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#### **MATHEMATICAL** FINANCE

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# **Equilibria of time-inconsistent stopping for one-dimensional diffusion processes**

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#### **Abstract**

We consider three equilibrium concepts proposed in the literature for time-inconsistent stopping problems, including mild equilibria (introduced in Huang and Nguyen-Huu (2018)), weak equilibria (introduced in Christensen and Lindensjö (2018)), and strong equilibria (introduced in Bayraktar et al. (2021)). The discount function is assumed to be log subadditive and the underlying process is one-dimensional diffusion. We first provide necessary and sufficient conditions for the characterization of weak equilibria. The smooth-fit condition is obtained as a by-product. Next, based on the characterization of weak equilibria, we show that an optimal mild equilibrium is also weak. Then we provide conditions under which a weak equilibrium is strong. We further show that an optimal mild equilibrium is also strong under a certain condition. Finally, we provide several examples including one showing a weak equilibrium may not be strong, and another one showing a strong equilibrium may not be optimal mild.

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## **1 INTRODUCTION**

On a filtered probability space,  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t>0}, \mathbb{P})$  consider the optimal stopping problem,

$$
\sup_{\tau \in \mathcal{T}} \mathbb{E}[\delta(\tau) f(X_{\tau})],\tag{1}
$$

where  $\delta(\cdot)$  is a discount function,  $X = (X_t)_t$  is a time-homogeneous one-dimensional strong Markov process, and  $f(\cdot)$  is a payoff function. It is well known that when  $\delta(\cdot)$  is not exponential, the problem can be time-inconsistent in the sense that an optimal stopping rule obtained today may no longer be optimal from a future's perspective.

( $\Omega$ ,  $F_i(F_i)_{z\geq0}$ ,  $P$ ) consider the optimal stopping problem,<br>sup  $E[\delta(\tau)f(X_{\tau})]$ ,<br>ion,  $X = (X_i)_i$  is a time-homogeneous one-dimensionayoff function. It is well known that when  $\delta(\cdot)$  is not expositent in the sense th  $\tau \in \mathcal{T}$ <br>  $= C$  action the probability of the probability of  $\mathcal{L}$ <br>  $\mathcal{L}$  and  $\mathcal{$  $\tau \in \mathcal{I}$  = (in t i and the p application of the p application of the p application of the space of the space of  $(201 \text{ at } 100) \geq \rho_S$  and  $\eta$  a  $E[\delta(\tau)f(X_{\tau})],$ <br>
(1)<br>  $E[\delta(\tau)f(X_{\tau})],$ <br>
(1), is a time-homogeneous one-dimensional strong<br>
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strespect to the init  $\delta(t)$  is a discount function,  $X = (X_t)$ , is a time-homogeneous one-dimensional strong<br>(i) is a discount function, It is well known that when  $\delta(t)$  is not exponential,<br>then can be time-inconsistent in the searc that an *f*( $\chi$ ) is a payoff function. It is well known that when  $\delta(\cdot)$  is not exponential,<br>ima-lineonistication. It is well known that when  $\delta(\cdot)$  is not exponential,<br>imal from a future's perspective.<br>thit hits ime-inconsi One way to deal with this time-inconsistency is to consider the precommitted strategy, that is, to derive a policy that is optimal with respect to the initial preference and stick to it over the whole planning horizon even if the preference changes later; see, for example, Agram and Djehiche [\(2021\)](#page-39-0); Miller [\(2017\)](#page-40-0). Another approach to address the time-inconsistency is to look for a subgame perfect Nash equilibrium; given the future selves follow the equilibrium strategy, the current self has no incentive to deviate from it. For equilibrium strategies, we refer to the works (Björk et al., [2021;](#page-39-0) Ekeland & Lazrak, 2010; Ekeland & Pirvu, [2008;](#page-40-0) He & Jiang, [2021;](#page-40-0) Ekeland & Lazrak, [2006;](#page-39-0) Hernández & Possamaï, [2020;](#page-40-0) Hamaguchi, [2021;](#page-40-0) Huang & Zhou, [2021;](#page-40-0) Wang & Yong, [2021;](#page-40-0) Wei et al., [2017\)](#page-40-0) among others for time-inconsistent control, and Christensen and Lindensjö [\(2020a,](#page-39-0) [2020b\)](#page-39-0); Ebert and Strack [\(2018\)](#page-39-0); He and Zhou [\(2022\)](#page-40-0); Huang and Nguyen-Huu [\(2018\)](#page-40-0); Liang and Yuan [\(2021\)](#page-40-0); Tan et al. (2021); Bodnariu et al. [\(2022\)](#page-39-0); Ebert et al. [\(2020\)](#page-39-0), and the references therein for time-inconsistent stopping.

How to properly define the notion of an equilibrium is quite subtle in continuous time. There are mainly two streams of research for equilibrium strategies of time-inconsistent stopping problems in continuous time. In the first stream of research, the following notion of equilibrium is considered.

**Definition 1.1.** A closed set  $S \subset \mathbb{X}$  is said to be a mild equilibrium, if

$$
\int f(x) \le J(x, S), \quad \forall x \notin S,\tag{2}
$$

$$
(f(x) \ge J(x, S), \quad \forall x \in S,
$$
\n<sup>(3)</sup>

where

This kind of equilibrium is first proposed and studied in stopping problems in the context of nonexponential discounting in Huang and Nguyen-Huu (2018). It is called *mild equilibrium* in Bayraktar et al. [\(2021\)](#page-39-0) to distinguish from other equilibrium concepts. Mild equilibria are further considered in Huang et al. [\(2020\)](#page-40-0) and Huang & Yu (2021) where the time inconsistency is caused by probability distortion and model uncertainty, respectively.

S ⊂ X is said to be a mild equilibrium, if<br>  $\int f(x) \leq J(x, S), \quad \forall x \notin S,$ <br>  $\int f(x) \geq J(x, S), \quad \forall x \in S,$ <br>  $\exists$ <br>  $\exists$  with  $\rho_S := \inf\{t > 0 : X_t \in S\}$  and<br>
s first proposed and studied in stopping<br>
in Huang and Nguyen-Huu (2018). It is<br>  $f(x) \leq J(x, S), \forall x \notin S,$  (2)<br>  $f(x) \geq J(x, S), \forall x \in S,$  (3)<br>
(3)<br>
(3)<br>
(4)<br>
th  $\rho_S := \inf\{t > 0 : X_t \in S\}$  and  $\mathbb{E}^x[\cdot] = \mathbb{E}[\cdot|X_0 = x].$  (4)<br>
proposed and studied in stopping problems in the context of<br>
ang and Nguyen-Huu (2018).  $f(x) \ge J(x, S), \forall x \in S,$  (3)<br>
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th  $\rho_S := \inf\{t > 0 : X_t \in S\}$  and  $\mathbb{E}^x[\cdot] = \mathbb{E}[\cdot|X_0 = x]$ . (4)<br>
proposed and studied in stopping problems in the context of<br>
ang and Nguyen-Huu (2018). It is called *mild equilibrium*  $J(x, S) := \mathbb{E}^x[\delta(\rho_S) f(X_{\rho_S})]$  with  $\rho_S := \inf\{t > 0 : X_t \in S\}$  and  $\mathbb{E}^x[\cdot] = \mathbb{E}[\cdot|X_0 = x]$ . (4)<br>This kind of equilibrium is first proposed and studied in stopping problems in the context of<br>nexponential discounting in Note that  $f(x)$  is the value for immediate stopping, and  $J(x, S) = \mathbb{E}^{x}[\delta(\rho_{S})f(X_{\rho_{S}})]$  is the value  $f(x)$  is the value for immediate stopping, and  $J(x, S) = \mathbb{E}^x[\delta(\rho_S)f(X_{\rho_S})]$  is the value mg as  $\rho_S$  is the first time to enter *S* after time 0. As a result, the economic meaning libria appears to be clear: in Equation for continuing as  $\rho_s$  is the first time to enter S after time 0. As a result, the economic meaning  $\rho_S$  is the first time to enter *S* after time 0. As a result, the economic meaning<br>appears to be clear: in Equation (2) when  $x \notin S$ , it is better to continue and get<br>than to stop and get the value *f*. In other words, t of mild equilibria appears to be clear: in Equation (2) when  $x \notin S$ , it is better to continue and get  $x \notin S$ , it is better to continue and get<br>words, there is no incentive to deviate<br>to also apply to the other case  $x \in S$ the value J rather than to stop and get the value  $f$ . In other words, there is no incentive to deviate *J* rather than to stop and get the value  $f$ . In other words, there is no incentive to deviate<br>action of "continuing." The same reasoning seems to also apply to the other case  $x \in S$ <br>discussed in the same reasoning seems from the action of "continuing." The same reasoning seems to also apply to the other case  $x \in S$ 

<span id="page-2-0"></span>in Equation (3), that is, no incentive for changing the action from "stopping" to "continuing." However, this is not really captured in Equation (3) after a second thought: In the one-dimensional diffusion (and continuous-time Markov chain) setting, under some very nonrestrictive condition, we have  $\rho_s = 0$  a.s., and thus Equation (3) holds trivially.<sup>1</sup> That is, there is no actual deviation from stopping to continuing captured in Equation (3).

Because of this issue, mild equilibria are indeed too "mild": the whole state space is always a mild equilibrium; in most of the examples provided in Huang and Nguyen-Huu [\(2018\)](#page-40-0); Huang and Zhou [\(2020\)](#page-40-0); Huang & Yu [\(2021\)](#page-40-0), there is a continuum of mild equilibria. As there are often too many mild equilibria in various models, it is natural to consider the problem of equilibrium selection.

**Definition 1.2.** A mild equilibrium S is said to be optimal, if for any other mild equilibrium  $R$ ,

$$
\mathbb{E}^x[\delta(\rho_S)f(X_{\rho_S})] \ge \mathbb{E}^x[\delta(\rho_R)f(X_{\rho_R})], \quad \forall x \in \mathbb{X}.
$$

 $\rho_S = 0$  a.s., and thus Equation (3) holds trivially.<sup>1</sup> That is, there is no actual deviation<br>
pipla to continuing captured in Equation (3).<br>
Even by this suse, mild equilibrial are indeed too "mild": the whole state spa S is said to be optimal, if for any other mild equilibrium R,<br>  $\epsilon_{\beta}$ )]  $\geq \mathbb{E}^x[\delta(\rho_R)f(X_{\rho_R})]$ ,  $\forall x \in \mathbb{X}$ .<br>
equilibrium is defined in the sense of pointwise dominance<br>
existence of optimal equilibria is first es  $E^x[\delta(\rho_S)f(X_{\rho_S})] \geq E^x[\delta(\rho_R)f(X_{\rho_R})], \quad \forall x \in \mathbb{X}.$ <br>tiy of a mild equilibrium is defined in the sense condition. The existence of optimal equilibria is fire<br>ter time models. The existence result is furthonal case in Huang Note that the optimality of a mild equilibrium is defined in the sense of pointwise dominance, which is a very strong condition. The existence of optimal equilibria is first established in Huang and Zhou [\(2019\)](#page-40-0) in discrete time models. The existence result is further extended to diffusion models for one-dimensional case in Huang and Zhou [\(2020\)](#page-40-0) and multidimensional case in Huang and Wang [\(2021\)](#page-40-0). In particular, for the one-dimensional diffusion case, Huang and Zhou [\(2020\)](#page-40-0) shows that under some general assumptions an optimal mild equilibrium exists and is given by the intersection of all mild equilibria (also see Lemma [4.1](#page-18-0) below). Huang and Zhou [\(2020\)](#page-40-0) also provide an example indicating that, in general, there may exist multiple optimal mild equilibria.

In the second stream of the research for equilibrium strategies for time-inconsistent stopping in continuous time, the following notion of equilibrium is introduced.

**Definition 1.3.** A closed set  $S \subset \mathbb{X}$  is said to be a weak equilibrium, if

$$
\begin{cases}\nf(x) \leq J(x, S), & \forall x \notin S, \\
f(x) - \mathbb{E}^x[\delta(\rho_S^{\varepsilon})f(X_{\rho_S^{\varepsilon}})]\n\end{cases} \tag{5}
$$

$$
\left\{\liminf_{\varepsilon \searrow 0} \frac{f(x) - \mathbb{E}\left[\mathcal{O}(\rho_S^c) f(x_{\rho_S^c})\right]}{\varepsilon} \ge 0, \quad \forall x \in S,\right\}
$$
\n
$$
(6)
$$

where

$$
\rho_S^{\varepsilon} := \inf \{ t \ge \varepsilon : X_t \in S \}.
$$

S ⊂ X is said to be a weak equilibrium, if<br>  $\frac{1}{\pm}$  is said to be a weak equilibrium, if<br>  $\frac{f(x) - \mathbb{E}^x[\delta(\rho_S^{\varepsilon})f(X_{\rho_S^{\varepsilon}})]}{\varepsilon} \geq 0$ ,  $\forall x \in S$ <br>
on  $\frac{f(x) - \mathbb{E}^x[\delta(\rho_S^{\varepsilon})f(X_{\rho_S^{\varepsilon}})]}{\varepsilon} \geq 0$ ,  $\forall x \in$  $f(x) \leq J(x, S),$ <br>  $\forall x \notin S,$  (5)<br>  $\liminf_{\varepsilon \searrow 0} \frac{f(x) - \mathbb{E}^x[\delta(\rho_S^{\varepsilon})f(X_{\rho_S^{\varepsilon}})]}{\varepsilon} \geq 0, \quad \forall x \in S,$  (6)<br>  $\rho_S^{\varepsilon} := \inf\{t \geq \varepsilon : X_t \in S\}.$ <br>
concept for time inconsistent stopping is proposed in Christensen and<br>
tr  $\varepsilon \searrow 0$ <br>concernet some only infinite incomplementary to  $\exists x \in S^\circ$ , if it is also infinite incomplementary in the set of  $x \in S^\circ$ , if it is also interest on  $\exists x \in S^\circ$ .  $\frac{1}{\pi}$  once the property millible  $\frac{1}{\pi}$  and  $\frac$  $f(x) = \frac{\varepsilon}{\varepsilon} \log(\rho_S)$ <br>  $\frac{\varepsilon}{\varepsilon}$ <br>  $\frac{\varepsilon}{\varepsilon}$ <br>
(i)  $\frac{\varepsilon}{\varepsilon}$ <br>
(i) that is Equation<br>
(i) Compared to a using a first-dominary and then  $\rho_S = 0$ ,  $\mathbb{P}^x$ -a.s<br>
(i) on then  $\rho_S = 0$ ,  $\mathbb{P}^x$ -a.s<br>
(i) G E  $\leq \varepsilon$  :  $X$ <br>
subseted  $\geq \varepsilon$  :  $X$ <br>
subsetential of  $\frac{1}{\varepsilon}$ <br>
subsetential order condex on  $\frac{1}{\varepsilon}$ <br>
subseted is music.<br>
if  $x \in \mathbb{R}$ <br>
subsetential of is music.  $X$  and  $Y$  a  $\frac{1}{t}$  is a view of  $\frac{1}{s}$  or  $\frac{1}{s}$  or  $\frac{s}{s}$  defined as  $\frac{s}{s}$ f n h t l rs  $\frac{1}{x}$  ) 9,  $\pi$  =  $\pi$ , ( $\pi$ )<br>ping is proposed in Christensen and<br>Lindensjö (2020a); Liang and Yuan<br>*p* holds for one-dimensional process,<br>oria, the condition (3) is replaced by<br>ion. This is analog to the first-order<br>> 0, the co  $\rho_{\tilde{S}}$  or lie, a<br>long single string  $\frac{1}{\beta}$  or  $\frac{1}{\beta}$  o rea arint in the part of the control of th : = inf{ $t \ge \varepsilon$  :  $X_t \in S$ }.<br>
ime inconsistent stopp<br>
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Equation (3) trivially l<br>
inpared to mild equilibr<br>
g a first-order condition<br>
int control. As  $\rho_S^{\varepsilon} \ge \varepsilon$  ><br>
iuing, and is much stro<br>
= The weak equilibrium concept for time inconsistent stopping is proposed in Christensen and Lindensjö [\(2018\)](#page-39-0), and further studied in Christensen and Lindensjö (2020a); Liang and Yuan [\(2021\)](#page-40-0); Tan et al. [\(2021\)](#page-40-0). Obviously, as Equation (3) trivially holds for one-dimensional process, a weak equilibrium is also mild. Compared to mild equilibria, the condition (3) is replaced by Equation (6) for weak equilibria using a first-order condition. This is analog to the first-order condition criterion in time-inconsistent control. As  $\rho_{S}^{\varepsilon} \geq \varepsilon > 0$ , the condition (6) does capture  $\rho$ <sub>S</sub><br>and<br> $\frac{1}{\beta}$ <br>and  $\frac{\varepsilon}{S}$  ≥<br>ich<br> $\frac{\partial S}{\partial S}$ , t<sup>i</sup>  $\varepsilon > 0$ , the condition (6) does capture<br>stronger than Equation (3). However,<br>een the identity  $\rho_S = 0$  requires some regularity<br>may not be trivial. the deviation from stopping to continuing, and is much stronger than Equation (3). However,

<sup>&</sup>lt;sup>1</sup> In multidimensional setting, if  $x \in S^0$ , then  $\rho_S = 0$ ,  $\mathbb{P}^x$ -a.s.; if  $x \in \partial S$ , then the identity  $\rho_S = 0$  requires some regularity  $x \in S^o$ , then  $\rho_S = 0$ ,  $\mathbb{P}^x$ -a.s.; if  $x \in \partial S$ , then the identity  $\rho_S = 0$  requires some regularity fication of Equation (3) on the boundary may not be trivial. of  $\partial S$ , and consequently, the verification of Equation (3) on the boundary may not be trivial.  $\partial S$ , and consequently, the verification of Equation (3) on the boundary may not be trivial.<br>

<span id="page-3-0"></span>there is still a drawback for Equation (6): when the limit is equal to zero, it is possible that for all  $\varepsilon > 0$ , we have  $f(x) < \mathbb{E}^{x}[\delta(\rho_{s}^{\varepsilon})f(X_{\rho_{s}^{\varepsilon}})]$ , and thus there is an incentive to deviate (see Björk et al. [\(2017,](#page-39-0) Remark 3.5), Huang & Zhou [\(2021\)](#page-40-0); Bayraktar et al. [\(2021\)](#page-39-0); He & Jiang [\(2021\)](#page-40-0) for more details). Roughly speaking, this is similar to a critical point not necessarily being a local maximum in calculus.

Recently, Bayraktar et al. [\(2021\)](#page-39-0) investigated the relation between the equilibrium concepts in these two streams of research we described above, and proposed an additional notion of equilibria.

**Definition 1.4.** A closed set  $S \subset \mathbb{X}$  is said to be a strong equilibrium, if

$$
\begin{cases}\nf(x) \leq J(x, S), & \forall x \notin S, \\
\exists \varepsilon(x) > 0, \text{ s.t. } \forall \varepsilon' \leq \varepsilon(x), f(x) - \mathbb{E}^x[\delta(\rho_S^{\varepsilon'})f(X_{\rho_S^{\varepsilon'}})] \geq 0, & \forall x \in S.\n\end{cases} \tag{7}
$$

Note that in the definition of strong equilibrium, the first-order condition (6) is replaced by a local maximum condition (7). This remedies the issue of weak equilibria mentioned in the above, and captures the economic meaning of "equilibrium" more accurately. Such kind of equilibria is also studied in Huang and Zhou [\(2021\)](#page-40-0); He and Jiang [\(2021\)](#page-40-0) for time inconsistent control. Obviously, a strong equilibrium must be weak. In Bayraktar et al. [\(2021\)](#page-39-0) under continuous-time Markov chain models with nonexponential discounting, a complete relation between mild, optimal mild, weak, and strong equilibria is obtained:

optimal mild 
$$
\subsetneq
$$
 strong  $\subsetneq$  weak  $\subsetneq$  mild. (8)

In this paper, we aim to establish the result (8) for one-dimensional diffusion models under nonexponential discounting. Compared to Bayraktar et al. (2021), the analysis in this paper is much more delicate. The proof in Bayraktar et al. (2021) crucially relies on the discrete state space of the Markov chain setting, and many critical ideas and steps therein cannot be applied in our diffusion framework, where novel approaches are needed for the characterizations of weak and strong equilibria. Here we list the main contributions of our paper as follows.

- $\varepsilon > 0$ , we have  $f(x) < \mathbb{E}^x[\delta(\rho_{\text{S}}^k, 1\ldots, 2017,$  Remark 3.5), Huang & ails). Roughly speaking, this is sails). Roughly speaking, this is scalculus.<br>
Recently, Bayraktar et al. (2021) se two streams of research we  $\frac{1}{2}$   $\frac{1}{2}$  (chian in a certain later a certain de later), certain later a certain de la certain de la certain d )], and thus there is an incentive to deviate (see Björke)<br>2021); Baynaktar et al. (2021): He & Jiang (2021) for more<br>of a critical point not necessarily being a local maximum<br>gated the relation between the equilibrium co  $S \subset \mathbb{X}$  is said to be a strong equilibrium, if<br>
),<br>  $S \subset \mathbb{X}$  is said to be a strong equilibrium, if<br>  $S \subset \mathbb{X}^{\epsilon}$   $\leq \epsilon(x)$ ,  $f(x) = \mathbb{E}^{x}[\delta(\rho_{S}^{\epsilon'})f(X_{\rho_{S}^{\epsilon'}})] \geq 0$ ,<br>
of strong equilibrium, the first-order  $f(x) \leq J(x, S)$ ,<br>  $\forall x \notin S$ ,<br>  $\exists z(x) > 0$ , s.t.  $\forall e' \leq \epsilon(x), f(x) - \mathbb{E}^{x}[\delta(\rho_{S}^{c})f(X_{\rho_{S}^{c}})] \geq 0$ ,  $\forall x \in S$ .<br>
the definition of strong equilibrium, the first-order condition (6) is<br>
condition (7). This remedies the issue ∃ $\varepsilon(x) > 0$ , s.t.  $\forall \varepsilon' \leq \varepsilon(x), f(x) - \mathbb{E}^x[\delta(\rho_{\varepsilon}^{\varepsilon})$ <br>
the definition of strong equilibrium, the frondition (7). This remedies the issue of veconomic meaning of "equilibrium" more economic meaning of "equilibr f induce in 2011 is to Fig. 1 in the fig. 1 is in the fig. 1 For  $\rho_S^s$ <br>irst-ord<br>irst-ord<br>irst-ord<br>veak eq<br>re accu<br>veak eq<br>re accu<br>1) for leads<br>the relat<br>there in must<br>chare there in a form must<br>chare there in a form must<br>chare in The "lo related raktar<br>veak eq<br>in the "lo related ccurlu n olesinic sa di sa antistica di sa ant )] ≥ 0,  $\forall x \in S$ . (7)<br>
er condition (6) is replaced by a<br>
uilibria mentioned in the above,<br>
rately. Such kind of equilibria is<br>
time inconsistent control. Obvi-<br>
under continuous-time Markov<br>
on between mild, optimal mil ⊊ strong ⊊ weak ⊊ mild. (8)<br>  $\subseteq$  strong ⊊ weak ⊊ mild. (8)<br>
(8) for one-dimensional diffusion models under non-<br>
rraktar et al. (2021), the analysis in this paper is much<br>
(2021) crucially relies on the discrete state • We provide a complete characterization (necessary and sufficient conditions) of weak equilibria. As a by-product, we show that any weak equilibrium must statisfy the smooth-fit conditions when the pay-off function  $f$  i ria. As a by-product, we show that any weak equilibrium must satisfy the smooth-fit condition when the pay-off function  $f$  is smooth. This gives a much sharper result in a much more gen $f$  is smooth. This gives a much sharper result in a much more gen-<br>the smooth-fit result obtained in Tan et al. (2021). (See Remark 3.3<br>r, in our paper  $f$  need not to be smooth, and our result also indi-<br>ondition is a s eral setting as compared to the smooth-fit result obtained in Tan et al. (2021). (See Remark [3.3](#page-10-0) for more details.) Moreover, in our paper  $f$  need not to be smooth, and our result also indicates that the smooth-fit condition is a special case of the "local convexity" property of weak equilibria. See Remark 3.4. Undoubtedly, such results related to smooth-fit condition have no correspondence in the Markov chain framework in Bayraktar et al. (2021).
- $f$  need not to be smooth, and our result also indi-<br>ecial case of the "local convexity" property of weak<br>such results related to smooth-fit condition have no<br>ework in Bayraktar et al. (2021).<br>m is also a weak equilibrium • We show that an optimal mild equilibrium is also a weak equilibrium. This proves that the set of weak equilibria is not empty. In terms of the mathematical method, in Bayraktar et al. (2021), the technique for the proof set of weak equilibria is not empty. In terms of the mathematical method, in Bayraktar et al. [\(2021\)](#page-39-0), the technique for the proof of such result relies on the fact that removing a point from a stopping region changes the stopping time, which is no longer applicable in the diffusion context. A different approach is developed to overcome this difficulty.
- We provide a sufficient condition under which a weak equilibrium is also strong. The condition is easy to verify as suggested by our examples. We also show that one may remove some "inessential" part of an optimal mild e tion is easy to verify as suggested by our examples. We also show that one may remove some "inessential" part of an optimal mild equilibrium, and the remaining part is still optimal mild

<span id="page-4-0"></span>

**FIGURE 1** Relations between results in Sections [3–5](#page-10-0) of this paper.  $A \rightarrow B$  means that statement A is used in the proof of statement  $B$ .

(and thus weak), and in fact strong under an additional assumption. In particular, this result implies that the smallest mild equilibrium essentially has no "inessential" parts and thus is strong. See Theorem [5.2](#page-26-0) and Remark [5.3.](#page-26-0)

 $A \rightarrow B$  means that statement *A* is used in<br>no "inessential" parts and thus is<br>introduces the notation and main<br>deffrequently throughout the paper.<br>equilibrium. In Section 4, we show<br>in Section 5, we provide a sufficient<br>o B. as represented at the S search of the S search of the S search of the S search of the S  $\mathbb{R}$ The rest of the paper is organized as follows. Section 2 introduces the notation and main assumptions, as well as some auxiliary results that will be used frequently throughout the paper. In Section [3,](#page-10-0) we provide a complete characterization of a weak equilibrium. In Section [4,](#page-18-0) we show that an optimal mild equilibrium is a weak equilibrium. Next, in Section [5,](#page-26-0) we provide a sufficient condition for a weak equilibrium to be strong. We also demonstrate how to construct a strong equilibrium from an optimal mild equilibrium by removing "inessential" parts. In particular, we show that the smallest mild equilibrium is strong under a mild assumption. Finally, three examples are provided in Section [6.](#page-32-0) The first example shows that a weak equilibrium may not be strong, while the second example shows that a strong equilibrium may not be optimal mild. The final example is about finding equilibria for the stopping problem of an American put option, which is used to demonstrate the usefulness of the results in Section [5.](#page-26-0) Figure 1 summarizes relations between the results in this paper.

## **2 SETUP AND SOME AUXILIARY RESULTS**

Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t>0}, \mathbb{P})$  be a filtered probability space, which supports a standard Brownian motion  $\Omega$ ,  $\mathcal{F}$ ,  $(\mathcal{F}_t)_{t\geq 0}$ ,  $\mathbb{P}$ ) be a filtered probability space, which supports a standard Brownian motion<br>  $\{W_t\}_{t\geq 0}$ . Let  $X = (X_t)_{t\geq 0}$  be a one-dimensional diffusion process with the dynamics<br>  $dX$  $W = (W_t)_{t \ge 0}$ . Let  $X = (X_t)_{t \ge 0}$  be a one-dimensional diffusion process with the dynamics<br> $dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$ ,

$$
dX_t = \mu(X_t)dt + \sigma(X_t)dW_t,
$$
\n(9)

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and take values in an interval  $X \subset \mathbb{R}$ . Let  $\mathbb{P}^x$  be the probability measure given  $X_0 = x$  and denote stopping times.

Let B be the family of all Borel subsets within X. For any  $A \in B$ , denote  $A^c := X \setminus A$  and within X. Denote  $B(x, r) := (x - r, x + r) \cap X$ . For  $A \in B$ , we define the first hitting and exit times

$$
\rho_A := \inf\{t > 0 : X_t \in A\} \quad \text{and} \quad \tau_A := \inf\{t > 0 : X_t \notin A\} = \rho_{A^c}.
$$
 (10)

Given a stopping region  $A \in \mathcal{B}$ , we define the value function  $V(t, x, A) : [0, \infty) \times \mathbb{X} \to \mathbb{R}$  as

$$
V(t, x, A) := \mathbb{E}^x[\delta(t + \rho_A)f(X_{\rho_A})]. \tag{11}
$$

Recall function *J* defined in Equation (4), we have  $J(x, A) = V(0, x, A)$ .

X  $\subset \mathbb{R}$ . Let  $\mathbb{P}^3$  the the probability measure given  $X_0 = x$  and denote by  $\top$  the set of  $\mathbb{R}$  the focus different substituting  $X$ . For any  $A \in B$ , denote  $A^c := \mathbb{X} \setminus A$  and interior of  $A$  and  $\overline{A}$   $E^x[\cdot] = E[\cdot|X_0 = x]$ . Let  $L_t^x$ <br>stopping times.<br>Let *B* be the family of a<br>stopping times.<br> $\partial A := \overline{A} \setminus A^o$ , where  $A^o$  is<br>within  $X$ . Denote  $B(x, r)$ :<br>withins<br>within  $X$ . Denote  $B(x, r)$ :<br>times<br> $\rho_A := \inf\{t : \overline{A} = \int A^o$  $x_t$  be the local time of<br>
all Borel subsets wit<br>
the interior of A and<br>  $:= (x - r, x + r) \cap \mathbb{X}$ <br>  $> 0 : X_t \in A$  and<br>  $\in B$ , we define the v<br>  $V(t, x, A) := \mathbb{E}^x$ <br>
in Equation (4), we has<br>
and  $\mathbb{Z} := \{0, \pm 1, \pm 2,$ <br>
of function X at point x up to time *t*. Denote by *T* the set of<br>
in X. For any  $A \in B$ , denote  $A^c := \mathbb{X} \setminus A$  and<br>
d is the closure of *A* under the Euclidean topology<br>
For  $A \in B$ , we define the first hitting and exit<br>  $\tau_A := \inf\{t >$ X. For any  $A \in B$ , denote  $A^c := \mathbb{X} \setminus A$  and<br>the closure of A under the Euclidean topology<br>r  $A \in B$ , we define the first hitting and exit<br> $A \in B$ , we define the first hitting and exit<br> $:= \inf\{t > 0 : X_t \notin A\} = \rho_{A^c}$ . (10)<br>f  $\beta$ . (2),  $\beta$ , where  $\lambda$ <sup>2</sup> is the interior of A and *A* is the closure of A and  $\lambda$  is the interior of A and *i*, is the interior of A and *τ*, is the interior of A and *τ*, is the interior of A and *τ*, is the int X. Denote  $B(x, r) := (x - r, x + r) \cap X$ . For  $A \in B$ , we define the first hitting and exit<br>  $\rho_A := \inf\{r > 0 : X_i \in A\}$  and  $\tau_A := \inf\{r > 0 : X_i \notin A\} = \rho_A$ . (40)<br>
stopping region  $A \in B$ , we define the value function  $V(t, x, A) : [0, \infty) \times X \rightarrow R$   $\beta_A$  : = inf {*t* > 0 :  $X_t \in A$ } and  $\tau_A$  : = inf {*t* > 0 :  $X_t \notin A$ } =  $\rho_A$ . (10)<br>
ing region  $A \in B$ , we define the value function  $V(t, x, A)$  : [0, ∞) ×  $X \rightarrow$  R as<br>  $V(t, x, A)$  :  $E^*[\delta(t + \rho_A)f(X_{\rho_A})]$ . (11)<br> *n I* define *A* ∈ *B*, we define the value function  $V(t, x, A)$  : [0, ∞) × × → R as<br>  $V(t, x, A) := E^x[\delta(t + \rho_A)f(X_{\rho_A})]$ .<br>
in Equation (4), we have  $J(x, A) = V(0, x, A)$ ,<br>
and  $\mathbb{Z}$  : = {0, ±1, ±2, ...}. Given  $E \in B$  and  $k \in \mathbb{N} \cup \{0\}$ , de  $V(t, x, A) := \mathbb{E}^x[\delta(t + \rho_A)f(X_{\rho_A})].$  (11)<br>
(11)<br>
quation (4), we have  $J(x, A) = V(0, x, A)$ . (2)<br>  $Z : \equiv \{0, \pm 1, \pm 2, ... \}$ . Given  $E \in B$  sand  $k \in \mathbb{N} \cup \{0\}$ , denote by<br>
ancitions  $v(t, x)$  that are continuously differentiable w *J* defined in Equation (4), we have  $J(x, A) = V(0, x, A)$ .<br>
{1,2, ...} and  $\mathbb{Z}$  :  $= \{0, 1, 1, 2, ... \}$ . Given  $E \in B$  and<br>
the family of functions  $v(t, x)$  that are continuously difficult<br>
imes continuously differentiable w. Denote  $\mathbb{N} := \{1, 2, ...\}$  and  $\mathbb{Z} := \{0, \pm 1, \pm 2, ...\}$ . Given  $E \in \mathcal{B}$  and  $k \in \mathbb{N} \cup \{0\}$ , denote by N : = {1,2,...} and *ℤ* : = {0, ± = {0, ± + 2, ...}. Given *E* ∈ *E* and *k* ∈ N∪ {0}, denote by <br>*x E*) the family of functions  $v(t, x)$  that are continuously differentiable with respect to *K* i Mickines continuously  $C^{1,k}([0,\infty)\times E)$  the family of functions  $v(t,x)$  that are continuously differentiable with respect to <sup>1</sup>/<sub>2</sub>((0, α) × *E*) the family of functions (*i*(*t*, *x*) that are continuously differentiable with respect to form  $(x, y)$  that are  $x$ -function  $y(x)$ , that are *k*-functions ( $x$ ) that are *k*-functions ( $x$ ) that a (w.r.t.) t and k-times continuously differentiable w.r.t. x when restricted to  $[0, \infty) \times E$ , and  $C^{k}(E)$ and k-times continuously differentiable w.r.t. when restricted to [0, ∞) × E, and C<sup>6</sup> (E),  $\sum_{i=1}^{\infty} \int_{i=1}^{\infty} \int_{i=1}^{\infty} \int_{i=1}^{\infty} \int_{i=1}^{\infty} \int_{i=1}^{\infty} \int_{i=1}^{\infty} \int_{i=1}^{\infty} \int_{i=1}^{\infty} \int_{i=1}^{\infty} \int_{i=1}^$ the family of functions  $v(x)$  that are k-times continuously differentiable when restricted to  $E^2$  For  $v(x)$  that are *k*-times continuously differentiable when restricted to *E*.<br>  $\infty$ )  $\times E \rightarrow \mathbb{R}$ ,  $v_x$ ,  $v_{xx}$  (resp.  $v_i$ ) denote the first- and second-order derivative w.r.t. *t*) if the derivatives exist. Moreover, d a function  $v(t, x)$ :  $[0, \infty) \times E \to \mathbb{R}$ ,  $v_x, v_{xx}$  (resp.  $v_t$ ) denote the first- and second-order deriva-(*x*(*x*) : [0, ∞) × *E* → [*R*, *ν*<sub>2</sub>, *ν*<sub>*x*x</sub> (resp. *to*) denote the first- and second-order deriva-<br>
(c (resp. the first- order derivative w.r.t. *t*) if the derivatives exist. Moreover, denote by<br>
esp. *v<sub>x</sub>*(*t* tives w.r.t  $x$  (resp. the first-order derivative w.r.t.  $t$ ) if the derivatives exist. Moreover, denote by x (resp. the first-order derivative w.r.t. *i*) if the derivatives exist. Moreover, denote by  $v_x(t, x+)$ , then derivative scale  $\log_{xx}(t, x)$ ,  $\log_{xx}(t, x)$ ,  $v_{xx}(t, x+)$ . For convenience, we denote  $v_t(0, x)$  as the right derivat  $v_z(t, x-)$  (resp.  $v_z(t, x+)$ ) the left (resp. right) derivative of *v* w.r.t. *x* at point (*t*, *x*). Similar nota-<br>
w.f. *t* at time  $t = 0$ . We further define the parabolic operator<br>  $\mathcal{L}v(t, x) := v_t(t, x) + \mu(x)v_x(t, x) + \frac{1}{2}\$ tion applies to  $v_{xx}(t, x-)$ ,  $v_{xx}(t, x+)$ . For convenience, we denote  $v_t(0, x)$  as the right derivative w.r.t. t at time  $t = 0$ . We further define the parabolic operator

$$
\mathcal{L}v(t,x) := v_t(t,x) + \mu(x)v_x(t,x) + \frac{1}{2}\sigma^2(x)v_{xx}(t,x) \quad \text{for a function } v \in C^{1,2}([0,\infty) \times E).
$$

Let us also use the following notation involving left or right derivatives w.r.t.  $x$ :

$$
\mathcal{L}v(t, x\pm) := v_t(t, x) + \mu(x)v_x(t, x\pm) + \frac{1}{2}\sigma^2(x)v_{xx}(t, x\pm), \quad \forall t \ge 0
$$

We now introduce the main assumptions in this paper. The first assumption concerns  $\mu$  and  $\sigma$ .

#### **Assumption 2.1.**

- (i)  $\mu, \sigma : \mathbb{X} \to \mathbb{R}$  are Lipschitz continuous.
- (ii)  $\sigma^2(x) > 0$  for all  $x \in \mathbb{X}$ .

 $\sigma^2(x) > 0$  for all  $x \in \mathbb{X}$ .<br>
ark 2.2. Assumption 2.<br>
=  $x \in \mathbb{X}$ . Assumption 2.<br>  $\mathbb{P}^x \left( \min_{0 \le s \le t} X_s < \mathbb{I} \right)$ <br>
tinuous differentiability is ext<br>
[a, b] ⊂ X, we say  $g \in C^{1,2}$ ([d] *Remark* 2.2. Assumption 2.1(i) guarantees that Equation (9) has a unique strong solution given

tion applies to 
$$
v_{xx}(t, x-)
$$
,  $v_{xx}(t, x+)$ . For convenience, we denote  $v_t(0, x)$  as the right derivative  
w.r.t. *t* at time *t* = 0. We further define the parabolic operator  
\n
$$
\mathcal{L}v(t, x) := v_t(t, x) + \mu(x)v_x(t, x) + \frac{1}{2}\sigma^2(x)v_{xx}(t, x)
$$
 for a function  $v \in C^{1,2}([0, \infty) \times E)$ .  
\nLet us also use the following notation involving left or right derivatives w.r.t. *x*:  
\n
$$
\mathcal{L}v(t, x \pm) := v_t(t, x) + \mu(x)v_x(t, x \pm) + \frac{1}{2}\sigma^2(x)v_{xx}(t, x \pm), \quad \forall t \ge 0.
$$
  
\nWe now introduce the main assumptions in this paper. The first assumption concerns  $\mu$  and  $\sigma$ .  
\n**Assumption 2.1.**  
\n(i)  $\mu, \sigma : \mathbb{X} \to \mathbb{R}$  are Lipschitz continuous.  
\n(ii)  $\sigma^2(x) > 0$  for all  $x \in \mathbb{X}$ .  
\nRemark 2.2. Assumption 2.1(i) guarantees that Equation (9) has a unique strong solution given  
\n $X_0 = x \in \mathbb{X}$ . Assumption 2.1(i)(ii) together imply that for any  $x \in \mathbb{X}$  and  $t > 0$ ,  
\n
$$
\mathbb{P}^x\left(\min_{0 \le s \le t} X_s < x\right) = \mathbb{P}^x\left(\max_{0 \le s \le t} X_s > x\right) = 1
$$
, and thus  $\rho_{\{x\}} = 0$ ,  $\mathbb{P}^x$ -a.s.. (12)  
\n
$$
\mathbb{P}^x\left(\min_{0 \le s \le t} v_s < x\right) = \mathbb{P}^x\left(\max_{0 \le s \le t} X_s > x\right) = 1
$$
, and thus  $\rho_{\{x\}} = 0$ ,  $\mathbb{P}^x$ -a.s.. (12)  
\n
$$
\mathbb{P}^x\left(\min_{0 \le s \le t} v_s < x\right) = \mathbb{P}^x\left(\max_{0 \le s \le t} X_s > x\right)
$$

ሥ<br>Fel<br>Հ  $0 \leq s \leq n$ tiabil  $\frac{0 \leq s \leq t}{\text{total}}$  $=$   $\mathbb{P}^n$ <br>ed to the  $\mathbb{R}$   $\times$  [a,  $0 \leq s \leq t$ <br>bound  $\frac{0 \leq s \leq t}{0}$ <br>bound<br>bound, if g <sup>2</sup> Continuous differentiability is extended to the boundary in a natural way if the boundary is included in  $E$ . For example, E. For example,<br>X). given  $[a, b] \subset \mathbb{X}$ , we say  $g \in C^{1,2}([0, \infty) \times [a, b])$ , if  $g = g_1$  on  $[0, \infty) \times [a, b]$  for some  $g_1 \in C^{1,2}([0, \infty) \times \mathbb{X})$ .  $[a, b] \subset \mathbb{X}$ , we say  $g \in C^{1,2}([0, \infty) \times [a, b])$ , if  $g = g_1$  on  $[0, \infty) \times [a, b]$  for some  $g_1 \in C^{1,2}([0, \infty) \times \mathbb{X})$ .

A quick proof for Equation [\(12\)](#page-5-0) is relegated in Appendix [A.](#page-41-0)

Notice that a (time-homogeneous Markovian) stopping policy can be characterized by a stopping region  $S \subset \mathbb{X}$ . For  $S \in \mathcal{B}$ , Equation (12) implies that  $\rho_S = \rho_{\overline{S}} \mathbb{P}^{\chi}$ -a.s. for any  $x \in \mathbb{X}$ . Also, stopping," not matter x belongs to S or not. Therefore, it suffices to work on stopping regions that are closed.

**Definition 2.3.**  $S \in B$  is called an admissible stopping policy, if S is closed (w.r.t the Euclidean topology within  $\mathbb{X}$ ) and for any  $x \in \partial S$ , one the following two cases holds:

- (a)  $x \in \partial(S^{\circ})$ , that is,  $\exists h > 0$  such that either  $(x h, x) \subset S^{\circ}$  and  $(x, x + h) \subset S^c$ , or  $(x h, x) \subset$
- (b) x is an isolated point, that is,  $B(x, h) \setminus \{x\} \subset S^c$  for some  $h > 0$ .

*Remark* 2.4. Except cases (a) and (b), the rest situation for a boundary point  $x \in \partial S$  is the following:

(c) There exist two sequences  $(x_n)_{n\in\mathbb{N}} \subset S$  and  $(y_n)_{n\in\mathbb{N}} \subset S^c$  such that both  $(x_n)_{n\in\mathbb{N}}$  and  $(y_n)_{n\in\mathbb{N}}$ approach to x from the left, or both approach to x from the right.<sup>3</sup>

Stopping regions containing boundary case (c) lack economic meaning, since it is not practical for an agent to follow a stopping policy classified as case (c). Mathematically, the regularity of not exist for S being the cantor set on  $[0,1]$ ; this would cause serious issue to establish our main results later as they crucially rely on the regularity of  $V(t, x, S)$  (e.g., the characterization of weak equilibria).

S C. X. For S ∈ B. Equation [\(12\)](#page-5-0) implies that  $S_2 = S_1^{\text{max}}$  E. S. Also, any ∈. E. X. Also, any ∈. S. Also, any ∈. S. Also, any ∈. S. Also, any ∈. S. Also, or an ( $f(x)$ , S) for all x e S, and a boundary point  $x \in S$  corresponds to the action<br>torophys<sup>-</sup> not matter x belongs to S or rot. Therefore, it suffices to work on stopping-regimes that<br>are closed.<br>Definition 2.3. S e  $\beta$  is x belongs to *S* or not. Therefore, it suffices to work on stopping regions that<br>
B is called an admissible stopping policy, if *S* is closed (w.r.t the Euclidean<br>
d for any  $x \in \partial S$ , one the following two cases holds:<br>  $S ∈ B$  is called an admissible stopping policy; if *S* is closed (w.r.t the Euclidean<br>
s) and for  $\sin y \times ∈ \partial S$ , one the following two cases holds:<br>
statis. ∃*h* > 0 such that either  $(x - h, x) ∈ S$  and  $(x, x + h) ∈ S$ , or  $(x - h, x) ∈ S$ X) and for any  $x ∈ \partial S$ , one the following two cases holds:<br>hat is,  $\exists h > 0$  such that either  $(x - h, x) ⊂ S^\circ$  and  $(x, x + h) ⊆ S^\circ$ ;<br>ted point, that is,  $B(x, h) \setminus \{x\} ⊂ S^\circ$  for some  $h > 0$ .<br>ccept cases (a) and (b), the rest si  $x ∈ \theta(S^*)$ , that is, ∃ $B > 0$  such that either ( $x = h, x$ ) ⊂  $S^>$  and ( $x, x + h$ ) ∈  $S^*$ , or ( $x = h, x$ ) ⊂ sin solated point, that is,  $B(x, h) \setminus \{x\} \subset S^c$  for some  $h > 0$ .<br>  $x^*$  an isolated point, that is,  $B(x, h) \setminus \{x\} \subset S$ S<sup>c</sup> and  $(x, x + h) \subset S^\circ$ ;<br>  $x$  is an isolated point, t<br>  $x$  is a.e.g.<br>
Similarly the summer to follow a sto x is an isolated point, that is,  $B(x, h) \{x\} \subset S^c$  for some  $h > 0$ .<br>
ark 2.4. Except cases (a) and (b), the rest situation for a both<br>
wing:<br>
There exist two sequences  $(x_n)_{n \in \mathbb{N}} \subset S$  and  $(y_n)_{n \in \mathbb{N}} \subset S^c$  such tiap  $x \in \partial S$  is the<br>
Notated Manuson and  $(y_n)_{n \in \mathbb{N}}$ <br>
is not practical<br>
experimentally of  $(t, 0+, S)$  may<br>
by lish our main<br>
and a stopping<br>
the case stud-<br>
rt et al. (2020);<br>
is Strack (2018)<br>
ples of Huang<br>
ples of Huang  $(x_n)_{n \in \mathbb{N}}$  ⊂ *S* and  $(y_n)_{n \in \mathbb{N}}$  ⊂ *S*<sup>c</sup> such that both  $(x_n)_{n \in \mathbb{N}}$  and  $(y_n)_{n \in \mathbb{N}}$ , t, or both approach to *x* from the right.<sup>3</sup><br>cu, or both approach to *x* from the right.<sup>3</sup><br>cu, or both approach to x from the left, or both approach to x from the right.<sup>3</sup><br>containing boundary case (c) lack economic meanin<br>illow a stopping policy classified as case (c). Mather<br>o be missing when *S* contains boundary case (c), for<br>ing  $V(t, x, S)$  may also be missing when *S* contains boundary case (c), for example,  $V_s(t, 0+S)$  may also be missing to ensight about the contains boundary case (c), it as they crucially rely on the regularity of  $V(t, x, S)$  (e. S being the cantor set on [0,1]; this would cause serious issue to establish our mains they crucially rely on the regularity of  $V(t, x, S)$  (e.g., the characterization of weak on admissible stopping policies is also well-al *V*(*t*, *x*, *S*) (e.g., the characterization of weak<br>Il-aligned with the literature, and a stopping<br>applications. For instance, all the case stud-<br>sen and Lindensjö (2018); Ebert et al. (2020);<br>libria. The results in Eb Focusing on admissible stopping policies is also well-aligned with the literature, and a stopping policy containing boundary case (c) is rarely studied in applications. For instance, all the case studies in Bodnariu et al. (2022); Tan et al. [\(2021\)](#page-40-0); Christensen and Lindensjö (2018); Ebert et al. (2020); Huang et al. [\(2020\)](#page-40-0) only focus on threshold-type equilibria. The results in Ebert and Strack (2018) mainly focus on two threshold stopping regions. The mild equilibria in all the examples of Huang and Nguyen-Huu (2018) have boundaries of cases only (a) and (b). All mild equilibria provided in Huang and Zhou (2020, Sections 6.1 and 6.2) are all admissible, so are the mild equilibria in the case study of Huang & Yu [\(2021,](#page-40-0) Section 4).

Let us also point out that all the interesting equilibria (i.e., optimal mild, weak, strong equilibria) provided in all the examples in this paper are admissible. Specifically, in the case study in Section 6.3, which can be thought of as a continuation of Huang and Zhou [\(2020,](#page-40-0) Sections 6.3), all the weak, strong, and optimal mild equilibria are admissible, and any mild equilibria is either admissible or has an admissible alternative (see Remark 6.6).

To sum up, focusing on cases (a) and (b) is economically meaningful, mathematically necessary, well-aligned with the literature, and general enough for applications.

<sup>&</sup>lt;sup>3</sup> Indeed, let  $x \in \partial S$ . Suppose x does not satisfy case (c). Then there exists  $h > 0$  such that either  $(x - h, x) \subset S$  or  $(x - h, x)$  $x \in \partial S$ . Suppose x does not satisfy case (c). Then there exists  $h > 0$  such that either  $(x - h, x) \subset S$  or  $(x - h, x)$  is the interval  $(x, x + h)$ . If  $(x - h, x) \subset S^c$  and  $(x, x + h) \subset S^c$ , then x satisfies case (b); otherwise x satisfi *h*, *x*) ⊂ *S<sup>c</sup>*, so is the interval (*x*, *x* + *h*). If (*x* − *h*, *x*) ⊂ *S<sup>c</sup>* and (*x*, *x* + *h*) ⊂ *S<sup>c</sup>*, then *x* satisfies case (b); otherwise *x* satisfies case (a). case (a).

# <span id="page-7-0"></span>**804 | WII FY** BAYRAKTAR ET AL.

Let  $\delta(\cdot) : [0, \infty) \to [0, 1]$  be a discount function that is nonincreasing, continuously differentiable, and  $\delta(0) = 1$ ,  $\delta(t) < 1$  for  $t > 0$ . We assume that  $\delta$  satisfies the following condition.

**Assumption 2.5.**  $\delta$  is log subadditive:

$$
\delta(t+s) \ge \delta(t)\delta(s), \quad \forall s, t \ge 0.
$$
\n(13)

 $\mathcal{S}(·)$  i.e., ⇒ [(a, 0, l) be a discount function that is nonincreasing, continuously differen-<br>
and  $\delta(0) = 1$ ,  $\delta(t) < 1$  for  $t > 0$ . We assume that  $\delta$  satisfies the following condition.<br> **aption 2.5.**  $\delta$  is log  $\delta(0) = 1$ ,  $\delta(t) < 1$  for  $t > 0$ . We assume that  $\delta$  satisfies the following condition.<br> **on 2.5.**  $\delta$  is log subadditive:<br>  $\delta(t + s) \geq \delta(t)\delta(s)$ ,  $\forall s, t \geq 0$ .<br>
Condition (13) see the disconsisted as the so-called decreas δ is log subadditive:<br>  $δ(t +$ <br>
ition (13) can be interaind the sudo-exponential din-Huu, 2018, Assum<br>
denotes the right d<br>
proof is relegated in<br>
ssumption 2.5 hold.<br>  $δ'(t) ≥ δ(t)δ'(0)$ ,<br>
nction  $f(x) : \mathbb{X} \to \mathbb{I}$ <br>
wing as  $\delta(t + s) \geq \delta(t)\delta(s)$ ,  $\forall s, t \geq 0$ . (13)<br>
se interpreted as the so-called decreasing impatience in finance<br>
mential discount functions, including hyperbolic, generalized<br>
trial discounting, satisfy Equation (13). See the *Remark* 2.6. Condition (13) can be interpreted as the so-called decreasing impatience in finance and economics. Many nonexponential discount functions, including hyperbolic, generalized hyperbolic, and pseudo-exponential discounting, satisfy Equation (13). See the discussion below (Huang and Nguyen-Huu, [2018,](#page-40-0) Assumption 3.12) for a more detailed explanation.

Recall that  $\delta'(0)$  denotes the right derivative of  $\delta(t)$  at  $t = 0$ . The following lemma is a quick result for  $\delta$  and the proof is relegated in the Appendix A.

**Lemma 2.7.** *Let Assumption 2.5 hold. Then,*

$$
\delta'(t) \ge \delta(t)\delta'(0), \text{ and } 1 - \delta(t) \le |\delta'(0)|t, \quad \forall t \ge 0.
$$

Let the payoff function  $f(x): \mathbb{X} \to \mathbb{R}$  be non-negative and continuous. We further assume that

**Assumption 2.8.** (i) For any  $x \in \mathbb{X}$ ,

$$
\lim_{t \to \infty} \delta(t) f(X_t) = 0, \quad \mathbb{P}^x - a.s., \tag{14}
$$

and there exists  $\zeta > 0$  such that

$$
\mathbb{E}^{x}\left[\sup_{t\geq 0} \left(\delta(t)f(X_{t})\right)^{1+\zeta}\right] < \infty. \tag{15}
$$

od Le off fc 2. ts gs i a hts (0) denotes the right derivative of  $\delta(t)$  at  $t = 0$ . The following lemma is a quick<br>the proof is relegated in the Appendix A.<br> *t Assumption 2.5 hold. Then,*<br>  $\delta'(t) \geq \delta(t)\delta'(0)$ , and  $1 - \delta(t) \leq |\delta'(0)|t$ ,  $\forall t \geq 0$ .<br>
funct δ and the proof is relegated in the Appendix [A.](#page-41-0)<br>
2.7. Let Assumption 2.5 hold. Then,<br>  $δ'(t) ≥ δ(t)δ'(0)$ , and  $1 - δ(t) ≤$ <br>
payoff function  $f(x) : X → ℝ$  be non-negative<br>
the following assumptions.<br> **ion 2.8.** (i) For any  $x ∈ X$ ,<br> o iong<br>iong<br>Fc<br>su<br>p<br>+1<br>+1 () ≥ ()′ (0), and  $1 - \delta(t) \le |\delta'|$ <br>
→ ℝ be non-negative an<br>
ns.<br>  $\leq$ ,<br>  $\frac{1}{\delta(t)} f(X_t) = 0$ ,  $\mathbb{P}^x - \frac{1}{\delta(t)} \left\{ \sup_{t \ge 0} (\delta(t) f(X_t))^{1+\zeta} \right\} <$ <br>
That is, there exists an e ll  $n \in I$ , such that  $f \in \mathbb{P}$ <br>
and denote<br>  $\mathcal{G} := \mathbb{$ (0)|*t*,  $\forall t \ge 0$ .<br>d continuous. V<br>*a.s.*,<br> $\infty$ .<br>ther finite or c<br> $C^2([\theta_n, \theta_{n+1}])$  f<br>8, which is an e<br>ies that<br> $x \in \mathbb{X}$ .  $f(x) : X \to \mathbb{R}$  be non-negative and continuous. We further assume that<br>
ssumptions.<br>  $\text{any } x \in X$ ,<br>  $\lim_{t \to \infty} \delta(t) f(X_t) = 0, \quad \mathbb{P}^x - a.s,$ <br>  $\text{any } \delta(t) f(X_t) = 0, \quad \mathbb{P}^x - a.s,$ <br>  $\text{any } \delta(t) f(X_t) = 0, \quad \mathbb{P}^x - a.s.$ <br>  $\text{any } \delta(t) f(X$ f satisfies the following assumptions.<br> **Assumption 2.8.** (i) For any  $x \in \mathbb{X}$ ,<br>  $\lim_{t \to \infty} \xi$ <br>
and there exists  $\zeta > 0$  such that<br>  $\mathbb{E}^{x}$ <br>
(ii)  $f(x)$  belongs to  $C^{2}$  piecewisely. Th<br>
X, with  $I \subset \mathbb{Z}$  and  $x \in \mathbb{X}$ ,<br>  $\lim_{t \to \infty}$ <br>
at  $\mathbb{E}^x$ <br>
isely. T<br>
for all  $\mathbb{X}$  o and  $(15)$  will<br>
Moreov<br>  $\mathbb{E}^x \left[ \text{su} \atop t \geq 0 \right]$ lim→∞ (14)<br>
lim→∞ ( $\mathbb{P}^X = a.s.,$  (14)<br>  $\mathbb{E}^X \left[ \sup_{t \ge 0} (\delta(t) f(X_t))^{1+\zeta} \right] < \infty.$  (15)<br>
ly. That is, there exists an either finite or countable set  $(\theta_n)_{n \in I} \subset \pi$  all  $n \in I$ , such that  $f \in C^2([\theta_n, \theta_{n+1}])$  for any  $n$  $\xi > 0$  such that<br>  $\partial_n$  is to  $C^2$  piecewise<br>  $\partial_n$  is  $\partial_{n+1}$  for  $\partial_{n+1}$  for  $\partial_{n+1}$  is  $\partial_{n+1}$ <br>  $\partial_{n+1}$  is assumption (1)<br>
in the paper. M<br>
E t  $\geq 0$ <br>aat i  $n \in \mathbb{d}$ <br>d d d  $\mathbb{G}$ :<br>be er, l  $\delta$  (*i* t ≥0 at  $n \in$  d d  $\beta$  : be er,  $\delta$ ( s, there exist<br> *I*, such that<br>  $I$ , such that<br>  $= \mathbb{X} \setminus \{\theta_n :$ <br>
used for Lending (1)<br>  $f(X_t)$  < (3)  $1+\zeta$ <br>ts are  $f$ <br>ts are  $n \in$ <br> $n \in$ <br>mma<br> $5)$  im < ∞. (15)<br>
either finite or countable set  $(\theta_n)_{n \in I}$  ⊂<br>  $\in C^2([\theta_n, \theta_{n+1}])$  for any  $n \in I$ . We also<br>
(16)<br>
3.8, which is an essential lemma for all<br>
plies that<br>
∀x ∈ X. (17) (ii)  $f(x)$  belongs to  $C^2$  piecewisely. That is, there exists an either finite or countable set  $(\theta_n)_{n\in I} \subset$ f(x) belongs to  $C^2$  piecewisely. That is, there exists an either finite or countable set  $(\theta_n)_{n \in I} \subset$ <br>with  $I \subset \mathbb{Z}$  and  $\theta_n < \theta_{n+1}$  for all  $n \in I$ , such that  $f \in C^2([\theta_n, \theta_{n+1}])$  for any  $n \in I$ . We also<br>ume that  $\mathbb{X}$ , with  $I \subset \mathbb{Z}$  and  $\theta_n < \theta_{n+1}$  for all  $n \in I$ , such that  $f \in C^2([\theta_n, \theta_{n+1}])$  for any  $n \in I$ . We also<br>assume that inf<sub>n∈I</sub>( $\theta_{n+1} - \theta_n$ ) > 0 and denote<br> $G := \mathbb{X} \setminus \{\theta_n : n \in I\}$ . (16)<br>*Remark* 2.9. The ass assume that  $\inf_{n \in I} (\theta_{n+1} - \theta_n) > 0$  and denote

$$
\mathcal{G} := \mathbb{X} \setminus \{\theta_n : n \in I\}.
$$
 (16)

inf<sub>*n∈I*</sub>( $\theta_{n+1} - \theta_n$ ) > 0 and denote<br>  $G := \mathbb{X} \setminus$ <br>
The assumption (15) will be used<br>
ults in the paper. Moreover, Equat<br>  $\mathbb{E}^x \left[ \sup_{t \ge 0} \delta(t) f(X) \right]$ *Remark* 2.9. The assumption (15) will be used for Lemma [3.8,](#page-13-0) which is an essential lemma for all the main results in the paper. Moreover, Equation (15) implies that

$$
G := \mathbb{X} \setminus \{\theta_n : n \in I\}. \tag{16}
$$
\n(15) will be used for Lemma 3.8, which is an essential lemma for all Moreover, Equation (15) implies that

\n
$$
\mathbb{E}^x \left[ \sup_{t \ge 0} \delta(t) f(X_t) \right] < \infty, \quad \forall x \in \mathbb{X}. \tag{17}
$$

<span id="page-8-0"></span>This together with Equation [\(14\)](#page-7-0) guarantees the well-posedness of  $V(t, x, S)$  for any stopping pol- $V(t, x, S)$  for any stopping pol-<br>minated convergence theorem<br>, Equations (17) and (14) also<br>ted in Huang and Zhou (2020,<br>ed in Huang and Zhou (2020,<br> $\infty$ ) × [a, b]), and<br> $\forall x \in \mathbb{X}$ . (18)<br>fficient condition for Assump-<br> icy S. $^4$  Equations [\(17\)](#page-7-0) and [\(14\)](#page-7-0) will also be used for applying the dominated convergence theorem in some localization arguments in the proofs later. Furthermore, Equations [\(17\)](#page-7-0) and [\(14\)](#page-7-0) also ensure the existence of an optimal mild equilibrium as demonstrated in Huang and Zhou [\(2020,](#page-40-0) Theorem 4.12) (also see Lemma [4.1](#page-18-0) in this paper).

Let us make an assumption on  $V(t, x, S)$ .

**Assumption 2.10.** For any admissible stopping policy S and  $a, b \in \mathbb{X}$  with  $a < b$  and  $(a, b) \subset S^c$ .

$$
\limsup_{t \searrow 0} \frac{1}{\sqrt{t}} |V_x(t, x\pm, S) - V_x(0, x\pm, S)| = 0, \quad \forall x \in \mathbb{X}.
$$
 (18)

*Remark* 2.11. It turns out that Assumption 2.10 is quite general. A sufficient condition for Assumption 2.10 is that  $\delta(t)$  is a weighted discount function as shown in the lemma below. One may also directly verify this assumption given the probability density functions of exit time

$$
p(x,t) := \mathbb{P}^x \Big( \tau_{(c,d)} \in dt, X_{\tau_{(c,d)}} = c \Big) \quad \text{and} \quad q(x,t) := \mathbb{P}^x \Big( \tau_{(c,d)} \in dt, X_{\tau_{(c,d)}} = d \Big) \tag{19}
$$

S.<br>
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30:02  $V(t, x, S)$ .<br>sible stop<br>(with A =<br> $V_x(t, x \pm, S)$ <br>mption 2.<br>discount i<br>en the production<br>the production fold and f<br>sumption<br>hold and f<br> $\delta(t) =$ <br>umulative<br> $\delta(t) =$ <br>umulative<br> $\delta(t) =$ <br>umulative<br>a and the<br>sumption<br>a and the<br>a and th S and *a*, *b* ∈ X with *a* < *b* and (*a*, *b*) ⊂ *S<sup>c</sup>*,<br>s to *C*<sup>1,2</sup>([0, ∞) × [*a*, *b*]), and<br>±, *S*)| = 0, ∀ *x* ∈ X. (18)<br>eneral. A sufficient condition for Assumpshown in the lemma below. One may also<br>nsity func  $U(t, x, S)$  defined in Equation [\(11\)](#page-5-0) (with  $A = S$ ) belongs to  $C^{1,2}(0, \infty) \times [a, b]$ ), and<br>  $\limsup_{\gamma \searrow 0} \frac{1}{\sqrt{t}} [V_x(t, x \pm, S) - V_x(0, x \pm, S)] = 0$ ,  $\forall x \in X$ .<br> *Remark* 2.11. It turns out that Assumption 2.10 is quite general. A *t*  $\sqrt{0}$  and that is a weid sumptify  $r_{(c,d)} \in$ <br>*F*<sub>(*c,d*)</sub> ∈ <br>*h*. For e werify the more g<br>sumptify more g<br>sumptify for and 2.12<br>and 2.12<br>and 2.12<br>ort et al weak equanting s ut t a w<br>
ump (d)<br>
Fol ify ore map is fit in the value of the valu  $\overline{\sqrt{2}}$  A  $\overline{11}$  and  $\overline{12}$  is a set of  $\overline{12}$  if  $\overline{12}$  if  $\overline{12}$  if  $\overline{12}$  if  $\overline{12}$  if  $\overline{12}$ ster at the control of the  $V_x(t, x\pm, S) - V_x(0, x\pm, S) = 0$ ,  $\forall x \in \mathbb{X}$ . (18)<br>
umption 2.10 is quite general. A sufficient condition for Assump-<br>
discount function as shown in the lemma below. One may also<br>
ven the probability density functions of ex  $\delta(t)$  is a weighted discount function as shown in the lemma below. One may also<br>
its assumption given the probability density functions of exit time<br>  $P^{\times}\left(\tau_{(c,d)} \in dt, X_{\tau_{(c,d)}} = c\right)$  and  $q(x, t) := P^{\times}\left(\tau_{(c,d)} \in dt, X_{\tau_{(c,d$  $p(x, t) := P$ <br>
g regular enou, then we can<br>
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baper.<br> **ma 2.12.** *Let*.<br>
ion of the foll<br>
e F(r) : [0, ∞<br>  $p$ <br>  $p$  F(r) : [0, ∞<br>  $p$ <br>  $p$ <br>  $p$  F(r) : [0, ∞<br>  $p$ <br>  $p$ <br>  $p$  F(r) : [0, ∞<br>  $p$ <br>  $p$ <br>  $\tau_{(c,d)} \in dt, X_{\tau_{(c,d)}} = c$ <br>
(h. For example, if *X* is<br>
verify that Assumptio<br>
more general sufficie<br>
ssumption 2.1 hold and<br>
ving form:<br>  $\delta(t)$  :<br>
→ [0, 1] is a cumulativ<br>  $\lim_{t \searrow 0} \frac{1}{\sqrt{t}} \int_0$ <br>
10 holds.<br>
ma 2.12 is  $q(x, t) := \mathbb{P}$ <br>
ian motion o<br>
olds by using<br>
tion for Assur<br> *nded on* X. *St*<br> *nded on* X. *St*<br>  $\int_0^{-rt} dF(r)$ ,<br> *x* A.<br>  $x$  $\tau_{(c,d)} \in dt, X_{\tau_{(c,d)}} = d$ <br>  $\mathbb{X} = \mathbb{R}, \delta(t) = \frac{1}{1+t}$ , an<br>
Equation (19) for the B<br>
ption 2.10 is out of the<br>
prose  $\delta(t)$  is a weighted<br>
prose  $\delta(t)$  is a weighted<br>
and  $\int_0^\infty r dF(r) <$ <br>
are studied in detail. That<br>
inti being regular enough. For example, if X is a Brownian motion on  $\mathbb{X} = \mathbb{R}$ ,  $\delta(t) = \frac{1}{1+t}$ , and  $f(x) =$ *X* is a Brownian motion on  $X = \mathbb{R}$ ,  $\delta(t) = \frac{1}{14}$ <br>ption 2.10 holds by using Equation (19) for<br>icient condition for Assumption 2.10 is out c<br>and *f* be bounded on X. Suppose  $\delta(t)$  is a weig<br>( $t$ ) =  $\int_0^\infty e^{-rt} dF(r)$  $\frac{1}{1+t}$ , and<br>r the Br<br>t of the s<br>ighted d<br>ighted d<br>ighted d<br>fined or<br>defined or by a symmian<br>
cope of iscount<br>
(20)<br>  $\circ$  and<br>
(21)<br>  $\circ$  and<br>
(21)<br>  $\circ$  and<br>
(21)<br>  $\circ$  and<br>  $\circ$  for  $\circ$   $\circ$ motion. Providing a more general sufficient condition for Assumption 2.10 is out of the scope of this paper.

**Lemma [2.1](#page-5-0)2.** Let Assumption 2.1 hold and f be bounded on  $X$ . Suppose  $\delta(t)$  is a weighted discount function *of the following form:*

$$
\delta(t) = \int_0^\infty e^{-rt} dF(r),\tag{20}
$$

*where*  $F(r)$ :  $[0, \infty) \to [0, 1]$  *is a cumulative distribution function satisfying*  $\int_0^\infty r dF(r) < \infty$  *and* 

$$
\lim_{t \searrow 0} \frac{1}{\sqrt{t}} \int_0^\infty r(1 - e^{-rt}) dF(r) = 0.
$$
 (21)

*Then, Assumption 2.10 holds.*

The proof of Lemma 2.12 is included in Appendix A.

0∨, x, then we can verify that Assumption 2.10 holds by using Equation (19) for the Brownian<br>motion. Providing a more general sufficient condition for Assumption 2.10 is out of the scope of<br>this paper.<br> **Lemma 2.12**. Le *f* be bounded on  $\infty$ . Suppose  $\delta(t)$  is a weighted discount<br>  $\int_0^\infty e^{-rt} dF(r)$ , (20)<br> *e* distribution function satisfying  $\int_0^\infty r dF(r) < \infty$  and<br>  $\int_0^\infty r(1 - e^{-rt}) dF(r) = 0$ . (21)<br>
Appendix A.<br>
ted discount functions are  $(0(t)) = \int$ <br>
ulative d:<br>  $\frac{1}{\sqrt{t}} \int_0^\infty r(t) dt$ <br>
ed in Approximation density<br>
is of  $V(t, x, t)$ <br>  $\frac{1}{\sqrt{t}}$  of  $V(t, x, t)$ tr<br>L -<br>er<br>di∘ou<br>∪ - ∞ 0  $e^{-rt}dF(r)$ , (20)<br>
bution function satisfying  $\int_0^\infty r dF(r) < \infty$  and<br>  $-e^{-rt}dF(r) = 0$ . (21)<br>
dix A.<br>
count functions are studied in detail. Tan et al.<br>
outh-fit condition for time-inconsistent stopping<br>
int functions, includ  $F(r)$  : [0, ∞) → [0, 1] *is a cumulative distribution function satisfying*  $\int_0^\infty$ <br>  $\lim_{t\searrow 0} \frac{1}{\sqrt{t}} \int_0^\infty r(1 - e^{-rt}) dF(r) = 0$ .<br>
Assumption 2.10 holds.<br>
proof of Lemma 2.12 is included in Appendix A.<br>
k 2.13. In Eber  $\frac{1}{10}$  $r dF(r) < \infty$  and<br>(2)<br>m detail. Tan et a<br>posistent stoppir<br>ential, hyperboli<br>ell-defined on  $\rho_S =$ t∖0<br>nclu<br>020)<br>bria<br>ng.<br>sedn nclu<br>020)<br>bria<br>ng.<br>sedn |
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|  $\frac{1}{\epsilon}$   $\int$ <br>d ii<br>veig<br>d any<br>of l  $A$  at the dine dine  $\frac{1}{(t,0)}$  $\begin{bmatrix} \text{d} \\ \text{e} \end{bmatrix}$   $\begin{bmatrix} \text{d} \\ \text{e} \end{bmatrix}$  $r(1 - e^{-rt})dF(r) = 0.$  (21)<br>ppendix A.<br>d discount functions are studied in detail. Tan et al.<br>smooth-fit condition for time-inconsistent stopping<br>scount functions, including exponential, hyperbolic,<br> $(x, S)$  because otherwise *Remark* 2.13. In Ebert et al. (2020), weighted discount functions are studied in detail. Tan et al. [\(2021\)](#page-40-0) investigates weak equilibria and the smooth-fit condition for time-inconsistent stopping in a weighted discounting setting. Many discount functions, including exponential, hyperbolic,

<sup>&</sup>lt;sup>4</sup> Equation [\(14\)](#page-7-0) is used for the well-posedness of  $V(t, x, S)$  because otherwise  $\delta(t) f(X_t)$  is not well-defined on  $\rho_S = \infty$  $V(t, x, S)$  because otherwise  $\delta(t) f(X_t)$  is not well-defined on  $\rho_S = \infty$ ; the upper limit lim sup<sub> $t \to \infty$ </sub>  $\delta(t) f(X_t)$ . (unless we do some extension, e.g., by considering the upper limit lim  $\sup_{t\to\infty} \delta(t) f(X_t)$ . lim sup<sub>t→∞</sub>  $\delta(t) f(X_t)$ .

generalized hyperbolic, and pseudo-exponential discounting, satisfy Equations [\(20\)](#page-8-0) and [\(21\)](#page-8-0). For example, a generalized hyperbolic discount function can be written as

$$
\delta(t) = \frac{1}{(1+\beta t)^{\frac{\gamma}{\beta}}} = \int_0^\infty e^{-rt} \frac{r^{\frac{\gamma}{\beta}-1}e^{-\frac{r}{\beta}}}{\beta^{\frac{\gamma}{\beta}}\Gamma(\frac{\gamma}{\beta})}dr = \int_0^\infty e^{-rt}dF(r), \quad \text{with } \frac{dF(r)}{dr} = \frac{r^{\frac{\gamma}{\beta}-1}e^{-\frac{r}{\beta}}}{\beta^{\frac{\gamma}{\beta}}\Gamma(\frac{\gamma}{\beta})},
$$

where  $\beta$ ,  $\gamma > 0$  are constants and  $\Gamma(\cdot)$  is the gamma function (see Tan et al. (2021, Section 2.1)). A direct calculation shows that

$$
\int_0^\infty r(1-e^{-rt})dF(r) = \int_0^\infty r(1-e^{-rt})\frac{r^{\frac{\gamma}{\beta}-1}e^{-\frac{r}{\beta}}}{\beta^{\frac{\gamma}{\beta}}\Gamma(\frac{\gamma}{\beta})}dr = \gamma - \gamma \frac{1}{(1+\beta t)^{\frac{\gamma}{\beta}+1}} \leq \gamma(\gamma+\beta)t \quad \forall t > 0,
$$

which implies Equation [\(21\)](#page-8-0).

The next lemma summarizes several preliminary properties of  $V(t, x, S)$ , which will be used to establish the main results in later sections.

**Lemma 2.14.** *Let Assumptions [2.1,](#page-5-0) 2.5, [2.8\(](#page-7-0)ii), [2.10](#page-8-0) hold and Then,*

d I a religion de la proporcional de la proporciona (a)  $V(t, x, S)$  belongs to  $C^{1,2}([0, \infty) \times \overline{S^c})$ , and  $V(t, x, S) = \delta(t) f(x)$  for any  $(t, x) \in [0, \infty) \times S$ . *Moreover,*

$$
\mathcal{L}V(t, x, S) \equiv 0, \quad \forall (t, x) \in [0, \infty) \times S^c. \tag{22}
$$

*(b)*  $\mathcal{L}V(t, x\pm, S)$  exists for all  $(t, x) \in [0, \infty) \times \mathbb{X}$ . For any  $h > 0$  and  $x_0 \in \mathbb{X}$  such that  $\overline{B(x_0, h)} \subset \mathbb{X}$ , *we have that*

$$
\sup_{(t,x)\in[0,\infty)\times\overline{B(x_0,h)}}|\mathcal{L}V(t,x\pm,S)|<\infty.
$$

The proof of Lemma 2.14 is provided in Appendix A. Throughout this paper, we will keep using the following local time integral formula provided in Peskir (2007).

**Lemma 2.15.** Let  $a, x_0, b \in \mathbb{R}$  with  $a < x_0 < b$ . Suppose  $g(t, y) : [0, \infty) \times \mathbb{R} \to \mathbb{R}$  such that  $g \in$  $C^{1,2}((0,\infty) \times (a,x_0])$ ,  $g \in C^{1,2}((0,\infty) \times [x_0,b))$ . Then, for  $X_0 = x \in (a,b)$ , we have that

$$
\delta(t) = \frac{1}{(1+\beta t)^{\frac{7}{2}}} = \int_{0}^{\infty} e^{-rt} \frac{r^{\frac{5}{2}-1}e^{-\frac{7}{\beta}}}{\beta^{\frac{7}{2}}\Gamma(\frac{7}{\beta})} dr = \int_{0}^{\infty} e^{-rt} dF(r), \text{ with } \frac{dF(r)}{dr} = \frac{r^{\frac{5}{2}-1}e^{-\frac{7}{\beta}}}{\beta^{\frac{7}{2}}\Gamma(\frac{7}{\beta})},
$$
  
\ntherefore  $\beta, \gamma > 0$  are constants and  $\Gamma(\cdot)$  is the gamma function (see Tan et al. (2021, Section 2.1)). A  
\nreirc calculation shows that  
\n
$$
\int_{0}^{\infty} r(1 - e^{-rt}) dF(r) = \int_{0}^{\infty} r(1 - e^{-rt}) \frac{r^{\frac{7}{2}-1}e^{-\frac{7}{\beta}}}{\beta^{\frac{7}{2}}\Gamma(\frac{7}{\beta})} dr = \gamma - \gamma \frac{1}{(1+\beta t)^{\frac{7}{\beta}+1}} \leq \gamma(\gamma+\beta)t \quad \forall t > 0,
$$
  
\nwhich implies Equation (21).  
\nThe next lemma summarizes several preliminary properties of  $V(t, x, S)$ , which will be used to  
\nstabilsht the main results in later sections.  
\n**emm. 2.14.** Let Assumptions 2.1, 2.5, 2.8(ii), 2.10 hold and *S* be an admissible stopping policy.  
\nthen,  
\nthen,  
\n
$$
V(t, x, S) \text{ belongs to } C^{1,2}([0, \infty) \times \overline{S^c}), \text{ and } V(t, x, S) = \delta(t)f(x) \text{ for any } (t, x) \in [0, \infty) \times S.
$$
  
\nMoreover,  
\n
$$
LV(t, x, S) = 0, \quad \forall (t, x) \in [0, \infty) \times S^c.
$$
  
\n
$$
LV(t, x, S) = 0, \quad \forall (t, x) \in [0, \infty) \times S^c.
$$
  
\n
$$
LV(t, x, S) = 0, \quad \forall (t, x) \in [0, \infty) \times S^c.
$$
  
\n
$$
TV(t, x \pm, S) = 0, \quad \forall (t, x) \
$$

<span id="page-9-0"></span>

# <span id="page-10-0"></span>**3 CHARACTERIZATION FOR WEAK EQUILIBRIA**

In this section, we provide the characterization for weak equilibria. Such characterization is critical to study of the relations between mild, weak, and strong equilibria. Below is the main result of this section.

**Theorem 3.1.** *Let Assumptions [2.1](#page-5-0)[–2.10](#page-8-0) hold. Suppose a weak equilibrium if and only if the followings are satisfied.*

$$
V(0, x, S) \ge f(x) \quad \forall x \notin S; \tag{23}
$$

$$
V_x(0, x-, S) \ge V_x(0, x+, S) \quad \forall x \in S; \tag{24}
$$

$$
\mathcal{L}V(0, x-, S) \vee \mathcal{L}V(0, x+, S) \le 0 \quad \forall x \in \mathbb{X}.\tag{25}
$$

The proof of Theorem 3.1 will be presented in the next subsection. A consequence of Theorem [3.11](#page-15-0) is the following smooth-fit condition of V at the boundary  $\partial S$  when f is smooth.

**Corollary 3.2** (Smooth-fit condition for weak equilibria when  $f$  is smooth). Let Assumptions 2.1– *[2.10](#page-8-0) hold, and let*

*Proof.* Take an arbitrary  $x \in \partial S$ . Take  $x \in \partial S$ . By Theorem 3.11, it suffices to prove that

Recall  $\mathcal G$  defined in Equation (16). For boundary case (a), without loss of generality, we assume  $(x, x + h) \subset (S^{\circ} \cap G)$  and  $(x - h, x) \subset S^{\circ}$  for some  $h > 0$ . Since  $V(0, x, S) \ge f(x)$  on  $S^{\circ}$  by Equation (5) and  $V(0, x, S) = f(x)$  on S by Lemma 2.14 (a), we have that for  $\varepsilon > 0$  small enough,

$$
\frac{V(0,x-\varepsilon,S)-V(0,x,S)}{\varepsilon}\geq \frac{f(x-\varepsilon)-f(x)}{\varepsilon}.
$$

By the differentiability of V on  $[0, \infty) \times \overline{S^c}$  (due to Lemma 2.14(a)) and existence of  $f'(x)$ , the above inequalities implies that

$$
V_x(0, x-, S) \le f'(x-) = f'(x+) = V_x(0, x+, S),
$$

where the last equality follows from  $V(0, x, S) = f(x)$  on  $(x, x + h) \subset (S^{\circ} \cap G)$ .

For boundary case (b), we can choose a constant  $h > 0$  such that  $(B(x, h) \setminus \{x\}) \subset (S^c \cap G)$ . Then

$$
V_x(0, x-, S) \le f'(x-) = f'(x+) \le V_x(0, x+, S)
$$

by an argument similar to that for boundary case (a).

*S* is an admissible stopping policy. Then *S* is<br> *ifted.*<br>  $\forall x \notin S$ ; (23)<br>  $S$   $\forall x \in S$ ; (24)<br>  $\leq 0 \forall x \in \mathbb{X}$ . (25)<br>
the next subsection. A consequence of<br>  $\forall$  at the boundary  $\partial S$  when  $f$  is smooth.<br>
ia when  $f$  $V(0, x, S) \ge f(x)$   $\forall x \notin S$ ; (23)<br>  $x-, S$   $\ge V_x(0, x+, S)$   $\forall x \in S$ ; (24)<br>  $-, S$   $\lor$   $\mathcal{L}V(0, x+, S) \le 0$   $\forall x \in \mathbb{X}$ . (25)<br>
1 be presented in the next subsection. A consequence of<br>
th-fit condition of  $V$  at the boundary  $\partial$  $V_x(0, x-, S) \ge V_x(0, x+, S)$   $\forall x \in S;$  (24)<br>
(0,  $x-, S$ )  $\vee$   $L V(0, x+, S) \le 0$   $\forall x \in X.$  (25)<br>
1 will be presented in the next subsection. A consequence of<br>
smooth-fit condition of  $V$  at the boundary  $\partial S$  when  $f$  is smooth.<br>  $V(0, x-, S) \vee CV(0, x+, S) \le 0$  V $x \in X$ . (25)<br>
1.1 will be presented in the next subsection. A consequence of<br>
3.1 will be presented in the next subsection. A consequence of<br>
3.5 smooth-fit condition of V at the boundary *BS* V at the boundary  $\partial S$  when  $f$  is smooth.<br>
ia when  $f$  is smooth). Let Assumptions 2<br>
uppose S is a weak equilibrium. Then for a<br>
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y Theorem 3.11, it suffices to prove that<br>
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y case (a), without loss of *f* is smooth). *Let Assumptions* 2.1-<br> *s* a weak equilibrium. Then for any<br>
m 3.11, it suffices to prove that<br>
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(), without loss of generality, we<br>
0. Since  $V(0, x, S) \ge f(x)$  on  $S^c$  by<br>
have that for  $\varepsilon > 0$  small e *b be an admissible stopping policy. Suppose <i>S* is a weak equilibrium. Then for any<br>
artists, then  $V_x(0, x-, S) = V_x(0, x+, S)$ .<br>
arbitrary  $x \in \partial S$ . Rex  $x \in \partial S$ . By Theorem 3.11, it suffices to prove that<br>
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li (*x*) *exists, then*  $V_x(0, x-, S) = V_x(0, x+, S)$ .<br>
e an arbitrary  $x \in \partial S$ . Take  $x \in \partial S$ . By<br>  $\leq V_x(0, x+, S)$  for both boundary cases (a)<br>
defined in Equation (16). For boundary cases (a)<br>  $t \cdot h$ )  $\subset$  (S°  $\cap$  *G*) and  $(x - h, x)$  $x \in \partial S$ . Take  $x \in \partial S$ . By Theorem 3.11, it suffices to prove that<br>for both boundary cases (a) and (b).<br>Introduction (16). For boundary case (a), without loss of generality, we<br>and  $(x - h, x) \subset S^c$  for some  $h > 0$ . Since  $V_x(0, x-, S) \le V_x(0, x+, S)$  for both boundary cases (a) and (b).<br>
Recall  $G$  defined in Equation (16). For boundary cases (a)<br>
assume  $(x, x + h) \subset (S^{\circ} \cap G)$  and  $(x - h, x) \subset S^{\circ}$  frome  $h > 0$ <br>
Equation (5) and  $V(0, x, S) = f(x)$  on ( $x, x + h$ ) C ( $S^{\circ} \cap G$ ) and  $(x - h, x) \in S^{\circ}$  for some  $h > 0$ . Since  $V(0, x, S) \ge f(x)$  on  $S^{\circ}$  by<br>  $f(5)$  and  $V(0, x, S) = f(x)$  on  $S$  by Lemma 2.14 (a), we have that for  $z > 0$  small enough,<br>  $\frac{V(0, x - \varepsilon, S) - V(0, x, S)}{\varepsilon}$  $V(0, x, S) = f(x)$  on *S* by Lemma 2.14 (a), we have that for  $\varepsilon > 0$  small enough,<br>  $\frac{V(0, x - \varepsilon, S) - V(0, x, S)}{\varepsilon} \ge \frac{f(x - \varepsilon) - f(x)}{\varepsilon}$ .<br>
bility of *V* on  $[0, \infty) \times \overline{S}^c$  (due to Lemma 2.14(a)) and existence of  $f'(x)$ ( $\sqrt{6}$ )  $\sqrt{6}$ )  $\sqrt{6}$  ( $V$  on  $[0, \infty) \times \overline{S^c}$  (due to  $\sinh(x)$ )  $\sqrt{x}(0, x-, S) \le f'(x-) = f'$ <br>
llows from  $V(0, x, S) = f(x)$ <br>  $\in B(x, h) \setminus \{x\}$ , which im  $V_x(0, x-, S) \le f'(x-) = f$ <br>
that for boundary case (a (2021), it is shown that ) ) vu e [-> ∨l wu si li a; ilt fi : . n ( $\frac{1}{\epsilon}$ ) =  $V_x(0, x+, S)$ <br>
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-fit cond V on  $[0, \infty) \times S^c$  (due to Lemma 2.14(a)) and existence of  $f'$ <br>that<br>(0,  $x-, S$ )  $\le f'(x-) = f'(x+) = V_x(0, x+, S)$ ,<br>ows from  $V(0, x, S) = f(x)$  on  $(x, x + h) \subset (S^{\circ} \cap G)$ .<br>e can choose a constant  $h > 0$  such that  $(B(x, h) \setminus \{x\}) \subset (S^c \cap g)$ (*x*), the<br>∂). Then<br>ometric<br>ak equi-<br>ooth-fit<br>sharper<br>tisfy the<br>th some  $V_x(0, x-, S) \le f'$ <br>
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see a constant  $h > 0$  such that  $(B(x, h) \setminus \{x\}$ <br>
{ $x$ }, which implies that<br>  $0 \le f'(x-) = f'(x+) \le V_x(0, x+, S)$ <br>
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shown that with the underlying process<br>
dition together w *h* > 0 such that (*B*(*x*, *h*) \{*x*}) ⊂ (*S<sup>c</sup>* ∩ *G*). Then<br>mplies that<br>*f'*(*x*+) ≤ *V<sub><i>x*</sub>(0, *x*+, *S*)<br>(a). □<br><br>with the underlying process being a geometric<br>er with some inequalities provides a weak equi-<br>Tan et  $V(0, y, S) \ge f(y)$  for all  $y \in B(x, h) \setminus \{x\}$ , which implies that<br>  $V_x(0, x-, S) \le f'(x-) = f'(x+) \le 1$ <br>
by an argument similar to that for boundary case (a).<br> *Remark* 3.3. In Tan et al. (2021), it is shown that with the u<br>
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ooth, any weak equ<br>
he smooth-fit condi □<br>ric ui-<br>fit ak<br>er he *Remark* 3.3. In Tan et al. (2021), it is shown that with the underlying process being a geometric Brownian motion, the smooth-fit condition together with some inequalities provides a weak equilibrium; in addition, the real options example in Tan et al. (2021) indicates that when smooth-fit condition fails, there is no weak equilibrium. This, however, does not indicate whether any weak equilibrium must satisfy the smooth-fit condition. Here, we are able to provide a much sharper result in a much more general setting: given  $f$  is smooth, any weak equilibrium must satisfy the  $f$  is smooth, any weak equilibrium must satisfy the smooth-fit condition together with some  $\theta$ smooth-fit condition, and may be constructed by the smooth-fit condition together with some

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other related inequalities. Let us also mention that smooth-fit result is also established in a very recent paper (Bodnariu et al., [2022\)](#page-39-0) for mixed weak equilibrium under a general setting.

*Remark* 3.4. In our paper, the payoff function  $f$  is only required to be piecewisely smooth. The inequality in Equation [\(24\)](#page-10-0) and Corollary [3.2](#page-10-0) show that the smooth-fit condition is a specially case of the "local convexity" property for a weak equilibrium  $S$ : the left derivative w.r.t. x of the value function  $V(0, x, S)$  must be bigger than or equal to its right derivative for any  $x \in S$ . In particular, if the payoff function is smooth at a point  $x \in S$ , such convexity property is reduced to the smooth-fit condition.

f is only required to be piecewisely smooth. The<br>show that the smooth-fit condition is a specially<br>behow that the smooth-fit condition is a specially<br>or equal to its right derivative for any  $x \in S$ . In<br>onit  $x \in S$ , such c S: the left derivative w.r.t. x of the<br>right derivative for any  $x \in S$ . In<br>right derivative for any  $x \in S$ . In<br>the current one-dimensional diffu-<br>variational inequalities. As is well<br>ssumptions) the optimal stopping<br>ties.  $V(0, x, S)$  must be bigger than or equal to its right derivative for *any*  $x \in S$ . In must be proportional diffuority from the proportion is monother to any *s* reduced to modifion.<br>
the proportion is expected in the case *x* ∈ *S*, such convexity property is reduced to<br>ential in the current one-dimensional diffu-<br>yield the variational inequalities. As is well<br>suitable assumptions) the optimal stopping<br>in such is an optimal stopping<br>in il *Remark* 3.5. Suppose the discount function is exponential in the current one-dimensional diffusion context. Then Equations [\(23\)](#page-10-0) and [\(25\)](#page-10-0) together yield the variational inequalities. As is well known in classical optimal stopping theory, (under suitable assumptions) the optimal stopping value and strategy can be characterized by variational inequalities. Therefore, when the discount function is exponential, Theorem [3.1](#page-10-0) indicates that any weak equilibrium is an optimal stopping region in the classical sense, so are strong and optimal mild equilibrium (as we will show later that an optimal mild equilibrium is also weak). On the other hand, a mild equilibrium is not necessarily a classical optimal stopping region, for example, the whole state space  $X$  is a mild equilibrium but may not be an optimal stopping region in general.

# **3.1 Proof of Theorem [3.1](#page-10-0)**

To characterize a weak equilibrium, one shall consider the two conditions (5) and (6) in Definition [1.3.](#page-2-0) Equation (5) is the same as Equation [\(23\)](#page-10-0) and thus we will focus on condition (6). By *V* defined in Equation (11), Equation (6) can be rewritten as

$$
\limsup_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \Big( \mathbb{E}^x [\delta(\rho_S^{\varepsilon}) f(X_{\rho_S^{\varepsilon}})] - f(x) \Big) = \limsup_{\varepsilon \searrow 0} \frac{1}{\varepsilon} (\mathbb{E}^x [V(\varepsilon, X_{\varepsilon}, S)] - V(0, x, S)) \le 0, \quad x \in S.
$$
\n(26)

 $\epsilon \searrow 0$ <br>
ace  $X_{\epsilon}$  and<br>
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imates  $\frac{1}{2}$  and  $\frac{1}{2}$  a  $\frac{1}{\varepsilon}$  and  $\frac{1}{\varepsilon}$  and  $\frac{1}{\varepsilon}$  in  $\frac{1}{\varepsilon}$  in  $\frac{1}{\varepsilon}$  in  $\frac{1}{\varepsilon}$  in  $\frac{1}{\varepsilon}$ n eenderstaanderstaanderstaanderstaanderstaanderstaanderstaanderstaanderstaanderstaanderstaanderstaanderstaand<br>D Since  $X_{\varepsilon}$  and thus  $V(\varepsilon, X_{\varepsilon}, S)$  are not uniformly bounded, we will apply some localization argument and restrict X within a bounded ball  $B(x, h)$ . Moreover, as  $x \mapsto V(t, \cdot, S)$  is only piecewisely smooth, we will choose  $h > 0$  small enough, such that  $V_x$  is only (possibly) discontinuous at the center of the ball  $B(x, h)$ , in order to apply Lemma 2.15 to V in Equation (26). By doing so, we will end up with

\n classical optimal stopping region, for example, the whole state space 
$$
\times
$$
 is a mild equilibrium *ay* not be an optimal stopping region in general.\n

\n\n**Proof of Theorem 3.1**\n

\n\n**Proof of Theorem 3.1**\n

\n\n In  $1.3$ . Equation (5) is the same as Equation (23) and thus we will focus on condition *V* defined in Equation (11), Equation (6) can be rewritten as\n 
$$
\sup_{x \searrow 0} \frac{1}{\varepsilon} \left( \mathbb{E}^x[\delta(\rho_{S}^{\varepsilon})f(X_{\rho_{S}^{\varepsilon}})] - f(x) \right) = \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \left( \mathbb{E}^x[V(\varepsilon, X_{\varepsilon}, S)] - V(0, x, S) \right) \leq 0, \quad x \in S.
$$
\n

\n\n A *z* and thus *V*(*z*, *X<sub>ε</sub>*, *S*) are not uniformly bounded, we will apply some localization argument-*st* with an bounded ball *B*(*x*, *h*). Moreover, as *x* → *V*(*t*, *, S*) is only piecewisely! and restrict *X* within a bounded ball *B*(*x*, *h*). Moreover, as *x* → *V*(*t*, *, S*) is only piecewisely! and restrict *X* with a bounded ball *B*(*x*, *h*). Moreover, as *x* → *V*(*t*, *, S*) is only piecewisely! and restrict *X* with *W<sub>x</sub>* is only (possibly) discontinuous at the of the ball *B*(*x*, *h*), in order to apply Lemma 2.15 to *V* in Equation (26). By doing so, we will put the  $\mathbb{E}^x[V(\varepsilon, X_{\varepsilon}, S) - V(0, x, S)] \approx \mathbb{E}^x[V(\varepsilon \wedge \tau_{B(x, h)}, X_{\varepsilon \wedge \tau_{B(x, h)}}) - V(0, x, S)]$ \n

\n\n =  $\mathbb{E}^x \left[$ 

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| 1<br>| 1<br>| 2 2 + ∟<sup>...</sup><br>on ir<br>uatio<br>egral<br>ed loo  $\frac{1}{2}$  E (er 1 2 *∫*<br>Equ (28<br>erm<br>ll tiı  $\frac{1}{\sinh(2)}$ <br>tion (2)<br>is due<br>on the<br> $e \mathbb{E}^x[L]$ - 122)<br>2020<br>2020 ( $v_x$ (s, x+, s) –  $v_x$ (s, x–, s)) $aL_s$ <br>
(7) will be made rigorous in Len<br>
to Lemma 2.15. Then the condi<br>
right-hand side (RHS) of Equa<br>  $\sum_{x \in \Lambda \tau_{B(x,h)}}$  (see Lemma 3.9, which is -<br>I<br>a<br>is (28)<br>ma 3.8, which is built<br>on (26) boils down to<br>on (28). This requires<br>wilt upon Lemmas 3.6 where the approximation in Equation (27) will be made rigorous in Lemma [3.8,](#page-13-0) which is built on Lemma [3.6,](#page-12-0) and Equation (28) is due to Lemma 2.15. Then the condition (26) boils down to comparing the two integral terms on the right-hand side (RHS) of Equation (28). This requires estimates for the expected local time  $\mathbb{E}^{x}[L_{\epsilon \wedge \tau_{R(x,b)}}^{x}]$  (see Lemma 3.9, which is built upon Lemmas 3.6  $\mathbb{E}^{\cdot\cdot}$   $\lfloor L_{\varepsilon} \rfloor$  $\sum_{\varepsilon \wedge \tau_{B(x,h)}}$ ] (see Lemma 3.9, which is built upon Lemmas [3.6](#page-12-0)

<span id="page-12-0"></span>and 3.7) and the growth of  $V_x(t, x \pm S)$  w.r.t. t (see Lemma 3.10). This is the overall idea on how we obtain Theorem [3.1.](#page-10-0)

Throughout this section, we shall also take advantage of the following standard estimate for moments of diffusions (see, e.g., Karatzas and Shreve [\(1991,](#page-40-0) Problem 3.15 on page 306)): Given a process  $Z_t$  satisfying  $dZ_t = \beta(Z_t)dt + \delta(Z_t)dW_t$  with  $\beta, \delta$  being Lipschitz and  $Z_0 = z \in \mathbb{X}$ , for all

$$
\mathbb{E}^{x} \left[ \sup_{0 \le s \le t} |Z_{s}|^{2m} \right] < \infty \quad \forall t \in (0, \infty), \tag{29}
$$

$$
\mathbb{E}^{x}\left[|Z_{\varepsilon}-z|^{2m}\right] \le K(1+|z|^{2m})\varepsilon^{m},\tag{30}
$$

where K is a constant independent of  $\varepsilon$ .

We first provide two Lemmas dealing with the probability of  $X$  exiting a ball, and the first-order moment related to  $X$  over a small time horizon  $\varepsilon$ . They will be used for proofs in both the current and later sections.

**Lemma 3.6.** *Let Assumption 2.1 hold. For any fixed*  $a > 0$  *we have that* 

$$
\mathbb{P}^x(\tau_{B(x,h)} \le \varepsilon) = o(\varepsilon^a), \quad \text{for } \varepsilon > 0 \text{ small enough.} \tag{31}
$$

*Proof.* Fix  $a > 0$ . We invoke the "change of space" method in Peskir and Shiryaev (2006, Section 5.2). Consider the process

$$
Y_t := \phi(X_t), Y_0 := \phi(x) \text{ with } \phi(y) := \int_0^y \exp\left(-\int_0^l \frac{2\mu(z)}{\sigma^2(z)} dz\right) dl. \tag{32}
$$

 $V_x(t, x\pm, S)$  w.r.t. *t* (see Lemma [3.10\)](#page-15-0). This is the overall idea on how<br>
we shall also take advantage of the following standar estimate forms,<br>  $\rho(z, z, t, z, t, \lambda)$  w.r.t.  $\beta(z, 2dt + \delta(z), dW_t$  with  $\beta, \delta$  being Lipschitz and L, satisfying  $dZ_i = \beta(Z_i)dt + \delta(Z_i)dW_i$  with  $\beta, \delta$  being Lipschitz and  $Z_n = x \in X$ , for all  $\text{and } m \ge 1$ , it holds that<br>  $\mathbb{E}^x \left[ \sum_{0 \le j \le k} |Z_j|^{2m} \right] < \infty \quad \forall t \in (0, \infty)$ , (29)<br>  $\mathbb{E}^x \left[ |Z_i - z|^{2m} \right] \le K(1 + |z|^{2m})e^m$ 0 ≤ ≤ 1 and ≥ 1, it holds that  $\begin{aligned} [Z_{\varepsilon} \text{ is } & 1 \leq \varepsilon \leq \varepsilon \end{aligned}$  is  $\leq \varepsilon$ ,  $\infty$  is  $\leq \varepsilon$ ,  $\infty$  is  $\leq \varepsilon$ ,  $\infty$  is  $\mathbb{R}$  is  $\geq \varepsilon$  in  $\mathbb{R}$  is  $\ge$  $0 \le s \le t$ <br>  $[1Z_{\varepsilon}$  t of ε<br>  $\iota$  t of ε<br>
aling<br>
time<br>  $old.$ <br>  $\leq \varepsilon$ )<br>
e "cl<br>  $\phi(x)$ <br>  $\geq$  ll-de<br>  $\phi(x)$ <br>  $\geq$  ll  $\phi(x)$ <br>  $\geq$   $\begin{vmatrix} 2 \ 1 \end{vmatrix}$  with  $\begin{vmatrix} 2 \ 1 \end{vmatrix}$  is a set of  $\begin{vmatrix} 2 \ 1 \end{vmatrix}$  of  $\begin{vmatrix} 2 \ 1 \end{vmatrix}$  and  $\begin{vmatrix} 2 \ 1 \end{vmatrix}$  a  $\begin{bmatrix} 2m \ 2m \end{bmatrix}$ <br>  $\begin{bmatrix} 2m \ 1 \end{bmatrix}$ <br>  $\begin{bmatrix} 2m \ k \end{bmatrix}$ <br>  $\begin{bmatrix}$ < ∞ *Vt* ∈ (0, ∞), (29)<br>
≤ K(1 + |z|<sup>2*m*</sup>)ε<sup>*m*</sup>, (30)<br>
e probability of X exiting a ball, and the first-order<br>
e. They will be used for proofs in both the current<br>
fixed a > 0 we have that<br>
for ε > 0 small enough. (31  $E^x$  [ent deall times]<br>all times and the set of the set of the set of the set of  $a$  -<br> $\left(\begin{array}{c} 1 & a \\ s & a \end{array}\right)$ <br>is the set of the set of  $a$  -<br> $\left(\begin{array}{c} 1 & a \\ s & a \end{array}\right)$ <br>is from  $1(i)$  with of ε.<br>
ling with the horical state of ε.<br>
ling with the horical state of  $\varepsilon$ .<br>  $\infty$  if  $\varepsilon$  is  $\varepsilon$  is  $\varepsilon$  of the difference of  $Y_t$  =<br>  $\infty$  if  $Y_t$  =<br>
in the sof Equatic  $\tilde{h}^{2\tilde{a}}$  of Equatic  $\tilde{h}^{2\tilde{$  $\begin{bmatrix} 2m \ 1 \end{bmatrix}$ <br>th tlizor<br> $\begin{bmatrix} 2m \ 2m \end{bmatrix}$ <br> $\begin{bmatrix} \epsilon^a \end{bmatrix}$ <br> $\begin{bmatrix} e \\ e \end{bmatrix}$ <br> $\begin{bmatrix} e \\ e \end{bmatrix}$ <br> $\begin{bmatrix} e \\ e \end{bmatrix}$ <br> $\begin{bmatrix} e \\ -e \end{bmatrix}$ <br> $\begin{bmatrix} 1 \end{bmatrix}$ <br> $\begin{bmatrix} 1 \end{bmatrix}$ <br> $\begin{bmatrix} 1 \end{bmatrix}$ <br> $\begin{bmatrix} 1 \end{bmatrix}$ <br> $\begin{bmatrix}$ probabilit<br>
orobabilit<br>
They will<br>
ed a > 0 1<br>
for  $\varepsilon$  > 0<br>
space" me<br>  $\therefore$  =  $\int_0^y e^x$ <br>
ctly increasily<br>  $\phi'(X_t) dV$ <br>  $(x - h), \phi$ <br>  $\leq 1$ . We h<br>  $\tilde{h}$  =  $\mathbb{P}^1$ <br>
al  $\overline{B(\phi(x))}$ <br>  $\frac{1}{\tilde{R}^{2\tilde{a}}}$ <br>
30), and <sup>2*m*</sup>)<sub>8</sub>*m*, (30)<br>
y of *X* exiting a ball, and the first-order<br>
be used for proofs in both the current<br>
we have that<br>
small enough. (31)<br>
ethod in Peskir and Shiryaev (2006,<br>
xp( $-\int_0^l \frac{2\mu(z)}{\sigma^2(z)}dz$ )dl. (32)<br>
ssing, *K* is a constant independent of *ε*.<br>
irst provide two Lemmas dealing<br>
th related to *X* over a small time l<br>
er sections.<br> **ia 3.6.** *Let Assumption 2.1 hold. F*<br>  $\mathbb{P}^x(\tau_{B(x,h)} \leq \varepsilon)$ :<br>
Fix *a* > 0. We invoke the X exiting a ball, and the first-order<br>
used for proofs in both the current<br>
ave that<br>
all enough. (31)<br>
d in Peskir and Shiryaev (2006,<br>  $\left(-\int_0^l \frac{2\mu(z)}{\sigma^2(z)}dz\right)dl.$  (32)<br>
g, and has first and second deriva-<br>
and the e X over a small time horizon *ε*. They will be used for proofs in both the current<br>
ssumption 2.1 hold. For any fixed  $a > 0$  we have that<br>  $\mathbb{P}^x(\tau_{B(x,h)} \leq \epsilon) = o(\epsilon^a)$ , for  $\epsilon > 0$  small enough. (31)<br>
We invoke the "chan *a* > 0 *we have that*<br> *r*  $\varepsilon$  > 0 *small enoug*<br>
ce" method in Pe<br>  $\int_0^y \exp\left(-\int_0^l \frac{2}{c} dt\right) dt$ <br> *i* increasing, and h<br>  $(X_t)dW_t$ , and the  $-h$ ,  $\phi(x + h)$ . Set<br>  $\therefore$  We have that<br>  $\int = \mathbb{P}^{Y_0}\left(\sup_{0 \le t \le \varepsilon} |Y_t\right) dt$  $\mathbb{P}^x(\tau_{B(x,h)} \leq \varepsilon) = o(\varepsilon^a)$ , *for*  $\varepsilon > 0$  *small enough.* (31)<br>
invoke the "change of space" method in Peskir and Shiryaev (2006,<br>
e process<br>  $\zeta_i$ ,  $Y_0 := \phi(x)$  with  $\phi(y) := \int_0^y \exp\left(-\int_0^1 \frac{2\mu(z)}{\sigma^2(z)} dz\right) dl$ .  $\alpha > 0$ . We invoke the "change of space" method in Peskir and Shiryaev [\(2006,](#page-40-0)<br>
Consider the process<br>  $Y_t := \phi(X_t), Y_0 := \phi(x)$  with  $\phi(y) := \int_0^y \exp\left(-\int_0^t \frac{2\mu(z)}{\sigma^2(z)}dz\right)dl$ . (32)<br>
ssumption 2.1,  $\phi$  is well-defined, strictly  $Y_t := \phi(X_t), Y_0 := \phi(x)$  with  $\phi(y) := \int$ <br>sumption 2.1,  $\phi$  is well-defined, strictly in<br>calculation shows that  $dY_t = \sigma(X_t)\phi'(X)$ <br>the exit time of  $Y_t$  to the interval  $[\phi(x - k)!) > 0$  and  $\tilde{\alpha} := \alpha + 1$ . Let  $0 < \varepsilon \le 1$ . W<br> $B(x, h) \le \$   $\begin{bmatrix} 0 & t \\ 0 & -t \\ 0 & 0 \end{bmatrix}$   $\begin{bmatrix} 0 & -t \\ -t & 0 \\ 0 & 0 \end{bmatrix}$   $\begin{bmatrix} 0 & 0 & t \\ -t & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$  $\exp\left(\frac{\text{easing}}{W_t, \text{ and }W_t} \right)$ <br>  $\phi(x + \text{have }t)$ <br>  $\exp\left(\frac{s}{\sigma^2}\right)$ <br>  $\hat{K}$  is a  $\hat{K}$  is a  $\hat{K}$  is a  $\theta$ <br>  $\exp\left(\frac{s}{\sigma^2}\right)$ <br>  $\hat{K}$  is a  $\theta$ <br>  $\phi(x)W_t$ − *]*<br>
, and t *h*)].<br>
hat up<br>
st≤t≤ε<br>
, wde t <br>
≤  $\frac{\hat{K}}{-}$  pos<br>
the *we*  0  $\frac{P(2)}{\sigma^2(z)}$ <br>has fir exit t<br>exit t<br>fi :=<br> $t - Y_0$ <br>an th<br> $\frac{1}{\tilde{h}^{2\tilde{a}}}$ <br>we conct tha<br> $\vec{R}_t$  :=  $\left(\frac{2\mu(z)}{\sigma^2(z)}dz\right)$ <br>
and first and exit time to<br>
exit time to<br>
the time to<br>  $\left(\frac{1}{\tilde{h}}\right)^{2\tilde{a}} \geq$ <br>
and then a<br>
to<br>
the then a<br>  $\frac{1}{\tilde{h}^{2\tilde{a}}}$ <br>
we constand<br>
that  $\tilde{a} >$ <br>
we that<br>  $\tilde{c}_t := X_t$ dl. (32)<br>
d second deriva-<br>  $\overline{B(x, h)}$  of  $X_t$  is<br>  $x + h$ ) –  $\phi(x)$ )  $\wedge$ <br>  $\geq \tilde{h}^{2\tilde{a}}$ <br>  $\Rightarrow$   $\tilde{h}^{2\tilde{a}}$ <br>  $\Rightarrow$   $\tilde{h}^{2\tilde{a}}$ <br>
pply the Doob's<br>  $\overline{\tilde{x}}_t^{\tilde{a}}$ , (34)<br>
t independent of<br> *a*.  $\overline{X}_t$ . Thanks to Assumption [2.1,](#page-5-0)  $\phi$  is well-defined, strictly increasing, and has first and second deriva- $\phi$  is well-defined, strictly increasing, and has first and <u>second</u> deriva-<br>hows that  $dY_t = \sigma(X_t)\phi'(X_t)dW_t$ , and the exit time to  $\overline{B(x, h)}$  of  $X_t$  is<br> $Y_t$  to the interval  $[\phi(x - h), \phi(x + h)]$ . Set  $\overline{h} := (\phi(x + h) - \phi(x)) \wedge$ <br> $\overline$ tives. A direct calculation shows that  $dY_t = \sigma(X_t) \phi'(X_t) dW_t$ , and the exit time to  $B(x, h)$  of  $X_t$  is  $ar_t = o(\lambda_t) \varphi$ <br>
interval  $[\phi(x - t) \leq \lambda_t]$ <br>  $[Y_t - Y_0] \geq \tilde{h}$ <br>  $|Y_t - Y_0| \geq \tilde{h}$ <br>  $\tilde{h}$ <br>  $(X_t)dW_t$ , and the exit time to  $B(x, h)$  of  $X_t$  is<br>  $-h$ ,  $\phi(x + h)$ . Set  $\tilde{h} := (\phi(x + h) - \phi(x))$   $\wedge$ <br>  $\therefore$  We have that<br>  $\Big) = \mathbb{P}^{Y_0}\Big(\sup_{0 \le t \le \varepsilon} |Y_t - Y_0|^{2\tilde{a}} \ge \tilde{h}^{2\tilde{a}}\Big).$  (33)<br>  $\frac{\overline{B(\phi(x), \tilde{h})}}{\overline{B(2\tilde{$ equivalent to the exit time of  $Y_t$  to the interval  $[\phi(x-h), \phi(x+h)]$ . Set  $\tilde{h} := (\phi(x+h) - \phi(x)) \wedge$ 

equivalent to the exit time of 
$$
Y_t
$$
 to the interval  $[\phi(x - h), \phi(x + h)]$ . Set  $h := (\phi(x + h) - \phi(x)) \land$   
\n $(\phi(x) - \phi(x - h)) > 0$  and  $\tilde{a} := a + 1$ . Let  $0 < \varepsilon \le 1$ . We have that  
\n
$$
\mathbb{P}^X(\tau_{B(x, h)} \le \varepsilon) \le \mathbb{P}^{Y_0} \left( \sup_{0 \le t \le \varepsilon} |Y_t - Y_0| \ge \tilde{h} \right) = \mathbb{P}^{Y_0} \left( \sup_{0 \le t \le \varepsilon} |Y_t - Y_0|^{2\tilde{a}} \ge \tilde{h}^{2\tilde{a}} \right).
$$
\n(33)  
\nNotice that Y is a martigale (within the interval  $\overline{B}(\phi(x), \tilde{h})$ ), we can then apply the Doob's  
\nsubmartingale inequality to the RHS of Equation (33) to conclude that  
\n
$$
\mathbb{P}^{Y_0} \left( \sup_{0 \le t \le \varepsilon} |Y_t - Y_0|^{2\tilde{a}} \ge \tilde{h}^{2\tilde{a}} \right) \le \frac{\mathbb{E}^{Y_0}[|Y_{\varepsilon} - Y_0|^{2\tilde{a}}]}{\tilde{h}^{2\tilde{a}}} \le \frac{\tilde{K}(1 + \phi^{2\tilde{a}}(x))\varepsilon^{\tilde{a}}}{\tilde{h}^{2\tilde{a}}},
$$
\n(34)  
\nwhere the last inequality follows from Equation (30), and  $\tilde{K}$  is a positive constant independent of  
\n $\varepsilon$ . Then Equation (31) follows from Equations (33) and (34) and the fact that  $\tilde{a} > a$ .  
\n**Lemma 3.7.** Let Assumption 2.1(i) hold. For  $\varepsilon > 0$  small enough we have that  
\n
$$
\mathbb{E}^{\chi}[|\tilde{X}_{\varepsilon}|] = O(\varepsilon), \quad \text{with } \tilde{X}_t := x + \mu(x)t + \sigma(x)W_t \text{ and } \tilde{X}_t := X_t - \tilde{X}_t.
$$
\n(35)

Notice that Y is a martigale (within the interval  $B(\phi(x), \tilde{h})$ ), we can then apply the Doob's submartingale inequality to the RHS of Equation (33) to conclude that

$$
x(\tau_{B(x,h)} \leq \varepsilon) \leq \mathbb{P}^{Y_0} \Big( \sup_{0 \leq t \leq \varepsilon} |Y_t - Y_0| \geq \tilde{h} \Big) = \mathbb{P}^{Y_0} \Big( \sup_{0 \leq t \leq \varepsilon} |Y_t - Y_0|^{2\tilde{a}} \geq \tilde{h}^{2\tilde{a}} \Big).
$$
 (33)  
at *Y* is a martigale (within the interval  $\overline{B(\phi(x), \tilde{h})}$ ), we can then apply the Doob's  
gale inequality to the RHS of Equation (33) to conclude that  

$$
\mathbb{P}^{Y_0} \Big( \sup_{0 \leq t \leq \varepsilon} |Y_t - Y_0|^{2\tilde{a}} \geq \tilde{h}^{2\tilde{a}} \Big) \leq \frac{\mathbb{E}^{Y_0} \Big[ |Y_{\varepsilon} - Y_0|^{2\tilde{a}} \Big]}{\tilde{h}^{2\tilde{a}}} \leq \frac{\tilde{K}(1 + \phi^{2\tilde{a}}(x))\varepsilon^{\tilde{a}}}{\tilde{h}^{2\tilde{a}}},
$$
 (34)  
as  
at inequality follows from Equation (30), and  $\tilde{K}$  is a positive constant independent of  
uation (31) follows from Equations (33) and (34) and the fact that  $\tilde{a} > a$ .  $\square$   
**7.** Let Assumption 2.1(i) hold. For  $\varepsilon > 0$  small enough we have that  

$$
\mathbb{E}^{x}[|\tilde{X}_{\varepsilon}|] = O(\varepsilon), \quad \text{with } \tilde{X}_t := x + \mu(x)t + \sigma(x)W_t \text{ and } \tilde{X}_t := X_t - \tilde{X}_t.
$$
 (35)

ast i<br>aatic<br>7.  $L \to \mathbb{R}^x$  $\sum_{\substack{\epsilon \leq t \leq \epsilon}}$ <br>qua<br> $(31)$ <br> $\sum_{\substack{\epsilon \leq t \leq t}}$ 0≤t ≤ε<br>equal (31)<br>Assu<br> $\bar{X}_{\varepsilon}$ |]  $T_t = T_0$ <br>ty follow<br>ollows from the propriation 2.<br> $= O(\varepsilon)$ ,  $^{2\tilde{a}} \ge$ <br>s fro<br>om I<br> $l(i)$  k<br>with n E<br>n E<br>qua<br>*old.*<br> $\widetilde{X}_t$  $\frac{d^2 - 11^2 \epsilon - 101}{\tilde{h}^{2\tilde{a}}}$ <br>on (30), and  $\tilde{R}$ <br>(33) and (34)<br>: > 0 small end<br>+  $\mu(x)t + \sigma(x)$ 2̃ --<br>。<br>8<br>И  $\frac{\varepsilon - I_0 \|B\|}{\tilde{h}^{2\tilde{a}}} \leq \frac{K(1 + \varphi)}{\tilde{h}^{2\tilde{a}}}$ <br>
b, and  $\tilde{K}$  is a positive cornal (34) and the fact that<br>
mall enough we have that<br>  $\int (t + \sigma(x)) W_t$  and  $\tilde{X}_t :=$  $\text{M}(\vec{x})$ <br>  $\mathbf{X}(\vec{a}) > c$ <br>  $\mathbf{X}_t - \mathbf{X}_t$  $\frac{1}{\tilde{h}^{2\tilde{a}}}$ , (34)<br>
constant independent of<br>
that  $\tilde{a} > a$ .  $\square$ <br>
that<br>  $:= X_t - \tilde{X}_t$ . (35) where the last inequality follows from Equation (30), and  $\tilde{K}$  is a positive constant independent of  $\hat{K}$  is a positive constant independent of<br>
) and the fact that  $\tilde{a} > a$ .  $\Box$ <br>
nough we have that<br>  $(x)W_t$  and  $\bar{X}_t := X_t - \tilde{X}_t$ . (35)

**Lemma 3.7.** Let Assumption 2.1(i) hold. For  $\varepsilon > 0$  small enough we have that

\n
$$
\varepsilon
$$
. Then Equation (31) follows from Equations (33) and (34) and the fact that  $\tilde{a} > a$ .\n

\n\n**Lemma 3.7.** Let Assumption 2.1(i) hold. For  $\varepsilon > 0$  small enough we have that\n

\n\n $\mathbb{E}^x[|\bar{X}_{\varepsilon}|] = O(\varepsilon)$ , with  $\tilde{X}_t := x + \mu(x)t + \sigma(x)W_t$  and  $\tilde{X}_t := X_t - \tilde{X}_t$ .\n

\n\n(35)\n

*Proof.* Throughout the proof, C will serve as a generic constant may change from line to line but is independent of  $\varepsilon$ . Let  $0 < \varepsilon \leq 1$ . First, we have

$$
\mathbb{E}^x|\bar{X}_{\varepsilon}| \leq \mathbb{E}^x \left| \int_0^{\varepsilon} (\mu(X_t) - \mu(x)) dt \right| + \mathbb{E}^x \left| \int_0^{\varepsilon} (\sigma(X_t) - \sigma(x)) dW_t \right|.
$$
 (36)

By applying Equation [\(30\)](#page-12-0) on  $X_t$  with  $m = 1$ , we have  $\mathbb{E}^{X}[|X_{\varepsilon} - x|^2] \leq C \varepsilon$ . This together with the Lipschitz continuity of  $\mu$  implies

$$
\mathbb{E}^{x} \left| \int_{0}^{\varepsilon} (\mu(X_{t}) - \mu(x)) dt \right| \leq \mathbb{E}^{x} \left[ \int_{0}^{\varepsilon} \frac{1}{2} (1 + |\mu(X_{t}) - \mu(x)|^{2}) dt \right]
$$
  

$$
\leq \frac{1}{2} \varepsilon + \frac{1}{2} \int_{0}^{\varepsilon} C \mathbb{E}^{x} [X_{t} - x|^{2}] dt = O(\varepsilon).
$$
 (37)

Similarly, we can estimate the second term in Equation (36) as follows:

$$
\mathbb{E}\left| \int_{0}^{R(X_{t})} \frac{\mu(x_{t})}{\mu(x_{t})} dx \right| \leq \mathbb{E}\left[ \int_{0}^{\frac{\pi}{2}} \frac{1}{2} \int_{0}^{\varepsilon} \mathbb{C}[\mathbb{E}^{X}[\|X_{t}-x|^{2}] dt = O(\varepsilon). \right]
$$
\n
$$
\leq \frac{1}{2} \varepsilon + \frac{1}{2} \int_{0}^{\varepsilon} \mathbb{C}[\mathbb{E}^{X}[\|X_{t}-x|^{2}] dt = O(\varepsilon).
$$
\nIn estimate the second term in Equation (36) as follows:\n
$$
\mathbb{E}^{X} \left| \int_{0}^{\varepsilon} (\sigma(X_{t}) - \sigma(x)) dW_{t} \right| \leq \left( \mathbb{E}^{X} \left[ \int_{0}^{\varepsilon} (\sigma(X_{t}) - \sigma(x))^{2} dt \right] \right)^{1/2} \leq \left( \int_{0}^{\varepsilon} \mathbb{E}[\mathbb{E}^{X}[\|X_{t}-x|^{2}] dt] \right)^{1/2} = O(\varepsilon).
$$
\n
$$
\text{Im} \text{Equations (37) and (38) into Equation (36), we have } \mathbb{E}^{X}[\|\bar{X}_{\varepsilon}\|] = O(\varepsilon). \qquad \Box
$$
\n
$$
\text{Im} \text{a concerns the approximation in Equation (27).}
$$
\n
$$
\text{Im} \text{an} \text{concerns the approximation in Equation (27).}
$$
\n
$$
\text{Im} \text{Ansumptions 2.1 and 2.8(i) hold. Let } S \in B, x \in X \text{ and } h > 0. Then, for \varepsilon > 0 \text{ small}
$$
\n
$$
\mathbb{E}^{X}[V(\varepsilon, X_{\varepsilon}, S)] = \mathbb{E}^{X}[V(\varepsilon \wedge \tau_{B(x,h)}, X_{\varepsilon \wedge \tau_{B(x,h)}}, S)] + o(\varepsilon). \tag{39}
$$
\n
$$
\text{On all } x \in X. \text{ Recall the constant } \zeta \text{ in Equation (15). We have that}
$$
\n
$$
0 \leq \mathbb{E}^{X}[V(\varepsilon, X_{\varepsilon}, S)] \frac{1}{1+\zeta} \cdot \left( \mathbb{
$$

Then, by plugging Equations (37) and (38) into Equation (36), we have  $\mathbb{E}^{x}[|\bar{X}_{\varepsilon}|] = O(\varepsilon)$ .

The next lemma concerns the approximation in Equation [\(27\)](#page-11-0).

**Lemma 3.8.** *Let Assumptions* 2.1 *and* 2.8(*i*) *hold. Let*  $S \in B$ ,  $x \in \mathbb{X}$  *and*  $h > 0$ . *Then, for*  $\varepsilon > 0$  *small enough,*

$$
\mathbb{E}^{x}[V(\varepsilon, X_{\varepsilon}, S)] = \mathbb{E}^{x}[V(\varepsilon \wedge \tau_{B(x,h)}, X_{\varepsilon \wedge \tau_{B(x,h)}}, S)] + o(\varepsilon). \tag{39}
$$

*Proof.* Let  $h > 0$  and  $x \in \mathbb{X}$ . Recall the constant  $\zeta$  in Equation (15). We have that

equation of the proof, C will serve as a generic constant may change from line to line but  
\n
$$
E^x | \mathcal{R}_{\varepsilon} | \leq E^x \left| \int_0^{\varepsilon} (\mu(X_t) - \mu(x)) dt \right| + E^x \left| \int_0^{\varepsilon} (\sigma(X_t) - \sigma(x)) dW_t \right|.
$$
\n(36)

\nEquation (30) on *X\_t* with *m* = 1, we have 
$$
E^x \left[ |X_{\varepsilon} - x|^2 \right] \leq C\varepsilon.
$$
 This together with the  
\n*n* continuity of *μ* implies

\n
$$
E^x \left| \int_0^{\varepsilon} (\mu(X_t) - \mu(x)) dt \right| \leq E^x \left[ \int_0^{\varepsilon} \frac{1}{2} (1 + |\mu(X_t) - \mu(x)|^2) dt \right]
$$
\n
$$
\leq \frac{1}{2} \varepsilon + \frac{1}{2} \int_0^{\varepsilon} C E^x \left[ |X_t - x|^2 \right] dt = O(\varepsilon).
$$
\nwe can estimate the second term in Equation (36) as follows:

\n
$$
E^x \left| \int_0^{\varepsilon} (\sigma(X_t) - \sigma(x)) dW_t \right| \leq \left( E^x \left[ \int_0^{\varepsilon} (\sigma(X_t) - \sigma(x))^2 dt \right] \right)^{1/2} = O(\varepsilon).
$$
\nIt is to assume that the second term in Equation (36), we have 
$$
E^x \left[ |X_{\varepsilon}| \right] = O(\varepsilon).
$$

\nIt is that 
$$
E^x \left| \int_0^{\varepsilon} (\sigma(X_t) - \sigma(x)) dW_t \right| \leq \left( E^x \left[ \int_0^{\varepsilon} (\sigma(X_t) - \sigma(x))^2 dt \right] \right)^{1/2} = O(\varepsilon).
$$

\nIt is that 
$$
E^x \left[ |X_{\varepsilon}(X_t, S)| \right] = E^x \left[ |Y(\varepsilon \wedge \tau_{B(x, h)}, X_{\varepsilon, x_{B(x, h)}}) | S| \right] + o(\varepsilon).
$$

\nBut 
$$
E^x \left[ |Y(\varepsilon, X_{\varepsilon}, S)| = E^x \left[ |Y
$$

 $O(1) \cdot ($ <br>y follow<br>follow:  $\mathbb{P}^{x}$  (<br>s frc<br>s frc  $\tau_{B(x,h)} \leq \varepsilon$ <br>om  $f \geq 0$ , t<br>om Jensen  $\frac{3}{4}$  $1$  se  $\frac{1}{10}$ where the first inequality follows from  $f \ge 0$ , the second inequality follows from Hölder's inequal $f \ge 0$ , the second inequality follows from Hölder's inequal-<br>Jensen's inequality, and the last inequality follows from ity, the third inequality follows from Jensen's inequality, and the last inequality follows from

<span id="page-13-0"></span>

<span id="page-14-0"></span>Equation [\(15\)](#page-7-0). Applying Lemma [3.6](#page-12-0) with  $a = \frac{1+\zeta}{\zeta}$  to Equation (40), we have

$$
E^{x}\Big[V(\varepsilon, X_{\varepsilon}, S) \cdot 1_{\{\varepsilon > \tau_{B(x,h)}\}}\Big] = o(\varepsilon).
$$
\n(41)

Similarly, we can show that

$$
\mathbb{E}^{x}\bigg[V(\varepsilon \wedge \tau_{B(x,h)}, X_{\varepsilon \wedge \tau_{B(x,h)}}, S) \cdot 1_{\{\varepsilon > \tau_{B(x,h)}\}}\bigg] = o(\varepsilon).
$$

This together with Equation (41) implies Equation [\(39\)](#page-13-0).

Recall that  $L_t^x$  is the local time of X at position x up to time t. We have the following result.

**Lemma 3.9.** *Let Assumption [2.1](#page-5-0) hold. Then, for any*  $x \in \mathbb{X}$  *and*  $h > 0$ *,* 

$$
\lim_{\varepsilon \searrow 0} \frac{\mathbb{E}^x \left[ L_{\varepsilon \wedge \tau_{B(x,h)}}^x \right]}{\sqrt{\varepsilon}} = \sqrt{\frac{2}{\pi}} \cdot |\sigma(x)|. \tag{42}
$$

*Proof.* Let  $h > 0$  and  $x \in \mathbb{X}$ . Thanks to Assumption 2.1(i) and (29) (with  $m = \frac{p+2}{2} > 1$ ), it holds for any  $p, t > 0$  that

$$
\mathbb{E}^x\left[\sup_{0\leq s\leq t}|X_s|^p\right]\leq 1+\mathbb{E}^x\left[\sup_{0\leq s\leq t}|X_s|^{p+2}\right]<\infty.
$$

This enables us to apply an argument similar to the proof of Lemma 3.8 and get that

$$
\mathbb{E}^{x}[|X_{\varepsilon}-x|] + o(\varepsilon) = \mathbb{E}^{x}[|X_{\varepsilon \wedge \tau_{B(x,h)}} - x|]. \tag{43}
$$

Applying Lemma [2.15](#page-9-0) on  $[0, \varepsilon \wedge \tau_{B(x,h)}]$  with  $g(t, y) := |y - x|$  and then taking expectation, and using Equation (43), we have that

$$
\mathbb{E}^{x}[|X_{\varepsilon}-x|] + o(\varepsilon) = \mathbb{E}^{x}\bigg[\int_{0}^{\varepsilon \wedge \tau_{B(x,h)}} \text{sgn}(X_{s}-x)\mu(X_{s})ds\bigg] + \mathbb{E}^{x}\bigg[L_{\varepsilon \wedge \tau_{B(x,h)}}^{x}\bigg].\tag{44}
$$

By Assumption [2.1\(](#page-5-0)i), the first term on the RHS of Equation (44) can be estimated as follows:

$$
\left| \mathbb{E}^{x} \left[ \int_{0}^{\varepsilon \wedge \tau_{B(x,h)}} sgn(X_{s} - x) \mu(X_{s}) ds \right] \right| \leq \sup_{y \in \overline{B(x,h)}} |\mu(y)| \varepsilon = O(\varepsilon). \tag{45}
$$

As for the left-hand side (LHS) of Equation (44), by Lemma 3.7 we have that

quation (15). Applying Lemma 3.6 with 
$$
\alpha = \frac{z_1 z_1}{\zeta}
$$
 to Equation (40), we have  
\n
$$
\mathbb{E}^{\kappa} \Big[ V(\varepsilon, X_{\varepsilon}, S) \cdot 1_{[\varepsilon > \tau_{B(x,h)}]} \Big] = o(\varepsilon). \tag{41}
$$
\nmilary, we can show that  
\n
$$
\mathbb{E}^{\kappa} \Big[ V(\varepsilon \wedge \tau_{B(x,h)}, X_{\varepsilon \wedge \tau_{B(x,h)}} S) \cdot 1_{\{\varepsilon > \tau_{B(x,h)}\}} \Big] = o(\varepsilon). \tag{41}
$$
\nmilary, we can show that  
\n
$$
\mathbb{E}^{\kappa} \Big[ V(\varepsilon \wedge \tau_{B(x,h)}, X_{\varepsilon \wedge \tau_{B(x,h)}} S) \cdot 1_{\{\varepsilon > \tau_{B(x,h)}\}} \Big] = o(\varepsilon).
$$
\nRecall that  $L_i^{\kappa}$  is the local time of X at position x up to time t. We have the following result.  
\n**lemma 3.9.** Let Assumption 2.1 hold. Then, for any  $x \in \mathbb{X}$  and  $h > 0$ ,  
\n
$$
\lim_{\varepsilon \to 0} \frac{\mathbb{E}^{\kappa} \Big[ L_{\varepsilon \wedge \tau_{B(x,h)}}^{\kappa} \Big]}{\sqrt{\varepsilon}} = \sqrt{\frac{2}{\pi}} \cdot |\sigma(x)|. \tag{42}
$$
\n
$$
\text{conf. Let } h > 0 \text{ and } x \in \mathbb{X}. \text{ Thanks to Assumption 2.1(i) and (29) (with } m = \frac{p+2}{2} > 1), \text{ it holds}
$$
\n
$$
\text{r any } p, t > 0 \text{ that}
$$
\n
$$
\mathbb{E}^{\kappa} \Big[ \sup_{0 \le s \le t} |X_s|^p \Big] \le 1 + \mathbb{E}^{\kappa} \Big[ \sup_{0 \le s \le t} |X_s|^{p+2} \Big] < \infty.
$$
\n
$$
\text{his enables us to apply an argument similar to the proof of Lemma 3.8 and get that}
$$
\n
$$
\mathbb{E}^{\kappa}[|X_{\varepsilon} - x|] + o(\varepsilon) = \mathbb{E}^{\kappa} \Big[ |X_{\
$$

 $\frac{1}{2}$ Then Equation (42) follows from plugging Equations (45) and (46) into Equation (44).

 $\Box$ 

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**Lemma 3.10.** *Let Assumptions* [2.1,](#page-5-0) [2.5,](#page-7-0) [2.10](#page-8-0) *hold. Let*  $S \in B$  *and*  $x \in \partial S$ *, and suppose*  $(x - h, x) \subset$ 

$$
V_x(t, x-, S) \le \delta(t) V_x(0, x-, S) \quad (resp. \ V_x(t, x+, S) \ge \delta(t) V_x(0, x+, S)). \tag{47}
$$

*Proof.* Notice that Assumption [2.10](#page-8-0) gives the existence of  $V_x(t, y-, S)$  for  $y \in (x - h, x]$  (resp.

$$
V(t, y, S) = \mathbb{E}^y[\delta(t + \rho_S)f(X_{\rho_S})] \ge \delta(t)\mathbb{E}^y[\delta(\rho_S)f(X_{\rho_S})] = \delta(t)V(0, y, S).
$$

Suppose  $(x - h, x) \subset S^c$ . Then by the fact that  $V(t, x, S) = \delta(t) f(x)$  (due to Lemma 2.14 (a)) and the above inequality, we have that

$$
V_x(t, x-, S) = \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} (V(t, x, S) - V(t, x - \varepsilon, S))
$$
  

$$
\leq \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} (\delta(t) (f(x) - V(0, x - \varepsilon, S))) = \delta(t) V_x(0, x-, S).
$$

Similar argument is applied for the result of  $V_x(t, x+0)$ , when  $(x, x + h) \subset S^c$ .

Lemmas 3.9 and 3.10 together indicate that, as long as  $V_r(0, x+, S) - V_r(0, x-, S) \neq 0$ , the local time integral is the dominating term on the RHS of Equation [\(28\)](#page-11-0). Thus, to make the LHS of Equation [\(28\)](#page-11-0) nonpositive in the limit,  $V_x(0, x+, S) - V_x(0, x-, S)$  shall be nonpositive. Based on this and recalling Equation (26), we now prove the necessary conditions for a weak equilibrium in the following proposition. The sufficiency part follows next.

**Proposition 3.11.** *Let Assumptions 2.1–2.10 hold. Suppose a weak equilibrium, then*

$$
\begin{cases} V_x(0, x+S) \le V_x(0, x-S) & \forall x \in S, \\ \mathcal{L}V(0, x+S) \vee \mathcal{L}V(0, x-S) \le 0 & \forall x \in \mathbb{X}. \end{cases}
$$
(48)

*Proof.* We verify the first inequality in Equation (48) by contradiction. Take  $x \in S$  and suppose

$$
a := V_x(0, x + 0, S) - V_x(0, x - 0, S) > 0. \tag{49}
$$

Recall G defined in Equation (16). Choose  $h > 0$  such that  $(x - h, x) \cup (x, x + h)$  is contained in  $C^{1,2}([0,\infty) \times [x,x+h))$ . Then, we can apply Lemma 2.15 to get

**Lemma 3.10.** *Let Assumptions 2.1, 2.5, 2.10 hold. Let S* ∈ *B* and *x* ∈ *δS*, and suppose 
$$
(x - h, x) \subset S^c
$$
 (resp.  $(x, x + h) \subset S^c$ ) for some  $h > 0$ . Then,  
\n $V_x(t, x -, S) \leq \delta(t)V_x(0, x -, S)$  (resp.  $V_x(t, x +, S) \geq \delta(t)V_x(0, x +, S)$ ). (47)  
\nProof. Notice that Assumption 2.10 gives the existence of  $V_x(t, y -, S)$  for  $y \in (x - h, x]$  (resp.  
\n $V_x(t, y +, S)$  for  $y \in [x, x + h)$ ) when  $(x - h, x) \subset S^c$  (resp. when  $(x, x + h) \subset S^c$ ). For any  $y \in X$ ,  
\n $V_x(t, y +, S) \subset Y^c$ . Then by the fact that  $V(t, x, S) = \delta(t)F^c[\delta(\rho_S)f(X_{\rho_S})] = \delta(t)V(0, y, S)$ .  
\nSuppose  $(x - h, x) \subset S^c$ . Then by the fact that  $V(t, x, S) = \delta(t)F(x)$  (due to Lemma 2.14 (a)) and  
\nthe above inequality, we have that  
\n $V_x(t, x -, S) = \lim_{\epsilon \searrow 0} \frac{1}{\epsilon}(V(t, x, S) - V(t, x - \epsilon, S))$   
\n $\leq \lim_{\epsilon \searrow 0} \frac{1}{\epsilon}(\delta(t) f(x) - V(0, x - \epsilon, S))$   
\n $\leq \lim_{\epsilon \searrow 0} \frac{1}{\epsilon}(\delta(t) f(x) - V(0, x - \epsilon, S))$   
\n $\leq \lim_{\epsilon \searrow 0} \frac{1}{\epsilon}(\delta(t) f(x) - V(0, x - \epsilon, S))$   
\nSimilarly, we have  $\text{line integral is the domain of the result of } V_x(t, x +, S) = V_x(0, x -, S) \neq 0$ , the local time integral is the nonpositive. Thus, to make the LHS of Equation (28), Thus, to make the LHS of the final time. This of Equation (28), Thus, to make the L

<span id="page-16-0"></span>Let  $\varepsilon \in (0,1)$ , notice that the diffusion integrand above is bounded on  $[0,1] \times B(x,h)$ . Taking expectation on both sides of Equation [\(50\)](#page-15-0) and then applying Lemma [3.8,](#page-13-0) we have that

*t* ∈ ∈ (0, 1), notice that the diffusion integrand above is bounded on [0, 1] × B(*x*, *h*). Taking  
prediction on both sides of Equation (50) and then applying Lemma 3.8, we have that  

$$
E^{x}[V(s, X_{\epsilon}, S) - V(0, x, S)] = E^{x} \left[ \int_{0}^{\epsilon \wedge x_{B(x,h)}} \frac{1}{2} (LV(s, X_{s}-, S) + LV(s, X_{s}+, S)) ds \right]
$$
(51)  
+ 
$$
E^{x} \left[ \frac{1}{2} \int_{0}^{\epsilon \wedge x_{B(x,h)}} (V_{x}(s, x+, S) - V_{x}(s, x-, S)) dt \right] + o(\epsilon).
$$
  
Lemma 3.10 and Equation (49),  

$$
V_{x}(t, x+, S) - V_{x}(t, x-, S) \ge \delta(t) (V_{x}(0, x+, S) - V_{x}(0, x-, S)) = a\delta(t), \quad \forall t \ge 0.
$$
  
the above inequality and the continuity of  $\delta$ , we can take *T* > 0 such that  

$$
V_{x}(s, x+, S) - V_{x}(s, x-, S) \ge \frac{a}{2}, \quad \forall s \in [0, T].
$$
  
then, for  $\epsilon \in [0, T \land 1]$ , the second term on the RHS of Equation (51) can be estimated as follows:  

$$
E^{x} \left[ \frac{1}{2} \int_{0}^{\epsilon \wedge x_{B(x,h)}} (V_{x}(s, x+, S) - V_{x}(s, x-, S)) dt \right]_{s}^{x} \right] \ge \frac{a}{4} E^{x}[L_{\epsilon_{B(x,h)}}^{x} \wedge t].
$$
(52)  
Lemma 2.14(b), we have  

$$
(U_{y})\epsilon[0,1]\sqrt{B(x,h)}]
$$

$$
LV'(t, y-, S) + LV'(t, y+, S)] < \infty,
$$
  
and thus the first term on the RHS of Equation (51) is of order *O*( $\epsilon$ ). Plugging this and Equation  
(51) and then applying Lemma 3.9, we have  

$$
\lim_{\epsilon \searrow 0} \frac{1}{\epsilon} E^{x}[V(\epsilon, X_{\epsilon}, S) - V(0, x, S)] \ge O(1) + \frac{a}{4} \lim_{\epsilon \searrow 0} \frac{1}{2} E^{x}[
$$

By Lemma [3.10](#page-15-0) and Equation [\(49\)](#page-15-0),

$$
V_x(t, x+, S) - V_x(t, x-, S) \ge \delta(t) (V_x(0, x+, S) - V_x(0, x-, S)) = a\delta(t), \quad \forall t \ge 0.
$$

By the above inequality and the continuity of  $\delta$ , we can take  $T > 0$  such that

$$
V_x(s, x+, S) - V_x(s, x-, S) \ge \frac{a}{2}, \quad \forall s \in [0, T].
$$

Then, for  $\varepsilon \in [0, T \wedge 1]$ , the second term on the RHS of Equation (51) can be estimated as follows:

$$
\mathbb{E}^{x}\left[\frac{1}{2}\int_{0}^{\varepsilon\wedge\tau_{B(x,h)}}(V_{x}(s,x+s)-V_{x}(s,x-s))dL_{s}^{x}\right]\geq\frac{a}{4}\mathbb{E}^{x}[L_{\tau_{B(x,h)}\wedge\varepsilon}^{x}].
$$
\n(52)

By Lemma 2.14(b), we have

$$
\sup_{(t,y)\in[0,1]\times\overline{B(x,h)}}|\mathcal{L}V(t,y-,S)+\mathcal{L}V(t,y+,S)|<\infty,
$$

and thus the first term on the RHS of Equation (51) is of order  $O(\varepsilon)$ . Plugging this and Equation (52) into Equation (51) and then applying Lemma 3.9, we have

$$
\liminf_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \mathbb{E}^{x} [V(\varepsilon, X_{\varepsilon}, S) - V(0, x, S)] \ge O(1) + \frac{a}{4} \liminf_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \mathbb{E}^{x} [L_{\tau_{B(x, h)} \wedge \varepsilon}^{x}] = \infty,
$$

which contradicts S being a weak equilibrium. Hence,  $V_x(0, x + S) - V_x(0, x - S) \le 0$ .

Next, we verify the second inequality in Equation (48). Take  $x \in \mathbb{X}$  and we consider three cases. Case (i)  $x \in S^c$ . Lemma 2.14(a) shows that  $LV(0, x, S) = 0$ .

+ E<br>9),  $S$  >  $\geq$  cont<br> $+$ ,  $S$  and te<br> $\frac{1}{x}$  of  $\frac{1}{x}$  o  $\frac{1}{2}$  i( it - m +  $|I|$  Ein 0 lij vi u ),  $V$  -  $\frac{1}{2}$  B(  $\frac{1}{2}$  B(  $\frac{1}{2}$  B),  $\frac{1}{2}$  $\frac{1}{2}$  *J*<br>  $\frac{1}{2}$  *J*<br>  $\frac{1}{2}$  *V* and  $\frac{1}{2}$  *V* Equing 0, x<br>  $\frac{1}{2}$  ibly in ws t uch  $\frac{1}{2}$  and  $\frac{1}{2}$  *V* (s  $V_x(0, x \cdot \text{of } \delta, \text{ w})$ <br>
of  $\delta$ , w<br>
(s, x –,<br>
the RI<br>  $V - V_x(1, y - 0.5) \geq 0$ <br>  $V(t, y - 0.5) \geq 0$ <br>  $V(t, y, \delta)$ <br>  $V(t, y, \delta)$ <br>  $V(t, y, \delta)$ (7 y x r b) (2 l 1 ;, ri h l (c)) , c) 。 ( $V_x(S, A+, S) - V_x(0, X-, S)) = a\delta(t),$ <br>  $+$ ,  $S$ ) −  $V_x(0, X-, S)$ ) =  $a\delta(t),$ <br>  $\epsilon$  can take  $T > 0$  such that<br>  $S$ ) ≥  $\frac{a}{2}$ ,  $\forall s \in [0, T].$ <br>
IS of Equation (51) can be estima<br>  $s, x-, S)$ ) $dL_s^x$  ≥  $\frac{a}{4} E^x [L_{\tau_{B(x,h)} \wedge \epsilon}^x].$ <br>  $\therefore$  $\mathbf{a}$  . The contract of the contract of  $\mathbf{b}$  ,  $\mathbf{b}$  ,  $\mathbf{b}$  ,  $\mathbf{b}$  $\geq 0.$ <br>ed as fo<br>md Equ<br> $\infty$ ,<br>f three<br> $\sum_{n=1}^{\infty} \frac{\delta(x, h)}{\delta(x, h)}$  $V_x(t, x+, S) - V_x(t, x-, S) \ge \delta(t) (V_x(0, x+, S) - V_x(0, x-, S)) \equiv a\delta(t), \quad \forall t \ge 0.$ <br>
bove inequality and the continuity of  $\delta$ , we can take  $T > 0$  such that<br>  $V_x(s, x+, S) - V_x(s, x-, S) \ge \frac{a}{2}$ ,  $\forall s \in [0, T]$ .<br>  $v \in [0, T \wedge 1]$ , the second term on t δ, we can take *T* > 0 such that<br>  $x-, S$ ) ≥  $\frac{a}{2}$ ,  $\forall s \in [0, T]$ .<br>
e RHS of Equation (51) can be  $V_x(s, x-, S)$ ) $dL_s^x$   $\bigg]$  ≥  $\frac{a}{4}$   $\mathbb{E}^x[L_{\tau_{B(x)}}^x$ <br>  $v-, S$ ) +  $\mathcal{L}V(t, y+, S)$  | < ∞,<br>
on (51) is of order *O*(ε).  $V_x(s, x+, S) - V_x(s, x-, S)$  ≥<br>
he second term on the RHS of<br>  $\pi_{B(x,h)}$ <br>  $(V_x(s, x+, S) - V_x(s, x-$ <br>
ve<br>
we<br>
sup  $|CV(t, y-, S) +$ <br>
1 the RHS of Equation (51) is<br>
d then applying Lemma 3.9,<br>  $\varepsilon, X_{\varepsilon}, S$ ) –  $V(0, x, S)$ ] ≥  $O(1)$ <br>
g a weak equi 2<sup>2</sup><br>
Equation (51) can<br> *S*)) $dL_s^x$  ≥  $\frac{a}{4}$  E<sup>x</sup><br>  $\frac{a}{2}$  E<sup>x</sup><br>  $\frac{a}{4}$  E<sup>x</sup><br>  $\frac{a}{4}$  lim<sub>ε</sub> inf  $\frac{1}{2}$  E<sup>x</sup> [*L*<br>  $\frac{a}{4}$  lim<sub>ε</sub> inf  $\frac{1}{2}$  E<sup>x</sup> [*L*<br>  $V_x(0, x +, S) - V$ <br> *S*. Take  $x \in \mathbb{X}$  and<br>  $V_x(0$  $\varepsilon \in [0, T \wedge 1]$ , the second term on the RHS of Equation (51) can be estimated as follows:<br>  $\mathbb{E}^x \left[ \frac{1}{2} \int_0^{t \wedge T_{B(x,b)}} (V_x(s, x+, S) - V_x(s, x-, S)) dI_x^* \right] \geq \frac{a}{4} \mathbb{E}^x [L_{T_{B(x,b)}^* \wedge t}^3].$  (52)<br>
a.2.14(b), we have<br>
su ピインストランド rst<br>ion nf<br>ict<br>ify se is derived if of the condition of the condition of the conditional derivation of the condition of the co 1  $\frac{1}{2}$  ∫ we ern  $(51)$  E<sup>x</sup><br>E<sup>x</sup> S be s<br>Len  $S^{\circ}$  A li s<br>is  $\lim_{\varepsilon \to 0}$ , s ave strategive and the  $\ell(t,y) \in [0, 0)$  on the and the  $\ell(\varepsilon, X_{\varepsilon}, \text{mg a w cond in } 2.12)$ <br>  $\max_{\varepsilon \to 0} \sum_{\varepsilon=0}^{\infty} \frac{1}{\varepsilon} \int_{0}^{\varepsilon} \rho_{(t,y) \in [0, 0]}$ <br>  $\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbb{E}$  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  or  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  is  $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$  or  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  or  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  is  $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$  or  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ( $v_x$ (s, x+, s) –  $v_x$ (s, x –, s)) $aL_s$ <br>
(p<br>
(p<br>
( $\mathcal{L}V(t, y-, S) + \mathcal{L}V(t, y+$ <br>
RHS of Equation (51) is of order<br>
m applying Lemma 3.9, we have<br>  $S$ ) –  $V(0, x, S)$ ]  $\geq O(1) + \frac{a}{4} \lim_{\varepsilon \to 0}$ <br>
eak equilibrium. Hence,  $V_x$  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$   $\frac{1}{4}$   $\mathbb{E} \left[ L_{\tau} \right]$ <br>  $\leq \infty$ ,<br>  $\therefore$  Plugg<br>  $\mathbb{E}^{x} [L_{\tau_{B(}}^{x} - V_{x}(0)) - V_{x}(0)]$ <br>  $\leq C^{1,2}$ <br>  $\mathbb{P}^{x} - \mathbb{E}^{x}$ <br>  $\mathbb{P}^{x} - \mathbb{E}^{x}$ <br>  $\leq C^{1,2}$ <br>  $\mathbb{P}^{x} - \mathbb{E}^{x}$ <br>  $\leq C^{1,2}$  $x_{B(x,h) \wedge \varepsilon}^x$ ]. (52)<br>ging this and Equation<br> $B(x,h) \wedge \varepsilon$ ] = ∞,<br>0, *x*−, *S*) ≤ 0.<br>we consider three cases.<br>at *V*(*t*, *y*, *S*) = *δ*(*t*)*f*(*y*)<br>-<sup>2</sup>([0, ∞) × *B*(*x*, *h*)) and<br>s..<br>s..<br>apply the dominated (53)  $\begin{bmatrix} 0,1 \end{bmatrix}$  as R and  $\begin{bmatrix} 0,1 \end{bmatrix}$  as  $\begin{bmatrix} 0,1 \end{bmatrix}$  as  $\begin{bmatrix} 14(38) \\ 0 \end{bmatrix}$  .  $\begin{bmatrix} 0,1 \end{bmatrix}$  is  $\begin{bmatrix} 0,1 \end{bmatrix}$  . on the RHS of<br>on the RHS of<br>and then apply<br>/(ε, X<sub>ε</sub>, S) – V<br>mg a weak equ<br>cond inequalit<br>ma 2.14(a) sho<br>Choose *h* > 0;<br>sumptions 2.1(<br>ontinuous on [<br> $\frac{1}{\delta}$   $\frac{1}{\epsilon}$   $\int_0^{\epsilon \wedge \tau_{B(x,h)}} L$ <br> $P_{(t,y) \in [0,\infty) \times \overline{B(x, \$  $V(t, y-, S) + \mathcal{L}V(t, y+, S)| < \infty$ ,<br>
quation (51) is of order  $O(\varepsilon)$ . Plug<br>
g Lemma 3.9, we have<br>  $x, S$ )] ≥  $O(1) + \frac{a}{4} \liminf_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \mathbb{E}^x[L_{\varepsilon}^x]$ <br>
orium. Hence,  $V_x(0, x+, S) - V_x$ <br>
n Equation (48). Take  $x \in \mathbb$  $O(\varepsilon)$ . Plugging this and Equation<br>
on  $\int_{0}^{1} \frac{1}{\varepsilon} \mathbb{E}^{x} [L_{\tau_{B(x,h)} \wedge \varepsilon}^{x}] = \infty$ ,<br>  $\cdot$ , S)  $- V_{x}(0, x-, S) \le 0$ .<br>  $\in \mathbb{X}$  and we consider three cases.<br>  $\therefore$  Notice that  $V(t, y, S) = \delta(t) f(y)$ <br>  $y, S) \in C^{1,2}([0,$  $\varepsilon \searrow 0$ <br>tradicte verify<br> $x \in S$ <br> $x \in S$ <br> $\Rightarrow$  Then,<br> $V(t, y, \text{a} 2.14)$ <br>ce the  $\text{addi} \in \mathbb{R}$ <br> $\text{vei} \in \mathbb{R}$ <br> $\text{wei} \in \mathbb{R}$ <br> $(t, y)$ <br>2.1  $\overline{\varepsilon}$  ,  $\overline{\varepsilon}$  , - ^ t ( C b r)<br>2)<br>n  $\mathbb{E}^{x}[V(\varepsilon, X_{\varepsilon}, S) - V(0, x, S)] \geq O(1) + \frac{a}{4}$ <br>
S being a weak equilibrium. Hence,  $V_{x}$ <br>
ne second inequality in Equation (48). T<br>
Lemma 2.14(a) shows that  $\mathcal{L}V(0, x, S) = S^{\circ}$ . Choose  $h > 0$  such that  $B(x, h) \subset G$ 4  $\varepsilon \searrow 0$ <br>  $x(0, x+, \cdot)$ <br>
Take  $x \in$ <br>  $= 0$ .<br>  $\mathcal{G} \cap S^{\circ}$ . I<br>
we  $V(t, y, \cdot)$ <br>
us,<br>  $V(0, x, S)$ <br>  $\circ$ . Then,<br>  $S = \mathcal{L} V(\cdot)$  $x + e$ <br>  $\infty$  $\begin{bmatrix} -\varepsilon \\ \varepsilon \end{bmatrix}$  )  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$  ) and  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ) >> C<br>C<br>S<br>D<br>D  $E^{-1}[L_{\tau}]$ <br>  $-V_x(\text{and }x)$ <br>  $\text{tice th}$ <br>  $0 \in C^1$ <br>  $D^x - a$ <br>  $\text{re can}$ <br>  $x, S$ ).  $\tau_{B(x,h)}(0, x-, S) \leq 0$ <br>
we consider t<br>
hat  $V(t, y, S)$ <br>
a.s..<br>
a.s..<br>
n apply the S being a weak equilibrium. Hence,  $V_x(0, x+, S) - V_x(0, x-, S) \le 0$ .<br>
he second inequality in Equation (48). Take  $x \in \mathbb{X}$  and we consider th<br>
Lemma 2.14(a) shows that  $LV(0, x, S) = 0$ .<br>  $S^\circ$ . Choose  $h > 0$  such that  $B(x, h) \subset G \$ x ∈ × and we consider three cases.<br>
5°. Notice that  $V(t, y, S) = \delta(t) f(y)$ <br>  $t, y, S$ ) ∈  $C^{1,2}([0, \infty) \times B(x, h))$  and<br>  $s, S$ ),  $\mathbb{P}^x$  – a.s..<br>
en, we can apply the dominated<br>  $CV(0, x, S)$ . (53)  $x \in S^c$ . Lemma 2.14(a) shows that  $\mathcal{L}V(0, x, S) = 0$ .<br>  $x \in \mathcal{G} \cap S^{\circ}$ . Choose  $h > 0$  such that  $B(x, h) \subset \mathcal{G} \cap$ <br>
Then, by Assumptions 2.1(i) and 2.8(ii), we have  $V(x, y, S)$  is continuous on  $[0, \infty) \times \overline{B(x, h)}$ . Th Case (ii)  $x \in \mathcal{G} \cap S^{\circ}$ . Choose  $h > 0$  such that  $B(x, h) \subset \mathcal{G} \cap S^{\circ}$ . Notice that  $V(t, y, S) = \delta(t) f(y)$  $x \in G \cap S^{\circ}$ . Choose  $h > 0$  such that  $B(x, h) \subset G \cap S^{\circ}$ . Notice that  $V(t, y, S) = \delta(t) f(y)$ <br>
Then, by Assumptions 2.1(i) and 2.8(ii), we have  $V(t, y, S) \in C^{1,2}([0, \infty) \times B(x, h))$  and<br>  $V(t, y, S)$  is continuous on  $[0, \infty) \times \overline{B(x, h)}$ for  $y \in S^{\circ}$ . Then, by Assumptions 2.1(i) and 2.8(ii), we have  $V(t, y, S) \in C^{1,2}([0, \infty) \times B(x, h))$  and  $y \in S^\circ$ . Then, by Assumptions 2.1(i) and 2.8(ii), we have  $V(t, y, S) \in C^{1,2}([0, \infty) \times B(x, h))$  and<br>  $y) \mapsto \mathcal{L}V(t, y, S)$  is continuous on  $[0, \infty) \times \overline{B(x, h)}$ . Thus,<br>  $\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^{\varepsilon \wedge \tau_{B(x, h)}} \mathcal{L}V(s, X_s, S) ds$ 

$$
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^{\varepsilon \wedge \tau_{B(x,h)}} \mathcal{L}V(s,X_s,S)ds = \mathcal{L}V(0,x,S), \ \mathbb{P}^x - \text{a.s.}
$$

 $(t, y) \mapsto \mathcal{L}V(t, y, S)$  is continuous on  $[0, \infty) \times \overline{B(x, h)}$ . Thus,<br>  $\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^{\epsilon \wedge \tau_{B(x, h)}} \mathcal{L}V(s, X_s, S) ds = \mathcal{L}V(0)$ <br>
By Lemma 2.14(b),  $\sup_{(t, y) \in [0, \infty) \times \overline{B(x, h)}} |\mathcal{L}V(t, y, S)| < \infty$ .<br>
convergence theore  $\varepsilon \rightarrow 0$ <br>sup<br>n to<br> $\prod_{\ell}$  $\frac{1}{2}$ By Lemma 2.14(b),  $\sup_{(t,y)\in[0,\infty)\times\overline{B(x,h)}}|\mathcal{L}V(t,y,S)| < \infty$ . Then, we can apply the dominated convergence theorem to derive

$$
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^{\infty} \mathcal{L}V(s, X_s, S)ds = \mathcal{L}V(0, x, S), \mathbb{P}^x - \text{a.s.}.
$$
\n
$$
\sup_{(t, y) \in [0, \infty) \times \overline{B(x, h)}} |\mathcal{L}V(t, y, S)| < \infty. \text{ Then, we can apply the dominated}
$$
\n
$$
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbb{E}^x \left[ \int_0^{\varepsilon \wedge \tau_{B(x, h)}} \mathcal{L}V(s, X_s, S)ds \right] = \mathcal{L}V(0, x, S). \tag{53}
$$

Notice that Equation [\(51\)](#page-16-0) is valid and the local time integral term in Equation [\(51\)](#page-16-0) vanishes in this case. Then, Equations [\(51\)](#page-16-0) and [\(53\)](#page-16-0) together lead to

$$
\lim_{\varepsilon\searrow 0}\frac{1}{\varepsilon}\mathbb{E}^{x}[V(\varepsilon, X_{\varepsilon}, S)-V(0, x, S)]=\lim_{\varepsilon\searrow 0}\frac{1}{\varepsilon}\mathbb{E}^{x}\left[\int_{0}^{\varepsilon\wedge\tau_{B(x,h)}}\mathcal{L}V(s, X_{s}, S)ds\right]=\mathcal{L}V(0, x, S).
$$

Since S is a weak equilibrium, we have  $\mathcal{L}V(0, x, S) \leq 0$ .

Case (iii)  $x \in S \setminus (G \cap S^{\circ})$ . As S is admissible, we can pick  $h > 0$  such that  $(x - h, x)$  is contained in either  $G \cap S^{\circ}$  or  $S^c$ . By the results in Cases (i) and (ii), as well as the continuity of

$$
\mathcal{L}V(0, x-, S) = \lim_{\varepsilon \searrow 0} \mathcal{L}V(0, (x - \varepsilon), S) \le 0.
$$

Similarly,  $\mathcal{L}V(0, x+, S) \leq 0$ .

*Proof of Theorem* 3.1. The necessity is implied by Proposition [3.11.](#page-15-0) Let us prove the sufficiency.

Take  $x \in S$ . Since S is admissible, by Lemma 2.14 and Assumption 2.8(ii), no matter  $x \in S^{\circ}$ or  $x \in \partial S$ , we can choose  $h > 0$  such that  $V(t, x, S) \in C^{1,2}([0, \infty) \times (x - h, x])$  and  $V(t, x, S) \in$  $C^{1,2}([0,\infty) \times [x,x+h])$ . By a similar argument as that for Equation (51) (with Lemmas 2.15 and [3.8](#page-13-0) applied), we have that

lim 
$$
\frac{1}{5}E^{X}[V(\varepsilon, X_{\varepsilon}, S) - V(0, x, S)] = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} E^{X} \left[ \int_{0}^{\varepsilon / \varepsilon_{B(x,\delta)}} \mathcal{L}V(s, X_{s}, S) ds \right] = \mathcal{L}V(0, x, S).
$$
  
\nSince S is a weak equilibrium, we have  $U'(0, x, S) \leq 0$ .  
\nCase (iii)  $x \in S \setminus (G \cap S^{\circ})$ . As S is admissible, we can pick  $h > 0$  such that  $(x - h, x)$  is con-  
\ntained in either  $G \cap S^{\circ}$  or  $S^{\circ}$ . By the results in Cases (i) and (ii), as well as the continuity of  
\n $x \mapsto \mathcal{L}V(0, x-, S)$  on  $(x - h, x]$ , we have that  
\n $\mathcal{L}V(0, x-, S) = \lim_{\varepsilon \to 0} \mathcal{L}V(0, (x - \varepsilon), S) \leq 0$ .  
\nSimilarly,  $\mathcal{L}V(0, x+, S) \leq 0$ .  
\n $\text{Proof of Theorem 3.1. The necessity is implied by Proposition 3.11. Let us prove the sufficiency,\nTake  $x \in S$ . Since S is admissible, by Lemma 2.4d and Assuming 2.4d and Assuming 2.4d in, no matter  $x \in S^{\circ}$ , we can choose  $h > 0$  such that  $V(t, x, S) \in C^{1,2}([0, \infty) \times (x - h, x])$  and  $V(t, x, S) \in C^{1,2}([0, \infty) \times [x, x + h)).$  By a similar argument as that for Equation (51) (with Lemma 2.15 and  
\n3.8 applied), we have that  
\n
$$
\frac{1}{\varepsilon} \left[ \mathbb{E}^{x} \left[ \frac{1}{2} \int_{0}^{\varepsilon \wedge \tau_{B(x, h)}} \frac{1}{2} (\mathcal{L}V(s, X_{s}, -S) + \mathcal{L}V(s, X_{s} + S)) ds \right] + \frac{1}{\varepsilon} \left[ \mathbb{E}^{x} \left[ \frac{1}{2} \int_{0}^{\varepsilon \wedge \tau_{
$$$ 

By Equation (25) and the (left/right) continuity of  $(s, y) \mapsto LV(s, y \pm, S)$  at  $(0, x)$ , for  $\mathbb{P}$ -a.s.  $\omega \in \Omega$ ,

$$
\limsup_{s\searrow 0}\frac{1}{2}(\mathcal{L}V(s,X_s(\omega)-,S)+\mathcal{L}V(s,X_s(\omega)+,S))\leq 0,
$$

which leads to

$$
\limsup_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_0^{\varepsilon \wedge \tau_{B(x,h)}} \frac{1}{2} (\mathcal{L}V(s, X_s(\omega) - S) + \mathcal{L}V(s, X_s(\omega) + S)) ds \le 0.
$$
 (55)

By Lemma 2.14 (b),

$$
\sup_{(t,y)\in[0,1]\times B(x,h)}|\mathcal{L}V(t,y-,S)+\mathcal{L}V(t,y+,S)|<\infty.
$$

This enables us to apply Fatou's lemma for Equation (55) and get

$$
+\frac{1}{\varepsilon}\mathbb{E}x\left[\frac{1}{2}\int_{0}^{x} (V_{x}(s,x+S)-V_{x}(s,x-,S))dL_{s}^{x}\right]+o(1)
$$
\n(54)  
\n5) and the (left/right) continuity of (s, y)  $\mapsto LV(s, y\pm, S)$  at (0, x), for P-a.s.  $\omega \in \Omega$ ,  
\n
$$
\limsup_{s\searrow 0} \frac{1}{2}(LV(s, X_{s}(\omega)-, S)+LV(s, X_{s}(\omega)+, S)) \leq 0,
$$
\n
$$
\limsup_{\varepsilon\searrow 0} \frac{1}{\varepsilon}\int_{0}^{\varepsilon\wedge\tau_{B(x,h)}} \frac{1}{2}(LV(s, X_{s}(\omega)-, S)+LV(s, X_{s}(\omega)+, S))ds \leq 0.
$$
\n(55)  
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\n(5

By Equation (18),

$$
\begin{aligned}\n &\varepsilon \searrow 0^{\text{T}} \varepsilon \quad [J_0 \quad 2^{(1/2)(1/2)} \leq \varepsilon \quad \text{and} \quad \varepsilon \to 0, \quad \varepsilon \to 0, \quad \varepsilon \to 0, \quad \varepsilon \to 0, \\
 &\varepsilon \searrow 0 \quad \varepsilon \to 0, \\
 &\varepsilon \searrow 0 \quad \varepsilon \to 0, \\
 &\varepsilon \searrow 0 \quad \varepsilon \to 0, \\
 &\varepsilon \searrow 0 \quad \varepsilon \to 0, \quad \
$$

<span id="page-17-0"></span>

<span id="page-18-0"></span>This together with Equation [\(24\)](#page-10-0) implies that

$$
\frac{1}{\varepsilon} \mathbb{E}^x \left[ \frac{1}{2} \int_0^{\varepsilon \wedge \tau_{B(x,k)}} (V_x(s, x +, S) - V_x(s, x -, S)) dL_s^x \right]
$$
\n
$$
\leq \frac{1}{\varepsilon} \mathbb{E}^x \left[ \frac{1}{2} \int_0^{\varepsilon \wedge \tau_{B(x,k)}} (V_x(0, x +, S) - V_x(0, x -, S) + o(\sqrt{\varepsilon})) dL_s^x \right]
$$
\n
$$
\leq \frac{1}{2\varepsilon} \cdot o(\sqrt{\varepsilon}) \cdot \mathbb{E}^x \left[ L_{\varepsilon \wedge \tau_{B(x,k)}}^x \right]
$$
\n
$$
= \frac{1}{2\varepsilon} \cdot o(\sqrt{\varepsilon}) \cdot O(\sqrt{\varepsilon}) = o(1),
$$
\nne follows from Lemma 3.9. Then by Equations (54), (56), and (57), we have that  
\n
$$
\limsup_{\varepsilon \searrow 0} \frac{1}{\varepsilon} (\mathbb{E}^x [V(\varepsilon, X_{\varepsilon}, S)] - V(0, x, S)) \leq 0.
$$
\n
$$
\Box
$$
  
\n**1AL MIL DEQUILIBRIA ARE WEAK EQUILIBRIA**\n
$$
= \text{Re} \text{how that an optimal mild equilibrium is a weak equilibrium.}
$$
\nlet us point out that optimal mild equilibrium is a weak equilibrium.  
\nlet us point out that optimal mild equilibrium exist for one-dimensional diffusions  
\nthat is involved in Huang and Zhou (2020, Theorem 4.12), and we summarize it  
\nthat S50.

where the last line follows from Lemma [3.9.](#page-14-0) Then by Equations [\(54\)](#page-17-0), [\(56\)](#page-17-0), and (57), we have that

$$
\limsup_{\varepsilon \searrow 0} \frac{1}{\varepsilon} (\mathbb{E}^x [V(\varepsilon, X_\varepsilon, S)] - V(0, x, S)) \le 0.
$$

П

# **4 OPTIMAL MILD EQUILIBRIA ARE WEAK EQUILIBRIA**

In this section, we show that an optimal mild equilibrium is a weak equilibrium.

To begin with, let us point out that optimal mild equilibria exist for one-dimensional diffusions. Such existence result is provided in Huang and Zhou (2020, Theorem 4.12), and we summarize it in the current context as follows.

**Lemma 4.1.** *Let Assumptions 2.1, [2.5,](#page-7-0) and Equations (14) and (17) hold. Then,*

$$
S^* := \cap_{S \in \mathcal{E}} S \tag{58}
$$

*is an optimal mild equilibrium, where*  $\varepsilon$  *is the set containing all mild equilibria.* 

 $=$   $\frac{1}{28}$ <br>e fol<br>e fol<br>e she usult<br>e she usult<br>text<br> $\frac{1}{28}$ <br>  $\frac{1}{2\varepsilon}$  ⋅ *o*( $\sqrt{\varepsilon}$ ) ⋅ *O*( $\sqrt{\varepsilon}$ ) = *o*(1),<br>ollows from Lemma 3.9. T<br>lim sup  $\frac{1}{\varepsilon}$ ( $E^x$ [*V*( $\varepsilon$ , *X*<br>L<br>*MILD EQUILIBRI*<br>how that an optimal mild us point out that optimal n<br>the us point out that o LD EQ<br>at an optout the video in the video is then it is rem 4.3 of this 2. **DE** an anut ded ws as 2. *n*, *n* uat 1 eq. *n*, *n* uat 1 eq. *n*, *n* it is *n* it m 4 ons  $\frac{1}{\varepsilon}$   $\mathbf{L}$  time  $\mathbf{L}$  is the  $\mathbf{L}$  is the  $\mathbf{L}$  is the  $\mathbf{L}$  is the  $\mathbf{L}$  in the  $\mathbf{L}$ ( $E^{\times}[V(\varepsilon, X_{\varepsilon}, S)] - V(0, x, S)) \leq 0.$ <br> **IILIBRIA ARE WEAK EQ**<br>
mal mild equilibrium is a weak e<br>
optimal mild equilibria exist for c<br>
Huang and Zhou (2020, Theorem<br>
5, and Equations (14) and (17) hold<br>  $S^* := \cap_{S \in \mathcal{E}} S$ 12.<br>13.<br>13.<br>13.<br>22.<br>19.<br>29.  $S^* := \bigcap_{S \in \mathcal{E}} S$  (58)<br>the set containing all mild equilibria.<br>s assumed in Huang and Zhou (2020, Theorem 4.12)<br>1, which is guaranteed by Assumption 2.1 as stated in<br>duced from Assumption 2.8(i) as stated in Remark 2.9 *Remark* 4.2. Notice that Equation [\(12\)](#page-5-0) is assumed in Huang and Zhou [\(2020,](#page-40-0) Theorem 4.12) for  $S^*$  being an optimal mild equilibrium, which is guaranteed by Assumption 2.1 as stated in Remark [2.2.](#page-5-0) Also, Equation [\(17\)](#page-7-0) can be deduced from Assumption 2.8(i) as stated in Remark [2.9.](#page-7-0)

Below is the main result of this section.

 $S^*$  being an optimal mild equilibrium, which is guaranteed by Assumption [2.1](#page-5-0) as stated in ank 2.2. Also, Equation (17) can be deduced from Assumption 2.8(i) as stated in Remark 2.9.<br>
below is the main result of this sec **Theorem 4.3.** Let Assumptions [2.1](#page-5-0)[–2.10](#page-8-0) hold and S be an admissible stopping policy. If S is an *S* be an admissible stopping policy. If *S* is an ibrium.<br>ibrium.<br>pwing.<br>ose *S*\* is admissible. Then *S*\* is also weak. *optimal mild equilibrium, then it is also a weak equilibrium.*

By Lemma 4.1 and Theorem 4.3, we have the following.

**Corollary 4.4.** *Let Assumptions [2.1](#page-5-0)[–2.10](#page-8-0) hold. Suppose* ∗ *is admissible. Then* ∗ *is also weak.*

<span id="page-19-0"></span>**816 I M**/II  $F$  **V** <u>**BAYRAKTAR ET AL.**</u>

*Remark* 4.5. The above corollary also provides the existence of weak equilibria (ignoring admissibility) as a by-product. Moreover, since any weak equilibrium is also mild, we can see that  $S^*$  is optimal among all mild and weak equilibria.

# **4.1 Proof of Theorem [4.3](#page-18-0)**

 $S^*$  is<br>es to<br>dadic-<br>etter<br>25) is<br>ation<br>hich<br>hich<br>(60)<br>ick a<br> $\sigma(x_0, l)$ <br>(00)<br>(-1,<br>(61)<br>that<br>(62) To show that an optimal mild equilibrium  $S$  is a weak equilibrium, by Theorem 3.1 it suffices to S is a weak equilibrium, by Theorem [3.1](#page-10-0) it suffices to  $\sin \alpha(24)$  will be proved in Proposition 4.7 by contradics  $S) - V_x(0, x_0 - S) > 0$ , then a mild equilibrium better hole  $B(x_0, h)$ " out of S. The proof of Equation (25) is verify Equations [\(24\)](#page-10-0) and [\(25\)](#page-10-0) for S. Equation (24) will be proved in Proposition 4.7 by contradic-S. Equation [\(24\)](#page-10-0) will be proved in Propositon [4.7](#page-21-0) by contradic-<br>  $\int_{X_2}(x_0x_0,tS) - V_x(0,x_0-s) > 0$ , then a mild equilibrium better<br>
ig a small hole  $B(x_0, h)$ " out of S. The proof of Equation (25) is<br>
by finding a better mil tion. In particular, if we assume  $V_x(0, x_0+, S) - V_x(0, x_0-, S) > 0$ , then a mild equilibrium better than S can be construct by "digging a small hole  $B(x_0, h)$ " out of S. The proof of Equation (25) is also carried out via contradiction by finding a *better* mild equilibrium.

Such construction of a *better* mild equilibrium requires the comparison between the expectation of a local time integral before the exit time  $\tau_{B(x,h)}$  and the expectation of  $\tau_{B(x,h)}$  for small h, which is stated in the following lemma.

**Lemma 4.6.** *Suppose Assumptions [2.1](#page-5-0) and [2.5](#page-7-0) hold. For*  $x_0 \in \mathbb{X}$ *, we have that* 

$$
\frac{\mathbb{E}^{x_0+rh}\left[\int_0^{\tau_{B(x_0,h)}} \delta(t) dL_t^{x_0}\right] \cdot h}{\mathbb{E}^{x_0+rh}[\tau_{B(x_0,h)}]}\xrightarrow{h \searrow 0} \frac{\sigma^2(x_0)}{1+|r|} \text{ uniformly for } r \in (-1,1). \tag{59}
$$

*Proof.* We first prove

$$
\frac{\mathbb{E}^{x_0+rh}[\tau_{B(x_0,h)}]}{(1-r^2)h^2} \xrightarrow{h\searrow 0} \frac{1}{\sigma^2(x_0)} \quad \text{uniformly for } r \in (-1,1)
$$
 (60)

by using an argument similar to that for Christensen and Lindensjö [\(2020a,](#page-39-0) Lemma A.5). Pick a constant *a* and consider the function  $g(t, z) := a(z - x_0)^2 - t$ . We have that

$$
\mathcal{L}g(t,z) = -1 + a\sigma^2(z) + \mu(z)2a(z - x_0).
$$

 $V_x(0, x_0, t, S) - V_x(0, x_0, -S) > 0$ , then a mild equilibrium *better*<br>  $\lim_{\delta \to 0} \sin \theta + B(x_0, h)^n$  out of *S*. The proof of Equation (25) is<br>
by finding a better mild equilibrium<br>
cid equilibrium requires the comparison between Stan be construct by "digging a small hole R( $x_0$ ,  $y_0$ )" out of S. The proof of Equation [\(25\)](#page-10-0) is<br>arried out via construction by finding a better mild equilibrium.<br>
ch construction of a derivernild equilibrium requires  $\tau_{B(x,h)}$  and the expectation of  $\tau_{B(x,h)}$  for small *h*, which<br>  $l \ge 2.5$  hold. For  $x_0 \in X$ , we have that<br>  $\frac{l}{1+|r|}$  uniformly for  $r \in (-1, 1)$  (59)<br>  $\frac{1}{1+|r|}$  uniformly for  $r \in (-1, 1)$  (60)<br>  $\frac{1}{1+|r|}$  uniform  $x_0 \in \mathbb{X}$ , we have that<br>
uniformly for  $r \in (-1, 1)$ <br>
d Lindensjö (2020a,<br>  $\big)$ <sup>2</sup> − *t*. We have that<br>
z)2a(z − x<sub>0</sub>).<br>  $\frac{1}{\sigma^2(x_0)}$ , by the continu<br>
g(*t*, z) ≥ 0 for any z  $\in$ <br>  $\int_0^{\tau_{B(x_0,h)}} \mathcal{L}g(t, X_t)dt$ <br>
1)  $E^{x_0 + rh}$   $E$ <br>t prove<br> $E$ <br>gument<br>d consid<br>n 2.1(ii),<br>> 0, whi<br>o g(t,  $X_t$ <br> $-x_0$ )<sup>2</sup>  $h$ ), rewr<br> $E[X(t) \leq t]$ ,  $\infty$ <br> $B(x_0, t)$  $\int_{0}^{\pi}$ <br> $\frac{1}{\pi}$ <br> 1.0 0 0 0 1 0 1 1 7 cm b 1 1 , b 1 t 6 , t 6 ,  $\frac{B(x_0, h)}{B(x_0, h)}$ <br>  $\frac{B(x_0, h)}{h^2}$ <br>
to that f<br>
to that f<br>
inction g<br>  $y(t, z) =$ <br>
> 0. For a<br>
depends<br>
we that<br>  $(x_0, h)$ ] –<br>  $x_0 + rh$  f<br>  $x_{B(x_0, h)}$  –<br>
<  $\tilde{a} < \frac{1}{\sigma^2}$ <br>  $r\tilde{h}$  [ $\tau_{B(x_0, h)}$  $\frac{1}{1}$   $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ , 7 + 1 y α α (y - sc))<sup>2</sup> = 0, 2  $\frac{\mathbb{E}^{x_0+rh}[\tau_{B(x_0,h)}]}{(1-r^2)h^2} \xrightarrow{h\searrow 0} \frac{1}{\sigma^2(x_0)}$  unif<br>
t similar to that for Christensen a<br>
der the function  $g(t, z) := a(z - x$ <br>  $\mathcal{L}g(t, z) = -1 + a\sigma^2(z) + \mu$ <br>  $\phi, \sigma^2(x_0) > 0$ . For any constant  $a >$ <br>
inch only depen 1 + |r| *uniformly for*  $r \in (-1, 1)$ . (59)<br>
1 + |r| *uniformly for*  $r \in (-1, 1)$  (60)<br>
1 msen and Lindensjö (2020a, Lemma A.5). Pick a<br>  $t(z - x_0)^2 - t$ . We have that<br>  $z + \mu(z)2a(z - x_0)$ .<br>
Inta  $z \frac{1}{\sigma^2(x_0)}$ , by the continuit  $\frac{C \cdot B(x_0, h)}{(1 - r^2)h^2}$ <br>to similar to that<br>der the function<br> $\mathcal{L}g(t, z)$ <br>),  $\sigma^2(x_0) > 0$ . For<br>inch only depen<br> $K_t$ ), we have that<br> $\left[-\mathbb{E}^y[\tau_{B(x_0, h)}] - \mathbb{E}^y[\tau_{B(x_0, h)}]\right]$ <br>rite  $y = x_0 + r\lambda$ <br> $\int_{0}^{x_0 + r\lambda} [\tau_{B(x_0$  $\frac{1}{(1 - r^2)h^2}$  →  $\frac{1}{\sigma^2(z)}$ <br>  $\frac{1}{(1 - r^2)h^2}$  →  $\frac{1}{\sigma^2(z)}$ <br>  $\frac{1}{\sigma^2(z)}$ <br>  $\frac{1}{\sigma^2(z)}$ <br>  $\frac{1}{\sigma^2(z_0)} > 0$ . For any con<br>  $h$  only depends on *a*, s<br>  $\frac{1}{(1 - r^2)h^2}$  +  $\frac{1}{\sigma^2(z_0)}$  =  $\frac{1}{\sigma^2(z_0)}$ , w (60)<br>
2(x<sub>0</sub>) uniformly for  $r \in (-1, 1)$  (60)<br>
2) :=  $a(z - x_0)^2 - t$ . We have that<br>  $+ a\sigma^2(z) + \mu(z)2a(z - x_0)$ .<br>
constant  $a > \frac{1}{\sigma^2(x_0)}$ , by the continuity of  $\mu(x)$  and  $\sigma(x)$ ,<br>
a, such that  $\mathcal{L}g(t, z) \ge 0$  for any  $z \in B$ *a* and consider the function  $g(t, z) := a(z - x_0)$ <br>  $\mathcal{L}g(t, z) = -1 + a\sigma^2(z) + \mu(z)$ <br>
aption 2.1(ii),  $\sigma^2(x_0) > 0$ . For any constant  $a > -\frac{1}{\sigma}$ <br>
and  $h > 0$ , which only depends on *a*, such that  $\mathcal{L}g$ <br>
and  $h > 0$ , which on ):  $\frac{1}{2}$  ( c) .  $\mu$  ,  $\frac{1}{2}$ − *t*. We have that<br>  $a(z - x_0)$ .<br>
<sub> $x_0$ </sub>, by the continu<br>
, z) ≥ 0 for any z<br>  $x_{B(x_0, h)}$   $\mathcal{L}g(t, X_t)dt$ <br>
Then the above ii<br>  $x^2 - ar^2h^2 = a(1 - \text{which only deep})$ <br>  $\forall r \in (-1, 1)$ . > 0. For any constant  $a > \frac{1}{\sigma^2(x_0)}$ , by the<br>  $y$  depends on a, such that  $\mathcal{L}g(t, z) \ge 0$  for<br>
ave that<br>  $B(x_0, h)$ ] –  $a(y - x_0)^2 = \mathbb{E}^y \left[ \int_0^{\tau_{B(x_0, h)}} \mathcal{L}g(t, z) dx_0 + rh$  for some  $r \in (-1, 1)$ . Then the  $\mathcal{L}g(t, z_$ By Assumption [2.1\(](#page-5-0)ii),  $\sigma^2(x_0) > 0$ . For any constant  $a > \frac{1}{\sigma^2(x_0)}$ , by the continuity of  $\sigma^2(x_0) > 0$ . For any constant  $a >$ <br>ch only depends on a, such that  $\mu$ <br>), we have that<br> $-\mathbb{E}^y[\tau_{B(x_0,h)}] - a(y - x_0)^2 = \mathbb{E}^y$ <br>ite  $y = x_0 + rh$  for some  $r \in (-1, +rh)[(X_{\tau_{B(x_0,h)}} - x_0)^2] - a(rh)^2 =$ <br>stant  $0 < \tilde{a} < \frac{1}{\sigma^2(x_0)}$  $\frac{1}{t}$   $\mu(x)$  and  $\sigma(x)$ ,<br>  $_0$ , h). Applying<br>  $\forall y \in B(x_0, h)$ <br>
lity leads to<br>  $^2$ ,  $\forall r \in (-1,$ <br>  $(61)$ <br>
n  $\tilde{a}$ , such that<br>  $(62)$ we can find  $h > 0$ , which only depends on a, such that  $\mathcal{L}g(t, z) \ge 0$  for any  $z \in B(x_0, h)$ . Applying Ito's formula to  $g(t, X_t)$ , we have that

we can find 
$$
h > 0
$$
, which only depends on a, such that  $\mathcal{L}g(t, z) \ge 0$  for any  $z \in B(x_0, h)$ . Applying  
\nIto's formula to  $g(t, X_t)$ , we have that\n
$$
a\mathbb{E}^y \Big[ (X_{\tau_{B(x_0, h)}} - x_0)^2 \Big] - \mathbb{E}^y[\tau_{B(x_0, h)}] - a(y - x_0)^2 = \mathbb{E}^y \Big[ \int_0^{\tau_{B(x_0, h)}} \mathcal{L}g(t, X_t) dt \Big] \ge 0, \quad \forall y \in B(x_0, h)
$$
\nFor  $y \in B(x_0, h)$ , rewrite  $y = x_0 + rh$  for some  $r \in (-1, 1)$ . Then the above inequality leads to\n
$$
\mathbb{E}^{x_0 + rh} [\tau_{B(x_0, h)}] \le a\mathbb{E}^{x_0 + rh} \Big[ (X_{\tau_{B(x_0, h)}} - x_0)^2 \Big] - a(rh)^2 = ah^2 - ar^2h^2 = a(1 - r^2)h^2, \quad \forall r \in (-1, 1)
$$
\nSimilarly, for any constant  $0 < \tilde{a} < \frac{1}{\sigma^2(x_0)}$ , we can find  $\tilde{h}$ , which only depends on  $\tilde{a}$ , such that\n
$$
\mathcal{L}g(t, z) \le 0
$$
 on  $B(x_0, \tilde{h})$ , and\n
$$
\mathbb{E}^{x_0 + r\tilde{h}}[\tau_{B(x_0, \tilde{h})}] \ge \tilde{a}(1 - r^2)\tilde{h}^2, \quad \forall r \in (-1, 1).
$$
\n(62)

For  $y \in B(x_0, h)$ , rewrite  $y = x_0 + rh$  for some  $r \in (-1, 1)$ . Then the above inequality leads to

$$
\mathbb{E}^{x_0+rh}[\tau_{B(x_0,h)}] \le a \mathbb{E}^{x_0+rh} \Big[ (X_{\tau_{B(x_0,h)}} - x_0)^2 \Big] - a(rh)^2 = ah^2 - ar^2h^2 = a(1-r^2)h^2, \quad \forall r \in (-1,1).
$$
\n(61)

For  $\mathbb{E}^{x_0}$ .<br>Sim  $\mathcal{L}g$  $(e^{t}B(x_0, h))$   $(e^{t})$ <br>  $\in B(x_0, h)$ , re<br>  $(h[\tau_{B(x_0, h)}] \le a$ <br>
arly, for any c<br>  $z) \le 0$  on  $B(x_0, h)$ 2<br>2<br>2<br>2<br>2<br>2 ite  $y = x_0 + rh$  for some  $r_0$ <br>
ite  $y = x_0 + rh$  for some  $r_0$ <br>  $e^{n+h} \left[ (X_{\tau_{B(x_0,h)}} - x_0)^2 \right] - a$ <br>
stant  $0 < \tilde{a} < \frac{1}{\sigma^2(x_0)}$ , we  $\tilde{a}$ <br>  $\tilde{a}$ ), and<br>  $\mathbb{E}^{x_0 + rh} [\tau_{B(x_0,\tilde{h})}] \ge \tilde{a}(1$ .<br>22<br>- $=$   $(-1)^n$ <br> $= (-1)^n$ <br> $=$   $(n+1)^2$   $=$  n fin Then<br> $u^2 - a$ <br>whic<br> $\forall r \in$ )<br>|}<br>h ality leads to<br>  $dh^2$ ,  $\forall r \in (-1, 1)$ <br>
(61)<br>
on  $\tilde{a}$ , such that<br>
(62)  $y \in B(x_0, h)$ , rewrite  $y = x_0 + rh$  for some  $r \in (-1, 1)$ . Then the above inequality leads to  $\mathbb{R}^{r}[\tau_{B(x_0, h)}] \le a \mathbb{E}^{x_0 + rh} \Big[ (X_{\tau_{B(x_0, h)}} - x_0)^2 \Big] - a(rh)^2 = ah^2 - ar^2h^2 = a(1 - r^2)h^2$ ,  $\forall r \in (-1, 1]$  on  $B(x_0, \tilde{h})$ , and  $\$  $\mathbb{E}^{x_0 + rh}[\tau_{B(x_0, h)}] \le a \mathbb{E}^{x_0 + rh}$ <br>
Similarly, for any constant<br>  $\mathcal{L}g(t, z) \le 0$  on  $B(x_0, \tilde{h})$ , an<br>  $\mathbb{E}^{x_0 + rh}$ ( $0 < \tilde{a} < \frac{1}{\sigma^2(x_0, \tilde{h})}$ <br>
( $0 < \tilde{a} < \frac{1}{\sigma^2(x_0)}$ <br>
(d  $\frac{1}{2}$   $\geq$  $- u(rn)^2 = an^2 - ar^2n^2 = a(1 - r^2)n^2$ ,  $\forall r \in (-1, 1).$  (61)<br>we can find  $\tilde{h}$ , which only depends on  $\tilde{a}$ , such that<br> $\tilde{a}(1 - r^2)\tilde{h}^2$ ,  $\forall r \in (-1, 1).$  (62) Similarly, for any constant  $0 < \tilde{a} < \frac{1}{\gamma}$ , we can find 0 <  $\tilde{a} < \frac{1}{\sigma^2(z)}$ <br>
1<br>
0<sup>+*rh*</sup>[ $\tau_{B(x_0,\tilde{h})}$ ] *h*, which only depends on  $\tilde{a}$ , such that<br>  $\forall r \in (-1, 1).$  (62)  $\mathcal{L}g(t, z) \leq 0$  on  $B(x_0, \tilde{h})$ , and  $g(t, z) \le 0$  on  $B(x_0, \tilde{h})$ , and<br> $\mathbb{E}^{x_0}$ 

and  
\n
$$
\mathbb{E}^{x_0 + r\tilde{h}}[\tau_{B(x_0,\tilde{h})}] \ge \tilde{a}(1 - r^2)\tilde{h}^2, \quad \forall r \in (-1, 1).
$$
\n(62)

<span id="page-20-0"></span>By [\(61\)](#page-19-0) and [\(62\)](#page-19-0), for any  $H \in (0, h \wedge \tilde{h}]$  we have that

$$
\tilde{a} \le \frac{\mathbb{E}^{x_0 + rH}[\tau_{B(x_0, H)}]}{(1 - r^2)H^2} \le a, \quad \text{for all } r \in (-1, 1), \ \tilde{a} \in \left(0, \frac{1}{\sigma^2(x_0)}\right) \text{ and } a > \frac{1}{\sigma^2(x_0)}
$$

Let  $\tilde{a} = \frac{1}{\sigma^2(x_0)} - \varepsilon$  and  $a = \frac{1}{\sigma^2(x_0)} + \varepsilon$  for any  $\varepsilon > 0$  and then take  $H \setminus 0$  for the above inequality. By the arbitrariness of  $\varepsilon$ , Equation (60) follows.

Next, we prove Equation [\(59\)](#page-19-0). Consider the function  $g(t, z) := \delta(t)|z - x_0|$ . For  $h > 0$  and  $y \in$ 

$$
\mathbb{E}^{y}[\delta(\tau_{B(x_0,h)})|X_{\tau_{B(x_0,h)}}-x_0|]-|y-x_0|=\mathbb{E}^{y}\left[\int_0^{\tau_{B(x_0,h)}}\frac{1}{2}(\mathcal{L}g(t,X_t-)+\mathcal{L}g(t,X_t+))dt\right] + \mathbb{E}^{y}\left[\int_0^{\tau_{B(x_0,h)}}\frac{1}{2}(1-(-1))\delta(t)dt_t^{x_0}\right].
$$
\n(63)

By Lemma [2.7,](#page-7-0)  $|\delta'(t)| \leq |\delta'(0)|\delta(t) \leq |\delta'(0)|$ . This implies that

$$
\left|\frac{1}{2}(\mathcal{L}g(t,z-)+\mathcal{L}g(t,z+))\right| \leq |\delta'(t)| \cdot |z-x_0| + \delta(t)|\mu(z)| \leq |\delta'(0)| \cdot |z-x_0| + |\mu(z)|. \tag{64}
$$

By Equations  $(63)$  and  $(64)$ , we have that

$$
h\mathbb{E}^{y}[\delta(\tau_{B(x_0,h)})] - |y - x_0| - \mathbb{E}^{y} \left[ \int_0^{\tau_{B(x_0,h)}} (|\delta'(0)| |X_t - x_0| + |\mu(X_t)|) dt \right]
$$
  
\n
$$
\leq \mathbb{E}^{y} \left[ \int_0^{\tau_{B(x_0,h)}} \delta(t) dL_t^{x_0} \right]
$$
  
\n
$$
\leq h\mathbb{E}^{y}[\delta(\tau_{B(x_0,h)})] - |y - x_0| + \mathbb{E}^{y} \left[ \int_0^{\tau_{B(x_0,h)}} (|\delta'(0)| |X_t - x_0| + |\mu(X_t)|) dt \right].
$$
\n(65)

Notice that  $|X_t - x| \le h$  for  $t \le \tau_{B(x_0, h)}$  and  $\sup_{z \in B(x_0, 1)} |\mu(z)| \le K$  for some constant  $K > 0$  that depends on  $x_0$ . Then, for  $h \le 1$ , by rewriting  $y = x_0 + rh$  in Equation (65) we have that

$$
h\mathbb{E}^{x_0+rh}[\delta(\tau_{B(x_0,h)})] - h|r| - (h|\delta'(0)| + K) \cdot \mathbb{E}^{x_0+rh}[\tau_{B(x_0,h)}]
$$
  
\n
$$
\leq \mathbb{E}^{x_0+rh} \left[ \int_0^{\tau_{B(x_0,h)}} \delta(t) dL_t^{x_0} \right]
$$
  
\n
$$
\leq h\mathbb{E}^{x_0+rh}[\delta(\tau_{B(x_0,h)})] - h|r| + (h|\delta'(0)| + K) \cdot \mathbb{E}^{x_0+rh}[\tau_{B(x_0,h)}], \quad \forall r \in (-1,1).
$$

Then,

By (61) and (62), for any 
$$
H \in (0, h \wedge h]
$$
 we have that  
\n
$$
a \leq \frac{\mathbb{E}^{x_0 + rH}[\tau_{B(x_0, H)}}{(1 - r^2)H^2} \leq a, \text{ for all } r \in (-1, 1), a \in (0, \frac{1}{\sigma^2(x_0)}) \text{ and } a > \frac{1}{\sigma^2(x_0)}.
$$
\nLet  $\tilde{a} = \frac{1}{\sigma^2(x_0)} - \varepsilon$  and  $a = \frac{1}{\sigma^2(x_0)} + \varepsilon$  for any  $\varepsilon > 0$  and then take  $H \searrow 0$  for the above inequality.  
\nBy the arbitrariness of  $\varepsilon$ , Equation (65)) Consider the function  $\mathfrak{g}(t, z) := \delta(t)|z - x_0|$ . For  $h > 0$  and  $y \in B(x_0, h)$ , applying Lemma 2.15 to  $\mathfrak{g}(t, X_t)$ , we have that  
\n
$$
\mathbb{E}^y[\delta(\tau_{B(x_0, h)})|X_{\tau_{B(x_0, h)}} - x_0|] - |y - x_0| = \mathbb{E}^y \left[ \int_0^{\tau_{B(x_0, h)}} \frac{1}{2}(L\mathfrak{g}(t, X_t -) + L\mathfrak{g}(t, X_t +))dt \right] + \mathbb{E}^y \left[ \int_0^{\tau_{B(x_0, h)}} \frac{1}{2}(1 - (-1))\delta(t)dt_x^{x_0} \right].
$$
\nBy Lemma 2.7,  $|\delta'(t)| \leq |\delta'(0)|\delta(t) \leq |\delta'(0)|$ . This implies that  
\n
$$
\left| \frac{1}{2}(L\mathfrak{g}(t, z -) + L\mathfrak{g}(t, z +)) \right| \leq |\delta'(t)| \cdot |z - x_0| + \delta(t)|\mu(z)| \leq |\delta'(0)| \cdot |z - x_0| + |\mu(z)|.
$$
\n(64)  
\nBy Equations (63) and (64), we have that  
\n
$$
h\mathbb{E}^y[\delta(\tau_{B(x_0, h)})] - |y - x_0| - \mathbb{E}^y \left[ \int_0^{\tau_{B(x_0, h)}} (|\delta'(0)| |X_t - x_0| + |\mu
$$

By the second inequality in Lemma [2.7,](#page-7-0) it holds uniformly in  $r \in (-1, 1)$  that

$$
|\mathbb{E}^{x_0+rh}[\delta(\tau_{B(x_0,h)})] - 1| = \mathbb{E}^{x_0+rh}[1 - \delta(\tau_{B(x_0,h)})] \leq |\delta'(0)| \cdot \mathbb{E}^{x_0+rh}[\tau_{B(x_0,h)}] \stackrel{h\searrow 0}{\longrightarrow} 0.
$$

This together with Equation [\(60\)](#page-19-0) implies that

$$
\frac{h^2(\mathbb{E}^{x_0+rh}[\delta(\tau_{B(x_0,h)})] - |r|)}{\mathbb{E}^{x_0+rh}[\tau_{B(x_0,h)}]} \xrightarrow{h \searrow 0} \frac{\sigma^2(x_0)}{1-r^2} \cdot (1-|r|) = \frac{\sigma^2(x_0)}{1+|r|}
$$
 uniformly for  $r \in (-1,1)$ . (67)

Notice that  $\lim_{h\setminus 0} (\frac{\delta'(0)}{h^2 + Kh}) = 0$ . This together with Equations (66) and (67) implies Equation [\(59\)](#page-19-0).

Now we are ready to deal with Equation (24) in the following proposition. The verification for Equation [\(25\)](#page-10-0) follows next.

**Proposition 4.7.** *Let Assumptions 2.1–2.10 hold and optimal mild equilibrium, then*

$$
V_x(0, x-, S) \ge V_x(0, x+, S) \quad \forall x \in S.
$$

□ *Proof.* Notice that Assumption [2.8\(](#page-7-0)ii) and Lemma 2.14 guarantees the existence of  $V_x(t, x \pm S)$ and  $\mathcal{L}V(t, x\pm, S)$  for any  $(t, x) \in [0, \infty) \times \mathbb{X}$ . We prove the desired result by contradiction. Take

$$
a := V_x(0, x_0 +, S) - V_x(0, x_0 -, S) > 0.
$$
\n<sup>(68)</sup>

Recall  $G$  defined in Equation [\(16\)](#page-7-0). To reach to a contradiction, we will construct a new mild equilibrium, which is strictly better than  $S$ , for each of the three cases:

(i)  $x_0 \in \partial S$  for boundary case (a); (ii)  $x_0 \in \partial S$  for boundary case (b); (iii)  $x_0 \in S^\circ$ .

 $r \in (-1, 1)$  that<br>  $l'(0) \cdot E^{x_0 + rh} [r_B$ <br>  $\frac{1}{l}$  uniformly for<br>  $\frac{1}{l}$  equations (64<br>  $\log$  proposition.<br>  $\log$  proposition.<br>  $\log$  proposition.<br>  $\log$  and  $\log$  is the exist exist exist exist exist by S) > 0.<br>  $\log$  ion, w  $E^{x_0+h}[ \delta(\tau_{B(x_0,h)})] - 1] = E^{x_0+h}[h - \delta(\tau_{B(x_0,h)})] \leq |\delta'|$ <br>
ether with Equation (60) implies that<br>  $e^{x+h}[\delta(\tau_{B(x_0,h)})] - |r|) \xrightarrow{h \searrow 0} \sigma^2(\chi_0) \cdot (1 - |r|) = \frac{\sigma^2(\chi_0 + \chi_0)}{1 + |r|}$ <br>
that  $\lim_{h \searrow 0} \delta(\delta'(0)|h^2 + Kh) = 0$ . This together (0)| ⋅ <u>L</u> +  $\left[\frac{P(B(x_0, h))}{P(B(x_0, h))}\right]$  or 0.<br>
informly for  $r \in (-1, 1]$ <br>
Equations (66) and (67)<br>
ing proposition. The verifica<br>
idmissible stopping policy. If<br>  $\in S$ .<br>
interes the existence of  $V_x(t)$ <br>
sired result by co  $(-1)$  (67 erifidicy of  $V_1$  divided that a that if  $y \geq 0$ ℎ2(0+ℎ[((0,ℎ))] − ||) Ex<sub>0</sub>+rh[ $\tau_{B(X_0,h)}$   $\longrightarrow$   $\frac{1-\gamma^2}{1-\gamma^2}$ <br>
hat  $\lim_{h\searrow 0}(|\delta'(0)|h^2 + Kh) = 0$ .<br>
(59).<br>
we are ready to deal with Equation<br>
(25) follows next.<br>
tion 4.7. Let Assumptions 2.1-2.1d<br>
mild equilibrium, then<br>  $V_x(0, x-, S)$ <br>
(otice 1−2<sup>2</sup> · (1 − |r|) =  $\frac{x}{1+|r|}$ <br>
1−2<sup>2</sup> · (1 − |r|) =  $\frac{x}{1+|r|}$ <br>
1=0. This together with<br>
puation (24) in the followir<br>
2.1-2.10 hold and S be an a<br>
x-, S) ≥  $V_x(0, x+, S)$  ∀x<br>
ii) and Lemma 2.14 guara<br>
∞) × X. We pro 1 +  $\frac{V}{1 + |r|}$  uniformly for  $r \in (-1, 1)$ . (67)<br>
∴ with Equations (66) and (67) implies<br>
∴<br>
Illowing proposition. The verification for<br> *ve an admissible stopping policy. If S is an*<br>
(a)  $\forall x \in S$ .<br>
guarantees the exi mm<sub>h</sub> <sub>\o</sub>(|*o*)<br>
i.e ready to<br>
() follows n<br>
4.7. Let A<br>
equilibrium<br>
e that Assued that Assued that Assued in Equilibrium<br>
i.e of the B which is started in Equilibrium<br>
for bound<br>
for bound<br>  $\therefore$ <br>  $\begin{aligned}\n\therefore \mathbb{P}^y \Big[$ (0)| $h^2 + Kh$ ) = 0. This together with Equations [\(66\)](#page-20-0) and (67) implies<br>
deal with Equation (24) in the following proposition. The verification for<br>
ext.<br>
stamptions 2.1-2.10 hold and S be an admissible stopping policy. If S *S* be an admissible stopping policy. If *S* is an  $\cdot$ , *S*)  $\forall x \in S$ .<br>
14 guarantees the existence of  $V_x(t, x \pm, S)$ <br>
we the desired result by contradiction. Take<br>
(0,  $x_0$ –, *S*) > 0. (68)<br>
ntradiction, we will construc  $V_x(0, x-, S) \ge V_x(0, x+, S)$  ∀  $x \in S$ .<br>
on 2.8(ii) and Lemma 2.14 guarantees<br>  $0 \in [0, \infty) \times \mathbb{X}$ . We prove the desired<br>  $\iota := V_x(0, x_0+, S) - V_x(0, x_0-, S) > 0$ <br>
(1(6). To reach to a contradiction, v<br>
better than *S*, for each of th iction. Take<br>
(68)<br>
a new mild<br>
(68)<br>
a new mild<br>  $(x_0, x_0 + y \in S)$ , and<br>
(69)<br>  $\ge 0.$  (70)<br>
(71)  $V(t, x±, S)$  for any  $(t, x) ∈ [0, ∞) × X$ . We prove the desired result by contradiction. Take<br>
and suppose<br>  $a := V_x(0, x_0+, S) - V_x(0, x_0-, S) > 0.$  (68)<br>  $G$  defined in Equation (16). To reach to a contradiction, we will construct a n  $x_0 \in S$  and suppose<br>
Recall *G* defined in<br>
equilibrium, which<br>
(i)  $x_0 \in \partial S$  for bot<br>
(ii)  $x_0 \in \partial S$  for bot<br>
(iii)  $x_0 \in S^\circ$ .<br> **Case (i)**  $x_0 \in \partial S$ <br>  $h_0$ ) ⊂ ( $S^\circ \cap G$ ) and (<br>
note that *l* can be —<br> *Step 1*. We  $a: = V_x(0, x_0 +, S) - V_x(0, x_0 -, S) > 0.$  (68)<br>
n (16). To reach to a contradiction, we will construct a new mild<br>
better than *S*, for each of the three cases:<br>
ase (a);<br>
adary case (a). Without loss of generality, we assume t S, for each of the three cases:<br>
(1). Without loss of generality,<br>
some  $h_0 > 0$ . Denote  $l := \sup$ <br>
(2006) for this case in three step<br>
(2007),  $h_0$  such that,<br>
(2007) =  $f(y) > 0$ ,  $\forall y \in (x_0 - k_0)$ <br>
(30) that<br>
(30) that<br>
(30  $x_0 \in \partial S$  for boundary case (a);<br>  $x_0 \in \partial S$  for boundary case (b);<br>  $x_0 \in S^\circ$ .<br> **se (i)**  $x_0 \in \partial S$  for boundary c<br>  $\vdots$  (S° ∩ *G*) and  $(x_0 - h_0, x_0) \subset S$ <br>
that *l* can be  $-\infty$ . We proceed<br> *ep 1*. We show that there  $x_0 \in \partial S$  for boundary case (b);<br>  $x_0 \in S^\circ$ .<br> **se (i)**  $x_0 \in \partial S$  for boundary c<br>  $\vdots$  (S° ∩ *G*) and  $(x_0 - h_0, x_0) \subset S$ <br>
that *l* can be  $-\infty$ . We proceed<br> *ep 1*. We show that there exists<br>  $\mathbb{E}^y \left[ V(\tau_{B(x_0,h)}, X_{\tau$  $x_0 \in S^\circ$ .<br> **sse (i)** x<br>  $\therefore (S^\circ \cap G)$ <br>
that l ca<br>
ep 1. We :<br>  $\therefore$ <br>  $\therefore$  e from I<br>  $V_x(t, x_0 - t) \in (0, h_0)$ **Case (i)**  $x_0 \in \partial S$  for boundary case (a). Without loss of generality, we assume that  $(x_0, x_0 +$  $x_0 \in \partial S$  for boundary case (a). Without loss of generality, we assume that  $(x_0, x_0 + G)$  and  $(x_0 - h_0, x_0) \subset S^c$  for some  $h_0 > 0$ . Denote  $l := \sup\{y \le x_0 - h_0 : y \in S\}$ , and an be  $-\infty$ . We proceed the proof for this case i *h*<sub>0</sub>) ⊂ (S° ∩ *G*) and (x<sub>0</sub> − *h*<sub>0</sub>, x<sub>0</sub>) ⊂ S<sup>c</sup> for some *h*<sub>0</sub> > 0. Denote *l* : = sup{*y* ≤ *x*<sub>0</sub> − *h*<sub>0</sub> : *y* ∈ *S*}, and note that *l* can be −∞. We proceed the proof for this case in three steps.<br> *Step 1*. note that  $l$  can be  $-\infty$ . We proceed the proof for this case in three steps.

*Step 1*. We show that there exists  $h \in (0, h_0)$  such that,

*l* can be 
$$
-\infty
$$
. We proceed the proof for this case in three steps.  
\nWe show that there exists  $h \in (0, h_0)$  such that,  
\n
$$
\mathbb{E}^y \Big[ V(\tau_{B(x_0, h)}, X_{\tau_{B(x_0, h)}}, S) \Big] - f(y) > 0, \quad \forall y \in (x_0 - h, x_0 + h).
$$
\n(69)  
\n
$$
\text{Im Lemma 3.10 and Equation (68) that}
$$
\n
$$
x_0+, S) - V_x(t, x_0-, S) \ge \delta(t) (V_x(0, x_0+, S) - V(0, x_0-, S)) = a\delta(t), \quad \forall t \ge 0.
$$
\n(70)  
\n
$$
\text{D}, h_0 \text{ and pick an arbitrary } x \in B(x_0, h). \text{ For all } n \in \mathbb{N}, \text{ write}
$$
\n
$$
\tau_n := \tau_{B(x_0, h)} \wedge n \tag{71}
$$

Notice from Lemma [3.10](#page-15-0) and Equation (68) that

$$
\mathbb{E}\left[V(t_{B(x_0,h)}, X_{\tau_{B(x_0,h)}}, S)\right] - J(y) > 0, \quad \forall y \in (x_0 - h, x_0 + h). \tag{69}
$$
\nce from Lemma 3.10 and Equation (68) that

\n
$$
V_x(t, x_0+, S) - V_x(t, x_0-, S) \ge \delta(t)(V_x(0, x_0+, S) - V(0, x_0-, S)) = a\delta(t), \quad \forall t \ge 0. \tag{70}
$$
\n
$$
u \in (0, h_0) \text{ and pick an arbitrary } x \in B(x_0, h). \text{ For all } n \in \mathbb{N}, \text{ write}
$$
\n
$$
\tau_n := \tau_{B(x_0, h)} \wedge n \tag{71}
$$

Fix  $h \in (0, h_0)$  and pick an arbitrary  $x \in B(x_0, h)$ . For all  $n \in \mathbb{N}$ , write  $h \in (0, h_0)$  and pick an arbitrary  $x \in B(x_0, h)$ . For all  $n \in \mathbb{N}$ , write<br> $\tau_n := \tau_{B(x_0, h)} \wedge n$ 

$$
\tau_n := \tau_{B(x_0, h)} \wedge n \tag{71}
$$

<span id="page-21-0"></span>

<span id="page-22-0"></span>for short. We apply Lemma [2.15](#page-9-0) to  $V(t, X_t, S)$  on  $[0, \tau_n]$  and take expectation; the diffusion term vanishes under expectation due to Lemma [2.14](#page-9-0) and continuity of  $\sigma$ . Then combining with Equation [\(70\)](#page-21-0), we have that

$$
\mathbb{E}^{x}\left[V\left(\tau_{n}, X_{\tau_{n}}, S\right)\right] - V(0, x, S)
$$
\n
$$
\geq \mathbb{E}^{x}\left[\int_{0}^{\tau_{n}} \frac{1}{2} (\mathcal{L}V(t, X_{t}-, S) + \mathcal{L}V(t, X_{t}+, S))dt\right] + a\mathbb{E}^{x}\left[\int_{0}^{\tau_{n}} \delta(t) dL_{t}^{x_{0}}\right].
$$
\n(72)

By Lemma [2.14\(](#page-9-0)b), we have

$$
M := \sup_{(t,y)\in[0,\infty)\times B(x_0,h_0)} \frac{1}{2}(|\mathcal{L}V(t,y-,S)| + |\mathcal{L}V(t,y+S)|) < \infty.
$$

This together with Equation (72) implies that

$$
\mathbb{E}^{x}[V(\tau_{n}, X_{\tau_{n}}, S)] - V(0, x, S) \ge -M \mathbb{E}^{x}[\tau_{n}] + a \mathbb{E}^{x} \left[ \int_{0}^{\tau_{n}} \delta(t) dL_{t}^{x_{0}} \right] \quad \forall n \in \mathbb{N}.
$$
 (73)

For the LHS of Equation (73), Equation [\(17\)](#page-7-0) readily implies that

$$
\lim_{n \to \infty} \mathbb{E}^{x_0} [V(\tau_n, X_{\tau_n}, S) = \mathbb{E}^{x_0} [V(\tau_{B(x_0, h)}, X_{\tau_{B(x_0, h)}}, S)]. \tag{74}
$$

 $V(t, X_t, S)$  on  $[0, \tau_n]$  and take expectation; the diffusion<br>to Lemma 2.14 and continuity of  $\sigma$ . Then combining with<br>to Lemma 2.14 and continuity of  $\sigma$ . Then combining with<br>x, S)<br> $S$  +  $\mathcal{L}V(t, X_t +, S)dt$  +  $\mathbf{d}\mathbb{E}$  $\sigma$ . Then combining with<br>  $\{\sigma_t\}_{\sigma}$  find  $\delta(t) dL_t^{x_0}$ . (72)<br>  $\sigma_t^{x_0}$  (72)<br>  $\forall n \in \mathbb{N}$ . (73)<br>  $\forall n \in \mathbb{N}$ . (73)<br>  $\sigma_t^{x_0}$   $\forall n \in \mathbb{N}$ . (73)<br>  $t \geq \tau_{B(x_0, h)}, X_t \in S$ . We<br>
for all  $n \in \mathbb{N}$ , and<br>
s.. Then by  $\mathbb{E}^x$   $\geq$   $\mathbb{I}$ <br> $\leq$   $\mathbb{I}$   $\mathbb{I}$ <br> $(\tau_n, \tau_n), \mathbb{I}$ <br> $(\tau_n, \tau_n), \mathbb{I}$ <br> $(\tau_n, \tau_n), \mathbb{I}$ <br> $\geq$   $\geq$   $\geq$   $\geq$   $\mathbb{E}^x$ <br> $\in$   $\mathbb{$  $\begin{bmatrix} x^x & b \end{bmatrix}$  and  $\begin{bmatrix} x & d \end{bmatrix}$  and  $\begin{bmatrix} x & f \end{bmatrix}$  and  $\begin{bmatrix} x & f \end{bmatrix}$  and  $\begin{bmatrix} x & f \end{bmatrix}$  and  $\begin{bmatrix} x & h \end{bmatrix}$  and  $\begin{bmatrix} x & h \end{bmatrix}$  and  $\begin{bmatrix} g(x) & h \end{bmatrix}$  $\tau_n, X_{\tau_n}, S$ ] –  $V(0, x, S)$ <br>  $\int_0^{\tau_n} \frac{1}{2} (CV(t, X_t-, S) +$ <br>
we have<br>  $:=$  sup  $\lim_{(t,y)\in[0,\infty)\times B(x_0,h_0)} \frac{1}{2}$ <br>
Equation (72) implies t<br>  $,X_{\tau_n}, S$ ] –  $V(0, x, S) \ge$ <br>
aation (73), Equation (17<br>  $\lim_{n\to\infty} \mathbb{E}^{x_0} [V(\tau_n, X_{\tau$  0  $\frac{1}{2}$  av  $\frac{1}{2}$  ti  $S$  (  $\frac{1}{2}$  ti  $\frac{1}{2}$  dia h  $\frac{1}{2}$  ti  $\frac{1}{2}$  ) a(  $\frac{1}{2}$  e) ave the Second of the Change of the Cha ( $\mathcal{L}V(t, X_t-, S) + \mathcal{L}V(t, X_t+, S))dt$ <br>  $\downarrow$  e<br>  $\text{sup}$ <br>  $\text{sup}$ <br>  $\frac{1}{2}(|\mathcal{L}V(t, y-, S)| + |t|$ <br>
on (72) implies that<br>  $|J| - V(0, x, S) \ge -M\mathbb{E}^x[\tau_n] + a\mathbb{E}^x$ <br>  $\uparrow$  73), Equation (17) readily implies that<br>  $\downarrow$   $\mathbb{E}^{x_0}[$  $\mathcal{L}V(t, y)$ <br> $\int_0^{\tau_n} \delta$ <br>aat<br> $\int_0^{\tau_n} \mathcal{L}_{\tau_{B(x_0)}}$ <br> $\eta := \eta_n \int_0^{\tau_n} \mathcal{L}_{\tau_n}$ <br>to get fiequa<br> $\int_0^{\tau_n} \mathcal{L}_{\tau_n}$ <br> $\int_0^{\tau_n} \mathcal{L}_{\tau_n}$ <br>aall en<br> $B(x_0, h)$ )  $L_t^2$   $\frac{L_t^2}{2}$   $\frac{1}{2}$   $\frac{1}{2}$ S d s f ) a n s r (o v g ) )  $\begin{aligned} \mathcal{L}_0(t) &\text{d} t \leq \infty. \end{aligned}$ <br>  $\begin{aligned} \geq \tau_{B(x_0,h)} \text{ for all } \\ \text{for all } \mathcal{L}_t \leq \mathbb{E}^{x_0}[\delta] \text{ and } \\ &\text{if } \mathcal{L}_0 \leq \mathcal{L}_t \leq \mathcal{L}_t$  $\frac{1}{2}$  . The contract of  $\frac{1}{2}$  is the co . (*t*,*y*)∈[0,∞)×<br>
th Equation (72)<br>  $\tau_n$ ,  $X_{\tau_n}$ ,  $S$ )] –  $V$ ((<br>
(quation (73), Eq<br>  $\lim_{n\to\infty} \mathbb{E}^{x_0}[V$ <br>  $:= \inf\{t \geq \tau_n$ ,  $X_t$ <br>  $X_{\tau_n}$ ,  $S$ )] =  $\mathbb{E}^{x}[\mathbb{E}^{x}$ <br>  $X_{\tau_n}$ ,  $S$ )] =  $\mathbb{E}^{x_0}[\mathbb{E}^{x}$ <br>
that (,)∈[0,∞)×(0,ℎ0) 1 2 (| $\mathcal{L}V(t, y-, S)| + |\mathcal{L}V(t, y+, S)|$ ) < ∞.<br>
hat<br>
hat<br>  $-M\mathbb{E}^{\times}[\tau_n] + a\mathbb{E}^{\times} \left[ \int_0^{\tau_n} \delta(t) dL_t^{x_0} \right]$  ∀,<br>  $\mathcal{L}Y$ ) readily implies that<br>  $S) = \mathbb{E}^{x_0}[V(\tau_{\mathcal{B}(x_0, h)}, X_{\tau_{\mathcal{B}(x_0, h)}}, S)]$ ,<br>
or all  $n \in \mathbb{N}$ , an  $E^{\times}[V(\tau_n, X_{\tau_n}, S)] - V(0, x, S) \ge -M E^{\times}[\tau_n] + \alpha E^{\times}$ <br>
IS of Equation (73), Equation (17) readily implies<br>  $\lim_{n\to\infty} E^{\times_0}[V(\tau_n, X_{\tau_n}, S)] = E^{\times_0}[V(\tau_{B(x_0)})$ <br>
et  $\eta_n := \inf\{t \ge \tau_n, X_t \in S\}$  for all  $n \in \mathbb{N}$ , and<br>  $V(\tau_n, X_{\tau_n$  $\tau_B$  :  $\eta$   $\eta$   $\mathbf{g}$   $\mathbf{q}$   $\mathbf{$ 0  $\begin{aligned} \mathcal{L}_{(x_0,h)}, S) &\Big| \&= \inf \{ t \geq \begin{cases} X_{\eta_n} \end{cases} \Big| \begin{cases} 1 & \text{if } 1 \leq x_0 \\ \text{if } \mathbb{P}^x\text{-a.s.}. \end{cases} \Big| \& \text{if } \mathbb{P}^x\text{-a.s.} \Big| \& \text{if } \mathbb{P}^x\text$   $\forall n \in \mathbb{N}.$  (73)<br>  $(x_0, h), X_t \in S$ }. We<br>
all  $n \in \mathbb{N}$ , and<br>
en by Assumption<br>  $x_0[\delta(\eta_n)f(X_{\eta_n})] =$ <br>  $n {\eta = \infty}$ }.)<br>
ad combining with<br>  $\delta(t) dL_t^{x_0}$ . (75)<br>  $\delta(t) dL_t^{x_0}$ . (76)<br>
1, 1).<br>
that<br>  $-1, 1$ .<br>  $\forall r \in (-1, 1)$ ,<br>  $\$ lime  $E^{x_0}[V(\tau_n, X_{\tau_n}, S) = E^{x_0}[V(\tau_{B(x_0, h)}, X_{\tau_{B(x_0, h)}}, S)].$ (74)<br>  $\pi_{\tau_{\infty}}$  [ε'  $\geq \tau_n, X_i \in S$ } for all  $n \in \mathbb{N}$ , and  $\eta := \inf\{i \geq \tau_{B(x_0, h)}, X_i \in S\}$ . We<br>  $\pi_{\tau_{\infty}}$  [E' $E^{x}[E^{x}[\delta(\eta_n)f(X_{\eta_n}) | \mathcal{F}_{\tau_n}]] = E^{x}[[\delta(\eta_n$ Indeed, set  $\eta_n := \inf\{t \geq \tau_n, X_t \in S\}$  for all  $n \in \mathbb{N}$ , and  $\eta := \inf\{t \geq \tau_{B(x_0, h)}, X_t \in S\}$ . We  $n_n$  : = inf{ $t \ge \tau_n$ ,  $X_t \in S$ } for all  $n \in \mathbb{N}$ , and  $\eta$  : = inf{ $t \ge \tau_{B(x_0,h)}, X_t \in S$ }. We<br>  $\tau_n$ ,  $X_{\tau_n}$ , 5)] =  $E^{(k)}[\delta(n_n)/f(X_{\eta_n})], F_{n}$ ;  $\prod_{i=1}^{N} E^{(i)}[\delta(n_n)/f(X_{\eta_n})],$  for all  $n \in \mathbb{N}$ , and<br>  $\pi_n X_{\tau_n}$ , s) = have  $\mathbb{E}^{x}[V(\tau_n, X_{\tau_n}, S)] = \mathbb{E}^{x}[\mathbb{E}^{x}[\delta(\eta_n)f(X_{\eta_n})|\mathcal{F}_{\tau_n}]] = \mathbb{E}^{x}[\delta(\eta_n)f(X_{\eta_n})]$  for all  $n \in \mathbb{N}$ , and  $\mathbb{E}^{x_0}[V(\tau_{B(x_0,h)}, X_{\tau_{B(x_0,h)}}, S)] = \mathbb{E}^{x_0}[\delta(\eta)f(X_{\eta})]$ . As  $n \to \infty$ ,  $\eta_n \to \eta$ ,  $\mathbb{P}^{x}$ -a.s  $E^{\infty}[V(\tau_n, X_{\tau_n}, S)] = E^{\infty}[E^{\infty}[\delta(\eta_n) f(X_{\tau_n})] | \mathcal{F}_{\tau_n}] = E^{\infty}[\delta(\eta_n) f(X_{\tau_n})]$  for all  $n \in \mathbb{N}$ , and  $\pi(\pi_{K(\tau_n, \mu)}, S)$  = [ $\pi(\rho(\eta) f(X_{\eta}))$ ] =  $\exp(-\pi \eta, \mathbb{R}^n)$  =  $\pi, \mathbb{R}^n$ ,  $\pi, \mathbb{R}^n$ ,  $\pi, \mathbb{R}^n$ ,  $\pi, \mathbb{$ [2.8,](#page-7-0) we can apply the dominated convergence theorem to get  $\lim_{n\to\infty} \mathbb{E}^{x_0}[\delta(\eta_n)f(X_{\eta_n})]=$ 

Applying the monotone convergence theorem to the RHS of Equation (73) and combining with Equation (74), we have that

$$
\mathbb{E}^{x}[V(\tau_{B(x_0,h)}, X_{B(x_0,h)}, S)] - V(0, x, S) \ge -M \mathbb{E}^{x}[\tau_{B(x_0,h)}] + a \mathbb{E}^{x} \left[ \int_{0}^{\tau_{B(x_0,h)}} \delta(t) dL_t^{x_0} \right].
$$
 (75)

By the arbitrariness of  $x \in B(x_0, h)$ ,

$$
\mathbb{E}^{x_0}[V(\tau_{B(x_0,h)}, X_{\tau_{B(x_0,h)}}, S)] = \mathbb{E}^{x_0}[\delta(\eta)f(X_{\eta})]. \text{ As } n \to \infty, \eta_n \to \eta, \mathbb{P}^x \text{-a.s.. Then by Assumption 2.8, we can apply the dominated convergence theorem to get } \lim_{n \to \infty} \mathbb{E}^{x_0}[\delta(\eta_n)f(X_{\eta_n})] = \mathbb{E}^{x_0}[\delta(\eta_n)f(X_{\eta_n})]
$$
 that is, Equation (74) holds. (Note that Equation (14) is used on  $\{\eta = \infty\}$ .) Applying the monotone convergence theorem to the RHS of Equation (73) and combining with Equation (74), we have that\n
$$
\mathbb{E}^{x}[V(\tau_{B(x_0,h)}, X_{B(x_0,h)}, S)] - V(0, x, S) \ge -M\mathbb{E}^{x}[\tau_{B(x_0,h)}] + a\mathbb{E}^{x}\left[\int_{0}^{\tau_{B(x_0,h)}} \delta(t) dL_{t}^{x_0}\right].
$$
 (75)\nBy the arbitrariness of  $x \in B(x_0, h)$ ,\n
$$
\mathbb{E}^{x_0 + rh}[V(\tau_{B(x_0,h)}, X_{B(x_0,h)}, S)] - V(0, x_0 + rh, S)
$$
\n
$$
\ge -M\mathbb{E}^{x_0 + rh}[\tau_{B(x_0,h)}] + a\mathbb{E}^{x_0 + rh}\left[\int_{0}^{\tau_{B(x_0,h)}} \delta(t) dL_{t}^{x_0}\right], \quad \forall r \in (-1, 1).
$$
\nBy Lemma 4.6 and  $|\sigma(x_0)| > 0$ , we can choose the above *h* small enough such that\n
$$
\mathbb{E}^{x_0 + rh}\left[\int_{0}^{\tau_{B(x_0,h)}} \delta(t) dL_{t}^{x_0}\right] \ge \left(\frac{M}{a} + 1\right) \mathbb{E}^{x_0 + rh}[\tau_{B(x_0,h)}], \quad \forall r \in (-1, 1).
$$
\nConsequently, Equation (76) leads to\n
$$
\mathbb{E}^{x_0 + rh}[V(\tau_{B(x_0,h)}, X_{\tau_{B(x_0,h)}, S})] - V(0, x_0 + rh, S) \ge a\mathbb{E}^{x_0 + rh}[\tau_{B(x_0,h)}]
$$

By Lemma [4.6](#page-19-0) and  $|\sigma(x_0)| > 0$ , we can choose the above h small enough such that

$$
\geq -M\mathbb{E}^{x_0+rh}[\tau_{B(x_0,h)}] + a\mathbb{E}^{x_0+rh}\left[\int_0^{\tau_{B(x_0,h)}} \delta(t) dL_t^{x_0}\right], \quad \forall r \in (-1,1).
$$
  
6 and  $|\sigma(x_0)| > 0$ , we can choose the above *h* small enough such that  

$$
\mathbb{E}^{x_0+rh}\left[\int_0^{\tau_{B(x_0,h)}} \delta(t) dL_t^{x_0}\right] \geq \left(\frac{M}{a} + 1\right) \mathbb{E}^{x_0+rh}[\tau_{B(x_0,h)}], \quad \forall r \in (-1,1).
$$
  
ly, Equation (76) leads to  

$$
\forall (\tau_{B(x_0,h)}, X_{\tau_{B(x_0,h)}}, S)] - V(0, x_0 + rh, S) \geq a\mathbb{E}^{x_0+rh}[\tau_{B(x_0,h)}] > 0, \quad \forall r \in
$$
  
Equation (69).

Consequently, Equation (76) leads to

$$
\mathbb{E}^{x_0+rn} \Big[ \int_0^{\delta(t) dL_t^{\infty}} \Big] \ge \Big( \frac{...}{a} + 1 \Big) \mathbb{E}^{x_0+rn} [\tau_{B(x_0,h)}], \quad \forall r \in (-1, 1).
$$
  
sequently, Equation (76) leads to  

$$
\mathbb{E}^{x_0+rh} [V(\tau_{B(x_0,h)}, X_{\tau_{B(x_0,h)}}, S)] - V(0, x_0 + rh, S) \ge a \mathbb{E}^{x_0+rh} [\tau_{B(x_0,h)}] > 0, \quad \forall r \in (-1, 1),
$$
  
ch gives Equation (69).

which gives Equation (69).

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*Step 2*. In the rest part of case (i), we take h such that Equation (69) holds and write  $S_h$  :=

$$
J(y, S_h) \ge \mathbb{E}^y [V(\tau_{B(x_0, h)}, X_{\tau_{B(x_0, h)}}, S)], \quad \forall y \in [x_0, x_0 + h). \tag{77}
$$

Suppose

$$
\alpha := \inf_{y \in [x_0, x_0 + h]} \left( J(y, S_h) - \mathbb{E}^y [V(\tau_{B(x_0, h)}, X_{\tau_{B(x_0, h)}}, S)] \right) < 0. \tag{78}
$$

As  $x_0 + h \in S^\circ$ , by Lemma 2.14(a),

$$
J(x_0 + h, S_h) = f(x_0 + h) = \mathbb{E}^{x_0 + h}[V(\tau_{B(x_0, h)}, X_{\tau_{B(x_0, h)}}, S)].
$$

By the continuity of functions  $y \mapsto J(y, S_h)$  and  $y \mapsto \mathbb{E}^y[V(\tau_{B(x_0, h)}, X_{\tau_{B(x_0, h)}}, S)]$  on  $[x_0, x_0 + h]$ , there exists  $z^* \in [x_0, x_0 + h)$  such that the infimum in Equation (78) is attained at  $z^*$ , that is,

$$
J(z^*, S_h) - \mathbb{E}^{z^*}[V(\tau_{B(x_0, h)}, X_{\tau_{B(x_0, h)}}, S)] = \alpha.
$$
 (79)

Define

$$
\nu := \inf \{ t \ge \tau_{B(x_0, h)} : X_t \in S \} \quad \text{and} \quad A := \{ X_{\tau_{B(x_0, h)}} = x_0 - h, X_{\nu} = x_0, \ \nu < \infty \}
$$

Notice that  $\rho_{S_h} = \nu$ ,  $\mathbb{P}^{z^*}$ -a.s. on both sets have that

Step 2. In the rest part of case (i), we take *h* such that Equation (69) holds and write 
$$
S_h
$$
 :=  
\n $S \setminus B(x_0, h)$  for short. In this step, we prove by contradiction that  
\n $J(y, S_h) \ge E^y[V(f_{B(x_0, h)}, X_{T_{B(x_0, h)}}, S)], \quad \forall y \in [x_0, x_0 + h).$  (77)  
\nSuppose  
\n
$$
\alpha := \inf_{y \in [x_0, x_0 + h]} (J(y, S_h) - E^y[V(f_{B(x_0, h)}, X_{T_{B(x_0, h)}}, S)] ) < 0.
$$
 (78)  
\nAs  $x_0 + h \in S^\circ$ , by Lemma 2.14(a),  
\n $J(x_0 + h, S_h) = f(x_0 + h) = E^{x_0 + h}[V(\tau_{B(x_0, h)}, X_{T_{B(x_0, h)}}, S)]$ .  
\nBy the continuity of functions  $y \mapsto J(y, S_h)$  and  $y \mapsto E^y[V(f_{B(x_0, h)}, X_{T_{B(x_0, h)}}, S)]$  on  $[x_0, x_0 + h]$ ,  
\nthere exists  $z^* \in [x_0, x_0 + h)$  such that the infimum in Equation (78) is attained at  $z^*$ , that is,  
\n $J(z^*, S_h) - E^z^* [V(\tau_{B(x_0, h)}, X_{T_{B(x_0, h)}}, S)] = \alpha$ . (79)  
\nDefine  
\n $y := \inf\{t \ge \tau_{B(x_0, h)} : X_t \in S\}$  and  $A := \{X_{T_{B(x_0, h)}} = x_0 - h, X_y = x_0, v < \infty\}$ .  
\nNotice that  $\rho_{S_h} = v, P^{z^*} - a.s.$  on both sets  $\{X_{T_{B(x_0, h)}} = x_0 + h\}$  and  $\{X_{T_{B(x_0, h)}} = x_0 - h, X_y < x_0\}$ . We  
\nhave that  
\n $J(z^*, S_h) - E^{z^*}[V(\tau_{B(x_0, h)}, X_{T_{B(x_0, h)}}, S)] = E^{z^*} [1_A \delta(v) \cdot (\sqrt{E^* \delta(s_h)} - \delta(v)f(X_{P_{S_h}}) - f(X_y))]$   
\n $\ge E^{z^*$ 

>  $E^2$ <br>
≥  $E^{z^*}$ <br>
>  $\alpha$ , com E<br>
and and b, the follow<br>
herefold eqn b) ⊂<br>
y) = J  $[1_A \delta(\nu)] \cdot ($ <br>  $[1_A \delta(\nu)] \cdot \alpha$ <br>
quation (13)<br>
the fact tha<br>
fifth (in)equ<br>
s from the f<br>
ore, Equatio<br>
ilibrium ar<br>  $S^\circ$ , we have<br>  $(y, S)$ ,  $\forall y$  $J(x_0, S_h) - \mathbb{E}^{x_0}[V(\tau_{B(x_0, h)}, X_{\tau_{B(x_0, h)}}, S)]$ <br>
and  $f \ge 0$ , the third (in)equality follow:<br>
t  $X_v = x_0$  on *A*, the fourth (in)equality<br>
ality follows from the definition of *α* in<br>
act that  $v \ge \tau_{B(x_0, h)} > 0$  and  $\delta$  $E^2$  α, n E d<br>nd he llov<br>ref(eq) ⊂ J quation (13)<br>the fact that<br>fifth (in)equence, Equation<br>illibrium are  $S^{\circ}$ , we have<br> $(y, S)$ ,  $\forall y$  $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ , the following the following  $\begin{pmatrix} -h \\ h \end{pmatrix}$  ( $\begin{pmatrix} -h \\ h \end{pmatrix}$ ) where the second (in)equality follows from Equation (13) and  $f \ge 0$ , the third (in)equality follows  $f \ge 0$ , the third (in)equality follows<br>=  $x_0$  on A, the fourth (in)equality<br>r follows from the definition of  $\alpha$  in<br>hat  $\nu \ge \tau_{B(x_0,h)} > 0$  and  $\delta(t) < 1$  for<br>r) holds.<br>strictly better than S. By Equations<br> $0, x_0 + h$ ). ( from the strong Markov property of X and the fact that  $X_v = x_0$  on A, the fourth (in)equality X and the fact that  $X_v = x_0$  on A, the fourth (in)equality<br>  $x_0$ , the fifth (in)equality follows from the definition of  $\alpha$  in<br>
y follows from the fact that  $v \ge \tau_{B(x_0, h)} > 0$  and  $\delta(t) < 1$  for<br>
Therefore, Equation (77) follows from Equation (69) with  $y = x_0$ , the fifth (in)equality follows from the definition of  $\alpha$  in  $y = x_0$ , the fifth (in)equality follows from the definition of  $\alpha$  in uality follows from the fact that  $v \ge \tau_{B(x_0, h)} > 0$  and  $\delta(t) < 1$  for (79). Therefore, Equation (77) holds.<br>is a mild equilibrium and is strictly bet Equation (78), and the last (in)equality follows from the fact that  $\nu \ge \tau_{B(x_0,h)} > 0$  and  $\delta(t) < 1$  for  $\nu \ge \tau_{B(x_0, h)} > 0$  and  $\delta(t) < 1$  for<br>olds.<br>ctly better than *S*. By Equations<br><sub>0</sub> + *h*). (80)

*t* > 0. This contradicts Equation (79). Therefore, Equation (77) holds.<br> *Step 3*. Now we prove that *S<sub>h</sub>* is a mild equilibrium and is strictly (69) and (77) and noticing that  $(x_0, x_0 + h) \subset S^\circ$ , we have  $J(y, S_h) > f(y) = J(y,$ *Step 3*. Now we prove that  $S_h$  is a mild equilibrium and is strictly better than *S*. By Equations  $S_h$  is a mild equilibrium and is strictly better than *S*. By Equations<br>  $t(x_0, x_0 + h) \subset S^\circ$ , we have<br>  $S_h$ ) >  $f(y) = J(y, S)$ ,  $\forall y \in [x_0, x_0 + h)$ . (80)<br>
ave that [\(69\)](#page-21-0) and (77) and noticing that  $(x_0, x_0 + h) \subset S^{\circ}$ , we have

g that 
$$
(x_0, x_0 + h) \subset S^\circ
$$
, we have  
\n $J(y, S_h) > f(y) = J(y, S)$ ,  $\forall y \in [x_0, x_0 + h)$ . (80)  
\nwe have that

Then for any  $y \in (l, x_0)$ , we have that  $y \in (l, x_0)$ , we have that

$$
J(y, S_h) - J(y, S) = \mathbb{E}^y \left[ 1_{\{X_{\rho_S} = x_0, \rho_S < \infty \}} \left( \delta(\rho_{S_h}) f(X_{\rho_{S_h}}) - \delta(\rho_S) f(x_0) \right) \right]
$$
\n
$$
\geq \mathbb{E}^y \left[ 1_{\{X_{\rho_S} = x_0, \rho_S < \infty \}} \delta(\rho_S) \left( \mathbb{E}^y \left[ \delta(\rho_{S_h} - \rho_S) f(X_{\rho_{S_h}}) | P_{\rho_S} \right] - f(x_0) \right) \right]
$$
\n
$$
= \mathbb{E}^y \left[ 1_{\{X_{\rho_S} = x_0, \rho_S < \infty \}} \delta(\rho_S) (J(x_0, S_h) - f(x_0)) \right]
$$
\n
$$
\geq 0.
$$
\n(81)\n6: the second (in)equality follows again from Equation (13) and the non-negativity of *f*, the (in)equality follows from the strong Markov property of *X*, and the last (in)equality follows  
\nEquation (80) with  $y = x_0$ . As *S* is a mild equilibrium, above inequality implies\n
$$
J(y, S_h) \geq J(y, S) \geq f(y), \quad \forall y \in (l, x_0).
$$
\n(82)\n6: the equilibrium and is strictly better than *S*.\n6: (ii)  $x_0 \in \delta S$  for boundary case (b). We denote\n
$$
\sup\{y < x_0, y \in S\}, \quad r := \inf\{y > x_0, y \in S\}, \quad \text{and} \quad \tau_n := \tau((l, r) \cap B(x_0, n)) \land n \text{ for } n \in \mathbb{N}.
$$
\n4:  $\sup\{y < x_0, y \in S\}, \quad r := \inf\{y > x_0, y \in S\}, \quad \text{and} \quad \tau_n := \tau((l, r) \cap B(x_0, n)) \land n \text{ for } n \in \mathbb{N}.$ \n5:  $\sup\{y' \in \tau, X_{\tau_n}, S\} \right] - V(0, x_0, S)$ \n
$$
\geq \mathbb{E}^{x_0} \left[ \int_0^{\tau_n} \frac{1}{2} \mathcal{L}V(s,
$$

where the second (in)equality follows again from Equation (13) and the non-negativity of  $f$ , the third (in)equality follows from the strong Markov property of  $X$ , and the last (in)equality follows from Equation (80) with  $y = x_0$ . As S is a mild equilibrium, above inequality implies

$$
J(y, S_h) \ge J(y, S) \ge f(y), \quad \forall y \in (l, x_0). \tag{82}
$$

This together with Equation [\(80\)](#page-23-0) and the fact  $J(\cdot, S_h) = J(\cdot, S)$  on  $\mathbb{X} \setminus (l, x_0 + h)$  implies that  $S_h$  is a mild equilibrium and is strictly better than S.

**Case (ii)**  $x_0 \in \partial S$  for boundary case (b). We denote

$$
l := \sup\{y < x_0, y \in S\}, \quad r := \inf\{y > x_0, y \in S\}, \quad \text{and} \quad \tilde{\tau}_n := \tau_{((l,r) \cap B(x_0, n))} \land n \text{ for } n \in \mathbb{N}.
$$

By a similar discussion through Equations (70)–(72) (with Lemmas 2.15 and [3.10](#page-15-0) applied), we have that

$$
\geq \mathbb{E}^{x_0}[V(x_{r,s},x_{r,s},\rho_{s}<\infty) \cup (P_{s})[\mathbf{E}^{x_0}[\mathbf{U}(x_{r,s}-\rho_{s})]\bigg) \text{ and the non-negative of } \mathbf{U}(x_{r,s}) = \mathbf{U}(x_{r,s}) \cup (x_{r,s}) = \mathbf{U}(x_{r,s})
$$
\n
$$
\geq 0. \tag{81}
$$
\nwhere the second (in)equality follows again from Equation (13) and the non-negativity of  $f$ , the  
\n
$$
\text{and (inequality follows from the strong Markov property of  $X$ , and the last (in)equality follows from the strong Markov property of  $X$ , and the last (in)equality follows by the equation (80) with  $y = x_0$ . As  $S$  is a mild equilibrium, above inequality implies  
\n
$$
J(y, S_k) \geq J(y, S) \geq f(y), \quad \forall y \in (l, x_0).
$$
\n(82)

\nthis together with Equation (80) and the fact  $J(\cdot, S_k) = J(\cdot, S)$  on  $\mathbb{X} \setminus \{l, x_0 + h\}$  implies that  $S_k$  is  
\n
$$
\text{and equilibrium and is strictly better than } S.
$$
\n**Case (ii)**  $x_0 \in \partial S$  for boundary case (b). We denote

\n
$$
l := \sup\{y < x_0, y \in S\}, \quad r := \inf\{y > x_0, y \in S\}, \quad \text{and } \tilde{\tau}_n := \tau((t, r) \cap B(x_0, n)) \land n \text{ for } n \in \mathbb{N}.
$$
\n**Case (iii)**  $x_0 \in \partial S$  for boundary case (b). We denote

\n
$$
l := \sup\{y < x_0, y \in S\}, \quad r := \inf\{y > x_0, y \in S\}, \quad \text{and } \tilde{\tau}_n := \tau((t, r) \cap B(x_0, n)) \land n \text{ for } n \in \mathbb{N}.
$$
\n**Case (iv)**  $\left| \int_{0}^{\tilde{\tau}_n} \delta(V(\tilde{\tau}_n, X_{\tilde{\tau}_n}, S) - V(0, x_0, S) \right| \geq \sigma \left$
$$

By Lemma 2.14(a),  $\mathcal{L}V(t, x, S) = 0$  for any  $(t, x) \in [0, \infty) \times S^c$ , and thus the first term on the RHS of Equation (83) vanishes for all  $n \in \mathbb{N}$ . As a result, we can rewrite Equation (83) as

$$
\mathbb{E}^{x_0}[V(\tilde{\tau}_n, X_{\tilde{\tau}_n}, S)] - V(0, x_0, S) \ge a \mathbb{E}^{x_0} \left[ \int_0^{\tilde{\tau}_n} \delta(t) dL_t^{x_0} \right] \ge a \mathbb{E}^{x_0} \left[ \int_0^{\tilde{\tau}_1} \delta(t) dL_t^{x_0} \right] > 0, \quad \forall n \in \mathbb{N}.
$$

Meanwhile, similar to Equation (74), Assumption 2.8 implies that  $\mathbb{E}^{x_0}[V(\tilde{\tau}_n, X_{\tilde{\tau}_n}, S)] \to$ 

$$
J(x_0, S \setminus \{x_0\}) - f(x_0) = \mathbb{E}^{x_0}[V(\tau_{(l,r)}, X_{\tau_{(l,r)}}, S)] - V(0, x_0, S) \ge a \mathbb{E}^{x_0} \left[ \int_0^{\tilde{\tau}_1} \delta(t) dL_t^{x_0} \right] > 0.
$$
\n(84)

L (a)  $(4(a)^3)$ ,  $S$  simple  $\pi$  (*x*<sub>(*x*</sub>, *z*<sub>(*x*</sub>, *z*<sub>(*x*)</sub> (*z*) Due  $x_0$ , 1  $V$  t a  $\begin{pmatrix} x \\ y \end{pmatrix}$  as  $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$  as c  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$  or  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ 1 2 ( $\mathcal{L}V(s, X_s-, S) + \mathcal{L}V(s, X_s+, S))ds$ <br>  $x, S$ ) = 0 for any ( $t, x$ ) ∈ [0, ∞) × S<sup>c</sup><br>
for all  $n \in \mathbb{N}$ . As a result, we can re<br>  $x_0, S$ ) ≥  $a\mathbb{E}^{x_0} \left[ \int_0^{\tilde{\tau}_n} \delta(t) dL_t^{x_0} \right] \ge a\mathbb{I}$ <br>
Equation (74), Assumption 2.8 and the E<br>
and the E<br>  $\lim_{x \to 0} \left[ \int_0^x f(x) \right]$ <br>  $\lim \text{plies } x \to 0$ <br>  $\lim \left( \int f(x) \right)$  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 &$ tian (control de la provincia de la control de la provincia de la provincia de la provincia de la provincia de<br>Desembla de la provincia de la  $\begin{aligned} \text{dist}(t) & \text{dist}(t) \\ \text{dist}(t) & \text{dist}(t) \\ \text{dist}(t) & \text{dist}(t) \\ & \text{dist}(t) \end{aligned}$  .  $V(t, x, S) = 0$  for any  $(t, x) \in [0, \infty) \times S^c$ , and thus the first term on the RHS<br>
sines for all  $n \in \mathbb{N}$ . As a result, we can rewrite Equation (83) as<br>  $V(0, x_0, S) \ge \alpha \mathbb{E}^{x_0} \left[ \int_0^{t_n} \delta(t) dL_t^{x_0} \right] \ge \alpha \mathbb{E}^{x_0} \$ *n* ∈ ℕ. As a result, we can rewrite Equation (83) as<br>  $\ge a \mathbb{E}^{x_0} \left[ \int_0^{\tilde{\tau}_n} \delta(t) dL_t^{x_0} \right] \ge a \mathbb{E}^{x_0} \left[ \int_0^{\tilde{\tau}_1} \delta(t) dL_t^{x_0} \right] >$ <br>
on (74), Assumption 2.8 implies that  $\mathbb{E}^{x_0}[V(C)$ <br>
This together w  $\mathbb{E}^{x_0}[V(\tilde{\tau}_n, X_{\tilde{\tau}_n}, S)] - V(0, x_0, S) \ge a \mathbb{E}^{x_0}$ <br>
Meanwhile, similar to Equation (7<br>  $\mathbb{E}^{x_0}[V(\tau_{(l,r)}, X_{\tau_{(l,r)}}, S)]$  as  $n \to \infty$ . This to<br>  $J(x_0, S \setminus \{x_0\}) - f(x_0) = \mathbb{E}^{x_0}[V(\tau_{(l,r)})]$ <br>
Now set  $\tilde{S} := S \setminus \{$ As  $\overline{a}$ ,  $\overline{b}$ ,  $\overline{c}$ ,  $\overline{c}$  and  $\overline{c}$  and  $\overline{c}$ t  $\mathfrak{g}(\mathfrak{g})$  is the sum of  $\mathfrak{g}(\mathfrak{g})$  $o(t)dL_t$ <br>sumption<br>r with th<br>s)] –  $V(t)$ <br> $\Rightarrow$ <br> $V(t)$ . We consider that  $B(x_0, t)$ <br> $\Rightarrow$   $J(t)$ ,  $S(t)$ <br> $\Rightarrow$   $f(y)$ ,  $\begin{align*}\n\begin{cases}\n3 & \text{in } 3 \\
\text{in } 3 \text{ in } 3 \\
\text{in } 3 \text{ in } 3\n\end{cases} \\
\text{apply}\n\begin{cases}\n1 & \text{if } 3 \text{ in } 3 \\
\text{if } 3 \text{ in } 3\n\end{cases} \\
\text{S.} \\
\text{S.}$ s al  $x_0$  are  $x_1$  are  $\alpha$  ),  $\mathcal{G}$  on e U 正 ai r ( i h  $\int_0^{\tilde{\tau}_1} \delta(\tilde{\tau}) d\tilde{\tau}_t$ <br>that  $\mathbb{E}^{x}$ <br>ty implies<br> $\int_0^{\tilde{\tau}_1} \delta(\tilde{\tau}) d\tilde{\tau}_t$ <br>summent a that  $J(\tilde{\tau})$ <br>we have  $\mathcal{U}(\tilde{\tau}_n, X_{\tilde{\tau}_n}, S)$  =  $\text{that}$ <br>  $dL_t^{x_0}$  > 0. (84<br>
inilar to that ir  $(\tilde{\tau}_0) - J(y, S) \geq 0$ <br>
hat  $\tilde{S}$  is a milon in  $h \leq h_0$ , which  $\mathbb{E} \left[\n\begin{bmatrix}\nV(t_n, X_{\tau_n}, S) \\
\end{bmatrix}\n\right] > 0.$ <br>
(84)<br>
ont similar to that in<br>  $J(y, \widetilde{S}) - J(y, S) \ge 0.$ <br>
ave that  $\widetilde{S}$  is a mild<br>
Following the argu-<br>  $0 < h \le h_0$ , which  $\mathbb{E}^{x_0}[V(\tau_{(l,r)}, X_{\tau_{(l,r)}}, S)]$  as  $n \to \infty$ . This together with the above inequality implies that<br>  $J(x_0, S \setminus \{x_0\}) - f(x_0) = \mathbb{E}^{x_0}[V(\tau_{(l,r)}, X_{\tau_{(l,r)}}, S)] - V(0, x_0, S) \ge a\mathbb{E}^{x_0} \left[ \int_0^{\tilde{\tau}_1} \delta(t) dL_t^{x_0}$ <br>
Now set  $J(x_0, S \setminus \{x_0\}) - f(x_0) = \mathbb{E}^{x_0}[V(\tau_{(l,r)}, X_{\tau_{(l,r)}}, S)] - V(0, x_0, S) \ge a\mathbb{E}^{x_0}$ <br>
w set  $\tilde{S} := S \setminus \{x_0\}$  and pick any  $y \in (l, r)$ . We can apply an ar<br>
uation (81), by using Equation (84) and replacing  $S_h$  with  $\tilde{$ eı<br>utha<br>).<br>ne n<br>1<br>0<br>01  $U(t)$   $\frac{dU(t)}{dt}$ <br>t similar<br> $U(y, \tilde{S})$  -<br>we that<br>Followin  $0 < h \leq$  o th $y, S$ <br>is a<br>the<br> $k_0, v$ Now set  $\tilde{S} := S \setminus \{x_0\}$  and pick any  $y \in (l, r)$ . We can apply an argument similar to that in S : = S \{x<sub>0</sub>} and pick any  $y \in (l, r)$ . We can apply an argument similar to that in (81), by using Equation (84) and replacing  $S_h$  with  $\overline{S}$ , to reach that  $J(y, \overline{S}) - J(y, S) \ge 0$ . (y,  $\overline{S}$ )  $\ge f(y)$  for  $y \in (l, r)$ Equation (81), by using Equation (84) and replacing  $S_h$  with  $\tilde{S}$ , to reach that  $J(y, \tilde{S}) - J(y, S) \ge 0$ .  $\widetilde{S}$ , to reach that  $J(y, \widetilde{S}) - J(y, S) \ge 0$ .<br>  $\mathbb{X} \setminus (l, r)$ , we have that  $\widetilde{S}$  is a mild  $\{x_0\} \subset (\mathcal{G} \cap S^\circ)$ . Following the argu-<br>  $\{1, 69\}$  for some  $0 < h \le h_0$ , which  $B(x_0, h)$ . Hence,  $J(y, \tilde{S}) \ge f(y)$  for  $y \in (l, r)$ . As  $J(\cdot, \tilde{S}) = J(\cdot, S)$  on  $\mathbb{X} \setminus (l, r)$ , we have that  $\tilde{S}$  is a mild equilibrium. Due to Equation (84),  $\tilde{S}$  is strictly better than S.

 $S_h$  with S<br>
, S) on  $\mathbb{X}$ <br>
t than S.<br>  $\{b_0, h_0\} \setminus \{x\}$ <br>
Equation<br>  $\forall y \in B$  $J(y, \tilde{S}) \ge f(y)$  for  $y \in (l, r)$ . As  $J(\cdot, \tilde{S}) = J(\cdot, S)$  on  $\mathbb{X} \setminus (l, r)$ , we have that  $\tilde{S}$ <br>ium. Due to Equation (84),  $\tilde{S}$  is strictly better than  $S$ .<br>**(iii)**  $x_0 \in S^\circ$ . Choose  $h_0 > 0$  such that  $B(x_0, h_0)$ S is a mild<br>g the argu-<br> $\sum h_0$ , which .<br>ur<br>B S is strictly better than S.<br>
0 such that  $B(x_0, h_0) \setminus \{$ <br>
in again reach Equation<br>  $B(x_0, h)) > f(y)$ ,  $\forall y \in$ **Case (iii)**  $x_0 \in S^\circ$ . Choose  $h_0 > 0$  such that  $B(x_0, h_0) \setminus \{x_0\} \subset (G \cap S^\circ)$ . Following the argu $x_0 \in S^\circ$ . Choose  $h_0 > 0$  such that  $B(x_0, h_0) \setminus \{x_0\} \subset (G \cap S^\circ)$ . Following the argu-<br>
1 of case (i), we can again reach Equation (69) for some  $0 < h \le h_0$ , which<br>  $J(y, S \setminus B(x_0, h)) > f(y)$ ,  $\forall y \in B(x_0, h)$ . ment in *Step 1* of case (i), we can again reach Equation (69) for some  $0 < h \le h_0$ , which  $0 < h \le h_0$ , which indicates

$$
J(y, S \setminus B(x_0, h)) > f(y), \quad \forall y \in B(x_0, h).
$$

<span id="page-25-0"></span>**822 IM/II EV BAYRAKTAR ET AL.** 

As  $J(\cdot, S \setminus B(x_0, h)) = J(\cdot, S)$  on  $X \setminus B(x_0, h)$ , we have that  $S \setminus B(x_0, h)$  is a mild equilibrium and is strictly better than  $S$ .

*Proof of Theorem* 4.3. Thanks to Lemma [2.14\(](#page-9-0)a), Theorem [3.1,](#page-10-0) and Proposition [4.7,](#page-21-0) we only need to show Equation [\(25\)](#page-10-0) for  $x \in S$ . Recall G defined in Equation (16). Let  $x_0 \in S$  and we consider three cases: (i)  $x_0 \in (S^\circ \cap G)$ , (ii)  $x_0 = \theta_n \in S^\circ \setminus G$  for some  $n \in I$ , and (iii)  $x_0 \in \partial S$ .

**Case (i)**  $x_0 \in (S^\circ \cap \mathcal{G})$ . We prove Equation (25) by contradiction. Suppose  $\mathcal{L}V(0, x_0, S) = a >$ 

$$
\mathcal{L}V(0, x, S) = \delta'(0)f(x) + \mu(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x) \ge \frac{a}{2}, \quad \forall x \in B(x_0, h). \tag{85}
$$

Then for any  $(t, x) \in [0, \infty) \times B(x_0, h)$ , we have that

$$
\mathcal{L}V(t, x, S) = \delta'(t)f(x) + \delta(t)\left(\mu(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x)\right)
$$
  

$$
\geq \delta(t)\left(\delta'(0)f(x) + \mu(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x)\right) \geq \delta(t)\frac{a}{2},
$$
\n(86)

where the first inequality above follows from Lemma [2.7](#page-7-0) and the non-negativity of  $f$ . Let us reuse the notation  $\tau_n$  defined in Equation (71). By Equation (86) and an argument similar to that for Equations (72) and (74) (notice that the local time integral in Lemma 2.15 vanishes in the current case), we have that for any  $x \in B(x_0, h)$ ,

$$
\begin{cases} \mathbb{E}^{x}[V(\tau_{n},X_{\tau_{n}},S)]-V(0,x,S)=\mathbb{E}^{x}\left[\int_{0}^{\tau_{n}}\mathcal{L}V(s,X_{s},S)\right]\geq\mathbb{E}^{x}\left[\int_{0}^{\tau_{n}}\delta(t)\frac{a}{2}dt\right]>0, & \forall n\in\mathbb{N},\\ \mathbb{E}^{x}[V(\tau_{B(x_{0},h)},X_{\tau_{B(x_{0},h)}},S)]=\lim_{n\to\infty}\mathbb{E}^{x}[V(\tau_{n},X_{\tau_{n}},S)] \end{cases}
$$

This implies that

$$
\mathbb{E}^{x}[V(\tau_{B(x_0,h)}, X_{\tau_{B(x_0,h)}}, S) - V(0, x, S)] \geq \mathbb{E}^{x} \left[ \int_0^{\tau_{B(x_0,h)}} \delta(t) \frac{a}{2} dt \right] > 0, \quad \forall x \in B(x_0, h).
$$

Now consider  $\widetilde{S} = S \setminus B(x_0, h)$ . The above inequality implies

$$
J(x,\widetilde{S}) - f(x) = \mathbb{E}^{x}[V(\tau_{B(x_0,h)}, X_{\tau_{B(x_0,h)}}, S)] - V(0, x, S) > 0 \quad \forall x \in B(x_0, h). \tag{87}
$$

Obviously,  $J(\cdot, \tilde{S}) = J(\cdot, S)$  on  $\mathbb{X} \setminus B(x_0, h)$ . This together with Equation (87) shows that  $\tilde{S}$  is an equilibrium and is strictly better than S, a contradiction. Hence,  $\mathcal{L}V(0, x_0, S) \le 0$ , as desired.

 $I_t^t$ , S,  $\hat{R}(x, \hat{p}, \hat{0}) = I_t^t(S)$  on  $X^t$ ,  $R(x_0, h)$ , we have that  $X$   $R(x_0, h)$  is a mild equilibrium and  $g$  of Theorem 5.1, and Proposition 4.7, we only need out  $\hat{p}(x, h)$  is a mild equilibrium and  $g$  of Theo S.<br>
Thanks to Lemma 2.14(a), Theorem 3.1, and Proposition 4.7, we only need<br>
Thanks to Lemma 2.14(a), Theorem 3.1, and Proposition 4.7, we only need<br>
So<sup>o</sup> Co<sub></sub> (1), (i)  $x_0 = \delta_0$ ,  $C_0$  (for some  $\epsilon = 1$ , and (ii)  $x_0 \$ x ∈. S. Recall  $Q$  defined in Equation [\(16\)](#page-7-0). Let  $x_0 \in S$  and we consider<br>
x ∈. S. Recall  $Q$  defined in Equation (16). Let  $x_0 \in S$  and we consider<br>
We grow Equation (25) by contradiction. Suppose  $L^V(0, x_0, S) = a >$ <br>
we  $x_0 \in (S^{\circ} \cap G)$ , (ii)  $x_0 = \theta_0 \in S^{\circ} \setminus G$  for some π ∈ *I*, and (iii)  $x_0 = 8$ ,  $x_0 \in S^{\circ}$  (i) for some π ∈ *I*, and (iii)  $x_0 = 8$ .<br>
(c)  $S^{\circ} \cap G$ ), We prove Equation (25) by contradiction. Suppose  $CV(0)$ <br>
(ion  $x_0 \in (S^o \cap G)$ . We prove Equation [\(25\)](#page-10-0) by contradiction. Suppose  $\mathcal{L}V(0, x_0, S) = a > 0$  prion 2.8(ii), we can choose  $h > 0$  such that  $V(t, x, S) = \delta(t) f(x) \in C^{\perp 2}(B(x_0, h)$ .<br>
(85)  $V(0, x, S) = \delta'(0) f(x) + \mu(x) f'(x) + \frac{1}{2}\sigma^2(x) f$ 0. By Assumption [2.8\(](#page-7-0)ii), we can choose  $h > 0$  such that  $V(t, x, S) = \delta(t)f(x) \in C^{1,2}(B(x_0, h) \times C(x_0, S))$ <br>  $\mathcal{L}V(0, x, S) = \delta'(0)f(x) + \mu(x)f'(x) + \frac{1}{2}\sigma^2(x)f'(x) \geq \frac{\alpha}{2}, \quad \forall x \in B(x_0, h).$  ((s5)<br>
Then for any  $(t, x, S) \in [0, \infty) \times B(x_0, h)$ , w (0, ∞)) and<br>
Then for ar<br>
Then for ar<br>
Then for ar<br>
Then for ar<br>
Equations (<br>
case), we ha<br>  $\begin{cases} \mathbb{E}^x[V(\tau_h\\ \mathbb{E}^x[V(\tau_h\\ \mathbb{E}^x[V(\tau_h\\ \text{This implies } \mathbb{E}^x[V\\ \text{Now consider } \mathbb{E}^x[V\\ \text{Now consider } \mathbb{E}^x[V(\tau_h\\ \text{Now consider } \mathbb{E}^x[V(\tau_h\\ \text{Now } \mathbb{E}^x[\$ (0, , ) = ′ (0)  $f(x) + \mu(x)$ <br>  $\rightarrow B(x_0, h)$ , we h<br>  $\delta'(t) f(x) + \delta(t)$ <br>  $\delta(t) \left( \delta'(0) f(x) - \delta(t) \right)$ <br>
bove follows from<br>
Equation (71). B<br>
tice that the loca<br>  $x \in B(x_0, h)$ ,<br>  $x, S$ ) =  $\mathbb{E}^x \left[ \int_0^{\tau_n} f(x_0, h) \right]$ <br>  $\rightarrow$ ,  $x, S$ ) =  $\mathbb{E}^x \left[$  $(x) + \frac{1}{2}$ <br>ave tha<br> $\mu(x)f'$ <br> $\mu(x)f$ <br> $\vdash \mu(x)f$ <br> $\vdash \mu(x),f$ <br> $\vdash$ a f f matin K X [ j it S pati( ; u e v i) ). o h x li  $\sigma^2(x) f''(x) \ge$ <br>  $\downarrow$ <br>  $(x) + \frac{1}{2}\sigma^2(x)$ <br>  $\downarrow$   $(2x) + \frac{1}{2}\sigma^2(x)$ <br>  $\downarrow$   $(2x) + \frac{1}{2}\sigma^2(x)$ <br>  $\downarrow$   $(3x) + \frac{1$  $\frac{1}{2}$   $\frac{1}{2}$  27.  $\forall x \in B(x_0, h).$  (85)<br>  $''(x)$   $\geq \delta(t) \frac{a}{2}$ , (86)<br>  $^{(n)}(x)$   $\geq \delta(t) \frac{a}{2}$ ,<br>  $\geq$   $\geq \delta(t) \frac{a}{2}$ ,<br>  $\geq \delta(t) \frac{a}{2}$ ,<br>  $\geq \delta(t) \frac{a}{2}$   $\geq 0$ ,  $\geq \delta(t) \frac{a}{2}$   $\leq \delta(t) \frac{a}{2}$   $\leq \delta(t) \frac{a}{2}$   $\leq \delta(t) \frac{a}{2}$ (*t*, *x*) ∈ [0, ∞) × B(*x*<sub>0</sub>, *h*), we have that<br>  $\mathcal{L}V(t, x, S) = \delta'(t) f(x) + \delta(t) (\mu(x) f'(x))$ <br>  $\geq \delta(t) (\delta'(0) f(x) + \mu(x) f'(x))$ <br>  $\leq \delta(t) (\delta'(0) f(x) + \mu(x) f'(x))$ <br>
to tinequality above follows from Lemmar,<br>  $\tau_n$  defined in Equation (71)  $\geq \delta$ <br>
(nequality abo<br>
defined in Equality abo<br>
defined in Equality abo<br>
defined in Equality (notic<br>
hat for any x<br>  $\{S_n, S_n\} - V(0, x)$ <br>  $\{S_n, X_{\tau_{B(x_0, h)}}, S_n\}$ <br>  $\{S_n, X_{\tau_{B(x_0, h)}}, S_n\}$ <br>  $\{S_n = S \setminus B(x_0, k, \tilde{S}) - f(x) = \mathbb$ (t)  $f(x) + \delta(t)$ <br>
(t)  $\left(\delta'(0) f(x) + \delta(t)\right)$ <br>
(d)  $\delta'(0) f(x) + \delta(t)$ <br>
(d)  $\delta'(0) f(x)$  = we follows from<br>
quation (71). By<br>  $\delta \in B(x_0, h)$ ,<br>  $\delta \in B[x_0, h)$ ,<br>  $\delta \in B[x_0, h]$ <br>  $\delta \in B[x_0, h]$ . The above ine<br>  $\delta(x_0, h)$ . The above ine<br>  $\delta$  $\mu(x) f'$ <br>Lemm<br>Equat<br>time ir<br> $V(s, X, \lambda)$ <br> $\gamma(\tau_n, X) \geq \mathbb{E}^x \left[ \int \lambda(x_0, h) \cdot S \right]$ <br> $\lambda(s_0, h) \cdot S$ .<br>his tog<br>are  $I$ .<br> $\lambda(s_0, h) \cdot S$ .<br>his tog<br> $n \in I$ .<br> $> 0$  surface  $i \in I$ .<br> $> 0$  surface  $i \in I$ .<br> $\lambda_0 + h$ <br>on  $(x_0, h)$  the appli  $(x) + \frac{1}{2}$ <br>  $(x) + \frac{1}{2}$ <br>  $\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right] \ge \frac{1}{2}$ <br>  $\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right] \ge \frac{1}{2}$ <br>  $\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right] \ge \frac{1}{2}$ <br>  $\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right] \ge \frac{1}{2}$ <br>  $\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right] \$  $\frac{1}{2}$  a  $\frac{1$  $\sigma^2(x)f''(x)$ <br>  $\sigma^2(x)f''(x)$ <br>
d the non-neg<br>
and an argu<br>
n Lemma 2.15<br>  $\mathbb{E}^x\left[\int_0^{\tau_n} \delta(t) \frac{a}{2} dt\right] > 0$ <br>  $\delta(t) \frac{a}{2} dt$   $> 0$ <br>  $\delta(t)$  (<br>  $\delta(t)$  (<br>  $\delta(t)$  =  $\delta$  +  $\delta$  =  $\delta$ olic (x = m V he (t \in d m d x e \in d m d x e \in (0)  $f(x) + \mu(x)$ <br>
(over from Lemma<br>
in (71). By Equati<br>
the local time in<br>  $\sum_{i=0}^{n} h$ ,<br>  $\mathbb{E}^{x} \left[ \int_{0}^{\tau_{n}} \mathcal{L}V(s, X_{s}) \mathbb{E}^{x} \left[ V(\tau_{n}, X_{t}) \mathbb{H}_{n \to \infty} \mathbb{E}^{x} \left[ V(\tau_{n}, X_{t}) \mathbb{H}_{n \to \infty} \mathbb{E}^{x} \left[ V(\tau_{n}, X_{t}) \mathbb$  $(x) + \frac{1}{2}$ <br>a 2.7 and<br>ion (86)<br>tegral i<br>for (86)<br> $\left[\text{total}\right] \geq \sum_{n} S$ <br> $\left[\text{total}\right] \geq \sum_{n} S$ <br> $\left[\text{d}\right] \geq \sum_{n} \left(\text{total}\right) \geq \sum_{n} \left(\text{total}\right)$  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  is the contract of  $\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}$  of  $\begin{bmatrix} 0 & x \\ x & 0 \end{bmatrix}$  or  $\begin{bmatrix} 0 & x \\ x & 0 \end{bmatrix}$  $\sigma^2(x) f''(x)$ <br>
d the non-neg<br>
and an argun<br>
n Lemma 2.15<br>  $\mathbb{E}^x \left[ \int_0^{\tau_n} \delta(t) \frac{a}{2} dt \right] > 0$ <br>  $\delta(t) \frac{a}{2} dt \right] > 0$ <br>
s<br>  $(x, x, S) > 0 \quad \forall$ <br>
ith Equation<br>
nnce,  $\mathcal{L}V(0, x_0$ <br>
at loss of  $\{x_0, x_0 + h\} \subset$ <br>
( $x_0, x_0 + h$  $\delta(t)$   $\frac{1}{2}$ <br>vity of sin<br>mishe <br>> 0,<br> $\forall x \in$ <br> $\equiv B(x)$ <br> $\geq 0$ ,<br> $\geq 0$ ,<br>eralit:<br> $\land \cap (\theta_1)$ <br> $\geq a/2$ <br>and<br> $\land S) \lor I$ <br> $h_0, x_0$ <br>e,  $CV$ 2 fm m es <br>  $\vdots$   $x_0$  ov  $\vdots$   $x_0$  ov  $\vdots$   $x_1$  are  $x_2$  and  $x_1$  are  $\vdots$   $x_0$  ov  $V($ f. Let us reuse<br>ilar to that for<br>in the current<br> $\forall n \in \mathbb{N}$ ,<br> $B(x_0, h)$ . (87)<br>ws that  $\overline{S}$  is an<br>as desired.<br>, we assume<br> $,\theta_{n+1}$ ). By the<br>nd the fact that<br> $> 0$  for all  $y \in$ <br> $V(0, x_0+, S) >$ <br> $\subset S^c$  for some<br> $(0, x_0+,$  $r_n$  defined in Equation (71). By Equation (86) and an argument similar to that for<br>  $\Omega$ ) and (74) (notice that the local time integral in Lemma 2.15 vanishes in the current<br>  $\Omega_{r_n}$ ,  $S$ )] –  $V(0, x, S) = E^x \left[ \int_0^{r_n} \mathcal$  $x \in B(x_0, h),$ <br>  $x, S$ ) =  $\mathbb{E}^{x}$   $\left[$   $\right)$ <br>  $\left[$   $\right]$  =  $\lim_{n \to \infty}$ <br>  $\left[$ ,  $S$ ) –  $V(0, x),$ <br>  $\left[$ ,  $h$ ). The abov<br>  $\mathbb{E}^{x}$   $\left[ V(\tau_{B(x_0, h)}, \sigma) \times \right]$   $\left[ \mathcal{E}(X_0, h) \right]$ <br>  $\left[ \sigma \right]$   $\left[ \mathcal{E}(X_0, h) \right]$ <br>  $\left[ \sigma$  $E = [V(t_n, A_{\tau_n}, S)] - V(0, X, S)] = E$ <br>  $E^X [V(\tau_{B(x_0, h)}, X_{\tau_{B(x_0, h)}}, S)] = \lim_{n \to \infty}$ <br>
simplies that<br>  $E^X [V(\tau_{B(x_0, h)}, X_{\tau_{B(x_0, h)}}, S) - V(0, S)]$ <br>  $W$  consider  $\tilde{S} = S \setminus B(x_0, h)$ . The at<br>  $J(x, \tilde{S}) - f(x) = E^X [V(\tau_{B(x_0, h)}],$ <br>  $J(x, \tilde{S}) - J$  $\begin{bmatrix} \frac{1}{2}x \\ y \end{bmatrix}$ .  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ .  $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ .  $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ .  $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ .  $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ .  $\begin{aligned} \n\mathbb{E}^{\mathcal{X}}[l] \mathcal{S} &= \mathbb{E}^{\mathcal{X}}[l] \mathcal{S} \n\end{aligned}$ <br>  $\mathbb{E}^{\mathcal{X}}[l]$ <br>  $\mathcal{S}$  a conne  $\mathcal{X}_{\tau_B}$ <br>  $\mathcal{X}_{\tau_B}$ <br>  $\mathcal{Y}$  a conne<br>  $\mathcal{X}_{\tau_B}$ <br>  $\mathcal{Y}$  a conne<br>  $\mathcal{Y} \mathcal{Y}$ <br>  $\mathcal{Y} \mathcal{Y}$ <br>  $\mathcal{Y} \$  $V(s, X_s, S)$ <br>  $Y(\tau_n, X_{\tau_n}, S)$ <br>  $\geq \mathbb{E}^{x} \left[ \int_{0}^{\tau_{B(x)}}$ <br>
equality implementation.<br>  $\pi \in I$ . Wit<br>  $> 0$  such th<br>  $n \in I$ . Wit<br>  $> 0$  such th<br>  $0 < \tilde{h} < h$  such  $0 < \tilde{h} < h$  suppose aga<br>  $x_0 + h_0 \subset C$ <br>
on  $(x_0 - h)$ <br>
e appli  $\delta$  (s is a set of  $\alpha$  in  $(x_0 + b_0)$  is a set of  $\alpha$  is a set of  $\begin{bmatrix} d \\ d \end{bmatrix}$  as  $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$  $\frac{a}{2}$  ()  $\frac{b}{2}$  ()  $\frac{c}{2}$  ()  $\frac{c}{2}$  ()  $\frac{d}{2}$  ()  $\frac{c}{2}$  ()  $\frac{d}{2}$  ()  $\frac{d}{2}$  $\left\{\begin{array}{c} \n\frac{1}{2} \\
\frac{1}{2} \\$  $\begin{array}{ccc} 1 & \text{if} &$ dt ]<br>
9,  $f(x \in (87, 8), 8)$ <br>
1,  $f(x \in S)$  $\forall x \in B(x_0, h).$ <br>  $B(x_0, h).$ <br>  $\leq B(x_0, h).$ <br>  $\leq 0$ , as desired rality, we ass<br>  $\cap (\theta_n, \theta_{n+1}).$  B<sub>i</sub>(ii), and the fact<br>  $\geq a/2 > 0$  for al and a contradia<br>  $\cap$   $\vee$   $\angle V(0, x_0+, s)$ <br>  $\cap$   $\vee$   $\angle V(0, x_0+, s)$ E<sup>x</sup>[V(τ<sub>B(x<sub>0</sub>,h), X<sub>τB(x<sub>0</sub>,h), 274 and t<sub>1</sub> < n)  $\iota_{\eta}$ ,  $\iota_{\eta}$ ,  $\iota_{\eta}$ <br>
simplies that<br>
E<sup>x</sup>[V(τ<sub>B(x<sub>0</sub>,h)</sub>, X<sub>τB(x<sub>0</sub>,h)</sub>, S) – V(0, x, S)] ≥ E<sup>x</sup>[ $\iint_0^{\tau_{B(x_0)}}$ <br>
w consider  $\tilde{S} = S \setminus B(x_0, h)$ . The abov</sub></sub>  $E^x[V(\tau_{B(x_0,h)}, X_{\tau_{B(x_0,h)}}, S) - V(0, x, S)] ≥ E^x$ <br>
consider  $\widetilde{S} = S \setminus B(x_0, h)$ . The above inequa<br>  $J(x, \widetilde{S}) - f(x) = E^x[V(\tau_{B(x_0,h)}, X_{\tau_{B(x_0,h)}})]$ <br>
usly,  $J(\cdot, \widetilde{S}) = J(\cdot, S)$  on  $\mathbb{X} \setminus B(x_0, h)$ . This<br>
brium and is strictly bet implie  $-V(0)$ <br>ther von. H<br>Withcom. H<br>with that  $\lambda$  Assure  $\lambda$  such again  $\subset (S - h_0, h_0)$ <br>again  $\lambda$  to get y )] (e ti c ) (e ) c (e ) n ) n )  $\delta(t)$   $\frac{1}{2}$ <br>ss<br>, x, S)<br>ith E ence,<br>it are  $(x_0, x_0)$ <br>that  $I$ <br> $(x_0 + \tilde{h})$ <br>that  $I$ <br> $\cap G$ ), are  $\begin{bmatrix} a \\ 2 \end{bmatrix}$ <br>  $\begin{bmatrix} 2 \end{bmatrix}$ <br>  $\begin$  $\forall x \in B(x_0, h).$ <br>
ion (87) shows that  $\hat{S}$ <br>  $(0, x_0, S) \le 0$ , as desire for generality, we as  $n) \subset S^\circ \cap (\theta_n, \theta_{n+1}).$ <br>  $1(i), 2.8(ii),$  and the fall  $(0, y, S) \ge a/2 > 0$  for  $a \in S^\circ \cap G$ , and a contrad  $(0, x_0 - h_0, x_0) \subset S^c$  for  $(x$  $S = S \setminus B(x_0, h)$ . The above inequality implies<br>  $\widetilde{S}$ ) -  $f(x) = \mathbb{E}^{x}[V(\tau_{B(x_0, h)}, X_{\tau_{B(x_0, h)}}, S)] - V(0,$ <br>  $\widetilde{S}$ ) =  $J(\cdot, S)$  on  $\mathbb{X} \setminus B(x_0, h)$ . This together wid is strictly better than  $S$ , a contradiction. Her  $J(x, S) - f(x) = \mathbb{E}^x [V(\tau_{B(x_0, h)}, X_{\tau_{B(x_0, h)}}, S)] - V(0, x, S) > 0 \quad \forall x \in B(x_0, h).$  (87)<br>  $J(\cdot, \overline{S}) = J(\cdot, S)$  on  $\mathbb{X} \setminus B(x_0, h)$ . This together with Equation (87) shows that  $\overline{S}$  is an<br>
an and is strictly better than *S*,  $J(\cdot, S) = J(\cdot, S)$  on  $\mathbb{X} \setminus B(x_0, h)$ . This together with Equation (87) shows that *S* and is strictly better than *S*, a contradiction. Hence,  $\mathcal{L}V(0, x_0, S) \le 0$ , as desired i)  $x_0 = \theta_n \in S^\circ \setminus G$  for some  $n \in I$ . W S is an ed.<br>ssume<br>By the act that<br>all  $y \in$ <br>diction<br> $+, S) >$ <br>r some<br> $S) > 0$ . S, a contradiction. Hence,  $\mathcal{L}V(0, x_0, S) \le 0$ , as desired.<br>some  $n \in I$ . Without loss of generality, we assicle  $h > 0$  such that  $(x_0, x_0 + h) \subset S^\circ \cap (\theta_n, \theta_{n+1})$ . By  $x_0 + h$ ) (due to Assumptions 2.1(i), 2.8(ii), and the **Case (ii)**  $x_0 = \theta_n \in S^\circ \setminus G$  for some  $n \in I$ . Without loss of generality, we assume  $x_0 = \theta_n \in S^\circ \setminus G$  for some  $n \in I$ . Without loss of generality, we assume<br>  $= a > 0$ . Then we can pick  $h > 0$  such that  $(x_0, x_0 + h) \subset S^\circ \cap (\theta_n, \theta_{n+1})$ . By the<br>  $\rightarrow$   $\mathcal{L}V(0, x+, S)$  on  $[x_0, x_0 + h)$  (due to Assumptions 2.1(  $\mathcal{L}V(0, x_0 +, S) = a > 0$ . Then we can pick  $h > 0$  such that  $(x_0, x_0 + h) \subset S^{\circ} \cap (\theta_n, \theta_{n+1})$ . By the  $V(0, x_0 +, S) = a > 0$ . Then we can pick  $h > 0$  such that  $(x_0, x_0 + h) \subset S^\circ \cap (\theta_n, \theta_{n+1})$ . By the ontinuity of  $x \to \mathcal{L}V(0, x +, S)$  on  $[x_0, x_0 + h)$  (due to Assumptions 2.1(i), 2.8(ii), and the fact that  $(t, x, S) = \delta(t)f(x)$  for  $x \$ continuity of  $x \to \mathcal{L}V(0, x +, S)$  on  $[x_0, x_0 + h]$  (due to Assumptions 2.1(i), 2.8(ii), and the fact that  $x \to \mathcal{L}V(0, x+, S)$  on [ $x_0$ ,  $x_0 + h$ ) (due to Assumptions 2.1(i), 2.8(ii), and the fact that  $(t)f(x)$  for  $x \in S$ ), we can find  $0 < h < k$  such that  $\mathcal{L}V(0, y, S) \ge a/2 > 0$  for all  $y \in$ Set  $\tilde{x} := (2x_0 + \tilde{h})/2$ . Then  $B(\til$ can be reached by the same argument as in case (i).

 $V(t, x, S) = \delta(t) f(x)$  for  $x \in S$ ), we can find  $0 < \tilde{h} < h$  such that  $LV(0, y, S) \ge a/2 > 0$  for all  $y \in (x_0, x_0 + \tilde{h})$ . Set  $\tilde{x} := (2x_0 + \tilde{h})/2$ . Then  $B(\tilde{x}, \tilde{h}/4) \subset (x_0, x_0 + \tilde{h}) \subset S^\circ \cap G$ , and a contradiction can be re  $(x_0, x_0 + \tilde{h})$ . Set  $\tilde{x} := (2x_0 + \tilde{h})/2$ . Then  $B(\tilde{x}, \tilde{h}/4) \subset (x_0, x_0 + \tilde{h}) \subset S^\circ \cap G$ , and a contradiction<br>can be reached by the same argument as in case (i).<br>**Case (iii)**  $x_0 \in \partial S$ . For boundary case (a), suppose **Case (iii)**  $x_0 \in \partial S$ . For boundary case (a), suppose again that  $\mathcal{L}V(0, x_0, S) \vee \mathcal{L}V(0, x_0, S) >$  $x_0 \in \partial S$ . For boundary case (a), suppose again that  $\mathcal{L}V(0, x_0-, S) \vee \mathcal{L}V(0, x_0+, S)$  > sss of generality, we assume  $(x_0, x_0 + h_0) \subset (S^\circ \cap G)$  and  $(x_0 - h_0, x_0) \subset S^c$  for some emma 2.14(a),  $\mathcal{L}V(0, x-, S) \equiv 0$  on 0. Without loss of generality, we assume  $(x_0, x_0 + h_0) \subset (S^\circ \cap G)$  and  $(x_0 - h_0, x_0) \subset S^c$  for some  $h_0 > 0$ . By Lemma 2.14(a),  $\mathcal{L}V(0, x-, S) \equiv 0$  on  $(x_0 - h_0, x_0]$ , and therefore,  $\mathcal{L}V(0, x_0+, S) > 0$ .<br>Then the same a  $h_0 > 0$ . By Lemma 2.14(a),  $LV(0, x-, S) \equiv 0$  on  $(x_0 - h_0, x_0]$ , and therefore,  $LV(0, x_0+, S) > 0$ .<br>Then the same argument as in case (ii) can be applied to get a contradiction. Then the same argument as in case (ii) can be applied to get a contradiction.

<span id="page-26-0"></span>For boundary case (b), Lemma [2.14\(](#page-9-0)a) directly tells that  $LV(0, x-, S) \vee LV(0, x+, S) = 0$ , and the proof is complete. П

# **5 WHEN WEAK OR OPTIMAL MILD EQUILIBRIA ARE STRONG**

After establishing the relation between optimal mild and weak equilibria, we take a further step to study whether a weak or optimal mild equilibrium is strong.

(0, −, ) ∨ (0, +, ) = 0, and company and the manufacture of the same of We already know that an admissible weak or optimal mild equilibrium S satisfies the two con-S satisfies the two con-<br>irst-order condition (6)<br>ussion at the beginning<br>iof Equation (28) being<br>ualities (24) and (25) is<br>d also be strong. To this<br> $\langle 29 \rangle > V_x(0, x+, S)$ }. (88)<br>ire provided in the next<br>ong.<br>equilibrium. I ditions [\(24\)](#page-10-0) and [\(25\)](#page-10-0) in Theorem [3.1.](#page-10-0) To make S a strong equilibrium, the first-order condition (6) S a strong equilibrium, the first-order condition (6)<br>ondition (7). Recall the discussion at the beginning<br>on for Equation (7) is the LHS of Equation (28) being<br>at least one of the two inequalities (24) and (25) is<br>equili needs to be upgraded to the local maximum condition [\(7\)](#page-3-0). Recall the discussion at the beginning of Section [3.1.](#page-11-0) Intuitively, a sufficient condition for Equation [\(7\)](#page-3-0) is the LHS of Equation (28) being negative for all  $\varepsilon$  small enough. As a result, if at least one of the two inequalities (24) and (25) is strict for all the points in the weak or optimal equilibrium  $S$ , then  $S$  should also be strong. To this end, let us define for any admissible  $S \in \mathcal{B}$ ,

$$
\mathfrak{S}_S := \{ x \in S : \mathcal{L}V(0, x-, S \wedge \mathcal{L}V(0, x+, S) < 0 \} \cup \{ x \in S : V_x(0, x-, S) > V_x(0, x+, S) \}. \tag{88}
$$

Theorems 5.1 and 5.2 are the main results of this section, and their proofs are provided in the next subsection. The first main result concerns when a weak equilibrium is strong.

**Theorem 5.1.** *Let Assumptions* [2.1](#page-5-0)[–2.10](#page-8-0) *hold and S be an admissible weak equilibrium. If*  $S = \mathfrak{S}_{S}$ , *then S is also strong.* 

The next result regards the relation between optimal mild and strong equilibria.

**Theorem 5.2.** *Let Assumptions [2.1](#page-5-0)[–2.10](#page-8-0) hold.*

- (a) For any admissible optimal mild equilibrium *S*, if  $\mathfrak{S}_S$  is admissible and closed, then  $\mathfrak{S}_S$  is a *strong equilibrium.*
- *(b)* Recall  $S^*$  defined in Equation (58). We have  $\overline{\mathfrak{S}}_{S^*} = S^*$ . Hence, if  $\mathfrak{S}_{S^*}$  is closed and admissible, *then*  $S^*$  *is a strong equilibrium.*

*Remark* 5.3. Theorem 5.2 indicates that  $S^*$  and  $\mathfrak{S}_{S^*}$  are almost the same, and roughly speaking,  $S^*$ is a strong equilibrium possibly except some points in  $\mathfrak{S}_{S^*}\setminus \mathfrak{S}_{S^*}$ . In many cases, we indeed have

s mall enough. As a result, if at least one of the two inequalities (24) and [\(25\)](#page-10-0) is<br>points in the weak or optimal equilibrium S, then S should also be strong. To this<br>foreign for any admissible  $S \in B$ ,<br> $\therefore LV(0, x-, S \land LV(0, x$ S, then S should also be strong. To this<br>
S :  $V_x(0, x-, S) > V_x(0, x+, S)$ }. (88)<br>
and their proofs are provided in the next<br>
quilibrium is strong.<br> *dmissible weak equilibrium. If*  $S = \mathfrak{S}_S$ ,<br>
ild and strong equilibria.<br> *is*  $S \in B$ ,<br>(0, x +,<br>sults of<br>cerns \times \text{\sults of<br>cerns \text{\sults of<br>2.10 hol<br>n betw<br>2.10 hol<br>! equili<br>equili {\sults }). We l<br>at S\* a<br>pt some<br>trong.<br>2.1-2.10<br>times \text{\surft{\surft{\surft{\surft{\surft{\surft{  $\mathfrak{S}_5 := \{x \in S : L^V(0, x-, S \wedge L^V(0, x+, S) < 0\} \cup \{x \in S : V_x(0, x-, S) > V_x(0, x-, S)\}$ <br>corens 5.1 and 5.2 are the main results of this section, and their proofs are provided in the<br>section. The first main result concerns when a w *S* be an admissible weak equilibrium. If  $S = \mathfrak{S}_S$ ,<br>
timal mild and strong equilibria.<br> *S*, if  $\mathfrak{S}_S$  is admissible and closed, then  $\mathfrak{S}_S$  is a<br>  $\overline{S_{S^*}} = S^*$ . Hence, if  $\mathfrak{S}_{S^*}$  is closed and admiss *S* is also strong.<br> **i**e next result re<br> **ior any admissistrong equilibrit**<br> **ior any admissistrong equilibrit**<br> **ions**  $S^*$  is a strong equilibrit<br> **ions**  $S^*$  is a strong equilibrit<br>  $\mathfrak{S}_{S^*}$ , as a resul<br>  $ark$  5. *S*, if  $\mathfrak{S}_S$  is admissible and closed, then  $\mathfrak{S}_S$  is a<br>  $\frac{1}{S^{s}} = S^*$ . Hence, if  $\mathfrak{S}_{S^*}$  is closed and admissible,<br>
are almost the same, and roughly speaking,  $S^*$ <br>  $\sin \overline{\mathfrak{S}_{S^*}} \setminus \mathfrak{S}_{S^*}$ . In *S*<sup>\*</sup> *defined in Equation [\(58\)](#page-18-0). We have*  $\mathfrak{S}_{S^*} = S^*$ *. Hence, if*  $\mathfrak{S}_{S^*}$  *is closed and admissible, is a strong equilibrium.*<br> *i.* Theorem 5.2 indicates that *S*<sup>\*</sup> and  $\mathfrak{S}_{S^*}$ , are almost the same *s*<sup>\*</sup> *is a strong equilibrium.*<br> *i.3.* Theorem 5.2 indicates<br> *g* equilibrium possibly exe, as a result of which *S*<sup>\*</sup> *i*:<br> *i.4.* Suppose Assumption<br>
ed point at which *f* is c<br>  $S^* = \overline{\mathfrak{S}}_{S^*}$  and *f* is smoo *S*<sup>\*</sup> and  $\mathfrak{S}_S$ <sup>\*</sup> are almost the same, and roughly speaking, *S*<sup>\*</sup><br>some points in  $\mathfrak{S}_{S^*} \setminus \mathfrak{S}_{S^*}$ . In many cases, we indeed have<br>ong. This is demonstrated in all the examples in Section 6.<br>-2.10 hold a  $\mathfrak{S}_{S^*} \setminus \mathfrak{S}_{S^*}$ . In many cases, we indeed have<br>nonstrated in all the examples in Section 6.<br> $S^*$  is admissible. Then  $S^*$  cannot contain<br>ntiable. Indeed, suppose x is an isolated<br>\*. On the other hand, since  $S^* = \mathfrak{S}_{S^*}$ , as a result of which  $S^*$  is strong. This is demonstrated in all the examples in Section [6.](#page-32-0)<br> *Remark* 5.4. Suppose Assumptions 2.1–2.10 hold and  $S^*$  is admissible. Then  $S^*$  cannot contain<br>
an isola *Remark* 5.4. Suppose Assumptions [2.1–](#page-5-0)2.10 hold and  $S^*$  is admissible. Then  $S^*$  cannot contain *S*<sup>\*</sup> is admissible. Then *S*<sup>\*</sup> cannot contain intiable. Indeed, suppose *x* is an isolated ... On the other hand, since  $\mathcal{L}(0, x-, S^*)$  =  $V_x(0, x+, S)$  by Corollary 3.2, we would an isolated point at which  $f$  is continuously differentiable. Indeed, suppose  $x$  is an isolated f is continuously differentiable. Indeed, suppose x is an isolated<br>mooth at x. Then  $x \in \mathfrak{S}_{S^*}$ . On the other hand, since  $\mathcal{L}(0, x-, S^*) =$ <br>2.14(a), and  $V_x(0, x-, S^*) = V_x(0, x+, S)$  by Corollary 3.2, we would<br>n. point of  $S^* = \mathfrak{S}_{S^*}$  and f is smooth at x. Then  $x \in \mathfrak{S}_{S^*}$ . On the other hand, since  $\mathcal{L}(0, x-, S^*) =$  $S^* = \overline{\mathfrak{S}}_{S^*}$  and  $f$  is smooth at  $x$ . Then  $x \in \mathfrak{S}_{S^*}$ . On the other hand, since  $\mathcal{L}(0, x-, S^*) = S^*$  = 0 by Lemma 2.14(a), and  $V_x(0, x-, S^*) = V_x(0, x+, S)$  by Corollary 3.2, we would  $\mathfrak{S}_{S^*}$ , a contradictio  $\mathcal{L}(0, x +, S^*) = 0$  by Lemma 2.14(a), and  $V_x(0, x-, S^*) = V_x(0, x+, S)$  by Corollary 3.2, we would (0, x+, S<sup>\*</sup>) = 0 by Lemma 2.14(a), and  $V_x(0, x-, S^*) = V_x(0, x+, S)$  by Corollary 3.2, we would<br>ave  $x \notin \mathfrak{S}_{S^*}$ , a contradiction. have  $x \notin \mathfrak{S}_{S^*}$ , a contradiction.  $x \notin \mathfrak{S}_{S^*}$ , a contradiction.

# <span id="page-27-0"></span>**824 IM/II FV** BAYRAKTAR ET AL.

## **5.1 Proofs of Theorems [5.1](#page-26-0) and [5.2](#page-26-0)**

As discussed above, we aim to achieve the negativity in the RHS of Equation [\(28\)](#page-11-0) for  $\varepsilon$  small enough; when  $V_x(s, x+, S) - V_x(s, x-, S) = 0$ , the integral on  $\frac{1}{2}(CV(s, X_s-, S) + LV(s, X_s+, S))$ on the RHS of Equation [\(28\)](#page-11-0) should be negative. Since only one of the two values  $\mathcal{L}V(s,X_{s+},S)$  is required to be negative in the definition of  $\mathfrak{S}_S$ , we will estimate the probability that X goes to the left/right from the starting point. Such probability estimation is provided in the following lemma.

**Lemma 5.5.** *Let Assumption [2.1](#page-5-0) hold. Then,*

$$
\lim_{t \searrow 0} \mathbb{P}^{x_0}(X_t > x_0) = \lim_{t \searrow 0} \mathbb{P}^{x_0}(X_t < x_0) = \frac{1}{2}, \quad \forall x_0 \in \mathbb{X}.
$$
 (89)

*Proof.* Let  $X_0 = x_0 \in \mathbb{X}$ . Recall  $\overline{X}$  and  $\overline{X}$  defined in Equation (35). Denote  $R_\varepsilon := \mu(x_0)\varepsilon + \overline{X}_\varepsilon$ . Then,

$$
X_{\varepsilon} = x_0 + R_{\varepsilon} + \sigma(x_0)W_{\varepsilon}.
$$
\n(90)

By Lemma [3.7,](#page-12-0) there exists some constant  $C > 0$  such that for any  $\varepsilon > 0$  small enough,  $\mathbb{E}^{x_0}[|R_{\varepsilon}|] \leq$ 

$$
\mathbb{P}^{x_0}\left(|R_{\varepsilon}| \ge \frac{1}{2}\varepsilon^{3/4}\right) \le \frac{2\mathbb{E}^{x_0}[|R_{\varepsilon}|]}{\varepsilon^{3/4}} \le 2C \cdot \varepsilon^{1/4}.\tag{91}
$$

By Equations (90) and (91), for  $\varepsilon > 0$  small enough

As discussed above, we aim to achieve the negativeity in the RHS of Equation (28) for 
$$
\varepsilon
$$
 small  
enough; when  $V_x(s, x, x, S) - V_x(s, x, S) = 0$ , the integral on  $\frac{2}{2}(CV(s, X_x +, S))$  is  
on the RHS of Equation (28) should be negative. Since only one of the two values  $\mathcal{LV}(s, X_x +, S)$  is  
required to be negative in the definition of  $\mathfrak{G}_S$ , we will estimate the probability that  $X$  goes to the  
left/right from the starting point. Such probability estimation is provided in the following lemma.  
**Lemma 5.5.** Let Assumption 2.1 hold. Then,  

$$
\lim_{t \to 0} P^{\chi_0}(X_t > x_0) = \lim_{t \to 0} P^{\chi_0}(X_t < x_0) = \frac{1}{2}
$$
,  $\forall x_0 \in \mathbb{X}$ .  
*Proof.* Let  $X_0 = x_0 \in \mathbb{X}$ . Recall  $\overline{X}$  and  $\overline{X}$  defined in Equation (35). Denote  $R_\varepsilon := \mu(x_0)\varepsilon + \overline{X}_\varepsilon$ .  
Then,  

$$
X_\varepsilon = x_0 + R_\varepsilon + \sigma(x_0)W_\varepsilon.
$$
 (90)  
By Lemma 3.7, there exists some constant  $C > 0$  such that for any  $\varepsilon > 0$  small enough,  $\mathbb{E}^{x_0}[|R_\varepsilon|] \leq$   
 $C\varepsilon$ , which leads to  

$$
\mathbb{P}^{x_0}(|R_\varepsilon| \geq \frac{1}{2}\varepsilon^{3/4}) \leq \frac{2\mathbb{E}^{x_0}[|R_\varepsilon|]}{\varepsilon^{3/4}} \leq 2C \cdot \varepsilon^{1/4}.
$$
 (91)  
By Equations (90) and (91), for  $\varepsilon > 0$  small enough  

$$
\mathbb{P}^{x_0}(|R_\varepsilon| \geq \frac{1}{2}\varepsilon^{3/4})
$$

$$
\geq \mathbb{P}^{x_0}(\sigma(x_0)W_\varepsilon > \varepsilon^{3/4}, R_\varepsilon > -\frac{1}{2}\varepsilon^{3/4})
$$

$$
\geq 1 - \Phi\left(\frac{\v
$$

1 – Φ $\left(\frac{\varepsilon^{1/4}}{\sigma(x_0)}\right)$ <br>
• Φ is the cu<br>
lim inf<sub>t \ni</sub>  $\mathbb{P}^1$ <br>
w we are rea<br>
of Theorem 5<br>
S is a weak of the current state<br>of  $\mathbb{P}^{x_0}$ <br>examples and contained the contact of  $\mathbb{P}^{x_0}$  $-2C\epsilon^{1/4}$  → 1 –  $\Phi(0)$  – 0 =  $\frac{1}{2}$ <br>
alative distribution function 1<br>  $V_t > x_0$ ) ≥  $\frac{1}{2}$ . Similarly, lim in<br>
co prove Theorem 5.1.<br>
To prove the desired result, w<br>  $(x_0) > 0$ , s.t.  $\forall \epsilon' \le \epsilon(x_0), f(x_0)$ <br>
dilibriu  $\frac{1}{2}$ <br>in  $\frac{1}{2}$ , as  $\varepsilon \searrow 0$ ,<br>
for the stand<br>  $f_{t\searrow 0} \mathbb{P}^{x_0}(X_t)$ <br>
e need to ver<br>  $0 - \mathbb{E}^{x_0} [\delta(\rho_S^{\varepsilon'}))$ <br>  $0, \quad \forall x$ . where  $\Phi$  is the cumulative distribution function for the standard normal distribution. There-Φ is the cumulative distribution function for the standard normal distribution. There-<br>
m inf<sub>t\2</sub>  $\mathbb{P}^{x_0}(X_t > x_0) \ge \frac{1}{2}$ . Similarly, lim inf<sub>t\2</sub>  $\mathbb{P}^{x_0}(X_t < x_0) \ge \frac{1}{2}$ . Thus, Equation (89)<br>  $\Box$ <br>
we are fore,  $\liminf_{t\searrow 0} \mathbb{P}^{x_0}(X_t > x_0) \geq \frac{1}{2}$ . Similarly,  $\liminf_{t\searrow 0} \mathbb{P}^{x_0}(X_t < x_0) \geq \frac{1}{2}$ . Thus, Equation (89) lim inf<sub>t</sub><sub>\so</sub>  $\mathbb{P}^{x_0}(X_t > x_0) \ge \frac{1}{2}$ .<br>
w we are ready to prove The<br>
of Theorem 5.1. To prove the<br>  $\exists \varepsilon(x_0) > 0$ , s.t<br>
S is a weak equilibrium, by<br>  $V_x(0, x_0)$ lim inf<sub>t</sub><sub>\so</sub>  $\mathbb{P}^{x_0}(X_t < x_0) \ge \frac{1}{2}$ <br>ult, we need to verify that fo<br> $f(x_0) - \mathbb{E}^{x_0}[\delta(\rho_S^{\varepsilon'})f(X_{\rho_S^{\varepsilon'}})]$ <br>,<br>,<br>,<br>,<br>, x<sub>0</sub>+, S)  $\ge$  0,  $\forall x_0 \in S$ . holds.

Now we are ready to prove Theorem 5.1.

*Proof of Theorem* 5.1. To prove the desired result, we need to verify that for any  $x_0 \in S$ ,

By to prove Theorem 5.1.

\n1. To prove the desired result, we need to verify that for any 
$$
x_0 \in S
$$
,

\n
$$
\exists \varepsilon(x_0) > 0, \text{ s.t. } \forall \varepsilon' \leq \varepsilon(x_0), f(x_0) - \mathbb{E}^{x_0} [\delta(\rho_{S}^{\varepsilon'}) f(X_{\rho_{S}^{\varepsilon'}})] \geq 0.
$$
\n(92)

\nequilibrium, by Theorem 3.1,

\n
$$
V_x(0, x_0-, S) - V_x(0, x_0+, S) \geq 0, \quad \forall x_0 \in S.
$$

Since S is a weak equilibrium, by Theorem  $3.1$ ,  $S$  is a weak equilibrium, by Theorem 3.1,<br>  $\label{eq:V_X(0, X_0-, S) - V_X(0)} V_x(0, x_0-, S) - V_x(0)$ 

$$
V_x(0, x_0-, S) - V_x(0, x_0+, S) \ge 0, \quad \forall x_0 \in S.
$$
  

$$
V_x(0, x_0-, S) - V_x(0, x_0+, S) \ge 0, \quad \forall x_0 \in S.
$$

<span id="page-28-0"></span>Recall Equation [\(88\)](#page-26-0) and G defined in Equation [\(16\)](#page-7-0). Pick  $x_0 \in \mathfrak{S}_S$ , and we shall verify Equation [\(92\)](#page-27-0) for two cases: (i)  $V_x(0, x_0-, S) - V_x(0, x_0+, S) > 0$ , and (ii)  $V_x(0, x_0-, S)$  $V_r(0, x_0 +, S) = 0.$ 

**Case (i)** Suppose  $a := V_x(0, x_0-, S) - V_x(0, x_0+, S) > 0$ . By the continuity of  $t \mapsto$  $V_x(t, x_0 \pm, S)$ , we take  $\varepsilon > 0$  small enough such that  $\delta(t) > \frac{1}{2}$  for all  $t \in (0, \varepsilon)$ , and

$$
V_x(t, x_0 +, S) - V_x(t, x_0 -, S) < -\frac{a}{2}, \quad \forall \, t \in (0, \varepsilon). \tag{93}
$$

Let  $h > 0$  such that both  $(x_0 - h, x_0)$  and  $(x_0, x_0 + h)$  belong to G. Then for  $\varepsilon$  small enough,

all Equation (88) and G defined in Equation (16). Pick 
$$
x_0 \in \mathfrak{S}_5
$$
, and we shall verify  
ation (92) for two cases: (i)  $V_x(0, x_0-, S) - V_x(0, x_0+, S) > 0$ , and (ii)  $V_x(0, x_0-, S) -$   
 $x_0, x_0+, S) = 0$ .  
**Case (i)** Suppose  $a := V_x(0, x_0-, S) - V_x(0, x_0+, S) > 0$ . By the continuity of  $t \mapsto$   
 $x, x_0 \pm$ , S), we take  $\varepsilon > 0$  small enough such that  $\delta(t) > \frac{1}{2}$  for all  $t \in (0, \varepsilon)$ , and  
 $V_x(t, x_0+, S) - V_x(t, x_0-, S) < -\frac{a}{2}$ ,  $\forall t \in (0, \varepsilon)$ .  
 $h > 0$  such that both  $(x_0 - h, x_0)$  and  $(x_0, x_0 + h)$  belong to G. Then for  $\varepsilon$  small enough,  

$$
E^{x_0}[V(\varepsilon, X_\varepsilon, S)] - V(0, x_0, S) + o(\varepsilon) = E^{x_0}[V(\varepsilon \land r_{B(x_0,h)}, X_{\varepsilon, x_{B(x_0,h)}}, S)] - V(0, x_0, S)
$$

$$
\leq E^{x_0}\left[\int_0^{\varepsilon \land \tau_{B(x_0,h)}} \frac{1}{2}(CV(x, X_s-, S) + LV(x, X_s+, S))ds\right] = \frac{a}{4}E^{x_0}[L^{x_0}_{\tau_{B(x_0,h)}, A}].
$$
  
are the first (in)equality follows from Lemma 3.8, the second (in)equality follows from  
mean 2.15 and Equation (93) (the diffusion term vanishes after taking expectation due to the  
nodes of  $V_x\sigma$  on  $[0, \varepsilon] \times \overline{B(x_0, h)}$ ). By Lemma 2.14(b), there exists a constant  $K > 0$  such  
( $0,0,0,0)$  and  $|\sigma(x_0)| > 0$ , we can take  $\varepsilon_0 \in (0, 1)$  such that for any  $\varepsilon \in (0, \varepsilon_0)$ ,  

$$
E^{x_0}[L^{
$$

 $V_x(0, x_0+, S) = 0.$ <br> **Case (i)** Sup<br>  $V_x(t, x_0 \pm, S)$ , we t<br>
Let  $h > 0$  such that<br>  $\mathbb{E}^{x_0}[V(\varepsilon, X_{\varepsilon}, S)]$ <br>  $\leq \mathbb{E}^{x_0} \left[ \int_0^{\varepsilon \wedge \tau}$ <br>
where the first (<br>
Lemma 2.15 and l<br>
boundedness of  $V$ <br>
that<br>
Then by Lemma  $V_x(t, x_0 \pm, S)$ , we take  $\varepsilon > 0$  small enough such that  $\delta(t) > \frac{1}{2}$ <br>  $V_x(t, x_0 +, S) - V_x(t, x_0 -, S) < -\frac{a}{2}$ ,<br>
Let  $h > 0$  such that both  $(x_0 - h, x_0)$  and  $(x_0, x_0 + h)$  belong<br>  $E^{x_0}[V(\varepsilon, X_{\varepsilon}, S)] - V(0, x_0, S) + o(\varepsilon) = E^{x_0}[V$ where the first (in)equality follows from Lemma 3.8, the second (in)equality follows from Lemma 2.15 and Equation (93) (the diffusion term vanishes after taking expectation due to the boundedness of  $V_x \sigma$  on  $[0, \varepsilon] \times B(x_0, h)$ . By Lemma 2.14(b), there exists a constant  $K > 0$  such that

$$
\sup_{(t,y)\in[0,1]\times B(x_0,h)}\frac{1}{2}|\mathcal{L}V(t,y+,S)+\mathcal{L}V(t,y-,S)|\leq K,
$$

Then by Lemma 3.9 and  $|\sigma(x_0)| > 0$ , we can take  $\varepsilon_0 \in (0,1)$  such that for any  $\varepsilon \in (0,\varepsilon_0)$ ,  $\frac{a}{4\varepsilon} \mathbb{E}^{x_0}[L_{\tau_{B(x_0,h)}\wedge \varepsilon}^{x_0}] \ge (K+1)$  and the term  $o(\varepsilon)$  in Equation (94) satisfies  $|o(\varepsilon)| \le \frac{1}{2}\vare$ Equation (94) leads to

$$
\mathbb{E}^{x_0}[\delta(\rho_S^{\varepsilon})f(X_{\rho_S^{\varepsilon}})] - f(x_0) = \mathbb{E}^{x_0}[V(\varepsilon, X_{\varepsilon}, S)] - V(0, x_0, S)
$$
  

$$
\leq K\varepsilon - \frac{a}{4}\mathbb{E}^{x_0}[L_{\tau_{B(x_0, h)} \wedge \varepsilon}^{x_0}] + \frac{1}{2}\varepsilon \leq -\frac{1}{2}\varepsilon, \quad \forall \varepsilon \leq \varepsilon_0.
$$

**Case (ii)** Suppose  $V_x(0, x_0-, S) - V_x(0, x_0+, S) = 0$ . Then, by Equation (18),

$$
|V_x(t, x_0+, S) - V_x(t, x_0-, S)| = o(\sqrt{t})
$$
 for  $t > 0$  small enough.

This together with Lemma 3.9 leads to

$$
\mathbb{E}^{x_0} \left| \frac{1}{2} \int_0^{\varepsilon \wedge \tau_{B(x_0, h)}} (V_x(s, x_0 +, S) - V_x(s, x_0 -, S)) dL_s^{x_0} \right| = o(\sqrt{\varepsilon}) \cdot \mathbb{E}^{x_0} \left[ L_{\varepsilon \wedge \tau_{B(x_0, h)}}^{x_0} \right] = o(\varepsilon). \tag{95}
$$

Ed<br>Cl<br>ar<br>Ed  $\mathbb{C}^{\infty}[L_{\tau_B}]\n$ quation<br>  $\mathbb{E}^{x_0} \left| \frac{1}{2} \right|$ <br>  $\mathbb{E}^{x_0} \left| \frac{1}{2} \right|$ <br>  $\mathbb{E}^{x_0} \left[ \text{equation} \right]$  $\frac{1}{2}$ <br>oose *h*<br>aati Choose  $h_0 > 0$  such that  $(x_0 - h_0, x_0) \cup (x_0, x_0 + h_0)$  is contained in  $(S^{\circ} \cap G) \cup (\mathbb{X} \setminus S)$ . For any  $h \in (0, h_0)$ , similar to Equation (94), we apply Lemmas 2.15, 3.8 and then combine with Equation (95) to get

$$
\begin{aligned}\n\mathbb{E}^{\mathbf{x}_{0}}[\int_{0}^{\varepsilon_{\Lambda}x_{B}}(x_{0},y_{0}) + \mathbf{0}(x) = \mathbf{E}^{\mathbf{x}_{0}}[\mathbf{v}(\mathbf{x},\mathbf{x}_{B}(\mathbf{x}_{0},h)},\mathbf{P},\mathbf{P},\mathbf{P},\mathbf{Q},\mathbf{X},\mathbf{P},\mathbf{Q},\mathbf{X})] &\quad \times (\mathbf{x},\mathbf{x}_{0},\mathbf{P},\mathbf{X},\mathbf{P},\mathbf{Q},\mathbf{X}) \\
&\leq \mathbb{E}^{\mathbf{x}_{0}}\left[\int_{0}^{\varepsilon_{\Lambda}x_{B}}(x_{0},y_{0},y_{0}) + \mathbf{0}(\mathbf{v}(\mathbf{x}) = \mathbf{E}^{\mathbf{x}_{0}}[\mathbf{v}(\mathbf{x},\mathbf{x}_{0},\mathbf{x}),\mathbf{X},\mathbf{X})]ds\right] &-\frac{a}{4}\mathbf{E}^{\mathbf{x}_{0}}[\mathbf{L}_{(B)}^{\mathbf{x}_{B}}(x_{0},h),\mathbf{Y})\n\end{aligned}
$$
\nwhere the first (in)equality follows from Lemma 3.8, the second (in)equality follows from the terms 2.15 and Equation (9.3) (the diffusion term vanishes after taking expectation due to the boundary of the equation (9.4) and the term of the equation (9.4) and the form of the equation (9.4) satisfies  $|\sigma(\mathbf{x})| \leq \frac{a}{2}\mathbf{E}^{\mathbf{x}_{0}}[\mathbf{L}_{(B)}^{\mathbf{x}_{0}}(x_{0},y_{0}) = 2$ , we can take  $\varepsilon_{0} \in (0,1)$  such that for any  $\varepsilon \in (0,\varepsilon_{0})$ ,  $\frac{a}{4}\mathbf{E}^{\mathbf{x}_{0}}[\mathbf{L}_{(B)}^{\mathbf{x}_{0}}(x_{0},y_{0}) = 2$ , we can take  $\varepsilon_{0} \in (0,1)$  such that for any  $\varepsilon \in (0,\varepsilon_{0})$ ,  $\frac{a}{4}\mathbf{E}^{\mathbf{x}_{0}}[\mathbf{L}_{(B)}^{\mathbf{x}_{0}}(x_{0},y_{0})] = \mathbf{I}(\mathbf{x}_{0}) = \mathbf{E}^{\mathbf{x}_{0}}[\mathbf{V}(\mathbf$ 

<span id="page-29-0"></span>Since  $V_x(0, x_0-, S) - V_x(0, x_0+, S) = 0$  and  $x_0 \in \mathfrak{S}_S$ , we have

$$
\mathcal{L}V(0, x_0-, S) \wedge \mathcal{L}V(0, x_0+, S) < 0. \tag{97}
$$

Without loss of generality, we can assume that

$$
-A := \mathcal{L}V(0, x_0+, S) < 0 \quad \text{and} \quad \mathcal{L}V(0, x_0-, S) \le 0.
$$

By the (left/right) continuity of  $(t, x) \mapsto \mathcal{L}V(t, x \pm S)$  at  $(0, x_0)$ , we can choose  $h \in (0, h_0)$  and  $\varepsilon_0 >$ 

$$
\mathcal{L}V(t,x,S) = \mathcal{L}V(t,x+S) \le -\frac{A}{2} \quad \text{and} \quad \mathcal{L}V(t,y,S) = \mathcal{L}V(t,y-,S) \le \frac{A}{8}.\tag{98}
$$

Then for  $\varepsilon \in (0, \varepsilon_0)$  small enough, the first inequality in Equation (98) implies that

Since 
$$
V_x(0, x_0-, S) - V_x(0, x_0+, S) = 0
$$
 and  $x_0 \in \mathfrak{S}_5$ , we have  
\n
$$
\mathcal{L}V(0, x_0-, S) \land \mathcal{L}V(0, x_0+, S) < 0.
$$
\nWithout loss of generality, we can assume that  
\n
$$
-A := \mathcal{L}V(0, x_0+, S) < 0 \text{ and } \mathcal{L}V(0, x_0-, S) \leq 0.
$$
\nBy the (left/right) continuity of  $(t, x) \mapsto \mathcal{L}V(t, x, \pm, S)$  at  $(0, x_0)$ , we can choose  $h \in (0, h_0)$  and  $\varepsilon_0 > 0$  small enough, such that for any  $t \in [0, \varepsilon_0]$ ,  $x \in (x_0, x_0 + h)$  and  $y \in (x_0 - h, x_0)$ ,  
\n
$$
\mathcal{L}V(t, x, S) = \mathcal{L}V(t, x+, S) \leq -\frac{A}{2} \text{ and } \mathcal{L}V(t, y, S) = \mathcal{L}V(t, y-, S) \leq \frac{A}{8}.
$$
\n(98) Then for  $\varepsilon \in (0, \varepsilon_0)$  small enough, the first inequality in Equation (98) implies that  
\n
$$
\mathbb{E}^{x_0} \left[ \int_0^{\varepsilon_0} \left[ \frac{1}{(1 - (x_1, x_0)) \cdot 1} \left[ \frac{1}{(1 - (x_1, x_0)) \cdot 1} \right] \right] \leq -\frac{A}{2} \mathbb{E}^{x_0} \left[ \int_0^{\varepsilon_0} \left[ \frac{1}{(1 - (x_1, x_0)) \cdot 1} \left[ \frac{1}{(1 - (x_1, x_0)) \cdot 1} \right] \right] \right]
$$
\n
$$
= -\frac{A}{2} \int_0^{\varepsilon} \left[ \frac{1}{(1 - (x_1, x_0)) \cdot 1} \left[ \frac{1}{(1 - (x_1, x_0)) \cdot 1} \right] \left[ \frac{1}{(1 - (x_1, x_0)) \cdot 1} \right] \right]
$$
\n
$$
\leq -\frac{A}{
$$

where the forth (in)equality above follows from Lemma 5.5, and the sixth (in)equality follows from Lemma 3.6. In addition, the second inequality in Equation (98) implies

$$
\mathbb{E}^{x_0}\left[\int_0^{\tau_{B(x_0,h)}\wedge \varepsilon} \mathcal{L}V(t,X_t,S)1_{\{X_t < x_0\}}dt\right] \leq \frac{A}{8}\varepsilon. \tag{100}
$$

Therefore, by plugging Equations (99) and (100) into Equation (96), we have that for  $\varepsilon > 0$  small enough,

$$
\mathbb{E}^{x_0}[\delta(\rho_S^{\varepsilon})f(X_{\rho_S^{\varepsilon}})] - f(x_0) = \mathbb{E}^{x_0}[V(\varepsilon, X_{\varepsilon}, S)] - V(0, x_0, S) \le -\frac{A}{6}\varepsilon + \frac{A}{8}\varepsilon + o(\varepsilon) < -\frac{A\varepsilon}{25},
$$

and the proof is complete.

− <del>5</del><br>1 (ir<br>6. It<br>6. It<br>lugg<br>f(X<sub>6</sub><br>cor<br>m th<br>whi<br>f f 6. I 5<br>in)eq<br>In ad<br> $\left[ K_{\rho_S^{\varepsilon}} \right)$ ]<br>ompl<br>he pr<br>from<br> $Let$  inila  $\frac{1}{2}$  and  $\frac{1}{2}$  a 2<br>
aality and ality and it<br>
ition,  $E^{x_0}$ <br>
Equation<br>  $F(x_0)$ <br>
Equation<br>  $f(x_0)$ <br>
te.<br>  $\log f$  and  $\log f$ <br>  $\log f$ <br>  $\log f$ <br>  $\log f$  $\tau_{B(x_0, h)} < \varepsilon$ )<br>
bove follows<br>
he second in<br>  $\int_0^{\tau_{B(x_0, h)} \wedge \varepsilon} L'$ <br>
ns (99) and<br>  $= \mathbb{E}^{x_0}[V(\varepsilon, \lambda)]$ <br>
eorem 5.2, l<br>  $\mathfrak{S}_S$  actually<br>
emaining pa<br>
ions 2.1–2.10<br>
tion, if  $\mathfrak{S}_S$  is  $=-\frac{}{5}$ <br>from i<br>equality<br>equality<br>(t,  $X_t$ , 100) in<br> $\epsilon$ , S)] -<br>et us il<br>forms<br>t is sti<br>hold. I<br>admis  $5$ <br>ity ir Len<br>ity ir  $\left(1, S\right)$ <br>into<br> $\left| -V \right|$ <br>illust s the<br>fill of For  $\frac{1}{2}$  m E<br>  $\frac{1}{2}$  m E<br>  $\frac{1}{2}$  $\frac{A}{2} \varepsilon \cdot o(\varepsilon) \le -\frac{A}{6}$ <br>ma 5.5, and the<br>Equation (98) i<br> $x_t < x_0$ ; $dt$   $\le \frac{A}{8} \varepsilon$ <br> $u(t)$   $\le 0$ ,  $x_0, S$   $\le -\frac{A}{6} \varepsilon$ <br>ate a property c<br>tiessential" part<br>timal mild.<br>*ny admissible c*, then  $\overline{\mathfrak{S}}_S$  is 6  $\frac{1}{6}$  are s im  $\frac{1}{6}$   $\varepsilon$ . We consider the optimal  $\frac{1}{6}$  optimal optimal optimal optimal optimal optimal optimal optimal optimal optim  $\mathbf{u}$  and  $\mathbf{u}$  and  $\mathbf{u}$  and  $\mathbf{u}$  and  $\mathbf{u}$  and  $\mathbf{u}$ s (99) ar<br>  $\mathbb{E}^{x_0}[V(x_0), \dots, x_n]$ <br>
orem 5.<br>  $x_5$  actual<br>
variating<br>  $\frac{x_2}{x_1-x_2}$ <br>
on, if  $\frac{x_1}{x_2}$  $\frac{1}{\epsilon}$  and  $\frac{1}{\epsilon}$  and  $\frac{1}{\epsilon}$  and  $\frac{1}{\epsilon}$  and  $\frac{1}{\epsilon}$  and  $\frac{1}{\epsilon}$  $V(t, X_t, S)1_{\{X_t < x_0\}}dt$ <br>
(100) into Equation (<br>  $[X_{\varepsilon}, S] - V(0, x_0, S) \le$ <br>
let us illustrate a prop<br>
the "essential"<br>
or forms the "essential"<br>
or forms the "essential"<br>
or forms de "essential"<br>
or for any admission or for  $\overline{8}$  ,  $\overline{A}6$  y are let  $\overline{R}$  $\frac{A}{8} \varepsilon$ . (100)<br>
, we have that for  $\varepsilon > 0$  small<br>  $\frac{A}{6} \varepsilon + \frac{A}{8} \varepsilon + o(\varepsilon) < -\frac{A\varepsilon}{25}$ ,<br>  $\square$ <br>
y of an arbitrary optimal mild<br>
art of *S*, and by removing the<br> *e optimal mild equilibrium S*,<br> *weak equili*  $\varepsilon > 0$  small<br> $-\frac{A\varepsilon}{25}$ ,<br>ptimal mild<br>moving the<br>*dilibrium S*,  $\mathbb{E}^{\sim}[\circ(\rho_{\widetilde{S}})]$ <br>the proof<br>prepare<br>iibrium  $S$ <br>ssential"<br>**position**<br>s also opt  $\frac{1}{2}$ <br>  $\frac{1}{2}$  is complete the set of the set of the set of the set of part from the set of th )] −  $f(x_0) = \mathbb{E}^{x_0}[V(\varepsilon, X_\varepsilon, S)] - V(0, x_0, S) \le -\frac{A}{6}$ <br>plete.<br>proof of Theorem 5.2, let us illustrate a property<br>h says that  $\mathfrak{S}_S$  actually forms the "essential" pair<br>por S, the remaining part is still optimal 6<br>y of a<br>art of<br>e opt<br>u wea  $\frac{1}{8}$ <br>a  $\frac{1}{8}$ ,<br>m e  $8^{\epsilon}$  +  $o(\epsilon)$  <  $-\frac{1}{25}$ <br>a arbitrary optima<br>S, and by removing<br>and mild equilibrian. 25 '<br>mal<br>bvin<br>brit 」<br>ild<br>be<br>S, To prepare for the proof of Theorem 5.2, let us illustrate a property of an arbitrary optimal mild equilibrium S, which says that  $\mathfrak{S}_S$  actually forms the "essential" part of S, and by removing the "inessential" part from  $S$ , the remaining part is still optimal mild.

S, which says that  $\mathfrak{S}_S$  actually forms the "essential" part of S, and by removing the<br>
' part from S, the remaining part is still optimal mild.<br>
15.6. Let Assumptions 2.1–2.10 hold. For any admissible optimal mild e S, the remaining part is still optimal mild.<br>
Issumptions 2.1–2.10 hold. For any admissi<br>
In addition, if  $\overline{\mathfrak{S}}_S$  is admissible, then  $\overline{\mathfrak{S}}_S$  is **Proposition 5.6.** *Let Assumptions 2.1–2.10 hold. For any admissible optimal mild equilibrium ,*  $\mathfrak{S}_S$  is also optimal mild. In addition, if  $\mathfrak{S}_S$  is admissible, then  $\mathfrak{S}_S$  is a weak equilibrium.<br> $\mathfrak{S}_S$ 

<span id="page-30-0"></span>*Proof.* Step 1. We first characterize  $S \setminus \mathfrak{S}_S$ . As S is admissible, we can write S as a union of disjoint closed intervals

$$
S = \bigcup_{n \in \Lambda_1} [\alpha_{2n-1}, \alpha_{2n}], \text{ where } \alpha_{2n-1} \le \alpha_{2n} < \alpha_{2n+1}^5. \tag{101}
$$

where  $\Lambda_1 \subset \mathbb{Z}$  is either a finite or countable subset. Since S is closed, we have that  $\overline{\mathfrak{S}_S} \subset S$ . For each  $n \in \Lambda_1$ , by the closeness of  $\mathfrak{S}_S$ , we can see that  $[\alpha_{2n-1}, \alpha_{2n}] \setminus \mathfrak{S}_S$  consists of at most countably many disjoint intervals  $(I_{n_k})_k$  of the following four forms:

1. 
$$
[\alpha_{2n-1}, \gamma);
$$
 2.  $(\gamma', \alpha_{2n});$  3.  $(\beta, \beta');$  4.  $[\alpha_{2n-1}, \alpha_{2n}].$  (102)

For each  $I_{n_k}$  of the four forms in Equation (102), we define an open interval  $(l_{n_k}, r_{n_k})$  as follows:

Step *I*. We first characterize *S* \e S<sub>x</sub>. As *S* is admissible, we can write *S* as a union of disjoint intervals\n
$$
S = \bigcup_{n \in A_1} [\alpha_{2n-1}, \alpha_{2n}],
$$
 where  $\alpha_{2n-1} \leq \alpha_{2n} < \alpha_{2n+1}$ .<sup>5</sup> (101)\n
$$
\Lambda_1 \subset \mathbb{Z}
$$
 is either a finite or countable subset. Since *S* is closed, we have that  $\overline{\mathcal{C}_S} \subset S$ .\n
$$
\Lambda_1 \subset \mathbb{Z}
$$
 is either a finite or countable subset. Since *S* is closed, we have that  $\overline{\mathcal{C}_S} \subset S$ .\n
$$
\Lambda_1 \subset \mathbb{Z}
$$
 is either a finite or countable subset. Since *S* is closed, we have that  $\overline{\mathcal{C}_S} \subset S$ .\n
$$
\Lambda_1 \subset \mathbb{Z}_1
$$
, by the closeness of  $\overline{\mathcal{C}_S}$ , we can see that  $[\alpha_{2n-1}, \alpha_{2n}] \setminus \overline{\mathcal{C}_S}$  consists of at most  $\overline{\mathcal{C}_S} \subset S$ .  $I$ , we can see that  $[\alpha_{2n-1}, \alpha_{2n}] \setminus \overline{\mathcal{C}_S}$  consists of at most  $\overline{\mathcal{C}_S} \subset \mathbb{Z}_2$ , we can see that  $[\alpha_{2n-1}, \alpha_{2n}]$ . (102)\n
$$
\Lambda_{n_k}
$$
 of the four forms in Equation (102), we define an open interval  $(l_{n_k}, r_{n_k})$  as follows:\n
$$
\begin{cases}\n1 \cdot l_{n_k} := \sup \{y < \alpha_{2n-1}, y \in \mathbb{Z}_S\}, & r_{n_k} := \overline{r}; \\
2 \cdot l_{n_k} := \sup \{y < \alpha_{2n-1}, y \in \mathbb{Z}_S\}, & r_{n_k} := \overline{r}; \\
3 \cdot l_{n_k} := \sup \{y < \alpha_{2n-1}, y \in \mathbb{Z}_S\}, & r_{n_k} := \overline{r}; \\
4 \cdot l_{n_k} := \sup \{y < \alpha_{2
$$

2. ∶= ′  $r_{n_k} := \beta',$ <br>  $r_{n_k} := \beta',$ <br>
pfy <  $\alpha_{2n-1}, y \in \mathfrak{S}_5$ ,  $r_{n_k} := \inf \{y > \alpha_{2n}, y \in \mathfrak{S}_5\}$ ,<br>
linf  $\emptyset := \sup X$  if it happens. Notice that each two of<br>
rdisjoint or identical, and  $l_{n_k}$  can be  $-\infty$  (resp.  $r_{n_k}$  can<br>
reals 3.  $t_{n_k} = \rho$ ,  $t_{n_k} = \rho$ <br>
4.  $l_{n_k} := \sup\{y < \alpha_{2n-1}, y \in \mathfrak{S}_5\}$ ,  $r_{n_k} := \inf$ <br>
4.  $l_{n_k} := \sup\{y < \alpha_{2n-1}, y \in \mathfrak{S}_5\}$ ,  $r_{n_k} := \inf$ <br>  $\lim_{k \to k}$  are either disjoint or identical, and  $l_{n_k}$  can be<br>
there disiploint or contain mentain and a contain design to the mental of the first and a contain to the second of the second to  $i_{rk}$   $i_{rk}$   $j_{rk}$   $k_{rk}$   $k_{rk}$   $l_{rk}$   $l$ and set sup  $\emptyset := \inf \mathbb{X}$  and  $\inf \emptyset := \sup \mathbb{X}$  if it happens. Notice that each two of those open intersup  $\theta$  : = inf X and inf  $\theta$  : = sup X if it happens. Notice that each two of those open inter-<br>  $x_k, r_{n_k}$ ) $_{n,k}$  are either disjoint or identical, and  $k_n$  can be  $-\infty$  (resp.  $r_{n_k}$  can be  $\infty$ ). Since the<br>
mber o vals  $((l_{n_k}, r_{n_k}))_{n,k}$  are either disjoint or identical, and  $l_{n_k}$  can be  $-\infty$  (resp.  $r_{n_k}$  can be  $\infty$ ). Since the (( $h_k$ ,  $r_m$ ) $h_k$ , are citient disjoint or identical, and  $h_k$  can be −∞ (resp.  $r_m$  can be ∞). Since the<br>number of these intervals  $((h_{n_k}, r_{n_k}))_{n \in \mathbb{N}}$  is at most countable, we omit the repeating ones and<br>dex them as total number of these intervals  $((l_{n_k}, r_{n_k}))_{n,k}$  is at most countable, we omit the repeating ones and  $((l_{n_k}, r_{n_k}))_{n,k}$  is at most countable, we omit the repeating ones and<br>
uch that they are disjoint and  $\Lambda \subset \mathbb{Z}$  is either a finite or countable<br>  $=\overline{\mathfrak{G}}_S$ .<br>  $h k \in \Lambda$ ,<br>  $S \setminus (l_k, r_k) = J(x, S)$ ,  $\forall x \in (l_k, r_k)$ .<br>
(104)<br> re-index them as  $((l_k, r_k))_{k \in \Lambda}$  such that they are disjoint and  $\Lambda \subset \mathbb{Z}$  is either a finite or countable subset. Then  $S \setminus (\cup_{k \in \Lambda} (l_k, r_k)) = \mathfrak{S}_S$ .

*Step 2*. We prove that for each  $k \in \Lambda$ ,

$$
J(x, S \setminus (l_k, r_k)) = J(x, S), \quad \forall x \in (l_k, r_k). \tag{104}
$$

Fix  $k \in \Lambda$ . Step 1 tells that for any  $x \in (l_k, r_k)$ , x either belongs to  $S \setminus \overline{\mathfrak{S}}_S$  or belongs to  $S^c$ . (1) If  $x \in S^c$  or  $x \in \partial S$  for boundary case (b), Lemma 2.14 tells that

$$
\mathcal{L}V(t, x +, S) \equiv \mathcal{L}V(t, x -, S) \equiv 0 \quad \forall t \in [0, \infty).
$$
 (105)

(( $l_k$ ,  $r_k$ ))<sub>ε∈Λ</sub> such that they are disjoint and Λ ⊂ ℤ is either a finite or countable<br>
ve that for ach k ∈ Λ,<br>  $J(x, S \setminus (l_k, r_k)) = J(x, S), \quad \forall x \in (l_k, r_k).$  (104)<br>
tells that for any  $x \in (l_k, r_k)$ ,  $x$  either belongs to  $S \setminus$ S \ (U<sub>k∈Λ</sub>(l<sub>k</sub>, r<sub>k</sub>)) =  $\mathfrak{S}_S$ .<br>
prove that for each  $k \in$ <br>  $J(x, S \setminus (l$ <br>
ep 1 tells that for any  $x \in$ <br>  $c$  or  $x \in \partial S$  for boundar<br>  $\mathcal{L}V(t, x+, S)$ <br>  $\in S^\circ \setminus \overline{\mathfrak{S}}_S$ . By the fact th<br>
also weak. Then, Equati<br>  $k \in \Lambda$ ,<br> $\delta \setminus (l_k,$ <br> $y \times \in ($ <br>(mdary of  $+$ ,  $S$ ) = act tha<br>aquation  $V(0, x$ .<br> $\blacksquare$  Equation  $-$ ,  $S$ )  $\land$ <br> $\blacksquare$  and  $\blacksquare$ ) and ones res  $J(x, S \setminus (l_k, r_k)) = J(x, S), \quad \forall x \in (l_k, r_k).$  (104)<br>
or any  $x \in (l_k, r_k)$ , x either belongs to  $S \setminus \overline{\mathfrak{S}_S}$  or belongs to  $S^c$ .<br>
the boundary case (b), Lemma 2.14 tells that<br>  $V(t, x+, S) \equiv LV(t, x-, S) \equiv 0 \quad \forall t \in [0, \infty).$  (105)<br>
the *k* ∈ Λ. *Step 1* tells that for any  $x \in (l_k, r_k)$ , *x* either belongs to  $S \setminus \mathfrak{S}_S$  or belongs to  $S^c$ .<br>
b) If  $x \in S^c$  or  $x \in \partial S$  for boundary case (b), Lemma 2.14 tells that<br>  $\mathcal{L}V(t, x+, S) \equiv \mathcal{L}V(t, x-, S) \equiv 0 \quad \forall t$  $x \in S^c$  or  $x \in \partial S$  for boundary case (b), Lemma [2.14](#page-9-0) tells that<br>  $\mathcal{L}V(t, x+, S) \equiv \mathcal{L}V(t, x-, S) \equiv 0 \quad \forall t \in [0, \infty)$ <br>
oose  $x \in S^{\circ} \setminus \overline{\mathfrak{S}_S}$ . By the fact that *S* is an admissible optimal mi<br>
t *S* is also weak. T  $V(t, x+, S) \equiv \mathcal{L}V(t, x-, S) \equiv 0 \quad \forall t \in [0, \infty).$  (105)<br>
y the fact that *S* is an admissible optimal mild equilibrium, Theorem 4.3<br>
en, Equation (25) together with the definition of  $\mathfrak{S}_S$  leads to<br>  $S(x) = \mathcal{L}V(0, x+, S) = 0;$ (2) Suppose  $x \in S^{\circ} \setminus \mathfrak{S}_S$ . By the fact that S is an admissible optimal mild equilibrium, Theorem 4.3 tells that S is also weak. Then, Equation (25) together with the definition of  $\mathfrak{S}_S$  leads to

$$
\mathcal{L}V(0, x-, S) = \mathcal{L}V(0, x+, S) = 0; \quad V(t, x, S) = \delta(t)f(x) \quad \forall t \ge 0.
$$

Then by a similar argument as in Equation (86) (with  $\frac{a}{2}$  replaced by 0), we reach that

$$
\mathcal{L}V(t, x-, S) \wedge \mathcal{L}V(t, x+, S) \ge 0 \quad \forall t \in [0, \infty). \tag{106}
$$

 $x \in S^\circ \setminus \mathfrak{S}_S$ . By the fact that *S* is an admissible optimal mild equilibrium, Theorem [4.3](#page-18-0)<br>
s also weak. Then, Equation (25) together with the definition of  $\mathfrak{S}_S$  leads to<br>  $\mathcal{L}V(0, x-, S) = \mathcal{L}V(0, x+, S) = 0$ ; S is also weak. Then, Equation (25) together with the definition of  $\mathfrak{S}_S$  leads to  $\mathcal{L}V(0, x-, S) = \mathcal{L}V(0, x+, S) = 0$ ;  $V(t, x, S) = \delta(t)f(x)$   $\forall t \ge 0$ .<br>
a similar argument as in Equation (86) (with  $\frac{a}{2}$  replaced by 0  $V(0, x-, S) = \mathcal{L}V(0, x+, S) = 0;$   $V(t, x, S) = \delta(t)f(x)$   $\forall t \ge 0.$ <br>
r argument as in Equation (86) (with  $\frac{a}{2}$  replaced by 0), we reach  $\mathcal{L}V(t, x-, S) \wedge \mathcal{L}V(t, x+, S) \ge 0$   $\forall t \in [0, \infty).$ <br>  $\in \partial S \setminus \overline{\mathfrak{S}_S}$  of boundary ca  $\frac{1}{2}$   $\geq$  6  $\frac{2}{2}$  replaced by 0), we reach that<br>  $\ge 0 \quad \forall t \in [0, \infty).$ <br>
for this case, we can also deduced and the case, we can also deduced and the case, we can also deduced and  $[\alpha_{2n-1},$  $V(t, x-, S) \land LV(t, x+, S) \ge 0 \quad \forall t \in [0, \infty).$  (106)<br>
For of boundary case (a), and for this case, we can also deduce Equation<br>
rases (1) and (2).<br>
as the ones restricted in X in a natural way, for example, one  $[\alpha_{2n-1}, \alpha_{2n}]$  co (3) Otherwise,  $x \in \partial S \setminus \overline{\mathfrak{S}}_S$  of boundary case (a), and for this case, we can also deduce Equation  $x \in \partial S \setminus \mathfrak{S}_S$  of boundary case (a), and for this case, we can also deduce Equation<br>pination of cases (1) and (2).<br>re understood as the ones restricted in  $X$  in a natural way, for example, one  $[\alpha_{2n-1}, \alpha_{2n}]$  could (106) by a combination of cases (1) and (2).

<sup>&</sup>lt;sup>5</sup>These intervals are understood as the ones restricted in X in a natural way, for example, one  $[\alpha_{2n-1}, \alpha_{2n}]$  could be  $\chi$  in a natural way, for example, one [α<sub>2n−1</sub>, α<sub>2n</sub>] could be  $[\alpha_{2n-1}, \infty)$  if sup  $\mathbb{X} = \infty$ .

In sum, we have

$$
\frac{1}{2}(\mathcal{L}V(t,x-,S) + \mathcal{L}V(t,x+,S)) \ge 0 \quad \forall (t,x) \in [0,\infty) \times (l_k,r_k). \tag{107}
$$

Recall  $(\theta_i)_{i \in I}$  defined in Assumption 2.8(ii). By Proposition 4.7 and the definition of  $\mathfrak{S}_S$ ,  $V_x(0, \theta_i+, S) = V_x(0, \theta_i-, S)$  for each  $\theta_i \in ((l_k, r_k) \cap S)$ . Then for any  $n \in \mathbb{N}$  and  $\theta_i \in (l_k, r_k) \cap S$ and Lemma [3.10,](#page-15-0) we have that

$$
V_x(t, \theta_i +, S) - V_x(t, \theta_i -, S) \ge \delta(t) (V_x(0, \theta_i +, S) - V_x(0, \theta_i -, S)) = 0, \quad \forall t \ge 0.
$$
 (108)

Note that for each  $n \in \mathbb{N}$  the interval  $B(x_0, n) \cap (l_k, r_k)$  contains at most finite points  $\theta_i$ . Now take

Recall 
$$
(θ_i)_{i \in I}
$$
 defined in Assumption 2.8(ii). By Proposition 4.7 and the definition of  $\mathfrak{S}_{\mathcal{S}}$ ,  $V_x(0, \theta, +, S) = V_x(0, \theta, -, S)$  for each  $\theta_i \in \mathfrak{C}(l_k, r_k)$ . (107)  $\mathfrak{F}_{\mathcal{S}}(0, \theta, -, S) = V_x(0, \theta, -, S)$  for each  $\theta_i \in (\mathbb{I}_{k}, r_k) \cap S$ . Then for any  $n \in \mathbb{N}$  and  $\theta_i \in (\mathbb{I}_{k}, r_k) \cap K_{\mathcal{S}}$ ,  $W$ , no matter  $\theta_i$  (e.g.,  $n$ ), no matter  $\theta_i$  (f.e.,  $n$ ), no matter  $\theta_i$  (g.e.,  $n$ ), no matter  $\theta_i$  (h.e.,  $n$ ), no after  $\theta_i$  is the  $V_x(t, \theta_i +, S) - V_x(t, \theta_i -, S) \geq \delta(t) (V_x(0, \theta_i +, S) - V_x(0, \theta_i -, S)) = 0, \forall t \geq 0.$  (108) Note that for each  $n \in \mathbb{N}$  the interval  $B(x_0, n) \cap (l_k, r_k)$  contains at most finite points  $\theta_i$ . Now take  $x_0 \in (l_k, r_k)$  and denote  $r_n := \tau_{(l_k, r_k) \cap B(x_0, n) \cap R}$  for  $\pi \in \mathbb{N}$ . By Lemma 2.15,  $V(\tau_n, X_{\tau_n}, S) - V(0, x_0, S) = \int_0^{\tau_n} \frac{1}{2} (LV(t, X_{\tau}, S) + LV(t, X_{\tau} +, S)) dt + \int_0^{\tau_n} V_x(t, X_{\tau}, S) \cap (X_0, S) \geq 1.$  Similarly, if  $\theta_i \in \mathbb{N}[V(\tau_n, X_{\tau_n}, S)] = V(0,$ 

Taking expectation for the above and combining with Equations (107) and (108), we have that

$$
\mathbb{E}^{x_0}[V(\tau_n, X_{\tau_n}, S)] - V(0, x_0, S) \ge 0.
$$

Similar to Equation (74), we can show that  $\lim_{n\to\infty} \mathbb{E}^{x_0}[V(\tau_n, X_{\tau_n}, S)] = \mathbb{E}^{x_0}[V(\tau_{(l_k, r_k)}, X_{\tau_{(l_{k-1})}}, S)].$ This together with the above inequality implies that

$$
J(x_0, S \setminus (l_k, r_k)) - J(x_0, S) = \mathbb{E}^{x_0}[V(\tau_{(l_k, r_k)}, X_{\tau_{(l_k, r_k)}}, S)] - V(0, x_0, S) \ge 0.
$$

+ *∫*<br>ing (ilar<br>ilar ; tog<br>he *i* ep *i*<br>tep *i*<br>stru is no ce, `<br>(x, ` im so a han  $\begin{bmatrix} x \\ y \end{bmatrix}$  is the set of  $\begin{bmatrix} x \\ y \end{bmatrix}$  is  $\begin{bmatrix} y \\ y \end{bmatrix}$  is e ture ai ( r si c t v l l g r l k  $V_x(t, X_t, S)\sigma(X_s) \cdot 1_{\{X_t \neq \theta_t, \forall t\}} dW_t$  +<br>
ectation for the above and combini<br>  $\mathbb{E}^{x_0}[V(\tau_n, X_{\tau_n}, S)]$ <br>
(quation (74), we can show that  $\lim_t$ <br>
er with the above inequality implies<br>  $J(x_0, S \setminus (l_k, r_k)) - J(x_0, S) = \mathbb{E}^{x_0$  $\begin{bmatrix} 2 \\ 4 \\ 9 \end{bmatrix}$  as  $\begin{bmatrix} 7 \\ 8 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} 3t \\ 1t \\ 2t \end{bmatrix}$  at  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  by  $\begin{bmatrix} 3t \\ 1 \end{bmatrix}$  by  $\begin{bmatrix} 3t \\ 1 \end{bmatrix}$ 2 with Eq. =  $V(0, \frac{1}{\infty})$ <br>with Eq. =  $V(0, \frac{1}{\infty})$ <br> $\mathbb{E}^{x_0}[V]$ <br>hat<br> $\tau_{(l_k, r_k)}, \frac{S}{\infty}$ ,  $(l_k, \frac{S}{\infty})$ , Then<br> $(104)$  for  $p$  2, Equation (103<br>est part<br> $\tau_{\epsilon}, r_k)$ ) =  $J(x, \frac{S}{\infty})$ ,  $\forall$ <br>ium. By ti  $S$ ,  $\frac{1}{2}$ ,  $\frac{$ context to the context your first context of the context ( $v_x(t, \sigma_i +, S) - v_x(t, \sigma_i -, S)$ ) $aL_t$ <br>
ons (107) and (108), we have that<br>
≥ 0.<br>  $[X_{\tau_n}, S]$ ] =  $\mathbb{E}^{x_0}[V(\tau_{(l_k, r_k)}, X_{\tau_{(l_k, r_k)}}, S)] - V(0, x_0, S) \ge 0.$ <br>
≥  $J(x, S)$  for all  $x \in (l_k, r_k)$ . Mea<br>
he optimality of  $S, S \setminus (l_k, r_k)$  is a ,  $\frac{1}{\sqrt{2}}$  ,  $\frac{1}{\sqrt{$  $E^{x_0}[V(\tau_n, X_{\tau_n}, S)] - V(0, x_0, S) \ge 0.$ <br>
n show that  $\lim_{n\to\infty} E^{x_0}[V(\tau_n, X_{\tau_n}, S)]$ <br>
nequality implies that<br>  $J(x_0, S) = E^{x_0}[V(\tau_{(l_k,r_k)}, X_{\tau_{(l_k,r_k)}}, S)]$ <br>  $\zeta, r_k$ , we have  $J(x, S \setminus (l_k, r_k)) \ge J(\zeta)$  for  $x \in X \setminus (l_k, r_k)$ . Then b lim<sub>n→∞</sub>  $L$  [ $V$ ( $\mathfrak{c}_{h_i}, X_{\tau_{h_i}}, S$ )] =  $L$  [ $V$ ( $\mathfrak{c}_{(l_k, r_k)}, X_{\tau_{(l_k, r_k)}}$ )<br>plies that<br> $\mathbb{E}^{x_0}[V(\tau_{(l_k, r_k)}, X_{\tau_{(l_k, r_k)}}, S)] - V(0, x_0, S) \ge 0.$ <br>we  $J(x, S \setminus (l_k, r_k)) \ge J(x, S)$  for all  $x \in (l_k, r_k)$ . Not  $\mathfrak{c}_{(l_k, r_k)}$ . , S)].<br>[ean-also<br>n the<br>hem<br>en it rariness of  $x_0 \in (x_0, b) - 0$  ( $x_k, r_k$ ))  $f(x_k, r_k) = 0$  ( $x_k, r_k$ ), we have  $J(x, S \setminus (l_k, r_k)) \ge \sum_{k=1}^{\infty} (l_k, r_k)$ ) =  $J(x, S)$  for  $x \in X \setminus (l_k, r_k)$ . Then by the nild equilibrium, and thus Equation (104) follows.<br>
show that  $\over$ , S)] –  $V(0, x_0, S) \ge 0$ .<br>  $\geq J(x, S)$  for all  $x \in (l_k$ <br>
e optimality of S, S \ (<br>
(104) holds for all  $k \in$ <br>
an see that removing<br>
is, for any  $k \in \Lambda$ ,<br>  $(l_k, r_k)$ .<br>  $\forall x \in (l_k, r_k)$ .<br>  $\forall x \in (l_k, r_k)$ .<br>  $\therefore$ <br>
em 4.3, if  $\overline$ By the arbitrariness of  $x_0 \in (l_k, r_k)$ , we have  $J(x, S \setminus (l_k, r_k)) \ge J(x, S)$  for all  $x \in (l_k, r_k)$ . Mean $x_0 \in (l_k, r_k)$ , we have  $J(x, S \setminus (l_k, r_k)) \geq J(x, S)$  for all  $x \in (l_k, r_k)$ . Mean-<br>  $= J(x, S)$  for  $x \in \mathbb{X} \setminus (l_k, r_k)$ . Then by the optimality of  $S, S \setminus (l_k, r_k)$  is also<br>  $\overline{\mathfrak{S}}_S$  is optimal mild. By *Step* 2, Equation ( while,  $J(x, S \setminus (l_k, r_k)) = J(x, S)$  for  $x \in \mathbb{X} \setminus (l_k, r_k)$ . Then by the optimality of  $S, S \setminus (l_k, r_k)$  is also an optimal mild equilibrium, and thus Equation (104) follows.

 $J(x, S \setminus (l_k, r_k)) = J(x, S)$  for  $x \in \mathbb{X} \setminus (l_k, r_k)$ . Then by the optimality of  $S$ ,  $S \setminus (l_k, r_k)$  is also<br>mial mild. equilibrium, and thus Equation (104) follows.<br>3. We show that  $\overline{\mathfrak{S}}_S$  is optimal mild. By *Step 2 Step 3*. We show that  $\mathfrak{S}_S$  is optimal mild. By *Step 2*, Equation (104) holds for all  $k \in \Lambda$ . From the  $\mathfrak{S}_S$  is optimal mild. By *Step 2*, Equation (104) holds for all  $k \in \Lambda$ . From the rvals  $(l_k, r_k)_{k \in \Lambda}$  in Equation (103), we can see that removing one of them tues of function *J* on the rest parts, that is, for a construction of the intervals  $(l_k, r_k)_{k \in \Lambda}$  in Equation (103), we can see that removing one of them does not change the values of function J on the rest parts, that is, for any  $k \in \Lambda$ ,

$$
J(x, \mathfrak{S}_S) = J(x, S \setminus (l_k, r_k)) \quad \forall x \in (l_k, r_k).
$$

Hence, we can conclude that for any  $k \in \Lambda$ ,

$$
J(x, \overline{\mathfrak{S}_{S}}) = J(x, S \setminus (l_{k}, r_{k})) = J(x, S), \quad \forall x \in (l_{k}, r_{k})
$$

As  $J(x, \overline{\mathfrak{S}_{S}}) = f(x) = J(x, S)$  for all  $x \in \overline{\mathfrak{S}_{S}}$ ,

$$
J(x, \mathfrak{S}_S) = J(x, S), \quad \forall x \in \mathbb{X}.
$$

 $(l_k, r_k)_{k \in \Lambda}$  in Equation (103), we can see that removing one of them<br>f function *J* on the rest parts, that is, for any  $k \in \Lambda$ ,<br> $\overline{x}, \overline{\mathscr{G}_S} = J(x, S \setminus (l_k, r_k)) \quad \forall x \in (l_k, r_k)$ .<br>for any  $k \in \Lambda$ ,<br> $\overline{x}$ ,  $\overline{S_S} = J(x, S \$ *J* on the rest parts, that is, for any  $k \in \Lambda$ ,<br>  $J(x, S \setminus (l_k, r_k)) \quad \forall x \in (l_k, r_k)$ .<br>
∈Λ,<br>  $\setminus (l_k, r_k)) = J(x, S), \quad \forall x \in (l_k, r_k)$ .<br>  $\equiv \overline{\mathfrak{G}_S}$ ,<br>  $\overline{\mathfrak{s}_S}$  =  $J(x, S)$ ,  $\forall x \in \mathbb{X}$ .<br>
d equilibrium. By Theorem 4.3, if  $\$ at for any  $k \in \Lambda$ ,<br>  $\overline{\mathfrak{s}}_S$ ) =  $J(x, S \setminus (l_k, r_k)) = J(x, S)$ ,  $\forall x \in (l_k, r_k)$ <br>
S) for all  $x \in \overline{\mathfrak{S}}_S$ ,<br>  $J(x, \overline{\mathfrak{S}}_S) = J(x, S)$ ,  $\forall x \in \mathbb{X}$ .<br>
optimal mild equilibrium. By Theorem 4.3<br>
nd Proposition 5.6, we are r  $k \in \Lambda$ ,<br>  $S \setminus (l_k)$ <br>  $\vdots \in \overline{\mathfrak{S}}_S$ <br>  $\overline{\mathfrak{S}}_S$  = ild equ<br>
ild equ<br>
ition 5  $J(x, S)$  for all  $x \in \overline{\mathfrak{S}}_S$ ,<br>  $J(x, \overline{\mathfrak{S}}_S) = J(x, S)$ ,  $\forall x \in \mathbb{X}$ .<br>
is an optimal mild equilibrium. By Theorem 4.3, if  $\overline{\mathfrak{S}}$ <br>
ium.<br>
1 5.1 and Proposition 5.6, we are ready to prove Theorem  $J(x, \mathfrak{S}_S) = f(x) = J(x, S)$  for all  $x \in \mathfrak{S}_S$ ,<br> $J(x, \overline{\mathfrak{S}_S}) =$ <br>is implies that  $\overline{\mathfrak{S}_S}$  is an optimal mild equ<br>lso a weak equilibrium.<br>Thanks to Theorem 5.1 and Proposition 5. (a)  $\approx$  15  $\frac{1}{2}$  (e)  $\approx$  5  $\frac{1}{2}$  (e)  $\approx$  6  $\frac{1}{2}$  (e)  $\approx$  6  $\frac{1}{2}$  (e)  $\approx$  6  $\frac{1}{2}$  (e)  $\approx$  6  $\frac{1}{2}$  (e)  $\approx$  6 This implies that  $\overline{\mathfrak{S}}_S$  is an optimal mild equilibrium. By Theorem 4.3, if  $\overline{\mathfrak{S}}_S$  is admissible then it  $\mathfrak{S}_S$  is an optimal mild equilibrium. By Theorem 4.3, if  $\mathfrak{S}_S$  is admissible then it<br>illibrium.<br>Drem 5.1 and Proposition 5.6, we are ready to prove Theorem 5.2. is also a weak equilibrium. □

Thanks to Theorem 5.1 and Proposition 5.6, we are ready to prove Theorem 5.2.

<span id="page-32-0"></span>*Proof of Theorem* 5.2. **Part (a)**: Suppose S is an optimal mild equilibrium and  $\mathfrak{S}_S$  is closed and admissible. Proposition [5.6](#page-29-0) tells that  $\mathfrak{S}_{S} = \mathfrak{S}_{S}$  is both an optimal mild and weak equilibrium. Then by Theorem [5.1,](#page-26-0) to prove that  $\mathfrak{S}_S$  is strong, it is sufficient to verify that  $\mathfrak{S}_S = \mathfrak{S}_{\mathfrak{S}_S}$ . Notice that  $\mathfrak{S}_{\mathfrak{S}_{\mathcal{S}}} \subset \mathfrak{S}_{\mathcal{S}}$ . Take  $x_0 \in \mathfrak{S}_{\mathcal{S}}$  and we show  $x_0 \in \mathfrak{S}_{\mathfrak{S}_{\mathcal{S}}}$ . If  $V_x(0, x_0, \mathfrak{S}_{\mathcal{S}}) > V_x(0, x_0, \mathfrak{S}_{\mathcal{S}})$ , then  $x_0 \in$ 

$$
\mathcal{L}V(0, x_0^-, \mathfrak{S}_S) \wedge \mathcal{L}V(0, x_0^+, \mathfrak{S}_S) < 0. \tag{109}
$$

Since both S and  $\mathfrak{S}_S$  are optimal mild, we have

$$
V(0, x, \mathfrak{S}_S) \equiv J(x, \mathfrak{S}_S) \equiv J(x, \overline{\mathfrak{S}_S}) \equiv J(x, S) \equiv V(0, x, S) \quad \forall x \in \mathbb{X}.
$$

Then,

$$
V_x(0, x_0-, S) - V_x(0, x_0+, S) = V_x(0, x_0-, \mathfrak{S}_S) - V_x(0, x_0+, \mathfrak{S}_S) = 0.
$$
 (110)

Since  $x_0 \in \mathfrak{S}_S$ , by the definition of  $\mathfrak{S}_S$ , Equation (110) leads to that

$$
\mathcal{L}V(0, x_0-, S) \wedge \mathcal{L}V(0, x_0+, S) < 0. \tag{111}
$$

This together with Equation (22) implies that  $x_0$  cannot be an isolated point of S. We consider the following two cases.

(1) Suppose  $x_0 \in S^\circ$ . Note that  $V(t, x, S) = \delta(t) f(x)$  on S. Then by Equation (111), without loss of generality, we assume  $\mathcal{L}V(0, x_0+, S) = \mathcal{L}(\delta f)(0, x_0+) < 0$ . By the right continuity of  $x \mapsto$  $\mathcal{L}(\delta f)(0, x_+)$  at  $x_0$ , we can find  $h > 0$  small enough such that  $[x_0, x_0 + h) \subset (S^{\circ} \cap \mathcal{G})$  (recall  $\mathcal{G}$ defined in Equation 16) and

$$
\mathcal{L}(\delta f)(0, x_+) < 0, \quad \forall x \in [x_0, x_0 + h). \tag{112}
$$

Hence,  $[x_0, x_0 + h) \subset \mathfrak{S}_S$ , and thus  $\mathcal{L}V(0, x_0 +, \mathfrak{S}_S) = \mathcal{L}(\delta f)(0, x_+) < 0$ .

(2) Otherwise,  $x_0 \in \partial(S^{\circ})$  for boundary case (a). Without loss of generality, we assume  $\mathcal{L}V(0, x_0 +, S) = \mathcal{L}(\delta f)(0, x_0 +) < 0$ . A similar discussion as in case (1) implies  $\mathcal{L}V(0, x_0 +, \mathcal{S}_S)$  =  $\mathcal{L}(\delta f)(0, x_0 +) < 0.$ 

In sum, Equation (109) holds, and the proof of part (a) is complete.

**Part (b)**: Lemma 4.1 indicates that  $S^*$  is an optimal mild equilibrium. Then by Proposition 5.6,

### **6 EXAMPLES**

S is an optimal mild equilibrium and  $\Phi_S$  is closed and<br>  $\equiv \overline{\Phi_S}$  is both an optimal mild equilibrium.<br>
strong, it is sufficient to verify that  $\overline{\Phi_S} = \overline{\Phi_{\Theta_S}}$ . Notice,<br>  $\sin \Phi_S = \overline{\Phi_{\Theta_S}}$ . Notice,  $\overline{\Phi_S} \geq V_{\$  $\mathfrak{S}_5 = \mathfrak{S}_6$  is both an optimal mild and weak equilibrium.<br>  $\mathfrak{S}_8$  is  $\mathfrak{S}_8$  is both an optimal mild and weak equilibrium.<br>
So is strong, it is sufficient to verify that  $\mathfrak{S}_5 = \mathfrak{S}_{\mathfrak{S}_7}$ . Notice<br>  $\mathcal{E}_5 = \text{is strong, it is sufficient to verify that } \mathcal{E}_5 = \mathcal{E}_{\mathcal{E}_6}$ . Notice<br>  $\mathcal{E}_6 = \text{is strong, if } V_x(0, x_0, \mathcal{E}_5) > V_x(0, x_0, \mathcal{E}_5)$ , then  $x_0 \in \mathcal{E}_6$ , if  $V_y(0, x_0, \mathcal{E}_5)$ , and it remains to verify that<br>  $x_0 - \mathcal{E}_5$ )  $\wedge$   $\mathcal{L}V(0, x_0$  $\mathcal{B}_{\infty} \subset \mathcal{B}_{\infty}$ . Take  $x_0 \in \mathcal{B}_{\infty}$  and we show  $x_0 \in \mathcal{B}_{\infty}$ . If  $V_1(0, x_0, \mathcal{B}_0)$ ,  $V_1(0, x_0, \mathcal{B}_0)$ , then  $x_0 \in \mathcal{B}_{\infty}$ . (I(19))<br>
be both S and  $\mathcal{B}_{\infty}$  are optimal mild, we have<br>  $L'(0,$  $\mathfrak{S}_{\mathfrak{S}_0}$ . Otherwise,  $V_x(0, x_0, \mathfrak{S}_0) = V_x(0, x_0, \mathfrak{S}_0)$ , and it remains to verify that<br>  $\mathcal{L}V(0, x_0 - \mathfrak{S}_0) \wedge \mathcal{L}V(0, x_0 + \mathfrak{S}_0) < 0$ .<br>
Since both *S* and  $\mathfrak{S}_S$  are optimal mild, we have<br>  $V(0$  $V(0, x_0 - \mathscr{E}_S) \wedge \mathcal{L}V(0, x_0 + \mathscr{E}_S) < 0.$  (109)<br>
al mild, we have<br>
al mild, we have<br>  $(x, \mathscr{E}_S) \equiv J(x, \overline{\mathscr{E}_S}) \equiv J(x, S) \equiv V(0, x, S)$   $\forall x \in \mathbb{X}.$ <br>  $(x, (0, x_0 +, S) = V_x(0, x_0 -, \mathscr{E}_S) - V_x(0, x_0 +, \mathscr{E}_S) = 0.$  (110)<br>
on o S and  $\mathfrak{S}_S$  are optimal mild, we have<br>  $V(0, x, \mathfrak{S}_S) \equiv J(x, \mathfrak{S}_S) \equiv J(x, \mathfrak{S}_S)$ <br>  $V_x(0, x_0-, S) - V_x(0, x_0+, S) = V$ <br>  $\mathfrak{S}_S$ , by the definition of  $\mathfrak{S}_S$ , Equation<br>  $\mathcal{L}V(0, x_0-, S) \wedge$ <br>
er with Equation (22) i  $V(0, x, \mathcal{L}_S) \equiv J(x, \mathcal{L}_S) \equiv J(x, \mathcal{L}_S) \equiv J(x, S) \equiv V(x, S) \equiv V(0, x, S)$  Vx ∈ X.<br>  $\chi$ (0,  $x_0$ -,  $S$ ) –  $V_x(0, x_0 +, S) = V_x(0, x_0 -, \mathcal{L}_S) - V_x(0, x_0 +, \mathcal{L}_S) = 0$ <br>
by the definition of  $\mathcal{L}_S$ , Equation (110) leads to that<br>  $\$  $V_x(0, x_0 - S) - V_x(0, x_0 + S) = V_x(0, x_0 - \mathfrak{S}_S) - V_x(0, x_0 + S_S) = 0.$  (110)<br>
(110)<br>
(b) the definition of  $\mathfrak{S}_9$ , Equation (110) leads to that<br>  $\mathcal{L}V(0, x_0 - S) \land \mathcal{L}V(0, x_0 + S) < 0.$  (111)<br>
with Equation (22) implies that  $x_0 \in \mathfrak{S}_S$ , by the definition of  $\mathfrak{S}_S$ , Equation (110) leads to that<br>  $\mathcal{L}V(0, x_0-, S) \land \mathcal{L}V(0, x_0+, S) < 0$ .<br>
opgether with Equation (22) implies that  $x_0$  cannot be an isolat<br>
imp two cases.<br>
Suppose  $x_0 \$  $V(0, x_0-, S) \wedge LV(0, x_0+, S) < 0.$  (111)<br>
implies that  $x_0$  cannot be an isolated point of *S*. We consider the<br>
at  $V(t, x, S) = \delta(t) f(x)$  on *S*. Then by Equation (III), without<br>  $h > 0$  small enough such that  $[x_0, x_0 + h) \subset (S^\circ \cap$  $x_0$  cannot be an isolated point of *S*. We consider the<br>  $= \delta(t) f(x)$  on *S*. Then by Equation (111), without<br>  $\epsilon \mathcal{L}(\delta f)(0, x_0+) < 0$ . By the right continuity of  $x \mapsto$ <br>
enough such that  $[x_0, x_0 + h) \subset (S^\circ \cap G)$  (recall  $G$ *x*<sub>0</sub> ∈ *S*<sup>0</sup>. Note that  $V(t, x, S) = \delta(t) f(x)$  on *S*. Then by Equation (111), without,  $y$ , we assume  $\mathcal{L}V(0, x_0+, S) = \mathcal{L}(\delta f)(0, x_0+) < 0$ . By the right continuity of  $x \mapsto x_0$ , we assume  $\mathcal{L}V(0, x_0+, S)$  small eno  $V(0, x_0 +, S) = \mathcal{L}(\delta f)(0, x_0 +) < 0$ . By the right continuity of  $x \mapsto h > 0$  small enough such that  $[x_0, x_0 + h) \subset (S^\circ \cap G)$  (recall  $\mathcal{G}(\delta f)(0, x_+) < 0$ ,  $\forall x \in [x_0, x_0 + h)$ . (112)<br>hus  $\mathcal{L}V(0, x_0 +, \mathfrak{S}_S) = \mathcal{L}(\delta f)(0, x_+)$ ( $\delta f$ )(0,  $x_*$ ) at  $x_0$ , we can find  $h > 0$  small enough such that  $[x_0, x_0 + h) \subset (S^> \cap G)$  (recall  $G$ <br>fined in Equation 16) and<br> $\mathcal{L}(\delta f)(0, x_+) < 0$ ,  $\forall x \in [x_0, x_0 + h)$ . (112)<br>ence,  $[x_0, x_0 + h) \subset \mathfrak{G}_S$ , and thus  $\mathcal$ ( $\delta f$ )(0,  $x_+$ ) < 0,  $\forall x \in [x_0, x_0 + h)$ . (112)<br>thus  $\mathcal{L}V(0, x_0 + , \mathfrak{S}_S) = \mathcal{L}(\delta f)(0, x_+) < 0$ .<br>for boundary case (a). Without loss of generality, we assume<br>0 small enough. Then by Equations (111) and (22), we again [ $x_0, x_0 + h$ )  $\subset \mathfrak{S}_5$ , and thus  $\mathcal{L}V(0, x_0 + , \mathfrak{S}_5) = \mathcal{L}(\delta f)(0, x_+) < 0$ .<br>
Otherwise,  $x_0 \in \partial(S^\circ)$  for boundary case (a). Without loss of  $h$ )  $\subset (S^\circ \cap G)$  for  $h > 0$  small enough. Then by Equations (111)  $(0$  $x_0 \in \partial(S^\circ)$  for boundary case (a). Without loss of generality, we assume  $\cap$  *G*) for *h* > 0 small enough. Then by Equations (111) and (22), we again have ( $\delta f$ )(0,  $x_0 +$ ) < 0. A similar discussion as in case (1) i ( $x_0$ ,  $x_0 + h$ ) ⊂ (S° ∩ G) for  $h > 0$  small enough. Then by Equations (111) and (22), we again have  $\mathcal{LV}(0, x_0 +, S) = \mathcal{L}(\delta f)(0, x_0 +) < 0$ . A similar discussion as in case (1) implies  $\mathcal{LV}(0, x_0 +, \mathfrak{S}_S) = \mathcal{L}(\delta f)(0$  $V(0, x_0 +, S) = \mathcal{L}(\delta f)(0, x_0 +) < 0$ . A similar discussion as in case (1) implies  $\mathcal{L}V(0, x_0 +, \mathfrak{S}_S) = (\delta f)(0, x_0 +) < 0$ .<br>
In sum, Equation (109) holds, and the proof of part (a) is complete.<br> **In sum, Equation (109) hol** ( $\delta f$ )(0,  $x_0$ +) < 0.<br>
In sum, Equatio<br> **Part (b)**: Lemm<br>  $\sum_{S^*}$  C  $S^*$  is an opt<br>
\*. The rest statem<br>
\*. The rest statem<br>
1 **EXAMP**<br>
1 this section, we prove strong equilibries<br>
ay not be optima<br>
rong. The third e<br> *S*<sup>\*</sup> is an optimal mild equilibrium. Then by Proposition [5.6,](#page-29-0) um. As *S*<sup>\*</sup> is the smallest optimal mild equilibrium,  $\mathfrak{S}_{S^*}$  = s from part (a).<br>  $\square$ <br>
ples to demonstrate our results. In the first example, we hav  $\mathfrak{S}_{s^*} \subset S^*$  is an optimal mild equilibrium. As  $S^*$  is the smallest optimal mild equilibrium,  $\mathfrak{S}_{s^*} = S^*$ . The rest statement directly follows from part (a).<br> **6**  $\perp$  **EXAMPLES**<br>
In this section, we provid  $S^*$ . The rest statement directly follows from part (a).<br>  $\Box$ <br>
6  $\Box$  **EXAMPLES**<br>
In this section, we provide three examples to demonstrate our results. In the first example, we have<br>
two strong equilibria, one of which In this section, we provide three examples to demonstrate our results. In the first example, we have two strong equilibria, one of which is not optimal mild. This indicates that an strong equilibrium may not be optimal mild. In the second example, we show that a weak equilibrium may not be strong. The third example is the stopping for an American put option on a geometric Brownian motion, in which we provide all three types of equilibria.

# <span id="page-33-0"></span>**6.1 An example showing optimal mild**  $\subsetneq$  strong

In this subsection, we construct an example where the set of optimal mild equilibria is strictly contained (i.e.,  $\subsetneq$ ) in the set of strong equilibria. Let  $dX_t = dW_t$  and thus X is a Brownian motion with  $X = \mathbb{R}$ . Take discount function  $\delta(t) = \frac{1}{1+\delta t}$ . Let

$$
\frac{\int_0^\infty e^{-s} \frac{\sqrt{2\beta s}}{\sinh((b-a)\sqrt{2\beta s})} ds}{\sqrt{\frac{\pi \beta}{2}} + \int_0^\infty e^{-s} \sqrt{2\beta s} \coth((b-a)\sqrt{2\beta s}) ds} < \frac{c}{d} < \int_0^\infty e^{-(s + (b-a)\sqrt{2\beta s})} ds.
$$
(113)

Notice that such parameters do exist, for example, let  $b - a = 1$ , then for Equation (113), we have LHS  $\approx 0.3952 < \frac{c}{4} < 0.4544 \approx$  RHS.

Define

$$
J_b(x) := d\mathbb{E}^x[\delta(\rho_{\{b\}})] = d\int_0^\infty \frac{p(t)}{1+\beta t}dt = d\int_0^\infty \int_0^\infty e^{-(1+\beta t)s}p(t)dsdt
$$
  
=  $d\int_0^\infty e^{-s}\mathbb{E}^x[e^{-\beta s\rho_{\{b\}}}]ds = d\int_0^\infty e^{-s}e^{-|x-b|\sqrt{2\beta s}}ds, \quad x \in \mathbb{X}.$  (114)

where the second line uses the formula in Borodin and Salminen [\(2002,](#page-39-0) 2.0.1 on page 204). We further define

1 An example showing optimal mild ⊆ strong  
\nis subsection, we construct an example where the set of optimal mild equilibria is strictly  
\nlined (i.e., ⊆) in the set of strong equilibria. Let dX<sub>i</sub> = dW<sub>i</sub> and thus X is a Brownian motion  
\n× = ℝ. Take discount function δ(t) = 
$$
\frac{1}{1+\beta i}
$$
. Let a < b, 0 < c < d such that  
\n
$$
\frac{\int_0^\infty e^{-s} \frac{\sqrt{2\beta s}}{\sinh((b-a)\sqrt{2\beta s})} ds}{\sqrt{\frac{\pi \beta}{2} + \int_0^\infty e^{-s} \sqrt{2\beta s} \coth((b-a)\sqrt{2\beta s})} ds} < \frac{c}{d} < \int_0^\infty e^{-(s+(b-a)\sqrt{2\beta s})} ds.
$$
\n(113)  
\n∴ the sum of the parameters do exist, for example, let b – a = 1, then for Equation (113), we have  
\n⇒ 0.3952  $\leq \frac{k}{d} < 0.4544 \approx$  RHS.  
\nSince the sum of the terms of the series of  $\frac{1}{2} = \frac{1}{2} = \int_0^\infty \frac{f(t)}{1 + \beta t} dt = d \int_0^\infty \int_0^\infty e^{-(1+\beta)ts} p(t) ds dt$   
\n
$$
= d \int_0^\infty e^{-s} \mathbb{E}^x [e^{-\beta s \rho_{[s]}}] ds = d \int_0^\infty e^{-s} e^{-|x-b|\sqrt{2\beta s}} ds, \quad x \in \mathbb{X}.
$$
\n(114)  
\nWe the second line uses the formula in Borodin and Salminen (2002, 2.0.1 on page 204). We  
\nor define  
\n
$$
J_{ab}(x) := c \mathbb{E}^x [\delta(\rho_{[a,b]}) \cdot 1_{[\rho_{[a,b] \to a]}}] + d \mathbb{E}^x [\delta(\rho_{[a,b]}) \cdot 1_{[\rho_{[a,b] \to b}]}]
$$
\n
$$
= \begin{cases}\n\frac{c}{\rho_0} e^{-s} e^{-|x-a|\sqrt{2\beta s}}}{e^{-s} \frac{\sinh((s-a)\sqrt{2\beta s})}{\sinh((s-a)\sqrt{2\beta s})} ds + d \int_0^\infty e^{-s} \frac{\sinh((s-a)\sqrt{2\beta s})}{\sinh((s-a)\sqrt{2\
$$

ss ol<br>Sa<br>(a  $\frac{1}{2}$  ( $\frac{1}{2}$  or  $\frac{1}{2}$ ) sinh((*b*-*a*)√2 $\beta$ s)<br>
on [*a*, *b*] is ob<br>
combined wi<br>
sumption 2.8<br>
= *d*;  $f(x)$  ·<br>  $c < d \int_0^\infty e^{-t} dt$ y<br>31<br>0<br>√ is the formula in Borodin and Sa<br>
age 218) combined with an argument similar to that in Equatisfying Assumption 2.8 such that<br>  $= c, f(b) = d; \quad f(x) < \min\{J_b(x), J_{ab}(x)\}, \quad \forall x \in \mathbb{X} \setminus \{a, b\}.$ <br>  $c < d \int_0^\infty e^{-(s + (b-a)\sqrt{2\beta s})} ds = J_b(a),$ where the expression for  $J_{ab}$  on [a, b] is obtained by the formula in Borodin and Salminen (2002,  $J_{ab}$  on [a, b] is obtained by the formula in Borodin and Salminen [\(2002,](#page-39-0)<br>218) combined with an argument similar to that in Equation (114). Let  $f$ <br>3 Assumption 2.8 such that<br> $f(b) = d; \quad f(x) < \min\{J_b(x), J_{ab}(x)\}, \quad \forall x \in \mathbb{X} \setminus \$ 3.0.5 (a) and (b) on page 218) combined with an argument similar to that in Equation (114). Let  $f$ be any function satisfying Assumption 2.8 such that

$$
f(a) = c, f(b) = d; \quad f(x) < \min\{J_b(x), J_{ab}(x)\}, \quad \forall x \in \mathbb{X} \setminus \{a, b\}. \tag{116}
$$

Note that

n satisfying Assumption 2.8 such that  
\n
$$
f(a) = c, f(b) = d; \quad f(x) < \min\{J_b(x), J_{ab}(x)\}, \quad \forall x \in \mathbb{X} \setminus \{a, b\}. \tag{116}
$$
\n
$$
c < d \int_0^\infty e^{-(s + (b - a)\sqrt{2\beta s})} ds = J_b(a), \tag{117}
$$

<span id="page-34-0"></span>which shows such function  $f(x)$  indeed exists.<sup>6</sup>

One can easily verify that Assumptions [2.1](#page-5-0)[–2.8](#page-7-0) hold. Moreover, Assumption [2.10](#page-8-0) is also satisfied due to Lemma [2.12](#page-8-0) and Remark [2.13.](#page-8-0) We have the following result.

**Proposition 6.1.**  $\{b\}$  is the unique optimal mild equilibrium, while both  $\{b\}$  and  $\{a, b\}$  are *strong equilibria.*

*Proof.* Recall  $S^*$  defined in Equation (58). First notice that

$$
J_b(x) = J(x, {b})
$$
 and  $J_{ab}(x) = J(x, {a, b})$ ,  $\forall x \in \mathbb{X}$ .

f(x) indeed exists.<sup>6</sup><br>Assumptions 2.1-2.8<br>ark 2.13. We have the<br>unique optimal<br>Equation (58). First<br>J(x, {b}) and  $J_{at}$ <br>d(117), it is easy to<br>um of f, any mild othat is,  $S^* = \{b\}$ . It<br>ins (116) and (117) ag<br>unique opt {*is*} *is the unique optimal mild equilibrium, while both {<i>b*} and {*a*, *b*} are<br>ind in Equation (58). First notice that<br> $I_0(x) = I(x, \{b\})$  and  $I_{ab}(x) = I(x, \{a, b\})$ ,  $\forall x \in \mathbb{X}$ .<br>(116) and (117), it is easy to see that S\* defined in Equation [\(58\)](#page-18-0). First notice that<br>  $J_b(x) = J(x, \{b\})$  and  $J_{ab}(x) = J(x, \{b\})$  and  $J_{ab}(x) = J(x, \{b\})$  and  $(Jx)$ , it is easy to see that t global maximum of f, any mild equilibrium<br>
equilibrium, that is,  $S^* = \{b\}$ .  $J_0(x) = J(x, {b})$  and  $J_{ab}(x) = J(x, {a, b})$ ,  $\forall x \in \mathbb{X}$ .<br>
(116) and (117), it is easy to see that both  ${a, b}$  and  ${b}$ )<br>
Imaximum of  $f$ , any mild equilibrium must contain b.<br>
Imaximum of  $f$ , any mild equilibrium must cont Then by Equations [\(116\)](#page-33-0) and [\(117\)](#page-33-0), it is easy to see that both  $\{a, b\}$  and  $\{b\}$  are mild equilibria. {*a*, *b*} and {*b*} are mild equilibria.<br>
sit contain *b*. Therefore, {*b*} is the<br>
om Lemma 4.1 that {*b*} is optimal<br>
hat  $f(x) < J(x, \{b\})$  for any  $x \neq b$ ,<br>
s for the optimal mild equilibrium<br>
> 0,<br>
<br>
> 0,<br>  $J'(b+, \{b\}) > 0$ Since *b* is the global maximum of f, any mild equilibrium must contain *b*. Therefore,  $\{b\}$  is the smallest mild equilibrium, that is,  $S^* = \{b\}$ . It then follows from Lemma 4.1 that  $\{b\}$  is optimal mild. Moreover, by Equations [\(116\)](#page-33-0) and [\(117\)](#page-33-0) again, we have that  $f(x) < J(x, {b})$  for any  $x \neq b$ , which implies that  ${b}$  is the unique optimal mild equilibrium.

Now we verify that both  ${b}$  and  ${a, b}$  are strong equilibria. As for the optimal mild equilibrium

$$
J'(b-, \{b\}) = d \int_0^\infty e^{-s} \sqrt{2\beta s} ds > 0,
$$

and by symmetry, we have  $J'(b+, \{b\}) < 0$ . Then,

$$
V_x(0, b-, \{b\}) - V_x(0, b+, \{b\}) = J'(b-, \{b\}) - J'(b+, \{b\}) > 0.
$$

is is the global maximum of f, any mild equilibrium must contain. D. Therefore,  $\{b\}$  is the global maximum of f, any mild equilibrium,<br>
is mild equilibrium, that is,  $S^* = \{b\}$ . It then follows from Lemma 4.1 that  $\{$  $S^* = {b}$ ; It then follows from Lemma [4.1](#page-18-0) that  ${b}$  is optimal<br>
and (117) again, we have that  $f(x) < J(x, {b})$  for any  $x \neq b$ ,<br>
and (117) again, we have that  $f(x) < J(x, {b})$  for any  $x \neq b$ ,<br>  $a, b$ } are strong equilibria. As f  $f(x) < J(x, {b})$  for any  $x \neq b$ ,<br>
r the optimal mild equilibrium<br>
(b+, {b}) > 0.<br>
Therefore, we have  $\mathfrak{S}_{\{b\}} = \{b\}$ <br>
5.2(b) tells that {b} is a strong<br>
lculations from Equation (115)<br>  $\frac{\pi \beta}{2}c$ ,<br>  $s \frac{\cosh((x - a)\sqrt{2\beta s})$ {*b*} is the unique optimal mild equilibrium.<br>
at both {*b*} and {*a*, *b*} are strong equilibria. A<br>
tion from Equation (114) shows that<br>  $J'(b-, \{b\}) = d \int_0^\infty e^{-s} \sqrt{2\beta s} ds$ .<br>
we have  $J'(b+, \{b\}) < 0$ . Then,<br>  $x(0, b-, \{b\}) - V_x$ {b} and {a, b} are strong equilibria. As for the optimal mild equilibrium<br>
If Equation (114) shows that<br>  $J'(b-, \{b\}) = d \int_0^\infty e^{-s} \sqrt{2\beta s} ds > 0$ ,<br>  $J'(b+, \{b\}) < 0$ . Then,<br>  $J'(b+, \{b\}) < 0$ . Then,<br>  $\{a\}, \{b\} \} = J'(b-, \{b\}) = J'(b-, \{b$ {*b*}, a direct calculation from Equation [\(114\)](#page-33-0) shows that<br>  $J'(b-, \{b\}) = d \int_0^\infty e^{-s} \sqrt{2}$ <br>
and by symmetry, we have  $J'(b+, \{b\}) < 0$ . Then,<br>  $V_x(0, b-, \{b\}) - V_x(0, b+, \{b\}) = J'(b)$ <br>
Meanwhile, by Lemma 2.14(a), we have  $LV(t, x\pm, \{b$  $\left(\begin{array}{ccc}b&b\\b&c\end{array}\right)$ , is  $\left(\begin{array}{ccc}c&b\\b&c\end{array}\right)$  is  $\left(\begin{array}{ccc}a&b&b\\b&c\end{array}\right)$ -, {b}) - J'(<br>
= 0 on X.<br>
Theorem<br>
Direct cal<br>  $\frac{1}{38}$ <br>  $\frac{1}{$ }<br>{{<br>4({b} t<br>({b} t<br>((competed)  $(b+, {b}) < 0$ . Then,<br>  $(b, {b}) = V_x(0, b+, {b})$ <br>
(a), we have  $LV(t, x)$ <br>
(b) a), we have  $LV(t, x)$ <br>
(a) is closed and adm<br>
he mild equilibirum<br>  $a-, {a, b}) = c \int_0^\infty a$ <br>  $\frac{\sinh((b - a)\sqrt{2\beta s})\sqrt{2\beta s}}{\sinh((b - a)\sqrt{2\beta s})}$ <br>  $X$  and Equation (117), o  $v_x(0, b-, \{0\}) = v_x(0, b+, \{0\}) = 3$ <br>
emma 2.14(a), we have  $LV(t, x \pm, \{0\})$ <br>
emma 2.14(a), we have  $LV(t, x \pm, \{0\})$ <br>  $\therefore$  (onsider the mild equilibirum  $\{a\}$ <br>  $J'(a-, \{a, b\}) = c \int_0^\infty e^{-s}$ ,<br>  $a, b$ ),<br>  $\therefore$   $\int_0^\infty e^{-s} \frac{\cosh((b-x)\sqrt{2$ ( $b-, {b}$ ) = 0 on X.<br>
b} = 0 on X.<br>
ble, Theorem<br>
b). Direct ca<br>  $\sqrt{2\beta s}ds = \sqrt{\frac{2\beta s}}{s}}$ <br>  $ds + d \int_0^\infty e^{-\frac{1}{2s}}$ <br>  $x \le a$ ,<br>  $x > b$ . Therefore, w<br>5.2(b) tells tl<br>liculations from<br> $\frac{\pi \beta}{2}$ c,<br> $\frac{\pi \beta}{2}$ c,<br> $\frac{\cosh((x - a) \sinh((b - a))}{\sinh((b - a))})$ Meanwhile, by Lemma [2.14\(](#page-9-0)a), we have  $\mathcal{L}V(t, x \pm \{b\}) \equiv 0$  on X. Therefore, we have  $\mathfrak{S}_{\{b\}} = \{b\}$ V(t, x+, {b}) = 0 on X. Therefore, we have  $\mathfrak{S}_{\{b\}} = \{b\}$ <br>
d admissible, Theorem 5.2(b) tells that {b} is a strong<br>
ibirum {a, b}. Direct calculations from Equation (II5)<br>  $\int_0^\infty e^{-s} \sqrt{2\beta s} ds = \sqrt{\frac{\pi \beta}{2}} c$ ,<br>  $\int_0$ from Equation [\(88\)](#page-26-0). Since  ${b}$  is closed and admissible, Theorem 5.2(b) tells that  ${b}$  is a strong equilibrium. Now consider the mild equilibirum  $\{a, b\}$ . Direct calculations from Equation (115) show that

$$
J'(a-, \{a, b\}) = c \int_0^\infty e^{-s} \sqrt{2\beta s} ds = \sqrt{\frac{\pi \beta}{2}} c,
$$

and for any  $x \in (a, b)$ ,

from Equation (88). Since *{b}* is closed and admissible, Theorem 5.2(b) tells that *{b}* is a strong  
equilibrium. Now consider the mild equilibrium *{a, b}*. Direct calculations from Equation (115)  
show that  

$$
J'(a-, {a, b}) = c \int_0^\infty e^{-s} \sqrt{2\beta s} ds = \sqrt{\frac{\pi \beta}{2}} c,
$$
and for any  $x \in (a, b)$ ,  

$$
J'(x, {a, b}) = -c \int_0^\infty e^{-s} \frac{\cosh((b - x)\sqrt{2\beta s})\sqrt{2\beta s}}{\sinh((b - a)\sqrt{2\beta s})} ds + d \int_0^\infty e^{-s} \frac{\cosh((x - a)\sqrt{2\beta s})\sqrt{2\beta s}}{\sinh((b - a)\sqrt{2\beta s})} ds.
$$
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$$
J'(b-, \{b\}) = d \int_0^{\infty} e^{-s} \sqrt{2\beta s} ds > 0,
$$
  
\n
$$
(b+, \{b\}) < 0.
$$
 Then,  
\n
$$
(b+, \{b\}) = J'(b-, \{b\}) - J'(b+, \{b\}) > 0.
$$
  
\na), we have  $\mathcal{L}V(t, x\pm, \{b\}) \equiv 0$  on X. Therefore, we have  $\mathfrak{S}_{\{b\}} = \{b\}$   
\n $\}$  is closed and admissible, Theorem 5.2(b) tells that  $\{b\}$  is a strong  
\nthe mild equilibrium  $\{a, b\}$ . Direct calculations from Equation (115)  
\n
$$
a-, \{a, b\} = c \int_0^{\infty} e^{-s} \sqrt{2\beta s} ds = \sqrt{\frac{\pi \beta}{2}} c,
$$
  
\n
$$
\frac{\operatorname{sh}((b-x)\sqrt{2\beta s})\sqrt{2\beta s}}{\sinh((b-a)\sqrt{2\beta s})} ds + d \int_0^{\infty} e^{-s} \frac{\cosh((x-a)\sqrt{2\beta s})\sqrt{2\beta s}}{\sinh((b-a)\sqrt{2\beta s})} ds.
$$
  
\n
$$
X \text{ and Equation (117), one can easily check that } J_{ab}(x) < J_b(x) \text{ for } x < b. \text{ Hence, a}
$$
  
\n
$$
J(x) := \begin{cases} \frac{1}{1+(a-x)} J_{ab}(x), & x \leq a, \\ \frac{1}{1+(a-x)} J_{ab}(x), & x > b. \end{cases}
$$
  
\n
$$
J(x) = \begin{cases} \frac{1}{1+(a-x)} J_{ab}(x), & x \leq a, \\ \frac{1}{1+(a-x)} J_{b}(x), & x > b. \end{cases}
$$

 $\frac{1}{\pi}$ <sup>6</sup> By the strong Markov property of X and Equation (117), one can easily check that  $J_{ab}(x) < J_b(x)$  for  $x < b$ . Hence, a quick example for such  $f$  would be  $f$  would be

<span id="page-35-0"></span>By taking  $x = a +$  in Equation (118) and combining with the first inequality in Equation (113), we have that

taking *x* = *a* + in Equation (118) and combining with the first inequality in Equation (113), we  
\nthe *x*  
\n*f*'(*a* +{*a*, *b*}) = -*c* 
$$
\int_0^\infty e^{-s} \coth((b - a)\sqrt{2\beta s})\sqrt{2\beta s}ds + d \int_0^\infty e^{-s} \frac{\sqrt{2\beta s}}{\sinh((b - a)\sqrt{2\beta s})} ds
$$
  
\n $<\sqrt{\frac{\pi \beta}{2}} c = J'(a-, \{a, b\}).$   
\ntaking *x* = *b* – in Equation (118) and the fact that 0 *c c d*, we have that  
\n*f*'(*b*–{*a*, *b*}) = -*c*  $\int_0^\infty e^{-s} \frac{\sqrt{2\beta s}}{\sinh((b - a)\sqrt{2\beta s})} ds + d \int_0^\infty e^{-s} \coth((b - a)\sqrt{2\beta s})\sqrt{2\beta s}ds$   
\n $>$  > 0 > *f*'(*b* + {*b*}) = *J*'(*b* + {*a*, *b*}).  
\nence,  
\n*V<sub>x</sub>*(0, *x* – {*a*, *b*}) > *V<sub>x</sub>*(0, *x* + {*a*, *b*}).  
\n $Vx(0, x – {a, b}) > Vx(0, x + {a, b}).\n $Vx(0, x – {a, b}) > Vx(0, x + {a, b}).\n $Vx(0, x – {a, b}) = 0$  on *X* {*a*, *b*. Therefore, by Theorem 3.1,  
\nthe example **showing strong** ⊆ weak  
\nthis subsectorion, we give an example in which a$$ 

By taking  $x = b$  in Equation (118) and the fact that  $0 < c < d$ , we have that

′ (+, {, }) = − ∫ 0 2 + ∫ 0 − sinh(( − )√2) < 2 =′ (−, {, }). = − in Equation [\(118\)](#page-34-0) and the fact that 0<<, we have that ′ (−, {, }) = − ∫ ∞ √2 ∞ − coth(( − )√ 2)√ 2

Hence,

$$
V_x(0, x - \{a, b\}) > V_x(0, x + \{a, b\}), \text{ for both } x = a, b. \tag{119}
$$

 $V_x($ ma 2.14<br>  $V_x($ ma 2.14<br>  $\mu$ uilibriu<br>
1 that {*a*<br> **ample**<br>
1, we give ak equi<br>
Let  $\delta(t$ <br>  $(0, \infty)$  and a s<br>  $\mu$ <br>  $\begin{cases} L \\ \mu \\ \nu \end{cases}$ <br>
eorem 3  $V_x(0, x-, {a, b}) > V_x(0, x+, {a, b})$ , for both  $x = a, b$ . (119)<br>
2.4(a) tells that  $LV(t, x, {a, b}) \equiv 0$  on  $X \setminus {a, b}$ . Therefore, by Theorem 3.1,<br>
brium. Moreover, by Equations (119) and (88),  $\mathfrak{G}_{[a,b]} = {a, b}$ . It then follows<br>
{ Meanwhile, Lemma [2.14\(](#page-9-0)a) tells that  $\mathcal{LV}(t, x, \{a, b\}) \equiv 0$  on  $\mathbb{X} \setminus \{a, b\}$ . Therefore, by Theorem 3.1,  $V(t, x, {a, b}) \equiv 0$  on  $X \setminus {a, b}$ . Therefore, by Theorem [3.1,](#page-10-0)<br>by Equations (119) and (88),  $\mathfrak{S}_{[a,b]} = {a, b}$ . It then follows<br>equilibrium.<br>**Strong**  $\subsetneq$  **weak**<br>in which a weak equilibrium is not strong, and thus {stron from Theorem 5.1 that  $\{a, b\}$  is a strong equilibrium.

# **6.2 An example showing strong**  $\subsetneq$  weak

In this subsection, we give an example in which a weak equilibrium is not strong, and thus {strong} equilibria}  $\subsetneq$  {weak equilibria}. Let *X* be a geometric Brownian motion:

$$
dX_t = \mu X_t dt + \sigma X_t dW_t \tag{120}
$$

{*a*, *b*} is a weak equilibrium. Moreover, by Equations (119) and [\(88\)](#page-26-0),  $\mathcal{E}_{[a,b]} = \{a,b\}$ . It then follows<br>from Theorem 5.1 that  $\{a,b\}$  is a strong equilibrium.<br>  $\Box$ <br>
6.2 **1 An example showing strong**  $\subsetneq$  weak<br>
I {*a*, *b*} is a strong equilibrium. <br> **le showing strong**  $\subsetneq$  **weak**<br>
ive an example in which a weak equilibrium is not strong, and thus {strong<br>
uilibria}. Let *X* be a geometric Brownian motion:<br>  $dX_t = \mu X_t dt + \sigma X_t dW_t$  {strong<br>(120)<br>aat  $\mu =$ <br>optimal<br>a direct<br>or  $\varepsilon > 0$  $dX_t = \mu X_t dt + \sigma X_t dW_t$  (120)<br>
d  $f(x) = x \wedge K$  for some constant  $K > 0$ . Assume that  $\mu =$ <br>
quilibrium but not strong, while  $[K, \infty)$  is the unique optimal<br>
vium.<br>
r (0,  $\infty$ ). Notice that  $V(t, x, (0, \infty)) \equiv \delta(t) f(x)$ , then direct<br> with  $X = (0, \infty)$ . Let  $\delta(t) = \frac{1}{1 + \delta t}$  and  $X = (0, ∞).$  Let  $\delta(t) = \frac{1}{1+t}$ .<br> **osition 6.2.**  $(0, ∞)$  is a wegallibrium and a strong equilibrium and a strong equili  $f(x) = x \land K$  for some constant  $K > 0$ . Assume that  $\mu =$ <br>
illibrium but not strong, while [ $K$ , ∞) is the unique optimal<br>
um.<br>
(0, ∞). Notice that  $V(t, x, (0, \infty)) \equiv \delta(t) f(x)$ , then direct<br>
(0, ∞). Notice that  $V(t, x, (0, \infty)) \equiv \delta$ 

**Proposition 6.2.**  $(0, \infty)$  is a weak equilibrium but not strong, while  $[K, \infty)$  is the unique optimal *mild equilibrium and a strong equilibrium.*

*Proof.* We first verify the result for  $(0, \infty)$ . Notice that  $V(t, x, (0, \infty)) \equiv \delta(t) f(x)$ , then direct calculations show

1(a, b) 
$$
y = -c \int_{0}^{c} e^{-\frac{1}{\sinh((b - a)\sqrt{2\beta s})}} \cos \theta + u \int_{0}^{c} e^{-\cot(h(u - a)\sqrt{2\beta s})\sqrt{2\beta s}} \cos \theta
$$
  
\n $\Rightarrow 0 > J'(b+, \{b\}) = J'(b+, \{a, b\}).$  (119)  
\n10. Let, Lemma 2.14(a) tells that  $\mathcal{L}V(t, x, \{a, b\}) = 0$  on  $\mathbb{N} \setminus \{a, b\}$ . Therefore, by Theorem 3.1, weak equilibrium. Moreover, by Equations (119) and (88),  $\mathfrak{S}_{[a,b]} = \{a, b\}$ . It then follows  
\n11. Let  $\{a, b\}$  is a strong equilibrium.  
\n22. Let  $\mathfrak{S}(a, b)$  is a strong equilibrium.  
\n23. Let  $\mathfrak{S}(a, b)$  is a strong equilibrium.  
\n24. Let  $\mathcal{L}V = \mathfrak{L} \setminus \mathfrak{L} \cup \mathfrak{L} \setminus \mathfrak{L} \times \mathfrak{L} \cup \mathfrak{L} \cup \mathfrak{L} \times \mathfrak{L} \times \mathfrak{L} \cup \mathfrak{L} \cup \mathfrak{L} \times \mathfrak{L} \times \mathfrak{L} \cup \mathfrak{L} \cup \mathfrak{L} \times \mathfrak{L} \times \mathfrak{L} \times \mathfrak{L} \times \mathfrak{L} \times \math$ 

 $\chi(0, K-, (0, \infty)) = 1 > 0 = V_X(0, K+, (0, \infty))$ <br>  $(1, (0, \infty))$  is a weak equilibrium. For  $x \in \delta(\varepsilon)X_{\varepsilon}1_{\{X_{\varepsilon} \le K\}} = \frac{e^{\mu\varepsilon}}{1 + \mu\varepsilon}N(d_{\varepsilon}) \cdot x$ 3.1, (0, ∞) is a weak equilibrium. For  $x \in (0, \frac{1}{2}[\delta(\varepsilon)X_{\varepsilon}1_{\{X_{\varepsilon} \leq K\}}] = \frac{e^{\mu \varepsilon}}{1 + \mu \varepsilon} N(d_{\varepsilon}) \cdot x$ Therefore, by Theorem [3.1,](#page-10-0)  $(0, \infty)$  is a weak equilibrium. For  $x \in (0, K)$ , we have that for  $\varepsilon > 0$ (0, ∞) is a weak equilibrium. For  $x \in (0, K)$ , we have that for  $\varepsilon > 0$ <br> $\sum_{\varepsilon} X_{\varepsilon} 1_{\{X_{\varepsilon} \le K\}}$ ] =  $\frac{e^{\mu \varepsilon}}{1 + \mu \varepsilon} N(d_{\varepsilon}) \cdot x$ small enough,

$$
\beta > 0.
$$
\n**Proposition 6.2.**  $(0, \infty)$  is a weak equilibrium but not strong  
\nmid equilibrium and a strong equilibrium.  
\n**Proof.** We first verify the result for  $(0, \infty)$ . Notice that V  
\ncalculations show\n
$$
\begin{cases}\n\mathcal{L}V(0, x, (0, \infty)) = (-\beta + \mu)x = 0, \\
\mathcal{L}V(0, x, (0, \infty)) = -\beta K < 0, \\
V_x(0, K-, (0, \infty)) = 1 > 0 = V_x(0, K)\n\end{cases}
$$
\nTherefore, by Theorem 3.1,  $(0, \infty)$  is a weak equilibrium. F  
\nsmall enough,  
\n
$$
\mathbb{E}[\delta(\rho_{(0,\infty)}^{\varepsilon})f(X_{\rho_{(0,\infty)}^{\varepsilon}})] > \mathbb{E}[\delta(\varepsilon)X_{\varepsilon}1_{\{X_{\varepsilon} \leq K\}}] = \frac{e^{\mu \varepsilon}}{1 + \mu \varepsilon}N(d_{\varepsilon}) \cdot x
$$

 $\geq \left(1+\mu\varepsilon+\frac{1}{2}\mu^{2}\varepsilon^{2}+o(\varepsilon^{2})\right)\left(1-\mu\varepsilon+\mu^{2}\varepsilon^{2}+o(\varepsilon^{2})\right)$  $\sqrt{2}$ √ ⋅  $\geq \left(1 + \frac{1}{2}\mu^2 \varepsilon^2 + o(\varepsilon^2)\right) \left(1 + o(\varepsilon^2)\right) \cdot x = \left(1 + \frac{1}{2}\mu^2 \varepsilon^2 + o(\varepsilon^2)\right)$ 

where  $d_{\varepsilon} := \frac{\varepsilon}{\sigma_1}$ Now we verify the result for  $[K, \infty)$ . By Ito's formula,

$$
d\left(\frac{X_t}{1+\beta t}\right)=\frac{X_t}{1+\beta t}\left(-\frac{\beta}{1+\beta t}+\mu\right)dt+\frac{\sigma X_t}{1+\beta t}dW_t=\frac{\mu^2tX_t}{(1+\mu t)^2}dt+\frac{\sigma X_t}{1+\mu t}dW_t.
$$

Then, by the facts that  $\rho_{[K,\infty)} > 0 \mathbb{P}^x$ -a.s. and  $X_t > 0$  for  $x \in (0,K)$ , we have that

$$
J(x,[K,\infty)) - f(x) = \mathbb{E}^x \left[ \frac{X_{\rho_{[K,\infty)}}}{1 + \beta \rho_{[K,\infty)}} \right] - x = \mathbb{E}^x \left[ \int_0^{\rho_{[K,\infty)}} \frac{\mu^2 t X_t}{(1 + \mu t)^2} dt \right] > 0, \quad \forall x \in (0,K), \tag{121}
$$

<span id="page-36-0"></span>1 +  $\frac{1}{2}\mu^2$ <br>  $\frac{1}{2$  $\frac{1}{2}$  2 m  $\infty$   $\frac{1}{2}$   $\frac{1}{2}$  2 n  $\infty$   $\frac{1}{\beta}$  x  $\frac{1}{\alpha}$   $\mu \epsilon + o(\epsilon^2)$ <br>  $+ o(\epsilon^2)$  (1 -<br>
dicates that<br>  $\frac{1}{t} + \mu \frac{dt}{t} + \frac{1}{t} dt +$ <br>
a.s. and  $X_t$ :<br>  $\frac{1}{(x-\lambda)^2}$  =  $x =$ <br>
ilibrium. On S must con<br>
ion. Therefo<br>
ilibrium S su<br>
(b), which im<br>
mild equilib<br>
(b) =  $\mathcal{L}(\delta(t)H)$ <br>
e  $(\varepsilon^2)$ ) ·  $x = \left(1 + \frac{1}{2}\mu^2 \varepsilon\right)$ <br>  $(\varepsilon^2)$ ) ·  $x = \left(1 + \frac{1}{2}\mu^2 \varepsilon\right)$ <br>  $(\varepsilon^2)$  is not a strong equal a,<br>  $\frac{X_t}{\beta t} dW_t = \frac{\mu^2 tX_t}{(1 + \mu t)^2}$ <br>
for  $x \in (0, K)$ , we have  $\left[\int_0^{\rho_{[K,\infty)}} \frac{\mu^2 tX_t}{(1 + \mu t)^2} dt\right]$  $\cdot$  o( $\cdot$ ) brit +  $\frac{1}{1}$  that  $\cdot$  0,  $\infty$ ) i ideation on g<br>  $\cdot$  0,  $\cdot$  0,  $\cdot$  1 ideation on g<br>  $\cdot$  0. In the  $\cdot$  4 equals on g<br>  $\cdot$  0. In the  $\cdot$  4 equals on g<br>  $\frac{1}{2}$ .  $\frac{1}{2}$   $\frac{1}{2}$   $\frac{1}{2}$   $\frac{1}{2}$   $\frac{1}{6}$   $\frac{1}{6}$   $\frac{1}{8}$   $\frac{1}{8}$   $\frac{1}{8}$   $\frac{1}{8}$   $\frac{1}{8}$   $\frac{1}{2}$   $\begin{cases} \n\frac{1}{2}x > 0 \\
\frac{1}{2}x < 0\n\end{cases}$ <br>  $\begin{cases} \n\frac{1}{2}x < 0 \\
\frac{1}{2}x < 0\n\end{cases}$ <br>  $\begin{cases} \n\frac{1}{2}x < 0 \\
\frac{1}{2}x < 0\n\end{cases}$ <br>  $\begin{cases} \n\frac{1}{2}x < 0 \\
\frac{1}{2}x < 0\n\end{cases}$ <br>  $\begin{cases} \n\frac{1}{2}x < 0 \\
\frac{1}{2}x < 0\n\end{cases}$  $e^{-a_{\varepsilon}}$ <br> $x$ ,  $\frac{1}{\varepsilon}$ <br> $\frac{1}{\varepsilon}$ <br> $\frac{1}{\varepsilon}$   $\frac{1}{\vare$  $\mathbf{R}$  ,  $\mathbf{S}$  ,  $\mathbf{R}$  ,  $\mathbf{S}$  ,  $\mathbf{R}$  ,  $\mathbf{R}$  ,  $\mathbf{R}$  ,  $\mathbf{R}$ ∕2) <sup>2</sup>) $\varepsilon$ . t fo<br>  $\frac{1}{3}$ t fo<br>  $\frac{1}{3}$ t fo<br>  $\frac{1}{3}$ t fo<br>  $\frac{1}{3}$  a a r<br>  $\frac{1}{3}$  a a r<br>  $\frac{1}{3}$  a a r<br>  $\frac{1}{3}$ , a r<br>  $\$ 1 2 bis indicates<br>  $K, \infty$ ). By Ito<br>  $\frac{\beta}{1 + \beta t} + \mu$ <br>  $0 \frac{\beta}{P^x - a.s.}$  and<br>  $\frac{X_{\rho_{[K,\infty)}}}{\beta \rho_{[K,\infty)}}$ <br>  $\frac{X_{\rho_{[K,\infty)}}}{\beta \rho_{[K,\infty)}}$ <br>  $\frac{X_{\rho_{[K,\infty)}}}{\beta \rho_{[K,\infty)}}$ <br>  $\frac{X_{\rho_{[K,\infty)}}}{\beta \rho_{[K,\infty)}} = C$ <br>  $\frac{X_{\rho_{[K,\infty)}}}{\beta \rho_{[K,\infty$ (1 +  $o(\varepsilon^2)$ ) ·  $x =$  (<br>
hat (0,  $\infty$ ) is not a s<br>
s formula,<br>  $t + \frac{\sigma X_t}{1 + \beta t} dW_t =$ <br>  $X_t > 0$  for  $x \in (0, 1]$ <br>  $x = \mathbb{E}^x \left[ \int_0^{\rho_{[K,\infty)}} \frac{t}{(1 - t)} dt \right]$ <br>
... On the other hand<br>
contain  $[K, \infty)$ , for refore,  $[K, \infty)$  tro:  $\frac{\mu^2}{(1 + K)}$ ,  $\frac{\mu^2 t}{1 + \mu}$ , si e sr  $\infty$ ) is  $\kappa$ , Se shi e sulhou  $\in$  ( $K, \nu$ )  $\frac{1}{2}$  and  $\frac{1}{2}$  by  $\frac{1}{2}$  and  $\frac{1}{2}$  an 2 equilibrium<br>  $\left[\frac{x_t}{(t^2)^2}dt + \frac{\sigma^2}{1+t^2}\right]$ <br>  $\left[\frac{1}{2}at\right] > 0,$ <br>  $\left[\frac{$  $\frac{1}{t}dW_t$ .<br>  $x \in (0, 0)$ <br>  $\leq K =$ <br>
ilibriu<br>
21) ince equilil<br>
shows<br>
tricula<br>
2). The<br>
orientical state  $z'$  we v  $d\left(\frac{y}{1+1}\right)$ <br>by the  $[K, \infty)$ <br>shows a of  $j$ <br>shows a of  $j$ <br> $(x \in K, \infty)$ <br>othinal  $(x) < 1$ ,  $[K, \infty)$ <br>tells till<br>set the netric on is contained in the set of  $K$ <br>a 6.11, a 6.3.<br> $S$  is  $a = (0, 0, 0)$ ln(*K*/x)–( $\mu$ + <sub>2</sub><br>
ln(*K*/x)–( $\mu$ + 2<br>
rify the rest<br>
rify the rest<br>
rify the rest<br>
rify the rest<br>
facts that  $\rho$ [<br>  $\beta t$  ) =  $\frac{X}{1 + \beta}$ <br>
facts that  $\rho$ [<br>  $\alpha$  ) –  $f(x) = 0$ <br>
that [*K*, ∞), any mild<br>  $\binom{C}{K}$ , ∞ 2 1) It f  $\frac{t}{\beta t}$   $\left[\frac{z}{\beta t}\right]$  is a equ or  $x$  is a equ or  $x$  is a equ or  $x$  is a a compute  $0, x$  if  $k$  and  $x$  is  $3, z$  it  $u$  m  $m$  is  $z = 1$ . ue III ha x Kini, fu ku Kini xia xia xia yi,  $\frac{X_t}{\epsilon}$ . This indicates that<br>
result for  $[K, \infty)$ . By Ito's forthulation is the sult for  $[K, \infty) > 0$   $\mathbb{P}^x$ -a.s. and  $X_t$ <br>  $= \mathbb{E}^x \left[ \frac{X_{\rho_{[K,\infty)}}}{1 + \beta_t} \right] - x =$ <br>  $\infty$ ) is a mild equilibrium. Onld equilibrium (0, ∞) is not a strong equilibrium.<br>
rmula,<br>  $\frac{\sigma X_t}{1 + \beta t} dW_t = \frac{\mu^2 t X_t}{(1 + \mu t)^2} dt + \frac{\sigma X}{1 + \beta t}$ <br>
> 0 for  $x \in (0, K)$ , we have that<br>  $\mathbb{E}^x \left[ \int_0^{\rho_{[K,\infty)}} \frac{\mu^2 t X_t}{(1 + \mu t)^2} dt \right] > 0$ ,  $\forall$ <br>
n the other hand, since  $[K$ [ $K$ ,  $\infty$ ). By Ito's formula,<br>  $-\frac{\beta}{1 + \beta t} + \mu \frac{\sigma X_t}{1 + \beta t}$ <br>  $\cdot 0 \, \mathbb{P}^{x}$ -a.s. and  $X_t > 0$  for<br>  $\frac{X_{\rho_{[K,\infty)}}}{1 + \beta t}$ <br>  $\frac{X_{\rho_{[K,\infty)}}}{1 + \beta \rho_{[K,\infty)}}$ ]  $-x = \mathbb{E}^{x} \left[ \int_{\mathcal{C}}$ <br>
iid equilibrium. On the o<br>
prium b  $\begin{bmatrix} 1 & s & s \\ 0 & s & s \\ 0 & 0 & s \end{bmatrix}$  t and  $\begin{bmatrix} 1 & 0 & s \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$  t and  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$  $\overline{a}$  and  $\frac{X_t}{1 + \beta t}$ <br>
the facts<br>
∞)) – *f*<br>
wus that<br>
f *f*, any<br>
∈ *S<sup>c</sup>* ∩ [*i*<br>
mal. Now<br>
< *J*(*x*, [*I*<br>
mal. Now<br>
∠ *J*(*x*, [*I*<br>
f, ∞) is tl<br>
s that ©<br> **Stoppi**<br>
the Ame<br>
s define<br>
To begi<br>
11, Corol<br>
5.3.<br>
s *a mild* =  $\frac{1}{1 + \frac{1}{1}}$ <br>that  $\rho_{[k]}$ <br> $(x) = \mathbb{E}$ <br> $K, \infty$ ) i mild  $\epsilon$ <br> $K, \infty$ ), a for any<br> $\frac{1}{1 + \frac{1}{1}}$ <br> $\frac{1}{1 + \frac{1}{1}}$  $\frac{X_t}{1 + \beta t}$ <br>  $\frac{1}{1 + \beta$  $-\frac{1}{1+}$ <br>> 0  $\mathbb{P}$ <br>> 0  $\mathbb{P}$ <br> $\frac{X_{\rho_{[K]}}}{+\beta \rho_{[]}}$ <br>anild equinity intradiced in the set of  $[K, \rho_{[K]}]$ <br>o  $\mathbb{P}$ . T<br>American set of  $[K, \rho_{[K]}]$ <br>o  $\mathbb{P}$ <br> $\mathbb{P}$ <br> $\mathbb{P}$  and  $\mathbb{P}$   $\mathbb{P}$ <br> $\mathbb{P}$ <br> $\mathbb{$  $\frac{\beta}{1 + \beta t} + \mu$ <br>  $0 \mathbb{P}^{x}$ -a.s. and<br>  $\frac{X_{\rho_{[K,\infty)}}}{\beta \rho_{[K,\infty)}}$ <br>  $\frac{1}{\beta \rho_{[K,\infty)}}$ <br>
and equilibrium<br>
rium *S* mus<br>
adiction. The equilibrium<br>  $[K, \infty)$ , which imal mild eq<br>  $[K, \infty)$ ) =  $\mathcal{L}$ <br>  $(K, \infty)$ . Then by T<br> at +  $\frac{1}{1 + \beta}$ <br>  $\frac{1}{X_t} > 0$  fo<br>  $\frac{1}{X_t} > 0$  fo<br>  $\frac{1}{X_t} > 0$  fo<br>  $\frac{1}{X_t} > 0$  fo<br>  $\frac{1}{X_t} > 0$ <br>  $\frac{1}{X_t} >$  $\frac{1}{1 + \beta t} aW_t = \frac{1}{(1 + \mu t)}$ <br>  $\geq 0$  for  $x \in (0, K)$ , we  $\mathbb{E}^x \left[ \int_0^{\rho_{[K,\infty)}} \frac{\mu^2 t X_t}{(1 + \mu t)^2} \right]$ <br>
the other hand, since<br>
tain  $[K, \infty)$ , for other<br>
thence,  $[K, \infty)$  is the small<br>
thence,  $[K, \infty) \neq \emptyset$ <br>
belies t  $\frac{u^2tX_t}{(1 + \mu t)^2}at + \frac{u^2tX_t}{1 + \mu t^2}dt$ <br>  $K$ ), we have that<br>  $\frac{u^2tX_t}{1 + \mu t^2}dt > 0$ ,  $\forall$ <br>
d, since  $[K, \infty)$  is there only is the original equal on the smallest mild equation (or an optimal mild mild r, direct calcul 1 +  $\mu t$ <br>at<br>at<br>b,  $\forall x \in (0$ <br>is the set of<br> $(x, S) < K$ :<br>d equilibriu<br>ion (121) include quililiation shows<br>g.<br>g.<br>In particula<br>all three ty<br>all three ty<br>equilibria st:  $\rho_{[K,\infty)} > 0$   $\mathbb{P}^x$ -a.s. and  $X_t > 0$  for  $x \in (0, K)$ , we have that<br>  $E \mathbb{E}^x \left[ \frac{X_{\rho_{[K,\infty)}}}{1 + \beta \rho_{[K,\infty)}} \right] - x = E^x \left[ \int_0^{\rho_{[K,\infty)}} \frac{\mu^2 t X_t}{(1 + \mu t)^2} dt \right] > 0$ ,<br>
b) is a mild equilibrium. On the other hand, since ( $f(x, [\alpha, \infty)) - f(x) = \pm \sin \alpha$ <br>
hich shows that  $[K, \infty)$  is<br>
axima of f, any mild eq<br>
r any  $x \in S^c \cap [K, \infty)$ , a<br>
us optimal. Now for any<br>
at  $f(x) < J(x, [K, \infty))$  or<br>
ence,  $[K, \infty)$  is the uniqu<br>  $LV(0,$ <br>
hich tells that  $\mathfrak{S}_{[K$  $\frac{1}{\beta \rho_{[K]}}$ <br> $\frac{1}{\beta \rho_{[K]}}$  and equality is equaller to the equal of  $[K, \infty)$ . The **merical state of the equality of the example state of the equality of**  $(K - 1)$  **and**  $\frac{1}{2}$  **and**  $\frac{1}{2}$  **and**  $\frac{1}{2}$  **and \frac{1}{2** mild equilib<br>ilibrium *S* n<br>inild equilibri<br>ilibrium *S* n<br>ontradiction.<br>ild equilibri<br> $S \setminus [K, \infty)$ , w<br>optimal mild<br>+,  $[K, \infty)$ ) =<br>,  $\infty$ ). Then b<br>**America**<br>example in 1<br>on given by 1<br>:=  $(K - x)^+$ <br>following le<br>and Proposi  $-x = \pm 1$ <br>um. On the state of the state her h<br>
her h<br>  $\therefore$ , ∞)<br>
∞) is  $S \setminus [$ <br>
aat S<br>
Moree<br>
3K,<br>
b), [k<br>
1<br>
wu (20<br>
with<br>  $\frac{1}{+\beta t}$ ,<br>
zes the g an<br>  $a \geq \frac{1}{2}$ 0  $\mu + \mu t$ <br>
and, sin or other sm<br>
and as in  $\infty$  and and are sm other sm<br>  $\infty$  and are sm  $\infty$  and are sm  $\infty$  is a set of  $x \in [0, 1]$ <br>
by Section small scale is a set of  $\infty$  is a set of  $\infty$  is a set of  $\infty$  is a  $\left[\mu^2 tX_t\right]$ <br>
(1 +  $\mu t$ )<sup>2</sup> dt <br>
and, since [*K*<br>
for otherwis<br>
the smallest<br> *K*,  $\infty$ )  $\neq \emptyset$ , E*K*<br>
s not an optii<br>
ver, direct ca<br>  $\forall x \in [K, \infty)$ <br>
,  $\infty$ ) is also st<br>  $\forall x \in [0, \infty)$ .<br> *N*e shall prov<br>
e results > 0, ∀ ∈ (0, ), which shows that  $[K, \infty)$  is a mild equilibrium. On the other hand, since  $[K, \infty)$  is the set of global [ $K$ , $\infty$ ) is a mild equilibrium. On the other hand, since [ $K$ , $\infty$ ), for explobal<br>  $\iota$ , mild equilibrium. On the other hand, since [ $K$ ,  $\infty$ ), for cherwise,  $J(x, S) < K = f(x)$ ,<br>  $K$ ,  $\infty$ ), a contradiction. Therefor maxima of f, any mild equilibrium S must contain  $[K, \infty)$ , for otherwise,  $J(x, S) < K = f(x)$ for any  $x \in S^c \cap [K, \infty)$ , a contradiction. Therefore,  $[K, \infty)$  is the smallest mild equilibrium and thus optimal. Now for any mild equilibrium S such that  $S \setminus [K, \infty) \neq \emptyset$ , Equation (121) indicates that  $f(x) < J(x, [K, \infty))$  on  $S \setminus [K, \infty)$ , which implies that S is not an optimal mild equilibrium. Hence,  $[K, \infty)$  is the unique optimal mild equilibrium. Moreover, direct calculation shows that

$$
\mathcal{L}V(0, x +, [K, \infty)) = \mathcal{L}(\delta(t)K) = -\beta K, \quad \forall x \in [K, \infty),
$$

which tells that  $\mathfrak{S}_{[K,\infty)} = [K,\infty)$ . Then by Theorem 5.2(b),  $[K,\infty)$  is also strong.

### **6.3 Stopping of an American put option**

f, any mild equilibrium *S* must contain [ $K$ , ∞), for otherwise,  $f(x, S) \le K \le f(x)$ ,  $S' \in K$ ,  $S$ ,  $S' \notin \emptyset$ ,  $\frac{1}{2}$ ,  $\frac{1}{2}$ ,  $\frac{1}{2}$ ,  $\frac{1}{2}$ ,  $\frac{1}{2}$ ,  $\frac$ *x* ∈ S<sup>c</sup> ∩ [*K*, ∞), a contradiction. Therefore, [*K*, ∞o) is the smallest mild equilibrium and<br>
timal. Now for any mild equilibrium S such that *S* \[*K*, ∞) ≠ *β*, Equation (121) indicates<br>  $c$ ) < *J*(*x*, [*K*, ∞)) S such that  $S \setminus [K, \infty) \neq \emptyset$ , Equation (121) indicates<br>h implies that S is not an optimal mild equilibrium.<br>uilibrium. Moreover, direct calculation shows that<br> $\delta(t)K$  =  $-\beta K$ ,  $\forall x \in [K, \infty)$ ,<br>heorem 5.2(b),  $[K, \infty)$  is  $f(x) < J(x, [K, \infty))$  on  $S \setminus [K, \infty)$ , which implies that  $S$  is not an optimal mild equilibrium.<br>  $\mathcal{L}V(0, x +, [K, \infty)) = \mathcal{L}(\delta(t)K) = -\beta K$ ,  $\forall x \in [K, \infty)$ ,<br>
thells that  $\mathfrak{E}_{[K, \infty)} = [K, \infty)$ . Then by Theorem 5.2(b),  $[K, \in$ [ $K$ , ∞) is the unique optimal mild equilibrium. Moreover, direct calculation shows that  $LV(0, x +, [K, \infty)) = L(\delta(t)K) = -\beta K$ ,  $\forall x \in [K, \infty)$ ,<br>ells that  $\mathfrak{S}_{[K,\infty)} = [K, \infty)$ . Then by Theorem 5.2(b), [ $K, \infty$ ) is also strong.  $V(0, x+, [K, \infty)) = L(\delta(t)K) = -\beta K, \quad \forall x \in [K, \infty),$ <br>  $\exists$  = [K, ∞). Then by Theorem 5.2(b), [K, ∞) is also stated by Theorem 5.2(b), [K, ∞) is also stated by Theorem 5.2(b), [K, ∞) is also stated by Theorem 5.2(b), [K, ∞) is al  $\mathfrak{S}_{[K,\infty)} = [K, \infty)$ . Then by Theorem 5.2(b),  $[K, \infty)$  is also strong. □<br> **ping of an American put option**<br>
merican put example in Huang and Zhou (2020, Section 6.3). In particular, *X* is<br>
whaia motion given by Equat Consider the American put example in Huang and Zhou (2020, Section 6.3). In particular,  $X$  is X is<br>yoff<br>s of<br>d in<br>122) a geometric Brownian motion given by Equation (120) with  $\mathbb{X} := (0, \infty)$ . Let  $\mu \ge 0$ . The payoff  $X := (0, ∞).$  Let  $μ ≥ 0$ . The payoff<br>We shall provide all three types of<br>e results of mild equilibria stated in<br>d Zhou (2020).<br> $e a ∈ (0, K].$ <br> $\frac{λ}{+λ} K$ , where<br>0,  $v := \frac{u}{σ^2} - \frac{1}{2}$ . (122) function is defined as  $f(x) := (K - x)^+$ , and  $\delta(t) := \frac{1}{1 + \beta t}$ . We shall provide all three types of  $f(x) := (K - x)^+$ , and  $\delta(t) := \frac{1}{1+t}$ ,<br>h, the following lemma summarize<br>6.13, and Proposition 6.15 in Huang<br>brium, then  $S \cap (0, K] = (0, a]$  for s<br>is mild equilibrium if and only if a<br> $\lambda := \int_0^\infty e^{-s} \left(\sqrt{v^2 + 2\beta s / \sigma^2} + v\right)$  $\frac{1}{1+\beta t}$ . We shall provide all three types of<br>izes the results of mild equilibria stated in<br>ng and Zhou (2020).<br> $r$  some  $a \in (0, K]$ .<br> $\int a \ge \frac{\lambda}{1+\lambda} K$ , where<br> $\psi$ ) > 0,  $\psi := \frac{u}{\sigma^2} - \frac{1}{2}$ . (122) equilibria. To begin with, the following lemma summarizes the results of mild equilibria stated in Lemma 6.11, Corollary 6.13, and Proposition 6.15 in Huang and Zhou (2020).

### **Lemma 6.3.**

- *(i)* If *S* is a mild equilibrium, then  $S \cap (0, K] = (0, a]$  for some  $a \in (0, K]$ .
- (ii)  $S = (0, a] \subset (0, K]$  is mild equilibrium if and only if  $a \geq \frac{\lambda}{1 + \lambda} K$ , where

If S is a mild equilibrium, then S 
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\cap
$$
 (0, K] = (0, a] for some  $a \in (0, K]$ .  
\n
$$
S = (0, a] \subset (0, K] \text{ is mild equilibrium if and only if } a \ge \frac{\lambda}{1 + \lambda} K, \text{ where}
$$
\n
$$
\lambda := \int_0^\infty e^{-s} \left( \sqrt{\nu^2 + 2\beta s / \sigma^2} + \nu \right) > 0, \quad \nu := \frac{u}{\sigma^2} - \frac{1}{2}.
$$
\n(122)

(iii)  $S^* = (0, \frac{\lambda}{1+\lambda}K]$  is the intersection of all mild equilibria and is the unique optimal mild *equilibria.*

Following from Lemma [6.3\(](#page-36-0)ii), we shall call the mild equilibria that belong to the family  $\{(0, a):$ equilibria, all other mild equilibria take the same form:  $(0, a] \cup D$  that satisfies a certain condition, and we shall call this family of mild equilibria "type II" mild equilibria.

**Proposition 6.4.** *Except the "type I" mild equilibria in Lemma [6.3\(](#page-36-0)ii), all other mild equilibria take form:*  $(0, a] \cup D$  *such that* 

$$
-(K-a)\int_0^\infty e^{-s}\left(\frac{\nu}{a} + \frac{\sqrt{\nu^2 + 2\beta s/\sigma^2}}{a} \cdot \frac{(b/a)^{\sqrt{\nu^2 + 2\beta s/\sigma^2}} + (a/b)^{\sqrt{\nu^2 + 2\beta s/\sigma^2}}}{(b/a)^{\sqrt{\nu^2 + 2\beta s/\sigma^2}} - (a/b)^{\sqrt{\nu^2 + 2\beta s/\sigma^2}}}\right)ds \ge -1, \quad (123)
$$

*where D* is a closed subset of  $[K, \infty)$  and  $b := \inf\{x \in D\}$  satisfying  $b > a$ .

 $a \geq \frac{A}{1+}\nequilit$ <br>equilit and we<br>propo<br>form: (<br> $-(K$ <br>where<br> $Proof.$ <br>the form of  $S = (0)$ <br>ately g<br>Equati When  $\frac{1}{J}$ <br>where Salmir *Proof.* Lemma [6.3\(](#page-36-0)i)(ii) together imply that any mild equilibrium is either of type I or takes the form:  $(0, a] \cup D$  with D being a closed subset of  $[K, \infty)$ . Consider a closed set of such form ately gives that S is a mild equilibrium. Notice that  $-(K - a) \ge 0$  and the integrand in the LHS of Equation (123) is positive, so Equation (123) holds.

When  $a < K$ , we have

$$
J(x, S) = (K - a) \int_0^{\infty} \frac{p(t)}{1 + \beta t} dt = (K - a) \int_0^{\infty} e^{-s} \mathbb{E}^x [e^{-\beta s \tau_{(a,b)}} \cdot 1_{\{X_{\tau_{(a,b)}} = a\}}] ds
$$
  
=  $(K - a) \int_0^{\infty} e^{-s} \left(\frac{a}{x}\right)^{\nu} \frac{(b/x)^{\sqrt{\nu^2 + 2\beta s/\sigma^2}} - (x/b)^{\sqrt{\nu^2 + 2\beta s/\sigma^2}}}{(b/a)^{\sqrt{\nu^2 + 2\beta s/\sigma^2}} - (a/b)^{\sqrt{\nu^2 + 2\beta s/\sigma^2}}} ds, \quad \forall x \in (a, b),$ 

where  $p(t) := \mathbb{P}^{x}(\tau_{(a,b)} \in dt, X_{\tau_{(a,b)}} = a)$ , and the second line above follows from Borodin and Salminen [\(2002,](#page-39-0) 3.0.5 (a) on page 633). Direct calculations show that for any  $x \in (a, b)$ 

(0, 
$$
\frac{N}{1+\lambda}K
$$
) is the intersection of all mild equilibria and is the unique optimal mild  
bria.  
By from Lemma 6.3(ii), we shall call the mild equilibria that belong to the family {0, a} :  
all other mild equilibria.7 The following proposition shows that, except "type I" mild  
all call this family of mild equilibria are the same form: (0, a] ∪D that satisfies a certain condition,  
and other mild equilibria take the same form: (0, a] ∪D that satisfies a certain condition,  
1) ∪  $\int_{0}^{\infty} e^{-s} \left( \frac{v}{a} + \frac{\sqrt{v^2 + 2\beta s/\sigma^2}}{a} \cdot \frac{(b/a)^{\sqrt{v^2 + 2\beta s/\sigma^2}} + (a/b)^{\sqrt{v^2 + 2\beta s/\sigma^2}}}{(b/a)^{\sqrt{v^2 + 2\beta s/\sigma^2}} - (a/b)^{\sqrt{v^2 + 2\beta s/\sigma^2}}}\right) ds ≥ -1, (123)$   
a closed subset of [K, ∞) and b := inf{x ∈ D}3 satisfying b > a.  
mma 6.3(i)(ii) together imply that any mild equilibrium is either of type I or takes  
(0, a] ∪D with D being a closed subset of [K, ∞). Consider a closed set of such form  
U D with b is : inf{x ∈ D}}> a. When a ≥ K, the fact that f = 0 on [K, ∞) immediately  
that S is a mild equilibrium. Notice that  $-(K - a) ≥ 0$  and the integrand in the LHS of  
123) is positive, so Equation (123) holds.  
 $< K$ , we have  
 $S$ ) =  $(K - a) \int_{0}^{\infty} \frac{p(t)}{1 + \beta t} dt = (K - a) \int_{0}^{\infty} e^{-s} \frac{E^{x}}{[e^{-\beta s\tau_{(a,b)}} + 1_{\{X_{\tau_{(a,b)}} = a\}}] ds$   
 $= (K - a) \int_{0}^{\infty} e^{-s} \left( \frac{a}{x} \right)^{y} \frac{(b/x)^{\sqrt{y^2 + 2\beta s/\sigma^2}} - (x/b)^{\sqrt{y^2 + 2\beta s/\sigma^2}}}{(b/a)^{\sqrt{y^2 + 2\beta s/\sigma^2}} - (a/b)^{\sqrt{y^2 + 2\beta s/\sigma^2}}}$   
 $= (K - a) \int_{0}^{\infty} e^{-s} \left( \frac{a}{x} \right)^{y} \$ 

(iii) S<sup>•</sup> = (0, 
$$
\frac{1}{1+x}
$$
 K] is the intersection of all *m*id equilibria and is the unique optimal *m*id  
equilibria.  
Pollowing from Lemma 6.3(ii), we shall call the mild equilibria that belong to the family {0, a]  
 $α \ge \frac{2}{1+x}$  R a e "type I" mild equilibria. The following proposition shows that, except "type I" mild  
equilibria, all other mild equilibria is the same form: (0, a] ∪ D that satisfies a certain condition,  
and we shall call this family of mild equilibria in Lemma 6.3(ii), all other *m*ild equilibria take  
form: (0, a] ∪ D such that  
 $-(K-a)\int_0^\infty e^{-s}\left(\frac{v}{a} + \frac{\sqrt{v^2 + 2\beta s/\sigma^2}}{a} \cdot \frac{(b/a)^{\sqrt{v^2 + 2\beta s/\sigma^2}}}{(b/a)^{\sqrt{v^2 + 2\beta s/\sigma^2}}} + (a/b)^{\sqrt{v^2 + 2\beta s/\sigma^2}}\right) ds ≥ -1, (123)$   
where D is a closed subset of  $[K, \infty)$  and b := inf{x ∈ D} satisfying b > a.  
Proof. Lemma 6.3(i)(ii) together imply that any mild equilibrium is either of type I or takes  
the form: (0, a] ∪ D with D being a closed subset of  $[K, \infty)$ . Consider a closed set of such form  
S = (0, a] ∪ D with b := inf{x ∈ D} > a. When a ≥ K, the fact that f = 0 on [K, ∞) immediate  
equation (123) is positive, so Equation (123) holds.  
When a  $K$ , we have  

$$
J(x, S) = (K - a)\int_0^\infty \frac{p(t)}{1+\beta t}dt = (K - a)\int_0^\infty e^{-s} \frac{E^x}{1-\beta t} [e^{-\beta s\tau_{(a,b)}} \cdot 1_{(X_{\tau_{(a,b)}} = a)}]ds
$$

$$
= (K - a)\int_0^\infty e^{-s}\left(\frac{a}{x}\right)^s \frac{(b/x)^{\sqrt{v^2 + 2\beta s/\sigma^2}}}{(b/a)^{\sqrt{v^2 + 2\beta s/\sigma^2}} - (a/b)^{\sqrt{v^2 + 2\beta s/\sigma^2}}}] ds, \forall x ∈ (a, b),
$$
  
where  $p(t) := P^x(\tau_{(a,b)} = d)$ ,

<sup>7</sup> It contains the trivial mild equilibrium  $X$  by setting  $a = \infty$  and  $X = X \cap (0, \infty)$ .  $\chi$  by setting  $a = \infty$  and  $\chi = \chi \cap (0, \infty]$ .

<span id="page-37-0"></span>

Recall  $\nu$  in Equation (122), we have that

$$
\nu + \sqrt{\nu^2 + 2\beta s/\sigma^2} > 0 \quad \text{and} \quad (2\nu^2 + \nu + 2\beta s/\sigma^2) + \left( (2\nu + 1)\sqrt{\nu^2 + 2\beta s/\sigma^2} \right) > 0.
$$

This together with

$$
0<(b/x)^{\sqrt{\nu^2+2\beta s/\sigma^2}}-(x/b)^{\sqrt{\nu^2+2\beta s/\sigma^2}}<(b/x)^{\sqrt{\nu^2+2\beta s/\sigma^2}}+(x/b)^{\sqrt{\nu^2+2\beta s/\sigma^2}}
$$

implies that both the integrands on the RHS of Equations [\(124\)](#page-37-0) and [\(125\)](#page-37-0) are positive. Therefore, if  $J'(a+, S) \ge -1$ . From Equation (124), we have

$$
J'(a+,S) = -(K-a)\int_0^\infty e^{-s}\left(\frac{\nu}{a} + \frac{\sqrt{\nu^2 + 2\beta s/\sigma^2}}{a} \cdot \frac{(b/a)\sqrt{\nu^2 + 2\beta s/\sigma^2} + (a/b)\sqrt{\nu^2 + 2\beta s/\sigma^2}}{(b/a)\sqrt{\nu^2 + 2\beta s/\sigma^2} - (a/b)\sqrt{\nu^2 + 2\beta s/\sigma^2}}\right)ds,
$$

so S is a mild equilibrium if and only if Equation (123) holds. Notice that  $J'(a+, S)$  converges to 0 when  $a \nearrow K$ . Then for any  $b > K$ , by the continuity of function  $a \mapsto J'(a+, S)$ , there exists a constant  $a_h < K$  such that for all  $a \in [a_h, K)$ , Equation (123) indeed holds and S is a mild equilibrium.

**Proposition 6.5.**  $S^* = (0, \frac{\lambda}{1+\lambda}K]$  is the unique weak and the unique strong equilibrium.

*Proof.* We first find all weak equilibria. Since a weak equilibrium is also mild, by Proposition [6.4,](#page-37-0) it is sufficient to select weak equilibria from the two types of mild equilibria. Given a mild equilibrium S that is weak, no matter which type it is, S must not contain K. Otherwise, by Lemma  $6.3(i)$ ,

$$
V_x(0, K-, S) = -1 < 0 = V_x(0, K+, S),
$$

which contradicts Equation [\(24\)](#page-10-0) in Theorem 3.1.

Consider an arbitrary type I mild equilibrium  $(0, a]$  with  $\frac{\lambda}{1 + \lambda} K \le a \lt K$ . By the smooth-fit condition in Corollary 3.2,  $S = (0, a]$  is a weak equilibrium if and only if

$$
V_x(0, a+, (0, a]) = J'(a+, (0, a]) = -1
$$

*v* in Equation [\(122\)](#page-36-0), we have that<br>  $v + \sqrt{v^2 + 2\beta s/\sigma^2} > 0$  and (2<br>
gether with<br>  $0 < (b/x)^{\sqrt{v^2 + 2\beta s/\sigma^2}} - (x/b)$ <br>
s that both the integrands on the<br>  $0 < 0$  and  $J''(x, S) > 0$  for  $x \in (a$ <br>
This together with the shape of *f*  $\nu + \sqrt{}$ <br>gether<br> $0 <$ <br>s that  $0 < 0 <$ <br>s that  $0 < 0 <$ <br>This to  $+$ ,  $S$   $>$   $\geq$ <br> $-$ ,  $S$   $>$   $=$   $\alpha$  mil<br>hen  $\alpha$ <br>tant  $\alpha$ <br>tant  $\alpha$ <br>sition<br>We fir<br>fficien<br>that is<br> $\subset S$ , w<br>contra sider and  $\alpha$ <br>is  $\in S$ , w<br>contra sider a  $\frac{1}{2^{2} + 2\beta s/\sigma^{2}}$  > 0 and  $(2\nu^{2} + \nu + 2\beta s/\sigma^{2})$  + (<br>
vith<br>
c (b/x)  $\sqrt{\nu^{2}+2\beta s/\sigma^{2}} - (x/b)\sqrt{\nu^{2}+2\beta s/\sigma^{2}} < (b/x)\sqrt{\nu^{2}}$ <br>
both the integrands on the RHS of Equations (12<br>
of  $M''(x, S) > 0$  for  $x \in (a, b)$ , and thus  $J$  $(2\nu + 1)\sqrt{\frac{1+2\beta s/\sigma^2}{1+2\beta s/\sigma^2}} + ($ <br>4) and  $(12\frac{\beta s}{\sqrt{1+2\beta s/\sigma^2}})$ <br>to strictly d<br> $\sqrt{\frac{\nu^2+2\beta s/\sigma^2}{1+\lambda^2}}$ <br>ds. Notice<br>iunction *a*<br>123) indee<br>*unique stro*<br>ium is also<br>ild equilit<br>tain *K*. Otl<br> $\zeta +$ , *S*),<br> $\frac{\lambda}{$ x/b)  $\sqrt{v^2+2\beta s}$ <br>
5) are positiv<br>
lecreasing and equilibriu:<br>  $+(a/b)\sqrt{v^2+2\beta s}$ <br>  $-(a/b)\sqrt{v^2+2\beta s}$ <br>  $-(a/b)\sqrt{v^2+2\beta s}$ <br>  $-(a/b)\sqrt{v^2+2\beta s}$ <br>
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at both th<br>
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weak externed to that<br>  $= (0, \frac{\lambda}{1+i})$ <br>  $\frac{1}{1+i}$ <br>  $\frac{1}{1+i}$ <br>  $\frac{1}{1+i}$ <br>  $\$ ds on the or  $x \in (0, 1, 2)$ <br>ds on the or  $x \in (0, 1, 2)$ <br>nape of  $j$ <br>on (124)<br> $s \left( \frac{v}{a} + \frac{1}{a} \right)$ <br>and only  $b > K$ <br>for all  $a$ <br> $K$ ] is the puilibria<br>uilibria<br>uilibria<br>which ty<br> $K$ ] is the puilibria<br>which ty<br> $f = 0$ <br> $s(x($ RHS of E<br> *b*), and<br>
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hus  $J(x, \text{ndicates})$ <br>  $\frac{\sqrt{\sigma^2}}{\sigma^2} \cdot \frac{b}{b}$ <br>  $\frac{b}{b}$ <br>
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ing and<br>  $\frac{\partial y}{\partial x^2 + 2\beta}$ <br>  $\frac{\partial y}{\partial y}$ J(caif scoto a P. Phitri(C) we cover Fissally A Ti S' ( $x, S$ ) < 0 and  $T'(x, S) > 0$  for  $x \in (a, b)$ , and thus ( $x, S$ ) is strictly decreasing and convex on<br>  $f'(a + S) = -(K - a) \int_0^\infty e^{-s} \left( \frac{a}{a} + \frac{\sqrt{y^2 + 2\beta s/a^2}}{a} + \frac{(b/a)\sqrt{y^2 + 3b/a^2}}{(b/a)\sqrt{y^2 + 3b/a^2}} + \frac{(a/b)\sqrt{y^2 + 3b/a^2}}{(b/a)\sqrt{y^2$ (*a, b*). This together with the shape of f on (*a, b*) indicates that *S* is a mild equilibrium if and only<br>  $J'(a+, S) = -(K - a) \int_0^\infty e^{-s} \left( \frac{2}{a} + \frac{\sqrt{v^2 + 2\beta s/\sigma^2}}{a} \cdot \frac{(b/a)^{\sqrt{v^2 + 2\beta s/\sigma^2}}}{(b/a)^{\sqrt{v^2 + 2\beta s/\sigma^2}}} \cdot \frac{(a/b$ ′ (a+, S) ≥ −1. From Equation [\(124\)](#page-37-0), we have<br>  $a+$ , S) = −(K − a)  $\int_0^\infty e^{-s} \left( \frac{v}{a} + \frac{\sqrt{v^2 + 2\beta}}{a} \right)$ <br>
is a mild equilibrium if and only if Equation<br>
when  $a \nearrow K$ . Then for any  $b > K$ , by the constant  $a_b < K$  such t )<br>online<br>conductions of the conductions of the conductions of the conductions of the conduction ( $a+$ ,  $b$ ) = −( $x - a$ )  $f$ <br>
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onstant  $a_b < K$  such<br>
dilibrium.<br> **oposition 6.5.**  $S^* = (of$ . We first find all w<br>
sufficient to select we m S that is weak, no n<br>  $K$ ] ⊂ S, which t n or or both data at a state of the sta 0  $e^{-s}$  (if and any l<br>if and any l<br>at for<br>any l<br>at for<br> $\frac{\lambda}{\pm \lambda} K$ ]<br>i.equi equil<br>equil equil<br>equil equil<br> $V_x$  ((24) i.e. I r<br> $V_x$  ((24) i.e.  $\overline{a}$  and  $\overline{a}$  is interested by the contract of  $\overline{a}$  is interested by  $\overline{a}$  . If  $\overline{a}$  is  $\overline{b}$  ,  $\overline{b}$  is  $\begin{bmatrix} 1 & 1 & 1 \ 1 & 1 & 1 \end{bmatrix}$  is a contract of  $\begin{bmatrix} 1 & 1 \ 1 & 1 \end{bmatrix}$  is  $\begin{bmatrix} 1 & 1 \ 1 & 1 \end{bmatrix}$  is  $\begin{bmatrix} 1 & 1 \ 1 & 1 \end{bmatrix}$  is  $\begin{bmatrix} 1 & 1 \ 1 & 1 \end{bmatrix}$  $\frac{y + 2p\sqrt{3}}{a}$ <br> *a*<br> *f* Equation (by the continual formulation to the continuation of the continuation of  $[ a_b, K )$ , Equivalent in the two to eit is, *S* muss i [*K*, ∞) imp  $) = -1 < 0$  = or are all illuminum (C weak e *a*<br>
uation (12<br>
ie continu, *K*), Equ<br> *i*, *K*), Equ<br> *ie* weak *a*<br> *i* a weak e<br> *i* a weak e<br> *i* a weak e<br> *i* a weak e<br> *i*  $\infty$ ) impli<br>  $-1 < 0 =$ <br>
3.1.<br> *i* a (0, *i* a mpli)<br> *i* a (12 in *I* a (12 in *I* a (124) (*b*/*a*)<br>(*b*/*a*)<br>3) holity of<br>ity of<br>ity of<br>ity of<br>differentially be pulled at the luminos of r.<br>5) that luming if (0, *a*]<br>luang <br>3) is the riving in strip be pulled at the luminos of r.<br> $\frac{2\beta s}{c}$ <br>(*a*/*b*)<br> $x = l$ 2+2∕2 + (∕)  $\frac{1}{\sqrt{2+2\beta s/\sigma^2}}$ <br>+, S) con<br>5), then<br>1, S) con<br>1, S), then<br>1 and S is<br>1 a mild explored in a mild explored in a mild explored in the smood in a mild explored in the small  $\mathcal{V}_x(0, b+)$ <br>t weak. If (*b*/*a*)√ $\frac{y^2+2\beta s/\sigma^2}$  – (*a*/*b*)√ $\frac{y^2+2\beta s/\sigma^2}$ <br>3) holds. Notice that *J'*(*a*+, *S*) condity of function *a* → *J'*(*a*+, *S*), then tion (123) indeed holds and *S* is *d* the unique strong equilibrium.<br>quil erge xis mi<br>
let 16.<br>
let 16.<br>
let 11<br>
in 13<br>
on g the 11<br>
on g the 11<br>
b.<br>
sur S is a mild equilibrium if and only if Equation (123) holds. Notice that *I''*<br>
or bomen  $a \nearrow K$ . Then for any  $b > K$ , by the continuity of function  $a \mapsto J'(a)$ <br>
onstant  $a_b < K$  such that for all  $a \in [a_b, K)$ , Equation (123) (*a*+, *S*) converges<br>  $(u+$ , *S*), there exists<br>
s and *S* is a mild<br>
librium.<br>
by Proposition 6.4,<br>
yen a mild equilib-<br>
i, by Lemma 6.3(i),<br>
By the smooth-fit<br>
such condition is<br>
ibrium among the<br>  $v = V_x(0, b+, S)$ .<br>
and we *a* ∕ *K*. Then for any *b* > *K*, by the continuity of function *a* → *J'*<br> *a<sub>b</sub>* < *K* such that for all *a* ∈ [*a<sub>b</sub>*, *K*), Equation (123) indeed ho<br>
n.<br> **n.**<br> **n.**<br> **o.5.**  $S^* = (0, \frac{\lambda}{1+\lambda}K)$  *is the unique weak a* (+, ), there exists *u*<sub>c</sub> *s* such that for all *a* ∈ [*a<sub>b</sub>*, *K*), Equation [\(123\)](#page-37-0) indeed holds and *S* is a mild<br> **n** 6.5. *S*<sup>\*</sup> = (0,  $\frac{\lambda}{1+\lambda}K$  | *is the unique weak and the unique strong equilibrium.*<br>
irst find all weak equilibria .4, .4, b-<br>i), i), fit<br>is is he<br>)}.<br>m,  $S^* = (0, \frac{1}{1+})$ <br>d all weak  $\epsilon$ <br>lect weak e<br>k, no matte:<br>ogether wi<br>l<br>Equation (<br>oitrary type<br>lary 3.2, S :<br>ion in the  $\frac{1}{1+}$ <br>y if  $a = \frac{\lambda}{1+}$ <br>ria. Now pii<br> $a < K$ . Th<br> $-\frac{K-a}{b^{\nu+1}} \int_{0}^{K}$ <br>h-fit condi  $\frac{K}{1+2}K$  is the unique weak and the unique strong equilibrium.<br>
k equilibria. Since a weak equilibrium is also mild, by Prope<br>
equilibria from the two types of mild equilibria. Given a mi<br>
ter which type it is, *S* mu S that is weak, no matter which type it is, S must not contain K. Otherwise, by Lemma [6.3\(](#page-36-0)i),  $\subset S$ , which together with  $f = 0$  on  $[K, \infty)$  implies that<br>  $V_x(0, K-, S) = -1 < 0 \cong V_x(0, K+, S)$ ,<br>
a contradicts Equation (24) in Theo (0, K]  $\subset$  *S*, which together with  $f = 0$  on  $[K, \infty)$  implies that<br>  $V_x(0, K-, S) = -1 < 0 = V_x(0, i)$ <br>
which contradicts Equation (24) in Theorem 3.1.<br>
Consider an arbitrary type I mild equilibrium (0, *a*] with<br>
condition in C (24) in Theorem 3.1.<br>
e I mild equilibrium (0, a] with  $\frac{\lambda}{1+\lambda}$ <br>
= (0, a] is a weak equilibrium if and or<br>  $V_x(0, a+, (0, a]) = J'(a+, (0, a]) = -1$ .<br>
proof of Lemma 6.12 in Huang and Zl<br>  $\frac{1}{\lambda} K$ . Hence,  $S^* = (0, \frac{\lambda}{1+\lambda} K]$  is (0, *a*] with  $\frac{A}{1 + A}$ <br>
ilibrium if and<br>  $a +$ , (0, *a*]) = -<br>
in Huang and<br>  $\frac{\lambda}{4\lambda}K$ ] is the on<br>
ilibrium *S* =<br>
we have<br>  $\frac{\lambda^2 + 2\beta s/\sigma^2}{(a/b)\sqrt{\nu^2 + 2\rho^2}}$ <br>  $\frac{\lambda}{2} - (a/b)\sqrt{\nu^2 + 2\rho^2}$ <br>  $\frac{\lambda}{2}$ <br>  $\frac{\lambda}{2}$  and  $\begin{aligned} \n\text{and} \quad \mathbf{r} &= -\n\end{aligned}$   $\text{and} \quad \begin{aligned} \n\text{and} \quad \mathbf{r} &= \n\end{aligned}$  $K \le a < K$ . By the smooth-fit<br>only if<br>1.<br>Zhou (2020), such condition is<br>y weak equilibrium among the<br>0, a]  $\cup D$  with  $b := \inf\{x \in D\}$ .<br> $\frac{1}{3s/\sigma^2} ds < 0 = V_x(0, b+, S)$ .<br>hence *S* is not weak. In sum,  $S = (0, a]$  is a weak equilibrium if and only if<br>  $V_x(0, a+, (0, a]) = J'(a+, (0, a]) = -1.$ <br>
e proof of Lemma 6.12 in Huang and Zhou ( $\frac{\lambda}{1+\lambda}K$ . Hence,  $S^* = (0, \frac{\lambda}{1+\lambda}K]$  is the only weal<br>
pick any type II mild equilibrium  $S = (0, a]$  $v_x$ (0, *a*+, (0, *a*]) = *J*<br>proof of Lemma 6.12<br> $\frac{1}{\lambda}$ *K*. Hence, *S*<sup>\*</sup> = (0, <br>ck any type II mild eq<br>en by Equation (124)<br> $\int_0^\infty e^{-s} \frac{2a^{\nu}\sqrt{b/a}}{(b/a)^{\sqrt{\nu^2+2\beta s}}}$ <br>tion fails at the bour<br>weak equilibrium. in Huang and Zh<br>  $\frac{\lambda}{1+\lambda}K$  is the only<br>
utilibrium  $S = (0,$ , we have<br>  $v^2 + 2\beta s/\sigma^2$ <br>  $v^2 - (a/b)\sqrt{v^2 + 2\beta s}$ <br>
dary  $x = b$ , and l From the calculation in the proof of Lemma 6.12 in Huang and Zhou [\(2020\)](#page-40-0), such condition is satisfied if and only if |+i<br>|+i<br>|h<br>|| *K*. Hence,  $S^* = (0, \frac{A}{1 + A})$ <br>
k any type II mild equently to the position (124),  $\infty$ <br>  $e^{-s} \frac{2a^{\nu}\sqrt{\nu^2}}{(b/a)^{\sqrt{\nu^2 + 2\beta s/\sigma^2}}}$ <br>
on fails at the bound<br>
veak equilibrium.  $\frac{x}{1+\lambda}K$ ] is the only weak equilibrium among the<br>
uulibrium  $S = (0, a] \cup D$  with  $b := \inf\{x \in D\}$ ,<br>  $\frac{v^2 + 2\beta s/\sigma^2}{\sigma^2 - (a/b)^{\sqrt{v^2 + 2\beta s/\sigma^2}}} ds < 0 = V_x(0, b+, S)$ .<br>
Indary  $x = b$ , and hence *S* is not weak. In sum, type I mild equilibria. Now pick any type II mild equilibrium  $S = (0, a] \cup D$  with  $b := \inf\{x \in D\}$ . As  $K \notin S$ , we have  $a \lt K$ . Then by Equation (124), we have

e I mild equilibria. Now pick any type II mild equilibrium 
$$
S = (0, a] \cup D
$$
 with  $b := \inf\{x \in D\}$ .  
\n $K \notin S$ , we have  $a < K$ . Then by Equation (124), we have  
\n
$$
V_x(0, b-, S) = -\frac{K-a}{b^{\nu+1}} \int_0^\infty e^{-s} \frac{2a^{\nu}\sqrt{\nu^2 + 2\beta s/\sigma^2}}{(b/a)^{\sqrt{\nu^2 + 2\beta s/\sigma^2}} - (a/b)^{\sqrt{\nu^2 + 2\beta s/\sigma^2}}} ds < 0 = V_x(0, b+, S).
$$
\nat is, the smooth-fit condition fails at the boundary  $x = b$ , and hence S is not weak. In sum,  
\n $= (0, \frac{\lambda}{1+\lambda} K]$  is the unique weak equilibrium.

 $V_x(0, b-, S) = -\frac{b^y + 1}{b^y + 1}$ <br>t is, the smooth-fit cor<br>=  $(0, \frac{\lambda}{1 + \lambda} K]$  is the unique  $\frac{b^{\nu+1}}{b^{\nu+1}}$  *J*<br>it condi<br>unique io<br>ve it<br>7  $e^{-s}$ <br>  $\frac{2a}{(b/a)^{\sqrt{v^2+2\beta}}}$ <br>
a fails at the book equilibrium.  $\frac{b^2 + 2p^2b^2 - (a/b)^{\sqrt{2}}}{b^2 - (a/b)^{\sqrt{2}}}$ dary  $x = b$ ,  $(b/a)^{\sqrt{\nu^2+2\beta s/\sigma^2}} - (a/b)^{\sqrt{\nu^2+2\beta s/\sigma^2}}$ <br>ils at the boundary  $x = b$ , and her quilibrium.  $\cos S$  is not weak. In s That is, the smooth-fit condition fails at the boundary  $x = b$ , and hence S is not weak. In sum,  $x = b$ , and hence *S* is not weak. In sum,  $S^* = (0, \frac{1}{1+1})$  $\frac{1}{1+\lambda}K$  is the unique weak equilibrium.

<span id="page-39-0"></span>Finally, a direct calculation shows that

$$
\mathcal{L}V(0, x-, S^*) = -\beta(K - x) - \mu x < 0, \quad \forall x \in \left(0, \frac{\lambda}{1 + \lambda} K\right],
$$

so  $S^* = \mathfrak{S}_{S^*}$ . Then, by Theorem 5.1 and the fact that  $S^*$  is the unique weak equilibrium, we can conclude that  $S^*$  is the unique strong equilibrium.

 $V(0, x-, S^*) = -\beta(K - x) - \mu x < 0$ , ∀ x ∈ (<br>by Theorem 5.1 and the fact that S<sup>\*</sup> is the unite<br>the unique strong equilibrium.<br>this example, we do not restrict equilibria to be equilibrium (0,  $\frac{1}{1 + K}$ [I turns out to be indee 0,  $\frac{1}{1+}$ <br>que w admi<br>d admi<br>d admi<br>d ame J<br>ame J<br>tion u<br>tion u<br>tion d with f<br>d stopp<br>s and J<br>s and J<br>s and J<br>d stopp<br>s and J<br>d stopp<br>d with f<br>d stopp<br>d s and J<br>d s increase for the with of with<br>outline the set of virt  $1 + \lambda$ <br>e weak<br>dmissib<br>dmissib<br>dmissib<br>m  $S =$ <br>ne  $J$  vali<br>ne  $J$  vali<br>nn unde<br>ne  $J$ <br>nn unde<br>topping<br>nd form.<br>for time-<br>ms and i<br>stiment a ch Cente<br>thout co<br>versity, , check of the contract of th  $S^2 = \mathbb{Q}_{g \times r}$ . Then, by Theorem [5.1](#page-26-0) and the fact that ε<sup>γ</sup> is the unique weak equilibrium, we can<br>conclude that  $S^*$  is the unique strong equilibrium.<br> *Mark 6.6.* Within this coample, we do not restrict equilibriu <sup>5</sup> is the unique strong equilibrium.<br>
<sup>17</sup> if the security correlated equilibria to be admissible. The unique weak<br>
if dimit this example, we do not restrict equilibria to be indeed admissible. Moreover, type 1<br>
al mid *Remark* 6.6. Within this example, we do not restrict equilibria to be admissible. The unique weak, strong, optimal mild equilibrium  $(0, \frac{\lambda}{1+\lambda} K]$  turns out to be indeed admissible. Moreover, type I (0,  $\frac{1}{1+}$  while the while the while the state of  $[0, a]$  by the  $\Gamma A T$  used.<br>rg/00  $\frac{1}{1+}$  while the state of  $[0, 0]$  and  $\frac{1}{1+\lambda}K$ ] turns out to be indeed admissible. Moreover, type I<br>tile any type II mild equilibrium  $S = (0, a] \cup D$  with  $b :=$ <br> $1 \cup [1, \infty)$ , which share the same *J* value and is admissible.<br>the National Science Foundation un mild equilibria are all admissible, while any type II mild equilibrium  $S = (0, a] \cup D$  with  $b :=$ 

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### **DATA AVAILABILITY STATEMENT**

Not applicable since no data were used.

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under Grant DMS2106556<br>
sistent stopping problems in<br>
l in continuous time. *Finance*<br>
finance applications. *Springer*<br>
ping and smooth fit fo inf  $[x \in D] > \alpha$  has an alternative (0,  $a$  | ∪ [b, ∞), which share the same *J* value and is admissible.<br> **ACNOWEE DCMENTS CONDIVISTING SCIENCIPS** CONDIVISTING TO HAND THE EVALUATION OF THE ALTERATION OF THE ANNEL SURVEY O Christensen, S., & Lindensjö, K. (2020b). Time-inconsistent stopping, myopic adjustment and equilibrium stability: With a mean-variance application. In *Stochastic modeling and control*, *Banach Center Publications* (Vol. 122, pp. 53–76). Polish Academy of Sciences, Institute of Mathematics.
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### <span id="page-41-0"></span>**APPENDIX A: PROOF FOR RESULTS IN SECTION [2](#page-4-0)**

*Proof* (of [\(12\)](#page-5-0) in Remark 2.2). Let  $X_0 = x \in \mathbb{X}$  and  $h > 0$  be small enough such that  $[x - h, x + h]$ 

*Proof* (of (12) in Remark 2.2). Let 
$$
X_0 = x \in X
$$
 and  $h > 0$  be small enough such that  $[x - h, x + h] \subset X$ . Let  $Y = (Y_t)_{t\geq 0}$  follows  $dY_t = \hat{\mu}(Y_t)dt + \hat{\sigma}(Y_t)dW_t$  with  $Y_0 = x$ , where  
\n $\hat{\mu}(y) := \begin{cases} \mu(y), & x - h \leq y \leq x + h, \\ \mu(x + h), & y < x - h, \\ \mu(x + h), & y > x + h, \end{cases}$  and  $\hat{\sigma}(y) := \begin{cases} \sigma(y), & x - h \leq y \leq x + h, \\ \sigma(x + h), & y < x - h, \\ \sigma(x + h), & y > x + h. \end{cases}$   
\nThen by Huang et al. (2020, Lemma A.1), for any  $t > 0$ ,  
\n
$$
\mathbb{P}^{\times}\left(\max_{0 \leq s \leq t} Y_s > x\right) = \mathbb{P}^{\times}\left(\min_{0 \leq s \leq t} Y_s < x\right) = 1.
$$
  
\nNote that  $Y_s = X_s$  for  $s \leq r_{\beta(x,h)}$ . Then for a.s.  $\omega \in \{r_{\beta(x,h)} > 1/n\}$ ,  
\n
$$
\max_{0 \leq s \leq t} X_s(\omega) > x \text{ and } \min_{0 \leq s \leq t} X_s(\omega) < x, \quad \forall t \in (0, 1/n)
$$
 and thus  $\forall t > 0$ .  
\nThen Equation (12) follows from the arbitrariness of  $n \in \mathbb{N}$ .  
\nProof of Lemma 2.7. By Equation (13), for any  $t, r \geq 0$   
\n
$$
\delta(t + r) - \delta(t) \geq \delta(t)\delta(t) - \delta(0)
$$
,  
\n
$$
1 - \delta(t) = \int_0^t -\delta'(s)ds \leq \int_0^t -\delta(s)\delta'(0)ds \leq \int_0^t |\delta'(0)|ds = |\delta'(0)|t.
$$
  
\n
$$
\text{Proof of Lemma 2.12. Take an admissible stopping policy } S. Let a, b \in X \text{ such that } [a, b] \subset X \text{ and } (a
$$

Then by Huang et al. [\(2020,](#page-40-0) Lemma A.1), for any  $t > 0$ ,

$$
\mathbb{P}^{x}\left(\max_{0\leq s\leq t}Y_{s}>x\right)=\mathbb{P}^{x}\left(\min_{0\leq s\leq t}Y_{s}
$$

Note that  $Y_s = X_s$  for  $s \leq \tau_{B(x,h)}$ . Then for a.s.  $\omega \in {\tau_{B(x,h)} > 1/n}$ ,

$$
\max_{0 \le s \le t} X_s(\omega) > x \quad \text{and} \quad \min_{0 \le s \le t} X_s(\omega) < x, \quad \forall \, t \in (0, 1/n) \text{ and thus } \forall \, t > 0.
$$

Then Equation (12) follows from the arbitrariness of  $n \in \mathbb{N}$ .

*Proof of Lemma* 2.7. By Equation [\(13\)](#page-7-0), for any  $t, r \ge 0$ 

$$
\delta(t+r) - \delta(t) \ge \delta(t)(\delta(r) - \delta(0)),
$$

This together with the differentiability of  $\delta(t)$  implies that  $\delta'(t) \geq \delta(t)\delta'(0)$ . As  $\delta'(t) \leq 0$ ,

$$
\begin{aligned}\n\mu(y), & x-h \le y \le x+h, & \text{and } \phi(y) := \begin{cases}\n\sigma(y), & x-h \le y \le x+h, \\
\mu(x+h), & y < x-h, \\
\mu(x+h), & y > x+h, \\
\mu(x+h), & y > x+h, \\
\sigma(x+h), & y > x+h.\n\end{cases}
$$
\nIt is a a 1.2020, Lemma A.1), for any  $t > 0$ ,

\n
$$
\begin{aligned}\n\mathbb{P}^x\left(\max_{0 \le s \le t} Y_s > x\right) &= \mathbb{P}^x\left(\min_{0 \le s \le t} Y_s < x\right) = 1.\n\end{aligned}
$$
\nIt's a 2.5, for  $s \le \tau_{B(x,h)}$ . Then for a.s.  $\omega \in \{\tau_{B(x,h)} > 1/n\}$ ,  
\n
$$
\begin{aligned}\n\frac{\partial s}{\partial s} = X_s \text{ for } s \le \tau_{B(x,h)}. \text{ Then for a.s. } \omega \in \{\tau_{B(x,h)} > 1/n\}, \\
\frac{\partial s}{\partial s} = X_s \text{ for } s \le \tau_{B(x,h)}. \text{ Then for a.s. } \omega \in \{\tau_{B(x,h)} > 1/n\}, \\
\frac{\partial s}{\partial s} = X_s \text{ for } s \le \tau_{B(x,h)}. \text{ Then for a.s. } \omega \in \{\tau_{B(x,h)} > 1/n\}, \\
\frac{\partial s}{\partial s} = X_s \text{ for } s \le \tau_{B(x,h)}. \text{ Then, for a.s. } \omega \in \{\tau_{B(x,h)} > 1/n\}, \\
\frac{\partial s}{\partial s} = X_s \text{ for } s \le \tau_{B(x,h)}. \text{ Therefore, } \omega \in \mathbb{R}^n.\n\end{aligned}
$$
\nThen,  $\omega(12)$  follows from the arbitrariness of  $n \in \mathbb{N}$ .

\nThen,  $\omega(12)$  follows from the  $n \in \mathbb{N}$  to  $n \in \mathbb{N}$ .

\nThen,  $\omega(12)$  follows from the  $n \in \mathbb{N}$  to  $n \in \mathbb{N}$ .

\nThen,  $\omega \in \mathbb{N}$ ,  $\omega \in \mathbb{N$ 

*Proof of Lemma* 2.12. Take an admissible stopping policy *S*. Let  $a, b \in \mathbb{X}$  such that  $[a, b] \subset \mathbb{X}$  and one line to another and is independent of  $r$ .

() is the that of the that of the that of the that of the single singled in the set  $\log b \subset S^c$  is the set  $v(x, [a, b], A)$  is  $\ge 0$ , and in the set  $v(x, [a, b])$  is call the verse function  $\log b$  is  $\log b$  is  $\log b$  is  $\log b$  is  $\$  $\frac{1}{\cos \theta}$ <br>ion ( $\theta$ )<br> $\theta$ )<br> $\theta$ )<br> $\theta$  and  $\theta$ <br> $\theta$ )<br> $\theta$  and  $\theta$ )<br> $\theta$  and  $\theta$ <br> $\theta$ )<br> $\theta$  and  $\theta$ <br> $\theta$  and  $\theta$ <br> $\theta$  and  $\theta$ <br> $\theta$ <br> $\theta$  and  $\theta$ <br> $\theta$ <br> $\theta$  and  $\theta$ 0≤s≤t<br>ion (<br>mma<br>er wi<br>1 –<br>mma<br>Thro<br>Thro<br>issum<br>itrary<br>tisfi  $na 2.12.$  Take<br>  $ha a 2.12.$  Take<br>  $hroughout$  t<br>  $i.e.,$ <br>  $i.e.,$ <br>  $j.e.,$ <br>  $f = E^x[e^{-t}]{i.e.,}$ <br>  $m p tion 2.10$ <br>  $m q trop 2.10$ e h i'r (i 、 y c ( ) i tf −′′<br>n ac prc<br>dep<br>f(X and<br>sur<br>la st;<br>ing<br>) + //<br>v(c<br>g fu<br>e fi (s)ds ≤  $\int$ <br>lmissible<br>oof, C > 0<br>endent of<br>the boun<br>the boun<br> $\sum_{r>s} |v(x, y)|$ <br>andard prediction<br>elliptic e<br> $u(x)u'(x)$ <br>andard prediction  $u(x)u'(x)$ <br>anction  $y =$ <br>unction  $\tilde{u}$ s、critic reg (1 - (  $-o(s)$ *o*<br>ppping p<br>ill serve<br>st provid<br>dness o<br>5)]  $\leq \sup_{x \in [0,1]}$ <br>aabilistic<br>ation:<br> $\frac{1}{2}σ^2(x)$ <br> $= v(b,$ <br> $\neq (x)$  def<br> $= u(\zeta)$ (0)ds ≤  $\int$ <br>
oolicy *S*. L<br>
as a gene<br>
de an estiif *f* gives t<br>
up  $|v(x, (a, b)]$ <br>
c argumen<br>  $u''(x) = 0$ <br>
r, *S*).<br>
ined in E<br>  $b^{-1}(y)$ ) (i. e an home is a communication of the communication of t  $\begin{bmatrix} 1, l \\ 0, l \end{bmatrix}$  or  $x$  this  $u(l)$  $\nu \in \mathbb{X}$  such that<br>
such that<br>
for  $|\nu_x(x, r, r)|$ -posednes<br>  $\leq C$ .<br>
Let can derive  $\in (a, b)$ ,<br>
on (32) and  $(x) = \tilde{u}(\phi(x))$ that  $\left[\frac{1}{2} \max_{\text{max}} |S| + \frac{1}{2} \max_{\text{max}} |S| \right]$  or  $\left[\frac{1}{2} \max_{\text{max}} |S| \right]$  or □ nd m ^) || all 1 ) ( )<br>3 ) || all 1 ) ( 2 ) he ) ] . S. Let  $a, b \in \mathbb{X}$  such that  $[a, b] \subset \mathbb{X}$  and<br>generic constant that may change from<br>estimate for  $|v_x(x, r, S)| + |v_{xx}(x, r, S)|$ <br>ves the well-posedness of  $v(\cdot, r, S)$  for all<br> $v(x, 0, S)| \le C$ . (A.1)<br>ument, one can derive that  $v(x$ (*a*, *b*) ⊂ *S*<sup>c</sup>. Throughout the proof, *C* > 0 will serve as a generic constant that may change from<br>one line to another and is independent of *r*.<br>
Set *v*(*x*, *r*, *S*) = E<sup>x</sup>[*e*<sup>-*r*'*s f*</sup>(*X<sub>p</sub>*<sub>S</sub>)]. We fir . Set  $v(x,r,S) := \mathbb{E}^{x} [e^{-r\rho s} f(X_{\rho s})]$ . We first provide an estimate for  $|v_x(x,r,S)| + |v_{xx}(x,r,S)|$  $v(x, r, S) := \mathbb{E}^x[e^{-r\rho_S}f(X_{\rho_S})]$ . We first provide an estimate for  $|v_x(x, r, S)| + |v_{xx}(x, r, S)|$ <br>
b]. Assumption 2.1(i) and the boundedness of f gives the well-posedness of  $v(\cdot, r, S)$  for all<br>
and<br>  $\sup_{x \in [a,b], r \ge 0} |v(x, r, S)| \le$ on [a, b]. Assumption 2.1(i) and the boundedness of f gives the well-posedness of  $v(\cdot, r, S)$  for all

$$
\sup_{x \in [a,b], r \ge 0} |v(x,r,S)| \le \sup_{x \in [a,b]} |v(x,0,S)| \le C.
$$
 (A.1)

 $r \geq 0$ , and<br>For an ari $C^2([a, b])$ <br>Recall the<br>inverse fu For an arbitrary  $r \ge 0$ , by a standard probabilistic argument, one can derive that  $v(x, r, S) \in$  $C^2([a, b])$  satisfies the following elliptic equation:

[*a*, *b*]. Assumption 2.1(*i*) and the boundedness of *f* gives the well-posedness of *v*(·, *r*, *S*) for all 0, and  
\n
$$
\sup_{x \in [a,b], r \ge 0} |v(x, r, S)| \le \sup_{x \in [a,b]} |v(x, 0, S)| \le C.
$$
\n(A.1)  
\n∴ an arbitrary *r* ≥ 0, by a standard probabilistic argument, one can derive that *v*(*x*, *r*, *S*) ∈ [*a*, *b*]) satisfies the following elliptic equation:  
\n
$$
\begin{cases}\n-ru(x) + \mu(x)u'(x) + \frac{1}{2}\sigma^2(x)u''(x) = 0, & x \in (a, b), \\
u(a) = v(a, r, S), u(b) = v(b, r, S).\n\end{cases}
$$
\n(A.2)  
\nand the strictly increasing function *y* = *φ*(*x*) defined in Equation (32) and denote by *φ*<sup>-1</sup> the  
\nerse function of *φ*. Define function  $\tilde{u}(y) := u(\phi^{-1}(y))$  (i.e., *u*(*x*) =  $\tilde{u}(\phi(x))$ ) on [*φ*(*a*), *φ*(*b*)].

<sup>2</sup>([*a*, *b*]) satisfies the following elliptic equation:<br>  $\begin{cases}\n-ru(x) + \mu(x)u'(x) + \frac{1}{2}\sigma^2(x) \\
u(a) = v(a, r, S), u(b) = v(b)\n\end{cases}$ <br>
ecall the strictly increasing function  $y = \phi(x)$  d<br>
werse function of  $\phi$ . Define function  $\tilde{u}(y)$  $=$ <br> $\frac{1}{2}$  $= v(b, r, S).$ <br>
x) defined in Equation (32) a<br>  $:= u(\phi^{-1}(y))$  (i.e.,  $u(x) = \tilde{u}(\phi^{-1}(x))$ (A.2)<br>  $u(a) = v(a, r, S), u(b) = v(b, r, S).$ <br>
reasing function  $y = \phi(x)$  defined in Equation (32) and denote by  $\phi^{-1}$  the<br>
. Define function  $\tilde{u}(y) := u(\phi^{-1}(y))$  (i.e.,  $u(x) = \tilde{u}(\phi(x))$ ) on  $[\phi(a), \phi(b)]$ . Recall the strictly increasing function  $y = \phi(x)$  defined in Equation (32) and denote by  $\phi^{-1}$  the  $y = \phi(x)$  defined in Equation (32) and denote by  $\phi^{-1}$  the<br>  $\tilde{u}(y) := u(\phi^{-1}(y))$  (i.e.,  $u(x) = \tilde{u}(\phi(x))$ ) on  $[\phi(a), \phi(b)]$ . inverse function of  $\phi$ . Define function  $\tilde{u}(y) := u(\phi^{-1}(y))$  (i.e.,  $u(x) = \tilde{u}(\phi(x))$ ) on  $[\phi(a), \phi(b)]$ .  $\phi$ . Define function  $\tilde{u}(y) := u(\phi^{-1}(y))$  (i.e.,  $u(x) = \tilde{u}(\phi(x))$ ) on  $[\phi(a), \phi(b)]$ .<br>

<span id="page-42-0"></span>Then  $\tilde{u} \in C^2([\phi(a), \phi(b)])$ , and Equation (A.2) leads to

$$
\begin{cases}\n-r\tilde{u}(y) + \frac{1}{2}\tilde{\sigma}^2(y)\tilde{u}''(y) = 0, & y \in (\phi(a), \phi(b)), \\
\tilde{u}(\phi(a)) = v(a, r, S), & \tilde{u}(\phi(b)) = v(b, r, S),\n\end{cases}
$$
\n(A.3)

with  $\tilde{\sigma}(y) := \sigma(\phi^{-1}(y))\phi'(\phi^{-1}(y))$ . Then Equation (A.1) together with the maximum principle implies that

$$
\sup_{y \in (\phi(a), \phi(b))} |\tilde{u}(y)| \le v(a, r, S) \vee v(b, r, S) \le C \qquad \forall r \ge 0.
$$

This together with the fact that  $\tilde{u}'' = 2\tilde{u}r/(\tilde{\sigma}^2)$  and uniform ellipticity of  $\tilde{\sigma}$  on  $(\phi(a), \phi(b))$  leads to

$$
\sup_{y \in (\phi(a), \phi(b))} |\tilde{u}''(y)| \le rC \qquad \forall r \ge 0. \tag{A.4}
$$

By the mean value theorem, there exists  $y_0 \in (\phi(a), \phi(b))$  such that

$$
|\tilde{u}'(y_0)| = \left|\frac{v(b, r, S) - v(a, r, S)}{\phi(b) - \phi(a)}\right| \le \frac{2C}{\phi(b) - \phi(a)} \qquad \forall r \ge 0.
$$
 (A.5)

Then by Equations (A.4) and (A.5), for any  $y \in (\phi(a), \phi(b))$  and  $r \ge 0$ , we have

$$
|\tilde{u}'(y)| \le |\tilde{u}'(y_0)| + \int_{y_0}^y |\tilde{u}''(l)|dl \le \frac{2C}{\phi(b) - \phi(a)} + \int_{\phi(a)}^{\phi(b)} rCdl = \frac{2C}{\phi(b) - \phi(a)} + rC(\phi(b) - \phi(a)).
$$

Therefore,

$$
\sup_{y \in (\phi(a), \phi(b))} (|\tilde{u}'(y)| + |\tilde{u}''(y)|) \le C(1+r) \qquad \forall r \ge 0.
$$

This together with the fact that

$$
\sup_{x \in (a,b)} (|u'(x)| + |u''(x)|) \le \sup_{y \in (\phi(a), \phi(b))} (|\tilde{u}'(y)| + |\tilde{u}''(y)|) \cdot \sup_{x \in (a,b)} (|\phi'(x)| + |\phi'(x)|^2 + |\phi''(x)|)
$$

implies

$$
\sup_{x \in (a,b)} (|v_x(x,r,S)| + |v_{xx}(x,r,S)|) = \sup_{x \in (a,b)} (|u'(x)| + |u''(x)|) \le C(1+r) \qquad \forall r \ge 0. \tag{A.6}
$$

 $\bar{u} \in C^2([\phi(a), \phi(b)]),$  and Equation [\(A.2\)](#page-41-0) leads to<br>  $\begin{cases} -r\bar{u}(y) + \frac{1}{2}\sigma^2(y)\bar{u}''(y) = 0, \\ \bar{u}(\phi(a)) = v(a, r, S), \bar{u}(\phi(b)) = 0, \\ \bar{u}(\phi(a)) = v(a, r, S), \bar{u}(\phi(b)) = 0, \end{cases}$ <br>  $\sigma(y) := \sigma(\phi^{-1}(y))\phi'(\phi^{-1}(y)).$  Then Equation (A<br>  $\sigma(y) = \frac{|u(x) - u$  $-vu(y) + \frac{1}{2}$ <br>  $\tilde{u}(\phi(a)) = v$ <br>  $\psi'(\phi^{-1}(y))$ .<br>
up  $|\tilde{u}(y)|$ <br>
a), $\phi(b)$ <br>
ict that  $\tilde{u}''$ <br>
su v∈( $\phi(a)$ <br>
em, there e<br>  $|v| = \left| \frac{v(b, r)}{\phi} \right|$ <br>
and (A.5),<br>  $\tilde{u}''(l)|dl \le$ <br>
sup ( $|\tilde{u}$ <br>  $(\phi(a), \phi(b))$ <br>
ct that<br>  $|v| \leq \sup_{$ 1 . ^. じ.a oi c.jpb , 、 ,. ) C at W = ( of *v*(*a,r, S*),  $\tilde{u}(\phi(b)) = v(b, r, S)$ ,<br>
Then Equation (A.1) together with<br>  $| \le v(a, r, S) \vee v(b, r, S) \le C$ <br>  $\forall i$ <br>  $| \le v(a, r, S) \vee v(b, r, S) \le C$ <br>  $\forall j$ <br>  $= 2\tilde{u}r/(\tilde{\sigma}^2)$  and uniform ellipticity<br>  $\frac{\tilde{v}(\phi(b))}{(b)}$ <br>  $\frac{\tilde{v}(\phi(b$ (A.3)<br>  $B'(\phi^{-1}(y))$ . Then Equation (A.1) together with the maximum principle<br>  $B'(\phi^{-1}(y))$ . Then Equation (A.1) together with the maximum principle<br>  $B'(\phi^{-1}(y))$ . Then Equation (A.1) together with the maximum principle<br>  $\psi(\$ ( $\partial$ o(y) : = ( $\partial$ (y)) $\varphi$ <br>
es that<br>
su<br>
su<br>
su<br>
together with the fac<br>
e mean value theore<br>  $|\tilde{u}'(y_0)|$ <br>
by Equations (A.4) a<br>  $|\tilde{u}'(y_0)|$ <br>
by Equations (A.4) a<br>  $|\tilde{u}'(y_0)| + \int_{y_0}^y |\tilde{u}$ <br>
fore,<br>  $y \in (\phi)$ <br>
for ( $\phi^{-1}(y)$ ). Then Equation [\(A.1\)](#page-41-0) together with the maximum principle<br>
p<br>
p<br>
p<br>
p<br>
( $\text{a}(y)$ )  $\le v(a, r, S) \vee v(b, r, S) \le C$   $\forall r \ge 0$ .<br>
(A.4)<br>
that  $a'' = 2ar/(\sigma^2)$  and uniform ellipticity of  $\sigma$  on  $(\phi(a), \phi(b))$  leads<br>  $\sup_{y \in (\phi(a$  $s(a), s(a), s(b)$ <br>fact rem  $s(b)$  are  $|\tilde{u}'|$ <br> $s(b)$  are  $|\tilde{u}'|$ <br> $\Rightarrow s(b)$  are  $|v_x|$ <br> $|v_x|$  are  $|v_y|$  are  $|v_y|$  are  $|v_y|$ the fact that<br>
the fact that<br>
the fact that<br>  $|f'(y_0)| = \left|\frac{1}{2}\right|$ <br>  $\frac{1}{2}$ <br> () ̃ | ≤ (, , ) ∨ (, , ) ≤ ∀ ≥ 0.  $\hat{u}'' = 2\hat{u}r/(\hat{\sigma}^2)$  and uniform ellipticity of  $\hat{\sigma}$  on  $(\phi(a), \phi(b))$  leads<br>
sup<br>
sup<br>  $|\hat{u}''(y)| \le rC$   $\forall r \ge 0.$  (A.4)<br>
re exists  $y_0 \in (\phi(a), \phi(b))$  such that<br>  $\phi(b) - \phi(a)$   $\le \frac{2C}{\phi(b) - \phi(a)}$   $\forall r \ge 0.$  (A.5)<br>
5), for  $\begin{align*}\n\frac{r}{\sqrt{a}} &\text{exi} \\
\frac{r}{\sqrt{b}} &\text{ex$ nere exists  $\frac{v(b,r,S)-\phi(b)-\Delta}{\phi(b)-\Delta}$ <br>
(A.5), for an <br>  $|dl \leq \frac{1}{\phi(b)}$ <br>
( $|\tilde{u}'(y)|$ <br>
( $|\tilde{u}'(y)|$ <br>
( $\Rightarrow$ )<br>
( $\Rightarrow$ )<br>
( $\Rightarrow$ )<br>
( $\phi(a),\phi(b)$ )<br>
( $\Rightarrow$ )<br>
( $\$  $\left.\begin{aligned} &\tilde{u}''(y)| \leq rC &\forall r \geq 0. \end{aligned}\right. \qquad (A.4)$ <br>  $\left.\begin{aligned} &\tilde{u}''(y)| \leq rC &\forall r \geq 0. \end{aligned}\right. \qquad (A.5)$ <br>  $\left.\begin{aligned} &\tilde{u}(a,r,S)| \leq \frac{2C}{\phi(b)-\phi(a)} &\forall r \geq 0. \end{aligned}\right. \qquad (A.5)$ <br>  $\left.\begin{aligned} &\tilde{v} \leq \left(\phi(a),\phi(b)\right) \text{ and } r \geq 0, \text{ we have} \end{aligned}\$  $y_0 \in (\phi(a), \phi(b))$  such that<br>  $\left|\frac{v(a, r, S)}{\phi(a)}\right| \leq \frac{2C}{\phi(b) - \phi(a)}$ <br>
my  $y \in (\phi(a), \phi(b))$  and  $r \geq \frac{2C}{-\phi(a)} + \int_{\phi(a)}^{\phi(b)} rC dl = \frac{2C}{\phi(b)}$ <br>  $+ |\tilde{u}''(y)| \leq C(1+r)$ <br>  $|\tilde{u}'(y)| + |\tilde{u}''(y)| \cdot \sup_{x \in (a,b)} (|\tilde{u}(x)| + |\tilde{u}''(x)|) \leq \sup_{x \in ($  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ <br> $\begin{bmatrix} b & b \\ c & d \end{bmatrix}$ <br> $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ <br> $\begin{bmatrix} b & b \\ c & d \end{bmatrix}$ <br> $\begin{bmatrix} b & b \\ c & d \end{bmatrix}$  $(y_0)| = |$ <br>A.4) and<br> $\int_{y_0}^{y} |\tilde{u}''(l)|$ <br> $\lim_{y \in (\phi(a), \phi(a))}$ <br>e fact that<br> $(x)| \le \lim_{y \to (\phi(a), \phi(a))}$ <br> $| + |v_{xx}(l)|$ <br>Figure in Equation (A.5), for any  $y \in (\phi$ <br>
(A.5), for any  $y \in (\phi$ <br>
(A.5), for any  $y \in (\phi$ <br>
(a)  $d$  ≤  $\frac{2C}{\phi(b) - \phi(a)}$ <br>
(b)  $\frac{(\vert \tilde{u}'(y) \vert + \vert \tilde{u}''(y) \vert)}{(\vert \tilde{u}'(y) \vert + \vert \tilde{u}''(y) \vert + \vert \tilde{u}''(y) \vert)}$ <br>
(e( $\phi(a), \phi(b)$ ))<br>
(c, r, S)|) = sup (), for any y  $\infty$ <br>
(), for any y  $\infty$ <br>  $\leq \frac{2C}{\phi(b) - \phi(b)}$ <br>
( $\tilde{u}'(y)$ ) +  $|\tilde{u}'(y)|$ <br>
( $\tilde{u}'(y)$ ),  $\phi(b)$ )<br>
( $\tilde{u}'(y)$ ),  $\phi(b)$ )<br>
(S)) = sup<br>  $\sup_{x \in (a,b)}$ <br>  $\in C^2([a, b])$ <br>  $= \int_0^\infty e^{-rt}$ <br>
(i.6) and the  $\frac{1}{x}$  and  $\frac{1}{x}$  and  $\frac{1}{x}$  and  $\frac{1}{x}$  and  $\frac{1}{x}$  and  $\frac{1}{x}$  both  $\frac{1}{x}$  and  $\frac{1}{x}$  (A.5)<br>  $\phi(b)$  and  $r \ge 0$ , we have<br>  $\phi(b)$  and  $r \ge 0$ , we have<br>  $\phi(b)$ <br>  $rCdl = \frac{2C}{\phi(b) - \phi(a)} + rC(\phi(b) - \phi(a))$ <br>  $C(1 + r)$   $\forall r \ge 0$ .<br>  $\psi(c)$ <br>  $\psi$  $y \in (\phi(a), \phi(b))$  and  $r \ge 0$ , we have<br>  $\frac{d\phi(a)}{d\phi(a)} + \int_{\phi(a)}^{\phi(b)} rCdl = \frac{2C}{\phi(b) - \phi(a)}$ <br>  $|\tilde{u}''(y)| \le C(1+r) \quad \forall r \ge 0.$ <br>  $(y)| + |\tilde{u}''(y)| \cdot \sup_{x \in (a,b)} (|\phi'(x)| + |\phi(x)|)$ <br>  $\lim_{x,b)} (|u'(x)| + |u''(x)|) \le C(1+r)$ <br>  $\lim_{a,b)}$ <br>  $b]$ ). For any  $r \ge 0$  $\Gamma$ h<br>Ch<br>Ch<br> $\chi$ <sup>1</sup><br>Ch<br> $\chi$ <br>He<br>He<br>Ch<br>Ch<br>Ch<br>Ch  $(y)| \le |\tilde{u}'|$ <br>erefore,<br>is togethe<br>sup  $(|u'(\cdot)|)$ <br>plies<br>sup  $(|v|)$ <br> $x \in (a,b)$ <br>xt, we ve<br> $(a+, r, S)$ <br>nce, we c<br>is, togeth<br> $^2([0, \infty))$ (y<sub>0</sub>)| + *j*<br>
r with th<br>
x)| + |u''<br>  $\left.\left(x, r, S\right)\right|$ <br>
rrify that<br>
,  $v_{xx}(b-$ <br>
onclude t<br>
er with<br>  $\left[a, b\right]$ ). ) a f t r l r l c c c  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 &$  $\tilde{u}''(l)| dl \le$ <br>sup  $(|i \n\phi(a), \phi(b))$ <br>ct that<br> $|) \leq \sup_{y \in (\phi(a))}$ <br> $v_{xx}(x, r, S)$ <br> $\in C^{1,2}([0, S))$  all exis<br> $v(x, r, S)$ <br> $V(t, x, S)$  $\frac{1}{2}$  +  $|\tilde{u}'|$  surverse  $|a, \infty$  e d  $\phi(b) - \phi(a) + \int$ <br>  $\phi(b) + |\tilde{u}''(y)|$  :<br>  $\phi(b)$ <br>  $\phi(b)$ <br>  $\phi(b)$ <br>  $\phi(c) = \sup_{x \in (a,b)} (|u'(x))$ <br>  $\phi(c) \times [a, b])$ . For and satisfy the<br>  $\int_0^\infty e^{-rt}v(x, r, t)$ <br>  $\phi(c) = \int_0^\infty e^{-rt}v(x, r, t)$ <br>  $\phi(c) = \int_0^\infty e^{-rt}v(x, r, t)$ <br>  $\phi(c) = \int_0^\infty e^{-rt}v(x, r, t$ (a)<br> $C($ <br> $(y)|$ <br> $|$  + any<br> $\lim_{y \to 0}$ <br> $|y|$ <br> $|$   $|$   $|$   $|$  $\leq C$ <br> $''(y)$ <br> $\Rightarrow$  an san ubi<br> $S$ )c mp = 2  $\phi(b) - \phi(a)$ <br>  $\forall r \ge 0.$ <br>  $(|\phi'(x)| + |\phi'(x)|^2 + |\phi''(x)|)$ <br>  $\le C(1+r)$   $\forall r \ge 0.$  (A.6)<br>  $\nu_x(a+r, S), \quad \nu_x(b-r, S)$  (resp.<br>
as the RHS of Equation (A.6).<br>
in, Equation (20) leads to<br>  $\in \mathbb{X}.$  (A.7)<br>  $dF(r) < \infty$ , implies that  $V \in$ sup Example 1.  $(x, y)$ <br>  $\leq$   $\frac{1}{y \in (d)}$ <br>  $\Rightarrow$   $\frac{1}{y \in (d)}$ <br>  $\Rightarrow$   $\frac{1}{y \in (d)}$ <br>  $\leq$   $\frac{1}{y \in$ (|u]<br>sup<br>(a), 4<br>(3) = (3) = (4)<br>sxist<br>(5) = (4) ()| + |̃′′()|) ≤ (1 + ) ∀ ≥ 0.  $\in$ (*a*,*l*)<br>plie su<br>su<br> $x \in$ ((*x*)<br>xt,  $\in$ (*a*)<br>is,  $\in$ 2([( mplies<br>sup<br> $x \in (a, b)$ <br> $\infty$ <br> $\infty$ <br> $(x, x)$ <br> $(x + b)$  $\begin{array}{c} \n\mathbf{y} & \n\mathbf{$  $(x)| + |u''(x)| \le \sup_{y \in (\phi(a), g)}$ <br>  $|v_x(x, r, S)| + |v_{xx}(x, r, S)|$ <br>
verify that  $V \in C^{1,2}([0, \infty))$ ,  $v_{xx}(b-, r, S)$  all exist<br>
conclude that  $v(x, r, S) \in V(t, x, S)$  =<br>
her with Equation (A.6<br>  $\times [a, b]$ ).  $[x, r, S]$  =  $[x, r, S]$  =  $[x, r, S]$  =  $\int_{0}^{1,2} ([0, \infty) \times I, r, S) \in C^{2}$ <br> $(x, s, S) = \int_{0}^{1,2} f(x, t) \, dt$  $\begin{array}{c} \n\cdot \quad \text{st} \\
\cdot \quad \text{st} \\
\$ (y)| + |*u*<sup>(y)</sup>|)  $\frac{f^{(0)}}{x \in (a,b)}$ <br>
(lp (|*u'*(x)| + |*u''*(x)|)<br>
(*a,b*)<br>
(*b*]). For any  $r \ge 0$ , tisfy the same bound<br> *b*]). By Fubini theore<br>  $\frac{-rt}{v(x,r,S)}$ dF(r)  $\forall$ <br>
the assumption  $\int_0^\infty$  $\mathcal{L}(x)$ <br>  $\geq 0, \quad \text{and}$ <br>  $\geq 0, \quad \text{and}$ <br>  $\text{deorem}$ <br>  $\int_0^\infty r^2 dx$ ( $|\varphi|$   $\leq C$   $\qquad \qquad$  $\varphi_x(a)$  as t  $\in \mathbb{R}$   $\qquad \qquad$  $dF($  $(1 + r)$ <br>+, r, S), the RHS condition (2<br> $\leq r$ )  $\lt \infty$ , if  $r$ )  $\lt \infty$ , if  $r$  $\forall x$ <br> $\forall x$  (b)<br> $\oint f$  Ec Eq D(b) 1 r<br>-<br>11<br>|i  $\geq 0.$  (A. $(t, r, S)$  (respectively)<br>attion (A.6)<br>ads to (A. $\geq$  that V  $\in$  (*a*,*l*<br> $\downarrow$ , w<br>*a* +, ce, ,<br>, tc xt, we<br>
xt, we<br>  $(a+, r)$ <br>
mce, w<br>
is, tog<br>  $\frac{1}{2}([0, \infty))$ ( $|v_x(x, r, s)| + |v_{xx}(x, r, s)|$ ) = sup<br>  $x \in (a, b)$ <br>  $\therefore$  S),  $v_{xx}(b-, r, S)$  all exist and satisfy<br>  $\therefore$  S),  $v_{xx}(b-, r, S)$  all exist and satisfy<br>  $V(t, x, S) = \int_0^\infty e^{-rt}$ <br>  $\therefore$  ether with Equation (A.6) and the polyton (A.6) and the [a, b])<br>satisf<br>[a, b])<br> $e^{-rt}$ <br>dd the  $($ |u  $)$ . F  $\frac{1}{2}$  the  $y$  th  $By$   $\nu(x, y)$  $(x)| + |u''(x)| \le C(1+r) \quad \forall r \ge 0.$  (A.6)<br>
or any  $r \ge 0$ ,  $v_x(a+, r, S)$ ,  $v_x(b-, r, S)$  (resp.<br>
e same bound as the RHS of Equation (A.6).<br>
Fubini theorem, Equation (20) leads to<br>  $r, S)dF(r) \quad \forall x \in \mathbb{X}.$  (A.7)<br>
sumption  $\int_0^\infty r dF(r) < \$ Next, we verify that  $V \in C^{1,2}([0,\infty) \times [a,b])$ . For any  $r \ge 0$ ,  $v_x(a+,r,S)$ ,  $v_x(b-,r,S)$  (resp.  $V \in C^{1,2}([0, \infty) \times [a, b])$ . For any  $r \ge 0$ ,  $v_x(a+, r, S)$ ,  $v_x(b-, r, S)$  (resp.<br>  $(r, S)$ ) all exist and satisfy the same bound as the RHS of Equation (A.6).<br>
hat  $v(x, r, S) \in C^2([a, b])$ . By Fubini theorem, Equation (20) leads to<br> Hence, we conclude that  $v(x, r, S) \in C^2([a, b])$ . By Fubini theorem, Equation (20) leads to

$$
v(x, r, S) \in C^{2}([a, b]).
$$
 By Fubini theorem, Equation (20) leads to  

$$
V(t, x, S) = \int_{0}^{\infty} e^{-rt} v(x, r, S) dF(r) \quad \forall x \in \mathbb{X}.
$$
 (A.7)  
uation (A.6) and the assumption  $\int_{0}^{\infty} r dF(r) < \infty$ , implies that  $V \in$ 

 $v_{xx}(a+, r, S), v_{xx}(b-, r, S))$  all exist and satisfy the same bound as the RHS of Equation (A.6).<br>
Hence, we conclude that  $v(x, r, S) \in C^2([a, b])$ . By Fubini theorem, Equation (20) leads to<br>  $V(t, x, S) = \int_0^\infty e^{-rt}v(x, r, S)dF(r) \quad \forall x \in \mathbb{X$  $(v(t, x, S)) = \int$ <br>aation (A.6)  $\frac{1}{\sqrt{2}}$  $e^{-rt}v(x, r, S)dF(r)$   $\forall x \in \mathbb{X}$ . (A.7)<br>the assumption  $\int_0^\infty r dF(r) < \infty$ , implies that  $V \in$ This, together with Equation (A.6) and the assumption  $\int_0^\infty r dF(r) < \infty$ , implies that  $V \in$ ∞  $rdF(r) < \infty$ , implies that  $V \in$  $C^{1,2}([0,\infty)\times [a,b])$ .  $1,2([0,\infty) \times [a, b]).$ 

<span id="page-43-0"></span>Finally, we prove Equation [\(18\)](#page-8-0) for  $V_x(t, x+, S)$  (the verification for  $V_x(t, x-, S)$  is similar and thus omitted). For  $(t, x) \in [0, \infty) \times [a, b)$ , by Equations (A.6), (A.7) and the assumption  $\int_{0}^{\infty} r dF(r) < \infty$ ,

$$
V_x(t, x+, S) = \int_0^\infty e^{-rt} v_x(x+, r, S) dF(r).
$$
 (A.8)

Now take any  $x \in \mathbb{X}$ . If there exists some  $h > 0$  such that  $(x, x + h) \subset S^c$ , then by Equations (21),  $(A.6)$ , and  $(A.8)$ , we have that

inally, we prove Equation (18) for 
$$
V_x(t, x+, S)
$$
 (the verification for  $V_x(t, x-, S)$  is similar  
thus omitted). For  $(t, x) \in [0, \infty) \times [a, b)$ , by Equations (A.6), (A.7) and the assumption  
 $rdF(r) < \infty$ ,  
 $V_x(t, x+, S) = \int_0^\infty e^{-rt}v_x(x+, r, S)dF(r)$ . (A.8)  
 $v$  take any  $x \in \mathbb{X}$ . If there exists some  $h > 0$  such that  $(x, x + h) \subset S^c$ , then by Equations (21),  
and (A.8), we have that  
 $|V_x(t, x+, S) - V_x(0, x+, S)| \le \int_0^\infty |v_x(x, r, S)||e^{-rt} - 1|dF(r)$   
 $\le C \int_0^\infty (1 + r)(1 - e^{-rt})dF(r) = o(\sqrt{t})$ , as  $t \to 0$ .  
erwise, since S is admissible, there exists some  $\bar{h} > 0$  such that  $(x, x + \bar{h}) \subset S$ . Then,  
 $|V_x(t, x+, S) - V_x(0, x+, S)| = |\delta(t)f(x) - \delta(0)f(x)| \le |\delta'(0)|t f(x) = o(\sqrt{t})$ , as  $t \to 0$ ,  
are the above inequality follows from Lemma 2.7. In sum, Equation (18) holds for  
 $x, x, +, S$ ).  
 $\int_0^x f(x, t, s, S) = V(x, t, s)$  asymptot 2.10 guarantees that  $V(t, x, S) \in C^{1,2}([0, \infty) \times \overline{S^c})$ ,  
 $\int_0^x f(x, x, S) = 0$   $V(t, x, S) \in (0, \infty) \times S^c$ .  
 $(0, \infty) \times S$ . Now we prove that  
 $\mathcal{L}V(t, x, S) \equiv 0$   $V(t, x) \in [0, \infty) \times S^c$ .  
 $(0, \infty) \times S$ . Now we prove that  
 $\mathcal{L}V(t, x, S) \in C^{1,2}([0, \infty) \times \overline{S^c})$ ,  $(s, x) \mapsto \mathcal{L}V(s, x, S)$  is continuous on

Otherwise, since S is admissible, there exists some  $\bar{h} > 0$  such that  $(x, x + \bar{h}) \subset S$ . Then,

$$
|V_x(t, x+S) - V_x(0, x+S)| = |\delta(t)f(x) - \delta(0)f(x)| \le |\delta'(0)|t f(x) = o(\sqrt{t}), \text{ as } t \to 0,
$$

where the above inequality follows from Lemma [2.7.](#page-7-0) In sum, Equation (18) holds for

*Proof of Lemma* 2.14. **Part (a)**: Assumption 2.10 guarantees that  $V(t, x, S) \in C^{1,2}([0, \infty) \times \overline{S^c})$ . Equation [\(12\)](#page-5-0) implies that  $\mathbb{P}^{x}(\rho_S = 0) = 1$  for any  $x \in S$ , so  $V(t, x, S) = \delta(t) f(x)$  for any  $(t, x) \in$ 

$$
\mathcal{L}V(t, x, S) \equiv 0 \quad \forall (t, x) \in [0, \infty) \times S^c. \tag{A.9}
$$

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∞ W, e <br>  $\frac{1}{c_0}$ <br>  $x$ <br>
⇒ DN ts  $x$  m  $\frac{1}{2}$ ,  $\frac{1}{2}$ ,  $\frac{1}{2}$  d  $\frac{1}{2}$ ,  $\$ (1 + r)(1 – e<sup>-rt</sup>)dF(r) = o( $\sqrt{ }$ <br>
ome  $\bar{h} > 0$  such that (x, x +  $\bar{h}$ <br>
- δ(0)f(x)| ≤ |δ'(0)|tf(x) = c<br>
Lemma 2.7. In sum, Equa<br>
2.10 guarantees that  $V(t, x, S)$ <br>
any  $x \in S$ , so  $V(t, x, S) = \delta(t)$ <br>  $\forall (t, x) \in [0, \infty) \times S^$ t), as  $t \to 0$ .<br>  $\bigcirc$  C S. Then,<br>  $p(\sqrt{t})$ , as  $t -$ <br>  $p(\sqrt{t})$ , as  $t -$ <br>  $\bigcirc$  (18) h<br>  $\bigcirc \in C^{1,2}([0, \infty))$ <br>  $f(x)$  for any<br>  $x_0 - h, x_0 +$ <br>
is continuou<br>
on term vani<br>  $\bigcirc$  s,  $X_s$ , S)ds S is admissible, there exists some *h* > 0 such that (x, x + *h*) C S. Then,<br>  $-V_x(0, x+, S)$  = |δ(t)f(x) - δ(0)f(x)| ≤ |δ'(0)|tf(x) = o(√t), as t -<br>
e inequality follows from Lemma 2.7. In sum, Equation (18) h<br>
2.14. **Part**  $V_x(t, x+, S) - V_x(0, x+, S)$  = | $\delta(t) f(x) - \delta(0) f(x)$  ≤ | $\delta'$ <br>
e the above inequality follows from Lemma 2.7. In<br>  $x+, S$ ).<br>  $\delta f$  Lemma 2.14. **Part (a)**: Assumption 2.10 guarantees tion (12) implies that  $\mathbb{P}^x(\rho_S = 0) = 1$  for an (0)|t  $f(x) = o(\sqrt{\sqrt{\sqrt{\frac{1}{5}}}})$ <br>
sum, Equation<br>
chat  $V(t, x, S) \in$ <br>  $t, x, S$ ) =  $\delta(t) f(x)$ <br>  $\leq S^c$ .<br>
(b) such that  $[x_0 - \sqrt{L}V(s, x, S)]$  is consider that  $[x_0 - \sqrt{L}V(s, x, S)]$  is considered. t), as  $t \to 0$ ,<br>
1 (18) holds<br>  $C^{1,2}([0, \infty)$  ><br>
(b) for any (t,<br>  $-h, x_0 + h$ ]<br>
ontinuous o:<br>
(c)<br>
erm vanishes  $V_x(t, x+, S)$ . □<br>
Epoof of Lemma 2.14. **Part (a):** Assumption 2.10 guarantees that  $V(t, x, S) \in C^{1,2}(0, \infty) \times \overline{S^c}$ .<br>
Equation (12) implies that  $P^x(\rho_S = 0) = 1$  for any  $x \in S$ , so  $V(t, x, S) = \delta(t) f(x)$  for any  $(t, x) \in$ <br>
[0,  $\in$  $V(t, x, S) \in C^{1,2}([0, \infty) \times \overline{S^c}).$ <br>  $(S) = \delta(t) f(x)$  for any  $(t, x) \in$ <br>
(A.9)<br>
ch that  $[x_0 - h, x_0 + h] \subset S^c.$ <br>  $V(s, x, S)$  is continuous on the<br>
ediffusion term vanishes due<br>  $\mathcal{CV}(t + s, X_s, S) ds$ . (A.11)<br>  $\mathcal{CV}(t + s, X_s, S) ds$ . (A.11)  $P<sup>x</sup>(ρ<sub>S</sub> = 0) = 1$  for any  $x \in S$ , so  $V(t, x, S) = δ(t)f(x)$  for any  $(t, x) \in$ <br>hat<br>  $CV(t, x, S) \equiv 0$   $V(t, x) \in [0, ∞) × S<sup>c</sup>.$  (A.9)<br>
Since S<sup>c</sup> is open, we can take  $h > 0$  such that  $[x<sub>0</sub> - h, x<sub>0</sub> + h] \subset S<sup>c</sup>.$ <br>  $\frac{\gamma(t, x,$ [0, ∞) × *S*. Now we prove that<br>  $\Gamma$ <br>
Take  $(t, x_0) \in [0, \infty) \times S^c$ . Sin<br>
By Assumption 2.1(i) and  $V(t, \text{compact set } [t, t+1] \times \overline{B(x_0, h)}$ <br>
Applying Ito's formula to  $V(t$ <br>
to the boundedness of  $V_x \sigma$  on<br>  $\mathbb{E}^{x_0}[V(t + \varepsilon \wedge \tau_{$  $V(t, x, S) \equiv 0 \quad \forall (t, x) \in [0, \infty) \times S^c.$  (A.9)<br>  $\forall x, S \in S^c \text{ is open, we can take } h > 0 \text{ such that } [x_0 - h, x_0 + h] \subset S^c.$ <br>  $(x, x, S) \in C^{1,2}([0, \infty) \times \overline{S^c}), (s, x) \mapsto \mathcal{L}V(s, x, S) \text{ is continuous on the }$ <br>  $\exists y, T \in \mathbb{R}^n$ ,  $\exists y, T \in \mathbb{R}^n$ ,  $|\mathcal{L}V(s, x, S)| < \infty.$  Take  $(t, x_0) \in [0, \infty) \times S^c$ . Since  $S^c$  is open, we can take  $h > 0$  such that  $[x_0 - h, x_0 + h] \subset S^c$ . (*t*, *x*<sub>0</sub>) ∈ [0, ∞) × *S<sup>c</sup>*. Since *S<sup>c</sup>* is open, we can take *h* > 0 such that  $[x_0 - h, x_0 + h]$  C *S<sup>c</sup>*.<br>sumption 2.1(i) and *V*(*t*, *x*, *S*) ∈  $C^{1,2}([0, \infty) \times S^c)$ ,  $(s, x) \mapsto LV(s, x, S)$  is continuous on the<br>act set [ By Assumption 2.1(i) and  $V(t, x, S) \in C^{1,2}([0, \infty) \times \overline{S^c})$ ,  $(s, x) \mapsto LV(s, x, S)$  is continuous on the compact set  $[t, t + 1] \times B(x_0, h)$ . Then,

$$
\sup_{(s,x)\in[t,t+1]\times\overline{B(x_0,h)}}|\mathcal{L}V(s,x,S)|<\infty.
$$
\n(A.10)

Applying Ito's formula to  $V(t + s, X_s, S)$  and taking expectation, the diffusion term vanishes due to the boundedness of  $V_x \sigma$  on  $[t, t + 1] \times B(x_0, h)$ , we have that

y Assumption 2.1(i) and 
$$
V(t, x, S) \in C^{1,2}([0, \infty) \times \overline{S^c})
$$
,  $(s, x) \mapsto \mathcal{L}V(s, x, S)$  is continuous on the  
\ncompact set  $[t, t + 1] \times \overline{B(x_0, h)}$ . Then,  
\n
$$
\sup_{(s,x) \in [t, t+1] \times \overline{B(x_0, h)}}
$$
 $|\mathcal{L}V(s, x, S)| < \infty.$  (A.10)  
\n
$$
\text{pplying Ito's formula to } V(t + s, X_s, S) \text{ and taking expectation, the diffusion term vanishes due\nt the boundedness of  $V_x \sigma$  on  $[t, t + 1] \times \overline{B(x_0, h)}$ , we have that  
\n
$$
\mathbb{E}^{x_0}[V(t + \varepsilon \wedge \tau_{B(x_0, h)}, X_{\varepsilon \wedge \tau_{B(x_0, h)}}, S)] - V(t, x_0, S) = \mathbb{E}^{x_0} \left[ \int_0^{\varepsilon \wedge \tau_{B(x_0, h)}} \mathcal{L}V(t + s, X_s, S) ds \right].
$$
 (A.11)  
\nleanwhile, by the continuity of  $\mathcal{L}V(s, x, S)$  on  $[t, t + 1] \times \overline{B(x_0, h)},$   
\n
$$
\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_0^{\varepsilon \wedge \tau_{B(x_0, h)}} \mathcal{L}V(t + s, X_s, S) ds = \mathcal{L}V(t, x_0, S), \quad \mathbb{P}^{x_0} \text{-a.s.}.
$$
$$

Meanwhile, by the continuity of  $LV(s, x, S)$  on  $[t, t + 1] \times B(x_0, h)$ ,

$$
\mathbb{E}^{x_0}[V(t+\varepsilon \wedge \tau_{B(x_0,h)}, X_{\varepsilon \wedge \tau_{B(x_0,h)}}, S)] - V(t, x_0, S) = \mathbb{E}^{x_0} \left| \int_0^{\infty} \mathcal{L}V(t+s, \varepsilon) \right|
$$
  
canwhile, by the continuity of  $\mathcal{L}V(s, x, S)$  on  $[t, t+1] \times \overline{B(x_0, h)},$   

$$
\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_0^{\varepsilon \wedge \tau_{B(x_0, h)}} \mathcal{L}V(t+s, X_s, S) ds = \mathcal{L}V(t, x_0, S), \quad \mathbb{P}^{x_0} \text{-a.s.}.
$$

Thanks to Equation [\(A.10\)](#page-43-0), we can apply the dominated convergence theorem to above equality and get that

$$
\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \mathbb{E}^{x_0} \left[ \int_0^{\varepsilon \wedge \tau_{B(x_0, h)}} \mathcal{L} V(t+s, X_s, S) ds \right] = \mathcal{L} V(t, x_0, S). \tag{A.12}
$$

On the other hand, for any  $\varepsilon > 0$ , it is obvious that

$$
\mathbb{E}^{x_0}[V(t+\varepsilon \wedge \tau_{B(x_0,h)}, X_{\varepsilon \wedge \tau_{B(x_0,h)}}, S)] = \mathbb{E}^{x_0}[\delta(t+\rho_S)f(X_{\rho_S})] = V(t, x_0, S). \tag{A.13}
$$

Then by Equations [\(A.11\)](#page-43-0)–(A.13), we have that

$$
0 = \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \Big( \mathbb{E}^{x_0} [V(t + \varepsilon \wedge \tau_{B(x_0, h)}, X_{\varepsilon \wedge \tau_{B(x_0, h)}}, S)] - V(t, x_0, S) \Big) = \mathcal{L}V(t, x_0, S),
$$

and thus Equation [\(A.9\)](#page-43-0) holds.

**Part (b)**: The existence of  $LV(t, x \pm, S)$  on  $[0, \infty) \times \mathbb{X}$  follows from part (a), the differentiability of  $\delta$  and Assumption 2.8(ii). Take  $x_0 \in \mathbb{X}$  and  $h > 0$  such that  $[x_0 - h, x_0 + h] \subset \mathbb{X}$ . We show that

$$
\sup_{(t,x)\in[0,\infty)\times\overline{B(x_0,h)}}|\mathcal{L}V(t,x-,S)|<\infty,\tag{A.14}
$$

 $\mathcal{L}V(t, x_0, S).$  (A.12)<br>  $f(X_{\beta_S})] = V(t, x_0, S).$  (A.13)<br>  $(t, x_0, S) = \mathcal{L}V(t, x_0, S),$  (A.13)<br>  $(t, x_0, S) = \mathcal{L}V(t, x_0, S),$ <br>
ws from part (a), the differentiability<br>  $[x_0 - h, x_0 + h] \subset \mathbb{X}$ . We show that<br>  $( $\infty$ ,  
\n $\rightarrow$  \mathcal{$  $E^{*2}[V(t + \varepsilon \wedge \tau_{NC_0,h}), X_{\varepsilon \wedge NC_0,h}, Y_{\varepsilon \wedge NC_0,h}, Y_{\varepsilon})] = E^{*2}[\delta(t + \rho_S)f(X_{\rho_S})] = V(t, x_0, S).$ Equations (A.11)-(A.13), we have that<br>  $0 = \lim_{\varepsilon \wedge 0} \frac{1}{\varepsilon} \left\{ F^{*2}[V(t + \varepsilon \wedge \tau_{RC_0,h}), X_{\varepsilon \wedge \tau_{RC_0,h}}) S] - V(t, x_0, S) \right\}$  $V(t, x, \pm, S)$  on [0, ∞) × X follows from part (a), the differentiability<br>kk  $x_0 \in X$  and  $h > 0$  such that  $[x_0 - h, x_0 + h] \subset X$ . We show that<br> $(x_0) \in \text{Lip}(x, x -, S)| < \infty$ , (A.14)<br>follows from a similar argument. Let  $x \in B(x_0, h)$  $\delta$  and Assumption 2.8(ii). Take  $x_0 \in \mathbb{X}$  and *h* > 0 such that  $|x_0 - h, x_0 + h] \subset \mathbb{X}$ . We show that  $\frac{0.000 \text{ m/s} - 0.000 \text{ m/s} - 0.000 \text{ m/s}^2}{(0.000 \text{ m/s})(0.000 \text{ m/s}^2)(0.000 \text{ m})}$  (A.14) (*0.7*(1(*x* + , *k*) V(t, x-, S)| < ∞, (A.14)<br>
illar argument. Let  $x \in B(x_0, h)$ . If  $(x - h', x) \in$ <br>
thinuity of  $y \mapsto LV(t, y-, S)$  at  $y = x$  and Equa-<br>
then wise, since S is admissible, there exists  $\bar{h}$  exhat  $V(t, x, S) = \delta(t) f(x)$  on  $[0, \infty) \times (x - \bar{h$ and the result for  $\mathcal{L}V(t, x + S)$  follows from a similar argument. Let  $x \in B(x_0, h)$ . If  $(x - h', x) \in$ Equa-<br> $\bar{h} \in \bar{h}, x$ ),<br>i) and tion [\(22\)](#page-9-0) in part (a), we have  $LV(t, x-, S) = 0$ . Otherwise, since S is admissible, there exists  $\bar{h} \in$ and we have that

$$
\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \mathbb{E}^{x_0} \left[ \int_0^{\varepsilon x_1(x_0, h)} \mathcal{L}V(t+s, X_s, S) ds \right] = \mathcal{L}V(t, x_0, S). \tag{A.12}
$$
\nOn the other hand, for any  $\varepsilon > 0$ , it is obvious that\n
$$
\mathbb{E}^{x_0}[V(t+\varepsilon \wedge \tau_{B(x_0, h)}, X_{\varepsilon \wedge \tau_{B(x_0, h)}}, S)] = \mathbb{E}^{x_0}[\delta(t+\rho_S)f(X_{\rho_S})] = V(t, x_0, S). \tag{A.13}
$$
\nThen by Equations (A.11)–(A.13), we have that\n
$$
0 = \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \left[ \mathbb{E}^{x_0}[V(t+\varepsilon \wedge \tau_{B(x_0, h)}, X_{\varepsilon \wedge \tau_{B(x_0, h)}}, S)] - V(t, x_0, S) \right] = \mathcal{L}V(t, x_0, S),
$$
\nand thus Equation (A.9) holds.\nPart (b): The existence of  $\mathcal{L}V(t, x \pm, S)$  on  $[0, \infty) \times X$  follows from part (a), the differentialility\nof  $\delta$  and Assumption 2.8(ii). Take  $x_0 \in X$  and  $h > 0$  such that  $[x_0 - h, x_0 + h] \subset X$ . We show that\n
$$
\sup_{(t,x) \in [0,\infty) \times B(x_0, h)} | \mathcal{L}V(t, x-, S) | < \infty, \tag{A.14}
$$
\nand the result for  $\mathcal{L}V(t, x+, S)$  follows from a similar argument. Let  $x \in B(x_0, h)$ . If  $(x - h', x) \in S$  for some constant  $h' > 0$ , then by the left continuity of  $y \mapsto \mathcal{L}V(t, y-, S)$  at  $y = x$  and Equation (22) in part (a),  $Y \in \mathcal{L}V(t, x-, S) = 0$ . Otherwise, since  $S$  is admissible, there exists  $h \in [0, 0, 0] \times (x - h, x)$ ,  
\nand we have that\n

 $\frac{1}{B(x_0)}$ <br>b fol  $.14$  $\frac{1}{2}$ <br>Sove follow<br>(A.14) h o<br>from:<br>ls.  $(0)$ || $J(y)$ | +  $|D(y)$ ]<br>bm the first inequali  $(y -)$ | +  $\frac{1}{2}$  $\frac{1}{2}$  $2$  (y)| $J'$  (y)| $J'$  (y)| $J'$  $\frac{1}{2}$  and  $\frac{1}{2}$ . where the inequality above follows from the first inequality in Lemma 2.7, Assumptions 2.1(i) and [2.8\(](#page-7-0)ii). Hence, Equation (A.14) holds.