Stretched Root Systems and the Geometry of Shard Modules

by

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To everyone in the Acknowledgements section

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ABSTRACT

A classic theorem of Gabriel states that, for a finite type Dynkin diagram G, the indecomposable representations of any quiver orienting G are in bijection with reflecting hyperplanes of the associated Coxeter group W. This is the starting point for a rich web of connections between the representation theory of algebras and the combinatorics and geometry of Coxeter groups.

Recent work of Iyama, Reading, Reiten, and Thomas constructs a similar correspondence between brick modules of the preprojective algebra Π_G for a finite type Dynkin diagram G and a combinatorially useful partition of the hyperplanes into cones called shards. A paper of the author, Speyer, and Thomas generalizes this beyond finite type Dynkin diagrams by defining a class of bricks of Π_G called shard modules which correspond to shards for arbitrary diagrams G. Although harder to understand than in the finite type case, shard modules provide a potential categorical tool for studying infinite Coxeter groups and cluster algebras.

In this thesis, we study how the relative position of shards affects the properties of their associated shard modules. We generalize beyond finite type a result of Iyama, Reading, Reiten, and Thomas showing that, when three shards meet in a certain configuration, their shard modules fit into a short exact sequence. We pay specific attention to "stretched" families of graphs obtained by inserting a path into a fixed diagram, describing recurring structure in the shards as the path grows. We use this structure to generalize patterns appearing in the shard modules for the A_n and D_n families of diagrams to any family of diagrams with tails.

CHAPTER I
Introduction

I.1: What this thesis is about, in 3 paragraphs

The principal objects of study of this thesis are **quiver representations**. A **quiver** is a diagram with points called **vertices** connected by arrows[1]. A **representation** of that quiver assigns to each arrow a simple rule for transforming data at its starting vertex into data at its end. For example, the quiver $\bullet \to \bullet$ has a representation

$$k^2 \xrightarrow{(3\ 2)} k$$

which expresses the rule "given two coordinates x and y, evaluate the combination $3x + 2y$".

For a special class of quiver representations called **shard modules**, the different ways they can behave are classified by **shards**, a highly symmetrical arrangement of wedges in space. The simplest example of this is shown in Figure 1. One can think of the arrangement of shards as a kind of "map" of the shard modules (in the cartographic rather than mathematical sense), with each shard the realm of a different type of behavior. The question that motivates this thesis is: how does a shard module's location on this map inform the details of its inner workings, and its relationships with its neighbors? This turns out to be much easier to understand than actual geopolitics, but it's still pretty tough.

In this thesis, we show that whenever shards meet in a configuration like the one in Figure 1, the modules associated to the shards on the left and right can each be obtained by combining together the modules associated to the two shards cutting through. We also study what the arrangement of shards looks like in the particular case of a quiver with a long tail, such as the one shown here:

[1] It's called a quiver because it's full of arrows. It took the author 2 years to get this pun, and we point it out now to prevent the reader experiencing a similar fate.

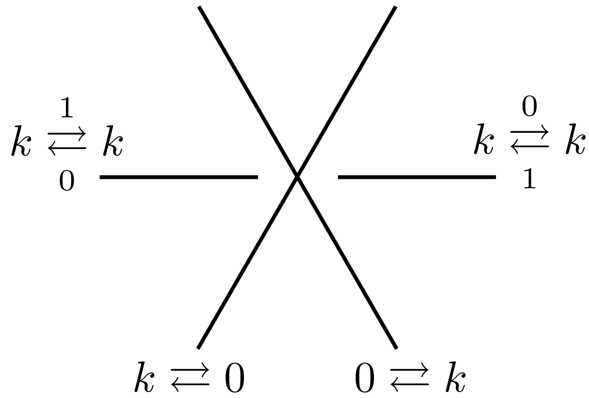
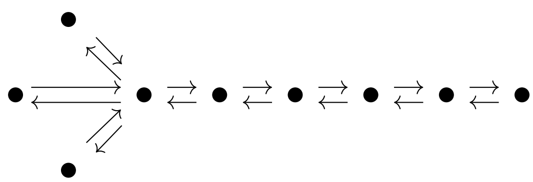


Figure 1: The shard modules of the quiver $\bullet \rightleftarrows \bullet$, and the shards they correspond to. In a sense, the goal of this entire thesis is to generalize the patterns in this picture to more complicated contexts.



Finally, we put these things together and show that, for a shard module with a tail, we can make some precise statements about how the arrows along the tail behave based on where its shard is.

I.2: The backstory: quiver representations, root systems, and shard modules

I.2.1: Quiver representations

Let k be any field. Throughout this thesis, vector spaces are assumed to be finite-dimensional over k unless stated otherwise. In particular, when we refer to the category of modules over a k-algebra, we mean finite-dimensional modules.

The story behind this thesis traces back to a 1972 theorem of Gabriel [Gab72]. In turn, that result stems from a basic theorem of linear algebra.

Theorem I.1. *Let $f : V \to W$ be a linear map of vector spaces. Then we can choose bases* *for V and W such that the matrix of f with respect to those bases has the block form*

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

where I is an identity matrix.

Essentially, this theorem describes the fundamentally different ways a map between two otherwise unrelated vector spaces can behave. In particular, it shows that, up to changes of coordinates on either side, a map's only distinguishing feature is its rank — the size of the identity matrix appearing in this block form.

There are a couple of natural directions for generalizing this theorem. On one hand, we could place extra structure on the vector spaces involved. For example, if V and W both have inner products and we restrict ourselves to choosing orthonormal bases, the different ways linear maps can behave are instead classified by singular value decomposition. On the other hand, we could consider multiple maps between plain vector spaces. For example, in a configuration such as

$$U \xrightarrow{f} V \xrightarrow{g} W$$

is it possible to choose bases of U, V, and W such that the maps f and g simultaneously assume some sort of nice canonical form? Is there a statistic like rank that fully describes such a pair of maps up to changes of coordinates?

In order to answer this question, we introduce the language of quiver representations. A **quiver** Q consists of a finite set Q_0 of vertices and a finite set Q_1 of arrows, where each arrow a joins two vertices, its **source** $s(a)$ and its **target** $t(a)$. Then a **representation** M of a quiver over a field k assigns a k-vector space M_i to each vertex i and to each arrow a assigns a linear map $M_{s(a)} \to M_{t(a)}$. In this language, a map between unrelated vector spaces is a representation of $\bullet \to \bullet$, while a chain of 2 linear maps $U \xrightarrow{f} V \xrightarrow{g} W$ is a representation of $\bullet \to \bullet \to \bullet$.

Quiver representations are a useful framework for linear algebra problems because they form a category. A **morphism** $\varphi : M \to N$ of two representations of the same quiver is a collection of linear maps $\varphi_i : M_i \to N_i$ for each vertex i which are compatible with the maps along the arrows. Specifically, we require that, for each arrow a, the square

$$\begin{tikzcd} M_{s(a)} \arrow[r, "M(a)"] \arrow[d, "\varphi_{s(a)}"] & M_{t(a)} \arrow[d, "\varphi_{t(a)}"] \\ N_{s(a)} \arrow[r, "N(a)"] & N_{t(a)} \end{tikzcd}$$

commutes.

In particular, we can rephrase Theorem I.1 in this language. Choosing bases of V and W is equivalent to choosing isomorphisms $\varphi_1 : V \xrightarrow{\sim} k^r$ and $\varphi_2 : W \xrightarrow{\sim} k^s$. When we say that a linear map $f : V \to W$ is given by a matrix A in this basis, that means that the diagram

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\varphi_1"] & W \arrow[d, "\varphi_2"] \\ k^r \arrow[r, "A"] & k^s \end{tikzcd}$$

commutes. By definition, this means that $V \xrightarrow{f} W$ and $k^r \xrightarrow{A} k^s$ are isomorphic. So Theorem I.1 becomes:

Theorem I.2. *Every representation of* $\bullet \to \bullet$ *is isomorphic to one of the form*

$$k^r \xrightarrow{\left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)} k^s$$

for an identity matrix I*.*

From this perspective, there is a clear generalizing question:

Question I.3. *For which quivers can we classify their representations up to isomorphism?*

In any classification question like this, it's useful to have a way of breaking objects down into simpler pieces. For quiver representations, this is done with **direct sum**: given two representations M and N of a quiver, the direct sum $M \oplus N$ associates to each vertex i the space $M_i \oplus N_i$ and to each arrow a the direct sum map $M(a) \oplus N(a)$, which we can view as a block matrix $\left(\begin{smallmatrix} M(a) & 0 \\ 0 & N(a) \end{smallmatrix}\right)$.

In particular, the representation

$$k^r \xrightarrow{\text{id}} k^r$$

can be viewed as a direct sum of r copies of $k \xrightarrow{1} k$, and more generally, any representation of the form

$$k^r \xrightarrow{\left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)} k^s$$

can be obtained by summing copies of $k \xrightarrow{1} k$, $k \to 0$, and $0 \to k$ (as summing with the latter two pads out the matrix with zero columns and rows). This makes our classification a little more elegant.

Theorem I.4. *Every representation of* $\bullet \to \bullet$ *is isomorphic to a direct sum of copies of* $k \xrightarrow{1} k$*,* $k \to 0$*, and* $0 \to k$*.*

Finally, just as it's often useful to have an operation of decomposition, it's useful to study the atomic objects which can't be decomposed further. Accordingly, we say a representation is **indecomposable** if it is not isomorphic to a direct sum of two nonzero representations. For example, the three representations appearing in Theorem I.4 are indecomposable. In the case of $k \to 0$ or $0 \to k$, this is just because the vector space k is indecomposable. In the case of $k \xrightarrow{1} k$, the only nontrivial direct sum that has the correct dimension at each vertex is $(k \to 0) \oplus (0 \to k) \cong k \xrightarrow{0} k$, and this is not isomorphic to $k \xrightarrow{1} k$ because rescaling the domain and codomain of a zero map cannot produce a nonzero map.

On the other hand, Theorem I.4 says that every other module can be decomposed into these three, so we arrive at one last rephrasing of the theorem we started with:

Theorem I.5. *The indecomposable representations of* $\bullet \to \bullet$, *up to isomorphism, are* $k \xrightarrow{1} k$, $k \to 0$, *and* $0 \to k$.

And our motivating question becomes:

Question I.6. *For which quivers can we classify the indecomposable representations up to isomorphism?*

In particular, some desirable features of the theorem we started with hinge on the quiver $\bullet \to \bullet$ having *finitely many* (namely 3) indecomposable representations. We say that Q is **of finite representation type**. For example, in this perspective the all-important statistic of rank just measures how many copies of $k \xrightarrow{1} k$ appear in the decomposition of a representation; in general, if a quiver is of finite representation type, then any representation can be boiled down to a finite list of nonnegative integers this way. So we'd like to know which quivers have this property. This is the question that Gabriel's theorem answers.

Theorem I.7 ([Gab72]). *A quiver is of finite representation type if and only if, ignoring the direction of its arrows, it is a disjoint union of the following graphs:*

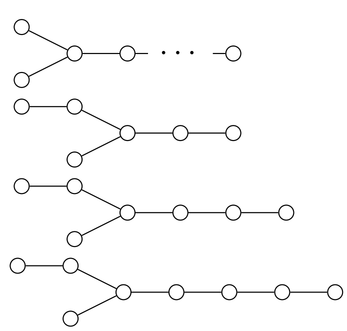
A_n

D_n

E_6

E_7

E_8



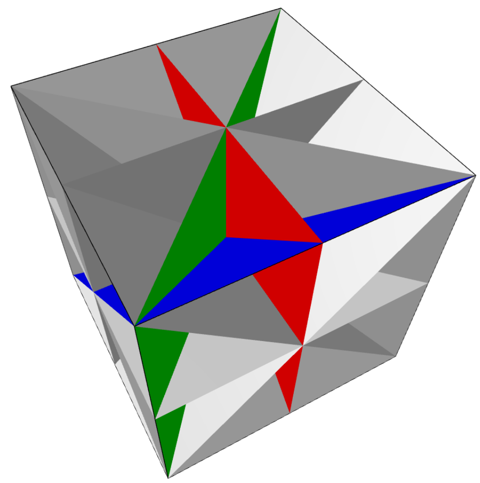


Figure 2: The reflection symmetries of a cube, illustrated here by the planes they fix, generate a Coxeter group. In fact, just the reflections over the three planes in color up front suffice to generate the group.

This classification has two features of immediate interest. The first is that the property of finite representation type is independent of the orientation of arrows in the quiver. The second is this particular list of graphs, which is far from unique to this situation: they are the **simply laced finite type Dynkin diagrams**, which appear in contexts ranging from Lie theory to cluster algebras to singularities of algebraic surfaces. In particular, these diagrams are also involved in the classification of finite Coxeter groups, whose connection to quiver representations runs much deeper. We now introduce their side of the story.

I.2.2: Coxeter groups and root systems

A **Coxeter group** is essentially a group generated by reflections. By a **reflection**, we mean a linear transformation $s : V \to V$ which is an involution ($s^2 = 1$) and fixes a hyperplane (a codimension-1 subspace) pointwise. Classic motivational examples include the symmetry groups of regular polytopes; each symmetry group is generated by reflection symmetries, which can be visualized in terms of the planes of symmetry they fix, as shown in Figure 2.

In general, we can define a Coxeter group using a **Cartan matrix**. This is an $n \times n$ matrix A such that:

- $A_{ii} = 2$ for all i;
- $A_{ij} \leq 0$ for all $i \neq j$;
- $A_{ij} = 0$ if and only if $A_{ji} = 0$;

- either $A_{ij}A_{ji} = 4\cos^2(\pi/m)$ for some integer m, or $A_{ij}A_{ji} \geq 4$.

For applications to representation theory, one generally also assumes that the entries of A are integers, in which case we say that the matrix is **crystallographic**.

Given a Cartan matrix, we set up a vector space V with a distinguished basis $\alpha_1, \ldots, \alpha_n$, and we define linear transformations $s_1, \ldots, s_n : V \to V$ in terms of this basis by

$$s_i(\alpha_j) = \alpha_j - A_{ij}\alpha_i$$

These are the reflections which generate the Coxeter group W. For example, in the case of the symmetry group of a regular polytope, we have an underlying inner product $(-,-)$, and each reflection will have the form

$$s(x) = x - 2\frac{(\alpha, x)}{(\alpha, \alpha)}\alpha$$

for a vector α normal to the hyperplane of symmetry. If we look at the arrangement of all hyperplanes of symmetry, just the reflections over hyperplanes bounding a single region of this arrangement (as shown in Figure 2) will suffice to generate the group. In this case, we choose the α_i to be normal vectors to these hyperplanes pointing toward the region, and the Cartan matrix is defined by

$$A_{ij} = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$$

In general, we can heuristically think of the A_{ij} as tracking the angles between the generating reflections.

Of course, our generating reflections may generate additional reflections — for any $w \in W$, ws_iw^{-1} is also a reflection. We keep track of all the reflections using a **root system**: we declare the basis elements $\alpha_1, \ldots, \alpha_n$ to be **simple roots**, and then say that a **root** is any element of the form $w\alpha_i$ for $w \in W$. It turns out that any root is either a nonnegative or nonpositive linear combination of simple roots, and in the former case we say it is a **positive root**. In the case of the symmetry group of a polytope, just as the α_i are normal vectors to the hyperplanes fixed by our generating reflections, the root system consists of two normal vectors (a positive root and a negative root) for *every* reflection.

The classification of finite root systems (which correspond to finite Coxeter groups) is of particular interest. For this purpose, we typically visualize the data of a Cartan matrix in the more compact form of a **Dynkin diagram**. For an $n \times n$ matrix, the Dynkin diagram is a graph with vertices labeled $1, \ldots, n$. If $A_{ij} = A_{ji} \neq 0$, we draw $-A_{ij}$ edges between vertices i and j, and if $A_{ij} \neq A_{ji} \neq 0$ for $i < j$ we draw a single edge and label it with the pair $(-A_{ij}, -A_{ji})$. An example (in the case of the cube) is shown in Figure 3.

$$\begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

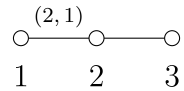
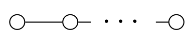
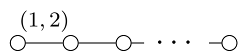


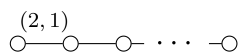
Figure 3: On the left is a Cartan matrix for the symmetry group of the cube. On the right is its Dynkin diagram.

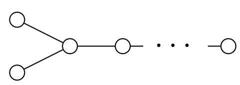
With this notation, we can state the classification.

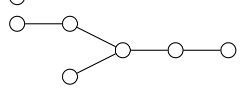
Theorem I.8 ([Hum72])**.** *The finite crystallographic root systems are exactly those associated to the following Dynkin diagrams (and disjoint unions of them):*

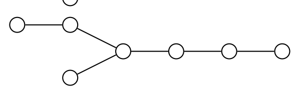
A_n 

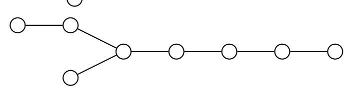
B_n 

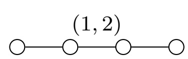
C_n 

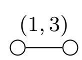
D_n 

E_6 

E_7 

E_8 

F_4 

G_2 

If we restrict our focus to the unlabeled diagrams on this list — which correspond to symmetric Cartan matrices — we see exactly the same list as the one appearing in Gabriel's theorem! In fact, not only can we use root systems to classify the quivers of finite type, we can also use them to classify the indecomposable representations themselves, as we now explain.

If M is a representation of a quiver Q, we define the **dimension vector** $\underline{\dim} M := (\dim M_i)_{i \in Q_0}$. If we forget the orientation of Q and view it as a Dynkin diagram, its vertices can be identified with the simple roots of the root system, and we can write the dimension vector as $\sum_{i=1}^{n} (\dim M_i)\alpha_i$. We now state the second part of Gabriel's theorem.

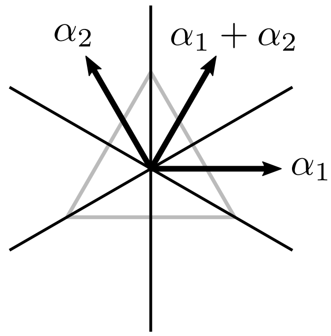


Figure 4: The three positive roots of the symmetry group of a triangle.

Theorem I.9 ([Gab72])**.** *Suppose a quiver Q is of finite representation type. Then the map $M \mapsto \underline{\dim}\, M$ gives a bijection between isomorphism classes of indecomposable representations and positive roots of the root system whose Dynkin graph is the undirected graph underlying Q.*

For example, the symmetry group of a triangle has Dynkin diagram $\circ\!\!-\!\!\!-\!\!\circ$ and three positive roots normal to its three lines of symmetry, as shown in Figure 4. In the basis of simple roots, these are α_1, α_2, and $\alpha_1+\alpha_2$ — which corresponds to the three indecomposable representations of $\bullet \to \bullet$: $k \to 0$, $0 \to k$, and $k \xrightarrow{1} k$.

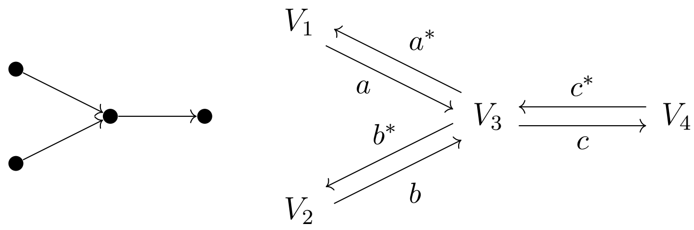
At first glance, quiver representations and root systems appear unrelated. The key connection between them is a powerful collection of operations called **reflection functors**, which transform representations into new ones while, on the level of dimension vectors, acting by reflections.

Thus, not only do root systems play a critical role in the classification of quiver representations, quiver representations "categorify" root systems — all the information of a root system is there in the dimension vectors, together with rich additional structure provided by representation theory.

Once it's clear how close the connection between these two fields is, a new line of inquiry emerges: are there other structures appearing in the context of Coxeter groups which have counterparts in the world of quiver representations, or vice versa?

I.2.3: The preprojective algebra

One way to refine this correspondence involves bringing the orientation of quivers into the picture. As noted above, both parts of Gabriel's theorem are independent of orientation. Thus, if we pick a specific quiver and use its representations to study the appropriate root system, we're making a seemingly arbitrary choice, and the results may not tell us the full



$$a^*a = b^*b = -cc^* = -aa^* - bb^* + c^*c = 0$$

Figure 5: Starting from the quiver on the left, modules over the preprojective algebra are given by representations of the form on the right obeying the relations shown.

story.

A more natural approach is to somehow consider all orientations at once. Starting with a quiver Q, for every arrow $s(a) \xrightarrow{a} t(a)$ we add a reverse arrow $t(a) \xrightarrow{a^*} s(a)$, creating a **double quiver** $\overline{Q}$. We then define a **module over the preprojective algebra** or Π_Q**-module** to be a representation M of the double quiver which satisfies the relation

$$\sum_{\substack{\text{arrows}\\ i \xrightarrow{a} \cdot}} M(a^*)M(a) = \sum_{\substack{\text{arrows}\\ \cdot \xrightarrow{a} i}} M(a)M(a^*) \quad \text{for each vertex } i$$

An example of this is shown in Figure 5.

We'll be able to say a little more about why we impose this relation in Section V.2.2. For the moment, we note that this does unify all the different orientations we could have chosen in the following two ways. First, the category of modules we get is independent of the orientation we started with: starting with a different orientation merely flips some signs in the defining relations and produces an equivalent category. We could (and will) just as well talk about Π_G-modules for an undirected graph G. Second, if we pick any quiver Q' obtained by orienting the arrows of Q differently and any representation M of Q', we can realize M as a Π_Q-module: we attach the maps of M to the appropriate arrows in the double quiver $\overline{Q}$, and have all the other arrows of $\overline{Q}$ act by 0. This is illustrated in Figure 6. The resulting representation certainly satisfies the above relations, because each individual term is 0 in this case. (However, there will be modules besides these ones, which are not tied to a specific orientation.)

The preprojective algebra has appeared in several other contexts in which it turns out to be the "correct" orientation-agnostic variation on the representation theory of quivers. It was introduced as an actual algebra Π_Q by Gel'fand and Ponomarev [GP79], who showed

$$k \xrightarrow{1} k \xrightarrow{1} k \quad \Rightarrow \quad k \underset{0}{\overset{1}{\rightleftarrows}} k \underset{0}{\overset{1}{\rightleftarrows}} k$$

$$k \xrightarrow{1} k \xleftarrow{1} k \quad \Rightarrow \quad k \underset{0}{\overset{1}{\rightleftarrows}} k \underset{1}{\overset{0}{\rightleftarrows}} k$$

Figure 6: A representation of any orientation of a quiver can be realized as a module over the preprojective algebra, as shown in two examples here.

that it breaks down as a direct sum of the important class of preprojective representations of the quiver Q. It has since been useful in category-theoretic constructions of concepts from Lie theory. Lusztig used Π_G-modules to construct a well-behaved basis (called the "canonical basis") of part of the universal enveloping algebra of the Lie algebra associated to G [Lus91]. Geiß, Leclerc, and Schröer similarly used Π_G-modules to construct cluster algebras [GLS06].

The connection between Coxeter groups and quiver representations also has a parallel for preprojective algebras. It hinges not on a classification of all indecomposable modules, but instead an important subclass. We say that a module is a **brick** if any nonzero endomorphism is invertible. Bricks are indecomposable, because any decomposable module $M \oplus N$ admits a noninvertible endomorphism $M \oplus N \to M \to M \oplus N$; however, being a brick is a stronger condition.

Just as indecomposable representations of a quiver correspond to roots of a root system, bricks of a preprojective algebra correspond to a more refined structure on the side of Coxeter groups called shards.

I.2.4: Shards

To define shards, it helps to switch from the discussion of root systems to a dual picture. Recall that V is the vector space spanned by the simple roots, on which we originally defined the action of a Coxeter group W. Then there is also an action on the dual space V^*: letting $\langle -, - \rangle : V^* \times V \to k$ be the natural pairing, this action is defined by

$$\langle wx, v \rangle = \langle x, w^{-1}v \rangle \quad w \in W, x \in V^*, v \in V$$

Each root $\beta \in V$ corresponds to a hyperplane $\beta^\perp := \{x \in V^* \mid \langle x, \beta \rangle = 0\} \subset V^*$, and just like roots, these hyperplanes correspond to reflections. Specifically, consider the action of the reflection ws_iw^{-1} on V^*: we have

$$\langle ws_iw^{-1}x, v \rangle = \langle x, ws_iw^{-1}v \rangle = \langle x, w(w^{-1}v - c\alpha_i) \rangle = \langle x, v \rangle - c\langle x, w\alpha_i \rangle$$

for some number c depending on v, and so $ws_iw^{-1}x = x$ exactly when $\langle x, w\alpha_i\rangle = 0$. This shows that the hyperplane $(w\alpha_i)^\perp$ is the subset fixed pointwise by ws_iw^{-1}, playing the same role as a plane of symmetry in the polytope example. Taken together, these hyperplanes form the **reflecting hyperplane arrangement**.

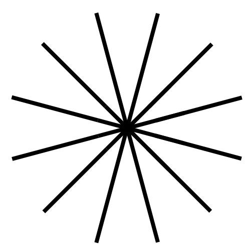
The **shards** are then defined by breaking these hyperplanes into convex cones. We roughly outline this process here and refer the reader to Chapter III for details:

- First, let $D \subset V^*$ be the set

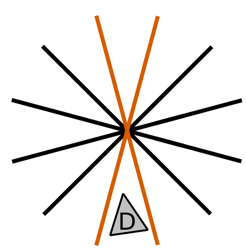
$$D = \{x \in V^* \mid \langle \alpha_i, x\rangle \geq 0, 1 \leq i \leq n\}$$

  This is a single **region** (a component of the complement of the reflecting hyperplane arrangement), with the hyperplanes $\alpha_i^\perp$ as walls.

- Then, given any two reflecting hyperplanes, consider the collection of all reflecting hyperplanes which contain their intersection, which we call a **rank 2 subarrangement**. This will be a symmetrical arrangement of hyperplanes meeting in a codimension-2 subspace:



- From the rank 2 subarrangement, we select the two adjacent hyperplanes which contain the region D between them[2], and which we call the **fundamental hyperplanes**:



- Finally, we break all of the non-fundamental hyperplanes along this codimension-2 intersection, replacing each one by half-planes on either side of the intersection:

[2]Such hyperplanes exist even if the subarrangement is infinite. For this, we refer to Proposition III.7 in Chapter III and [RS11, Theorem 2.7(i) and Proposition 2.11].

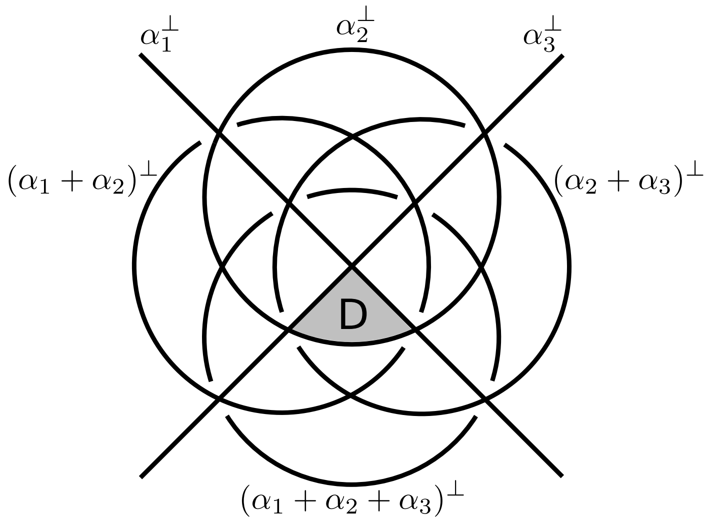
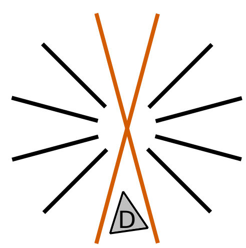


Figure 7: The stereographic projection of the shards of a tetrahedron's planes of symmetry. Note that, wherever 3 planes meet, the one furthest from the distinguished region D is broken.



When we do this for every rank 2 subarrangement, most hyperplanes will be subdivided multiple times, partitioning each one into a collection of convex cones. We refer to the resulting cones as shards.

For example, Figure 7 illustrates how the planes of symmetry of a tetrahedron are partitioned into shards. To make the picture two-dimensional, we intersect the planes with a sphere (turning them into great circles) and stereographically project them onto the plane, obtaining circles and lines. Some planes remain intact, while the bottom circle is broken into 4 shards.

Shards were originally introduced by Nathan Reading to study the lattice structure of weak ordering on the Coxeter group [Rea03], but they turn out to also classify bricks of preprojective algebras. Recall that Gabriel's Theorem uses the dimension vector of an indecomposable representation M to obtain a positive root, which in the dual picture we can identify with the hyperplane $(\underline{\dim}\, M)^\perp$. We can similarly associate shards to bricks, but since a hyperplane has multiple shards, we need to record a little more information.

Specifically, we also need to consider the brick's submodules. Given a module M, we define the **stability domain** $\mathrm{Stab}(M) \subset V^*$ to be

$$\mathrm{Stab}(M) := \{\theta \in V^* \mid \langle \theta, \underline{\dim} M \rangle = 0 \text{ and } \langle \theta, \underline{\dim} N \rangle \geq 0 \text{ for all } N \hookrightarrow M\}$$

Then there is a classification very much like Gabriel's theorem.

Theorem I.10 ([Tho18, Theorem 6]). *Suppose a quiver Q is of finite representation type. Then the map $M \mapsto \mathrm{Stab}(M)$ gives a bijection between isomorphism classes of bricks of the preprojective algebra associated to Q and shards of the reflecting hyperplane arrangement associated to the Dynkin diagram underlying Q.*

I.2.5: Shard modules

To reach the context of this thesis, we turn to a further question that has been lurking in the background of the whole discussion above: what can we say about quivers that *aren't* of finite representation type? The story here is far murkier. Most quivers have "wild" representation categories [KJ16, Chapter 7], meaning that fully classifying their indecomposable representations is effectively impossible[3]. However, there are still strong connections with Coxeter groups. In particular, the dimension vectors of indecomposable representations need no longer be roots — but if we limit our focus to representations which *do* have this property, one consequence of a theorem of Victor Kac gives the same bijection as Gabriel's theorem.

Theorem I.11 ([Kac80, Theorem 2]). *For any symmetric Cartan matrix, any orientation Q of its Dynkin diagram, and any root β of its root system, there is an indecomposable representation M with $\underline{\dim} M = \beta$ which is unique up to isomorphism.*

In a recent paper with David Speyer and Hugh Thomas [DST23], we obtain a similar result generalizing the preprojective situation. There, we likewise restrict from the class of bricks to a class of **shard modules**. A shard module of a preprojective algebra is a brick M which additionally satisfies two conditions:

- Its dimension vector $\underline{\dim} M$ is a root, as we supposed in stating Kac's theorem above. This turns out to equate to the categorical property that $\mathrm{Ext}^1(M, M) = 0$.
- Its stability domain $\mathrm{Stab}(M)$ has dimension $n-1$, where n is the number of vertices of the underlying quiver. Since the stability domain is contained in $(\underline{\dim} M)^\perp$, a subspace of dimension $n-1$, this is just saying that its dimension is as large as possible.

[3]More specifically, for a fixed wild quiver Q, *any* module category over a finite-dimensional algebra A admits a functor to the category of representations of Q which realizes A's indecomposable modules as a subclass of those of Q. As a result, a classification of representations of Q would enable the classification of representations of any finite-dimensional algebra, which isn't realistic.

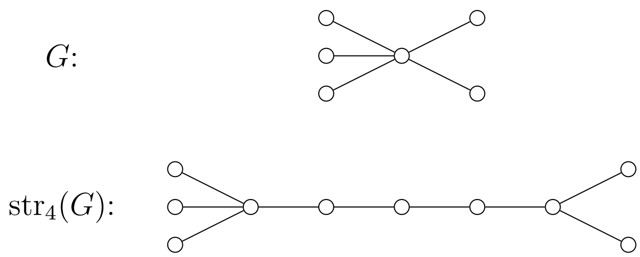


Figure 8: An example of stretching a Dynkin diagram.

Once again, we have a correspondence.

Theorem I.12 ([DST23, Theorem 5.7])**.** *Let A be any symmetric Cartan matrix with Dynkin diagram G. The map $M \mapsto \mathrm{Stab}(M)$ gives a bijection between isomorphism classes of shard modules of the preprojective algebra of G and shards of the reflecting hyperplane arrangement associated to G.*

The hope behind realizing shards with representation theory this way is that it should provide new tools for working with the weak ordering on Coxeter groups and cluster algebras, both of which become dramatically more complicated outside the finite type case. But in this thesis, we consider a more immediate question: what do these shard modules look like up close?

I.3: The themes and structure of this thesis

By the previous theorem, we know that every shard K corresponds to a unique shard module $M(K)$ with K as its stability domain. The basic motivating question here is: how does the convex geometry of K and its position within the larger arrangement of shards affect the properties of $M(K)$? We're interested both in the internal structure of $M(K)$ (i.e., what are the maps constituting it?) and its behavior in the category of Π_G-modules.

A specific setting we're interested in analyzing, both as a tractable starting point and as a natural generalization of the finite type case, is families of **stretched** Dynkin diagrams. Given an arbitrary Dynkin diagram G, together with a fixed vertex j and a partition of its neighbors into a left side and a right side, we define a stretched diagram $\mathrm{str}_m(G)$ by replacing j with a path of m edges, hooking its left and right ends up to j's left and right neighbors. This is shown in Figure 8.

To such a stretched diagram we can also associate a stretched Cartan matrix $\mathrm{str}_m(A)$ and root system $\mathrm{str}_m(\Phi)$. In doing this, we aim to mimic the infinite families of finite root systems A_n, B_n, C_n, and D_n shown in Theorem I.8, each of which can be obtained with this construction.

Several other sources have studied stretching Dynkin diagrams in this way. Among them, Richard Hepworth [Hep16] proved homological stability for families of Coxeter groups obtained by appending a tail onto an arbitrary Dynkin diagram; Victor Reiner [Rei95] computed generating functions for combinatorial statistics in any family of Coxeter groups obtained through stretching; and Chen-Krause [CK11] and Hochenegger-Kalck-Ploog [HKP19] studied relationships between categories of quiver representations (which they called, respectively, "expansion" and "A_n-insertion") which correspond to stretching the underlying quiver. The special cases of A_n, B_n, C_n, and D_n and their behavior as $n \to \infty$ have been studied in too many contexts to cite; what's special about the level of generality provided by stretching is that it does not require the resulting Coxeter groups to be finite.

Part of the appeal of this construction in our case is that the shards and shard modules of the A_n root systems are especially easy to describe. Thus, one goal is to find patterns in the shards and shard modules of $\mathrm{str}_m(G)$ and its preprojective algebra, persisting across different values of m, which somehow exhibit behavior analogous to the type A family.

We now preview what's to come in the rest of this thesis.

I.3.1: Results on root systems and shards

In Chapter II, we introduce the basic theory of Coxeter groups and root systems, starting with finite root systems and then explaining how they motivate the infinite case.

In Chapter III, we discuss more of the motivation behind shards and formally define them. In particular, we bring in a useful recursive formula for computing shards, which relates the shards of a hyperplane $\beta^\perp$ to the shards of $s_i\beta^\perp$ for a reflection s_i. This will enable arguments by induction which form the backbone of the rest of the thesis.

Chapter IV studies the operation of stretching and its interaction with shards purely combinatorially, featuring results that originally appeared in [Dan21]. We first show how to relate the root systems $\mathrm{str}_m(\Phi)$ for different values of m: if stretching replaces the vertex j with vertices $j_0, \ldots, j_m$, then every root $\beta \in \Phi$ has a counterpart $\mathrm{str}_m(\beta) \in \mathrm{str}_m(\Phi)$ obtained by replacing α_j with $\alpha_{j_0} + \ldots + \alpha_{j_m}$. We then proceed to our main goal of the section, showing that for a root $\beta \in \Phi$ and sufficiently large m, the shards of $\mathrm{str}_m(\beta)^\perp$ admit a uniform description independent of m:

Theorem I.13 (Theorem IV.13)**.** *Let $\beta \in \Phi$ be a positive root. Then there exist:*

- *a nonnegative integer r;*
- *two lists of linear forms $f_1, \ldots, f_s$ and $g_1, \ldots, g_t$*

such that for m sufficiently large, the shards of $\mathrm{str}_m(\beta)^\perp$ *are cut out by its intersections with the hyperplanes*

$$\begin{aligned} f_u = 0 \quad & 1 \leq u \leq s \\ g_u - x_r - x_{r+1} - \ldots - x_v = 0 \quad & 1 \leq u \leq t, r+1 \leq v \leq m-r-1 \end{aligned}$$

where $x_v := \langle -, \alpha_v \rangle$.

In particular, this has an enumerative consequence:

Corollary I.14 (Corollary IV.20)**.** *The number of shards of* $\mathrm{str}_m(\beta)^\perp$ *is a linear combination of exponential functions of m.*

We draw parallels to the case of A_n, where the hyperplane $(\alpha_1 + \ldots + \alpha_n)^\perp$ has 2^{n-1} shards.

We make particular note of the case that all of j's neighbors are on the left, in which case $\mathrm{str}_m(G)$ is obtained by just appending a tail onto j. In this case we can make a stronger statement. Suppose that the vertices of the tail are grouped into $c+1$ blocks, with the pth block containing $m_p + 1$ vertices for $0 \leq p \leq c$. Let $\underline{m}$ denote the tuple $(m_0, m_1, \ldots, m_p)$, and for any root $\beta \in \mathrm{str}_c(\Phi)$ define the root $\mathrm{str}_{\underline{m}}(\beta)$ by replacing α_{j_p} with the sum of the simple roots associated to the pth block. For example, if $c = 2$ and $\underline{m} = (2, 1, 2)$,

$$\mathrm{str}_{\underline{m}}(5\alpha_{j_0} + 3\alpha_{j_1} + 2\alpha_{j_2}) = 5(\alpha_{j_0} + \alpha_{j_1} + \alpha_{j_2}) + 3(\alpha_{j_3} + \alpha_{j_4}) + 2(\alpha_{j_5} + \alpha_{j_6} + \alpha_{j_7}).$$

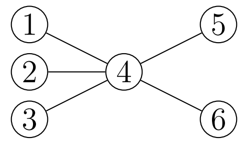
In this case, we can describe the arrangement of shards quite explicitly.

Theorem I.15 (Theorem IV.25)**.** *Let $\beta = \mathrm{str}_{\underline{m}}(\overline{\beta})$ for some $\overline{\beta} \in \mathrm{str}_c(\Phi)$. Then the shards of $\beta^\perp$ are cut out by its intersections with*

- *hyperplanes of the form $\mathrm{str}_{\underline{m}}(\overline{\gamma})^\perp$, where $\overline{\gamma}^\perp$ likewise cuts $\overline{\beta}^\perp$, together with*
- *a list of hyperplanes $(\gamma^r_{pp'})^\perp$ (to be defined in Chapter IV), indexed by $0 \leq p \leq p' \leq c$ and $1 \leq r \leq m_p$, which depends only on the tuple $\underline{m}$.*

Finally, as an aside, we analyze the **root poset** of a stretched root system from the same perspective. This is an ordering of positive roots where we say that $\beta < s_i\beta$ if, expressing both roots in the basis of simple roots, applying s_i to β increases one of its coefficients. For a given root β, we study the **downsets** $\downarrow \mathrm{str}_m(\beta)$, consisting of all roots below $\mathrm{str}_m(\beta)$ in this poset. Effectively, the downset encapsulates all minimal-length expressions of $\mathrm{str}_m(\beta)$ in terms of starting from a simple root and applying generating reflections. We obtain a characterization of this downset which is uniform in m.

Key to this characterization is the notion of a **stretching class**, a collection of roots in $\bigsqcup_m \mathrm{str}_m(\Phi)$ which matches a pattern. For example, starting from the diagram



there is a stretching class denoted

$$\begin{array}{ccccccccc} 2 & \diagdown & & & & & & \diagup & 1 \\ 2 & - & 7^* & - & 4^* & - & 2 & & \\ 2 & \diagup & & & & & & \diagdown & 1 \end{array}$$

which consists of all roots, across all stretches of the root system, of the form

$$2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 7(\alpha_{40} + \ldots + \alpha_{4(p-1)}) + 4(\alpha_{4p} + \ldots + \alpha_{4(m-1)}) + 2\alpha_{4m} + \alpha_5 + \alpha_6.$$

In general, we specify a stretching class by allowing some consecutive subset of coefficients along the stretched path to repeat any number of times, as with 7 and 4 here.

Theorem I.16 (Theorem IV.30)**.** *For a root β, there is a finite list of stretching classes such that, for all sufficiently large m, the downset $\downarrow\mathrm{str}_m(\beta)$ consists of precisely the roots of $\mathrm{str}_m(\Phi)$ belonging to these stretching classes.*

This will also have an enumerative consequence:

Corollary I.17 (Theorem IV.35)**.** *For sufficiently large m, the size of the downset $\downarrow\mathrm{str}_m(\beta)$ is a polynomial in m.*

Again, we draw parallels to the case of A_n, which has $n(n+1)/2$ positive roots.

I.3.2: Results on quiver representations

In Chapter V, we move on to the representation-theoretic background. We formally introduce the category of quiver representations and the preprojective algebra, and cite some important homological properties of the latter. We also introduce reflection functors, operations on Π_G-modules which are of fundamental importance to the connection with Coxeter groups, as they act on dimension vectors by reflections.

In Chapter VI, we introduce some basic facts about bricks and stability domains and discuss the correspondence between bricks and shards given by Theorems I.10 and I.12. We describe the details of this correspondence for the A_n Dynkin diagrams, as this is an

$$\begin{aligned} x_4 &\geq 0 \\ x_3 + x_4 &\leq 0 \\ x_2 + x_3 + x_4 &\leq 0 \\ x_1 + x_2 + x_3 + x_4 &= 0 \end{aligned}$$

$$k \leftarrow k \leftarrow k \rightarrow k$$

Figure 9: Above are the inequalities defining a shard of the A_4 reflecting hyperplane arrangement. Below is the corresponding shard module (with maps acting by 0 omitted). The color-coding shows the correspondence between inequality directions and arrow directions.

important motivational example for later results: specifically, we note that each shard of the hyperplane $(\alpha_1 + \ldots + \alpha_n)^\perp$ is determined by its position relative to $n-1$ independent hyperplanes, which can be specified by a list of $n-1$ signs, and in the associated brick these signs correspond to the orientations of arrows with nonzero maps. This correspondence is illustrated in Figure 9.

We also show in this chapter that the second part of the definition of a shard module (a stability domain of full dimension) is necessary, by exhibiting a brick whose dimension vector is a real root but whose stability domain is too small to be a shard. This is a substantial obstacle preventing Theorem I.12 from being as nice as it could be.

Finally, we explicitly describe all the shard modules of the preprojective algebra associated to a cycle graph, which is in some ways the nicest Dynkin diagram which isn't finite type. We do this by constructing a "covering functor" through which we lift modules over the preprojective algebra of the cycle to modules over the preprojective algebra of a path — the type A case, which is well-understood.

Chapter VII features our core result relating shard modules of *any* preprojective algebra based on the relative positions of their shards, generalizing a theorem of Iyama-Reading-Reiten-Thomas [IRRT18, Proposition 4.3] which was specific to the finite type case. In essence, whenever a shard is sliced at one of its walls (the codimension-1 cones forming its boundary) by two other shards, the three associated shard modules fit into a short exact sequence. More specifically:

Theorem I.18 (Theorem VII.9)**.** *Let β be a root, let K be a shard of $\beta^\perp$, and choose a wall of the cone K. Let $\gamma_1^\perp$ and $\gamma_2^\perp$ be the fundamental hyperplanes cutting $\beta^\perp$ into shards along that wall, with γ_1 and γ_2 positive roots, and suppose $\beta = c_1\gamma_1 + c_2\gamma_2$. Assume without loss of generality that K lies on the positive side of $\gamma_1^\perp$. Let L_1 and L_2 be the shards of $\gamma_1^\perp$ and $\gamma_2^\perp$, respectively, which meet K at that wall, and let $M(L_1)$, $M(L_2)$, and $M(K)$ be the associated*

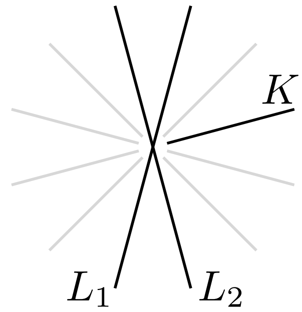


Figure 10: The configuration of shards considered in Theorem I.18, in which case there is a short exact sequence relating $M(L_1)$, $M(K)$, and $M(L_2)$.

shard modules. Then there is an exact sequence

$$0 \to M(L_1)^{\oplus c_1} \to M(K) \to M(L_2)^{\oplus c_2} \to 0.$$

At the end of this chapter, we show that $M(K)$ is in some sense the generic extension of $M(L_2)^{\oplus c_2}$ by $M(L_1)^{\oplus c_1}$.

Theorem I.19 (Theorem VII.17)**.** *The* $\mathrm{Aut}(M(L_2)^{\oplus c_2})^{\mathrm{op}} \times \mathrm{Aut}(M(L_1)^{\oplus c_1})$*-orbit in* $\mathrm{Ext}^1(M(L_2)^{\oplus c_2}, M(L_1)^{\oplus c_1})$ *of the short exact sequence in Theorem I.18 is Zariski-open, and in particular dense.*

This suggests an alternative way of reasoning recursively with shard modules — while the other arguments in this thesis typically rely on inductive proofs relating β to $s_i\beta$ for a generating reflection s_i, these theorems allow us to start with a shard module and, by picking one of its shard's walls, express it as a generic extension of smaller shard modules.

Finally, in Chapter VIII, we put this new recursive tool to use alongside the results on stretching from Chapter IV, to show how the placement of a shard is reflected in its shard module in the case of a diagram with a tail.

As before, we construct $\mathrm{str}_m(G)$ by appending a tail of m edges to a diagram G, and for a tuple $\underline{m} = (m_0, \ldots, m_c)$ we turn a root $\overline{\beta} \in \mathrm{str}_c(\Phi)$ into a root $\mathrm{str}_{\underline{m}}(\overline{\beta})$ by repeating its pth tail coefficient across a block of $m_p + 1$ vertices. Theorem I.15 gives a description of the shards of $\mathrm{str}_{\underline{m}}(\overline{\beta})^{\perp}$ in terms of the shards of $\overline{\beta}^{\perp}$ and a fixed list of additional hyperplanes $(\gamma^r_{pp'})^{\perp}$. Our goal in this chapter is to show one more instance of "type A behavior" in the relationship between a shard K and shard module $M(K)$. The full result is somewhat technical to state in words, but we give a rough statement here along with an illustration and a numerical consequence.

$$
\begin{tikzcd}
 & k \arrow[d, shift left, "\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)"] & & & & & \\
k \arrow[r, shift left, "\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)"] & k^3 \arrow[l, shift left, "\left(\begin{smallmatrix}0&0&1\end{smallmatrix}\right)"] \arrow[u, shift left, "\left(\begin{smallmatrix}0&1&0\end{smallmatrix}\right)"] \arrow[d, shift left, "\left(\begin{smallmatrix}1&0&0\end{smallmatrix}\right)"] \arrow[r, shift left, "\left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}\right)"] & k^3 \arrow[l, shift left, "\left(\begin{smallmatrix}0&1&1\\0&0&0\\0&0&0\end{smallmatrix}\right)"] \arrow[r, shift left, "\left(\begin{smallmatrix}0&1&1\\0&0&0\\0&1&1\end{smallmatrix}\right)"] & k^3 \arrow[l, shift left, "\left(\begin{smallmatrix}1&0&0\\0&1&0\\0&1&0\end{smallmatrix}\right)"] \arrow[r, shift left, "\left(\begin{smallmatrix}0&0&1\end{smallmatrix}\right)"] & k \arrow[l, shift left, "\left(\begin{smallmatrix}0\\0\\0\end{smallmatrix}\right)"] \arrow[r, shift left, "0"] & k \arrow[l, shift left, "1"] \arrow[r, shift left, "0"] & k \arrow[l, shift left, "1"] \\
 & k \arrow[u, shift left, "\left(\begin{smallmatrix}0\\0\\0\end{smallmatrix}\right)"] & & & & &
\end{tikzcd}
$$

$$
\begin{aligned}
x_{41} + x_{42} &\geq 0 \\
x_{42} &\leq 0 \\
x_{41} + x_{42} + x_{50} + x_{51} + x_{52} &\geq 0 \\
x_{42} + x_{50} + x_{51} + x_{52} &\geq 0 \\
x_{51} + x_{52} &\leq 0 \\
x_{52} &\leq 0
\end{aligned}
$$

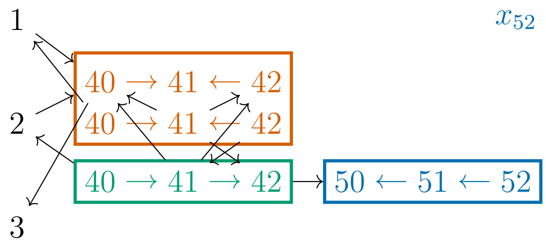


Figure 11: Above is a shard module with dimension vector $\alpha_1 + \alpha_2 + \alpha_3 + 3(\alpha_{40} + \alpha_{41} + \alpha_{42}) + (\alpha_{50} + \alpha_{51} + \alpha_{52})$. Below it are inequalities describing the position of its shard relative to the hyperplanes $(\gamma^r_{pp'})^\perp$ appearing in Theorem I.15. By selecting a basis of each vector space, we can depict the module by the picture at the bottom, showing the effect of each map on each basis element. Theorem I.20 says that the signs of the inequalities correspond to the directions of certain arrows in this picture, as color-coded here.

Theorem I.20 (Theorem VIII.5, roughly)**.** *In the situation above, a shard module $M(K)$ admits a filtration whose subquotients supported on the tail can be identified with Π_{A_n}-modules which correspond to the signs of $\langle -, \gamma^r_{pp'}\rangle$ on K in the same fashion as in Figure 9.*

We can visualize this result by using the filtration to pick a basis of the vector space at each vertex and representing each map with arrows denoting where each basis element is sent. In this case, the subquotients described in Theorem I.20 appear as "layers" in the diagram. This is shown in Figure 11.

While this filtration is far from a complete description of the module's structure, it does have concrete consequences for the maps assigned to the arrows. For example, if we focus on a specific arrow and look at the ranks of the maps each subquotient assigns to that arrow, the rank of the map assigned by the full module is bounded below by their sum.

Corollary I.21 (Corollary VIII.9). *Suppose that the rth edge in the pth block joins tail vertices j_q and j_{q+1}. Let K be a shard, and let $M(K)$ be its shard module. Let $\overline{\beta}(j_p)$ be the coefficient of α_{j_p} in $\overline{\beta}$. Then*

$$\operatorname{rank}(M(K)(j_q \to j_{q+1})) \geq \sum_{\substack{p' \geq p \\ \langle -, \gamma^r_{pp'} \rangle \geq 0 \text{ on } K}} \overline{\beta}(j_{p'}) - \overline{\beta}(j_{p'+1})$$

$$\operatorname{rank}(M(K)(j_{q+1} \to j_q)) \geq \sum_{\substack{p' \geq p \\ \langle -, \gamma^r_{pp'} \rangle \leq 0 \text{ on } K}} \overline{\beta}(j_{p'}) - \overline{\beta}(j_{p'+1})$$

Finally, we briefly explain why the methods of this chapter fail to apply to the general case of stretching (inserting a path between two portions of the graph, rather than just tacking on a tail).

At the end of the thesis is an appendix discussing how to adapt these results to the setting of non-symmetric Cartan matrices. All of the connections discussed above only work as stated for symmetric Cartan matrices, because for a non-symmetric Cartan matrix the Dynkin diagram carries the additional structure of edge labels, which have no counterpart in the definition of quiver representations or Π_G-modules. Since part of the underlying motivation here is to use representation theory to address questions about root systems and related objects, this inability to incorporate a large swath of root systems (including the B_n, C_n, F_4, and G_2 finite systems) is awkward. Fortunately, it has long been known that this theory can be adapted to non-symmetric Cartan matrices by replacing quivers with the slightly more general notion of **species**: a representation of a species is like a quiver representation, but with vector spaces over different fields at different vertices. In the appendix, we introduce the basic language of species and associated preprojective algebras, and describe the small changes needed to make the results of this thesis work in that context.

CHAPTER II
Coxeter Groups and Root Systems

II.1: Finite Coxeter groups

This thesis is fundamentally concerned with questions about infinite Coxeter groups. However, it will be useful to start with an introduction to the theory of finite Coxeter groups, both to motivate the definitions in the infinite case and to introduce everything that will go wrong once we move to the infinite setting. Our principal reference for this section's results is [Hum90].

II.1.1: Reflections

Let V be a real vector space with a positive definite inner product $(-,-) : V \times V \to \mathbb{R}$.

Definition II.1. Let $\beta \in V$ be any nonzero vector. The **reflection by** β is the linear transformation $s_\beta : V \to V$ defined by

$$s_\beta(v) = v - 2\frac{(v,\beta)}{(\beta,\beta)}\beta$$

It is straightforward to check that s_β is orthogonal and fixes the points of the perpendicular hyperplane $\beta^\perp$. Additionally, any orthogonal linear transformation that fixes a hyperplane is of the form above.

Definition II.2. A **finite Coxeter group** is a finite subgroup of $O(V)$ generated by reflections.

Two classic examples motivate this — and, in their own ways, serve as building blocks for the general theory of Coxeter groups:

- Let Dih_m be the dihedral group of symmetries of an m-gon. If we consider two adjacent planes of symmetry of the m-gon, as shown in Figure 12, and let s_1 and s_2 be the

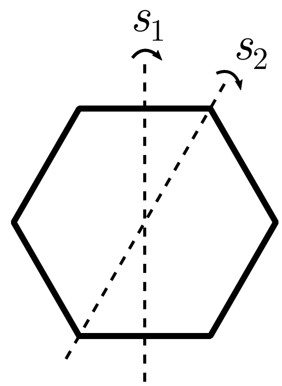


Figure 12: The reflections s_1 and s_2 generate the symmetry group of a hexagon.

reflections over them, then s_1 and s_2 generate the whole group, subject to the following relations:

$$\begin{aligned} s_1^2 &= 1 \\ s_2^2 &= 1 \\ (s_1 s_2)^m &= 1 \end{aligned}$$

- Let S_n be the symmetric group on n letters, and let s_i for $1 \leq i \leq n-1$ be the transposition swapping i and $i+1$. These transpositions generate the group, subject to the following relations:

$$\begin{aligned} s_i^2 &= 1 \\ s_i s_j &= s_j s_i \text{ if } |i-j| \geq 2 \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \end{aligned}$$

Now consider the action of S_n on $\mathbb{R}^n$ given by permuting coordinates:

$$w \cdot (x_1, \ldots, x_n) = (x_{w^{-1}(1)}, \ldots, x_{w^{-1}(n)})$$

In this context, the transpositions act by reflections: s_i fixes the points of the hyperplane $x_i = x_{i+1}$ and preserves the dot product. So this action realizes S_n as a Coxeter group.

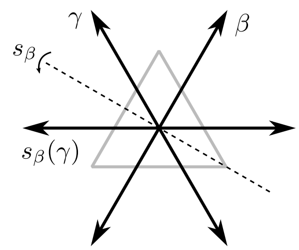


Figure 13: The A_2 root system, which consists of 6 vectors. Reflecting γ by β produces another vector in the system. The associated Coxeter group is Dih_3, the dihedral group of symmetries of a triangle.

II.1.2: Finite root systems

The first step towards systematically studying Coxeter groups is to consider all of their elements which act by reflections, rather than just a set of generators. The key observation here is that the set of vectors representing reflections is itself closed under the action of the group:

Proposition II.3 ([Hum90, Proposition 1.2])**.** *If W is a finite Coxeter group and $w \in W$, then $s_{w\beta}$ belongs to W whenever s_β does.*

Thus in order to classify the different possible arrangements of reflections in a Coxeter group, we axiomatize this closure with the notion of a root system.

Definition II.4. A **finite root system** is a finite set Φ of vectors called **roots** which satisfy the following properties:

- For each $\beta \in \Phi$, Φ also contains $-\beta$, but no other multiple of β.
- For any $\beta, \gamma \in \Phi$, $s_\beta(\gamma) \in \Phi$.

This definition is illustrated in Figure 13.

Any finite Coxeter group can be associated to a finite root system, and vice versa.

Proposition II.5 ([Hum90, Section 1.2])**.** *(a) Given any finite root system Φ, the reflections s_β for $\beta \in \Phi$ generate a finite Coxeter group.*

(b) For any finite Coxeter group W, there exists a finite root system such that W is obtained from the construction in part (a).

It is worth noting that a Coxeter group can be associated to multiple different root systems, because rescaling vectors does not change the reflections they induce. So a root system contains a little more information than a Coxeter group, and for our purposes it will be a slightly more useful notion.

Examples.

- The dihedral group Dih_m has as a root system a wheel of $2m$ evenly spaced unit vectors, perpendicular to the planes of symmetry of the m-gon. Figure 13 illustrates this for $m = 3$. If m is even, we can allow the roots to have two different lengths, as long as they alternate in length around the wheel; this is illustrated for $m = 6$ in Figure 15.

- Under the action of the symmetric group S_n introduced above, the elements which act by reflection are exactly the transpositions, with the transposition that swaps i and j fixing the hyperplane $x_i = x_j$. Perpendicular to this hyperplane are the vectors $e_i - e_j$ and $e_j - e_i$, where e_i denotes the ith standard basis vector. It is straightforward to verify that acting on the vector $e_i - e_j$ with an element of S_n produces another vector of that form. Thus the vectors $e_i - e_j$ for all $1 \leq i \neq j \leq n$ form a root system for S_n.

II.1.3: Compactly describing root systems: positive and simple roots

While root systems allow us to look at the reflections in Coxeter groups without privileging a particular generating set, having a generating set is often useful. In particular, it eliminates the redundancy arising from symmetry and allows us to describe a root system more compactly. In this section, we'll outline a systematic way of obtaining such a generating set for a root system Φ.

The first bit of redundancy to deal with is that every reflection is represented by two roots, pointing in opposite directions. In order to pick out one representative from each pair, choose a vector $v \in V$ which is not orthogonal to any root of Φ. Define sets of **positive roots** Φ^+ and **negative roots** Φ^- by

$$\Phi^+ := \{\beta \in \Phi \mid (v, \beta) > 0\}$$
$$\Phi^- := \{\beta \in \Phi \mid (v, \beta) < 0\}$$

Because we chose v to avoid pairing to 0 with any root, we have $\Phi = \Phi^+ \sqcup \Phi^-$, and $\Phi^- = -\Phi^+$.

From among the positive roots, we then pick a subset which is fundamental.

Theorem II.6 ([Hum90, Theorem 1.3])**.** *Given a partition $\Phi = \Phi^+ \sqcup \Phi^-$ as above, there exists a unique collection Δ of roots such that:*

(a) The roots of Δ form a basis of $\mathrm{span}(\Phi)$.

(b) Any root in Φ^+ is a linear combination of roots from Δ with nonnegative coefficients.

Definition II.7. The elements of Δ in the context of Theorem II.6 are called **simple roots**.

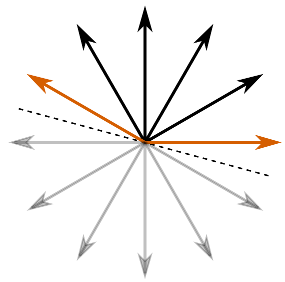


Figure 14: The root system associated to the dihedral group of a hexagon, Dih_6. The positive roots are defined to be lying on one side of the dashed line. Each positive root is a nonnegative linear combination of the simple roots (in orange), lying in the cone that they generate.

We now identify simple roots in our two running examples:

- For the root system of Dih_m, any sequence of m consecutive roots around the wheel can be realized as the positive roots by choosing an appropriate v. Then the simple roots are the two roots at either end of this sequence. This is illustrated in Figure 14.
- For the root system of S_n, we choose $v = (1, 2, \ldots, n)$. Then $(v, e_j - e_i) > 0$ precisely when $j > i$, which defines a set of positive roots. The simple roots are then those of the form $e_{i+1} - e_i$: we can write any positive root $e_j - e_i$ as

$$e_j - e_i = (e_j - e_{j-1}) + (e_{j-1} - e_{j-2}) + \ldots + (e_{i+1} - e_i).$$

While different choices of vector v will lead to different definitions of positive and simple roots, the choice ends up not mattering: any choice of positive and simple roots can be obtained from any other through the action of the Coxeter group ([Hum90, Theorem 1.4]). So from here on, for any given root system Φ, we will assume that we've fixed a particular Φ^+ and consequent Δ.

II.1.4: Generators and relations

The collection of simple roots is fundamental in two key ways. First, by definition, they form a basis for the space spanned by the roots, and in what follows we will generally find it helpful to express all roots in the basis of simple roots. However, they also give us a set of generators for the Coxeter group.

Theorem II.8 ([Hum90, Theorem 1.5]). *Let W be a finite Coxeter group with associated root system Φ. Then the reflections s_α for simple roots $\alpha \in \Delta$ generate W.*

We call the reflections associated to simple roots **simple reflections**.

The relations between these generators also admit a simple description. Because our group is finite, for any two simple roots $\alpha_i, \alpha_j \in \Delta$, the product $s_{\alpha_i} s_{\alpha_j}$ must have finite order, and we denote this order by m_{ij}. It turns out that these relations, together with the fact that reflections have order 2, are enough.

Theorem II.9 ([Hum90, Theorem 1.9])**.** *Let $\alpha_1, \ldots, \alpha_n$ be simple roots for a finite Coxeter group W, and let m_{ij} be the order of $s_{\alpha_i} s_{\alpha_j}$. Then*

$$W \cong \langle s_1, \ldots, s_n \mid s_i^2 = (s_i s_j)^{m_{ij}} = 1, 1 \leq i, j \leq n \rangle$$

In what follows, if a list of simple roots $\alpha_1, \ldots, \alpha_n$ is understood, we will write s_i for s_{α_i}.

It is straightforward to read off the numbers m_{ij} from the geometry of the root system using the following lemma.

Lemma II.10. *The angle between the roots $\alpha_i, \alpha_j \in \Delta$ is $\pi - \frac{\pi}{m_{ij}}$.*

Proof. Consider the subgroup $W_{ij} := \langle s_i, s_j \rangle \subset W$. Note that $\alpha_i^\perp \cap \alpha_j^\perp$ is a codimension-2 subspace fixed pointwise by W_{ij}, and so W_{ij} acts on the 2-dimensional space $V/(\alpha_i^\perp \cap \alpha_j^\perp)$. However, if two reflections acting on $\mathbb{R}^2$ generate a finite group, it must be a dihedral group. The product of two reflections generating Dih_m has order m, so in this case $W_{ij} \cong \mathrm{Dih}_{m_{ij}}$.

Now, we can get a root system for W_{ij} as a subset of Φ, consisting of the roots associated to reflections in W_{ij}; these roots will be orthogonal to $\alpha_i^\perp \cap \alpha_j^\perp$, and thus lie in $\mathrm{span}(\alpha_i, \alpha_j)$. We can likewise choose a set of positive roots for W_{ij} to be those which lie in Φ^+. Because α_i and α_j are simple in Φ, every positive root in W_{ij} is expressible as a nonnegative linear combination of α_i and α_j, so α_i and α_j are also simple roots for W_{ij}. On the other hand, we know what simple roots look like in $\mathrm{Dih}_{m_{ij}}$ (as shown in Figure 14) and can conclude that the angle between them is $\pi - \frac{\pi}{m_{ij}}$. □

Corollary II.11. *For α_i, α_j, and m_{ij} as above,*

$$\frac{(\alpha_i, \alpha_j)}{\|\alpha_i\| \|\alpha_j\|} = -\cos\left(\frac{\pi}{m_{ij}}\right).$$

II.1.5: Cartan matrices and Dynkin diagrams

The previous section shows that we can recover a Coxeter group and its root system from its simple roots and the angles between them. There are two common ways of recording this information: as a matrix, and as a decorated graph.

Definition II.12. The **Cartan matrix** of a root system Φ with simple roots $\alpha_1, \ldots, \alpha_n$ is the $n \times n$ matrix A with

$$A_{ij} = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$$

These matrix entries are chosen such that $s_j(\alpha_i) = \alpha_i - A_{ij}\alpha_j$.

For applications to representation theory, we often restrict to the case that the entries A_{ij} are integers. In this case, we say the root system is **crystallographic**. This imposes meaningful limitations on the types of Coxeter groups we can consider.

Lemma II.13. *If W admits a crystallographic Cartan matrix, then $m_{ij} \in \{2, 3, 4, 6\}$ for all $i \neq j$.*

Proof. Let θ_{ij} be the angle between the simple roots α_i and α_j. We have

$$A_{ij}A_{ji} = 4\frac{(\alpha_i, \alpha_j)^2}{(\alpha_i, \alpha_i)(\alpha_j, \alpha_j)} = 4\cos^2(\theta_{ij})$$

The crystallographic assumption implies that this is an integer; on the other hand, Lemma II.10 implies that θ_{ij} has the form $\pi - \frac{\pi}{m_{ij}}$. Together, these two conditions force m_{ij} to be in the given set. $\square$

In what follows, we will assume root systems are crystallographic unless stated otherwise.

Examples.

- Consider Dih_6, the dihedral group of a hexagon. A root system consisting of unit vectors, as shown in Figure 14, is not crystallographic, as we have

$$\frac{2(\alpha_1, \alpha_2)}{(\alpha_1, \alpha_1)} = \frac{2(\alpha_1, \alpha_2)}{(\alpha_2, \alpha_2)} = 2(\alpha_1, \alpha_2) = -2\cos(\frac{\pi}{6}) = -\sqrt{3}.$$

  However, we can scale every other root by a factor of $\sqrt{3}$, producing the system shown in Figure 15. Now $(\alpha_1, \alpha_1) = 1$, but $(\alpha_2, \alpha_2) = 3$. This is a crystallographic root system, because we now have

$$\frac{2(\alpha_1, \alpha_2)}{(\alpha_1, \alpha_1)} = 2(\alpha_1, \alpha_2) = 2\frac{-3}{2} = -3$$
$$\frac{2(\alpha_1, \alpha_2)}{(\alpha_2, \alpha_2)} = \frac{2}{3}(\alpha_1, \alpha_2) = -1$$

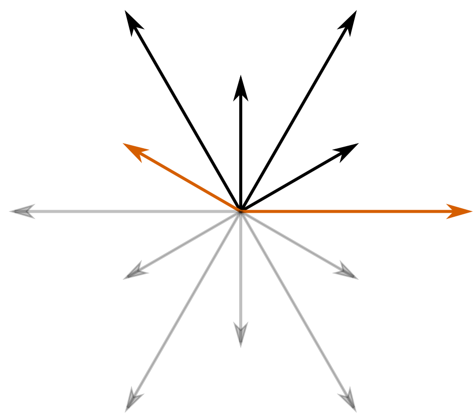


Figure 15: A crystallographic root system for Dih_6.

This root system has the Cartan matrix

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

- On the other hand, the root system for S_n we used above, given by $\alpha_i := e_{i+1} - e_i$ for $1 \leq i \leq n-1$, is crystallographic. If $|i-j| \geq 2$, we simply have $(\alpha_i, \alpha_j) = 0$. On the other hand,
$$2\frac{(\alpha_i, \alpha_{i+1})}{(\alpha_{i+1}, \alpha_{i+1})} = 2\frac{-1}{2} = -1.$$
This root system has the Cartan matrix
$$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

Ultimately, we want to connect root systems to quivers, so it will be useful to represent these matrices by graphs.

Definition II.14. The **Dynkin diagram** of a root system with an $n \times n$ Cartan matrix A is a graph with labeled edges, defined as follows:

- The vertices are $1, 2, \ldots, n$.

- If $A_{ij} = A_{ji} \neq 0$, we draw $-A_{ij}$ unlabeled edges between i and j.
- If $A_{ij} \neq A_{ji}$ and $i < j$, we draw an edge between i and j and label it with the ordered pair $(-A_{ij}, -A_{ji})$. (In drawing Dynkin diagrams, we will orient any such edge so that the vertex i is on the left.)

This is a less commonly used notation for Dynkin diagrams, based on the definition from [Kac80] (for example, it differs slightly from [Hum72]). This will avoid confusion later on.

Examples.

- The root system for Dih_6 in Figure 15 has Dynkin diagram

$$\bigcirc \overset{(1,3)}{\text{———}} \bigcirc$$

- The root system for S_n has Dynkin diagram

$$\bigcirc\text{——}\bigcirc\text{—}\cdots\text{—}\bigcirc$$

In addition to giving us a convenient way to represent root systems, Dynkin diagrams enable a systematic notation for the roots themselves. By construction, the vertices of the Dynkin diagram correspond to the simple roots. Since the simple roots form a basis of the space spanned by the roots, every root is expressible as a unique linear combination of simple roots. Thus we can denote a root by labeling the vertices of the Dynkin diagram with the coefficients of this expression. If Φ is crystallographic, these labels will be integers, and by Theorem II.6, the labels will either be entirely nonnegative or entirely nonpositive.

Example. Recall from above that the root system we chose for S_n has simple roots of the form $\alpha_i = e_{i+1} - e_i$, and any other positive root is of the form

$$e_j - e_i = (e_j - e_{j-1}) + (e_{j-1} - e_{j-2}) + \ldots + (e_{i+1} - e_i) = \alpha_{j-1} + \alpha_{j-2} + \ldots + \alpha_i$$

Representing these roots as labelings of the Dynkin diagram, they all have the form

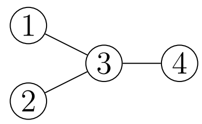
$$0 - \cdots - 0 — 1 - \cdots - 1 — 0 - \cdots - 0$$

labeling a consecutive string of vertices with 1's.

Definition II.15. The **support** of a root is the subset of the vertices of the Dynkin diagram (or the subdiagram induced by this subset) which are assigned nonzero coefficients by this notation.

We will take this to be our default notation for portraying roots, and in particular we will refer to the "coefficients" of a root, meaning its coefficients in this notation. Similarly, we may refer to applying s_i as **reflecting at vertex** i.

Example. Consider the D_4 diagram



Then we can write any combination of simple roots $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4$ as

$$\begin{array}{lll} c_1 & & \\ & \diagdown & \\ & & c_3 \;—\; c_4 \\ & \diagup & \\ c_2 & & \end{array}$$

When we apply s_3, reflecting at vertex 3, to any simple root, we get

$$s_3(\alpha_i) = \alpha_i - A_{i3}\alpha_3 = \begin{cases} \alpha_i + \alpha_3 & i = 1,2,4 \\ -\alpha_3 & i = 3 \end{cases}$$

Thus when we apply s_3 to a general linear combination as above, we get

$$s_3(c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4) = c_1\alpha_1 + c_2\alpha_2 + (c_1 + c_2 + c_4 - c_3)\alpha_3 + c_4\alpha_4$$

When we view this on the diagram, we see that this reflection can be summarized as "subtract the coefficient at vertex 3 from the sum of its neighbors." More generally, unwinding the definition of reflection like this implies:

Proposition II.16. *Suppose vertex j of a Dynkin diagram is adjacent to vertices $i_1, \ldots, i_\ell$. For a root β, let $\beta(i)$ be the α_i-coefficient of β. Then*

$$s_j(\beta)(i) = \begin{cases} \left(\sum_{p=1}^{\ell} -A_{i_p j}\beta(i_p)\right) - \beta(j) & i = j \\ \beta(i) & \textit{otherwise} \end{cases}$$

II.1.6: Classification

With a compact notation in hand, we can state the classification of finite crystallographic root systems.

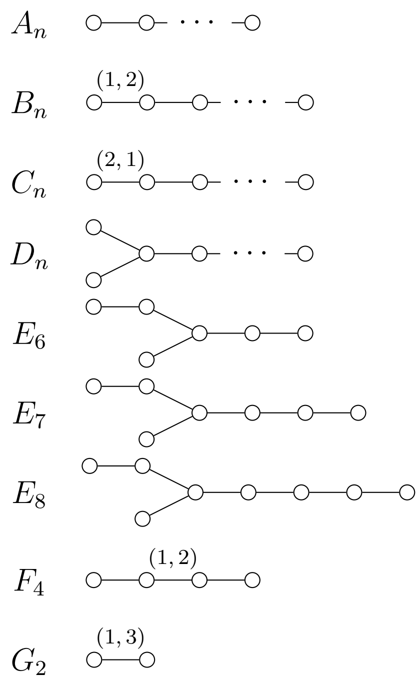


Figure 16: The Dynkin diagrams of finite crystallographic root systems.

Theorem II.17 ([Hum72, Theorem 11.4]). *The finite crystallographic root systems are exactly those associated to the Dynkin diagrams shown in Figure 16 (and disjoint unions of these diagrams).*

The diagrams on the list in Figure 16 are often simply called *the* Dynkin diagrams, but we will follow the convention of [Kac80] that any labeled graph of this kind is a Dynkin diagram. Thus we will also talk about the Dynkin diagrams of infinite root systems once they are introduced below. We call the diagrams in Figure 16 **finite type Dynkin diagrams**.

Each diagram has a number of vertices given by its subscript. In particular, the root system we attach to the symmetric group S_n is called A_{n-1}, because there are only $n-1$ simple generators.

By Lemma II.13, the only dihedral groups which admit crystallographic root systems are Dih_m for $m = 2, 3, 4, 6$. Dih_3 has root system A_2, while Dih_4 has root system $B_2 = C_2$. Dih_6 has root system G_2. Dih_2 has as its Dynkin diagram two unconnected vertices (representing two commuting reflections); we typically denote this $A_1 \times A_1$.

II.1.7: The reflecting hyperplane arrangement and its regions

Just as significant as the roots β whose reflections generate the Coxeter group are the perpendicular hyperplanes $\beta^\perp$ fixed by these reflections, which we call **reflecting hyperplanes**.

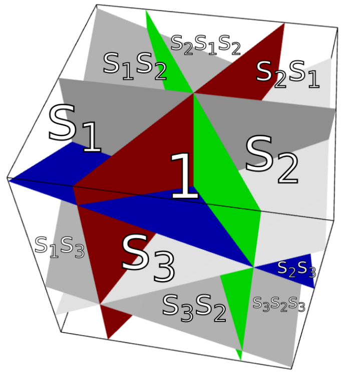


Figure 17: Once we select a base region D, its translations by the action of W are in bijection with the elements of W. Here, s_1, s_2, and s_3 act by reflection over the red, green, and blue planes, respectively.

The arrangement of these hyperplanes encodes important information about the group.

Consider the region

$$D := \{v \in V \mid (v, \alpha_i) \geq 0 \ \forall \alpha_i \in \Delta\}$$

- For Dih_n, this is a wedge enclosed by two adjacent planes of symmetry.
- For S_n with our preceding choice of simple roots, we have

$$D = \{(v_1, \ldots, v_n) \mid v_1 \leq v_2 \leq \cdots \leq v_n\}.$$

Theorem II.18 ([Hum90, Theorem 1.12]). *D is a fundamental domain for the action of W on V: every W-orbit in V contains exactly one point in D. Additionally, any point in the interior of D has trivial stabilizer.*

Corollary II.19. *The map $w \mapsto wD$ gives a bijection between the Coxeter group and the regions of the arrangement of reflecting hyperplanes.*

This correspondence is illustrated in Figure 17.

II.1.8: The weak order

The distinguished set of generators of a Coxeter group allows us to analyze elements in terms of how they can be expressed using those generators. In particular, the group admits a few natural orderings based on how many generators it takes to express a particular element.

Definition II.20. An element $w \in W$ can be written as a product of generators $s_{i_1} s_{i_2} \cdots s_{i_\ell}$. The **length** of w, denoted $\ell(w)$, is the smallest possible value of ℓ (that is, the smallest possible length of such an expression.) Any such expression of minimal length is a **reduced expression** for w.

Definition II.21. Let $w, w' \in W$ be any two elements of a Coxeter group. We say that $w \leq w'$ in the **right weak order** if we can write

$$w' = ws_{i_1} \cdots s_{i_k}$$

such that

$$\ell(ws_{i_1} \cdots s_{i_j}) = \ell(w) + j \text{ for all } 1 \leq j \leq k$$

In other words, for w' to be greater than w in the right weak order means we can obtain w' from w by multiplying on the right by a sequence of simple reflections, each of which increases length. We can also define a **left weak order** using left multiplication instead, but in what follows we'll just use **weak order** to refer to right weak order.

One key property of the weak order on a finite Coxeter group is that it is a **lattice**: any pair of elements w, v have a unique least upper bound or **join** $w \vee v$ and a unique greatest lower bound or **meet** $w \wedge v$ [BB05, Section 3.2].

The weak order also manifests itself geometrically in the arrangement of reflecting hyperplanes. Specifically, we can define a poset on the regions of the arrangement which, under the bijection from Corollary II.19, corresponds to weak order.

Definition II.22 ([Rea16b, Definition 1.13])**.** Given a hyperplane arrangement with a distinguished base region D, for any other region R, let $S(R)$ be the set of hyperplanes which separate R from D. Then the **poset of regions** is the partial order on regions of the arrangement where $Q \leq R$ if $S(Q) \subset S(R)$.

This poset has the base region as its unique minimal element, and crossing hyperplanes away from the base region corresponds to moving up in the poset.

Theorem II.23 ([Rea16a, Theorem 3.1])**.** *The bijection from Corollary II.19 gives an isomorphism between the poset of regions and the weak order.*

II.2: Infinite Coxeter groups and root systems

On the path to finite Coxeter groups, we started with a group concretely generated by reflections, attached a root system to it, and then produced a Cartan matrix, a Dynkin diagram, and a presentation by generators and relations. The infinite setting can be built up in reverse: we start with a group presentation, Dynkin diagram, or Cartan matrix, and use that information to construct an infinite root system and define an action of the group by reflections. Our principal reference for this section is [Kac80], with some results from [BB05].

Definition II.24. A **Coxeter group of rank** n is a group defined by generators and relations of the form

$$\langle s_1, \ldots, s_n \mid (s_i s_j)^{m_{ij}} = 1 \rangle$$

where:

- $m_{ii} = 1$ (so each generator has order 2)
- For $i \neq j$, $m_{ij} \in \{2, 3, \ldots, \infty\}$. (If $m_{ij} = \infty$, we simply omit that relation, so $s_i s_j$ has infinite order.)

In what follows, we fix a Coxeter group W with rank n and defining data m_{ij}. Our first task is to turn this abstract definition by generators and relations into an action on a space, mimicking the original example of reflection groups.

Definition II.25. Given the data defining a Coxeter group W as above, a **Cartan matrix** for the group is an $n \times n$ matrix A such that:

- $A_{ii} = 2$ for all i.
- $A_{ij} \leq 0$ for $i \neq j$, with equality if and only if $m_{ij} = 2$.
- If $3 \leq m_{ij} < \infty$, $A_{ij} A_{ji} = 4\cos^2(\pi/m_{ij})$.
- If $m_{ij} = \infty$, $A_{ij} A_{ji} \geq 4$.

We attach a **Dynkin diagram** to a Cartan matrix in the same manner as above.

Now let V be a real vector space with basis $\alpha_1, \ldots, \alpha_n$. Define linear maps $\varphi_i : V \to \mathbb{R}$ for $1 \leq i \leq n$ by $\varphi_i(\alpha_j) = A_{ij}$.

Definition II.26. The **simple reflections** are linear transformations $s_1, \ldots, s_n : V \to V$ defined by

$$s_i(\beta) = \beta - \varphi_i(\beta)\alpha_i$$

Theorem II.27 ([BB05, Theorem 4.2.7])**.** *W admits a faithful action on V through which the generators act by simple reflections.*

Definition II.28. The **simple roots** of a Cartan matrix are the basis vectors $\alpha_1, \ldots, \alpha_n$, and we denote by Δ the set of simple roots. In general, the **roots** are elements of the form $w\alpha_i$ for $w \in W$. We call the collection of all roots the **root system** of A, and denote it by Φ.

What we call "roots", representation-theoretic contexts such as [Kac80] often refer to as **real roots**, alongside an additional set of **imaginary roots**. As our focus is almost entirely on the real roots, we will assume that all roots are real unless stated otherwise.

While the theory of infinite root systems can be built up in great generality, in the context of preprojective algebras we'll want to impose a couple of additional conditions. As above, we say that a Cartan matrix is **crystallographic** if its entries are integers. Additionally:

Definition II.29. A Cartan matrix is **symmetrizable** if there exist integers $d_1, \dots, d_n$ such that $d_i A_{ij} = d_j A_{ij}$ for all i, j.

If the Cartan matrix is symmetrizable, we can define the simple reflections using a bilinear form which will be useful later. Having fixed $d_1, \dots, d_n$, we can define a symmetric bilinear form $(-,-) : V \times V \to \mathbb{R}$ by $(\alpha_i, \alpha_j) := d_i A_{ij} = d_j A_{ji}$. Then define $\alpha_i^\vee := \alpha_i / d_i$. We have $(\alpha_i^\vee, \alpha_j) = A_{ij}$, and thus can write

$$s_i(\beta) = \beta - (\alpha_i^\vee, \beta)\alpha_i.$$

Importantly, this pairing is preserved by the action of the group: $(w\beta, w\gamma) = (\beta, \gamma)$ for all $w \in W$.

Going forward, we will assume Cartan matrices are crystallographic and symmetrizable unless stated otherwise, and use parentheses to denote the bilinear form just defined.

Finally, just as simple roots correspond to simple reflections, all other roots correspond to other reflections. Given $\beta = w\alpha_i \in \Phi$, let $\beta^\vee = 2\beta/(\beta, \beta)$, so that $(\beta^\vee, \beta) = 2$. Note also that $(\beta, \beta) = (\alpha_i, \alpha_i) = 2d_i$, so $\beta^\vee = w\alpha_i^\vee$. Then we define

$$s_\beta(\gamma) := w s_i w^{-1}(\gamma) = w(w^{-1}\gamma - (\alpha_i^\vee, w^{-1}\gamma)\alpha_i) = \gamma - (w\alpha_i^\vee, \gamma)\beta = \gamma - (\beta^\vee, \gamma)\beta$$

II.2.1: The reflecting hyperplane arrangement and the weak order in the infinite case

The fundamental difference between finite and infinite Coxeter groups is that the pairing $(-,-)$ may not be positive definite, and thus may not induce an inner product on the underlying space. Indeed, a Coxeter group is finite precisely when this pairing is positive definite [Hum90, Theorem 6.4].

This has consequences for how we define the arrangement of reflecting hyperplanes. While, in the finite case, roots β were associated to hyperplanes $\beta^\perp$ in the same space using the inner product, to get similar geometric properties here we need to think of these hyperplanes as lying in the dual space V^*.

We denote with angle brackets the natural pairing $\langle -, - \rangle : V^* \times V \to \mathbb{R}$ given by $\langle f, \beta \rangle = f(\beta)$. Then we define the **contragredient action** of W on V^* such that $\langle wf, w\beta \rangle = \langle f, \beta \rangle$ for any $w \in W$; equivalently, we can say $wf = f \circ w^{-1}$.

Proposition II.30. *The subset of V^* fixed by s_i is precisely the hyperplane $\alpha_i^\perp := \{x \mid \langle x, \alpha_i \rangle = 0$.*

Proof. We have

$$(s_i f)(\beta) = f(s_i \beta) = f(\beta - (\alpha_i^\vee, \beta)\alpha_i) = f(\beta) - (\alpha_i^\vee, \beta) f(\alpha_i)$$

This is equal to $f(\beta)$ for all β if and only if $f(\alpha_i) = 0$. $\square$

Thus we define a reflecting hyperplane arrangement in V^* consisting of the hyperplanes $\beta^\perp$ for $\beta \in \Phi$, and single out a base region D defined by

$$D := \{x \in V^* \mid \langle x, \alpha_i \rangle \geq 0 \; \forall \alpha_i \in \Delta\}$$

This shares some properties with the arrangement in the finite case, but with one key difference: W no longer acts transitively on the regions of the arrangement. Instead, let $T = \bigcup_{w \in W} wD$ be the set of all points we can reach by starting from the base region and acting with W. We call this the **Tits cone**. As long as we restrict ourselves to the Tits cone, we have results just like the finite case.

Theorem II.31. *[Hum90, Theorem 5.13] D is a fundamental domain for the action of W on T: every W-orbit in T contains exactly one point in D. Additionally, any point in the interior of D has trivial stabilizer.*

Corollary II.32. *The map $w \mapsto wD$ gives a bijection between the Coxeter group and the regions of the arrangement of reflecting hyperplanes which lie in T.*

We can also define the (right) weak order on an infinite Coxeter group just as we do for a finite one, and it has the same relationship to the regions of the Tits cone as in Theorem II.23. However, it's no longer a lattice: unlike a finite Coxeter group, an infinite one does not have a largest element, and it's possible that two elements may have no mutual upper bound at all. Nonetheless, the weak order is still a **complete meet-semilattice**: any collection of elements has a greatest lower bound [BB05, Theorem 3.2.1].

CHAPTER III
Shards

To fully detail the relationship between preprojective algebras and root systems, we will work with a partition of the reflecting hyperplanes into cones called **shards**. In this section, we briefly describe the original motivation for shards, define them, and detail a few tools for working with them. Our most important tool will be a recursive method for calculating shards, relating the shards of $\beta^\perp$ to those of $s_i\beta^\perp$ for a generating reflection s_i.

III.1: Motivation: lattice quotients of the weak order

Recall from the previous section that the weak order on a finite Coxeter group forms a **lattice**: any pair of elements w, v have a unique least upper bound or **join** $w \vee v$ and a unique greatest lower bound or **meet** $w \wedge v$ [BB05, Section 3.2]. Shards were introduced by Nathan Reading in order to study this structure.

One interesting aspect of the weak order as a lattice is that other combinatorially important lattices can be realized as **quotient lattices**. We now give an example briefly explaining what this means.

Consider the collection T_n of unambiguous parenthesizations of n elements: expressions such as

$$((\bullet(\bullet\bullet))\bullet)$$

in which each pair of parentheses encloses exactly two things (which can either be $\bullet$ or another expression in parentheses). There is a partial order on T_n called the **Tamari lattice** which captures the additional structure of the associative rule: there is a cover relation $x \lessdot y$ in this order whenever y is obtained from x by a replacement $(\circ\circ)\circ \to \circ(\circ\circ)$. For example, we have

$$(((\bullet\bullet)\bullet)\bullet) \lessdot ((\bullet(\bullet\bullet))\bullet) \lessdot (\bullet((\bullet\bullet)\bullet))$$

The Tamari lattice is the transitive closure of this relation, which turns out to be a lattice.

There is a map τ from permutations of $n-1$ letters to parenthesizations of n elements.

We denote a permutation w by the one-line notation $w(1)w(2)\cdots w(n-1)$. We then define τ with the following recursive formula:

- $\tau(\varnothing) = \bullet$;
- for a sequence c with last element c_ℓ, let $c_<$ be the subsequence of elements $< c_\ell$ and let $c_>$ be the subsequence of elements $> c_\ell$. Then $\tau(c) = (\tau(c_<)\tau(c_>))$

For example,

$$\begin{aligned}
\tau(2413) &= (\tau(21)\tau(4)) \\
&= ((\tau(\varnothing)\tau(2))(\tau(\varnothing)\tau(\varnothing))) \\
&= ((\bullet(\tau(\varnothing)\tau(\varnothing)))(\bullet\bullet)) \\
&= ((\bullet(\bullet\bullet))(\bullet\bullet))
\end{aligned}$$

The key result is that we can think of τ as a quotient map.

Proposition III.1 ([BW97, Section 9], [Rea06])**.** *Consider S_{n-1} as a lattice with the weak order. Then $\tau : S_{n-1} \to T_n$ is a surjective lattice homomorphism from permutations to parenthesizations; that is,*

$$\begin{aligned}
\tau(x \vee y) &= \tau(x) \vee \tau(y) \\
\tau(x \wedge y) &= \tau(x) \wedge \tau(y)
\end{aligned}$$

We can thus view the fibers of τ as a **lattice congruence** on the weak order: an equivalence relation compatible with the join and meet operations. Figure 18 shows this in a small case. The map $\tau : S_3 \to T_4$ sends two permutations to the same parenthesization: we can think of those elements as being equivalent modulo a lattice congruence, and we can view T_4 as the quotient of S_3 by this congruence.

What's special about the weak order in this context is that it can also be described as an ordering of the regions of a hyperplane arrangement (Theorem II.23). This gives us a different way of looking at lattice congruences. A congruence will identify elements together, and if we think of those elements as regions, this amounts to removing the walls between regions to merge them, as illustrated in Figure 19.

One problem of interest in the lattice theory of the weak order is to classify all its lattice congruences. Not every equivalence relation is a lattice congruence: sometimes, identifying two elements will force us to identify a different pair elsewhere. So in the hyperplane picture, we need to know which chunks of the reflecting hyperplanes can be removed in a way that respects the lattice structure. These chunks are shards.

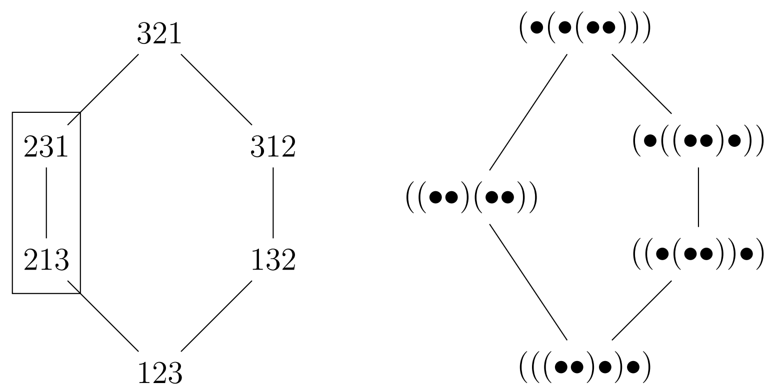


Figure 18: On the left, the weak order on the symmetric group S_3; on the right, the Tamari lattice T_4 ordering parenthesizations of 4 elements. The map $\tau : S_3 \to T_4$ identifies the boxed elements of S_3 — thus realizing the Tamari lattice as a lattice quotient of the weak order.

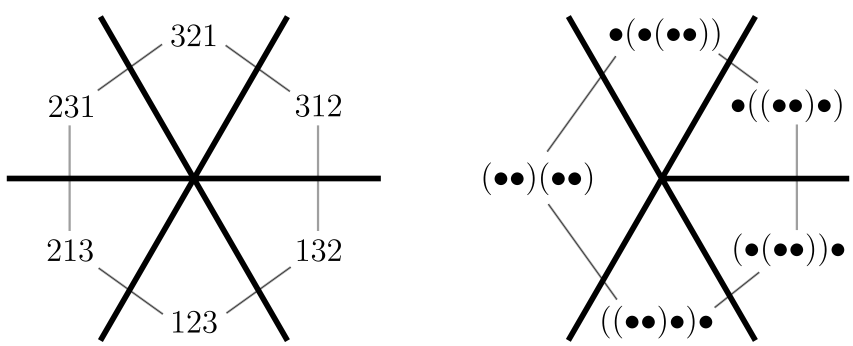


Figure 19: The identification shown in Figure 18 can be viewed as removing a wall from the reflecting hyperplane arrangement of S_3, merging two adjacent regions.

III.2: Rank 2 subsystems

The picture in Figure 19, illustrating the situation in 2 dimensions, is in some ways a template for the definition of shards in general.

Let Φ be a root system of rank n, let V be the space containing its roots, and let $\alpha_1, \ldots, \alpha_n$ be the simple roots.

Definition III.2. A subset $R \subset \Phi$ is a **rank 2 subsystem** if $\operatorname{span}(R)$ is 2-dimensional and $R = \Phi \cap \operatorname{span}(R)$.

Along with roots, we're interested in the corresponding reflecting hyperplanes, so we make a dual definition.

Definition III.3. Given a collection of hyperplanes $\mathcal{H}$ in V^* and a subset $S \subset \mathcal{H}$, let $\bigcap S$ be the intersection of the hyperplanes in S. Then S is a **rank 2 subarrangement** if $\bigcap S$ has codimension 2 in V^* and $S = \{H \in \mathcal{H} \mid H \supset \bigcap S\}$.

For any subset $R \subset \Phi$, let $R^* \subset \mathcal{H}$ be the collection of hyperplanes dual to the roots in R. Unpacking and dualizing the definitions, it is straightforward to check:

Proposition III.4. *A subset $S \subset \mathcal{H}$ is a rank 2 subarrangement if and only if it is R^* for a rank 2 subsystem R.*

Within each rank 2 subystem, we will highlight the roots which play the same role as simple roots do in the full system.

Definition III.5. The **fundamental roots** of a rank 2 subsystem R are roots γ_1, γ_2 such that any positive root $\beta \in \Phi^+ \cap R$ is a nonnegative linear combination of γ_1 and γ_2.

Exactly 2 fundamental roots exist [RS11, Theorem 2.7(i) and Proposition 2.11]. They can be visualized as lying on the edges of the cone generated by the positive roots in R. Again, there is also a dual picture.

Definition III.6. Let D be the base region of the full reflecting hyperplane arrangement (as defined in Section II.2.1). Let R^* be a rank 2 subarrangement, and let D' be the region of R^* which contains D. Then the **fundamental hyperplanes** of R^* are those containing the walls of D'.

Proposition III.7. *The fundamental hyperplanes of R^* are dual to the fundamental roots of R.*

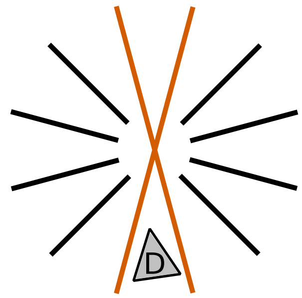


Figure 20: To produce shards, for each rank 2 subarrangement, we break all the non-fundamental hyperplanes.

Proof. By definition, D consists of the points which pair nonnegatively with the simple roots of Φ, and thus with all the positive roots. Then D', as the region of R^* containing D, consists of the points which pair nonnegatively with the positive roots in R. Say γ_1 and γ_2 are the fundamental roots of R. Then a point pairs nonnegatively with γ_1 and γ_2 if and only if it pairs nonnegatively with all positive roots in R, since they are nonnegative linear combinations of γ_1 and γ_2. Thus D' consists of points lying on the positive side of $\gamma_1^\perp$ and $\gamma_2^\perp$, and those hyperplanes contain the walls. □

Given any two roots β_1, β_2 with $\beta_2 \neq \pm\beta_1$, we denote the unique rank 2 subsystem containing them by $R(\beta_1, \beta_2)$.

III.3: Shards

Having defined fundamental roots, we can now define shards. The basic idea is as follows: for each rank 2 subarrangement of hyperplanes, break all non-fundamental hyperplanes along the subarrangement's intersection, as shown in Figure 20. For a rank 2 subsystem R, let $R^\perp$ denote the intersection of the hyperplanes in R^*.

Definition III.8. A rank 2 subsystem R **cuts** a reflecting hyperplane $\beta^\perp$ if $\beta \in R$, but β is not fundamental in R. In this case, we say that $R^\perp$ is a **fracture** of $\beta^\perp$. We will also say that a root γ or hyperplane $\gamma^\perp$ cuts $\beta^\perp$ if the rank 2 subsystem $R(\beta, \gamma)$ does.

If $\beta \in R$, then $R^\perp \subset \beta^\perp$. Since $R^\perp$ is codimension 2 in the whole of V^*, it is codimension 1 in $\beta^\perp$. Thus, we can think of the collection of fractures of $\beta^\perp$ as a hyperplane arrangement within $\beta^\perp$.

Definition III.9. The **shard arrangement** is the arrangement of fractures in $\beta^\perp$. The **shards** of $\beta^\perp$ are the closures of the regions of this arrangement.

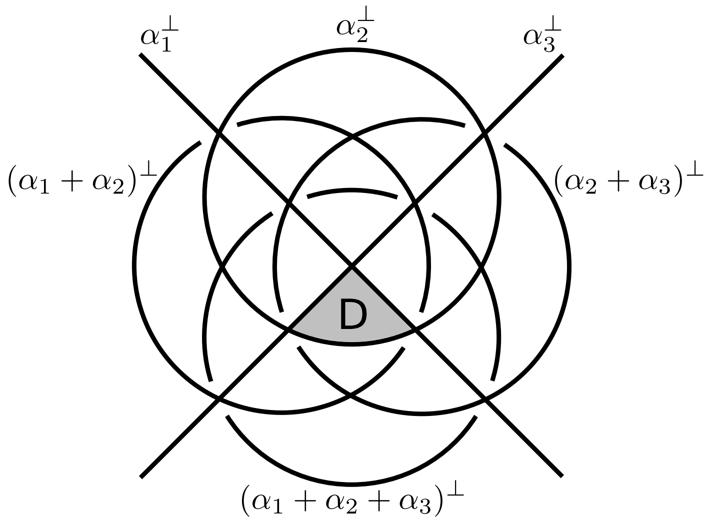


Figure 21: The stereographic projection of the shards of the A_3 root system. Note that, wherever 3 planes meet, the one furthest from D is broken.

We recall the example of the A_3 root system from the introduction: Figure 21 illustrates the division of its reflecting hyperplanes into shards. To render this in 2 dimensions, we intersect the arrangement with a sphere centered at the origin, then stereographically project that sphere to the plane. Thus lines and circles both represent planes in this arrangement.

We now revisit the motivational question from the beginning of this chapter to explain how shards answer it.

Proposition III.10 ([Rea16b, Proposition 8.3])**.** *In any lattice congruence of the weak order on a finite Coxeter group, equivalence classes can be identified with maximal cones of a fan obtained by removing shards from the reflecting hyperplane arrangement.*

In other words, a lattice congruence can't just remove arbitrary walls between regions — it must remove shards, each of which may consist of multiple walls.

We'd also like to know which collections of shards can be removed to produce a lattice congruence. Fortunately, there is also a simply stated geometric criterion for this.

Definition III.11 ([Rea16b, Definition 7.16])**.** The **shard digraph** is a directed graph whose vertices are shards of the reflecting hyperplane arrangement, with an edge $K_1 \to K_2$ if the hyperplane containing K_1 cuts the hyperplane containing K_2 and $K_1 \cap K_2$ has dimension $n - 2$.

The relations implied by this digraph tell us exactly which shards' removal forces the removal of other shards.

Theorem III.12 ([Rea16b, Theorem 7.18])**.** *There exists a congruence on the weak order on a finite Coxeter group removing a collection $\mathcal{K}$ of shards if and only if, when $K_1 \in \mathcal{K}$ and $K_1 \to K_2$ in the shard digraph, $K_2 \in \mathcal{K}$.*

In the case of an infinite Coxeter group, where the weak order does not always admit joins, it seems likely that a similar result should hold, but the particulars have not been worked out yet.

III.4: Positive expressions

The material of this section originally appeared in a slightly different form in [DST23].

Shards also admit a recursive description which is highly useful for inductive arguments. Specifically, we express a root in terms of starting from a simple root and applying simple reflections, and examine how these reflections affect the breakdown of a hyperplane into shards. Of key importance are the expressions which use as few reflections as possible.

Definition III.13. A **positive expression** for $\beta \in \Phi^+$ is an expression

$$\beta = s_{i_\ell} \cdots s_{i_2} s_{i_1}(\alpha_{i_0})$$

such that for all $1 \leq j \leq \ell$,

$$s_{i_j} s_{i_{j-1}} \cdots s_{i_1}(\alpha_{i_0}) - s_{i_{j-1}} \cdots s_{i_1}(\alpha_{i_0}) \in \mathbb{R}_{>0}\alpha_{i_j}$$

In other words, at each step in evaluating the expression, a coefficient increases.

Definition III.14. The **depth** of a positive root β is the smallest ℓ such that there exists an expression $\beta = s_{i_\ell} \cdots s_{i_1}(\alpha_{i_0})$

Lemma III.15 ([BB05, Lemma 4.6.2])**.** *For a simple reflection s_i and a root $\beta \in \Phi^+ - \{\alpha_i\}$,*

$$\operatorname{depth}(s_i\beta) = \begin{cases} \operatorname{depth}(\beta) - 1 & s_i\beta - \beta \in \mathbb{R}_{<0}\alpha_i \\ \operatorname{depth}(\beta) & s_i\beta = \beta \\ \operatorname{depth}(\beta) + 1 & s_i\beta - \beta \in \mathbb{R}_{>0}\alpha_i \end{cases}$$

Corollary III.16. *An expression $\beta = s_{i_\ell} \cdots s_{i_1}(\alpha_{i_0})$ is of minimal length among such expressions if and only if it is a positive expression.*

Proof. If the expression is positive, then by repeatedly applying Lemma III.15 we can conclude that $\operatorname{depth}(\beta) = \ell$, so ℓ is minimal. Conversely, if the expression were not positive, there would be some index j such that $s_{i_j}s_{i_{j-1}}\cdots s_{i_1}(\alpha_{i_0}) - s_{i_{j-1}}\cdots s_{i_1}(\alpha_{i_0}) \in \mathbb{R}_{\leq 0}\alpha_i$. But then Lemma III.15 implies that $\operatorname{depth}(s_{i_j}s_{i_{j-1}}\cdots s_{i_1}(\alpha_{i_0})) < j$, and so there exists a shorter expression for β. $\square$

Corollary III.17. *Any positive root has a positive expression.*

Constructing such an expression means repeatedly finding a vertex of β at which we can reflect such that the coefficient there decreases and doing so; the corollary guarantees that we can always find such a vertex.

We now explain how to determine shards of the hyperplane $\beta^\perp$ from a positive expression for β. The first step is to understand how applying a single simple reflection affects the fractures of a hyperplane.

Lemma III.18. *Let R be a rank 2 subsystem with fundamental roots γ_1 and γ_2. Suppose $\alpha_i \notin R$. Then the fundamental roots of s_iR are $s_i\gamma_1$ and $s_i\gamma_2$.*

Proof. First, note that s_i permutes the set of positive roots other than α_i. In particular, since neither R nor s_iR contains α_i, applying s_i sends the positive roots of one subsystem to those of the other. Then if β is a positive root in s_iR, $s_i\beta$ is a positive root in R, expressible as a nonnegative linear combination $c_1\gamma_1 + c_2\gamma_2$; thus $\beta = c_1s_i\gamma_1 + c_2s_i\gamma_2$, a nonnegative linear combination. This shows that $s_i\gamma_1$ and $s_i\gamma_2$ are fundamental in s_iR. $\square$

Lemma III.19 ([DST23, Corollary 2.16])**.** *Let β be any positive root, and s_i a reflection such that $\beta \neq \alpha_i$.*

- *If $s_i\beta - \beta \in \mathbb{R}_{>0}\alpha_i$, then the rank 2 subsystems which cut $s_i\beta$ are*

$$\{s_iR \mid R \text{ cutting } \beta\} \cup \{R(\alpha_i, \beta)\}$$

- *If $s_i\beta - \beta \in \mathbb{R}_{<0}\alpha_i$, then the rank 2 subsystems which cut $s_i\beta$ are a subset of $\{s_iR \mid R \text{ cutting } \beta\}$.*

Proof. Suppose that $s_i\beta - \beta \in \mathbb{R}_{>0}\alpha_i$, and let R be any rank 2 subsystem containing β but *not* α_i. By Lemma III.18, R cuts $\beta^\perp$ if and ony if s_iR cuts $(s_i\beta)^\perp$.

Then note that, by assumption, $s_i\beta$ is a nonnegative linear combination of β and α_i. Since these are all positive roots, $s_i\beta$ cannot be fundamental in $R(s_i\beta, \alpha_i)$, so $(s_i\beta)^\perp$ is cut by that system. Putting this and the previous statement together gives the claimed list of fractures.

Finally, suppose that $s_i\beta - \beta \in \mathbb{R}_{<0}\alpha_i$. Then $s_i(s_i\beta) - s_i\beta \in \mathbb{R}_{>0}\alpha_i$ By exchanging the roles of β and $s_i\beta$, we can apply the first part of the lemma to conclude that the fractures of

$\beta^\perp$ are obtained from the fractures of $(s_i\beta)^\perp$ as detailed above. The lemma's second claim follows. $\square$

We now apply this lemma recursively to a positive expression. We can frame the result in two ways: as a list of the subsystems which cut $\beta^\perp$, or as a list of the shards.

Definition III.20. Given an expression of the form $s_{i_\ell}\cdots s_{i_1}(\alpha_{i_0})$, its **truncation at** s_{i_j}, for $1 \le j \le \ell$, is the root

$$s_{i_\ell}\cdots s_{i_{j+1}}(\alpha_{i_j}).$$

Theorem III.21 ([DST23, Proposition 2.14])**.** *Let β be a positive root with an expression $\beta = s_{i_\ell}\cdots s_{i_1}(\alpha_{i_0})$. (Note that this expression need not be positive.) Then every rank 2 subsystem which cuts $\beta^\perp$ is of the form $R(\beta,\tau)$, where τ is some truncation of the given expression for β. If the given expression is positive, then every such $R(\beta,\tau)$ cuts $\beta^\perp$.*

Proof. We induct on ℓ, the length of the expression used. The base case $\ell = 0$ is vacuous: there are no truncations and α_{i_0}, being a simple root, is fundamental in every rank 2 subsystem containing it.

So suppose we know the theorem for expressions of length $\ell-1$. Let $\beta' = s_{i_{\ell-1}}\cdots s_{i_1}(\alpha_{i_0})$, so that $\beta = s_{i_\ell}\beta'$.

Then every rank 2 subsystem which cuts $(\beta')^\perp$ is of the form $R(\beta',\tau)$ for some truncation τ of $s_{i_{\ell-1}}\cdots s_{i_1}(\alpha_{i_0})$. Lemma III.19 then shows that every rank 2 subsytem which cuts β is either of the form $s_{i_\ell}R(\beta',\tau) = R(\beta, s_{i_\ell}\tau)$, or $R(\beta,\alpha_{i_\ell})$. Both $s_{i_\ell}\tau$ and α_{i_ℓ} are truncations of the expression for β, as desired.

Now suppose that the expression for β is positive, from which it follows that the expression for β' is also positive. Then for every truncation τ of the expression for β', $R(\beta',\tau)$ cuts β'. We also know that $s_{i_\ell}\beta' - \beta' \in \mathbb{R}_{>0}\alpha_{i_\ell}$, so by Lemma III.19, each subsytem $s_{i_\ell}R(\beta',\tau) = R(\beta, s_{i_\ell}\tau)$ cuts β, as well as $R(\beta,\alpha_{i_\ell})$. The roots $s_{i\ell}\tau$ and α_{i_ℓ} are all the truncations of our expression for β. $\square$

When our expression for β is positive, there is an alternative way to choose representatives of the cutting subsystems, which will be useful later.

Definition III.22. Given an expression of the form $s_{i_\ell}\cdots s_{i_1}(\alpha_{i_0})$, its **omission at** s_{i_j}, for $1 \le j \le \ell$, is the root

$$s_{i_\ell}\cdots s_{i_{j+1}}s_{i_{j-1}}\cdots s_{i_1}(\alpha_{i_0})$$

Proposition III.23. *Let $\beta = s_{i_\ell}\cdots s_{i_1}(\alpha_{i_0})$ be a positive expression. Let τ be the truncation of this expression at s_{i_j}, and let μ be the omission. Then*

$$R(\beta,\tau) = R(\beta,\mu) = R(\tau,\mu)$$

Proof. By the definition of the reflection s_{i_j}, there is a constant c such that

$$s_{i_j} s_{i_{j-1}} \cdots s_{i_1}(\alpha_{i_0}) = s_{i_{j-1}} \cdots s_{i_1}(\alpha_{i_0}) + c\alpha_{i_j}$$

Applying $s_{i_\ell} \cdots s_{i_{j+1}}$ to both sides then implies

$$\beta = \mu + c\tau$$

The assumption that our expression is positive further implies that $c \neq 0$. Thus all three roots lie in the same rank 2 subsystem and no two of them are linearly dependent, from which the result follows. □

Finally, we note that although the truncations of a positive expression produce all the rank 2 subsystems cutting a hyperplane, different truncations may induce the same rank 2 subsystem. Trivial examples come from the rank 2 root systems B_2 or G_2: each contains roots of depth greater than 1, but only has one rank 2 subsystem (its entirety).

We now rephrase Theorem III.21 in terms of shards.

Definition III.24. Given a convex cone $K \subset V^*$, define

$$\sigma^+(K) = s_i(K \cap \{x \in V^* \mid \langle x, \alpha_i \rangle \geq 0\})$$
$$\sigma^-(K) = s_i(K \cap \{x \in V^* \mid \langle x, \alpha_i \rangle \leq 0\})$$

Theorem III.25 ([DST23, Theorem 3.6])**.** *Let $\beta = s_{i_\ell} \cdots s_{i_1}(\alpha_{i_0})$. Then the shards of $\beta^\perp$ are precisely the cones of the form $\sigma^\pm_{i_\ell} \cdots \sigma^\pm_{i_1}(\alpha^\perp_{i_0})$ which have dimension $n-1$ (where the superscript signs can be chosen independently).*

Proof. We again proceed by induction on ℓ. The base case $\ell = 0$ is again vacuous. So let $\beta' = s_{i_{\ell-1}} \cdots s_{i_1}(\alpha_{i_0})$.

From Lemma III.19, we know that the fractures of $\beta^\perp$ come from applying s_{i_ℓ} to the fractures of $(\beta')^\perp$ and adding $\beta^\perp \cap \alpha^\perp_{i_\ell}$. Thus the shards of $\beta^\perp$ can be obtained from those of $(\beta')^\perp$ by applying s_{i_ℓ} and then further dividing each shard with the plane $\alpha^\perp_{i_\ell}$. This amounts to applying $\sigma^+_{i_\ell}$ and $\sigma^-_{i_\ell}$ to each shard of $(\beta')^\perp$ and taking the full-dimensional cones which result. Describing the shards of $(\beta')^\perp$ using the induction hypothesis, we get our result. □

These two theorems will be essential for our discussion of shards in the following chapters.

Example. We use this characterization of fractures to compute the shards of the A_n root systems. Recall from Section II.1.5 that every root of A_n has the form $\alpha_{i,j} := \alpha_i + \alpha_{i+1} +$

$\ldots + \alpha_j$ for $i \leq j$. A positive expression for this root is given by $s_j s_{j-1} \cdots s_{i+1}(\alpha_i)$. The truncations of this expression then give the roots $\alpha_{k,j}$ for $i < k \leq j$.

In particular, all of these roots are linearly independent. Thus the arrangement formed by the intersections of the $\alpha_{k,j}^\perp$ with $\alpha_{i,j}^\perp$ produces 2^{j-i} shards: the shard a point x belongs to is determined by the signs of $\langle x, \alpha_{k,j} \rangle$ for $i < k \leq j$, which are independent of each other. (In general, the shard arrangement will be more complicated than this.)

III.5: The root poset

Positive expressions for roots are closely analogous to reduced expressions for elements in the Coxeter group — both are expressions using a minimal number of generators. Reduced expressions for a group element w correspond to saturated chains from 1 to w in the weak order — ascending such a chain corresponds to iteratively multiplying by generators such that the expression obtained at each step is reduced. Similarly, it's interesting to consider an ordering of the set of positive roots in which saturated chains correspond to positive expressions.

Definition III.26. The **root poset** is a partial ordering of Φ^+ in which $\gamma \leq \beta$ if $\beta = s_{i_\ell} \cdots s_{i_1}(\gamma)$ such that $\mathrm{depth}(s_{i_j} \cdots s_{i_1}(\gamma)) = \mathrm{depth}(\gamma) + j$ for all $1 \leq j \leq \ell$.

It follows from Lemma III.15 that the cover relations in this poset are of the form $\beta \lessdot s_i\beta$, where $s_i\beta - \beta \in \mathbb{R}_{>0}\alpha_i$.

Example. Figure 22 shows the root poset of the A_3 root system. There are 4 different positive expressions for the root $\alpha_1 + \alpha_2 + \alpha_3$, which correspond to the 4 different paths from the top down to the bottom of the poset:

$$s_3 s_2(\alpha_1) = s_3 s_2(\alpha_2) = s_1 s_3(\alpha_2) = s_1 s_2(\alpha_3)$$

We note that there are two different structures called the "root poset". A more commonly used definition [Arm09, Definition 5.1.1] defines $\gamma \leq \beta$ if $\beta - \gamma$ is a nonnegative linear combination of simple roots. If $\gamma \leq \beta$ in our root poset, then that is also the case in this other root poset, but the two are rarely equivalent. We will not consider this other poset relation in this thesis.

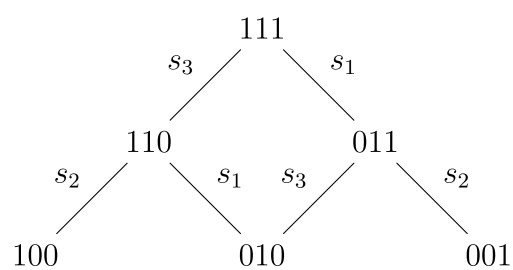


Figure 22: The root poset of A_3. Each cover is labeled with the simple reflection that induces it.

CHAPTER IV
Uniformity of Stretched Root Systems, Root Posets, and Shards

The material of this chapter originally appeared in [Dan21].

In this chapter, we study the operation of stretching a Dynkin diagram G to obtain a new diagram $\mathrm{str}_m(G)$ by replacing a vertex with a path of $m+1$ vertices. Alongside this, we get an operation of stretching on roots: if β is a root of the root system of G, then $\mathrm{str}_m(\beta)$ is a root of the root system of $\mathrm{str}_m(G)$, obtained by repeating the coefficient at the stretched vertex along all vertices of the resulting path.

Here we show three patterns which apply to the whole family of diagrams $\mathrm{str}_m(G)$ and roots $\mathrm{str}_m(\beta)$. First, we show that the depth of the root $\mathrm{str}_m(\beta)$ is linear in m. Secondly, we give a description of the shards of $\mathrm{str}_m(\beta)^\perp$ for all sufficiently large m. (In the case that we are appending a tail to the graph, rather than inserting a path in the middle, the shards are much simpler, and we work out this case in more explicit detail.) Finally, we describe all the roots which lie below $\mathrm{str}_m(\beta)$ in the root poset for sufficiently large m.

In [Dan21], these properties were referred to as "stability", inspired by the similar terminology in representation stability, but as this thesis features a different, unrelated notion also called stability, we instead refer to this chapter's phenomena as "uniformity". We're pretty sure no one has ever used this term for anything else.

IV.1: Stretching diagrams

Definition IV.1. Let G be a Dynkin diagram, j a vertex of G, and $L_j \sqcup R_j$ a partition of the neighbors of j into two subsets. We call j an **elastic vertex**, L_j and R_j its **left** and **right neighbors** respectively, and the tuple (j, L_j, R_j) **elastic data**.

Then the m**-stretched diagram** $\mathrm{str}_m(G)$ is obtained by:

- replacing the vertex j with vertices $j_0, \ldots, j_m$;

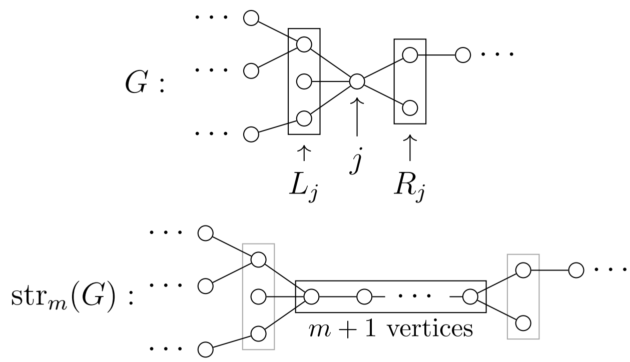


Figure 23: Above, we specify elastic data for G. Below, we illustrate the resulting family of stretched diagrams $\mathrm{str}_m(G)$.

- replacing the edges between j and L_j with correspondingly labeled edges between j_0 and L_j;
- replacing the edges between j and R_j with correspondingly labeled edges between j_m and R_j;
- and inserting unlabeled edges between j_p and j_{p+1} for $0 \leq p < m$.

We call the subdiagram induced by $j_0, \ldots, j_m$ the **stretched path**. This construction is illustrated in Figure 23.

There is a natural way to attach a Cartan matrix to the stretched diagram.

Definition IV.2. Let A be a Cartan matrix for the Coxeter diagram G. Then the m-**stretched Cartan matrix**, $\mathrm{str}_m(A)$, has rows and columns indexed by the vertices of $\mathrm{str}_m(G)$, with

$$\mathrm{str}_n(A)_{ik} = \begin{cases} A_{ik} & i, k \notin \{j_0, \ldots, j_m\} \\ A_{jk} & (i = j_0 \text{ and } k \in L_j) \text{ or } (i = j_m \text{ and } k \in R_j) \\ A_{ij} & (k = j_0 \text{ and } i \in L_j) \text{ or } (k = j_m \text{ and } i \in R_j) \\ 2 & i = k = j_p \\ -1 & i = j_p, k = j_{p\pm 1} \\ 0 & \text{otherwise} \end{cases}$$

If Φ is the root system associated to A, then the m-**stretched root system**, $\mathrm{str}_m(\Phi)$, is the root system associated to $\mathrm{str}_m(A)$.

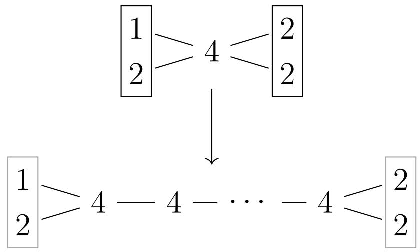


Figure 24: Stretching a root from Φ to produce a root in $\mathrm{str}_m(\Phi)$.

In what follows, assume that we have fixed a diagram G, an associated Cartan matrix A and root system Φ, and elastic data (j, L_j, R_j) for G.

One way of relating the different stretches of a root system is to embed $\mathrm{str}_{m'}(\Phi) \hookrightarrow \mathrm{str}_m(\Phi)$ for $m' < m$. There is a natural way of doing this.

Definition IV.3. Let β be an integer-valued function on the vertices of G. Then $\mathrm{str}_m(\beta)$ is the function on $\mathrm{str}_m(G)$ with value $\beta(j)$ at all the vertices j_p on the stretched path and with the same values as β elsewhere. This construction is illustrated in Figure 24.

For this to give an embedding of Φ into $\mathrm{str}_m(\Phi)$, we would need to know that when β is a root, $\mathrm{str}_m(\beta)$ is as well. This turns out to be true.

Proposition IV.4. *Let $\beta \in \Phi$. Then $\mathrm{str}_m(\beta) \in \mathrm{str}_m(\Phi)$.*

Proof. It will suffice to assume β is positive. We proceed by induction on $\mathrm{depth}(\beta)$. First consider the base case that β is simple: either $\mathrm{str}_m(\beta)$ is also simple, or it has value 1 on the stretched path and 0 elsewhere, and this is straightforward to obtain by reflections of a simple root.

Now consider any positive root β. If it is possible to reflect at a vertex other than j and decrease the value there, obtaining a root of lesser depth β', then we can perform the same operation to $\mathrm{str}_m(\beta)$ and get $\mathrm{str}_m(\beta')$. By the induction hypothesis, $\mathrm{str}_m(\beta')$ is a root, so $\mathrm{str}_m(\beta)$ is too.

Otherwise, we can reflect at j and decrease the value there to get $\beta' := s_j(\beta)$. Then

$$\begin{aligned}\mathrm{str}_m(\beta') &= s_{j_0} s_{j_1} \cdots s_{j_m} s_{j_{m-1}} \cdots s_{j_1} s_{j_0}(\mathrm{str}_m(\beta)) \\ &= s_{j_m} s_{j_{m-1}} \cdots s_{j_0} s_{j_1} \cdots s_{j_{m-1}} s_{j_m}(\mathrm{str}_m(\beta)).\end{aligned}$$

By the induction hypothesis, $\mathrm{str}_m(\beta')$ is a root, so $\mathrm{str}_m(\beta)$ is too. □

As an aside, we note that a reverse version of Proposition IV.4 also holds: roots with repeated coefficients can be squished to give roots of a smaller diagram.

Proposition IV.5. *Let β be any integer-valued function on the vertices of G such that $\mathrm{str}_1(\beta)$ is a root of $\mathrm{str}_1(\Phi)$. Then β is a root of Φ.*

Proof. We assume without loss of generality that $\mathrm{str}_1(\beta)$ is positive. Let $\widetilde{\beta} := \mathrm{str}_1(\beta)$, and define roots $\{\widetilde{\alpha}_i \mid i \in G\}$ in $\mathrm{str}_1(\Phi)$ by

$$\widetilde{\alpha}_i := \begin{cases} \alpha_i & i \neq j \\ \alpha_{j_0} + \alpha_{j_1} & i = j \end{cases}$$

Then $\widetilde{\beta}$ is a nonnegative linear combination of these.

We proceed by induction on $\mathrm{depth}(\widetilde{\beta})$. The base case is when $\widetilde{\beta} = \widetilde{\alpha}_i$, which is trivial. Now suppose $\widetilde{\beta}$ is different from these. We have $(\widetilde{\beta}^\vee, \widetilde{\beta}) = 2 > 0$. Thus, writing out $\widetilde{\beta}^\vee$ as a nonnegative linear combination of roots $\widetilde{\alpha}_i$, at least one of the pairings $(\widetilde{\alpha}_i, \widetilde{\beta})$ is positive.

If $(\widetilde{\alpha}_i, \widetilde{\beta}) > 0$ for some $i \neq j$, then $s_i(\widetilde{\beta}) - \widetilde{\beta} = -(\widetilde{\alpha}_i, \widetilde{\beta})\widetilde{\alpha}_i$ is a negative multiple of $\widetilde{\alpha}_i$, and so $s_i(\widetilde{\beta}) < \widetilde{\beta}$ in the root poset. We still have $s_i(\widetilde{\beta})(j_0) = s_i(\widetilde{\beta})(j_1)$, and so by the induction hypothesis there exists a root $\beta' \in \Phi$ such that $\mathrm{str}_1(\beta') = s_i(\widetilde{\beta})$. But then $s_i(\beta') = \beta$, so β is a root.

On the other hand, if $(\widetilde{\alpha}_j, \widetilde{\beta}) > 0$, then $s_{j_0}s_{j_1}s_{j_0}(\widetilde{\beta}) = \widetilde{\beta} - (\widetilde{\alpha}_j, \widetilde{\beta})\widetilde{\alpha}_j$ has two coefficients which are smaller than those of $\widetilde{\beta}$, so in applying $s_{j_0}s_{j_1}s_{j_0}$ we must have gone down in the root poset at least twice and up at most once, implying $\mathrm{depth}(s_{j_0}s_{j_1}s_{j_0}(\widetilde{\beta})) < \mathrm{depth}(\widetilde{\beta})$. As above, by the induction hypothesis there is some $\beta' \in \Phi$ such that $\mathrm{str}_1(\beta') = s_{j_0}s_{j_1}s_{j_0}(\widetilde{\beta})$, and direct computation shows that $s_j(\beta') = \beta$, implying β is also a root. $\square$

IV.2: Stretching and positive expressions

If we start with a positive expression for β and iterate the argument in Proposition IV.4, we get an expression for $\mathrm{str}_m(\beta)$ in terms of simple reflections applied to a simple root, but it may no longer be a positive expression. We examine when this happens, and obtain a result on the depth of stretched roots in the process.

Consider a positive root β such that $s_j(\beta) < \beta$ in the root poset. Let b be the coefficient at j. Let $i_1, \ldots, i_\ell$ be the left neighbors of j, with coefficients $a_1, \ldots, a_\ell$, and let $k_1, \ldots, k_r$ be the right neighbors, with coefficients $c_1, \ldots, c_r$. Let $S_L := \sum_q -A_{ji_q}a_q$ and $S_R := \sum_s -A_{jk_s}c_s$. Then to assume $s_j(\beta) < \beta$ means $S_L + S_R - b < b$. In particular, at least one of S_L and S_R must be less than b. The situation then splits into three cases:

(1) both S_L and S_R are less than b.

(2) one of S_L and S_R is equal to b.

(3) one of S_L and S_R is greater than b.

This trichotomy classifies the different possible relationships between $s_j(\beta)$ and β in the root poset.

Lemma IV.6. *(1) In Case 1,* $\mathrm{str}_m(s_j(\beta)) < \mathrm{str}_m(\beta)$*, and*

$$\mathrm{depth}(\mathrm{str}_m(\beta)) = \mathrm{depth}(\mathrm{str}_m(s_j(\beta))) + (2m+1).$$

(2) In Case 2 above, $\mathrm{str}_m(s_j(\beta)) < \mathrm{str}_m(\beta)$*, and*

$$\mathrm{depth}(\mathrm{str}_m(\beta)) = \mathrm{depth}(\mathrm{str}_m(s_j(\beta))) + (m+1).$$

(3) In Case 3, $\mathrm{str}_m(s_j(\beta))$ *and* $\mathrm{str}_m(\beta)$ *are incomparable, and*

$$\mathrm{depth}(\mathrm{str}_m(\beta)) = \mathrm{depth}(\mathrm{str}_m(s_j(\beta))) + 1.$$

Proof. Suppose without loss of generality that $S_L < b$. We know from the proof of Proposition IV.4 that

$$\mathrm{str}_m(s_j(\beta)) = s_{j_0} s_{j_1} \cdots s_{j_m} s_{j_{m-1}} \cdots s_{j_1} s_{j_0}(\mathrm{str}_m(\beta))$$

If $S_L \geq b$, we instead use the expression

$$\mathrm{str}_m(s_j(\beta)) = s_{j_m} s_{j_{m-1}} \cdots s_{j_0} s_{j_1} \cdots s_{j_{m-1}} s_{j_m}(\mathrm{str}_m(\beta))$$

We then check, in each case of the trichotomy, whether each of these simple reflections steps up or down in the root poset, and use the fact that the poset is graded by depth. Let $b' := S_L + S_R - b$ be the coefficient at j in $s_j(\beta)$. Then Figure 25 shows the result of applying each reflection in turn.

(1) If $S_L, S_R < b$, then $b' = S_L + S_R - b$ is less than both. Thus at each of the $2m+1$ steps in Figure 25, a coefficient decreases. So $\mathrm{str}_m(s_j(\beta)) < \mathrm{str}_m(\beta)$ and $\mathrm{depth}(\mathrm{str}_m(s_j(\beta))) = \mathrm{depth}(\mathrm{str}_m(\beta)) - (2m+1)$.

(2) Assuming $S_L < b$, this case happens when $S_R = b$, in which case $b' = S_L$. In particular, after we apply s_{j_m} halfway through Figure 25, we have already reached $\mathrm{str}_m(s_j(\beta))$ after $m+1$ steps. At each of those steps a coefficient decreases, so $\mathrm{str}_m(s_j(\beta)) < \mathrm{str}_m(\beta)$ and $\mathrm{depth}(\mathrm{str}_m(s_j(\beta))) = \mathrm{depth}(\mathrm{str}_m(\beta)) - (m+1)$.

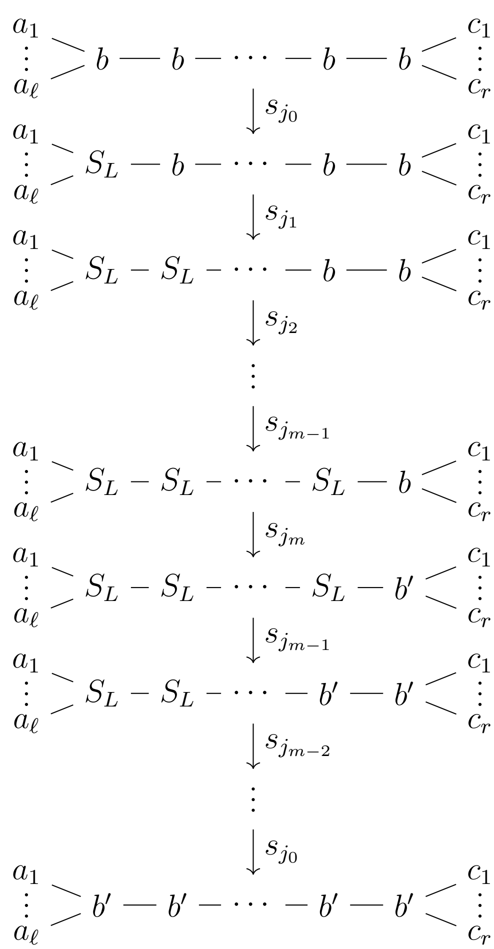


Figure 25: The sequence of roots appearing at each step of the expression $\mathrm{str}_m(s_j(\beta)) = s_{j_0}s_{j_1}\cdots s_{j_m}s_{j_{m-1}}\cdots s_{j_1}s_{j_0}(\mathrm{str}_m(\beta))$.

(3) Assuming $S_L < b$, this case happens when $S_R > b$, in which case $b' > S_L < b$. Thus the first $m+1$ steps in Figure 25 decrease coefficients, while the remaining m increase coefficients. Thus $\text{depth}(\text{str}_m(s_j(\beta))) = \text{depth}(\text{str}_m(\beta)) - 1$. In particular, if $\text{str}_m(s_j(\beta))$ were comparable to $\text{str}_m(\beta)$, it would be covered by $\text{str}_m(\beta)$, but since they differ in more than one coefficient this is not possible.

□

This gives us our first numerical result on stretching a root:

Theorem IV.7. *For any positive root β, there exists an integer t such that* $\text{depth}(\text{str}_m(\beta)) = tm + \text{depth}(\beta)$.

Proof. By induction on $\text{depth}(\beta)$. For any simple root based away from vertex j, the stretched root's depth is 1 (so $t = 0$), while for the simple root α_j it is $m + 1$ (so $t = 1$).

Then suppose we have a cover $s_i(\beta) \lessdot \beta$ in the root poset, such that $\text{depth}(\text{str}_m(s_i(\beta))) = t'm + \text{depth}(s_i(\beta))$. If $i \neq j$, then $\text{str}_m(s_i(\beta)) \lessdot \text{str}_m(\beta)$ is still a cover, and so

$$\text{depth}(\text{str}_m(\beta)) = t'm + \text{depth}(s_i(\beta)) + 1 = t'm + \text{depth}(\beta).$$

If $i = j$, then Lemma IV.6 implies

$$\text{depth}(\text{str}_m(\beta)) = t'm + \text{depth}(s_j(\beta)) + cm + 1 = (t' + c)m + \text{depth}(\beta),$$

where $c = 0$, 1, or 2. □

Definition IV.8. The **depth growth rate** of β is this integer t.

We end this section by looking in more detail at examples of stretching the roots in a cover relation.

In case (1), since $\text{str}_m(\beta)$ and $\text{str}_m(s_j(\beta))$ are comparable, we can consider the interval between them in the root poset. Figure 26 shows a cover relation exhibiting case (1), together with the interval between their 3-stretched versions. This reveals a bit of type A behavior. If we consider the root poset as an order relation on all roots, rather than just positive roots, this interval is the A_m root poset.

In case (2), $\text{str}_m(\beta)$ and $\text{str}_m(s_j(\beta))$ are still comparable, but now the interval between them is just the chain forming the top half of Figure 25: each reflection in that chain is the only one we can make while decreasing a coefficient on the stretched path.

In case (3), although $\text{str}_m(\beta)$ and $\text{str}_m(s_j(\beta))$ are incomparable, we can situate them in a sideways version of the interval in Figure 26, accounting for the fact that reflections which go down in the root poset in case (2) may go up in case (3). This is illustrated in Figure 27.

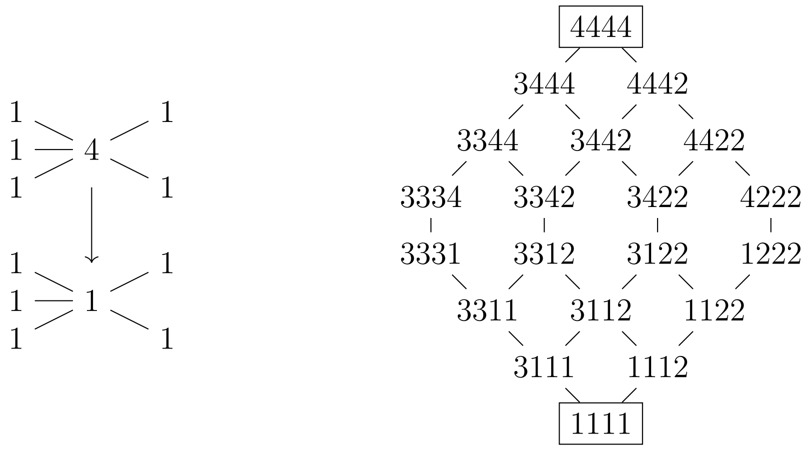


Figure 26: A cover exhibiting case (1) of Lemma IV.6 and the interval between the 3-stretched roots. Roots on the right are represented by their values on the stretched path.

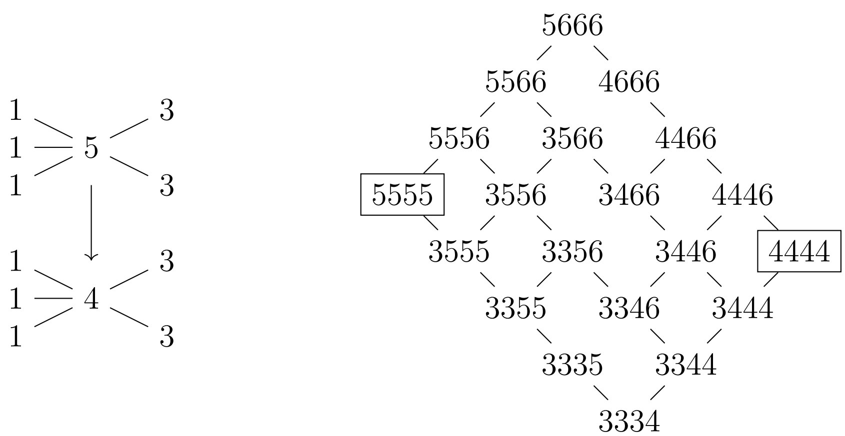


Figure 27: A cover exhibiting case (3) of Lemma IV.6 and the analog of the interval shown in Figure 26.

In all cases, we see the effect of stretching on depth stated in Lemma IV.6.

IV.3: Shards of stretched roots

For any construction associated to roots, we can look for a uniform description of what happens when we apply the construction to $\mathrm{str}_m(\beta)$ for sufficiently large m. In the context of this thesis, a natural such construction is the arrangement of shards of $\mathrm{str}_m(\beta)^\perp$.

Our best tool for computing shards is Theorem III.21, realizing fractures as truncations of a positive expression. Thus we'd like to systematically write positive expressions for the stretches $\mathrm{str}_m(\beta)$. As observed in the previous section, we can almost accomplish this by starting with a positive expression for β and replacing each instance of s_j with an appropriate sequence of reflections along the stretched path, but the resulting expression may not be positive.

However, we can avoid this obstruction by stretching a small amount first and then choosing a positive expression.

Definition IV.9. Let G be a Coxeter diagram with elastic data (j, L_j, R_j) and let j_p be a vertex on the stretched path of $\mathrm{str}_{m_0}(G)$. Then the **elastic data induced by** j_p for $\mathrm{str}_{m_0}(G)$ is

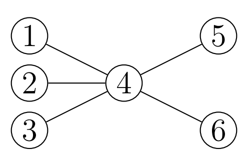
$$\begin{array}{ll} (j_0, L_j, \{j_1\}) & \text{if } p = 0 \\ (j_i, \{j_{p-1}\}, \{j_{p+1}\}) & \text{if } 0 < p < m_0 \\ (x_{m_0}, \{j_{m_0-1}\}, R_j) & \text{if } p = m_0 \end{array}$$

Recall that, in the previous section, we described a trichotomy of situations in which we reflect at the elastic vertex. In particular, say that a reflection applied to a root $\beta \in \Phi$ at the elastic vertex j is a **type (2) reflection** if one of the sums S_L or S_R is equal to $\beta(j)$.

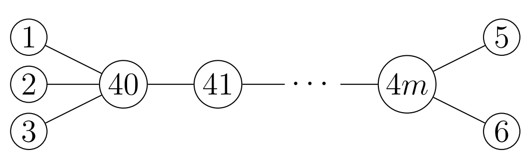
Lemma IV.10. *For any positive root β, there exists some m_0, a vertex j_p on the stretched path of $\mathrm{str}_{m_0}(G)$, and a positive expression for $\mathrm{str}_{m_0}(\beta)$ such that every reflection at the vertex j_p is type (2) with respect to the elastic data induced by j_p.*

We refer to such a positive expression as a **type (2) expression**.

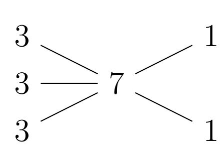
Example. Consider the following diagram.



We let 4 be the elastic vertex, with its left and right neighbors on the left and right as shown. We label the stretched diagram like so:



Then we claim the following root β does not admit a type (2) expression:



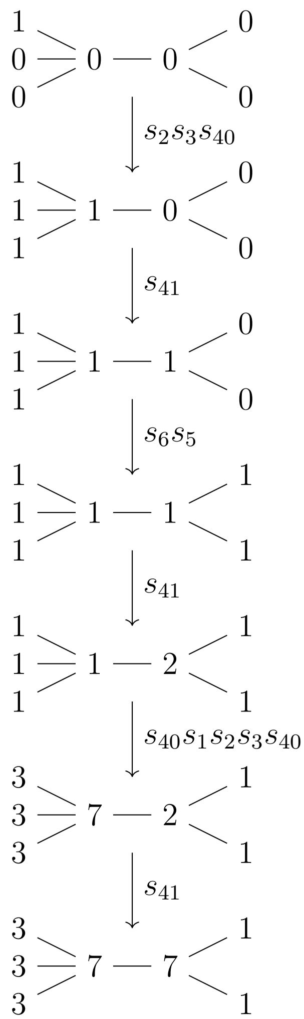


Figure 28: A type (2) expression, where vertex 41 (the right center one) is the elastic vertex. Note that whenever we reflect at vertex 41, its coefficient is equal to the sum of the coefficients on one side.

Reflecting anywhere other than vertex 4 increases a coefficient, so any positive expression must conclude with applying a reflection at vertex 4. Because $S_L = 3+3+3 > 7$, the cover $s_4(\beta) \lessdot \beta$ in the root poset falls under type (3) in the above trichotomy.

But now suppose we stretch the root once, and let 41 be the new elastic vertex. Then we claim that the expression

$$s_{41}s_{40}s_1s_2s_3s_{40}s_{41}s_6s_5s_{41}s_3s_2s_{40}(\alpha_1)$$

is a type (2) expression. This is illustrated in Figure 28.

Proof of Lemma IV.10. We proceed by induction on the coefficient at j. In the base case

$\beta(j) = 0$, choosing $m_0 = 0$ and any positive expression works vacuously.

Now consider a root $\beta \in \Phi$ with $\beta(j) > 0$. Recall that a positive expression for a root is equivalent to a saturated chain in the root poset from that root down to a simple root. So begin constructing a chain down from β in the root poset. Suppose that, having reached the root β', we reflect at j for the first time.

Suppose first that this reflection is type (2). Using the induction hypothesis, we obtain some stretch $\mathrm{str}_{m_0}(s_j(\beta'))$ and a type (2) expression for this root with respect to some j_p. We now use the following lemma:

Lemma IV.11. *If there exists a chain from β down to γ in the root poset for which every reflection at the elastic vertex is type (2), then for any stretch factor m and vertex j_p on the stretched path, there is a chain from $\mathrm{str}_m(\beta)$ down to $\mathrm{str}_m(\gamma)$ for which every reflection at j_p is type (2).*

Proof. Given the assumed chain, we can obtain a chain from $\mathrm{str}_m(\beta)$ down to $\mathrm{str}_m(\gamma)$ by replacing every reflection at j with either $s_{j_0}s_{j_1}\cdots s_{j_m}$ or $s_{j_m}s_{j_{m-1}}\cdots s_{j_0}$, as shown in the proof of Lemma IV.6 and the top half of Figure 25. Figure 25 also illustrates that each reflection on the stretched path is type (2). $\square$

In this case, we can apply the lemma to our chain from β down to β' and get a chain from $\mathrm{str}_{m_0}(\beta)$ down to $\mathrm{str}_{m_0}(s_j(\beta'))$ in which every reflection at j_p is type (2). Splicing this with the type (2) expression for $\mathrm{str}_{m_0}(s_j(\beta'))$ gives us a type (2) expression for $\mathrm{str}_{m_0}(\beta)$.

Now suppose instead that the reflection s_j applied to β' is not type (2). Assume without loss of generality that $S_L < \beta(j)$. Then the reflection s_{j_0} applied to $\mathrm{str}_1(\beta')$ is type (2). By the induction hypothesis, there is some m_0 and a type (2) expression for $\mathrm{str}_{m_0}(s_{j_0}(\mathrm{str}_1(\beta')))$ with respect to some j_{0p}. Then by the above lemma, we can turn the chain from $\mathrm{str}_1(\beta)$ down to $s_{j_0}(\mathrm{str}_1(\beta'))$ into a chain from $\mathrm{str}_{m_0+1}(\beta)$ down to $\mathrm{str}_{m_0}(s_{j_0}(\mathrm{str}_1(\beta')))$, which we can splice with a type (2) expression for $\mathrm{str}_{m_0}(s_{j_0}(\mathrm{str}_1(\beta')))$ to get a type (2) expression for $\mathrm{str}_{m_0+1}(\beta)$. $\square$

Once we have a type (2) expression for $\mathrm{str}_{m_0}(\beta)$ with respect to the elastic data induced by j_p, we can get a positive expression for any $\mathrm{str}_{m_0+m}(\beta) = \mathrm{str}_{m_0+m}(\beta)$ by replacing each instance of s_{j_p} with an appropriate choice of $s_{j_{p0}}s_{j_{p1}}\cdots s_{j_{pm}}$ or $s_{j_{pm}}s_{j_{p(m-1)}}\cdots s_{j_{p0}}$, as in part (1) of Lemma IV.6. If our expression starts with the simple root α_{j_p}, we also must replace it with $s_{j_{p0}}\cdots s_{j_{p(m-1)}}(\alpha_{j_{pm}})$. In particular, the proof of Theorem IV.7 implies:

Proposition IV.12. *The number of reflections at the elastic vertex in a type (2) expression for β, plus 1 if the simple root it starts at is α_j, is the depth growth rate of β.*

Thus we can write down positive expressions for stretched roots in a systematic way, and from that describe their fractures in a systematic way.

To enable cleaner formulas and highlight the analogy with the type A root systems, we write the roots of $\mathrm{str}_m(\Phi)$ in a different basis, still indexed by the vertices of G. We define

$$\gamma_i := \begin{cases} \alpha_i & i \notin \{j_0, \dots, j_m\} \\ \sum_{q=0}^{p} \alpha_{j_q} & i = j_p \end{cases}$$

or, inversely,

$$\alpha_i = \begin{cases} \gamma_i & i \notin \{j_1, \dots, j_m\} \\ \gamma_{j_p} - \gamma_{j_{p-1}} & i = j_p, p \geq 1 \end{cases}$$

This choice of basis echoes the interpretation of the type A Coxeter groups as symmetric groups, with root systems consisting of roots of the form $e_{i+1} - e_i$, as shown in Section II.1.3. In particular, the simple reflections s_{j_p} should act on the γ-basis in a manner resembling transpositions. We have

(IV.3.1)
$$s_{j_0}(\gamma_i) = \begin{cases} \gamma_i + \gamma_{j_0} & i \in L_j \\ -\gamma_{j_0} & i = j_0 \\ \gamma_i - \gamma_{j_0} & i = j_p,\ p \geq 1 \\ \gamma_i & \text{otherwise.} \end{cases}$$

For $1 \leq p \leq m-1$, we have

(IV.3.2)
$$s_{j_p}(\gamma_i) = \begin{cases} \gamma_{j_p} & i = j_{p-1} \\ \gamma_{j_{p-1}} & i = j_p \\ \gamma_i & \text{otherwise} \end{cases}$$

Finally, we have

(IV.3.3)
$$s_{j_m}(\gamma_i) = \begin{cases} \gamma_i + \gamma_{j_m} - \gamma_{j_{m-1}} & i \in R_j \\ \gamma_{j_m} & i = j_{m-1} \\ \gamma_{j_{m-1}} & i = j_m \\ \gamma_i & \text{otherwise} \end{cases}$$

Thus, away from the ends of the path, the s_{j_p} act by transpositions on our γ-basis.

With this property in mind, we examine the fractures of $\mathrm{str}_m(\beta)^\perp$ in the γ-basis.

Theorem IV.13. *Let β be a positive root with a type (2) expression. Then there exist:*

- *a nonnegative integer r;*
- *two lists of formal linear combinations $f_1, \ldots, f_s$ and $g_1, \ldots, g_t$ (where t is the depth growth rate of β) of the following terms:*

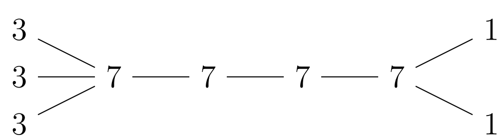
$$\gamma_i \text{ for } i \text{ a vertex of } G \text{ other than } j,$$
$$\gamma_{j_0}, \gamma_{j_1}, \ldots, \gamma_{j_r},$$
$$\gamma_{j_{m-r}}, \gamma_{j_{m-r+1}}, \ldots, \gamma_{j_m};$$

such that for $m \geq 2r$, the fractures of $\mathrm{str}_m(\beta)^\perp$ are precisely its intersections with the hyperplanes

$$\{(f_u)^\perp \mid 1 \leq u \leq s\} \cup \{(g_u - \gamma_{j_v})^\perp \mid 1 \leq u \leq t, r+1 \leq v \leq m-r-1\}.$$

In what follows, we say that a linear form is **unsupported** at a variable γ_{j_p} if its coefficient of γ_{j_p} is 0. Thus, we require the forms f_u and g_u to be unsupported at γ_{j_p} for $r < p < m-r$.

Example. We illustrate what this theorem states for the example introduced earlier in this section. Consider the root $\mathrm{str}_3(\beta)$:



Using the type (2) expression from that previous example, but replacing each instance of s_{41} with an appropriate choice of $s_{43}s_{42}s_{41}$ or $s_{41}s_{42}s_{43}$, we get a positive expression for this root:

$$(s_{43}s_{42}s_{41})s_{40}s_1s_2s_3s_{40}(s_{41}s_{42}s_{43})s_6s_5(s_{43}s_{42}s_{41})s_3s_2s_{40}(\alpha_1)$$

By Theorem III.21, the truncations of this expression give roots γ such that $R(\gamma, \mathrm{str}_3(\beta))$ cuts $\mathrm{str}_3(\beta)^\perp$. The statement of Theorem IV.13 is that these roots take two different forms. On one side are the roots which assign the same coefficient to every vertex on the stretched path.

$$3\alpha_1 + 2\alpha_2 + 2\alpha_3 + 6\alpha_{40} + 6\alpha_{41} + 6\alpha_{42} + 6\alpha_{43} + \alpha_5 + \alpha_6$$
$$\alpha_1 + \alpha_3 + \alpha_{40} + \alpha_{41} + \alpha_{42} + \alpha_{43}$$
$$\alpha_1 + \alpha_2 + \alpha_{40} + \alpha_{41} + \alpha_{42} + \alpha_{43}$$
$$\alpha_1 + \alpha_2 + \alpha_3 + 3\alpha_{40} + 3\alpha_{41} + 3\alpha_{42} + 3\alpha_{43} + \alpha_5$$
$$\alpha_1 + \alpha_2 + \alpha_3 + 3\alpha_{40} + 3\alpha_{41} + 3\alpha_{42} + 3\alpha_{43} + \alpha_6$$
$$\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_{40} + 2\alpha_{41} + 2\alpha_{42} + 2\alpha_{43}$$
$$\alpha_3 + \alpha_{40} + \alpha_{41} + \alpha_{42} + \alpha_{43}$$
$$\alpha_2 + \alpha_{40} + \alpha_{41} + \alpha_{42} + \alpha_{43}$$
$$\alpha_1 + \alpha_{40} + \alpha_{41} + \alpha_{42} + \alpha_{43}$$
$$\alpha_{40} + \alpha_{41} + \alpha_{42} + \alpha_{43}$$

When we rewrite these roots in the γ-basis, they are unsupported at γ_{40}, γ_{41}, and γ_{42}. These are the forms f_u in the theorem statement:

$$\begin{aligned}
f_1 &= 3\gamma_1 + 2\gamma_2 + 2\gamma_3 + 6\gamma_{43} + \gamma_5 + \gamma_6 \\
f_2 &= \gamma_1 + \gamma_3 + \gamma_{43} \\
f_3 &= \gamma_1 + \gamma_2 + \gamma_{43} \\
f_4 &= \gamma_1 + \gamma_2 + \gamma_3 + 3\gamma_{43} + \gamma_5 \\
f_5 &= \gamma_1 + \gamma_2 + \gamma_3 + 3\gamma_{43} + \gamma_6 \\
f_6 &= \gamma_1 + \gamma_2 + \gamma_3 + 2\gamma_{43} \\
f_7 &= \gamma_3 + \gamma_{43} \\
f_8 &= \gamma_2 + \gamma_{43} \\
f_9 &= \gamma_1 + \gamma_{43} \\
f_{10} &= \gamma_{43}
\end{aligned}$$

The remaining roots, which aren't constant on the stretched path, fall into three neat families.

$$
\begin{array}{l}
\alpha_1 + \alpha_2 + \alpha_3 + 3\alpha_{40} + 4\alpha_{41} + 4\alpha_{42} + 4\alpha_{43} + \alpha_5 + \alpha_6 \\
\alpha_1 + \alpha_2 + \alpha_3 + 3\alpha_{40} + 3\alpha_{41} + 4\alpha_{42} + 4\alpha_{43} + \alpha_5 + \alpha_6 \\
\alpha_1 + \alpha_2 + \alpha_3 + 3\alpha_{40} + 3\alpha_{41} + 3\alpha_{42} + 4\alpha_{43} + \alpha_5 + \alpha_6 \\
\hline
\alpha_1 + \alpha_2 + \alpha_3 + 3\alpha_{40} + 3\alpha_{41} + 3\alpha_{42} + 2\alpha_{43} \\
\alpha_1 + \alpha_2 + \alpha_3 + 3\alpha_{40} + 3\alpha_{41} + 2\alpha_{42} + 2\alpha_{43} \\
\alpha_1 + \alpha_2 + \alpha_3 + 3\alpha_{40} + 2\alpha_{41} + 2\alpha_{42} + 2\alpha_{43} \\
\hline
\alpha_{41} + \alpha_{42} + \alpha_{43} \\
\alpha_{42} + \alpha_{43} \\
\alpha_{43}
\end{array}
$$

Again, once we write things in the γ-basis (and flip the signs of some roots, which doesn't change the hyperplanes they define), these are the roots defined using the forms g_u from the theorem statement:

$$
\begin{aligned}
g_1 - \gamma_{4v} &= \gamma_1 + \gamma_2 + \gamma_3 + 4\gamma_{43} + \gamma_5 + \gamma_6 - \gamma_{4v} && v = 0, 1, 2 \\
g_2 - \gamma_{4v} &= -(\gamma_1 + \gamma_2 + \gamma_3 + 2\gamma_{43}) - \gamma_{4v} && v = 0, 1, 2 \\
g_3 - \gamma_{4v} &= \gamma_{43} - \gamma_{4v} && v = 0, 1, 2
\end{aligned}
$$

If we instead consider $\mathrm{str}_m(\beta)$ for some larger m, the list of cutting roots can be described in almost the same way — we just replace γ_{43} with γ_{4m} and let v range from 0 to $m-1$.

Proof of Theorem IV.13. We proceed by induction on the length of our type (2) expression. For simple roots, the proposition is vacuously true. Now suppose that the last reflection applied in our type (2) expression for β is s_i, and let $\beta' = s_i\beta$. Based on our above discussion of how to obtain a reduced expression for $\mathrm{str}_m(\beta)$, we have $\mathrm{str}_m(\beta) = s_i\,\mathrm{str}_m(\beta')$ (if $i \neq j$) or $s_{j_0}s_{j_1}\cdots s_{j_m}\,\mathrm{str}_m(\beta')$ or $s_{j_m}s_{j_{m-1}}\cdots s_{j_0}(\beta')$ (if $i = j$). Our induction hypothesis is that the fractures of $\mathrm{str}_m(\beta')^{\perp}$ admit a uniform description given by some integer r' and linear forms f'_u and g'_u as in the theorem. Then we must show that after applying s_i, $s_{j_0}s_{j_1}\cdots s_{j_m}$, or $s_{j_m}s_{j_{m-1}}\cdots s_{j_0}$ to $\mathrm{str}_m(\beta')$, there exist r, $\{f_u\}$, and $\{g_u\}$ which describe the list of fractures of the resulting hyperplane in the same way.

First, suppose that i is a vertex off the stretched path. By Lemma III.19, the fractures of $\mathrm{str}_m(\beta)^{\perp}$ are its intersections with $s_i f'^{\perp}_u$ and $s_i(g'_u - \gamma_{j_v})^{\perp}$, together with $\alpha_i^{\perp}$.

Because each linear form f'_u is unsupported at γ_{j_p} for all $r' < p < m - r'$, the same is true of $s_i f'_u$, so we can choose $f_u = s_i f'_u$ for $1 \leq u \leq s$.

Additionally, for $0 \leq p \leq m-1$, we have

$$s_i(\gamma_{j_p}) = \begin{cases} \gamma_{j_p} + \gamma_i & i \in L_j \\ \gamma_{j_p} & i \notin L_j \end{cases}$$

Thus we can set $g_u = s_i g'_u$ or $g_u = s_i g'_u - \gamma_i$ for $1 \leq u \leq t$, so that $s_i(g'_u - \gamma_{j_v}) = g_u - \gamma_{j_v}$.

Finally, we note that we can define $f_{s+1} = \alpha_i = \gamma_i$, which is also unsupported on the stretched path, incorporating the one additional fracture $\alpha_i^\perp$ into our description.

In particular, note that stepping from $\mathrm{str}_m(\beta')$ to $\mathrm{str}_m(\beta)$ in this way doesn't increment the number t of collections of fractures of the form $(g_u - \gamma_{j_v})^\perp$, which is consistent with the number of reflections at j in the original expression (and thus, by Proposition IV.12, the depth growth rate) staying the same.

Thus it remains to show that applying $s_{j_0} \cdots s_{j_m}$ or $s_{j_m} \cdots s_{j_0}$ also preserves the uniform description of the fractures.

In the former case, by repeated application of Lemma III.19, the fractures of $\mathrm{str}_m(\beta)^\perp$ are its intersections with $s_{j_0} \cdots s_{j_m} {f'}_u^\perp$ and $s_{j_0} \cdots s_{j_m}(g'_u - \gamma_{j_v})^\perp$, together with the hyperplanes

$$\begin{aligned} \alpha_{j_0}^\perp &= \gamma_{j_0}^\perp \\ s_{j_0}(\alpha_{j_1})^\perp &= \gamma_{j_1}^\perp \\ &\vdots \\ s_{j_0} s_{j_1} \cdots s_{j_{m-1}}(\alpha_{j_m})^\perp &= \gamma_{j_m}^\perp \end{aligned}$$

By combining equations IV.3.1, IV.3.2, and IV.3.3, we note that

$$s_{j_0} s_{j_1} \cdots s_{j_m}(\gamma_i) = \begin{cases} \gamma_i + \gamma_{j_0} & i \in L_j \\ \gamma_{j_{p+1}} - \gamma_{j_0} & i = j_p,\ 0 \leq p \leq m-1 \\ -\gamma_{j_0} & i = j_m \\ \gamma_i + \gamma_{j_m} & i \in R_j \\ \gamma_i & \text{otherwise} \end{cases}$$

Let $f_u = s_{j_0} \cdots s_{j_m} f'_u$ for $1 \leq u \leq s$. Then, since f'_u is unsupported at γ_p for $r' < p < m - r'$, each f_u is unsupported at γ_p for $r' + 1 < p < m - r' + 1$. Accordingly, let $r = r' + 1$.

Similarly, we can choose g_u for $1 \leq u \leq t$ such that $s_{j_0} \cdots s_{j_m}(g'_u - \gamma_{j_v}) = g_u - \gamma_{j_{v+1}}$, where g_u is unsupported at γ_p for $r < p < m - r$. Because of our new choice of r, some fractures defined by $(g_u - \gamma_{j_v})$ are no longer supported at γ_p for $r < p < m - r$ and can be added to the list of f's.

Finally, let $g_{t+1} = 0$; then the additional fractures $\gamma_{j_0}^{\perp}$, $\gamma_{j_1}^{\perp}, \dots \gamma_{j_m}^{\perp}$ can either be added to the list of f's or are defined by $(g_{t+1} - \gamma_{j_v})^{\perp}$ for $r < v < m - r$.

Altogether, this shows that the fractures of $\mathrm{str}_m(\beta)^{\perp}$ can be described in the manner given in the theorem. In particular, stepping from $\mathrm{str}_m(\beta')$ to $\mathrm{str}_m(\beta)$ in this way increments the number t of collections of fractures of the form $(g_u - \gamma_{j_v})^{\perp}$ by 1, which is consistent with the number of reflections at j in the original expression (and thus, by Proposition IV.12, the depth growth rate) going up by 1 from β' to β.

The case of applying $s_{j_m} \cdots s_{j_0}$ is similar. The fractures of $\mathrm{str}_m(\beta)^{\perp}$ are its intersections with $s_{j_m} \cdots s_{j_0} f_u'^{\perp}$ and $s_{j_m} \cdots s_{j_0}(g_u' - \gamma_{j_v})^{\perp}$, together with the hyperplanes

$$\begin{aligned} \alpha_{j_m}^{\perp} &= (\gamma_{j_m} - \gamma_{j_{m-1}})^{\perp} \\ s_{j_m}(\alpha_{j_{m-1}})^{\perp} &= (\gamma_{j_m} - \gamma_{j_{m-2}})^{\perp} \\ &\vdots \\ s_{j_m} s_{j_{m-1}} \cdots s_{j_2}(\alpha_{j_1})^{\perp} &= (\gamma_{j_m} - \gamma_{j_0})^{\perp} \\ s_{j_m} s_{j_{m-1}} \cdots s_{j_1}(\alpha_{j_0})^{\perp} &= \gamma_{j_m}^{\perp} \end{aligned}$$

By combining equations IV.3.1, IV.3.2, and IV.3.3, we note that

$$s_{j_m} s_{j_{m-1}} \cdots s_{j_0}(\gamma_i) = \begin{cases} \gamma_i + \gamma_{j_m} & i \in L_j \\ -\gamma_{j_m} & i = j_0 \\ \gamma_{j_{p-1}} - \gamma_{j_m} & i = j_p,\ 1 \le p \le m \\ \gamma_i + \gamma_{j_m} - \gamma_{j_{m-1}} & i \in R_j \\ \gamma_i & \text{otherwise} \end{cases}$$

Let $f_u = s_{j_m} \cdots s_{j_0} f_u'$ for $1 \le u \le s$. Then, since f_u' is unsupported at γ_p for $r' < p < m - r'$, each f_u is unsupported at γ_p for $r' - 1 < p < m - r' - 1$. Accordingly, let $r = r' + 1$.

Similarly, we can choose g_u for $1 \le u \le t$ such that $s_{j_m} \cdots s_{j_0}(g_u' - \gamma_{j_v}) = g_u - \gamma_{j_{v-1}}$, where g_u is unsupported at γ_p for $r < p < m - r$. Because of our new choice of r, some fractures defined by $(g_u - \gamma_{j_v})$ are no longer supported at γ_p for $r < p < m - r$ and can be added to the list of f's.

Finally, let $g_{t+1} = \gamma_{j_m}$; then the remaining fractures can either be added to the list of f's or are defined by $(g_{t+1} - \gamma_{j_v})^{\perp}$ for $r < v < m - r$.

Again, this shows that the fractures of $\mathrm{str}_m(\beta)^{\perp}$ can be described in the manner given in the theorem. And again, stepping from $\mathrm{str}_m(\beta')$ to $\mathrm{str}_m(\beta)$ in this way increments the number t of collections of fractures of the form $(g_u - \gamma_{j_v})^\perp$ by 1, which is consistent with the depth growth rate going up by 1 from β' to β. $\square$

Combining this result with Lemma IV.10, we draw a conclusion for arbitrary roots:

Corollary IV.14. *Let β be any positive root. Then for sufficiently large n, the fractures of $\mathrm{str}_n(\alpha)^\perp$ admit a uniform description as in Theorem IV.13.*

From this uniform description of shard arrangements, we can in particular consider how the number of shards depends on the stretch factor, which emphasizes the analogy with shards in type A. More specifically, we describe the characteristic polynomial, which is somewhat like a q-analog of this enumeration, because why not?

Definition IV.15 ([Sta12, Section 3.7])**.** Let P be any poset. The **Möbius function** $\mu : P \times P \to \mathbb{Z}$ is recursively defined by

$$\begin{aligned} \mu(x, x) &= 1 \\ \mu(x, y) &= 0 \text{ if } x \not< y \\ \mu(x, y) &= -\sum_{x \leq z < y} \mu(x, z) \text{ if } x < y \end{aligned}$$

Definition IV.16 ([Ath96])**.** Let $\mathcal{A}$ be a finite collection of hyperplanes through the origin in $\mathbb{R}^n$. Let $L_{\mathcal{A}}$ be the collection of subspaces obtained by intersecting subsets of these hyperplanes, partially ordered by reverse inclusion. We include the intersection of 0 hyperplanes, which is all of R^n and which we denote $\widehat{0}$; this is the minimal element of $L_{\mathcal{A}}$. Then let μ be the Möbius function of $L_{\mathcal{A}}$.

The **characteristic polynomial** of $\mathcal{A}$ is

$$\chi_{\mathcal{A}}(q) := \sum_{x \in L_{\mathcal{A}}} \mu(\widehat{0}, x) q^{\dim(x)}$$

The characteristic polynomial connects back to more concrete measurements on hyperplane arrangements in a couple of key ways.

Lemma IV.17 ([Ath96, Theorem 1.1])**.** *A hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^n$ has $(-1)^n \chi_{\mathcal{A}}(-1)$ regions.*

Lemma IV.18 ([Ath96, Theorem 2.2])**.** *Let $\mathcal{A}$ be a hyperplane arrangement defined over the integers. Then for q any sufficiently large prime, $\chi_{\mathcal{A}}(q)$ is the number of points in the complement of $\mathcal{A}$ over $\mathbb{F}_q$.*

Thus, the question of counting shards of $\mathrm{str}_m(\beta)^\perp$ can be subsumed in calculating the characteristic polynomial of its shard arrangement. Here, we show how it admits a description uniform in m.

Theorem IV.19. *Suppose the positive root β has depth growth rate t. Let $\chi_m(q)$ be the characteristic polynomial of the shard arrangement of $\mathrm{str}_m(\beta)$. Then there exist polynomials $p_1(q), \ldots, p_t(q)$ and an integer e such that*

$$\chi_n(q) = \sum_{k=1}^{t} p_k(q)(q-k)^{m-e}$$

Proof. By Lemma IV.18, to show $\chi_n(q)$ has the claimed form, it will suffice to show that the point counts over $\mathbb{F}_q$ eventually do.

Choosing a point in the complement of the hyperplanes in Theorem IV.13 amounts to:

- choosing all the coordinates except $\gamma_{j_{r+1}}, \ldots, \gamma_{j_{m-r-1}}$ such that $f_1, \ldots, f_s$ are nonzero;
- plugging these coordinates into $g_1, \ldots, g_t$ and choosing the coordinates $\gamma_{j_{r+1}}, \ldots, \gamma_{j_{m-r-1}}$ independently, each subject to the condition that they are different from all the values of the g's, which excludes at most t values.

To count these points, consider the subarrangement formed by the f's. We stratify its complement according to the number of distinct values assumed by the g's. Each stratum is built up from the hyperplanes $(f_u)^\perp$ and $(g_{u_1} - g_{u_2})^\perp$ through complementation, union, and intersection, and so we can repeatedly use Lemma IV.18 to conclude that the number of points in the kth stratum over $\mathbb{F}_q$ is given by a polynomial $p_k(q)$ for large primes q. Then we independently choose $m-2r$ variables avoiding k values, which can be done in $(q-k)^{m-2r}$ ways. Combining these observations gives the claimed form for $\chi_n(q)$.

∎

Corollary IV.20. *Let n be the number of vertices of G. Then for sufficiently large m, the number of shards of $\mathrm{str}_m(\beta)^\perp$ is*

$$(-1)^{n-e-1} \sum_{k=1}^{t} p_k(-1)(k+1)^{m-e}$$

In particular, it is $O((t+1)^m)$.

Proof. This follows directly from Theorem IV.19 and Lemma IV.17. ∎

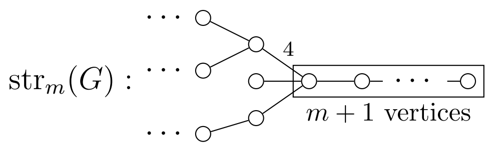
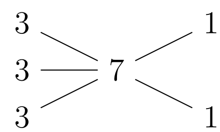


Figure 29: Stretching in the case $R_x = \varnothing$. The vertices in the box form the tail, and all other vertices form the body.

Example. In our running example of the root β given by



our type (2) expression for $\mathrm{str}_1(\beta)$ included 3 reflections at the elastic vertex. Thus by Proposition IV.12, the depth growth rate is 3. Corollary IV.20 then implies that, for m sufficiently large that Theorem IV.13 applies, the number of shards of $\mathrm{str}_m(\beta)^\perp$ is a linear combination of 2^m, 3^m, and 4^m. We can work out this expression just by computing the number of shards for 3 such values of m and solving for the coefficients, from which we conclude that $\mathrm{str}_m(\beta)^\perp$ has

$$462 \cdot 4^m - 172 \cdot 3^m$$

shards.

This exponential growth in the number of shards can be seen as a generalization of the behavior in type A — recall that that top root of A_n has 2^{n-1} shards. The key generalization here is that the base of this exponential expression is linked to the depth growth rate of the original root.

IV.4: Shards of roots with a tail

We now restrict our attention to the case that $R_j = \varnothing$ — in other words, the case in which we append a tail to the Coxeter diagram, as illustrated in Figure 29. In this case, we can describe shards of stretched roots more precisely.

In this context, we will refer to the stretched path of $\mathrm{str}_m(G)$ as the **tail**, and the vertices off the stretched path (not including j_0) as the **body**. We will be classifying roots based on which piece(s) they have nonzero coefficients in, so we make a definition:

Definition IV.21. The **support** of a root β is the subset of the vertices of its Coxeter diagram where it has nonzero coefficients. Say β **is supported at** a vertex i if i is in β's support.

When our diagram has a tail, it imposes inequalities on the roots.

Lemma IV.22. *If β is any root of $\mathrm{str}_m(G)$, then either:*

(1) β is only supported on the tail (and thus can be identified with a root of the A_m root system), or

(2) the sequence of coefficients $\beta(j_0), \beta(j_1), \ldots, \beta(j_m)$ is nonincreasing.

Proof. We proceed by induction on depth. The base case consists of the simple roots, for which the proposition is clear one way or the other.

Now suppose β' is a root satisfying the proposition, and $\beta = s_i(\beta')$ has a greater coefficient at i. If β' is in case (1) and i is in the tail, then β is also in case (1). If β' is in case (1) but i is in the body, then β' must be supported at j_0 in order for β to have greater depth. Since β' can be identified with a root of A_m, its coefficients on the tail (and thus also those of β) must consist of some number of 1's followed by some number of 0's, so they are nonincreasing.

Now suppose β' is in case (2). If i is in the body, nothing changes on the tail and the proposition is also true of β. If $i = j_0$, β differs from β' only in having a larger coefficient at x_0, and the lemma holds. Finally, suppose $i = j_p$ for $p \geq 1$. Then we have

$$\beta'(j_{p+1}) \geq \beta'(j_p) \geq \beta'(j_{p-1})$$

and so

$$\beta(j_p) = \beta'(j_{p+1}) + \beta'(j_{p-1}) - \beta'(j_p) \leq \beta'(j_{p+1}) = \beta(j_{p+1})$$

and

$$\beta(j_p) = \beta'(j_{p-1}) + \beta'(j_{p+1}) - \beta'(j_p) \geq \beta'(j_{p-1}) = \beta(j_{p-1}).$$

□

Because we'll be able to obtain much more explicit results on shards in this case, we can work in slightly greater generality: rather than only considering roots with constant coefficient on the entire stretched path, we can allow them to have multiple blocks of repeated coefficients.

To that end, fix a stretch factor c, and let β be a root of $\mathrm{str}_c(\Phi)$. Then let $\underline{m} = (m_0, \ldots, m_c)$ be any tuple of nonnegative integers, and let $m := m_0 + \ldots + m_c + c$. We define $\mathrm{str}_{\underline{m}}(\beta)$ to be the root of $\mathrm{str}_m(\Phi)$ obtained by stretching β at each vertex on the tail, by a factor of m_p at vertex j_p. See Figure 30.

Definition IV.23. Given $\underline{m}$ as above, a root β of $\mathrm{str}_m(\Phi)$ is **blocky** if it has the form $\mathrm{str}_{\underline{m}}(\overline{\beta})$. It is **strictly blocky** if all coefficients of $\overline{\beta}$ on the tail of $\mathrm{str}_c(\Phi)$ are distinct.

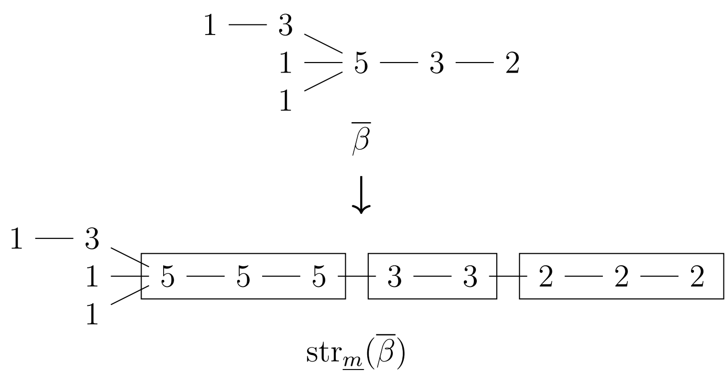


Figure 30: An instance of stretching a root at multiple points on the tail. Here $c = 2$ and $\underline{m} = (2, 1, 2)$. The resulting root is strictly blocky for this choice of $\underline{m}$, because the coefficients 5, 3, and 2 are all distinct.

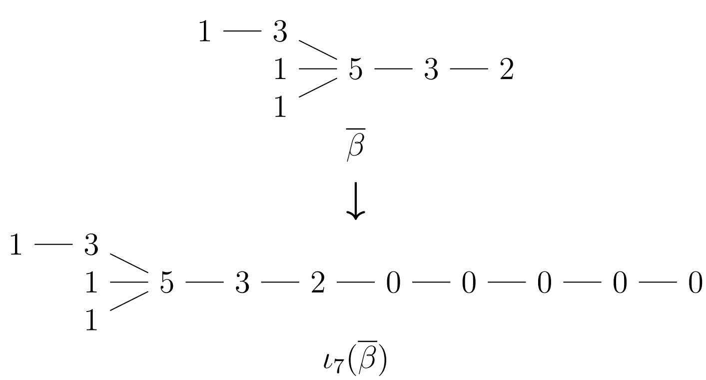


Figure 31: The operation ι_7 embeds $\mathrm{str}_2(\Phi)$ into $\mathrm{str}_7(\Phi)$ by padding the tail with zeroes.

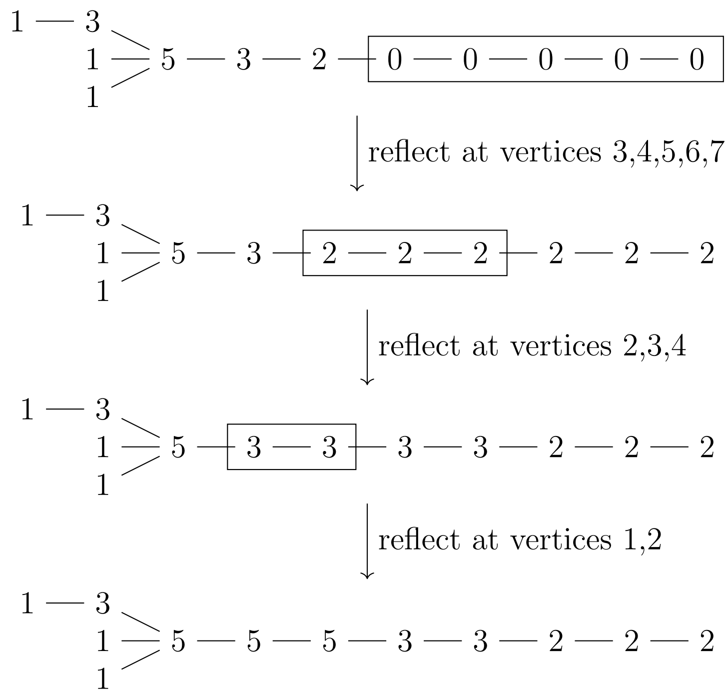


Figure 32: The sequence of reflections described in Lemma IV.24. At each step, we reflect at the boxed vertices from left to right.

Let $q_p := m_0 + \ldots + m_p + p$ for $0 \leq p \leq c$. Additionally, define $q_{-1} = -1$. Note that j_{q_p} is the last of the block of vertices obtained from stretching the pth vertex of $\mathrm{str}_c(G)$. In what follows, we refer to the simple reflections $s_{j,p} := s_{j_p}$ and simple roots $\alpha_{j,p} := \alpha_{j_p}$ to avoid triply nesting subscripts.

Lemma IV.24.

$$\mathrm{str}_{\underline{m}}(\overline{\beta}) = (s_{j,q_0} s_{j,q_0-1} \cdots s_{j,1})(s_{j,q_1} s_{j,q_1-1} \cdots s_{j,2}) \cdots (s_{j,q_c} s_{j,q_c-1} \cdots s_{j,c+1})(\iota_m(\overline{\beta}))$$

Additionally, if $\mathrm{str}_{\underline{m}}(\overline{\beta})$ *is strictly blocky, then applying these reflections increases a coefficient at each step. In particular, if we replace* $\iota_m(\overline{\beta})$ *with one of its positive expressions (which we can identify with a positive expression for* $\overline{\beta}$*), we get a positive expression for* $\mathrm{str}_{\underline{m}}(\overline{\beta})$.

Proof. Both statements follow from direct calculation. The procedure of applying these reflections is illustrated in Figure 32. $\square$

Theorem IV.25. *Let* $\beta = \mathrm{str}_{\underline{m}}(\overline{\beta})$ *for some* $\overline{\beta} \in \mathrm{str}_c(\Phi)$*. The rank 2 subsystems which cut* $\beta^{\perp}$ *are* $R(\beta, \gamma)$*, where* γ *belongs to one of the following sets:*

- *roots of the form* $\mathrm{str}_{\underline{m}}(\overline{\gamma})$*, where* $\overline{\gamma}$ *cuts* $\overline{\beta}^{\perp}$.

- *some subset of the roots of the form*

$$\gamma^r_{pp'} := \alpha_{j,q_{p-1}+r+1} + \alpha_{j,q_{p-1}+r+2} + \ldots + \alpha_{j,q_{p'}}$$

for $0 \leq p \leq p' \leq c$, $1 \leq r \leq m_p$. *If* β *is strictly blocky, all of these roots work.*

This classification of fractures is illustrated in Figures 33 and 34, for the example of the root introduced in Figure 30.

Proof. Lemma IV.24 shows that β admits the expression

$$\mathrm{str}_{\underline{m}}(\overline{\beta}) = (s_{j,q_0}s_{j,q_0-1}\cdots s_{j,1})(s_{j,q_1}s_{j,q_1-1}\cdots s_{j,2})\cdots(s_{j,q_c}s_{j,q_c-1}\cdots s_{j,c+1})(\iota_m(\overline{\beta}))$$

We can obtain a list including all the γ for which $R(\beta,\gamma)$ cuts $\beta^\perp$ by replacing $\iota_m(\overline{\beta})$ with a positive expression for $\overline{\beta}$ and truncating the full expression, by Theorem III.21. The two different bullet points in the statement of the corollary come from whether we truncate within the positive expression for $\iota_m(\overline{\beta})$ or in the portion shown in Lemma IV.24.

In the first case, having chosen a particular reflection in the positive expression for $\overline{\beta}$, let $\overline{\gamma}$ be the truncation there, so that $\overline{\gamma}$ cuts $\overline{\beta}$. Then by Lemma IV.24, truncating the expression for β at the corresponding point produces the root $\mathrm{str}_{\underline{m}}(\overline{\gamma}) = \gamma$. The fundamental roots of $R(\beta,\gamma)$ are obtained by stretching the fundamental roots of $R(\overline{\beta},\overline{\gamma})$, so $\beta^\perp$ is not a fundamental hyperplane and is cut.

The second case considers a truncation of the form

$$\mathrm{str}_{\underline{m}}(\overline{\beta}) = (s_{j,q_0}s_{j,q_0-1}\cdots s_{j,1})(s_{j,q_1}s_{j,q_1-1}\cdots s_{j,2})\cdots s_{j,q_{p'}}s_{j,q_{p'}-1}\cdots s_{j,q+1}(\alpha_{j,q})$$

for any $0 \leq p' \leq c$ and $p'+1 \leq q \leq q_{p'}$. Let p be the smallest index such that $q \leq q_p + (p'-p)$; note that our constraints on q force $0 \leq p \leq p'$. Then let $r = q - (q_{p-1} + p' - (p-1))$. By assumption, $q_{p-1} + (p' - (p-1)) < q \leq q_p + (p'-p)$, so $0 < r \leq m_p$. Then we claim that this truncation is equal to $\gamma^r_{pp'}$.

We can verify this by carefully working through the blocks of reflections defining the truncation. In this argument, we will be working entirely with reflections and roots supported on the tail, where the roots are quite simple. Every such root has the form

$$\alpha_{j,x} + \alpha_{j,x+1} + \ldots + \alpha_{j,y}.$$

Applying $s_{j,x-1}$ or $s_{j,y+1}$ to this root adds $\alpha_{j,x-1}$ or $\alpha_{j,y+1}$, respectively, while applying $s_{j,x}$ or $s_{j,y}$ removes $\alpha_{j,x}$ or $\alpha_{j,y}$ respectively. All other simple reflections have no effect.

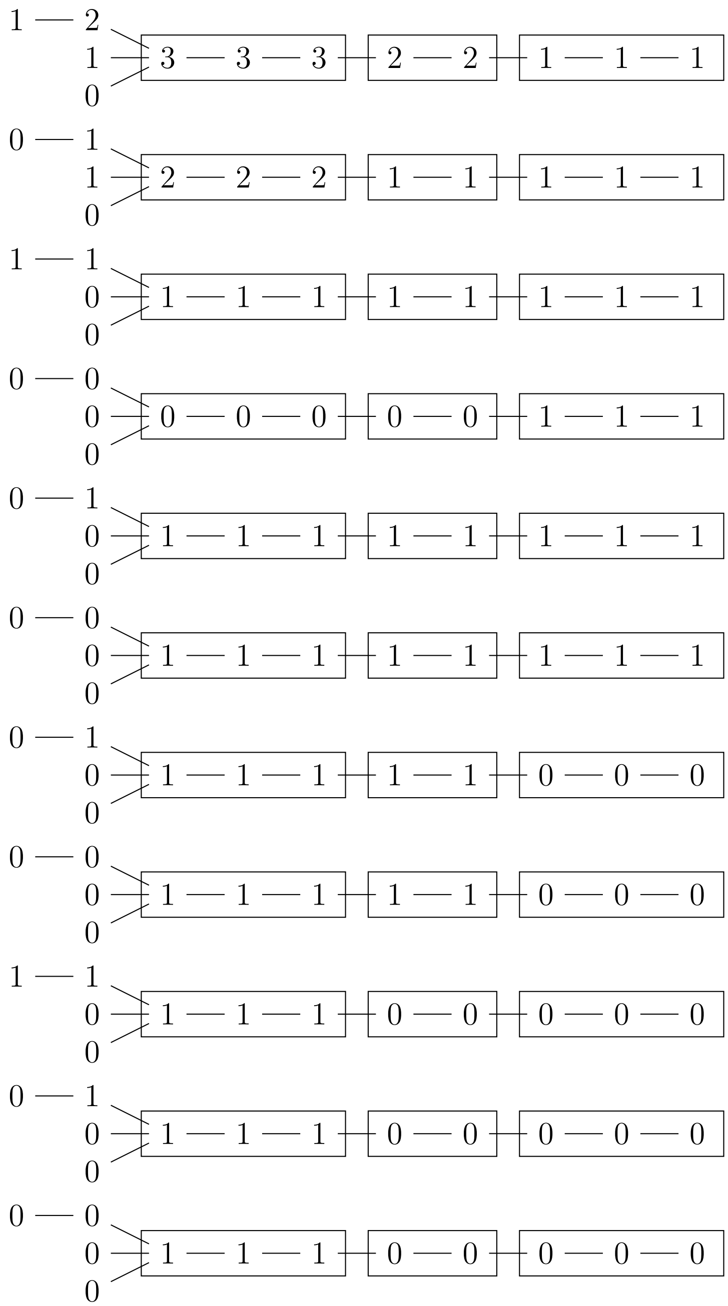


Figure 33: The first part of the list of roots produced by Theorem IV.25, where β is the root appearing in Figure 30. These are obtained by stretching the roots cutting $\overline{\beta}^{\perp}$.

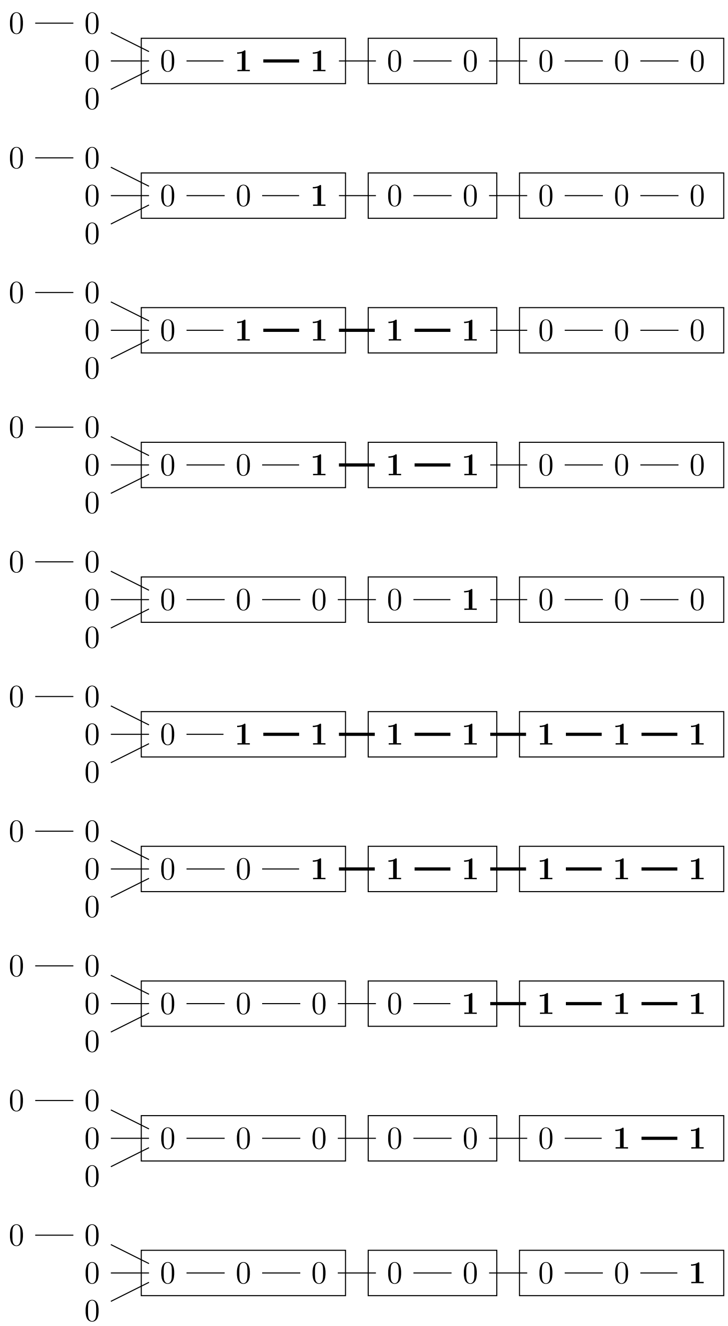


Figure 34: The second part of the list of roots produced by Theorem IV.25, where β is the root appearing in Figure 30. These are the roots which do not arise from stretching, which are supported only on the tail; the support of each is bolded. This list depends only on the tuple of stretch factors $\underline{m}$.

With these rules in mind, we first compute

$$s_{j,q_{p'}} s_{j,q_{p'}-1} \cdots s_{j,q+1}(\alpha_{j,q}) = \alpha_{j,q} + \alpha_{j,q+1} + \ldots + \alpha_{j,q_{p'}}.$$

Then, we apply the block of reflections $s_{j,q_{p'-1}} s_{j,q_{p'-1}-1} \cdots s_{j,p'}$. If $q > q_{p'-1} + 1$, then these reflections are all too far to the left have an effect. On the other hand, if $p' < q \leq q_{p'-1} + 1$, these reflections have no effect *until* we apply s_{q-1}, producing $\alpha_{j,q-1} + \ldots + \alpha_{j,q_{p'}}$, at which point they continue to have no effect.

Similarly, we then apply the block of reflections $s_{j,q_{p'-2}} s_{j,q_{p'-2}-1} \cdots s_{j,p'-1}$. If $q - 1 > q_{p'-2} + 1$ (in particular, if $q > q_{p'-1} + 1$), the reflections are all too far to the left to have an effect. But if $q - 1 \leq q_{p'-2} + 1$ (that is, $q \leq q_{p'-2} + 2$), the reflection s_{q-2} will add $\alpha_{j,q-2}$ to our root and then the others will have no effect.

In general, when we apply our ath block of reflections, we check whether $q \leq q_{p'-a} + a$. If it is not, we know that all remaining reflections are too far to the left to have any effect. If it is, then applying that ath block will augment the intermediate root obtained thus far by adding support at one more vertex on its left end. If p is the smallest index such that $q \leq q_p + (p' - p)$, then we will perform $(p' - p)$ such augmentations. The result is

$$\alpha_{j,q-(p'-p)} + \alpha_{j,q-(p'-p)+1} + \ldots + \alpha_{j,q_{p'}}.$$

which is exactly $\gamma^r_{pp'}$ for the chosen values of p, p', and r. □

IV.5: Downsets in stretched root posets

We conclude this section by returning to the general case of stretching (allowing $R_j \neq \varnothing$ again) and descibing how stretching affects a different structure: the root poset. Although the material of this section isn't directly relevant to the discussion of shards or representation theory elsewhere in this thesis, it is still a window into how stretching affects root systems and positive expressions.

Again, an underlying goal here is to find analogs of the behavior of the type A root posets in the general setting of stretching. Usually, the root system $\mathrm{str}_m(\Phi)$ will be infinite, so in order to carve out finite pieces of interest, we consider downsets defined by stretched roots.

Definition IV.26. For a positive root β, the **downset generated by** β is

$$\downarrow\beta := \{\gamma \in \Phi^+ \mid \gamma \leq \beta\}$$

In the A_n root poset, there is a unique maximal root $\alpha_1 + \ldots + \alpha_n$, which can be viewed as $\mathrm{str}_{n-1}(\alpha_1)$, the stretch of the unique positive root of A_1. The rest of the root poset is then the downset generated by this root. Accordingly, for general root systems with elastic data, we consider downsets $\downarrow\mathrm{str}_m(\beta)$. We'll prove the following result:

Theorem IV.27. *Let β be a positive root. Then there is a polynomial $p(m)$ such that*

$$\# \downarrow\mathrm{str}_m(\beta) = p(m)$$

for sufficiently large m.

We'll do this by constructing a single, finitely specified structure which describes $\downarrow\mathrm{str}_m(\beta)$ for all sufficiently large m.

Example. Consider the D_n root systems. Each of these also has a unique maximal root, given by stretching the maximal root β for D_4:

$$\begin{matrix} 1 \\ 1 \end{matrix} > 2 - \cdots - 2 - 1$$

Thus the entire root poset for D_n is of the form $\downarrow\mathrm{str}_{n-4}(\beta)$.

The roots of D_n for any n fit a finite list of patterns. For example, anything of the form

$$\begin{matrix} 1 \\ 1 \end{matrix} > 2 - \cdots - 2 - 1 - \cdots - 1 - 1$$

is a root. We can compactly describe the roots of this form with the notation

$$\begin{matrix} 1 \\ 1 \end{matrix} > 2^* - 1^* - 1$$

using an asterisk on a coefficient to mean that it can repeat any nonzero number of times. (We distinguish the rightmost vertex because it is not part of the stretched path.) Then we can write down a list of expressions like this which describe the roots of every D_n, show in Figure 35.

In general, however, describing downsets is not quite as simple as allowing values to repeat freely on the stretched path. For example, the downset of the top root in Figure 36 contains roots of the form on the left but not those of the form on the right, as suggested by Figure 27.

Thus, to describe downsets using patterns as in Figure 35, we need our patterns to allow for some values at the ends of the stretched path to not repeat.

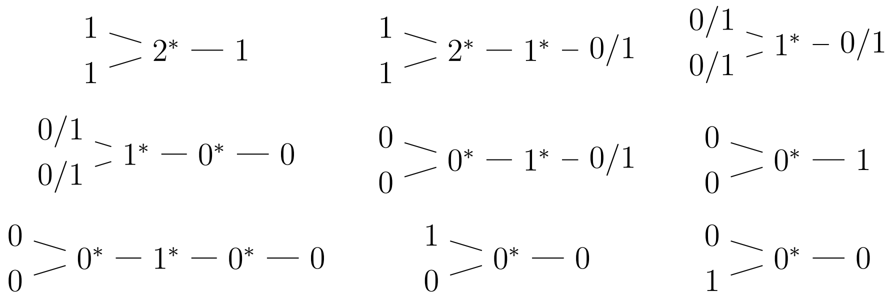


Figure 35: The roots of D_n are precisely the combinations of simple roots which fit these patterns.

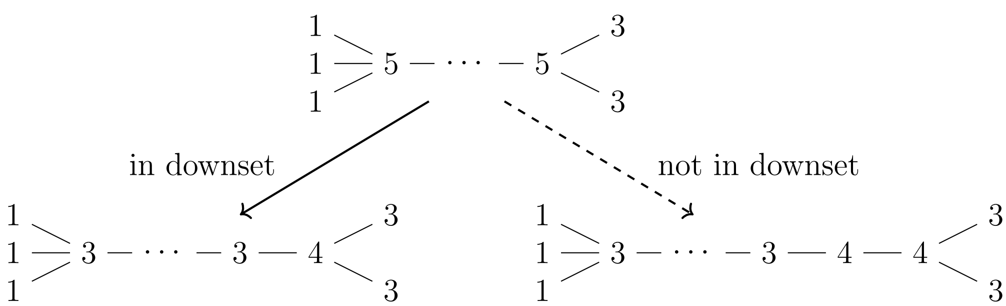


Figure 36: The downset of the top root contains roots of the form on the left, but not those of the form on the right.

Definition IV.28. Let $\overline{\beta}$ be a integer-valued function on the vertices of some stretch of G, together with a marking of some consecutive vertices on the stretched path by asterisks, subject to the constraint that no adjacent asterisked values are the same.

Then the **stretching class** determined by $\overline{\beta}$ consists of all functions in $\bigsqcup_m \mathbb{R}^{\mathrm{str}_m(G)}$ which assume the non-asterisked values at the prescribed places, and which assume the asterisked values along the stretched path in the prescribed order, each repeated any nonzero number of times.

Note that Propositions IV.4 and IV.5 imply that if one element of a stretching class is a root, they all are, so we can also think of stretching classes as subsets of $\bigsqcup_m \mathrm{str}_m(\Phi)$.

As we denote individual roots by Greek letters, we denote stretching classes by barred Greek letters, such as $\overline{\beta}$. We denote the set of roots in $\overline{\beta}$ defined on a specific stretch $\mathrm{str}_m(G)$ by $\overline{\beta}[m]$. We emphasize that, despite this name and notation, stretching classes are not equivalence classes, since they can intersect nontrivially.

Example Start with D_4 and let $\overline{\beta}$ be the stretching class

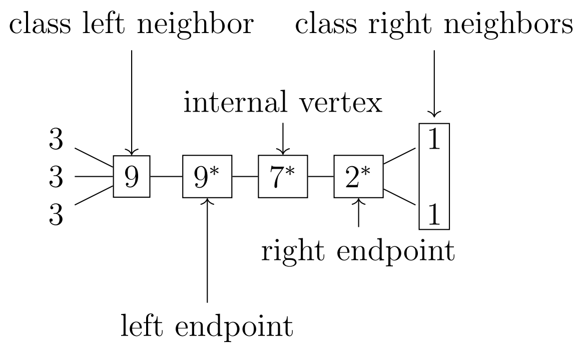


Figure 37: An example of the terminology we use with stretching classes.

$$\begin{matrix} 1 & & \\ & > & 2^* - 1^* - 1 \\ 1 & & \end{matrix}$$

Then $\overline{\beta}[3]$ consists of 3 roots for $\mathrm{str}_3(D_4) = D_7$:

$$\begin{matrix} 1 & & \\ & > & 2 - 1 - 1 - 1 - 1 \\ 1 & & \end{matrix}$$

$$\begin{matrix} 1 & & \\ & > & 2 - 2 - 1 - 1 - 1 \\ 1 & & \end{matrix}$$

$$\begin{matrix} 1 & & \\ & > & 2 - 2 - 2 - 1 - 1 \\ 1 & & \end{matrix}$$

In what follows we will consider how applying a reflection to a root affects the stretching classes it belongs to. To do this, we need to distinguish whether this reflection is happening on or off the path of repeatable values marked by asterisks, or at the path's ends.

We will freely talk about the vertices and coefficients of a stretching class, by which we mean the vertices and coefficients in the defining notation. In this language, we define the **left endpoint** of a stretching class to be the leftmost vertex with an asterisk, and define the **right endpoint** to be the rightmost such vertex. The **internal vertices** will be the other vertices with asterisks. The **class left neighbors** will be the neighbors to the left of the left endpoint, and we define the **class right neighbors** similarly. (Note that these may differ from L_j and R_j, since in general not every vertex on the stretched path will have an asterisk. If the set of class left/right neighbors is not L_j/R_j, it will consist of a single other vertex on the path.) We illustrate these terms in Figure 37.

We may also talk about reflecting at a vertex of a stretching class, which amounts to applying that reflection to the defining notation as if it were an ordinary root on $\mathrm{str}_m(G)$, ignoring the asterisks.

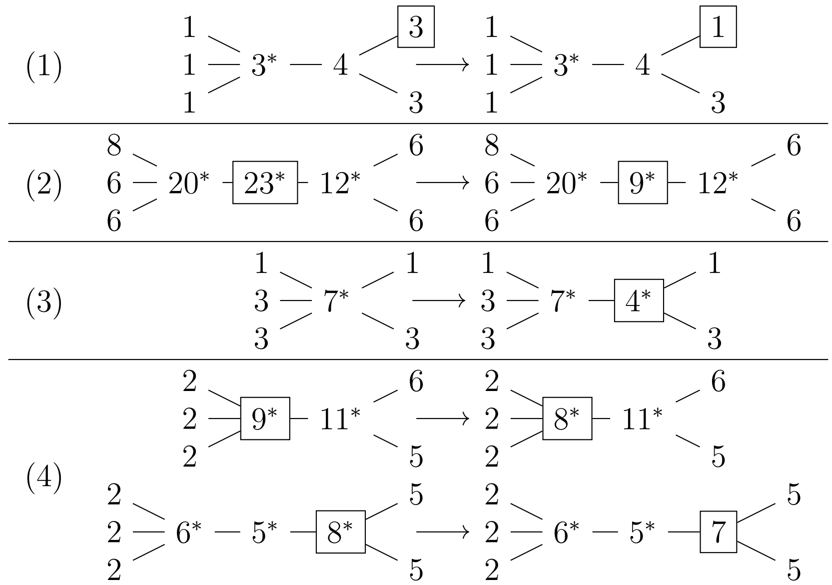


Figure 38: The four operations we can perform on stretching classes which correspond to reflections on their roots.

Finally, let j_ℓ be the left endpoint and let $i_1, \ldots, i_k$ be the class left neighbors. Then the **weighted left sum** of $\overline{\beta}$ is $\sum_q -A_{j_\ell i_q}\overline{\beta}(i_q)$. Note that if $j_\ell = j_0$, this is the quantity S_L from section IV.2, and otherwise it is $\overline{\beta}(j_{\ell-1})$. We likewise define the **weighted right sum**.

Now fix a positive root β. We iteratively construct a directed graph P whose vertices represent stretching classes which, for sufficiently large m, describe $\downarrow\mathrm{str}_m(\beta)$. First, define a stretching class β^* by placing an asterisk on the value of β at j, so that β^* consists of the stretches of β. We make β^* a vertex of P.

Then in each step of constructing P, for each of its vertices $\overline{\gamma}$, we add arrows $\overline{\gamma} \to \overline{\gamma}'$, where $\overline{\gamma}'$ can be obtained from $\overline{\gamma}$ by the following operations:

(1) Reflect at a vertex without an asterisk, such that its coefficient decreases.

(2) Reflect at an internal vertex, such that its coefficient decreases.

(3) If the weighted left (right) sum is less than the coefficient of the left (right) endpoint, insert that sum with an asterisk as the new left (right) endpoint.

(4) If there is more than one vertex with an asterisk, reflect at the left/right endpoint such that the coefficient there decreases. If the new coefficient on the left/right endpoint is greater than or equal to the asterisked coefficient next to it, remove the asterisk from the left/right endpoint.

Figure 38 shows examples of these operations.

In each case, we don't add the arrow if the operation results in a negative coefficient. This allows for the construction of P to eventually stop, and we now show this happens.

Lemma IV.29. *P is finite and acyclic.*

Proof. We track the following tuple of numbers associated to a stretching class in lexicographic order, from most to least significant:

- The sum of the weighted left and right sums.
- The sum of the coefficients at the left and right endpoints.
- The sum of all the coefficients.

We check that the operations used to define P can only decrease this tuple.

(1,2) Operations 1 and 2 don't lengthen the diagram, and they decrease one of the coefficients. Then either the first quantity decreases, or the first two stay the same while the third decreases.

(3) This operation leaves the class left/right neighbors untouched, but decreases the coefficient at either the left or right endpoint, so it keeps the first quantity the same while decreasing the second.

(4) If the reflected coefficient on the left endpoint is greater than or equal to the asterisked coefficient next to it, then it must also be less than the weighted left sum, or else the reflection would not have decreased it. By removing the left endpoint's asterisk, we make it the sole class left neighbor, and so we have decreased the weighted left sum.

On the other hand, if we don't remove the left endpoint's asterisk, then the sum of the weighted left and right sums stays the same while the sum of the left and right endpoints' coefficients decreases.

Thus P has no oriented cycles and (since we require every vertex to have all nonnegative coefficients) no infinite paths. Since each vertex has only finitely many arrows emanating from it, we also know P is finite. $\square$

Now we show the main result of this section.

Theorem IV.30. *Let the graph P be constructed from a root β as above. Let m_0 be the smallest value such that every stretching class in P with a single asterisk has an element defined on $\mathrm{str}_{m_0}(G)$. Then for $m \geq m_0 + 1$,*

$$\downarrow\mathrm{str}_m(\beta) = \bigcup_{\overline{\gamma} \in P} \overline{\gamma}[m]$$

We present each direction of containment as a separate lemma.

Lemma IV.31. *Let β, P, m_0 be as above, and $m \geq m_0 + 1$. Then*

$$\downarrow\mathrm{str}_m(\beta) \subset \bigcup_{\overline{\gamma} \in P} \overline{\gamma}[m]$$

Proof. Certainly $\mathrm{str}_m(\beta)$ is in the latter set, because the first element we added to P in defining it was the stretching class consisting of all $\mathrm{str}_m(\beta)$. We then show that, for any cover relation $\delta' \lessdot \delta$ in the root poset, if $\delta \in \bigcup_{\overline{\gamma} \in P} \overline{\gamma}[m]$, then so is δ'. Let $\overline{\delta}$ be a stretching class in P which contains δ. We claim that δ' belongs either to a stretching class obtained by applying to $\overline{\delta}$ one of the operations used to define P, or to $\overline{\delta}$ itself.

We proceed by cases. Say that a coefficient of δ is **repeatable** if it is represented by an asterisked vertex in $\overline{\delta}$, and say that a repeatable coefficient is **alone** if it is the only coefficient of δ represented by that vertex (i.e., both of its neighbors in δ are different).

(0) If δ' is obtained from δ by reflecting at a repeatable coefficient which is not alone, and it is not the furthest left or furthest right repeatable coefficient, then the reflection there changes the quantities of repeated coefficients but not which ones appear:

$$\begin{array}{c} \cdots - a - b - b - \cdots \\ \downarrow \\ \cdots - a - a - b - \cdots \end{array}$$

Thus δ' is also in $\overline{\delta}$.

(1) If δ' is obtained from δ by reflecting at a non-repeatable coefficient, then δ' belongs to a stretching class obtained by applying operation 1 above.

(2) If we reflect at an alone coefficient other than the furthest left or furthest right repeatable coefficient, then δ' lies in the stretching class obtained by applying operation 2 above.

(3) If we reflect at the furthest left or furthest right repeatable coefficient and it is not alone, then δ' lies in the stretching class obtained by applying operation 3.

(4) If we reflect at the furthest left or furthest right repeatable coefficent and it is alone, then because $m \geq m_0 + 1$, $\overline{\delta}$ must have more than one coefficient with an asterisk. Then δ' lies in the stretching class obtained by applying operation 4.

□

Lemma IV.32. *Let β, P be as above. Then*

$$\bigcup_{\overline{\gamma}\in P} \overline{\gamma}[m] \subset \downarrow \mathrm{str}_m(\beta)$$

In particular, this second containment is true for all m.

Proof. We know that $\mathrm{str}_m(\beta)$ is the sole member of $\beta^*[m]$, and it is in $\downarrow \mathrm{str}_m(\beta)$. Then we will show that, for each arrow $\overline{\gamma} \to \overline{\delta}$ of P and $\delta \in \overline{\delta}[m]$, there is some $\gamma \in \overline{\gamma}$ such that $\delta \leq \gamma$. We check this for each of the four operations:

(1) If $\overline{\delta}$ is obtained from $\overline{\gamma}$ by a reflection at a non-asterisked vertex, then any root in $\overline{\delta}$ is obtained from one in $\overline{\gamma}$ with the same amount of each asterisked coefficient, just by performing that reflection.

(2) Suppose $\overline{\delta}$ is obtained from $\overline{\gamma}$ by a reflection at an internal vertex where $\overline{\delta}$ has coefficient b, and let $\delta \in \overline{\delta}[m]$. If b is alone in δ, then reflecting there will bring us up to a root in $\overline{\gamma}$. Otherwise, we know that some neighbor of b^* in $\overline{\delta}$ must have a coefficient $b' > b$, or else b could not be smaller than the value of $\overline{\gamma}$ at the same vertex. By repeatedly reflecting at the instances of b which neighbor instances of b', we decrease the number of repetitions of b while moving up in the root poset:

$$\begin{array}{c}
\cdots - b - b' - b' - b' - \cdots \\
\uparrow \\
\cdots - b - b - b' - b' - \cdots \\
\uparrow \\
\cdots - b - b - b - b' - \cdots
\end{array}$$

Thus we reduce to the case that b is alone.

(3) Suppose that $\overline{\delta}$ is obtained from $\overline{\gamma}$ by inserting the weighted (without loss of generality) left sum, which we call b, as the new left endpoint, and let $\delta \in \overline{\delta}[m]$. As in case (2), if b is alone on the left end of the stretched path, reflecting there moves back up to an element of $\overline{\gamma}$, while if b is not alone, the next coefficient to the right on the path will be larger, and we can move up the root poset to a case where b is alone.

(4) Suppose that $\overline{\delta}$ is obtained from $\overline{\gamma}$ by reflecting at the (without loss of generality) left endpoint, resulting in the value b. Let a be the weighted left sum, and let c be the asterisked coefficient immediately to the right of b.

    If $c \leq b$, then by the definition of operation (4), b has no asterisk, so any $\delta \in \overline{\delta}[m]$ has only one instance of b preceding c. Reflecting there returns us to a root in $\overline{\gamma}$. If $c > b$, b may appear multiple times in δ; however, since $c > b$, we can apply the same reasoning as in cases (2) and (3) to move up the root poset to an element of $\overline{\delta}[m]$ with only one instance of b, whereupon we fall back to the previous reasoning.

$\square$

Thus we can describe $\downarrow\mathrm{str}_m(\beta)$ for sufficiently large m. We now derive the consequence that $\# \downarrow\mathrm{str}_m(\beta)$ is eventually polynomial in m.

First, we must deal with redundancy between our stretching classes, since they may nontrivially overlap. Fortunately, those overlaps are also stretching classes.

Lemma IV.33. *The intersection of two stretching classes is either empty, a single root, or a stretching class.*

Proof. If the classes assume different values off the stretched path, then their intersection is empty. Thus it suffices to consider only the requirements the classes impose on the stretched path and assume they agree elsewhere. It will clarify matters to introduce a slightly different notation. For a symbol a and positive integer n, let $a^{\geq n}$ denote the set of words consisting of at least n copies of a and let a^n denote the singleton set containing the word consisting of exactly n copies of a. Then for multiple symbols $a_1, \ldots, a_k$ and integers $n_1, \ldots, n_k$, we denote by

$$a_1^{(\geq)n_1} a_2^{(\geq)n_2} \cdots a_k^{(\geq)n_k}$$

the set of all words obtained by concatenating words from these sets.

In particular, by collapsing together repeated values, we see that the sequences of values which a stretching class allows to appear on the stretched path are described by an expression of the form

$$a_1^{n_1} \cdots a_r^{n_r} a_{r+1}^{\geq n_{r+1}} a_{r+2}^{\geq 1} \cdots a_{s-1}^{\geq 1} a_s^{\geq n_s} a_{s+1}^{n_{s+1}} \cdots a_k^{n_k}$$

in which no consecutive a_i's are the same.

For the sets defined by two such expressions to intersect nontrivially, their a_i values must be the same. In this case, we can find their intersection by computing it for each a_i

individually. We have

$$a^m \cap a^{m'} = \begin{cases} a^m & m = m' \\ \varnothing & m \neq m' \end{cases}$$
$$a^{\geq m} \cap a^{m'} = \begin{cases} a^{m'} & m \leq m' \\ \varnothing & m > m' \end{cases}$$
$$a^{\geq m} \cap a^{\geq m'} = a^{\geq \max\{m,m'\}}$$

Using these rules, one can check that intersecting two sets of the above form produces the empty set, a singleton (if no $a^{\geq m}$ terms remain), or another set of that form. $\square$

The final step is to compute the size of a single stretching class, which is a straightforward counting problem.

Lemma IV.34. *Suppose $\overline{\beta}$ is a stretching class defined on $\mathrm{str}_c(G)$ with ℓ asterisked vertices. Then*

$$|\overline{\beta}[m]| = \binom{m-c+\ell-1}{\ell-1}$$

Example We return to the example following Definition IV.28, which featured a stretching class $\overline{\beta}$ defined on $\mathrm{str}_1(D_4)$ with 2 asterisked vertices:

$$\begin{matrix} 1 \\ 1 \end{matrix} > 2 - \cdots - 2 - 1 - \cdots - 1 - 1$$

An element of this class on a particular $\mathrm{str}_m(D_4)$ is determined by where its coefficients switch from 2 to 1, and there are $m = \binom{m-1+2-1}{2-1}$ ways of doing this.

Combining Theorem IV.30 with the last two lemmas and the inclusion-exclusion principle allows us to conclude:

Theorem IV.35. *Let α, P, m_0 be as above. Let ℓ be the largest value such that there is a stretching class in P with ℓ asterisked vertices. Then there is a polynomial $p(m)$ of degree $\ell - 1$ such that $|\downarrow\mathrm{str}_m(\alpha)| = p(m)$ for $m \geq m_0 + 1$.*

Again, this loosely generalizes the behavior seen in finite type, where the sizes of the A_n and D_n root systems (each of which can be realized as the downset of a single maximal, stretched root) are given by quadratic polynomials. However, in more complicated infinite type situations, it it easy for polynomials of higher degree to appear.

CHAPTER V
Quiver Representations and Preprojective Algebras

We now review basic facts about quiver representations and Gabriel's theorem from the introduction and formally define the preprojective algebra. We additionally introduce two important tools for working with the preprojective algebra which connect it to a root system: reflection functors, which act on the dimension vectors of modules by reflections, and the Crawley-Boevey identity, which relates the dimensions of certain Hom-spaces to the bilinear pairing $(,)$ on the space containing the roots.

V.1: Quiver representations and path algebras

Definition V.1. A **quiver** Q consists of

- a finite set Q_0 of **vertices**;
- a finite set Q_1 of **arrows**, where each a is associated to two vertices, its **source** $s(a)$ and **target** $t(a)$.

Definition V.2. Let k be a field. A **representation** M of a quiver Q over k consists of:

- a k-vector space M_i for each vertex $i \in Q_0$, and
- linear maps $M(a) : M_{s(a)} \to M_{t(a)}$ for each edge $a \in Q_1$.

For example, a representation of the quiver $\bullet \to \bullet$ is simply a linear map between two vector spaces.

Representations of a quiver also naturally form a category. We define a morphism between representations by maps on the level of vertices which are compatible with the maps corresponding to the arrows.

Definition V.3. Let M and N be representations of a quiver Q. Then a **morphism** $\varphi : M \to N$ consists of maps $\varphi_i : M_i \to N_i$ for each vertex $i \in Q_0$ such that, for each arrow $a \in Q_1$, the square

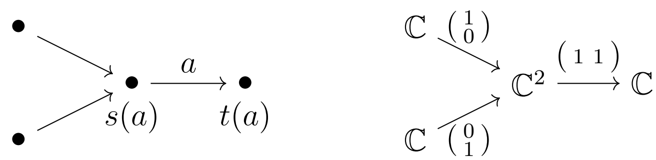


Figure 39: On the left is a quiver, where we have marked an edge a and its source and target. On the right is a particular representation of that quiver over $\mathbb{C}$.

$$\begin{tikzcd}
M_{s(a)} \arrow[r, "M(a)"] \arrow[d, "\varphi_{s(a)}"] & M_{t(a)} \arrow[d, "\varphi_{t(a)}"] \\
N_{s(a)} \arrow[r, "N(a)"] & N_{t(a)}
\end{tikzcd}$$

commutes.

This definition gives us a category $\mathrm{Rep}_k(Q)$ of representations of Q over k.

V.1.1: Path algebras

At first, the notion of a quiver representation appears quite different from other notions of "representation". However, the category of representations of a quiver turns out to be equivalent to the category of modules over a certain algebra called the path algebra. This allows us to use preexisting tools of module theory to study quiver representations.

All modules will be left modules unless stated otherwise.

Definition V.4. Let Q be a quiver. A **path** in Q is a sequence of arrows $a_\ell \cdots a_2 a_1$ such that $t(a_i) = s(a_{i+1})$ for all i. We define the source and target of a path by $s(a_\ell \cdots a_2 a_1) = s(a_1)$ and $t(a_\ell \cdots a_2 a_1) = t(a_\ell)$. A path can be empty, in which case it associated to staying put at a vertex. We write e_i for the empty path at vertex i, and say that $s(e_i) = t(e_i) = i$.

Definition V.5. The **path algebra** kQ consists of formal k-linear combinations of paths in Q, together with a multiplication on paths defined by concatenation:

$$(b_m \cdots b_1)(a_\ell \cdots a_1) := \begin{cases} b_m \cdots b_1 a_\ell \cdots a_1 & t(a_\ell) = s(b_1) \\ 0 & \text{otherwise} \end{cases}$$

In particular,

$$e_i p = \begin{cases} p & t(p) = i \\ 0 & \text{otherwise} \end{cases} \qquad p e_i = \begin{cases} p & s(p) = i \\ 0 & \text{otherwise} \end{cases}$$

It follows that the identity element of kQ is $\sum_{i \in Q_0} e_i$.

Now we describe the equivalence between kQ-modules and representations of Q over k.

First, given a kQ-module M, we must construct a representation of Q. We associate to vertex i the vector space e_iM. Now let a be any arrow. We note that, because $a = e_{t(a)}a$ as a path, multiplication by a maps $e_{s(a)}M$ into $e_{t(a)}M$, and in this way we define the maps associated to the arrows of the quiver.

Conversely, given a representation M of Q, we construct a kQ-module. Let $V := \bigoplus_{i\in Q_0} M_i$. Then we define a path $a_\ell \cdots a_1$ to act on V by sending every summand other than $M_{s(a_1)}$ to 0 and mapping $M_{s(a_1)}$ into $M_{t(a_\ell)}$ with the composition $M(a_\ell)\cdots M(a_1)$.

Given a morphism φ of quiver representations, we define a map between the associated kQ-modules by $\bigoplus_{i\in Q_0} \varphi_i$. And given a map φ between kQ-modules, we define a map between representations of Q by restricting φ to each of the spaces e_iM.

Theorem V.6 ([DW17])**.** *The above construction defines an equivalence between kQ-mod and* $\mathrm{Rep}_k(Q)$*.*

Because of this equivalence, in what follows we will treat quiver representations and modules over the path algebra interchangeably.

This equivalence shows us that $\mathrm{Rep}_k(Q)$ is a well-behaved abelian category, with notions of quotients, subobjects, kernels, cokernels, and direct sums. When we translate these notions into the language of quiver representations, they're generally described by applying the notion to each vertex separately in a way that's compatible with the arrows.

Examples.

- A subrepresentation $N \subset M$ is a collection of subspaces $N_i \subset M_i$ which are mapped into each other by the maps associated to the arrows.

  Consider the representation of $\bullet \to \bullet$ given by $k \xrightarrow{\mathrm{id}} k$. This has $0 \to k$ as a subrepresentation, because the diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & k \arrow[d, "\mathrm{id}"] \\ k \arrow[r, "\mathrm{id}"] & k \end{tikzcd}$$

  vacuously commutes. On the other hand, $k \to 0$ is not a subrepresentation, because

$$\begin{tikzcd} k \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \arrow[d] \\ k \arrow[r, "\mathrm{id}"] & k \end{tikzcd}$$

does not commute. In general, if a subrepresentation contains a subspace, it must also contain everything "downstream" from that subspace – everything resulting from applying arrow maps.

- A direct sum of representations $M \oplus M'$ associates to each vertex the space $M_i \oplus M'_i$ and to each arrow the map $M(a) \oplus M'(a)$.

  On the level of matrices, this can be viewed as constructing a block diagonal matrix for each arrow, built up from the matrices of the constituent representations' arrows. For example, the direct sum $(k \xrightarrow{\text{id}} k) \oplus (k \xrightarrow{\text{id}} k) \oplus (k \to 0)$ is given by

$$k^3 \xrightarrow{\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)} k^2$$

V.1.2: Indecomposables and Gabriel's theorem

The notion of direct sum is particularly useful for classifying representations: we can use it to break them down into pieces. From this perspective, the important representations are those which can't be broken down any further.

Definition V.7. A representation M is **indecomposable** if it is not isomorphic to a direct sum of two nonzero representations.

In addition to knowing what the indivisible pieces with respect to the direct sum operation are, we need to know that we can express everything else in terms of those pieces.

Theorem V.8 (Krull-Remak-Schmidt [DW17, Theorem 1.7.4])**.** *Every representation of a quiver is isomorphic to a direct sum of indecomposable representations, and this decomposition is unique up to isomorphism and reordering of terms.*

We now recall from the introduction the definition of dimension vector and the statement of Gabriel's theorem.

Definition V.9. Let Q be a quiver and let M be a representation over k. Let V be a vector space over $\mathbb{R}$ with a distinguished basis $(\alpha_i)_{i \in Q_0}$ indexed by the vertices of Q. The **dimension vector** of M, denoted $\underline{\dim}\, M$, is the vector $\sum_{i \in Q_0} (\dim_k M_i)\alpha_i \in V$.

Theorem V.10 ([Gab72])**.**

- *A quiver is of finite representation type if and only if, ignoring orientation, it is one of the simply laced finite type Dynkin diagrams (A_n, D_n, E_6, E_7, or E_8) or a disjoint union of same.*

- *In this case, the map $M \mapsto \underline{\dim}\, M$ gives a bijection between isomorphism classes of indecomposable representations and positive roots of the associated root system, where α_i is the simple root associated to vertex i.*

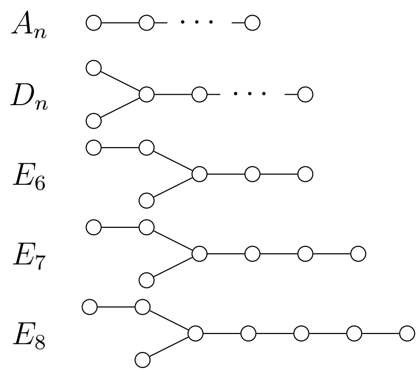


Figure 40: The simply laced finite type Dynkin diagrams. Gabriel's theorem states that the quivers of finite representation type are exactly those obtained by orienting these and taking disjoint unions.

V.2: Preprojective algebras

As noted in the introduction, an interesting aspect of Gabriel's theorem is that the connection between a root system and representations of a quiver does not depend on that quiver's orientation. The fullest picture of this connection will come from somehow considering all of the quiver's orientations at once.

We accomplish this by constructing a particular algebra associated to a Dynkin diagram called the preprojective algebra which, for any orientation Q of that diagram, has kQ has a quotient.

Definition V.11. Let G be a graph. Construct a **double quiver** $\overline{G}$ by replacing each edge with a pair of arrows going opposite directions, and for any arrow a let a^* be its reversed partner. Choose a map $\operatorname{sgn} : \overline{G}_1 \to \{\pm 1\}$ such that $\operatorname{sgn}(a^*) = -\operatorname{sgn}(a)$. The **preprojective algebra** Π_G is the quotient of the path algebra $k\overline{G}/\langle c\rangle$, where $\langle c\rangle$ is the two-sided ideal generated by

$$c := \sum_{a\in\overline{G}_1} \operatorname{sgn}(a)a^*a$$

While this definition includes an arbitrary choice of sgn, different choices will produce isomorphic algebras [Kü17, Lemma 4.9].

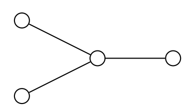
The above is the most concise form of the defining relation, but we can clarify it a bit by noting that each term a^*a or aa^* is a path which begins and ends at the same vertex, and we can isolate all the terms which begin and end at i by multiplying e_ice_i. Quotienting out by c is then equivalent to quotienting out by all the $e_i c e_i$. We have

$$e_i c e_i = \sum_{\substack{a \in \overline{G}_1 \\ s(a)=i}} \operatorname{sgn}(a) a^* a$$

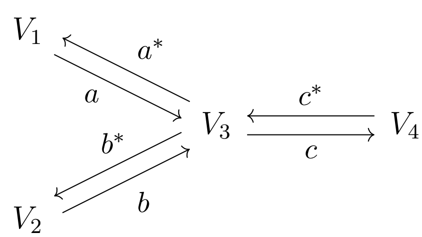
Modules over Π_G can be identified with $k\overline{G}$-modules on which the elements of $\langle c \rangle$ act by 0. Thus we can view modules over Π_G as representations of $\overline{G}$ in which certain combinations of maps are zero. Specifically, our expression for $e_i c e_i$ shows that, for each vertex, we want a signed sum of the "out-and-back" maps originating from that vertex to be 0.

Examples.

- Let G be the D_4 Dynkin diagram.



  Suppose we define the sgn map on the double quiver to send rightward arrows to $+1$ and leftward arrows to -1. Then a Π_{D_4}-module can be viewed as a representation



  subject to the relations

$$a^* a = b^* b = -cc^* = -aa^* - bb^* + c^* c = 0$$

- For any graph G, we can define the following modules:

  Definition V.12. Let S_i be the Π_G-module which assigns a 1-dimensional space to vertex i and 0 spaces elsewhere (so that all maps are 0). This is the **simple module at** i.

  These are simple objects in the category of Π_G-modules, and will play an important role throughout the rest of the thesis.

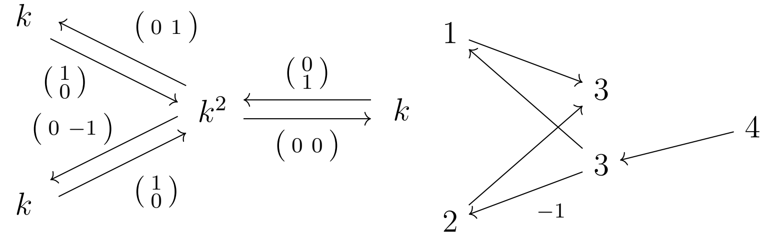


Figure 41: On the left is a Π_{D_4}-module which is not a representation of any orientation of D_4. On the right is an exploded view.

- Suppose Q is a quiver with underlying undirected graph G, and let M be a representation of Q. We can view M as a representation of the double quiver $\overline{G}$ by assigning the maps of Q to their corresponding arrows in $\overline{G}$ and assigning 0 maps to all other arrows. This satisfies the relation defining the preprojective algebra: for any two paired opposite arrows in $\overline{G}$, one of them will act by 0, and so each term in the relation will individually be 0. Thus the category of Π_G-modules includes all representations of any orientation of G. On the other hand, there exist Π_G-modules which can't be realized in this way, such as the example in Figure 41.

Usually, the Π_G-modules we consider will be representable in a suitable basis by fairly sparse matrices with entries 0, 1, and -1. In this case, we can visualize the representation more clearly by tracking the individual basis elements, in a format we call the **exploded view**. We construct a directed graph with vertices corresponding to basis elements of the spaces constituting the representation, labeling each basis element of V_i by i. Then if one of the maps of the representation sends v to $c_1w_1 + c_2w_2 + \ldots + c_rw_r$, we draw arrows from v to $w_1, \ldots, w_r$, labeling the arrow to w_j by c_j if $c_j \neq 1$. This is illustrated in Figure 41.

V.2.1: Duality

As is usual in math, we'd prefer to make half as many arguments, and a duality operation on Π_G-mod allows us to do this. Define a contravariant functor $D : \Pi_G\text{-mod} \to \Pi_G\text{-mod}$ by

$$DM := \mathrm{Hom}_{\Pi_G}(M, k)$$

which we equip with the structure of a Π_G-module by defining, for any arrow $a \in \overline{G}_1$,

$$af(m) := f(a^*m)$$

As a quiver representation, DM is obtained from M by setting $(DM)_i := M_i^*$ and letting $DM(a)$ be the adjoint map $M(a^*)^*$. There is a natural isomorphism $D(DM) \cong M$, so D is a duality of categories. In particular, D establishes bijections between injections and surjections, between projective and injective objects, etc. which we will use freely in what follows.

V.2.2: Reflection functors

Two crucial tools illuminate the connection between preprojective algebras and Coxeter groups. First, to each vertex i of G we associate a pair of **reflection functors** $\Sigma_i^\pm : \Pi_G\text{-mod} \to \Pi_G\text{-mod}$ which, on the level of dimension vectors, mostly act by reflections. These were originally introduced by Baumann and Kamnitzer in [BK12], based on similar functors defined for path algebras by Bernstein, Gelfand, and Ponomarev in [BGP73].

Given a Π_G-module M, we consider the direct sum over the vertices adjacent to i (multiply counting vertices with multiple edges to i).

$$M_{\partial i} := \bigoplus_{\substack{\text{arrows} \\ j \xrightarrow{a} i}} M_j$$

We then define a map $M(i, \mathrm{in}) : M_{\partial i} \to M_i$ by combining the maps $\mathrm{sgn}(a)M(a) : M_j \to M_i$ for each arrow $j \xrightarrow{a} i$, and a map $M(i, \mathrm{out}) : M_i \to M_{\partial i}$ by combining the maps $M(a^*) : M_i \to M_j$ for each such arrow. Crucially, the relation defining the preprojective algebra is equivalent to the composition $M(i, \mathrm{in}) \circ M(i, \mathrm{out})$ being 0. (So from one perspective, the preprojective algebra is defined the way it is because that's precisely what we need to make the tools of this section work.)

Then we define a new module $\Sigma_i^+(M)$, first on vertices by

$$\Sigma_i^+(M)_j := \begin{cases} \ker(M(i, \mathrm{in})) & j = i \\ M_j & j \neq i \end{cases}$$

To define the maps associated to arrows pointing in and out of a, it's equivalent to define $\Sigma_i^+(M)(i, \mathrm{in})$ and $\Sigma_i^+(M)(i, \mathrm{out})$:

- We say $\Sigma_i^+(M)(i, \mathrm{out}) : \ker(M(i, \mathrm{in})) \to M_{\partial i}$ is simply the inclusion of the kernel, so the individual maps out of i are given by projection to the summands of $M_{\partial i}$.

- Then note that, because $M(i, \mathrm{in}) \circ M(i, \mathrm{out}) = 0$, the composition $M(i, \mathrm{out}) \circ M(i, \mathrm{in}) : M_{\partial i} \to M_{\partial i}$ takes values in $\ker(M(i, \mathrm{in}))$. So we take this to be $\Sigma_i^+(M)(i, \mathrm{in})$.

Finally, any arrow not incident to i is assigned the same map by $\Sigma_i^+(M)$ as it was assigned by M.

The definition of $\Sigma_i^-(M)$ is dual, in the sense that $\Sigma_i^-(M) := D\Sigma_i^+(DM)$. Explictly, we can define it on vertices by

$$\Sigma_i^-(M)_j := \begin{cases} \operatorname{coker}(M(i,\text{out})) & j = i \\ M_j & j \neq i \end{cases}$$

And to define the maps pointing in and out of i:

- We say $\Sigma_i^-(M)(i,\text{in}) : M_{\partial i} \to \operatorname{coker}(M(i,\text{out}))$ is the quotient map to the cokernel, so the individual maps into i are given by applying this map to the summands of $M_{\partial i}$.

- Because $M(i,\text{in}) \circ M(i,\text{out}) = 0$, the composition $M(i,\text{out}) \circ M(i,\text{in}) : M_{\partial i} \to M_{\partial i}$ factors through a map $\operatorname{coker}(M(i,\text{out})) \to M_{\partial i}$. So we take this to be $\Sigma_i^-(M)(i,\text{out})$.

We can tersely summarize these definitions by insisting that the following diagram commutes:

$$\begin{tikzcd}[column sep=large]
 & & \Sigma_i^-(M)_i := \operatorname{coker}(M(i,\text{out})) \arrow[d, dashed] \arrow[dr, "\Sigma_i^-(M)(i,\text{out})"] & & \\
M_i \arrow[r, "M(i,\text{out})"'] & M_{\partial i} \arrow[r, "M(i,\text{in})"'] \arrow[ur, two heads, "\Sigma_i^-(M)(i,\text{in})"] \arrow[dr, "\Sigma_i^+(M)(i,\text{in})"'] & M_i \arrow[r, "M(i,\text{out})"'] \arrow[d, dashed] & M_{\partial i} \arrow[r, "M(i,\text{in})"'] & M_i \\
 & & \Sigma_i^+(M)_i := \ker(M(i,\text{in})) \arrow[ur, hook, "\Sigma_i^+(M)(i,\text{out})"'] & &
\end{tikzcd}$$

Here the upper left and lower right arrows are the natural projection and inclusion, respectively. The dotted arrows factor $M(i,\text{in})$ (respectively, $M(i,\text{out})$) through $\operatorname{coker}(M(i,\text{out}))$ (respectively, $\ker(M(i,\text{in}))$), and their existence follows from $M(i,\text{in}) \circ M(i,\text{out}) = 0$. In particular, we can also define natural maps $\Sigma_i^-(M) \xrightarrow{f_i^-} M \xrightarrow{f_i^+} \Sigma_i^+(M)$ by the dotted arrows at vertex i (and identity maps elsewhere).

We can also verify from this diagram that the relation defining the preprojective algebra is still satisfied by $\Sigma_i^+(M)$ and $\Sigma_i^-(M)$. We have $\Sigma_i^\pm(M)(i,\text{out}) \circ \Sigma_i^\pm(M)(i,\text{in}) = M(i,\text{out}) \circ M(i,\text{in})$, which shows that for any arrow $j \xrightarrow{a} i$, the map attached to a^*a is unchanged. We can also check directly from definitions that $\Sigma_i^\pm(M)(i,\text{in}) \circ \Sigma_i^\pm(M)(i,\text{out}) = 0$.

We'd like reflection functors to "reflect" in the following senses: they should undo each other, and, since we can interpret dimension vectors as belonging to the space spanned by simple roots, they should act as reflections on the level of dimension vectors. These aren't quite true, but they are once we restrict attention to a subcategory.

Recall that S_i is the simple representation from Definition V.12. Let NoQuot_i be the full subcategory of Π_G-mod consisting of modules which do not have S_i as a quotient module, and let NoSub_i be the full subcategory of modules which do not have S_i as a submodule. Note by unwinding definitions that $M \in \mathrm{NoQuot}_i$ is equivalent to $M(i, \mathrm{in})$ being surjective, while $M \in \mathrm{NoSub}_i$ is equivalent to $M(i, \mathrm{out})$ being injective.

Proposition V.13 ([BK12, Proposition 2.5])**.** *(1) The functors* $\Sigma_i^+ : \mathrm{NoQuot}_i \to \mathrm{NoSub}_i$ *and* $\Sigma_i^- : \mathrm{NoSub}_i \to \mathrm{NoQuot}_i$ *are inverse equivalences of categories.*

(2) For $M \in \mathrm{NoQuot}_i$*, we have* $\underline{\dim}\, \Sigma_i^+(M) = s_i(\underline{\dim}\, M)$*.*

(3) For $M \in \mathrm{NoSub}_i$*, we have* $\underline{\dim}\, \Sigma_i^-(M) = s_i(\underline{\dim}\, M)$*.*

We briefly note why (2) (and, dually, (3)) are true: since $M \in \mathrm{NoQuot}_i$ means that $M(i, \mathrm{in})$ is surjective, the dimension of $\Sigma_i^+(M)_i := \ker(M(i, \mathrm{in}))$ is given by subtracting the dimension of M_i from the dimensions of neighboring spaces. This is exactly reflection at vertex i.

Finally, while membership in NoQuot_i implies nice properties of Σ_i^+, it also has a consequence for Σ_i^-.

Lemma V.14. *(1) For* $M \in \mathrm{NoQuot}_i$*, the map* $f_i^- : \Sigma_i^-(M) \to M$ *defined using the dashed arrows in the above diagram is surjective.*

(2) For $M \in \mathrm{NoSub}_i$*, the map* $f_i^+ : M \to \Sigma_i^+(M)$ *is injective.*

Proof. (1) Since $M \in \mathrm{NoQuot}_i$, the map $M(i, \mathrm{in}) : M_{\partial i} \to M_i$ is surjective. Adding this information to the above commutative diagram, we get

$$\begin{tikzcd} & \Sigma_i^-(M)_i \arrow[d, "(f_i^-)_i"] \arrow[dr, "\Sigma_i^-(M)(i,\mathrm{out})"] & \\ M_{\partial i} \arrow[ur, two heads, "\Sigma_i^-(M)(i,\mathrm{in})"] \arrow[r, two heads, "M(i,\mathrm{in})"'] & M_i \arrow[r, "M(i,\mathrm{out})"'] & M_{\partial i} \end{tikzcd}$$

Looking at the left triangle, we see that $(f_i^-)_i$ is also surjective. Since the map f_i^- is given by isomorphisms at every vertex other than i, it is surjective overall.

(2) This proceeds dually to the proof of part (1).

□

If a module is in both NoQuot_i *and* NoSub_i, these maps complete to exact sequences, which will form the base case of a much more general result in Chapter VII.

Lemma V.15. *If* $M \in \mathrm{NoQuot}_i \cap \mathrm{NoSub}_i$*, then:*

(1) There is an exact sequence

$$0 \to S_i^{\oplus -(\alpha_i, \underline{\dim} M)} \xrightarrow{g} \Sigma_i^-(M) \xrightarrow{f_i^-} M \to 0$$

and the map g *is universal in the following sense: given any map* $h : S_i \to \Sigma_i^-(M)$*, there is a unique map* $\overline{h} : S_i \to S_i^{\oplus -(\alpha_i, \underline{\dim} M)}$ *such that* $h = g \circ \overline{h}$*.*

(2) There is an exact sequence

$$0 \to M \xrightarrow{f_i^+} \Sigma_i^+(M) \xrightarrow{g} S_i^{\oplus -(\alpha_i, \underline{\dim} M)} \to 0$$

and the map g *is universal in the following sense: given any map* $h : \Sigma_i^+(M) \to S_i$*, there is a unique map* $\overline{h} : S_i^{\oplus -(\alpha_i, \underline{\dim} M)} \to S_i$ *such that* $h = \overline{h} \circ g$*.*

Proof. (1) By the previous lemma, since $M \in \mathrm{NoQuot}_i$, we know f_i^- is surjective. Then since $M \in \mathrm{NoSub}_i$, we know by Proposition V.13(3) that

$$\underline{\dim}\, \Sigma_i^-(M) = (\underline{\dim}\, M) - (\alpha_i, \underline{\dim}\, M)\alpha_i$$

so the kernel of this map has dimension vector $-(\alpha_i, \underline{\dim}\, M)\alpha_i$, which forces it to be $S_i^{\oplus -(\alpha_i, \underline{\dim} M)}$.

Then consider an arbitrary $h : S_i \to \Sigma_i^-(M)$. Because $M \in \mathrm{NoSub}_i$, the composition $f_i^- \circ h$ must be 0. The claimed universality then follows from the universal property of kernel.

(2) This proceeds dually to the proof of part (1).

□

We end this section by noting that, although we gave an ad hoc definition of reflection functors above, they can also be described in terms of common module operations. Let $I_i \subset \Pi_G$ be the annihilator of S_i. Since any path other than the stationary path e_i will pass through some other vertex and act by 0 on S_i, I_i is just the space spanned by paths other than e_i, which we can also realize as the two-sided ideal $\Pi_G(1 - e_i)\Pi_G$. Note that I_i may be an infinite-dimensional Π_G-module; this requires caution under the hood in some homological proofs but does not meaningfully affect anything we do here.

Lemma V.16 ([BK12, Remark 2.4(iii)])**.** *The functors* $\Sigma_i^+(-)$ *and* $\Sigma_i^-(-)$ *are naturally isomorphic to* $\mathrm{Hom}_{\Pi_G}(I_i, -)$ *and* $I_i \otimes_{\Pi_G} -$*, respectively.*

One nice feature of this lemma is that it allows us to simply describe the derived functors of Σ_i^+ and Σ_i^- — they are $\mathrm{Ext}^j_{\Pi_G}(I_i, -)$ and $\mathrm{Tor}_j^{\Pi_G}(I_i, -)$. In the next section, we'll say a little more about the properties of these functors.

V.2.3: Crawley-Boevey's identity and other homological properties

The second tool for connecting preprojective algebras with Coxeter groups comes from their homological properties. In what follows, all Hom and Ext functors are computed in the category of Π_G-modules unless stated otherwise.

Proposition V.17 ([CB00, Lemma 1])**.** *For any $M, N \in \Pi_G$-mod,*

$$\dim_k \mathrm{Hom}(M,N) - \dim_k \mathrm{Ext}^1(M,N) + \dim_k \mathrm{Hom}(N,M) = (\underline{\dim}\, M, \underline{\dim}\, N)$$

where $(-,-)$ is the symmetric bilinear form associated to the Cartan matrix of G.

Closely related is one of the few ways in which the preprojective algebras of infinite-type Dynkin diagrams are actually *better*-behaved than in the finite type case:

Proposition V.18 ([BIRS09, Proposition II.1.3(c)])**.** *Suppose G is not one of the finite type Dynkin diagrams. Then for any two Π_G-modules M, N and $i \in \{0, 1, 2\}$, we have an isomorphism of vector spaces*

$$\mathrm{Ext}^i(M,N)^* \cong \mathrm{Ext}^{2-i}(N,M)$$

We say that in this case the category of Π_G-modules is **2-Calabi-Yau**. We won't need the full force of this property in this thesis, and will be satisfied with the following weaker fact which holds for all preprojective algebras:

Proposition V.19. *For any Π_G-modules M and N, there is an injective map $\mathrm{Ext}^2(M,N) \hookrightarrow \mathrm{Hom}(N,M)^*$.*

Proof (sketch). The proof of Proposition V.17 in [CB00] involves constructing the start of a projective resolution

$$P_2 \to P_1 \to P_0 \to M \to 0$$

and then showing that the cokernel of the map $\mathrm{Hom}(P_1, N) \to \mathrm{Hom}(P_2, N)$ is isomorphic to $\mathrm{Hom}(N, M)^*$. The homology of $\mathrm{Hom}(P_\bullet, N)$ in degree 2, which computes $\mathrm{Ext}^2(M, N)$, is a submodule of this. $\square$

Based on the properties above, we'll use the following facts about reflection functors.

Lemma V.20. *(1) Σ_i^+ is left exact and Σ_i^- is right exact.*

(2) If $M \in \mathrm{NoQuot}_i$, then $\mathrm{Ext}^1(I_i, M) = 0$; if $M \in \mathrm{NoSub}_i$, then $\mathrm{Tor}_1(I_i, M) = 0$.

(3) As long as i is not an isolated vertex of G, $\mathrm{Ext}^1(I_i, S_i) \cong \mathrm{Tor}_1(I_i, S_i) \cong S_i$.

Proof. (1) This is the case for all functors defined by Hom and $\otimes$ as the reflection functors are.

(2) We can apply $\mathrm{Hom}(-, M)$ to the exact sequence

$$0 \to I_i \to \Pi_G \to S_i \to 0$$

and get an exact sequence

$$\mathrm{Ext}^1(\Pi_G, M) \to \mathrm{Ext}^1(I_i, M) \to \mathrm{Ext}^2(S_i, M) \to \mathrm{Ext}^2(\Pi_G, M).$$

Because Π_G is projective, the end terms are 0, so $\mathrm{Ext}^1(I_i, M) \cong \mathrm{Ext}^2(S_i, M)$.

Now suppose $M \in \mathrm{NoQuot}_i$. By Proposition V.19, $\mathrm{Ext}^2(S_i, M) \subset \mathrm{Hom}(M, S_i)^* = 0$.

The second half of the result follows from duality, by applying the first half to DM.

(3) Choose an arrow $j \xrightarrow{a} i$, and consider the Π_G-module M_a which assigns 1-dimensional spaces to j and i, the identity map to a, and 0 spaces and maps elsewhere. Likewise define M_{a^*} using $i \xrightarrow{a^*} j$. We have a short exact sequence

$$0 \to S_i \to M_a \to S_j \to 0$$

Applying Σ_i^+, we get a sequence

$$0 \to 0 \to S_j \to M_{a^*} \to \mathrm{Ext}^1(I_i, S_i) \to \mathrm{Ext}^1(I_i, M_a)$$

By (2), the last term is 0, which implies the result for Ext^1. To see the result for Tor_1, instead apply Σ_i^- to $0 \to S_j \to M_{a^*} \to S_i \to 0$.

$\square$

CHAPTER VI
Shard Modules

In the previous chapter, we introduced Gabriel's theorem, which gives a correspondence between indecomposable representations of a finite type quiver and reflecting hyperplanes of the corresponding root system. In this chapter, we refine this correspondence to one involving the preprojective algebra on one side and shards on the other. It's not realistic to classify all indecomposable modules of the preprojective algebra even for most finite type Dynkin diagrams, so we need to limit our focus to a more specific class of modules: we start by introducing the class of **bricks**, which suffices for the finite type case, and then add a couple technical conditions to the general case which define the class of **shard modules**. At the end of the chapter, we demonstrate this correspondence in the case of the $A_n^{(1)}$ Dynkin diagrams, which are in some ways the simplest diagrams which are not finite type.

VI.1: Stability and bricks

VI.1.1: Stability

Throughout this chapter, G is a Dynkin diagram with n vertices, Φ is the associated root system, and V is a vector space containing both roots of Φ and dimension vectors of Π_G-modules, as in the setting of Gabriel's theorem. We will identify $V \cong \mathbb{R}^n$ using the basis of simple roots, and likewise identify $V^* \cong \mathbb{R}^n$ using the dual basis.

Definition VI.1. Let $\theta \in V^*$. A Π_G-module M is **θ-semistable** if:

- $\langle \theta, \underline{\dim}\, M \rangle = 0$, and
- $\langle \theta, \underline{\dim}\, N \rangle \geq 0$ for all $N \subset M$

We could equivalently replace the second condition by requiring $\langle \theta, \underline{\dim}\, N \rangle \leq 0$ for all $M \twoheadrightarrow N$.

Example. We consider the quiver representation $M := k \xrightarrow{1} k$ as a Π_{A_2}-module by letting the leftward arrow act by 0. M has dimension vector $\alpha_1 + \alpha_2$, and its only proper subrepresentation is $0 \to k$, which has dimension vector α_2. Thus, for $\theta = (\theta_1, \theta_2)$, M is θ-semistable if and only if $\theta_1 + \theta_2 = 0$ and $\theta_2 \geq 0$.

This concept of stability was introduced by King [Kin94], originally in the context of constructing moduli spaces of representations up to isomorphism. King's intent was to consider, for a fixed θ, the collection of θ-semistable modules, and construct an appropriate moduli space classifying them.

However, it turns out to also be interesting to fix a module M and consider the set of all weights θ for which it is semistable. We refer to this as the **stability domain** of M, and denote it by $\mathrm{Stab}(M)$. For example, the ray of points (θ_1, θ_2) satisfying the two properties in the above example is the stability domain of the module considered there.

In general, although M may have infinitely many non-isomorphic submodules, they will only have finitely many dimension vectors among them. So $\mathrm{Stab}(M)$ is defined by one linear equation and a finite list of linear inequalities, and it is a polyhedral convex cone.

VI.1.2: Bricks

Definition VI.2. An object M in a k-linear category is a **brick** if $\mathrm{End}(M)$ is a division algebra.

A brick is indecomposable, because any decomposable module $M_1 \oplus M_2$ admits a noninvertible endomorphism given by projection to a summand. However, not all indecomposable modules are bricks.

Example. We consider Π_{A_3} (with $\mathrm{sgn}(a) = 1$ for the rightward arrows). Let M be the following module:

$$\begin{tikzcd} k \arrow[r, shift left, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] & k^2 \arrow[l, shift left, "\left(\begin{smallmatrix}0&1\end{smallmatrix}\right)"] \arrow[r, shift left, "\left(\begin{smallmatrix}0&1\end{smallmatrix}\right)"] & k \arrow[l, shift left, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] \end{tikzcd}$$

We can also visualize M with an exploded view:

$$\begin{tikzcd} & 2 & \\ 1 \arrow[ur] & & 3 \arrow[ul] \\ & 2 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

We claim that M is indecomposable. It cannot have a simple module S_i as a summand, because this would consist of a 1-dimensional subspace at some vertex which is annihilated by all arrows, which does not happen. Instead, a nontrivial decomposition would need to consist of modules with dimension vectors $(1, 1, 0)$ and $(0, 1, 1)$. But if this were the case, the images of the two inward-pointing arrows would be distinct, a contradiction.

On the other hand, M is not a brick. This is easiest to see in the exploded view. M has S_2 as both a quotient (the quotient by the submodule spanned by the top 3 basis elements in the picture) and a submodule (spanned by the top basis element):

$$\begin{tikzcd}[row sep=small, column sep=small]
 & 2 & \\
1 \arrow[ur] & & 3 \arrow[ul] \\
 & 2 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

S_2 as a quotient

$$\begin{tikzcd}[row sep=small, column sep=small]
 & 2 & \\
1 \arrow[ur] & & 3 \arrow[ul] \\
 & 2 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

S_2 as a submodule

Putting these together, we get a map $M \twoheadrightarrow S_2 \hookrightarrow M$ which is nonzero but not invertible.

Despite this discrepancy, from the perspective of stability, bricks are the fundamental building blocks of interest in the following sense:

Proposition VI.3. *Let M be a Π_G-module and let $\varphi : M \to M$ be an endomorphism. Then*

$$\mathrm{Stab}(M) = \mathrm{Stab}(\ker(\varphi)) \cap \mathrm{Stab}(\mathrm{im}(\varphi))$$

Proof. First, suppose that M is θ-semistable. The exact sequence

$$0 \to \ker(\varphi) \to M \to \mathrm{im}(\varphi) \to 0$$

implies that

$$\langle \theta, \underline{\dim}\, M \rangle = \langle \theta, \underline{\dim} \ker(\varphi) \rangle + \langle \theta, \underline{\dim}\, \mathrm{im}(\varphi) \rangle$$

By the definition of semistability, the left side is 0 while the terms on the right are both nonnegative, and thus also 0. Additionally, any submodule N of $\ker(\varphi)$ or $\mathrm{im}(\varphi)$ is also a submodule of M, implying $\langle \theta, \underline{\dim}\, N \rangle \geq 0$. This shows that $\ker(\varphi)$ and $\mathrm{im}(\varphi)$ are both θ-semistable.

Conversely, suppose $\ker(\varphi)$ and $\mathrm{im}(\varphi)$ are both θ-semistable. Then both terms on the right of the above equation are 0, so $\langle \theta, \underline{\dim}\, M \rangle = 0$ as well. If $N \subset M$ is any submodule, we have an exact sequence

$$0 \to N \cap \ker(\varphi) \to N \to \varphi(N) \to 0$$

which implies that

$$\langle \theta, \underline{\dim}\, N \rangle = \langle \theta, \underline{\dim}(N \cap \ker(\varphi)) \rangle + \langle \theta, \underline{\dim}\, \varphi(N) \rangle.$$

The assumption that $\ker(\varphi)$ and $\operatorname{im}(\varphi)$ are θ-semistable implies that both terms on the right are nonnegative, so the left side is also nonnegative. This shows that M is also θ-semistable. □

In particular, since any nonbrick admits an endomorphism with nontrivial kernel and image, its stability domain can be expressed in terms of those smaller modules. Thus we're primarily interested in the stability domains of bricks.

VI.2: Bricks and shards: the finite type case

The bricks of Π_G admit a classification similar to the classification of indecomposables of the path algebra given by Gabriel's theorem. Now, instead of positive roots being on the other side, we use shards.

Theorem VI.4 ([Tho18, Theorems 5 and 6])**.** *Let G be a simply laced finite type Dynkin diagram: A_n, D_n, E_6, E_7, or E_8. Then the operation* $\mathrm{Stab}(-)$ *gives a bijection between bricks of Π_G and shards of the reflecting hyperplane arrangement associated to G.*

Examples. Let $G = \bullet - \bullet$. Then Π_G has four bricks:

$$k \to 0 \quad 0 \to k \quad k \underset{0}{\overset{1}{\rightleftarrows}} k \quad k \underset{1}{\overset{0}{\rightleftarrows}} k$$

The stability domains of these bricks are, respectively,

$$\begin{aligned}
&\{(x_1, x_2) \mid x_1 = 0\} \\
&\{(x_1, x_2) \mid x_2 = 0\} \\
&\{(x_1, x_2) \mid x_1 + x_2 = 0 \text{ and } x_2 \geq 0\} \\
&\{(x_1, x_2) \mid x_1 + x_2 = 0 \text{ and } x_1 \geq 0\}
\end{aligned}$$

These are exactly the shards of the A_2 reflecting hyperplane arrangement. We can see this in the plot in Figure 42. Note that this plot is viewed in the basis of simple roots, obscuring the symmetry of the A_2 root system, but a suitable change of coordinates will restore the symmetry.

More generally, let G be the A_n diagram. Consider the unique root $\alpha_{1,n}$ supported on all vertices:

$$1 — 1 — \cdots — 1$$

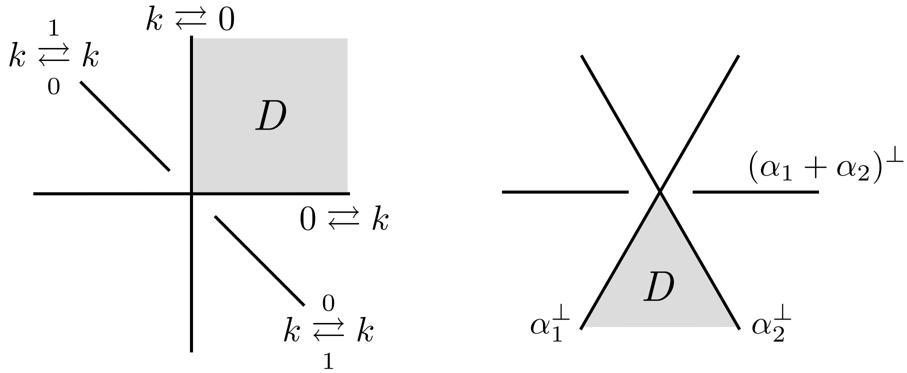


Figure 42: On the left, the stability domains of bricks of Π_{A_2}. The base region D used to define shards is shaded. Note that the domains are plotted in the basis of simple roots, obscuring the symmetry of the A_2 root system; a suitable change of coordinates produces the more typical picture on the right.

Then choose any tuple $a \in \{+,-\}^{n-1}$, which we use to define a Π_{A_n}-module M_a with dimension vector $\alpha_{1,n}$. We assign a 1-dimensional space to each vertex, and for each edge we pick one arrow to act by the identity and one to act by 0. If the jth sign in a is $+$, we pick the rightward of the arrows along the jth edge to be the nonzero map, and if the sign is $-$, we pick the leftward arrow. For example, the Π_{A_4}-module M_{-++} is given by

$$k \overset{0}{\underset{1}{\rightleftarrows}} k \overset{1}{\underset{0}{\rightleftarrows}} k \overset{1}{\underset{0}{\rightleftarrows}} k.$$

which we typically depict by omitting the zero maps:

$$k \leftarrow k \rightarrow k \rightarrow k$$

This is a brick: an endomorphism amounts to multiplication by a scalar at each vertex, and the requirement of commuting with the arrows implies that all these scalars are the same.

It's also straightforward to compute the stability domains of these modules. Pick any edge: then if its nonzero map points left, the spaces to the left of the arrow form a submodule, while if the nonzero map points right, the spaces to the right of the arrow form a submodule:

$$\boxed{k} \longleftarrow k \longrightarrow k \longrightarrow k \qquad k \longleftarrow k \longrightarrow \boxed{k \longrightarrow k} \qquad k \longleftarrow k \longrightarrow k \longrightarrow \boxed{k}$$

Thus, for $(x_1, \ldots, x_n) \in \mathrm{Stab}(M_a)$, we must have

$$\begin{aligned} x_1 + x_2 + \ldots + x_i &\geq 0 \text{ if } a_i = - \\ x_{i+1} + x_{i+2} + \ldots + x_n &\geq 0 \text{ if } a_i = + \end{aligned}$$

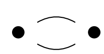
Finally, we note that, combined with the condition $x_1 + \ldots + x_n = 0$, the former inequality is equivalent to $x_{i+1} + x_{i+2} + \ldots + x_n \leq 0$. While M_a has other submodules, it's straightforward to check that any further inequalities they impose on the stability domains are implied by these ones.

But now we recall from the example following Theorem III.25 that this list of inequalities is exactly of the form required to specify a shard of the hyperplane $\alpha_{1,n}^{\perp}$. Thus each of these stability domains is a shard. In particular, since there are 2^{n-1} choices of tuple a, and 2^{n-1} shards of $\alpha_{1,n}^{\perp}$, by Theorem VI.4 this is a complete list of bricks of dimension $\alpha_{1,n}$.

One point to emphasize is that, in this case, there is a more fine-grained correspondence between the signs of linear inequalities defining a shard and the orientations of arrows defining the associated brick, both of them summarized by the sign vector a. While the behavior of the A_n case is unusually nice in this regard, it will serve as a model for our study of the structure of bricks in Chapter VIII.

VI.3: Shard modules

It would be great if we could generalize Theorem VI.4 to graphs beyond the finite type Dynkin diagrams. However, there is a simple reason why the correspondence fails. Consider the example of the $A_1^{(1)}$ diagram



The root system is easy to describe — labeling the vertices 1 and 2, each root is obtained by applying an alternating sequence of s_1 and s_2 to α_1 or α_2. Direct computation then shows that each root has the form $n\alpha_1 + (n \pm 1)\alpha_2$.

On the other hand, we can specify a representation of $\Pi_{A_1^{(1)}}$ by giving a representation of the quiver $\bullet \rightrightarrows \bullet$ and defining the backward maps to be 0. The representation

$$k \overset{1}{\underset{0}{\rightrightarrows}} k$$

is certainly a brick.

However, its dimension vector $\alpha_1 + \alpha_2$ is not a positive root! In particular, its stability domain must lie in the hyperplane $x_1 + x_2 = 0$: this is not part of the reflecting hyperplane arrangement of $A_1^{(1)}$, and can't contain a shard.

So in order to establish a correspondence like Theorem VI.4, we need to restrict the class of modules we're looking at. As the above example shows, one necessary condition is that the dimension vector is a root.

Definition VI.5. A **real brick**[1] is a brick of Π_G whose dimension vector is a root of Φ.

This definition looks somewhat contrived as-is, but real bricks admit both a useful recursive description and a natural categorical one.

First, we can build up real bricks using reflection functors. Recall that, while reflection functors can be defined for arbitrary Π_G-modules, they give equivalences $\mathrm{NoQuot}_i \leftrightarrow \mathrm{NoSub}_i$. Thus we'll usually want to ensure that the input to the reflection functor Σ_i^+ (respectively, Σ_i^-) is in NoQuot_i (respectively, NoSub_i)[2]

Definition VI.6. An expression

$$\Sigma_{i_\ell}^{\pm_\ell} \cdots \Sigma_{i_1}^{\pm_1}(M)$$

is **well-behaved** if, for $1 \leq r \leq \ell$,

$$\Sigma_{r-1}^{\pm_{r-1}} \cdots \Sigma_{i_1}^{\pm_1}(M) \in \begin{cases} \mathrm{NoQuot}_{i_r} & \pm_{i_r} = + \\ \mathrm{NoSub}_{i_r} & \pm_{i_r} = - \end{cases}$$

Such expressions have nice properites which follow from repeated application of Proposition V.13. And it turns out that they give another way to describe real bricks.

Proposition VI.7 ([DST23, Theorem 5.1])**.** *Let β be a positive root and let $s_{i_\ell} \cdots s_{i_1}(\alpha_{i_0})$ be a positive expression for β. The bricks with dimension vector β are precisely the modules given by well-behaved expressions $\Sigma_{i_\ell}^{\pm_\ell} \cdots \Sigma_{i_1}^{\pm_1}(S_{i_0})$.*

We refer to [DST23] for the proof, but we note here why such an expression gives a brick with dimension vector β. By Proposition V.13(2) and (3), the reflection functors all act on the dimension vector by reflections, so the dimension vector of the full expression is $s_{i_\ell} \cdots s_{i_1}(\alpha_{i_0}) = \beta$. And by Proposition V.13(1), the reflection functors can all be treated as equivalences of categories, so

$$\mathrm{End}(\Sigma_{i_\ell}^{\pm_\ell} \cdots \Sigma_{i_1}^{\pm_1}(S_{i_0})) \cong \mathrm{End}(S_{i_0}) \cong k$$

This last observation is important to the following chapter, so we frame it as a lemma.

Lemma VI.8. *If M is a real brick, then* $\mathrm{End}(M) \cong k$.

In general, if k is not algebraically closed, the endomorphism ring of a brick could be some other division algebra over k.

[1] Not to be confused with the building material, which we refer to as an **actual physical literal brick**.

[2] [DST23] deals with this by simply defining Σ_i^+ and Σ_i^- to take inputs in NoQuot_i and NoSub_i, respectively, and declaring them to be ill-defined otherwise. However, at one point later we will want to apply a reflection functor to a module outside of the appropriate subcategory.

On the other hand, part of what makes real bricks interesting is that they admit a description that isn't obviously combinatorial.

Proposition VI.9 ([DST23, Proposition 4.14]). *A brick B is real if and only if* $\operatorname{Ext}^1(B,B) = 0$ *(in which case the brick is also called* ***rigid****).*

Again, we explain the "only if" half of this statement and refer to [DST23] for the other half. Let B be a real brick, which by Proposition VI.7 has the form $\Sigma_{i_\ell}^{\pm_\ell} \cdots \Sigma_{i_1}^{\pm_1}(S_{i_0})$. By Crawley-Boevey's identity (Proposition V.17), we know that

$$2 \dim_k \operatorname{Hom}(B,B) - \dim_k \operatorname{Ext}^1(B,B) = (\underline{\dim}\, B, \underline{\dim}\, B)$$

On one hand, $\operatorname{Hom}(B,B) \cong k$ by Lemma VI.8. On the other hand, since $\underline{\dim}\, B$ is a root, $(\underline{\dim}\, B, \underline{\dim}\, B) = 2$. Inserting these values on both sides of the identity implies $\operatorname{Ext}^1(B,B) = 0$.

A second simple necessary condition for the stability domain of a brick to be a shard is that the stability domain must be big enough. By their definition as regions of a hyperplane arrangement in an $(n-1)$-dimensional space, shards are cones of dimension $n-1$. The stability domains of the modules we consider are also cones in an $(n-1)$-dimensional space, but in principle, the inequalities defining a stability domain could cut it down to a smaller dimension. Thus we impose one more condition on bricks.

Definition VI.10. A **shard module** is a real brick with stability domain of dimension $n-1$.

VI.4: Shard modules and shards: the general case

Having imposed two straightforwardly necessary conditions on bricks to define the class of shard modules, it turns out that those conditions are all we need to generalize Theorem VI.4:

Theorem VI.11 ([DST23, Theorem 5.7]). *The operation* $\operatorname{Stab}(-)$ *gives a bijection between shard modules of* Π_G *and shards of the reflecting hyperplane arrangement associated to G.*

The key step in proving this theorem is to observe how reflection functors affect stability domains. The answer lies in the operations

$$\sigma_i^+(K) = s_i(K \cap \{x \in V^* \mid \langle x, \alpha_i \rangle \geq 0\})$$
$$\sigma_i^-(K) = s_i(K \cap \{x \in V^* \mid \langle x, \alpha_i \rangle \leq 0\})$$

which we previously defined in order to state Theorem III.25. The most general form of this relationship is expressed by the following lemma (and a dual counterpart), which we will use later.

Lemma VI.12 ([DST23, Lemma 5.6])**.** *Let β and β' be positive roots with $\beta = s_i\beta'$. Let B' be a brick of dimension β' in* NoQuot_i*. Then*

$$\sigma_i^+(\mathrm{Stab}(B')) = \mathrm{Stab}(\Sigma_i^+ B') \cap \{\theta : \langle\theta, \alpha_i\rangle \leq 0\}$$

In the context of a positive expression, this statement becomes cleaner:

Lemma VI.13 ([DST23, Proposition 5.4])**.** *In the context of the previous lemma, suppose further that $\beta - \beta' \in \mathbb{R}_{>0}\alpha_i$. Then*

$$\sigma_i^+(\mathrm{Stab}(B')) = \mathrm{Stab}(\Sigma_i^+ B')$$

Proof. The assumption $\beta - s_i\beta \in \mathbb{R}_{>0}$ is equivalent to saying $(\alpha_i^\vee, \beta) > 0$, and thus $(\alpha_i, \beta) > 0$.

Now let $B = \Sigma_i^+(B')$. By Proposition V.17,

$$\dim_k \mathrm{Hom}(S_i, B) - \dim_k \mathrm{Ext}^1(S_i, B) + \dim_k \mathrm{Hom}(B, S_i) > 0$$

On the other hand, a module in the image of Σ_i^+ must be in NoSub_i, so the first term is 0. It follows that $\mathrm{Hom}(B, S_i) \neq 0$, so B has S_i as a quotient. In particular, any point θ in $\mathrm{Stab}(B)$ must satisfy $\langle\theta, \alpha_i\rangle \leq 0$. Thus the right side of Lemma VI.12 reduces to $\mathrm{Stab}(B)$. ☐

Inductively applying this lemma and the dual statement for Σ_i^-, we can conclude:

Theorem VI.14 ([DST23, Theorem 5.3])**.** *Let β be a positive root with positive expression $s_{i_\ell} \cdots s_{i_1}(\alpha_{i_0})$. Let $B = \Sigma_{i_\ell}^{\pm_\ell} \cdots \Sigma_{i_1}^{\pm_1}(S_{i_0})$ be a well-behaved expression giving a real brick. Then* $\mathrm{Stab}(B) = \sigma_{i_\ell}^{\pm_\ell} \cdots \sigma_{i_1}^{\pm_1}(\alpha_{i_0}^\perp)$.

This shows how stability domains produce shards; the other important aspect of the theorem is that every shard is a stability domain.

Theorem VI.15. *Let $K = \sigma_{i_\ell}^{\pm_\ell} \cdots \sigma_{i_1}^{\pm_1}(\alpha_{i_0}^\perp)$ be a shard defined using a positive expression as above. Then $\Sigma_{i_\ell}^{\pm_\ell} \cdots \Sigma_{i_1}^{\pm_1}(S_{i_0})$ is a well-behaved expression for the unique shard module B with stability domain K.*

Proof. We show that the expression is well-behaved by induction on ℓ. The base case of $\ell = 0$ is vacuous.

Suppose that $\pm_\ell = -$; the case of $\pm_\ell = +$ proceeds similarly. By the induction hypothesis, $B' := \Sigma_{i_{\ell-1}}^{\pm_{\ell-1}} \cdots \Sigma_{i_1}^{\pm_1}(S_{i_0})$ is a well-behaved expression, so it remains to show that $B' \in \mathrm{NoSub}_{i_\ell}$. Suppose instead that B' has S_{i_ℓ} as a submodule; then $\langle -, \alpha_{i_\ell} \rangle \geq 0$ on $\mathrm{Stab}(B') = \sigma_{i_{\ell-1}}^{\pm_{\ell-1}} \cdots \sigma_{i_1}^{\pm_1}(\alpha_{i_0}^\perp)$. However, when we apply $\sigma_{i_\ell}^-$ to $\mathrm{Stab}(B')$, we first intersect it with $\{\theta \mid \langle \theta, \alpha_{i_\ell} \rangle \leq 0\}$, which in this case would restrict it to $\alpha_{i_\ell}^\perp$, reducing its dimension. This implies that $\sigma_{i_\ell}^{\pm_\ell} \cdots \sigma_{i_1}^{\pm_1}(\alpha_{i_0}^\perp)$ is not a shard, a contradiction.

It then follows from Theorem VI.14 that B has stability domain K. □

VI.5: Not every real brick is a shard module

The condition of being a real brick ended up equating to the nice algebraic property of rigidity. We don't know whether the requirement that shard modules have stability domains of maximal dimension has a similar characterization — at this time it seems to be as contrived as it looks on first glance.

However, it's surprisingly difficult to find an example of a real brick which *doesn't* fulfill the condition on its stability domain. For a while, it was an open question whether such modules existed, and much of the work towards this thesis was done with an eye towards the possibility of proving that they don't.

Nonetheless, there are real bricks which aren't shard modules. The smallest one we could find is a Π_G-module, where G is the following graph:

$$\begin{tikzcd} 1 \arrow[dr] & & & \\ 2 \arrow[r] & 4 \arrow[dl] \arrow[r] & 5 \arrow[r] & 6 \\ 3 & & & \end{tikzcd}$$

Here the orientations represent the arrows a for which we set $\mathrm{sgn}(a) = 1$ in defining Π_G.

The brick in question is given by

$$B := \Sigma_1^- \Sigma_2^+ \Sigma_4^+ \Sigma_3^+ \Sigma_5^- \Sigma_4^- \Sigma_1^- \Sigma_2^+ \Sigma_4^+ \Sigma_5^+ (S_6).$$

and has dimension vector $(3, 3, 2, 4, 2, 1)$. An exploded view of the brick is shown in Figure 43.

In Figure 44, we illustrate four submodules of this brick (by highlighting the basis elements spanning each one).

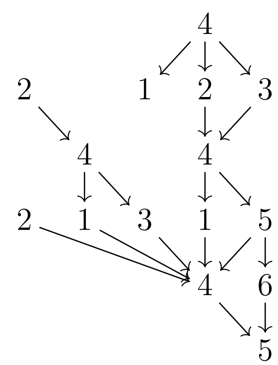


Figure 43: The exploded view of a real brick which is not a shard module.

These submodules have dimension vectors

$$(1, 2, 1, 2, 1, 1)$$
$$(1, 2, 1, 2, 1, 0)$$
$$(3, 1, 2, 4, 2, 1)$$
$$(1, 0, 0, 0, 0, 0)$$

which satisfy the equation

$$2 \cdot (1, 2, 1, 2, 1, 1) + 2 \cdot (1, 2, 1, 2, 1, 0) + (3, 1, 2, 4, 2, 1) + 2 \cdot (1, 0, 0, 0, 0, 0) = 3 \cdot (3, 3, 2, 4, 2, 1)$$

In particular, if $\theta \in \mathrm{Stab}(B)$, it must pair nonnegatively with each of the vectors on the left side of this equation — but it also must pair to 0 with the right side, which is a multiple of $\underline{\dim}(B)$. This implies that θ must pair to 0 with all four of the above vectors, cutting down the dimension of $\mathrm{Stab}(B)$ and preventing B from being a shard module.

This brick was found through a computer search using SageMath code documented at [Dan]. The strategy is to find a sequence of signs $\pm_1, \ldots, \pm_\ell$ such that the expression

$$\Sigma_{i_\ell}^{\pm_\ell} \cdots \Sigma_{i_1}^{\pm_1}(S_{i_0})$$

is well-behaved, but the corresponding cone

$$\sigma_{i_\ell}^{\pm_\ell} \cdots \sigma_{i_1}^{\pm_1}(\alpha_{i_0}^{\perp})$$

is not a shard.

We conduct this search by:

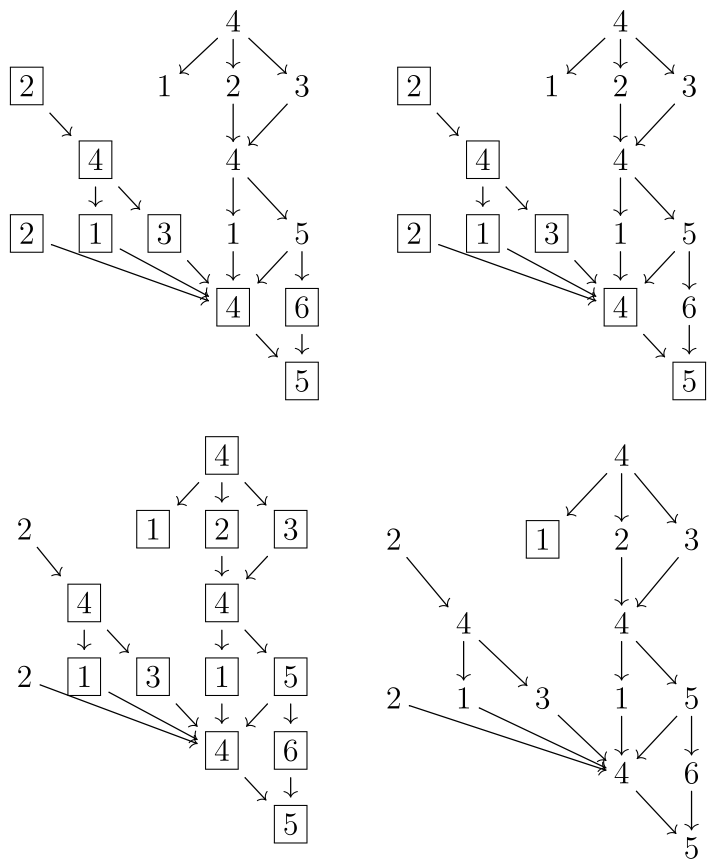
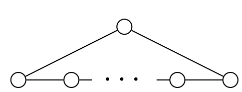


Figure 44: Four submodules of the module in Figure 43. Each picture depicts a submodule because it's closed under following arrows.

- producing a list of all tuples of signs which represent shards;
- for each one, flipping one sign at a time and checking whether the resulting tuple gives a well-behaved expression for a brick;
- if it does, checking whether it also appears in the list of shards.

VI.6: Shard modules of $A_{n-1}^{(1)}$

The **affine type A** root system $A_{n-1}^{(1)}$ has an n-cycle as its Dynkin diagram:



We conclude this chapter with a classification of the real bricks of the corresponding preprojective algebras (which turn out to all be shard modules). To our knowledge, this classficiation has not yet been documented.

We label the vertices 0 through $n-1$ in cyclic order. Because the Dynkin diagram is a cycle, we can pull off a unique trick relating it to the finite type A case discussed at the end of Section VI.2.

Let A_∞ be an infinite path, with vertices indexed by $\mathbb{Z}$. We can define a preprojective algebra Π_{A_∞} just as we did for finite graphs; while this algebra is a horrible beast, we restrict our attention to the category $(\Pi_{A_\infty}\text{-mod})^b$ of finite-dimensional modules with finite support. Any such module can be viewed as a Π_{A_m}-module for some m, and $(\Pi_{A_\infty}\text{-mod})^b$ can thus be viewed as a direct limit of the categories Π_{A_m}-mod using all the natural inclusions $\Pi_{A_{m'}}\text{-mod} \to \Pi_{A_m}\text{-mod}$ for $m' < m$.

Then, thinking of $A_{n-1}^{(1)}$ as a circle, and A_∞ as a line, our key tool is a sort of covering map:

Definition VI.16. The **covering functor** $\pi : (\Pi_{A_\infty}\text{-mod})^b \to \Pi_{A_{n-1}^{(1)}}\text{-mod}$ is defined on objects by

$$\pi(M)_i := \bigoplus_{j \equiv i (\mathrm{mod}\ n)} M_j$$

$$\pi(M)(i \pm 1 \leftarrow i) := \bigoplus_{j \equiv i (\mathrm{mod}\ n)} M(j \pm 1 \leftarrow j)$$

and on morphisms by

$$\pi(f : M \to N)_i = \bigoplus_{j \equiv i (\mathrm{mod}\ n)} f_j$$

The key property of this functor is that it interacts nicely with reflection functors.

Lemma VI.17.

$$\Sigma_i^\pm(\pi(M)) \cong \pi(\cdots \Sigma_{i-n}^\pm \Sigma_i^\pm \Sigma_{i+n}^\pm \cdots M)$$

Note that the expression on the right is well-defined because M is only supported at finitely many vertices, so only finitely many of the reflection functors have any effect.

Proof. It follows straightforwardly from definitions that

$$\pi(M)(i, \mathrm{in}) = \bigoplus_{j \equiv i (\mathrm{mod}\ n)} M(j, \mathrm{in})$$

and thus

$$\Sigma_i^+(\pi(M))_i = \ker(\pi(M)(i, \mathrm{in})) = \bigoplus_{j \equiv i (\mathrm{mod}\ n)} \ker(M(j, \mathrm{in})) = \bigoplus_{j \equiv i (\mathrm{mod}\ n)} \Sigma_j^+(M)_j$$

This is the same space that we get by applying Σ_j^+ to M for all $j \equiv i(\text{mod } n)$ and then applying π; importantly, the order in which we apply these functors doesn't matter, since no two choices of j label adjacent vertices, and a reflection functor only affects a single vertex and the maps incident to it.

Checking that the maps between vertices are affected in the same way by π, and that Σ_i^- behaves the same way, is similar. □

Lemma VI.18. *If $\pi(M) \in \mathrm{NoQuot}_i$ (respectively, NoSub_i), then $M \in \mathrm{NoQuot}_j$ (respectively, NoSub_j) for all $j \equiv i(mod\ n)$.*

Proof. Contrapositively, given a nonzero map $S_j \to M$ or $M \to S_j$, its image under the functor π is clearly also nonzero. □

Proposition VI.19. *Any real brick M of $\Pi_{A_{n-1}^{(1)}}$ is isomorphic to $\pi(\widetilde{M})$ for some real brick $\widetilde{M}$ in $(\Pi_{A_\infty}\text{-}mod)^b$.*

Proof. By Proposition VI.7, M can be obtained from a well-behaved sequence of reflection functors applied to some simple module S_i. By lifting $S_i \in \Pi_{A_{n-1}^{(1)}}$-mod to $S_i \in (\Pi_{A_\infty}\text{-mod})^b$ and repeatedly applying Lemma VI.17, we can obtain some $\widetilde{M} \in (\Pi_{A_\infty}\text{-mod})^b$ with $M \cong \pi(\widetilde{M})$ which also results from a sequence of reflection functors applied to S_i. By repeatedly applying Lemma VI.18, we can further conclude that this sequence is well-behaved. Thus we can apply Proposition VI.7 in the other direction to conclude that $\widetilde{M}$ is a real brick. □

This is a powerful observation because, following the example in Section VI.2, we know exactly what a real brick of Π_{A_m} looks like: a sequence of 1-dimensional spaces and, for each edge, a choice of orientation specifying which arrow along that edge is nonzero. Applying the covering functor π to such a representation gives us a way to depict any real brick of $\Pi_{n-1}^{(1)}$ in exploded view, as shown in Figure 45.

We now need to check which exploded views of the form in Figure 45 actually denote real bricks. The first task is to determine the roots. Throughout what follows, we will use the cyclic symmetry of the $A_{n-1}^{(1)}$ diagram to make statements "up to rotation". We also note that any reflection or reflection functor at a vertex will take into account its two neighbors in cyclic order.

Lemma VI.20. *The positive roots of the $A_1^{(1)}$ are, up to rotation, exactly the vectors of the form*

$$c\alpha_0 + c\alpha_1 + \ldots + c\alpha_{j-1} + (c+1)\alpha_j + \ldots + (c+1)\alpha_{n-1}$$

for $c \geq 0$, where the coefficients c and $(c+1)$ both appear at least once.

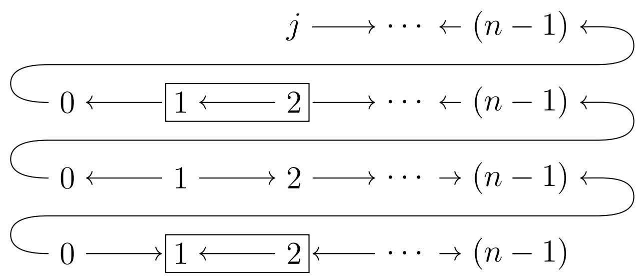


Figure 45: Any real brick of $\Pi_{A_1^{(1)}}$ has, up to rotation, an exploded view that looks like this. *A priori*, the start and end points and the orientations of arrows can be chosen independently. However, this may not produce a brick: here, the boxed sections show the module $B := 1 \leftarrow 2$ arising as both a quotient (top) and a submodule (bottom), and so there is a noninvertible endomorphism $M \twoheadrightarrow B \hookrightarrow M$.

Proof. It is straightforward to calculate that applying any reflection to a vector of this form produces another vector of this form (though one must take care to distinguish between the case where the coefficients c and $(c+1)$ both appear at least twice and the case where one of them appears only once).

Then we can show by induction on depth that every root has this form. Certainly any simple root α_i does, with $c = 0$; given any other root β and a reflection s_i which decreases a coefficient, $s_i\beta$ has this form by the induction hypothesis, so β does as well.

Similarly, we can show that every vector of this form is a root by induction on the sum of the coefficients. The base case of coefficients summing to 1 clearly gives a simple root. Given a vector of this form, let i be an index such that α_i has coefficient $(c+1)$ but one of its neighbors has coefficient c; then applying s_i decreases the coefficient there, producing a root by the induction hypothesis, and implying the vector we started with was also a root. $\square$

Now given a root, rotated to have the form above, we refer to the edges $(n-1) \leftrightarrow 0$ and $(j-1) \leftrightarrow j$, connecting vertices with coefficient c to those with coefficient $c+1$, as **interface edges**. Abusing terminology slightly, we also use this term to refer to all arrows in the exploded view corresponding to the maps along these edges.

Theorem VI.21. *A module with the exploded view in Figure 45 is a brick if and only if:*

(1) all interface edges are oriented away from the larger coefficients, in which case

- *if any arrow along a non-interface edge between i and $i+1$ is oriented $i \leftarrow (i+1)$, all following arrows between those two vertices are also oriented that way, OR*

(2) all interface edges are oriented toward the larger coefficients, in which case

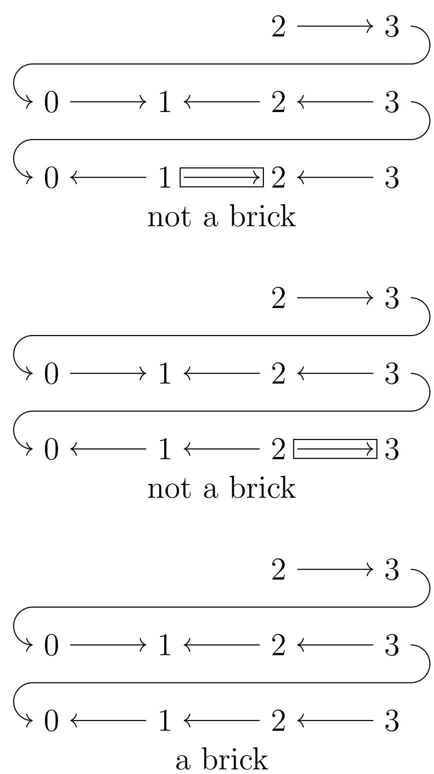


Figure 46: Some exploded forms which do and don't meet the conditions of Theorem VI.21. In the first picture, three of the interface edges are oriented away from the larger coefficients, but the boxed edge is not. In the second case, the interface edges are consistently oriented, but the boxed arrow $2 \to 3$ switches back to pointing right after the previous arrow $2 \leftarrow 3$ points left. The third picture is valid.

- *if any arrow along a non-interface edge between i and $i+1$ is oriented $i \to (i+1)$, all following arrows between those two vertices are also oriented that way.*

This result is illustrated in Figure 46.

Proof. Analogously to the previous result, we proceed by induction on the depth of the dimension vector as a root. The base case consists of the simple modules S_i, which are bricks and vacuously satisfy the required conditions.

Now consider a module M with a form fulfilling condition (1). (If M instead satisfies condition (2), the dual DM, which has all arrows reversed, satisfies condition (1), so we can assume this without loss of generality.)

First, we claim that either $M \in \mathrm{NoSub}_j$ or $M \in \mathrm{NoQuot}_j$, where the index j is as in Lemma VI.20. Condition (1) implies that all arrows between $j-1$ and j in the exploded view point left, which means that the kernel of the map $M_j \to M_{j-1}$ is spanned by the

$$
\begin{array}{ccccc}
 & & j & \longrightarrow & j+1 \\
j-1 & \longleftarrow & j & \dashleftarrow\dashrightarrow & j+1 \\
j-1 & \longleftarrow & j & \dashleftarrow\dashrightarrow & j+1 \\
\vdots & & \vdots & & \vdots
\end{array}
\qquad\qquad
\begin{array}{ccccc}
 & & j & \longleftarrow & j+1 \\
j-1 & \longleftarrow & j & \longleftarrow & j+1 \\
j-1 & \longleftarrow & j & \longleftarrow & j+1 \\
\vdots & & \vdots & & \vdots
\end{array}
$$

Figure 47: At vertex j, an exploded view satisfying condition (1) must either have the form on the left, in which case $M \in \mathrm{NoSub}_j$, or the form on the right, in which case $M \in \mathrm{NoQuot}_j$.

M	$\Sigma_j^-(M)$
$0 \qquad k \longrightarrow k$	$0 \qquad 0 \qquad k$
$k \longleftarrow k \longleftarrow k$	$k \longleftarrow k \longleftarrow k$
$k \longleftarrow k \longrightarrow k$	$k \longrightarrow k \longleftarrow k$

Figure 48: The effect of reflection functors applied to the center vertex of A_3.

top instance of j. If the top arrow between j and $j+1$ points right, it means that the map $M_j \to M_{j+1}$ is nonzero on this top basis element, implying that $\ker M(j, \mathrm{out}) = 0$ and $M \in \mathrm{NoSub}_j$. But if the top arrow between j and $j+1$ points left, the second part of condition (1) implies that *all* arrows between j and $j+1$ point left; this means that the map $M_{j+1} \to M_j$ is the identity, and $\operatorname{im} M(j, \mathrm{in}) = M_j$, so $M \in \mathrm{NoQuot}_j$. These two cases are illustrated in Figure 47.

Thus it is well-behaved to compute one of $\Sigma_j^-(M)$ or $\Sigma_j^+(M)$. By Lemma VI.17, we can just apply Σ^- or Σ^+ separately at each vertex labeled j in the exploded view. The possible outcomes at each vertex are documented in Figure 48. When we do this, we see that the result has a lesser jth coefficient and still satifies either condition (1) or condition (2), as shown in Figure 49. Thus by the induction hypothesis, it is a brick. Applying the inverse reflection functor, by Proposition VI.7, we see that M is also a brick.

Conversely, suppose that M is a real brick. By Proposition VI.7, it has the form $\Sigma_i^\pm(M')$, where M' is a real brick with lesser dimension vector. By rotation and the induction hypothesis, we can assume without loss of generality that M' has dimension vector $c\alpha_0 + \ldots + c\alpha_j + (c+1)\alpha_{j+1} + \ldots + (c+1)\alpha_{n-1}$ and satisfies condition (1). In this setup, the only vertices i at which the reflection s_i will increase a coefficient are 0 and j. We can assume without loss of generality that $i = j$; if $i = 0$, we can reverse the ordering of the vertices ("reflecting" the picture of $A_{n-1}^{(1)}$) and rotate back into the form of Lemma VI.20.

Then we claim that $M = \Sigma_j^\pm(M')$ also satisfies condition (1). This proceeds along the same lines as the converse direction, and essentially amounts to reading Figure 49 from right

$$\begin{array}{ccccc} & & j & \longrightarrow & j+1 \\ j-1 & \longleftarrow & j & \longrightarrow & j+1 \\ j-1 & \longleftarrow & j & \longleftarrow & j+1 \\ \vdots & & \vdots & & \vdots \end{array} \quad \xrightarrow{\Sigma_j^-} \quad \begin{array}{ccccc} & & & & j+1 \\ j-1 & \longrightarrow & j & \longleftarrow & j+1 \\ j-1 & \longleftarrow & j & \longleftarrow & j+1 \\ \vdots & & \vdots & & \vdots \end{array}$$

$$\begin{array}{ccccc} & & n-1 & \longrightarrow & 0 \\ n-2 & \longleftarrow & n-1 & \longrightarrow & 0 \\ \vdots & & \vdots & & \vdots \\ n-2 & \longleftarrow & n-1 & \longrightarrow & 0 \\ n-2 & \longleftarrow & n-1 & & \end{array} \quad \xrightarrow{\Sigma_{n-1}^-} \quad \begin{array}{ccccc} & & & & 0 \\ n-2 & \longrightarrow & n-1 & \longleftarrow & 0 \\ \vdots & & \vdots & & \vdots \\ n-2 & \longrightarrow & n-1 & \longleftarrow & 0 \\ n-2 & & & & \end{array}$$

$$\begin{array}{ccccc} & & j & \longleftarrow & j+1 \\ j-1 & \longleftarrow & j & \longleftarrow & j+1 \\ j-1 & \longleftarrow & j & \longleftarrow & j+1 \\ \vdots & & \vdots & & \vdots \end{array} \quad \xrightarrow{\Sigma_j^+} \quad \begin{array}{ccccc} & & & & j+1 \\ j-1 & \longleftarrow & j & \longleftarrow & j+1 \\ j-1 & \longleftarrow & j & \longleftarrow & j+1 \\ \vdots & & \vdots & & \vdots \end{array}$$

Figure 49: If a module satisfies condition (1) and lies in NoSub_j, then applying Σ_j^- produces another module satisfying condition (1) (shown for $j \neq n-1$ above and $j = n-1$ in the middle). If the module instead lies in NoQuot_j, applying Σ_j^+ produces another module satisfying condition (1) (below).

to left. ☐

Thus we have classified the real bricks of $A_{n-1}^{(1)}$ by exploding them.

Finally, we note that these real bricks are all shard modules. One can explicitly work out the stability domains of these bricks, but that would be unenlightening to describe in detail here. We settle for showing that the number of real bricks is equal to the number of shards, alluding to but not proving the bijection underlying this equality.

The first step is to enumerate shards, which means the zeroth step is to describe shards.

Lemma VI.22. *Consider the expression*

$$s_{n-1}s_{n-2}\cdots s_1 s_0 s_{n-1}s_{n-2}\cdots s_{h+1}(\alpha_h),$$

consisting of ℓ reflections applied to a simple root in cyclic order, ending with s_{n-1}. Then this expression is positive, and produces the root

$$c\alpha_0+\ldots+c\alpha_{j-1}+(c+1)\alpha_j+\ldots+(c+1)\alpha_{n-1}$$

where $c=\lfloor \ell/(n-1)\rfloor$ and $j=(n-1)-\ell \pmod{n-1}$.

Proof. By induction on ℓ. The base case $\ell=0$ is just α_{n-1}, as required. Then suppose the expression of length ℓ produces a root of the expected form. Note that we can step from the expression of length ℓ to the one of length $\ell+1$ by applying s_0 and then shifting all indices back one step in cyclic order. If $\ell\not\equiv n-2 \pmod{n-1}$, so $j\geq 2$, this produces the root

$$c\alpha_0+\ldots+c\alpha_{j-2}+(c+1)\alpha_{j-1}+\ldots+(c+1)\alpha_{n-1}$$

increasing a coefficient and producing the required value. If $\ell\equiv n \pmod{n-1}$, so $j=1$, this produces the root

$$(c+1)\alpha_0+\ldots+(c+1)\alpha_{n-2}+(c+2)\alpha_{n-1}$$

increasing a coefficient and producing the required value. ☐

Lemma VI.23. *The fractures of $\beta^\perp$, where*

$$\beta:=c\alpha_0+\ldots+c\alpha_{j-1}+(c+1)\alpha_j+\ldots+(c+1)\alpha_{n-1}$$

are its intersections with the hyperplanes dual to

$$\gamma_i^{(b)}:=\alpha_i+\alpha_{i+1}+\ldots+\alpha_{n-1}+b\delta$$

where $1 \leq i \leq n-1$, $\delta = \alpha_0 + \ldots + \alpha_{n-1}$, *and* $0 \leq b \leq c-1$, *or* $b = c$ *if* $i \geq j+1$.

Proof. The preceding lemma gives us a positive expression for β. By Theorem III.21, the truncations of this expression will give a list of roots which induce the fractures of $\beta^\perp$. However, note that the truncations of this expression also have the form appearing in the lemma. It follows that these truncations are exactly the roots of the form

$$b\alpha_0 + \ldots + b\alpha_{i-1} + (b+1)\alpha_i + \ldots + (b+1)\alpha_{n-1}$$

with lesser depth than β, which is exactly the claimed list. □

Once we have a list of roots inducing fractures, we can specify a shard of a root $\beta^\perp$ by a list of signs, indicating whether it pairs positively or negatively with these roots.

Proposition VI.24. *Let the roots* β *and* $\gamma_i^{(b)}$ *be as in the previous lemma. Then a shard* K *of* $\beta^\perp$ *is specified by any choice of signs for* $\langle -, \gamma_i^{(b)} \rangle$ *such that*

(1) $\langle -, \gamma_j^{(b)} \rangle \geq 0$ *for all* b*, and*
  - *if* $\langle -, \gamma_i^{(b_0)} \rangle \geq 0$ *for some* $i \neq j$*, then* $\langle -, \gamma_i^{b} \rangle \geq 0$ *for all* $b \geq b_0$*, OR*

(2) $\langle -, \gamma_j^{(b)} \rangle \leq 0$ *for all* b*, and*
  - *if* $\langle -, \gamma_i^{(b_0)} \rangle \leq 0$ *for some* $i \neq j$*, then* $\langle -, \gamma_i^{b} \rangle \leq 0$ *for all* $b \geq b_0$

If $c = 0$*, the top-level choice becomes irrelevant.*

Proof. First, note that $\gamma_j^{(b)} = \beta - (c-b)\delta$. Thus, on $\beta^\perp$, the sign of $\langle -, \gamma_j^{(b)} \rangle$ is equal to that of $\langle -, -\delta \rangle$ for all b.

On the other hand, the roots $\gamma_i^{(0)}$ for $i \neq j$ and δ are independent mod β, so for any list of real numbers we can find a point of $\beta^\perp$ at which the forms $\langle -, \gamma_i^{(0)} \rangle$ for $i \neq j$ and $\langle -, \delta \rangle$ assume those values.

Suppose that $\langle -, \delta \rangle \geq 0$ on K. Then

$$\langle -, \gamma_i^{(0)} \rangle \leq \langle -, \gamma_i^{(1)} \rangle \leq \cdots \leq \langle -, \gamma_i^{(c)} \rangle$$

on K. In particular, if $\langle -, \gamma_i^{(b_0)} \rangle \geq 0$ on K for some index b_0, then we must also have $\langle -, \gamma_i^{(b)} \rangle \geq 0$ for all $b \geq b_0$; however, because we can realize any combination of values of $\langle -, \gamma_i^{(0)} \rangle$ for $i \neq j$ and $\langle -, \delta \rangle$, we can choose the value of b at which the sign switches from negative to positive.

This produces exactly the shards described by part (1) of the proposition. Part (2) covers the case that $\langle -, \delta \rangle \leq 0$ on K instead.

Finally, note that if $c = 0$, then the only $\gamma_i^{(b)}$ cutting $\beta^\perp$ have $i \geq j+1$ and $b = 0$. In particular, $\gamma_j^{(b)}$ does not cut $\beta^\perp$ for any b, justifying the last statement in the proposition. $\square$

At this point, note that the data required to specify a shard closely relates to the data required to specify a real brick, as described in Theorem VI.21 — fixing the signs of inequalities in the former setting corresponds to fixing the directions of arrows in the latter. Although we only mention the enumerative consequences here, we'll investigate phenomena like this in more detail in chapter VIII.

Corollary VI.25. *The root*

$$\beta = c\alpha_0 + \ldots + c\alpha_{j-1} + (c+1)\alpha_j + \ldots + (c+1)\alpha_{n-1}$$

has $2(c+1)^{j-1}(c+2)^{n-1-j}$ *shards if* $c \geq 1$, *or* 2^{n-1-j} *shards if* $c = 0$.

Proof. In order to define a shard in accordance with Proposition VI.24, we must pick whether $\langle -, \gamma_j^{(b)} \rangle$ is ≥ 0 or ≤ 0 for all b. In the former case, we then choose for each $i \neq j$ the smallest b such that $\langle -, \gamma_i^{(b)} \rangle \geq 0$; there are $c+1$ options for this if $i < j$, and $c+2$ options if $i > j$. The latter case is the same with inequalities reversed. If $c = 0$, we do not count the two cases separately. $\square$

Lemma VI.26. $\Pi_{A_{n-1}^{(1)}}$ *has* $2(c+1)^{j-1}(c+2)^{n-1-j}$ *real bricks of dimension vector*

$$\beta = c\alpha_0 + \ldots + c\alpha_{j-1} + (c+1)\alpha_j + \ldots + (c+1)\alpha_{n-1}.$$

Proof. In the case $c = 0$, this reduces to the type A classification, so we assume $c \geq 1$.

By Theorem VI.21, to define a real brick we must first pick whether the interface edges point away from or towards the larger coefficient. In the former case, we then choose, for each pair $(i-1, i)$ with $1 \leq i \leq n-1$ and $i \neq j$, the earliest arrow $(i-1) \leftarrow i$ in the exploded view which points right to left; there are $c+1$ options for this if $i < j$ and $c+2$ options if $i > j$. The latter case is the same with arrow directions reversed. $\square$

Corollary VI.27. *Every real brick of* $\Pi_{A_{n-1}^{(1)}}$ *is a shard module.*

Proof. If this were not the case, there would be more real bricks than there are shards, but the counts agree. $\square$

CHAPTER VII
Short Exact Sequences of Shard Modules

Now that we have a correspondence between shard modules and shards for any root system, it's natural to try to refine this correspondence by considering what relationships between shards tell us about relationships between their corresponding shard modules. A useful starting point for this is a result of Iyama, Reading, Reiten, and Thomas [IRRT18] for finite type preprojective algebras: when shards meet in a certain configuration, there exists a short exact sequence relating the corresponding bricks. In this chapter, we introduce this result in its original context, explain how it connects to the context of shards, and then generalize it to shard modules for any preprojective algebra.

Throughout this section:

- A is a crystallographic symmetrizable Cartan matrix, of size $n \times n$.
- G is its Coxeter diagram.
- Φ is the associated root system, with simple roots $\alpha_1, \ldots, \alpha_n$.
- W is the associated Coxeter group, with generators $s_1, \ldots, s_n$.
- Given any two roots $\beta_1, \beta_2 \in \Phi$ with $\beta_1 \neq \pm\beta_2$, the rank 2 subsystem containing them is denoted $R(\beta_1, \beta_2)$.

Additionally, following Theorems VI.4 and VI.11, given a shard K, we let $M(K)$ be the unique shard module with stability domain K.

VII.1: The finite type case

The source [IRRT18] for the result which motivates this section discusses the same classification of bricks as [Tho18], but rather than using the concept of shards, it refers to other aspects of the lattice theory of Coxeter groups. We state their result in its original form, then explain how to translate it into the language of shards. In this section, G is assumed to be one of the simply laced Dynkin diagrams A_n, D_n, E_6, E_7, or E_8.

Specifically, for any cover relation $w \lessdot ws_h$ in the weak order, one can define a corresponding **layer module**. Suppose that $s_{i_1} \cdots s_{i_\ell}$ is a positive expression for w, and recall the ideals $I_i := \Pi_G(1 - e_i)\Pi_G$ which appeared in Section V.2.2. Then the layer module is the quotient

$$(I_{i_1} \cdots I_{i_\ell})/(I_{i_1} \cdots I_{i_\ell} I_h)$$

Theorem VII.1 ([IRRT18, Proposition 4.3]). *For an interval in weak order of the form*

$$\begin{tikzcd}
& ws_is_js_i & \\
ws_is_j \arrow[ur, dash, "Y"] & & ws_js_i \arrow[ul, dash, "X"'] \\
ws_i \arrow[u, dash, "E"] & & ws_j \arrow[u, dash, "F"'] \\
& w \arrow[ul, dash, "X"] \arrow[ur, dash, "Y"'] &
\end{tikzcd}$$

(0) The layer modules X, Y, E, and F are as shown.

(1) There exist short exact sequences

$$0 \to X \to E \to Y \to 0$$
$$0 \to Y \to F \to X \to 0$$

(2) $\mathrm{Ext}^1(X, Y)$ and $\mathrm{Ext}^1(Y, X)$ are 1-dimensional, spanned by these extensions.

In this case, [IRRT18] calls the modules E and F a **doubleton**.

The layer modules are bricks [IRRT18, Theorem 1.2], and it is with layer modules that [Tho18] makes the connection between bricks and shards, specifically using the following result:

Theorem VII.2 ([Tho18, Proof of Theorem 6]). *Let D be the base region of the reflecting hyperplane arrangement. For any cover $w \lessdot ws_j$ in the weak order, let $B(w \lessdot ws_j)$ be the layer module. Then $-\mathrm{Stab}(B(w \lessdot ws_j))$ contains*[1] *the wall separating wD and ws_jD; in particular, it is the shard containing that wall.*

[1]The definition of stability in [Tho18] uses an inequality going the other way, which accounts for the negative sign here.

The six elements in the statement of Theorem VII.1 correspond to 6 regions in the reflecting hyperplane arrangement, and so the modules X, Y, E, and F in the theorem are the shard modules corresponding to the negations of the shards which separate these regions. Thus we can restate the theorem in the following way.

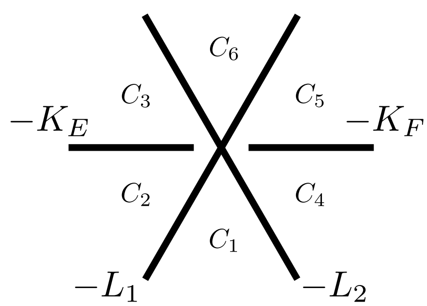
Theorem VII.3. *Suppose a hyperplane $\beta^\perp$ is cut by a rank 2 subsystem with fundamental roots γ_1 and γ_2. Let L_1 and L_2 be shards of $\gamma_1^\perp$ and $\gamma_2^\perp$, respectively, whose intersection has codimension 2. Since $\beta^\perp$ is cut by this subsystem, there exist two shards K_E and K_F of $\beta^\perp$ whose intersection with $L_1 \cap L_2$ is codimension 2. Suppose wihout loss of generality that $\langle \gamma_1, - \rangle \geq 0$ on K_E. Then:*

(1) There exist short exact sequences

$$0 \to M(L_1) \to M(K_E) \to M(L_2) \to 0$$
$$0 \to M(L_2) \to M(K_F) \to M(L_1) \to 0$$

(2) $\mathrm{Ext}^1(M(L_2), M(L_1))$ and $\mathrm{Ext}^1(M(L_1), M(L_2))$ are 1-dimensional, spanned by these extensions.

Proof. If we choose a generic point in the intersection $-(K_E \cap K_F \cap L_1 \cap L_2)$ and examine how a small neighborhood of that point meets the reflecting hyperplane arrangement and its shards, we see that it touches 6 regions arranged like so:



These six regions will then correspond to an interval of 6 elements as in Proposition VII.1. We claim that, in this interval, C_1 is the least element (which will show that our figure is oriented in the same way as the figure in VII.1).

Recall from the context of Theorem II.23 that for a region C, $S(C)$ denotes the set of hyperplanes separating C from the base region D, and that the ordering of regions is defined by $C \leq C'$ if $S(C) \subset S(C')$.

By assumption, K_E lies on the positive side of $\gamma_1^\perp$ (by which we mean that its points pair nonnegatively with γ_1). By definition, D does as well. Thus $\gamma_1^\perp$ separates $-K_E$ and D. As a result, it also separates regions C_2, C_3, and C_6 from D.

On the other hand, $-K_F$ lies on the positive side of $\gamma_1^\perp$. Because γ_1 and γ_2 are fundamental in this rank 2 subsystem, β is a positive linear combination of them. For any $x \in -K_F$, $\langle x, \beta \rangle = 0$ but $\langle x, \gamma_1 \rangle \geq 0$, so $\langle x, \gamma_2 \rangle \leq 0$. Thus $\gamma_2^\perp$ separates $-K_F$, and thus the regions C_4, C_5, and C_6, from D. It follows that C_1, as the only region here not separated from D by $\gamma_1^\perp$ or $\gamma_2^\perp$, is the least element in this interval.

We can now apply Theorem VII.1 to this situation. Note that by Theorem VII.2, the roles of X, Y, E, and F are played by $M(L_1)$, $M(L_2)$, $M(K_E)$, and $M(K_F)$ respectively. $\square$

VII.2: Fundamental shards

In generalizing Theorem VII.3 to the infinite setting, we'd like to apply it to the task of systematically breaking down a shard module as follows. Suppose we're given a shard K, and we pick some wall of K (i.e., one of its facets — an $(n-2)$-dimensional cone). That wall exists because the hyperplane containing K is cut by a rank 2 subsystem R there: in other words, the wall is $K \cap R^\perp$, and we say that the wall is **sliced by** R. If we select from each of the fundamental hyperplanes of R its shard which meets K, we'll be in the situation of Theorem VII.3 and can realize $M(K)$ as an extension of smaller shard modules.

However, if we try to frame this as an algorithm which takes a shard and one of its walls as input and spits out a short exact sequence, we run into a small issue: *a priori*, each fundamental hyperplane could have multiple shards meeting K, introducing an arbitrary choice.

It turns out that this doesn't happen. In this section, we will prove:

Theorem VII.4. *Let β be a root, R a rank 2 subsystem cutting $\beta^\perp$, and γ_1 and γ_2 the fundamental roots of R. Let $K \subset \beta^\perp$ be a shard with a wall sliced by R. Then for $i = 1, 2$ there exists a unique shard $L_i \subset \gamma_i^\perp$ such that $L_i \supset K \cap R^\perp$.*

Without loss of generality, we'll prove the case $i = 1$. The strategy here is to show that, for any fracture of $\gamma_1^\perp$, there is a fracture of $\beta^\perp$ which meets the subspace $R^\perp$ in the same way. In particular, if two different shards of $\gamma_1^\perp$ met $K \cap R^\perp$, any fracture separating them would correspond to a fracture of $\beta^\perp$ passing through K, a contradiction since K is a shard of $\beta^\perp$. We now explain the details of this approach.

The key step is the following lemma.

Lemma VII.5. *Let $\beta, R, \gamma_1, \gamma_2$ be as in Theorem VII.4. Let T be a rank 2 subsystem cutting $\gamma_1^\perp$. Then there exists a rank 2 subsystem S cutting $\beta^\perp$, such that*

$$\mathrm{span}(R, S) = \mathrm{span}(R, T)$$

Proof. We proceed by induction on the depth of β. The base case of depth 0 is vacuous, since then β is a simple root and is fundamental in any rank 2 subsystem. We proceed to the inductive step.

Let $\beta = s_{i_\ell} \cdots s_{i_1}(\alpha_{i_0})$ be a positive expression for β. By Theorem III.21, there is a truncation $\tau = s_{i_\ell} \cdots s_{i_{j+1}}(\alpha_{i_j})$ such that $R = R(\beta, \tau)$. Letting $\mu = s_{i_\ell} \cdots s_{i_{j+1}} s_{i_{j-1}} \cdots s_{i_1}(\alpha_{i_0})$ be the corresponding omission, we additionally know by Proposition III.23 that $R = R(\beta, \mu)$.

The proof now splits into three cases.

(a) First, suppose $\tau = \gamma_1$. By Theorem III.21, $T = R(\tau, \tau')$ for some truncation $\tau' = s_{i_\ell} \cdots s_{i_{k+1}}(\alpha_{i_k})$ of the expression defining τ. But then τ' is also a truncation of our original positive expression for β, so $S = R(\beta, \tau')$ cuts $\beta^\perp$ and has the required property.

(b) Next, suppose $\mu = \gamma_1$. By Theorem III.21, $T = R(\mu, \tau')$ for some truncation τ' of the expression defining μ. This will take the form

$$s_{i_\ell} \cdots s_{i_{k+1}}(\alpha_{i_k})$$

or

$$s_{i_\ell} \cdots s_{i_{j+1}} s_{i_{j-1}} \cdots s_{i_{k+1}}(\alpha_{i_k})$$

depending on whether $k > j$ or $k < j$.

In the former case, we can take $S = R(\beta, \tau')$ for the same reason as in the previous case. In the latter case, let

$$\tau'' = s_{i_\ell} \cdots s_{i_{j+1}} s_{i_j} s_{i_{j-1}} \cdots s_{i_{k+1}}(\alpha_{i_k})$$

This is a truncation of the positive expression for β, so $R(\beta, \tau'')$ cuts β. Then let $S = R(\beta, \tau'')$. By the definition of s_{i_j} we have

$$s_{i_j} s_{i_{j-1}} \cdots s_{i_{k+1}}(\alpha_{i_k}) = s_{i_{j-1}} \cdots s_{i_{k+1}}(\alpha_{i_k}) + c\alpha_{i_j}$$

for some constant c. Applying $s_{i_\ell} \cdots s_{i_{j+1}}$ to both sides implies

$$\tau'' = \tau' + c\tau$$

and thus

$$\operatorname{span}(R, S) = \operatorname{span}(\beta, \tau, \tau'') = \operatorname{span}(\beta, \tau, \tau') = \operatorname{span}(\beta, \mu, \tau') = \operatorname{span}(R, T).$$

(c) Suppose neither τ nor μ is γ_1, and additionally $\tau \neq \gamma_2$. Then τ is not fundamental in R. Since τ has lesser depth than β, we can apply the induction hypothesis to find a rank 2 subsystem T_τ cutting $\tau^\perp$ such that $\operatorname{span}(R, T_\tau) = \operatorname{span}(R, T)$. But then we can apply the reasoning of part (a) (which works even if τ is not fundamental) to find a rank 2 subsystem S cutting $\beta^\perp$ such that $\operatorname{span}(R, S) = \operatorname{span}(R, T_\tau)$.

(d) Finally, suppose neither τ nor μ is γ_1 but $\tau = \gamma_2$. Then $\mu \neq \gamma_2$ (since we know τ and μ generate a rank 2 subsystem) and so it is not fundamental in R. Again, μ has lesser depth than β, so we can apply the induction hypothesis to find a rank 2 subsystem T_μ cutting $\mu^\perp$ such that $\operatorname{span}(R, T_\mu) = \operatorname{span}(R, T)$. Then we can apply the reasoning of part (b) (which, again, works even if μ is not fundamental) to find a rank 2 subsystem S cutting $\beta^\perp$ such that $\operatorname{span}(R, S) = \operatorname{span}(R, T_\mu)$.

□

Now we can prove Theorem VII.4.

Proof. Let $F = K \cap R^\perp$ be the wall under consideration. We first claim that there exists a shard L_1 of $\gamma_1^\perp$ containing F. Specifically, we show that for any rank 2 subsystem T cutting $\gamma_1^\perp$, F lies entirely on one side of the fracture $T^\perp$. By Lemma VII.5, there exists a rank 2 subsystem S cutting $\beta^\perp$ such that $\operatorname{span}(R, S) = \operatorname{span}(R, T)$. Then K lies on one side of $S^\perp$, and so $F = K \cap R^\perp$ lies on one side of $R^\perp \cap S^\perp = R^\perp \cap T^\perp$. Since F doesn't cross any fractures of $\gamma_1^\perp$, it must be contained in a shard L_1.

Then we show that L_1 is unique in this regard. If there were some other shard L_1' also containing F, then F would be contained in the fracture separating L_1 and L_1', which is a subspace of dimension $n-2$. However, F is a cone of dimension $n-2$, so the only possible such subspace is the span of F, which is $R^\perp$. This cannot be a fracture of $\gamma_1^\perp$ because, by definition, γ_1 is fundamental in R. □

Definition VII.6. In the setting of Theorem VII.4, we say that L_1 and L_2 are the **fundamental shards meeting K at the wall $R^\perp$.**

We note that the relationship between a shard and the fundamental shards at a wall can be recast in terms of the shard digraph, which was defined in Section III.3. The main result of this section becomes:

Corollary VII.7. *Let K be any shard. For each wall of K, there are exactly two arrows out of K in the shard digraph, pointing to the fundamental shards meeting K at that wall.*

Finally, we state a lemma on the positioning of fundamental shards which will be useful in what follows.

Lemma VII.8. *Let L_1 and L_2 be the fundamental shards of $\gamma_1^\perp$ and $\gamma_2^\perp$ meeting K at a wall $R^\perp$. Then L_1 contains points on both sides of $\gamma_2^\perp$.*

Proof. This can be seen in Figure 50. More precisely, if L_1 lay entirely to one side of $\gamma_2^\perp$, then the cone $L_1 \cap R^\perp \subset \gamma_2^\perp$ would belong to the boundary of L_1, so it would lie in a fracture of $\gamma_1^\perp$. But since $L_1 \cap R^\perp$ contains the wall $K \cap R^\perp$, it has dimension $n-2$, and its span is all of $R^\perp$, which must thus fracture $\gamma_1^\perp$. This contradicts that γ_1 is fundamental in R. □

VII.3: The doubleton short exact sequence

This section is dedicated to proving the following generalization of Theorem VII.3 beyond finite type.

Theorem VII.9. *Let β be a root and let K be a shard of $\beta^\perp$. Let R be a rank 2 subsystem slicing a wall of K, and let γ_1 and γ_2 be the fundamental roots of R. Without loss of generality, suppose $\langle -, \gamma_1 \rangle \geq 0$ on K. Let c_1, c_2 be constants such that $\beta = c_1\gamma_1 + c_2\gamma_2$. Let L_1 and L_2 be the fundamental shards of $\gamma_1^\perp$ and $\gamma_2^\perp$, respectively, meeting K at R. Then:*

(1) There exists a short exact sequence

$$0 \to M(L_1)^{\oplus c_1} \xrightarrow{f} M(K) \xrightarrow{g} M(L_2)^{\oplus c_2} \to 0$$

(2) $\dim \operatorname{Hom}(M(K), M(L_1)) = \dim \operatorname{Hom}(M(L_2), M(K)) = 0$,
$\dim \operatorname{Hom}(M(L_1), M(K)) = c_1$, *and* $\dim \operatorname{Hom}(M(K), M(L_2)) = c_2$.

There is a useful characterization of the maps in this sequence. We view f as a row vector of c_1 component maps $f_i : M(L_1) \to M(K)$, and likewise view g as a column vector of c_2 component maps $g_i : M(K) \to M(L_2)$.

Corollary VII.10. *For any exact sequence of the form in Theorem VII.9, the components of f are a basis of* $\operatorname{Hom}(M(L_1), M(K))$ *and the components of g are a basis of* $\operatorname{Hom}(M(K), M(L_2))$.

Proof. Because the number of components of f equals the dimension of $\operatorname{Hom}(M(L_1), M(K))$, it suffices to show they are linearly independent. If not, there would be a relation $\sum_{i=1}^{c_1} a_i f_i = 0$. But then for any element $m \in M(L_1)$, the element $(a_1 m, \ldots, a_{c_1} m)$ would be sent to 0 by f, contradicting that f is injective. The proof for g proceeds dually. □

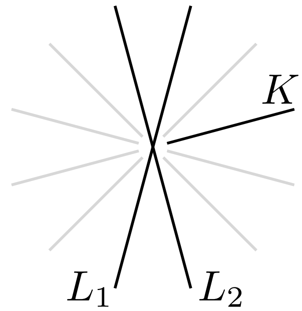


Figure 50: Theorem VII.9 states that, whenever three shards meet in this configuration, there is a short exact sequence relating $M(L_1)$, $M(K)$, and $M(L_2)$.

The setting in which the theorem applies is shown in Figure 50. Following the language of [IRRT18], we call this sequence the **doubleton sequence** or **doubleton extension** of $M(L_2)^{\oplus c_2}$ by $M(L_1)^{\oplus c_1}$.

Throughout this proof, we let $\beta = s_{i_\ell} \cdots s_{i_1}(\alpha_{i_0})$ be a positive expression for β. Then by Theorem III.25, we can choose signs such that $K = \sigma_{i_\ell}^{\pm_\ell} \cdots \sigma_{i_1}^{\pm_1}(\alpha_{i_0}^\perp)$. This will be a proof by induction, and in preparation for this we additionally define

$$\begin{aligned} \beta' &:= s_{i_\ell}\beta = s_{i_{\ell-1}} \cdots s_{i_1}(\alpha_{i_0}) \\ K' &:= \sigma_{i_{\ell-1}}^{\pm_{\ell-1}} \cdots \sigma_{i_1}^{\pm_1}(\alpha_{i_0}^\perp) \\ R' &:= s_{i_\ell} R \end{aligned}$$

By Theorem III.21, there is some truncation $\tau = s_{i_\ell} \cdots s_{i_{j+1}}(\alpha_{i_j})$ such that $R = R(\beta, \tau)$. Let j be the smallest index for which this happens. Then we proceed by induction on $\ell - j$.

VII.3.1: The base case

We first treat the base case of $j = \ell$. This is the case that $R = R(\beta, \alpha_{i_\ell})$, and no truncation besides the one at s_{i_ℓ} lies in R. To prove the theorem in this case, we need to compute the fundamental roots and shards.

Lemma VII.11. *In the case $j = \ell$:*

(1) The fundamental roots of $R(\beta, \alpha_{i_\ell})$ are α_{i_ℓ} and β'.

(2) The fundamental shards meeting K at the wall $R^\perp$ are $\alpha_{i_\ell}^\perp$ and K'.

Proof. (1) A simple root is always fundamental in any rank 2 subsystem, so α_{i_ℓ} is fundamental. Suppose for a contradiction that β' is not. Then $(\beta')^\perp$ is cut by α_{i_ℓ}, so by Theorem III.21 there is some truncation $\tau' = s_{i_{\ell-1}} \cdots s_{i_{j+1}}(\alpha_{i_j})$ such that $\alpha_{i_\ell} \in R(\beta', \tau')$.

But then, applying s_{i_ℓ}, we see that $s_{i_\ell}\tau'$ is a truncation of the expression for β, at an index smaller than ℓ, such that $\alpha_{i_\ell} \in R(\beta, s_{i_\ell}\tau')$. This contradicts the assumption $j = \ell$.

(2) Because $\alpha_{i_\ell}^\perp$ is the *only* shard of $\alpha_{i_\ell}^\perp$, it certainly meets K. Note that $R^\perp \subset \alpha_{i_\ell}^\perp$, so it is fixed pointwise by s_{i_ℓ}. Thus we have

$$K \cap R^\perp = \sigma_{i_\ell}^{\pm_\ell}(K' \cap R^\perp) \subset s_{i_\ell}(K' \cap R^\perp) = K' \cap R^\perp$$

which implies K' is the fundamental shard of $(\beta')^\perp$ meeting K.

$\square$

We can now state what Theorem VII.9 says in this case. Suppose without loss of generality that $K = \sigma_{i_\ell}^+(K')$. (The case of $K = \sigma_{i_\ell}^-(K')$ proceeds dually.) Then $\langle -, \alpha_{i_\ell}\rangle \leq 0$ on K, so we have $L_1 = K'$ and $L_2 = \alpha_{i_\ell}^\perp$. Additionally, by the definition of reflection

$$\beta' = s_{i_\ell}(\beta) = \beta - (\alpha_{i_\ell}, \beta)\alpha_{i_\ell}$$

so we set $c_1 = 1$, $c_2 = (\alpha_{i_\ell}, \beta)$. To finish the base case, we thus need to show:

Lemma VII.12. *In the context of this section,*

(1) There exists a short exact sequence

$$0 \to M(K') \xrightarrow{f} M(K) \xrightarrow{g} S_{i_\ell}^{\oplus(\alpha_{i_\ell},\beta)} \to 0$$

(2) $\dim \operatorname{Hom}(M(K), M(K')) = \dim \operatorname{Hom}(S_{i_\ell}, M(K)) = 0$*,*
$\dim \operatorname{Hom}(M(K'), M(K)) = 1$*, and* $\dim \operatorname{Hom}(M(K), S_{i_\ell}) = (\alpha_{i_\ell}, \beta)$*.*

Proof. (1) Because $K = \sigma_{i_\ell}^+(K')$, it follows from Theorem VI.15 that $M(K) = \Sigma_{i_\ell}^+(M(K'))$ and that this expression is well-behaved, so $M(K') \in \mathrm{NoQuot}_{i_\ell}$. Then the result will follow from Lemma V.15 if we show that $M(K')$ is also in NoSub_{i_ℓ}. If $M(K')$ had S_{i_ℓ} as a submodule, it would impose $\langle -, \alpha_{i_\ell}\rangle \geq 0$ on the stability domain K'. However, this would imply that K' lies entirely on one side of $\alpha_{i_\ell}^\perp$, contradicting Lemma VII.8.

(2) If there were a nonzero map $M(K) \to M(K')$, we could compose it with the map $M(K') \hookrightarrow M(K)$ to get a noninvertible endomorphism of $M(K)$, contradicting that it is a brick, so $\dim \operatorname{Hom}(M(K), M(K')) = 0$.

Any output of $\Sigma_{i_\ell}^+$ lies in NoSub_{i_ℓ}, so since $M(K) = \Sigma_{i_\ell}^+(M(K'))$, we have
$\dim \operatorname{Hom}(S_{i_\ell}, M(K)) = 0$.

Applying $\mathrm{Hom}(M(K'), -)$ to the sequence from part (1) gives an exact sequence

$$0 \to \mathrm{Hom}(M(K'), M(K')) \to \mathrm{Hom}(M(K'), M(K)) \to \mathrm{Hom}(M(K'), S_{i_\ell}).$$

Since $M(K') \in \mathrm{NoQuot}_{i_\ell}$, the last term is 0, and since $M(K')$ is a real brick, the first term is 1-dimensional by Lemma VI.8. Thus $\dim \mathrm{Hom}(M(K'), M(K)) = 1$.

Finally, viewing the map $g : M(K) \to S_{i_\ell}^{\oplus(\alpha_i,\beta)}$ from part (1) as a collection of (α_i, β) component maps $M(K) \to S_{i_\ell}$, we claim that these form a basis for $\mathrm{Hom}(M(K), S_{i_\ell})$. By part (2) of Lemma V.15, for any map $h : M(K) \to S_{i_\ell}$, there is a unique map $\overline{h} : S_{i_\ell}^{\oplus(\alpha_{i_\ell},\beta)} \to S_{i_\ell}$ such that $\overline{h} \circ g = h$. Because $\mathrm{Hom}(S_{i_\ell}, S_{i_\ell}) \cong k$, this is equivalent to realizing h as a unique linear combination of the components of g.

□

VII.3.2: Preparing for the inductive step

We now move on to the inductive step of the proof, so suppose that $\ell > j$. Our goal here is to reduce to considering the shard K'. We first claim that K' has a wall sliced by R'. Because $j < \ell$, we can take the truncation $\tau' = s_{i_{\ell-1}} \cdots s_{i_{j+1}}(\alpha_{i_j})$ and write $R' = R(\beta', \tau')$. Thus R' cuts $(\beta')^\perp$ by Theorem III.21. To see that $K' \cap (R')^\perp$ is actually a wall of K', we show that it has dimension $n-2$. We have

$$K \cap R^\perp = \sigma_{i_\ell}^{\pm_\ell}(K' \cap (R')^\perp) \subset s_{i_\ell}(K' \cap (R')^\perp)$$

and since the left side has dimension $n-2$, the right side does as well.

Next, we note that in the context of the shard K' and its wall sliced by R', the quantity $\ell - j$ is smaller. The quantity j has not changed: as observed above, truncating the positive expression at index j produces a root τ' such that $R' = R(\beta', \tau')$, and if any truncation at a smaller index produced a root in R', the corresponding truncation of the expression for β would produce a root in R, contradicting the original definition of j. However, the depth ℓ has gone down. This means that, by the induction hypothesis, Theorem VII.9 applies to the shard K' and its wall sliced by R'.

However, the particulars of getting from here back to a statement about K and R are slightly different depending on whether R contains the simple root α_{i_ℓ}. We thus divide this step into two cases.

VII.3.3: Inductive step, case 1: R does not contain α_{i_ℓ}

We first state exactly what the induction hypothesis tells us in this case. The key fact special to this case is that, by Lemma III.18, $\gamma_1' := s_{i_\ell}(\gamma_1)$ and $\gamma_2' := s_{i_\ell}(\gamma_2)$ are the fundamental roots of R'. In particular, $\beta' = c_1\gamma_1' + c_2\gamma_2'$. Let L_1' and L_2' be the fundamental shards of $(\gamma_1')^\perp$ and $(\gamma_2')^\perp$, respectively, which meet K' at the wall sliced by R'. Finally, just as K lies on the positive side of $\gamma_1^\perp$, we claim that K' lies on the positive side of $(\gamma_1')^\perp$: if instead it lay on the negative side, since $K \subset s_{i_\ell}(K')$ we would have $\langle \theta, \gamma_1 \rangle = \langle s_{i_\ell}\theta, \gamma_1' \rangle \leq 0$ for any $\theta \in K$, a contradiction. Putting this together, the induction hypothesis gives an exact sequence

$$0 \to M(L_1')^{\oplus c_1} \to M(K') \to M(L_2')^{\oplus c_2} \to 0$$

Then we claim that applying $\Sigma_{i_\ell}^{\pm_\ell}$ to this sequence produces the sequence in Theorem VII.9. We need to check that each term of this sequence is sent to the appropriate term of the new sequence, and that the sequence remains exact afterwards. Assume without loss of generality that $\pm_\ell = +$; the other case proceeds dually.

Lemma VII.13. *For L_1' and L_2' as defined in this section, $M(L_1')$ and $M(L_2')$ both lie in NoQuot_{i_ℓ}.*

Proof. If $M(L_1')$ instead has S_{i_ℓ} as a quotient, then its stability domain L_1' satisfies the constraint $\langle -, \alpha_{i_\ell} \rangle \leq 0$. Thus it will suffice to show that both L_1' and L_2' contain points where $\langle -, \alpha_{i_\ell} \rangle > 0$. By definition, L_1' and L_2' both contain $K' \cap (R')^\perp$, so in turn it will suffice to show that this wall contains such points. If it didn't, we would have

$$K \cap R^\perp = \sigma_{i_\ell}^+(K' \cap (R')^\perp) = s_{i_\ell}(K' \cap \{\theta \mid \langle -, \alpha_{i_\ell} \rangle \geq 0\} \cap (R')^\perp) \subset \alpha_{i_\ell}^\perp \cap R^\perp$$

But since R doesn't contain α_{i_ℓ}, the right side has dimension $n-3$, contradicting that $K \cap R^\perp$ is a wall of K. □

Lemma VII.14. *For L_1' and L_2' as defined in this section, $\Sigma_{i_\ell}^+(M(L_1')) \cong M(L_1)$ and $\Sigma_{i_\ell}^+(M(L_2')) \cong M(L_2)$.*

Proof. We focus on L_1'; the case of L_2' is identical. We first show that $\Sigma_{i_\ell}^+(M(L_1'))$ is a shard module. Then, we show that its stability domain is L_1, at which point Theorem VI.11 will imply that it is actually $M(L_1)$.

As observed in our discussion of Proposition VI.7, applying $\Sigma_{i_\ell}^+$ to a real brick in NoQuot_{i_ℓ} produces another real brick. Thus $\Sigma_{i_\ell}^+(M(L_1'))$ is a real brick. Then we check that its stability domain has dimension $n-1$, so that it is a shard module. By Lemma VI.12,

$$\mathrm{Stab}(\Sigma^+_{i_\ell}(M(L'_1))) \supset \sigma^+_{i_\ell}(\mathrm{Stab}(M(L'_1))) = \sigma^+_{i_\ell}(L'_1)$$

As shown in the previous lemma, L'_1 contains points with $\langle -, \alpha_{i_\ell}\rangle > 0$, so its dimension does not go down when we intersect it with $\{\theta \mid \langle \theta, \alpha_{i_\ell}\rangle \geq 0\}$. Thus $\sigma^+_{i_\ell}(L'_1)$ is also a cone of dimension $n-1$.

Finally, we note that since $L'_1 \supset K' \cap (R')^\perp$,

$$\mathrm{Stab}(\Sigma^+_{i_\ell}(M(L'_1))) \supset \sigma^+_{i_\ell}(L'_1) \supset \sigma^+_{i_\ell}(K' \cap (R')^\perp) = K \cap R^\perp$$

Thus $\mathrm{Stab}(\Sigma^+_{i_\ell}(M(L'_1)))$, which we now know must be a shard of $\gamma_1^\perp$, is the fundamental shard meeting K at the wall R, namely L_1. □

Now that we know $\Sigma^+_{i_\ell}$ will send the terms of our sequence where they're supposed to go, we complete this case of the inductive step.

Lemma VII.15. *In the context of this section:*

(1) There exists a short exact sequence

$$0 \to M(L_1)^{\oplus c_1} \to M(K) \to M(L_2)^{\oplus c_2} \to 0$$

(2) $\dim \mathrm{Hom}(M(K), M(L_1)) = \dim \mathrm{Hom}(M(L_2), M(K)) = 0$,
$\dim \mathrm{Hom}(M(L_1), M(K)) = c_1$, *and* $\dim \mathrm{Hom}(M(K), M(L_2)) = c_2$.

Proof. (1) By Lemma VII.14, applying $\Sigma^+_{i_\ell}$ to the exact sequence given by the induction hypothesis produces the sequence

$$0 \to M(L_1)^{\oplus c_1} \to M(K) \to M(L_2)^{\oplus c_2} \to \mathrm{Ext}^1(I_{i_\ell}, M(L'_1)^{\oplus c_1}).$$

By Lemma V.20, this last term is 0.

(2) By the induction hypothesis and the fact that $\beta' = c_1\gamma'_1 + c_2\gamma'_2$, we have

$$\begin{aligned}
\dim \mathrm{Hom}(M(K'), M(L'_1)) &= 0\\
\dim \mathrm{Hom}(M(L'_2), M(K')) &= 0\\
\dim \mathrm{Hom}(M(L'_1), M(K')) &= c_1\\
\dim \mathrm{Hom}(M(K'), M(L'_2)) &= c_2
\end{aligned}$$

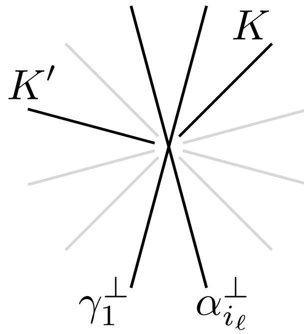


Figure 51: The situation of step 2 of the induction. K and K' are both cut by R, but lie on opposite sides of it.

Because $\Sigma_{i_\ell}^+$ acts as an equivalence of categories on NoQuot_{i_ℓ}, applying it doesn't change any of these dimensions.

□

VII.3.4: Inductive step, case 2: R contains α_{i_ℓ}

The key factor differentiating this case from the previous one is that the subsystems R and $R' := s_{i_\ell}R$ are now the same. This is clearest in the dual picture: $R^\perp \subset \alpha_{i_\ell}^\perp$, so it is fixed pointwise by s_{i_ℓ}. In particular, the wall $K \cap R^\perp$ is also a wall of K'. In contrast with the previous case, when we apply the induction hypothesis to K', we will use the same fundamental roots and shards.

As before, we treat the case that $\pm_\ell = +$, and the other case proceeds dually. Since α_{i_ℓ} is a simple root, it is fundamental in any rank 2 subsystem that contains it, including R, and $\alpha_{i_\ell}^\perp$ is a single shard. Since $K = \sigma_{i_\ell}^+(K')$, we have $\langle -, \alpha_{i_\ell}\rangle \leq 0$ on K, so $\alpha_{i_\ell} = \gamma_2$. The situation is illustrated in Figure 51.

Because K' meets the same wall as K, the induction hypothesis will give an exact sequence relating it to $M(L_1)$ and $M(L_2) = S_{i_\ell}$. However, because $K = \sigma_{i_\ell}^+(K') = s_{i_\ell}(K' \cap \{\theta \mid \langle \theta, \alpha_{i_\ell}\rangle\})$, we have $\langle -, \alpha_{i_\ell}\rangle \geq 0$ on K'. Additionally,

$$\beta' = s_{i_\ell}(\beta) = s_{i_\ell}(c_1\gamma_1 + c_2\alpha_{i_\ell}) = c_1\gamma_1 + (c_2 - c_1(\gamma_1, \alpha_{i_\ell})\alpha_{i_\ell}$$

Accordingly, we let $c_1' = c_2 - c_1(\gamma_1, \alpha_{i_\ell})$ and $c_2' = c_1$.

Thus, by the induction hypothesis, there is an exact sequence

$$0 \to S_{i_\ell}^{\oplus c_1'} \to M(K') \to M(L_1)^{\oplus c_2'} \to 0$$

We now complete the inductive step.

Lemma VII.16. *In the context of this section:*

(1) There exists a short exact sequence

$$0 \to M(L_1)^{\oplus c_1} \to M(K) \to S_{i_\ell}^{\oplus c_2} \to 0$$

(2) $\dim \operatorname{Hom}(M(K), M(L_1)) = \dim \operatorname{Hom}(S_{i_\ell}, M(K)) = 0$,
$\dim \operatorname{Hom}(M(L_1), M(K)) = c_1$, *and* $\dim \operatorname{Hom}(M(K), S_{i_\ell}) = c_2$.

Proof. (1) Applying $\Sigma^+_{i_\ell}$ to the exact sequence given by the induction hypothesis produces the sequence

$$0 \to \Sigma^+_{i_\ell}(S_{i_\ell})^{\oplus c'_1} \to M(K) \to \Sigma^+_{i_\ell}(M(L_1))^{\oplus c'_2} \to \operatorname{Ext}^1(I_{i_\ell}, S_{i_\ell})^{\oplus c'_1} \to \operatorname{Ext}^1(I_{i_\ell}, M(K')).$$

Direct calculation shows that $\Sigma^+_{i_\ell}(S_{i_\ell}) = 0$, while Lemma V.20 shows that the last two terms are $S_{i_\ell}^{\oplus c'_1}$ and 0 (because $M(K') \in \mathrm{NoQuot}_{i_\ell}$). Thus this reduces to a short exact sequence

$$0 \to M(K) \xrightarrow{j} \Sigma^+_{i_\ell}(M(L_1))^{\oplus c'_2} \xrightarrow{h} S_{i_\ell}^{\oplus c'_1} \to 0.$$

Next, we observe that $M(L_1) \in \mathrm{NoQuot}_{i_\ell} \cap \mathrm{NoSub}_{i_\ell}$. If it had S_{i_ℓ} as a quotient, it would impose $\langle -, \alpha_{i_\ell}\rangle \geq 0$ on the stability domain L_1. Likewise, if it had S_{i_ℓ} as a submodule, it would impose $\langle -, \alpha_{i_\ell}\rangle \leq 0$. In either case, this would contradict Lemma VII.8. Then by Lemma V.15, there is another exact sequence

$$0 \to M(L_1) \xrightarrow{f''} \Sigma^+_{i_\ell}(M(L_1)) \xrightarrow{g''} S_{i_\ell}^{\oplus -(\alpha_i, \gamma_1)} \to 0$$

By part (2) of Lemma V.15, there is a unique map $\overline{h} : S_{i_\ell}^{\oplus - c'_2(\alpha_i, \gamma_1)} \to S_{i_\ell}^{\oplus c'_1}$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & M(L_1)^{\oplus c'_2} \arrow[r, "f''"] & \Sigma^+_{i_\ell}(M(L_1))^{\oplus c'_2} \arrow[r, "g''"] \arrow[d, equal] & S_{i_\ell}^{\oplus - c'_2(\alpha_{i_\ell}, \gamma_1)} \arrow[r] \arrow[d, dashed, "\exists!\ \overline{h}"] & 0 \\
0 \arrow[r] & M(K) \arrow[r, "j"] & \Sigma^+_{i_\ell}(M(L_1))^{\oplus c'_2} \arrow[r, "h"] & S_{i_\ell}^{\oplus c'_1} \arrow[r] & 0
\end{tikzcd}$$

In turn, this implies that $h \circ f'' = \overline{h} \circ g'' \circ f'' = 0$, so f'' factors through a map $f : M(L_1)^{\oplus c'_2} \to \ker(h) \cong M(K)$. Since $c'_2 = c_1$, this is the first map of the desired sequence; since f'' is injective, so is f.

Then since $\beta = c_1\gamma_1 + c_2\alpha_{i_\ell}$, the cokernel of f has dimension $c_2\alpha_{i_\ell}$, so it must be $S_{i_\ell}^{\oplus c_2}$. This produces the desired sequence.

(2) Applying $\mathrm{Hom}(-, S_{i_\ell})$ to the sequence we just obtained gives the sequence

$$0 \to \mathrm{Hom}(S_{i_\ell}, S_{i_\ell})^{\oplus c_2} \to \mathrm{Hom}(M(K), S_{i_\ell}) \to \mathrm{Hom}(M(L_1), S_{i_\ell})^{\oplus c_1} = 0$$

because $M(L_1) \in \mathrm{NoQuot}_{i_\ell}$, as previously observed. Thus $\dim \mathrm{Hom}(M(K), S_{i_\ell}) = c_2$.

Likewise, applying $\mathrm{Hom}(M(L_1), -)$ to the sequence gives the sequence

$$0 \to \mathrm{Hom}(M(L_1), M(L_1))^{\oplus c_1} \to \mathrm{Hom}(M(L_1), M(K)) \to \mathrm{Hom}(M(L_1), S_{i_\ell})^{\oplus c_2} = 0$$

and so, because $M(L_1)$ is a real brick,

$$\dim \mathrm{Hom}(M(L_1), M(K)) = c_1 \dim \mathrm{Hom}(M(L_1), M(L_1)) = c_1$$

by Lemma VI.8.

We know that $\mathrm{Hom}(S_{i_\ell}, M(K)) = 0$ because $M(K)$ is an output of $\Sigma_{i_\ell}^+$, and thus in NoSub_{i_ℓ}.

Finally, suppose there exists a nonzero homomorphism $\varphi : M(K) \to M(L_1)$. Let $\iota : M(L_1) \to M(L_1)^{\oplus c_1}$ be any inclusion map. Then the composition $f \circ \iota \circ \varphi : M(K) \to M(K)$ would be a nonzero endomorphism of $M(K)$. But because it factors through a smaller submodule, it cannot be invertible, contradicting that $M(K)$ is a brick.

□

Having finished both cases of the induction, the proof of Theorem VII.9 is complete.

VII.4: $M(K)$ is the generic extension of $M(L_2)^{\oplus c_2}$ by $M(L_1)^{\oplus c_1}$

We recall now that there was a second part to Theorem VII.3: in that case, not only do $M(K_E)$, $M(L_1)$, and $M(L_2)$ slot into an exact sequence, but that extension spans $\mathrm{Ext}^1(M(L_2), M(L_1))$, which is only 1-dimensional. In particular, this means that $M(K_E)$ is the only module arising as a non-split extension of $M(L_2)$ by $M(L_1)$.

In our more general setting of Theorem VII.9, we can't have exactly the same property: we'd like to make a statement about $\mathrm{Ext}^1(M(L_2)^{\oplus c_2}, M(L_1)^{\oplus c_1})$, whose dimension is necessarily divisible by c_1c_2. The appropriate generalization turns out to be that $M(K)$ is somehow the *generic* extension of $M(L_2)^{\oplus c_2}$ by $M(L_1)^{\oplus c_1}$. In this section, we make this precise.

First, note that by the functoriality of Ext, there is an action of $\mathrm{Aut}(M(L_2)^{\oplus c_2})^{\mathrm{op}} \times \mathrm{Aut}(M(L_1)^{\oplus c_1})$ on $\mathrm{Ext}^1(M(L_2)^{\oplus c_2}, M(L_1)^{\oplus c_1})$. Because $M(L_1)$ and $M(L_2)$ are real bricks, we can identify their endomorphism rings with k by Lemma VI.8, recasting this as an action of $\mathrm{GL}_{c_2}(k)^{\mathrm{op}} \times \mathrm{GL}_{c_1}(k)$. To clean up notation, we write $\mathrm{GL}_{c_i} := \mathrm{GL}_{c_i}(k)$ in what follows.

Precisely, given an extension $0 \to M(L_1)^{\oplus c_1} \to N \to M(L_2)^{\oplus c_2} \to 0$ and an element $h_2 \in \mathrm{Aut}(M(L_2)^{\oplus c_2}) \cong \mathrm{GL}_{c_2}$, acting by h_2 gives the extension obtained by pullback:

$$\begin{tikzcd}
0 \arrow[r] & M(L_1)^{\oplus c_1} \arrow[r] \arrow[d, equal] & N \times_{M(L_2)^{\oplus c_2}} M(L_2)^{\oplus c_2} \arrow[r] \arrow[d] & M(L_2)^{\oplus c_2} \arrow[r] \arrow[d, "h_2"] & 0 \\
0 \arrow[r] & M(L_1)^{\oplus c_1} \arrow[r, "f"] & N \arrow[r, "g"] & M(L_2)^{\oplus c_2} \arrow[r] & 0
\end{tikzcd}$$

However, it is straightforward to verify from either the universal property or explicit construction that this pullback square can be realized by

$$\begin{tikzcd}
0 \arrow[r] & M(L_1)^{\oplus c_1} \arrow[r, "f"] \arrow[d, equal] & N \arrow[r, "h_2^{-1}\circ g"] \arrow[d, equal] & M(L_2)^{\oplus c_2} \arrow[r] \arrow[d, "h_2"] & 0 \\
0 \arrow[r] & M(L_1)^{\oplus c_1} \arrow[r, "f"] & N \arrow[r, "g"] & M(L_2)^{\oplus c_2} \arrow[r] & 0
\end{tikzcd}$$

Dually, we can show that acting by an element $h_1 \in \mathrm{Aut}(M(L_1)^{\oplus c_1}) \cong \mathrm{GL}_{c_1}$ produces the sequence

$$0 \to M(L_1)^{\oplus c_1} \xrightarrow{f\circ h_1^{-1}} N \xrightarrow{g} M(L_2)^{\oplus c_2} \to 0.$$

From our perspective, applying this action to the doubleton short exact sequence just produces a different instance of the sequence — it's changing the bases of $M(L_1)^{\oplus c_1}$ and $M(L_2)^{\oplus c_2}$, rather than altering their relationship with $M(K)$. Thus our claim that the doubleton extension is generic becomes the following theorem.

Theorem VII.17. *Viewing* $\mathrm{Ext}^1(M(L_2)^{\oplus c_2}, M(L_1)^{\oplus c_1})$ *as affine space, the* $\mathrm{GL}_{c_2}^{\mathrm{op}} \times \mathrm{GL}_{c_1}$*-orbit of the doubleton exact sequence is Zariski-open.*

Proof. Our strategy is to use orbit-stabilizer reasoning to show that the orbit has the same dimension as the full Ext space. In the setting of algebraic geometry, the role of the orbit-stabilizer theorem is played by a couple of facts from the theory of algebraic groups.

Proposition VII.18 ([Mil17, Proposition 5.23])**.** *Let G be an algebraic group and H an algebraic subgroup. Then*

$$\dim G = \dim H + \dim G/H.$$

Proposition VII.19 ([Mil17, Propositions 7.12 and 7.17])**.** *Let $G \times X \to X$ be the action of a smooth algebraic group on a separated scheme X over k. For any k-valued point x of X, let G_x be its stabilizer and O_x its orbit. Then the quotient G/G_x exists, and the map $g \mapsto gx$ induces an immersion $G/G_x \to X$, with image O_x. In particular, O_x is open in a closed subscheme.*

We first claim that the stabilizer of the doubleton extension is 1-dimensional. An element $(h_1, h_2) \in \mathrm{GL}_{c_1}^{\mathrm{op}} \times \mathrm{GL}_{c_2}^{\mathrm{op}}$ is in the stabilizer when there exists an automorphism $j : M(K) \to M(K)$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & M(L_1)^{\oplus c_1} \arrow[r, "f"] \arrow[d, equal] & M(K) \arrow[r, "g"] \arrow[d, "j"] & M(L_2)^{\oplus c_2} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M(L_1)^{\oplus c_1} \arrow[r, "f\circ h_1^{-1}"] & M(K) \arrow[r, "h_2^{-1}\circ g"] & M(L_2)^{\oplus c_2} \arrow[r] & 0
\end{tikzcd}$$

However, $M(K)$ is a real brick, so by Lemma VI.8, j must act by scalar multiplication by some $a \in k^\times$. Since $af = fa = f \circ h_1^{-1}$ and f is injective, we know that h_1^{-1} is multiplication by a; likewise, since $ga = ag = h_2^{-1} \circ g$ and g is surjective, h_2^{-1} is multiplication by a. It follows that the stabilizer is isomorphic to $k^\times$, and thus 1-dimensional.

We also note that $\dim(\mathrm{GL}_{c_2}^{\mathrm{op}} \times \mathrm{GL}_{c_1}) = c_1^2 + c_2^2$. So by Proposition VII.18, the quotient by the stabilizer has dimension $c_1^2 + c_2^2 - 1$. By Proposition VII.19, the orbit of the doubleton extension is open in a closed subscheme of this dimension.

Now we claim that

$$\dim \mathrm{Ext}^1(M(L_2)^{\oplus c_2}, M(L_1)^{\oplus c_1}) = c_1^2 + c_2^2 - 1.$$

We first observe that

$$\mathrm{Hom}(M(L_1), M(L_2)) = \mathrm{Hom}(M(L_2), M(L_1)) = 0.$$

To see that $\mathrm{Hom}(M(L_1), M(L_2)) = 0$, we apply $\mathrm{Hom}(M(L_1), -)$ to the doubleton sequence to get the exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}(M(L_1), M(L_1)^{\oplus c_1}) \to \mathrm{Hom}(M(L_1), M(K)) \to \mathrm{Hom}(M(L_1), M(L_2)^{\oplus c_2}) \\ \to \mathrm{Ext}^1(M(L_1), M(L_1)^{\oplus c_1}) = 0 \end{aligned}$$

where the last term is 0 by Proposition VI.9. Then since $M(L_1)$ is a real brick, the first nonzero term has dimension c_1 by Lemma VI.8, and by Theorem VII.9(a), the second term also has dimension c_1. It follows that the third term is 0.

To see that $\mathrm{Hom}(M(L_2), M(L_1)) = 0$, note that any nonzero map $M(L_2) \to M(L_1)$ could be composed with the injective $f : M(L_1) \to M(K)$ to give a nonzero map $M(L_2) \to M(K)$, contradicting Theorem VII.9(a).

It follows from Proposition V.17 that

$$(c_2\gamma_2, c_1\gamma_1) = (\underline{\dim}\, M(L_2)^{\oplus c_2}, \underline{\dim}\, M(L_1)^{\oplus c_1}) = -\dim \mathrm{Ext}^1(M(L_2)^{\oplus c_2}, M(L_1)^{\oplus c_1})$$

On the other hand, because β, γ_1, and γ_2 are all roots, we have

$$\begin{aligned} 2 &= (\beta, \beta) = (c_1\gamma_1 + c_2\gamma_2, c_1\gamma_1 + c_2\gamma_2) \\ &= c_1^2(\gamma_1, \gamma_1) + 2(c_2\gamma_2, c_1\gamma_1) + c_2^2(\gamma_2, \gamma_2) = 2c_1^2 + 2c_2^2 + 2(c_2\gamma_2, c_1\gamma_1) \end{aligned}$$

and thus

$$(c_2\gamma_2, c_1\gamma_1) = 1 - c_1^2 - c_2^2.$$

Putting these last two statements together gives the claimed dimension for Ext^1.

In particular, the orbit of the doubleton extension is open in a closed subscheme of dimension equal to the affine space it lies in, which can only be the entire space; thus it must be Zariski-open. □

CHAPTER VIII
Filtrations of Shard Modules with Tails

In Section VI.2, we saw the classification of bricks of type A and how they correspond to shards. The key takeaway there was that each shard can be expressed by a sign vector — recording, for each fracture, which side the shard lies on — while each brick can be expressed by an orientation of the type A diagram, and signs in one context correspond to arrow directions in the other.

A key motivation behind this thesis has been the question of whether we can observe this behavior in other families of diagrams. In Chapter IV, we tackled this question on the side of shards, finding a uniform description of the shards within any stretched family.

In this chapter, we consider the other side of the story. In the case of a diagram with a tail, we show that a shard module's structure along the tail reflects how its shard is described in the framework of Chapter IV. Our methods don't always work for more general stretched diagrams, with L_j and R_j both nonempty: in the last part of the chapter, we explain what we'd like to say in this situation and what goes wrong.

VIII.1: Motivation: shard modules in types A and D

We first recall the classification of bricks of Π_{A_n} from Section VI.2. Let $\alpha_{i,j}$ denote the root $\alpha_i + \alpha_{i+1} + \ldots + \alpha_j$ for $1 \leq i < j \leq n$.

For fixed i and j, let $a : \{i, \ldots, j-1\} \to \{+, -\}$ be a sign vector, and define a module M_a to have dimension vector $\alpha_{i,j}$, with a nonzero map $(M_a)_h \to (M_a)_{h+1}$ if $a(h) = +$ and a nonzero map $(M_a)_{h+1} \to (M_a)_h$ if $a(h) = -$.

Proposition VIII.1. *M_a is the shard module associated to the shard defined by*

$$\langle -, \alpha_{i,j} \rangle = 0$$
$$\langle -, \alpha_{h,j} \rangle \begin{cases} \geq 0 & \text{if } a(h-1) = + \\ \leq 0 & \text{if } a(h-1) = - \end{cases}$$

$$
\begin{aligned}
x_4 &\geq 0 \\
x_3 + x_4 &\leq 0 \\
x_2 + x_3 + x_4 &\leq 0 \\
x_1 + x_2 + x_3 + x_4 &= 0
\end{aligned}
$$

$$k \leftarrow k \leftarrow k \rightarrow k$$

Figure 52: The correspondence described in Proposition VIII.1, for the module M_{--+}. The orientation of an arrow corresponds to the sign of the inequality of the same color.

This result, and in particular the correspondence between signs and orientations, is illustrated in Figure 52.

The other simply laced finite type family is type D. The bricks of Π_{D_n} were classified by Asai in [Asa22]. The classification is somewhat complex and is phrased in the language of weak ordering rather than the language of shards, so we do not reproduce it in full here; however, the way he formats bricks reveals important similarities to the type A case.

Asai labels the vertices of the D_n diagram by

$$
\begin{tikzcd}
1 \arrow[dr, no head] & & & & \\
 & 2 \arrow[r, no head] & 3 \arrow[r, no head] & \cdots \arrow[r, no head] & (n-1) \\
-1 \arrow[ur, no head] & & & &
\end{tikzcd}
$$

He then represents bricks in exploded form as built up from two layers, like so:

$$
\begin{tikzcd}
1 \arrow[dr] & 2 \arrow[r] & 3 & 4 \arrow[l] \arrow[r] & 5 & 6 \arrow[l] & 7 \arrow[l] \arrow[r] & 8 \\
-1 \arrow[ur] & 2 \arrow[l] \arrow[r] & 3 \arrow[ul] \arrow[ur] & 4 \arrow[l] \arrow[r] & 5 \arrow[r] \arrow[ul] \arrow[ur] & 6 & &
\end{tikzcd}
$$

Note that the portion of the module supported on the tail divides into two layers which can be viewed as type A bricks, and the arrows between these two layers (at least, beyond vertex 2) all point in the same direction. One way to summarize this observation is that the brick admits a filtration: the layer which the arrows point towards can be realized as a submodule, and the quotient by that submodule contains the other layer:

$$
\begin{tikzcd}
1 \arrow[dr] & 2 \arrow[r] & 3 & 4 \arrow[l] \arrow[r] & 5 & 6 \arrow[l] & 7 \arrow[l] \arrow[r] & 8 \\
-1 \arrow[ur] & 2 \arrow[l] \arrow[r] & 3 \arrow[ul] \arrow[ur] & 4 \arrow[l] \arrow[r] & 5 \arrow[r] \arrow[ul] \arrow[ur] & 6 & &
\end{tikzcd}
$$

The similarity to the type A case is illuminated further when we examine the stability domains of type D bricks, which will be shards. Consider, for example, the root of D_6

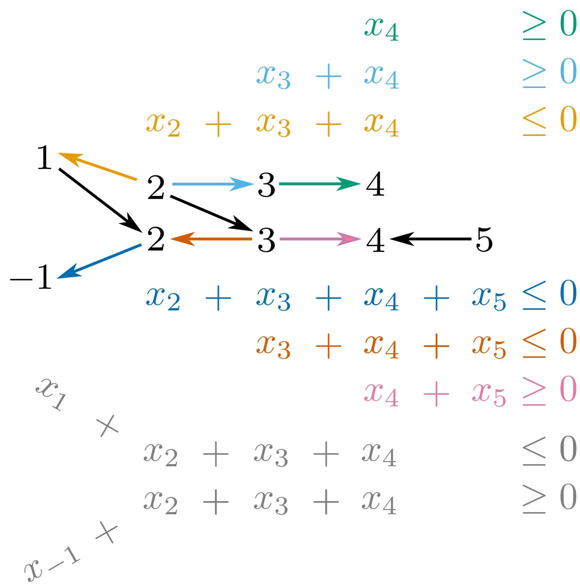
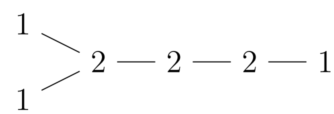


Figure 53: A shard module of Π_{D_6} is shown alongside the inequalities defining its stability domain. Again, there is a neat correspondence between orientations of arrows and signs of inequalities.



This has a reduced expression

$$s_4 s_3 s_2 s_5 s_4 s_3 s_{-1} s_1(\alpha_2)$$

whose truncations are the roots

$$\begin{array}{cc}
\alpha_4 & \alpha_4 + \alpha_5 \\
\alpha_3 + \alpha_4 & \alpha_3 + \alpha_4 + \alpha_5 \\
\alpha_2 + \alpha_3 + \alpha_4 & \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 \\
\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 & \alpha_{-1} + \alpha_2 + \alpha_3 + \alpha_4
\end{array}$$

When we line up an example brick alongside the inequalities defining its stability domain in terms of these truncations, we again see that the signs of the inequalities line up with orientations of arrows in the two layers, as illustrated in Figure 53.

This is the pattern we'll pursue in the rest of this chapter: given a shard module for a diagram with a tail, we'll construct a filtration by modules resembling the bricks of A_n. The orientations of the nonzero maps within the subquotients of our filtration will correspond to certain inequalities defining the shard.

VIII.2: The general result

Our general result will apply to preprojective algebras of diagrams with a tail. We briefly recall the definitions and results we need from Section IV.4. We start with a diagram G associated to a root system Φ, choose an elastic vertex j, and pick elastic data such that $R_j = \varnothing$. We fix a stretch factor c and a tuple $\underline{m} = (m_0, \ldots, m_c)$ of nonnegative integers, letting $m := m_0 + \ldots + m_c + c$. In this setup, for any root $\overline{\beta} \in \mathrm{str}_c(\Phi)$, we define a root $\mathrm{str}_{\underline{m}}(\beta) \in \mathrm{str}_m(\Phi)$ by stretching the coefficient at j_p by a factor of m_p for all $0 \leq p \leq c$. We then let $q_p := m_0 + \ldots + m_p + p$ for any $0 \leq p \leq c$; then j_{q_p} is the last vertex in the pth stretched block.

Definition VIII.2. Given $\underline{m}$ as above, a root β of $\mathrm{str}_m(\Phi)$ is **blocky** if it has the form $\mathrm{str}_{\underline{m}}(\overline{\beta})$. It is **strictly blocky** if all coefficients of $\overline{\beta}$ on the tail of $\mathrm{str}_c(G)$ are distinct.

The key result of Section IV.4 was a description of the fractures of hyperplanes defined by blocky roots.

Theorem VIII.3. *Let $\beta = \mathrm{str}_{\underline{m}}(\overline{\beta})$ for some $\overline{\beta} \in \mathrm{str}_c(\Phi)$. The rank 2 subsystems which cut $\beta^\perp$ are $R(\beta, \gamma)$, where γ belongs to one of the following sets:*

- *roots of the form $\mathrm{str}_{\underline{m}}(\overline{\gamma})$, where $\overline{\gamma}$ cuts $\overline{\beta}^\perp$.*
- *some subset of the roots of the form*

$$\gamma^r_{pp'} := \alpha_{j,q_{p-1}+r+1} + \alpha_{j,q_{p-1}+r+2} + \ldots + \alpha_{j,q_{p'}} \qquad (\alpha_{j,q} := \alpha_{j_q})$$

for $0 \leq p \leq p' \leq c$, $1 \leq r \leq m_p$. If β is strictly blocky, all of these roots work.

Definition VIII.4. The **stretched fractures** of $\beta^\perp$ are the fractures induced by the roots in the first bullet point above. The **tail roots** are the roots $\gamma^r_{pp'}$ described by the second bullet point, and the **tail fractures** of $\beta^\perp$ are the fractures induced by these roots.

We now prepare to state the theorem which is the main objective of this chapter, by defining $\Pi_{\mathrm{str}_m(G)}$-modules resembling the bricks of type A. Fix some p, $0 \leq p \leq c$. Let $a : \{q_{p-1}+1, q_{p-1}+2, \ldots, q_p - 1\} \to \{+, -\}$ be an assignment of signs to each edge between vertices in the pth block. Then define the $\Pi_{\mathrm{str}_m(G)}$-module $M_{p,a}$ to have 1-dimensional spaces at the vertices $j_{q_{p-1}+1}, \ldots, j_{q_p}$, 0-dimensional spaces elsewhere, and nonzero maps $M_{p,a}(j_q) \to M_{p,a}(j_{q+1})$ if $a(q) = +$ and $M_{p,a}(j_q) \leftarrow M_{p,a}(j_{q+1})$ if $a(q) = -$. A couple of examples are shown in Figure 54.

$$
\begin{tikzcd}[column sep=small, row sep=tiny]
\cdots\ 0 \arrow[dr, dash] & & & & & & & & & \\
\cdots\ 0 \arrow[r, dash] & k & k \arrow[l] \arrow[r] & k \arrow[r] & k \arrow[r, dash] & 0 \arrow[r, dash] & 0 \arrow[r, dash] & 0 \arrow[r, dash] & 0 \arrow[r, dash] & 0 \\
\cdots\ 0 \arrow[ur, dash] & & & & & & & & &
\end{tikzcd}
$$

$M_{0,-++}$

$$
\begin{tikzcd}[column sep=small, row sep=tiny]
\cdots\ 0 \arrow[dr, dash] & & & & & & & & & \\
\cdots\ 0 \arrow[r, dash] & 0 \arrow[r, dash] & 0 \arrow[r, dash] & 0 \arrow[r, dash] & 0 \arrow[r, dash] & k \arrow[r] & k & k \arrow[l] \arrow[r, dash] & 0 \arrow[r, dash] & 0 \\
\cdots\ 0 \arrow[ur, dash] & & & & & & & & &
\end{tikzcd}
$$

$M_{1,+-}$

Figure 54: Here, $\underline{m} = (3, 2, 1)$, resulting in blocks of 4, 3, and 2 vertices. Two examples of $M_{p,a}$ in this setting are shown.

Theorem VIII.5. *Let $\beta = \operatorname{str}_{\underline{m}}(\overline{\beta})$ be a blocky root and let K be a shard of $\beta^\perp$. Then the associated shard module $M(K)$ admits a filtration with the following properties:*

(1) The subquotients of the filtration are either simple modules at vertices in the body, or modules of the form $M_{p,a}$.

(2) For each pair of block indices $0 \le p \le p' \le c$, the list of subquotients has $\overline{\beta}(j_{p'}) - \overline{\beta}(j_{p'+1})$ copies of $M_{p,a}$, where

$$
a(q_{p-1} + r) = \begin{cases} + & \langle -, \gamma^r_{pp'} \rangle \ge 0 \text{ on } K \\ - & \langle -, \gamma^r_{pp'} \rangle \le 0 \text{ on } K \end{cases} \qquad 1 \le r \le m_p
$$

and we define $\overline{\beta}(j_{c+1}) = 0$.

(3) Let $\overline{K}$ be the unique shard of $\overline{\beta}^\perp$ such that, for each fracture $\overline{\gamma}^\perp$ of $\overline{\beta}^\perp$, the sign of $\langle -, \overline{\gamma} \rangle$ on $\overline{K}$ matches the sign of $\langle -, \operatorname{str}_{\underline{m}}(\overline{\gamma}) \rangle$ on K. Then there exists a filtration of $M(\overline{K})$ by simple modules, whose subquotients are obtained from the subquotients of the filtration of $M(\overline{K})$ in order by replacing $M_{p,a}$ with S_p.

VIII.2.1: An example of Theorem VIII.5

The statement of this theorem is quite complicated, so before moving on to a proof we illustrate what it's saying pictorially. We let $\overline{G}$ be the diagram

$$
\begin{tikzcd}[column sep=small, row sep=tiny]
1 \arrow[dr, dash] & & \\
2 \arrow[r, dash] & 4 \arrow[r, dash] & 5 \\
3 \arrow[ur, dash] & &
\end{tikzcd}
$$

and stretch it with stretch factors $\underline{m} = (2, 2)$ to get the diagram G, with its vertices labeled like so:

$$\begin{array}{l} 1 \diagdown \\ 2 - 40 - 41 - 42 - 50 - 51 - 52 \\ 3 \diagup \end{array}$$

We then focus on the blocky dimension vector

$$\begin{array}{l} 1 \diagdown \\ 1 - 3 - 3 - 3 - 1 - 1 - 1 \\ 1 \diagup \end{array}$$

and specify a particular shard K. By Theorem VIII.3, a shard is determined by two sets of inequalities. First, we specify its position relative to the stretched fractures.

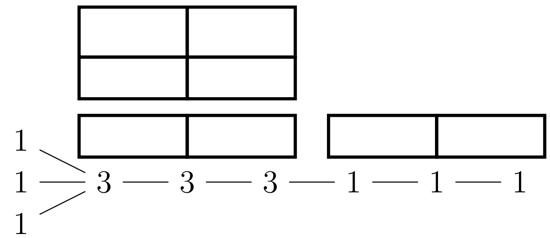
$$\begin{aligned} x_2 + x_3 + 2(x_{40} + x_{41} + x_{42}) + (x_{50} + x_{51} + x_{52}) &\geq 0 \\ (x_{40} + x_{41} + x_{42}) + (x_{50} + x_{51} + x_{52}) &\leq 0 \\ x_2 + (x_{40} + x_{41} + x_{42}) &\leq 0 \\ x_3 + (x_{40} + x_{41} + x_{42}) &\geq 0 \\ (x_{40} + x_{41} + x_{42}) &\leq 0 \end{aligned}$$

Secondly, and more importantly for our purposes, we specify its position relative to the tail fractures:

$$\begin{aligned} \langle -, \gamma_{01}^1 \rangle = x_{41} + x_{42} + x_{50} + x_{51} + x_{52} &\geq 0 \\ \langle -, \gamma_{01}^2 \rangle = x_{42} + x_{50} + x_{51} + x_{52} &\geq 0 \\ \langle -, \gamma_{11}^1 \rangle = x_{51} + x_{52} &\leq 0 \\ \langle -, \gamma_{11}^2 \rangle = x_{52} &\leq 0 \\ \langle -, \gamma_{00}^1 \rangle = x_{41} + x_{42} &\geq 0 \\ \langle -, \gamma_{00}^2 \rangle = x_{42} &\leq 0 \end{aligned}$$

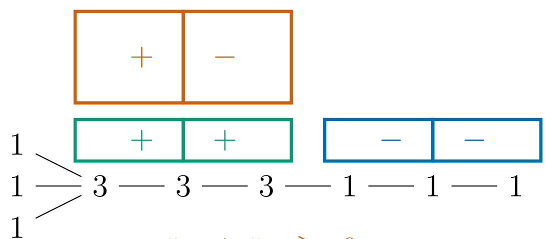
The theorem's claim is that the associated shard module admits a filtration whose subquotients capture exactly the information provided by this second set of inequalities. So we next describe how to use this information to construct the associated graded module of this filtration.

First, we stack boxes above the edges in the visualization of the dimension vector. Over every tail edge between two vertices with the same coefficient, we stack a number of boxes equal to that coefficient. We additionally introduce a horizontal break across the whole grid at the top of each stack, as shown here.



The stack over the pth block will be broken into $c - p$ components this way, and the ith component from the top (indexed starting from 0) will be $\overline{\beta}(j_{p+i}) - \overline{\beta}(j_{p+i+1})$ boxes tall. In this example, the components over the block of 3s have heights $3 - 1 = 2$ and $1 - 0 = 1$. This quantity is the multiplicity appearing in part (2) of the theorem statement.

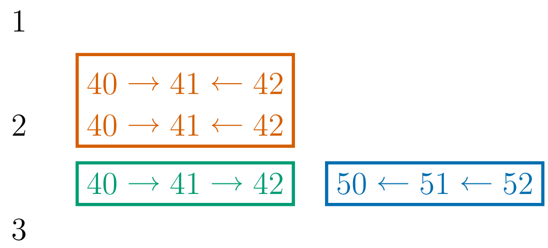
We then fill each component of the stack of boxes with signs determined by the second set of inequalities above:



$$
\begin{aligned}
x_{41} + x_{42} &\geq 0\\
x_{42} &\leq 0\\
x_{41} + x_{42} + x_{50} + x_{51} + x_{52} &\geq 0\\
x_{42} + x_{50} + x_{51} + x_{52} &\geq 0\\
x_{51} + x_{52} &\leq 0\\
x_{52} &\leq 0
\end{aligned}
$$

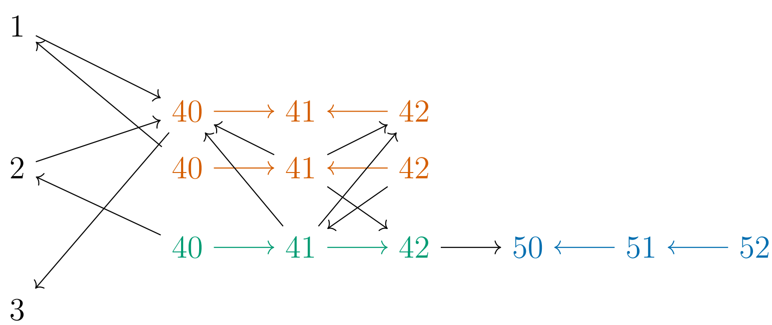
The ith component over the pth block is associated with the roots $\gamma^r_{p(p+i)}$ for $1 \leq r \leq m_p$, and we fill it from left to right with $+$ and $-$ according as these roots pair positively or negatively with the points of K.

Finally, we fill the boxes over the pth block with copies of $M_{p,a}$, where the sign vector a is determined by the signs we just put in. We additionally include simple modules at the body vertices in the quantities necessary to produce the dimension vector. The theorem claims that this is an exploded view of the associated graded module of a filtration of the shard module.



$$\begin{aligned}
x_{41} + x_{42} &\geq 0 \\
x_{42} &\leq 0 \\
x_{41} + x_{42} + x_{50} + x_{51} + x_{52} &\geq 0 \\
x_{42} + x_{50} + x_{51} + x_{52} &\geq 0 \\
x_{51} + x_{52} &\leq 0 \\
x_{52} &\leq 0
\end{aligned}$$

To see that that is the case here, here is an exploded view of the entire shard module.



This module has a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_7 = M(K)$$

with subquotients

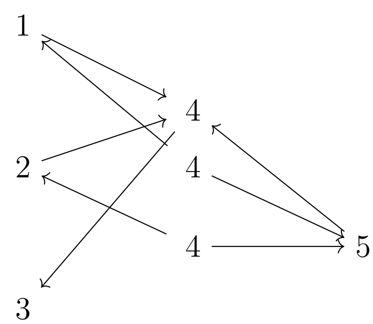
$$\begin{aligned}
M_1/M_0 &= S_3 \\
M_2/M_1 &= M_{0,+-} \\
M_3/M_2 &= S_2 \\
M_4/M_3 &= S_1 \\
M_5/M_4 &= M_{1,--} \\
M_6/M_5 &= M_{0,++} \\
M_7/M_6 &= M_{0,+-}.
\end{aligned}$$

We can verify this by first noting that S_3 is a submodule (since there are no arrows out of the vertex labeled 3) and quotienting out (removing the vertex); then noting that the top copy of $M_{0,+-}$ is a submodule of what remains, and quotienting that out; and so on.

Finally, we illustrate part (3) of the theorem. We turn our attention to the *first* list of inequalities defining K, the stretched ones, and collapse each stretched block down to a single vertex to obtain a list of inequalities defining a shard $\overline{K}$ of the root system of $\overline{G}$.

$$\begin{aligned} x_2 + x_3 + 2x_4 + x_5 &\geq 0 \\ x_4 + x_5 &\leq 0 \\ x_2 + x_4 &\leq 0 \\ x_3 + x_4 &\geq 0 \\ x_4 &\leq 0 \end{aligned}$$

The resulting shard module turns out to be



Part (3) of the theorem states that this module has a filtration whose subquotients appear in the order obtained by replacing $M_{0,+-}$ and $M_{0,++}$ with S_4 and $M_{1,--}$ with S_5 in the list of subquotients of our above filtration. In this case, that order is $S_3, S_4, S_2, S_1, S_5, S_4, S_4$. Such a filtration indeed exists here: we can verify this in the exploded view by finding a vertex labeled 3 which is a sink (which determines an S_3 submodule), removing it, finding a vertex labeled 4 which is a sink, and so on.

However, we note that the relationship between $M(K)$ and $M(\overline{K})$ stated in part (3) of the theorem is not as strong as it might initially appear. In this example, the map $M(K)_{50} \to M(K)_{42}$ between the two blocks of $M(K)$ is zero, while the map $M(\overline{K})_5 \to M(\overline{K})_4$ between the corresponding vertices in $M(\overline{K})$ is not. (If it were, the relation defining the preprojective algebra wouldn't be satisfied.) Thus we can't obtain $M(K)$ from $M(\overline{K})$ simply by replacing the spaces at vertices 4 and 5 with representations of Π_{A_3} and leaving everything else intact; there are additional discrepancies between the two representations.

VIII.2.2: Proof of Theorem VIII.5

Our strategy for proving this theorem revolves around the doubleton extension sequence developed in the previous chapter. Starting from the shard K, we select a wall and use it to express $M(K)$ as an extension of simpler shard modules. We then use this process inductively to build a filtration, with the following basic fact: given a short exact sequence

$$0 \to N_1 \xrightarrow{\iota} M \xrightarrow{\pi} N_2 \to 0$$

and filtrations

$$0 = N_1^{(0)} \subset N_1^{(1)} \subset \cdots \subset N_1^{(h)} = N_1$$
$$0 = N_2^{(0)} \subset N_2^{(1)} \subset \cdots \subset N_2^{(\ell)} = N_2$$

we obtain a filtration

$$0 = N_1^{(0)} \subset N_1^{(1)} \subset \cdots \subset N_1^{(h)} = \pi^{-1}(N_2^{(0)}) \subset \pi^{-1}(N_2^{(1)}) \cdots \subset \pi^{-1}(N_2^{(\ell)}) = M$$

whose subquotients are exactly those of the two filtrations we started with, put together.

With these tools in place, the result will come together fairly quickly. The one thing we need to be careful about is that this process — specifically, the step of selecting a wall of our shard — respects the stretched structure underlying everything. This will be taken care of by the following lemma.

Lemma VIII.6. *Suppose $\beta = \mathrm{str}_{\underline{m}}(\overline{\beta})$ for some $\overline{\beta} \in \mathrm{str}_c(\Phi)$, and that β is supported at some vertex other that the tail vertices $j_0, \ldots, j_m$. Let K be a shard of $\beta^\perp$. Then K has a wall contained in a stretched fracture.*

The key steps in proving this are a basic fact about hyperplane arrangements and an observation about the span of the subsystems cutting a root.

Lemma VIII.7. *Let V be a real vector space, V^* the dual space, and $\gamma_1, \ldots, \gamma_a \in V$ a collection of vectors. Let $R \subset V^*$ be a region of the arrangement of hyperplanes $\gamma_1^\perp, \ldots, \gamma_a^\perp$. Let $\gamma_{i_1}^\perp, \ldots, \gamma_{i_b}^\perp$ be the subset of these hyperplanes which contain the walls of R. Then*

$$\mathrm{span}(\gamma_{i_1}, \ldots, \gamma_{i_b}) = \mathrm{span}(\gamma_1, \ldots, \gamma_a)$$

Proof. Let $K = \bigcap_{i=1}^{a} \gamma_i^\perp$, and let $K_R = \bigcap_{j=1}^{b} \gamma_{i_j}^\perp$. Note that $K_R \subset R$. Suppose for a contradiction that

$$\mathrm{span}(\gamma_{i_1}, \ldots, \gamma_{i_b}) \subsetneq \mathrm{span}(\gamma_1, \ldots, \gamma_a).$$

Then, dually, $K \subsetneq K_R$. Choose some point $x \in K_R - K$. Then there exists some γ_i such that $\langle x, \gamma_i \rangle \neq 0$. Then $\langle x, \gamma_i \rangle$ and $\langle -x, \gamma_i \rangle$ have different signs. But since $x, -x \in K_R \subset R$, and either $\langle -, \gamma_i \rangle \geq 0$ or $\langle -, \gamma_i \rangle \leq 0$ on R, this is a contradiction. □

Lemma VIII.8. *Let β be a root, and let I be the set of vertices in its support. Suppose $R_1, \ldots, R_\ell$ are the rank 2 subsystems cutting $\beta^\perp$. Then*

$$\operatorname{span}(R_1 \cup \cdots \cup R_\ell) = \operatorname{span}(\{\alpha_i\}_{i \in I})$$

Proof. Let $\beta = s_{i_\ell} \cdots s_{i_1}(\alpha_{i_0})$ be a positive expression, and let $\gamma_k = s_{i_\ell} \cdots s_{i_{k+1}}(\alpha_{i_k})$ be the truncation of this expression for each $0 \leq k \leq \ell$. (Note that $\gamma_0 = \beta$.) By Theorem III.21, the left side of the desired equation is equal to $\operatorname{span}(\gamma_0, \gamma_1, \ldots, \gamma_\ell)$.

Because applying s_{i_k} in the course of evaluating the positive expression will cause the coefficient at vertex i_k to increase without changing the others, the support I of β is precisely the set $\{i_0, i_1, \ldots, i_\ell\}$. The truncated expression for γ_k may no longer be positive, but for similar reasons the support of γ_k is contained in the set $\{i_k, i_{k+1}, \ldots, i_\ell\}$. It follows that

$$\operatorname{span}(\gamma_0, \gamma_1, \ldots, \gamma_\ell) \subset \operatorname{span}(\{\alpha_i\}_{i \in I})$$

Now consider the set A of indices $1 \leq k \leq \ell$ such that i_k is the leftmost appearance of the vertex i_k in the positive expression for β. (For example, we always have $\ell \in A$ and $\ell - 1 \in A$; we have $\ell - 2 \in A$ exactly when $i_{\ell-2} \neq i_\ell$; and so on.) This set has one element for each vertex in I, corresponding to that vertex's leftmost appearance. For $k \in A$, we know that the support of γ_{i_k} contains the vertex i_k, since the expression $\gamma_{i_k} = s_{i_\ell} \cdots s_{i_{k+1}}(\alpha_{i_k})$ does not feature any reflections at i_k and thus retains a coefficient of 1 there.

Now write each γ_k for $k \in A$ as a linear combination of the simple roots α_{i_k} for $k \in A$, arranged in ascending order by k, and construct a matrix whose columns are the resulting vector representations of γ_k in ascending order by k. It follows from the above observations on support that this matrix is lower triangular with 1's along the diagonal. (For example, $\gamma_\ell = \alpha_{i_\ell}$, while $\gamma_{\ell-1} = \alpha_{i_{\ell-1}} + c\alpha_{i_\ell}$ for some coefficient c, and so on.) Thus its columns span the space $\operatorname{span}(\{\alpha_{i_k}\}_{k \in A}) = \operatorname{span}(\{\alpha_i\}_{i \in I})$, from which it follows that

$$\operatorname{span}(\gamma_0, \gamma_1, \ldots, \gamma_\ell) \supset \operatorname{span}(\{\alpha_i\}_{i \in I})$$

□

Proof of Lemma VIII.6. Suppose for a contradiction that $\beta^\perp$ has some shard without any walls contained in stretched fractures. By Theorem VIII.3, every wall of that shard is contained in $\gamma^\perp$ for γ a tail root. Then by Lemma VIII.7 applied to the arrangement of fractures in $\beta^\perp$, it follows that the span in $V/\operatorname{span}(\beta)$ of all rank 2 subsystems cutting $\beta^\perp$ is contained in the span of all tail roots in $V/\operatorname{span}(\beta)$.

However, it would then follow from Lemma VIII.8 that, letting I be the support of β, $\operatorname{span}(\{\alpha_i\}_{i\in I})$ is contained in the span of the tail roots together with β. This is only possible if the support of β consists of the tail vertices $j_1, \ldots, j_m$ together with at most one additional vertex. In turn, this is only possible if that additional vertex is j_0: because each root is built up from a simple root by increasing coefficients at vertices adjacent to vertices with nonzero coefficients, the support of a root must induce a connected subgraph of $\operatorname{str}_m(G)$. This contradicts our assumption that β is supported off the tail. $\square$

Proof of Theorem VIII.5. We proceed by induction on the total dimension of $M(K)$. For the base case, we suppose that β is not supported on the body. A brick supported only on the tail can be identified with a brick of Π_{A_n} for some n. In this case, the result follows from Proposition VIII.1.

So consider an arbitrary $\beta = \operatorname{str}_{\underline{m}}(\overline{\beta})$. By Lemma VIII.6, K has a wall sliced by a stretched fracture. Let R be the associated rank 2 subsystem cutting $\beta^\perp$, let δ_1, δ_2 be its fundamental roots, and suppose $\beta = c_1\delta_1 + c_2\delta_2$. Let L_1 and L_2 be the fundamental shards of δ_1 and δ_2, respectively, meeting K. By Theorem VII.9, there exists a short exact sequence

$$0 \to M(L_1)^{\oplus c_1} \to M(K) \to M(L_2)^{\oplus c_2} \to 0$$

(up to switching L_1 and L_2).

We chose our wall such that there exist roots $\overline{\delta}_1$, $\overline{\delta}_2$ of $\overline{G}$ with $\delta_i = \operatorname{str}(\overline{\delta}_i)$. So by the induction hypothesis, $M(L_1)$ and $M(L_2)$ admit filtrations with the desired properties.

We can splice c_1 copies of the filtration of $M(L_1)$ and c_2 copies of the filtration of $M(L_2)$ together to obtain a filtration of $M(K)$. To finish the proof, we must show that this filtration also has the properties stated in the theorem.

First, given any $0 \le p \le p' \le c$ and $1 \le r \le m_p$, suppose $\langle -, \gamma^r_{pp'}\rangle \ge 0$ on K. Then we claim there are points of L_1 and L_2 at which $\langle -, \gamma^r_{pp'}\rangle > 0$. Indeed, K intersects L_1 and L_2 in a codimension-2 cone. If $\langle -, \gamma^r_{pp'}\rangle$ vanished at every point in their intersection, it would imply $\gamma^r_{pp'}$ was in the rank 2 subsystem $R(\delta_1, \delta_2)$, a contradiction since $\gamma^r_{pp'}$ is not blocky while δ_1 and δ_2 are. Thus there must be some point in the intersection $K \cap L_1 \cap L_2$ where $\langle -, \gamma^r_{pp'}\rangle$ is nonzero, and thus positive.

In particular, the sign sequence a which the theorem statement attaches to our choice of p and p' is the same for K as it is for L_1 and L_2. We additionally know by the induction hypothesis that this choice of p and p' accounts for $\overline{\delta}_i(j_{p'}) - \overline{\delta}_i(j_{p'+1})$ copies of $M_{p,a}$ in the filtration of $M(L_i)$, for $i = 1, 2$. Thus when we combine the filtrations to get one for $M(K)$, this accounts for

$$c_1(\overline{\delta}_1(j_{p'}) - \overline{\delta}_1(j_{p'+1})) + c_2(\overline{\delta}_2(j_{p'}) - \overline{\delta}_2(j_{p'+1})) = \overline{\beta}(j_{p'}) - \overline{\beta}(j_{p'+1})$$

copies of $M_{p,a}$, which verifies property (2).

Now, let $\overline{K}$ be the shard of $\overline{\beta}$ defined in property (3). We can likewise define shards $\overline{L}_1$ and $\overline{L}_2$ of $\overline{\delta}_1$ and $\overline{\delta}_2$ such that, for each fracture $\overline{\varepsilon}$ of $\overline{\delta}_i$, the sign of $\langle -, \overline{\varepsilon}\rangle$ on $\overline{L}_i$ matches the sign of $\langle -, \mathrm{str}_{\underline{m}}(\overline{\varepsilon})\rangle$ on L_i. The roots $\overline{\delta}_1$ and $\overline{\delta}_2$ are still fundamental in the rank 2 subsystem they span, and this subsystem still defines a wall of $\overline{K}$, so we also have a short exact sequence

$$0 \to M(\overline{L}_1)^{\oplus c_1} \to M_{\overline{K}} \to M(\overline{L}_2)^{\oplus c_2} \to 0$$

By the induction hypothesis, $M(\overline{L}_1)$ and $M(\overline{L}_2)$ admit filtrations by simples which are compatible with the filtrations used above, as dictated by property (3), and by concatenating these filtrations together we get a filtration of $M(\overline{K})$ which is compatible with the filtration of $M(K)$ we constructed above. This verifies property (3). □

VIII.2.3: A corollary on rank

To show that the existence of this filtration does imply something substantial, we use it to obtain a concrete numerical fact about the maps along the tail of the shard module $M(K)$.

Corollary VIII.9. *Let $\beta = \mathrm{str}_{\underline{m}}(\overline{\beta})$ be a blocky root and let K be a shard of $\beta^\perp$. Choose $0 \leq p \leq c$ and $1 \leq r \leq m_p$. Then*

$$\mathrm{rank}(M(K)(j_{q_p+r} \to j_{q_p+r+1})) \geq \sum_{\substack{p' \geq p \\ \langle -, \gamma^r_{pp'}\rangle \geq 0 \text{ on } K}} \overline{\beta}(j_{p'}) - \overline{\beta}(j_{p'+1})$$

$$\mathrm{rank}(M(K)(j_{q_p+r+1} \to j_{q_p+r})) \geq \sum_{\substack{p' \geq p \\ \langle -, \gamma^r_{pp'}\rangle \leq 0 \text{ on } K}} \overline{\beta}(j_{p'}) - \overline{\beta}(j_{p'+1})$$

This follows directly from combining the filtration from Theorem VIII.5 with the following lemma:

Lemma VIII.10. *Let A be a k-algebra, M a finite-length A-module, and $0 = M_0 \subset M_1 \subset \cdots \subset M_\ell = M$ a filtration. For an element $a \in A$ and module N, let $a|_N : N \to N$ be* *multiplication by a. Then*

$$\mathrm{rank}(a|_M) \geq \sum_{i=1}^{\ell} \mathrm{rank}(a|_{M_i/M_{i-1}})$$

Proof. Given k-bases of M_i/M_{i-1} for $1 \leq i \leq \ell$, we combine them into a basis of M by lifting the basis of M_i/M_{i-1} to M_i. In the resulting basis, the action of a on M can be viewed as a block upper triangular matrix:

$$a|_M = \begin{pmatrix} a|_{M_1/M_0} & * & * & \cdots & * \\ 0 & a|_{M_2/M_1} & * & \cdots & * \\ 0 & 0 & a|_{M_3/M_2} & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a|_{M_\ell/M_{\ell-1}} \end{pmatrix}$$

If we select, from each matrix along the diagonal, a collection of linearly independent columns, then extend each of these columns to the corresponding columns of the full matrix, they remain linearly independent. Thus the rank of the full matrix is bounded below by the sum of the ranks of the matrices on the diagonal. $\square$

For our purposes, we take the element a to be the edge $(j_{q_p+r} \to j_{q_p+r+1})$ or $(j_{q_p+r+1} \to j_{q_p+r})$ in the preprojective algebra.

VIII.3: Why this proof doesn't work for stretching with $R_j \neq \varnothing$

In Chapter IV we described shards in the general context of stretching, not just a diagram with a tail. In that context, recall from Theorem IV.13 that (up to adjusting where the stretched path starts and ends) the fractures of a stretched root β fall into two forms: the fractures $f_u^\perp$, where f_u is also a stretched root, and the fractures $(g_u - \gamma_v)^\perp$, where $\gamma_v = \alpha_{j_0} + \ldots + \alpha_{j_v}$ and v is allowed to vary through the vertices of the stretched path independently of u. The original hope for this project was that, just as in the tail situation, a shard module $M(K)$ of dimension β should admit a filtration by modules M_a for sign vectors a determined by the position of K relative to the hyperplanes $(g_u - \gamma_v)^\perp$. (These are defined the same way as the modules $M_{p,a}$, but for simplicity we only consider the case that the stretched path is a single block, so $p = 0$.) In fact, such a theorem almost works, and we know exactly what it *should* say; the only difference is the multiplicity factor that was given by $\overline{\beta}(j_{p'}) - \overline{\beta}(j_{p'+1})$ above.

Definition VIII.11. Let β be a root of any root system and let R be a rank 2 subsystem containing β. Let γ_1, γ_2 be the fundamental roots of R, and let $\beta = c_1\gamma_1 + c_2\gamma_2$. The **multiplicity** of R with respect to β is

$$\mathrm{mult}(\beta, R) := c_1 + c_2 - 1.$$

Conjecture VIII.12. *Let $\beta = \mathrm{str}_m(\overline{\beta})$ and let K be a shard of $\beta^\perp$. Then the associated shard module $M(K)$ admits a filtration with the following properties:*

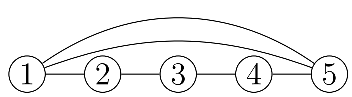
(1) The subquotients of the filtration are either simple modules at vertices in the body, or modules of the form M_a.

(2) For each of the forms g_u, the list of subquotients has $\mathrm{mult}(\beta, R(\beta, g_u - \gamma_v))$ (a quantity independent of v) copies of M_a, where

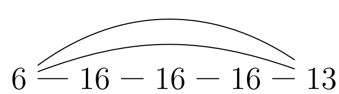
$$a(v) = \begin{cases} + & \langle -, g_u - \gamma_v \rangle \geq 0 \text{ on } K \\ - & \langle -, g_u - \gamma_v \rangle \leq 0 \text{ on } K \end{cases}$$

Indeed, one can verify that this statement is true in the case of the stretched family $A_n^{(1)}$, following the classification in Section VI.6.

The fatal flaw with attempting to apply our above proof is that Lemma VIII.6 may no longer hold: there can exist shards of $\beta^\perp$, for β a stretched root, which do not have a wall defined by a stretched root. This obstructs the approach of breaking down a shard module as an extension of smaller ones in a way that respects the stretched structure. The simplest example we have of this occurs for the diagram



and the root



Calculations in SageMath show that this root has a shard given by

$$\sigma_4^+\sigma_3^+\sigma_2^+\sigma_3^-\sigma_4^+\sigma_5^-\sigma_1^+\sigma_4^-\sigma_3^-\sigma_2^-\sigma_3^+\sigma_4^+\sigma_5^+(\alpha_1^\perp)$$

whose walls are dual to the roots

$$\begin{aligned}
&\alpha_1 + 3\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 \\
&\alpha_2 + \alpha_3 \\
&\alpha_2 \\
&\alpha_4
\end{aligned}$$

none of which assign the same coefficient to vertices 2, 3, and 4.

Curiously, however, the shard module associated to this shard does actually satisfy Conjecture VIII.12. Thus, it is still an open question whether an alternate proof strategy could be used to construct filtrations for the general stretched case.

APPENDIX
Nonsymmetric Cartan Matrices and Species

In this appendix, we discuss the framework which can be used to generalize the results of this thesis to root systems with non-symmetric (but symmetrizable) Cartan matrices. In this case, the Dynkin diagram is decorated with edge labels, and in the language of quiver representations and preprojective algebras we don't have a way to incorporate those. This is solved by the introduction of species, which are like quivers but with this extra data; a representation of a species is like a quiver representation, but with vector spaces over potentially different fields.

A.1: Species

Since Gabriel's theorem shows such a neat connection between quiver representations and the A_n, D_n, and E_n Dynkin diagrams, it's natural to wonder there is a variation of the notion of a quiver representation for which the other finite root systems — B_n, C_n, F_4, and G_2 — play a similar role.

Recall that in the finite case, for a Cartan matrix to be symmetric means that the simple roots satisfy

$$2\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = 2\frac{(\alpha_j, \alpha_i)}{(\alpha_j, \alpha_j)}$$

for all $i \neq j$, which in turn implies that they all have the same length.

Thus symmetrizable Cartan matrices correspond to root systems with simple roots of varying lengths. In order to express this nonuniformity on the side of representation theory, Dlab and Ringel considered representations of quivers in which the vector spaces at different vertices can be defined with respect to different ground fields.

Definition A.1 ([DR75]). Let k be a field. A k**-species** is specified by:

- A finite index set I.
- A collection of fields K_i for $i \in I$ which are finite-dimensional extensions of k.

- For each pair $i, j \in I$, a K_j-K_i-bimodule $E(j \leftarrow i)$.

A **representation** M of this k-species is specified by:

- For each $i \in I$, a K_i-vector space M_i.
- For each pair $i, j \in I$, a map ${}_j\varphi_i : E(j \leftarrow i) \otimes_{K_i} M_i \to M_j$

k-species are natural generalizations of quivers if we think of the K_i as vertices and $E(j \leftarrow i)$ as the collection of arrows from K_i to K_j; indeed, quivers can be identified with species in which all of the K_i are the same.

To be precise, suppose Q is a quiver and choose a field k. Then we can define a species with index set Q_0, $K_i = k$ for all $i \in Q_0$, and $E(j \leftarrow i)$ the space of formal linear combinations of arrows from i to j. Representations of this species are the same thing as representations of Q over k: if there are r arrows $i \to j$, then $E(j \leftarrow i) \cong k^r$, and ${}_j\varphi_i$ can be identified with a map $V_i^{\oplus r} \to V_j$ which records the action of all these arrows at once.

There is also an analog of the path algebra for k-species. Given a k-species as above, we define the **tensor algebra** T to be

$$T := \bigoplus_{i \in I} K_i \oplus \bigoplus_{\substack{\text{sequences} \\ i_1, i_2, \ldots, i_m \in I}} E(i_m \leftarrow i_{m-1}) \otimes_{K_{i_{m-1}}} E(i_{m-1} \leftarrow i_{m-2}) \otimes_{K_{i_{m-2}}} \cdots \otimes_{K_{i_2}} E(i_2 \leftarrow i_1)$$

Thinking of each $E(j \leftarrow i)$ as the "space of edges" from i to j as above, each summand in the right sum can be thought of as the "space of paths" going through the indices i_1,i_2,$\ldots$, i_m in order, while each summand K_i in the left sum is the space spanned by the "stationary path" at i. Accordingly, we define multiplication in this algebra by concatenating tensors with compatible endpoints ($ab := a \otimes b$), just like how multiplication in the path algebra is defined by concatenating paths. As before, the category of representations of a species is equivalent to the category of left modules over T, and so it behaves like any other module category.

Now we make precise the bridge back to root systems. To a quiver, we associated an unlabeled Dynkin diagram simply by forgetting its orientation. We associate a general Dynkin diagram to a species as follows[1]:

- The vertices are labeled by the index set I;

[1]This differs slightly from how Dlab and Ringel associate a diagram to a species in [DR75]; however, their result still holds using this definition, and it is more consistent with later work such as [DX03].

- Choose an arbitrary ordering of I, and suppose $i < j \in I$. If $E(j \leftarrow i)$ or $E(i \leftarrow j)$ is nonzero, we draw an edge between i and j, and label it with

$$(\dim_{K_i} E(j \leftarrow i) + \dim_{K_i} E(i \leftarrow j), \dim_{K_j} E(j \leftarrow i) + \dim_{K_j} E(i \leftarrow j))$$

Example. Consider an $\mathbb{R}$-species with $I = \{1, 2\}$, $K_1 = \mathbb{C}$, $K_2 = \mathbb{R}$, $E(2 \leftarrow 1) = \mathbb{C}$, and $E(1 \leftarrow 2) = 0$. Then a representation is given by an $\mathbb{R}$-linear map $f : V_1 \to V_2$, where V_1 is a $\mathbb{C}$-vector space and V_2 is an $\mathbb{R}$-vector space. Because $E(2 \leftarrow 1)$ is 1-dimensional over $\mathbb{C}$ but 2-dimensional over $\mathbb{R}$, this species corresponds to the Dynkin diagram

$$\circ \overset{(1,2)}{\text{———}} \circ$$

The last ingredient we need to generalize Gabriel's theorem is a definition of dimension vector:

Definition A.2. Let $(K_i, E(j \leftarrow i))_{i,j \in I}$ be a k-species and let M be a representation of it. Let V be a vector space over $\mathbb{R}$ with a distinguished basis $(\alpha_i)_{i \in Q_0}$ indexed by I. The **dimension vector** of M, denoted $\underline{\dim}\, M$, is the vector $\sum_{i \in I} (\dim_{K_i} M_i)\alpha_i \in V$.

Theorem A.3. *[DR75] A k-species has finitely many indecomposable representations if and only if the associated Dynkin diagram is a disjoint union of finite type diagrams. In this case, the map $M \mapsto \underline{\dim}\, M$ gives a bijection between isomorphism classes of indecomposable representations and positive roots of the associated root system, where α_i is the simple root associated to vertex i.*

Example. The example species immediately above corresponds to the B_2 Dynkin diagram, which denotes a root system for Dih_4. Thus we expect it to have 4 indecomposable representations, since a square has 4 lines of symmetry. These are $\mathbb{C} \to 0$, $0 \to \mathbb{R}$, $\mathbb{C} \xrightarrow{\mathrm{Re}} \mathbb{R}$, and $\mathbb{C} \xrightarrow{(\mathrm{Re},\mathrm{Im})} \mathbb{R}^2$.

A.2: Preprojective algebras of species

The material of this section originally appeared in [DST23].

As in section A.1, it's good to have an analog of the preprojective algebra associated to general Dynkin diagrams rather than just unlabeled graphs. Such an analog is provided by Julian Külshammer's recent theory of **pro-species**. Here, we introduce Külshammer's general definition of the preprojective algebra associated to a species and then explain how to apply it to our situation. The foundational paper [Kü17] provides most of the tools we need, with the exception of an analog of Proposition V.17, which is proven in the appendix of [DST23].

Starting from a k-species[2], Külshammer constructs a double species with isomorphisms of K_i-K_j-bimodules $E(i \leftarrow j) \cong \mathrm{Hom}_{K_i}(E(j \leftarrow i), K_i) \cong \mathrm{Hom}_{K_j}(E(j \leftarrow i), K_j)$. This is equivalent to specifying a K_i-linear map $\langle -, - \rangle_i : E(i \leftarrow j) \otimes_{K_j} E(j \leftarrow i) \to K_i$ and a K_j-linear map $\langle -, - \rangle_j : E(j \leftarrow i) \otimes_{K_i} E(i \leftarrow j) \to K_j$.

Then for each pair of indices $j \leftarrow i$, we choose a K_i-basis $e_1, \dots, e_r$ of $E(j \leftarrow i)$ and let $f_1, \dots, f_r$ be the dual K_i basis of $E(i \leftarrow j)$ under the pairing $\langle -, - \rangle_i$. We define the **Casimir element**

$$c_{j \leftarrow i \leftarrow j} := \sum_{t=1}^{r} e_t \otimes f_t \in E(j \leftarrow i) \otimes_{K_i} E(i \leftarrow j)$$

which is independent of the chosen basis.

Finally, we define a function $\mathrm{sgn} : \{(j \leftarrow i) \mid i \neq j \in I\} \to \{\pm 1\}$ such that $\mathrm{sgn}(i \leftarrow j) = -\mathrm{sgn}(j \leftarrow i)$, as in the definition of the original preprojective algebra. The **preprojective algebra** of this setup is the quotient of the tensor algebra of the double species by the two-sided ideal generated by

$$c := \sum_{i \neq j \in I} \mathrm{sgn}(j \leftarrow i) c_{j \leftarrow i \leftarrow j}$$

As with the original preprojective algebra, note that $e_j c e_j$ (where e_j is the idempotent in the tensor algebra associated to index j) is the sum of just the terms indexed by a specific j, representing the paths going out of and back to a specific vertex. Quotienting out by c is equivalent to quotienting out by each of these separately.

Now let A be a crystallographic Cartan matrix of rank n. Here it is also important that we assume A is symmetrizable, meaning that there are positive integers $d_1, \dots, d_n$ such that $d_i A_{ij} = d_j A_{ji}$ for any i and j. We will construct a preprojective algebra as above with reflection functors and a Crawley-Boevey identity linking it to the root system of A.

Starting with the symmetrizing integers d_i, let $L = \mathrm{LCM}(d_1, \dots, d_n)$ and let $d_{ij} = \mathrm{LCM}(d_i, d_j)$ for any pair i, j. Let $k(L)/k$ be a Galois field extension with Galois group cyclic of order L, so that there exists a unique intermediate field $k(d)$ of degree d over k for all $d \mid L$. As seen in the example of species above, for $L = 2$ we could let $k(L)/k = \mathbb{C}/\mathbb{R}$, while for general L, we could let $k(L)/k = \mathbb{F}_{p^L}/\mathbb{F}_p$ for some prime p.

[2]Külshammer actually works with **pro-species of algebras**, for which the fields K_i are allowed to be finite-dimensional algebras and the bimodules $E(j \leftarrow i)$ are only required to be projective on either side. We will not need this level of generality.

Let Γ be the quiver with vertices $1, \ldots, n$ and $-d_i A_{ij}/d_{ij}$ arrows $i \to j$. This quantity is symmetrical in i and j, so we can choose a bijection $a \leftrightarrow a^*$ between arrows $i \to j$ and $j \to i$ such that $(a^*)^* = a$. Then let $E(j \leftarrow i)$ be a $k(d_{ij})$-vector space with a basis given by arrows $i \to j$, and let $\langle -, - \rangle_{ij} : E(j \leftarrow i) \times E(i \leftarrow j) \to k(d_{ij})$ be the $k(d_{ij})$-bilinear pairing for which $a \mapsto a^*$ sends each basis to its dual.

We have field trace maps $\mathrm{tr}_{k(d_i)} : k(d_{ij}) \to k(d_i)$ and $\mathrm{tr}_{k(d_j)} : k(d_{ij}) \to k(d_j)$; thus we can define the requisite pairings

$$\begin{aligned}
\langle -, - \rangle_i &:= \mathrm{tr}_{k(d_i)}(\langle -, - \rangle_{ji}) : E(i \leftarrow j) \otimes_{k(d_j)} E(j \leftarrow i) \to k(d_i) \\
\langle -, - \rangle_j &:= \mathrm{tr}_{k(d_j)}(\langle -, - \rangle_{ij}) : E(j \leftarrow i) \otimes_{k(d_i)} E(i \leftarrow j) \to k(d_j)
\end{aligned}$$

Next, we work out the Casimir elements corresponding to these pairings. For any pair $i \neq j$, let $b_1^{ij}, \ldots, b_r^{ij}$ be a $k(d_i)$-basis of $k(d_{ij})$, and let $(b_1^{ij})^*, \ldots, (b_r^{ij})^*$ be the dual basis under the trace pairing $(b, b') \mapsto \mathrm{tr}_{k(d_i)}(bb')$. Then we have

$$c_{j \leftarrow i \leftarrow j} = \sum_{j \xrightarrow{a} i} \sum_t (b_t^{ij})^* a^* \otimes a b_t^{ij}$$

Thus the preprojective algebra is the tensor algebra of the species we've constructed, quotiented by the relations

$$\sum_{i \neq j} \mathrm{sgn}(j \leftarrow i) \sum_{j \xrightarrow{a} i} \sum_t (b_t^{ij})^* a^* \otimes a b_t^{ij} = 0 \quad \text{for each fixed } j \in I$$

Example. Following our example of species, consider the B_2 Cartan matrix $\left(\begin{smallmatrix} 2 & -1 \\ -2 & 2 \end{smallmatrix}\right)$. This is symmetrized by $d_1 = 2, d_2 = 1$, so we choose $k = k(d_2) = \mathbb{R}$, $k(d_1) = \mathbb{C}$, and $E(2 \leftarrow 1) \cong E(1 \leftarrow 2) \cong \mathbb{C}$.

A representation of the preprojective algebra is given by a $\mathbb{C}$-vector space M_1 and an $\mathbb{R}$-vector space M_2, together with an $\mathbb{R}$-linear map $f : \mathbb{C} \otimes_{\mathbb{C}} M_1 \cong M_1 \to M_2$ and a $\mathbb{C}$-linear map $g : \mathbb{C} \otimes_{\mathbb{R}} M_2 \to M_1$ (which we can identify with an $\mathbb{R}$-linear map $M_2 \to M_1$). These maps must satisfy the relations

$$\begin{aligned}
fg &= 0 \\
\frac{1}{2} gf - \frac{i}{2} gfi &= 0
\end{aligned}$$

Külshammer also defines reflection functors for preprojective algebras of species. This follows the same setup as before; we just need to update the definitions of $M_{\partial i}$, $M(i, \mathrm{in})$, and

$M(i, \text{out})$ to this new context. We define

$$M_{\partial i} := \bigoplus_{j \neq i} E(i \leftarrow j) \otimes_{k(d_j)} M_j.$$

We define the map $M(i, \text{in}) : M_{\partial i} \to M_i$ on the jth summand to be $\text{sgn}(i \leftarrow j) M_{i \leftarrow j}$. To define the map $M(i, \text{out}) : M_i \to M_{\partial i}$, we note that the duality between $E(j \leftarrow i)$ and $E(i \leftarrow j)$, together with tensor-hom adjunction, gives a natural isomorphism

$$(-)^\vee : \text{Hom}_{k(d_j)}(E(j \leftarrow i) \otimes_{k(d_i)} M_i, M_j) \xrightarrow{\sim} \text{Hom}_{k(d_i)}(M_i, E(i \leftarrow j) \otimes_{k(d_j)} M_j)$$

Thus we define $M(i, \text{out})$ to map into the jth summand of $M_{\partial i}$ by $M_{j \leftarrow i}^\vee$.

With these definitions in place, we can define reflection functors using the same commutative diagrams as in Section V.2.2. The results of that section still hold [Kü17, Section 5]. Proposition V.17, Proposition V.19, and Lemma V.20 also generalize to this context, following from [DST23, Appendix].

A.3: The aspects of the thesis which change when we're working with species

With one exception, the results of this thesis also apply to preprojective algebras of species. For the most part, what needs to be changed involves recognizing that the role of the ground field k is now played by a collection of extension fields $k(d_i)$ associated to the different vertices. We summarize the differences in this section.

In what follows, for any root β we define $d_\beta := \frac{(\beta,\beta)}{2}$. In particular, we have $d_{\alpha_i} = \frac{(\alpha_i,\alpha_i)}{2} = d_i \frac{(\alpha_i^\vee,\alpha_i)}{2} = d_i$ by definition. Thus, given any expression $\beta = s_{i_\ell} \cdots s_{i_1}(\alpha_{i_0})$, because the pairing $(-,-)$ is preserved by the group action, we have $d_\beta = d_{i_0}$.

A.3.1: Chapter VI

In Section VI.3, we observed that, since any real brick can be obtained by applying reflection functors to a simple module, the endomorphism ring of a real brick must be k. In this context we need to be a little more careful. Suppose that our real brick is given by

$$M := \Sigma_{i_\ell}^{\pm_\ell} \cdots \Sigma_{i_1}^{\pm_1}(S_{i_0})$$

Then the same argument implies that its endomorphism ring is isomorphic to $\text{End}(S_{i_0}) \cong k(d_{i_0}) = k(d_{\underline{\dim} M})$.

Similarly, a factor of d_β appears on both sides of the argument following Proposition VI.9: for a real brick B, we have

$$2\dim_k \operatorname{Hom}(B,B) - \dim_k \operatorname{Ext}^1(B,B) = (\underline{\dim}\, B, \underline{\dim}\, B)$$

and both $2\dim_k \operatorname{Hom}(B,B)$ and $(\underline{\dim}\, B, \underline{\dim}\, B)$ now equal $2d_\beta$.

A.3.2: Chapter VII

In this chapter, we need to be careful with statements about the dimensions of vector spaces, since it is no longer clear which fields those dimensions are taken over. However, with the appropriate dimensions taken into account, the proofs are the same.

The correct statement of Theorem VII.9 becomes:

Theorem A.4. *Let β be a root and let K be a shard of $\beta^\perp$. Let R be a rank 2 subsystem slicing a wall of K, and let γ_1 and γ_2 be the fundamental roots of R. Without loss of generality, suppose $\langle -, \gamma_1\rangle \geq 0$ on K. Let c_1, c_2 be constants such that $\beta = c_1\gamma_1 + c_2\gamma_2$. Let L_1 and L_2 be the fundamental shards of $\gamma_1^\perp$ and $\gamma_2^\perp$, respectively, meeting K at R. Then:*

(1) There exists a short exact sequence

$$0 \to M(L_1)^{\oplus c_1} \xrightarrow{f} M(K) \xrightarrow{g} M(L_2)^{\oplus c_2} \to 0$$

(2) $\dim_k \operatorname{Hom}(M(K), M(L_1)) = \dim_k \operatorname{Hom}(M(L_2), M(K)) = 0$,
$\dim_k \operatorname{Hom}(M(L_1), M(K)) = d_{\gamma_1} c_1$, *and* $\dim_k \operatorname{Hom}(M(K), M(L_2)) = d_{\gamma_2} c_2$.

Here the only change is in the dimensions in part (2).

We then note that the space $\operatorname{Hom}_k(M(L_1), M(K))$ inherits an $\operatorname{End}(M(L_1))$-module structure, and thus is a $k(d_{\gamma_1})$-vector space. Likewise, $\operatorname{Hom}_k(M(K), M(L_2))$ is a $k(d_{\gamma_2})$-vector space. So the correct statement of Corollary VII.10 becomes:

Corollary A.5. *For any exact sequence of the form in Theorem A.4, the components of f are a $k(d_{\gamma_1})$-basis of $\operatorname{Hom}_k(M(L_1), M(K))$, and the components of g are a $k(d_{\gamma_2})$-basis of $\operatorname{Hom}_k(M(K), M(L_2))$.*

Finally, we haven't worked out the details of Theorem VII.17, the genericity of the doubleton sequence, in this context. Although it seems plausible that a similar result should hold, the algebro-geometric complications of mixing multiple fields together seemed too far afield from the focus of this thesis.

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