \begin{pmatrix} \theta_1^2 - |a_3|^2 - |a_2|^2 & \theta_1 a_3 - \theta_2 a_3 - \bar{a}_1 \bar{a}_2 & \theta_1 \bar{a}_2 - a_3 a_1 - \theta_3 \bar{a}_2 \\ -\theta_1 \bar{a}_3 + \theta_2 \bar{a}_3 + a_1 a_2 & \theta_2^2 - |a_3|^2 + |a_1|^2 & -\bar{a}_2 \bar{a}_3 + \theta_2 a_1 + \theta_3 a_1 \\ -\theta_1 a_2 + \bar{a}_3 \bar{a}_1 + \theta_3 a_2 & -a_2 a_3 + \theta_2 \bar{a}_1 + \theta_3 \bar{a}_1 & \theta_3^2 - |a_2|^2 + |a_1|^2 \end{pmatrix}.$$

The condition that $\mathfrak{X}^2 = \mathfrak{X}$ implies that

$$a_j a_{j+1} = m_{j+2} \bar{a}_{j+2}, \quad m_j \in \mathbb{R}, j \in \mathbb{Z} \bmod 3.$$

Case 1: If $a_3 = a_2 = 0$, then $\theta_1^2 = \theta_1$ implies that $\theta_1 = 1$ or 0 . When $\theta_1 = 1$, $\text{tr} \mathfrak{X} = \theta_1^2 + \theta_2^2 + \theta_3^2 + 2|a_1|^2 = 1$ implies $\theta_2 = \theta_3 = a_1 = 0$. Hence $\mathfrak{X} = I_{1,2}(1, 0, 0)^*(1, 0, 0)$. When $\theta_1 = 0$, we have a natural diffeomorphism

$$\mathfrak{J}_{1,0} \rightarrow \{\mathfrak{X} \in \text{Mat}_{2 \times 2}(\mathbb{O}) : \mathfrak{X}^2 = \mathfrak{X}, \text{tr} \mathfrak{X} = 1, \mathfrak{X}^* = \mathfrak{X}\}$$

by forgetting the first row and the first column, where the latter one is $\mathbb{O}\mathbb{P}^1 \cong S^8$ by [Baez02, 3. Octonionic projective geometry].

Case 2: If $a_3 = 0$ and $a_2 \neq 0$ (the case when $a_2 = 0$ and $a_3 \neq 0$ is similar), then $\mathfrak{X}^2 = \mathfrak{X}$ implies the following.

- (1). $\theta_1^2 - |a_2|^2 = \theta_1 \neq 0$;
- (2). $-\theta_1 + \theta_3 = -1$;
- (3). $\theta_2^2 + |a_1|^2 = -\theta_2$;
- (4). $\theta_3^2 - |a_2|^2 + |a_1|^2 = -\theta_3$.

Notice that $1 = \theta_1 - \theta_3$, $\text{tr} \mathfrak{X} = 1$ implies that $\theta_2 = a_1 = 0$ and that $|a_2|^2 = \theta_1 \theta_3$. Hence we can choose $\theta = \sqrt{|\theta_1|}$, $b = 0$ and $c = \text{sgn}(\theta_1) \bar{a}_2 / \theta$ and (II.2.1) holds.

Case 3: If a_2, a_3 both not equal to 0, then

$$|a_j|^2 = m_{j+1}m_{j+2}, \quad j = 1, 2, 3 \text{ mod } 3.$$

Therefore $\mathfrak{X}^2 = \mathfrak{X}$ and $\text{tr}\mathfrak{X} = 1$ imply the following.

- (1). $\theta_1 - \theta_2 - m_3 = 1$;
- (2). $\theta_1 - \theta_3 - m_2 = 1$;
- (3). $m_1 - \theta_2 - \theta_3 = 1$;
- (4). $\theta_1 - \theta_2 - \theta_3 = 1$.

Hence $m_j = \theta_j$, where $j = 1, 2, 3$. Choose $\theta = \sqrt{|\theta_1|}$, $b = \text{sgn}(\theta_1)a_3/\theta$ and $c = \text{sgn}(\theta_1)\bar{a}_2/\theta$ and (II.2.1) holds. Uniqueness of the vector is trivial. \square

Remark. It is easy to see that the condition “ $\text{tr}\mathfrak{X} = 1$ ” and the condition “all 2×2 subdeterminants of \mathfrak{X} vanish” introduced in [Most73, §19. Spaces of R -rank-1, page 137] are equivalent in this setting.

Therefore we have the following alternative definition for the Cayley hyperbolic space (also see [Park08, page 87]).

Definition II.5 (“Vector model”). The Cayley hyperbolic space can be alternatively defined as

$$\mathbb{O}\mathbb{H}^2 = \{(\theta, a, b) \in \mathbb{R}_+ \times \mathbb{O}^2 : \theta^2 - |a|^2 - |b|^2 = 1\}.$$

Remark. It follows from simple computations that for any $\mathfrak{X} \in \mathbb{O}\mathbb{H}^2$, $\mathfrak{X}_{11} \geq 1$.

The descriptions here coincide with [Most73, §19. Spaces of \mathbb{R} -rank 1] in the following way. For any $v = (\theta, b, c) \in \mathbb{R}_{\geq 0} \times \mathbb{O}^2$, define $\mathfrak{X}_v = I_{1,2}v^*v \in \mathfrak{J}(1, 2, \mathbb{O})$. Let $\mathfrak{J}_0 = \{\mathfrak{X} \in \mathfrak{J}(1, 2, \mathbb{O}) : \mathfrak{X}^2 = 0, \mathfrak{X} \neq 0\}$ denotes the collection of all *nilpotents*. It follows from a similar argument as in the above proof that all elements in \mathfrak{J}_0 can be written as $\pm\mathfrak{X}_v$ for some $v \in \mathbb{R}_{\geq 0} \times \mathbb{O}^2$ satisfying $vI_{1,2}v^* = 0$. Define

$$\mathfrak{J}_1 = \{\mathfrak{X} \in \mathfrak{J}(1, 2, \mathbb{O}) : \mathfrak{X}^2 = \mathfrak{X}, \text{tr}\mathfrak{X} = 1\}.$$

Then for any $\mathfrak{X} \in \mathfrak{J}_1$, the above proposition implies that either $\mathfrak{X} = \mathfrak{X}_v$ for some $v \in \mathbb{R}_{\geq 0} \times \mathbb{O}^2$ satisfying $vI_{1,2}v^* = 1$ (when $\mathfrak{X}_{11} \geq 1$) or $\mathfrak{X} = -\mathfrak{X}_w$ for some $w \in \mathbb{R}_{\geq 0} \times \mathbb{O}^2$ satisfying $wI_{1,2}w^* = -1$ (when $\mathfrak{X}_{11} \leq 0$). (Given the remark of Proposition II.4, one can check that the set $\mathfrak{J}_0 \cup \mathfrak{J}_1$ is naturally identified with the Cayley projective plane $\mathbb{O}\mathbb{P}^2$ described in [Most73, §19. Spaces of \mathbb{R} -rank 1, page 137].)

Following the definition in [Most73], a point $\mathfrak{X} \in \mathfrak{J}_1$ is called an *inner point* if for any $\mathfrak{Y} \in \mathfrak{J}_0 \cup \mathfrak{J}_1$ such that $\text{tr} \mathfrak{X} \circ \mathfrak{Y} = 0$, there exists some $\mathfrak{X}_0 \in \mathfrak{J}_0$ satisfying $\text{tr} \mathfrak{X}_0 \circ \mathfrak{Y} = 0$, where $\mathfrak{X} \circ \mathfrak{Y}$ is the Jordan multiplication defined as $(\mathfrak{X}\mathfrak{Y} + \mathfrak{Y}\mathfrak{X})/2$. Otherwise \mathfrak{X} is an *outer point*. In [Most73, §19. Spaces of \mathbb{R} -rank 1, page 138], the Cayley hyperbolic space is defined to be the collection of all inner points in \mathfrak{J}_1 . Hence, the equivalence of our model for the Cayley hyperbolic space and the model in [Most73, §19. Spaces of \mathbb{R} -rank 1] boil down to the following proposition.

Proposition II.6. *The “matrix model” for $\mathbb{O}\mathbb{H}^2$ defined in Definition II.3 is the collection of all inner points in \mathfrak{J}_1 .*

Proof. For any $0 \neq v = (1, a, b), w = (1, c, d) \in \mathbb{R}_{\geq 0} \times \mathbb{O}^2$, we have

$$\text{tr} \mathfrak{X}_v \circ \mathfrak{X}_w = 1 - 2\langle a, c \rangle - 2\langle b, d \rangle + |ac|^2 + |bd|^2 + 2\langle \bar{a}b, \bar{c}d \rangle.$$

When $a \neq 0$, the first Moufang identity implies that

$$\langle \bar{a}b, \bar{c}d \rangle = \langle b, a(\bar{c}d) \rangle = \frac{1}{|a|^2} \langle ba, (a(\bar{c}d))a \rangle = \frac{1}{|a|^2} \langle ba, (a\bar{c})(da) \rangle = \frac{1}{|a|^2} \langle c\bar{a}, (da)(\bar{b}a) \rangle.$$

Therefore

$$\text{tr} \mathfrak{X}_v \circ \mathfrak{X}_w = \begin{cases} \left| 1 - c\bar{a} - \frac{(da)(\bar{b}a)}{|a|^2} \right|^2, & a \neq 0, \\ |1 - d\bar{b}|^2, & a = 0. \end{cases} \quad (\text{II.2.2})$$

Case 1: When $a = b = 0$, the quantity in (II.2.2) never equal to 0. The only $w \in \mathbb{R}_{\geq 0} \times \mathbb{O}^2$ such that $\text{tr} \mathfrak{X}_v \circ \mathfrak{X}_w = 0$ is in the form of $w = (0, c, d)$ with $c, d \in \mathbb{O}$. Let $u = (1, d/|(c, d)|, -c/|(c, d)|)$. One can verify that $\mathfrak{X}_u \in \mathfrak{J}_0$ and $\text{tr} \mathfrak{X}_u \circ \mathfrak{X}_w = 0$. Therefore \mathfrak{X}_v is an inner point. Denote by x_0 this very special inner point, i.e.,

$$x_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Simple computation shows that for any $u \in \mathbb{R}_{\geq 0} \times \mathbb{O}^2$, $\mathfrak{X}_u \circ x_0 = 0$ implies that $u \in \{0\} \times \mathbb{O}^2$. Let $[\mathfrak{X}_u]$ be the unique element in $\mathbb{R}\mathfrak{X}_u \cap \mathfrak{J}_1$ for those $\mathfrak{X}_u \notin \mathfrak{J}_0$. Therefore $[\mathfrak{X}_u]$ is an outer point for any $u \in \{0\} \times \mathbb{O}^2 \setminus \{0\}$.

Case 2: When $(a, b) \neq 0$, without loss of generality we can assume that $a \neq 0$. If $|a|^2 + |b|^2 < 1$, $\text{tr} \mathfrak{X}_v \circ \mathfrak{X}_w = 0$ implies that $|c|^2 + |d|^2 \geq 1$, following (II.2.2). If $c = 0$ (or similarly $d = 0$), it is easy to find a $u = (1, \lambda, \xi d)$ for some real numbers λ, ξ such that $\mathfrak{X}_u \in \mathfrak{J}_0$ and $\text{tr} \mathfrak{X}_u \circ \mathfrak{X}_w = 0$. If $c, d \neq 0$, let $\lambda, \xi \in \mathbb{R}$ and $u = (1, \lambda c, \xi d)$. Then the set of equations with respect to λ and ξ given

by

$$\begin{cases} \mathfrak{X}_u \in \mathfrak{J}_0; \\ \text{tr} \mathfrak{X}_u \circ \mathfrak{X}_w = 0 \end{cases} \iff \begin{cases} 1 = \lambda^2 |c|^2 + \xi^2 |d|^2; \\ 1 = \lambda |c|^2 + \xi |d|^2 \end{cases} \quad (\text{II.2.3})$$

has real solutions

$$\begin{cases} \lambda = \frac{1 \pm \sqrt{\frac{|d|^2}{|c|^2} (|c|^2 + |d|^2 - 1)}}{|c|^2 + |d|^2}; \\ \xi = \frac{1 \mp \sqrt{\frac{|c|^2}{|d|^2} (|c|^2 + |d|^2 - 1)}}{|c|^2 + |d|^2} \end{cases}$$

if and only if $|c|^2 + |d|^2 \geq 1$. Therefore we can conclude that $[\mathfrak{X}_v]$ is an inner point when $vI_{1,2}v^* > 0$.

On the other hand, we define $\hat{v} = (1, a/(|a|^2 + |b|^2), b/(|a|^2 + |b|^2))$ for any $v = (1, a, b) \in \mathbb{R}_{\geq 0} \times \mathbb{O}^2$ such that $vI_{1,2}v^* < 0$. One can verify that $\text{tr} \mathfrak{X}_v \circ \mathfrak{X}_{\hat{v}} = 0$ and that $\hat{v}I_{1,2}\hat{v}^* > 0$. Therefore $\text{tr} \mathfrak{X} \circ \mathfrak{X}_{\hat{v}} \neq 0$ for any $\mathfrak{X} \in \mathfrak{J}_0$ by (II.2.3), which implies that $[\mathfrak{X}_v]$ is an outer point when $vI_{1,2}v^* < 0$.

Hence we can conclude from the above discussions that $\mathfrak{X} \in \mathfrak{J}_1$ is an inner point if $\mathfrak{X}_{11} \geq 1$ and is an outer point if $\mathfrak{X}_{11} \leq 0$, which proves that our definition of $\mathbb{O}\mathbf{H}^2$ coincide with the definition in [Most73, §19. Spaces of \mathbb{R} -rank 1]. \square

We refer to [Most73, §19. Spaces of \mathbb{R} -rank 1] and [Baez02] for an overview of related subjects. A very detailed and general theory can be found in [SV63], [SV00] by T. A. Springer and F. D. Veldkamp for further reference.

Denote by g_0 the symmetric metric on $\mathbb{O}\mathbf{H}^2$ and $M = (\mathbb{O}\mathbf{H}^2, g_0)$ such that the distance function on $\mathbb{O}\mathbf{H}^2$ is given by

$$\cosh(2d(\mathfrak{X}, \mathfrak{Y})) = 2\text{tr}(\mathfrak{X} \circ \mathfrak{Y}) - 1, \quad \forall \mathfrak{X}, \mathfrak{Y} \in \mathbb{O}\mathbf{H}^2$$

as in [Most73]. We identify $T_{x_0}M_0$ with \mathbb{O}^2 such that each unit vector $v = (a, b) \in \mathbb{O}^2$ corresponds to the initial vector of $\gamma_v(t) := (\cosh(t), a \sinh(t), b \sinh(t))$. We will compute the Riemannian curvature data at x_0 by understanding the *geodesic hinge* $\angle \mathfrak{X}x_0\mathfrak{Y}$ for any $\mathfrak{X}, \mathfrak{Y} \in M \setminus \{x_0\}$, where $\angle \mathfrak{X}x_0\mathfrak{Y}$ consists of two geodesic segments $\overline{\mathfrak{X}x_0}$, $\overline{\mathfrak{Y}x_0}$ and the angle $\angle \mathfrak{X}x_0\mathfrak{Y}$. A *comparison hinge* of $\angle \mathfrak{X}x_0\mathfrak{Y}$ in some space form M' is a geodesic hinge $\angle \mathfrak{X}'x'_0\mathfrak{Y}'$ in M' with the same angle such that the lengths of geodesic segments $\overline{\mathfrak{X}x_0}$, $\overline{\mathfrak{Y}x_0}$ and $\overline{\mathfrak{X}'x'_0}$, $\overline{\mathfrak{Y}'x'_0}$ are equal respectively.

Proposition II.7. *The following hold for the Cayley hyperbolic space.*

- (1). For any unit vector $v = (a, b) \in \mathbb{O}^2$, $\gamma_v(t)$ gives a unit speed geodesic starting at x_0 with initial vector v ;
- (2). The Riemannian metric at x_0 is given by the Euclidean inner product on \mathbb{O}^2 . Hence the map $\chi : \mathbb{O}^2 \rightarrow M$ such that

$$\chi(a, b) = \left(\cosh(|(a, b)|), \frac{a}{|(a, b)|} \sinh(|(a, b)|), \frac{b}{|(a, b)|} \sinh(|(a, b)|) \right)$$

gives the geodesic normal coordinates centered at x_0 (hence $d\chi : \mathbb{O}^2 \rightarrow T_{x_0}M_0$ is the isometric correspondence from \mathbb{O}^2 with Euclidean inner product to $T_{x_0}M_0$ mentioned above)

- (3). For any $v = (a, b) \in \mathbb{O}^2$, denote by

$$\text{Cay}(v) = \begin{cases} \mathbb{O} \cdot (1, a^{-1}b), & a \neq 0; \\ \mathbb{O} \cdot (0, 1), & a = 0 \end{cases}$$

the Cayley line containing v . Then for any non-parallel pair of non-zero vectors $(a, b), (c, d)$ contained in the same Cayley line and any pair of points $\mathfrak{X} \in \gamma_{(a,b)}(\mathbb{R}_+)$ and $\mathfrak{Y} \in \gamma_{(c,d)}(\mathbb{R}_+)$, the comparison hinge $\angle \mathfrak{X}'x_0\mathfrak{Y}'$ of $\angle \mathfrak{X}x_0\mathfrak{Y}$ in a space form of constant sectional curvature -4 satisfies $d(\mathfrak{X}', \mathfrak{Y}') = d(\mathfrak{X}, \mathfrak{Y})$;

- (4). For any $a, b \in \mathbb{O} \setminus \{0\}$, any pair of points $\mathfrak{X} \in \gamma_{(a,0)}(\mathbb{R}_+)$ and $\mathfrak{Y} \in \gamma_{(0,b)}(\mathbb{R}_+)$, the comparison hinge $\angle \mathfrak{X}'x_0\mathfrak{Y}'$ of $\angle \mathfrak{X}x_0\mathfrak{Y}$ in a space form of constant sectional curvature -1 satisfies $d(\mathfrak{X}', \mathfrak{Y}') = d(\mathfrak{X}, \mathfrak{Y})$.

Proof. (1). This follows easily by direct computations.

- (2). Let $(a, b), (c, d)$ be unit vectors in \mathbb{O}^2 . Hence the inner product of these two vectors is given by

$$\begin{aligned} - \frac{d}{dt} \Big|_{t=0} d(\gamma_{(a,b)}(t), \gamma_{(c,d)}(1)) &= - \frac{\frac{d}{dt} \Big|_{t=0} (2\text{tr}\gamma_{(a,b)}(t) \circ \gamma_{(c,d)}(1) - 1)}{2 \sinh(2)} \\ &= \frac{2 \sinh(1) \cosh(1) (\langle a, c \rangle + \langle b, d \rangle)}{2 \sinh(2)} = \langle a, c \rangle + \langle b, d \rangle. \end{aligned}$$

- (3). Let a, b, c be unit octonions and $\theta \in [0, \pi/2)$. Let $v = (\cos \theta, a \sin \theta)$. For any $t_1, t_2 > 0$, we

have

$$\begin{aligned}
& \cosh(2d(\gamma_{bv}(t_1), \gamma_{cv}(t_2))) \\
&= 2\text{tr}\gamma_{bv}(t_1) \circ \gamma_{cv}(t_2) - 1 \\
&= 2\cosh^2(t_1)\cosh^2(t_2) - 4\cosh(t_1)\sinh(t_1)\cosh(t_2)\sinh(t_2)\langle bv, cv \rangle - 1 \\
&\quad + 2\sinh^2(t_1)\sinh^2(t_2)(\cos^4\theta + \sin^4\theta) + 4\sinh^2(t_1)\sinh^2(t_2)\cos^2\theta\sin^2\theta \\
&= 2\cosh^2(t_1)\cosh^2(t_2) + 2\sinh^2(t_1)\sinh^2(t_2) - \sinh(2t_1)\sinh(2t_2)\langle bv, cv \rangle - 1 \\
&= \cosh(2t_1)\cosh(2t_2) - \sinh(2t_1)\sinh(2t_2)\langle bv, cv \rangle.
\end{aligned}$$

Notice that $t_1 = d(\gamma_{bv}(t_1), E_1)$ and $t_2 = d(\gamma_{cv}(t_2), E_1)$, the above equation coincides with the law of cosine in a space form with constant sectional curvature -4 .

(4). Let a, b be unit octonions and $t_1, t_2 > 0$. Write $v = (a, 0)$ and $w = (0, b)$. Then

$$\cosh(2d(\gamma_v(t_1), \gamma_w(t_2))) = 2\cosh^2(t_1)\cosh^2(t_2) - 1,$$

which implies that

$$\cosh(d(\gamma_v(t_1), \gamma_w(t_2))) = \cosh(t_1)\cosh(t_2).$$

The above equation coincides with the law of cosine in a space form with constant sectional curvature -1 . \square

A direct corollary of the above proposition is the following.

Corollary II.8. *For any non-zero $v, w \in \mathbb{O}^2$ the following hold.*

- (1). *If v, w belong to the same Cayley line and $v \notin \mathbb{R}w$, then the sectional curvature of the 2-dimensional plane spanned by $d\chi(v), d\chi(w)$ is -4 ;*
- (2). *If $\text{Cay}(v) \perp \text{Cay}(w)$, then the sectional curvature of the 2-dimensional plane spanned by $d\chi(v), d\chi(w)$ is -1 .*
- (3). *Recall that from classical results that $\mathbb{O}\mathbf{H}^2 = F_4^{-20}/\text{Spin}(9)$ with $\text{Spin}(9)$ the stabilizer of x_0 . Since F_4^{-20} acts by isometries on $M = (\mathbb{O}\mathbf{H}^2, g_0)$, identifying $T_{x_0}M_0$ with \mathbb{O}^2 , one can view $\text{Spin}(9)$ as a subgroup of $\text{SO}(\mathbb{O}^2) \cong \text{SO}(16)$. In particular, $\text{Spin}(9)$ maps Cayley lines to Cayley lines and acts transitively on the set of all Cayley lines.*

Remark. Let p be a fixed point in $\mathbb{C}\mathbf{H}^n$. Recall that for any unit vector $v \in T_p^1\mathbb{C}\mathbf{H}^n$ with respect to the symmetric metric, there exists a linear map $J \in \text{End}(T_p\mathbb{C}\mathbf{H}^n)$ such that the sectional curvature between v and $J(v)$ is -4 . The same argument holds true in quaternionic hyperbolic spaces but **NOT** in the Cayley hyperbolic space. This is due to the non-associativity

of octonionic multiplication. In fact, if such a map J exists, without loss of generality we can assume that $p = x_0$ and identify the tangent space at x_0 with \mathbb{O}^2 via $d\chi$ as in Proposition II.7. If $J(1, 0) = (a, 0)$ for some unit octonion $a \in \mathbb{O} \setminus \mathbb{R}$, then for any unit octonion b and any $\theta \in [0, \pi/2)$, $J(\cos \theta, b \sin \theta) = a(\cos \theta, b \sin \theta)$. In particular, $J(0, 1) = (0, a)$. Similarly we have $J(b \sin \theta, \cos \theta) = a(b \sin \theta, \cos \theta)$ for any unit octonion b and any $\theta \in [0, \pi/2)$, which implies that $J(v) = av$ for any $v \in \mathbb{O}^2$. Therefore, for any octonions $b \neq 0, c$, $J(b, bc) = (ab, a(bc)) \in \mathcal{Cay}(b, bc) = \mathcal{Cay}(1, c)$, which implies that $(ab)c = a(bc)$. Notice that if $(ab)c = a(bc)$ for any $b, c \in \mathbb{O}$, then a must be real. This contradicts the assumption that $a \in \mathbb{O} \setminus \mathbb{R}$.

CHAPTER III

Proving Filling Minimality: a Projection Argument

III.1: A toy model

Before we move onto the complicated case of symmetric spaces, we first look at a very special example.

Let $D \subset M := \mathbb{R}^n \subset \mathcal{L} := \mathbb{R}^N$ be a compact region with smooth boundary ∂D in \mathbb{R}^n equipped with the natural Euclidean metric, where $n < N$ and M is a vector subspace of \mathcal{L} . Let $D' \subset \mathbb{R}^N$ be an embedded n -dimensional smooth Riemannian manifold with boundary $\partial D' = \partial D$. Then we have the following result.

Proposition III.1. *Under the above assumptions, the following holds.*

(1). $\text{bd}_{D'} \geq \text{bd}_D$;

(2). $\text{Vol}(D') \geq \text{Vol}(D)$.

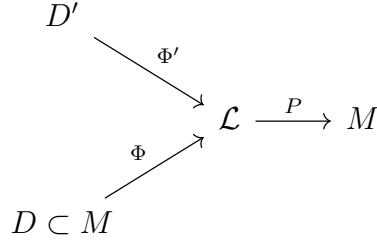
Proof. Let $P : \mathcal{L} \rightarrow M$ be the orthogonal projection. For any $x, y \in \partial D = \partial D'$ and any rectifiable curve segment $\gamma : [0, T] \rightarrow D'$ connecting x, y , we have

$$\text{length}(\gamma) \geq \text{length}(P \circ \gamma) \geq d_D(x, y) = \text{bd}_D(x, y).$$

Taking the infimum on all such γ proves the first assertion. Since $P|_{\partial D} = \text{Id}_{\partial D}$, we have $D \subset P(D')$. Notice that the n -dimensional Jacobian $\text{Jac}P \leq 1$. We have that P does not increase n -dimensional volume. Hence the second assertion follows from

$$\text{Vol}(D') \geq \text{Vol}(P(D')) \geq \text{Vol}(D). \quad \square$$

The general argument presented in this paper is very similar to the proof of the above proposition, which is essentially the following diagram



such that

- (1). $\Phi : M \rightarrow \mathcal{L}$ is an “embedding”, $\Phi' : D' \rightarrow \mathcal{L}$ is a map such that $\Phi|_{\partial D} = \Phi'|_{\partial D'}$;
- (2). $P \circ \Phi = \text{Id}$ and $P \circ \Phi'$ is volume non-increasing;
- (3). $D \subset P \circ \Phi'(D')$. When $P \circ \Phi'$ is continuous, this is a corollary of $\Phi|_{\partial D} = \Phi'|_{\partial D'}$ and $P \circ \Phi = \text{Id}$.

In this special example, an embedding Φ , a map Φ' , an “ambient space” \mathcal{L} are all given in the assumptions. Also, the construction of a “projection” P and verifications of the required properties are more straightforward than later constructions in this chapter.

III.2: Proof of filling minimality for compact regions in Euclidean spaces

In this section, we sketch Burago-Ivanov’s proof of filling minimality for compact regions in a Euclidean space $M = \mathbb{R}^n$ with the standard Euclidean metric. Namely

Proposition III.2 (A naive corollary of [BI10, Theorem 1]). *Let $D \subset M := \mathbb{R}^n$ be a compact region with smooth boundary ∂D in \mathbb{R}^n equipped with the natural Euclidean metric. Then D is a minimal filling.*

Remark. In their original paper [BI10], their method was used to prove strict minimal filling for almost Euclidean regions. See [BI10] for more details.

Proof for Proposition III.2. We want to use the projection argument in the previous section, but we have additional difficulties:

- (Q1). \mathcal{L} is not given in the assumptions. Which ambient space \mathcal{L} should we use?
- (Q2). For an arbitrary D' with $\partial D = \partial D'$ and $\text{bd}_{D'} \geq \text{bd}_D$, how to construct Φ' and Φ so that we can compare D and D' in \mathcal{L} ?
- (Q3). How to construct P after we construct \mathcal{L} , Φ' and Φ ?
- (Q4). How to prove that our constructions satisfy the required properties?

The answers given by Burago-Ivanov to the above questions are the following: Let $S := S^{n-1} \subset M$ be the unit sphere equipped with the standard spherical probability measure ds . Then \mathcal{L} is chosen to be $L^\infty(S, ds)$. For simplicity, we write $L^\infty(S)$ if there is no ambiguity of the measure class. Denoted by $\langle \cdot, \cdot \rangle$ the standard Euclidean inner product on $M = \mathbb{R}^n$. The “embedding” map $\Phi : M \rightarrow \mathcal{L}$ and the map $\Phi' : D' \rightarrow \mathcal{L}$ are defined as follows.

$$\Phi_\theta(x) := \Phi(x)(\theta) = -\langle x, \theta \rangle; \quad \Phi'_\theta(p) := \Phi'(p)(\theta) := \chi_N \left(\inf_{y \in \partial D} \{d_{D'}(p, y) + \Phi_\theta(y)\} \right),$$

where $\chi_N(t) := \max\{\min\{t, -N\}, N\}$ for some N sufficiently large. It is not hard to verify that Φ is an isometric embedding of M into \mathcal{L} as a metric space, Φ' is a 1-Lipschitz map and $\Phi|_{\partial D} = \Phi'|_{\partial D'}$. Notice that the image of Φ is a n -dimensional vector subspace, we can define \widehat{P} to be the $L^2(S, ds)$ -orthogonal projection onto $\Phi(M)$ and $P = \Phi^{-1} \circ \widehat{P}$. If we fix an orthonormal basis in M and write $x = (x_1, \dots, x_n)$ to be its coordinates, then

$$P \circ \Phi'(p) = - \left(n \int_S \Phi'_\theta(p) \theta_1 ds(\theta), \dots, n \int_S \Phi'_\theta(p) \theta_n ds(\theta) \right), \quad \forall p \in D'.$$

It is easy to see that $P \circ \Phi = \text{Id}$ and that $P \circ \Phi'$ is continuous (actually even Lipschitz). Therefore it remains for us to check whether $P \circ \Phi'$ is volume non-increasing.

By Rademacher’s theorem, for every $\theta \in S$, $\Phi'_\theta(\cdot)$ is differentiable almost everywhere in D' . Hence for almost any $p \in D'$ and any $v \in T_p D'$, a notion of “weak derivative” $d_p \Phi' : T_p D' \rightarrow T_{\Phi'(p)} \mathcal{L} \cong \mathcal{L}$ is a well-defined 1-Lipschitz linear map (see [BI10, Lemma 5.3]). Moreover

$$d_p(P \circ \Phi')(v) = - \left(n \int_S d_p \Phi'_\theta(v) \theta_1 ds(\theta), \dots, n \int_S d_p \Phi'_\theta(v) \theta_n ds(\theta) \right)$$

for almost every $p \in D'$ (see [BI10, Lemma 5.4]). For such $p \in D'$, let v_1, \dots, v_n be an orthonormal basis in $T_p D'$. For any $a_1, \dots, a_n \in \mathbb{R}$ such that $a_1^2 + \dots + a_n^2 = 1$, we have

$$\|a_1 d_p \Phi'(v_1) + \dots + a_n d_p \Phi'(v_n)\|_{L^\infty(S)} \leq 1,$$

which implies that

$$\|(d_p \Phi'(v_1))^2 + \dots + (d_p \Phi'(v_n))^2\|_{L^\infty(S)} \leq 1.$$

Hence, by Cauchy-Schwarz inequality and the fact that $\sqrt{n}\theta_j$ are orthonormal in $L^2(S, ds)$, we

have

$$\begin{aligned} \sum_{j=1}^n \|d_p(P \circ \Phi')(v_j)\|^2 &= n \sum_{j=1}^n \sum_{i=1}^n \left(\int_S d_p \Phi'_\theta(v_j) \sqrt{n} \theta_i ds(\theta) \right)^2 \\ &\leq n \sum_{j=1}^n \left[\int_S d_p (\Phi'_\theta(v_j))^2 ds(\theta) \right] \leq n. \end{aligned} \quad (\text{III.2.1})$$

We recall the following linear algebra fact.

Lemma III.3. *For any $m \times m$ real matrix $H = (h_1 \ h_2 \ \dots \ h_m)$, assuming $k_1, k_2, \dots, k_m \geq 0$ are eigenvalues of $H^T H$, we have*

$$|\det H| \leq \sqrt{\left(\frac{1}{m} \sum_{i=1}^m k_i \right)^m} = \sqrt{\left(\frac{1}{m} \sum_{i=1}^m h_i^T h_i \right)^m}.$$

Proof.

$$|\det H| = \sqrt{\det H^T H} = \sqrt{\prod_{i=1}^m k_i} \leq \sqrt{\left(\frac{1}{m} \sum_{i=1}^m k_i \right)^m} = \sqrt{\left(\frac{1}{m} \sum_{i=1}^m h_i^T h_i \right)^m}. \quad \square$$

By the above lemma, the Jacobian of $d_p(P \circ \Phi')$, denoted by $\text{Jac}(d_p(P \circ \Phi'))$ or simply $\text{Jac}(P \circ \Phi')(p)$, is bounded above by 1. This proves that $P \circ \Phi'$ is volume non-increasing. Filling minimality of D then follows from the projection argument in Proposition III.1, the above constructions and verifications of necessary properties. \square

Remark. The connections between this proof and the proof for the rank-1 symmetric spaces (which will be presented in the rest of this paper) can be summarized as follows.

- (1). Section III.3 explains how we can use the projection argument mentioned in Section III.1 to prove Theorem I.5 and subsequently Corollary I.6;
- (2). The key step (III.2) in the estimate of Jacobian of $P \circ \Phi'$ is generalized in Section III.4. The main idea is to introduce a notion of ‘‘Riemannian metric’’ on an open subset of \mathcal{L} such that the n -dimensional Jacobian of any 1-Lipschitz map Φ' is bounded above by 1. Some technical definitions required in our constructions will also be introduced in Section III.4.
- (3). In the Euclidean setting, the projection P from \mathcal{L} to M can also be defined as follows: For any $\phi \in S$, $P(\Phi)$ is the unique critical point of the convex function

$$F_\phi : M \rightarrow \mathbb{R}, \quad F_\phi(y) = \int_S \left[\frac{1}{2} (\Phi_\theta(y))^2 - \phi(\theta) \Phi_\theta(y) \right] ds(\theta).$$

Moreover, if we view S , the unit sphere in $M = \mathbb{R}^n$, as the *boundary at infinity* for M , then $\Phi_\theta(y)$ are exactly *Busemann functions* on M and ds is the *visual measure* for any point on M . These concepts will be introduced in Section IV.1 when we discuss the constructions for rank-1 symmetric spaces. As one may expect, some general ideas behind these constructions will be similar.

III.3: Proof of Theorem I.5 and Corollary I.6

This section is a review of [BI13, Section 2. Proof of the theorems]. The purpose of this section is to prove Theorem I.5 and Corollary I.6 from Proposition III.4, which is the $\mathbb{K}\mathbf{H}^n$ version of [BI13, Proposition 2.1]. This proposition asserts that we can “embed” $\mathbb{K}\mathbf{H}^n$ into $L^\infty(\partial_\infty M) = L^\infty(S^{dn-1})$ ($d = 1, 2, 4, 8$ when $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, respectively) and also “project” an open subset of $L^\infty(S^{dn-1})$ to $\mathbb{K}\mathbf{H}^n$ with certain properties. The rest of the thesis provides the technical details of these two maps along with verification of properties.

Let g_0 denote the standard metric on $\mathbb{K}\mathbf{H}^n$ such that sectional curvature on $\mathbb{K}\mathbf{H}^n$ lies in the interval $[-4, -1]$. Let $D \subset \mathbb{K}\mathbf{H}^n$ be a region with smooth boundary. Let g be a Riemannian metric on D which is C^r -sufficiently close to $g_0|_D$ for a suitable r . (See the remark after Proposition IV.34.)

Fix a point $x_0 \in \mathbb{K}\mathbf{H}^n$. Let $B_{x_0}(R)$ be the ball of radius R in $\mathbb{K}\mathbf{H}^n$ centered at x_0 with respect to the symmetric Riemannian metric g_0 . Fix an $R > 0$ such that $D \subset B_{x_0}(R/5)$. By [Horm90, Theorem 2.3.6] the metric g can be smoothly extended from D to $\mathbb{K}\mathbf{H}^n$ which coincides with g_0 outside $B_{x_0}(R/2)$. Moreover, the extension can be constructed in such a way that it converges to g_0 as g converges to $g_0|_D$.

We denote the extension by the same letter g and let $M = (\mathbb{K}\mathbf{H}^n, g)$. We also assume in addition that g is sufficiently close to g_0 such that $B_{x_0}(R/5)$ is contained in the ball of radius $R/4$ centered at x_0 with respect to the metric g . Our goal is to prove that for any region $D \subset B_{x_0}(R/5)$, the space $(D, g) \subset M$ is a minimal filling and is boundary rigid.

Let $S = S^{dn-1}$ and $\mathcal{L} = L^\infty(S)$. For any $r > 0$, let $\mathcal{B}(r)$ be the ball of radius r in \mathcal{L} centered at the origin.

The technical results established in the rest of the thesis can be summarized by the following proposition.

Proposition III.4. *If g is sufficiently close to g_0 , then there exists a distance preserving map $\Phi : M \rightarrow \mathcal{L}$ such that $\Phi(x_0) = 0 \in \mathcal{L}$ and a Lipschitz map*

$$P_\sigma : \mathcal{B}(R) \cup \Phi(M) \rightarrow M$$

satisfying the following properties

- (1). $P_\sigma \circ \Phi = \text{Id}_M$.
- (2). For every dn -dimensional Riemannian manifold N and every 1-Lipschitz map $f : N \rightarrow \mathcal{B}(R)$, the composition $P_\sigma \circ f$ does not increase dn -dimensional volumes.
- (3). For any N and f as above, $f(N) \subset \Phi(M)$ provided the composition $P_\sigma \circ f : N \rightarrow M$ preserves volumes of all measurable sets.

Proof of Theorem 1.5 from Proposition III.4. The proof is the same as the proof of [BI13, Theorem 1.6] from [BI13, Proposition 2.1]. We will present the proof for reader's convenience.

Let g be sufficiently close to g_0 so that the maps Φ and P_σ from Proposition III.4 exist. Let D' be a smooth compact manifold with boundary $\partial D' = \partial D$ and g' be a metric on D' such that

$$d_{(D',g')}(x, y) \geq d_{(D,g)}(x, y), \quad \forall x, y \in \partial D.$$

For simplicity we write $M' = (D', g')$. Notice that $(D, g) \subset M$, we have

$$d_{(D,g)}(x, y) \geq d_M(x, y), \quad \forall x, y \in \partial D.$$

Therefore

$$d_{M'}(x, y) \geq d_M(x, y), \quad \forall x, y \in \partial D.$$

Since Φ is distance preserving with respect to M , $\Phi|_{\partial D}$ is 1-Lipschitz with respect to the metric on M' . Therefore we can apply the method in [Ivan09, Proposition 1.6] (or [BI10, Proposition 4.9]) to construct $\Phi' : M' \rightarrow \mathcal{L}$ as an extension of $\Phi|_{\partial D}$. Here is an explicit formula for Φ' :

$$\Phi'(x)(s) = \chi_R(\inf\{\Phi(y)(s) + d_{M'}(x, y) : y \in \partial D\}), \quad \forall x \in M', s \in S,$$

where $\chi_R : \mathcal{L} \rightarrow \mathcal{L}$ is a cutoff function given by

$$\chi_R(\phi)(s) = \min\{R/2, \max\{-R/2, \phi(s)\}\}.$$

(Since we assumed that $D \subset B_{x_0}(R/5)$ and that for any $x \in B_{x_0}(R/5)$, $d_g(x, x_0) < R/4$ at the beginning of this section, the cutoff function does not change anything when $x \in \partial D$. Therefore $\Phi|_{\partial D} = \Phi'|_{\partial D}$)

Consider a map $\pi = P_\sigma \circ \Phi' : M' \rightarrow M$. The first assertion in Proposition III.4 implies that $\pi|_{\partial M'} = \text{Id}_{\partial D}$, therefore $D \subset \pi(M')$. The second assertion of Proposition III.4 implies that $\text{Vol}(M') \geq \text{Vol}(D, g)$. Therefore (D, g) is a minimal filling.

To prove that (D, g) is a strict minimal filling, suppose that $\text{Vol}(M') = \text{Vol}(D, g)$. Then π is volume-preserving. By the third assertion of Proposition III.4 we have $\Phi'(M') \subset \Phi(M)$. Therefore $\pi = \Phi^{-1} \circ \Phi'$. Since Φ' is 1-Lipschitz and Φ is distance preserving, π is therefore a 1-Lipschitz volume-preserving map. It follows from [BI10, Lemma 9.1] that π is an isometry. Hence (D, g) is a strict minimal filling. \square

Proof of Corollary I.6 from Theorem I.5. The prove is also a $\mathbb{K}\mathbf{H}^n$ version of [BI13, Proof of Theorem 1.3]. For reader's convenience we will present the proof here.

Let D and g be as above, and let g' be a Riemannian metric on D' such that D' and D share the same boundary and g' induces the same boundary distance function as (D, g) . By Theorem I.5 (D, g) is a strict minimal filling. Hence it suffices to show that $\text{Vol}(D, g) = \text{Vol}(D', g')$. Since D is a region in contained in a large ball in M , it satisfies the SGM condition introduced by C. B. Croke [Crok91] if g is sufficiently close to g_0 (for example, g having negative sectional curvature). Hence $\text{Vol}(D, g) = \text{Vol}(D', g')$ by [Crok91, Lemma 5.1]. \square

III.4: General setup and computations

Recall that in Proposition III.4 we need existence of an “embedding” $\Phi : M \rightarrow \mathcal{L}$ and a “projection” $P_\sigma : \mathcal{B}(R) \cup \Phi(M) \rightarrow M$ satisfying certain properties. We will adopt the general setup introduced in [BI13, Section 3, General computations] in order to help us understand these maps. For reader's convenience, we will list their major concepts and results without proof.

Notation III.5. In this section, we assume that (M, g) is a n -dimensional manifold where any two points are connected by a geodesic realizing the distance. This is always the case when (M, g) is complete. Let $S = S^{n-1}$ and $\mathcal{L} := L^\infty(S)$. We equip S with the standard (Haar) probability measure ds . In the rest of the paper, we let $L^2(S) = L^2(S, ds)$.

We denote by T^1M the unit tangent bundle of M and by T_x^1M its fiber over $x \in M$. Let ds_x be the standard probability measure on T_x^1M with respect to the Riemannian metric g . We will use $ds_x = ds_{x,g}$ for simplicity when there is no ambiguity of metric.

Definition III.6 (Special embedding). A map $\Phi : M \rightarrow \mathcal{L}$ is a *special embedding* if there is a family $\{\Phi_s\}_{s \in S}$ of real-valued functions on M such that the following holds:

- (1). For every $x \in M$, the image $\Phi(x)$ is a function $s \rightarrow \Phi_s(x)$ which belongs to \mathcal{L} .
- (2). The function $(x, s) \rightarrow \Phi_s(x)$ is smooth on $M \times S$.
- (3). Every function $\Phi_s : M \rightarrow \mathbb{R}$ is distance-like; that is, $|\text{grad}\Phi_s| \equiv 1$.
- (4). For every $x \in M$, the map $s \rightarrow \text{grad}\Phi_s(x)$ is a diffeomorphism between S and T_x^1M .

Remark. Every special embedding Φ is a distance preserving map. The third assertion in Definition III.6 implies that Φ is 1-Lipschitz. To prove that it is distance preserving for all $x, y \in M$, consider a unit speed geodesic γ connecting x and y . By the fourth assertion in Definition III.6, there exists some $s \in S$ such that $\text{grad}\Phi_s(x)$ is the initial velocity vector of γ . Since Φ_s is distance-like, its gradient curves are geodesics, which implies that γ is a gradient curve of Φ_s . Therefore

$$\Phi_s(y) - \Phi_s(x) = d_M(x, y)$$

and hence $\|\Phi(x) - \Phi(y)\|_{L^\infty} \geq d_M(x, y)$. Thus Φ is distance-preserving.

Notation III.7. Let $\alpha_x : T_x^1 M \rightarrow S$ be the inverse of $s \rightarrow \text{grad}\Phi_s(x)$ and $\alpha : T^1 M \rightarrow S$ be a map such that $\alpha|_{T_x^1 M} = \alpha_x$ for every $x \in M$.

We define a probability measure $d\mu_x$ by the pushforward of the standard probability measure ds_x on $T_x^1 M$ to S . In other words,

$$d\mu_x = (\alpha_x)_* ds_x.$$

We denote by $\lambda(x, s)$ the density of $d\mu_x$ at $s \in S$ with respect to ds . The second and the fourth assertion in Definition III.6 imply that $\lambda : M \times S \rightarrow \mathbb{R}$ is smooth and positive.

Definition III.8 (Scalar product, Riemannian metric and special Riemannian metric). A symmetric bilinear form G on \mathcal{L} is called a *scalar product* on \mathcal{L} if it is L^2 -compatible (with respect to ds). In other words, there exists some positive constants c, C such that

$$c\|u\|_{L^2(S)}^2 \leq G(u, u) \leq C\|u\|_{L^2(S)}^2, \quad \forall u \in \mathcal{L}.$$

A *Riemannian metric* in an open subset $\mathcal{U} \subset \mathcal{L}$ is a smooth family $G = \{G_\phi\}_{\phi \in \mathcal{U}}$ of scalar products on \mathcal{L} . In other words, for any point $\phi \in \mathcal{U}$, there is a scalar product G_ϕ defined on $T_\phi \mathcal{L} = \mathcal{L}$ which depends smoothly on the base point ϕ .

Let $\Phi : M \rightarrow \mathcal{L}$ be a special embedding and G be a Riemannian metric in an open subset $\mathcal{U} \subset \mathcal{L}$ containing $\Phi(M)$. We say that G is *special* with respect to Φ if the following hold:

- (1). For every $\phi \in \mathcal{U}$, the scalar product G_ϕ has the form

$$G_\phi(X, Y) = n \int_S X(s)Y(s)d\nu_\phi(s), \quad \forall X, Y \in \mathcal{L},$$

where ν_ϕ is a probability measure on S .

- (2). Every measure ν_ϕ has positive density bounded away from zero with respect to ds ; these densities depend smoothly on ϕ .

(3). If $\phi = \Phi(x)$ for an $x \in M$, then $\nu_\phi = \mu_x$.

Notation III.9. Let G be a scalar product in \mathcal{L} and V be a n -dimensional Euclidean space. Let $T : \mathcal{L} \rightarrow V$ be a linear map bounded with respect to G . Denote by $\text{Jac}_{G, \mathcal{W}} T$ the Jacobian of $T|_{\mathcal{W}}$, where $\mathcal{W} \subset \mathcal{L}$ is an arbitrary n -dimensional subspace. We define Jacobian of T as

$$\text{Jac}_G T := \sup_{\mathcal{W} \subset \mathcal{L}, \dim(\mathcal{W})=n} \text{Jac}_{G, \mathcal{W}} T.$$

Let M be an arbitrary n -dimensional Riemannian manifold. For any smooth map $F : \mathcal{L} \rightarrow M$, we denote by $d_\phi F : T_\phi \mathcal{L} \rightarrow T_{F(\phi)} M$ the tangent map of F at ϕ .

For simplicity, we will use $T_x \Phi = \Phi_*(T_x M)$ for any $x \in M$ and $\Phi : M \rightarrow \mathcal{L}$.

Definition III.10 (L^2 -smooth). Let $\mathcal{U} \subset \mathcal{L}$ be an open subset of \mathcal{L} . We say that a map $P : \mathcal{U} \rightarrow M$ is L^2 -smooth if it is differentiable with respect to the L^∞ structure and its derivative at every point $\phi \in \mathcal{U}$ can be extended to a bounded linear map from L^2 to a fiber of TM which depends smoothly on ϕ .

Definition III.11 (Projection). Let $\Phi : M \rightarrow \mathcal{U} \subset \mathcal{L}$ be a smooth isometric immersion with respect to a Riemannian metric G on \mathcal{U} . We say that a map $P : \mathcal{U} \rightarrow M$ is a *projection* if it is L^2 -smooth and satisfies the following two properties.

- (1). $P \circ \Phi = \text{Id}_M$;
- (2). for every $x \in M$, $d_{\Phi(x)} P(V) = 0$ for every vector $V \in \mathcal{L}$ orthogonal (with respect to G) to $T_x \Phi$.

Proposition III.12. [BI13, Proposition 3.13] Let $\Phi : M \rightarrow \mathcal{U} \subset \mathcal{L}$ be a special embedding and G be a Riemannian metric with respect to Φ . Let $P : \mathcal{U} \rightarrow M$ be a projection in the sense of Definition III.11. Then for every $x \in M$ and every $V \in \mathcal{L}$ orthogonal to $T_x \Phi$, we have

$$d_{\Phi(x)} \text{Jac}_G P(V) = 0.$$

Proposition III.13. [BI13, Lemma 3.14, Lemma 3.15] Let N be a dn -dimensional Riemannian manifold (with volume form $d\text{vol}_N$) and $f : N \rightarrow \mathcal{L}$ be a 1-Lipschitz map. Suppose that G is a special Riemannian metric in an open subset $\mathcal{U} \subset \mathcal{L}$ with respect to a special embedding $\Phi : M \rightarrow \mathcal{L}$ and $f(N) \subset \mathcal{U}$. Assume $P : \mathcal{U} \rightarrow M$ is an L^2 -smooth map. Then we have the following inequalities

- (1). f does not increase n -dimensional volume. In other words,

$$\text{Vol}_G(f) := \int_N d\text{vol}_{f^*G} \leq \text{Vol}(N),$$

where $d\text{vol}_{f^*G}$ denotes the volume form on N with respect to the Riemannian metric f^*G .

(2).

$$\text{Vol}(P \circ f) := \int_N (P \circ f)^* d\text{vol}_M \leq \int_N \text{Jac}_G P(f(x)) d\text{vol}_N(x).$$

CHAPTER IV

Technical Constructions and the Proof of Proposition III.4

IV.1: The construction in $\mathbb{K}\mathbf{H}^n$

Recall in Subsection III.3 we proved Theorem I.5 and Corollary I.6 assuming Proposition III.4, which asserts the existence of an “embedding” map Φ and a “projection” map P_σ . In this section, we will present the main construction behind these aforementioned maps. We will adopt the same construction as in [BI13, Section 4, The construction] for $\mathbb{K}\mathbf{H}^n$ with a metric g close to the symmetric metric.

To simplify exposition, we do not track the dependence on g and its derivatives in our proof. We say that a dependence on g is *smooth* if for every integer $k > 0$ there exists an $r > 0$ such that this dependence is k -times differentiable with respect to the C^r -norm on a neighborhood of g_0 in the space of metrics.

Notation IV.1. Let $B_{x_0}(r)$ the ball of radius r in $\mathbb{K}\mathbf{H}^n$ centered at x_0 with respect to the symmetric metric g_0 , where $x_0 \in \mathbb{K}\mathbf{H}^n$ is a fixed point. Recall that in Section III.3 we assumed that $D \subset B_{x_0}(R/5)$ with metric g smoothly extended to the whole $\mathbb{K}\mathbf{H}^n$ (also denoted as g). Moreover, g is C^r close to g_0 and $g \equiv g_0$ outside $B_{x_0}(R/2)$. Let $M = (\mathbb{K}\mathbf{H}^n, g)$ and $M_0 = (\mathbb{K}\mathbf{H}^n, g_0)$. Notice that M_0 and M share the same underlying manifold, we denote by d_{g_0}, d_g the distance functions on M_0 and M respectively.

Let \mathcal{O}_M be the set of geodesic rays in M . In other words

$$\mathcal{O}_M := \{\gamma : [0, \infty) \rightarrow M \mid d_M(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|\}.$$

Two geodesic rays γ_1, γ_2 are equivalent if $\lim_{t \rightarrow \infty} d(\gamma_1(t), \gamma_2(t)) < \infty$. In a Euclidean space, geodesics are straight lines and two geodesic rays are equivalent if and only if they are parallel. The *boundary at infinity* for M , denoted as $\partial_\infty M$, is defined to be the equivalence classes of geodesic rays. One can check that for Euclidean spaces, the boundary at infinity can be canonically identified with the unit sphere centered at an arbitrary point. More generally, for non-positively curved manifolds of dimension m , the boundary at infinity is homeomorphic to the sphere of

dimension $m - 1$. If $\gamma \in \mathcal{O}_M$ is in the equivalence class of $s \in \partial_\infty M$, then we say s is the positive infinity endpoint of γ , denoted by $\gamma(\infty) = s$.

Since g and g_0 coincide outside a compact set, boundaries at infinity for both M and M_0 are canonically identified. Let $S = S^{dn-1} = \partial_\infty M$ be the boundary at infinity and $\mathcal{L} := L^\infty(S)$ as in Section III.3, where $d = \dim_{\mathbb{R}} \mathbb{K}$. For every $s \in S$, we denote by $\Phi_s : M \rightarrow \mathbb{R}$ the Busemann function of a geodesic ray starting at x_0 towards $s \in S$. To be more precise, if we let $\gamma(t)$ be the unit speed geodesic such that $\gamma(0) = x_0$ and $\gamma(\infty) = s$, we have

$$\Phi_s(x) = \lim_{t \rightarrow \infty} d_g(x, \gamma(t)) - t, \quad x \in M, s \in S.$$

We define the ‘‘embedding map’’ $\Phi : M \rightarrow \mathcal{L}$ such that

$$\Phi(x)(s) = \Phi_s(x), \quad x \in M, s \in S.$$

Lemma IV.2. *The map Φ defined above depends smoothly on x, s and g . If g is sufficiently close to g_0 , then Φ is a special embedding in the sense of Definition III.6.*

We recall some notations introduced in the Subsection III.4 before we give a proof for the above lemma.

Notation IV.3. Let $d_{S_{x,g}}$ be the standard probability measure on $T_x^1 M$ with respect to the Riemannian metric g . Let $\alpha_{x,g} : T_x^1 M \rightarrow S$ be the inverse of $s \rightarrow \text{grad} \Phi_s(x)$ and $\alpha_g : T^1 M \rightarrow S$ satisfying $\alpha_g|_{T_x^1 M} = \alpha_{x,g}$ for every $x \in M$. For any $v \in T^1 M$, denoted by $v(t) \in T^1 M$ the image of v after applying the geodesic flow for time t . Then for any $v \in T^1 M$ we have $\alpha_g(v) = v(-\infty)$, the negative infinity endpoint of the geodesic with initial vector v .

We define a probability measure $\mu_{x,g}$, known as the *visual measure* at x on M , by the pushforward of the standard probability measure on $T_x^1 M$ to S . In other words,

$$\mu_{x,g} = (\alpha_{x,g})_* d_{S_{x,g}}.$$

We denote by $\lambda_g(x, s)$ the density of $\mu_{x,g}$ at $s \in S$ with respect to the Haar measure $ds = (\alpha_{x_0, g_0})_* d_{S_{x_0, g_0}}$. Lemma IV.2 and Definition III.6 imply that $\lambda_g : M \times S \rightarrow \mathbb{R}$ is smooth and positive. For simplicity we will use $ds_x, \alpha_x, \alpha, \mu_x$ and $\lambda(x, s)$ instead of $d_{S_{x,g}}, \alpha_{x,g}, \alpha_g, \mu_{x,g}$ and $\lambda_g(x, s)$ when there is no ambiguity on the choice of the metric g .

Proof of Lemma IV.2. Let $H_{s,c}$ be the horosphere in M_0 at $s \in S$ such that

- (1). $H_{s,c}$ is tangent to $B_{x_0}(c)$ with $c > R > 0$;
- (2). Any geodesic ray starting at $H_{s,c}$ towards s does not intersect the interior of $B_{x_0}(c)$.

Since g and g_0 coincide on $\mathbb{K}\mathbf{H}^n \setminus B_{x_0}(R/2)$, horospheres of M contained in $\mathbb{K}\mathbf{H}^n \setminus B_{x_0}(R/2)$ coincide with those of M_0 . Therefore,

$$\Phi_s(x) = d_g(x, H_{s,c_1}) - d_g(x_0, H_{s,c_2}) - c_1 + c_2, \quad x \in M, s \in S, c_1, c_2 \gg 1. \quad (\text{IV.1.1})$$

The first and the third conditions in Definition III.6 follow immediately from the definition of Busemann functions. To verify the second and the fourth conditions we first recall that in the proof of Corollary I.6 we assumed that g has negative sectional curvature. Notice that α_{g_0} is smooth, for any $v \in T^1M$, smoothness of the map $\alpha_g(v) = \alpha_g(v(-T))$ follows from choosing arbitrarily large $T > 0$ and the smoothness of α_{g_0} . (This is because $\alpha_g(v(-T)) = \alpha_{g_0}(v(-T))$ when $T \gg 1$.) This verifies the fourth condition in Definition III.6 and also gives a smooth diffeomorphism from T^1M to $M \times S$ by identifying $v \in T_xM$ with $(x, \alpha_g(v))$. Denoted by $\beta : T^1M_0 \rightarrow \mathbb{R}$ a smooth map such that $\beta(v) = c$ for any v orthogonal to $H_{v(-\infty),c}$. The same map is also defined on those points in T^1M where $g = g_0$. For sufficiently large $T > 0$, by (IV.1.1) we have

$$\Phi_s(x) = -\beta(\alpha_{x,g}^{-1}(s)(-T)) + \beta(\alpha_{x_0,g}^{-1}(s)(-T)), \quad x \in M, s \in S.$$

Since all maps involved in the above formula are smooth, we have $\Phi_s(x) : M \times S \rightarrow \mathbb{R}$ is smooth. This verifies the second assertion in Definition III.6. \square

Lemma IV.4. *If $g = g_0$, then*

$$\lambda(x, s) = e^{-\delta(M_0)\Phi_s(x)} = e^{-(dn+d-2)\Phi_s(x)}, \quad \forall x \in M, s \in S,$$

where $\delta(M_0)$ is the volume growth entropy of any compact quotient of M_0 defined as

$$\delta(M_0) = \overline{\lim}_{r \rightarrow \infty} \frac{\ln(\text{Vol}(\{x \in M_0 | d_{M_0}(x, x_0) < r\}))}{r}.$$

Proof. The proof can be found in [BCG95]. \square

Remark. For more general g , we can assume that g and g_0 are sufficiently close such that

$$\frac{1}{2}e^{-(dn+d-2)\Phi_s(x)} \leq \lambda(x, s) \leq 2e^{-(dn+d-2)\Phi_s(x)}, \quad \forall x \in B_{x_0}(R_0), s \in S, \quad (\text{IV.1.2})$$

for some choice of positive real number $R_0 > 0$ to be determined. This will be useful in later computations when we choose a specific R_0 depending only on R and n to help verifying some properties in our construction.

Let $\mathcal{B}(R)$ be the ball of radius R centered at 0 in \mathcal{L} (with respect to L^∞ -norm). We define a projection as the following.

Definition IV.5. Let \mathcal{U} be a neighborhood of $\Phi(M) \cup \mathcal{B}(R)$ in \mathcal{L} . For any $\phi \in \mathcal{U}$, we define a map $\Omega_{\phi,g} : M \rightarrow T^*M$ as

$$\Omega_{\phi,g}(x) = \int_S e^{(dn+d)[\Phi_s(x)-\phi(s)]} d\Phi_s(x) d\mu_{x,g}(s). \quad (\text{IV.1.3})$$

Let $P : \mathcal{U} \rightarrow M$ be such that $\Omega_{\phi,g}(P(\phi)) = 0$.

We first prove that it is well-defined, which is the $\mathbb{K}\mathbf{H}^n$ version of [BI13, Lemma 4.4]. For reader's convenience, we provide a slightly different proof.

Lemma IV.6. *If g is sufficiently close to g_0 , then there exists a smooth map P satisfying Definition IV.5 such that*

- (1). $P(\Phi(x)) = x$ for all $x \in M$;
- (2). There exists some constant $R_1 = R_1(n, R)$ depending only on n and R such that for any $\phi \in \mathcal{B}(R)$, $P(\phi) \in B_{x_0}(R_1)$ and $d_g(P(\phi), x_0) \leq R_1$.

Hence as a direct corollary of the second assertion, $\Phi(P(\phi)) \in \mathcal{B}(R_1)$ for any $\phi \in \mathcal{B}(R)$.

Proof. If $\phi = \Phi(x)$, then we define $P(\phi) = x$ and it satisfies the requirements in Definition IV.5. In the rest of the proof we extend P to a neighborhood of $\Phi(M)$ containing $\mathcal{B}(R)$.

Consider a map $E : \mathcal{L} \rightarrow L^2(S)$ given by

$$E(\phi) = e^{-(dn+d)\phi(s)}.$$

Let $\phi \in \mathcal{L}$ and $\psi = E(\phi)$. Then the equation $\Omega_{\phi,g}(x) = 0$ takes the form

$$\int_S \psi(s) e^{(dn+d)\Phi_s(x)} d\Phi_s(x) d\mu_x(s) = 0.$$

Notice that when $\phi = \Phi(x)$, spherical symmetry implies that

$$\int_S E \circ \Phi_s(x) e^{(dn+d)\Phi_s(x)} d\Phi_s(x) d\mu_x(s) = 0.$$

Hence the equation $\Omega_{\phi,g}(x) = 0$ is equivalent to the following

$$\int_S (\psi(s) - E \circ \Phi_s(x)) e^{(dn+d)\Phi_s(x)} d\Phi_s(x) d\mu_x(s) = 0,$$

which is equivalent to

$$\int_S (\psi(s) - E \circ \Phi_s(x)) e^{2(dn+d)\Phi_s(x)} d_x(E \circ \Phi)(s) d\mu_x(s) = 0.$$

Define $\mathcal{E} : M \times L^2(S) \rightarrow T^*M$ as

$$\mathcal{E}(x, \psi) = \int_S (\psi(s) - E \circ \Phi_s(x)) e^{2(dn+d)\Phi_s(x)} d_x(E \circ \Phi)(s) d\mu_x(s).$$

By the Implicit Function Theorem applied to $\mathcal{E}(x, \psi) = 0$, there exists a smooth map $\tilde{P} : \tilde{\mathcal{U}} \rightarrow M$ defined on a neighborhood $\tilde{\mathcal{U}}$ of $E(\Phi(M))$ such that $\mathcal{E}(\tilde{P}(\psi), \psi) = 0$. Therefore we can extend P to $E^{-1}(\tilde{\mathcal{U}})$ by setting $P = \tilde{P} \circ E$.

It remains for us to extend P to $\mathcal{B}(R)$. Let

$$\omega_g(x) = e^{(dn+d)\Phi_s(x)} \lambda_g(x, s) d\Phi_s(x) \in T_x^*M$$

and hence

$$\Omega_{\phi, g}(x) = \int_S E(\phi(s)) \omega_g(x) ds.$$

When $g = g_0$, Lemma IV.4 implies that $\omega_{g_0} = d(e^{2\Phi_s}/2)$. A classic result from [BCG95] (to be more precise, the first assertion of Lemma IV.11) implies that

$$\nabla \omega_{g_0} > e^{2\Phi_s} g_0.$$

Hence

$$\nabla \Omega_{\phi, g_0} \geq \int_S E(\phi(s)) e^{2\Phi_s} g_0 ds > 0.$$

For general g close to g_0 , we denote the induced quadratic form of $\nabla \omega_g$ and $\nabla \Omega_{\phi, g}$ by the same notations. Similar to the remark for Lemma IV.4, we can assume that g is sufficiently close to g_0 in the sense that

$$\nabla \omega_g \geq \frac{1}{2} e^{2\Phi_s} g, \quad \forall x \in B_{x_0}(\tilde{R}_0) \tag{IV.1.4}$$

for some choice of $\tilde{R}_0 = \tilde{R}_0(n, R) > 2R > 0$ which will be determined in later part of this proof. Notice that $B_{x_0}(r)$ is convex in M when $r > 2R$. Then

$$\nabla \Omega_{\phi, g}|_x \geq \frac{1}{2} \int_S E(\phi(s)) e^{2\Phi_s(x)} g|_x ds > 0, \quad \forall x \in B_{x_0}(\tilde{R}_0).$$

Therefore for any unit speed geodesic segment γ in $B_{x_0}(\tilde{R}_0)$, the function $\Omega_{\phi, g}(\dot{\gamma}(t))$ has derivative equal to $\nabla \Omega_{\phi, g}(\dot{\gamma}, \dot{\gamma})$ and hence is strictly increasing, which implies that $\Omega_{\phi, g} = 0$ has at most one solution in $B_{x_0}(\tilde{R}_0) \subset M$. Moreover, since $\nabla \Omega_{\phi, g}$ is non-degenerate, we can apply the Implicit Function Theorem to $\Omega_{\phi, g} = 0$, which proves that we can extend $x = P(\phi)$ smoothly if the

equation $\nabla\Omega_{\phi,g} = 0$ has a solution for any $\phi \in \mathcal{B}(R)$.

To prove the existence of such a solution, we first claim that $\inf_{x \notin B_{x_0}(r)} \int_S e^{2\Phi_s(x)} ds \rightarrow \infty$ as r tends to infinity (independent of the choice of g). Define

$$\text{Shadow}_g(N, p) = \{s \in S \mid \text{Im}\gamma_{s,p} \cap N \neq \emptyset\}, \quad N \subset M, p \in M \sqcup S,$$

where $\gamma_{s,p}$ denotes the geodesic ray (with respect to g) starting at p towards s . Since in Lemma IV.2 we assumed that g is negatively curved, for any $r > R/2$ and any $x \in \mathbb{KH}^n \setminus B_{x_0}(R/2)$, we have $\text{Shadow}_g(B_{x_0}(r), x) = \text{Shadow}_{g_0}(B_{x_0}(r), x)$. Define

$$D_x = \text{Shadow}_{g_0}(B_{x_0}(2R), x) \setminus \text{Shadow}_{g_0}(B_{x_0}(R), x) \subset S, \quad \forall x \in M \sqcup S.$$

Notice that for any $r > R > 0$ and any $s \in S$, the set

$$\text{Shadow}_{g_0}(B_{x_0}(r), s) := \lim_{x \rightarrow s} \text{Shadow}_{g_0}(B_{x_0}(r), x)$$

has a fixed positive area only depending on r and n with respect to the Haar measure ds on S . Moreover, the area of $\text{Shadow}_{g_0}(B_{x_0}(r), x)$ only depends on $d_{g_0}(x, x_0)$ for any $x \in M \sqcup S$. Hence in particular $\text{Area}(D_s, ds)$ equal to some positive constant $I(n, R) > 0$ for any $s \in S$ and there exists a continuous non-negative function $\mathcal{A} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ such that $\text{Area}(D_x, ds) = \mathcal{A}(d_{g_0}(x, x_0)) \rightarrow I(n, R)$ as $d_{g_0}(x, x_0) \rightarrow \infty$. Notice that $\Phi_s(x) \geq d_g(x, x_0) - 2\text{diam}_g(B_{x_0}(2R))$ for any $s \in \text{Shadow}_{g_0}(B_{x_0}(2R), x)$ and any $x \in M$. Without loss of generality we can assume that g is sufficiently close to g_0 such that $\text{diam}_g(B_{x_0}(2R)) < 5R$. Then we have

$$\int_S e^{2\Phi_s(x)} ds \geq \mathcal{A}(d_{g_0}(x, x_0)) e^{2(d_g(x, x_0) - 10R)} \geq \mathcal{A}(d_{g_0}(x, x_0)) e^{2(d_{g_0}(x, x_0) - 12R)}, \quad \forall x \in M,$$

which proves that $\inf_{x \notin B_{x_0}(r)} \int_S e^{2\Phi_s(x)} ds \rightarrow \infty$ as r tends to infinity. Therefore

$$\inf_{x \notin B_{x_0}(r)} \int_S E(\phi(s)) e^{2\Phi_s(x)} ds \geq e^{-(dn+d)R} \inf_{x \notin B_{x_0}(r)} \int_S e^{2\Phi_s(x)} ds \geq \widehat{\mathcal{A}}(n, R, r) \rightarrow \infty$$

as r tends to infinity, where $\widehat{\mathcal{A}}(n, R, \cdot)$ is a positive continuous function defined on $\mathbb{R}_{\geq 0}$ depending on n and R . Hence there exists some constant $r_0 = r_0(n, R) > 0$ such that

$$\nabla\Omega_{\phi,g} \geq r_0 g, \quad \forall x \in B_{x_0}(\widetilde{R}_0)$$

for $\widetilde{R}_0 > 0$ as in (IV.1.4). Let $\mathcal{X}_{\phi,g}$ be the dual of $\Omega_{\phi,g}$ with respect to the metric g and define a function $\mathcal{F} = |\mathcal{X}_{\phi,g}|_g^2 = g(\mathcal{X}_{\phi,g}, \mathcal{X}_{\phi,g})$ on M . Notice that $\phi \in \mathcal{B}(R)$ implies $\mathcal{F}(x_0) \leq e^{2(dn+d)R}$ and

that

$$\sqrt{\mathcal{F}(\gamma(t))} \geq \left| \frac{g(\mathcal{X}_{\phi,g}, \dot{\gamma}(t))}{\|\dot{\gamma}(t)\|_g} \right| = |\Omega_{\phi,g}(\dot{\gamma}(t))| \geq \int_0^t \nabla \Omega_{\phi,g}(\dot{\gamma}(\tau), \dot{\gamma}(\tau)) d\tau - |\Omega_{\phi,g}(\dot{\gamma}(0))|,$$

where γ is any unit speed geodesic segment in M . Choose $R_1(n, R) = 2e^{2(dn+d)R}/r_0 > 0$ and g sufficiently close to g_0 such that $\tilde{R}_0 = R_1 + 1$, we have

$$\mathcal{F}(x) > \mathcal{F}(x_0), \quad \forall x \in B_{x_0}(\tilde{R}_0) \setminus B_{x_0}(R_1).$$

Hence $\mathcal{F}|_{B_{x_0}(\tilde{R}_0)}$ achieves minimum at some point $x_{\min} \in B_{x_0}(R_1)$. In particular

$$0 = \mathcal{X}_{\phi,g} \mathcal{F}(x_{\min}) = 2g(\nabla_{\mathcal{X}_{\phi,g}} \mathcal{X}_{\phi,g}, \mathcal{X}_{\phi,g}) = 2(\nabla \Omega_{\phi,g})(\mathcal{X}_{\phi,g}, \mathcal{X}_{\phi,g}) \geq 2r_0 g(\mathcal{X}_{\phi,g}, \mathcal{X}_{\phi,g}),$$

which implies that $\mathcal{X}_{\phi,g}(x_{\min}) = 0$. Hence P is well-defined and smooth. \square

Before we state Lemma IV.8, we first introduce some notations.

Notation IV.7. For any $x \in B_{x_0}(\tilde{R}_0)$ and $s \in S$, we define a linear operator $A_{x,s} : T_x M \rightarrow T_x M$ by

$$A_{x,s}(\xi) = e^{-(dn+d)\Phi_s(x)} \lambda(x, s)^{-1} \nabla_{\xi} [e^{(dn+d)\Phi_s(x)} \lambda(x, s) \text{grad} \Phi_s(x)], \quad (\text{IV.1.5})$$

where ∇_{ξ} denotes the Levi-Civita derivative along ξ .

Recall that in the previous proof, we set

$$\omega_g(x) = e^{(dn+d)\Phi_s(x)} \lambda(x, s) d\Phi_s(x) \in T_x^* M.$$

Hence for any g sufficiently close to g_0 in the sense of (IV.1.4) and $0 \neq \xi \in T_x M$, we have

$$\langle A_{x,s}(\xi), \xi \rangle = e^{-(dn+d)\Phi_s(x)} \lambda(x, s)^{-1} (\nabla \omega_g)(\xi, \xi) > 0. \quad (\text{IV.1.6})$$

Let $\phi \in \mathcal{L}$ and $x = P(\phi)$. Denoted by ρ_{ϕ} a function on S such that

$$\rho_{\phi}(s) = e^{(dn+d)[\Phi_s(x) - \phi(s)]} \quad (\text{IV.1.7})$$

Let $\bar{\rho}_{\phi}$ be the same function normalized with respect to the measure μ_x . In other words,

$$\bar{\rho}_{\phi} = \frac{\rho_{\phi}}{\int_S \rho_{\phi} d\mu_x}. \quad (\text{IV.1.8})$$

Assuming $x = P(\phi) \in B_{x_0}(\tilde{R}_0)$, we define a linear operator $A_\phi : T_x M \rightarrow T_x M$ by

$$A_\phi = \int_S \bar{\rho}_\phi(s) A_{x,s} d\mu_x(s), \quad (\text{IV.1.9})$$

which is invertible due to (IV.1.6).

Lemma IV.8. *Let $\phi \in \mathcal{U}$ and $x = P(\phi)$. Then differentiating P yields*

$$d_\phi P = A_\phi^{-1} \circ E_\phi,$$

for any ϕ such that $x \in B_{x_0}(\tilde{R}_0)$. The linear map $E_\phi : \mathcal{L} \rightarrow T_x M$ is given by

$$E_\phi(X(s)) = (dn + d) \int_S X(s) \bar{\rho}_\phi(s) \text{grad} \Phi_s(x) d\mu_x(s) \quad (\text{IV.1.10})$$

for any $X(s) \in T_\phi \mathcal{L} = \mathcal{L}$.

Proof. By Definition IV.5, We have

$$\Omega_{\phi,g}(P(\phi)) = 0, \quad \forall \phi \in \mathcal{U}.$$

Let $x = P(\phi)$. For any $X \in T_\phi \mathcal{L} = \mathcal{L}$, $\xi = d_\phi P(X) \in T_x M$ and any vector field $\tilde{\eta}$ on M , we differentiate the above equation evaluated at $\tilde{\eta}$ and obtain

$$\int_S D_\phi(\rho_\phi(s))(X) \langle \text{grad} \Phi_s(x), \tilde{\eta} \rangle d\mu_x(s) + \xi [\Omega_{\phi,g}(P(\phi))(\tilde{\eta})] = 0.$$

Notice that

$$\xi [\Omega_{\phi,g}(P(\phi))(\tilde{\eta})] = \xi \langle \mathcal{X}_{\phi,g}, \tilde{\eta} \rangle = \langle \nabla_\xi \mathcal{X}_{\phi,g}, \tilde{\eta} \rangle + \langle \mathcal{X}_{\phi,g}, \nabla_\xi \tilde{\eta} \rangle,$$

where $\mathcal{X}_{\phi,g}$ is the dual of $\Omega_{\phi,g}$ with respect to g as in the proof of Lemma IV.6 and hence vanish at $P(\phi)$. By arbitrariness of $\tilde{\eta}$, we have

$$\int_S D_\phi(\rho_\phi(s))(X) \text{grad} \Phi_s(x) d\mu_x(s) + \nabla_\xi \mathcal{X}_{\phi,g} = 0, \quad (\text{IV.1.11})$$

By (IV.1.7), (IV.1.8) and (IV.1.10), the first term in (IV.1.11) takes the form

$$\begin{aligned} & \int_S D_\phi(\rho_\phi(s))(X) \text{grad} \Phi_s(x) d\mu_x(s) \\ &= - (dn + d) \int_S X(s) \rho_\phi(s) \text{grad} \Phi_s(x) d\mu_x(s) = - \left(\int_S \rho_\phi(s) d\mu_x(s) \right) E_\phi(X). \end{aligned}$$

A direct computation yields

$$\begin{aligned}
\nabla_{\xi} \mathcal{X}_{\phi,g} &= \int_S e^{-(dn+d)\phi(s)} \nabla_{\xi} [e^{(dn+d)\Phi_s(x)} \lambda(x,s) \text{grad} \Phi_s(x)] ds \\
&= \int_S \rho_{\phi}(s) e^{-(dn+d)\Phi_s(x)} \nabla_{\xi} [e^{(dn+d)\Phi_s(x)} \lambda(x,s) \text{grad} \Phi_s(x)] ds \\
&= \int_S \rho_{\phi}(s) e^{-(dn+d)\Phi_s(x)} \lambda(x,s)^{-1} \nabla_{\xi} [e^{(dn+d)\Phi_s(x)} \lambda(x,s) \text{grad} \Phi_s(x)] d\mu_x(s) \\
&= \int_S \rho_{\phi}(s) A_{x,s}(\xi) d\mu_x(s) = \left(\int_S \rho_{\phi}(s) d\mu_x(s) \right) A_{\phi}(\xi),
\end{aligned}$$

where the last two equalities follows from our notations in (IV.1.5) and (IV.1.9).

Summarizing the above, (IV.1.11) implies that

$$A_{\phi}(\xi) = E_{\phi}(X).$$

Therefore,

$$d_{\phi}P(X) = \xi = A_{\phi}^{-1} \circ E_{\phi}(X).$$

By the arbitrariness of X , we have

$$d_{\phi}P = A_{\phi}^{-1} \circ E_{\phi}. \quad \square$$

Definition IV.9. Let G be a Riemannian metric on $P^{-1}(B_{x_0}(\tilde{R}_0))$, We require that for any $\phi \in P^{-1}(B_{x_0}(\tilde{R}_0))$, the scalar product G_{ϕ} on $T_{\phi}\mathcal{L} = \mathcal{L}$ is defined by

$$G_{\phi}(X, Y) = nd \int_S X(s)Y(s)\bar{\rho}_{\phi}(s) d\mu_x(s), \quad \forall X, Y \in \mathcal{L}, \quad (\text{IV.1.12})$$

where $x = P(\phi)$.

Remark. There are a number of different inner products mentioned in the latter half of this paper. In order to avoid cumbersome notations we denote by $\langle \cdot, \cdot \rangle$ the inner product of a Euclidean space or a tangent space for a particular Riemannian manifold. The distinction between different settings will be indicated via different notations of Riemannian manifolds/metrics or verbal descriptions. We also denote by $\langle \cdot, \cdot \rangle_{G_{\phi}}$ the above mentioned scalar product G_{ϕ} . The norm induced by G_{ϕ} is denoted by $\| \cdot \|_{G_{\phi}}$.

Lemma IV.10. Let P be the map defined in Definition IV.5. Then we have the following.

- (1). G is a special metric with respect to Φ .
- (2). P is a projection with respect to G and Φ in the sense of Definition III.11.

Proof. (1). The first two requirements of Definition III.8 follow immediately. For any $\phi = \Phi(x)$, we have

$$\rho_\phi(s) = e^{(dn+d)(\Phi_s(x)-\Phi_s(x))} \equiv 1.$$

Therefore

$$\mathcal{X}_{\phi,g}(x) = \int_S \text{grad}\Phi_s(x) d\mu_x(s) = \int_{T_x^1 M} v ds_x(v) = 0,$$

which is equivalent to $\Omega_{\phi,g}(x) = 0$ and therefore $P(\phi) = x$. Hence $P \circ \Phi = \text{Id}_M$. Meanwhile, the scalar product G_ϕ on $T_\phi \mathcal{L} = \mathcal{L}$ is defined by

$$G_\phi(X, Y) = nd \int_S X(s)Y(s) d\mu_x(s).$$

Direct computations imply that it satisfies the third requirement of Definition III.8.

(2). The fact that P is L^2 -smooth in the sense of Definition III.10 follows from Lemma IV.6 and Lemma IV.8. Since we proved the first requirement of Definition III.11 in our previous assertion, it remains to verify the second requirement, that is, for every $x \in M$, $d_{\Phi(x)}P(V) = 0$ for every vector $V \in \mathcal{L}$ orthogonal (with respect to G) to $T_x \Phi$. Let $\phi = \Phi(x)$ for simplicity. By Lemma IV.8, it suffices to show that $E_\phi(X) = 0$ for any $X \in T_\phi \mathcal{L}$ perpendicular to $T_x \Phi$ with respect to G_ϕ .

Let $v \in T_x M$ be an arbitrary vector in $T_x M$. We have

$$\begin{aligned} \langle E_\phi(X), v \rangle &= \left\langle (dn + d) \int_S X(s) \text{grad}\Phi_s(x) d\mu_x(s), v \right\rangle \\ &= (dn + d) \int_S X(s) \langle \text{grad}\Phi_s(x), v \rangle d\mu_x(s) \\ &= (dn + d) \int_S X(s) d_x \Phi(v)(s) d\mu_x(s) \\ &= \frac{(dn + d)}{dn} \cdot dn \int_S X(s) d_x \Phi(v)(s) d\mu_x(s) = \frac{(n + 1)}{n} G_\phi(X, d_x \Phi(v)) = 0. \end{aligned}$$

By arbitrariness of $v \in T_x M$, we have $E_\phi(X) = 0$. Hence $d_\phi P(X) = 0$. \square

IV.2: Approximating $d_\phi P$

IV.2.1: The construction of \widehat{A}_ϕ when $\mathbb{K} \neq \mathbb{O}$

Recall that in Section IV.1 we wrote the derivative map $d_\phi P$ as a composition of two operators A_ϕ^{-1} and E_ϕ for any $\phi \in \mathcal{B}(R)$. A direct application of the above facts gives the following lemma on the operator A_ϕ when $g = g_0$.

Lemma IV.11. *Let Φ be as in the previous section. If $g = g_0$, then we have the following:*

(1). *We can compute the Hessian of $e^{2\Phi_s(x)}/2$:*

$$\text{Hess}_{g_0} \frac{1}{2} e^{2\Phi_s(x)} = e^{2\Phi_s(x)} \left[g_0 + \sum_{t=0}^{d-1} (d\Phi_s \circ \mathcal{J}_{t,g_0})^2 \right].$$

The maps $\mathcal{J}_{t,g_0} : TM \rightarrow TM$ come from the complex structure and quaternionic structure on M . In other words, $\mathcal{J}_{0,g_0} = \text{Id}$, $\mathcal{J}_{1,g_0}(v) = vi$ when $\mathbb{K} = \mathbb{C}$ or \mathbb{H} , $\mathcal{J}_{2,g_0}(v) = vj$ and $\mathcal{J}_{3,g_0}(v) = vk$ when $\mathbb{K} = \mathbb{H}$.

(2). *The operator $A_{x,s}$ defined in (IV.1.5) has the following explicit formula:*

$$A_{x,s}(\xi) = \xi + \sum_{t=0}^{d-1} \langle \xi, \mathcal{J}_{t,g_0} \text{grad}\Phi_s(x) \rangle \mathcal{J}_{t,g_0} \text{grad}\Phi_s(x), \quad \forall \xi \in T_x M. \quad (\text{IV.2.1})$$

Proof. (1). This can be found in [BCG95, page 751].

(2). By Lemma IV.4, we have

$$\lambda(x, s) = e^{-(dn+d-2)\Phi_s(x)}, \quad \forall x \in M, s \in S,$$

when $g = g_0$. Therefore, we can further simplify (IV.1.5) to the following.

$$\begin{aligned} A_{x,s}(\xi) &= e^{-(dn+d)\Phi_s(x)} \lambda(x, s)^{-1} \nabla_\xi [e^{(dn+d)\Phi_s(x)} \lambda(x, s) \text{grad}\Phi_s(x)] \\ &= e^{-2\Phi_s(x)} \nabla_\xi [e^{2\Phi_s(x)} \text{grad}\Phi_s(x)] = e^{-2\Phi_s(x)} \nabla_\xi \text{grad} \left(\frac{1}{2} e^{2\Phi_s(x)} \right). \end{aligned}$$

Hence for any vector $\eta \in T_x M$, we have

$$\begin{aligned} \langle A_{x,s}(\xi), \eta \rangle &= \frac{1}{2} \langle e^{-2\Phi_s(x)} \nabla_\xi \text{grad}(e^{2\Phi_s(x)}), \eta \rangle \\ &= \frac{1}{2} e^{-2\Phi_s(x)} \langle \nabla_\xi \text{grad}(e^{2\Phi_s(x)}), \eta \rangle \\ &= \frac{1}{2} e^{-2\Phi_s(x)} \text{Hess}_{g_0} e^{2\Phi_s(x)}(\xi, \eta) \\ &= \langle \xi, \eta \rangle + \sum_{t=0}^{d-1} \langle \xi, \mathcal{J}_{t,g_0} \text{grad}\Phi_s(x) \rangle \langle \eta, \mathcal{J}_{t,g_0} \text{grad}\Phi_s(x) \rangle \\ &= \left\langle \xi + \sum_{t=0}^{d-1} \langle \xi, \mathcal{J}_{t,g_0} \text{grad}\Phi_s(x) \rangle \mathcal{J}_{t,g_0} \text{grad}\Phi_s(x), \eta \right\rangle. \end{aligned}$$

Therefore,

$$A_{x,s}(\xi) = \xi + \sum_{t=0}^{d-1} \langle \xi, \mathcal{J}_{t,g_0} \text{grad} \Phi_s(x) \rangle \mathcal{J}_{t,g_0} \text{grad} \Phi_s(x), \quad \forall \xi \in T_x M. \quad \square$$

The above result relies on the almost complex structure (almost quaternionic structure resp.) on TM when $g = g_0$ and the explicit formulae for density functions $\lambda_g(x, s)$. In order to understand the more general case when g is sufficiently close to g_0 , similarly we construct an almost complex structure (almost quaternionic structure resp.) which preserves the Riemannian metric g .

Notation IV.12. Let V, W be any real vector spaces with bases $v = \{v_1, v_2, \dots, v_m\}$ and $w = \{w_1, w_2, \dots, w_l\}$, respectively. Let $T : V \rightarrow W$ be a linear map. We denote by ${}_w[T]_v$ the matrix of T under bases v and w . In other words, if

$$T(v_i) = \sum_{j=1}^l a_{ji} w_j, \quad 1 \leq i \leq m,$$

then

$${}_w[T]_v = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq l}.$$

We start with a collection of vector fields $\{\tilde{\xi}_{l,t,g_0} | 1 \leq l \leq n, 0 \leq t \leq d-1\}$ such that for any $x \in \mathbb{K}\mathbf{H}^n$, $\{\tilde{\xi}_{l,t,g_0}(x) | 1 \leq l \leq n, 0 \leq t \leq d-1\}$ is an orthonormal basis for $T_x \mathbb{K}\mathbf{H}^n$ under the standard complex hyperbolic metric g_0 . In other words, we have the following

$$\langle \tilde{\xi}_{l_1,t_1,g_0}(x), \tilde{\xi}_{l_2,t_2,g_0}(x) \rangle_{g_0} = \delta_{l_1 l_2} \delta_{t_1 t_2},$$

where $1 \leq l_1, l_2 \leq n$ and $0 \leq t_1, t_2 \leq d-1$. By Corollary II.2, we can further assume that these vector fields satisfy the following equations.

$$\mathcal{J}_{t,g_0} \tilde{\xi}_{l,0,g_0} = \tilde{\xi}_{l,t,g_0}, \quad 1 \leq l \leq n \text{ and } 0 \leq t \leq d-1.$$

We can therefore apply a Gram-Schmidt process to obtain a new collection of vector fields $\{\tilde{\xi}_{l,t,g} | 1 \leq l \leq n, 0 \leq t \leq d-1\}$ such that for any $x \in M = (\mathbb{K}\mathbf{H}^n, g)$, $\{\tilde{\xi}_{l,t,g}(x) | 1 \leq l \leq n, 0 \leq t \leq d-1\}$ is an orthonormal basis for $T_x M$ under g . In other words, we have

$$\left(\tilde{\xi}_{1,0,g}, \dots, \tilde{\xi}_{1,d-1,g}, \dots, \tilde{\xi}_{n,0,g}, \dots, \tilde{\xi}_{n,d-1,g} \right) = \left(\tilde{\xi}_{1,0,g_0}, \dots, \tilde{\xi}_{1,d-1,g_0}, \dots, \tilde{\xi}_{n,0,g_0}, \dots, \tilde{\xi}_{n,d-1,g_0} \right) \cdot \mathcal{N}_g,$$

where \mathcal{N}_g is a $dn \times dn$ upper triangular matrix recording the Gram-Schmidt process, and

$$\langle \tilde{\xi}_{l_1,t_1,g}(x), \tilde{\xi}_{l_2,t_2,g}(x) \rangle = \delta_{l_1 l_2} \delta_{t_1 t_2},$$

where $1 \leq l_1, l_2 \leq n$ and $0 \leq t_1, t_2 \leq d - 1$.

We denote by $\tilde{\Xi}_g$ the ordered basis $\{\tilde{\xi}_{1,0,g}, \dots, \tilde{\xi}_{1,d-1,g}, \dots, \tilde{\xi}_{n,0,g}, \dots, \tilde{\xi}_{n,d-1,g}\}$ and for simplicity $\tilde{\Xi}$ when there is no ambiguity of the metric. We construct an almost complex structure or almost quaternionic structure $\mathcal{J}_{t,g}$ on TM such that

$$\tilde{\Xi}_g[\mathcal{J}_{t,g}]_{\tilde{\Xi}_g} = \tilde{\Xi}_{g_0}[\mathcal{J}_{t,g_0}]_{\tilde{\Xi}_{g_0}}, \quad \forall 0 \leq t \leq d - 1.$$

It is clear that $\mathcal{J}_{t,g}$ preserves g . Moreover, we have the following lemma.

Lemma IV.13. *If $\|g - g_0\|_{C^r} \leq \epsilon \ll 1$, then $\|\mathcal{J}_g - \mathcal{J}_{g_0}\|_{C^r} \leq \mathcal{K}_0(n, r)\|g - g_0\|_{C^r}$ for some constant $\mathcal{K}_0(n, r)$ depending only on n and r .*

We define a new operator $\hat{A}_{x,s} : T_x M \rightarrow T_x M$ as follows

$$\hat{A}_{x,s}(\xi) = \xi + \sum_{t=0}^{d-1} \langle \xi, \mathcal{J}_{t,g} \text{grad} \Phi_s(x) \rangle \mathcal{J}_{t,g} \text{grad} \Phi_s(x), \quad \forall \xi \in T_x M. \quad (\text{IV.2.2})$$

One should expect that $\hat{A}_{x,s}$ is close to $A_{x,s}$ if g is close to g_0 since $\hat{A}_{x,s}$ and $A_{x,s}$ on M (when the metric is g) are close to their counterparts on M_0 (when the metric is g_0) and $\hat{A}_{x,s} = A_{x,s}$ when $g = g_0$. If $x = P(\phi)$, we have a corresponding $\hat{A}_\phi : T_x M \rightarrow T_x M$ defined as

$$\hat{A}_\phi = \int_S \bar{\rho}_\phi(s) \hat{A}_{x,s} d\mu_x(s). \quad (\text{IV.2.3})$$

\hat{A}_ϕ is close to A_ϕ when g is close to g_0 . In particular, $\hat{A}_\phi = A_\phi$ when $g = g_0$.

IV.2.2: The construction of \hat{A}_ϕ when $\mathbb{K} = \mathbb{O}$ and $n = 2$

Since \mathbb{O} is not associative, the models for real, complex and quaternionic hyperbolic spaces do not work for the Cayley hyperbolic space. In particular, the remark after Corollary II.8 suggests that we cannot find any fiberwise linear map $\mathcal{J} : TM_0 \rightarrow TM_0$ such that for any $v \in TM_0$, the sectional curvature $K_{M_0}(v, \mathcal{J}(v)) = -4$. Such \mathcal{J} maps exist for complex and quaternionic hyperbolic spaces and was used in the constructions when $\mathbb{K} \neq \mathbb{O}$ (see Subsection IV.2.1). Hence we need a different way to construct \hat{A}_ϕ in the Cayley hyperbolic setting.

Recall that in Proposition II.7 we define the *Cayley line* of a vector $0 \neq v = (a, b) \in \mathbb{O}^2$ as

$$\text{Cay}(v) = \begin{cases} \mathbb{O} \cdot (1, a^{-1}b), & a \neq 0; \\ \mathbb{O} \cdot (0, 1), & a = 0. \end{cases}$$

Let $F_4^{-20} = \mathcal{KAN}$ be the Iwasawa decomposition of F_4^{-20} . Denoted by $v_{l,t} = (\delta_{1t}e_t, \delta_{2t}e_t)$ with

$l = 1, 2$, δ_{lm} the Kronecker delta, $0 \leq t \leq 7$ and $\{e_t\}_{0 \leq t \leq 7}$ the standard orthonormal basis for \mathbb{O} . Then we can construct \mathcal{AN} -invariant vector fields $\tilde{\xi}_{l,t,g_0}$ such that

$$\tilde{\xi}_{l,t,g_0}(x_0) = \Psi(v_{l,t}) \quad l = 1, 2 \text{ and } 0 \leq t \leq 7,$$

where Ψ is the same as $d\chi$ in Proposition II.7. Define $\Psi_{x,g_0} : \mathbb{O}^2 \rightarrow T_x M_0$ such that

$$\Psi_{x,g_0}(v_{l,t}) = \tilde{\xi}_{l,t,g_0}(x), \quad l = 1, 2 \text{ and } 0 \leq t \leq 7.$$

Denoted by

$$\mathcal{Cay}_{g_0}(\xi) = \Psi_{x,g_0}(\mathcal{Cay}(\Psi_{x,g_0}^{-1}(\xi))), \quad 0 \neq \xi \in T_x M_0$$

the Cayley line containing ξ . Then following the fact that the Cayley hyperbolic space is Riemannian symmetric, the same arguments in Corollary II.8 can be applied to all points in M_0 . Namely,

Corollary IV.14. *For any $x \in M_0$ and any non-zero $\xi, \xi' \in T_x M_0$.*

- (1). *If ξ, ξ' belong to the same Cayley line and $\xi \notin \mathbb{R}\xi'$, then the sectional curvature $K_{M_0}(\xi, \xi') = -4$;*
- (2). *If $\mathcal{Cay}_{g_0}(\xi) \perp \mathcal{Cay}_{g_0}(\xi')$, then the sectional curvature $K_{M_0}(\xi, \xi') = -1$.*

Therefore we have following the Cayley hyperbolic version of Lemma IV.11.

Lemma IV.15. *Let Φ be as in the previous section and $\mathbb{KH}^n = \mathbb{OH}^2$. If $g = g_0$, then we have the following:*

- (1). *We can compute the Hessian of $e^{2\Phi_s(x)}/2$:*

$$\text{Hess}_{g_0} \frac{1}{2} e^{2\Phi_s(x)} = e^{2\Phi_s(x)} [g_0 + g_0 (\pi_{\mathcal{Cay}_{g_0}(\xi)}(\cdot), \pi_{\mathcal{Cay}_{g_0}(\xi)}(\cdot))].$$

- (2). *The operator $A_{x,s}$ defined in (IV.1.5) has the following explicit formula:*

$$A_{x,s}(\xi) = \xi + \pi_{\mathcal{Cay}_{g_0}(\text{grad}\Phi_s(x))}(\xi), \quad \forall \xi \in T_x M, \quad (\text{IV.2.4})$$

where $\pi_{\mathcal{Cay}_{g_0}(\xi)}$ denotes the orthogonal projection onto $\mathcal{Cay}_{g_0}(\xi)$.

Proof. (1). See [CF03, page 36] and [Eber96, page 47].

- (2). Same as Lemma IV.11.

□

As in the statement of the above lemma, we heavily used the concept of Cayley lines similar to the way we used almost complex and almost quaternionic structure in complex and quaternionic hyperbolic spaces. Therefore, similar to the previous subsection, the construction of the operator \widehat{A}_ϕ approximating A_ϕ relies on a notion of Cayley lines for M .

We first apply a Gram-Schmidt process to $\{\tilde{\xi}_{l,t,g_0}\}$ with respect to the perturbed metric g and obtain $\{\tilde{\xi}_{l,t,g}\}$ as a collection of orthonormal vector fields on M which gives an orthonormal basis at every point in M . Define $\Psi_{x,g} : \mathbb{O}^2 \rightarrow T_x M$ such that

$$\Psi_{x,g}(v_{l,t}) = \tilde{\xi}_{l,t,g}(x) \quad l = 1, 2 \text{ and } 0 \leq t \leq 7.$$

For any $x \in M$ and $\xi \in T_x M$, we define the *Cayley line* containing ξ by

$$\text{Cay}_g(\xi) = \Psi_{x,g}(\text{Cay}(\Psi_{x,g}^{-1}(\xi))) \quad 0 \neq \xi \in T_x M.$$

Then we can mimic the case when $g = g_0$ and define a new operator $\widehat{A}_{x,s} : T_x M \rightarrow T_x M$ such that

$$\widehat{A}_{x,s}(\xi) = \xi + \pi_{\text{Cay}_g(\text{grad}\Phi_s(x))}(\xi), \quad \forall \xi \in T_x M, \quad (\text{IV.2.5})$$

where $\pi_{\text{Cay}_g(\text{grad}\Phi_s(x))}$ refers to the orthogonal projection onto $\text{Cay}_g(\text{grad}\Phi_s(x))$. For any $\phi \in P^{-1}(\mathcal{B}(R))$ and $x = P(\phi)$, we have a corresponding $\widehat{A}_\phi : T_x M \rightarrow T_x M$ defined as

$$\widehat{A}_\phi = \int_S \bar{\rho}_\phi(s) \widehat{A}_{x,s} d\mu_x(s). \quad (\text{IV.2.6})$$

In the case when $g = g_0$, $\widehat{A}_\phi = A_\phi$ and they are close when g is close to g_0 similar to the complex and quaternionic hyperbolic cases.

IV.2.3: E_ϕ and Jacobian inequalities

Notation IV.16. For any $\phi \in \mathcal{B}(R)$ fixed, we let $x = P(\phi)$. For simplicity we write $\xi_{l,t,g} = \tilde{\xi}_{l,t,g}(x)$ for all $1 \leq l \leq n$ and $0 \leq t \leq d-1$. Denoted by Ξ short for $\tilde{\Xi}(x)$. Since $\mathcal{J}_{t,g}$ preserves g , $\mathcal{J}_{t,g}\Xi$ is also an (ordered) orthonormal basis of $T_x M$.

We define the following vectors in $T_\phi \mathcal{L} = \mathcal{L}$:

$$X_{l,t,g}(s) = \langle \text{grad}\Phi_s(x), \xi_{l,t,g} \rangle = d\Phi_s(\xi_{l,t,g}), \quad 1 \leq l \leq n \text{ and } 0 \leq t \leq d-1.$$

They are linearly independent due to Ξ being a basis and the second requirement of G being a special Riemannian metric (see Definition III.8 and Definition IV.9). Let $\mathcal{V}_\phi := \text{span}\{X_{l,t,g} | 1 \leq$

$l \leq n, 0 \leq t \leq d-1$. We write

$$Z_\phi := \{X_{1,0,g}, \dots, X_{1,d-1,g}, \dots, X_{n,0,g}, \dots, X_{n,d-1,g}\} = d\Phi(\Xi),$$

which is a(n) (ordered) basis for \mathcal{V}_ϕ . Define $K_{x,g} \subset \text{SO}(T_x M)$ such that

$$K_{x,g} = \{\Psi_{x,g} \circ T \circ \Psi_{x,g}^{-1} | T \in \text{Spin}(9) \subset \text{SO}(\mathbb{O}^2) = \text{SO}(16)\},$$

where the inclusion of $\text{Spin}(9) \subset \text{SO}(16)$ is given by the third assertion in Corollary II.8.

We are now ready to take a closer look at \widehat{A}_ϕ and E_ϕ . Write

$$J_t = \Xi[\text{Id}]_{\mathcal{J}_{t,g}\Xi}, \quad \forall 1 \leq t \leq d-1$$

and

$$\widehat{Q}_\phi = \Xi[n(\widehat{A}_\phi - \text{Id})]_\Xi.$$

Then we have the following

Lemma IV.17. (1). *When $\mathbb{K} \neq \mathbb{O}$, we have the following formula for \widehat{A}_ϕ defined in (IV.2.3):*

$$\Xi[\widehat{A}_\phi]_\Xi = \text{Id} + \frac{1}{n}\widehat{Q}_\phi = \text{Id} + \frac{1}{dn} \sum_{t=0}^{d-1} J_t Q_\phi J_t^{-1},$$

where $Q_\phi = (Q_{ml,\phi})_{1 \leq m,l \leq n}$ is a $dn \times dn$ real symmetric matrix such that

$$Q_{ml,\phi} = \begin{pmatrix} \langle X_{m,0,g}, X_{l,0,g} \rangle_{G_\phi} & \cdots & \langle X_{m,0,g}, X_{l,d-1,g} \rangle_{G_\phi} \\ \vdots & \ddots & \vdots \\ \langle X_{m,d-1,g}, X_{l,0,g} \rangle_{G_\phi} & \cdots & \langle X_{m,d-1,g}, X_{l,d-1,g} \rangle_{G_\phi} \end{pmatrix} \quad (\text{IV.2.7})$$

for any $1 \leq m, l \leq n$;

(2). *When $\mathbb{K} = \mathbb{O}$ and $n = 2$, for any $0 \neq \xi \in T_x M$, the normalized inner product $\langle \widehat{A}_\phi(\xi), \xi \rangle / \|\xi\|^2$ is constant along $\text{Cay}_g(\xi)$. Equivalently, for any $O \in K_{x,g}$, the matrix $O^T \widehat{Q}_\phi O$ has the form*

$$O^T \widehat{Q}_\phi O = \begin{pmatrix} \lambda \text{Id} & L \\ L^T & \eta \text{Id} \end{pmatrix} \in \text{Mat}_{16 \times 16}(\mathbb{R}),$$

where $\lambda, \eta \in \mathbb{R}_{\geq 0}$ and every block in the above matrix has size 8×8 .

(3). *When $\mathbb{K} = \mathbb{O}$ and $n = 2$, for any $0 \neq \xi \in T_x M$, there exist orthogonal linear maps*

$I_{t,\xi} \in \text{SO}(\text{Cay}_g(\xi))$, $t = 0, 1, \dots, 7$, such that $I_{0,\xi} = \text{Id}$ and

$$\widehat{Q}_\phi|_{\text{Cay}_g(\xi)} = \frac{1}{8} \sum_{t=0}^7 I_{t,\xi} Q_\phi|_{\text{Cay}_g(\xi)} I_{t,\xi}^{-1}.$$

Equivalently, for any $O \in K_{x,g}$, write

$$O^T \widehat{Q}_\phi O = \begin{pmatrix} \widehat{Q}_{11,O,\phi} & \widehat{Q}_{12,O,\phi} \\ \widehat{Q}_{21,O,\phi} & \widehat{Q}_{22,O,\phi} \end{pmatrix} \text{ and } O^T Q_\phi O = \begin{pmatrix} Q_{11,O,\phi} & Q_{12,O,\phi} \\ Q_{21,O,\phi} & Q_{22,O,\phi} \end{pmatrix},$$

where every block in the above matrix has size 8×8 . Then there exist $I_{t,O,\phi} \in \text{SO}(8)$, $0 \leq t \leq 7$, such that $I_{0,O,\phi} = \text{Id}$ and

$$\widehat{Q}_{11,O,\phi} = \frac{1}{8} \sum_{t=0}^7 I_{t,O,\phi} Q_{11,O,\phi} I_{t,O,\phi}^{-1}.$$

Proof. (1). Since Ξ is an orthonormal basis for $T_x M$, it suffices to calculate elements of the form

$$\langle \widehat{A}_\phi(\xi), \eta \rangle, \quad \xi, \eta \in \Xi.$$

Notice that

$$\begin{aligned} \langle \widehat{A}_\phi(\xi_{l_1,t_1,g}), \xi_{l_2,t_2,g} \rangle &= \left\langle \int_S \bar{\rho}_\phi(s) \widehat{A}_{x,s}(\xi_{l_1,t_1,g}) d\mu_x(s), \xi_{l_2,t_2,g} \right\rangle \\ &= \int_S \bar{\rho}_\phi(s) \langle \xi_{l_1,t_1,g}, \xi_{l_2,t_2,g} \rangle d\mu_x(s) \\ &\quad + \sum_{t=0}^{d-1} \int_S \bar{\rho}_\phi(s) \langle \xi_{l_1,t_1,g}, \mathcal{J}_{t,g} \text{grad} \Phi_s(x) \rangle \langle \xi_{l_2,t_2,g}, \mathcal{J}_{t,g} \text{grad} \Phi_s(x) \rangle d\mu_x(s) \\ &= \int_S \bar{\rho}_\phi(s) \langle \xi_{l_1,t_1,g}, \xi_{l_2,t_2,g} \rangle d\mu_x(s) \\ &\quad + \sum_{t=0}^{d-1} \int_S \bar{\rho}_\phi(s) \langle \mathcal{J}_{t,g} \xi_{l_1,t_1,g}, \text{grad} \Phi_s(x) \rangle \langle \mathcal{J}_{t,g} \xi_{l_2,t_2,g}, \text{grad} \Phi_s(x) \rangle d\mu_x(s) \\ &= \delta_{l_1 l_2} \delta_{t_1 t_2} + \frac{1}{dn} \sum_{t=0}^{d-1} \langle d\Phi(\mathcal{J}_{t,g} \xi_{l_1,t_1,g}), d\Phi(\mathcal{J}_{t,g} \xi_{l_2,t_2,g}) \rangle_{G_\phi}. \end{aligned}$$

Therefore the first assertion follows.

(2). Recall the definition of \widehat{A}_ϕ in (IV.2.5) and (IV.2.6), it suffices to show that

$$\langle \xi, \pi_{\mathcal{Cay}_g(\eta)}(\xi) \rangle / \|\xi\|^2 \quad (\text{IV.2.8})$$

is constant along $\mathcal{Cay}_g(\xi)$ for any choice of $\xi, \eta \in T_x M$. By applying $K_{x,g}$ actions we can assume without loss of generality that $\xi = \Psi_{x,g}(1, 0)$ and $\eta = \Psi_{x,g}(b \cos \theta, ba \sin \theta)$, where $\theta \in \mathbb{R}$ and a, b are unit octonions. Then for any unit octonion c and any $\xi_c = \Psi_{x,g}(c, 0)$, we have

$$\pi_{\mathcal{Cay}_g(\eta)}(\xi_c) = \Psi_{x,g}(c \cos^2 \theta, ca \sin \theta \cos \theta).$$

Hence

$$\langle \xi_c, \pi_{\mathcal{Cay}_g(\eta)}(\xi_c) \rangle = \langle (c, 0), (c \cos^2 \theta, ca \sin \theta \cos \theta) \rangle = \cos^2 \theta,$$

which proves that (IV.2.8) is constant along $\mathcal{Cay}_g(\xi)$.

(3). The computations in the first assertion imply that the operator on $T_x M$ defined by

$$\xi \rightarrow \int_S \bar{\rho}_\phi(s) \pi_{\text{grad}\Phi_s(x)}(\xi) d\mu_x(s)$$

has matrix expression exactly equal to $Q_\phi/16$ under the basis Ξ , where $\pi_{\text{grad}\Phi_s(x)}$ is the orthogonal projection onto $\mathbb{R}\text{grad}\Phi_s(x)$. It suffices to show that for any $0 \neq \xi, \eta \in T_x M$, there exists some $I_{t,\xi,\phi} \in \text{SO}(\mathcal{Cay}_g(\xi))$ with $0 \leq t \leq 7$ such that

$$\langle \xi', \pi_{\mathcal{Cay}_g(\eta)}(\xi') \rangle = \sum_{t=0}^7 \langle I_{t,\xi,\phi}(\xi'), \pi_\eta(I_{t,\xi,\phi}(\xi')) \rangle, \quad \forall \xi' \in \mathcal{Cay}_g(\xi). \quad (\text{IV.2.9})$$

By applying $K_{x,g}$ actions we can assume without loss of generality that $\xi = \Psi_{x,g}(1, 0)$ and $\eta = \Psi_{x,g}(b \cos \theta, ba \sin \theta)$, where a, b are unit octonions. Let $e_0 = 1, e_1, \dots, e_7$ be the standard orthonormal basis for \mathbb{O} . Then

$$\{\Psi_{x,g}(\bar{e}_t b \cos \theta, (\bar{e}_t b) a \sin \theta) : 0 \leq t \leq 7\}$$

is an orthonormal basis for $\mathcal{Cay}_g(\eta)$. Hence

$$\langle \xi', \pi_{\mathcal{Cay}_g(\eta)}(\xi') \rangle = \sum_{t=0}^7 \langle \xi', \pi_{\Psi_{x,g}(\bar{e}_t b \cos \theta, (\bar{e}_t b) a \sin \theta)}(\xi') \rangle, \quad \forall \xi' \in \mathcal{Cay}_g(\xi).$$

Notice that any ξ' has the form $\Psi_{x,g}(c, 0)$ for some unit octonion c and

$$\begin{aligned} & \langle \xi', \pi_{\Psi_{x,g}(\bar{e}_t b \cos \theta, (\bar{e}_t b) a \sin \theta)} \xi' \rangle \\ &= \langle \Psi_{x,g}(c, 0), \Psi_{x,g}(\bar{e}_t b \cos \theta, (\bar{e}_t b) a \sin \theta) \rangle^2 \\ &= \langle c, \bar{e}_t b \cos \theta \rangle^2 \\ &= \langle e_t c, b \cos \theta \rangle^2 = \langle \Psi_{x,g}(e_t c, 0), \eta \rangle^2 = \langle \Psi_{x,g}(e_t \Psi_{x,g}^{-1}(\xi')), \pi_\eta(\Psi_{x,g}(e_t \Psi_{x,g}^{-1}(\xi'))) \rangle, \quad 0 \leq t \leq 7. \end{aligned}$$

Choose $I_{t,\xi}(\cdot) = \Psi_{x,g}(e_t \Psi_{x,g}^{-1}(\cdot))$ and (IV.2.9) follows. \square

Remark. A similar version of the second and the third assertions also hold for real, complex and quaternionic hyperbolic spaces. In particular, we can choose the orthogonal maps $I_{t,\xi}$ in the third assertion to be restrictions of $\mathcal{J}_{t,g}$ onto \mathbb{K} -lines when $\mathbb{K} \neq \mathbb{O}$. It is not hard to verify that the first assertion implies the counterparts for the second and third assertion in the complex and quaternionic case. However, due to the remark of Corollary II.8, there is no obvious way to find $\mathcal{J}_{t,g}$ due to non-associativity of octonionic multiplication, making the Cayley hyperbolic case more complicated.

In order to understand $E_\phi : T_\phi \mathcal{L} \rightarrow T_x M$, recall that $\mathcal{V}_\phi = \text{span} Z_\phi$. We first claim that

$$E_\phi|_{\mathcal{V}_\phi^\perp} = 0,$$

where \mathcal{V}_ϕ^\perp denotes the orthogonal complement in $T_\phi \mathcal{L} = \mathcal{L}$ with respect to G_ϕ . This is because for any $X \in \mathcal{V}_\phi^\perp$ and any $\xi_{l,t,g} \in \Xi$,

$$\begin{aligned} \langle E_\phi(X), \xi_{l,t,g} \rangle &= (dn + d) \int_S \bar{\rho}_\phi(s) X(s) \langle \text{grad} \Phi_s(x), \xi_{l,t,g} \rangle d\mu_x(s) \\ &= (dn + d) \int_S \bar{\rho}_\phi(s) X(s) X_{l,t,g}(s) d\mu_x(s) = \frac{n+1}{n} \langle X, X_{l,t,g} \rangle_{G_\phi} = 0. \end{aligned}$$

Therefore

$$E_\phi|_{\mathcal{V}_\phi^\perp} = 0$$

since Ξ is a basis for $T_x M$.

Now it suffices to understand $E_\phi|_{\mathcal{V}_\phi}$. Let W_ϕ be an ordered orthonormal basis for \mathcal{V}_ϕ . We have the following result.

Lemma IV.18.

$$\Xi[E_\phi|_{\mathcal{V}_\phi}]_{Z_\phi} = \frac{n+1}{n} Q_\phi$$

and

$$\Xi[E_\phi|_{\mathcal{V}_\phi}]_{W_\phi} = \frac{n+1}{n} w_\phi [\text{Id}]_{Z_\phi}^T,$$

where $w_\phi[\text{Id}]_{Z_\phi}^T$ denotes the transpose of $w_\phi[\text{Id}]_{Z_\phi}$.

Proof. Applying a change of basis, we have

$$\Xi[E_\phi|_{\mathcal{V}_\phi}]_{W_\phi} = \Xi[E_\phi|_{\mathcal{V}_\phi}]_{Z_\phi} \cdot z_\phi[\text{Id}]_{W_\phi}.$$

Since Ξ is orthonormal in $T_x M$, $\Xi[E_\phi|_{\mathcal{V}_\phi}]_{Z_\phi}$ has its entries in the form of $\langle E_\phi(X), \xi \rangle$, where $X \in Z_\phi$ and $\xi \in \Xi$. Notice that

$$\begin{aligned} \langle E_\phi(X_{l_1, t_1, g}), \xi_{l_2, t_2, g} \rangle &= (dn + d) \int_S \bar{\rho}_\phi(s) X_{l_1, t_1, g}(s) \langle \text{grad} \Phi_s(x), \xi_{l_2, t_2, g} \rangle d\mu_x(s) \\ &= (dn + d) \int_S \bar{\rho}_\phi(s) X_{l_1, t_1, g}(s) X_{l_2, t_2, g}(s) d\mu_x(s) = \frac{n+1}{n} \langle X_{l_1, t_1, g}, X_{l_2, t_2, g} \rangle_{G_\phi}. \end{aligned}$$

Therefore by (IV.2.7)

$$\Xi[E_\phi|_{\mathcal{V}_\phi}]_{Z_\phi} = \frac{n+1}{n} Q_\phi.$$

Since W_ϕ is orthonormal in \mathcal{V}_ϕ , we have

$$Q_\phi = \left(\begin{array}{ccc} \langle X_{m, 0, g}, X_{l, 0, g} \rangle_{G_\phi} & \dots & \langle X_{m, 0, g}, X_{l, d-1, g} \rangle_{G_\phi} \\ \dots & \dots & \dots \\ \langle X_{m, d-1, g}, X_{l, 0, g} \rangle_{G_\phi} & \dots & \langle X_{m, d-1, g}, X_{l, d-1, g} \rangle_{G_\phi} \end{array} \right)_{1 \leq m, l \leq n} = w_\phi[\text{Id}]_{Z_\phi}^T \cdot w_\phi[\text{Id}]_{Z_\phi}.$$

Therefore

$$\begin{aligned} \Xi[E_\phi|_{\mathcal{V}_\phi}]_{W_\phi} &= \Xi[E_\phi|_{\mathcal{V}_\phi}]_{Z_\phi} \cdot z_\phi[\text{Id}]_{W_\phi} \\ &= \frac{n+1}{n} w_\phi[\text{Id}]_{Z_\phi}^T \cdot w_\phi[\text{Id}]_{Z_\phi} \cdot z_\phi[\text{Id}]_{W_\phi} = \frac{n+1}{n} w_\phi[\text{Id}]_{Z_\phi}^T \cdot w_\phi[\text{Id}]_{W_\phi} = \frac{n+1}{n} w_\phi[\text{Id}]_{Z_\phi}^T. \end{aligned}$$

Remark. For simplicity we write $U_\phi = w_\phi[\text{Id}]_{Z_\phi}$ and hence $Q_\phi = U_\phi^T U_\phi$.

Notation IV.19. For any $m \in \mathbb{Z}_+$ and any matrix $H \in \text{Mat}_{m \times m}(\mathbb{R})$, we define its norm by

$$\|H\| = \left(\sup_{v \neq 0, v \in \text{Mat}_{m \times 1}(\mathbb{R}) = \mathbb{R}^m} \frac{v^T H^T H v}{v^T v} \right)^{\frac{1}{2}} = \sup_{\|v\|=1} \|Hv\|.$$

where $\|v\| = \sqrt{v^T v}$ denotes the Euclidean norm of a real vector $v \in \text{Mat}_{m \times 1}(\mathbb{R})$. In particular, H is $\|H\|$ -Lipschitz, regarded as an endomorphism on a finite dimensional Euclidean space (with a prescribed orthonormal basis).

Before estimating determinants, we first recall a linear algebra fact from [BCG95, B.2 Lemme].

Lemma IV.20 (Besson-Courtois-Gallot, [BCG95]). *The determinant function is log-concave on positive semi-definite matrices with real entries. As a corollary, let A_1, A_2, \dots, A_n be real positive definite $m \times m$ matrices such that $\det(A_1) = \det(A_2) = \dots = \det(A_n) = c > 0$. Then*

$$\det \left(\frac{1}{n} \sum_{j=1}^n A_j \right) \geq c$$

with equality holds only when $A_1 = A_2 = \dots = A_n$.

Proof. We will prove this lemma by induction on n . Clearly the lemma holds when $n = 1$. If the case $n - 1$ holds true, replacing A_i by $A_1^{-1/2} A_i A_1^{-1/2}$ for $1 \leq i \leq n$, WLOG we can assume that $A_1 = \text{Id}$ and hence $c = 1$. Let

$$B = \frac{1}{n-1} \sum_{j=2}^n A_j.$$

The conclusion in the case $n - 1$ implies that $\det B \geq 1$ and $\det B = 1$ only when $A_2 = A_3 = \dots = A_n$. Let $\lambda_1, \dots, \lambda_m > 0$ be eigenvalues of B . We have

$$\begin{aligned} & \det \left(\frac{1}{n} \sum_{j=1}^n A_j \right) \\ &= \det \left(\frac{1}{n} \text{Id} + \frac{n-1}{n} B \right) \\ &= \prod_{j=1}^m \left(\frac{1}{n} + \frac{n-1}{n} \lambda_j \right) \\ &= \sum_{j=0}^m \left[\left(\frac{1}{n} \right)^{m-j} \left(\frac{n-1}{n} \right)^j \sum_{1 \leq i_1 < i_2 < \dots < i_j \leq m} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_j} \right] \\ &\geq \sum_{j=0}^m \left[\left(\frac{1}{n} \right)^{m-j} \left(\frac{n-1}{n} \right)^j \binom{m}{j} \left(\prod_{i=1}^n \lambda_i \right)^{\binom{m-1}{j-1} / \binom{m}{j}} \right] \\ &= \sum_{j=0}^m \left[\left(\frac{1}{n} \right)^{m-j} \left(\frac{n-1}{n} \right)^j \binom{m}{j} (\det B)^{\frac{j}{m}} \right] \geq \sum_{j=0}^m \left(\frac{1}{n} \right)^{m-j} \left(\frac{n-1}{n} \right)^j \binom{m}{j} = 1, \end{aligned}$$

in which the first inequality is due to the inequality between arithmetic mean and geometric mean. Equality holds only when $\det B = 1$ and all λ_i are the same. Therefore $B = \text{Id}$ and hence $\text{Id} = A_1 = A_2 = \dots = A_n = B$. This proves the inductive step and the lemma follows from induction on n . \square

Lemma IV.21. (1).

$$\det \left(\text{Id} + \frac{1}{n} \widehat{Q}_\phi \right) \geq \det \left(\text{Id} + \frac{1}{n} U_\phi^T U_\phi \right);$$

(2). The trace of \widehat{Q}_ϕ and $U_\phi^T U_\phi$ are both equal to dn . Moreover, let $0 \leq \eta_1 \leq \dots \leq \eta_{dn}$ be eigenvalues of \widehat{Q}_ϕ and $0 \leq \lambda_1 \leq \dots \leq \lambda_{dn}$ be eigenvalues of $U_\phi^T U_\phi$, then

$$\lambda_1 \leq \eta_1 \leq 1 \leq \eta_{dn} \leq \lambda_{dn} \leq dn. \quad (\text{IV.2.10})$$

(3).

$$\det \left(\text{Id} + \frac{1}{n} U_\phi^T U_\phi \right) \geq \left(\frac{n+1}{n} \right)^{dn} \det(U_\phi^T U_\phi)^{\frac{1}{n+1}}$$

and

$$\det(U_\phi^T U_\phi) \leq \text{Id}.$$

Equality in both inequalities holds if and only if $U_\phi^T U_\phi = \text{Id}$.

(4). $\|U_\phi\|, \|U_\phi^T\| \leq \sqrt{dn}$.

Proof. (1). When $\mathbb{K} \neq \mathbb{O}$, this follows from Lemma IV.20 and the first assertion in Lemma IV.17. It remains to prove for the Cayley hyperbolic case. Let ξ be a unit eigenvector of \widehat{A}_ϕ having the largest eigenvalue denoted by η_{\max} . Since the second assertion in Lemma IV.17 implies that $\langle \xi, \widehat{A}_\phi(\xi) \rangle / \|\xi\|^2$ is constant on $\mathcal{Cay}_g(\xi)$, all non-zero vectors in $\mathcal{Cay}_g(\xi)$ are eigenvectors of \widehat{A}_ϕ with eigenvalue η_{\max} . Let $\mathcal{Cay}_g(\xi')$ be the orthogonal complement of $\mathcal{Cay}_g(\xi)$. Since \widehat{A}_ϕ is self-adjoint, $\mathcal{Cay}_g(\xi')$ is an invariant subspace of \widehat{A}_ϕ and non-zero vectors in this subspace are all eigenvectors of \widehat{A}_ϕ for some common eigenvalue η_{\min} due to the second assertion in Lemma IV.17. Therefore there exists some $O \in K_{x,g} \subset \text{SO}(T_x M)$ such that

$$O^T \left(\text{Id} + \frac{1}{n} \widehat{Q}_\phi \right) O = \begin{pmatrix} \eta_{\max} \text{Id} & 0 \\ 0 & \eta_{\min} \text{Id} \end{pmatrix},$$

where every block in the above matrix has size 8×8 . The assertion then follows from Lemma IV.20, the third assertion in Lemma IV.17 and the fact that for any positive semi-definite matrix $\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ with $A, B, C \in \text{Mat}_{m \times m}(\mathbb{R})$, we have

$$\det \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \leq \det(A) \det(C).$$

(2). Notice that

$$\begin{aligned}
\mathrm{tr}(U_\phi^T U_\phi) &= \sum_{l=1}^n \sum_{t=0}^{d-1} \langle X_{l,t,g}, X_{l,t,g} \rangle_{G_\phi} \\
&= \sum_{l=1}^n \sum_{t=0}^{d-1} dn \int_S \bar{\rho}_\phi(s) \langle \mathrm{grad} \Phi_s, \xi_{l,t,g} \rangle^2 d\mu_x(s) \\
&= dn \int_S \bar{\rho}_\phi(s) \langle \mathrm{grad} \Phi_s, \mathrm{grad} \Phi_s \rangle^2 d\mu_x(s) = dn \int_S \bar{\rho}_\phi(s) d\mu_x(s) = dn.
\end{aligned}$$

Therefore the trace of \widehat{Q}_ϕ and $U_\phi^T U_\phi$ are both equal to dn , which follows from Lemma IV.17. The fact that $\lambda_1 \leq \eta_1 \leq \eta_{dn} \leq \lambda_{dn}$ follows directly from Lemma IV.17 and two linear algebra facts listed below:

- (a). Let A be a positive semi-definite matrix and O_j be orthogonal matrices with $1 \leq j \leq m$. Denoted by k_{\max}, k_{\min} the largest and the smallest eigenvalues of A . Let l_{\max}, l_{\min} be the largest and the smallest eigenvalues of $\left(\sum_{j=1}^m O_j A O_j^T\right) / m$. Then $k_{\min} \leq l_{\min} \leq l_{\max} \leq k_{\max}$. (This is used when $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or \mathbb{O} .)
- (b). Let $\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ be a positive semi-definite matrix with $A, B, C \in \mathrm{Mat}_{m \times m}(R)$. Let k_{\max}, k_{\min} be the largest and the smallest eigenvalues of $\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ respectively. Denoted by l_{\max}, l_{\min} the largest and the smallest eigenvalues of $\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}$ respectively. Then $k_{\min} \leq l_{\min} \leq l_{\max} \leq k_{\max}$. (This is used when $\mathbb{K} = \mathbb{O}$.)

(3). Following the notation in the second assertion, we have

$$\sum_{t=1}^{dn} \lambda_t = dn. \tag{IV.2.11}$$

Therefore by Lemma IV.20

$$\begin{aligned}
\det \left(\text{Id} + \frac{1}{n} U_\phi^T U_\phi \right) &= \prod_{l=1}^{dn} \left(1 + \frac{1}{n} \lambda_l \right) \\
&= \prod_{l=1}^{dn} \frac{n+1}{n} \cdot \frac{n \cdot 1 + \lambda_l}{n+1} \\
&\geq \left(\frac{n+1}{n} \right)^{dn} \prod_{l=1}^{dn} (1^n \cdot \lambda_l)^{\frac{1}{n+1}} \\
&= \left(\frac{n+1}{n} \right)^{dn} \left(\prod_{l=1}^{dn} \lambda_l \right)^{\frac{1}{n+1}} = \left(\frac{n+1}{n} \right)^{dn} \det(U_\phi^T U_\phi)^{\frac{1}{n+1}}.
\end{aligned}$$

The inequality $\det(U_\phi^T U_\phi) \leq \text{Id}$ follows directly by applying the arithmetic mean-geometric mean inequality to (IV.2.11). Equal signs in both inequalities are achieved if and only if all $\lambda_l = 1$. In other words, $U_\phi^T U_\phi = \text{Id}$.

- (4). By (IV.2.11), $\|U_\phi^T U_\phi\| \leq dn$. Hence $\|(U_\phi^T U_\phi)^{\frac{1}{2}}\| \leq \sqrt{dn}$. Since $U_\phi = K(U_\phi^T U_\phi)^{\frac{1}{2}}$ for some $K \in O(dn)$. Therefore, by the fact that any element in $O(dn)$ has norm 1, $\|U_\phi\|, \|U_\phi^T\| \leq \sqrt{dn}$. \square

Corollary IV.22. *The maps E_ϕ and $\widehat{A}_\phi^{-1} \circ E_\phi : T_\phi \mathcal{L} \rightarrow T_x M$ is $(n+1)\sqrt{d/n}$ -Lipschitz and the latter one is volume non-increasing. In particular, when $g = g_0$, P is $(n+1)\sqrt{d/n}$ -Lipschitz and volume non-increasing and the first two assertion in Proposition III.4 holds when $g = g_0$.*

Proof. Since $\text{Id} + \frac{1}{n} \widehat{Q}_\phi \geq \text{Id}$, by Lemma IV.17 \widehat{A}_ϕ^{-1} is 1-Lipschitz. It follows from Lemma IV.18 and the second assertion in Lemma IV.21 that E_ϕ is $(n+1)\sqrt{d/n}$ -Lipschitz. Hence their composition $\widehat{A}_\phi^{-1} \circ E_\phi : T_\phi \mathcal{L} \rightarrow T_x M$ is $(n+1)\sqrt{d/n}$ -Lipschitz.

It remains to show that

$$\text{Jac}_G(\widehat{A}_\phi^{-1} \circ E_\phi) = \left| \det \widehat{A}_\phi^{-1} \right| J_G E_\phi \leq 1.$$

Since $E_\phi|_{\mathcal{V}_\phi^\perp} = 0$, we have $E_\phi = E_\phi|_{\mathcal{V}_\phi} \circ P_{\mathcal{V}_\phi}$, where $P_{\mathcal{V}_\phi} : \mathcal{L} \rightarrow \mathcal{V}_\phi$ is the orthogonal projection onto \mathcal{V}_ϕ . Hence

$$\text{Jac}_G E_\phi = \text{Jac}_{G, \mathcal{V}_\phi} E_\phi \cdot \text{Jac}_G P_{\mathcal{V}_\phi}.$$

We claim that $\text{Jac}_G P_{\mathcal{V}_\phi} = 1$. This is because on the one hand, $\text{Jac}_{G, \mathcal{V}_\phi} P_{\mathcal{V}_\phi} = \det \text{Id}_{\mathcal{V}_\phi} = 1$. On the other hand, for any arbitrary dn -dimensional subspace $\mathcal{W} \subset \mathcal{L}$ with an orthonormal basis

$\{\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_{dn}\}$, we have the following

$$\text{Jac}_{G, \mathcal{W}} P_{\mathcal{V}_\phi} = \text{Jac}_G(P_{\mathcal{V}_\phi} |_{\mathcal{W}}) \leq \left(\frac{1}{dn} \sum_{i=1}^{dn} \|P_{\mathcal{V}_\phi}(\tilde{X}_i)\|_{G_\phi}^2 \right)^{\frac{dn}{2}} \leq \left(\frac{1}{dn} \sum_{i=1}^{dn} \|\tilde{X}_i\|_{G_\phi}^2 \right)^{\frac{dn}{2}} = \sqrt{1^{dn}} = 1. \quad (\text{IV.2.12})$$

The second inequality sign follows from the fact that $P_{\mathcal{V}_\phi}$ is 1-Lipschitz and the first inequality sign follows from Lemma III.3. Therefore

$$\text{Jac}_G E_\phi = \text{Jac}_{G, \mathcal{V}_\phi} E_\phi = \left| \det \frac{n+1}{n} U_\phi^T \right| = \left(\frac{n+1}{n} \right)^{dn} (\det U_\phi^T U_\phi)^{\frac{1}{2}}.$$

On the other hand

$$|\det \hat{A}_\phi^{-1}| = \left| \det \Xi[\hat{A}_\phi]\Xi \right|^{-1} = \det \left(\text{Id} + \frac{1}{n} \hat{Q}_\phi \right)^{-1}.$$

Hence by the first assertion in Lemma IV.21,

$$\begin{aligned} \text{Jac}_G(\hat{A}_\phi^{-1} \circ E_\phi) &= \frac{\left(\frac{n+1}{n}\right)^{dn} (\det U_\phi^T U_\phi)^{\frac{1}{2}}}{\det \left(\text{Id} + \frac{1}{n} \hat{Q}_\phi \right)} \\ &\leq \frac{\left(\frac{n+1}{n}\right)^{dn} (\det U_\phi^T U_\phi)^{\frac{1}{2}}}{\left(\frac{n+1}{n}\right)^{dn} (\det U_\phi^T U_\phi)^{\frac{1}{n+1}}} = (\det U_\phi^T U_\phi)^{\frac{1}{2} - \frac{1}{n+1}} \leq 1^{\frac{1}{2} - \frac{1}{n+1}} = 1. \end{aligned} \quad (\text{IV.2.13})$$

In particular, when $g = g_0$, $A_\phi = \hat{A}_\phi$. Hence the above result implies that $\text{Jac}_G P \leq 1$. \square

IV.3: Estimating the correction factor

In the previous section, we introduced an operator \hat{A}_ϕ for any $\phi \in P^{-1}(B_{x_0}(R_1))$ which has an explicit formula and coincides with A_ϕ when $g = g_0$. We will first estimate the difference of these two operators.

Notation IV.23. Denoted by $\epsilon = \epsilon(g, r)$ the C^r -norm of $g - g_0$.

Let $x = P(\phi)$. Define

$$\phi_t(s) = \Phi_s(x) - \frac{1}{dn+d} \ln \left(1 - t + t e^{(dn+d)(\Phi_s(x) - \phi(s))} \right).$$

Then

$$e^{(dn+d)(\Phi_s(x) - \phi_t(s))} = 1 - t + t \rho_\phi(s) = 1 + t(\rho_\phi(s) - 1).$$

Lemma IV.24. For any $\phi \in \mathcal{B}(R)$, we have the following facts

- (1). $P(\phi_t) = P(\phi) = x$ for any ϕ_t in the domain of P ;
- (2). $A_{\phi_0} = \widehat{A}_{\phi_0} = \frac{n+1}{n}\text{Id}$;
- (3). $\frac{d}{dt}\big|_{t=0} \det A_{\phi_t} = \frac{d}{dt}\big|_{t=0} \det \widehat{A}_{\phi_t} = 0$.

Proof. (1). Let $x = P(\phi)$. Notice that

$$\begin{aligned} \Omega_{\phi_t, g}(x) &= \int_S e^{(dn+d)(\Phi_s(x) - \phi_t(s))} d\Phi_s(x) d\mu_x(s) \\ &= \int_S (1 - t + t\rho_\phi(s)) d\Phi_s(x) d\mu_x(s) = \int_S t\rho_\phi(s) d\Phi_s(x) d\mu_x(s) = t\Omega_{\phi, g}(x) = 0. \end{aligned}$$

Hence $P(\phi_t) = P(\phi) = x$ by Definition IV.5. In particular,

$$\rho_{\phi_t}(s) = 1 - t + t\rho_\phi(s) = 1 + t(\rho_\phi(s) - 1).$$

- (2). Since $\phi_0 = \Phi_s(x)$, we have $\rho_{\phi_0} \equiv 1$, which implies that $U_{\phi_0}^T U_{\phi_0} = \text{Id}$ given the formula (IV.2.7). Therefore Lemma IV.17 and Lemma IV.21 imply that $\widehat{A}_{\phi_0} = \frac{n+1}{n}\text{Id}$. Since Lemma IV.18 suggests that $E_{\phi_0}|_{\mathcal{V}_{\phi_0}}$ is surjective and $d_x\Phi(\Xi) = Z_{\phi_0}$ by their definition in the previous section (equal in an order preserving way), it suffices to show that

$$\text{Id} = \frac{n}{n+1} E_{\phi_0} \circ d_x\Phi,$$

given that $\text{Id} = d_{\phi_0}P \circ d_x\Phi = A_{\phi_0}^{-1} \circ E_{\phi_0} \circ d_x\Phi$ from Definition III.11, Lemma IV.10 and Lemma IV.8.

By Lemma IV.18 we have

$$\Xi[E_{\phi_0}]_{Z_{\phi_0}} = \frac{n+1}{n} U_{\phi_0}^T U_{\phi_0} = \frac{n+1}{n} \text{Id}.$$

Therefore $\frac{n}{n+1} E_{\phi_0}(Z_{\phi_0}) = \Xi$ (equal in an order-preserving way) and

$$\frac{n}{n+1} E_{\phi_0} \circ d_x\Phi(\Xi) = \frac{n}{n+1} E_{\phi_0}(Z_{\phi_0}) = \Xi$$

(equal in an order preserving way), which implies that

$$\text{Id} = \frac{n}{n+1} E_{\phi_0} \circ d_x\Phi.$$

(3). Let ψ be an arbitrary element in $\mathcal{B}(R)$. We assume that \widehat{Q}_ψ has eigenvalues $0 \leq \eta_1 \leq \eta_2 \leq \dots \leq \eta_{dn}$. Lemma IV.21 shows that

$$\sum_{s=1}^{dn} \eta_s = dn.$$

Hence

$$\det \widehat{A}_\psi = \det \left(\text{Id} + \frac{1}{n} \widehat{Q}_\psi \right) = \prod_{t=1}^{dn} \left(1 + \frac{1}{n} \eta_t \right) \leq \left[\frac{1}{dn} \sum_{t=1}^{dn} \left(1 + \frac{1}{n} \eta_t \right) \right]^{dn} = \left(\frac{n+1}{n} \right)^{dn}.$$

Lemma IV.21 implies that the equality holds when $U_\psi^T U_\psi = \text{Id}$. In particular

$$\det \widehat{A}_{\phi_0} = \left(\frac{n+1}{n} \right)^{dn}.$$

Therefore

$$\left. \frac{d}{dt} \right|_{t=0} \det \widehat{A}_{\phi_t} = 0.$$

Notice that $d_{\phi_t} P = A_{\phi_t}^{-1} \circ E_{\phi_t}$. We have

$$\det A_{\phi_t} = \frac{\text{Jac}_G E_{\phi_t}}{\text{Jac}_G P(\phi_t)}.$$

Lemma IV.18 and the third assertion in Lemma IV.21 imply that $\text{Jac}_G E_{\phi_t}$ attains maximum at $t = 0$. On the other hand, for any $\xi \in T_x M$,

$$\begin{aligned} & \left\langle \left. \frac{d}{dt} \right|_{t=0} \phi_t(s), \langle \text{grad} \Phi_s(x), \xi \rangle \right\rangle_{G_{\Phi(x)}} \\ &= \frac{-dn}{dn+d} \int_S (\rho_\phi(s) - 1) \langle \text{grad} \Phi_s(x), \xi \rangle d\mu_x(s) \\ &= \frac{-dn}{dn+d} [\Omega_{\phi,g}(x)(\xi) - \Omega_{\Phi_s(x),g}(x)(\xi)] \\ &= \frac{-dn}{dn+d} [\Omega_{\phi,g}(P(\phi))(\xi) - \Omega_{\Phi_s(x),g}(P(\Phi_s(x)))(\xi)] = 0, \end{aligned}$$

where the last three equal signs follow from Definition IV.5 and the first assertion in Lemma IV.6. Therefore $\left. \frac{d}{dt} \right|_{t=0} \phi_t(s) \in T_{\Phi_s(x)} \mathcal{L}$ is perpendicular to $T_x \Phi = \Phi_*(T_x M)$. By Proposi-

tion III.12 and Lemma IV.10, $t = 0$ is a critical point for $\text{Jac}_G P(\phi_t)$. Hence

$$\left. \frac{d}{dt} \right|_{t=0} \det A_{\phi_t} = 0. \quad \square$$

Proposition IV.25. *There exist positive constants r and $C_0 = C_0(n, R)$ such that for any g sufficiently close to g_0 (without loss of generality, assume $\epsilon < 1$) and every $\phi \in \mathcal{B}(R)$ one has*

$$\|A_\phi - \widehat{A}_\phi\| \leq C_0 \epsilon \|\phi - \Phi(P(\phi))\|_{L^2(S)}$$

and

$$|\det A_\phi - \det \widehat{A}_\phi| \leq C_0 \epsilon \|\phi - \Phi(P(\phi))\|_{L^2(S)}^2.$$

Proof. Let

$$b(t) = \int_S \rho_{\phi_t}(s) d\mu_x(s).$$

Notice that

$$\begin{aligned} A_{\phi_t} - \widehat{A}_{\phi_t} &= \frac{1}{b(t)} \int_S \rho_{\phi_t}(s) (A_{x,s} - \widehat{A}_{x,s}) d\mu_x(s) \\ &= \frac{1}{b(t)} \int_S (A_{x,s} - \widehat{A}_{x,s}) d\mu_x(s) + \frac{t}{b(t)} \int_S (\rho_\phi(s) - 1) (A_{x,s} - \widehat{A}_{x,s}) d\mu_x(s) \\ &= \frac{t}{b(t)} \int_S (\rho_\phi(s) - 1) (A_{x,s} - \widehat{A}_{x,s}) d\mu_x(s). \end{aligned} \quad (\text{IV.3.1})$$

The last equality follows from the first two assertions in Lemma IV.24 and the fact that

$$\frac{1}{b(t)} \int_S (A_{x,s} - \widehat{A}_{x,s}) d\mu_x(s) = \frac{1}{b(t)} \int_S \bar{\rho}_{\phi_0} (A_{x,s} - \widehat{A}_{x,s}) d\mu_x(s) = \frac{1}{b(t)} (A_{\phi_0} - \widehat{A}_{\phi_0}) = 0.$$

Since $\phi \in \mathcal{B}(R)$, Lemma IV.6 implies $\Phi(x) = \Phi(P(\phi)) \in \mathcal{B}(R_1)$. Hence

$$c_0^{-1} \leq \rho_\phi(s) = e^{(dn+d)(\Phi_s(x) - \phi(s))} \leq c_0 \quad (\text{IV.3.2})$$

and

$$\|\rho_\phi - 1\|_{L^2(S)} = \|e^{(dn+d)(\Phi_s(x) - \phi(s))} - 1\|_{L^2(S)} \leq c_0 \|\Phi_s(x) - \phi(s)\|_{L^2(S)},$$

where

$$c_0 = c_0(n, R) = e^{(dn+d)(R+R_1)}(dn + d).$$

Since $x \in B_{x_0}(R_1)$, $A_{x,s}$ and $\widehat{A}_{x,s}$ depend smoothly on g and they coincide when $g = g_0$, we have

$$\|A_{x,s} - \widehat{A}_{x,s}\|_{L^2(S)} \leq c_1 \epsilon$$

for some $c_1 = c_1(n, R)$. Therefore by the Cauchy-Schwarz inequality, (IV.3.1) implies

$$\begin{aligned} \|A_{\phi_t} - \widehat{A}_{\phi_t}\| &= \frac{t}{b(t)} \left\| \int_S (\rho_\phi(s) - 1)(A_{x,s} - \widehat{A}_{x,s}) d\mu_x(s) \right\| \\ &\leq \frac{t}{b(t)} \|\rho_\phi - 1\|_{L^2(S, d\mu_x)} \cdot \|A_{x,s} - \widehat{A}_{x,s}\|_{L^2(S, d\mu_x)}. \end{aligned}$$

Choose $R_0 = 1 + R_1$ in (IV.1.2) and Lemma IV.6 implies that

$$\frac{1}{2} e^{-(dn+d-2)R_1} \leq |\lambda(x, s)| \leq 2e^{(dn+d-2)R_1} \quad (\text{IV.3.3})$$

for any $\phi \in \mathcal{B}(R)$ and $x = P(\phi) \in B_{x_0}(R_1)$. Hence by (IV.3.2) and (IV.3.3) we have

$$\begin{aligned} &\frac{t}{b(t)} \|\rho_\phi - 1\|_{L^2(S, d\mu_x)} \|A_{x,s} - \widehat{A}_{x,s}\|_{L^2(S, d\mu_x)} \\ &\leq 2e^{(dn+d-2)R_1} \frac{t}{b(t)} \|(\rho_\phi - 1)\|_{L^2(S)} \|A_{x,s} - \widehat{A}_{x,s}\|_{L^2(S)} \\ &\leq 2te^{(dn+d-2)R_1} c_0 \|(\rho_\phi - 1)\|_{L^2(S)} \|A_{x,s} - \widehat{A}_{x,s}\|_{L^2(S)} \\ &\leq 2te^{(dn+d-2)R_1} c_0^2 c_1 \epsilon \|\Phi_s(x) - \phi(s)\|_{L^2(S)}. \end{aligned}$$

Summarizing up, we have

$$\|A_{\phi_t} - \widehat{A}_{\phi_t}\| \leq 2te^{(dn+d-2)R_1} c_0^2 c_1 \epsilon \|\Phi_s(x) - \phi(s)\|_{L^2(S)}.$$

In particular, when $t = 1$, we have

$$\|A_\phi - \widehat{A}_\phi\| \leq 2e^{(dn+d-2)R_1} c_0^2 c_1 \epsilon \|\Phi_s(x) - \phi(s)\|_{L^2(S)} \quad (\text{IV.3.4})$$

In order to estimate the difference in determinants, we first introduce a linear algebra lemma.

Lemma IV.26. *Let H_1, H_2 be $n \times n$ trace-free matrices such that $\|H_1\|, \|H_2\| \leq R$. Then we have*

$$|\det(\text{Id} + H_1 + H_2) - \det(\text{Id} + H_1)| \leq \mathcal{K}(n, R) \|H_2\| (\|H_1\| + \|H_2\|),$$

for some $\mathcal{K}(n, R) > 0$.

Proof of Lemma IV.26. Let $H_1 = (a_{ij})_{1 \leq i, j \leq n}$ and $H_2 = (b_{ij})_{1 \leq i, j \leq n}$. Then we have

$$|a_{ij}| \leq \|H_1\| \leq R \quad \text{and} \quad |b_{ij}| \leq \|H_2\| \leq R, \quad 1 \leq i, j \leq n.$$

Direct computation shows that $|\det(\text{Id} + H_1 + H_2) - \det(\text{Id} + H_1)|$ can be written as the sum of

at most 3^n monomials in a_{ij} and b_{kl} with coefficients ± 1 . These monomials have degree at least 2 due to trace-free assumptions and always have positive degrees with respect to some b_{kl} . More precisely,

$$\begin{aligned}
& |\det(\text{Id} + H_1 + H_2) - \det(\text{Id} + H_1)| \\
&= \left| \sum_{\sigma \in S_n} (-1)^{\text{sgn}(\sigma)} \left[\prod_{i=1}^n (\delta_{i\sigma(i)} + a_{i\sigma(i)} + b_{i\sigma(i)}) - \prod_{i=1}^n (\delta_{i\sigma(i)} + a_{i\sigma(i)}) \right] \right| \\
&= \left| \sum_{\sigma \in S_n, \text{Fix}(\sigma)^c \subset \Omega \subset S_n, |\Omega| \geq 2 \text{ if } \sigma = \text{Id}} (-1)^{\text{sgn}(\sigma)} \left(\prod_{i \in \Omega} (a_{i\sigma(i)} + b_{i\sigma(i)}) - \prod_{i \in \Omega} a_{i\sigma(i)} \right) \right| \\
&\leq \sum_{\sigma \in S_n, \text{Fix}(\sigma)^c \subset \Omega \subset S_n, |\Omega| \geq 2 \text{ if } \sigma = \text{Id}} \left| \prod_{i \in \Omega} (a_{i\sigma(i)} + b_{i\sigma(i)}) - \prod_{i \in \Omega} a_{i\sigma(i)} \right| \\
&\leq \sum_{\sigma \in S_n, \text{Fix}(\sigma)^c \subset \Omega \subset S_n, |\Omega| \geq 2 \text{ if } \sigma = \text{Id}} (2^{|\Omega|} - 1) \|H_2\| (\|H_1\| + \|H_2\|)^{|\Omega|-1} \\
&\leq \sum_{\sigma \in S_n, \text{Fix}(\sigma)^c \subset \Omega \subset S_n} (2^n - 1) (2R + 1)^{n-2} \|H_2\| (\|H_1\| + \|H_2\|) \\
&\leq |S_n| 2^n (2^n - 1) (2R + 1)^{n-2} \|H_2\| (\|H_1\| + \|H_2\|),
\end{aligned}$$

where S_n is the group of symmetry over $\{1, 2, 3, \dots, n\}$. Choose $\mathcal{K}(n, R) = |S_n| 2^n (2^n - 1) (2R + 1)^{n-2}$ and the lemma follows. \square

Back to the proof of Proposition IV.25, notice that Lemma IV.21 implies that

$$\text{tr} \left(\widehat{A}_\psi - \frac{n+1}{n} \text{Id} \right) = 0,$$

for any $\psi \in \mathcal{B}(R)$.

Lemma IV.24 and the above fact suggest that

$$\begin{aligned}
0 &= \text{tr} \frac{d}{dt} \Big|_{t=0} \left(A_{\phi_t} - \frac{n+1}{n} \text{Id} \right) \\
&= \text{tr} \frac{d}{dt} \Big|_{t=0} \left(\widehat{A}_{\phi_t} - \frac{n+1}{n} \text{Id} \right) + \text{tr} \frac{1}{b(0)} \int_S (\rho_\phi(s) - 1) (A_{x,s} - \widehat{A}_{x,s}) d\mu_x(s) \\
&= \text{tr} \frac{1}{b(0)} \int_S (\rho_\phi(s) - 1) (A_{x,s} - \widehat{A}_{x,s}) d\mu_x(s).
\end{aligned}$$

Let

$$H_1 = \widehat{A}_\phi - \frac{n+1}{n} \text{Id}$$

and

$$H_2 = A_\phi - \widehat{A}_\phi = A_\phi - \widehat{A}_\phi - \frac{1}{b(1)}(A_{\phi_0} - \widehat{A}_{\phi_0}) = \frac{1}{b(1)} \int_S (\rho_\phi(s) - 1)(A_{x,s} - \widehat{A}_{x,s}) d\mu_x(s).$$

In order to estimate $\|H_1\|$, we first recall its matrix under the orthonormal basis $\Xi \subset T_x M$.

$$\Xi[H_1]\Xi = \frac{1}{n} \left(\widehat{Q}_\phi - \text{Id} \right).$$

The second assertion of Lemma IV.21 implies that

$$\|H_1\| \leq \left\| \frac{1}{n} (U_\phi^T U_\phi - \text{Id}) \right\|$$

Hence it suffices to consider components in $\frac{1}{dn} (U_\phi^T U_\phi - \text{Id})$. Notice that

$$\begin{aligned} & \frac{1}{dn} \left(\langle X_{l_1, t_1, g}, X_{l_2, t_2, g} \rangle_{G_\phi} - \delta_{l_1 l_2} \delta_{t_1 t_2} \right) \\ &= \int_S (\bar{\rho}_\phi(s) - 1) X_{l_1, t_1, g}(s) X_{l_2, t_2, g}(s) d\mu_x(s) \\ &= \frac{1}{\int_S \rho_\phi(s) d\mu_x(s)} \int_S (\rho_\phi(s) - 1) X_{l_1, t_1, g}(s) X_{l_2, t_2, g}(s) d\mu_x(s) \\ &\quad - \frac{\int_S (\rho_\phi(s) - 1) d\mu_x(s)}{\int_S \rho_\phi(s) d\mu_x(s)} \int_S X_{l_1, t_1, g}(s) X_{l_2, t_2, g}(s) d\mu_x(s). \end{aligned}$$

Therefore (IV.3.2) and (IV.3.3) imply that

$$\begin{aligned} & \left| \frac{1}{dn} \left(\langle X_{l_1, t_1, g}, X_{l_2, t_2, g} \rangle_{G_\phi} - \delta_{l_1 l_2} \delta_{t_1 t_2} \right) \right| \\ & \leq \left| \frac{1}{\int_S \rho_\phi(s) d\mu_x(s)} \int_S (\rho_\phi(s) - 1) X_{l_1, t_1, g}(s) X_{l_2, t_2, g}(s) d\mu_x(s) \right| \\ & \quad + \left| \frac{\int_S (\rho_\phi(s) - 1) d\mu_x(s)}{\int_S \rho_\phi(s) d\mu_x(s)} \int_S X_{l_1, t_1, g}(s) X_{l_2, t_2, g}(s) d\mu_x(s) \right| \\ & \leq c_0 \left| \int_S (\rho_\phi(s) - 1) X_{l_1, t_1, g}(s) X_{l_2, t_2, g}(s) d\mu_x(s) \right| \\ & \quad + c_0 \left| \left(\int_S (\rho_\phi(s) - 1) d\mu_x(s) \right) \int_S X_{l_1, t_1, g}(s) X_{l_2, t_2, g}(s) d\mu_x(s) \right| \\ & \leq 2c_0 \|(\rho_\phi - 1)\|_{L^1(S, d\mu_x)} \leq 4c_0 e^{(dn+d-2)R_1} \|(\rho_\phi - 1)\|_{L^1(S)} \leq 4c_0 e^{(dn+d-2)R_1} \|(\rho_\phi - 1)\|_{L^2(S)} \end{aligned}$$

for any $1 \leq l_1, l_2 \leq n$ and $0 \leq t_1, t_2 \leq d - 1$. Hence by (IV.3.2) we have

$$\begin{aligned} \|H_1\| &\leq \left\| \frac{1}{n} (U_\phi^T U_\phi - \text{Id}) \right\| \\ &\leq dn \sup_{1 \leq l_1, l_2 \leq n, 0 \leq t_1, t_2 \leq d-1} \left| \frac{1}{n} (\langle X_{l_1, t_1, g}, X_{l_2, t_2, g} \rangle_{G_\phi} - \delta_{l_1 l_2} \delta_{t_1 t_2}) \right| \\ &\leq 4d^2 n c_0 e^{(dn+d-2)R_1} \|(\rho_\phi - 1)\|_{L^2(S)} \leq 4d^2 n c_0^2 e^{(dn+d-2)R_1} \|\Phi_s(x) - \phi(s)\|_{L^2(S)} \end{aligned}$$

Recall that (IV.3.4) implies

$$\|H_2\| \leq 2e^{(dn+d-2)R_1} c_0^2 c_1 \epsilon \|\Phi_s(x) - \phi(s)\|_{L^2(S)}.$$

Choose $R_3 = R_3(n, R) = (R + R_1)R_2$ and

$$R_2 = R_2(n, R) = (2e^{(dn+d-2)R_1} c_0^2 c_1 + 4d^2 n c_0^2 e^{(dn+d-2)R_1}).$$

Hence $\|H_1\| + \|H_2\| \leq \min\{R_3, R_2\} \|\Phi_s(x) - \phi(s)\|_{L^2(S)}$ by the assumption of $\phi \in \mathcal{B}(R)$, $\epsilon < 1$ and Lemma IV.6. It follows from Lemma IV.26 that

$$\|\det A_\phi - \det \widehat{A}_\phi\| \leq \mathcal{K}_1(n, R_3) \|H_2\| (\|H_1\| + \|H_2\|).$$

Hence

$$\frac{\|\det A_\phi - \det \widehat{A}_\phi\|}{\epsilon \|\Phi_s(x) - \phi(s)\|_{L^2(S)}^2} \leq \mathcal{K}_1(n, R_3) (R_2 + 1)^2.$$

Based on the above inequality and (IV.3.4), we can choose $C_0(n, R)$ satisfying

$$C_0(n, R) > \max\{\mathcal{K}_1(n, R_3) (R_2 + 1)^2, 2e^{(dn+d-2)R_1} c_0^2 c_1\}.$$

Then we have

$$\|A_\phi - \widehat{A}_\phi\| \leq C_0 \epsilon \|\phi - \Phi(P(\phi))\|_{L^2(S)}$$

and

$$|\det A_\phi - \det \widehat{A}_\phi| \leq C_0 \epsilon \|\phi - \Phi(P(\phi))\|_{L^2(S)}^2. \quad \square$$

Corollary IV.27. *Let $\phi \in \mathcal{B}(R)$. If g is C^r sufficiently close to g_0 , then A_ϕ is invertible. Moreover, we have*

$$\left\| A_\phi^{-1} - \widehat{A}_\phi^{-1} \right\| \leq \widetilde{C}_0 \epsilon \|\phi - \Phi(P(\phi))\|_{L^2(S)}$$

and

$$\left| \det A_\phi^{-1} - \det \widehat{A}_\phi^{-1} \right| \leq \widetilde{C}_0 \epsilon \|\phi - \Phi(P(\phi))\|_{L^2(S)}^2,$$

for some constant $\widetilde{C}_0 = \widetilde{C}_0(n, R) > 0$.

Proof. When $\phi \in \mathcal{B}(R)$, Lemma IV.6 implies $\Phi(P(\phi)) \in \mathcal{B}(R_1)$. Since Lemma IV.17 implies that $\text{Id} \leq \widehat{A}_\phi \leq (d+1)\text{Id}$, we have

$$1 \leq \det \widehat{A}_\phi \leq (d+1)^{dn}.$$

Therefore, when $\epsilon(g, r) \leq \min\{1/[2C_0(R+R_1)], 1/[2C_0(R+R_1)^2]\}$, by the norm estimate and the determinant estimate in Proposition IV.25 we have

$$\frac{1}{2}\text{Id} \leq A_\phi \leq \frac{2d+3}{2}\text{Id}$$

and

$$\frac{1}{2} \leq \det A_\phi \det \widehat{A}_\phi \leq (d+1)^{dn} \left[(d+1)^{dn} + \frac{1}{2} \right],$$

which imply that

$$\left\| A_\phi^{-1} - \widehat{A}_\phi^{-1} \right\| \leq \|A_\phi^{-1}\| \left\| A_\phi - \widehat{A}_\phi \right\| \left\| \widehat{A}_\phi^{-1} \right\| \leq 2C_0 \epsilon \|\phi - \Phi(P(\phi))\|_{L^2(S)}$$

and that

$$\left| \det A_\phi^{-1} - \det \widehat{A}_\phi^{-1} \right| \leq 2C_0 \epsilon \|\phi - \Phi(P(\phi))\|_{L^2(S)}^2.$$

The corollary follows from choosing $\widetilde{C}_0 = 2C_0$. □

We need one further estimate on the Jacobian of $\widehat{A}_\phi^{-1} \circ E_\phi$.

Proposition IV.28. *For any $\phi \in \mathcal{B}(R)$ (assuming $x = P(\phi)$), any dn -dimensional subspace $\mathcal{W} \subset \mathcal{L}$ and any $X \in \mathcal{W}$ such that $\langle X, X \rangle_{G_\phi} = 1$, we have*

$$\det(\widehat{A}_\phi^{-1}) \text{Jac}_{G, \mathcal{W}} E_\phi \leq 1 - C_1 \|X - d\Phi \circ \widehat{A}_\phi^{-1} \circ E_\phi(X)\|_{L^2(S)}^2$$

for some $C_1 = C_1(n, R)$.

Proof. First, we proof the case when $\mathcal{W} = \mathcal{V}_\phi = \text{span} Z_\phi$, where \mathcal{V}_ϕ, Z_ϕ are defined in the previous chapter.

Special case: $\mathcal{W} = \mathcal{V}_\phi$.

Since the left hand side of the inequality can be written in terms of eigenvalues of $U_\phi^T U_\phi$, our main idea is to estimate $\|X - d\Phi \circ \widehat{A}_\phi^{-1} \circ E_\phi(X)\|_{L^2(S)}^2$ by eigenvalues of $U_\phi^T U_\phi$.

Recall that in Lemma IV.17 and Lemma IV.18 we have

$$\Xi[\widehat{A}_\phi]\Xi = \text{Id} + \frac{1}{n}\widehat{Q}_\phi$$

and

$$\Xi[E_\phi|_{\mathcal{V}_\phi}]_{W_\phi} = \frac{n+1}{n}U_\phi^T,$$

we can conclude that

$$\Xi[\widehat{A}_\phi^{-1} \circ E_\phi|_{\mathcal{V}_\phi}]_{W_\phi} = \frac{n+1}{n} \left(\text{Id} + \frac{1}{n}\widehat{Q}_\phi \right)^{-1} U_\phi^T. \quad (\text{IV.3.5})$$

Let $\phi_0 := \Phi(P(\phi))$. We define a map $i_\phi : T_{\phi_0}\mathcal{L} \rightarrow T_\phi\mathcal{L}$ such that, if we identify both tangent spaces as \mathcal{L} , $i_\phi(Y) = Y$ for all $Y \in \mathcal{L} = T_{\phi_0}\mathcal{L}$. In particular, $i_\phi(Z_{\phi_0}) = Z_\phi$. Notice that $d\Phi(\Xi) = Z_{\phi_0}$, (IV.3.5) implies that

$$Z_\phi[i_\phi \circ d\Phi \circ \widehat{A}_\phi^{-1} \circ E_\phi|_{\mathcal{V}_\phi}]_{W_\phi} = \frac{n+1}{n} \left(\text{Id} + \frac{1}{n}\widehat{Q}_\phi \right)^{-1} U_\phi^T.$$

Hence

$$\begin{aligned} & w_\phi[i_\phi \circ d\Phi \circ \widehat{A}_\phi^{-1} \circ E_\phi|_{\mathcal{V}_\phi}]_{W_\phi} \\ &= \frac{n+1}{n} w_\phi[\text{Id}]_{Z_\phi} \left(\text{Id} + \frac{1}{n}\widehat{Q}_\phi \right)^{-1} U_\phi^T = \frac{n+1}{n} U_\phi \left(\text{Id} + \frac{1}{n}\widehat{Q}_\phi \right)^{-1} U_\phi^T. \end{aligned}$$

Therefore

$$w_\phi[\text{Id}|_{\mathcal{V}_\phi} - i_\phi \circ d\Phi \circ \widehat{A}_\phi^{-1} \circ E_\phi|_{\mathcal{V}_\phi}]_{W_\phi} = \text{Id} - \frac{n+1}{n} U_\phi \left(\text{Id} + \frac{1}{n}\widehat{Q}_\phi \right)^{-1} U_\phi^T.$$

For simplicity, let

$$\mathcal{G}(U_\phi) = \text{Id} - \frac{n+1}{n} \left(\text{Id} + \frac{1}{n}\widehat{Q}_\phi \right)^{-1} U_\phi^T U_\phi \in \text{Mat}_{dn \times dn}(\mathbb{R})$$

and hence

$$w_\phi[\text{Id} - i_\phi \circ d\Phi \circ \widehat{A}_\phi^{-1} \circ E_\phi|_{\mathcal{V}_\phi}]_{W_\phi} = U_\phi \mathcal{G}(U_\phi) U_\phi^{-1}. \quad (\text{IV.3.6})$$

Assume that \widehat{Q}_ϕ has eigenvalues $0 \leq \eta_1 \leq \eta_2 \leq \dots \leq \eta_{dn}$ and $U_\phi^T U_\phi$ has eigenvalues $0 \leq \lambda_1 \leq$

$\lambda_2 \leq \dots \leq \lambda_{dn}$. Lemma IV.21 shows that

$$\sum_{t=1}^{dn} \lambda_t = \sum_{t=1}^{dn} \eta_t = dn$$

and

$$\lambda_1 \leq \eta_1 \leq 1 \leq \eta_{dn} \leq \lambda_{dn} \leq dn.$$

Therefore

$$\begin{aligned} \|\mathcal{G}(U_\phi)\| &= \left\| \text{Id} - \frac{n+1}{n} \left(\text{Id} + \frac{1}{n} \widehat{Q}_\phi \right)^{-1} U_\phi^T U_\phi \right\| \\ &\leq \left\| \text{Id} - \frac{n+1}{n} \left(\text{Id} + \frac{1}{n} \widehat{Q}_\phi \right)^{-1} \right\| + \left\| \frac{n+1}{n} \left(\text{Id} + \frac{1}{n} \widehat{Q}_\phi \right)^{-1} (U_\phi^T U_\phi - \text{Id}) \right\| \\ &\leq \left\| \text{Id} - \frac{n+1}{n} \left(\text{Id} + \frac{1}{n} \widehat{Q}_\phi \right)^{-1} \right\| + \frac{n+1}{n} \| (U_\phi^T U_\phi - \text{Id}) \| \\ &\leq \frac{n+1}{n} \left(\frac{1}{1 + \frac{1}{n} \eta_1} - \frac{1}{1 + \frac{1}{n} \eta_{dn}} + \lambda_{dn} - \lambda_1 \right) \\ &\leq \frac{n+1}{n} \left(\frac{1}{1 + \frac{1}{n} \lambda_1} - \frac{1}{1 + \frac{1}{n} \lambda_{dn}} + \lambda_{dn} - \lambda_1 \right) \leq \left(\frac{n+1}{n} \right)^2 (\lambda_{dn} - \lambda_1), \quad (\text{IV.3.7}) \end{aligned}$$

where the third inequality follows from the fact that if a symmetric matrix has non-positive and non-negative eigenvalues, then its norm is controlled by the difference between its largest and smallest eigenvalues. In order to estimate $\|U_\phi \mathcal{G}(U_\phi) U_\phi^{-1}\|$, we need the following result.

Lemma IV.29. *For any $\phi \in \mathcal{B}(R)$ (assuming $x = P(\phi)$), there exists a positive constant $C_2 = C_2(n, R) > 0$ such that*

$$\lambda_1 \geq C_2,$$

where $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{dn}$ are eigenvalues of $U_\phi^T U_\phi$.

Proof of Lemma IV.29. Let $v \in \mathbb{R}^{dn}$ be an arbitrary unit vector in an Euclidean space. Define

$$c_2(n) = \frac{\text{Area}(\{w \in S^{dn-1} : \langle w, v \rangle_{\mathbb{R}^{dn}} \geq 1/2\}, ds)}{\text{Area}(S^{dn-1}, ds)} > 0,$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^{dn}}$ denotes the standard Euclidean inner product. c_2 is independent of the choice of v due to spherical symmetry. For any unit vector $v = (a_{1,0}, \dots, a_{1,d-1}, \dots, a_{n,0}, \dots, a_{n,d-1})^T \in \text{Mat}_{dn \times 1}(\mathbb{R}) = \mathbb{R}^{dn}$, we construct a vector (see Subsection IV.2.1 and IV.2.2 for definitions of

$\xi_{l,t,g}$ and $X_{l,t,g}$)

$$\xi_v = \sum_{l=1}^n \sum_{t=0}^{d-1} a_{l,t} \xi_{l,t,g} \in T_x M.$$

Then (IV.2.7) implies that

$$\begin{aligned} v^T U_\phi^T U_\phi v &= dn \int_S \bar{\rho}_\phi(s) \left(\sum_{l=1}^n \sum_{t=0}^{d-1} a_{l,t} X_{l,t,g}(s) \right)^2 d\mu_x(s) \\ &= dn \int_S \bar{\rho}_\phi(s) \langle \text{grad} \Phi_s(x), \xi_v \rangle^2 d\mu_x(s) \\ &= dn \int_{T_x^1 M} \bar{\rho}_\phi(\alpha_g^{-1}(s)) \langle \xi, \xi_v \rangle^2 ds_x(\xi) \\ &\geq dn \int_{\{w \in T_x^1 M : \langle \xi, \xi_v \rangle \geq 1/2\}} \bar{\rho}_\phi(\alpha_g^{-1}(s)) \langle \xi, \xi_v \rangle^2 ds_x(\xi) \\ &\geq \frac{dn}{4} \int_{\{w \in T_x^1 M : \langle w, v \rangle \geq 1/2\}} \bar{\rho}_\phi(\alpha_g^{-1}(s)) ds_x(\xi). \end{aligned} \quad (\text{IV.3.8})$$

Since (IV.3.2) implies that

$$c_0^{-2} \leq \bar{\rho}_\phi \leq c_0^2. \quad (\text{IV.3.9})$$

Therefore we have

$$\frac{dn}{4} \int_{\{\xi \in T_x^1 M : \langle \xi, \xi_v \rangle \geq 1/2\}} \bar{\rho}_\phi(\alpha^{-1}(s)) ds_x(\xi) \geq \frac{dn}{4} c_2(n) c_0^{-2}.$$

As a consequence of (IV.3.8), we have

$$v^T U_\phi^T U_\phi v \geq \frac{dn}{4} c_2(n) c_0^{-2},$$

which implies that

$$\|U_\phi^T U_\phi\| \geq \frac{dn}{4} c_2(n) c_0^{-2}.$$

In particular, let $C_2 = C_2(n, R) = \frac{dn}{4} c_2(n) c_0^{-2}$, we have $\lambda_1 \geq C_2$. □

Going back to the proof of Proposition IV.28, Lemma IV.29, (IV.2.10) and (IV.3.7) imply that

$$\begin{aligned} \left\| w_\phi [\text{Id} - i_\phi \circ d\Phi \circ \widehat{A}_\phi^{-1} \circ E_\phi]_{W_\phi} \right\| &= \|U_\phi \mathcal{G}(U_\phi) U_\phi^{-1}\| \leq \|U_\phi\| \|\mathcal{G}(U_\phi)\| \|U_\phi^{-1}\| \\ &\leq \lambda_{dn}^{1/2} \cdot \left(\frac{n+1}{n}\right)^2 (\lambda_{dn} - \lambda_1) \cdot \lambda_1^{-1/2} \\ &\leq \sqrt{dn} \cdot \left(\frac{n+1}{n}\right)^2 C_2^{-1/2} (\lambda_{dn} - \lambda_1), \end{aligned}$$

which implies that

$$\|X - d\Phi \circ \widehat{A}_\phi^{-1} \circ E_\phi(X)\|_{G_\phi} \leq \sqrt{dn} \cdot \left(\frac{n+1}{n}\right)^2 C_2^{-1/2} (\lambda_{dn} - \lambda_1).$$

Notice that $\|\cdot\|_{G_\phi} = \|\cdot\|_{L^2(S, dn\bar{\rho}_\phi d\mu_x)} = \|\cdot\|_{L^2(S, dn\bar{\rho}_\phi \lambda(x,s) ds)}$, it follows from (IV.3.3) and (IV.3.9) that

$$\sqrt{\frac{dn}{2} c_0^{-2} e^{-(dn+d-2)R_1}} \|\cdot\|_{L^2(S)} \leq \|\cdot\|_{G_\phi} \leq \sqrt{2dnc_0^2 e^{(dn+d-2)R_1}} \|\cdot\|_{L^2(S)}. \quad (\text{IV.3.10})$$

Write $c_3 = c_3(n, R) = \sqrt{2(dn)^{-1} c_0^2 e^{(dn+d-2)R_1}} \cdot \sqrt{dn} \cdot \left(\frac{n+1}{n}\right)^2 C_2^{-1/2}$. Therefore

$$\|X - d\Phi \circ \widehat{A}_\phi^{-1} \circ E_\phi(X)\|_{L^2(S)} \leq c_3 (\lambda_{dn} - \lambda_1). \quad (\text{IV.3.11})$$

Following (IV.2.13) and Lemma IV.29, we have

$$\begin{aligned} \det(\widehat{A}_\phi^{-1}) \text{Jac}_{G, \nu} E_\phi &\leq (\det U_\phi^T U_\phi)^{\frac{1}{2} - \frac{1}{n+1}} \\ &= \left(\lambda_1 \lambda_{dn} \prod_{t=2}^{dn-1} \lambda_t \right)^{\frac{1}{2} - \frac{1}{n+1}} \\ &= \left[\left(\frac{\lambda_1 + \lambda_{dn}}{2} \right)^2 \prod_{t=2}^{dn-1} \lambda_t - \frac{1}{4} (\lambda_1 - \lambda_{dn})^2 \prod_{t=2}^{dn-1} \lambda_t \right]^{\frac{1}{2} - \frac{1}{n+1}} \\ &\leq \left[\left(\frac{\sum_{t=1}^{dn} \lambda_t}{dn} \right)^{dn} - \frac{1}{4} (\lambda_1 - \lambda_{dn})^2 \lambda_1^{dn-2} \right]^{\frac{1}{2} - \frac{1}{n+1}} \\ &\leq \left[1 - \frac{1}{4} C_2^{dn-2} (\lambda_1 - \lambda_{dn})^2 \right]^{\frac{1}{2} - \frac{1}{n+1}} \leq 1 - \left(\frac{1}{2} - \frac{1}{n+1} \right) \frac{1}{4} C_2^{dn-2} (\lambda_1 - \lambda_{dn})^2 \end{aligned}$$

Choose

$$C'_1 = C'_1(n, R) = \left(\frac{1}{2} - \frac{1}{n+1} \right) \frac{1}{4} C_2^{dn-2} c_3^{-2}.$$

Then it follows from (IV.3.11) that

$$\begin{aligned} \det(\widehat{A}_\phi^{-1})\text{Jac}_{G,\mathcal{V}}E_\phi &\leq 1 - \left(\frac{1}{2} - \frac{1}{n+1}\right) \frac{1}{4} C_2^{dn-2} (\lambda_1 - \lambda_{dn})^2 \\ &\leq 1 - C'_1 c_3^2 (\lambda_1 - \lambda_{dn})^2 \leq 1 - C'_1 \|X - d\Phi \circ \widehat{A}_\phi^{-1} \circ E_\phi(X)\|_{L^2(S)}^2. \end{aligned} \quad (\text{IV.3.12})$$

Now we return to the general case when \mathcal{W} is an arbitrary dn -dimensional subspace of \mathcal{L} . For any $X \in \mathcal{W}$, we write $X = X^\parallel + X^\perp$, where $X^\perp \in \mathcal{V}_\phi^\perp$ and $X^\parallel \in \mathcal{V}_\phi$. We first make the following claim.

Lemma IV.30. *Let $P_{\mathcal{V}_\phi} : T_\phi \mathcal{L} = \mathcal{L} \rightarrow \mathcal{V}_\phi$ be the orthogonal projection onto \mathcal{V}_ϕ . Then there exists some constant $C_3 = C_3(n) > 0$ such that*

$$\text{Jac}_{G,\mathcal{W}}P_{\mathcal{V}_\phi} = \text{Jac}_G(P_{\mathcal{V}_\phi}|_{\mathcal{W}}) \leq 1 - C_3 \|X^\perp\|_{G_\phi}^2.$$

Proof of Lemma IV.30. We follow the same idea as in Corollary IV.22 and (IV.2.12). Choose an orthonormal basis $\{\tilde{X}_1 = X, \tilde{X}_2, \dots, \tilde{X}_{dn}\}$ in \mathcal{W} . Then (IV.2.12) implies that

$$\begin{aligned} &\text{Jac}_{G,\mathcal{W}}P_{\mathcal{V}_\phi} \\ &= \text{Jac}_G(P_{\mathcal{V}_\phi}|_{\mathcal{W}}) \\ &\leq \left(\frac{1}{dn} \sum_{t=1}^{dn} \|P_{\mathcal{V}_\phi}(\tilde{X}_t)\|_{G_\phi}^2\right)^{\frac{dn}{2}} \leq \left(1 - \frac{\|X^\perp\|_{G_\phi}^2}{dn}\right)^{\frac{dn}{2}} \leq 1 - \frac{1}{2} \left(1 - \frac{1}{dn}\right)^{\frac{dn}{2}-1} \|X^\perp\|_{G_\phi}^2. \end{aligned}$$

The last inequality follows from the Mean Value Theorem and the fact that $\|X^\perp\|_{G_\phi} \leq 1$. Choose $C_3(n) = \frac{1}{2} \left(1 - \frac{1}{dn}\right)^{\frac{dn}{2}-1}$ and the lemma follows. \square

Going back to the proof of Proposition IV.28, (IV.3.10) and (IV.3.12) imply that

$$\begin{aligned}
\det(\widehat{A}_\phi^{-1})\text{Jac}_{G,\mathcal{W}}E_\phi &= \det(\widehat{A}_\phi^{-1})\text{Jac}_G(E_\phi|_{\mathcal{V}_\phi} \circ P_{\mathcal{V}_\phi}) \\
&= \det(\widehat{A}_\phi^{-1})\text{Jac}_{G,\mathcal{V}_\phi}E_\phi \cdot \text{Jac}_G(P_{\mathcal{V}_\phi}|_{\mathcal{W}}) \\
&\leq \det(\widehat{A}_\phi^{-1})\text{Jac}_{G,\mathcal{V}_\phi}E_\phi \left(1 - C_3\|X^\perp\|_{G_\phi}^2\right) \\
&\leq \det(\widehat{A}_\phi^{-1})\text{Jac}_{G,\mathcal{V}_\phi}E_\phi \left(1 - \frac{dn}{2}c_0^{-2}e^{-(dn+d-2)R_1}C_3\|X^\perp\|_{L^2(S)}^2\right) \\
&\leq \left(1 - \frac{C'_1}{\|X^\parallel\|_{G_\phi}^2} \left\|X^\parallel - d\Phi \circ \widehat{A}_\phi^{-1} \circ E_\phi(X^\parallel)\right\|_{L^2(S)}^2\right) \left(1 - c_4\|X^\perp\|_{L^2(S)}^2\right) \\
&\leq \left(1 - C'_1 \left\|X^\parallel - d\Phi \circ \widehat{A}_\phi^{-1} \circ E_\phi(X^\parallel)\right\|_{L^2(S)}^2\right) \left(1 - c_4\|X^\perp\|_{L^2(S)}^2\right) \\
&= \left(1 - C'_1 \left\|X^\parallel - d\Phi \circ \widehat{A}_\phi^{-1} \circ E_\phi(X)\right\|_{L^2(S)}^2\right) \left(1 - c_4\|X^\perp\|_{L^2(S)}^2\right) \\
&\leq 1 - \frac{1}{2} \left(C'_1 \left\|X^\parallel - d\Phi \circ \widehat{A}_\phi^{-1} \circ E_\phi(X)\right\|_{L^2(S)}^2 + c_4\|X^\perp\|_{L^2(S)}^2\right),
\end{aligned}$$

where $c_4 = c_4(n, R) = \frac{dn}{2}c_0^{-2}e^{-(dn+d-2)R_1}C_3$ and the last inequality follows from the fact that $(1 - 2a)(1 - 2b) \leq 1 - a - b$ provided $0 \leq 1 - 2a, 1 - 2b \leq 1$. Take

$$C_1 = C_1(n, R) = \frac{1}{2} \min \left\{ \frac{1}{2}C'_1, \frac{1}{2}c_4 \right\},$$

we can summarize that

$$\begin{aligned}
\det(\widehat{A}_\phi^{-1})\text{Jac}_{G,\mathcal{W}}E_\phi &\leq 1 - \frac{1}{2} \left(C'_1 \left\|X^\parallel - d\Phi \circ \widehat{A}_\phi^{-1} \circ E_\phi(X)\right\|_{L^2(S)}^2 + c_4\|X^\perp\|_{L^2(S)}^2\right) \\
&\leq 1 - 2C_1 \left(\left\|X^\parallel - d\Phi \circ \widehat{A}_\phi^{-1} \circ E_\phi(X)\right\|_{L^2(S)}^2 + \|X^\perp\|_{L^2(S)}^2\right) \\
&\leq 1 - C_1 \left\|X^\perp + X^\parallel - d\Phi \circ \widehat{A}_\phi^{-1} \circ E_\phi(X)\right\|_{L^2(S)}^2 \\
&= 1 - C_1 \left\|X - d\Phi \circ \widehat{A}_\phi^{-1} \circ E_\phi(X)\right\|_{L^2(S)}^2. \quad \square
\end{aligned}$$

IV.4: A compression trick

This subsection is a review of [BI13, Section 7, A compression trick] in the more general case of $\mathbb{K}\mathbf{H}^n$. Recall in Proposition III.4, we need a "projection" map P_σ . In the previous two sections, we constructed a projection map P , but whether it is area non-increasing is not clear. In this subsection, we will give our construction of P_σ as a small perturbation of P . Notice that the second assertion in

Definition III.11 suggests that P is a locally “orthogonal” projection in the sense that $d_\phi P$ vanish on the orthogonal complement of $\text{Im}(d_{P(\phi)}\Phi)$. Therefore P_σ constructed as a small perturbation of P can be viewed as a “almost locally orthogonal” projection.

Notation IV.31. We define a “height” map $h(\phi) : \mathcal{B}(R) \rightarrow \mathbb{R}_+$ as follows.

$$h(\phi) = \|\phi - \Phi(P(\phi))\|_{L^2(S)}.$$

Similar to [BI13], we construct a map $F_c : \mathcal{B}(R) \rightarrow M \times \mathbb{R}_+$ by

$$F_c(\phi) = (P(\phi), ch(\phi)).$$

Since h is smooth on $\mathcal{B}(R) \setminus \Phi(M)$, so is F_c . We continue with the notation that $\epsilon = \epsilon(g, r)$ as in Section IV.3.

Lemma IV.32. *There exist positive constants $C_5 = C_5(n, R), C_6 = C_6(n, R) > 0$ such that for any $0 \leq c \leq C_5$ and any $\phi \in \mathcal{B}(R) \setminus \Phi(M)$*

$$\text{Jac}_G F_c \leq 1 + C_6 \epsilon h^2(\phi)$$

provided $\epsilon \ll 1$.

Proof. Let $\mathcal{W} \subset T_\phi \mathcal{L} = \mathcal{L}$ be any dn -dimensional subspace. Denoted by $\{\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_{dn}\}$ an orthonormal basis in \mathcal{W} and $\{\omega_1, \omega_2, \dots, \omega_{dn}\}$ an orthonormal basis in $T_x M$, where $x = P(\phi)$. Without loss of generality, we can assume that

- (1). $d_\phi h(\tilde{X}_l) = 0$ for any $l \geq 2$;
- (2). $d_\phi P|_{\mathcal{W}}$ has upper triangular matrix $(a_{sl})_{1 \leq s, l \leq dn}$ with non-negative diagonal under the above choices of bases.

Then

$$\text{Jac}_G F_c|_{\mathcal{W}} = \sqrt{a_{11}^2 + t^2} \prod_{l=2}^{dn} a_{ll}, \quad (\text{IV.4.1})$$

where

$$t = d_\phi(ch(\tilde{X}_1)) = \frac{c}{2h(\phi)} d_\phi(h^2)(\tilde{X}_1).$$

Therefore

$$|t| \leq c \|\tilde{X}_1 - d\Phi \circ d_\phi P(\tilde{X}_1)\|_{L^2(S)}. \quad (\text{IV.4.2})$$

The first assertion in Corollary IV.27 suggests that

$$\|A_\phi^{-1} - \widehat{A}_\phi^{-1}\| \leq \widetilde{C}_0 \epsilon \|\phi - \Phi(P(\phi))\|_{L^2(S)}.$$

Therefore $\|A_\phi^{-1}\| \leq 2$ provided $\epsilon \leq [\widetilde{C}_0(R+R_1)]^{-1}$ (which is actually weaker than $\epsilon \leq \min\{[2C_0(R+R_1)]^{-1}, [2C_0(R+R_1)^2]^{-1}\}$ mentioned in the proof of Corollary IV.27). Combined with Corollary IV.22, we have P is $2(dn+d)/\sqrt{dn}$ -Lipschitz. In particular,

$$0 \leq a_{ii} \leq \frac{2(dn+d)}{\sqrt{dn}}. \quad (\text{IV.4.3})$$

By (IV.1.2) and (IV.3.10) we have

$$\begin{aligned} & \|\widetilde{X}_1 - d\Phi \circ d_\phi P(\widetilde{X}_1)\|_{L^2(S)} \\ &= \|\widetilde{X}_1 - a_{11}d\Phi(\omega_1)\|_{L^2(S)} \\ &\leq \|\widetilde{X}_1\|_{L^2(S)} + a_{11}\|d\Phi(\omega_1)\|_{L^2(S)} \\ &\leq \|\widetilde{X}_1\|_{L^2(S)} + \frac{2(dn+d)}{\sqrt{dn}}\|d\Phi(\omega_1)\|_{L^2(S)} \\ &\leq \sqrt{2(dn)^{-1}c_0^2e^{(dn+d-2)R_1}}\|\widetilde{X}_1\|_{G_\phi} + \frac{2(dn+d)}{\sqrt{dn}}\sqrt{2e^{(dn+d-2)R_1}}\|d\Phi(\omega_1)\|_{L^2(S, d\mu_x)} \\ &= \sqrt{2(dn)^{-1}c_0^2e^{(dn+d-2)R_1}}\|\widetilde{X}_1\|_{G_\phi} + \frac{2(dn+d)}{dn}\sqrt{2e^{(dn+d-2)R_1}}\|d\Phi(\omega_1)\|_{G_{\Phi(x)}} \\ &= \sqrt{2(dn)^{-1}c_0^2e^{(dn+d-2)R_1}}\|\widetilde{X}_1\|_{G_\phi} + \frac{2(n+1)}{n}\sqrt{2e^{(dn+d-2)R_1}}\|\omega_1\| \\ &= \sqrt{2(dn)^{-1}c_0^2e^{(dn+d-2)R_1}} + \frac{2(n+1)}{n}\sqrt{2e^{(dn+d-2)R_1}} =: c_5(n, R) = c_5. \end{aligned} \quad (\text{IV.4.4})$$

Without loss of generality, assume that

$$C_5 = C_5(n, R) \leq \max \left\{ c_5^{-1} \left[2(dn+d)/\sqrt{dn} \right]^{-dn}, 1 \right\}.$$

It follows from (IV.4.2) and the assumption of $0 \leq c \leq C_5$ that

$$|t| \leq \left[2(dn+d)/\sqrt{dn} \right]^{-dn}.$$

We consider two different cases.

Case 1. Assume that $a_{11} < \left[2(dn+d)/\sqrt{dn} \right]^{-dn}$. Then

$$\sqrt{a_{11}^2 + t^2} \leq \sqrt{2} \left[\frac{2(dn+d)}{\sqrt{dn}} \right]^{-dn}.$$

It follows from (IV.4.1) and (IV.4.3) that

$$\begin{aligned} \text{Jac}_G F_c|_{\mathcal{W}} &= \sqrt{a_{11}^2 + t^2} \prod_{l=2}^{dn} a_{ll} \\ &\leq \sqrt{2} \left[\frac{2(dn+d)}{\sqrt{dn}} \right]^{-dn} \prod_{l=2}^{dn} a_{ll} \leq \sqrt{2} \left[\frac{2(dn+d)}{\sqrt{dn}} \right]^{-dn} \left[\frac{2(dn+d)}{\sqrt{dn}} \right]^{dn-1} = \frac{\sqrt{2dn}}{2(dn+d)} < 1. \end{aligned}$$

Case 2. Assume that $a_{11} \geq \left[2(dn+d)/\sqrt{dn} \right]^{-dn}$. Then

$$\sqrt{a_{11}^2 + t^2} \leq a_{11} + \frac{t^2}{2a_{11}}$$

and

$$\begin{aligned} \text{Jac}_G F_c|_{\mathcal{W}} &= \sqrt{a_{11}^2 + t^2} \prod_{l=2}^{dn} a_{ll} \leq \left(a_{11} + \frac{t^2}{2a_{11}} \right) \prod_{l=2}^{dn} a_{ll} \\ &= \text{Jac}_{G, \mathcal{W}} P + \frac{t^2}{2a_{11}} \prod_{l=2}^{dn} a_{ll} \\ &\leq \text{Jac}_{G, \mathcal{W}} P + \frac{t^2}{2 \left[\frac{2(dn+d)}{\sqrt{dn}} \right]^{-dn}} \left[\frac{2(dn+d)}{\sqrt{dn}} \right]^{dn-1} \\ &= \text{Jac}_{G, \mathcal{W}} P + \frac{1}{2} \left[\frac{2(dn+d)}{\sqrt{dn}} \right]^{2dn-1} t^2. \end{aligned} \tag{IV.4.5}$$

Lemma IV.21, Corollary IV.27 and Proposition IV.28 imply that

$$\begin{aligned}
\text{Jac}_{G,\mathcal{W}}P &= \det(A_\phi^{-1})\text{Jac}_{G,\mathcal{W}}E_\phi \\
&\leq \left(\det(\widehat{A}_\phi^{-1}) + \widetilde{C}_0\epsilon\|\phi - \Phi(P(\phi))\|_{L^2(S)}^2 \right) \text{Jac}_{G,\mathcal{W}}E_\phi \\
&= \det(\widehat{A}_\phi^{-1})\text{Jac}_{G,\mathcal{W}}E_\phi + \widetilde{C}_0\epsilon h^2(\phi)\text{Jac}_{G,\mathcal{W}}E_\phi \\
&\leq \det(\widehat{A}_\phi^{-1})\text{Jac}_{G,\mathcal{W}}E_\phi + \widetilde{C}_0\epsilon h^2(\phi)\text{Jac}_G E_\phi \\
&= \det(\widehat{A}_\phi^{-1})\text{Jac}_{G,\mathcal{W}}E_\phi + \widetilde{C}_0\epsilon h^2(\phi) \left| \det\left(\frac{n+1}{n}U_\phi^T\right) \right| \\
&= \det(\widehat{A}_\phi^{-1})\text{Jac}_{G,\mathcal{W}}E_\phi + \widetilde{C}_0\epsilon h^2(\phi) \det\left[\frac{n+1}{n}(U_\phi^T U_\phi)^{\frac{1}{2}}\right] \\
&\leq \det(\widehat{A}_\phi^{-1})\text{Jac}_{G,\mathcal{W}}E_\phi + \left(\frac{n+1}{n}\right)^{dn} \widetilde{C}_0\epsilon h^2(\phi) \\
&\leq 1 - C_1\|\widetilde{X}_1 - d\Phi \circ \widehat{A}_\phi^{-1} \circ E_\phi(\widetilde{X}_1)\|_{L^2(S)}^2 + \left(\frac{n+1}{n}\right)^{dn} \widetilde{C}_0\epsilon h^2(\phi).
\end{aligned}$$

It follows from (IV.4.5) that

$$\begin{aligned}
&\text{Jac}_G F_c|_{\mathcal{W}} \\
&\leq 1 - C_1\|\widetilde{X}_1 - d\Phi \circ \widehat{A}_\phi^{-1} \circ E_\phi(\widetilde{X}_1)\|_{L^2(S)}^2 + \left(\frac{n+1}{n}\right)^{dn} \widetilde{C}_0\epsilon h^2(\phi) + \frac{t^2}{2} \left[\frac{2(dn+d)}{\sqrt{dn}} \right]^{2dn-1}.
\end{aligned} \tag{IV.4.6}$$

Corollary IV.22, Corollary IV.27, (IV.3.3) and (IV.4.2) imply that

$$\begin{aligned}
|t| &\leq c\|\widetilde{X}_1 - d\Phi \circ d_\phi P(\widetilde{X}_1)\|_{L^2(S)} \\
&\leq c\left\| \widetilde{X}_1 - d\Phi \circ \widehat{A}_\phi^{-1} \circ E_\phi(\widetilde{X}_1) \right\|_{L^2(S)} + c\left\| d\Phi \circ \left(A_\phi^{-1} - \widehat{A}_\phi^{-1} \right) \circ E_\phi(\widetilde{X}_1) \right\|_{L^2(S)} \\
&\leq c\left\| \widetilde{X}_1 - d\Phi \circ \widehat{A}_\phi^{-1} \circ E_\phi(\widetilde{X}_1) \right\|_{L^2(S)} + c\sqrt{2e^{(dn+d-2)R_1}} \left\| d\Phi \circ \left(A_\phi^{-1} - \widehat{A}_\phi^{-1} \right) \circ E_\phi(\widetilde{X}_1) \right\|_{L^2(S, d\mu_x)} \\
&= c\left\| \widetilde{X}_1 - d\Phi \circ \widehat{A}_\phi^{-1} \circ E_\phi(\widetilde{X}_1) \right\|_{L^2(S)} + c\sqrt{\frac{2}{dn}e^{(dn+d-2)R_1}} \left\| d\Phi \circ \left(A_\phi^{-1} - \widehat{A}_\phi^{-1} \right) \circ E_\phi(\widetilde{X}_1) \right\|_{G_{\Phi(x)}} \\
&\leq c\left\| \widetilde{X}_1 - d\Phi \circ \widehat{A}_\phi^{-1} \circ E_\phi(\widetilde{X}_1) \right\|_{L^2(S)} + c\sqrt{\frac{2}{dn}e^{(dn+d-2)R_1}} \|d\Phi\| \left\| \left(A_\phi^{-1} - \widehat{A}_\phi^{-1} \right) \right\| \|E_\phi\| \|\widetilde{X}_1\|_{G_\phi} \\
&\leq c\left(\left\| \widetilde{X}_1 - d\Phi \circ \widehat{A}_\phi^{-1} \circ E_\phi(\widetilde{X}_1) \right\|_{L^2(S)} + \frac{(n+1)}{n} \sqrt{2e^{(dn+d-2)R_1}} \widetilde{C}_0\epsilon\|\phi - \Phi(P(\phi))\|_{L^2(S)} \right) \\
&= c\left(\left\| \widetilde{X}_1 - d\Phi \circ \widehat{A}_\phi^{-1} \circ E_\phi(\widetilde{X}_1) \right\|_{L^2(S)} + \frac{(n+1)}{n} \sqrt{2e^{(dn+d-2)R_1}} \widetilde{C}_0\epsilon h(\phi) \right).
\end{aligned}$$

Since $C_5 \leq 1$ by definition, therefore $c < 1$ and

$$t^2 \leq 2c^2 \left\| \tilde{X}_1 - d\Phi \circ \hat{A}_\phi^{-1} \circ E_\phi(\tilde{X}_1) \right\|_{L^2(S)}^2 + c_6 \epsilon h^2(\phi), \quad (\text{IV.4.7})$$

where

$$c_6 = c_6(n, R) = 2 \left(\frac{(n+1)}{n} \sqrt{2e^{(dn+d-2)R_1} \tilde{C}_0} \right)^2,$$

and $\epsilon < 1$ by our assumption. Let

$$C_6 = C_6(n, R) = \frac{1}{2} \left[\frac{2(dn+d)}{\sqrt{dn}} \right]^{2dn-1} c_6 + \left(\frac{n+1}{n} \right)^{dn} \tilde{C}_0.$$

(IV.4.6) and (IV.4.7) can be summarized as

$$\text{Jac}_G F_c|_{\mathcal{W}} \leq 1 - \left\{ C_1 - \left[\frac{2(dn+d)}{\sqrt{dn}} \right]^{2dn-1} c^2 \right\} \left\| \tilde{X}_1 - d\Phi \circ \hat{A}_\phi^{-1} \circ E_\phi(\tilde{X}_1) \right\|_{L^2(S)}^2 + C_6 \epsilon h^2(\phi).$$

Choose C_5 in Lemma IV.32 such that

$$C_5 = \min \left\{ C_1^{\frac{1}{2}} \left[\frac{2(dn+d)}{\sqrt{dn}} \right]^{\frac{1-2dn}{2}}, c_5 \left[\frac{2(dn+d)}{\sqrt{dn}} \right]^{-dn}, 1 \right\}.$$

Then

$$\text{Jac}_G F_c|_{\mathcal{W}} \leq 1 + C_6 \epsilon h^2(\phi)$$

and the lemma follows. \square

We follow the idea in [BI13] to construct a "homothety" $\mathcal{A}_t : M \rightarrow M$ by

$$\mathcal{A}_t(p) = \exp_{x_0}(t \exp_{x_0}^{-1}(p)), \quad \forall p \in M,$$

and a map $\mathcal{Q}_\sigma : M \times \mathbb{R}_+ \rightarrow M$ by

$$\mathcal{Q}_\sigma(p, h) = \mathcal{A}_{1+\sigma h^2}(p), \quad \forall p \in M.$$

Lemma IV.33. *If $d(p, x_0) \leq 4\sigma^{-1/2}$, assuming g sufficiently close to g_0 such that (M, g) has strictly negative sectional curvature, then the dn -dimensional Jacobian of \mathcal{Q}_σ at (p, h) is no greater than $(1 + \sigma h^2)^{-1}$.*

Proof. The proof can be found in [BI13, Lemma 7.2]. \square

Now we define $P_\sigma(\phi) = \mathcal{Q}_\sigma(P(\phi), \sigma h(\phi)) = \mathcal{Q}_\sigma(F_\sigma(\phi))$ on $\mathcal{B}(R)$. By definition P_σ is smooth.

Proposition IV.34. *For every $R > 0$, there exists a $\sigma > 0$, $c > 0$ and $\epsilon_0 > 0$ such that the dn -dimensional Jacobian $J(\phi) := \text{Jac}_G P_\sigma(\phi)$ with respect to G at any point $\phi \in \mathcal{B}(R)$ satisfies*

$$J(\phi) \leq 1 - ch^2(\phi)$$

provided $\epsilon(g, r) \leq \epsilon_0$.

Proof. Choose σ such that $\sigma < C_5(n, R)$ in Lemma IV.32 and $(4\sigma)^{-1/2} > R_1(n, R)$ in Lemma IV.6.

For any $\phi \in \Phi(M)$, $h(\phi) = 0$, Hence we have $\text{Jac } d_\phi P_\sigma = \text{Jac } d_\phi P = 1$, following the equality conditions in Lemma IV.21 and Corollary IV.22.

For any $\phi \in \mathcal{B}(R) \setminus \Phi(M)$, Lemma IV.32 and Lemma IV.33 implies

$$J(\phi) \leq \text{Jac } \mathcal{Q}_\sigma \cdot \text{Jac}_G F_\sigma \leq \frac{1 + C_6 \epsilon(g, r) h^2(\phi)}{1 + \sigma^3 h^2(\phi)}.$$

Choose $\sigma = \sqrt[3]{3C_6\epsilon_0}$ and ϵ_0 such that

- (1). ϵ_0 is smaller than all upper bounds for $\epsilon(g, r)$ mentioned prior to this proposition;
- (2). $3C_6\epsilon_0 h^2(\phi) \leq 3C_6\epsilon_0(R + R_1)^2 \leq 1$;
- (3). $\sigma < C_5$;
- (4). $(4\sigma)^{-1/2} \geq R_1$.

Let $c = C_6(n, R)\epsilon_0 = \sigma^3/3$, Hence

$$J(\phi) \leq \frac{1 + C_6 \epsilon(g, r) h^2(\phi)}{1 + \sigma^3 h^2(\phi)} \leq \frac{1 + \frac{\sigma^3}{3} h^2(\phi)}{1 + \sigma^3 h^2(\phi)} \leq 1 - ch^2(\phi). \quad \square$$

Now we are in position to complete the proof of Proposition III.4.

Proof of Proposition III.4. Let ϵ_0 and σ be as in Proposition IV.34. Assume that $\epsilon(g, r) < \epsilon_0$. Consider the map $P_\sigma : \mathcal{B}(R) \rightarrow M$ constructed above. Proposition III.13 and the inequality in Proposition IV.34 suggests that $P_\sigma \circ f$ does not increase volume for any Riemannian manifold N and any 1-Lipschitz map $f : N \rightarrow \mathcal{B}(R)$. In the case of equality, P_σ has Jacobian equal to 1 for almost all points in $f(N)$. Therefore by continuity and the above proposition, $h(\phi) = 0$ for all $\phi \in f(N)$, hence $f(N) \subset \Phi(M)$. Therefore the map P_σ possesses all properties claimed in Proposition III.4. \square

Remark. In the proof of Proposition IV.34, we mentioned three additional conditions on g . Here is a summary of all conditions posed on g before Proposition IV.34.

- (1). $d_g(x, x_0) \leq R/4$ for any $x \in B_{x_0}(R/5)$, introduced at the beginning of Section III.3;
- (2). g has negative sectional curvature, introduced in the proof of Corollary I.6;
- (3). (IV.1.2) with $R_0 > R_1$, introduced in the remark of Lemma IV.4 and (IV.3.3);
- (4). (IV.1.4) with $\tilde{R}_0 > R_1$, introduced in Lemma IV.6;
- (5). $\text{diam}_g(B_{x_0}(2R)) < 5R$, introduced in Lemma IV.6;
- (6). $\epsilon = \epsilon(g, r)$ should be small enough such that we can apply the Gram-Schmidt process mentioned in Section IV.2;
- (7). $\epsilon = \epsilon(g, r)$ should be small enough such that $\|\hat{A}_{x,s} - A_{x,s}\|_{L^2(S)} \leq c_1\epsilon$ for some $c_1 = c_1(n, R)$ and any $x \in B_{x_0}(R_1)$, introduced in the proof of Proposition IV.25;
- (8). $\epsilon = \epsilon(g, r) = \|g - g_0\|_{C^r} < \min\{[2C_0(R + R_1)]^{-1}, [2C_0(R + R_1)^2]^{-1}, 1\}$, introduced in Corollary IV.27 and Lemma IV.32.

Therefore Theorem I.5 and Corollary I.6 hold for any g with ϵ small enough such that g satisfies all 11 conditions. In particular, there exists some $g \neq g_0$ such that (D, g) is a strict minimal filling and boundary rigid.

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