

Using Mixed Hodge Modules to Study Singularities

by

Bradley Dirks

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Doctoral Committee:

Professor Mircea Mustața, Chair
Postdoctoral Assistant Professor Qianyu Chen
Assistant Professor Alexander Perry
Professor Christian Schnell
Professor James Tappenden

Bradley Dirks
bdirks@umich.edu
ORCID iD: 0000-0003-3634-3147

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To my family

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ABSTRACT

Morihiro Saito's theory of Hodge modules have made an incredible impact in the study of singularities. So far, the strongest results have been obtained in the case of hypersurface singularities, using the strong properties of the V -filtration along hypersurfaces which is built into the theory of Hodge modules.

This thesis extends two important tools from the case of hypersurfaces. The first is a compatibility property between the Hodge filtration of a mixed Hodge module and the V -filtration along a higher codimension subvariety. The second is a formula explaining how to restrict to a smooth subvariety of higher codimension using the V -filtration along that subvariety. The main tool at work in proving these theorems is the blow-up along the smooth subvariety.

There are two main applications of these theorems: the first is to analyze the Hodge and weight filtration on the local cohomology module along a singular locally complete intersection subvariety. We define the minimal exponent of a locally complete intersection variety and show that its value dictates when the Hodge filtration on local cohomology is equal to the pole order filtration. This shows that the minimal exponent understands information about k -du Bois singularities, and it turns out that the minimal exponent also understands k -rational singularities, by its relation to the weight filtration on local cohomology.

The second application is to the study of the Fourier-Laplace transform of monodromic mixed Hodge modules. These modules naturally arise through Verdier's specialization construction. We explicitly write out the Hodge and weight filtrations for such modules.

CHAPTER I

Introduction

There are various results in algebraic geometry which can be stated without any hypotheses on the characteristic of the ground field. In this way, one can view varieties in characteristic p and over the complex numbers as similar objects. In the complex setting, affine varieties are Stein spaces, and so their sheaf cohomology is well understood thanks to Cartan's theorems A and B. It turns out that affine algebraic varieties in arbitrary characteristic have well understood quasicoherent sheaf cohomology, thanks to Serre's algebraic analogues of those theorems [Ser55]. This is one way in which complex analytic geometry has influenced algebraic geometry.

As a complex algebraic variety is locally defined by polynomials in \mathbf{C}^n , it naturally has an underlying analytic space, endowed with the Euclidean topology. Serre's famous GAGA paper [Ser56] made this connection between complex algebraic varieties and complex analytic spaces even more precise, in the projective setting. By Chow's theorem, it was known that any projective complex analytic variety is actually algebraic, and Serre's result shows that not only is it algebraic as a space, but all of the coherent data is algebraic, too. In this way, complex analytic geometry can have a direct impact on complex algebraic geometry, as we will see below.

Of course, the topology on an algebraic variety and that on its underlying analytic space differ immensely. Constant sheaves in the former topology have no higher cohomology, whereas in the latter topology their cohomology computes the singular cohomology of the

analytic variety. So when speaking of constant sheaves or local systems (defined in the next section), it is most interesting to do so in the Euclidean topology.

I.1: Riemann-Hilbert Correspondence, \mathcal{D} -modules and Perverse Sheaves

Let X be a smooth complex algebraic variety. The derivative of a polynomial is a polynomial, and similarly if one differentiates a regular function on X , it remains a regular function. If Ω_X^1 is the algebraic cotangent bundle of X (algebraically, the sheaf of Kähler differentials), this gives a map $d : \mathcal{O}_X \rightarrow \Omega_X^1$ which satisfies the Leibniz rule: $d(fg) = fd(g) + gd(f)$.

Given a coherent sheaf \mathcal{F} on X , we define a *flat connection* on \mathcal{F} to be a map

$$\nabla : \mathcal{F} \rightarrow \Omega_X^1 \otimes_{\mathcal{O}} \mathcal{F}, \quad \text{satisfying } \nabla(fm) = d(f) \otimes m + f\nabla(m),$$

and so that $\nabla^2 : \mathcal{F} \rightarrow \Omega_X^2 \otimes_{\mathcal{O}} \mathcal{F}$ is 0. The map ∇ gives a way to differentiate sections of the sheaf \mathcal{F} (compatibly with the \mathcal{O} -action), and the condition $\nabla^2 = 0$ is essentially equivalent to Clairaut's theorem from calculus: $\partial_{x_i}\partial_{x_j}(m) = \partial_{x_j}\partial_{x_i}(m)$.

In the Euclidean topology, by the classical Riemann-Hilbert correspondence, it turns out that such an \mathcal{F} is always *locally trivial*, meaning there exists some integer k and an open cover $X = \bigcup_i U_i$ such that $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus k}$, with the trivial connection defined by d . An easy observation is that the kernel of d is the constant functions, and so for any flat connection ∇ on \mathcal{F} , the kernel of ∇ should be a local system, i.e., a sheaf F for which there is an open cover $X = \bigcup_i U_i$ as above such that $F|_{U_i} \cong \underline{\mathbf{C}}_{U_i}^{\oplus k}$. The correspondence sending a flat connection (\mathcal{F}, ∇) to the local system $\ker(\nabla)$ is an equivalence of categories.

More generally, one can consider arbitrary \mathcal{O} -modules (not necessarily finitely generated) with a flat connection, as defined above. Such modules are called \mathcal{D}_X -modules, because they are modules over the ring \mathcal{D}_X of differential operators on X . If the tangent bundle of X is locally trivialized by choice of coordinates $\partial_{x_1}, \dots, \partial_{x_n}$, then an element of \mathcal{D}_X is of the form $\sum_{\alpha \in \mathbf{N}^n} h_{\alpha} \partial_x^{\alpha}$ where each $h_{\alpha} \in \mathcal{O}_X$. We will focus only on those modules which are locally

finitely generated over \mathcal{D}_X , which are called *coherent* \mathcal{D}_X -modules. One can associate to any non-zero coherent \mathcal{D}_X -module \mathcal{M} a notion of dimension (defined precisely in chapter two), which is always an integer between $\dim X$ and $2 \dim X$. If the dimension is equal to $\dim X$, we say \mathcal{M} is *holonomic*.

Any flat connection is holonomic, but there are many more holonomic modules than there are flat connections. It turns out that for any holonomic \mathcal{D}_X -module \mathcal{M} , there exists a dense open subset $U \subseteq X$ such that $\mathcal{M}|_U$ is isomorphic to a flat connection. The complement of the largest such U is then called the *singular locus* of \mathcal{M} . In this way, flat connections are the *smooth objects* in the category of holonomic \mathcal{D} -modules.

The replacement for the correspondence $(\mathcal{F}, \nabla) \mapsto \ker(\nabla)$ is to send a holonomic \mathcal{D} -module to its *de Rham complex* $DR_X(\mathcal{M})$, which is the complex

$$\mathcal{M} \rightarrow \Omega_X^1 \otimes \mathcal{M} \rightarrow \Omega_X^2 \otimes \mathcal{M} \rightarrow \cdots \rightarrow \Omega_X^{\dim X} \otimes \mathcal{M},$$

placed in cohomological degrees $-\dim X, \dots, 0$.

Roughly, the fact that there exists $U \subseteq X$ such that $\mathcal{M}|_U$ is isomorphic to a flat connection leads to the fact that one can *stratify* X into a disjoint union of smooth subvarieties such that \mathcal{M} restricted (in a suitable sense) to each piece is a flat connection. Making this idea precise is the content of Kashiwara’s constructibility theorem [Kas75]. A sheaf of \mathbf{C} -vector spaces L on X is *constructible* if one can stratify X so that the restriction of L to each stratum is a local system. Kashiwara’s theorem says that if \mathcal{M} is holonomic, then $DR_X(\mathcal{M})$ has constructible cohomology sheaves, which we write as $DR_X(\mathcal{M}) \in D_{con}^b(X)$, where the latter is the bounded derived category of complexes with constructible cohomology sheaves.

There is a parallel aspect to this story which is to allow for certain constructible complexes $C^\bullet \in D_{con}^b(X)$ which have “mild singularities”. Goresky and MacPherson [GM83] define and study “intersection cohomology” for a singular variety. This is a certain constructible complex which satisfies Poincaré duality, which fails in the non-smooth setting. It is a better

behaved analogue of the constant sheaf $\underline{\mathbf{C}}_X$ on such singular spaces. In their landmark work [BBDG82], Beilinson, Bernstein, Deligne and Gabber formalized the concept of “perverse sheaves”, generalizing intersection cohomology complexes.

One can define a subcategory of the category of holonomic modules, which are the regular holonomic modules. The functor DR_X restricted to this subcategory is an equivalence of categories, which behaves nicely with respect to duality and several naturally defined functors from geometry, like pushforward and pullback. This fact is known as the Riemann-Hilbert correspondence, shown independently by [Meb89] and [Kas84].

There are some other naturally defined functors on constructible complexes. Given $i : H \rightarrow X$ the inclusion of a closed subvariety of codimension 1, Deligne [Del73] defined the *nearby and vanishing cycles* functors ψ, ϕ which refine the restriction functors $i^!, i^* : D_{con}^b(X) \rightarrow D_{con}^b(H)$ in the sense that there exist natural exact triangles

$$\begin{aligned} \psi C^\bullet &\xrightarrow{can} \phi C^\bullet \rightarrow i^* C^\bullet \xrightarrow{+1} \\ i^! C^\bullet &\rightarrow \phi C^\bullet \xrightarrow{var} \psi C^\bullet \xrightarrow{+1} . \end{aligned}$$

By their construction, ψC^\bullet and ϕC^\bullet come equipped with monodromy operators. Equipping $i^* C^\bullet$ and $i^! C^\bullet$ with the trivial monodromy operators, the complexes above preserve monodromy. These complexes and their monodromy are sheaf theoretic incarnations of the cohomology of the Milnor fiber of the hypersurface H , and so contain some data concerning the singularities of H .

Gabber showed that if C^\bullet is a perverse sheaf on X , then $\psi C^\bullet[-1]$ and $\phi C^\bullet[-1]$ are perverse on H (see [Bry86, Page 14]). A natural question arises: what do ψ and ϕ correspond to on the regular holonomic \mathcal{D}_X -module side of the Riemann-Hilbert correspondence? This was answered by Kashiwara and Malgrange using the theory of V -filtrations. As V -filtrations are a major tool used in this thesis, we defer their precise definition to Section II.2 below. Vaguely, for a \mathcal{D}_X -module \mathcal{M} and a smooth hypersurface H defined by $t \in \mathcal{O}_X$, the V -

filtration of \mathcal{M} along t is a \mathbf{Q} -indexed, decreasing, discrete filtration $V^\bullet \mathcal{M}$ satisfying

$$tV^\lambda \mathcal{M} \subseteq V^{\lambda+1} \mathcal{M}, \quad \partial_t V^\lambda \mathcal{M} \subseteq V^{\lambda-1} \mathcal{M},$$

and so that $\partial_t t - \lambda$ is nilpotent on $gr_V^\lambda \mathcal{M} := V^\lambda \mathcal{M} / V^{>\lambda} \mathcal{M}$.

Under the Riemann-Hilbert correspondence, we have

$$DR_H\left(\bigoplus_{\lambda \in (0,1]} gr_V^\lambda \mathcal{M}\right) = \psi_t DR_X(\mathcal{M}), \quad DR_H\left(\bigoplus_{\lambda \in [0,1)} gr_V^\lambda \mathcal{M}\right) = \phi_t DR_X(\mathcal{M}),$$

and the morphisms *can* and *var* above correspond to

$$\partial_t : gr_V^1 \mathcal{M} \rightarrow gr_V^0 \mathcal{M}, \quad t : gr_V^0 \mathcal{M} \rightarrow gr_V^1 \mathcal{M},$$

respectively.

I.2: Various aspects of Hodge Theory

For a smooth, projective variety X (viewed as a complex manifold), Hodge theory endows the singular cohomology $H^k(X^{an}, \mathbf{C})$ with a canonical bigrading

$$H^k(X^{an}, \mathbf{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

called the *Hodge decomposition*. Using the real structure induced by $H^k(X^{an}, \mathbf{R})$, these subspaces satisfy the conjugation symmetry $H^{p,q} = \overline{H^{q,p}}$. This is functorial, in the sense that any morphism of cohomology which comes from geometry between smooth, projective varieties must preserve this decomposition. Another way to think of this data is through the *Hodge filtration* $F^\bullet H^k(X^{an}, \mathbf{C}) = \bigoplus_{p \geq \bullet} H^{p, k-p}(X)$. Abstractly, one calls the resulting structure a *Hodge structure of weight k* .

Of course, $H^k(X^{an}, \mathbf{C})$ only depends on the underlying topological space of X^{an} . On the

other hand, $H^{p,q}(X)$ can be identified with $H^q(X, \Omega_X^p)$, which depends on the algebraic structure. In this way, Hodge theory gives a powerful dictionary to translate between topology and coherent data.

This was generalized in two ways. The first of these was the study of families of smooth, projective varieties. Again, we will think of the underlying complex analytic spaces. Given a smooth, projective morphism $f : \mathcal{X} \rightarrow S$, the higher direct image $R^i f_* \mathbf{Q}_{\mathcal{X}}$ is a local system on S whose stalk at a point $s \in S$ is the cohomology of the fiber X_s . For any k , the k th singular cohomology of any fiber, being a smooth, projective variety, admits a Hodge decomposition. The individual pieces $H^{p,q}(X_s)$ need not vary holomorphically in the family \mathcal{X} , but it turns out that the Hodge filtrations $F^\bullet H^k(X_s, \mathbf{C})$ do vary holomorphically.

By the classical Riemann-Hilbert correspondence, we obtain a flat connection \mathcal{E}^i from the local system $R^i f_* \mathbf{Q}_{\mathcal{X}}$ along with subbundles $F^\bullet \mathcal{E}^i$. This situation was studied by Griffiths [Gri68a], who called the resulting object a *Variation of Hodge structure of weight k* . He noticed the important property (Griffiths transversality), which says that, if $\nabla : \mathcal{E}^i \rightarrow \mathcal{E}^i \otimes \Omega_S^1$ is the connection, that $\nabla(F^\bullet \mathcal{E}^i) \subseteq F^{\bullet-1} \mathcal{E}^i \otimes \Omega_S^1$. We will see later that this implies that $F^\bullet \mathcal{E}^i$ gives a *good* filtration for the \mathcal{D}_S -module \mathcal{E}^i . Griffiths generalized this situation to certain filtered flat connections which do not necessarily come from a family of varieties, and studied them in [Gri68b, Gri70]. This object is known as a *variation of Hodge structures* or VHS. For details, see the textbook of Voisin [Voi02, Voi03].

A bit later, Deligne [Del71, Del74] investigated the problem of dropping the smooth and projective assumptions on the variety X . The result is that the cohomology of any complex algebraic variety naturally carries a *mixed Hodge structure*. The main insight is that these cohomology spaces should carry a *weight filtration* $W_\bullet H^k(X^{an}, \mathbf{Q})$ and a Hodge filtration such that for any $i, k \in \mathbf{Z}$, the vector space $gr_i^W H^k(X^{an}, \mathbf{C})$ with the induced Hodge filtration is a pure Hodge structure of weight i .

Another place in which mixed Hodge structures arise is in the degeneration of pure Hodge structures. If one has a VHS on Δ^* , the punctured unit disk in \mathbf{C} , the work of Schmid

[Sch73] showed that this extends to give a Hodge structure over 0, but the Hodge structure is mixed, with weight filtration coming from the *monodromy filtration* of the monodromy of the local system around 0. Similarly, the cohomology of the Milnor fiber of a hypersurface singularity naturally admits a Hodge structure [Ste77]. Its relation to the singularities of the hypersurface are a prototype for the relations we will see in the thesis concerning the Hodge filtration and the V -filtration along a hypersurface.

When studying variations of mixed Hodge structure, if one wants a nice theory of degenerations like in the pure case, it is necessary to actually assume the existence of a “relative monodromy filtration” (more on this in Section II.6 below). This leads to the notion of *admissible* variations of mixed Hodge structure, due to Steenbrink and Zucker [SZ85]. This problem was also studied by Zucker [Zuc79], Cattani, Kaplan and Schmid [CKS86, CKS87], and several others.

I.3: Hodge Modules

Saito [Sai88, Sai90] developed a striking generalization of the theory of variations of Hodge structures. We will give more details in Section II.3 below, so we just mention the main ideas. In essence, Saito uses the theory of regular holonomic \mathcal{D} -modules to define variations of Hodge structures “with singularities”. As in the definition of Hodge structures, one needs a \mathbf{Q} -structure. For variations of Hodge structure $(\mathcal{E}, \nabla, F^\bullet)$, this came from a \mathbf{Q} -local system L such that $\ker(\nabla) \cong L \otimes_{\mathbf{Q}} \mathbf{C}$. In analogy with this, Saito uses a \mathbf{Q} -perverse sheaf as \mathbf{Q} -structure, so if \mathcal{M} is a regular holonomic \mathcal{D}_X -module, he requires the existence of a \mathbf{Q} -perverse sheaf and an isomorphism $DR_X(\mathcal{M}) \cong \mathcal{K} \otimes_{\mathbf{Q}} \mathbf{C}$.

Moreover, the regular holonomic \mathcal{D}_X -module should come with a good filtration $F_\bullet \mathcal{M}$ which is subject to many compatibility properties with respect to V -filtrations along locally defined functions. Finally, Saito defines the category of $MHM(X)$ inductively on $\dim X$, by saying an object is a mixed Hodge module if its “refined restrictions” $\psi_f(M), \phi_f(M)$ underlie mixed Hodge modules on $\{f = 0\}$. The base case is that $MHM(pt)$ should be the

graded polarizable category of mixed Hodge structures. To show that this is a well-defined and interesting theory is a herculean task. For details, see the original papers of Saito, or the introductory article by Schnell [Sch14].

I.4: Main Results

As mentioned above, mixed Hodge modules are defined to satisfy many nice properties with respect to the V -filtration along *hypersurfaces*. When one is interested in the behavior of a mixed Hodge module with respect to a smooth subvariety $Z = V(t_1, \dots, t_r)$, it would be interesting to understand *iterated nearby and vanishing cycles* along the hypersurfaces defined by the functions t_1, \dots, t_r . This is, understandably, notoriously difficult, as it is rather optimistic in examples to expect to explicitly compute the V -filtration along a single hypersurface.

It would be nice, then, to have an understanding of a single V -filtration which contains much of the information concerning the behavior of a mixed Hodge module \mathcal{M} with respect to the higher codimension subvariety Z . This is the aim of the main two theorems of this thesis. One thing to note, however, is that one of the main tools to understand this V -filtration is using Verdier's specialization [Ver83] [BMS06, Section 1.3], which allows one to express this V -filtration using the V -filtration along a hypersurface. Then the tools from the theory of mixed Hodge modules can be immediately applied.

Of course, another way to relate the ideal of a smooth subvariety to that of a divisor is by *blowing up*. This trick is at play in the proof of both main theorems, to reduce to known results for hypersurfaces.

The first result is a higher codimension version of the compatibility between the Hodge and V -filtrations [Sai88, Section 3.2]. However, for each i , we have maps $t_i : gr_V^\lambda \mathcal{M} \rightarrow gr_V^{\lambda+1} \mathcal{M}$ and $\partial_{t_i} : gr_V^{\lambda+1} \mathcal{M} \rightarrow gr_V^\lambda \mathcal{M}$. The natural replacement of the *var* and *can* maps in $r = 1$ are then the corresponding Koszul-like complexes $A^\lambda(\mathcal{M}, F), B^\lambda(\mathcal{M}, F)$ (a filtered and associated graded complex with differentials given by t_i) and $C^\lambda(\mathcal{M}, F)$ (an associated

graded complex with differentials given by ∂_{t_i}). For the precise definition of these complexes (resp. filtered complexes), see Chapter II (resp. Chapter IV).

The first main theorem is the following:

Theorem I.1. *[CD21, Theorem 1.1] Let (\mathcal{M}, F) underlie a mixed Hodge module on $X \times \mathbf{A}^r$. Then for $\lambda > 0$ (resp. $\lambda < 0$), the complex $A^\lambda(\mathcal{M}, F)$ (resp. $C^\lambda(\mathcal{M}, F)$) is filtered acyclic.*

For the second main theorem, we set $B(\mathcal{M}, F) = B^0(\mathcal{M}, F)$ and $C(\mathcal{M}) = C^0(\mathcal{M}, F)$. It is not too hard to see (Theorem IV.5) that, at the \mathcal{D} -module level, these compute the functors $i^! \mathcal{M}$ and $i^* \mathcal{M}$, respectively. Our next main theorem is a strengthening of this to the category of mixed Hodge modules.

As $gr_V^i \mathcal{M}$ need not even be coherent over \mathcal{D}_X , it is impossible to hope that the terms of $B(\mathcal{M})$ and $C(\mathcal{M})$ underlie mixed Hodge modules. To remedy this, we use the language of *mixed Hodge complexes*, from [Sai00] (see Definition IV.9). This is the mixed Hodge module analogue of the usual theory of mixed Hodge complexes, see [PS08].

Theorem I.2. *[CD21, Theorem 1.2] Let (\mathcal{M}, F, W) underlie a mixed Hodge module on $X \times \mathbf{A}^r$. Then $B^0(\mathcal{M}, F) \cong i^!(\mathcal{M}, F)$ and $C^0(\mathcal{M}, F) \cong i^*(\mathcal{M}, F)$. Moreover, $B^0(\mathcal{M}, F, W)$ and $C^0(\mathcal{M}, F, W)$ are mixed Hodge complexes, where the filtration W is defined using the relative monodromy filtration on $gr_V^j(\mathcal{M}, F)$ for all $0 \leq j \leq r$. Moreover, for any $k, \ell \in \mathbf{Z}$, the quasi-isomorphisms above induce isomorphisms of pure Hodge modules of weight $k + \ell$:*

$$gr_k^W \mathcal{H}^\ell B(\mathcal{M}) \cong gr_{k+\ell}^W \mathcal{H}^\ell i^! \mathcal{M},$$

$$gr_k^W \mathcal{H}^\ell C(\mathcal{M}) \cong gr_{k+\ell}^W \mathcal{H}^\ell i^* \mathcal{M}.$$

As mentioned in Corollary IV.11, this result implies that $B(\mathcal{M}, F)$ and $C(\mathcal{M}, F)$ are strict with respect to the Hodge filtration. However, they are not necessarily strict with respect to the weight filtration. To obtain a strict complex, one uses Deligne's décalé construction.

Now, we explain some applications of the two main theorems.

One of the most important numerical measure of singularities in the minimal model program is the *log canonical threshold* of a pair (X, Z) , where Z is a closed subvariety of X . It is defined in terms of data coming from a log resolution of the pair (X, Z) . For us, we let $Z \subseteq X$ be a locally complete intersection subvariety of pure codimension r . We define a refinement of the log canonical threshold of Z , called the *minimal exponent of Z* and denoted $\tilde{\alpha}(Z)$. This agrees with the log canonical threshold when $\tilde{\alpha}(Z) \leq r$. When $r = 1$, it agrees with the definition of minimal exponent due to Saito [Sai94].

For the definition, first we assume Z is defined by a regular sequence $f_1, \dots, f_r \in \mathcal{O}_X$. Then we consider the hypersurface $g = \sum_{i=1}^r y_i f_i$ in $Y = X \times \mathbf{A}^r$.

The definition uses as motivation the main result of [Mus22], which relates the Bernstein-Sato polynomial of an ideal (f_1, \dots, f_r) to that of the linear combination hypersurface $g = \sum_{i=1}^r y_i f_i$. We define $\tilde{\alpha}(Z)$ to be $\tilde{\alpha}(g|_U)$, where $U = Y - (X \times \{0\})$.

The results concerning singularities of locally complete intersection singularities are expressed using the \mathcal{D} -module \mathcal{B}_f (resp. \mathcal{B}_g and $\tilde{\mathcal{B}}_g$). These will be carefully defined in Chapter V (resp. III). In terms of \mathcal{D} -modules, \mathcal{B}_f is the pushforward of \mathcal{O}_X along the graph embedding (f_1, \dots, f_r) , and \mathcal{B}_g is the pushforward of \mathcal{O}_Y along the graph of g . Saito [Sai94] defines the *microlocalization* $\tilde{\mathcal{B}}_g = \mathcal{B}_g[\partial_z^{-1}]$, where ∂_z is the differential along the fiber coordinate of the target of the graph embedding map $Y \rightarrow Y \times \mathbf{A}^1$. Saito showed that the latter module carries a V -filtration along z and a Hodge filtration (though it does not underlie a Hodge module), and elements in $\tilde{\mathcal{B}}_g$ have a “microlocal b -function”. The modules \mathcal{B}_f and \mathcal{B}_g underlie Hodge modules, and so they too have Hodge and V -filtrations (along t_1, \dots, t_r and z , respectively). Here t_1, \dots, t_r are the fiber coordinates of the target of the graph embedding $X \rightarrow X \times \mathbf{A}^r$.

One major tool in the study of this invariant is the following strengthening of the result in [Mus22]. Here $\tilde{\mathcal{B}}_g^{(0)}$ is an eigenspace of the operator $\partial_z z + \theta_y$ on $\tilde{\mathcal{B}}_g$, defined in Chapter V.

Theorem I.3. [CDMO22, Theorem 3.3, Prop. 3.4] *Using the notation as above we have a*

filtered \mathcal{D}_X -linear isomorphism

$$\varphi : (\tilde{\mathcal{B}}_g^{(0)}, V) \rightarrow (\mathcal{B}_f, V)$$

and, moreover, we have equality of b -functions

$$b_m(s) = b_{\varphi(m)}(s)$$

where on the left, we use the microlocal b -function for $m \in \tilde{\mathcal{B}}_g$.

Using this, we are able to show that the minimal exponent we define governs when certain pieces of the Hodge filtration are contained in $V^r \mathcal{B}_f$.

Theorem I.4. [[CDMO22](#), Theorem 1.2] *Let $Z \subseteq X$ be a complete intersection of pure codimension r in X , defined by $f_1, \dots, f_r \in \mathcal{O}_X$. Then*

$$\tilde{\alpha}(Z) \geq r + k \iff F_{k+r} \mathcal{B}_f \subseteq V^r \mathcal{B}_f.$$

We relate the condition in the previous theorem to the local cohomology $\mathcal{H}_Z^r(\mathcal{O}_X)$, which naturally has the structure of a mixed Hodge module. For definitions, see Chapter [V](#).

Theorem I.5. ([[CDMO22](#), Theorem 1.3, Theorem 1.4]) *Let $Z \subseteq X$ be a complete intersection of pure codimension r in X , defined by $f_1, \dots, f_r \in \mathcal{O}_X$. Then*

$$F_k \mathcal{H}_Z^r(\mathcal{O}_X) = \left\{ \left[\sum_{\alpha} \frac{h_{\alpha}}{f_1^{\alpha_1+1} \cdots f_r^{\alpha_r+1}} \right] \mid \sum_{|\alpha| \leq k} h_{\alpha} \partial_t^{\alpha} \delta_f \in V^r \mathcal{B}_f \right\}.$$

In particular, $F_k \mathcal{H}_Z^r(\mathcal{O}_X) = P_k \mathcal{H}_Z^r(\mathcal{O}_X) := \{m \in \mathcal{H}_Z^r(\mathcal{O}_X) \mid (f_1, \dots, f_r)^{k+1} m = 0\}$ iff $\tilde{\alpha}(Z) \geq r + k$.

In [[MP22a](#)], Mustața and Popa related the condition $F_k \mathcal{H}_Z^r(\mathcal{O}_X) = P_k \mathcal{H}_Z^r(\mathcal{O}_X)$ to the following property of Z . Recall that the du Bois complex of a complex algebraic variety

Z , defined in [dB81], is an object in the filtered derived category of \mathcal{O}_Z -modules, denoted $\underline{\Omega}_Z$. The p th du Bois complex is $\underline{\Omega}_Z^p := gr_p^F \underline{\Omega}_Z[-p]$, which, if Z is smooth, agrees with the Kähler differentials Ω_Z^p . In general, $\underline{\Omega}_Z^p$ need not be concentrated in degree 0, but it always admits a natural morphism $\Omega_Z^p \rightarrow \underline{\Omega}_Z^p$. Steenbrink [Ste83] defined and studied a class of singularities, *du Bois singularities*, to be those varieties Z for which $\mathcal{O}_Z \rightarrow \underline{\Omega}_Z^0$ is a quasi-isomorphism. Mustașă, Popa, Olano and Witaszek [MOPW21] studied hypersurfaces D for which $\Omega_D^p \rightarrow \underline{\Omega}_D^p$ is a quasi-isomorphism for all $p \leq k$. They relate this condition to the minimal exponent of D as in the “if” part of the last statement in Theorem V.6. This relationship was also proved in [JKSY22], with its converse, and in that paper they named any D having this property as having *k -du Bois singularities*. See [Sch07] for an alternative definition of the du Bois complex, and [KS11] for a survey article about du Bois singularities.

A stronger condition on singularities was defined [FL22a] and studied [FL22c, FL22b] by Friedman and Laza. This gives the class of *k -rational singularities*. Using a resolution of singularities for Z , one can define a natural map $\underline{\Omega}_Z^p \rightarrow R\mathcal{H}om_{\mathcal{O}_Z}(\underline{\Omega}_Z^{d_Z-p}, \omega_Z^\bullet)$, where $d_Z = \dim Z$ and ω_Z^\bullet is the dualizing complex for Z . One says Z has *k -rational singularities* if it has *k -du Bois singularities* and the composition

$$\Omega_Z^p \rightarrow \underline{\Omega}_Z^p \rightarrow R\mathcal{H}om_{\mathcal{O}_Z}(\underline{\Omega}_Z^{d_Z-p}, \omega_Z^\bullet)$$

is a quasi-isomorphism for all $p \leq k$. In [MP22a, Theorem E] and [FL22b], it was shown that for hypersurfaces, this property is equivalent to $\tilde{\alpha}(Z) > k + 1$. Our next main theorem is a generalization of this result to the locally complete intersection case.

Theorem I.6. [CDM22, Theorem 1.1] *Let $Z \subseteq X$ be a locally complete intersection of pure codimension r . Then*

$$\tilde{\alpha}(Z) > r + k \iff Z \text{ has } k\text{-rational singularities.}$$

As another application, we use our understanding of the restriction functors for mixed

Hodge modules to study the *Fourier-Laplace* transform of a monodromic mixed Hodge module on $E = X \times \mathbf{A}^r$. Let z_1, \dots, z_r be the fiber coordinates on E , with vector fields $\partial_{z_1}, \dots, \partial_{z_r}$. These define the Euler vector field $\theta_z = \sum_{i=1}^r z_i \partial_{z_i}$.

We say a \mathcal{D}_E -module is *monodromic* if, for all sections $m \in \mathcal{M}$, there exists a non-zero polynomial $b_m(s) \in \mathbf{C}[s]$ such that

$$b_m(\theta)m = 0.$$

Such \mathcal{D}_E -modules were studied in [Bry86].

Equivalently, there is a decomposition $\mathcal{M} = \bigoplus_{\chi \in \mathbf{C}} \mathcal{M}^\chi$, where

$$\mathcal{M}^\chi = \bigcup_{\ell \geq 1} \ker((\theta_z - \chi + r)^\ell).$$

For mixed Hodge modules, $\mathcal{M}^\chi = 0$ unless $\chi \in \mathbf{Q}$, so we will only consider this case.

A mixed Hodge module M on E is monodromic if its underlying \mathcal{D}_E -module is monodromic. These modules were studied by T. Saito in [Sai22a] in the $r = 1$ case and [Sai22b] in the general case.

Given any \mathcal{D}_E -module (not necessarily monodromic) \mathcal{M} , one can define the *Fourier Laplace* transform $\mathrm{FL}(\mathcal{M})$, which is a \mathcal{D} -module on the dual vector bundle E^\vee . The \mathcal{O}_X -module is the same, and the action of the coordinates w_1, \dots, w_r and vector fields $\partial_{w_1}, \dots, \partial_{w_r}$ is defined by

$$w_i m = -\partial_{z_i} m \quad \partial_{w_i} m := z_i m.$$

It is important to note that even if \mathcal{M} is regular holonomic, it is possible that $\mathrm{FL}(\mathcal{M})$ is not. For example, on \mathbf{A}^1 , $\mathcal{M} = \mathcal{D}/(\partial_z^2 z + 1)$ gives $\mathrm{FL}(\mathcal{M}) = \mathcal{D}/(w^2 \partial_w + 1)$, which is the \mathcal{D} -module corresponding to the essential singularity $e^{1/w}$.

However, Brylinski [Bry86] showed that if \mathcal{M} is monodromic and regular holonomic, then $\mathrm{FL}(\mathcal{M})$ is also regular holonomic. Of course, $\mathrm{FL}(\mathcal{M})$ is monodromic with respect to the variables w_1, \dots, w_r , with $\mathrm{FL}(\mathcal{M})^\chi = \mathcal{M}^{r-\chi}$.

Our main result concerning the Fourier-Laplace transform gives an explicit isomorphism of $\mathrm{FL}(\mathcal{M})$ for a monodromic, regular holonomic \mathcal{D}_E -module with a \mathcal{D}_{E^\vee} -module coming from geometry. Let $p : \mathcal{E} = E \times_X E^\vee \rightarrow E$ be the projection, $g = \sum_{i=1}^r z_i w_i \in \mathcal{O}(\mathcal{E})$, $\Gamma : \mathcal{E} \rightarrow \mathcal{E} \times \mathbf{A}^1$ the graph embedding along g , and finally, $\sigma : E^\vee \rightarrow \mathcal{E}$ induced by the zero section.

Theorem I.7. *[CD21, Theorem 1.4] Let \mathcal{M} be a monodromic regular holonomic \mathcal{D}_E -module. Then, using the above notation, there is a natural isomorphism*

$$\mathrm{FL}(\mathcal{M}) \cong \mathcal{H}^0 \sigma^* \phi_t \Gamma_+ p^1(\mathcal{M})[-r].$$

With this theorem, we see that if \mathcal{M} underlies a mixed Hodge module, then so does $\mathrm{FL}(\mathcal{M})$. Using Theorem I.2, we are able to study the Hodge and weight filtration on $\mathrm{FL}(\mathcal{M})$ under this isomorphism.

Theorem I.8. *[CD21, Theorem 1.4] Let (\mathcal{M}, F_\bullet) be a filtered \mathcal{D}_E -module underlying a mixed Hodge module on E . Then the Hodge filtration on $\mathrm{FL}(\mathcal{M})$ satisfies*

$$F_p \mathrm{FL}(\mathcal{M})^{r-\chi} = F_{p-[\chi]} \mathcal{M}^\chi$$

for all $p \in \mathbf{Z}$ and $\chi \in \mathbf{Q}$.

Before stating the result for the weight filtration, we mention an important tool concerning the weight filtration of monodromic mixed Hodge modules.

Theorem I.9. *[CD21, Theorem 1.5] Let (\mathcal{M}, W_\bullet) underlie a monodromic mixed Hodge module on E . Let $N = \bigoplus_{\chi \in \mathbf{Q}} (\theta_Z - \chi + r)$ be the nilpotent operator on \mathcal{M} . Then $W_\bullet \mathcal{M}$ is its own relative monodromy filtration with respect to N , i.e., $NW_\bullet \mathcal{M} \subseteq W_{\bullet-2} \mathcal{M}$ (the other condition being automatic).*

For a monodromic module $\mathcal{N} = \bigoplus_{\chi \in \mathbf{Q}} \mathcal{N}^\chi$, we define

$$\mathcal{N}^{\lambda+\mathbf{Z}} := \bigoplus_{k \in \mathbf{Z}} \mathcal{N}^{\lambda+k},$$

for any $\lambda \in [0, 1)$. Then the weight filtration on $\mathrm{FL}(\mathcal{M})$ is given by the following:

Theorem I.10. [[CD21](#), Theorem 1.4] *Let (\mathcal{M}, W) be a monodromic \mathcal{D} -module with weight filtration $W_\bullet \mathcal{M}$ underlying a mixed Hodge module on E . Then the weight filtration on $\mathrm{FL}(\mathcal{M})$ satisfies*

$$W_k \mathrm{FL}(\mathcal{M})^{\lambda+\mathbf{Z}} = \mathrm{FL}(W_{k+r+\lceil \lambda \rceil} \mathcal{M})^{\lambda+\mathbf{Z}}.$$

The weight filtration of the Fourier-Transform was studied in [[RW22](#), Section 4] for a specific class of monodromic mixed Hodge modules. We remark here that, a priori, it seems that our definition of the Fourier-Laplace transform does not agree with that in this paper, or that for constructible complexes in [[KS90](#)], however, it turns out that they do agree. See Remark [VI.5](#) in Chapter 6.

I.5: Layout

In Chapter [II](#), we define V -filtrations in the general setting due to Sabbah. We provide many examples, explain their dependence on the defining functions, and prove their uniqueness by relating them to \mathbf{Z} -indexed filtrations. We prove that certain Koszul-like complexes are acyclic. We then proceed with a brief introduction to the theory of mixed Hodge modules. We explain the inductive definition using V -filtrations along hypersurfaces, and various important structural results about functoriality of the category of mixed Hodge modules. We end with the Verdier specialization construction, which allows one to study ${}^L V$ -filtrations for $r > 1$ in terms of the V -filtration along a hypersurface, using deformations to the normal bundle.

In Chapter [III](#), we discuss hypersurface singularities. Many of the results of the thesis

are generalizations of what has been shown so far for hypersurface singularities, and so it will be helpful to see the results in their easiest to state form, as well as to explain the history behind those results. We define the Bernstein-Sato polynomial of a hypersurface, the minimal exponent, and Hodge ideals. We also explain Saito’s microlocalization of \mathcal{B}_f , which leads to the definition of microlocal multiplier ideals.

In Chapter IV, we begin by proving “topological” properties of ${}^L V$ -filtrations for regular holonomic \mathcal{D} -modules on $X \times \mathbf{A}^r$. For example, we show that one can compute the restriction functors $i^!$ and i^* in the category of regular holonomic \mathcal{D} -modules, where $i : X \times \{0\} \rightarrow X \times \mathbf{A}^r$ is the zero section. We also use ${}^L V$ -filtrations to characterize when a module has submodules or quotients supported on $X \times \{0\}$.

The remainder of Chapter IV is dedicated to the study of mixed Hodge modules. For mixed Hodge modules, at the moment the proofs only work for $L = (1, \dots, 1)$, i.e., the standard V -filtration along $X \times \{0\}$. The two main results are a filtered acyclicity of the Koszul-like complexes in Chapter II and a bifiltered version of the computation of $i^!$ and i^* in terms of the V -filtration along t_1, \dots, t_r . The basic idea for the proof is to blow-up along $X \times \{0\}$ and locally relate the various V -filtrations of higher codimension with that of the exceptional divisor.

Chapter V is devoted to the application of the main theorems in Chapter IV to the study of the mixed Hodge module with underlying \mathcal{D} -module equal to $\mathcal{H}_Z^r(\mathcal{O}_X)$, when $Z \subseteq X$ is a complete intersection of codimension r . We show that the V -filtration on \mathcal{B}_f along t_1, \dots, t_r can be used to study the Hodge and weight filtrations on $\mathcal{H}_Z^r(\mathcal{O}_X)$. As a result, this V -filtration can characterize when Z has k -rational and k -du Bois singularities. We define the minimal exponent for Z and, using the previous result, connect it to these classes of singularities. Interestingly, the definition of the minimal exponent uses a connection between the V -filtration on \mathcal{B}_f and the microlocal V -filtration on \mathcal{B}_g for $g = \sum_{i=1}^r y_i f_i$ a hypersurface on $X \times \mathbf{A}^r$.

Finally, Chapter VI is devoted to the study of the Fourier-Laplace transform of mon-

odromic mixed Hodge modules. Using the description of i^* in terms of the V -filtration, we can compute the Fourier-Laplace transform as a composition of functors coming from geometry. Then, we are able to study the Hodge and weight filtration on the Fourier transform.

I.6: Conventions

For algebraic varieties, we follow the conventions of [Har13], so varieties are reduced and irreducible. For \mathcal{D} -modules, we follow the conventions of [HTT08], using *left* \mathcal{D} -modules throughout the entire thesis. We will provide a brief review of the theory of \mathcal{D} -modules in Chapter II. Throughout, the ground field is \mathbf{C} .

CHAPTER II

Background on V -filtrations and Mixed Hodge Modules

Morihiro Saito's theory of algebraic mixed Hodge modules is a vast generalization of the theory of variations of Hodge structure which was studied in the 1980's. It allows for variations with singularities and admits a six-functor formalism. In this chapter, we explain what we will need from the basic theory of \mathcal{D} -modules, introduce V -filtrations on \mathcal{D} -modules, in their general version due to Sabbah, which are the backbone to Saito's theory. We then give a rough description of the category of mixed Hodge modules on a smooth variety X , and explain important results about them. The chapter concludes with a description of the specialization operation, which allows one to talk about general V -filtrations in terms of those along hypersurfaces, which are much better understood.

II.1: Preliminaries on \mathcal{D} -modules

Let X be a smooth, irreducible variety of dimension n . The *ring of differential operators on X* , denoted \mathcal{D}_X , is defined to be the subring of $\mathcal{E}nd_{\mathbf{C}}(\mathcal{O}_X)$ generated by \mathcal{O}_X , which acts on itself by multiplication, and the tangent bundle \mathcal{T}_X , which we view as \mathbf{C} -linear derivations on \mathcal{O}_X and so which naturally lies inside $\mathcal{E}nd_{\mathbf{C}}(\mathcal{O}_X)$. This ring is non-commutative. For example, by the Leibniz rule, if $\theta \in \mathcal{T}_X$ is a derivation, then $[\theta, h] = \theta(h)$ for any regular function $h \in \mathcal{O}_X$.

Locally, we trivialize the tangent bundle on X by sections $\partial_{x_1}, \dots, \partial_{x_n}$, giving an isomorphism

$$\mathcal{T}_X = \bigoplus_{i=1}^n \mathcal{O}_X \partial_{x_i},$$

and then on this open subset, sections of \mathcal{D}_X are of the form

$$\sum_{\alpha \in \mathbf{N}^n} h_\alpha \partial_x^\alpha, \quad h_\alpha \in \mathcal{O}_X.$$

The ring \mathcal{D}_X comes with a \mathbf{Z} -indexed filtration by locally free \mathcal{O}_X -submodules, called the *order filtration* and denoted $F_\bullet \mathcal{D}_X$. Locally, these are the sections $\sum_{|\alpha| \leq \bullet} h_\alpha \partial_x^\alpha$, where $|\alpha| = \alpha_1 + \dots + \alpha_n$. The filtered ring $(\mathcal{D}_X, F_\bullet)$ is *almost commutative*, in the sense that $gr^F \mathcal{D}_X$ is a graded commutative ring. In fact, if T^*X is the cotangent bundle with projection $\pi : T^*X \rightarrow X$, then there is a natural identification $gr^F \mathcal{D}_X \cong \pi_* \mathcal{O}_{T^*X}$, which will be of use to us below.

As \mathcal{D}_X is not commutative, when speaking of modules over it, one must specify if \mathcal{D}_X acts on the right or the left. Throughout this thesis, all modules will be *left* \mathcal{D}_X -modules. The theory of \mathcal{D}_X -modules allows for one to go from left modules to right modules without losing any information, so this is not a restrictive condition. The category of left \mathcal{D}_X -modules is abelian. We say a \mathcal{D}_X -module is coherent if it is quasi-coherent as an \mathcal{O}_X -module and if it is locally finitely generated over \mathcal{D}_X .

Example II.1. As $\mathcal{D}_X \subseteq \mathcal{E}nd_{\mathbf{C}}(\mathcal{O}_X)$, by definition it acts on \mathcal{O}_X on the left, and so \mathcal{O}_X is a \mathcal{D}_X -module. Trivially, \mathcal{D}_X is also a \mathcal{D}_X -module.

A more important example is that of *integrable connections*. These are finite rank vector bundles \mathcal{E} on X along with a connection $\nabla : \mathcal{E} \rightarrow \Omega_X^1 \otimes_{\mathcal{O}} \mathcal{E}$, which is not an \mathcal{O}_X -linear map but which satisfies

$$\nabla(fs) = df \otimes s + f\nabla(s),$$

where $d : \mathcal{O}_X \rightarrow \Omega_X^1$ is the usual exterior derivative, and which has $\nabla \circ \nabla = 0$. Without the

last condition, this says that we know how to apply \mathcal{T}_X to sections of \mathcal{E} . The last condition ensures that this action extends to the entire ring \mathcal{D}_X .

From a \mathcal{D}_X -module, one can associate a complex of \mathbf{C} -vector spaces $DR_X(\mathcal{M})$, by

$$\mathcal{M} \xrightarrow{\nabla} \Omega_X^1 \otimes_{\mathcal{O}} \mathcal{M} \xrightarrow{\nabla} \dots \xrightarrow{\nabla} \omega_X \otimes_{\mathcal{O}} \mathcal{M}.$$

The morphisms are not \mathcal{O} -linear, as they must satisfy the Leibniz rule.

Given a \mathcal{D}_X -module \mathcal{M} with an increasing filtration $F_{\bullet}\mathcal{M}$ by \mathcal{O}_X -submodules, we say the filtration $F_{\bullet}\mathcal{M}$ is *good* if $gr^F\mathcal{M}$ is a finitely generated module over $gr^F\mathcal{D}_X$. This implies in particular that $F_p\mathcal{M} = 0$ for $p \ll 0$, $\bigcup_p F_p\mathcal{M} = \mathcal{M}$ and $F_p\mathcal{D}_X \cdot F_q\mathcal{M} \subseteq F_{p+q}\mathcal{M}$.

A module \mathcal{M} admits a good filtration if and only if it is coherent. Given a coherent \mathcal{D}_X -module \mathcal{M} , we can thus find a good filtration $F_{\bullet}\mathcal{M}$ and associate to this module a coherent $gr^F\mathcal{D}_X$ -module $gr^F\mathcal{M}$. Using the isomorphism $\pi_*\mathcal{O}_{T^*X} \cong gr^F\mathcal{D}_X$ and the fact that π is an affine map, we can define a coherent sheaf of \mathcal{O}_{T^*X} -modules, which we denote again by $gr^F\mathcal{M}$.

The *characteristic variety* of a coherent \mathcal{D}_X -module is the reduced variety underlying $Ch(\mathcal{M}) := Supp(gr^F\mathcal{M}) \subseteq T^*X$. It turns out that it does not depend on choice of good filtration on \mathcal{M} . It is a conical (i.e., \mathbf{C}^* -invariant) subvariety, because $gr^F\mathcal{M}$ is graded.

An extremely important theorem concerning $Ch(\mathcal{M})$ is that if $\mathcal{M} \neq 0$, its dimension is always $\geq n = \dim X$. This is known as *Bernstein's inequality*. A special class of coherent \mathcal{D}_X -modules is the collection which has smallest dimension for $Ch(\mathcal{M})$. A coherent \mathcal{D}_X -module \mathcal{M} is called *holonomic* if $\dim Ch(\mathcal{M}) = \dim X = n$.

Example II.2. The module \mathcal{D}_X has $Ch(\mathcal{D}_X) = T^*X$, and so it is holonomic if and only if X is a point.

It is a fact that a coherent module \mathcal{M} has $Ch(\mathcal{M}) = T^*X_X$, the zero section in the cotangent bundle, if and only if \mathcal{M} is an integrable connection, if and only if it is locally free as a \mathcal{O}_X -module, if and only if it is coherent as \mathcal{O}_X -module. Hence, any integrable

connection is automatically holonomic.

The subcategory of holonomic \mathcal{D}_X -modules is an abelian subcategory. In fact, it has *finite length*, so every holonomic \mathcal{D}_X -module admits a composition series. One can define the dual of a \mathcal{D}_X -module $\mathbb{D}(\mathcal{M})$, which in general is a complex of \mathcal{D}_X -modules. However, \mathcal{M} is holonomic if and only if this complex has a single non-vanishing cohomology, and then that cohomology is also a holonomic \mathcal{D}_X -module.

Given a morphism $f : X \rightarrow Y$, there is a pushforward functor f_+ and a pullback functor $f^!$ which send a \mathcal{D}_X -module to a complex of \mathcal{D}_Y -modules. If the starting module is holonomic, then it turns out that $\mathcal{H}^j f_+(\mathcal{M})$ and $\mathcal{H}^j f^!(\mathcal{M})$ are holonomic, too. By conjugating with the duality operator, one can define $f_! = \mathbb{D}_Y \circ F_+ \circ \mathbb{D}_X$ and $f^* = \mathbb{D}_X \circ f^! \circ \mathbb{D}_Y$. Moreover, Kashiwara's theorem [Kas75] shows that $DR_X(\mathcal{M})$ is a complex with constructible cohomology if \mathcal{M} is holonomic. In fact, it is a \mathbf{C} -perverse sheaf [BBDG82].

A perverse sheaf over a field k on X is a bounded constructible k -complex C^\bullet such that

$$(II.1.1) \quad \dim \operatorname{supp} \mathcal{H}^j(C^\bullet) \leq -j \text{ for all } j \in \mathbf{Z}$$

$$(II.1.2) \quad \dim \operatorname{supp} \mathcal{H}^j(\mathbf{D}_X C^\bullet) \leq -j \text{ for all } j \in \mathbf{Z},$$

where \mathbf{D}_X is the Verdier dual operation on constructible complexes.

As mentioned in the introduction, there is a subcategory of holonomic \mathcal{D}_X -modules, called *regular holonomic* \mathcal{D}_X -modules, which is equivalent to the category of perverse sheaves under the de Rham functor. This is called the *Riemann-Hilbert correspondence*. All functors mentioned above preserve the property of being regular holonomic, and all \mathcal{D}_X -modules considered below are regular holonomic.

II.2: V -filtrations on \mathcal{D} -modules

Let X be a smooth, irreducible algebraic variety. Let $Y = X \times \mathbf{A}^r$ with fiber coordinates t_1, \dots, t_r and corresponding derivations $\partial_{t_1}, \dots, \partial_{t_r}$. For any non-zero $L = \sum_{i=1}^r a_i s_i$ a linear form with $a_i \in \mathbf{Z}_{\geq 0}$ for all i , we have a \mathbf{Z} -indexed filtration

$${}^L V^k \mathcal{D}_Y = \left\{ \sum_{\beta, \gamma} P_{\beta, \gamma} t^\beta \partial_t^\gamma \mid L(\beta) \geq L(\gamma) + k, P_{\beta, \gamma} \in \mathcal{D}_X \right\}.$$

For example, if $a_i = \begin{cases} 1 & i \in I \\ 0 & i \notin I \end{cases}$ for some nonempty subset $I \subseteq \{1, \dots, r\}$, then ${}^L V \mathcal{D}_{X \times \mathbf{A}^r}$

agrees with the usual notion of V -filtration along the subvariety defined by $\{t_i = 0 \mid i \in I\}$ [BMS06]. Let $s_i = -\partial_{t_i} t_i \in {}^L V^0 \mathcal{D}_{X \times \mathbf{A}^r}$, so we can consider the operator $L(s) \in {}^L V^0 \mathcal{D}_{X \times \mathbf{A}^r}$.

Let \mathcal{M} be a coherent \mathcal{D}_Y -module. An ${}^L V$ -filtration on \mathcal{M} is a decreasing, \mathbf{Q} -indexed filtration ${}^L V^\bullet \mathcal{M}$ which is discrete and left continuous. Here, discrete means that there exists a \mathbf{Z} -indexed increasing sequence $\alpha_i \in \mathbf{Q}$ with $\lim_{i \rightarrow \infty} \alpha_i = \infty$, $\lim_{i \rightarrow -\infty} \alpha_i = -\infty$ and so that ${}^L V^\alpha \mathcal{M}$ only depends on the interval $\alpha \in (\alpha_i, \alpha_{i+1}]$. Left-continuous means that, for all $\alpha \in \mathbf{Q}$, we have ${}^L V^\alpha \mathcal{M} = \bigcap_{\beta > \alpha} {}^L V^\beta \mathcal{M}$. The filtration must satisfy the following properties:

1. (Compatibility) ${}^L V^\ell \mathcal{D}_X \cdot {}^L V^\lambda \mathcal{M} \subseteq {}^L V^{\lambda+\ell} \mathcal{M}$.
2. (Coherence) Each ${}^L V^\lambda \mathcal{M}$ is a coherent ${}^L V^0 \mathcal{D}_X$ -module.
3. (Discreteness) There exists an integer $k \in \mathbf{Z}_{>0}$ such that ${}^L V^\lambda \mathcal{M} = {}^L V^\gamma \mathcal{M}$ if $\lambda, \gamma \in (\frac{i}{k}, \frac{i+1}{k}]$.
4. (Goodness) For $\lambda \gg 0$, we have ${}^L V^\lambda \mathcal{M} = \sum_{i=1}^r t_i {}^L V^{\lambda-a_i} \mathcal{M}$.
5. (Principle Property) For all $\lambda \in \mathbf{Q}$, the operator $L(s) + \lambda$ is nilpotent on $gr_L^\lambda \mathcal{M} := {}^L V^\lambda \mathcal{M} / {}^L V^{>\lambda} \mathcal{M}$, here ${}^L V^{>\lambda} \mathcal{M}$ is defined as $\bigcup_{\chi > \lambda} {}^L V^\chi \mathcal{M}$.

Remark II.3. When $L(s) = \sum_{i=1}^r s_i$, the ${}^L V$ -filtration just defined is equal to the V -filtration of [BMS06, Section 1.1]. In particular, it only depends on the ideal (t_1, \dots, t_r) rather than the choice of generators. Of course, this is not true for arbitrary L . In fact, even changing the order of the generators will affect the ${}^L V$ -filtration.

These ${}^L V$ -filtrations were defined and studied in [Sab87b, Sab90], and are used in the definition of the “canonical multi-indexed V -filtration”, which we do not discuss in this thesis. We say a coherent \mathcal{D}_Y -module \mathcal{M} is L -specializable if it admits an ${}^L V$ -filtration. The following theorem says that all \mathcal{D}_Y -modules we care about are L -specializable for any L .

Theorem II.4. ($L = \sum_{i=1}^r s_i$ case [Kas83, Mal83], general case [Sab87b, Théorème 3.1.1])
Assume \mathcal{M} is a regular holonomic \mathcal{D}_Y -module. Then \mathcal{M} is L -specializable.

Example II.5. Let \mathcal{N} be a coherent \mathcal{D}_X -module, and consider the push-forward $\mathcal{M} = i_+ \mathcal{N}$, where $i : X \rightarrow X \times \mathbf{A}^r$ is the inclusion of the zero section. By definition of the push-forward for \mathcal{D} -modules, we have

$$\mathcal{M} = \bigoplus_{\alpha \in \mathbf{N}^r} \mathcal{N} \partial_t^\alpha.$$

It is not hard to check that

$${}^L V^\lambda \mathcal{M} = \bigoplus_{L(\alpha) \leq \lfloor -\lambda \rfloor} \mathcal{N} \partial_t^\alpha.$$

In particular, ${}^L V^{>0} \mathcal{M} = 0$.

Example II.6. By Kashiwara’s equivalence, the formula in the previous example holds for any coherent $\mathcal{D}_{X \times \mathbf{A}^r}$ -module with support contained in X . For example, let $Z = \{f_1 = \dots = f_r = 0\} \subseteq X$ be a closed subvariety and consider the graph embedding $\Gamma : X \rightarrow X \times \mathbf{A}^r$ along (f_1, \dots, f_r) . Let \mathcal{P} be a \mathcal{D}_X -module supported on Z and let $\mathcal{M} = \Gamma_+ \mathcal{P} = \bigoplus_{\alpha \in \mathbf{N}^r} \mathcal{P} \partial_t^\alpha \delta_f$.

We can define naturally an isomorphism

$$\tau : \mathcal{P} \rightarrow \mathcal{M}_0 = {}^L V^0 \mathcal{M} = gr_L^0 \mathcal{M}$$

as follows. Note that by the previous example, ${}^L V^0 \mathcal{M} = \bigcap_{i=1}^r \ker(t_i : \mathcal{M} \rightarrow \mathcal{M})$, and so an element $u = \sum_{\alpha} u_{\alpha} \partial_t^{\alpha} \delta_f$ lies in ${}^L V^0 \mathcal{M}$ iff $u_{\alpha + e_i} = \frac{f_i u_{\alpha}}{(\alpha_i + 1)}$ for all $1 \leq i \leq r$ and α .

Hence, defining

$$\tau(u) = \sum \frac{1}{\alpha!} f^{\alpha} u \partial_t^{\alpha} \delta_f$$

gives the desired isomorphism. The inverse is simply given by sending $\sum u_{\alpha} \partial_t^{\alpha} \delta_f \in {}^L V^0 \mathcal{M}$ to u_0 .

If we instead express such an element as $u = \sum_{i=0}^m Q_i(s_1, \dots, s_r) u_i \delta_f$ for $Q_i \in \mathbf{C}[s_1, \dots, s_r]$ and $u_i \in \mathcal{P}$, then

$$\tau^{-1}(u) = \sum_{i=0}^m Q_i(0, \dots, 0) u_i.$$

Indeed, each Q_i can be written

$$Q_i = \sum_{\alpha \in \mathbf{N}^r} c_{\alpha}^{(i)} \binom{s_1}{\alpha_1} \cdots \binom{s_r}{\alpha_r}$$

and then

$$\begin{aligned} Q_i \delta_f &= \sum_{\alpha} \frac{(-1)^{|\alpha|} c_{\alpha}^{(i)}}{\alpha!} u_i \partial_{t_1}^{\alpha_1} t_1^{\alpha_1} \cdots \partial_{t_r}^{\alpha_r} t_r^{\alpha_r} \delta_f \\ &= \sum_{\alpha} \frac{(-1)^{|\alpha|} f^{\alpha} c_{\alpha}^{(i)}}{\alpha!} u_i \partial_t^{\alpha} \delta_f. \end{aligned}$$

Example II.7. ([BMS06, Prop. 2.2]) Let $Y = X \times \mathbf{A}^{r_1+r_2}$. Let \mathcal{M} be an $X \times \mathbf{A}^{r_1}$ -module which is L -specializable for some $L = \sum_{i=1}^{r_1} a_i s_i$. Let $\ell = \sum_{i=r_1+1}^{r_1+r_2} b_i s_i$ be another linear form. If $i : X \times \mathbf{A}^{r_1} \rightarrow X \times \mathbf{A}^{r_1+r_2}$ is the inclusion of the zero section, then $i_+ \mathcal{M}$ is $L + \ell$ -specializable. Moreover, if we write

$$i_+ \mathcal{M} = \bigoplus_{\alpha \in \mathbf{N}^{r_2}} \mathcal{M} \partial_t^{\alpha},$$

then it is easy to check that

$${}^{L+\ell}V^\lambda \mathcal{M} = \bigoplus_{\alpha \in \mathbf{N}^{r_2}} {}^L V^{\lambda+\ell(\alpha)} \mathcal{M} \partial_t^\alpha.$$

This recovers the formula of [BMS06, Prop. 2.2] by taking $L = \sum_{i=1}^{r_1} s_i$ and $\ell = \sum_{i=r_1+1}^{r_2} s_i$.

Example II.8. Let $\pi : X \times \mathbf{A}^r \rightarrow X$ be the projection, with coordinates t_1, \dots, t_r on \mathbf{A}^r factor. Given a coherent \mathcal{D}_X -module \mathcal{N} , the box-product $\mathcal{M} = \mathcal{N} \boxtimes \mathcal{O}_{\mathbf{A}^r}$ is isomorphic as \mathcal{O} -modules to $\mathcal{N}[t_1, \dots, t_r]$ with the obvious \mathcal{D} -action. One can check

$${}^L V^\lambda \mathcal{M} = \{m t^\alpha \mid m \in \mathcal{M}, L(\alpha) \geq \lambda - |L|\}.$$

We prove our first result concerning these filtrations.

We fix here some notation: for \mathcal{M} an L -specializable module, let

$$\begin{aligned} A^\alpha(\mathcal{M}) &= \left[{}^L V^\alpha \mathcal{M} e_0 \xrightarrow{t} \bigoplus_{|I|=1} {}^L V^{\alpha+L_I} \mathcal{M} e_I \xrightarrow{t} \bigoplus_{|I|=2} {}^L V^{\alpha+L_I} \mathcal{M} e_I \xrightarrow{t} \dots \xrightarrow{t} {}^L V^{\alpha+|L|} \mathcal{M} e \right] \\ B^\alpha(\mathcal{M}) &= \left[gr_L^\alpha \mathcal{M} e_0 \xrightarrow{t} \bigoplus_{|I|=1} gr_L^{\alpha+L_I} \mathcal{M} e_I \xrightarrow{t} \bigoplus_{|I|=2} gr_L^{\alpha+L_I} \mathcal{M} e_I \xrightarrow{t} \dots \xrightarrow{t} gr_L^{\alpha+|L|} \mathcal{M} e \right] \\ C^\alpha(\mathcal{M}) &= \left[gr_L^{\alpha+|L|} \mathcal{M} e_0 \xrightarrow{\partial_t} \bigoplus_{|I|=1} gr_L^{\alpha+|L|-L_I} \mathcal{M} e_I \xrightarrow{\partial_t} \bigoplus_{|I|=2} gr_L^{\alpha+|L|-L_I} \mathcal{M} e_I \xrightarrow{\partial_t} \dots \xrightarrow{\partial_t} gr_L^\alpha \mathcal{M} e \right] \end{aligned}$$

be the various Koszul-like complexes placed respectively in cohomological degrees $[0, r]$, $[0, r]$ and $[-r, 0]$. Here, for $I = \{i_1 < \dots < i_\ell\}$, we set $e_I = e_{i_1} \wedge \dots \wedge e_{i_\ell}$, $e = e_1 \wedge \dots \wedge e_r$ and $e_\emptyset = e_0$ to be formal symbols which help keep track of the differential. The differentials are, respectively

$$\begin{aligned} m e_I &\mapsto \sum_{i=1}^r t_i m e_i \wedge e_I, \text{ where } m \in {}^L V^{\alpha+L_I} \mathcal{M}, \\ \bar{m} e_I &\mapsto \sum_{i=1}^r t_i \bar{m} e_i \wedge e_I, \text{ where } \bar{m} \in gr_V^{\alpha+L_I} \mathcal{M}, \end{aligned}$$

$$\bar{m}e_I \mapsto \sum_{i=1}^r \partial_{t_i} \bar{m}e_i \wedge e_I, \text{ where } \bar{m} \in gr^{\alpha+|L|-L_I} \mathcal{M},$$

where $|L| = \sum_{i=1}^r a_i$ and $L_I = \sum_{i \in I} a_i$.

Lemma II.9. *For any $\alpha \neq 0$, the complexes $B^\alpha(\mathcal{M})$ and $C^\alpha(\mathcal{M})$ are acyclic.*

Proof. We let e_1^*, \dots, e_r^* be the dual basis of e_1, \dots, e_r , i.e., $e_i^*(e_j) = \delta_{ij}$, the Kronecker delta.

Then e_k^* acts on wedge products by the alternating sum

$$e_k^*(e_{i_1} \wedge \dots \wedge e_{i_\ell}) = \sum_{j=1}^{\ell} (-1)^{j-1} e_{i_1} \wedge \dots \wedge e_k^*(e_{i_j}) \wedge \dots \wedge e_{i_\ell}.$$

We prove the claim for $C^\alpha(\mathcal{M})$, the claim for $B^\alpha(\mathcal{M})$ being completely analogous. We shall construct an automorphism of the complex $C^\alpha(\mathcal{M})$ which is nullhomotopic. The $-r + \ell$ th term of the complex $C^\alpha(\mathcal{M})$ is

$$\bigoplus_{|I|=\ell} gr_L^{\alpha+|L|-L_I} \mathcal{M}e_I,$$

where $L_I = \sum_{i \in I} a_i$. Define a map s_ℓ from the $-r + \ell$ th term to the $-r + (\ell - 1)$ th term by

$$\eta e_I \mapsto \sum_{i=1}^r a_i t_i \eta e_i^*(e_I).$$

We compute $s \circ d + d \circ s$. Given such an element ηe_I with $\eta \in gr_L^{\alpha+|L|-L_I}$, we have

$$ds(\eta e_I) = \sum_{i=1}^r a_i d(t_i \eta e_i^*(e_I)) = \sum_{i=1}^r \sum_{a=1}^r a_i \partial_{t_a} t_i \eta e_a \wedge e_i^*(e_I),$$

and

$$sd(\eta e_I) = \sum_{j=1}^r s(\partial_{t_j} \eta e_j \wedge e_I) = \sum_{j=1}^r \sum_{k=1}^r a_k t_k \partial_{t_j} e_k^*(e_j \wedge e_I).$$

We show first that all terms with e_J for $J \neq I$ cancel out in the sum. Note that, by definition, such a J must be of the form $I - \{b\} \cup \{c\}$ for some $b \in I$ and $c \notin I$. In the first

term, this comes from $e_c \wedge e_b^*(e_I)$ and in the second term, it comes from $e_b^*(e_c \wedge e_I)$. The coefficient is $a_b \partial_{t_c} t_b$ in both cases, and they come with different signs, so they cancel in the sum.

The only remaining terms have e_I , and so we must have $a = i \in I$ in the first term and $j = k \notin I$ in the second term. By checking signs, we see that

$$(sd + ds)(\eta e_I) = (L(t\partial_t) - (|L| - L_I) + |L|)(\eta)e_I,$$

which is an automorphism of $gr_L^{\alpha+|L|-L_I}$ when $\alpha \neq 0$, as a unit plus a nilpotent is a unit. This proves $C^\alpha(\mathcal{M})$ is acyclic for $\alpha \neq 0$. \square

In Lemma II.16 below, we will strengthen the previous result for $B^\alpha(\mathcal{M})$ by showing that $A^\alpha(\mathcal{M})$ is acyclic for $\alpha > 0$.

Remark II.10. We will use Lemma II.9 and II.16 in the following way. By the vanishing of the rightmost cohomology in Lemma II.9 and Lemma II.16, we see that

$${}^L V^\chi \mathcal{M} = \sum_{i=1}^r \partial_{t_i} {}^L V^{\chi+a_i} \mathcal{M} + {}^L V^{>\chi} \mathcal{M} \text{ for } \chi \neq 0,$$

and

$${}^L V^\chi \mathcal{M} = \sum_{i=1}^r t_i {}^L V^{\chi-a_i} \mathcal{M} \text{ for } \chi > |L|.$$

II.2.1: Relation to \mathbf{Z} -indexed Filtrations

We explain here an alternative point of view of ${}^L V$ -filtrations which both proves they are unique and relates them to b -functions. Later we will give an argument similar to the standard way of arguing that the V -filtration is unique for $L = \sum_{i=1}^r s_i$. This subsection is based on the analogous results for $r = 1$, which can be found, for example, in [SS, Chapter 9] or [Sab87a].

Given a $\mathcal{D}_{X \times \mathbf{A}^r}$ -module \mathcal{M} , a \mathbf{Z} -indexed filtration $U^\bullet \mathcal{M}$ is *compatible* with the filtration

${}^L V^\bullet \mathcal{D}_{X \times \mathbf{A}^r}$ if for any $j, k \in \mathbf{Z}$, we have

$${}^L V^j \mathcal{D}_{X \times \mathbf{A}^r} \cdot U^k \mathcal{M} \subseteq U^{k+j} \mathcal{M}.$$

The filtration $U^\bullet \mathcal{M}$ is *good* if it is exhaustive, compatible and if there exist $m_1, \dots, m_\ell \in \mathcal{M}$ and integers $k_1, \dots, k_\ell \in \mathbf{Z}$ such that for all $j \in \mathbf{Z}$, we have

$$U^j \mathcal{M} = \sum_{i=1}^{\ell} {}^L V^{j-k_i} \mathcal{D}_{X \times \mathbf{A}^r} \cdot m_i.$$

By exhaustiveness, if a module admits a good filtration then it is coherent over $\mathcal{D}_{X \times \mathbf{A}^r}$, and conversely, by choosing generators for the module and integers $k_i \in \mathbf{Z}$, one can define a good filtration by the above formula. If $U_1^\bullet \mathcal{M}$ and $U_2^\bullet \mathcal{M}$ are two good filtrations, then there exists $k \in \mathbf{Z}_{\geq 0}$ such that

$$U_1^{\bullet+k} \mathcal{M} \subseteq U_2^\bullet \mathcal{M} \subseteq U_1^{\bullet-k} \mathcal{M}.$$

The following lemma is a result of the characterization of good filtrations in terms of the Rees modules, and the fact that the Rees ring $R_{L_V}(\mathcal{D}_{X \times \mathbf{A}^r})$ is Noetherian, which itself is proven by realizing this ring as the ring of relative differential operators on a deformation to the normal bundle \tilde{Y}^L . See, for example, [Wu21, Section 4] and [Sab87b].

Lemma II.11. *Let $\mathcal{N} \subseteq \mathcal{M}$ be a submodule of the coherent module \mathcal{M} . If $U^\bullet \mathcal{M}$ is a good filtration on \mathcal{M} , then $U^\bullet \mathcal{N} = \mathcal{N} \cap U^\bullet \mathcal{M}$ is a good filtration on \mathcal{N} .*

A good filtration $U^\bullet \mathcal{M}$ is *specializable* if there exists a non-zero polynomial of a single variable $b(w) \in \mathbf{C}[w]$ such that

$$b(L(s) + j) U^j \mathcal{M} \subseteq U^{j+1} \mathcal{M}.$$

The collection of these polynomials forms an ideal, so there exists a minimal monic polynomial $b_U(w)$ satisfying the relation, which we call the *b-function* for $U^\bullet \mathcal{M}$.

Lemma II.12. *Let $U_1^\bullet \mathcal{M}$ and $U_2^\bullet \mathcal{M}$ be two good filtrations on \mathcal{M} . Then $U_1^\bullet \mathcal{M}$ is specializable iff $U_2^\bullet \mathcal{M}$ is.*

Proof. Let $b_1(w)$ be a b -function for $U_1^\bullet \mathcal{M}$. Let k be an integer such that $U_1^{\bullet+k} \mathcal{M} \subseteq U_2^\bullet \mathcal{M} \subseteq U_1^{\bullet-k} \mathcal{M}$. Then

$$\prod_{\ell=-k}^k b_1(L(s) + j + \ell) U_2^j \mathcal{M} \subseteq \prod_{\ell=-k}^k b_1(L(s) + j + \ell) U_1^{j-k} \mathcal{M} \subseteq U_1^{j+k+1} \mathcal{M} \subseteq U_2^{j+1} \mathcal{M}.$$

The other direction is proven in the same way. □

This lemma implies that specializability is a property of the coherent module \mathcal{M} rather than a property of $U^\bullet \mathcal{M}$. We hence call a module \mathcal{M} specializable if it admits a specializable filtration $U^\bullet \mathcal{M}$.

Given a specializable filtration $U^\bullet \mathcal{M}$, write its b -function as $\prod_{\lambda \in R(U)} (w + \lambda)^{m_\lambda}$ where $m_\lambda > 0$ and $R(U)$ is a finite set, which we call the “roots” (though really, they are negations of the roots). For \mathcal{D} -modules underlying mixed Hodge modules, $R(U) \subseteq \mathbf{Q}$, so we will make this assumption throughout the rest of this subsection, for ease of notation, though it is not necessary. We call such modules \mathbf{Q} -specializable.

Lemma II.13. *Assume \mathcal{M} is \mathbf{Q} -specializable. Then there exists a unique good filtration ${}^L V^\bullet \mathcal{M}$ with b -function having roots in $[0, 1)$.*

Proof. Let $U^\bullet \mathcal{M}$ be any good filtration on \mathcal{M} with b -function $b_U(w) = \prod_{\lambda \in R(U)} (w + \lambda)^{m_\lambda}$. Order the roots $\lambda_1 < \dots < \lambda_\ell$. By simply shifting $U^\bullet \mathcal{M}$, we can assume $\lambda_1 < \dots < \lambda_\ell < 1$. If $\lambda_1 \geq 0$, we are done.

Otherwise, define a filtration $U_1^\bullet \mathcal{M}$ by the formula

$$U_1^\bullet \mathcal{M} = U^{\bullet+1} \mathcal{M} + (L(s) + \bullet + \lambda_1)^{m_{\lambda_1}} U^\bullet \mathcal{M}.$$

This is clearly good. This has b -function given by $(w + \lambda_1 + 1)^{m_{\lambda_1}} \prod_{i>\ell} (w + \lambda_i)^{m_{\lambda_i}}$. We can repeat this process finitely many times until all roots lie in $[0, 1)$, as desired. □

The filtration constructed in the previous lemma is unique. Indeed, we will show that it agrees with the *order filtration* defined in terms of b -functions. For any $m \in \mathcal{M}$, the b -function for m is the unique, non-zero monic polynomial of smallest degree $b_m(w)$ such that

$$b_m(L(s))m \in {}^L V^1 \mathcal{D}_{X \times \mathbf{A}^r} \cdot m.$$

Such b -functions exist for every section if \mathcal{M} is specializable. Moreover, their roots are in \mathbf{Q} if \mathcal{M} is \mathbf{Q} -specializable.

Writing $b_m(w) = \prod_{\lambda \in R(m)} (w + \lambda)^{m_\lambda}$, we define the order $\text{ord}_L(m) := \min R(m)$.

Proposition II.14. *Let \mathcal{M} be a \mathbf{Q} -specializable coherent $\mathcal{D}_{X \times \mathbf{A}^r}$ -module. If $U^\bullet \mathcal{M}$ is a good filtration satisfying $R(U) \subseteq [0, 1)$, then*

$$U^\bullet \mathcal{M} = \{m \in \mathcal{M} \mid \text{ord}_L(m) \geq \bullet\}$$

and so, such a filtration is unique.

Proof. Let $m \in U^j \mathcal{M}$ with b -function $b_m(w)$. The module $\mathcal{N} = \mathcal{D}_{X \times \mathbf{A}^r} \cdot m$ has two good filtrations: $U^\bullet \mathcal{N}$ and ${}^L V^\bullet \mathcal{D}_{X \times \mathbf{A}^r} \cdot m$. So there exists an integer k such that $U^{\bullet+k} \mathcal{N} \subseteq {}^L V^\bullet \mathcal{D}_{X \times \mathbf{A}^r} \cdot m \subseteq U^{\bullet-k} \mathcal{N}$.

Then $\prod_{\ell=j}^k b_U(L(s) + \ell)m \in U^{k+1} \mathcal{N} \subseteq {}^L V^1 \mathcal{D}_{X \times \mathbf{A}^r} \cdot m$. As $R(U) \subseteq [0, 1)$, this implies $\text{ord}_L(m) \geq j$. As the roots of $b_U(w)$ lie in $[0, 1)$, this implies $R(m) \in [j, k + 1)$, and so $\text{ord}_L(m) \geq j$.

Conversely, assume $\text{ord}_L(m) \geq j$. Let $m \in U^i \mathcal{M}$ for some $i \in \mathbf{Z}$. If $i \geq j$, we are done. Otherwise, note that ${}^L V^1 \mathcal{D}_{X \times \mathbf{A}^r} \cdot m \subseteq U^{i+1} \mathcal{M}$. But also $b_U(L(s) + i)m \in U^{i+1} \mathcal{M}$. As $j > i$, we know $b_m(w)$ and $b_U(w + i)$ are coprime. Hence, we see that $m \in U^{i+1} \mathcal{M}$. Repeating in this way, we conclude $m \in U^j \mathcal{M}$, as desired. \square

Moreover, this filtration is precisely the (integer part) of the ${}^L V$ -filtration defined above, by the next proposition.

Proposition II.15. *Let \mathcal{M} be a $\mathcal{D}_{X \times \mathbf{A}^r}$ -module which admits a \mathbf{Q} -indexed ${}^L V$ -filtration as defined above. Then ${}^L V^\bullet \mathcal{M}$ is the unique specializable ${}^L V$ -filtration satisfying $R({}^L V) \subseteq [0, 1)$.*

Proof. Of course this filtration is exhaustive and compatible. Let N be large enough that $(L(s) + \lambda)^N \text{gr}_L^\lambda \mathcal{M} = 0$ for all $\lambda \in [0, |L|]$. Note that such an N exists because there are only finitely many non-zero such $\text{gr}_L^\lambda \mathcal{M}$ in the interval $[0, |L|]$, by discreteness of the filtration.

This N satisfies $(L(s) + \lambda)^N \text{gr}_L^\lambda \mathcal{M} = 0$ for all $\lambda \in \mathbf{Q}$. Indeed, this claim is proven by increasing (resp. decreasing) induction for $\lambda > |L|$ (resp. $\lambda < 0$), using Remark II.10

$$\text{gr}_L^\lambda \mathcal{M} = \sum_{i=1}^r t_i \text{gr}_L^{\lambda - a_i} \mathcal{M} \text{ for } \lambda > |L|$$

$$\text{(resp.) } \text{gr}_L^\lambda \mathcal{M} = \sum_{i=1}^r \partial_{t_i} \text{gr}_L^{\lambda + a_i} \mathcal{M} \text{ for } \lambda < 0.$$

Let $0 = \lambda_1 < \lambda_2 < \dots < \lambda_\ell < 1$ be the finitely many indices for which $\text{gr}_L^{\lambda_i} \mathcal{M} \neq 0$. By the same argument, we see that $\text{gr}_L^\lambda \mathcal{M} \neq 0$ implies $\lambda - \lfloor \lambda \rfloor = \lambda_i$ for some i . Putting these observations together, we see that $\prod_{i=1}^\ell (w + \lambda_i)^N$ is a b -function for the \mathbf{Z} -indexed filtration ${}^L V^\bullet \mathcal{M}$. As its roots lie in $[0, 1)$, all that remains to be shown is that ${}^L V^\bullet \mathcal{M}$ is good.

For all $i \in [0, |L|] \cap \mathbf{Z}$, we know ${}^L V^i \mathcal{M}$ is finitely generated over $V^0 \mathcal{D}_{X \times \mathbf{A}^r}$. Choose generators $m_1^{(i)}, \dots, m_{b_i}^{(i)} \in {}^L V^i \mathcal{M}$.

Then ${}^L V^k \mathcal{M} \supseteq \sum_{i=0}^{|L|} \sum_{c=1}^{b_i} {}^L V^{k-i} \mathcal{D}_{X \times \mathbf{A}^r} m_c^{(i)} =: U^k \mathcal{M}$ for all k . Clearly, these filtrations agree for any $k \in [0, |L|]$. Both filtrations satisfy the result of Remark II.10 (using the corresponding property for the filtered ring $(\mathcal{D}_{X \times \mathbf{A}^r}, {}^L V)$, which is easy to check) so this implies inductively that they are the same filtration. \square

By the uniqueness shown above, this implies that the \mathbf{Q} -indexed filtration ${}^L V^\bullet \mathcal{M}$ is completely determined by the \mathbf{Z} -indexed part. Indeed, if $\lambda \in [j, j+1)$ for some integer j , then ${}^L V^\lambda \mathcal{M}$ is precisely the ${}^L V^0 \mathcal{D}_{X \times \mathbf{A}^r}$ -subspace of ${}^L V^j \mathcal{M}$ which contains ${}^L V^{j+1} \mathcal{M}$ and which, in the quotient ${}^L V^j \mathcal{M} / {}^L V^{j+1} \mathcal{M}$ gives the elements which are killed by some power of $L(s) + \lambda$.

The uniqueness of the filtration ${}^L V^\bullet \mathcal{M}$ implies that any morphism $\varphi : \mathcal{M} \rightarrow \mathcal{N}$ between specializable \mathcal{D} -modules is *strict* with respect to the ${}^L V$ -filtration, meaning for all $\lambda \in \mathbf{Q}$,

$$\text{im}(\varphi) \cap {}^L V^\lambda \mathcal{N} = \varphi({}^L V^\lambda \mathcal{M}).$$

This has the following useful implication: given a short exact sequence

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{M} \rightarrow \mathcal{N} \rightarrow 0$$

of specializable $\mathcal{D}_{X \times \mathbf{A}^r}$ -modules, the induced sequence

$$0 \rightarrow {}^L V^\lambda \mathcal{L} \rightarrow {}^L V^\lambda \mathcal{M} \rightarrow {}^L V^\lambda \mathcal{N} \rightarrow 0$$

is also exact, for any $\lambda \in \mathbf{Q}$.

To end this subsection, we give the promised proof that $A^\alpha(\mathcal{M})$ is acyclic for $\alpha > 0$.

Lemma II.16. *For any $\alpha > 0$, the complex $A^\alpha(\mathcal{M})$ is acyclic.*

Proof. As $B^\alpha(\mathcal{M})$ is acyclic, it suffices to prove the claim for $\alpha \gg 0$. We will show that, for any \mathbf{Z} -indexed good ${}^L V$ -filtration $U^\bullet \mathcal{M}$ that the corresponding complex $A^j(\mathcal{M})$ is acyclic for $j \gg 0$.

By goodness, there exists a strict surjection $\bigoplus_{i=1}^a (\mathcal{D}_{X \times \mathbf{A}^r}, {}^L V[k_i]) \rightarrow (\mathcal{M}, U) \rightarrow 0$ for some integers $k_1, \dots, k_a \in \mathbf{Z}$. The kernel (\mathcal{K}, U) with its induced filtration is also good. By strictness of these morphisms, we have a short exact sequence of complexes

$$0 \rightarrow A^j(\mathcal{K}, U) \rightarrow A^j\left(\bigoplus_{i=1}^a (\mathcal{D}_{X \times \mathbf{A}^r}, {}^L V[k_i])\right) \rightarrow A^j(\mathcal{M}, U) \rightarrow 0.$$

Note that for $j > \max\{k_i\}$, the middle complex is acyclic. This is a simple computation concerning the ${}^L V$ -filtration on the ring $\mathcal{D}_{X \times \mathbf{A}^r}$ and the fact that t_i form a regular sequence in that ring, which can be checked on the associated graded $gr^F \mathcal{D}_{X \times \mathbf{A}^r}$.

Hence, by looking at the long exact sequence in cohomology, we immediately obtain $\mathcal{H}^r A^j(\mathcal{M}, U) = 0$ for $j > \max\{k_i\}$, and we get isomorphisms

$$\mathcal{H}^{b-1} A^j(\mathcal{M}, U) \cong \mathcal{H}^b A^j(\mathcal{K}, U)$$

for all $0 \leq b \leq r$. But $U^\bullet \mathcal{K}$ is also a good filtration, so by possibly increasing j , we know $\mathcal{H}^r A^j(\mathcal{K}, U) = 0$, and so $\mathcal{H}^{r-1} A^j(\mathcal{M}, U) = 0$. Repeating in this way, increasing j finitely many times, we conclude the claim. \square

II.2.2: Monodromic \mathcal{D} -modules

Let $E = X \times \mathbf{A}^r$ be a trivial vector bundle over X with fiber coordinates t_1, \dots, t_r and corresponding vector fields $\partial_{t_1}, \dots, \partial_{t_r}$. A coherent \mathcal{D}_E -module \mathcal{M} is *L-monodromic* if, for any locally defined section $m \in \mathcal{M}$, there exists a non-zero polynomial $b(s) \in \mathbf{Q}[s]$ of a single variable such that $b(L(s))m = 0$. Such a module decomposes into generalized eigenspaces for the operator $L(s)$

$$\mathcal{M} = \bigoplus_{\chi \in \mathbf{Q}} \mathcal{M}^\chi \text{ where } \mathcal{M}^\chi = \bigcup_{\ell > 0} \ker(L(s) + \chi)^\ell.$$

Any subquotient of an L -monodromic \mathcal{D}_E -module is again L -monodromic. Moreover, if $\varphi : \mathcal{M} \rightarrow \mathcal{N}$ is a morphism of L -monodromic \mathcal{D}_E -modules, it satisfies

$$\varphi(\mathcal{M}^\chi) \subseteq \mathcal{N}^\chi.$$

The L V -filtration on L -monodromic \mathcal{D}_E -modules is particularly easy to describe:

$${}^L V^\lambda \mathcal{M} = \bigoplus_{\chi \geq \lambda} \mathcal{M}^\chi.$$

It is easy to check that modules in Example II.6 (setting $Y = X \times \{0\}$) and Example

II.8 are L -monodromic for *any* slope L .

II.3: Brief Definition of Mixed Hodge Modules

We give a very brief treatment of the important aspects of the theory of mixed Hodge modules on smooth, algebraic varieties. For more details, see [Sch14], [Sai88, Sai90]. A mixed Hodge module on a smooth algebraic variety X consists of the following data: a regular holonomic \mathcal{D}_X -module \mathcal{M} , an increasing good filtration $F_\bullet \mathcal{M}$ (called the ‘‘Hodge filtration’’), a finite, increasing filtration $W_\bullet \mathcal{M}$ by \mathcal{D}_X -submodules (called the ‘‘Weight filtration’’) and a \mathbf{Q} -perverse sheaf \mathcal{K} with an isomorphism $DR_X(\mathcal{M}) \cong \mathcal{K} \otimes_{\mathbf{Q}} \mathbf{C}$. These data are required to satisfy a laundry list of properties. For example, if X is a point, then a mixed Hodge module on X is simply a graded polarizable mixed Hodge structure.

For higher dimensional X , the required properties concern the compatibility of the Hodge filtration and the V -filtration of \mathcal{M} along any locally defined hypersurface $H = \{f = 0\} \subseteq X$. Let $i : X \rightarrow X \times \mathbf{A}^1$ be the graph embedding along f . As usual, we consider the V -filtration of $i_+ \mathcal{M}$ along the smooth hypersurface $X \times \{0\}$. Then the good filtration $F_\bullet \mathcal{M}$ induces a good filtration on $i_+ \mathcal{M}$ by

$$F_p i_+ \mathcal{M} = \bigoplus_{k \geq 0} F_{p-k-1} \mathcal{M} \partial_t^k \delta.$$

This filtration must satisfy the following compatibility relations:

$$(II.3.1) \quad F_p V^\lambda i_+ \mathcal{M} \xrightarrow{t} F_p V^{\lambda+1} i_+ \mathcal{M} \text{ is an isomorphism for all } p \in \mathbf{Z}, \lambda > 0$$

$$(II.3.2) \quad F_p gr_V^{\lambda+1} i_+ \mathcal{M} \xrightarrow{\partial_t} F_{p+1} gr_V^\lambda i_+ \mathcal{M} \text{ is an isomorphism for all } p \in \mathbf{Z}, \lambda < 0,$$

which are filtered versions of the $r = 1$ case of Lemma II.16 and Lemma II.9, respectively. Note that, if one looks at the ring $\mathcal{D}_{X \times \mathbf{A}^1}$ with the order filtration F_\bullet and the V -filtration along t , these isomorphisms are satisfied. They are *not* satisfied for $\lambda < 0$, respectively,

$\lambda > 0$. Roughly, these conditions allow for bifiltered free resolutions of $(i_+\mathcal{M}, F, V)$.

Moreover, if one sets

$$\psi(\mathcal{M}, F) := \bigoplus_{\lambda \in (0,1]} gr_V^\lambda(i_+\mathcal{M}, F), \psi_1(\mathcal{M}, F) := gr_V^1(i_+\mathcal{M}, F), \phi_1(\mathcal{M}, F) := gr_V^0(i_+\mathcal{M}, F[-1]),$$

then one requires the filtration induced by F_\bullet to be good on these modules, which are \mathcal{D}_X -modules supported on H . The inductive definition of mixed Hodge modules then requires that these objects, with Hodge and weight filtration defined in the next section, underlie mixed Hodge modules.

If $t \in \mathcal{O}_X$ defines a smooth hypersurface, and M is a mixed Hodge module on X , then the definition of the restriction functors to $H = \{t = 0\}$ are

$$(II.3.3) \quad i^!M = [\psi_{t,1}(M) \xrightarrow{var} \phi_{t,1}(M)], \quad \text{resp. } i^*M = \left[\phi_{t,1}(M) \xrightarrow{can} \psi_{t,1}(M) \right],$$

placed in cohomological degree 0, 1 (resp. $-1, 0$). The \mathcal{D} -module maps underlying these morphisms are $t \cdot$ (resp. $\partial_t \cdot$).

If Z is a singular variety, then by using local embeddings into smooth algebraic varieties, one can define the category of mixed Hodge modules on Z , too.

II.4: Important Results about Mixed Hodge Modules

II.4.1: Various Theorems for Mixed Hodge Modules

Given X a smooth algebraic variety, the category $\text{MHM}(X)$ is abelian. Moreover, if M is a pure Hodge module of weight d on X , it is polarizable, so there is an isomorphism $\mathbf{D}(M) = M(d)$, where $(-)(d)$ is the Tate twist. More generally, if M is a mixed Hodge module, we have a natural isomorphism

$$\mathbf{D}(gr_d^W M) \cong gr_{-d}^W \mathbf{D}M.$$

One of the most fundamental results about mixed Hodge modules is that any morphism between them is *bistrict* with respect to the Hodge filtration and the V -filtration along any hypersurface. In practice, this means that, given a short exact sequence

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

of mixed Hodge modules, the induced sequences

$$0 \rightarrow F_p V^\lambda \mathcal{L} \rightarrow F_p V^\lambda \mathcal{M} \rightarrow F_p V^\lambda \mathcal{N} \rightarrow 0$$

are exact, for all $p \in \mathbf{Z}, \lambda \in \mathbf{Q}$.

In general, a filtered morphism $\varphi : (\mathcal{M}, F) \rightarrow (\mathcal{N}, F)$ is *strict* if $\varphi(F_p \mathcal{M}) = F_p \mathcal{N} \cap \text{im}(\varphi)$. A filtered complex (K^\bullet, F) is strict if all morphisms in the complex are strict. The first main theorem of Saito is that, for push-forwards of Hodge modules along projective morphisms, the resulting filtered complex of \mathcal{D} -modules is always strict.

Theorem II.17. (*[Sai88]*) *Let $f : Y \rightarrow X$ be a projective morphism between two smooth complex algebraic varieties with $\ell \in H^2(Y, \mathbf{Z})$ the class of an f -ample line bundle. Let M be a pure Hodge module on Y of weight w with underlying filtered \mathcal{D} -module (\mathcal{M}, F) . Then*

1. *The filtered complex $f_+(\mathcal{M}, F)$ is strict and $\mathcal{H}^i f_+(\mathcal{M}, F)$ underlies a pure Hodge module on X of weight $w + i$.*
2. *The map $\ell^i : \mathcal{H}^{-i} f_+(\mathcal{M}, F) \rightarrow \mathcal{H}^i f_+(\mathcal{M}, F)(i)$ is an isomorphism of Hodge modules for all $i \geq 0$.*

Here the functor $(-)(i)$ is the *Tate twist*, which shifts the Hodge filtration by i and decreases the weight by $2i$.

When $f : Y \rightarrow \{*\}$ is the constant map, this strictness recovers the fact that the Hodge-de Rham spectral sequence degenerates at E_1 . Moreover, the strictness of $f_+(\mathcal{M}, F)$ is true

even if (\mathcal{M}, F) underlies a mixed Hodge module with non-trivial weight filtration, see [Sai90] and [KS21].

Example II.18. We will make particular use of this theorem when $f : Y = X \times Z \rightarrow X$ is the projection, where Z is a smooth projective variety. Then the complex $f_+(\mathcal{M})$ is given by the following: consider the relative de Rham complex

$$K^\bullet = \left(\mathcal{M} \xrightarrow{d} \mathcal{M} \otimes_{\mathcal{O}} \Omega_Z^1 \xrightarrow{d} \dots \xrightarrow{d} \mathcal{M} \otimes_{\mathcal{O}} \Omega_Z^{\dim Z} \right),$$

with a filtration

$$F_p K^\bullet = \left(F_p \mathcal{M} \xrightarrow{d} F_{p+1} \mathcal{M} \otimes_{\mathcal{O}} \Omega_Z^1 \xrightarrow{d} \dots \xrightarrow{d} F_{p+\dim Z} \mathcal{M} \otimes_{\mathcal{O}} \Omega_Z^{\dim Z} \right).$$

Then strictness tells us that the natural morphism

$$R^i f_*(F_p K^\bullet) \rightarrow R^i f_*(K^\bullet) = \mathcal{H}^i f_+(\mathcal{M})$$

is injective, and the image defines the Hodge filtration on $\mathcal{H}^i f_+(\mathcal{M})$.

As a corollary of Theorem II.17, Saito proves the following strengthening of the Decomposition Theorem of [BBDG82].

Corollary II.19. *Let $(\mathcal{M}, F_\bullet \mathcal{M})$ underlie a pure polarizable Hodge module on a smooth algebraic variety Y . Let $f : Y \rightarrow X$ be a projective morphism. Then*

$$f_+(\mathcal{M}, F_\bullet) = \bigoplus_{k \in \mathbf{Z}} \mathcal{H}^k f_+(\mathcal{M}, F_\bullet)[-k]$$

in the filtered derived category of \mathcal{D}_X -modules.

The second major theorem of Saito's theory is the structure theorem for pure Hodge modules. Built into the theory is the stipulation that any pure Hodge module M decomposes

by strict support, a property which can be detected through the V -filtration. Given an irreducible closed subset $Z \subseteq X$, a module N has *strict support* Z if it has no non-trivial subquotient supported on a closed subset of Z .

Then a pure Hodge module M decomposes into strict support if there exists a finite direct sum decomposition

$$M = \bigoplus_{Z \subseteq X} M_Z,$$

where M_Z has strict support Z .

Theorem II.20. ([Sai88]) *Let X be a smooth algebraic variety and $Z \subseteq X$ a closed irreducible subset. Then*

1. *Every polarizable variation of Hodge structure V of weight $w - \dim Z$ on a Zariski open subset of Z_{reg} extends uniquely to a polarized Hodge module of weight w on X with strict support Z .*
2. *Every polarized Hodge module of weight w on X with strict support Z arises in this way.*

This theorem gives a structure theorem for the category of polarizable pure Hodge modules. Another important aspect of this category is that it is *semisimple*. For this, it is important to focus on polarizable Hodge modules.

Mixed Hodge modules on algebraic varieties admit a six functor formalism in the sense of Grothendieck, see [Sai90, Section 4] and [Gal21]. The functors are compatible with the corresponding functors for regular holonomic \mathcal{D} -modules and perverse sheaves. For example, for any morphism of varieties $f : Y \rightarrow X$, there are functors f_* , f^* , $f^!$, $f_!$ so that f^* is left adjoint to f_* , $f_!$ is left adjoint to $f^!$, there is a natural morphism $f_! \rightarrow f_*$ which is an isomorphism if f is proper, and we have

$$f^* \mathbf{D}_X = \mathbf{D}_Y f^!, \quad f_* \mathbf{D}_Y = \mathbf{D}_X f_!.$$

Moreover, if $i : Z \rightarrow X$ is a closed subvariety with complement U , then for any $M^\bullet \in D^b(\text{MHM}(X))$, we have $i_* = i_!$ (as i is proper) which is exact, $j^* = j^!$ is also exact, which follows because j is étale. Moreover, $i^!j_* = i^*j_! = j^*i_* = 0$. Finally, there are exact triangles

$$i_!i^!M^\bullet \rightarrow M^\bullet \rightarrow j_*j^*M^\bullet \xrightarrow{+1}$$

$$j_!j^!M^\bullet \rightarrow M^\bullet \rightarrow i_*i^*M^\bullet \xrightarrow{+1}.$$

Given any variety Y with constant map $a : Y \rightarrow *$, by taking the trivial Hodge structure (viewed as a Hodge module on $*$), one obtains an element $\mathbf{Q}_Y^H \in D^b(\text{MHM}(Y))$. If Y is smooth, then $\mathbf{Q}_Y^H[\dim Y]$ is a pure Hodge module of weight $\dim Y$. In fact, the underlying filtered \mathcal{D} -module is \mathcal{O}_Y with filtration given by $gr_0^F \mathcal{O}_Y = \mathcal{O}_Y$.

By functoriality of pullbacks, if $f : Y \rightarrow X$ is any morphism, then $f^*\mathbf{Q}_X^H = \mathbf{Q}_Y^H$. Hence, for a closed embedding $i : Z \rightarrow X$, we have $i^*\mathbf{Q}_X^H = \mathbf{Q}_Z^H$, though, even if X is smooth (so $\mathbf{Q}_X^H[\dim X]$ is a pure Hodge module) it is certainly not the case that $\mathbf{Q}_Z^H[\dim Z]$ is pure, unless Z is also smooth. In any case, if X is smooth, then using the properties mentioned above and by choosing a polarization on $\mathbf{Q}_X^H[\dim X]$, we get an isomorphism

$$\mathbf{DQ}_Z^H \cong i^!\mathbf{Q}_X^H(\dim X)[2 \dim X].$$

It is known [Sai90, Formula (4.5.7)] that for any variety Z , \mathbf{Q}_Z^H is mixed of weight ≤ 0 , i.e.,

$$gr_i^W \mathcal{H}^j(\mathbf{Q}_Z^H) = 0 \text{ for all } i > j.$$

If Z has pure dimension d , then by [Sai90, Formula (4.5.6)], we have $\mathcal{H}^j(\mathbf{Q}_Z^H) = 0$ for $j > d$, and the *intersection cohomology module* $IC_Z \mathbf{Q}^H$ is defined as the pure Hodge module

$$gr_d^W \mathcal{H}^d(\mathbf{Q}_Z^H),$$

which is the unique extension of $\mathbf{Q}_{Z_{reg}}^H[\dim Z]$ to Z with no subquotient supported on Z_{sing} . If Z is irreducible, this is a simple object. If Z has N irreducible components, then

$$\mathrm{End}(IC_Z \mathbf{Q}^H) = \mathbf{Q}^N,$$

where the scalars must be rational so that they respect the \mathbf{Q} -structure. Moreover, such an endomorphism is uniquely determined by its restriction to Z_{reg} , see [Sai90, Formula (4.5.14)].

By definition of $IC_Z(\mathbf{Q}_X^H)$, there exists a canonical morphism (see [HTT08, Prop. 8.2.15] for the morphism of perverse sheaves)

$$\gamma_Z : \mathbf{Q}_Z^H[\dim Z] \rightarrow IC_Z \mathbf{Q}^H.$$

Let $i : Z \rightarrow X$ be the inclusion of a closed subvariety of pure dimension d into a smooth, irreducible variety X of dimension n . The following chain of isomorphisms is easy to check by what we have said already:

$$i_* \mathbf{D}(IC_Z \mathbf{Q}^H) = gr_{-d}^W i_* \mathcal{H}^{-d} \mathbf{D}(\mathbf{Q}_Z^H) = gr_{-d}^W i_* \mathcal{H}^{-d}(i_* i^! \mathbf{Q}_X^H(n)[2n]).$$

The underlying mixed Hodge module of $\mathcal{H}^p(i_* i^! \mathbf{Q}_X^H[n])$ is the local cohomology $\mathcal{H}_Z^p(\mathcal{O}_X)$. Hence, by taking out the cohomological shifts and the Tate twist, and setting $r = n - d$, we get that all modules in this chain of equalities are isomorphic to

$$gr_{n+r}^W \mathcal{H}_Z^r(\mathcal{O}_X)(n) = W_{n+r} \mathcal{H}_Z^r(\mathcal{O}_X)(n)$$

and $gr_p^W \mathcal{H}_Z^r(\mathcal{O}_X)(n) = 0$ for $p < n + r$. This lowest piece of the weight filtration has underlying \mathcal{D}_X -module given by the intersection cohomology \mathcal{D} -module of Brylinski-Kashiwara [BK81], which is the unique simple \mathcal{D} -submodule of $\mathcal{H}_Z^r(\mathcal{O}_X)$ if Z is irreducible.

Finally, we will also $\gamma_Z^\vee = \mathbf{D}(\gamma_Z)(d)$, which can be identified through all that we have

said with a morphism

$$\gamma_Z^\vee : \mathbf{D}(IC_Z \mathbf{Q}^H)(-d) \rightarrow i^! \mathbf{Q}_X^H[n+r](r).$$

Now, let X be a smooth algebraic variety with a mixed Hodge module M on X . The underlying filtered \mathcal{D}_X -module (\mathcal{M}, F_\bullet) gives, via the Riemann-Hilbert correspondence, the \mathbf{C} -perverse sheaf $DR_X(\mathcal{M})$, which in this case is endowed with a filtration defined by

$$F_p DR_X(\mathcal{M}) = [0 \rightarrow F_p \mathcal{M} \rightarrow \Omega_X^1 \otimes_{\mathcal{O}} F_{p+1} \mathcal{M} \rightarrow \cdots \rightarrow \omega_X \otimes_{\mathcal{O}} F_{p+\dim X} \mathcal{M}].$$

Note that the morphisms are not \mathcal{O} -linear, but they are after taking gr^F : hence, we obtain a bounded complex of \mathcal{O} -modules, for any $p \in \mathbf{Z}$, given by $gr_p^F DR_X(\mathcal{M})$. This construction is compatible with proper pushforwards and satisfies

$$(II.4.1) \quad R\mathcal{H}om_{\mathcal{O}}(gr_{-p}^F DR_X(\mathcal{M}), \omega_X[\dim X]) = gr_p^F DR_X(\mathbf{D}\mathcal{M}).$$

or, in other words, it interchanges the duality for mixed Hodge modules with Grothendieck duality.

II.4.2: Hodge and Weight Filtration Indexing

In this section, we collect the shifts of Hodge and weight filtration which are incurred when applying functors to a mixed Hodge module. Throughout this thesis, we use *left* \mathcal{D} -modules, and so these conventions are for those modules. A shift of filtration is necessary when going from left modules to right, but we will not be concerned with that in this paper. Throughout, let (\mathcal{M}, F, W) be a bifiltered \mathcal{D} -module underlying a mixed Hodge module M on a smooth complex variety X .

Tate twist: For any $k \in \mathbf{Z}$, the k th Tate twist of (\mathcal{M}, F, W) is the mixed Hodge module

$M(k)$, with the same underlying \mathcal{D}_X -module and \mathbf{Q} -structure, but satisfying

$$F_{\bullet}(\mathcal{M}(k)) := F[k]_{\bullet}\mathcal{M} := F_{\bullet-k}\mathcal{M}, \quad .W_{\bullet}(\mathcal{M}(k)) = W_{\bullet+2k}\mathcal{M},$$

so that, if M is pure of weight w , then $M(k)$ is pure of weight $w - 2k$.

Smooth pullback: Let $p : X \times Y \rightarrow X$ be a smooth projection, where Y is another smooth variety, of dimension r . By [Sai90, (4.4.2)], we set $p^*(M) = M \boxtimes \mathbf{Q}_Y^H$. As shifting this cohomologically to the left by r gives

$$p^*(M)[r] = M \boxtimes \mathbf{Q}_Y^H[r] \in \text{MHM}(X \times Y),$$

we see that $p^*(M)$ is concentrated in cohomological degree r . Similarly, $p^!(M) = M \boxtimes \mathbf{Q}_Y^H[2r](r)$, which, by shifting cohomologically to the right by r , gives the mixed Hodge module

$$(II.4.2) \quad p^!(M)[-r] = M \boxtimes \mathbf{Q}_Y^Hr = p^*(M)r.$$

In particular, the underlying \mathcal{D} -module of either mixed Hodge module is $p_{\mathcal{O}}^*(\mathcal{M})$, the \mathcal{O} - (and \mathcal{D} -) module pullback of \mathcal{M} along p . By [Sai90, (2.17.4)], we see that

$$F_k p^*(\mathcal{M})[r] = p_{\mathcal{O}}^*(F_k \mathcal{M}), \quad W_k p^*(\mathcal{M})[r] = p_{\mathcal{O}}^*(W_{k-r} \mathcal{M})$$

and by using the rule for Tate twists, this gives

$$(II.4.3) \quad F_k p^!(\mathcal{M})[-r] = p_{\mathcal{O}}^*(F_{k-r} \mathcal{M}), \quad W_k p^!(\mathcal{M})[-r] = p_{\mathcal{O}}^*(W_{k+r} \mathcal{M}).$$

Closed Embedding: Let $i : X \rightarrow Y$ be the inclusion of X as a smooth subvariety in the smooth variety Y , defined by a system of coordinates t_1, \dots, t_r with vector fields $\partial_{t_1}, \dots, \partial_{t_r}$.

Then, under the isomorphism

$$i_+ \mathcal{M} \cong \bigoplus_{\alpha \in \mathbf{N}^r} \mathcal{M} \partial_t^\alpha,$$

we have

$$F_p i_+ \mathcal{M} = \bigoplus_{\alpha \in \mathbf{N}^r} F_{p-|\alpha|-r} \mathcal{M} \partial_t^\alpha, \quad W_k i_+ \mathcal{M} = \bigoplus_{\alpha \in \mathbf{N}^r} W_k \mathcal{M} \partial_t^\alpha.$$

Nearby and Vanishing Cycles: Assume $t \in \mathcal{O}_X$ defines a smooth, nonempty hypersurface H . As mentioned above, we have the nearby cycles

$$\psi_t(M) = \bigoplus_{\lambda \in (0,1]} \psi_{t,\lambda}(M)$$

and the unipotent vanishing cycles

$$\phi_{t,1}(M),$$

with underlying \mathcal{D}_H -modules

$$\psi_t(\mathcal{M}) = \bigoplus_{\lambda \in (0,1]} gr_V^\lambda \mathcal{M},$$

respectively,

$$\phi_{t,1}(\mathcal{M}) = gr_V^0 \mathcal{M},$$

where $V^\bullet \mathcal{M}$ is the V -filtration of \mathcal{M} along the hypersurface H .

The Hodge filtration is defined as

$$(II.4.4) \quad F_k \psi_t(\mathcal{M}) = \bigoplus_{\lambda \in (0,1]} F_k gr_V^\lambda \mathcal{M} = \bigoplus_{\lambda \in (0,1]} \frac{F_k V^\lambda \mathcal{M}}{F_k V^{>\lambda} \mathcal{M}},$$

$$(II.4.5) \quad F_k \phi_t(\mathcal{M}) = F_{k+1} gr_V^0 \mathcal{M} = \frac{F_{k+1} V^0 \mathcal{M}}{F_{k+1} V^{>0} \mathcal{M}}.$$

For the weight filtration, we use the relative monodromy filtration as defined in Section II.6. Set $M_i \phi_{t,\lambda}(\mathcal{M}) = \phi_{t,\lambda}(W_{i+1} \mathcal{M})$ and $M_i \phi_{t,1}(\mathcal{M}) = \phi_{t,1}(W_i \mathcal{M})$. Then $\phi_{t,\lambda}(\mathcal{M})$ carries

a relative monodromy filtration for (M_\bullet, N) , where $N = \partial_t t - \lambda$ is the nilpotent operator, and similarly $\phi_{t,1}(\mathcal{M})$ carries a relative monodromy filtration for (M_\bullet, N) where $N = \partial_t t$. The weight filtration for $\psi_t(\mathcal{M})$ is defined by taking the direct sum of the various relative monodromy filtrations for $\phi_{t,\lambda}(\mathcal{M})$, and the weight filtration for $\phi_{t,1}(\mathcal{M})$ is the relative monodromy filtration.

We define the *total vanishing cycles* by setting $\phi_{t,\lambda}(M) = \psi_{t,\lambda}(M)$, so $\phi_t(M) = \bigoplus_{\lambda \in (0,1]} \phi_{t,\lambda}(M)$.

II.5: Specialization

As mentioned above, Saito made extensive use of the V -filtration along hypersurfaces in the definition of mixed Hodge modules. In this section, we describe a method, originally due to Verdier, which allows one to study the ${}^L V$ -filtrations using properties of V -filtrations along hypersurfaces. This was used to great effect in [BMS06].

For ease of notation, set $Y = X \times \mathbf{A}^r$. Let $L(w) = \sum_{i=1}^r a_i w_i$ be a non-degenerate slope. We define the *deformation to the normal bundle* of Y along $X \times \{0\}$ in the direction L as

$$\tilde{Y}^L := \underline{\text{Spec}}_Y \left(\bigoplus_{k \in \mathbf{Z}^r} \bigotimes (t_i)^{-k_i} \otimes u^{a_i k_i} \right),$$

where u is a new variable. This admits a smooth morphism $u : \tilde{Y}^L \rightarrow \mathbf{A}^1$, so \tilde{Y}^L is a smooth algebraic variety of dimension $\dim Y + 1$. Moreover, restricting this morphism to $\mathbf{A}^1 - \{0\} \subseteq \mathbf{A}^1$, the morphism is isomorphic to the projection $p : Y \times (\mathbf{A}^1 - \{0\}) \rightarrow \mathbf{A}^1 - \{0\}$. Let $j : \{u \neq 0\} = Y \times (\mathbf{A}^1 - \{0\}) \rightarrow \tilde{Y}$ be the open immersion.

Let \mathcal{M} be a regular holonomic $\mathcal{D}_{X \times \mathbf{A}^r}$ -module. We can define a $\mathcal{D}_{\tilde{Y}^L}$ -module by

$$\tilde{\mathcal{M}} = j_* p^! (\mathcal{M})[-1],$$

which agrees with the \mathcal{O} -module $j_* p^* (\mathcal{M}) = \bigoplus_{k \in \mathbf{Z}} \mathcal{M} u^k$.

We now describe the \mathcal{D} -module action. The variety Y has local coordinates $x_1, \dots, x_n, t_1, \dots, t_r$,

and so $Y \times (\mathbf{A}^1 - \{0\})$ has coordinates $x_1, \dots, x_n, t_1, \dots, t_r, u$. However, the variety \widetilde{Y}^L has coordinates $x_1, \dots, x_n, \frac{t_1}{u^{a_1}}, \dots, \frac{t_r}{u^{a_r}}, u$. We set $\widetilde{t}_i = \frac{t_i}{u^{a_i}}$, and when u is viewed as part of this latter system of coordinates, we denote it by \widetilde{u} .

Then the change of variables formula tells us that

$$\begin{aligned}\partial_{t_i} &= \partial_{t_i}(\widetilde{t}_i)\partial_{\widetilde{t}_i} = \frac{1}{u^{a_i}}\partial_{\widetilde{t}_i}, \\ \partial_u &= \partial_u(\widetilde{u})\partial_{\widetilde{u}} + \sum_{i=1}^r \partial_u(\widetilde{t}_i)\partial_{\widetilde{t}_i}.\end{aligned}$$

Rearranging these equalities, we have

$$\begin{aligned}\partial_{\widetilde{t}_i} &= u^{a_i}\partial_{t_i}, \\ \partial_{\widetilde{u}} &= \partial_u + \sum_{i=1}^r a_i t_i u^{-1}\partial_{t_i}.\end{aligned}$$

Hence, if $mu^k \in \widetilde{\mathcal{M}}$ is a section, then

$$\begin{aligned}\partial_{\widetilde{t}_i}(mu^k) &= \partial_{t_i}(m)u^{k+a_i}, \\ \partial_{\widetilde{u}}(mu^k) &= (k + \sum_{i=1}^r a_i t_i \partial_{t_i})(m)u^{k-1}.\end{aligned}$$

Using the latter formula, one can prove the following (see [BMS06, (1.3.1)]): let $V \bullet \widetilde{\mathcal{M}}$ be the V -filtration along u . Then

$$(II.5.1) \quad V^\lambda \widetilde{\mathcal{M}} = \bigoplus_{k \in \mathbf{Z}} {}^L V^{\lambda+|L|-k-1} \mathcal{M} u^k \text{ for all } \lambda \in \mathbf{Q}.$$

With this in hand, we can define the L -specialization of \mathcal{M} as

$$Sp_L(\mathcal{M}) := \psi_u(\widetilde{\mathcal{M}}),$$

which is a regular holonomic \mathcal{D} -module on $\{u = 0\} \cong N_X Y = X \times \mathbf{A}^r$. By definition, it is given by the formula

$$Sp_L(\mathcal{M}) = \bigoplus_{\lambda \in (0,1]} gr_V^\lambda \widetilde{\mathcal{M}} = \bigoplus_{\lambda \in (0,1]} \bigoplus_{k \in \mathbf{Z}} gr_L^{\lambda+|L|-k-1} \mathcal{M} u^k.$$

Now, assume (\mathcal{M}, F) is a filtered regular holonomic \mathcal{D}_Y -module underlying a mixed Hodge module on Y . Then, as $j_*(-)$ and $p^!(-1)[-1]$ preserve the category of mixed Hodge modules, $\widetilde{\mathcal{M}}$ also underlies a mixed Hodge module, now on \widetilde{Y}^L . Similarly, $Sp_L(\mathcal{M})$ underlies a mixed Hodge module on $X \times \mathbf{A}^r$. It is easy to check, using the commutativity of duality with vanishing cycles [MM04, Sai89], that Sp_L commutes with the dual functor on mixed Hodge modules.

One observation is that the Hodge filtration on $Sp_L(\mathcal{M})$ is the obvious one induced from the Hodge filtration on \mathcal{M} . More generally, we have:

Proposition II.21. *Let (\mathcal{M}, F) underlie a mixed Hodge module on $Y = X \times \mathbf{A}^r$. Then, for any $p \in \mathbf{Z}, \lambda \in \mathbf{Q}$, we have*

$$F_p V^\gamma \widetilde{\mathcal{M}} = \bigoplus_{k \in \mathbf{Z}} V^{\gamma+|L|-1-k} \mathcal{M} \cap \left(\sum_{q=0}^{\lfloor -\gamma \rfloor} (L(t\partial_t) + k + 1)(L(t\partial_t) + k + 2) \dots (L(t\partial_t) + k + q) F_{p+1-q} V^{|L|-1-q-k} \mathcal{M} \right) u^k,$$

where if $\gamma > -1$, the formula is given by

$$F_p V^\gamma \widetilde{\mathcal{M}} = \bigoplus_{k \in \mathbf{Z}} F_{p+1} V^{\gamma+|L|-1-k} \mathcal{M} u^k.$$

We first write out an immediate corollary concerning the Hodge filtration on $Sp_L(\mathcal{M})$. Recall that in the definition of nearby cycles for Hodge modules, a Tate twist by 1 is involved:

Corollary II.22. *Let (\mathcal{M}, F) underlie a mixed Hodge module on $Y = X \times \mathbf{A}^r$. Then for any $p \in \mathbf{Z}$, we have*

$$F_p Sp_L(\mathcal{M}) = \bigoplus_{\lambda \in (0,1]} \bigoplus_{k \in \mathbf{Z}} F_p gr_L^{\lambda+|L|-k-1} \mathcal{M} u^k.$$

Proof of Proposition II.21. We make use of Formula (3.2.3.2) of [Sai88], which in this situation tells us

$$F_p j_* (p^! (\mathcal{M}) [-1]) = \sum_{q \geq 0} \partial_{\tilde{u}}^q (V^0 \widetilde{\mathcal{M}} \cap j_* j^* (F_{p-q} \widetilde{\mathcal{M}})).$$

Note that $j^* \widetilde{\mathcal{M}} = p^! (\mathcal{M}) [-1]$, which, as p is smooth of relative dimension 1, has Hodge filtration given by

$$F_{\bullet} p^! (\mathcal{M}) [-1] = \bigoplus_{k \in \mathbf{Z}} F_{\bullet+1} \mathcal{M} u^k.$$

Putting this fact, the formula II.5.1 and the action of $\partial_{\tilde{u}}$ together, we see

$$F_p \widetilde{\mathcal{M}} = \bigoplus_{k \in \mathbf{Z}} \left(\sum_{q \geq 0} (L(t\partial_t) + k + 1) \dots (L(t\partial_t) + k + q) F_{p+1-q} {}^L V^{|L|-q-1-k} \mathcal{M} \right) u^k.$$

If we take $q = 0$ on the right hand side, we see that

$$F_p \widetilde{\mathcal{M}} \supseteq \bigoplus_{k \in \mathbf{Z}} F_{p+1} {}^L V^{|L|-1-k} \mathcal{M} u^k,$$

so, after intersecting with $V^\gamma \widetilde{\mathcal{M}}$, we see

$$F_p V^\gamma \widetilde{\mathcal{M}} \supseteq \bigoplus_{k \in \mathbf{Z}} F_{p+1} {}^L V^{\max\{|L|-1-k, |L|+\gamma-1-k\}} \mathcal{M} u^k.$$

Now, let $mu^k \in F_p V^\gamma \widetilde{\mathcal{M}}$, so we can write $m = \sum_{q=0}^N (L(t\partial_t) + k + 1) \dots (L(t\partial_t) + k + q) m_q$ for some $m_q \in F_{p+1-q} {}^L V^{|L|-q-1-k} \mathcal{M}$. Moreover, as $mu^k \in V^\gamma \widetilde{\mathcal{M}}$, we have $m \in {}^L V^{|L|+\gamma-k-1} \mathcal{M}$. We write m as two sums, stopping at $\lceil -\gamma \rceil$ in the first one:

$$m = \sum_{q=0}^{\lceil -\gamma \rceil} (L(t\partial_t) + k + 1) \dots (L(t\partial_t) + k + q) m_q + \sum_{\lceil -\gamma \rceil + 1}^N (L(t\partial_t) + k + 1) \dots (L(t\partial_t) + k + q) m_q.$$

Note that $|L| + \gamma - k - 1 \leq |L| - q - k - 1$ iff $\gamma \leq -q$ iff $-\gamma \geq q$ iff $\lceil -\gamma \rceil \geq q$. Hence, the first sum is contained in ${}^L V^{|L|+\gamma-k-1} \mathcal{M}$. Hence, as m also lies in this piece of ${}^L V$ by choice

of m , we see that

$$(L(t\partial_t)+k+1)\dots(L(t\partial_t)+k+\lfloor-\gamma\rfloor+1)\sum_{q=\lfloor-\gamma\rfloor+1}^N(L(t\partial_t)+k+\lfloor-\gamma\rfloor+2)\dots(L(t\partial_t)+k+q)m_q \in {}^L V^{|L|+\gamma-1-k}\mathcal{M},$$

too.

Now, write $m' = \sum_{q=\lfloor-\gamma\rfloor+1}^N(L(t\partial_t) + k + \lfloor-\gamma\rfloor + 2)\dots(L(t\partial_t) + k + q)m_q$. As $m_q \in {}^L V^{|L|-q-k-1}\mathcal{M} \subseteq {}^L V^{|L|-N-k-1}\mathcal{M}$, we see that $m' \in {}^L V^{|L|-N-k-1}\mathcal{M}$. By definition of the ${}^L V$ -filtration, there exists some power of $L(t\partial_t) - (|L| - N - k - 1) + |L| = L(t\partial_t) + N + k + 1$ which multiplies m' into ${}^L V^{>|L|-N-k-1}\mathcal{M}$. But also the operator $(L(t\partial_t) + k + 1)\dots(L(t\partial_t) + k + \lfloor-\gamma\rfloor + 1)$ does. These operators, as polynomials in $L(t\partial_t)$ are coprime, as $N > \lfloor-\gamma\rfloor$, so by Bézout's identity, $m' \in {}^L V^{>|L|-N-k-1}\mathcal{M}$. Repeating this argument finitely many times, by discreteness of the ${}^L V$ -filtration, implies that $m' \in {}^L V^{|L|-\lfloor-\gamma\rfloor-k-1}\mathcal{M}$. Moreover, as $L(t\partial_t)$ increases the Hodge filtration by 1, it lies in $F_{p+1-\lfloor-\gamma\rfloor}\mathcal{M}$. Hence, $m_{\lfloor-\gamma\rfloor} + m' \in F_{p+1-\lfloor-\gamma\rfloor}{}^L V^{|L|-\lfloor-\gamma\rfloor-k-1}\mathcal{M}$, proving the claim. \square

II.6: Relative Monodromy Filtrations

In this section, we gather various results about relative monodromy filtrations, for more details, see [Sai90, Section 1]. This will be useful in understanding the weight filtration on $\mathrm{Sp}_L(\mathcal{M})$, the weight filtration on $i^!, i^*$ and the weight filtration on $\mathrm{FL}(\mathcal{M})$ when \mathcal{M} is monodromic.

First of all, we recall the definition of the relative monodromy filtration. Let \mathcal{A} be an abelian category with an exact subcategory $\mathcal{C} \subseteq \mathcal{A}$ admitting an additive automorphism $S : \mathcal{C} \rightarrow \mathcal{C}$. Let $A \in \mathcal{C}$ be an object with a finite filtration $M_\bullet A$ and a nilpotent endomorphism

$$N : (A, M) \rightarrow S^{-1}(A, M).$$

Then the *relative monodromy filtration* of (A, M) with respect to N is the unique, increasing filtration $W_\bullet A$ which satisfies the following two conditions:

1. $N : (A, W) \rightarrow (A, W[2])$
2. $N^i : gr_{k+i}^W gr_k^M A \rightarrow gr_{k-i}^W gr_k^M A$ is an isomorphism for all i and k .

The relative monodromy is not guaranteed to exist, unless M satisfies $gr_k^M A = A$ for a unique $k \in \mathbf{Z}$, in which case it will be called the *monodromy filtration*. For us, the exact category will be a filtered category, and S will be shifting the filtration. The notion of relative monodromy filtration is fundamental in the work of Steenbrink and Zucker [SZ85] in the study of variations of mixed Hodge structure.

We first prove some general statements about relative monodromy filtrations before looking specifically at what happens for Hodge modules. For details, see [Sch01].

Let (A, M_\bullet) be a filtered object as above with the nilpotent endomorphism N , and assume the relative monodromy filtration $W_\bullet A$ exists. Moreover, assume there exists a *splitting operator* $Y : \mathcal{M} \rightarrow \mathcal{M}$ which is diagonalizable and has integer eigenvalues and which satisfies

$$W_k \mathcal{M} = \bigoplus_{\ell \leq k} E_\ell(Y),$$

where $E_\ell(Y)$ is the ℓ -eigenspace for Y . We say Y is *admissible* if

$$(II.6.1) \quad [Y, N] = -2N \quad \text{and} \quad Y M_i \mathcal{M} \subseteq M_i \mathcal{M} \quad \text{for all } i.$$

The first condition says that $N E_\ell(Y) \subseteq E_{\ell-2}(Y)$, and the second says that M_i splits into a direct sum over the eigenspaces for Y , for any i .

Assume moreover that there exists a splitting operator Y' for $M_\bullet \mathcal{M}$ which commutes with Y . Write $N = \sum_{i \in \mathbf{Z}} N_i$, where $[Y', N_i] = i N_i$. As N preserves M , we know that $N_i = 0$ for $i > 0$. Hence, N_0 is also nilpotent.

The pair $(N_0, Y - Y')$ satisfies $[Y - Y', N_0] = [Y, N_0] - [Y', N_0] = -2N_0$, and so we can extend the pair to an \mathfrak{sl}_2 -triple by defining an operator e on the eigenspaces of $Y - Y'$ in the

usual way. This should satisfy

$$[Y - Y', e] = 2e, \quad [Y - Y', N_0] = -2N_0, \quad [e, N_0] = Y - Y'.$$

For basics on \mathfrak{sl}_2 -representations, see for example [SS, Section 3.1]. We will use three facts:

1. If A is an \mathfrak{sl}_2 -representation, then $\text{End}(A)$ is, too.
2. For all $k \geq 0$, the map $N_0 : H_k \rightarrow H_{-k}$ is an isomorphism. We call $\ker(N_0^{k+1})$ the *primitive part* P_k of H_k . It is equal to $\ker(e) \cap H_k$.
3. For any $k \geq 0$, we have a the Lefschetz decomposition

$$H_k = \bigoplus_{j \geq 0} N_0^j P_{k+2j} \subseteq P_k + N_0 P_{k+2}.$$

We call the tuple (A, M, N, Y, Y') a *Deligne system* after [Sch01] if we have the relation

$$[e, N_j] = 0 \text{ for } j \neq 0,$$

or, equivalently, $[e, N] = [e, N_0]$.

The following theorem is the main result we will use concerning Deligne systems: it says that if Y is admissible and there exists any commuting splitting operator \tilde{Y} , then we can always find a splitting operator which completes the data to a Deligne system. In fact, the splitting operator Y' which makes a Deligne system is unique.

Remark II.23. We will use automorphisms $g : A \rightarrow A$ to iteratively alter the splitting operator \tilde{Y} . We spell out the details here. Specifically, let $T : A \rightarrow A$ be an operator with $\text{ad}(\tilde{Y})(T) = -kT$ for some $k > 0$. Then T is nilpotent, so $g = 1 + T$ is an automorphism of

A. We define a new operator $\tilde{Y}_g = g\tilde{Y}g^{-1}$, which is a splitting operator. Indeed, we have

$$A = \bigoplus_i E_i(\tilde{Y}_g), \quad E_i(\tilde{Y}_g) = \{gx \mid x \in E_i(\tilde{Y})\}.$$

We will only consider T which commutes with Y , i.e., $\text{ad}(Y)(T) = 0$. Note that the decomposition $N = \sum_{i \leq 0} N_i$ was in terms of \tilde{Y} -weights. We will be interested in computing the decomposition for N in terms of \tilde{Y}_g -weights, using this decomposition. We only care about the terms $-k \leq i \leq 0$ below.

For this, let $v = gx \in E_i(\tilde{Y}_g)$, so $x \in E_i(\tilde{Y})$. Write

$$Nx = \sum_{\ell \leq 0} N_\ell x, \quad NTx = \sum_{\ell \leq 0} N_\ell Tx,$$

in terms of \tilde{Y} -weights. Then $Nv = Nx + NTx$. It is clear that for $0 \leq j < k$, the \tilde{Y} -weight $i - j$ piece is $N_{-j}x$. We write this as

$$N_{-j}x + T(N_{-j}x) - T(N_{-j}x) = g(N_{-j}x) - T(N_{-j}x),$$

where the last term has \tilde{Y} -weight $i - j - k$, and so we do not concern ourselves with it unless $j = 0$. For $j = k$, the \tilde{Y} -weight $i - k$ piece is

$$N_{-k}x + N_0Tx - TN_0x = (N_{-k} + [N_0, T])x,$$

where on the left hand side, the third-most term comes from the case $j = 0$ above.

Hence, writing N_{-i}^g for decomposition of N in terms of \tilde{Y}_g weights, we have

$$N_{-i}^g = \begin{cases} gN_{-i}g^{-1} & 0 \leq i < k \\ g(N_{-k} + \text{ad}(N_0)(T))g^{-1} & i = k \end{cases}.$$

Moreover, it is easy to see that the new e (completing the \mathfrak{sl}_2 -triple) is simply geg^{-1} .

Theorem II.24. *Let Y be a splitting operator for W which is admissible as in condition II.6.1. If there exists a splitting operator \tilde{Y} for M which commutes with Y , then there exists a unique splitting operator Y' so that (A, M, N, Y, Y') is a Deligne system.*

Proof. Let \tilde{Y} be a splitting operator for M as in the theorem statement.

We construct, by induction on k , a splitting operator \tilde{Y}_k such that $[e, N_i] = 0$ for all $0 < i < k$, where e and the decomposition $N = \sum_{i \leq 0} N_i$ depend on the splitting operator \tilde{Y}_k .

We begin with some easy observations. By definition, $\text{ad}(\tilde{Y})(N_{-k}) = -kN_{-k}$ and $\text{ad}(Y)(N_{-k}) = -2N_{-k}$ by Property II.6.1. Hence, $\text{ad}(H)(N_{-k}) = \text{ad}(Y - \tilde{Y})(N_{-k}) = (k - 2)N_{-k}$.

Now, we construct \tilde{Y}_1 . We use the fact that

$$\text{ad}(N_0) : H_1 \rightarrow H_{-1} \text{ is an isomorphism,}$$

where H_\bullet is the \bullet -weight space of the $\text{ad}(H)$ action on $\text{End}(A)$.

Hence, $N_{-1} = \text{ad}(N_0)(N''_{-1})$ for some unique $N''_{-1} \in H_1$. As $\text{ad}(Y)(N_{-1}) = -2N_{-1}$, we see that

$$-2\text{ad}(N_0)(N''_{-1}) = \text{ad}(Y)\text{ad}(N_0)(N''_{-1}) = \text{ad}(N_0)\text{ad}(Y)(N''_{-1}) - 2\text{ad}(N_0)(N''_{-1})$$

again using Property II.6.1. Hence, $\text{ad}(N_0)\text{ad}(Y)(N''_{-1}) = 0$. But $\text{ad}(Y)(N''_{-1}) \in H_1$, too, as Y commutes with $Y - \tilde{Y}$. But $\text{ad}(N_0)$ is injective on H_1 , so we must have $\text{ad}(Y)(N''_{-1}) = 0$.

Set $g = 1 - N''_{-k}$. We are now in the situation of Remark II.23. For this splitting operator, as shown in that remark, we use geg^{-1} and $g(N_{-1} + \text{ad}(N_0)(N''_{-1}))g^{-1}$. We need to show the vanishing of the commutator. This follows from the following computation:

$$[geg^{-1}, g(N_{-1} + \text{ad}(N_0)(N''_{-1}))g^{-1}] = g[e, N_{-1} + \text{ad}(N_0)(N''_{-1})]g^{-1} = g\text{ad}(e)(N''_{-1})g^{-1} = 0,$$

proving that this new splitting operator is an improvement.

Now, let $k \geq 2$, and assume we have a splitting operator for M_\bullet giving a decomposition $N = \sum_{i \leq 0} N_i$ and an \mathfrak{sl}_2 -triple $(N_0, Y - \tilde{Y}, e)$ with $[e, N_{-i}] = 0$ for $0 < i < k$. We construct one with $[e, N_{-k}] = 0$, proving the claim by induction.

By definition, N_{-k} has $H = (Y - \tilde{Y})$ -degree $(k - 2) \geq 0$. Hence, by the Lefschetz decomposition, there exists a decomposition

$$N_{-k} = N'_{-k} + \text{ad}(N_0)(N''_{-k})$$

where N'_{-k} is $\text{ad}(e)$ -primitive and N''_{-k} has H -degree k . Using $\text{ad}(Y)N_{-k} = -2N_{-k}$ (by Property II.6.1), we have

$$\begin{aligned} -2N'_{-k} - 2\text{ad}(N_0)(N''_{-k}) &= \text{ad}(Y)N'_{-k} + \text{ad}(Y)\text{ad}(N_0)(N''_{-k}) \\ &= \text{ad}(Y)N'_{-k} + \text{ad}(N_0)\text{ad}(Y)(N''_{-k}) - 2\text{ad}(N_0)(N''_{-k}). \end{aligned}$$

Rearranging, we get

$$(\text{ad}(Y) + 2)N'_{-k} + \text{ad}(N_0)\text{ad}(Y)(N''_{-k}) = 0.$$

Hence, if we apply $\text{ad}(N_0)^{k-1}$ to both sides of the equality, we get

$$\text{ad}(N_0)^k \text{ad}(Y)(N''_{-k}) = 0,$$

but $\text{ad}(Y)(N''_{-k}) \in H_k$, so $\text{ad}(N_0)^k$ is injective on H_k , proving $\text{ad}(Y)(N''_{-k}) = 0$.

As in Remark II.23, we use $g = 1 - N''_{-k}$ to define a new splitting operator. We use geg^{-1} , $gN_{-i}g^{-1}$ and $g(N_{-k} + \text{ad}(N_0)(-N''_{-k}))g^{-1}$, and we must check the vanishing of commutators. But this is shown the same as above, proving the claim. \square

Corollary II.25. *Let $T : (A_1, M, N, Y, Y') \rightarrow (A_2, M, N, Y, Y')$ be a morphism of Deligne*

systems, i.e., $T : (A_1, M) \rightarrow (A_2, M)$, $TY = YT$ and $TN = NT$. Then

$$TY' = Y'T.$$

Proof. As T preserves M , we can write $T = \sum_{i \leq 0} T_i$ where $Y'T_i - T_iY' = iT_i$. We will prove the claim by showing $T_0 = T$, i.e., $T_i = 0$ for all $i < 0$. We proceed by induction on i . We will abuse notation and write $[T_i, N_j] = T_iN_j - N_jT_i$, where it is understood that in the first term, N_j is for A_1 and N_j in the second term is for A_2 . We do the same for e .

As $TN = NT$, by looking at Y' eigenspaces, we have $N_0T_{-1} + N_{-1}T_0 = T_0N_{-1} + T_{-1}N_0$.

We write this as

$$[N_0, T_{-1}] + [N_{-1}, T_0] = 0.$$

Using this, we compute $[e, [N_0, T_{-1}]] = [e, [T_0, N_{-1}]] = [[e, T_0], N_{-1}] + [T_0, [e, N_{-1}]]$, but $[e, T_0] = [e, N_{-1}] = 0$. Hence, we have shown that $\text{ad}(e)\text{ad}(N_0)(T_{-1}) = 0$. But $\text{ad}(Y - Y')(T_{-1}) = T_{-1}$ by definition of the decomposition $T = \sum T_j$. Hence, $T_{-1} = 0$.

Now, assume inductively $T_{-1} = T_{-2} = \dots = T_{-k+1} = 0$. We show $T_{-k} = 0$. The same argument decomposing $TN = NT$ into Y' -eigenspaces and using the inductive hypothesis shows

$$[N_0, T_{-k}] + [N_{-k}, T_0] = 0,$$

and the second computation is exactly the same. Then, use $\text{ad}(Y - Y')(T_{-k}) = kT_{-k}$ to conclude $T_{-k} = 0$, as $k > 0$. \square

For Hodge modules, the existence of relative monodromy filtrations is built into the definition of mixed Hodge modules. Indeed, one requires

1. For any W -filtered \mathcal{D} -module (\mathcal{M}, W_\bullet) underlying a mixed Hodge module and any locally defined, non-constant function $f \in \mathcal{O}_X$, the nearby cycles $\psi_f(\mathcal{M})$ and unipotent vanishing cycles $\phi_{f,1}(\mathcal{M})$ admit a relative monodromy filtration with respect to the

induced filtrations

$$M_{\bullet}\psi_f(\mathcal{M}) = \psi_f(W_{\bullet+1}\mathcal{M}), \quad M_{\bullet}\phi_{f,1}(\mathcal{M}) = \phi_{f,1}(W_{\bullet}\mathcal{M})$$

and the nilpotent operator N .

2. For $(\mathcal{M}, F_{\bullet}, W_{\bullet})$ underlying a mixed Hodge module, the following sequence is exact for all $\lambda \in \mathbf{Q}, k, p \in \mathbf{Z}$:

$$0 \rightarrow F_k V^{\lambda} W_{p-1} \mathcal{M} \rightarrow F_k V^{\lambda} W_p \mathcal{M} \rightarrow F_k V^{\lambda} gr_p^W \mathcal{M} \rightarrow 0,$$

where $V^{\bullet}\mathcal{M}$ is the V -filtration along f .

Using specialization, we see that the relative monodromy filtration on $gr_L^{\lambda}\mathcal{M}$ exists for any $\lambda \in \mathbf{Q}$. Moreover, using $\widetilde{\mathcal{M}}$ as in the specialization construction, we obtain short exact sequences for all $\lambda \in \mathbf{Q}, k, p \in \mathbf{Z}$:

$$0 \rightarrow F_k^L V^{\lambda} W_{p-1} \mathcal{M} \rightarrow F_k^L V^{\lambda} W_p \mathcal{M} \rightarrow F_k^L V^{\lambda} gr_p^W \mathcal{M} \rightarrow 0.$$

We are interested in the following statement concerning splitting operators:

Lemma II.26. *Let X be smooth and consider for $j = 1, 2$ bifiltered \mathcal{D} -modules $(\mathcal{M}, F_{\bullet}, W_{\bullet})$ and $(\mathcal{M}_j, F_{\bullet}, W_{\bullet})$ underlying mixed Hodge modules M, M_i on $X \times \mathbf{A}^r$ with coordinates t_1, \dots, t_r on \mathbf{A}^r . Let ${}^L V$ be the canonical ${}^L V$ -filtration along t_1, \dots, t_r . For $j = 1, 2$ (and for \mathcal{M}), let $M_{\bullet} gr_L^{\lambda}(\mathcal{M}_j) = gr_L^{\lambda}(W_{\bullet}\mathcal{M}_j)$ and $W_{\bullet} gr_L^{\lambda}(\mathcal{M}_j)$ the relative monodromy filtration for M_{\bullet} and the nilpotent operator $L(s) + \lambda$. Then*

1. For $1 \leq i \leq r$, we have that the induced filtered morphism

$$t_i : (gr_{\bullet}^W gr_L^{\lambda} \mathcal{M}, F) \rightarrow (gr_{\bullet}^W gr_L^{\lambda+a_i} \mathcal{M}, F)$$

splits along the decomposition induced by M_\bullet , i.e.,

$$t_i : (gr_\bullet^W gr_*^M gr_L^\lambda \mathcal{M}, F) \rightarrow (gr_\bullet^W gr_*^M gr_L^{\lambda+a_i} \mathcal{M}, F).$$

2. For $1 \leq i \leq r$, we have that the induced filtered morphism

$$\partial_{t_i} : (gr_\bullet^W gr_L^\lambda \mathcal{M}, F) \rightarrow (gr_\bullet^W gr_L^{\lambda-a_i} \mathcal{M}, F[-1])$$

splits along the decomposition induced by M_\bullet , i.e.,

$$\partial_{t_i} : (gr_\bullet^W gr_*^M gr_L^\lambda \mathcal{M}, F) \rightarrow (gr_\bullet^W gr_*^M gr_L^{\lambda-a_i} \mathcal{M}, F[-1]).$$

3. If $\varphi : M_1 \rightarrow M_2$ is a morphism of mixed Hodge modules, then

$$gr_\bullet^W \varphi : (gr_\bullet^W gr_L^\lambda \mathcal{M}_1, F) \rightarrow (gr_\bullet^W gr_L^\lambda \mathcal{M}_2, F)$$

splits along the decomposition induced by M_\bullet .

Proof. It will suffice, by Corollary II.25 to complete to a Deligne system and show that these morphisms induce morphisms of Deligne systems. Then each morphism will commute with the splitting operator for M_\bullet , which is exactly the claim.

Our object of interest is $A = gr_\bullet^W gr_L^\lambda \mathcal{M}$ (or $A_i = gr_\bullet^W gr_L^\lambda \mathcal{M}_i$), with $N = L(s) + \lambda$, $W_k gr_\bullet^W gr_L^\lambda \mathcal{M} = \bigoplus_{j \leq k} gr_j^W gr_L^\lambda \mathcal{M}$ and the obvious splitting operator for W . By specialization and [Sai90, Prop. 1.5], there exists a splitting operator for M_\bullet on $gr_\bullet^W gr_L^\lambda \mathcal{M}$.

Hence, by Theorem II.24, there exists a unique splitting operator which completes this data into a Deligne system. It is obvious that each morphism described respects N, M and the splitting Y , hence is a morphism of Deligne systems. \square

II.7: Technical Applications of the Specialization Construction

In this section, we prove several technical lemmas which will be fundamental in the proofs below. The main idea, following [BMS06], is that claims about higher codimension V -filtrations can be reduced to those for hypersurfaces, where they are better understood.

Lemma II.27. ([BMS06, Prop. 3.2]) *Let $Y = X \times \mathbf{A}^r$, and let W be a smooth projective variety. Denote by $p : Y \times W \rightarrow Y$ the projection to Y , which is smooth and projective. Let M be a mixed Hodge module on $Y \times W$, with underlying filtered \mathcal{D}_Y -module (\mathcal{M}, F) . Let ${}^L V^\bullet \mathcal{M}$ be the ${}^L V$ -filtration along $X \times W$. Then the bifiltered direct image is bistrict, and the induced F and ${}^L V$ filtrations are the Hodge and ${}^L V$ -filtrations on the \mathcal{D}_Y -module underlying $\mathcal{H}^k p_* M$.*

Proof. The pushforward is defined using the bifiltered relative de Rham complex whose $-i$ th term is

$$\Omega_W^{\dim W - i} \otimes_{\mathcal{O}} (\mathcal{M}, F[i], {}^L V).$$

Let \tilde{Y}^L be the deformation to the normal bundle considered above, and consider $\tilde{Y}^L \times W$. Using the method above, we obtain a mixed Hodge module \tilde{M} on $\tilde{Y}^L \times W$, which has underlying \mathcal{O} -module $\tilde{\mathcal{M}} = \bigoplus_{k \in \mathbf{Z}} \mathcal{M} u^k$. Let $\tilde{p} : \tilde{Y}^L \times W \rightarrow \tilde{Y}^L$. Then the pushforward is again defined using the bifiltered relative de Rham complex

$$\Omega_W^{\dim W - i} \otimes_{\mathcal{O}} (\tilde{\mathcal{M}}, F[i], V),$$

where V is the V -filtration along u .

By [Sai88, Prop. 3.3.17], we have bistrictness of $\tilde{p}_+(\tilde{\mathcal{M}}, F[-i], V)$, as this concerns only the V -filtration along a hypersurface. We conclude the desired claims now by decomposing along the u^k direct sum and by Proposition II.21. \square

The next claim concerns the weight filtration, the argument is standard:

Lemma II.28. *Let (\mathcal{M}, F) be a filtered \mathcal{D} -module underlying a mixed Hodge module on $Y \times W$. Let $p : Y \times W \rightarrow Y$ be the second projection, and let ${}^L V^\bullet \mathcal{M}$ be the ${}^L V$ -filtration along $X \times W$. Then*

1. *The spectral sequence associated to the relative monodromy filtration on $p_+(gr_L^\alpha \mathcal{M}, F)$ degenerates at the second page.*
2. *If (\mathcal{M}, F) underlies a polarizable pure Hodge module, then $E_2^{p,q}$ is a filtered direct summand of $E_1^{p,q}$ for all $p, q \in \mathbf{Z}$.*
3. *If (\mathcal{M}, F) underlies a polarizable pure Hodge module and $W_\bullet r_L^\alpha \mathcal{M}$ is the monodromy filtration, then the image of*

$$\mathcal{H}^i p_+(W_k gr_L^\alpha \mathcal{M}) \rightarrow \mathcal{H}^i p_+(gr_L^\alpha \mathcal{M})$$

is the monodromy filtration of $\mathcal{H}^i p_+(gr_L^\alpha \mathcal{M})$, which by the previous Lemma is $gr_L^\alpha \mathcal{H}^i p_+(\mathcal{M})$.

4. *We have the decomposition in the filtered derived category of \mathcal{D} -modules*

$$p_+(gr_k^W gr_L^\alpha \mathcal{M}, F) \cong \bigoplus_i (\mathcal{H}^i p_+ gr_k^W gr_L^\alpha \mathcal{M}, F)[-i].$$

Proof. As in the previous proposition, we have

$$\tilde{p}_+(gr_V^\alpha \widetilde{\mathcal{M}}, F) = \bigoplus_{k \in \mathbf{Z}} p_+(gr_L^{|L|+\alpha-k-1} \mathcal{M}, F[-1])u^k.$$

Let $E_r^{p,q}$ be the spectral sequence associated to the relative monodromy filtrations. Then d_r is compatible with the above decomposition, for example, because it is a morphism of \mathcal{D} -modules. As $(gr_V^\alpha \widetilde{\mathcal{M}}, F, W)$ underlies a (direct summand of a) mixed Hodge module for $\alpha \in [0, 1]$, the spectral sequence $E_r^{p,q}(gr_V^\alpha \widetilde{\mathcal{M}}, F)$ degenerates at the second page, and hence it does too for the direct summand $E_r^{p,q}(gr_L^{|L|+\alpha-k-1} \mathcal{M}, F)$ for any $k \in \mathbf{Z}$, proving (a).

If \mathcal{M} is polarizable, then $\mathrm{Sp}_L(\mathcal{M})$ is graded polarizable, and so for every $p, q \in \mathbf{Z}$ the module $E_1^{p,q}$ is pure polarizable, hence semisimple by [Sai88, 5.2.13]. As $E_2^{p,q}$ is a subquotient of this, it must be a direct summand. Now, decomposing along the u^k terms, we see that the same is true for the spectral sequence associated to \mathcal{M} , hence part (b).

By [Sai88, 5.3.4.2], we know that the image of $\mathcal{H}^i \tilde{p} W_\bullet \mathrm{gr}_V^\alpha \widetilde{\mathcal{M}}$ in $\mathcal{H}^i \tilde{p} \mathrm{gr}_V^\alpha \widetilde{\mathcal{M}}$ is the monodromy filtration. Again, we decompose along u^k , and obtain (c).

Finally, for part (d), we know $\mathrm{gr}_k^W \mathrm{gr}_V^\alpha \widetilde{\mathcal{M}}$ is a polarizable Hodge module, so by choosing an ample class on Y , we can use the Hard Lefschetz theorem to obtain an isomorphism

$$\mathcal{H}^{-i} \tilde{p}(\mathrm{gr}_k^W \mathrm{gr}_V^\alpha \widetilde{\mathcal{M}}, F) \cong \mathcal{H}^i(\mathrm{gr}_k^W \mathrm{gr}_V^\alpha \widetilde{\mathcal{M}}, F)(i),$$

and finally, decomposing along u^k , we obtain (d). □

CHAPTER III

Singularities of Hypersurfaces

In this chapter, we mention the known results in the case of hypersurfaces. We do not give any proofs, but these results are important to see the motivation for the theorems which come in the following chapters. Throughout, X is a smooth, irreducible complex algebraic variety of dimension n .

III.1: Bernstein-Sato polynomials of hypersurfaces

In the previous chapter, we defined b -functions of sections of specializable \mathcal{D} -modules. The motivating example for b -functions is the *Bernstein-Sato* polynomial, studied independently by Bernstein [Ber72] and Sato. This is defined as the monic polynomial of smallest degree $b_f(s) \in \mathbf{C}[s]$ such that there exists a differential operator $P(s) \in \mathcal{D}_X[s]$ satisfying

$$b_f(s)f^s = P(s)f^{s+1},$$

where f^s is a formal symbol on which a derivation acts via the power rule from differential calculus.

Many computer algebra systems, for example, Macaulay2, have algorithms which allow for the computation of Bernstein-Sato polynomials, see [BL10].

Kashiwara [Kas77] proved that the roots of $b_f(s)$ lie in $\mathbf{Q}_{<0}$. In fact, Kashiwara showed something more precise. We state here the stronger version due to Lichtin [Lic89]: let

$\pi : Y \rightarrow X$ be a log resolution of the pair $(X, \{f = 0\})$. This is a birational morphism where Y is smooth and $K_{Y/X} + \pi^*(f)$ is a simple normal crossings divisor. Let $\{E_i\}_{i \in I}$ be the exceptional divisors of the resolution. Then we can write

$$\pi^*(f) = \sum_{i \in I} a_i E_i, K_{Y/X} = \sum_{i \in I} k_i E_i.$$

Then we have

Theorem III.1. ([\[Lic89, Theorem 5\]](#)) *With this notation, every root of $b_f(s)$ is of the form $-\frac{k_i+1+\ell}{a_i}$ for some $i \in I$ and some $\ell \in \mathbf{Z}_{\geq 0}$.*

The fractions appearing in the theorem statement are related to classical invariants from birational geometry, the *multiplier ideals* and *log canonical threshold* of the hypersurface f . These are defined to be, for $\lambda > 0$,

$$\mathcal{I}(f^\lambda) = \pi_*(\mathcal{O}_Y(K_{Y/X} - \lfloor \lambda \pi^*(f) \rfloor)) \subseteq \mathcal{O}_X,$$

$$\text{lct}(f) := \min_i \frac{k_i + 1 + \ell}{a_i}.$$

It is not hard to see the following:

1. For $0 < \lambda \ll 1$ we have $\mathcal{I}(f^\lambda) = \mathcal{O}_X$.
2. For $\lambda = 1$, we have $\mathcal{I}(f^1) = (f)$.
3. For $\lambda \leq \mu$, we have $\mathcal{I}(f^\lambda) \supseteq \mathcal{I}(f^\mu)$.

It turns out that $\text{lct}(f) = \sup\{\lambda \in \mathbf{Q}_{>0} \mid \mathcal{I}(f^\lambda) = \mathcal{O}_X\}$. For details, see [\[Laz04, Chapter 9\]](#).

We see immediately from Lichtin's theorem that every root of $b_f(s)$ is $\leq -\text{lct}(f)$. However, an argument due to Kollár [\[Kol97\]](#), using the original, analytic definition of multiplier ideals and integration by parts, showed that $-\text{lct}(f)$ is always a root of $b_f(s)$. In fact, this

theorem is strengthened in [ELSV04], again using an integration by parts argument, to show that all numbers $\lambda \in (0, 1)$ such that

$$\mathcal{I}(f^{\lambda-\epsilon}) \supsetneq \mathcal{I}(f^\lambda)$$

satisfy $b_f(-\lambda) = 0$. These are called the *jumping numbers* of f in the interval $(0, 1)$.

The connection between the Bernstein-Sato polynomial and the b -function defined in the previous chapter is the following. Let $\Gamma : X \rightarrow X \times \mathbf{A}^1$ be the graph embedding along f and consider the direct image $\Gamma_+ \mathcal{O}_X = \mathcal{B}_f = \bigoplus_{k \geq 0} \mathcal{O}_X \partial_t^k \delta_f$, where δ_f is a formal symbol and t is the coordinate on \mathbf{A}^1 . By definition, a derivation $\tau \in \mathcal{T}_X$ acts on δ_f by

$$\tau(\delta_f) = -\tau(f) \partial_t \delta_f,$$

and t acts by

$$t \delta_f = f \delta_f.$$

The module \mathcal{B}_f naturally underlies a pure Hodge module, with Hodge filtration

$$F_p \mathcal{B}_f = \bigoplus_{k \leq p-1} \mathcal{O}_X \partial_t^k.$$

We can also consider the module $\mathcal{O}_X[s, \frac{1}{f}] f^s$ which is free over $\mathcal{O}_X[s]$ and which inherits a $\mathcal{D}_X[s]$ -action via the Leibniz rule and the power rule, meaning that for $\tau \in \mathcal{T}_X$, we have

$$\tau(f^s) = s \tau(f) f^{s-1}.$$

This relation is one motivation for, as in the previous chapter, defining $s := -\partial_t t$.

We can perform the same constructions with \mathcal{O}_X replaced by the localization along f , $\mathcal{O}_X(*D)$. These are related by the following. For details, see [MP20, Prop. 2.5] or [Mal83].

Proposition III.2. *Using the notation above, we have an isomorphism*

$$\mathcal{O}_X(*D)[s]f^s \rightarrow \Gamma_+\mathcal{O}_X(*D)$$

defined by

$$us^j f^s \mapsto u(-\partial_t t)^j \delta_f.$$

Using this isomorphism, we see that the Bernstein-Sato polynomial $b_f(s)$ can be rephrased as the b -function for the element $\delta_f \in \Gamma_+\mathcal{O}_X(*D)$. Interestingly, the proof of [Kas77] and [Lic89] can be strengthened to a computation of the roots of b -functions for other elements of $\Gamma_+\mathcal{O}_X \subseteq \Gamma_+\mathcal{O}_X(*D)$, as in the main result of [DM22b].

Recall that, in the previous chapter, we showed that the V -filtration on a $\mathcal{D}_{X \times \mathbf{A}^1}$ -module is related to b -functions. The result of Kashiwara on negativity of the roots of $b_f(s)$ says that $\delta_f \in V^{>0}\mathcal{B}_f$, and the result of Lichtin tells us that $\delta_f \in V^{\text{let}(f)}\mathcal{B}_f$. In fact, Kollár's result shows that $\delta_f \notin V^{>\text{let}(f)}\mathcal{B}_f$.

This relation of V -filtration and log canonical threshold was strengthened in Budur-Saito [BS05] to the following relation between the V -filtration on \mathcal{B}_f and the multiplier ideals:

Theorem III.3. [BS05, Theorem 0.1] *Using the notation above, for all $\alpha \in \mathbf{Q}$, we have*

$$\{h \mid h\delta_f \in V^\alpha \mathcal{B}_f\} = \mathcal{I}(f^{\alpha-\epsilon})$$

for $0 < \epsilon \ll 1$.

The same result is shown for arbitrary ideals in [BMS06, Theorem 1].

III.1.1: Hodge Ideals

Initially unrelated to the V -filtration, Mustașă and Popa [MP19] commenced the study of *Hodge ideals* for divisors, which put multiplier ideals into a $\mathbf{Z}_{\geq 0}$ -indexed family of ideal sheaves. The construction is as follows: let $U = \{f \neq 0\} \xrightarrow{j} X$ be the inclusion of the

complement of the hypersurface D defined by f . By applying the mixed Hodge module pushforward j_* to $\mathbf{Q}_U^H[\dim U]$, one sees that the underlying module is

$$j_*\mathcal{O}_U = \mathcal{O}_X(*D),$$

but the corresponding Hodge filtration $F_\bullet\mathcal{O}_X(*D)$ is rather subtle. There is another natural filtration on $\mathcal{O}_X(*D)$, the *pole order* filtration, denoted by

$$P_k\mathcal{O}_X(*D) = \{u \in \mathcal{O}_X(*D) \mid f^{k+1}u \in \mathcal{O}_X\} = \mathcal{O}_X((k+1)D).$$

Saito [Sai93] showed that there is always an inclusion

$$F_k\mathcal{O}_X(*D) \subseteq P_k\mathcal{O}_X(*D),$$

and so Mustaă and Popa [MP19] defined the *Hodge ideals* to be

$$\mathcal{I}_k(f) = F_k\mathcal{O}_X(*D) \otimes_{\mathcal{O}} \mathcal{O}_X(-(k+1)D) \subseteq \mathcal{O}_X.$$

Using the exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(*D) \rightarrow \mathcal{H}_D^1(\mathcal{O}_X) \rightarrow 0,$$

where $\mathcal{H}_D^1(\mathcal{O}_X)$ is the *local cohomology* module along D , and the fact that the Hodge filtration on \mathcal{O}_X is essentially trivial, the study of Hodge ideals is equivalent to the study of the Hodge filtration on local cohomology. This aspect of the theory has been studied by many authors [MP22a], [MP22b], [Rai21, Theorem 1.5] to name a few. This is the main subject matter of Chapter V, using V -filtrations for higher codimension subvarieties to study local cohomology.

Multiplier ideals also make sense for effective \mathbf{Q} -divisors, and in [MP20], Mustaă and Popa define and study Hodge ideals for \mathbf{Q} -divisors. Again, these generalize the multiplier

ideal, but they are rather mysterious. For example, there is not in general a containment

$$I_k(D) \subseteq I_{k-1}(D),$$

though there is for D a reduced divisor.

There are certain properties of Hodge ideals which are rather well understood and which are analogues of those properties for multiplier ideals. For example, subadditivity [MP18, Theorem B], restriction [MP18, Theorem A], and finite pushforward formulas [DM22a, Theorem 1.3] exist for these ideals. Moreover, they satisfy, in certain situations, analogues of the celebrated *Nadel vanishing theorem* for multiplier ideals [MP19, Theorem F], [MP20, Section C] and [Dut20].

An interesting refinement of Hodge ideals for reduced divisors were defined by Olano in [Ola22b] for the case $k = 0$ (i.e., for multiplier ideals) and in [Ola22a] for $k > 0$. These are defined by using the weight filtration on the mixed Hodge module $\mathcal{O}_X(*D)$ and intersecting with the Hodge filtration. For example, [Ola22b, Theorem A] gives a characterization of the adjoint ideal in terms of weighted multiplier ideals.

III.1.2: Microlocal V -Filtration

Given a hypersurface defined by a regular function $f \in \mathcal{O}_X(X)$ on a smooth complex algebraic variety X , we consider the \mathcal{D} -module $\mathcal{B}_f = \bigoplus_{k \geq 0} \mathcal{O}_X \partial_t^k \delta_f$. Saito [Sai94] defines the *partial algebraic microlocalization* to be

$$\tilde{\mathcal{B}}_f := \mathcal{B}_f[\partial_t^{-1}] = \bigoplus_{k \in \mathbf{Z}} \mathcal{O}_X \partial_t^k \delta_f,$$

with ‘‘Hodge’’ filtration

$$F_p \tilde{\mathcal{B}}_f := \bigoplus_{k \leq p-1} \mathcal{O}_X \partial_t^k \delta_f.$$

If f is not a unit, then $(s+1) \mid b_f(s)$, and so one can consider the *reduced Bernstein-*

Sato polynomial $\bar{b}_f(s) = \frac{b_f(s)}{(s+1)}$. Then Saito [Sai94] defines the *minimal exponent* of f to be the negation of the most positive root of $\bar{b}_f(s)$, denoted $\tilde{\alpha}(f)$. Saito [Sai94, Theorem 0.3] shows that, actually, $\bar{b}_f(s)$ can be interpreted as the *microlocal Bernstein-Sato polynomial* for f , defined in terms of a functional equation for δ_f in $\tilde{\mathcal{B}}_f$. This interpretation is crucial to proving many properties of the minimal exponent, for example, the bound $\tilde{\alpha}(f) \leq \frac{n}{2}$ when f is singular and the following Thom-Sebastiani type result:

Theorem III.4. [Sai94, Theorem 0.8] *Let $f \in \mathcal{O}_X$ and $g \in \mathcal{O}_Y$ be regular functions with a vector field $\tau \in \mathcal{T}_Y$ satisfying $\tau g = g$. Then*

$$\tilde{\alpha}(f + g) = \tilde{\alpha}(f) + \tilde{\alpha}(g).$$

When the hypersurface defined by f has isolated singularities, the minimal exponent is actually the smallest of the Steenbrink spectral numbers. The Steenbrink spectrum of a hypersurface with isolated singularities is a multi-set of rational numbers defined by studying the action of monodromy on the cohomology of the Milnor fiber of f and how it interacts with the Hodge structure defined in [Ste77]. Many important properties for the minimal exponent in the case of isolated singularities were obtained from this viewpoint in the 1980's. For example, an interesting result due to Varchenko [Var82] is that, for a family of hypersurfaces with isolated singularities and constant rank of their top cohomology of the Milnor fiber, the minimal exponent is constant. The spectrum was related in [BS05] to the V -filtration and Hodge filtration on \mathcal{B}_f .

Using the V -filtration on \mathcal{B}_f , one can define a V -filtration on $\tilde{\mathcal{B}}_f$ on which the usual properties hold and $\partial^{-1} : V^\bullet \mathcal{B}_f \rightarrow V^{\bullet+1} \mathcal{B}_f$. This V -filtration satisfies $gr_V^\lambda \mathcal{B}_f \rightarrow gr_V^\lambda \tilde{\mathcal{B}}_f$ is an isomorphism for all $\lambda < 1$.

This V -filtration can be computed explicitly in many cases: see [Sai16], [Sai09] and [Zha21].

Using the microlocal V -filtration, Saito [Sai16] defines the *microlocal multiplier ideals*

$\tilde{V}^\gamma \mathcal{O}_X$ by looking at the image of $V^\gamma \tilde{\mathcal{B}}_f$ under the projection map to $gr_0^F \tilde{\mathcal{B}}_f \cong \mathcal{O}_X$, in analogy with the relation between V -filtration on \mathcal{B}_f and multiplier ideals.

Interestingly, in loc. cit. Saito shows the following for a reduced hypersurface D defined by f :

Theorem III.5. [[Sai16](#), Thm 1] *For all $p \geq 0$, we have*

$$I_p(D) = \tilde{V}^{p+1} \mathcal{O}_X \text{ mod } (f).$$

This was strengthened to a similar relationship between the microlocal V -filtration and Hodge ideals for \mathbf{Q} -divisors in [[MP20](#)]. This result allows for an algorithmic approach to the computation of Hodge ideals, see [[Bla22](#)].

III.1.3: Local Cohomology and Classes of Singularities

As mentioned above, the study of Hodge ideals of a reduced divisor D is equivalent to the study of the Hodge filtration on local cohomology $\mathcal{H}_D^1(\mathcal{O}_X)$. The following theorem is a culmination of many of the main ideas in this story:

Theorem III.6. [[MOPW21](#)] [[JKSY22](#)] *Let $f \in \mathcal{O}_X$ define a singular hypersurface D . Let $\mathcal{H}_D^1(\mathcal{O}_X)$ be the local cohomology along D . Then*

$$\tilde{\alpha}(f) \geq k + 1 \iff F_k \mathcal{H}_D^1(\mathcal{O}_X) = P_k \mathcal{H}_D^1(\mathcal{O}_X) \iff F_{k+1} \mathcal{B}_f \subseteq V^1 \mathcal{B}_f$$

and

$$\tilde{\alpha}(f) > k + 1 \iff F_k W_{n+1} \mathcal{H}_D^1(\mathcal{O}_X) = P_k \mathcal{H}_D^1(\mathcal{O}_X) \iff F_{k+2} \mathcal{B}_f \subseteq V^{>0} \mathcal{B}_f.$$

Moreover, these properties are equivalent to D having k -du Bois (resp. k -rational) singularities.

As mentioned above, $\tilde{\alpha}(f) \leq \frac{n}{2}$ if f defines a singular hypersurface. We mention here an interesting result which says when equality can occur.

Theorem III.7. *[DM22a, Cor. 6.3] Let $f \in \mathcal{O}_X$ be such that $\tilde{\alpha}_x(f) = \frac{n}{2}$ for some $x \in X$. Then, up to analytic change of coordinates, we can write $f = x_1^2 + \cdots + x_n^2$ with (x_1, \dots, x_n) an analytic system of coordinates centered at x .*

CHAPTER IV

Higher Codimension Subvarieties

This chapter contains the main body of the thesis: the study of V -filtrations along higher codimension smooth subvarieties as found in [CD21]. It contains two main theorems. Before the theorems, general statements are shown concerning ${}^L V$ -filtrations, similar to those results for hypersurface V -filtrations in [Sai88, Section 3.1]. The theorems concern only the slope $L = \sum_{i=1}^r s_i$. The first shows that the Koszul-like complexes defined in Chapter II are filtered acyclic for filtered \mathcal{D} -modules underlying mixed Hodge modules. The second shows that one can compute i^* and $i^!$ for mixed Hodge modules using the Koszul-like complexes from Chapter II.

IV.0.1: Topological Properties of V -filtrations

In this subsection, let $X \times \{0\} \subseteq X \times \mathbf{A}^r$ be the zero section defined by t_1, \dots, t_r , with corresponding vector fields $\partial_{t_1}, \dots, \partial_{t_r}$. Let \mathcal{M} be a left regular holonomic $\mathcal{D}_{X \times \mathbf{A}^r}$ -module. Recall that in this case, \mathcal{M} admits a canonical ${}^L V$ -filtration, as in Chapter 2, for any slope $L = \sum_{i=1}^r a_i s_i$. We assume from here out that L is non-degenerate, so all a_i are nonzero.

Note that, although we state all results for $X \times \{0\} \subseteq X \times \mathbf{A}^r$, this is not really a restrictive setting. For any $Z \subseteq X$ a smooth subvariety, using local defining equations and the graph embedding, we can always reduce to the case at hand.

We first show that the ${}^L V$ -filtration allows one to detect sub-modules and quotient modules supported on $\{t_1 = \dots = t_r = 0\}$.

Lemma IV.1. *Let \mathcal{M} be a regular holonomic $\mathcal{D}_{X \times \mathbf{A}^r}$ -module. Then \mathcal{M} has no submodules supported on $X \times \{0\}$ if and only if the natural map*

$$gr_L^0 \mathcal{M} \xrightarrow{t} \bigoplus_{i=1}^r gr_L^{a_i} \mathcal{M}$$

is injective.

Proof. First, we see that if $m \in \mathcal{M}$ satisfies $t_i m = 0$ for $1 \leq i \leq r$, then $m \in {}^L V^0 \mathcal{M}$. Indeed, the assumption on m gives $(\sum_{i=1}^r a_i \partial_{t_i} t_i) m = 0$. If $m \in {}^L V^\lambda \mathcal{M} - {}^L V^{>\lambda} \mathcal{M}$ for some $\lambda \geq 0$, we are done. Otherwise, $\lambda < 0$, and by definition of the ${}^L V$ -filtration its class in $gr_L^\lambda \mathcal{M}$ is annihilated by $(\sum_{i=1}^r a_i \partial_{t_i} t_i) - \lambda = -(L(s) + \lambda)$. Hence, it is killed by $\lambda \neq 0$, so its class is 0 in the associated graded. By discreteness of the ${}^L V$ -filtration, we can repeat this argument finitely many times and arrive at $m \in {}^L V^0 \mathcal{M}$.

We have the short exact sequence of complexes

$$0 \rightarrow A^{>0}(\mathcal{M}) \rightarrow A^0(\mathcal{M}) \rightarrow B^0(\mathcal{M}) \rightarrow 0,$$

where the leftmost complex is acyclic by Lemma II.16. By the Snake Lemma, we see then that the natural map

$$\ker \left({}^L V^0 \mathcal{M} \xrightarrow{t} \bigoplus_{i=1}^r {}^L V^{a_i} \mathcal{M} \right) \rightarrow \ker \left(gr_L^0 \mathcal{M} \xrightarrow{t} \bigoplus_{i=1}^r gr_L^{a_i} \mathcal{M} \right)$$

is an isomorphism.

Putting this together, we see that \mathcal{M} has a submodule supported on $X \times \{0\}$ iff there exists an element $0 \neq m \in \mathcal{M}$ with $t_i m = 0$ for all $1 \leq i \leq r$ iff there exists an element $0 \neq m \in {}^L V^0 \mathcal{M}$ with $t_i m = 0$ for all $1 \leq i \leq r$ iff $\ker(gr_L^0 \mathcal{M} \xrightarrow{t} \bigoplus_{i=1}^r gr_L^{a_i} \mathcal{M}) \neq 0$. \square

Lemma IV.2. *Let \mathcal{M} be a regular holonomic $\mathcal{D}_{X \times \mathbf{A}^r}$ -module. Then \mathcal{M} has no quotient*

modules supported on $X \times \{0\}$ iff the natural map

$$\bigoplus_{i=1}^r gr_L^{a_i} \mathcal{M} \xrightarrow{\partial_t} gr_L^0 \mathcal{M}$$

is surjective.

More generally, if $U = X \times \mathbf{A}^r - X \times \{0\}$ and $\mathcal{M}' \subseteq \mathcal{M}$ is the smallest submodule such that $\mathcal{M}'|_U = \mathcal{M}|_U$, then

$$\mathcal{M}/\mathcal{M}' = i_+ \text{coker} \left(\bigoplus_{i=1}^r gr_L^{a_i} \mathcal{M} \xrightarrow{\partial_t} gr_L^0 \mathcal{M} \right).$$

Proof. The first claim follows from the second because if $\mathcal{M} \rightarrow \mathcal{N}$ is any quotient module supported on Z , then the projection map must factor $\mathcal{M} \rightarrow \mathcal{M}/\mathcal{M}' \rightarrow \mathcal{N}$.

So we prove the second claim. First, note that $\mathcal{M}' = \mathcal{D}_{X \times \mathbf{A}^r} \cdot {}^L V^\lambda \mathcal{M}$ for any $\lambda > 0$. Indeed, as $\mathcal{M}'|_U = \mathcal{M}|_U$, the quotient \mathcal{M}/\mathcal{M}' is supported on $X \times \{0\}$, and so satisfies ${}^L V^{>0}(\mathcal{M}/\mathcal{M}') = 0$ by Example II.5. Hence, ${}^L V^\lambda \mathcal{M}' = {}^L V^\lambda \mathcal{M}$ for any $\lambda > 0$. Hence, $\mathcal{D}_{X \times \mathbf{A}^r} \cdot {}^L V^\lambda \mathcal{M} = \mathcal{D}_{X \times \mathbf{A}^r} \cdot {}^L V^\lambda \mathcal{M}' \subseteq \mathcal{M}'$ for any $\lambda > 0$. For the other inclusion, use the minimality of \mathcal{M}' and the fact that ${}^L V^\lambda \mathcal{M}|_U = \mathcal{M}|_U$.

By Kashiwara's equivalence applied to the module \mathcal{M}/\mathcal{M}' , which is supported on $X \times \{0\}$, we have $\mathcal{M}/\mathcal{M}' = i_+ gr_L^0(\mathcal{M}/\mathcal{M}')$, where $i : X \times \{0\} \rightarrow X \times \mathbf{A}^r$ is the inclusion. But $gr_L^0(\mathcal{M}/\mathcal{M}') = gr_L^0(\mathcal{M})/gr_L^0(\mathcal{M}')$ by strictness. Now,

$$gr_L^0(\mathcal{M}') = \frac{{}^L V^0 \mathcal{M}'}{{}^L V^{>0} \mathcal{M}'} = \frac{{}^L V^0 \mathcal{M} \cap \mathcal{M}'}{{}^L V^{>0} \mathcal{M}},$$

where the second inequality follows from what we have already argued and strictness of the inclusion $(\mathcal{M}', {}^L V) \rightarrow (\mathcal{M}, {}^L V)$.

It thus suffices to prove the following

$${}^L V^0 \mathcal{M} \cap \mathcal{M}' = \sum_{i=1}^r \partial_{t_i} {}^L V^{a_i} \mathcal{M} + {}^L V^{>0} \mathcal{M}.$$

Define inductively $U^\lambda \mathcal{M}' = {}^L V^\lambda \mathcal{M}$ if $\lambda > 0$ and for $\lambda \leq 0$, define $U^\lambda \mathcal{M}' = \sum_{i=1}^r \partial_{t_i} U^{\lambda+a_i} \mathcal{M}' + U^{>\lambda} \mathcal{M}'$. Then ${}^L V^\lambda \mathcal{M}' = U^\lambda \mathcal{M}'$ by uniqueness of the ${}^L V$ -filtration, which proves the claim. \square

With these results in hand, we can give a characterization for a module to decompose. This is related to the decomposition to strict support for mixed Hodge modules.

Proposition IV.3. *Let \mathcal{M} be a regular holonomic $\mathcal{D}_{X \times \mathbf{A}^r}$ -module. Then \mathcal{M} decomposes as $\mathcal{M} = \mathcal{M}' \oplus \mathcal{M}''$ with $\text{supp}(\mathcal{M}') \subseteq X \times \{0\}$ and \mathcal{M}'' having no submodules or quotient modules supported on $X \times \{0\}$ if and only if*

$$gr_L^0 \mathcal{M} = \left(\ker(gr_L^0 \mathcal{M} \xrightarrow{t} \bigoplus_{i=1}^r gr_L^{a_i} \mathcal{M}) \right) \oplus \left(\sum_{i=1}^r \partial_{t_i} gr_L^{a_i} \mathcal{M} \right).$$

Proof. Assume $\mathcal{M} = \mathcal{M}' \oplus \mathcal{M}''$. By the previous two lemmas, we know

$$gr_L^0 \mathcal{M}'' = \sum_{i=1}^r \partial_{t_i} gr_L^{a_i} \mathcal{M}'', \quad \ker(gr_L^0 \mathcal{M}'' \xrightarrow{t} \bigoplus_{i=1}^r gr_L^{a_i} \mathcal{M}'').$$

Also, as \mathcal{M}' is supported on $X \times \{0\}$, we know $gr_L^0 \mathcal{M}' = \ker(gr_L^0 \mathcal{M}' \xrightarrow{t} gr_L^{a_i} \mathcal{M}')$, so the claim follows by applying gr_L^0 to $\mathcal{M}' \oplus \mathcal{M}''$.

For the other direction, we set $\mathcal{M}' = \mathcal{H}_X^0(\mathcal{M})$, the submodule of sections supported on $X \times \{0\}$. Also, set $\mathcal{M}'' = \mathcal{D}_{X \times \mathbf{A}^r} \cdot {}^L V^{>0} \mathcal{M}$, which by the proof of the previous lemma is the smallest submodule of \mathcal{M} which agrees with \mathcal{M} upon restricting to $U = X \times \mathbf{A}^r \setminus X \times \{0\}$.

This satisfies

$$\mathcal{M}'' = i_+ \text{coker} \left(\bigoplus_{i=1}^r gr_L^{a_i} \mathcal{M} \xrightarrow{\partial_t} gr_L^0 \mathcal{M} \right)$$

by the previous lemma.

By assumption, this cokernel is isomorphic to $\ker(gr_L^0 \mathcal{M} \xrightarrow{t} \bigoplus_{i=1}^r gr_L^{a_i} \mathcal{M})$, hence $\mathcal{M}/\mathcal{M}'' \cong \mathcal{M}'$. But the inclusion $\mathcal{M}' \rightarrow \mathcal{M}$ splits this map, proving the desired decomposition. \square

We conclude with the following theorem, which says that at the \mathcal{D} -module level, the

complexes $B^0(\mathcal{M})$ and $C^0(\mathcal{M})$ compute $i^!\mathcal{M}$ and $i^*\mathcal{M}$, respectively, where $i : X \times \{0\} \rightarrow X \times \mathbf{A}^r$ is the inclusion of the zero section. This will be enhanced to a statement for mixed Hodge modules later.

Finally, for the statement of i^* when $L = (1, \dots, 1)$, we will make use of the following result of Ginzburg:

Proposition IV.4. [*Gin86, Prop. 10.4*] *Let \mathcal{M} be monodromic regular holonomic on $X \times \mathbf{A}^r$. Then there is a natural quasi-isomorphism*

$$p_*(\mathcal{M}) \cong i^*(\mathcal{M})$$

for $i : X \times \{0\} \rightarrow X \times \mathbf{A}^r$ the inclusion of the zero section and $p : X \times \mathbf{A}^r \rightarrow X$ the projection.

Theorem IV.5. *Let \mathcal{M} be a regular holonomic $\mathcal{D}_{X \times \mathbf{A}^r}$ -module. Then there is a natural quasi-isomorphism*

$$B^0(\mathcal{M}) \cong i^!\mathcal{M}.$$

Moreover, when $L = (1, \dots, 1)$, we have a natural quasi-isomorphism $C^0(\mathcal{M}) \cong i^*(\mathcal{M})$.

Proof. By [*HTT08, Page 32*], we can compute $i^!\mathcal{M}$ as the derived \mathcal{O} -module pullback. As t_1, \dots, t_r form a regular sequence, a resolution of \mathcal{O}_X is given by the Koszul complex on t_1, \dots, t_r , and so we have

$$i^!\mathcal{M} = \left[\mathcal{M} \xrightarrow{t} \mathcal{M}^{\oplus r} \xrightarrow{t} \mathcal{M}^{\oplus \binom{r}{2}} \xrightarrow{t} \dots \xrightarrow{t} \mathcal{M} \right] =: \text{Kosz}(\mathcal{M}, t)$$

, placed in cohomological degrees $0, \dots, r$.

We know by Proposition [II.16](#) that the complex $A^\alpha(\mathcal{M})$ is acyclic for all $\alpha > 0$, and so the natural quotient map $A^0(\mathcal{M}) \rightarrow B^0(\mathcal{M})$ is a quasi-isomorphism. We will show that $A^0(\mathcal{M})$ is naturally quasi-isomorphic to $\text{Kosz}(\mathcal{M}, t)$, which will then finish the proof.

By Proposition II.9, $B^\alpha(\mathcal{M})$ is acyclic for all $\alpha \neq 0$, for $\lambda < 0$, the natural inclusion $A^0(\mathcal{M}) \rightarrow A^\lambda(\mathcal{M})$ is a quasi-isomorphism.

For any α , we have an inclusion of complexes $A^\alpha(\mathcal{M}) \rightarrow \text{Kosz}(\mathcal{M}, t)$, which is the identity upon taking the limit (or union) over all $\alpha \rightarrow -\infty$. Hence, as the limit is exact, the natural map

$$A^0(\mathcal{M}) \rightarrow \lim_{\lambda \rightarrow -\infty} A^\lambda(\mathcal{M}) = \text{Kosz}(\mathcal{M}, t)$$

is a quasi-isomorphism.

Now, assume $L = (1, \dots, 1)$. For $C^0(\mathcal{M}) \cong i^*\mathcal{M}$, it is easy to check that $i^*Sp(\mathcal{M}) = i^*\mathcal{M}$, and so by replacing \mathcal{M} with $Sp(\mathcal{M})$, we can assume \mathcal{M} is monodromic. But then, by Ginzburg's result, $i^*Sp(\mathcal{M}) = p_*Sp(\mathcal{M})$, which is computed using the relative de Rham complex for p . Using the choice of coordinates $\partial_{t_1}, \dots, \partial_{t_r}$, this is precisely the Koszul-like complex

$$\bigoplus_{\lambda \in \mathbf{Q}} C^\lambda(\mathcal{M}),$$

which, by Proposition II.9, is quasi-isomorphic to $C^0(\mathcal{M})$. □

IV.0.2: Koszul Complexes are Filtered Acyclic for Mixed Hodge Modules

Now, let (\mathcal{M}, F, W) be a left bifiltered regular holonomic $\mathcal{D}_{X \times \mathbf{A}^r}$ -module underlying a mixed Hodge module M . As mentioned in the introduction, for the remainder of the paper, we have to restrict our attention to the case $L = (1, \dots, 1)$. We consider throughout this section the filtered complexes

$$\begin{aligned} A^\alpha(\mathcal{M}, F) &= \left[V^\alpha(\mathcal{M}, F[-r])e_0 \xrightarrow{t} \bigoplus_{|I|=1} V^{\alpha+1}(\mathcal{M}, F[-r])e_I \xrightarrow{t} \bigoplus_{|I|=2} V^{\alpha+2}(\mathcal{M}, F[-r])e_I \xrightarrow{t} \dots \xrightarrow{t} V^{\alpha+r}(\mathcal{M}, F[-r])e \right] \\ B^\alpha(\mathcal{M}, F) &= \left[gr_V^\alpha(\mathcal{M}, F[-r])e_0 \xrightarrow{t} \bigoplus_{|I|=1} gr_V^{\alpha+1}(\mathcal{M}, F[-r])e_I \xrightarrow{t} \bigoplus_{|I|=2} gr_V^{\alpha+2}(\mathcal{M}, F[-r])e_I \xrightarrow{t} \dots \xrightarrow{t} gr_V^{\alpha+r}(\mathcal{M}, F[-r])e \right] \\ C^\alpha(\mathcal{M}, F) &= \left[gr_V^{\alpha+r}(\mathcal{M}, F)e_0 \xrightarrow{\partial_t} \bigoplus_{|I|=1} gr_V^{\alpha+r-1}(\mathcal{M}, F[-1])e_I \xrightarrow{\partial_t} \bigoplus_{|I|=2} gr_V^{\alpha+r-2}(\mathcal{M}, F[-2])e_I \xrightarrow{\partial_t} \dots \xrightarrow{\partial_t} gr_V^\alpha(\mathcal{M}, F[-r])e \right] \end{aligned}$$

where, for any filtration F_\bullet , the shifted filtration $F[k]_\bullet$ satisfies $F[k]_\bullet = F_{\bullet-k}$.

The main theorem of this section is the following:

Theorem IV.6. [[CD21](#), Theorem 1.1] Let (\mathcal{M}, F) underlie a mixed Hodge module on $X \times \mathbf{A}^r$. Then for $\lambda > 0$ (resp. $\lambda < 0$), the complex $A^\lambda(\mathcal{M}, F)$ (resp. $C^\lambda(\mathcal{M}, F)$) is filtered acyclic.

Before beginning the proof, we prove a lemma. We state the lemma in terms of general ${}^L V$ -filtrations, because it holds true in that generality:

Lemma IV.7. Let Y be a smooth projective variety and consider a filtered regular holonomic \mathcal{D} -module (\mathcal{M}, F) underlying a pure polarizable Hodge module on $X \times \mathbf{A}^r \times Y$. Let $p : X \times \mathbf{A}^r \times Y \rightarrow X \times \mathbf{A}^r$ be the projective, smooth projection. Let t_1, \dots, t_r be the coordinates on \mathbf{A}^r , and let ${}^L V^\bullet \mathcal{M}$ be the ${}^L V$ -filtration along t_1, \dots, t_r . Then

- If the complex $F_\ell A^\alpha(\mathcal{M})$ is acyclic for some $\ell \in \mathbf{Z}, \alpha \in \mathbf{Q}$, then $F_\ell A^\alpha(\mathcal{H}^k p_+(\mathcal{M}))$ is acyclic for all $k \in \mathbf{Z}$.
- If the complex $F_\ell C^\alpha(\mathcal{M})$ is acyclic for some $\ell \in \mathbf{Z}, \alpha \in \mathbf{Q}$, then $F_\ell C^\alpha(\mathcal{H}^k p_+(\mathcal{M}))$ is acyclic for all $k \in \mathbf{Z}$.

Proof. By Lemma [II.27](#), we know that the i th cohomology of

$$p_+(F_q {}^L V^\alpha) = R p_*(DR_{X \times Y/Y}(F_q {}^L V^\alpha \mathcal{M}))$$

is canonically isomorphic to $F_q V^\alpha \mathcal{H}^i p_+ \mathcal{M}$. Moreover, by choosing an ample class ℓ on W , the Hard Lefschetz theorem for polarizable Hodge modules, the Lefschetz isomorphism gives us that

$$(2\pi\sqrt{-1}\ell)^k : \mathcal{H}^{-k} p_+ \mathcal{M} \rightarrow \mathcal{H}^k p_+ \mathcal{M}(k)$$

is an isomorphism of polarizable Hodge modules, and so restricting to $F_q V^\alpha$, we get isomorphisms

$$(2\pi\sqrt{-1}\ell)^k : F_q V^\alpha \mathcal{H}^{-k} p_+ \mathcal{M} \rightarrow F_{q-k} V^\alpha \mathcal{H}^k p_+ \mathcal{M}.$$

By Deligne's formalism for decomposition theorems from Hard Lefschetz [[Del68](#)] and the fact that these are canonically isomorphic to $\mathcal{H}^k p_+(F_q V^\alpha \mathcal{M})$ (suitably shifted), we get a

decomposition

$$p_+(F_q V^\alpha \mathcal{M}) = \bigoplus \mathcal{H}^k(F_{q-k} V^\alpha \mathcal{M})[-k]$$

in $D_{coh}^b(X)$. □

Proof. By the short exact sequence 2 (which gives a short exact sequence of complexes), it suffices to check the claim for a pure polarizable Hodge module M of weight w . Also, the claims of the theorem statement may obviously be checked on direct summands. Using the decomposition by strict support, we can thus assume that the Hodge module has strict support.

If the support is contained in any hypersurface $\{t_i = 0\}$, then the claim follows by the inductive hypothesis, using Kashiwara's equivalence and the result of Example II.7. Hence, we can assume that \mathcal{M} has strict support which is not contained in any hypersurface $\{t_i = 0\}$.

For ease of notation, let $Y = X \times \mathbf{A}^r$ and denote by X the subvariety $X \times \{0\} \subseteq Y$. As acyclicity will be checked locally on Y , we will assume we have coordinates x_1, \dots, x_n on X , hence, coordinates $(x_1, \dots, x_n, t_1, \dots, t_r)$ on Y .

Let $\mathcal{B} \rightarrow \mathbf{A}^r$ be the blowup of \mathbf{A}^r along $\{0\}$ with exceptional divisor E . We view $\mathcal{B} \subseteq \mathbf{A}^r \times \mathbf{P}^{r-1}$, and factor the projection as

$$\begin{array}{ccc} X \times \mathcal{B} & \xrightarrow{i} & X \times \mathbf{A}^r \times \mathbf{P}^{r-1} \\ \downarrow \pi & \swarrow p & \\ X \times \mathbf{A}^r & & \end{array},$$

where i is a closed embedding and p is the projection to the first two factors (which is smooth and projective).

Now, as $\mathcal{B} - E \cong \mathbf{A}^r - \{0\}$, we can restrict \mathcal{M} to $\mathbf{A}^r - \{0\}$ and minimally extend to \mathcal{B} , getting $\widehat{\mathcal{M}}$, which underlies a pure Hodge module of weight w . Also, we know $\mathcal{H}^0 \pi_+(\widehat{\mathcal{M}})$ is a pure Hodge module by Theorem II.17, and \mathcal{M} is a direct summand of it by Theorem II.20 and the decomposition by strict support. We show that the property of the theorem holds for $i_+ \widehat{\mathcal{M}}$, which by Lemma IV.7 implies that it holds for $\mathcal{H}^0 \pi_+(\widehat{\mathcal{M}})$, and hence, for \mathcal{M} .

As the property of acyclicity is local, we work in one of the standard open charts of \mathbf{P}^{r-1} isomorphic to \mathbf{A}^{r-1} , say with coordinates u_2, \dots, u_r . In this chart, the blowup is defined by $z_2 = t_2 - u_2 t_1, \dots, z_r = t_r - u_r t_1$. In this chart, we write $i_+ \widehat{\mathcal{M}} = \bigoplus_{\alpha \in \mathbf{N}^{r-1}} \widehat{\mathcal{M}} \partial_z^\alpha$. Clearly we have an equality of ideals $(t_1, \dots, t_r) = (t_1, z_2, \dots, z_r)$, so as the V -filtration only depends on the ideal (by Remark II.3), we obtain the formula (as in Example II.7)

$$V^\lambda i_+ \widehat{\mathcal{M}} = \bigoplus_{\alpha \in \mathbf{N}^{r-1}} V^{\lambda+|\alpha|} \widehat{\mathcal{M}} \partial_z^\alpha,$$

where $V^\bullet \widehat{\mathcal{M}}$ is the V -filtration along t_1 (which defines E in this chart) and hence, by definition of the Hodge filtration for closed embeddings of codimension $r-1$, we have

$$F_\ell V^\lambda i_+ \widehat{\mathcal{M}} = \bigoplus_{\alpha \in \mathbf{N}^{r-1}} F_{\ell-|\alpha|-(r-1)} V^{\lambda+|\alpha|} \widehat{\mathcal{M}} \partial_z^\alpha.$$

Let $\lambda > 0$, then in the Koszul-like complex $F_\ell A^\lambda(i_+ \widehat{\mathcal{M}})$, one of the differentials is $t_1 : F_\ell V^{\lambda+j} i_+ \widehat{\mathcal{M}} \rightarrow F_\ell V^{\lambda+j+1} i_+ \widehat{\mathcal{M}}$. Decomposing along ∂_z^α (with which t_1 commutes), it is the map

$$t_1 : F_{\ell-|\alpha|-(r-1)} V^{\lambda+|\alpha|+j} \widehat{\mathcal{M}} \rightarrow F_{\ell-|\alpha|-(r-1)} V^{\lambda+|\alpha|+j+1} \widehat{\mathcal{M}}.$$

This map is an isomorphism, as $V^\bullet \widehat{\mathcal{M}}$ is the V -filtration along a hypersurface and $\lambda+|\alpha| > 0$ by assumption on λ . Hence, the complex $F_\ell A^\lambda(i_+ \widehat{\mathcal{M}})$ is acyclic for all $\ell, \lambda > 0$.

A simple computation for changing bases from $(\underline{x}, \underline{t}, \underline{u})$ to $(\underline{x}, t_1, \underline{z}, \underline{u})$ shows that $\partial_{z_2} = \partial_{t_2}, \dots, \partial_{z_r} = \partial_{t_r}$.

For $\lambda < 0$, the complex $F_\ell C^\lambda(i_+ \widehat{\mathcal{M}})$, we can again use the computation of $F_\bullet V^\bullet$, now taking associated graded pieces. The complex splits up along the ∂_z^α pieces, as $\partial_{z_i} = \partial_{t_i}$.

The main observation is that

$$\partial_{t_i} : F_{\ell-|\alpha|-(r-1)} gr_V^\lambda \widehat{\mathcal{M}} \partial_z^\alpha \rightarrow F_{\ell+1-|\alpha+e_i|-(r-1)} gr_V^\lambda \widehat{\mathcal{M}} \partial_z^{\alpha+e_i}$$

is an isomorphism. Any direct summand which involves one of these morphisms must then be acyclic.

Hence, the only possibly non-trivial part is $\partial_{t_1} : F_{\ell-(r-1)}gr_V^{\lambda+1}\widehat{\mathcal{M}}\partial_z^0 \rightarrow F_{\ell+1-(r-1)}gr_V^\lambda\widehat{\mathcal{M}}\partial_z^0$, which is an isomorphism as V is the V -filtration along the hypersurface defined by t_1 . This completes the proof. \square

IV.1: Restriction Functors

We set $B(\mathcal{M}) = B^0(\mathcal{M}, F)$ and $C(\mathcal{M}) = C^0(\mathcal{M}, F)$. We show, in analogy with the codimension one case (see Equation II.3.3), that the cohomology of these complexes computes $\mathcal{H}^k i^! \mathcal{M}$, respectively, $\mathcal{H}^k i^* (\mathcal{M})$. Here $i : X \times \{0\} \rightarrow X \times \mathbf{A}^r$ is the zero section. Specifically, we will prove

Theorem IV.8. *[CD21, Theorem 1.2] Let (\mathcal{M}, F, W) underlie a mixed Hodge module on $X \times \mathbf{A}^r$. Then $B^0(\mathcal{M}, F, W) \cong i^!(\mathcal{M}, F)$ and $C^0(\mathcal{M}, F)$. Moreover, $B^0(\mathcal{M}, F, W)$ and $C^0(\mathcal{M}, F, W)$ are mixed Hodge complexes, where the filtration W is defined using the relative monodromy filtration on $gr_V^j(\mathcal{M}, F)$ for all $0 \leq j \leq r$. Moreover, for any $k, \ell \in \mathbf{Z}$, the quasi-isomorphisms above induce isomorphisms of pure Hodge modules of weight $k + \ell$:*

$$gr_k^W \mathcal{H}^\ell B(\mathcal{M}) \cong gr_{k+\ell}^W \mathcal{H}^\ell i^! \mathcal{M},$$

$$gr_k^W \mathcal{H}^\ell C(\mathcal{M}) \cong gr_{k+\ell}^W \mathcal{H}^\ell i^* \mathcal{M}.$$

We begin by showing that we can naturally endow the complexes $B(\mathcal{M})$ and $C(\mathcal{M})$ with \mathbf{Q} -structure and a weight filtration so that they are *mixed Hodge complexes* [Sai00].

Definition IV.9. A mixed Hodge complex is a bifiltered complex of \mathcal{D} -modules (C^\bullet, F, W) where F is a filtration by \mathcal{O} -subcomplexes and W is a filtration by \mathcal{D} -subcomplexes, and a

\mathbf{Q} -structure $(C_{\mathbf{Q}}, W_{\mathbf{Q}})$, such that, as filtered complexes,

$$DR(C^{\bullet}, W) \cong (C_{\mathbf{Q}}, W_{\mathbf{Q}}) \otimes_{\mathbf{Q}} \mathbf{C}.$$

Moreover, we should have a decomposition

$$(gr_k^W C^{\bullet}, F) = \bigoplus_{\ell} (\mathcal{H}^{\ell} gr_k^W C^{\bullet}, F)[- \ell],$$

in the derived category of filtered \mathcal{D} -modules. Finally, $(\mathcal{H}^{\ell} gr_k^W C^{\bullet}, F)$ with the induced \mathbf{Q} -structure should underlie a pure polarizable Hodge module of weight $k + \ell$.

Let (\mathcal{M}, F, W) be a bifiltered \mathcal{D} -module underlying a mixed Hodge module on $X \times \mathbf{A}^r$. Each term $gr_V^j \mathcal{M}$ of the complex $B(\mathcal{M})$ (resp. $C(\mathcal{M})$) carries a relative monodromy filtration for the filtration induced by W_{\bullet} and the nilpotent operator $s + j$. As the differential in $B(\mathcal{M})$ (resp. $C(\mathcal{M})$) preserves the filtration induced by W_{\bullet} and commutes with the nilpotent operator, the term-wise relative monodromy filtration induces a filtration on $B(\mathcal{M})$ (resp. $C(\mathcal{M})$).

Theorem IV.10. *Let $(\mathcal{M}, F, W, \mathcal{K})$ be a bifiltered \mathcal{D} -module with \mathbf{Q} -structure $\mathcal{K} \in \text{Perv}_{\mathbf{Q}}(X \times \mathbf{A}^r)$. Then $(B(\mathcal{M}), F, W, i^! \mathcal{K})$ and $(C(\mathcal{M}), F, W, i^* \mathcal{K})$ are mixed Hodge complexes.*

Proof. First of all, by Theorem IV.5, we see that, indeed, $DR(B(\mathcal{M}), W) \cong i^!(\mathcal{K}, W) \otimes_{\mathbf{Q}} \mathbf{C}$ and $DR(C(\mathcal{M}), W) \cong i^*(\mathcal{K}, W) \otimes_{\mathbf{Q}} \mathbf{C}$. So the \mathbf{Q} -structure claim is OK. If \mathcal{M} is supported on $X \times \{0\}$, the claim is immediate, as both complexes are actually just mixed Hodge modules.

Case 1: Pure First, assume \mathcal{M} is a pure polarizable Hodge module of weight w with strict support not contained in $X \times \{0\}$. We blowup $\{0\} \in \mathbf{A}^r$ to get $\widehat{Y} = X \times \mathcal{B}$. As in the proof of Theorem IV.6, using the blowup, we can minimally extend \mathcal{M} to get a pure polarizable Hodge module $\widehat{\mathcal{M}}$ on \widehat{Y} . Then \mathcal{M} arises as a direct summand of $\mathcal{H}^0_{p_+ i_+} \widehat{\mathcal{M}}$, and so it suffices to prove the claim for this module. We do it for i_+ and show that it is preserved by $\mathcal{H}^0_{p_+}$.

Step 1: Closed Embedding As the property of being a mixed Hodge complex is local, we

can again restrict to the standard open cover of \mathbf{P}^{r-1} on which the blowup is defined (without loss of generality) by $z_2 = t_2 - u_2 t_1, \dots, z_r = t_r - u_r t_1$. A simple computation using change of basis shows that, on $gr_V^\lambda i_+ \widehat{\mathcal{M}}$, the nilpotent operator induced by $\theta - \lambda + r = \sum_{i=1}^r t_i \partial_{t_i} - \lambda + r$ is equal to the nilpotent operator $\theta' - \lambda + r = t_1 \partial_{t_1} + \sum_{i=2}^r z_i \partial_{z_i} - \lambda + r$.

Write $i_+ \widehat{\mathcal{M}} = \bigoplus_{\alpha \in \mathbf{N}^{r-1}} \widehat{\mathcal{M}} \partial_z^\alpha$. It is easy to check that

$$(t_1 \partial_{t_1} + \sum_{i=2}^r z_i \partial_{z_i} - \lambda + r)(m \partial_z^\alpha) = (t_1 \partial_{t_1} - (\lambda + |\alpha|) + 1)(m) \partial_z^\alpha.$$

Hence, we get an identification of monodromy filtrations

$$W_\bullet gr_V^\lambda i_+ \widehat{\mathcal{M}} = \bigoplus_{\alpha} W_\bullet gr_V^{\lambda + |\alpha|} \widehat{\mathcal{M}} \partial_z^\alpha.$$

The local quasi-isomorphisms described in the proof of Theorem IV.6 then also preserve the monodromy filtration. Hence, we have shown that, on this chart, $B(i_+ \widehat{\mathcal{M}})$ is bifiltered quasi-isomorphic to $B_{t_1}(\widehat{\mathcal{M}})$, and similarly for the complex C . But for V -filtrations along hypersurfaces, the resulting complex is of course a mixed Hodge complex (in fact, it is simply a morphism between mixed Hodge modules), which proves the claim in this step.

Step 2: Projection Assume $B_{X \times \{0\} \times Y}(\mathcal{N})$ is a mixed Hodge complex for \mathcal{N} a pure Hodge module on $X \times \mathbf{A}^r \times Y$ for Y smooth and projective. We show then that $B_{X \times \{0\}}(\mathcal{H}^k p_+(\mathcal{N}))$ is a mixed Hodge complex for all $k \in \mathbf{Z}$.

We have the decomposition $gr_k^W B(\mathcal{N}) = \bigoplus_{\ell} (\mathcal{H}^\ell gr_k^W B(\mathcal{N}))[-\ell]$ in the category of filtered \mathcal{D} -modules, hence, by applying p_+ , the decomposition remains. But on the right hand side, each $\mathcal{H}^\ell gr_k^W B(\mathcal{N})$ is a pure polarizable Hodge module. Hence, by Saito's Decomposition Theorem II.19, we know $p_+ \mathcal{H}^\ell gr_k^W B(\mathcal{N}) = \bigoplus_j (\mathcal{H}^j p_+ \mathcal{H}^\ell gr_k^W B(\mathcal{N}))[-j]$. Putting this together, we have

$$p_+ gr_k^W B(\mathcal{N}) \cong \bigoplus_{j, \ell} \mathcal{H}^j p_+ \mathcal{H}^\ell gr_k^W B(\mathcal{N})[-j - \ell].$$

Let $\mathcal{F}_{k, \ell}^i = \mathcal{H}^\ell p_+ gr_k^W B^i(\mathcal{N})$, where $B^i(\mathcal{N})$ is the i th term of the complex $B(\mathcal{N})$.

Note: by Lemma II.27, we have $\mathcal{H}^\ell p_+ B^i(\mathcal{N}) = B^i \mathcal{H}^\ell p_+(\mathcal{N})$.

Case 2: Mixed We use the splitting $gr^W gr_V^\lambda \mathcal{M} \cong gr^W gr^M gr_V^\lambda \mathcal{M}$ which, by Lemma II.26, is compatible with the morphisms t_1, \dots, t_r and $\partial_{t_1}, \dots, \partial_{t_r}$. Hence, $gr^W B(\mathcal{M}) \cong gr^W B(gr^W \mathcal{M})$, and $gr^M \mathcal{M}$ is pure, so we can conclude by the previous step. \square

By general properties of mixed Hodge complexes [Sai00, Prop. 2.3], we conclude the following.

Corollary IV.11. *The Hodge filtration on $B(\mathcal{M})$ and $C(\mathcal{M})$ is strict, and the weight spectral sequence of $B(\mathcal{M})$ (resp. $C(\mathcal{M})$) degenerates at E_2 .*

Using the Hodge filtration strictness, we show that these complexes compute the Hodge filtration on each cohomology.

Proof of Theorem. By [Sai90, Proof of Prop. 2.19], we can compute $i_* i^! M$ via the Čech complex

$$K(M) = \left(M \rightarrow \bigoplus_{i=1}^r M(*Z_i) \rightarrow \bigoplus_{1 \leq i < j \leq r} M(*(Z_i + Z_j)) \rightarrow \dots \rightarrow M(* \sum_{i=1}^r Z_i) \right)$$

placed in cohomological degrees $0, \dots, r$. We let $K(\mathcal{M})$ denote the underlying complex of filtered \mathcal{D} -modules. We consider the double complex $BK(\mathcal{M})$

(IV.1.1)

$$\begin{array}{ccccccc} gr_V^0 \mathcal{M} & \xrightarrow{\delta} & \bigoplus_{|I|=1} gr_V^1 \mathcal{M} & \xrightarrow{\delta} & \dots & \xrightarrow{\delta} & gr_V^r \mathcal{M} \\ \downarrow & & \downarrow & & & & \downarrow \\ \bigoplus_{j=1}^r gr_V^0 \mathcal{M}(*Z_j) & \xrightarrow{\delta} & \bigoplus_{j=1}^r (\bigoplus_{|I|=1} gr_V^1 \mathcal{M}(*Z_j)) & \xrightarrow{\delta} & \dots & \xrightarrow{\delta} & \bigoplus_{j=1}^r gr_V^r \mathcal{M}(*Z_j) \\ \downarrow & & \downarrow & & & & \downarrow \\ \vdots & & \dots & & \dots & & \dots \\ \downarrow & & & & & & \downarrow \\ gr_V^0 \mathcal{M}(* \sum_{j=1}^r Z_j) & \xrightarrow{\delta} & \bigoplus_{|I|=1} gr_V^1 \mathcal{M}(* \sum_{j=1}^r Z_j) & \xrightarrow{\delta} & \dots & \xrightarrow{\delta} & gr_V^r \mathcal{M}(* \sum_{j=1}^r Z_j) \end{array}$$

The top row is $B^0(\mathcal{M})$ and the leftmost column is $gr_V^0 K(\mathcal{M})$.

The j th column is $\bigoplus_{|J|=j} gr_V^j$ applied to $K(\mathcal{M})$. The cohomology of $K(\mathcal{M})$ being a mixed Hodge module supported on $t_1 = \cdots = t_r = 0$ implies that if $j > 1$ then $gr_V^j K(\mathcal{M})$ is bifiltered acyclic.

The i th row is $\bigoplus_{|J|=i} B^0(\mathcal{M}(*Z_J))$, where $Z_J = \sum_{j \in J} Z_j$. By Corollary IV.11, we know $B^0(\mathcal{M}(*Z_J))$ is strict with respect to the Hodge filtration. But the underlying complex of \mathcal{D} -modules is acyclic, hence each row is filtered acyclic.

A similar argument works for $CK(\mathcal{M})$. □

Finally, we need to compare the weight filtrations mentioned above to the canonically defined ones on $i^!M$ and i^*M . The statement we are after is the following: recall that the filtration W_\bullet on $B(\mathcal{M})$ (resp. $C(\mathcal{M})$) is induced by the relative monodromy filtration termwise on each $gr_V^i \mathcal{M}$.

Theorem IV.12. [CD21, Theorem 1.2] *The bifiltered complexes $B^0(\mathcal{M}, F, W)$ and $C^0(\mathcal{M}, F, W)$ are mixed Hodge complexes, where the filtration W is defined using the relative monodromy filtration on $gr_V^j(\mathcal{M}, F)$ for all $0 \leq j \leq r$. Moreover, for any $k, \ell \in \mathbf{Z}$, the quasi-isomorphisms $B^0(\mathcal{M}, F) \cong i^! \mathcal{M}$, $C^0(\mathcal{M}, F) \cong i^* \mathcal{M}$ induce isomorphisms of pure Hodge modules of weight $k + \ell$:*

$$\begin{aligned} gr_k^W \mathcal{H}^\ell B(\mathcal{M}) &\cong gr_{k+\ell}^W \mathcal{H}^\ell i^! \mathcal{M}, \\ gr_k^W \mathcal{H}^\ell C(\mathcal{M}) &\cong gr_{k+\ell}^W \mathcal{H}^\ell i^* \mathcal{M}. \end{aligned}$$

Remark IV.13. We remark, once more, that the filtered complexes $(B(\mathcal{M}), W_\bullet)$ and $(C(\mathcal{M}), W_\bullet)$ need not be strict, but the weight spectral sequence does degenerate at E_2 .

This has the following interpretation: for any $\emptyset \neq J \subseteq \{1, \dots, r\}$, we know $B(\mathcal{M}(*Z_J))$ is an acyclic complex. Hence, as $E_2 = E_\infty$ in the weight spectral sequence, we see that $E_2 = 0$ for the weight spectral sequence on $B(\mathcal{M}(*Z_J))$.

Proof. Recall that the quasi-isomorphisms in the theorem statement are induced by double complexes $BK(\mathcal{M})$ (resp. $CK(\mathcal{M})$), see the terms IV.1.1, with horizontal differential de-

noted δ . Let $\delta_0 : B^0K(\mathcal{M}) \rightarrow B^1K(\mathcal{M})$ be the morphism between the first two columns. In general, we view $BK(\mathcal{M})$ as a complex of complexes (with differential δ), and similarly for $CK(\mathcal{M})$. We write \mathcal{H}_δ^ℓ for the ℓ th cohomology complex of $BK(\mathcal{M})$ with respect to the differential δ .

Throughout, when we say “filtered”, we mean with respect to the Hodge filtration, and we suppress this from the notation. We thus begin with a lemma:

Lemma IV.14. *For all $\ell > 0$, the complex $\mathcal{H}_\delta^\ell gr^W BK(\mathcal{M})$ is filtered acyclic. The natural map $\ker gr^W \delta_0 = \mathcal{H}_\delta^0 gr^W BK(\mathcal{M}) \rightarrow gr^W B^0K(\mathcal{M})$ is a filtered quasi-isomorphism.*

Proof of Lemma. By Lemma II.26, we can choose canonical splittings so that the double complex $gr^W BK(\mathcal{M})$ decomposes into

$$\begin{array}{ccccccc}
gr^W gr^M gr_V^0 \mathcal{M} & \xrightarrow{\delta} & \bigoplus_{|I|=1} gr^W gr^M gr_V^1 \mathcal{M} & \xrightarrow{\delta} & \dots & \xrightarrow{\delta} & gr^W gr^M gr_V^r \mathcal{M} \\
\downarrow & & \downarrow & & & & \downarrow \\
\bigoplus_{j=1}^r gr^W gr^M gr_V^0 \mathcal{M}(*Z_j) & \xrightarrow{\delta} & \bigoplus_{j=1}^r (\bigoplus_{|I|=1} gr^W gr^M gr_V^1 \mathcal{M}(*Z_j)) & \xrightarrow{\delta} & \dots & \xrightarrow{\delta} & \bigoplus_{j=1}^r gr^W gr^M gr_V^r \mathcal{M}(*Z_j) \\
\downarrow & & \downarrow & & & & \downarrow \\
\vdots & & \vdots & & \dots & & \vdots \\
\downarrow & & \downarrow & & & & \downarrow \\
gr^W gr^M gr_V^0 \mathcal{M}(*\sum_{j=1}^r Z_j) & \xrightarrow{\delta} & \bigoplus_{|I|=1} gr^W gr^M gr_V^1 \mathcal{M}(*\sum_{j=1}^r Z_j) & \xrightarrow{\delta} & \dots & \xrightarrow{\delta} & gr^W gr^M gr_V^r \mathcal{M}(*\sum_{j=1}^r Z_j)
\end{array}$$

where $M_\bullet gr_V^i(\mathcal{M}(*Z_j)) = gr_V^i(W_\bullet \mathcal{M}(*Z_j))$.

As $K(\mathcal{M})$ is a complex of mixed Hodge modules on $X \times \mathbf{A}^r$, it is bistrict with respect to the Hodge and weight filtrations. In particular, $\mathcal{H}^\ell gr_i^W K(\mathcal{M}) = gr_i^W \mathcal{H}^\ell K(\mathcal{M})$. But the category of polarizable pure Hodge modules on $X \times \mathbf{A}^r$ is semisimple, so $\mathcal{H}^\ell gr_i^W K(\mathcal{M})$ is a filtered direct summand of $gr_i^W K^\ell(\mathcal{M})$. In particular, we have a containment

$$\mathcal{H}^\ell gr^W K(\mathcal{M}) \subseteq gr^W K^\ell(\mathcal{M}),$$

and so, applying gr_V^0 to both sides, we get

$$gr_V^0 \mathcal{H}^\ell gr^W K(\mathcal{M}) \subseteq gr_V^0 gr^W K^\ell(\mathcal{M}) = gr^M B^0 K^\ell(\mathcal{M}),$$

and so by taking gr^W (for the relative monodromy filtration), we have containements

$$gr^W gr_V^0 \mathcal{H}^\ell gr^W K(\mathcal{M}) \subseteq gr^W gr^M B^0 K^\ell(\mathcal{M}),$$

which is the ℓ th term of the first column of $BK(\mathcal{M})$.

Now, as $gr^W gr_V^0 \mathcal{H}^\ell gr^W K(\mathcal{M})$ is supported on $X \times \{0\}$, by the definition of δ_0 , we see that actually

$$gr^W \mathcal{H}^\ell gr^M B^0 K(\mathcal{M}) = gr^W gr_V^0 \mathcal{H}^\ell gr^W K(\mathcal{M}) \subseteq \ker gr^W \delta_0,$$

where the first equality uses strictness of the V -filtration and the definition of M_\bullet on gr_V^0 . But the left hand side is also $\mathcal{H}^\ell gr^W gr^M B^0 K(\mathcal{M})$, which is $\mathcal{H}^\ell gr^W B^0 K(\mathcal{M})$, proving the desired quasi-isomorphism.

Now, we write $B^i K(\mathcal{M}) = gr_V^i K(\mathcal{M})$, where $K(\mathcal{M})$ is the Čech complex for \mathcal{M} . In particular, the cohomology of $K(\mathcal{M})$ is supported on $X \times \{0\}$, so $gr_V^i K(\mathcal{M}) = B^i K(\mathcal{M})$ is a filtered acyclic complex when $i > 0$. The same is true for $gr^W B^i K(\mathcal{M})$, and so the total complex $gr^W BK(\mathcal{M})$ is filtered quasi-isomorphic to its first column $gr^W B^0 K(\mathcal{M}) = gr^W gr_V^0 K(\mathcal{M})$. We have just argued that this first column is quasi-isomorphic to $\mathcal{H}^0 \delta gr^W BK(\mathcal{M})$.

Finally, by Theorem IV.10, we have the decomposition

$$gr^W BK^j(\mathcal{M}) = \bigoplus_{\ell \in \mathbf{Z}} \mathcal{H}^\ell (gr^W BK^j(\mathcal{M}))[-\ell],$$

which implies the other cohomologies must be acyclic, as desired. \square

We handle the case of $B(\mathcal{M})$, the proof for $C(\mathcal{M})$ being completely analogous. For $0 \leq j \leq r$, consider the complex $BK^j(\mathcal{M})$ with differential δ , i.e., the j th row of the double complex IV.1.1. This is simply $\bigoplus_{|J|=j} B(\mathcal{M}(*Z_J))$. Hence, by Theorem IV.10, it is a mixed

Hodge complex, and hence by Corollary IV.11, its weight spectral sequence

$$E_1^{pq} = \mathcal{H}^{p+q} gr_{-p}^W BK^j(\mathcal{M}) \implies \mathcal{H}^{p+q} BK^j(\mathcal{M})$$

degenerates at E_2 . For any $k, \ell \in \mathbf{Z}$ and our fixed j , we have a complex from the first page

$$0 \rightarrow E_1^{-k-\ell, k+\ell} \rightarrow E_1^{-k-\ell-1, k+\ell} \rightarrow \dots \rightarrow E_1^{-k-\ell-r, k+\ell} \rightarrow 0,$$

written as

$$0 \rightarrow \mathcal{H}^0 gr_{k+\ell}^W BK^j(\mathcal{M}) \rightarrow \mathcal{H}^1 gr_{k+\ell-1}^W BK^j(\mathcal{M}) \rightarrow \dots \rightarrow \mathcal{H}^r gr_{k+\ell-r}^W BK^j(\mathcal{M}) \rightarrow 0.$$

By naturality, we obtain a double complex $D_{k,\ell}$, with this complex as its j th row.

If $j > 0$, then by Remark IV.13, as $E_2 = 0$, we see that the j th row of $D_{k,\ell}$ is exact.

Hence, the total complex $Tot(D_{k,\ell})$ is quasi-isomorphic to its first row,

$$(IV.1.2) \quad 0 \rightarrow \mathcal{H}^0 gr_{k+\ell}^W B(\mathcal{M}) \rightarrow \mathcal{H}^1 gr_{k+\ell-1}^W B(\mathcal{M}) \rightarrow \dots \rightarrow \mathcal{H}^r gr_{k+\ell-r}^W B(\mathcal{M}) \rightarrow 0.$$

By the Lemma, this is filtered quasi-isomorphic to $gr_{k+\ell}^W B^0 K(\mathcal{M}) = gr_{k+\ell}^W gr_V^0 K(\mathcal{M})$. Applying \mathcal{H}^ℓ gives $gr_{k+\ell}^W gr_V^0 \mathcal{H}^\ell K(\mathcal{M}) = gr_{k+\ell}^W \mathcal{H}^\ell i^!(\mathcal{M})$, where we use the fact that $K(\mathcal{M}) = i_* i^! \mathcal{M}$, and so $gr_V^0 \mathcal{H}^\ell K(\mathcal{M}) = \mathcal{H}^\ell i^! \mathcal{M}$, as the cohomology is supported on $X \times \{0\}$ (see Example II.6).

Finally, \mathcal{H}^ℓ applied to the complex IV.1.2 gives, by definition, $E_2^{-k-\ell+\ell, k+\ell} = E_2^{-k, k+\ell}$. By the E_2 -degeneration, this is $E_\infty^{-k, k+\ell} = gr_k^W \mathcal{H}^\ell B(\mathcal{M})$, proving the claim. \square

CHAPTER V

Minimal Exponent and Application to Local Cohomology

In this chapter, we will use the results of Chapter [IV](#) to study the local cohomology of a locally complete intersection subvariety $Z \subseteq X$ of codimension larger than one. The data of the Hodge ideals for reduced divisors is equivalent to the Hodge filtration on the local cohomology of \mathcal{O}_X along the divisor D . Similarly, the module $\mathcal{H}_Z^r(\mathcal{O}_X)$, which is familiar to commutative algebraists, carries a Hodge filtration. This can be compared to the pole order filtration in the same way as for hypersurfaces.

The main definition of this section is that of the *minimal exponent* for a local complete intersection subvariety, $\tilde{\alpha}(Z)$. Interestingly, we do not define it in exact analogy with the case of hypersurfaces, but instead we make use of an auxiliary construction used in Mustața's paper [\[Mus22\]](#). We show that this invariant does everything which the usual minimal exponent is known to do for hypersurfaces: it controls when the Hodge filtration and the order filtration on $\mathcal{H}_Z^r(\mathcal{O}_X)$ agree, which in turn gives information about higher du Bois singularities, and in fact it can also detect higher rational singularities in the sense of Laza and Friedman. At the end of this chapter, we will give some examples of local complete intersection varieties whose minimal exponents we can compute.

V.0.1: Definition of Local Cohomology

For more details, see [MP22a] and [Har67]. Let $Z \subseteq X$ be a closed subvariety of the smooth variety X . Working locally, we assume Z is defined by $f_1, \dots, f_r \in \mathcal{O}_X(X)$. Moreover, we assume Z is a complete intersection, meaning that $\text{codim}_X(Z) = r$, the number of defining equations. In general, the codimension is bounded above by r .

Let $H_i \subseteq X$ be the hypersurface defined by f_i , which is non-empty by assumption that Z is a complete intersection. We form the Čech complex

$$K(\mathcal{O}_X) = \mathcal{O}_X \rightarrow \bigoplus_{i=1}^r \mathcal{O}_X(*H_i) \rightarrow \bigoplus_{i<j} \mathcal{O}_X(*H_i + H_j) \rightarrow \dots \rightarrow \mathcal{O}_X(*H),$$

where $H = \sum_{i=1}^r H_i$ is the union of all H_i . It is well-known that the condition on the codimension of Z implies that this complex has only one non-vanishing cohomology $\mathcal{H}_Z^r(\mathcal{O}_X)$, which is the *local cohomology* of \mathcal{O}_X along Z .

All terms in the complex naturally underlie mixed Hodge modules on X , and the morphisms are all that of mixed Hodge modules. In particular, $\mathcal{H}_Z^r(\mathcal{O}_X)$ inherits the structure of a mixed Hodge module on X which is supported on Z . Hence, it has a Hodge filtration $F_\bullet \mathcal{H}_Z^r(\mathcal{O}_X)$ and a finite weight filtration $W_\bullet \mathcal{H}_Z^r(\mathcal{O}_X)$.

As $\mathcal{H}_Z^r(\mathcal{O}_X)$ is supported on Z , we can define another filtration on it, the *pole order* filtration, as

$$P_k \mathcal{H}_Z^r(\mathcal{O}_X) = \{m \in \mathcal{H}_Z^r(\mathcal{O}_X) \mid (f_1, \dots, f_r)^{k+1} \cdot m = 0\}.$$

In [MP22a], it is observed that $F_k \mathcal{H}_Z^r(\mathcal{O}_X) \subseteq P_k \mathcal{H}_Z^r(\mathcal{O}_X)$ for all k . As in the case of Hodge ideals, it is interesting to ask when we have equality $F_k \mathcal{H}_Z^r(\mathcal{O}_X) = P_k \mathcal{H}_Z^r(\mathcal{O}_X)$.

Remark V.1. In [MP22a], Mustață and Popa show the following facts regarding this question:

1. If $F_k \mathcal{H}_Z^r(\mathcal{O}_X) = P_k \mathcal{H}_Z^r(\mathcal{O}_X)$, then for all $p \leq k$ we have $F_p \mathcal{H}_Z^r(\mathcal{O}_X) = P_p \mathcal{H}_Z^r(\mathcal{O}_X)$.
2. If Z is singular, then $F_k \mathcal{H}_Z^r(\mathcal{O}_X) \subsetneq P_k \mathcal{H}_Z^r(\mathcal{O}_X)$ for all $k \geq n - r + 1$.

3. If we set $p(Z) = \sup\{-1, k \text{ such that } F_k \mathcal{H}_Z^r(\mathcal{O}_X) = P_k \mathcal{H}_Z^r(\mathcal{O}_X)\}$, this number does not depend on the embedding of Z into a smooth variety and it satisfies restriction and semicontinuity theorems.
4. The inequality $p(Z) \geq k$ holds for some $k \geq 0$ iff Z has at least k -du Bois singularities. These singularities are defined by the natural map

$$\Omega_Z^p \rightarrow \underline{\Omega}_Z^p := gr_F^{-p} \underline{\Omega}_Z[-p]$$

being a quasi-isomorphism for all $p \leq k$.

V.0.2: Relation to the V -filtration

Now, consider the graph embedding $i : X \rightarrow X \times \mathbf{A}^r$ defined by $x \mapsto (x, f_1(x), \dots, f_r(x))$, with coordinates t_1, \dots, t_r on \mathbf{A}^r . We set $\mathcal{B}_f := i_+ \mathcal{O}_X$, which as a set is equal to $\bigoplus_{\alpha \in \mathbf{N}^r} \mathcal{O}_X \partial_t^\alpha \delta$ and for which the action of $\mathcal{O}_X[\partial_{t_1}, \dots, \partial_{t_r}]$ is the obvious one and

$$t_i(h\partial_t^\alpha \delta) = f_i h\partial_t^\alpha \delta - \alpha_i h\partial_t^{\alpha - e_i} \delta, \text{ for all } 1 \leq i \leq r,$$

$$\tau(h\partial_t^\alpha \delta) = \tau(h)\partial_t^\alpha \delta - \sum_{i=1}^r \tau(f_i) h\partial_t^{\alpha + e_i} \delta, \text{ for all } \tau \in \mathcal{T}_X.$$

The module \mathcal{B}_f is a pure Hodge module of weight n on $X \times \mathbf{A}^r$. The Hodge filtration is

$$F_k \mathcal{B}_f = \bigoplus_{|\alpha| \leq k-r} \mathcal{O}_X \partial_t^\alpha \delta,$$

though in the literature it is often conventionally re-indexed. We will not reindex here.

V.0.3: Definition of Minimal Exponent and Relation to Singularities

Recall that in [Mus22] Mustaa uses the following construction: let y_1, \dots, y_r be new variables, and consider the hypersurface $g = \sum_{i=1}^r y_i f_i$ on $Y = X \times \mathbf{A}^r$. The main result of *loc.*

cit. is the relation between b -functions:

$$\tilde{b}_g(s) = b_f(s).$$

Our goal is to strengthen this result to a comparison between certain \mathcal{D} -modules. We will make use of the module $\mathcal{B}_g = \Gamma_+ \mathcal{O}_Y = \bigoplus_{k \geq 0} \mathcal{O}_Y \partial_z^k \delta_g$, where $\Gamma : Y \rightarrow Y \times \mathbf{A}^1$ is the graph embedding along g and z is the coordinate on \mathbf{A}^1 . Let $\theta_y = \sum_{i=1}^r y_i \partial_{y_i}$. Then $\theta_y g = g$, and so it is easy to check that

$$\theta_y \delta_g = -\partial_z z \delta_g = s \delta_g.$$

More generally,

$$\theta_y h y^\alpha \partial_z^k \delta_g = (s + |\alpha| - k)(h y^\alpha \partial_z^k \delta_g).$$

In fact, this relation holds in the microlocalization $\tilde{\mathcal{B}}_g = \mathcal{B}_g[\partial_z^{-1}] = \bigoplus_{k \in \mathbf{Z}} \mathcal{O}_Y \partial_z^k \delta_g$. Hence, $\tilde{\mathcal{B}}_g$ decomposes into eigenspaces for the $\theta_y - s$ operator:

$$\tilde{\mathcal{B}}_g = \bigoplus_{\ell \in \mathbf{Z}} E^{(\ell)}, \text{ where } E^{(\ell)} = \bigoplus_{\alpha \in \mathbf{N}^r} \mathcal{O}_X y^\alpha \partial_z^{|\alpha| - \ell} \delta_g.$$

By definition, if $m \in \tilde{\mathcal{B}}_g^{(\ell)}$, we have $(\theta_y - s)m = \ell m$. Let $V^\bullet \tilde{\mathcal{B}}_g$ be the microlocal V -filtration along z . Then, as $\theta_y - s$ preserves $V^\gamma \tilde{\mathcal{B}}_g$ for all $\gamma \in \mathbf{Q}$, we see that we have a decomposition

$$V^\gamma \tilde{\mathcal{B}}_g = \bigoplus_{\ell \in \mathbf{Z}} V^\gamma \tilde{\mathcal{B}}_g^{(\ell)}, \quad V^\gamma \tilde{\mathcal{B}}_g = V^\gamma \tilde{\mathcal{B}}_g \cap \tilde{\mathcal{B}}_g^{(\ell)}.$$

Now we define the map which allows for a comparison of \mathcal{D} -modules: let $\varphi : \tilde{\mathcal{B}}_g \rightarrow \mathcal{B}_f$ be the \mathcal{O}_X -linear map sending $y^\alpha \partial_z^k \delta_g \mapsto \partial_t^\alpha \delta$. Note that $\varphi_\ell = \varphi|_{\tilde{\mathcal{B}}_g^{(\ell)}}$ is an \mathcal{O}_X -linear isomorphism $\tilde{\mathcal{B}}_g^{(\ell)} \rightarrow \mathcal{B}_f$ for any $\ell \in \mathbf{Z}$.

Lemma V.2. *The map φ defined above satisfies the following properties:*

1. φ is \mathcal{D}_X -linear.

2. For all $m \in \widetilde{\mathcal{B}}_g$ and $k \in \mathbf{Z}$, we have $\varphi(m) = \varphi(\partial_z^k m)$.

3. For all $m \in \widetilde{\mathcal{B}}_g$, $1 \leq i \leq r$, we have $\varphi(y_i m) = \partial_{t_i} \varphi(m)$.

4. For all $m \in \widetilde{\mathcal{B}}_g$, $1 \leq i \leq r$, we have $\varphi(\partial_{y_i} m) = -t_i \varphi(m)$.

5. For all $m \in \widetilde{\mathcal{B}}_g^{(\ell)}$, we have $\varphi(sm) = (s - \ell)\varphi(m)$.

Proof. For 1., as φ is by definition \mathcal{O}_X -linear, it suffices to check that $\varphi(\tau m) = \tau \varphi(m)$ for any $\tau \in \mathcal{T}_X$ and $m \in \widetilde{\mathcal{B}}_g$. It suffices to check for $m = hy^\alpha \partial_z^k \delta_g$. By definition,

$$\tau(hy^\alpha \partial_z^k \delta_g) = \tau(h)y^\alpha \partial_z^k \delta_g - \left(\sum_{i=1}^r \tau(f_i) y_i \right) (hy^\alpha \partial_z^{k+1} \delta_g)$$

so applying φ to both sides yields

$$\varphi(\tau(hy^\alpha \partial_z^k \delta_g)) = \tau(h) \partial_t^\alpha \delta - \sum_{i=1}^r \tau(f_i) h \partial_t^{\alpha+e_i} \delta,$$

which is clearly τ applied to $\varphi(m) = h \partial_t^\alpha \delta$.

Properties 2. and 3. are clear by definition of φ .

For property 4., let $m = hy^\alpha \partial_z^k \delta_g$. Then

$$\partial_{y_i} m = \alpha_i hy^{\alpha-e_i} \partial_z^k \delta_g - f_i hy^\alpha \partial_z^{k+1} \delta_g,$$

and so applying φ to both sides, we get

$$\varphi(\partial_{y_i} m) = \alpha_i h \partial_t^{\alpha-e_i} \delta - f_i h \partial_t^\alpha \delta,$$

which is clearly $-t_i$ applied to $\varphi(m) = h \partial_t^\alpha \delta$.

Finally, let $m \in \widetilde{\mathcal{B}}_g^{(\ell)}$. Then $(\theta_y - s)m = \ell m$, so applying φ to both sides yields

$$\varphi(\theta_y m) - \varphi(sm) = \ell \varphi(m),$$

which by property 3. and 4. and $s = \sum_{i=1}^r -\partial_{t_i} t_i$ tells us

$$s\varphi(m) - \varphi(sm) = \ell\varphi(m),$$

finishing the proof. □

The main technical input of this section is the following Theorem comparing the V -filtration on \mathcal{B}_f and the microlocal V -filtration on $\tilde{\mathcal{B}}_g$.

Theorem V.3. [[CDMO22](#), Theorem 3.3, Prop. 3.4] *Let $f_1, \dots, f_r \in \mathcal{O}_X$ and define $g = \sum_{i=1}^r f_i y_i$ on $Y = X \times \mathbf{A}^r$. The map*

$$\varphi : (\tilde{\mathcal{B}}_g^{(0)}, V) \rightarrow (\mathcal{B}_f, V)$$

is a filtered \mathcal{D}_X -module isomorphism, and, moreover, we have equality of b -functions

$$b_m(s) = b_{\varphi(m)}(s)$$

where on the left, we use the microlocal b -function for $m \in \tilde{\mathcal{B}}_g$.

Proof. Let $W^\bullet \mathcal{B}_f := \varphi_0(V^\bullet \tilde{\mathcal{B}}_g^{(0)})$. This is an exhaustive, decreasing and discrete filtration, which by Lemma V.2 above is compatible with the V -filtration $V^\bullet \mathcal{D}_{X \times \mathbf{A}^r}$ and which has $s + \gamma$ acting nilpotently on $gr_W^\gamma \mathcal{B}_f$.

The proof that $W^\bullet \mathcal{B}_f \subseteq V^\bullet \mathcal{B}_f$ is similar to the usual proof [[Sai88](#), Lemme 3.1.2] that the V -filtration is unique if it exists. As we will use the main idea of this proof for the other inclusion, we omit the proof of this fact.

For the reverse inclusion, we need to show $\varphi_0(V^\gamma \tilde{\mathcal{B}}_g^{(0)}) \subseteq V^\gamma \mathcal{B}_f$. In fact, we prove the stronger statement that

$$V^\gamma \tilde{\mathcal{B}}_g \subseteq U^\gamma \tilde{\mathcal{B}}_g := \bigoplus_{m \in \mathbf{Z}} \partial_z^{-m} \varphi_0^{-1}(V^{\gamma-m} \mathcal{B}_f) \text{ for all } \gamma \in \mathbf{Q}.$$

This is immediate once we know that the right hand side satisfies most of the properties of the microlocal V -filtration on $\tilde{\mathcal{B}}_g^{(0)}$. In fact, we don't even need $U^\gamma \tilde{\mathcal{B}}_g$ to be finitely generated over V^0 . Clearly, U^\bullet is decreasing, discrete, left-continuous and exhaustive. It is trivial to check that $\partial_z^\ell U^\gamma = U^{\gamma-\ell}$, $zU^\gamma \subseteq U^{\gamma+1}$, and that $s + \gamma$ acts nilpotently on $gr_U^\gamma \tilde{\mathcal{B}}_g$.

We wish to show that $V^\bullet \tilde{\mathcal{B}}_g \subseteq U^\bullet \tilde{\mathcal{B}}_g$. If $\gamma \neq \gamma'$ are distinct rational numbers, then

$$(V.0.1) \quad gr_V^\gamma gr_U^{\gamma'} \tilde{\mathcal{B}}_g = \frac{V^\gamma U^{\gamma'} \tilde{\mathcal{B}}_g}{V^{>\gamma} U^{\gamma'} \tilde{\mathcal{B}}_g + V^\gamma U^{>\gamma'} \tilde{\mathcal{B}}_g}$$

is acted on nilpotently by $(s + \gamma)$ and $(s + \gamma')$, hence, by their difference $\gamma - \gamma' \neq 0$. So this quotient is 0.

We use this to see that

$$(V.0.2) \quad V^\gamma \tilde{\mathcal{B}}_g \subseteq U^\gamma \tilde{\mathcal{B}}_g + V^{>\gamma} \tilde{\mathcal{B}}_g$$

Indeed, let $m \in V^\gamma \tilde{\mathcal{B}}_g$. Then there exists some λ with $m \in U^\lambda \tilde{\mathcal{B}}_g$. We are done if $\lambda \geq \gamma$. Otherwise, by the vanishing of the quotient [V.0.1](#), we can write $m = u_1 + u_2$ with $u_1 \in V^{>\gamma} U^\lambda \tilde{\mathcal{B}}_g$ and $u_2 \in V^\gamma U^{>\lambda} \tilde{\mathcal{B}}_g$. We see that $m \in U^\gamma \tilde{\mathcal{B}}_g + V^{>\gamma} \tilde{\mathcal{B}}_g$ if and only if u_2 lies in that subspace. But then we can repeat this argument for u_2 , with λ replaced with some $\lambda' > \lambda$. Iterating this and using discreteness proves the containment.

Using discreteness of the V -filtration, and by iteratively applying the containment [V.0.2](#), we get for any $\gamma, \lambda \in \mathbf{Q}$, the containment

$$V^\gamma \tilde{\mathcal{B}}_g \subseteq W^\gamma \tilde{\mathcal{B}}_g + V^\lambda \tilde{\mathcal{B}}_g.$$

By definition of the microlocal V -filtration, there exists some $q_0 \geq 0$ such that, for all $q \geq q_0$, we have

$$V^{\gamma+q} \tilde{\mathcal{B}}_g = \partial^{-(q-q_0)} V^{\gamma+q_0} \tilde{\mathcal{B}}_g.$$

As $V^{\gamma+q_0} \tilde{\mathcal{B}}_g$ is finitely generated over $V^0 \mathcal{D}_{Y \times \mathbf{A}^1}[\partial^{-1}]$, there exists some β with $V^{\gamma+q_0} \tilde{\mathcal{B}}_g \subseteq$

$W^\beta \tilde{\mathcal{B}}_g$. We choose q large enough such that $q - q_0 + \beta \geq \gamma$. Finally, we have

$$V^\gamma \tilde{\mathcal{B}}_g \subseteq W^\gamma \tilde{\mathcal{B}}_g + V^{\gamma+q} \tilde{\mathcal{B}}_g \subseteq W^\gamma \tilde{\mathcal{B}}_g + \partial^{-(q-q_0)} V^{\gamma+q_0} \tilde{\mathcal{B}}_g = W^\gamma \tilde{\mathcal{B}}_g + W^{\beta+q-q_0} \tilde{\mathcal{B}}_g,$$

where the rightmost module is contained in $W^\gamma \tilde{\mathcal{B}}_g$ by choice of q . This proves the claim.

The proof for the equality of b -functions is easy and left to the reader. \square

Let $U = Y - (X \times \{0\})$. Then we define the minimal exponent $\tilde{\alpha}(Z)$ to be

$$\tilde{\alpha}(Z) := \tilde{\alpha}(g|_U).$$

We will make use of the following lemma:

Lemma V.4. *Let $\gamma \in \mathbf{Q}$ and $\alpha \in \mathbf{Z}_{\geq 0}^r$ be such that $y^\alpha \partial_z^{|\alpha|} \delta_g \in V^\gamma \tilde{\mathcal{B}}_g \setminus V^{>\gamma} \tilde{\mathcal{B}}_g$ and $y^\alpha \partial_z^{|\alpha|} \delta_U \in V^{>\gamma} \tilde{\mathcal{B}}_{g|_U}$. Then $\gamma \in \mathbf{Z}_{\geq r}$.*

Proof. The assumptions tell us that the class of $y^\alpha \partial_z^{|\alpha|} \delta_g$ is non-zero in $gr_V^\gamma \tilde{\mathcal{B}}_g$, but there exists some integer N such that $(y_1, \dots, y_r)^N \cdot [y^\alpha \partial_z^{|\alpha|} \delta_g] = 0$. In particular, there exists some $\beta \in \mathbf{Z}_{\geq 0}$ such that $v := y^\beta [y^\alpha \partial_z^{|\alpha|} \delta_g] \neq 0$ but $(y_1, \dots, y_r)v = 0$.

As $y^\alpha \partial_z^{|\alpha|} \delta_g \in \tilde{\mathcal{B}}_g^{(0)}$, we know $v \in gr_V^\gamma(\tilde{\mathcal{B}}_g^{(|\beta|)})$. Hence, $(\theta_y - s)v = |\beta|v$. But as $(y_1, \dots, y_r)v = 0$ and $\theta_y + r = \sum_{i=1}^r \partial_{y_i} y_i$, we see that

$$0 = (\theta_y + r)v = (s + r + |\beta|)v,$$

and so, since $(s + \gamma)$ also acts nilpotently on the element v , we get $\gamma = r + |\beta|$, proving the claim. \square

Now, we are in position to prove the main theorem of this subsection.

Theorem V.5. ([\[CDMO22\]](#)) *Let $Z \subseteq X$ be a local complete intersection of codimension r*

defined by f_1, \dots, f_r . Then for any $\gamma \in (0, 1]$ and $k \geq 0$, we have

$$\tilde{\alpha}(Z) \geq r + k + \gamma \iff F_{k+1}\mathcal{B}_f \subseteq V^{r-1+\gamma}\mathcal{B}_f.$$

Proof. First, assume $\tilde{\alpha}(Z) \geq r + k + \gamma$. Assume toward contradiction that there exists some α with $|\alpha| = k + 1$ and $\partial_t^\alpha \delta \notin V^{r-1+\gamma}\mathcal{B}_f$. Equivalently, by Lemma V.2, this implies $y^\alpha \partial_z^{|\alpha|} \notin V^{r-1+\gamma}\tilde{\mathcal{B}}_g$. Assume this element defines a non-zero element in $gr_V^\beta \tilde{\mathcal{B}}_g$. Then $\beta < r - 1 + \gamma \leq r$. However, by assumption, we have $\delta_g \in V^{r+k+\gamma}\tilde{\mathcal{B}}_g$ on U . Hence, by the previous lemma, we get $\beta \geq r$, a contradiction.

Conversely, assume $F_{k+1}\mathcal{B}_f \subseteq V^{r-1+\gamma}\mathcal{B}_f$. This is equivalent to $\partial_t^\alpha \delta \in V^{r-1+\gamma}\mathcal{B}_f$ for all $|\alpha| \leq k + 1$. By Lemma V.2, this is equivalent to $y^\alpha \partial_z^{|\alpha|} \delta_g \in V^{r-1+\gamma}\tilde{\mathcal{B}}_g$ for all $|\alpha| \leq k + 1$. Applying this for $\alpha = (k+1)e_i$, we see that $y_i^{k+1} \partial_z^{k+1} \delta_g \in V^{r-1+\gamma}\tilde{\mathcal{B}}_g$. Hence, on $U_i = \{y_i \neq 0\}$, we have $\partial_z^{k+1} \delta_g \in V^{r-1+\gamma}\tilde{\mathcal{B}}_g$, which is true if and only if $\delta_g \in V^{r+k+\gamma}\tilde{\mathcal{B}}_g$ on U_i . As this is true for all i , and $U = U_1 \cup \dots \cup U_r$, this shows that $\delta_g \in V^{r+k+\gamma}\tilde{\mathcal{B}}_g$ on U , and so $\tilde{\alpha}(Z) \geq r + k + \gamma$. \square

V.0.4: Relation to Local Cohomology

In this section, we relate the V -filtration on \mathcal{B}_f to the local cohomology Hodge module $\mathcal{H}_Z^r(\mathcal{O}_X)$. This gives a relation between k -du Bois singularities and $[\tilde{\alpha}(Z)]$, using Theorem V.5 and Theorem V.6 below. In the next subsection, we use similar ideas to relate k -rational singularities and $\tilde{\alpha}(Z)$.

Assume throughout that Z is defined by $f_1, \dots, f_r \in \mathcal{O}_X(X)$ and it has pure codimension r . The goal is the following theorem:

Theorem V.6. *Let X be a smooth, irreducible complex algebraic variety. Let Z be a complete intersection of codimension r defined by $f_1, \dots, f_r \in \mathcal{O}_X(X)$. Then*

$$F_k \mathcal{H}_Z^r(\mathcal{O}_X) = \left\{ \left[\sum_{|\alpha| \leq k} \frac{u_\alpha}{\alpha! f_1^{\alpha+1} \dots f_r^{\alpha+r}} \right] \mid \sum_{|\alpha| \leq k} u_\alpha \partial_t^\alpha \delta_f \in V^r \mathcal{B}_f \right\}.$$

Recall that $\mathcal{H}_Z^r(\mathcal{O}_X)$ is a \mathcal{D}_X -module which naturally underlies the mixed Hodge module $\mathcal{H}^r I^! \mathbf{Q}_X^H[\dim X]$, where $I : Z \rightarrow X$ is the inclusion. Let $\Gamma : X \rightarrow X \times \mathbf{A}^r$ be the graph embedding along f_1, \dots, f_r , and let $i : X \times \{0\} \rightarrow X \times \mathbf{A}^r$ be the inclusion of the zero section, defined by the coordinates t_1, \dots, t_r .

Then, by Saito's base-change [Sai90, Formula (4.4.3)], $i^! \Gamma_+ \mathcal{O}_X = I_+ I^! \mathcal{O}_X$. Hence, in order to compute local cohomology, we can use the module $\mathcal{B}_f := \Gamma_+ \mathcal{O}_X$. By Theorem I.2, we can compute $i^! \mathcal{B}_f$ by the Koszul-like complex

$$T(\mathcal{B}_f) = \left[gr_V^0(\mathcal{B}_f, F[-r])e_0 \xrightarrow{t} \bigoplus_{i=1}^r gr_V^1(\mathcal{B}_f, F[-r])e_i \xrightarrow{t} \dots \xrightarrow{t} gr_V^r(\mathcal{B}_f, F[-r])e_1 \wedge \dots \wedge e_r, \right]$$

where as before, the e_i are used to keep track of the Koszul differentials.

By definition, and the fact that we are using left \mathcal{D} -modules, the Hodge filtration on $\mathcal{B}_f = \bigoplus_{\alpha \in \mathbf{N}^r} \mathcal{O}_X \cdot \partial_t^\alpha \delta_f$ is defined by

$$F_p \mathcal{B}_f = \bigoplus_{|\alpha| \leq q-r} \mathcal{O}_X \cdot \partial_t^\alpha \delta_f.$$

By Theorem IV.6, we know that $T(\mathcal{B}_f)$ is filtered quasi-isomorphic to

$$\left[V^0(\mathcal{B}_f, F[-r])e_0 \xrightarrow{t} \bigoplus_{i=1}^r V^1(\mathcal{B}_f, F[-r])e_i \xrightarrow{t} \dots \xrightarrow{t} V^r(\mathcal{B}_f, F[-r])e_1 \wedge \dots \wedge e_r. \right] = A^0(\mathcal{B}_f, F)$$

Let $\mathcal{L} = \frac{V^r \mathcal{B}_f}{\sum_{i=1}^r t_i V^{r-1} \mathcal{B}_f}$, with its induced Hodge filtration. Then, by what we have said above, there is an isomorphism of \mathcal{D} -modules

$$\sigma : \mathcal{L} \rightarrow \mathcal{H}_Z^r(\mathcal{O}_X),$$

which we plan to make explicit, using the method of Theorem IV.5.

By the proof of that theorem, we saw that the inclusion of complexes

$$A^0(\mathcal{B}_f) \rightarrow \text{Kosz}(\mathcal{B}_f, t) =: T(\mathcal{B}_f)$$

was a quasi-isomorphism. Hence, \mathcal{L} is naturally isomorphic to $\mathcal{H}^r T(\mathcal{B}_f)$.

We also make use of the Čech complex

$$\check{C}(\mathcal{B}_f) = \left[\mathcal{B}_f \rightarrow \bigoplus_{i=1}^r \mathcal{B}_f(*H_i)\xi_i \rightarrow \cdots \rightarrow \mathcal{B}_f(*H)\xi_1 \wedge \cdots \wedge \xi_r, \right].$$

where $H_i = \{t_i = 0\}$ and $H = \sum_{i=1}^r H_i$. Note that, as a complex, this can be identified with

$$\iota_+ \left[\mathcal{O}_X \rightarrow \bigoplus_{i=1}^r \mathcal{O}_X\left[\frac{1}{f_i}\right] \rightarrow \cdots \rightarrow \mathcal{O}_X\left[\frac{1}{f_1, \dots, f_r}\right], \right]$$

where $\iota : X \rightarrow X \times \mathbf{A}^r$ is the graph embedding. Hence, $\check{C}(\mathcal{B}_f)$ is a resolution of $\iota_+ \mathcal{H}_Z^r(\mathcal{O}_X)$.

This is true by the identification $\mathcal{B}_f(*H_I) = i_+ \mathcal{O}_X[\frac{1}{f_I}]$, where $f_I = \prod_{i \in I} f_i$, which is defined by the formula

$$\frac{1}{t^\beta} u \mapsto \sum_{\alpha} \binom{s_1 - \beta_1}{\alpha_1} \cdots \binom{s_r - \beta_r}{\alpha_r} \frac{(-1)^{|\alpha|} \alpha! h_{\alpha}}{f^{\alpha+\beta}} \delta_f,$$

where $t^\beta = \prod_{i=1}^r t_i^{\beta_i}$, $u = \sum h_{\alpha} \partial_t^{\alpha} \delta_f \in \mathcal{B}_f$, $s_i = -\partial_{t_i} t_i$, and

$$\binom{s_i + k}{\ell} = \frac{(s_i + k) \cdots (s_i + k - \ell + 1)}{\ell!}.$$

Let $\mathcal{M} = \iota_+ \mathcal{H}_Z^r(\mathcal{O}_X)$. As $\mathcal{H}_Z^r(\mathcal{O}_X)$ is supported on Z , we know \mathcal{M} is supported on $X \times \{0\}$, and so we have a canonical isomorphism

$$\tau : \mathcal{H}_Z^r(\mathcal{O}_X) \rightarrow V^0 \mathcal{M} = gr_V^0(\mathcal{M}),$$

by Example II.6

Lemma V.7. *Let $\bar{u} \in \text{gr}_V^r \mathcal{B}_f$ be represented by some $u = \sum_{\alpha} h_{\alpha} \partial_t^{\alpha} \delta \in V^r \mathcal{B}_f$. Then*

$$\sigma(\bar{u}) = \sum_{\alpha} \left[\frac{\alpha! h_{\alpha}}{f_1^{\alpha_1+1} \cdots f_r^{\alpha_r+1}} \right] \in \mathcal{H}_Z^r(\mathcal{O}_X).$$

Proof. We make use of two double complexes. The first double complex mixes the complex A^{\bullet} and the Čech complex along t_1, \dots, t_r . For $0 \leq i, j \leq r$, the terms are

$$A^{i,j} = \bigoplus_{|I|=i} \bigoplus_{|J|=j} V^i \mathcal{B}_f(*H_J) e_I \otimes \xi_J.$$

Note that, except for the first, the rows and columns are acyclic. Indeed, we know $A^0(B_f(*H_J)) = i^! B_f(*H_J)$ is acyclic if $J \neq \emptyset$, so the columns are acyclic. Moreover, we know that the cohomology of the complex $\check{C}(\mathcal{B}_f)$ is supported on $t_1 = \cdots = t_r = 0$, so $V^i \mathcal{H}^k \check{C}(\mathcal{B}_f)$ vanishes for all k . But V^i is an exact functor, so $0 = V^i \mathcal{H}^k \check{C}(\mathcal{B}_f) = \mathcal{H}^k V^i \check{C}(\mathcal{B}_f)$ is the k th cohomology of the i th row.

The second double complex $K^{\bullet, \bullet}$ mixes the Koszul and Čech complexes along the t_1, \dots, t_r . Namely, for $0 \leq i, j \leq r$, the terms are

$$K^{i,j} \mathcal{B}_f = \bigoplus_{|I|=i} \bigoplus_{|J|=j} \mathcal{B}_f(*H_J) e_I \otimes \xi_J.$$

As mentioned before the lemma statement, the rows $K^{i, \bullet} \mathcal{B}_f$ give resolutions of $\bigoplus_{|I|=i} \iota_+ \mathcal{H}_Z^r(\mathcal{O}_X)$. Moreover, the columns compute $\bigoplus_{|J|=j} i^! \mathcal{B}_f(*H_J)$, so if $j \geq 1$, they are acyclic.

We have a natural inclusion of double complexes $A^{\bullet, \bullet} \rightarrow K^{\bullet, \bullet}$, hence, an inclusion of total complexes

$$T = \text{Tot}(A^{\bullet, \bullet}) \rightarrow Q = \text{Tot}(K^{\bullet, \bullet}).$$

We have morphisms of complexes going from these total complexes to the rows and

columns. This is given by the diagram

$$\begin{array}{ccccc} A^0(\mathcal{B}_f) & \longleftarrow & T & \longrightarrow & V^0K(\mathcal{B}_f) \\ \downarrow & & \downarrow & & \downarrow \\ \text{Kosz}(\mathcal{B}_f, t) & \longleftarrow & Q & \longrightarrow & \text{Kosz}(\mathcal{M}, t)[-r] \end{array},$$

where by the proof of Theorem IV.5, the leftmost vertical map is a quasi-isomorphism. The bottom right map is given by the diagram

$$\begin{array}{ccccccc} Q^{r-1} & \longrightarrow & Q^r & \longrightarrow & Q^{r+1} & \longrightarrow & \dots \longrightarrow Q^{2r} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow = \\ & & K^{0,r} & & K^{1,r} & & \dots & K^{r,r} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & \mathcal{M} & \longrightarrow & \bigoplus_{|I|=1} \mathcal{M} & \longrightarrow & \dots \longrightarrow & \mathcal{M}. \end{array}$$

where the first vertical map in any column is the projection, and since $K^{i,r} = \bigoplus_{|I|=i} \mathcal{B}_f(*H)$, the second vertical map in any column is the quotient map $\mathcal{B}_f(*H) \rightarrow \mathcal{M}$.

As $\mathcal{H}_Z^r(\mathcal{O}_X)$ is supported on Z , $\iota_+ \mathcal{H}_Z^r(\mathcal{O}_X)$ is supported on $t_1 = \dots = t_r = 0$, the Koszul complex is isomorphic to $\mathcal{H}_Z^r(\mathcal{O}_X)$ placed in degree 0 (via the isomorphism τ). Hence, taking \mathcal{H}^r of the above diagram, we have a diagram

$$\begin{array}{ccccc} \mathcal{L} = V^r \mathcal{B}_f / \sum_{i=1}^r t_i V^{r-1} \mathcal{B}_f & \longleftarrow & \mathcal{H}^r T & \longrightarrow & \mathcal{H}_Z^r(\mathcal{O}_X) \\ \downarrow & & \downarrow & & \downarrow = \\ \mathcal{B}_f / \sum_{i=1}^r t_i \mathcal{B}_f & \longleftarrow & \mathcal{H}^r Q & \longrightarrow & \mathcal{H}_Z^r(\mathcal{O}_X) \end{array}.$$

The morphism we are interested in is the composition $\mathcal{L} \rightarrow \mathcal{B}_f / \sum_{i=1}^r t_i \mathcal{B}_f \rightarrow \mathcal{H}_Z^r(\mathcal{O}_X)$, using the fact that the bottom left map of the diagram is an isomorphism. As the first map in the composition is the natural inclusion, we need only make precise the second map.

For any $u \in \mathcal{B}_f$ and $J \subseteq \{1, \dots, r\}$, we can consider $\frac{1}{t_J} u \in \mathcal{B}_f(*H_J)$. Now, define

$$\eta = \sum_I \frac{\text{sgn}(I, J)}{t_J} u e_I \otimes \xi_J \in Q^r,$$

where $I \sqcup J = \{1, \dots, r\}$.

The first projection of this element is $\bar{u} \in \mathcal{B}_f / \sum_{i=1}^r t_i \mathcal{B}_f$, while its projection to $\mathcal{H}_Z^r(\mathcal{O}_X)$ is $[\frac{1}{t_1 \dots t_r} u]$. Hence, we need only show that η is a cocycle.

The differential d of the total complex Q is defined as follows: first of all, it sends an element of the form η above inside Q^r to a tuple of elements which are indexed by subsets $K, \kappa \subseteq \{1, \dots, r\}$ such that $|K| + |\kappa| = r + 1$ and $K \cap \kappa = \{\ell\}$, a singleton set.

We write out the component corresponding to K, κ as

$$\begin{aligned} (d\eta)_{K,\kappa} &= \frac{\text{sgn}(K - \{\ell\}, \kappa) t_\ell}{t_\kappa} u e_\ell \wedge e_{K - \{\ell\}} \otimes \xi_\kappa + (-1)^{|K|} \frac{\text{sgn}(K, \kappa - \{\ell\})}{t_{\kappa - \{\ell\}}} u e_K \otimes \xi_\ell \wedge \xi_{\kappa - \{\ell\}} \\ &= \frac{1}{t_{\kappa - \{\ell\}}} u (\text{sgn}(K - \{\ell\}, \kappa) e_\ell \wedge e_{K - \{\ell\}} \otimes \xi_\kappa + (-1)^{|K|} \text{sgn}(K, \kappa - \{\ell\}) e_K \otimes \xi_\ell \wedge \xi_{\kappa - \{\ell\}}). \end{aligned}$$

Hence, it suffices to show that the term in the parentheses is 0. Assume ℓ is the i th element of K and the j th element of κ . Then the term in the parentheses is equal to (by rearranging the wedges)

$$\begin{aligned} &((-1)^{i-1} \text{sgn}(K - \{\ell\}, \kappa) + (-1)^{j-1+|K|} \text{sgn}(K, \kappa - \{\ell\})) e_K \otimes \xi_\kappa \\ &= ((-1)^{i+j+|K|-2} + (-1)^{i+j+|K|-3}) \text{sgn}(\{\ell\}, K - \{\ell\}, \kappa - \{\ell\}) e_K \otimes e_\kappa = 0. \end{aligned}$$

Hence, η is a cocycle, and finally we just need to use the morphism τ to complete the proof. But τ^{-1} is given by evaluating the expression at $s_1 = \dots = s_r = 0$ as shown in Example II.6, and so we achieve our claim. \square

From this, the description of the Hodge filtration is immediate:

Proof of Theorem V.6. We prove the equivalent statement that $F_k \mathcal{B}_f \subseteq V^r \mathcal{B}_f$ iff $F_k \mathcal{H}_Z^r(\mathcal{O}_X) = P_k \mathcal{H}_Z^r(\mathcal{O}_X)$.

Note that the elements

$$\left[\frac{1}{f_1^{\alpha_1+1} \dots f_r^{\alpha_r+1}} \right] \in \mathcal{H}_Z^r(\mathcal{O}_X)$$

for $|\alpha| = p$ generate $P_k \mathcal{H}_Z^r(\mathcal{O}_X)$ by [MP22a, Lemma 9.2]. Hence, if $F_k \mathcal{B}_f \subseteq V^r \mathcal{B}_f$, then $\partial_t^\alpha \delta_f \in V^r \mathcal{B}_f$ for all $|\alpha| \leq k$, and so by applying the theorem, we conclude $F_k \mathcal{H}_Z^r(\mathcal{O}_X) = P_k \mathcal{H}_Z^r(\mathcal{O}_X)$.

We prove the converse by induction on k . Note that the containment $F_k \mathcal{B}_f \subseteq V^r \mathcal{B}_f$ is automatic at points of X which do not lie in Z , since the V -filtration is all of \mathcal{B}_f away from Z . Hence, we need only show the containment at points of Z .

For $k = 0$, the assumption $F_0 \mathcal{H}_Z^r(\mathcal{O}_X) = P_0 \mathcal{H}_Z^r(\mathcal{O}_X)$ implies that the class $[\frac{1}{f_1 \dots f_r}] \in F_0 \mathcal{H}_Z^r(\mathcal{O}_X)$, so there exists some $h \in \mathcal{O}_X$ with $h - 1 \in (f_1, \dots, f_r)$ and such that $h \delta_f \in V^r \mathcal{B}_f$. In particular, at any point of Z , $\delta_f \in V^r \mathcal{B}_f$.

Now, work inductively. We $F_{k+1} \mathcal{H}_Z^r(\mathcal{O}_X) = P_{k+1} \mathcal{H}_Z^r(\mathcal{O}_X)$, hence we can assume $\partial_t^\alpha \delta_f \in V^r \mathcal{B}_f$ for all $|\alpha| \leq k$, and we want to show that the same is true for all $|\alpha| \leq k + 1$.

By [MP22a, Lemma 9.1,9.2], we know that $gr_{k+1}^P \mathcal{H}_Z^r(\mathcal{O}_X)$ is a free \mathcal{O}_Z -module with basis given by

$$v_\alpha = \left[\frac{\alpha!}{f_1^{\alpha+1} \dots f_r^{\alpha+r}} \right]$$

for $|\alpha| = k + 1$, because Z is a complete intersection.

The map $F_{k+1} V^r \mathcal{B}_f \rightarrow gr_{k+1}^P \mathcal{H}_Z^r(\mathcal{O}_X)$ is surjective, so for all α with $|\alpha| = k + 1$ there exists a lift of v_α , say $u_\alpha \in F_{k+1} V^r \mathcal{B}_f$. We express $u_\alpha = \sum_{|\beta| \leq k+1} h_{\alpha,\beta} \partial_t^\beta \in V^r \mathcal{B}_f$. The assumption that u_α maps to v_α means that $h_{\alpha,\beta} \in (f_1, \dots, f_r)$ for $\beta \neq \alpha$ and $h_{\alpha,\alpha} - 1 \in (f_1, \dots, f_r)$.

By induction, we can subtract the lower order terms from u_α , and so we see $\sum_{|\beta|=k+1} h_{\alpha,\beta} \partial_t^\beta \in V^r \mathcal{B}_f$. Hence, $\partial_t^\alpha \in V^r \mathcal{B}_f$ at all points of Z , proving the claim. \square

This gives an interpretation of k -du Bois singularities in terms of the V -filtration on \mathcal{B}_f .

V.0.5: Characterization of k -rational Singularities

Recall that, given a complex algebraic variety Z , Laza and Friedman [FL22a] have defined and studied [FL22c, FL22b] the class of k -rational singularities to be those for which the

natural map

$$\Omega_Z^p \rightarrow \underline{\Omega}_Z^p \rightarrow \mathbb{D}(\underline{\Omega}_Z^p)$$

is a quasi-isomorphism for all $p \leq k$. Here $\mathbb{D}(\underline{\Omega}_Z^p) = RHom_{\mathcal{O}_Z}(\underline{\Omega}_Z^p, \omega_Z^\bullet)$ is the shifted Grothendieck dual functor, with ω_Z^\bullet the dualizing complex. The second map is defined via a resolution of singularities $\pi : \tilde{Z} \rightarrow Z$ using functoriality of the du Bois complex and the fact that, on a smooth variety, $\Omega_{\tilde{Z}}^p = \underline{\Omega}_{\tilde{Z}}^p$, so the isomorphism follows from the usual fact for Kähler differentials on a smooth variety.

The goal of this section is to show that this property can be understood through the invariant $\tilde{\alpha}(Z)$, which has already been shown for hypersurface singularities by [FL22b], see also [MP20].

Theorem V.8. *Let Z be a local complete intersection of pure codimension r contained in the smooth, irreducible algebraic variety X . Then Z has k -rational singularities if and only if $\tilde{\alpha}(Z) > r + k$.*

In fact, we can prove more, relating this condition to the intersection cohomology mixed Hodge module via the weight filtration on local cohomology.

Throughout, let $d = n - r$ be the dimension of Z . We recall the pertinent notions from Section II.4. In that section, we constructed maps

$$\gamma_Z : \mathbf{Q}_Z^H[\dim Z] \rightarrow IC_Z \mathbf{Q}^H$$

and

$$\gamma_Z^\vee : \mathbf{D}(IC_Z \mathbf{Q}^H)(-d) \rightarrow i^! \mathbf{Q}_X^H[n+r](r).$$

Moreover, we identified

$$(V.0.3) \quad .i_* \mathbf{D}(IC_Z \mathbf{Q}^H) \cong gr_{-d}^W i_* \mathcal{H}^{-d}(i_* i^! \mathbf{Q}_X^H(n)[2n]).$$

Theorem V.9. *For any nonnegative integer k , the following are equivalent:*

1. $\tilde{\alpha}(Z) > r + k$.
2. $(s + r)F_{k+r}V^r\mathcal{B}_f \subseteq V^{>r}\mathcal{B}_f$.
3. $W_{n+r}F_k\mathcal{H}_Z^r(\mathcal{O}_X) = P_k\mathcal{H}_Z^r(\mathcal{O}_X)$.
4. *The morphism*

$$F_{p+r}i_*\mathbf{Q}_Z^H[d] \rightarrow F_{p+r}IC_Z(\mathbf{Q}_X^H)$$

induced by γ_Z and the composition

$$F_pW_{n+r}\mathcal{H}_Z^r(\mathcal{O}_X) \rightarrow F_p\mathcal{H}_Z^r(\mathcal{O}_X) \rightarrow P_p\mathcal{H}_Z^r(\mathcal{O}_X)$$

induced by γ^\vee are isomorphisms for all $p \leq k$.

5. *Z has k-rational singularities, i.e., Z has k-du Bois singularities and the natural map*

$$\psi_k : \Omega_Z^k \rightarrow R\mathcal{H}om_{\mathcal{O}_Z}(\underline{\Omega}_Z^{d-k}, \omega_Z)$$

is an isomorphism.

6. *The canonical morphism*

$$\Omega_Z^p \rightarrow gr_{-p}^F DR_X(i_*IC_X(\mathbf{Q}_X^H))[p - d]$$

is a quasi-isomorphism for all $p \leq k$.

Proof. The proof is in many steps.

Step 0: It is clear that 1) implies 2), as $\tilde{\alpha}(Z) > r + k$ is true if and only if

$$F_{k+r+1}\mathcal{B}_f \subseteq V^{>r-1}\mathcal{B}_f,$$

which implies

$$\sum_{i=1}^r t_i F_{k+r+1} \mathcal{B}_f \subseteq V^{>r} \mathcal{B}_f,$$

hence $(s+r)F_{k+r} \mathcal{B}_f \subseteq V^{>r} \mathcal{B}_f$.

Step 1: 2) \iff 3) Assume 2) holds.

As \mathcal{B}_f is a pure Hodge module of weight $n = \dim X$, we are concerned with the monodromy filtration $W_\bullet \text{gr}_V^\alpha \mathcal{B}_f$ which is characterized by the properties

1. $N = s + \alpha : (\text{gr}_V^\alpha \mathcal{B}_f, W) \rightarrow (\text{gr}_V^\alpha \mathcal{B}_f, W[2])$
2. $N^i : \text{gr}_{n+i}^W \text{gr}_V^\alpha \mathcal{B}_f \cong \text{gr}_{n-i}^W \text{gr}_V^\alpha \mathcal{B}_f$.

Explicitly, we have

$$W_{n+i} \text{gr}_V^\alpha \mathcal{B}_f = \sum_j \ker((s + \alpha)^{i+j+1}) \cap \text{Im}((s + \alpha)^j).$$

By Theorem 1.2, if $\sigma : \bigoplus_{i=1}^r (\text{gr}_V^{r-1} \mathcal{B}_f, W) \xrightarrow{t} (\text{gr}_V^r \mathcal{B}_f, W)$, then for every i , we have an isomorphism of filtered \mathcal{D} -modules

$$\text{gr}_{i+r}^W \mathcal{H}_Z^r(\mathcal{O}_X) \cong (\text{gr}_i^W \text{coker}(\sigma), F[-r]).$$

Recall that $W_{i+r} \mathcal{H}_Z^r(\mathcal{O}_X) = 0$ for all $i < n$, hence $\text{gr}_{n+r}^W \mathcal{H}_Z^r(\mathcal{O}_X) = W_{n+r} \mathcal{H}_Z^r(\mathcal{O}_X)$. By assumption, we see that

$$F_{k+r} \text{gr}_V^r \mathcal{B}_f \subseteq \ker((s+r)),$$

which by the description of the monodromy filtration implies $F_{k+r} \text{gr}_V^r \mathcal{B}_f \subseteq W_n \text{gr}_V^r \mathcal{B}_f$. Hence,

$$F_k W_{n+r} \mathcal{H}_Z^r(\mathcal{O}_X) = F_k \mathcal{H}_Z^r(\mathcal{O}_X) = P_k \mathcal{H}_Z^r(\mathcal{O}_X),$$

where the last equality follows from the description of the Hodge filtration on $\mathcal{H}_Z^r(\mathcal{O}_X)$ in terms of V . Hence, we have shown 3), and reading the proof backwards we see that 3)

implies 2).

Step 2: 3) \implies 4) We begin with two lemmas

Lemma V.10. *Assume $F_k W_{n+r} \mathcal{H}_Z^r(\mathcal{O}_X) = F_k \mathcal{H}_Z^r(\mathcal{O}_X)$. Then for all $p \leq k$, we have*

$$F_p W_{n+r} \mathcal{H}_Z^r(\mathcal{O}_X) = F_p \mathcal{H}_Z^r(\mathcal{O}_X).$$

Proof of the Lemma. As $W_{n+r} \mathcal{H}_Z^r(\mathcal{O}_X)$ is supported on Z , it is clear that $I_Z F_p W_{n+r} \mathcal{H}_Z^r(\mathcal{O}_X) \subseteq F_{p-1} W_{n+r} \mathcal{H}_Z^r(\mathcal{O}_X)$ for all p , see [Sai88, Prop. 3.2.6]. Moreover, by definition of the pole order filtration, we have

$$I_Z \cdot F_p \mathcal{H}_Z^r(\mathcal{O}_X) = F_{p-1} \mathcal{H}_Z^r(\mathcal{O}_X),$$

so this proves the claim by descending induction on p . □

Lemma V.11. *Assume $F_p W_{n+r} \mathcal{H}_Z^r(\mathcal{O}_X) = F_p \mathcal{H}_Z^r(\mathcal{O}_X)$ for some p . Then the surjection*

$$F_{p+r+1} i_* \mathbf{Q}_Z^H[d] \rightarrow F_{p+r+1} i_* IC_Z \mathbf{Q}^H$$

induced by γ_Z is an isomorphism.

Proof of Lemma. The assumed equality in the theorem statement is equivalent to $F_p gr_{n+r+j}^W \mathcal{H}_Z^r(\mathcal{O}_X) = 0$ for all $j > 0$. Each $gr_{n+r+j}^W \mathcal{H}_Z^r(\mathcal{O}_X)$ is a polarizable pure Hodge module of weight $n+r+j$, hence, there is an isomorphism of filtered \mathcal{D}_X -modules

$$(V.0.4) \quad \mathbf{D}_X(gr_{n+r+j}^W \mathcal{H}_Z^r(\mathcal{O}_X)) \cong gr_{n+r+j}^W \mathcal{H}_Z^r(\mathcal{O}_X)(n+r+j).$$

On the other hand,

(V.0.5)

$$\mathbf{D}_X(gr_{n+r+j}^W \mathcal{H}_Z^r(\mathcal{O}_X)) \cong gr_{-n-r-j}^W \mathbf{D}_X \mathcal{H}_Z^r(\mathcal{O}_X) \cong gr_{-n-r-j}^W i_* \mathbf{Q}_Z^H[d](n) \cong (gr_{d-j}^W i_* \mathbf{Q}_Z^H[d])(n),$$

by isomorphism V.0.3.

Putting the isomorphisms [V.0.4](#) and [V.0.5](#) together, we obtain an isomorphism of filtered \mathcal{D}_X -modules

$$gr_{n+r+j}^W \mathcal{H}_Z^r(\mathcal{O}_X)(r+j) \cong gr_{d-j}^W i_* \mathbf{Q}_Z^H[d].$$

Taking F_{p+r+1} of this isomorphism yields an isomorphism

$$F_{p+1-j} gr_{n+r+j}^W \mathcal{H}_Z^r(\mathcal{O}_X) \cong F_{p+r+1} gr_{d-j}^W i_* \mathbf{Q}_Z^H[d].$$

As $j > 0$, the left hand side is 0, so the right hand side is, too. As W is a bounded below filtration, this implies $F_{p+r+1} W_{d-1} i_* \mathbf{Q}_Z^H[d] = 0$, which completes the proof by definition of the map γ . \square

Putting these lemmas together immediately shows that 3) \implies 4).

Step 3: 4) \implies 5) and 6). By [[MP22a](#), Theorem F], the assumption of 5) implies Z is k -du Bois. As ψ_k is defined to be $gr_{-k}^F DR_X(i_* \psi)[k-d]$, the first is an isomorphism iff the second is. By duality and the fact that $\mathbf{D}\psi = \psi(d)$, the latter is an isomorphism iff $gr_{k-d}^F DR_X(i_* \psi)[d-k]$ is, hence we have 6).

Now, as Z is k -du Bois, to show 7) we need $gr_{-p}^F DR_X(i_* \gamma_Z)$ to be an isomorphism for all $p \leq k$. Again, by duality, this is equivalent to $gr_{p-d}^F DR_X(i_* \gamma_Z)$ being an isomorphism for all $p \leq k$. This is implied by the map $F_{p+r} i_* \gamma_Z^\vee$ being an isomorphism for all $p \leq k$ using the definition of the functor DR_X , but this is precisely the map $F_p W_{n+r} \mathcal{H}_Z^r(\mathcal{O}_X) \rightarrow F_p \mathcal{H}_Z^r(\mathcal{O}_X)$, proving the claim.

Step 4: 5) \implies 4) By the argument of [[MP22b](#)] (generalized to the reducible LCI case), the conditions in 6) imply that Z has rational singularities. Hence, the conditions for k imply them for all $p \leq k$, and in particular, we have that

$$\psi_p : gr_{p-d}^F DR_X(i_* \psi_Z)[p-d]$$

is a quasi-isomorphism for all $p \leq k$. Using Property [II.4.1](#) and the fact that $\mathbf{D}\psi_Z = \psi_Z(d)$,

we conclude also that $gr^F DR_X(i_*\psi_Z)$ is a quasi-isomorphism for any $p \leq k$.

Also, as Z is k -rational, it must be k -du Bois, too. Hence, $F_p \mathcal{H}_Z^r(\mathcal{O}_X) = P_p \mathcal{H}_Z^r(\mathcal{O}_X)$ for all $p \leq k$. Hence, the morphisms in 5) are quasi-isomorphisms, as desired.

Step 5: 6) \implies 3) We induce on k . For $k = 0$, note that by applying $R\mathcal{H}om(-, \omega_X[n])$ to the isomorphism in the assumption of 7), and by using Property II.4.1, we get an isomorphism

$$gr_{-n}^F DR_X(W_{n+r} \mathcal{H}_Z^r(\mathcal{O}_X)) \rightarrow gr_{-n}^F DR_X(\mathcal{H}_Z^r(\mathcal{O}_X)) \rightarrow \mathcal{E}xt_{\mathcal{O}_X}^r(\mathcal{O}_Z, \omega_x)$$

where the lattermost module is the only non-vanishing $\mathcal{E}xt$, as Z is a complete intersection of codimension r . By definition of the filtration on the de Rham complex and the fact that the Hodge filtration satisfies $F_{-1} = 0$ for the modules we are concerned with, we have

$$gr_{-n}^F DR_X(W_{n+r} \mathcal{H}_Z^r(\mathcal{O}_X)) = \omega_X \otimes_{\mathcal{O}} F_0 W_{n+r} \mathcal{H}_Z^r(\mathcal{O}_X),$$

$$gr_{-n}^F DR_X(\mathcal{H}_Z^r(\mathcal{O}_X)) = \omega_X \otimes_{\mathcal{O}} F_0 \mathcal{H}_Z^r(\mathcal{O}_X),$$

and the image of $\mathcal{E}xt^r(\mathcal{O}_Z, \omega_X)$ in $\omega_X \otimes_{\mathcal{O}} \mathcal{H}_Z^r(\mathcal{O}_X)$ is, by definition, $\omega_X \otimes_{\mathcal{O}} P_0 \mathcal{H}_Z^r(\mathcal{O}_X)$, proving the claim for $k = 0$.

For higher k , we apply induction to assume $F_p W_{n+r} \mathcal{H}_Z^r(\mathcal{O}_X) = P_p \mathcal{H}_Z^r(\mathcal{O}_X)$ for $p \leq k - 1$. Hence, Z has $k - 1$ -du Bois singularities, and so satisfies $\text{codim}_X Z_{\text{sing}} \geq 2(k - 1) + 1 = 2k - 1$ by [MP22a, Cor. 3.40]. We need to prove that the inclusion $F_k W_{n+r} \mathcal{H}_Z^r(\mathcal{O}_X) \hookrightarrow P_k \mathcal{H}_Z^r(\mathcal{O}_X)$ is an isomorphism. But by the inductive hypothesis, this is equivalent to showing that the natural morphism

$$gr_{k-n}^F DR_X(W_{n+r} \mathcal{H}_Z^r(\mathcal{O}_X)) \rightarrow gr_{k-n}^P DR_X(\mathcal{H}_Z^r(\mathcal{O}_X))$$

is a quasi-isomorphism. Using Property II.4.1 and by applying $R\mathcal{H}om(-, \omega_X[r + k])$ to the

isomorphism assumed in 7), we get that the composition

$$(V.0.6) \quad gr_{k-n}^F DR_X(W_{n+r} \mathcal{H}_Z^r(\mathcal{O}_X)) \rightarrow gr_{n-k}^F DR_X(\mathcal{H}_Z^r(\mathcal{O}_X)) \rightarrow R\mathcal{H}om_{\mathcal{O}_X}(\Omega_Z^k, \omega_X[r+k])$$

is an isomorphism. But as $\text{codim}_X(Z_{\text{sing}}) \geq 2k - 1 \geq k$, using that $k > 0$, we know by [MP22a, Section 5.2] that the second map in the composition V.0.6 is the canonical map

$$gr_{k-n}^F DR_X(\mathcal{H}_Z^r(\mathcal{O}_X)) \rightarrow gr_{k-n}^P DR_X(\mathcal{H}_Z^r(\mathcal{O}_X)),$$

proving the claim.

Step 6: 3) \implies 1) We use the following notation

$$\sigma : \bigoplus_{i=1}^r gr_V^{r-1} \mathcal{B}_f \xrightarrow{t_i} gr_V^r \mathcal{B}_f$$

$$\delta : gr_V^r \mathcal{B}_f \xrightarrow{\partial_{t_i}} \bigoplus_{i=1}^r gr_V^{r-1} \mathcal{B}_f.$$

By [CD21, Theorem 1.2], we have an isomorphism in the category of filtered \mathcal{D}_X -modules

$$gr_{n+i}^W \ker(\delta) \cong gr_{d+i}^W i_* \mathbf{Q}_Z^H[d],$$

which implies $W_n \ker(\delta) = \ker(\delta)$. Similarly, we have $W_{n-1} \text{coker}(\sigma) = 0$.

We have the canonical inclusion

$$W_n \ker(\delta) \subseteq W_n gr_V^r \mathcal{B}_f$$

which we can compose to get a morphism

$$(V.0.7) \quad gr_n^W \ker(\delta) \rightarrow W_n \text{coker}(\sigma) = \frac{W_n gr_V^r \mathcal{B}_f}{W_n gr_V^r \mathcal{B}_f \cap \text{im}(\sigma)}$$

which preserves the Hodge filtration. Indeed, we see that $W_{n-1} \ker(\delta) \mapsto 0$, as

$$W_{n-1} \ker(\delta) \subseteq W_{n-1} gr_V^r \mathcal{B}_f = (s+r)W_{n+1} gr_V^r \mathcal{B}_f = \sum_{i=1}^r t_i \partial_{t_i} gr_V^r \mathcal{B}_f \subseteq \text{im}(\sigma).$$

Note that, by Lemma V.11, we have

$$gr_{p+r}^F i_* \mathbf{Q}_Z^H[d] \rightarrow gr_{p+r}^F gr_d^W i_* \mathbf{Q}_Z^H[d]$$

is an isomorphism for all $p \leq k+1$, and so we see that the natural morphisms

$$gr_{k+r}^F \ker(\delta) \rightarrow gr_{k+r}^F gr_n^W \ker(\delta), \quad gr_{k+r}^F W_n \text{coker}(\sigma) \rightarrow gr_{k+r}^F \text{coker}(\sigma)$$

are isomorphisms. We wish to show that the morphism V.0.7 is a filtered isomorphism.

Note that it is a filtered isomorphism on the open subset $\tilde{U} \subseteq X$ such that $\tilde{U} \cap Z = Z_{reg}$. Indeed, in this case,

$$\ker(\delta)|_{\tilde{U}} = gr_V^r \mathcal{B}_f|_{\tilde{U}} = \text{coker}(\sigma)|_{\tilde{U}} = i_+ \mathcal{O}_{\tilde{U}},$$

where $i : \tilde{U} \rightarrow \tilde{U} \times \mathbf{A}^r$ is the graph embedding along f_1, \dots, f_r , and these equalities respect the Hodge filtration. The morphism V.0.7 has both source and target decomposing into simple \mathcal{D}_X -modules, corresponding to the irreducible components of Z . It is thus an isomorphism, and hence, as the Hodge filtration is determined by the restriction to the regular locus (see [Sai88, (3.2.2.2)]), it is a filtered isomorphism.

In summary, we have shown that the composition

$$gr_{k+r}^F \ker(\delta) \rightarrow gr_{k+r}^F gr_V^r \mathcal{B}_f \rightarrow gr_{k+r}^F \text{coker}(\sigma)$$

is an isomorphism. Now, the assertion in 4) implies $\tilde{\alpha}(Z) \geq r+k$ by Theorem V.6. Hence, we have

$$gr_{k+r}^F gr_V^r \mathcal{B}_f = gr_{k+r}^F \mathcal{B}_f / I_Z \cdot gr_{k+r}^F \mathcal{B}_f = gr_{k+r}^F \text{coker}(\sigma),$$

where the first equality follows from $F_{k+r}V^{>r}\mathcal{B}_f = \sum_{i=1}^r t_i F_{k+r}V^{>(r-1)}\mathcal{B}_f$ by Theorem IV.6. Hence, by what we have shown, we see that $\delta = 0$ on $gr_{k+r}^F\mathcal{B}_f$, as the composition and the second morphism are isomorphisms. Hence,

$$\partial_{t_i} F_{k+r}\mathcal{B}_f \subseteq F_{k+r}\mathcal{B}_f + V^{>r-1}\mathcal{B}_f \subseteq V^{>r-1}\mathcal{B}_f,$$

where we know as $\tilde{\alpha}(Z) \geq r + k$ that $F_{k+r}\mathcal{B}_f \subseteq V^r\mathcal{B}_f$. Hence, we have shown

$$F_{k+r+1}\mathcal{B}_f \subseteq V^{>r-1}\mathcal{B}_f,$$

which implies $\tilde{\alpha}(Z) > r + k$, as desired. □

V.0.6: $\tilde{\alpha}(Z)$ for locally complete intersections and a local variant

In this section, we define the minimal exponent for a locally complete intersection, show that $\tilde{\alpha}(Z) = \infty$ iff Z is smooth, and define $\tilde{\alpha}_x(Z)$ for any $x \in Z$.

Using Theorem V.5, we see that an alternative definition to the minimal exponent is the following:

$$\tilde{\alpha}(Z) = \begin{cases} \sup\{\lambda > 0 \mid \delta_f \in V^\lambda\mathcal{B}_f\} & \delta_f \notin V^r\mathcal{B}_f \\ \sup\{r - 1 + q + \gamma \mid F_q\mathcal{B}_f \subseteq V^{r-1+\gamma}\mathcal{B}_f, \gamma \in (0, 1]\} & \delta_f \in V^r\mathcal{B}_f \end{cases}.$$

Remark V.12. Note that if there are open subsets $U_1, \dots, U_N \subseteq X$ such that $Z \cap U_i \neq \emptyset$ and $Z \subseteq U_1 \cup \dots \cup U_N$, then

$$\tilde{\alpha}(Z) = \min_{1 \leq i \leq N} \tilde{\alpha}(Z \cap U_i).$$

Indeed, the containment $F_q\mathcal{B}_f \subseteq V^{r-1+\gamma}\mathcal{B}_f$ can be checked on these open subsets, as it trivially holds on $X - Z$.

Lemma V.13. *The definition of $\tilde{\alpha}(Z)$ does not depend on the choice of regular sequence defining Z in X .*

Proof. By the previous remark, we can check this on an affine open cover of X , so we assume X is affine. Let $(f_1, \dots, f_r) = (g_1, \dots, g_r)$ define Z in X . If $\delta_f \notin V^r \mathcal{B}_f$, then it lies in $V^{\text{lct}(Z)} \mathcal{B}_f$, and $\text{lct}(Z)$ does not depend on choice of generators.

Hence, we need only show that if $q > 0$ is such that $F_q \mathcal{B}_f \subseteq V^{r-1+\gamma} \mathcal{B}_f$, then $F_q \mathcal{B}_g \subseteq V^{r-1+\gamma} \mathcal{B}_g$ and conversely.

Write $g_i = \sum_{j=1}^r a_{ij} f_j$ for some $a_{ij} \in \mathcal{O}_X$. Let A be the matrix (a_{ij}) , with determinant defining a hypersurface D in X . By choice of the g 's, this hypersurface does not intersect Z . Hence, we can remove it by the previous remark, and assume the matrix A is invertible.

Now, we have an isomorphism

$$u : X \times \mathbf{A}^r \rightarrow X \times \mathbf{A}^r \quad (x, \underline{t}_i) \mapsto (x, \underbrace{\sum_{j=1}^r a_{ij} t_j}_{\underline{t}_i}),$$

through which the graph embedding along g_1, \dots, g_r factors, by first applying the graph embedding along f_1, \dots, f_r . In particular, we see that $u_+ \mathcal{B}_f = \mathcal{B}_g$ as Hodge modules.

This isomorphism u preserves $X \times \{0\}$ and induces an automorphism on $\mathcal{D}_{X \times \mathbf{A}^r}$ which maps t_1, \dots, t_r to linear forms of t_1, \dots, t_r . We view the identification $u_+ \mathcal{B}_f = \mathcal{B}_g$ as an isomorphism $\tau : \mathcal{B}_f \rightarrow \mathcal{B}_g$ where \mathcal{B}_f is a $\mathcal{D}_{X \times \mathbf{A}^r}$ -module acted on through the automorphism u^* of $\mathcal{D}_{X \times \mathbf{A}^r}$.

It is clear then that $\tau(F_q \mathcal{B}_f) = F_q \mathcal{B}_g$, and one can show easily using uniqueness of the V -filtrations and the fact that u preserves $X \times \{0\}$ that $\tau(V^\lambda \mathcal{B}_f) = V^\lambda \mathcal{B}_g$ for all λ . This proves the claim. \square

This lemma shows that the following definition does not depend on choices.

Definition V.14. Let Z be a locally complete intersection of pure codimension r . Let $U_1, \dots, U_N \subseteq X$ be open subsets such that $Z \subseteq U_1 \cup \dots \cup U_N$ and such that $Z \cap U_i \neq \emptyset$ is a complete intersection of codimension r . Then define

$$\tilde{\alpha}(Z) := \min_{1 \leq i \leq N} \tilde{\alpha}(Z \cap U_i).$$

Lemma V.15. *We have that $\tilde{\alpha}(Z) \leq \frac{n+r}{2}$ if and only if Z is singular.*

Proof. First of all, assume Z is smooth. Then f_1, \dots, f_r can be taken to be part of a system of coordinates on X . Let $g = \sum_{i=1}^r y_i f_i$ be the hypersurface defining the minimal exponent. It is easy to check that, in general, the singular locus of $g|_U$, where $U = (X \times \mathbf{A}^r) - (X \times \{0\})$ is contained in $Z_{\text{sing}} \times \mathbf{A}^r$.

Hence, $g|_U$ is smooth if Z is smooth, and so $\tilde{\alpha}(g|_U) = \tilde{\alpha}(Z) = \infty$.

Conversely, if Z is singular, there exists some f_i and a point $x \in Z$ with $\text{mult}_x(f_i) \geq 2$. We assume for ease that $i = 1$. Then for $p = (1, 0, \dots, 0) \in \mathbf{A}^r$, the point $(x, p) \in U$ satisfies $\text{mult}_{(x,p)}(g|_U) \geq 2$, and so $\tilde{\alpha}(g|_U) \leq \frac{n+r}{2}$ by [MP20, Theorem E(3)]. \square

We define now a local variant of the minimal exponent. Let $x \in Z$ be a point which is fixed throughout this discussion. We see immediately that if $x \in V \subseteq V'$ are two open subsets, then $\tilde{\alpha}(Z \cap V') \leq \tilde{\alpha}(Z \cap V)$. Assume that there exists a decreasing sequence of open neighborhoods of x , say $V_1 \supseteq V_2 \supseteq \dots$ so that $\tilde{\alpha}(Z \cap V_i)$ is a strictly increasing sequence. If this sequence increases to ∞ , then it eventually is larger than $\frac{n+r}{2}$, and so Z would be smooth at x , but then the sequence of minimal exponents stabilizes. If the limit strictly increases to a bounded value, this would contradict discreteness of the V -filtration.

Hence, we have argued that for any $x \in Z$, there exists an open neighborhood $x \in V \subseteq X$ such that if $x \in V' \subseteq V$, then

$$\tilde{\alpha}(Z \cap V') = \tilde{\alpha}(Z \cap V).$$

We define $\tilde{\alpha}_x(Z) := \tilde{\alpha}(Z \cap V)$ for this choice of neighborhood V . Alternatively, it can be defined as

$$\tilde{\alpha}_x(Z) := \max_{x \in V} \tilde{\alpha}(Z \cap V).$$

Remark V.16. By discreteness of the V -filtration and the fact that the possible values for $\tilde{\alpha}_x(Z)$ lie in the set $(0, \frac{n+r}{2}] \cup \{\infty\}$, it is easy to see that the set

$$\{\tilde{\alpha}_x(Z) \mid x \in Z\}$$

is finite. Moreover, we have

$$\tilde{\alpha}(Z) = \min_{x \in Z} \tilde{\alpha}_x(Z),$$

and the set

$$\{x \in Z \mid \tilde{\alpha}_x(Z) \geq \gamma\} \subseteq Z$$

is an open subset of Z for any $\gamma \in \mathbf{Q}_{>0}$.

V.0.7: Example: Cones over Smooth Complete Intersections in Projective Space

In this section we compute the minimal exponent for a class of complete intersection varieties, generalizing [CDMO22, Example 4.23].

Let $f_1, \dots, f_r \in \mathbf{C}[x_1, \dots, x_n]$ be weighted homogeneous of degree $d \geq 2$ for some weights (ρ_1, \dots, ρ_n) . So for all j , we have

$$\sum_{i=1}^n \rho_i x_i \partial_{x_i} f_j = d \cdot f_j.$$

Let $\theta = \sum_{i=1}^n \rho_i x_i \partial_{x_i}$.

The main result of this section is the following:

Theorem V.17. *Let $f_1, \dots, f_r \in \mathbf{C}[x_1, \dots, x_n]$ be weighted homogeneous with weights (ρ_1, \dots, ρ_n) of degree d . Assume $Z = \{f_1 = \dots = f_r = 0\} \subseteq \mathbf{A}^n$ is a complete intersection of codimension r with an isolated singularity at the origin. Then*

$$\tilde{\alpha}(Z) = \frac{\sum_{i=1}^n \rho_i}{d}.$$

We will prove this by studying the hypersurface $g = \sum_{i=1}^r f_i y_i$. First, we begin with an important lemma concerning the singular locus of g . This, in part, explains the choice of actually looking at $g|_U$, where $U = X \times \mathbf{A}^r - (X \times \{0\})$. The proof is left to the reader.

Lemma V.18. *Let $f_1, \dots, f_r \in \mathcal{O}_X$ define a variety Z of codimension r . Let $g = \sum_{i=1}^r y_i f_i$.*

Then

$$\text{Sing}(g) = \sqcup_{x \in Z} x \times W_x,$$

where $W_x = \ker J_f^t(x)$ is a linear subspace of \mathbf{A}^r and $J_f^t(x)$ is the transpose of the Jacobian matrix of f_1, \dots, f_r at x . In particular,

$$\text{Sing}(g|_U) = \sqcup_{x \in Z_{\text{sing}}} (x \times W_x - \{0\}).$$

Now, assume $Z = V(f_1, \dots, f_r)$ satisfies $Z_{\text{sing}} = \{0\}$ as in the theorem statement. The lemma tells us that $\text{Sing}(g|_U) = 0 \times \mathbf{A}^r$, as the Jacobian matrix vanishes at 0.

Note that $g = \sum_{i=1}^r y_i f_i$, so $\theta(g) = dg$. Hence, in \mathcal{B}_g and $\tilde{\mathcal{B}}_g$, we have

$$\theta \partial_z^k \delta_g = \partial_z^k (ds) \delta_g = d(s - k) \partial_z^k \delta_g,$$

for all k . The same relation holds in $\tilde{\mathcal{B}}_{g|_U}$, which we denote by $\tilde{\mathcal{B}}_U$.

The goal is to understand for which λ we know δ_g defines a non-zero element of $gr_V^\lambda \tilde{\mathcal{B}}_U$. This λ is, by definition, $\tilde{\alpha}(g|_U) = \tilde{\alpha}(Z)$.

For every $\alpha \in \mathbf{Q} \cap [0, 1)$, the filtered \mathcal{D}_U -module $(gr_V^\alpha \mathcal{B}_U, F)$ is a direct summand of a mixed Hodge module. In particular, it is regular and quasi-unipotent in the sense of Saito [Sai88, Section 3.2.1]. Moreover, we have a filtered isomorphism for such α

$$(gr_V^\alpha \mathcal{B}_U, F) \cong (gr_V^\alpha \tilde{\mathcal{B}}_U, F)$$

by [Sai94, (2.1.4)]. Also, by [Sai94, (2.2.3)], we have filtered isomorphisms for $\lambda = k + \alpha, k \in \mathbf{Z}$

$$\partial_z^k : (gr_V^\lambda \tilde{\mathcal{B}}_U, F) \rightarrow (gr_V^\alpha \tilde{\mathcal{B}}_U, F[-k]),$$

Hence, $(gr_V^\lambda \tilde{\mathcal{B}}_U, F)$ is also quasi-unipotent and regular. Moreover, it is supported on $\text{Sing}(g|_U)$, which is the vanishing locus of x_1, \dots, x_n .

By [Sai88, Lemme 3.2.6], we know that the smallest piece of the Hodge filtration on $gr_V^\lambda \tilde{\mathcal{B}}_U$ is killed by (x_1, \dots, x_n) . By definition of the Hodge filtration on $\tilde{\mathcal{B}}_U$, if δ defines a non-zero element of $gr_V^\lambda \tilde{\mathcal{B}}_U$, then it must lie in the lowest Hodge piece. Indeed, $F_{-1} \tilde{\mathcal{B}}_U \subseteq \sum_{i < 0} \mathcal{O}_X \partial_z^i \delta \subseteq V^{\lambda+1} \tilde{\mathcal{B}}_U$.

Hence, if λ is such that δ defines a non-zero element of $gr_V^\lambda \tilde{\mathcal{B}}_U$, we see then that $(x_1, \dots, x_n) \bar{\delta} = 0$ in this associated graded. But then $\theta + \sum_{i=1}^n \rho_i = \sum_{i=1}^n \rho_i \partial_{x_i} x_i$ also kills δ . So we have

$$0 = (\theta + \sum_{i=1}^n \rho_i) \bar{\delta} = (ds + \sum_{i=1}^n \rho_i) \bar{\delta}.$$

Finally, in $gr_V^\lambda \tilde{\mathcal{B}}_U$, the operator $(s + \lambda)$ is nilpotent, and so we finally get

$$\tilde{\alpha}(Z) = \tilde{\alpha}(g|_U) = \lambda = \frac{\sum_{i=1}^n \rho_i}{d},$$

as desired.

CHAPTER VI

Monodromic Mixed Hodge Modules and Fourier Transform

Here we apply the results on V -filtration for higher codimension subvarieties to the study of the Fourier-Lagrange transform of mixed Hodge modules. It is known that this operation need not preserve the category of regular singularities for the underlying \mathcal{D} -module, so we must restrict our attention to monodromic mixed Hodge modules. We give a complete description of the Hodge and weight filtrations for the Fourier-Laplace transform of a monodromic mixed Hodge module.

VI.1: Monodromic Mixed Hodge Modules

Let X be a smooth complex algebraic variety and let $E = X \times \mathbf{A}^r$ have fiber coordinates z_1, \dots, z_r and vector fields $\partial_{z_1}, \dots, \partial_{z_r}$ giving the Euler operator $\theta_z = \sum_{i=1}^r z_i \partial_{z_i}$. We say that a mixed Hodge module M on E is *monodromic* if its underlying \mathcal{D}_E -module is monodromic. Recall that this is equivalent to having a decomposition

$$(VI.1.1) \quad \mathcal{M} = \bigoplus_{\chi \in \mathbf{Q}} \mathcal{M}^\chi, \text{ where } \mathcal{M}^\chi = \bigcup_{\ell \geq 1} \ker((\theta_z - \chi + r)^\ell).$$

These monodromic pieces \mathcal{M}^χ are related by $z_i \mathcal{M}^\chi \subseteq \mathcal{M}^{\chi+1}$, $\partial_{z_i} \mathcal{M}^\chi \subseteq \mathcal{M}^{\chi-1}$. Moreover,

if $V^\bullet \mathcal{M}$ is the V -filtration on \mathcal{M} along $X \times \{0\}$, then

$$V^\lambda \mathcal{M} = \bigoplus_{\chi \geq \lambda} \mathcal{M}^\chi.$$

This class of mixed Hodge modules was studied by T. Saito [Sai22a] in the rank 1 case and later in the arbitrary rank case [Sai22b]. One interesting result (Theorem 0.1) of loc. cit. is that the Hodge filtration also decomposes along the monodromic decomposition

$$(VI.1.2) \quad F_p \mathcal{M} = \bigoplus_{\chi \in \mathbf{Q}} F_p \mathcal{M}^\chi, \text{ where } F_p \mathcal{M}^\chi = F_p \mathcal{M} \cap \mathcal{M}^\chi.$$

Now, we begin a study of the weight filtration of a monodromic mixed Hodge module. First, assume that M is pure. Then we have the following understanding of the nilpotent operator $N = \bigoplus_{\chi \in \mathbf{Q}} \theta_z - \chi + r$. This was shown in the rank 1 case by T. Saito.

Proposition VI.1. *Let \mathcal{M} be a \mathcal{D}_E -module underlying a monodromic pure Hodge module M . Then $N = 0$, i.e.,*

$$\mathcal{M}^\chi = \ker(\theta_z - \chi + r).$$

Proof. We can decompose M into simple summands, as the category of polarizable pure Hodge modules is semi-simple. Hence, we can assume M is simple. But then N being nilpotent implies it must be 0, as it cannot be an isomorphism. \square

Interestingly, this has the following consequence in rank 1:

Corollary VI.2. *Let \mathcal{M} underlie a monodromic pure Hodge module on $X \times \mathbf{A}^1$. Then*

$$z : \mathcal{M}^0 \rightarrow \mathcal{M}^1, \partial_z : \mathcal{M}^1 \rightarrow \mathcal{M}^0$$

are identically 0.

Proof. By the previous theorem, we know $N = z\partial_z : \mathcal{M}^1 \rightarrow \mathcal{M}^1$ and $N = \partial_z z : \mathcal{M}^0 \rightarrow \mathcal{M}^0$ are identically 0. So, in \mathcal{M}^0 , we have $\text{im}(\partial_z) \subseteq \ker(z)$. But, as \mathcal{M} allows a decomposition

by strict support, we have

$$\mathcal{M}^0 = \ker(z) \oplus \text{im}(\partial_z),$$

and so we must have $\text{im}(\partial_z) = 0$, and so $\ker(z) = \mathcal{M}^0$, proving the claim. \square

We now proceed to the case of non-trivial weight filtration. It is easy to see that, as \mathcal{D}_E -modules, we have an isomorphism $Sp(\mathcal{M}) = \mathcal{M}$ for any monodromic \mathcal{D}_E -module \mathcal{M} , using the isomorphism $E \cong T_{X \times \{0\}}E$.

Theorem VI.3. *[CD21, Theorem 1.5] Let (\mathcal{M}, W_\bullet) underlie a monodromic mixed Hodge module on E . Let $N = \bigoplus_{\chi \in \mathbb{Q}} \theta_z - \chi + r$ be the nilpotent operator on \mathcal{M} . Then $W_\bullet \mathcal{M}$ is its own relative monodromy filtration with respect to N , i.e., $NW_\bullet \mathcal{M} \subseteq W_{\bullet-2} \mathcal{M}$.*

Note that if \mathcal{M} is pure, this theorem is exactly saying that $N = 0$, the conclusion of Proposition VI.1. Also note that the other condition in the definition of relative monodromy filtration obviously holds for the filtration W itself.

Proof. Let (\mathcal{M}, W) underlie a monodromic mixed Hodge module on E . Note that the relative monodromy filtration for (\mathcal{M}, W_\bullet) with the nilpotent operator $N = \bigoplus \theta_z - \chi + r$ exists. Indeed, for any χ , we know from Chapter II the existence of the relative monodromy filtration on $gr_V^\chi \mathcal{M}$ with respect to $N = \theta_z - \chi + r$ and the induced filtration $M_\bullet gr_V^\chi \mathcal{M} = gr_V^\chi W_\bullet \mathcal{M}$. But $gr_V^\chi \mathcal{M} = \mathcal{M}^\chi$, so we can just take the direct sum of each of these relative monodromy filtrations to define the one on \mathcal{M} .

Now, $gr_i^W \mathcal{M}$ is monodromic for any i , as it is a \mathcal{D}_E -module subquotient of \mathcal{M} . It is also pure, so by Proposition VI.1, we see that $N = 0$ on $gr_i^W \mathcal{M}$. Hence, $\theta_z - \chi + r$ is 0 on $(gr_i^W \mathcal{M})^\chi$ for all χ , and so the relative monodromy filtration is equal to $W_\bullet \mathcal{M}$. \square

VI.2: Fourier-Laplace Transform

Let E^\vee be the dual bundle of E , which is also trivial. Say the fiber coordinates are w_1, \dots, w_r with derivations $\partial_{w_1}, \dots, \partial_{w_r}$. Given any \mathcal{D}_E -module \mathcal{M} , we can define the *Fourier-Laplace*

transform $\mathrm{FL}(\mathcal{M})$, which is a \mathcal{D}_{E^\vee} -module, as follows: as a \mathcal{D}_X -module, it agrees with the structure of \mathcal{M} . However, we have

$$w_i m = -\partial_{z_i} m, \partial_{w_i} m = z_i m$$

for all $1 \leq i \leq r$. In particular, if \mathcal{M} is monodromic, then $\mathrm{FL}(\mathcal{M})$ is, too, and if $\mathrm{FL}(\mathcal{M})^\chi$ is the χ th monodromic piece (along w_1, \dots, w_r), we have as \mathcal{D}_X -modules equality

$$\mathrm{FL}(\mathcal{M})^\chi = \mathcal{M}^{r-\chi},$$

where the right hand side is the $r - \chi$ th monodromic piece of \mathcal{M} .

The functor FL need not preserve the category of regular holonomic \mathcal{D} -modules, and in particular it need not preserve the category of mixed Hodge modules. However, Brylinski showed that if \mathcal{M} is regular holonomic and monodromic on \mathcal{D}_E , then $\mathrm{FL}(\mathcal{M})$ is regular holonomic and monodromic on \mathcal{D}_{E^\vee} . The main theorem is that $\mathrm{FL}(\mathcal{M})$ naturally underlies a mixed Hodge module on E^\vee if $\mathcal{M} \in \mathrm{MHM}_{\mathrm{mon}}(E)$.

Let $\mathcal{E} = E \times_X E^\vee$ which, as a variety, is isomorphic to $X \times \mathbf{A}^r \times \mathbf{A}^r$. Let $p : \mathcal{E} \rightarrow E$ be the projection and $i : E^\vee \rightarrow \mathcal{E}$ the inclusion of the zero section. Let $g : \mathcal{E} \rightarrow \mathbf{A}^1$ be the regular function $\sum_{i=1}^r z_i w_i$. For any $M \in \mathrm{MHM}(\mathcal{E})$, consider the total nearby cycles

$$\phi_g(M) = \bigoplus_{\lambda \in (0,1]} \phi_{g,\lambda}(M).$$

The goal of this section is

Theorem VI.4. *[CD21, Theorem 1.4] Let \mathcal{M} be a monodromic regular holonomic \mathcal{D}_E -module. Then, using the above notation, there is a natural isomorphism*

$$\mathrm{FL}(\mathcal{M}) \cong \mathcal{H}^0 \sigma^* \phi_t \Gamma_+ p^1(\mathcal{M})[-r].$$

Using this and the fact that $\mathcal{H}^0 i^*$, ϕ_g and $p^!$ preserve the category of mixed Hodge modules, we see that $\mathrm{FL}(\mathcal{M})$ naturally underlies a mixed Hodge module on E^\vee . We will explain the Hodge and weight filtration on this Hodge module after the proof of the theorem.

Remark VI.5. In [RW22, Definition 4.9] and [KS90, (10.3.31)], the Fourier-Laplace transform (called Fourier-Sato transform for monodromic objects) is defined in the following way: let $p : \mathcal{E} \rightarrow E$, $g \in \mathcal{O}_{\mathcal{E}}(\mathcal{E})$ and $\Gamma : \mathcal{E} \rightarrow \mathcal{E} \times \mathbf{A}^1$ be defined as in our notation. Then, let $\omega = (q \times id) \circ \Gamma$ where $q : \mathcal{E} \rightarrow E^\vee$ is the projection onto E^\vee (leaving \mathbf{A}^1 fixed). The Fourier-Sato transform of a monodromic mixed Hodge module is defined to be the composition

$$\phi_\xi \omega_* p^!(\mathcal{M})[-r],$$

where ξ is the coordinate on \mathbf{A}^1 .

However, this is precisely our definition. Indeed, writing $\omega_* = (q \times id)_* \Gamma_+$, we see easily that $\phi_t(q \times id)_* = q_* \phi_t$, as $q \times id$ does not affect t , the coordinate on \mathbf{A}^1 . Also, by Lemma VI.6, we know $\phi_\xi \Gamma_+ p^!(\mathcal{M})[-r]$ is z -monodromic. Then the claim boils down to a mixed Hodge module version of Proposition IV.4, which is immediate from the proof of Proposition IV.4. Indeed, the proof amounts to showing the vanishing of a certain \mathcal{D} -module, but the corresponding mixed Hodge module is 0 if the underlying \mathcal{D} -module is, too. Hence, $p_* = \sigma^*$ on $\phi_\xi \Gamma_+ p^!(\mathcal{M})$, proving the claim.

We introduce some notation to prove the theorem. We can write the underlying $\mathcal{D}_{\mathcal{E}}$ -module of $p^!(\mathcal{M})$ as $\mathcal{M}[w]$, with the obvious $\mathcal{D}_{\mathcal{E}}$ -module action. Now, g defines a singular hypersurface, so we must use the graph embedding $\Gamma : \mathcal{E} \rightarrow \mathcal{E} \times \mathbf{A}^1$ to understand ϕ_g . Let ξ be the coordinate on \mathbf{A}^1 with corresponding derivation ∂ . Recall that $\Gamma_+ \mathcal{M}[w] = \bigoplus_{k \geq 0} \mathcal{M}[w] \partial^k \delta_g$, and the action is given by

$$P(mw^\beta \partial^k \delta_g) = P(mw^\beta) \partial^k \delta_g \text{ for } P \in \mathcal{D}_X + \mathcal{O}_{\mathcal{E}},$$

$$\partial_{w_i}(mw^\beta \partial^k \delta_g) = \beta_i mw^{\beta-e_i} \partial^k \delta_g - z_i mw^\beta \partial^{k+1} \delta_g \text{ for } 1 \leq i \leq r,$$

$$\partial_{z_i}(mw^\beta \partial^k \delta_g) = \partial_{z_i}(m)w^\beta \partial^k \delta_g - mw^{\beta+e_i} \partial^{k+1} \delta_g \text{ for } 1 \leq i \leq r,$$

$$\xi(mw^\beta \partial^k \delta_g) = gmw^\beta \partial^k \delta_g - kmw^\beta \partial^{k-1} \delta_g$$

$$\partial(mw^\beta \partial^k \delta_g) = mw^\beta \partial^{k+1} \delta_g.$$

Hence, if we define $\tilde{\theta}_z$ by the formula

$$\tilde{\theta}_z(mw^\beta \partial^k \delta_g) = \theta_z(m)w^\beta \partial^k \delta_g,$$

we see that

$$\theta_z(mw^\beta \partial^k \delta_g) = (\tilde{\theta}_z - (k+1) - \xi\partial)(mw^\beta \partial^k \delta_g),$$

and similarly,

$$\theta_w(mw^\beta \partial^k \delta_g) = (|\beta| - (k+1) - \xi\partial)(mw^\beta \partial^k \delta_g).$$

Note that $\Gamma_+ \mathcal{M}[w]$ decomposes into eigenspaces for the operators $T = \theta_z + \xi\partial + 1$ and $S = \theta_w + \xi\partial + 1$. Indeed, every element can be written uniquely as a sum of elements of the form $mw^\beta \partial^k \delta_g$ for $m \in \mathcal{M}^\chi$ for some χ, β, j . For such an element, we have

$$(T - \lambda)^a(mw^\beta \partial^k \delta_g) = 0 \iff (\theta_z - k - \lambda)^a m = 0 \iff m \in \mathcal{M}^{k+\lambda+r},$$

$$(S - \lambda)^a(mw^\beta \partial^k \delta_g) = 0 \iff (|\beta| - k - \lambda)^a = 0 \iff |\beta| = k + \lambda.$$

As $|\beta|$ and k lie in \mathbf{Z} , this shows that the only eigenvalues of S are integers. Moreover, these operators obviously commute with each other, so $\Gamma_+ \mathcal{M}[w]$ decomposes into simultaneous eigenspaces. We shift these for ease of notation and denote

$$E_{\beta, \ell} = \sum_{\alpha \in \mathbf{N}^r, |\alpha| \geq -\ell} \mathcal{M}^{\beta+|\alpha|+\ell} w^\alpha \partial^{|\alpha|+\ell} \delta_g,$$

and we pick out the part with a fixed power ∂^j , denoted

$$F_{\beta,\ell}^j = \sum_{|\alpha|=j-\ell} \mathcal{M}^{\beta+|\alpha|+\ell} w^\alpha \partial^j \delta_g,$$

hence,

$$E_{\beta,\ell} = \bigoplus_{j \geq \ell} F_{\beta,\ell}^j.$$

It is trivial to check that these spaces are moved in the following way with the operators of $\mathcal{D}_{\mathcal{E} \times \mathbf{A}^1}$:

$$z_i E_{\beta,\ell} \subseteq E_{\beta+1,\ell}$$

$$\partial_{z_i} E_{\beta,\ell} \subseteq E_{\beta-1,\ell}$$

$$w_i E_{\beta,\ell} \subseteq E_{\beta,\ell-1}$$

$$\partial_{w_i} E_{\beta,\ell} \subseteq E_{\beta,\ell+1}$$

$$\xi E_{\beta,\ell} \subseteq E_{\beta+1,\ell-1}$$

$$\partial E_{\beta,\ell} \subseteq E_{\beta-1,\ell+1}.$$

For any $\lambda \in \mathbf{Q}$, the piece $V^\lambda \Gamma_+ \mathcal{M}[w]$ is invariant under the operators T and S , so it also decomposes into simultaneous eigenspaces. We write $E_{\beta,\ell}^\lambda = V^\lambda \Gamma_+ \mathcal{M}[w] \cap E_{\beta,\ell}$.

Lemma VI.6. *Let $\lambda \in \mathbf{Q}$. Then*

$$gr_V^\lambda \Gamma_+ \mathcal{M}[w] = \bigoplus_{\beta,\ell} E_{\beta,\ell}^\lambda / E_{\beta,\ell}^{>\lambda}.$$

Moreover, $gr_V^\lambda \Gamma_+ \mathcal{M}[w]$ is monodromic along z_1, \dots, z_r and along w_1, \dots, w_r . The monodromic pieces are given, respectively, by

$$(gr_V^\lambda \Gamma_+ \mathcal{M}[w])_z^\chi = \bigoplus_{\ell \in \mathbf{Z}} E_{\chi+\lambda,\ell}^\lambda / E_{\chi+\lambda,\ell}^{>\lambda}$$

$$(gr_V^\lambda \Gamma_+ \mathcal{M}[w])_w^\chi = \bigoplus_{\beta \in \mathbf{Q}} E_{\beta, r-\lambda-\chi}^\lambda / E_{\beta, r-\lambda-\chi}^{>\lambda}$$

Proof. The first claim is trivial. For the last two claims, use the fact that $gr_V^\lambda \Gamma_+ \mathcal{M}[w]$ decomposes into its simultaneous eigenspaces for T and S . Also, use the fact that $T = \theta_z + N + \lambda$ and $S = \theta_w + N + \lambda$, where $N = \xi \partial - \lambda + 1$ is the nilpotent operator on $gr_V^\lambda \Gamma_+ \mathcal{M}[w]$. Hence, if an element u lies in an eigenspace for T , and N acts nilpotently, it must also lie in an eigenspace for θ_z . Similarly for an element lying in an eigenspace for S and N acts nilpotently. From here it is just an index check. \square

Finally, we define morphisms $\varphi_{\beta, \ell} : E_{\beta, \ell} \rightarrow \mathcal{M}^{\beta+\ell}$ by

$$\sum m_\alpha w^\alpha \partial^{|\alpha|+\ell} \delta_g \mapsto (-1)^\ell \sum \partial_z^\alpha (m_\alpha).$$

By $m_\alpha w^\alpha \partial^{|\alpha|+\ell} \delta_g \in E_{\beta, \ell}$, we know $m_\alpha \in \mathcal{M}^{\beta+\ell+|\alpha|}$, and so the image does land in $\mathcal{M}^{\beta+\ell}$. Moreover, the morphism is \mathcal{D}_X -linear. It is easy to check that this family of morphisms has the following behavior with respect to the operators in $\mathcal{D}_{\mathcal{E} \times \mathbf{A}^1}$:

$$\varphi_{\beta-1, \ell} \circ \partial_{z_i} = 0.$$

$$\varphi_{\beta, \ell-1} \circ w_i = -\partial_{z_i} \circ \varphi_{\beta, \ell}$$

$$\varphi_{\beta, \ell+1} \circ \partial_{w_i} = z_i \circ \varphi_{\beta, \ell}$$

$$\varphi_{\beta+1, \ell-1} \circ \xi = -(\theta_z - \ell + r) \circ \varphi_{\beta, \ell}$$

$$\varphi_{\beta-1, \ell+1} \circ \partial = -\varphi_{\beta, \ell}.$$

We can strengthen the first property as follows:

Lemma VI.7. *Assume $\ell \geq 0$. Then*

$$\ker(\varphi_{\beta, \ell}) = \sum_{i=1}^r \partial_{z_i} E_{\beta+1, \ell}.$$

Proof. We clearly have the containment \supseteq .

Now, let $\eta = \sum_{|\alpha| \leq a} m_\alpha w^\alpha \partial^{|\alpha|+\ell} \delta_g$ lie in the kernel. We prove the claim by induction on a . Note that if $a = 0$, then $\varphi_{\beta,\ell}(m_0 \partial^\ell \delta_g) = (-1)^\ell m_0 = 0$ implies m_0 equals 0, so the base case is handled.

Assume the claim holds true for any sum with $|\alpha| \leq a - 1$ lying in the kernel. As $a > 0$, for any α with $|\alpha| = a$, there exists some non-zero index $\alpha_i > 0$. Choose such an i for each α , call it i_α . Then $\eta + \sum_{|\alpha|=a} \partial_{z_{i_\alpha}} (m_\alpha w^{\alpha - e_{i_\alpha}} \partial^{|\alpha|+\ell-1}) \delta_g$ has no terms w^α with $|\alpha| = a$. Also, it lies in the kernel, because it is a sum of two elements which lie in the kernel of $\varphi_{\beta,\ell}$. By induction, this term lies in $\sum_{i=1}^r \partial_{z_i}(E_{\beta+1,\ell})$, and so η does, too. \square

VI.2.1: Computing the V -filtration on $\Gamma_+ \mathcal{M}[w]$

To compute the V -filtration, we first break up $\mathcal{M} = \bigoplus_{\lambda \in [0,1)} \bigoplus_{j \in \mathbf{Z}} \mathcal{M}^{\lambda+j}$ and then compute the V -filtration on $\mathcal{M}^{\lambda+\mathbf{Z}} = \bigoplus_{j \in \mathbf{Z}} \mathcal{M}^{\lambda+j}$ in the two cases: $\lambda = 0$ and $\lambda \in (0, 1)$.

First of all, we make the following easy observations. They are

$$(VI.2.1) \quad \sum_{i=1}^r w_i F_{\beta,\ell-1}^j = F_{\beta,\ell}^j$$

$$(VI.2.2) \quad \chi + j \neq r - 1 \implies \sum_{i=1}^r F_{\chi,\ell}^j = F_{\chi+1,\ell}^j$$

$$(VI.2.3) \quad \chi + j \neq r - 1 \implies \sum_{i=1}^r \partial_{w_i} F_{\chi,j}^j = F_{\chi,j+1}^{j+1},$$

where the first follows from definition and the second two follow from the fact that $\mathcal{M}^\chi = \sum_{i=1}^r z_i \mathcal{M}^{\chi-1}$ for $\chi \neq r$, by Remark II.10.

We define a filtration $U^\bullet \Gamma_+ \mathcal{M}[w]$ by defining it for $\bullet \in [0, 1]$ explicitly and then induc-

tively defining

$$U^\lambda = \xi^j U^{\lambda-j} \text{ for } \lambda > 1, \lambda - j \in (0, 1],$$

$$U^\lambda = \partial^j U^{\lambda+j} + U^{>\lambda} \text{ for } \lambda < 0, \lambda + j \in [0, 1).$$

Then we need only check that the filtration is exhaustive and satisfies the following properties:

1. For $\lambda \in [0, 1]$, the module U^λ is coherent over $V^0 \mathcal{D}_{\mathcal{E} \times \mathbf{A}^1}$.
2. For $\lambda > \lambda'$, we have $U^\lambda \subseteq U^{\lambda'}$.
3. We have $\xi U^0 \subseteq U^1$.
4. We have $\partial U^1 \subseteq U^0$.
5. For each $\lambda \in [0, 1)$, there exists $a \geq 0$ such that $(\xi \partial - \lambda + 1)^a U^\lambda \in U^{>\lambda}$.

Note that in Property 5, we need not check the nilpotency for $\lambda = 1$ thanks to the previous conditions. Indeed, let a be such that $(\xi \partial + 1)^a U^0 \subseteq U^{>0}$. Then $(\xi \partial)^{a+1} U^1 = \xi (\partial \xi)^a \partial U^1 \subseteq \xi U^{>0} = U^{>1}$.

Case 1: $\lambda = 0$. Define

$$U^0 := V^0 \mathcal{D}_{\mathcal{E} \times \mathbf{A}^1} \cdot F_{0,0}^0 + V^0 \mathcal{D}_{\mathcal{E} \times \mathbf{A}^1} \cdot F_{0,r}^r,$$

$$U^1 := V^0 \mathcal{D}_{\mathcal{E} \times \mathbf{A}^1} \cdot F_{1,0}^0 + V^0 \mathcal{D}_{\mathcal{E} \times \mathbf{A}^1} \cdot F_{1,r-1}^{r-1}.$$

Exhaustive: let $\mathcal{U} = \bigcup_k U^k$. As \mathcal{U} is closed under the action of ∂ and w_1, \dots, w_r , it suffices to prove that $\mathcal{M}^\ell \delta_g \subseteq \mathcal{U}$ for all $\ell \in \mathbf{Z}$. Well $F_{0,0}^0 = \mathcal{M}^0 \delta_g \subseteq \mathcal{U}$ by definition, and so by Remark II.10 we get $\mathcal{M}^1 \delta_g, \dots, \mathcal{M}^{r-1} \delta_g \subseteq \mathcal{U}$. Moreover, by induction, we see that $\mathcal{M}^\ell \delta_g \subseteq \mathcal{U}$ for all $\ell \leq 0$. Assume $\mathcal{M}^\ell \delta_g \subseteq \mathcal{U}$ for some $\ell \leq 0$. Then $\mathcal{M}^\ell [w] \partial^k \delta_g \subseteq \mathcal{U}$ for all $k \geq 0$ and $\sum_{i=1}^r \partial_{z_i} (\mathcal{M}^\ell \delta_g) \subseteq \mathcal{U}$. But

$$\partial_{z_i} \cdot m \delta_g = \partial_{z_i} m \delta_g - w_i m \partial \delta_g,$$

and $w_i m \partial \delta_g \in \mathcal{U}$, so $\partial_{z_i}(m) \delta_g \in \mathcal{U}$. Hence, we see that (using Remark II.10) $\mathcal{M}^{\ell-1} \delta_g = (\sum_{i=1}^r \partial_{z_i} \mathcal{M}^\ell + \mathcal{M}^\ell) \delta_g \subseteq \mathcal{U}$, as desired.

Also, $\mathcal{M}^r \delta_g = F_{0,r}^r \subseteq \mathcal{U}$, and so using Remark II.10 and the z_i action, we get $\mathcal{M}^\ell \delta_g = (\sum_{i=1}^r z_i \mathcal{M}^{\ell-1} \delta_g) \subseteq \mathcal{U}$ for all $\ell > r$, too, which proves exhaustiveness.

1: To see U^0 is finitely generated over $V^0 \mathcal{D}_{\mathcal{E} \times \mathbb{A}^1}$, let m_1, \dots, m_N be finitely many $gr_V^0 \mathcal{D}_E$ generators of \mathcal{M}^0 and let η_1, \dots, η_M be generators for \mathcal{M}^r over $gr_V^0 \mathcal{D}_E$. Then these elements generate U^0 , by the following fact: given $m \partial^k \delta_g \in U^0$, we obtain $(gr_V^0 \mathcal{D}_E \cdot m) \partial^k \delta_g \subseteq U^0$. Indeed, we easily get $\mathcal{D}_X \cdot m \partial^k \delta_g$, and to get $z_i \partial_{z_j}(m) \delta_g$, we use

$$z_i \partial_{z_j}(m) \delta_g = z_i \partial_{z_j}(m \delta_g) + w_j \partial_{w_i}(m \delta_g),$$

which lies in U^0 . The same proof works for U^1 .

For the remaining conditions, we use the following lemma

Lemma VI.8. *We have containment $F_{\chi,\ell}^j \subseteq U^0$ for any triple satisfying either of the two conditions*

- $\chi \geq 0, j \geq r, \ell \leq j$.
- $0 \leq \chi \leq r-1, 0 \leq j < r-\chi, \ell \leq j$.

Also, we have $F_{\chi,\ell}^j \subseteq U^1$ in either of the following cases:

- $\chi \geq 1, j \geq r-1, \ell \leq j$
- $1 \leq \chi \leq r-1, 0 \leq j \leq r-\chi, \ell \leq j$.

In particular, we have

- $E_{\chi,\ell} \subseteq U^0$ for all $\chi \geq 0, \ell \geq r$,
- $E_{\chi,\ell} \subseteq U^1$ for all $\chi \geq 1, \ell \geq r-1$,
- $E_{1,\ell} \subseteq U^1$ for all $\ell \leq 0$.

Proof. We make use of the fact that U^0 is closed under z_i, ∂_{w_i} and w_i for all i .

Starting from $F_{0,r}^r \subseteq U^0$, by Formula VI.2.2 we get $F_{\chi,r}^r \subseteq U^0$ for all $\chi \geq 0$. Then by Formula VI.2.3 we get $F_{\chi,j}^j \subseteq U^0$ for all $j \geq r$. Finally, by Formula VI.2.1 we get $F_{\chi,\ell}^j \subseteq U^0$ for all $\chi \geq 0, j \geq r, \ell \leq j$.

Starting from $F_{0,0}^0 \subseteq U^0$, we get by Formula VI.2.2 $F_{\chi,0}^0 \subseteq U^0$ for all $0 \leq \chi \leq r-1$. By applying Formula VI.2.3 we get $F_{\chi,j}^j \subseteq U^0$ for all $0 \leq \chi \leq r-1$ and $\chi + j < r$. Finally, applying Formula VI.2.1 we get $F_{\chi,\ell}^j \subseteq U^0$ for all $0 \leq \chi \leq r-1, \chi + j < r$ and $\ell \leq j$.

Similarly, we argue for the containment of the other subsets in U^1 .

The last statements follow easily from these containments. For example, let $\ell \leq 0$, then $F_{1,\ell}^j \subseteq U^1$ for all $j \geq 0$. Indeed, if $j \geq r-1$, then this comes from the fact that $F_{1,r-1}^{r-1}$ is contained as argued above. If $0 \leq j < r-1$, then in particular, $1 + j = \chi + j < r$, so this follows from the fact that $F_{1,0}^0$ is contained as argued above. \square

2: Obvious, from the lemma and using the fact that $F_{1,r-1}^{r-1} = (\xi - g)F_{0,r}^r$.

3 Indeed, $\xi F_{0,0}^0 \subseteq E_{1,-1}$ and $\xi F_{0,r}^r \subseteq E_{1,r-1}$, so this follows from the lemma.

4 Indeed, $\partial F_{1,0}^0 = F_{0,1}^1 \subseteq E_{0,1}$, which is in U^0 by the lemma, and $\partial F_{1,r-1}^{r-1} = F_{0,r}^r$, which is in U^0 by definition.

5 Note that $\varphi_{0,0} \circ (\partial\xi)^a = (\theta + r)^a \circ \varphi_{0,0}$, so since $\varphi_{0,0}$ has image in \mathcal{M}^0 , $(\theta + r)^a$ kills this for $a \gg 0$. Similarly, $\varphi_{0,r} \circ (\partial\xi)^a = (\theta - r + r)^a \circ \varphi_{0,r}$, and $(\theta - r + r)^a$ kills \mathcal{M}^r for $a \gg 0$. Thus, we see that $(\partial\xi)^a$ multiplies $F_{0,0}^0$ and $F_{0,r}^r$ into $\ker(\varphi_{0,0})$ and $\ker(\varphi_{0,r})$, respectively. Well, by Lemma VI.7, these are

$$\sum_{i=1}^r \partial_{z_i}(E_{1,0}) \quad \text{and} \quad \sum_{i=1}^r \partial_{z_i}(E_{1,r})$$

respectively, and both of these are contained in U^1 by the lemma and the fact that U^1 is closed under ∂_{z_i} action.

This finishes the proof and shows that $U^\bullet = V^\bullet$ is the V -filtration along ξ .

Case 2: $\lambda \in (0, 1)$.

Define $U^0 = U^\lambda := V^0 \mathcal{D}_{\mathcal{E} \times \mathbf{A}^1} \cdot F_{\lambda,0}^0$ and $U^1 := V^0 \mathcal{D}_{\mathcal{E} \times \mathbf{A}^1} \cdot F_{\lambda+1,0}^0$.

Exhaustive: As $F_{\lambda,0}^0 = \mathcal{M}^\lambda \otimes 1$, the fact that the filtration is exhaustive is shown in exactly the same way as above (using the acyclicity of the Koszul-like complex).

1: By taking finitely many $gr_V^0 \mathcal{D}_E$ generators of \mathcal{M}^λ and $\mathcal{M}^{\lambda+1}$, we see that U^\bullet are $V^0 \mathcal{D}_{\mathcal{E} \times \mathbf{A}^1}$ -coherent.

2: This is obvious using the relation VI.2.2 above.

In a similar way to the lemma above, we see that $F_{\lambda+b,\ell}^j \subseteq U^0$ and $F_{\lambda+1+b,\ell}^j \subseteq U^1$ for all $b \geq 0, j \geq 0$ and $\ell \leq j$.

3, 4: Note that $\xi F_{\lambda,0}^0 \subseteq E_{\lambda+1,-1}$, which is contained in U^1 by the previous observation. Similarly, $\partial F_{\lambda+1,0} \subseteq E_{\lambda,1}$ which is contained in U^0 by the previous observation.

5: Finally, we need only check $(\partial \xi - \lambda)^a U^\lambda \subseteq U^1$ for some $a \gg 0$. Just as before, $(\partial \xi - \lambda)^a$ multiplies $F_{\lambda,0}^0$ into $\ker(\varphi_{\lambda,0}) = \sum \partial_{z_i}(E_{\lambda+1,0})$. By the above, this is contained in U^1 , as desired.

This completes the proof that this is indeed the V -filtration along ξ .

VI.2.2: Constructing the Isomorphism with $\mathrm{FL}(\mathcal{M})$

In this subsection, we construct the isomorphism $\mathcal{H}^0 \sigma^* \phi_\xi \Gamma_+(p^!(\mathcal{M})[-r]) \cong \mathrm{FL}(\mathcal{M})$, proving Theorem I.7. Recall that $\mathcal{N} := \phi_\xi \Gamma_+(p^!(\mathcal{M})[-r])$ is monodromic along the z 's (by Lemma VI.6), and so we know by Theorem I.2

$$(VI.2.4) \quad \mathcal{H}^0 \sigma^*(\mathcal{N}) = \mathrm{coker}\left(\bigoplus_{i=1}^r \mathcal{N}^1 \xrightarrow{\partial_{z_i}} \mathcal{N}^0\right).$$

As \mathcal{N} is also monodromic along the w 's (again by Lemma VI.6), this property gets inherited by $\mathcal{N}^1, \mathcal{N}^0$ and the maps ∂_{z_i} preserve the monodromic structure. In particular, $\mathcal{H}^0 \sigma^*(\mathcal{N})$ is also monodromic along the w 's, which is exactly what we expect, because $\mathrm{FL}(\mathcal{M})$ is monodromic, too. So we need only identify the individual monodromic pieces in such a way that the w_i and ∂_{w_i} maps are identified between pieces.

Lemma VI.6 tells us that we can decompose Equation VI.2.4 into the following:

$$(VI.2.5) \quad \text{coker} \left[\bigoplus_{\lambda \in [0,1)} \bigoplus_{\ell \in \mathbf{Z}, 1 \leq i \leq r} E_{\lambda+1,\ell}^\lambda / E_{\lambda+1,\ell}^{>\lambda} \xrightarrow{\partial_{z_i}} E_{\lambda,\ell}^\lambda / E_{\lambda,\ell}^{>\lambda} \right],$$

and applying it once more to decompose into w -monodromic pieces, we see that the $r - \chi$ th w -monodromic piece of $\mathcal{H}^0 \sigma^*(\mathcal{N})$ is

$$(VI.2.6) \quad (\mathcal{H}^0 \sigma^*(\mathcal{N}))^{r-\chi} = \text{coker} \left[\bigoplus_{i=1}^r E_{\lambda+1,\chi-\lambda}^\lambda / E_{\lambda+1,\chi-\lambda}^{>\lambda} \xrightarrow{\partial_{z_i}} E_{\lambda,\chi-\lambda}^\lambda / E_{\lambda,\chi-\lambda}^{>\lambda} \right],$$

where we necessarily have $\lambda = \chi - \lfloor \chi \rfloor$, as $\chi - \lambda$ must be an integer.

We have the maps

$$\varphi_{\lambda,\chi-\lambda} : E_{\lambda,\chi-\lambda}^\lambda \rightarrow \mathcal{M}^{\lambda+\chi-\lambda} = \mathcal{M}^\chi = \text{FL}(\mathcal{M})^{r-\chi},$$

and so we need to see that these induce isomorphisms on the cokernels in Equation VI.2.6.

Of course, the image of ∂_{z_i} lies in the kernel, so the only possible issue is the $E^{>\lambda}$ part.

To see that $E_{\lambda,\chi-\lambda}^{>\lambda}$ lies in the kernel, note that by our computation of the V -filtration above, $E_{\lambda,\chi-\lambda}^{>\lambda} = E_{\lambda,\chi-\lambda}^1$. Now, write an arbitrary element $P \in V^0 \mathcal{D}_{\mathcal{E} \times \mathbf{A}^1}$ as

$$P = \sum P_{\beta,\gamma,\rho,\alpha,j,k} \partial_z^\beta \partial_w^\gamma z^\rho w^\alpha (\xi \partial)^j \xi^k,$$

with $P_{\beta,\gamma,\rho,\alpha,j,k} \in \mathcal{D}_X$. Using the way in which the various operators move the eigenspaces, we see that

$$P_{\beta,\gamma,\rho,\alpha,j,k} \cdot E_{\lambda+1,\ell} \subseteq E_{\lambda+1-|\beta|+k+|\rho|,\ell+|\gamma|-|\alpha|-k}.$$

The only way in which $P_{\beta,\gamma,\rho,\alpha,j,k}$ can move $E_{\lambda+1,\ell}$ to $E_{\lambda,\ell'}$ for some $\ell' \in \mathbf{Z}$, then, is for $|\beta| > 0$, i.e., for $P_{\beta,\gamma,\rho,\alpha,j,k}$ to have some ∂_{z_i} . But by our explicit description of V^1 above, it is generated over $V^0 \mathcal{D}_{\mathcal{E} \times \mathbf{A}^1}$ by subspaces of the form $F_{\lambda+1,\ell}^j$ for some $j, \ell \in \mathbf{Z}$. In particular, those elements that land in $E_{\lambda,\chi-\lambda}$ must involve a ∂_{z_i} , and hence lie in the kernel of φ . So

we get well-defined maps as desired.

By Lemma VI.7, we see that, indeed, the induced morphisms $\mathcal{H}^0\sigma^*(\mathcal{N})^{r-\chi} \rightarrow \mathrm{FL}(\mathcal{M})^{r-\chi}$ are injective. We see that they are surjective as follows: recall that, for $\ell \geq 0$, we have seen that $F_{\lambda,\ell}^\ell = \mathcal{M}^{\lambda+\ell}\partial^\ell\delta_g \subseteq V^\lambda$. This will hit all of $\mathcal{M}^{\lambda+\ell}$ under φ . So we have surjectivity, and hence isomorphism, when $\chi - \lambda \geq 0$, i.e., $\chi \geq 0$. We obtain the other isomorphisms using the fact that both $\mathrm{FL}(\mathcal{M})$ and $\mathcal{H}^0\sigma^*(\mathcal{N})$ are w -monodromic, so they must satisfy the acyclicity of Lemma II.9. Thus, we have a morphism between acyclic complexes

$$\begin{array}{ccccccc} \mathcal{H}^0\sigma^*(\mathcal{N}) & \xrightarrow{w} & \mathcal{H}^0\sigma^*(\mathcal{N}) & \xrightarrow{w} & \dots & \xrightarrow{w} & \mathcal{H}^0\sigma^*(\mathcal{N})^{r-\chi} \\ \downarrow \varphi & & \downarrow \varphi & & & & \downarrow \varphi \\ \mathrm{FL}(\mathcal{M}) & \xrightarrow{w} & \mathrm{FL}(\mathcal{M}) & \xrightarrow{w} & \dots & \xrightarrow{w} & \mathrm{FL}(\mathcal{M})^{r-\chi} \end{array}$$

in which, inductively, all but the rightmost map is an isomorphism, so the rightmost map must also be an isomorphism. This proves the claim.

VI.2.3: Hodge and Weight Filtration Computations

Now, to understand the Hodge filtration on $\mathrm{FL}(\mathcal{M})$ given by this isomorphism, we need only track what happens to the Hodge filtration when applying the functors \mathcal{H}^0i^* , ϕ_g and $p^!$.

The statement we are after is the following:

Theorem VI.9. *[CD21, Theorem 1.4] Let (\mathcal{M}, F_\bullet) be a filtered \mathcal{D}_E -module underlying a mixed Hodge module on E . Then the Hodge filtration on $\mathrm{FL}(\mathcal{M})$ satisfies*

$$F_p\mathrm{FL}(\mathcal{M})^{r-\chi} = F_{p-\lceil\chi\rceil}\mathcal{M}^\chi,$$

for all $p \in \mathbf{Z}$ and $\chi \in \mathbf{Q}$.

Proof. As mentioned, we simply need to trace our way through the various functors and

keep track of the Hodge filtration. We have by Equation II.4.3

$$F_j p^!(\mathcal{M})[-r] = (F_{j-r} \mathcal{M})[w]$$

for any $k \in \mathbf{Z}$.

Then, applying the graph embedding along g , we have

$$F_j \Gamma_+ p^!(\mathcal{M})[-r] = \bigoplus_{k \geq 0} (F_{j-k-1} p^!(\mathcal{M})[-r]) \partial^k = \bigoplus_{k \geq 0} (F_{j-k-1-r} \mathcal{M})[w] \partial^k.$$

The vanishing cycles ϕ_u inherits this Hodge filtration, though there is a shift of the Hodge filtration on the $\phi_{g, \neq 1}$ part, by Equations II.4.5 and II.4.4.

Finally, in Theorem I.2, the complex is strict with respect to the Hodge filtration. We note, however, that there is a shift by r , as we are looking at the rightmost cohomology of σ^* . This undoes the shift coming from $p^!$. In summary, we will be interested in the subspace

$$F_p E_{\lambda, \ell}^\lambda = \sum_{|\alpha| \geq -\ell} F_{p-[\lambda]-\ell-|\alpha|} \mathcal{M}^{\lambda+\ell+|\alpha|} w^\alpha \partial^{|\alpha|+\ell} \delta_g,$$

Applying φ to an arbitrary element $\sum m_\alpha w^\alpha \partial^{|\alpha|+\ell} \delta_g$, we get

$$(-1)^\ell \sum_{\alpha} \partial_z^\alpha(m_\alpha) \subseteq F_{p-[\lambda]-\ell} \mathcal{M}^{\lambda+\ell}$$

We get the other containment for $\ell \geq 0$ easily. Again, in this case, $F_{\lambda, \ell}^\ell = \mathcal{M}^{\lambda+\ell} \partial^\ell \delta_g \subseteq V^\lambda$. For $\ell < 0$, we use induction and Filtered surjectivity of the Koszul-like complexes in the ∂_z 's for \mathcal{M} to conclude. \square

Finally, we handle the weight filtration and conclude this chapter.

Given a monodromic module $\mathcal{N} = \bigoplus \mathcal{N}^\lambda$, for any $\lambda \in \mathbf{Q} \cap [0, 1)$, denote

$$\mathcal{N}^{\lambda+\mathbf{Z}} = \bigoplus_{\ell \in \mathbf{Z}} \mathcal{N}^{\lambda+\ell}.$$

The goal is the following:

Theorem VI.10. *[CD21, Theorem 1.4] Let (\mathcal{M}, W) be a monodromic \mathcal{D} -module with weight filtration $W_{\bullet}\mathcal{M}$ underlying a mixed Hodge module on E . Then the weight filtration on $\mathrm{FL}(\mathcal{M})$ satisfies*

$$W_k \mathrm{FL}(\mathcal{M})^{\lambda+\mathbf{Z}} = \mathrm{FL}(W_{k+r+\lceil\lambda\rceil}\mathcal{M})^{\lambda+\mathbf{Z}}.$$

We first prove an easy containment: for $\lambda \in [0, 1)$, we have

$$(VI.2.7) \quad W_k \mathrm{FL}(\mathcal{M})^{\lambda+\mathbf{Z}} \subseteq \mathrm{FL}(W_{k+r+\lceil\lambda\rceil}\mathcal{M})^{\lambda+\mathbf{Z}}.$$

Again, we want to trace through the various functors and keep track of the weight filtration. By Equation II.4.3, we have

$$W_k p^!(\mathcal{M})[-r] = p^*(W_{k+r}\mathcal{M}).$$

For ϕ_g , we must use the relative monodromy filtration. Set

$$M_{\bullet}\phi_g p^!(\mathcal{M})[-r] = \bigoplus_{\lambda \in [0, 1)} \phi_{\xi, \lambda} \Gamma_+ p^!(W_{\bullet+r+\lceil\lambda\rceil}\mathcal{M})[-r].$$

Let $W_{1, \bullet}\phi_g p^!(\mathcal{M})[-r]$ be the relative monodromy filtration for M_{\bullet} and the nilpotent operator $N_1 = \xi\partial - \lambda + 1$, which computes the weight filtration on the mixed Hodge module $\phi_{\xi, \lambda} \Gamma_+ p^!(\mathcal{M})[-r]$.

Consider the 0th z -monodromic piece \mathcal{N}^0 of $\phi_{\xi, \lambda} \Gamma_+ p^!(W_{\bullet+r+\lceil\lambda\rceil}\mathcal{M})[-r]$. On here, $N_2 := \theta_z + r$ acts nilpotently. Moreover, it preserves the filtration $W_{1, \bullet}$, and we let $W_{2, \bullet}$ to be the relative monodromy filtration for $W_{1, \bullet}$ and N_2 .

Note that by definition of the \mathcal{D} -module action on $\Gamma_+ p^!(\mathcal{M})[-r]$, the two operators $\xi\partial - \lambda + 1$ and $\theta_z + r$ are related by a third operator $\tilde{\mathcal{N}}$, which acts on an element $m[w]\partial^k \delta_g \in \mathcal{M}^{\chi}[w]\partial^k \delta_g$ by $(\theta_z - \chi + r)$.

By Theorem I.9, we know that (the operator induced by) $\tilde{\mathcal{N}}$ maps $W_{1,\bullet}$ into $W_{\bullet,-2}^1$, because it does so for M_\bullet . Then, because it does this for $W_{1,\bullet}$, by the same reasoning it does so for $W_{2,\bullet}$.

But on $gr_k^{W_1}\mathcal{N}^0$, we know N_1 is identically 0, and so $\tilde{N} = N_2$. Hence, $W_{2,\bullet}$ is also the relative monodromy filtration for \tilde{N} and $W_{1,\bullet}$. But we have argued that \tilde{N} decreases $W_{1,\bullet}$ by 2, so actually $W_{1,\bullet}$ is its own relative monodromy filtration, i.e., $W_{2,\bullet} = W_{1,\bullet}$.

Finally, we have the quotient map $\mathcal{N}^0 \rightarrow \mathcal{H}^0\sigma^*(\phi_g p^!(\mathcal{M})[-r])$, and by Theorem I.2 (and the E_2 -degeneration mentioned in Corollary IV.11), the weight filtration on the target is induced by the filtration $W_{2,\bullet}$ on the domain. But we have argued that $W_{2,\bullet} = W_{1,\bullet}$. Finally, by functoriality of relative monodromy filtrations, this must be contained in the relative monodromy of the target with respect to the induced operator by N_1 and the filtration M_\bullet . But under φ this precisely maps to the relative monodromy filtration, so again by Theorem I.9, we conclude the desired containment.

To handle the opposite containment, we will make use of the inverse Fourier transform. Note that applying the Fourier-transform twice makes z_i act by $-z_i$ and ∂_{z_i} act by $-\partial_{z_i}$. This can be remedied using the involution $a : E \rightarrow E$ defined by $z_i \mapsto -z_i$. Hence, for underlying \mathcal{D} -modules, we have

$$\psi : a^*\mathrm{FL}_{E^\vee}\mathrm{FL}_E\mathcal{M} \cong \mathcal{M}.$$

By [KS90, Thm. 3.7.12(i)] and [Bry86, Prop. 6.13], we know that this isomorphism preserves the underlying \mathbf{Q} -structure. Hence, if we can define a Hodge and weight filtration for $\mathrm{FL}_{E^\vee}(-)$ which makes the above morphism bifiltered, it will automatically be bistrict, hence an isomorphism of mixed Hodge modules. As it is clear that whatever structure we define will act as the inverse of FL_E , we will call it the *inverse Fourier transform*, and denote it $\overline{\mathrm{FL}}_{E^\vee}$.

It turns out that the definition is very close to that of FL , except we need to Tate twist

on the individual components:

Definition VI.11. For $E = X \times \mathbf{A}^r$, using the notation above, we define the *inverse Fourier transform* $\overline{\text{FL}}_E : \text{MHM}(E) \rightarrow \text{MHM}(E^\vee)$ by the formula:

$$\overline{\text{FL}}_E(M) = \bigoplus_{\lambda \in [0,1)} \mathcal{H}^0 \sigma^* \phi_{g,\lambda} p^!(M)[-r](-[\lambda] - r).$$

Proposition VI.12. *The map $\psi : a^* \overline{\text{FL}}_{E^\vee} \text{FL}_E \mathcal{M} \rightarrow \mathcal{M}$ is an isomorphism of mixed Hodge modules.*

Proof. As the map ψ is an isomorphism of \mathcal{D} -modules, we really just need to compute the weight and Hodge filtrations of the domain and show that they are contained in the corresponding Hodge and weight filtrations for \mathcal{M} . We check the claim for monodromic pieces, so fix $\chi \in \mathbf{Q}$ with $\chi - [\chi] = \lambda \in [0, 1)$. We will check the claims for $\lambda \in (0, 1)$, the claim for $\lambda = 0$ is similar and left to the reader.

Note that $\overline{\text{FL}}$ is simply a Tate twist of the usual FL by $(-1 - r)$ in this case, so we can use Theorem I.8 and the containment in Equation VI.2.7, keeping track of the Tate twists.

Well, a^* being the pullback along an isomorphism does not affect the Hodge or weight filtrations. So we can ignore it. For the Hodge filtration, we have

$$\begin{aligned} F_p \overline{\text{FL}}(\text{FL}(\mathcal{M}))^\chi &= F_p \text{FL}_{E^\vee} \text{FL}_E(\mathcal{M})^\chi(-1 - r) \\ &= F_{p+r+1} \text{FL}_{E^\vee} \text{FL}_E(\mathcal{M})^\chi = F_{p+r+1-[\chi]} \text{FL}_E(\mathcal{M})^{r-\chi} = F_{p+r+1-[\chi]-[r-\chi]} \mathcal{M}^\chi, \end{aligned}$$

and $p + r + 1 - [\chi] - [r - \chi] = p + 1 - [\chi] - [-\chi] = p$.

Finally, for the weight filtration, we have

$$\begin{aligned} W_k \overline{\text{FL}}(\text{FL}(\mathcal{M}))^\chi &= W_k \text{FL}_{E^\vee} \text{FL}_E(\mathcal{M})^\chi(-1 - r) \\ &= W_{k-2-2r} \text{FL}_{E^\vee} \text{FL}_E(\mathcal{M})^\chi \subseteq \text{FL}_{E^\vee}(W_{k-1-r} \text{FL}(\mathcal{M})) \subseteq \text{FL}_{E^\vee} \text{FL}_E(W_k \mathcal{M}) = W_k \mathcal{M}, \end{aligned}$$

proving the claim. □

As this is a W -filtered isomorphism, we get the other containment in Equation [VI.2.7](#), so this completes the proof of Theorem [I.10](#).

BIBLIOGRAPHY

- [BBDG82] Alexander Beilinson, Joseph Bernstein, Pierre Deligne, and Ofer Gabber. Faisceaux pervers. *Astérisque*, 100:1–171, 1982.
- [Ber72] I. N. Bernstein. Analytic continuation of generalized functions with respect to a parameter. *Funkcional. Anal. i Priložen*, 6(4):26–40, 1972.
- [BK81] Jean-Luc Brylinski and Masaki Kashiwara. Kazhdan-Lusztig conjecture and holonomic systems. *Invent. Math.*, 64:387–410, 1981.
- [BL10] Christine Berkesch and Anton Leykin. Algorithms for Bernstein-Sato polynomials and multiplier ideals. pages 99–106. ISSAC, ACM, New York, 2010.
- [Bla22] Guillem Blanco. An algorithm for Hodge ideals. *Math. Comp.*, 91(338):2955–2967, 2022.
- [BMS06] Nero Budur, Mircea Mustața, and Morihiko Saito. Bernstein-Sato polynomials of arbitrary varieties. *Compos. Math.*, 142(3):779–797, 2006.
- [Bry86] Jean-Luc Brylinski. Transformations canoniques, dualité projective, théorie de Lefschetz, transformations de Fourier et sommes trigonométriques. *Astérisque*, 140(141):3–134, 1986.
- [BS05] Nero Budur and Morihiko Saito. Multiplier ideals, V-filtration, and spectrum. *J. Algebraic Geom.*, 14(2):269–282, 2005.
- [CD21] Qianyu Chen and Bradley Dirks. On V-filtration, Hodge filtration and Fourier transform. *arxiv*, 2111.04622, 2021.
- [CDM22] Qianyu Chen, Bradley Dirks, and Mircea Mustața. The minimal exponent and k -rationality for locally complete intersections. *arXiv math.AG*, 2212.01898, 2022.
- [CDMO22] Qianyu Chen, Bradley Dirks, Mircea Mustața, and Sebastián Olano. V-filtrations and minimal exponents for locally complete intersection singularities. *arXiv math.AG*, 2208.03277, 2022.
- [CKS86] Eduardo Cattani, Aroldo Kaplan, and Wilfried Schmid. Degeneration of Hodge structures. *Ann. of Math.*, 123(3):457–535, 1986.

- [CKS87] Eduardo Cattani, Aroldo Kaplan, and Wilfried Schmid. L^2 and intersection cohomologies for a polarizable variation of Hodge structure. *Invent. Math.*, 87(2):217–252, 1987.
- [dB81] Philippe du Bois. Complexe de de Rham filtré d’une variété singulière. *Bull. Soc. Math. France*, 109:41–81, 1981.
- [Del68] Pierre Deligne. Théorème de Lefschetz et critères de dégénérescence de suites spectrales. *Publications Mathématiques de l’IHÉS*, 35:5–57, 1968.
- [Del71] Pierre Deligne. Théorie de Hodge: II. *Inst. Hautes Études Sci. Publ. Math.*, (40):5–57, 1971.
- [Del73] Pierre Deligne. *Le formalisme des cycles évanescents*, volume 340 of *Groupes de Monodromie en Géométrie Algébrique*, pages 82–115 and 116–164. Springer, Berlin, 1973.
- [Del74] Pierre Deligne. Théorie de Hodge: III. *Publications Mathématiques de l’IHÉS*, 44:5–77, 1974.
- [DM22a] Bradley Dirks and Mircea Mustață. Minimal exponents of hyperplane sections: a conjecture of Teissier. *J. Eur. Math. Soc.*, To appear, 2022.
- [DM22b] Bradley Dirks and Mircea Mustață. Upper bounds for roots of B-functions, following Kashiwara and Lichtin. *Publ. Res. Inst. Math. Sci.*, 58(4):693–712, 2022.
- [Dut20] Yajnaseni Dutta. Vanishing for Hodge ideals on toric varieties. *Math. Nachr.*, 293(1), 2020.
- [ELSV04] Lawrence Ein, Robert Lazarsfeld, Karen Smith, and Dror Varolin. Jumping coefficients of multiplier ideals. *Duke Math. J.*, 123(3):469–503, 2004.
- [FL22a] Robert Friedman and Radu Laza. Deformations of singular Fano and Calabi-Yau varieties. *arXiv math.AG*, 2203.04823, 2022.
- [FL22b] Robert Friedman and Radu Laza. The higher Du Bois and higher rational properties for isolated singularities. *arXiv math.AG*, 2207.07566, 2022.
- [FL22c] Robert Friedman and Radu Laza. Higher Du Bois and higher rational singularities. *arXiv math.AG*, 2205.04729, 2022.
- [Gal21] Martin Gallauer. An introduction to six-functor formalisms. *arXiv preprint: 2112.10456*, 2021.
- [Gin86] Victor Ginzburg. Characteristic varieties and vanishing cycles. *Invent. Math.*, 84(2):327–402, 1986.
- [GM83] Mark Goresky and Robert MacPherson. Intersection homology ii. *Invent. Math.*, 72(1):77–129, 1983.

- [Gri68a] Philip Griffiths. Periods of integrals on algebraic manifolds I (Construction and Properties of the Modular Varieties). *American Journal of Mathematics*, 90(2):568–626, 1968.
- [Gri68b] Philip Griffiths. Periods of integrals on algebraic manifolds II (Local Study of the Period Mapping). *American Journal of Mathematics*, 90(3):808–865, 1968.
- [Gri70] Philip Griffiths. Periods of integrals on algebraic manifolds III (Some global differential-geometric properties of the period mapping). *Publications Mathématiques de l’IHÉS*, 38:125–180, 1970.
- [Har67] Robin Hartshorne. *Local cohomology*, volume 41. Springer, New York, Heidelberg, Berlin, 1967.
- [Har13] Robin Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics. Springer New York, 2013.
- [HTT08] Riyoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki. *D-modules, perverse sheaves, and representation theory*. Birkhäuser Boston, 2008.
- [JKSY22] Seung-Jo Jung, In-Kyun Kim, Morihiko Saito, and Youngho Yoon. Higher Du Bois singularities of hypersurfaces. *Proc. Lond. Math. Soc.*, 125(3):543–567, 2022.
- [Kas75] Masaki Kashiwara. On the maximally overdetermined system of linear differential equations I. *Publ. Res. Inst. Math. Sci.*, 10:563–579, 1974–1975.
- [Kas77] Masaki Kashiwara. B-functions and holonomic systems. *Inv. Math.*, 38:33–53, 1976/1977.
- [Kas83] Masaki Kashiwara. *Vanishing cycle sheaves and holonomic systems of differential equations*, volume 1016, pages 134–142. Springer, Berlin, (Tokyo/Kyoto, 1983.
- [Kas84] Masaki Kashiwara. The Riemann-Hilbert problem for holonomic systems. *Publ. Res. Inst. Math. Sci.*, 20(2):319–365, 1984.
- [Kol97] János Kollár. Singularities of Pairs. In *Algebraic Geometry — Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math*, pages 221–287, Providence, RI, 1997. Amer. Math. Soc.
- [KS90] Masaki Kashiwara and Pierre Schapira. *Sheaves on Manifolds*, volume 292 of *Grundlehren der mathematischen Wissenschaften*. Springer Berlin, Heidelberg, 1 edition, 1990.
- [KS11] Sándor Kovács and Karl Schwede. Hodge theory meets the minimal model program: a survey of log canonical and du bois singularities. In *Topology of stratified spaces.*, volume 58, pages 51–94. Cambridge University Press, Cambridge, 2011.

- [KS21] Stefan Kebekus and Christian Schnell. Extending holomorphic forms from the regular locus of a complex space to a resolution of singularities. *J. Amer. Math. Soc.*, 34:315–368, 2021.
- [Laz04] Robert Lazarsfeld. *Positivity in Algebraic Geometry II*, volume 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, Berlin, 2004.
- [Lic89] Benjamin Lichtin. Poles of $|f(z, w)|^2$ s and roots of the b-function. *Ark. Mat.*, 27(2):283–304, 1989.
- [Mal83] Bernard Malgrange. Polynômes de Bernstein-Sato et cohomologie évanescence. *Analysis and Topology on Singular Spaces*, pages 243–267, 1983.
- [Meb89] Zoghman Mebkhout. Le théorème de comparaison entre cohomologies de de Rham d’une variété algébrique complexe et le théorème d’existence de Riemann. *Inst. Hautes Études Sci. Publ. Math.*, 69:47–89, 1989.
- [MM04] Phillipe Maisonobe and Zoghman Mebkhout. Le théorème de comparaison pour les cycles évanescents. *Eléments de la théorie des systèmes différentiels géométriques*, pages 311–389, 2004.
- [MOPW21] Mircea Mustața, Sebastián Olano, Mihnea Popa, and Jakub Witaszek. The Du Bois complex of a hypersurface and the minimal exponent. *to appear in Duke Math. J.*, 2021.
- [MP18] Mircea Mustața and Mihnea Popa. Restriction, subadditivity, and semicontinuity theorems for Hodge ideals. *Int. Math. Res. Not. IMRN*, (11):3587–3605, 2018.
- [MP19] Mircea Mustața and Mihnea Popa. Hodge ideals. *Mem. Amer. Math. Soc.*, 262(1268):80 pp., 2019.
- [MP20] Mircea Mustața and Mihnea Popa. Hodge ideals for \mathbb{Q} -divisors, V -filtration, and minimal exponent. *Forum Math. Sigma*, 8(e19), 2020.
- [MP22a] Mircea Mustața and Mihnea Popa. Hodge filtration on local cohomology, Du Bois complex, and local cohomological dimension. *Forum Math. Pi*, 10(e22):58 pp., 2022.
- [MP22b] Mircea Mustața and Mihnea Popa. On k -rational and k -Du Bois local complete intersections. *arXiv math.AG*, 2207.08743, 2022.
- [Mus22] Mircea Mustața. Bernstein-Sato polynomials for general ideals vs. principal ideals. *Proc. Amer. Math. Soc. (to appear)*, 2022.
- [Ola22a] Sebastián Olano. Weighted Hodge ideals of reduced divisors. *arXiv math.AG*, 2208.03271, 2022.
- [Ola22b] Sebastián Olano. Weighted multiplier ideals of reduced divisors. *Math. Ann.*, 384(3–4):1091–1126, 2022.

- [PS08] Chris A. M. Peters and Joseph H. M. Steenbrink. *Mixed Hodge Structures*. H.M. Steenbrink Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. Springer Berlin, Heidelberg, 1 edition, 2008.
- [Rai21] Michael Perlman Claudiu Raicu. Hodge ideals for the determinant hypersurface. *Selecta Math.*, 27(1):22, 2021.
- [RW22] Thomas Reichelt and Uli Walther. Weight filtrations on GKZ-systems. *Amer. J. Math.*, 144(5):1437–1484, 2022.
- [Sab87a] Claude Sabbah. \mathcal{D} -modules et cycles évanescents (d’après Kashiwara et Malgrange). *Géométrie algébrique et applications III (La Rabida, 1984)*, 24:53–98, 1987.
- [Sab87b] Claude Sabbah. Proximité évanescents i: La structure polaire d’un \mathcal{D} -module. *Compos. Math.*, 62:283–288, 1987.
- [Sab90] Claude Sabbah. Modules d’Alexander et \mathcal{D} -modules. *Duke Mathematical Journal*, 60(3):729–814, 1990.
- [Sai88] Morihiko Saito. Modules de Hodge Polariables. *Publ. Res. Inst. Math. Sci.*, 24(6):849–995, 1988.
- [Sai89] Morihiko Saito. Duality for Vanishing Cycle Functors. *Publ. Res. Inst. Math. Sci.*, 25:889–921, 1989.
- [Sai90] Morihiko Saito. Mixed Hodge Modules. *Publ. Res. Inst. Math. Sci.*, 26(2):221–333, 1990.
- [Sai93] Morihiko Saito. On b-function, spectrum and rational singularity. *Math. Ann.*, 295(1):54–73, 1993.
- [Sai94] Morihiko Saito. On microlocal b-function. *Bull. Soc. Math. France*, 122(2):163–184, 1994.
- [Sai00] Morihiko Saito. Mixed Hodge complexes on algebraic varieties. *Math. Ann.*, 316:283–331, 2000.
- [Sai09] Morihiko Saito. On the Hodge filtration of Hodge modules. *Mosc. Math. J.*, 9(1):161–191, 2009.
- [Sai16] Morihiko Saito. Hodge ideals and microlocal V-filtration. *arxiv*, 1612.08667, 2016.
- [Sai22a] Takahiro Saito. A description of monodromic mixed Hodge modules. *to appear in Journal für die reine und angewandte Mathematik*, 2022.
- [Sai22b] Takahiro Saito. The Hodge filtrations of monodromic mixed Hodge modules and the irregular Hodge filtrations. *arxiv: 2204.13381*, 2022.

- [Sch73] Wilfried Schmid. Variation of Hodge structure: the singularities of the period mapping. *Invent. Math.*, 22:211–319, 1973.
- [Sch01] Christine Schwarz. Relative monodromy weight filtrations. *Math. Z.*, 236(1):11–21, 2001.
- [Sch07] Karl Schwede. A simple characterization of du Bois singularities. *Compos. Math.*, 143(4):813–828, 2007.
- [Sch14] Christian Schnell. *An overview of Morihiko Saito’s theory of mixed Hodge modules*, pages 27–80. Number 1405.3096. Int. Press, Somerville, MA, 2014.
- [Ser55] Jean-Pierre Serre. Faisceaux algébriques cohérents. *Ann. of Math.*, 61(2):197–278, 1955.
- [Ser56] Jean-Pierre Serre. Géométrie algébrique et géométrie analytique. *Annales de l’Institut Fourier*, 6:1–42, 1956.
- [SS] Claude Sabbah and Christian Schnell. Mixed Hodge Modules Project.
- [Ste77] J. H. M. Steenbrink. *Mixed Hodge structure on the vanishing cohomology*, volume 130. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.
- [Ste83] Joseph H. M. Steenbrink. Mixed Hodge structures associated with isolated singularities. In *Proc. Sympos. Pure Math.*, volume 40 of *Singularities, Part 2*, pages 513–536, Arcata, Calif., 1983. Amer. Math. Soc., Providence, RI.
- [SZ85] Joseph Steenbrink and Steven Zucker. Variation of mixed Hodge structure. I. *Invent. Math.*, 80(3):489–542, 1985.
- [Var82] A. N. Varchenko. The complex singularity index does not change along the stratum $\mu = \text{const}$. *Funktsional. Anal. i Prilozhen.*, 16(1):1–12, 96, 1982.
- [Ver83] Jean-Louis Verdier. Spécialisation de faisceaux et monodromie modérée. *Analysis and Topology on Singular Spaces, II, III (Luminy, 1981)*, pages 332–364, 1983.
- [Voi02] Claire Voisin. *Hodge Theory and Complex Algebraic Geometry I*, volume 1 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2002.
- [Voi03] Claire Voisin. *Hodge Theory and Complex Algebraic Geometry II*, volume 2 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2003.
- [Wu21] Lei Wu. Characteristic cycles associated to holonomic D-modules. *Mathematische Zeitschrift*, 301(2), 2021.
- [Zha21] Mingyi Zhang. Hodge filtration and Hodge ideals for \mathbb{Q} -divisors with weighted homogeneous isolated singularities. *Asian J. Math.*, 25(5):641–664, 2021.

[Zuc79] Steven Zucker. Hodge theory with degenerating coefficients. L_2 cohomology in the Poincaré metric. *Ann. of Math.*, 109(3):415–476, 1979.