Local Cohomology Modules and Motivic Chern Class Computations

by

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ABSTRACT

We give a new proof of a result by Puthenpurakal on Lyubeznik's conjecture regarding the associated primes of local cohomology modules. The conjecture states that if R is a regular ring and $I \subset R$ is an ideal, then, for all i, the local cohomology module $H_I^i(R)$ has finitely many associated primes. The result of interest is for $H_I^{n-1}(R)$ where R is of dimension n, contains $\mathbb{Q}$, and satisfies a certain condition on the singular loci of its reduced quotient rings. We use the theory of D-modules over formal power series rings to show this, and also give a result on the codimension of the support of $H_I^i(R)$ when R is further catenary, using the same methods.

Next, we compute the equivariant motivic Chern class for the nilpotent cone in M_n, the space of $n \times n$ matrices, and for the affine cone over a smooth hypersurface. In the nilpotent cone case, we consider the action of $\mathrm{GL}_n \times \mathbb{C}^\star$ acting on M_n by conjugation in GL_n and by scaling in $\mathbb{C}^\star$. With this action, the orbits of the nilpotent cone are the nilpotent orbits, indexed by partitions of n. Following the techniques of Feher, Rimanyi, and Weber, we compute the motivic Chern class of these nilpotent orbits. In the affine cone case, we consider the action of $\mathbb{C}^\star$ on $\mathbb{A}^{n+1}$ by scaling and compute the motivic Chern class of the affine cone $C(D)$, in $\mathbb{A}^{n+1}$, where $D \subset \mathbb{P}^n$ is a smooth hypersurface.

CHAPTER 1

Introduction

1.1: History and Motivation

1.1.1: D-modules and Local Cohomology Modules

Let R be a commutative noetherian ring and $I \subset R$ be an ideal. There is a functor $\Gamma_I(-)$ associated to I which maps an R-module M to $\Gamma_I(M) = \{u \in M | I^n u = 0 \text{ for some } n \geq 1\}$. This is a left exact functor and the local cohomology functors $H_I^i(-)$, for $i \geq 0$, are the derived functors of $\Gamma_I(-)$. These functors were first introduced in a seminar by Grothendieck in 1961 and recorded by Hartshorne in [Har67].

Example 1.1. Let $\mathscr{k}$ be a noetherian commutative ring and let $R = \mathscr{k}[x]$. Then $H_{(x)}^i(R) = 0$ for $i \neq 1$ and $H_{(x)}^1(R) = \frac{R_x}{R} \cong \bigoplus_{i<0} \mathscr{k} \cdot x^i$, where the free $\mathscr{k}$-module $\bigoplus_{i<0} \mathscr{k} \cdot x^i$ has the natural $\mathbb{Z}$-graded R-module structure given by $x \cdot x^{-1} = 0$ and $x \cdot x^i = x^{i+1}$ for all $i < 0$.

The local cohomology modules $H_I^i(M)$ for various rings R, ideals I, and R-modules M have been the subject of much study for the last 60 years. One area of study has been on the finiteness results that $H_I^i(R)$ exhibits. In most cases, such as in the example $H_{(x)}^1(\mathscr{k}[x])$ above, $H_I^i(R)$ is not a finitely generated R-module, but Huneke and Sharp proved in [HS93], that when R is regular and of characteristic $p > 0$, $H_I^i(R)$ has finitely many associated primes. Since their proof employed the action of the Frobenius on the local cohomology modules in positive characteristic, it could not be extended to the characteristic 0 case (or to the mixed characteristic case).

However, following their result, Lyubeznik showed in [Lyu93] that when $\Bbbk$ is a field of characteristic 0, and R is either a regular local ring with residue field $\Bbbk$ or a smooth $\Bbbk$-algebra, $H^i_I(R)$ has finitely many associated primes. His proof exploits the module structure of $H^i_I(R)$ over the ring of differential operators on R. Specifically, when R is a smooth $\Bbbk$-algebra or R is a ring of formal power series over $\Bbbk$, we have a well-behaving theory of $D_{R/\Bbbk}$-modules, where $D_{R/\Bbbk} \subset \operatorname{End}_\Bbbk(R)$ is the ring of $\Bbbk$-linear differential operators on R. The key point is that the local cohomology modules are finitely generated over $D_{R/\Bbbk}$ in these two settings. Lyubeznik poses the following question in [Lyu02, §6]:

Lyubeznik's Conjecture: If R is a regular ring containing $\mathbb{Q}$, then the set of the associated primes of $H^i_I(R)$ is finite.

The difficulty of extending Lyubeznik's result from the local and affine case to arbitrary regular rings containing $\mathbb{Q}$ lies in the lack of a D-module structure on the local cohomology modules when R is outside the settings that Lyubeznik examined. One cannot reduce to the local case either, since there could be infinitely many maximal ideals that are associated primes of $H^i_I(R)$. A result in the direction of extending Lyubeznik's result is due to Puthenpurakal.

Theorem.[Put16, Theorem 1.3] Let R be a regular excellent ring of dimension n containing $\mathbb{Q}$, and let $I \subset R$ be an ideal. Then $H^{n-1}_I(R)$ has finitely many associated primes.

One of the main results of this thesis is a new simpler proof to Puthenpurakal's result in Theorem 2.89 with a slightly weaker hypothesis than excellence. More precisely, we say that a Noetherian domain R is J-0 if there is a non-empty open set $U \subset \operatorname{Spec}(R)$ on which $R_\mathfrak{p}$ is regular for all $\mathfrak{p} \in U$. The following result is Theorem 2.89 in Chapter 2.

Theorem A. Let R be a regular ring of dimension n containing $\mathbb{Q}$ with the property that for every prime ideal $\mathfrak{p} \subset R$ of height $\geq n - 1$, the quotient ring $R/\mathfrak{p}$ is J-0. Let $I \subset R$ be an ideal. Then $H_I^{n-1}(R)$ has finitely many associated primes.

We also prove a result on the dimension of the support of the local cohomology modules $H_I^i(R)$, using similar techniques as in the proof of Theorem A. This is Theorem 2.87 in Chapter 2, stated below.

Theorem B. Let R be a regular ring of dimension n containing $\mathbb{Q}$ such that for every prime ideal $\mathfrak{p} \subset R$, the quotient ring $R/\mathfrak{p}$ is J-0. Let $I \subset R$ be an ideal of pure height d. Then $\operatorname{codim}\left(\overline{\operatorname{Supp}\left(H_I^i(R)\right)}\right) \geq i + 1$ for all $i > d$.

Excellent rings satisfy the J-0 conditions in the theorems above, and most rings are excellent rings. For example, rings of formal power series and smooth $\Bbbk$-algebras are all excellent rings. [Put16, Example 1.2] gives some examples of excellent regular ring of finite dimension where the finiteness of associated primes of local cohomology modules is not known from the current techniques.

We now make a few comments on the proofs of Theorems A and B. In Theorem 2.71 in Chapter 2, we provide a different proof of the following result, first shown by Lewis in [Lew22].

Theorem C. Let R be a noetherian ring, and let $I \subset R$ be an ideal with $\operatorname{ht}(I) = d$. Then there is $f \in R$ such that $\operatorname{ht}(I + (f)) > d$ and the natural transformation

$$H_{I+(f)}^i(-) \longrightarrow H_I^i(-)$$

is an isomorphism for $i > d + 1$ and gives an exact sequence

$$0 \longrightarrow H^1_{I+(f)} H^d_I(-) \longrightarrow H^{d+1}_{I+(f)}(-) \longrightarrow H^{d+1}_I(-) \longrightarrow 0.$$

Our proof differs from her proof in that it uses the spectral sequence of local cohomology modules. Theorem C allows us to use induction for $i > d + 1$ in the proofs of Theorems A and B. For the case $i = d + 1$, we look at the behavior of $H^{d+1}_{I+(f)}(R)$ when we localize and complete at prime ideals $\mathfrak{p}$, with $R_\mathfrak{p}/(I+(f))R_\mathfrak{p}$ regular. We use results on D-modules over regular complete local rings to analyze $H^{d+1}_{I+(f)}(R)$, thus proving Theorems A and B.

1.1.2: Motivic Chern Classes

Let X be a complex algebraic variety. If X is smooth, then an important variant of X is the total Chern class $c(T_X)$ of the tangent bundle T_X of X. This lives in the singular cohomology $H^\star(X, \mathbb{Q})$ of X, but there is also an algebraic invariant with values in the Chow ring $A^\star(X)$. The easiest version to describe is in the context of K-theory. Recall that the Grothendieck group $K_0(X)$ of any algebraic variety X is the quotient of the free Abelian group on isomorphism classes of coherent sheaves on X, modulo relations of the form $[\mathcal{F}] = [\mathcal{F}'] + [\mathcal{F}'']$, where

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is an exact sequence of coherent sheaves. If X is smooth, then the K-theoretic version of the Chern class of X is

$$\lambda_y(\Omega_{X/\mathbb{C}}) := \sum \left[\Omega^i_{X/\mathbb{C}}\right] \in K_0(X)[y].$$

Example 1.2. $K_0(\mathbb{P}^n) = \mathbb{Z}[h]/(1-h)^{n+1}$ where $h = [\mathcal{O}_{\mathbb{P}^n}(-1)]$. Using the Euler sequence, one can see that $\lambda_y(\Omega_{\mathbb{P}^n}) = \frac{(1+hy)^{n+1}}{1+y}$ in $K_0(\mathbb{P}^n)[y]$.

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Over the years, there has been a lot of interest in extending this definition to singular varieties such that suitable functoriality properties are satisfied. An important example is Macpherson's invariant $c_{SM}(X) \in A_\star(X)$ associated to any complex algebraic variety and known now as the Chern-Schwarz-Macpherson class, see [Mac74].

A more refined invariant, of a K-theoretic nature, was introduced by Brasselet, Schürmann, and Yokura in [BSY10]. Given a complex algebraic variety X, the motivic Chern class is a morphism of Abelian groups

$$\mathrm{mC}_y : K_0(\mathrm{Var}/X) \longrightarrow K_0(X)[y].$$

Here $K_0(\mathrm{Var}/X)$ is the Grothendieck group of varieties over X: it is the quotient of the free Abelian group on the set of isomorphism classes of varieties $Y \to X$ over X, modulo relations of the form

$$[Y \to X] = [Y' \to X] + [Y \setminus Y' \to X]$$

where Y' is a closed subvariety of Y. The motivic Chern class is uniquely characterized by the fact that it commutes with proper pushforward and if X is smooth, then

$$\mathrm{mC}_y(X \xrightarrow{1_X} X) = \lambda_y(\Omega_{X/\mathbb{C}}) \in K_0(X)[y].$$

Furthermore, it satisfies certain properties, including the projection formula.

Example 1.3. Let $Y \subset \mathbb{P}^2$ be the union of two lines L_1, L_2 in $\mathbb{P}^2$ intersecting at a point $p \in \mathbb{P}^2$. Then $Y = L_1 \setminus \{p\} \sqcup L_2 \setminus \{p\} \sqcup \{p\}$. We compute

$$\mathrm{mC}_y(Y \subset \mathbb{P}^2) = \mathrm{mC}_y(L_1 \subset \mathbb{P}^2) + \mathrm{mC}_y(L_2 \subset \mathbb{P}^2) - \mathrm{mC}_y(\{p\} \hookrightarrow \mathbb{P}^2) = 2\frac{(1 + hy)^2(1 - h)}{1 + y} - (1 - h)^2.$$

The motivic Chern class allows recovering of the Chern-Schwarz-Macpherson class via suit-

able specialiation of y using a Todd class transformation (see [BSY10]):

$$td_{(1+y)} : K_0(X)[y, y^{-1}] \longrightarrow A_\star(X)[y, y^{-1}].$$

Let G be a complex linear algebraic group. We say that G acts morphically on X if there is an algebraic morphism

$$\sigma : G \times X \to X$$

which gives an action of G on X. For example, the space of $n \times n$ matrices M_n has one GL_n-equivariant structure given by conjugation. Since the determinantal varieties (i.e. spaces of matrices of rank $\leq r$, for $0 \leq r \leq n$) and the nilpotent cone (i.e. space of nilpotent matrices) are closed subspaces of M invariant under this action, they are also examples of GL_n-equivariant varieties. A morphism $f : Y \to X$ between G-equivariant varieties is G-equivariant if it commutes with the G-actions on X and Y.

Furthermore, for every sheaf $\mathcal{F}$ on X, the action of G on X induces an isomorphism on its stalks. If these isomorphisms are of a global nature, i.e. if there is an isomorphism

$$\Phi : \mathrm{pr}_2^\star \mathcal{F} \cong \sigma^\star \mathcal{F},$$

where pr_2 is the projection of $G \times X$ onto X, which satisfies the cocyle conditions, we say that $(\mathcal{F}, \Phi)$ is a G-equivariant sheaf. The structure sheaf $\mathcal{O}_X$ and the cotangent sheaf $\Omega_{X/\mathbb{C}}$ are both G-equivariant sheaves. A G-equivariant homomorphism between G-equivariant sheaves $(\mathcal{F}', \Phi') \to (\mathcal{F}, \Phi)$ on X is a homomorphism of sheaves such that its pullback along pr_2 and σ commute with the isomorphisms Φ and Φ'.

Similarly to $K(\mathrm{Var}/X)$ and $K_0(X)$, one can define the Grothendieck group of G-varieties over X, $K^G(\mathrm{Var}/X)$, and the Grothendieck group of G-equivariant sheaves, $K_0^G(X)$. In [FRW21], they generalize the construction of motivic Chern classes to the G-equivariant

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motivic Chern class

$$\mathrm{mC}_y^G : K_0^G(\mathrm{Var}/X) \to K_0^G(X)[y]$$

and develop an axiomatic characterization of equivariant motivic Chern classes of orbits of G in X under certain conditions (e.g. G acts on X with finitely many orbits) analogously to the axiomatic system of Okonoukov's K-theoretic stable envelopes (see [Oko17]). Motivated by the appearance of enumerative classes of matrix Schubert cells as the "atoms" of the theory of multiple fundamental classes, including in the theory of Chern-Schwarz-Macpherson classes (see [FR18]), they also compute the equivariant motivic Chern classes of matrix Schubert cells as certain weight functions.

One of the goals in Chapter 3 is to use the methods of [FRW21] to compute the equivariant motivic Chern classes of the orbits of the nilpotent cone in M_n. Although the nilpotent cone has singularities, it has a stratification by nilpotent orbits, which are smooth and locally closed subsets. Furthermore, the nilpotent cone has a natural resolution of singularities by the cotangent bundle of the full flag variety of flags in $\mathbb{C}^n$, and the closure of each nilpotent orbit similarly has a natural resolution of singularities by the cotangent bundle of an appropriate partial flag variety. Due to their many properties, nilpotent cones and their orbits are interesting objects in representation theory and symplectic geometry. We use the geometric approach to computing these invariants, and write them in terms of the weight functions, i.e. the equivariant motivic Chern classes of the matrix Schubert cells.

Another reason one might want to look at equivariant motivic Chern classes instead of ordinary motivic Chern classes is that more information could be captured if the action of G is considered. For example, we have that $K_0(\mathbb{A}^n) = \mathbb{Z}$, since all locally free sheaves on $\mathbb{A}^n$ are trivial. However, if the action of $\mathbb{C}^\star$ by scaling on $\mathbb{A}^n$ is considered, we have $K^{\mathbb{C}^\star}(\mathbb{A}^n) = \mathbb{Z}[\zeta^{\pm 1}]$, where, if Z is a degree d hypersurface, and L_Z is the rank one locally free sheaf on X such that

$$0 \longrightarrow L_Z \longrightarrow \mathcal{O}_{\mathbb{A}^n} \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

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is a $\mathbb{C}^{\star}$-equivariant short exact sequence, then $[L_Z] = \zeta^d$. Thus, our second goal in Chapter 3 is to compute the $\mathbb{C}^{\star}$-equivariant motivic Chern classes of affine cones over smooth projective hypersurfaces.

1.2: Outline of Thesis

The thesis is divided into two parts, where Chapter 2 is on D-modules and local cohomology modules, and Chapter 3 is on motivic Chern classes. The proofs of the main results Theorem A, B, and C are given in Section 2.3. In Sections 2.1 and 2.2 we provide the necessary background on local cohomology and D-modules needed for the proofs in Section 2.3. In particular, in Section 2.2, after providing a brief introduction to the ring of differential operators, we will turn our focus to D-modules over the ring of formal power series.

In Section 3.1, give an introduction to motivic Chern classes. Sections 3.2 and 3.3 are dedicated to computing the equivariant motivic Chern class of the nilpotent Cone and of the affine cone over a smooth projective hypersurface, respectively. Section 3.3 is joint work with Sridhar Venkatesh.

1.3: Conventions and notations

$\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{C}$ are the ring of integers, field of rational numbers, and field of complex numbers, respectively. $\mathbb{N}$ is the set of non-negative integers.

All rings will be unital. If $\mathscr{k}$ is a field, $\mathscr{k}$-algebra will mean a (unital) commutative ring containing $\mathscr{k}$. For $n \geq 0$, $\mathbb{A}_{\mathscr{k}}^n$ and $\mathbb{P}_{\mathscr{k}}^n$ denotes the affine space and projective space of dimension n over $\mathscr{k}$.

Let R be a noetherian commutative ring. Considering R as an ideal in itself and the $\emptyset$ as a subset of $\mathrm{Spec}(R)$, we define $\mathrm{ht}(R) = \infty$ and $\mathrm{codim}(\emptyset) = \infty$.

CHAPTER 2

Local Cohomology and D-modules

2.1: Background on Local Cohomology

2.1.1: Local Cohomology

Througout this section, let R be a noetherian commutative ring. Let Mod_R denote the abelian category of R-modules. Let $I \subseteq R$ be an ideal. If M is an R-module with support in $V(I)$, then any submodule or quotient module of it will have support in $V(I)$. Consequently, any complex of R-modules with support in $V(I)$ will have cohomology with support in $V(I)$.

Lemma 2.1. Let $I \subset R$ be an ideal. Let M be an R-module. Then M has support in $V(I)$ if and only if for all $m \in M$, $\exists n \geq 0$ such that $I^n \cdot m = 0$.

Proof. Suppose M has support in $V(I)$. Let $f \in I$. If $\mathfrak{p} \subset R$ is a prime ideal such that $\mathfrak{p} \not\ni f$, then $p \not\supseteq I$, so $M_p = 0$. Then we must have that $M_f = 0$. But that means that for any $m \in M$, $\exists n \geq 0$ such that $f^n m = 0$. Since R is noetherian, I can be generated by elements $f_1, \ldots, f_k \in I$. We may pick $n >> 0$ such that $f_i^n m = 0$ for all $i = 1, \ldots, n$, and hence, $I^n \cdot m = 0$.

On the other hand, suppose that for any $m \in M$, there exists $n \geq 0$ such that $I^n \cdot m = 0$. Suppose $\mathfrak{p} \subset R$ is a prime ideal such that $\mathfrak{p} \not\supseteq I$. Let $f \in I \setminus \mathfrak{p}$. Then $M_f = 0$ since for every $m \in M$, $\exists n \geq 0$ such that $f^n m = 0$. Therefore $M_{\mathfrak{p}} = 0$. $\qquad\square$

To any ideal $I \subset R$, we associate a functor Γ_I on Mod_R as follows. For every R-module

M, we define the R-submodule

$$\Gamma_I(M) := \{m \in M | \exists n \text{ s.t } I^n m = 0\} \subseteq M.$$

For any homomorphism of R-modules $\phi : M \to M'$, the restriction of ϕ to $\Gamma_I(M)$ naturally factors through $\Gamma_I(M')$. Hence, we define $\Gamma_I(\phi)$ to be the restriction of ϕ to $\Gamma_I(M) \to \Gamma_I(M')$. Note that $\Gamma_I(M)$ has support in $V(I)$.

By Lemma 2.1, we get the following.

Lemma 2.2. Let $I \subset R$ be an ideal. If M is an $R-$ module with support in $V(I)$, then $\Gamma_I(M) = M$.

Γ_I is left exact, so it gives rise to the derived functor

$$R\Gamma_I : D^+(Mod_R) \to D^+(Mod_R),$$

where $D^+(Mod_R)$ denotes the derived category of complexes of R-modules that are bounded below. We denote

$$H_I^i(M) := R^i\Gamma_I(M)$$

the *i-th local cohomology module.* Note that these local cohomology modules all have support in $V(I)$, since we may compute $H_I^i(M) = \mathcal{H}^i\Gamma_I(Q^\bullet)$ where $M \to Q^\bullet$ is an injective resolution and $\Gamma_I(Q^\bullet)$ is a complex of R-modules with support in $V(I)$.

We will now discuss some properties regarding these local cohomology functors.

Proposition 2.3. [ILL$^+$07, Proposition 7.3] Let $I \subset R$ be an ideal. Then we have

$$R\Gamma_I(-) \cong R\Gamma_{\sqrt{I}}(-).$$

Proposition 2.4. [ILL$^+$07, Proposition 9.15] Suppose we have $\dim(R) = n$. Let $I \subset R$ be an ideal. Then for all finitely generated modules M, $H_I^i(M) = 0$ for all $i > n$.

Proposition 2.5. [Har67, Thm 3.8] Let $I \subset R$ be an ideal and let M be a finitely generated R-module. Let $d = \operatorname{depth}_I(M)$. Then $H^i_I(M) = 0$ for all $i < d$ and $H^d_I(M) \neq 0$.

Corollary 2.6. If R is a Cohen-Macaulay ring and $c = \operatorname{ht}(I)$, then $H^i_I(R) = 0$ and $H^c_I(R) \neq 0$.

Krull's Height Theorem tells us that the minimal number of generators of I is always $\geq \operatorname{ht}(I)$. It turns out that the minimal number of generators gives an upper bound for the largest non-vanishing degree of local cohomology modules. To see this, we will first need to describe local cohomology modules in a different way.

Definition 2.7 (Cech Complex). Let $I \subset R$ be an ideal. Let $f_1, \ldots, f_r$ be generators of I. For every R-module M, we define the *Čech complex* $\check{C}^\bullet(f_1, \ldots, f_k; M)$ as follows. For $l \geq 0$,

$$\check{C}^l(f_1, \ldots, f_k; M) := \bigoplus_{I = (1 \leq i_1 < \ldots < i_l \leq n)} M_{f_{i_1} \cdots f_{i_l}}$$

and the boundary maps d^i are given componentwise as

$$d^i_{I,J} = \begin{cases} 0 & I \not\subseteq J \\ (-1)^k \nu_{I,J} & I = J \setminus \{j_k\} \end{cases}$$

where $\nu_{J,I} : M_{f_{i_1} \cdots f_{i_l}} \to M_{f_{j_1} \cdots f_{j_{l+1}}}$ are the localization maps for $I \subseteq J$.

Proposition 2.8. [ILL$^+$07, Thm 7.13] Let $I \subset R$ be an ideal and $f_1, \ldots, f_k$ be a system of generators of I. Then we have natural isomorphisms of functors

$$H^i_I(-) \cong \mathcal{H}^i \check{C}(f_1, \ldots, f_k; -)$$

for all $i \geq 0$.

The Čech Complex description of local cohomology modules shows the following vanishing of top local cohomology modules.

Proposition 2.9. Let $I \subset R$ be an ideal and let k be the minimal number of generators of I. Then $H_I^i(R) = 0$ for all $i > k$.

Using the Čech Complex construction of local cohomology modules, we can compute the following basic examples.

Example 2.10. From Propositions 2.9 and 2.6, if $f_1, \ldots, f_c \in R$ is a regular sequence and $I = (f_1, \ldots, f_c)$, then $H_I^i(R) = 0$ for $i \neq c$.

Example 2.11. In the case that $R = \Bbbk[[x_1, \ldots, x_n]]$ is a formal power series ring and $I = (x_1, \ldots, x_n)$, the Čech complex describes $H_I^c(R)$ as:

$$H_I^n(R) = \bigoplus_{i_1, \ldots, i_n \geq 1} \Bbbk \frac{1}{x_1^{i_1} \ldots x_n^{i_n}}.$$

Lemma 2.12. Suppose $R \to A$ is a flat ring homomorphism. Then we have natural isomorphism of functors

$$\Phi^i : H_I^i(-) \otimes_R A \to H_{IA}^i((-) \otimes_R A)$$

for all $i \geq 0$.

Proof. Let $f_1, \ldots, f_k$ generate I. Then their images $\widetilde{f}_1, \ldots, \widetilde{f}_k$ in A generate IA. It is easy to see that there is a natural isomorphism of functors

$$\check{C}(f_1, \ldots, f_k; -) \otimes_R A \cong \check{C}(\widetilde{f}_1, \ldots, \widetilde{f}_k; M \otimes_R A).$$

Since $(-) \otimes_R A$ is an exact functor, it commutes with taking cohomology of complexes, i.e the natural transformation of functors on $D^+(Mod_R)$

$$\mathcal{H}^i(-) \otimes_R A \to \mathcal{H}^i((-) \otimes_R A)$$

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is an isomorphism for all i. Thus we get natural isomorphisms

$$H^i_{IA}((-) \otimes_R A) \cong \mathcal{H}^i \check{C}(\widetilde{f}_1, \ldots, \widetilde{f}_k; (-) \otimes_R A) \cong \mathcal{H}^i \left(\check{C}(f_1, \ldots, f_k; (-)) \otimes_R A \right)$$

$$\cong \left(\mathcal{H}^i \check{C}(f_1, \ldots, f_k; (-)) \right) \otimes_R A \cong H^i_I(-) \otimes_R A.$$

$\square$

Remark 2.13. It is easy to check that the natural isomorphism in Lemma 2.12 is independent of choice of generators $f_1, \ldots, f_k$ of I. Additionally, for every R-module M, we can explicitly describe the natural homomorphisms Φ^0_M. Tensoring the inclusion of R-modules

$$\Gamma_I(M) \subseteq M$$

by A, we get

$$
\begin{array}{ccc}
 & & \Gamma_I(M \otimes_R A) \\
 & \overset{\Phi^0_M}{\nearrow} & \cap \\
\Gamma_I(M) \otimes_R A & \longhookrightarrow & M \otimes_R A.
\end{array}
$$

Furthermore, it is easy to check that the family of functors from $R-$ modules to A-modules $\{H^i_I(-) \otimes_R A\}$ and $\{H^i_{IA}((-) \otimes_R A)\}$ are universal $\delta-$functors for the functors $\Gamma_I(-) \otimes_R A$ and $\Gamma_{IA}((-) \otimes_R A)$, respectively, and the isomorphism in Lemma 2.12 is a natural isomorphism of these δ-functors.

Definition 2.14. Let $I \subset J \subset R$ be ideals. Then we have a natural inclusion of functors on $R-$ modules

$$\Gamma_J(-) \hookrightarrow \Gamma_I(-).$$

This gives rise to a natural transformation of derived functors on $R-$ modules for all i

$$H^i_J(-) \longrightarrow H^i_I(-).$$

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Lemma 2.15. Suppose $R \to A$ is a flat ring homomorphism. Let $I \subset J \subset R$ be ideals. Then we have commutative diagram of natural transformations of functors from $R-$modules to A-modules for all i

$$
\begin{array}{ccc}
H_J^i(-) \otimes_R A & \longrightarrow & H_I^i(-) \otimes_R A \\
\| \wr & & \| \wr \\
H_{JA}^i((-) \otimes_R A) & \longrightarrow & H_{IA}^i((-) \otimes_R A)
\end{array}
$$

where the horizontal natural transformations arise from Definition 2.14 and the vertical natural transformations are those in Lemma 2.12.

Proof. By Remark 2.13 and the universal property of universal $\delta -$ functors, it is enough to check for $i = 0$. In this case it is clear from their descriptions in Remark 2.13 that they commute. $\qquad \square$

Definition 2.16 (Cech complex projection map). Let $I \subset J \subset R$ be ideals. Let $f_1, \ldots, f_k$ be a systems of generators of I. Extend this sequence to generators $f_1, \ldots, f_k, f_{k+1}, \ldots, f_n$ of J. Then we have for any R-module M, a morphism of Čech complexes

$$
\rho : \check{C}(f_1, \ldots, f_n; M) \longrightarrow \check{C}(f_1, \ldots, f_k; M),
$$

where ρ^l is the projection map

$$
\check{C}^l(f_1, \ldots, f_n; M) = \bigoplus_{I=(1 \leq i_1 < \ldots < i_l \leq k)} M_{f_{i_1} \ldots f_{i_l}} \oplus \bigoplus_{I=(k < i_1 < \ldots < i_l \leq n)} M_{f_{i_1} \ldots f_{i_l}} \twoheadrightarrow \check{C}^l(f_1, \ldots, f_k; M) =
$$

$$
\bigoplus_{I=(1 \leq i_1 < \ldots < i_l \leq k)} M_{f_{i_1} \ldots f_{i_l}}.
$$

Remark 2.17. With the same set-up as in the previous definition, $\mathcal{H}^i(\rho)$ coincides with the natural morphism $H_J^i(M) \to H_I^i(M)$ from Definition 2.14.

We also provide another local cohomology vanishing theorem below which will be used later in this chapter.

Theorem 2.18 (Hartshorne-Lichtenbaum Vanishing)**.** [BH94] Let $(R, \mathfrak{m})$ be a local Noetherian ring and let $I \subset R$ be an ideal. Set $d = \dim(R)$ and let $\hat{R}$ be the completion of R at the maximal ideal. Then the following statements are equivalent:

(i) $H_I^d(R) = 0$.

(ii) For all minimal prime ideals $\mathfrak{p} \subset \hat{R}$ such that $\dim(\hat{R}/\mathfrak{p}) = \dim(R)$, we have

$$\dim(\hat{R}/(I\hat{R} + \mathfrak{p})) > 0.$$

2.1.2: Spectral Sequences

We introduce the local cohomology spectral sequences in this section and make some remarks which will be relevant in the last section of this chapter. Our main reference is [Wit11, Section 2.5].

2.1.2.1   Brief Review of Spectral Sequences

Let $A^{\bullet,\bullet} = \left(A^{i,j}, \partial_\rightarrow^{i,j}, \partial_\uparrow^{i,j}\right)$ be a double complex of R-modules such that $A^{i,j} = 0$ for $i < 0$ or $j < 0$. We define the total complex $Tot(A^{\bullet,\bullet})$ given by

$$Tot^n(A^{\bullet,\bullet}) = \bigoplus_{i+j=n} A^{i,j}$$

with differential given by

$$\partial^n = \sum_{i+j=n} \partial_\rightarrow^{i,j} + (-1)^i \partial_\uparrow^{i,j}.$$

$Tot(A^{\bullet,\bullet})$ has a decreasing filtration given by

$$F^p Tot^n(A^{\bullet,\bullet}) = \bigoplus_{i \geq p} A^{i,n-i},$$

which induces the filtration $F^\bullet \mathcal{H}^n Tot(A^{\bullet,\bullet})$ on $\mathcal{H}^n Tot(A^{\bullet,\bullet})$ given by

$$F^p \mathcal{H}^n Tot(A^{\bullet,\bullet}) = \operatorname{im}(\mathcal{H}^n F^p Tot(A^{\bullet,\bullet}) \to \mathcal{H}^n Tot(A^{\bullet,\bullet})).$$

Let $E_0 = \operatorname{gr}_\bullet^F Tot(A^{\bullet,\bullet})$. Then $E_0^{p,q} = A^{p,q}$ with only the horizontal differentials which we denote $d_0^{p,q} := \partial_\to^{p,q}$. Next, let $E_1^{p,q} = \ker(d_0^{p,q}) = \mathcal{H}^q(A^{p,\bullet})$. The maps $\partial_\uparrow^{p,q}$ on $Tot(A^{\bullet,\bullet})$ induce differentials $d_1^{p,q} = d_\uparrow^{p,q}$ on $E_1^{p,q}$. We bundle this data together as $E_1 = (E_1^{p,q}, d_1^{p,q})$. Inductively we can define

$$E_r = \left(E_r^{p,q} = \frac{\ker(d_{r-1}^{p,q})}{\operatorname{im}(d_{r-1}^{p-r,q+r-1}}, \ d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1} \right)$$

where $d_r^{p,q}$ is induced by the differential $\partial_\leftarrow$ and $\partial_\uparrow$ on $Tot(A^{\bullet,\bullet})$. We call E_r the r-th page of the spectral sequence.

We can see that for each n, for all $r \geq n+2$, and for all $p+q = n$, we have $E_r^{p+r,q-r+1} = 0$ and $E_r^{p-r,q+r-1} = 0$, and hence $E_r^{p,q} = E_{r+1}^{p,q}$. Therefore, $E_r^{p,q} = E_{n+2}^{p,q}$. We will denote this by $E_\infty^{p,q}$.

Theorem 2.19. We have canonical isomorphisms

$$E_\infty^{p,q} \cong \operatorname{gr}_p^F \mathcal{H}^{p+q} Tot(A^{\bullet,\bullet})$$

for all p, q.

We say that E_r abuts to $\mathcal{H}^{p+q} Tot(A^{\bullet,\bullet})$ and denote this as

$$E_2^{p,q} \Rightarrow_p \mathcal{H}^{p+q} Tot(A^{\bullet,\bullet}).$$

Remark 2.20. Since for all n and $r > 0$, we have $E_r^{-r,n+r-1} = 0$, the above definitions tell

us that we have canonical inclusions

$$E_{r+1}^{0,n} = \ker(d_r^{0,n}) \subseteq E_r^{0,n},$$

giving us

$$E_\infty^{0,n} \subseteq E_r^{0,n}$$

for all n.

Remark 2.21. By the definition in Theorem 2.19, we also have canonical maps

$$\mathcal{H}^n(Tot(A^{\bullet,\bullet})) \twoheadrightarrow E_\infty^{0,n} \cong \operatorname{gr}_0^F \mathcal{H}^n Tot(A^{\bullet,\bullet}) = \frac{F_0 \mathcal{H}^n Tot(A^{\bullet,\bullet})}{F_1 \mathcal{H}^n Tot(A^{\bullet,\bullet})} = \frac{\mathcal{H}^n Tot(A^{\bullet,\bullet})}{F_1 \mathcal{H}^n Tot(A^{\bullet,\bullet})}.$$

If $B^{\bullet,\bullet}$ is another double complex and $\{E_r', d_r'\}$ is the spectral sequence associated to its vertical (resp. horizontal) filtration, a morphism of double complexes $\phi : A^{\bullet,\bullet} \longrightarrow B^{\bullet,\bullet}$ induces a homomorphism of spectral sequences

$$\{\phi_r\} : \{E_r, d_r\} \longrightarrow \{E_r', d_r'\}$$

and we have commmutativity

$$
\begin{array}{ccc}
\mathcal{H}^n(Tot(A^{\bullet,\bullet})) & \longrightarrow\!\!\!\!\rightarrow & E_\infty^{0,q} \\
\Big\downarrow{\scriptstyle \mathcal{H}^n(\phi)} & & \Big\downarrow{\scriptstyle \phi_\infty^{0,n}} \\
\mathcal{H}^n(Tot(B^{\bullet,\bullet})) & \longrightarrow\!\!\!\!\rightarrow & E_\infty'^{0,q}
\end{array}
$$

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2.1.2.2 The Local Cohomology Spectral Sequence

Let $I, J \subset R$ be two ideals. Let $f_1, \ldots, f_k$ generate I and $g_1, \ldots, g_l$ generate J. We define the double complex

$$\check{C}^{\bullet, \bullet}(f_1, \ldots, f_k; g_1, \ldots g_l; M) := \check{C}(f_1, \ldots, f_k; M) \otimes_R \check{C}(g_1, \ldots, g_l; R).$$

Then

$$Tot\left(\check{C}^{\bullet, \bullet}(f_1, \ldots, f_k; g_1, \ldots g_l; M)\right) = \check{C}(f_1, \ldots, f_k, g_1, \ldots, g_l; M).$$

as complexes. Furthermore, we see that the spectral sequence associated to $\check{C}^{\bullet, \bullet}(f_1, \ldots, f_k; g_1, \ldots g_l; M)$ is such that

$$E_1^{\bullet, q} = H_I^q(M) \otimes_R \check{C}(g_1, \ldots, g_l; R) \cong \check{C}(g_1, \ldots, g_l; H_I^q(M))$$

as complexes, so the second page is the following

$$
\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
E_2^{2,0} = H_J^2 H_I^0(M) \xrightarrow{d_2^{1,1}} & E_2^{2,1} = H_J^2 H_I^1(M) \xrightarrow{d_2^{1,2}} & E_2^{2,2} = H_J^2 H_I^2(M) & \ldots \\
E_2^{1,0} = H_J^1 H_I^0(M) \xrightarrow{d_2^{0,1}} & E_2^{1,1} = H_J^1 H_I^1(M) \xrightarrow{d_2^{0,2}} & E_2^{1,2} = H_J^1 H_I^2(M) & \ldots \\
E_2^{0,0} = H_J^0 H_I^0(M) & E_2^{0,1} = H_J^0 H_I^1(M) & E_2^{0,2} = H_J^0 H_I^2(M) & \ldots
\end{array}
$$

where

$$E_2^{p,q} = H_J^p H_I^q(M)$$

for all p, q. Thus we get

$$E_2^{p,q} = H_J^p H_I^q(M) \Rightarrow_p H_{I+J}^{p+q}(M).$$

We will call this the local cohomology spectral sequence for the ideals I, J and the R-module M.

Example 2.22. Let $I \subset R$ be an ideal with generators $f_1, \ldots, f_k \in I$. Let $J' \subset J \subset R$ be ideals. Let $g_1, \ldots, g_l, \ldots g_m \in J$ generate J such that $g_1, \ldots, g_l \in J'$ generate J'. Now let M be an R-module. The projection map

$$\rho : \check{C}(g_1, \ldots, g_m; R) \longrightarrow \check{C}(g_1, \ldots, g_l; R),$$

given in Definition 2.16 induces a morphism of complexes

$$\check{C}^{\bullet,\bullet}(f_1, \ldots, f_k; g_1, \ldots g_m; M) \overset{1 \otimes_R \rho}{\longrightarrow} \check{C}^{\bullet,\bullet}(f_1, \ldots, f_k; g_1, \ldots g_l; M).$$

Note that the induced map on the total complexes

$$Tot\left(\check{C}^{\bullet,\bullet}(f_1, \ldots, f_k; g_1, \ldots g_m; M)\right) = \check{C}(f_1, \ldots, f_k, g_1, \ldots, g_m; M) \overset{Tot(1 \otimes_R \rho)}{\longrightarrow}$$

$$Tot\left(\check{C}^{\bullet,\bullet}(f_1, \ldots, f_k; g_1, \ldots g_l; M)\right) = \check{C}(f_1, \ldots, f_k, g_1, \ldots, g_l; M)$$

coincides with the projection map in Definition 2.16 for the inclusion of ideals $J' + I \subset J + I$ and the R-module M. Then, by Remark 2.17, we get that the induced homomorphism on the cohomology of the total complex

$$H^n_{J+I}(M) \cong \mathcal{H}^n(Tot(\Gamma_J(F^{\bullet,\bullet})))(M) \overset{\mathcal{H}^n Tot(1 \otimes_R \rho)}{\longrightarrow} \mathcal{H}^n(Tot(\Gamma_{J'}((F^{\bullet,\bullet})))(M) \cong H^n_{J'+I}(M)$$

is the natural transformation in Definition 2.14 applied to $J' + I \subset J + I$ and M.

Furthermore, denoting E_r, E'_r as the local cohomology spectral sequence for the ideals I, J and the R-module M, and the ideals I, J' and the R-module M, respectively, the induced

morphism of spectral sequences induced by ϕ

$$\{\phi_r\} : \{E_r, d_r\} \longrightarrow \{E'_r, d_r\}$$

is such that

$$\phi_2^{p,q} : E_2^{p,q} = H_J^p H_I^q(M) \longrightarrow H_{J'}^p H_I^q(M)$$

is the natural transformation in Definition 2.14 applied to $H_I^q(M)$ for all $p, q \geq 0$.

In particular, in the case $J' = 0$, we get the commutative diagram

$$
\begin{array}{ccc}
H_{J+I}^n(M) \cong \mathcal{H}^n(Tot(\Gamma_J(F^{\bullet,\bullet})))(M) & \longrightarrow & H_I^n(M) \\
\downarrow & & \| \\
E_\infty^{0,n} \xrightarrow{\quad \phi_\infty^{0,n} \quad} & & E'^{0,n}_\infty \\
\cap & & \| \\
E_2^{0,n} = H_J^0 H_I^n(M) \xrightarrow{\quad \phi_2^{0,n} \quad} & & E'^{0,n}_2 = H_I^n(M)
\end{array}
$$

which coincides with the natural transformation in Definition 2.14 applied to M.

Lemma 2.23. Let $I, J \subset R$ be ideals. Let Q be an injective R-module. Then $\Gamma_I(Q)$ is Γ_J-acyclic.

Proof. Since Q is an injective R-module, we have $H_I^i(Q) = 0$ for all $i > 0$. Then the local cohomology spectral sequence for the ideals I, J and the R-module Q on the second page is:

$$E_2^{p,q} = H_J^p H_I^q(Q) = \begin{cases} 0 & q \neq 0 \\ H_J^p \Gamma_I(Q) & q = 0. \end{cases}$$

It is easy to see that, in this case, $E_r^{p,q} = E_2^{p,q}$ for all $r \geq 2$. Hence, $E_\infty^{p,q} = E_2^{p,q}$. Since

$$E_2^{p,q} = H_J^p H_I^q(Q) \Rightarrow_p H_{I+J}^{p+q}(Q)$$

and $E_\infty^{p,q} = E_2^{p,q} = 0$ if $q \neq 0$, by Theorem 2.19, $H_{I+J}^p(Q) = E_\infty^{p,0} = H_J^p \Gamma_I(Q)$. But

$H^p_{I+J}(Q) = 0$ for $p > 0$ since Q is an injective R-module. Thus we have for all $p > 0$, $H^p_J\Gamma_I(Q) = 0$, i.e Q is Γ_J-acyclic. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 2.23 allows us to extend Lemma 2.2 to the following.

Lemma 2.24. Let $I \subset R$ be an ideal. If M is an $R-$module supported on $V(I)$, then $R\Gamma_I(M) = M$.

Proof. By Lemma 2.2, we already have that $\Gamma_I(M) = M$. We need to show that $H^i_I(M) = 0$ for all $i > 0$. Let Q be an injective R-module such that we have an injective homomorphism $M \hookrightarrow Q$. Then we get an injective homomorphism $M = \Gamma_I(M) \hookrightarrow \Gamma_I(Q)$. Consider the short exact sequence

$$0 \longrightarrow M \longrightarrow \Gamma_I(Q) \xrightarrow{\ \pi\ } M' := \Gamma_I(Q)/M \longrightarrow 0. \qquad\qquad (\star)$$

Since $\Gamma_I(Q)$ has support in $V(I)$, so does M'. First, we get the exact sequence

$$\Gamma_I(\Gamma_I(Q)) \xrightarrow{\ \Gamma_I(\pi)\ } \Gamma_I(M') \longrightarrow H^1_I(M) \longrightarrow H^1_I(\Gamma_I(Q)) \overset{\text{Lemma } 2.23}{=} 0.$$

By Lemma 2.2, $\Gamma_I(\Gamma_I(Q)) = \Gamma_I(Q)$ and $\Gamma_I(M') = M'$, so $\Gamma_I(\pi) = \pi$ is surjective. Thus we get $H^1_I(M) = 0$.

Since M was picked arbitrarily, we get that for all R-modules N with support in $V(I)$, $H^1_I(N) = 0$. Now assume that for all such N and $1 \leq j < i$, we have $H^j_I(N) = 0$. From the short exact sequence $(\star)$ again, we get the exact sequence

$$H^{i-1}_I(M') \overset{\text{Ind hyp}}{=} 0 \longrightarrow H^i_I(M) \longrightarrow H^i_I\Gamma_I(Q) \overset{\text{Lemma } 2.23}{=} 0$$

Thus $H^i_I(M) = 0$. Hence, we are done by induction. $\qquad\qquad\qquad\qquad\qquad\square$

Proposition 2.25. Let $I, J \subset R$ be ideals. Then we have a natural isomorphism of functors

$$R\Gamma_{I+J} \cong R\Gamma_J \circ R\Gamma_I.$$

Proof. By Lemma 2.23, Γ_I takes injective R-modules to Γ_J-acyclic R-modules. The proposition then follows by [GM03, Theorem 3.7.1]. □

2.2: Background on D-modules

Fix $\Bbbk$ be a field of characteristic 0 and let A be a $\Bbbk$-algebra. We consider its ring of differential operators $D_{A/\Bbbk}$. While these rings are noncommutative, they exhibit good behavior and are well understood in two important cases of commutative algebra and algebraic geometry. The first case is when A is a smooth $\Bbbk$-algebra [HTT08] and the second case is when A is a ring of formal power series over k. Our primary focus in this chapter will be on the latter.

We begin this section by defining rings of differential operators. Then we turn our focus to D-modules on rings of formal power series. These D-modules share many similar properties to D-modules on smooth $\Bbbk$-algebras. We mainly use [Bj1] and [Cou95] as references, and provide details when we are unable to find a direct source.

2.2.1: Rings of Differential Operators

2.2.1.1 Derivations, Kahler Differential, and Prinicipal parts

Definition 2.26. Let M be an A-module. A $\Bbbk$-*derivation* from A to M is a $\Bbbk$-linear map $\phi : A \to M$ such that $\phi(ab) = a\phi(b) + \phi(a)b$. The set of all such derivation naturally has the structure of an A-module. Thus we can define a functor on A-modules $\mathrm{Der}_{\Bbbk}(A, -)$ which maps a module M to the module of $\Bbbk$-derivations from A to M.

In order to define the ring of differential operators, we want to generalize the notion of derivations $\mathrm{Der}_{\Bbbk}(A, -)$ to functors $D_{\Bbbk}^i(A, -) \subset \mathrm{Hom}_{\Bbbk}(A, -)$ on A-modules. Setting $D^i(A, -) = 0$ for $i < 0$, we inductively define $D_{\Bbbk}^i(A, -)$ as

$$M = A - \text{module} \longmapsto D_{\Bbbk}^i(A, M) := \{\phi \in \mathrm{End}_{\Bbbk}(A, M), [a, \phi] \in D_k^{i-1}(A, M) \text{ for all } a \in A\}$$

for $i \geq 0$. This gives an inclusion of functors $D_{\Bbbk}^i(A, -) \subset D_{\Bbbk}^{i+1}(A, -)$ with $D_{\Bbbk}^0(A, -) = \operatorname{Hom}_A(A, -)$. Note that $\operatorname{Hom}_A(A, -) + \operatorname{Der}_{\Bbbk}(A, -) = D_{\Bbbk}^1(A, -)$.

Now denote $T_{A/\Bbbk} := \operatorname{Der}_{\Bbbk}(A, A)$. When $A = \Bbbk[[x_1, \ldots, x_n]]$ is a ring of formal power series over $\Bbbk$, $T_{A/\Bbbk}$ is a free A-module of rank $\dim(A)$ with basis given by the derivations $\partial_i := \frac{\partial}{\partial x_i}$ (see [Now94, Theorem 1.5.2]). There is a natural A-linear map for all $k \geq 0$,

$$\operatorname{Sym}_A^k T_{A/\Bbbk} \longrightarrow D^k(A, A)$$

$$\xi_{i_1} \cdot \ldots \cdot \xi_{i_k} \longrightarrow \partial_{i_1}, \ldots, \partial_{i_k}.$$

This map is injective: Let $\zeta := \sum_{|\alpha|=k} f_\alpha \xi^\alpha \in \operatorname{Sym}_A^k T_{A/\Bbbk}$. Then its image in $D_{\Bbbk}^k(A, A)$ is

$$D = \sum_{|\alpha|=k} f_\alpha \partial^\alpha.$$

Let β such that $|\beta| = k$ and $f_\beta \neq 0$. Then

$$D(x^\beta) = \sum_{|\alpha|=k} f_\alpha \partial^\alpha(x^\beta) = f_\beta \partial^\alpha(x^\alpha) = \frac{1}{\prod_{i=1}^n \alpha_i!} f_\beta \neq 0.$$

Theorem 2.27. Let $A = \Bbbk[[x_1, \ldots, x_n]]$. For all $k \geq 0$, the above natural maps induce isomorphisms of A-modules

$$\operatorname{Sym}_A^k T_{A/\Bbbk} \xhookrightarrow{\quad} D^k(A, A) \xrightarrow{\quad\sim\quad} \frac{D^k(A,A)}{D^{k-1}(A,A)} \ .$$

Proof. We will first show the map is injective. It is enough to show that for all $k \geq 0$, the image of every nonzero element in $\operatorname{Sym}_A^k T_{A/\Bbbk}$ does not lie in $D^{k-1}(A, A)$. We will proceed by induction on k. The case for $k = 0$ is clear since $\operatorname{Sym}_A^0 T_{A/\Bbbk} = A = D_{\Bbbk}^0(A, A)$. Now let $k > 0$. Assume that for every $0 \leq j \leq k-1$, if $\zeta \neq 0 \in \operatorname{Sym}_A^j T_{A/\Bbbk}$, then its image in $D_{\Bbbk}^j(A, A)$ does not lie in $D_{\Bbbk}^{j-1}(A, A)$.

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Let $\zeta := \sum_{|\alpha|=k} f_\alpha \xi^\alpha \in \mathrm{Sym}_A^k T_{A/\Bbbk}$. Then its image in $D_\Bbbk^k(A,A)$ is

$$D = \sum_{|\alpha|=k} f_\alpha \partial^\alpha.$$

Since $D \neq 0$ and $D(1) = 0$, there exists i such that $[D, x_i] \neq 0$. Note that

$$[\partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}, x_i] = \begin{cases} 0 & \alpha_i = 0 \\ \alpha_i \partial_1^{\alpha_1} \ldots \partial_{i-1}^{\alpha_{i-1}} \partial_i^{\alpha_i-1} \partial_{i+1}^{\alpha_{i+1}} \ldots \partial_n^{\alpha_n} & else. \end{cases}$$

Hence, denoting $e_i = (\underbrace{0,\ldots,0}_{i-1}, 1, \underbrace{0,\ldots,0}_{n-i})$, we have

$$[D, x_i] = \sum_{|\alpha|=k-1} (\alpha_i + 1) f_{\alpha+e_i} \partial^\alpha \neq 0$$

in $D_\Bbbk^{k-1}(A,A)$. Since $[D, x_i]$ is the image of

$$\sum_{|\alpha|=k-1} (\alpha_i + 1) f_{\alpha+e_i} \xi^\alpha \ \in \mathrm{Sym}_A^{k-1} T_{A/\Bbbk},$$

it is not in $D_\Bbbk^{k-2}(A,A)$ from the induction hypothesis. Thus we can conclude that $D \notin D_\Bbbk^{k-1}(A,A)$.

To show surjectivity, it is enough to show that every differential operator $D \in D_{A/\Bbbk}$ is a sum of monomials in $\partial_1, \ldots, \partial_n$ with coefficients in A. The proof is the same as for the case of $T_{A/\Bbbk}$ in [Now94, Theorem 1.5.2].

Denote by $\mathfrak{m}$ the maximal ideal of A and set $\mathfrak{m}^i = A$ for $i \leq 0$. Recall that A has the $\mathfrak{m}$-adic topology given by the metric

$$d(f, g) = \frac{1}{2^i}$$

where $f, g \in A$ and i is such that $f - g \in \mathfrak{m}^i \setminus \mathfrak{m}^{i+1}$. The polynomial ring $\Bbbk[x_1, \ldots, x_n]$

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is dense in A with respect to the $\mathfrak{m}$-adic topology. If $D, D' \in \operatorname{End}_{\Bbbk}(A)$ are $\mathfrak{m}$-addically continuous and coincide on polynomials, then they coincide everywhere.

First, we show that every differential operator is $\mathfrak{m}$-adically continuous. Since differential operators are additive maps, it suffices to show that for any $k \geq 0$ and $i \geq 0$,

$$D^k(A, A) \cdot \mathfrak{m}^{i+k} \subseteq \mathfrak{m}^i.$$

We will show this by induction on k. For $k = 0$, it is clear that for every $\lambda \in D^0(A, A) = A$, $\lambda \cdot \mathfrak{m}^i \subseteq \mathfrak{m}^i$ for all $i \geq 0$. Let $k > 0$. Suppose that for every $D \in D^k(A, A)$, we have $D(\mathfrak{m}^{i+k}) \subseteq \mathfrak{m}^i$ for every $i \geq 0$. Let $D \in D^{k+1}(A, A)$. We will show by induction on i that $D(\mathfrak{m}^{i+k+1}) \subseteq \mathfrak{m}^i$.

For $i = 0$, it is clear that for every $f \in \mathfrak{m}^{k+1}$, $D(f) \in \mathfrak{m}^0 = R$. Let $i > 0$. Suppose that for every $f \in \mathfrak{m}^{i+k+1}$, $D(f) \in \mathfrak{m}^i$. Let $f \in \mathfrak{m}^{i+1+k+1}$. We can write $f = \sum g_j f_j'$ with $g_j \in \mathfrak{m}$ and $f_j' \in \mathfrak{m}^{i+k+1}$. Then

$$D(f) = \sum D(g_j f_j') = \sum g_j D(f_j') + \sum [D, g_j](f_j').$$

For all j, $[D, g_j] \in D^k(A, A)$ by definition. By induction hypothesis on k, $[D, g_j](f_j') \in \mathfrak{m}^{i+1}$ for all j. By induction hypothesis on i, $D(f_j') \in \mathfrak{m}^i$ so $g_j D(f_j') \in \mathfrak{m}^{i+1}$ for all j. Thus $D(f) \in \mathfrak{m}^{i+1}$. Hence we conclude that for any $k \geq 0$, for any $D \in D^k(A, A)$, we have $D(\mathfrak{m}^{i+k}) \subseteq \mathfrak{m}^i$ for all $i \geq 0$.

Our next steps are to show that every differential operator D coincides with a differential operator of the form $D' = \sum a_\alpha \partial^\alpha$, where $a_\alpha \in A$. Since differential operators are $\Bbbk$-linear, if $D(x^\alpha) = D'(x^\alpha)$ for every monomial α, then $D = D'$. Furthermore, we have the following.

Claim 2.28. Suppose $D \in D^m(A, A)$ is such that $D(x^\alpha) = 0$ for every $|\alpha| := \sum_{i=1}^n \alpha_i \leq m$. Then $D(x^\alpha) = 0$ for every monomial x^α.

Proof. Note that if $[D, x_i] = 0$ for all i, then D is A-linear: For every monomial x^α, we have

$$D(x^\alpha) = D(1) \cdot x^\alpha,$$

and by the previous paragraph, $D = D(1) \in A$.

The assertion for $m = 0$ is clear since in this case $D \in A$. Now assume that for all differential operators in $D^{m-1}(A, A)$, the claim is true. Then for every α with $|\alpha| \leq m - 1$, we have that for every $i = 1, \ldots, n$,

$$[D, x_i](x^\alpha) = D(x_i x^\alpha) - x_i D(x^\alpha) = 0.$$

Since $[D, x_i] \in D^{m-1}(A, A)$, the induction hypothesis tells us that

$$[D, x_i] = 0.$$

Since $D(1) = 0$, this implies that $D = 0$. $\qquad\square$

For any differential operator $D \in D_{A/\Bbbk}$ and $J \in \mathbb{N}^n$, define $P_{D,J} \in A$ inductively as $P_{D,0} = D(1)$ and

$$P_{D,J} = \frac{1}{\prod_{i=1}^n J_i!} \left(D(x^J) - \sum_{I < J} P_{D,I} \partial^I(x^J) \right).$$

Let $D \in D^k(A, A)$. Then we have that for every α with $|\alpha| \leq k$,

$$\left(\sum_{|I| \leq k} P_{D,I} \partial^I \right)(x^\alpha) = P_{D,\alpha} \partial^\alpha(x^\alpha) + \sum_{J < \alpha} P_{D,J} \partial^J(x^\alpha) = \left(\prod_{i=1}^n \alpha_i! \right) P_{D,\alpha} + \sum_{J < \alpha} P_{D,J} \partial^J(x^\alpha) = D(x^\alpha)$$

By Claim 2.28 we get

$$D = \sum_{|I| \leq k} P_{D,I} \partial^I.$$

$\qquad\square$

2.2.1.2 Grothendieck Definition of Rings of Differential Operators

We can construct the ring of differential operators $D_{A/\Bbbk}$ as a subring of the ring of $\Bbbk$-linear endomorphisms $\mathrm{End}_{\Bbbk}(A)$ with multiplication given by composition. Setting $D_{\Bbbk}^i(A) = 0$ for $i < 0$ and $D_{\Bbbk}^i(A) := D_{\Bbbk}^i(A, A)$ for $i \geq 0$, we obtain the following definition.

Definition 2.29. $D_{A/\Bbbk} := \bigcup D_{\Bbbk}^i(A) \subset \mathrm{End}_{\Bbbk}(A)$ is the *ring of differential operators*. The $A -$ submodules $D_{\Bbbk}^i(A) \subset D_{A/\Bbbk}$ naturally define a filtration by A-modules which respects the multiplication on $D_{A/\Bbbk}$, i.e $D_{\Bbbk}^i(A) \cdot D_{\Bbbk}^j(A) \subseteq D_{\Bbbk}^{i+j}(A)$ for all $i, j \geq 0$. This filtration is called the *order filtration*.

Hence the order filtration on $D_{A/\Bbbk}$ allows us to define the graded A-module

$$\mathrm{gr}\, D_{A/\Bbbk} = \bigoplus_{i \geq 0} \frac{D_{\Bbbk}^i(A)}{D_{\Bbbk}^{i-1}(A)}.$$

The multiplicative structure on $D_{A/\Bbbk}$ induces a multiplicative structure on $\mathrm{gr}\, D_{A/\Bbbk}$. Although $D_{A/\Bbbk}$ itself is noncommutative, $\mathrm{gr}\, D_{A/\Bbbk}$ is a commutative $A -$ algebra. This is because the obstruction to commutativity between differential operators of order d_1 and d_2, which is given by their lie bracket, lies in $F_{d_1+d_2-1} D_{A/\Bbbk}$ and so it vanishes when we pass to the graded ring. Hence, $\mathrm{gr}\, D_{A/\Bbbk}$ is in fact a graded A-algebra.

As a consequence of the Proposition 2.27, we get the following.

Proposition 2.30. Suppose that $A = \Bbbk[[x_1, \ldots, x_n]]$. Then $D_{A/\Bbbk}$ is generated by A and $T_{A/\Bbbk}$ and we have a canonical isomorphism of graded $A -$ algebras

$$\mathrm{gr}\, D_{A/\Bbbk} \cong Sym_A^{\bullet} T_{A/\Bbbk}.$$

In particular, $\mathrm{gr}\, D_{A/\Bbbk}$ is noetherian.

2.2.2: D-modules over rings of formal power series

Throughout this section, $\Bbbk$ will be a field of characteristic 0 and $\widehat{A}_n := \Bbbk[[x_1, \ldots, x_n]]$. We put $\widehat{D}_n := D_{\widehat{A}_n/\Bbbk}$ and $\partial_i := \frac{\partial}{\partial x_i}$ from now on. $D^{\bullet}_{\Bbbk}(\widehat{A}_n)$ is the order filtration on $\widehat{D}_n$. In this case we have $\operatorname{gr} \widehat{D}_n = \operatorname{Sym}^{\bullet}_{\widehat{A}_n} T_{\widehat{A}_n/\Bbbk} = \widehat{A}_n[\xi_1, \ldots, \xi_n]$ where $\xi_1, \ldots, \xi_n$ give a basis for $T_{\widehat{A}_n/\Bbbk}$.

Let M be an $\widehat{A}_n-$module. Giving a structure of a left (resp. right) $\widehat{D}_n$-module structure on M comes down to extending the corresponding $\widehat{A}_n -$ module structure $\widehat{A}_n \to \operatorname{End}_{\Bbbk}(M)$ to a ring homomorphism (resp. anti-homorphism) $\widehat{D}_n \to \operatorname{End}_{\Bbbk}(M)$.

Example 2.31. Since, by construction, we have $\widehat{D}_n \subset \operatorname{End}_{\Bbbk}(\widehat{A}_n)$, $\widehat{A}_n$ naturally has a left $\widehat{D}_n$-module structure.

Example 2.32 (Localization of a $\widehat{D}_n$-module)**.** Let M be a left $\widehat{D}_n$-module and let $f \in \widehat{A}_n$. We can extend the action of $\widehat{D}_n$ on M to M_f using the product rule from calculus: for every $m \in M$ and integer $k > 0$, we put

$$\partial_i \cdot \frac{1}{f^k} m = -k \frac{\partial_i(f)}{f^{k+1}} \cdot m + \frac{1}{f^k} \partial_i \cdot m.$$

One can check that with this definition M_f becomes a left $\widehat{D}_n$-module.

Example 2.33.   (a) Let $I \subset R$ be an ideal and $f_1, \ldots, f_k \in I$ be generators of I. For any left $\widehat{D}_n$-module M, the boundary maps d^i in the Čech complex $C(f_1, \ldots, f_k; M)$ from 2.7 are $\widehat{D}_n$-linear. Hence the local cohomology modules $H^i_I(M)$ are $\widehat{D}_n$-modules for all i. Furthermore, their $\widehat{D}_n$-module structure does not depend on the choice of generators $f_1, \ldots, f_k$ of I.

   (b) Let $I \subset J \subset R$ be ideals. Let $f_1, \ldots, f_k, f_{k+1} \ldots, f_n \in J$ be generators of J such that $f_1, \ldots, f_k \in I$ are generators of I. For any left $\widehat{D}_n$-module M, the natural maps

$$\rho : \check{C}(f_1, \ldots, f_n; M) \longrightarrow \check{C}(f_1, \ldots, f_k; M)$$

as in Definition 2.16 are $\widehat{D}_n$-linear. Hence, the induced natural maps on local coho-
mology module

$$H^i_J(M) \longrightarrow H^i_I(M)$$

are $\widehat{D}_n$-linear. By Remark 2.17, these natural maps on local cohomology modules
coincide with those in Definition 2.14.

We say M is a *coherent* (left) $\widehat{D}_n$-module if it is finitely generated over $\widehat{D}_n$. The following
are some examples of coherent left $\widehat{D}_n$-modules.

Example 2.34.　(a) Multiplication on the left equips $\widehat{D}_n$ with the the structure of a co-
herent left $\widehat{D}_n$-module. Let $I \subset \widehat{D}_n$ be a left ideal. Then I and $\frac{\widehat{D}_n}{I}$ are also coherent
left $\widehat{D}_n$-modules.

(b) Any quotient of a coherent (left) $\widehat{D}_n$-module is a coherent (left) $\widehat{D}_n$-module and since
$\widehat{D}_n$ is noetherian, any (left) $\widehat{D}_n$-submodule of a coherent (left) $\widehat{D}_n$-module is a coherent
(left) $\widehat{D}_n$-module.

(c) As a left $\widehat{D}_n$-module, $\widehat{A}_n \cong \frac{\widehat{D}_n}{\widehat{D}_n \cdot (\partial_1, \ldots, \partial_n)}$ is coherent.

(d) Let $E = H^n_{\mathfrak{m}}(\widehat{A}_n)$. We have the presentation

$$\frac{\widehat{D}_n}{\widehat{D}_n \cdot (x_1, \ldots, x_n)} \cong \Bbbk[\partial_1, \ldots, \partial_n] \xrightarrow{\ \sim\ } E$$

$$1 \longmapsto \frac{1}{x_1 \cdots x_n}$$

so E is a coherent left $\widehat{D}_n$-module.

Theorem 2.35. [Lyu93, Theorem 2.4] Let M be a coherent $\widehat{D}_n-$module. Then M has
finitely many associated primes in $\widehat{A}_n$.

Corollary 2.36. Let M be a coherent $\widehat{D}_n-$module. Then $\mathrm{Supp}(M)$ is closed in $\mathrm{Spec}(\widehat{A}_n)$.

2.2.2.1 Good filtrations and the characteristic cycle

Definition 2.37. Let M be a coherent $\widehat{D}_n$-module. A *good filtration* on M is a filtration $F_\bullet M$ by finitely generated $\widehat{A}_n$ − modules such that $F_i M = 0$ for $i << 0$, $\bigcup_i F_i M = M$,

$$D_{\Bbbk}^i(\widehat{A}_n) \cdot F_j M \subseteq F_{i+j} M \text{ for all } i, j \in \mathbb{Z},$$

and $\mathrm{gr}_F M := \bigoplus_{i \in \mathbb{Z}} \frac{F_i M}{F_{i-1} M}$ is finitely generated as a $\mathrm{gr}\,\widehat{D}_n$-module.

Proposition 2.38. [Bj1, Proposition 2.6.1] A left $\widehat{D}_n$-module is coherent if and only if it has a good filtration.

Definition 2.39. Let M be a coherent left $\widehat{D}_n$-module and let $F_\bullet M$ be a good filtration on it. We call $\mathbf{Ch}(M) = \mathrm{Supp}(\mathrm{gr}_F M)$ the characteristic variety of M. Denoting by $I(\mathbf{Ch}(M))$ the set of prime ideals corresponding to the irreducible components of $\mathbf{Ch}(M)$, we define by

$$CC(M) = \sum_{\mathfrak{p} \in I(\mathbf{Ch}(M))} \ell_{(\mathrm{gr}_F \widehat{D}_n)_\mathfrak{p}} (\mathrm{gr}_F M)_\mathfrak{p} \cdot [V(\mathfrak{p})]$$

the *characteristic cycle* of M.

Proposition 2.40. [Bj1, Lemma 2.6.2] The characteristic cycle is independent of the good filtration.

Example 2.41. Recall that if we consider on $\widehat{D}_n$ the order filtration, then we naturally have

$$\mathrm{gr}\,\widehat{D}_n \cong \widehat{A}_n[\xi_1, \ldots, \xi_n]$$

where $\xi_1, \ldots, \xi_n$ are a basis for $T_{\widehat{A}_n/\Bbbk}$ as a $\widehat{A}_n$-module. Then $\mathrm{Spec}\left(\mathrm{gr}\,\widehat{D}_n\right) \cong \mathrm{Spec}(\widehat{A}_n) \times \mathbb{A}_k^n$.

(a) The order filtration on $\widehat{D}_n$ is a clearly a good filtration. We have $CC(\widehat{D}_n) = [\mathrm{Spec}(\widehat{A}_n) \times \mathbb{A}_k^n]$.

(b) Define $F_i \widehat{A}_n = \widehat{A}_n$ for all $i \geq 0$ and $F_i \widehat{A}_n = 0$ for $i < 0$. We have that $\mathrm{gr}_F \widehat{A}_n = \dfrac{\mathrm{Sym}^\bullet_{\widehat{A}_n} T_{\widehat{A}_n/\Bbbk}}{\mathrm{Sym}^{\geq 1}_{\widehat{A}_n} T_{A/\Bbbk}}$, so $F_\bullet \widehat{A}_n$ is a good filtration. We have $CC(\widehat{A}_n^{\oplus k}) = k \cdot [\mathrm{Spec}(\widehat{A}_n) \times \{0\}]$ for any $k > 0$.

(c) Let $E = H^n_{\mathfrak{m}}(\widehat{A}_n)$. The order filtration on $\widehat{D}_n$ induces a good filtration on E

$$F_i E = \bigoplus_{\substack{i_1,\ldots,i_n \geq 1 \\ i_1+\ldots+i_n \leq i+n}} \Bbbk \cdot \frac{1}{x_1^{i_1} \ldots x_n^{i_n}}$$

via the map in Example 2.34(d). E is supported entirely on the closed point $\{0\} = V(x_1, \ldots, x_n)$. We have $CC(E^{\oplus k}) = k \cdot [\mathbf{T}_{\{0\}}]$ where $\mathbf{T}_{\{0\}} = \{0\} \times \mathbb{A}^n_k$ is the normal bundle associated to the embedding of the closed point into $\mathrm{Spec}(\widehat{A}_n)$.

2.2.2.2 Functors on the category of coherent $\widehat{D}_n$-modules

In this section we discuss some functors on (left) $\widehat{D}_n$-modules. The material here is standard for algebras of finite type over $\Bbbk$ (see [HTT08] or [Cou95]), but we give details since some of the results needed do not appear in the literature in our setting over the formal power series rings. First, we recall the following lemmas regarding complete local rings.

Lemma 2.42 (Cohen Structure Theorem). [Eis95, Proposition 10.16] Let $(R, \mathfrak{m})$ be a complete regular local ring of dimension n with residue field $\Bbbk = R/\mathfrak{m}$. If R contains a field, then $R \cong \Bbbk[[x_1 \ldots, x_n]]$ and the isomorphism can be chosen to send the variables x_i to any given regular system of parameters in R.

Lemma 2.43. Let $(R, \mathfrak{m})$ be a complete local ring. Then any quotient ring of R is a complete local ring with respect to $\mathfrak{m}$.

Proof. This follows immediately from [Eis95, Theorem 7.2(a)]. $\square$

As a consequence of the previous two lemmas, let $I \subset \widehat{A}_n$ is an ideal, with $k = \mathrm{ht}(I)$, such that $\widehat{A}_n/I$ is regular, then the surjective ring homomorphism $\widehat{A}_n \twoheadrightarrow \widehat{A}_n/I$ is isomorphic to

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the ring homomorphism

$$\widehat{A}_n = \Bbbk[[x_1,\ldots,x_n]] \cong \widehat{A}_{n-k}[[x_{n-k+1},\ldots,x_n]] \twoheadrightarrow \widehat{A}_{n-k} = \Bbbk[[x_1,\ldots,x_{n-k}]] \qquad (\star)$$

which sends x_i to x_i for $1 \leq n-k$ and to 0 for $i > n-k$. Hence, to define the following functors on coherent left $\widehat{D}_n$-modules, we will consider only ring homomorphisms from $\widehat{A}_n$ of this form, and also think of $\widehat{A}_n$ as formal power series ring in k variables over $\widehat{A}_{n-k}$.

Definition 2.44 (Direct Image). Let $\pi : \widehat{A}_n \twoheadrightarrow \widehat{A}_{n-k}$ as in $(\star)$. Recall that we have a $D_{\Bbbk[[x_{n-k+1},\ldots,x_n]]/\Bbbk}$-module structure on

$$\Bbbk[\partial_{n-k+1},\ldots,\partial_n] \cong \frac{D_{\Bbbk[[x_{n-k+1},\ldots,x_n]]/\Bbbk}}{D_{\Bbbk[[x_{n-k+1},\ldots,x_n]]/\Bbbk} \cdot (x_{n-k+1},\ldots,x_n)}$$

given by

$$x_j \cdot \partial^\alpha = \begin{cases} -\alpha_j \partial_1^{\alpha_1} \ldots \partial_{j-1}^{\alpha_{j-1}} \partial_j^{\alpha_j-1} \partial_{j+1}^{\alpha_{j+1}} \ldots \partial_n^{\alpha_n} & \alpha_j > 0 \\ \\ 0 & else \end{cases}$$

and

$$\partial_j \cdot \partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_{j-1}^{\alpha_{j-1}} \partial_j^{\alpha_j+1} \partial_{j+1}^{\alpha_{j+1}} \ldots \partial_n^{\alpha_n}$$

for every $j = n-k+1,\ldots n$ and $\alpha \in \mathbb{N}$. In particular, note that for every $D \in \Bbbk[\partial_{n-k+1},\ldots,\partial_n]$, there is an integer $L \geq 0$ such that for every $j_1,\ldots,j_k > L$,

$$x_{n-k+1}^{j_1} \ldots x_n^{j_n} \cdot D = 0.$$

Let M be a left $\widehat{D}_{n-k}$-module and consider the $\Bbbk$-vector space

$$M \otimes_\Bbbk \Bbbk[\partial_{n-k+1},\ldots,\partial_n].$$

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For very $f \in \widehat{A}_{n-k}$ and integers $j_1, \ldots, j_k \geq 0$, define

$$f x_{n-k+1}^{j_1} \ldots x_n^{j_k} \cdot (m \otimes_{\Bbbk} D) = (f \cdot m) \otimes_{\Bbbk} (x_{n-k+1}^{j_1} \ldots x_n^{j_k} \cdot D)$$

for every $m \in M$ and $D \in \Bbbk[\partial_{n-k+1}, \ldots, \partial_n]$. For every $g \in \widehat{A}_n$,

$$g = \sum_{j_1, \ldots j_k \geq 0} f_{j_1, \ldots, j_k} x_{n-k+1}^{j_1} \ldots x_n^{j_k}$$

for $f_{j_1, \ldots, j_k} \in \widehat{A}_{n-k}$. Let $L \geq 0$ such that $x_{n-k+1}^{j_1} \ldots x_n^{j_k} \cdot D = 0$ for all $j_1, \ldots, j_k \geq L$. Then

$$g \cdot (m \otimes_{\Bbbk} D) = \left(\sum_{L \geq j_1, \ldots, j_k \geq 0} x_{n-k+1}^{j_1} \ldots x_n^{j_k} \right) \cdot (m \otimes_{\Bbbk} D).$$

Hence, we obtain a $\widehat{A}_n$-module structure on $M \otimes_{\Bbbk} \Bbbk[\partial_{n-k+1}, \ldots, \partial_n]$. We can further extend this naturally to a $\widehat{D}_n$-module structure by

$$f \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n} \cdot (m \otimes_{\Bbbk} D) = f \cdot \left(\partial_1^{\alpha_1} \ldots \partial_{n-k}^{\alpha_{n-k}} \cdot m \otimes_{\Bbbk} \partial_{n-k+1}^{\alpha_{n-k+1}} \ldots \partial_n^{\alpha_n} \cdot D \right)$$

for $f \in \widehat{A}_n$, $\alpha_1, \ldots, \alpha_n \geq 0$, $m \in M$, and $D \in \Bbbk[\partial_{n-k+1}, \ldots, \partial_n]$.

Then we define the *direct image* functor

$$\pi_! = (-) \otimes_{\Bbbk} \Bbbk[\partial_1, \ldots, \partial_n]$$

from left $\widehat{D}_{n-k}$-modules to left $\widehat{D}_n$-modules with support in $V(x_{n-k+1}, \ldots, x_n)$. This functor is exact and preserves the coherence of left $\widehat{D}_n$-modules.

Remark 2.45. Let $0 \leq l \leq n$ and $0 \leq k \leq n - l$. Suppose we have surjective ring homomorphisms $\pi_1 : \widehat{A}_n \twoheadrightarrow \widehat{A}_{n-l}$ and $\pi_2 : \widehat{A}_{n-l} \twoheadrightarrow \widehat{A}_{n-l-k}$ as in $(\star)$. Then $(\pi_2 \circ \pi_1)_! \cong \pi_{1,!} \circ \pi_{2,!}$.

Definition 2.46 (Dimension of Module)**.** Let M be a coherent left $\widehat{D}_n$-module and F a good filtration on M. Recall that $\mathbf{Ch}(M) = \operatorname{Supp}(gr_F(M))$ is independent of F due to

Proposition 2.40. Then the dimension of M is $d(M) := \dim \mathbf{Ch}(M)$, and it is independent of F.

Lemma 2.47. Let $\pi : \widehat{A}_n \twoheadrightarrow \widehat{A}_{n-k}$ as $(\star)$. If M be is a coherent left $\widehat{D}_{n-k}$-module, then $d\left(\pi_! M\right) = d(M) + k$.

Proof. The proof is essentially the same as in [Cou95, Chapter 13], where it is shown for the case of polynomial rings over $\Bbbk$. By Remark 2.45, it is enough to show it for $k = 1$.

Let F be a good filtration on M. Then a good filtration on $M \otimes_{\Bbbk} \Bbbk[\partial_n]$ is given by

$$
\Gamma_l \left(M \otimes_{\Bbbk} \Bbbk[\partial_n]\right) = \bigoplus_{i+j=l} \left(F_i M \otimes_{\Bbbk} \sum_{k \leq j} \Bbbk \cdot \partial_n^k \right)
$$

Under this filtration, we see that

$$
F_{i-1} M \otimes_{\Bbbk} \sum_{k \leq j} \Bbbk \cdot \partial_n^k \subseteq F_i M \otimes_{\Bbbk} \sum_{k \leq j} \Bbbk \cdot \partial_n^k
$$

and

$$
F_i M \otimes_{\Bbbk} \sum_{k \leq j-1} \Bbbk \cdot \partial_n^k \subseteq F_i M \otimes_{\Bbbk} \sum_{k \leq j} \Bbbk \cdot \partial_n^k
$$

so

$$
\mathrm{gr}_\Gamma^l \left(M \otimes_{\Bbbk} \Bbbk[\partial_n^k]\right) = \bigoplus_{i+j=l} \frac{F_i M \otimes_{\Bbbk} \sum_{k \leq j} \Bbbk \cdot \partial_n^k}{F_{i-1} M \otimes_{\Bbbk} \sum_{k \leq j} \Bbbk \cdot \partial_n^k + F_i M \otimes_{\Bbbk} \sum_{k \leq j-1} \Bbbk \cdot \partial_n^k}.
$$

Recall that we naturally have

$$
\mathrm{gr}\, \widehat{D}_n = \widehat{A}_n[\xi_1, \ldots, \xi_n],
$$

where $\xi_1, \ldots, \xi_n$ are a basis for $T_{\widehat{A}_n/\Bbbk}$ as a $\widehat{A}_n$-module. For every $l \geq 0$ and $i + j = l$, the $\Bbbk$-linear map

$$
F_i M \otimes_{\Bbbk} \sum_{k \leq j} \Bbbk \cdot \partial_n^k \xrightarrow{\eta_{ij}} \mathrm{gr}_F^i M \otimes_{\Bbbk} \sum_{k \leq j} \Bbbk \cdot \xi_n^k
$$

$$
m \otimes_{\Bbbk} \partial_n^k \longmapsto \overline{m} \otimes_{\Bbbk} \xi_n^k
$$

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is surjective and has kernel given by

$$F_{i-1}M \otimes_{\Bbbk} \sum_{k \le j} \Bbbk \cdot \partial_n^k + F_i M \otimes_{\Bbbk} \sum_{k \le j-1} \Bbbk \cdot \xi_n^k.$$

Thus we get an isomorphism of $\Bbbk$-vector spaces

$$\mathrm{gr}_\Gamma^l \left(M \otimes_{\Bbbk} \Bbbk[\partial_n] \right) \longrightarrow \bigoplus_{i+j=l} \mathrm{gr}_F^i M \otimes_{\Bbbk} \xi_n^j$$

which gives us an isomorphism of $\Bbbk$-vector spaces

$$\mathrm{gr}_\Gamma \left(M \otimes_{\Bbbk} \Bbbk[\partial_n] \right) \xrightarrow{\ \eta\ } \mathrm{gr}_F M \otimes_{\Bbbk} \Bbbk[\xi_n] \ .$$

Since $\mathrm{gr}_F M$ is a module over $\mathrm{gr}\, \widehat{D}_{n-1} = \widehat{A}_{n-1}[\xi_1, \ldots, \xi_{n-1}]$, we get that $\mathrm{gr}_F M \otimes_{\Bbbk} \Bbbk[\xi_n]$ is naturally a module over the ring

$$\widehat{A}_{n-1}[\xi_1, \ldots, \xi_n] \cong \mathrm{gr}\, \widehat{D}_{n-1} \otimes_{\Bbbk} k[\xi_n]$$

where the natural action is given by $(f \otimes_{\Bbbk} \xi_n^i) \cdot (m \otimes_{\Bbbk} \xi_n^j) = (f \cdot m) \otimes_{\Bbbk} \xi^{i+j}$ for $f \in \widehat{A}_{n-1}$, $m \in \mathrm{gr}_F M$, and $i, j \ge 0$. Its support in $\mathrm{Spec}\left(\widehat{A}_{n-1}[\xi_1, \ldots, \xi_n] \right) = \mathrm{Spec}(\mathrm{gr}\, \widehat{D}_{n-1}) \times_{\Bbbk} \mathrm{Spec}\left(\Bbbk[\xi_n] \right)$ is

$$\mathrm{Supp}\left(\mathrm{gr}_F M \otimes_{\Bbbk} \Bbbk[\xi_n] \right) = \mathrm{Supp}(\mathrm{gr}_F M) \times_{\Bbbk} \mathrm{Spec}\left(\Bbbk[\xi_n] \right)$$

which has dimension $d(M) + 1$. The quotient ring homomorphism

$$\mathrm{gr}\, \widehat{D}_n = \widehat{A}_n[\xi_1, \ldots, \xi_n] \twoheadrightarrow \widehat{A}_{n-1}[\xi_1, \ldots, \xi_n]$$

induced by π extends the $\widehat{A}_{n-1}[\xi_1, \ldots, \xi_n]$-module structure on $\mathrm{gr}_F M \otimes_{\Bbbk} \Bbbk[\xi_n]$ to a $\mathrm{gr}\, \widehat{D}_n$-module structure. One can check that with this $\mathrm{gr}\, \widehat{D}_n$-module structure, η is then $\mathrm{gr}\, \widehat{D}_n$-

linear. Hence η is an isomorphism of gr $\widehat{D}_n$-modules, and we get that

$$d\left(\pi_! M\right) = \dim \operatorname{Supp}\left(\operatorname{gr}_\Gamma\left(M \otimes_{\Bbbk} \Bbbk[\partial_n]\right)\right) = d(M) + 1.$$

$\square$

Let $\pi : \widehat{A}_{n+1} = \Bbbk[[x_1, \ldots, x_n, t]] \twoheadrightarrow \widehat{A}_n = \Bbbk[[x_1, \ldots, x_n]]$ be the ring homomorphism sending x_i to x_i and t to 0. For any $\Bbbk$-linear map $\phi : \widehat{A}_n \to \widehat{A}_n$, we can lift it naturally to the $\Bbbk$-linear map $\widetilde{\phi} : \widehat{A}_{n+1} \to \widehat{A}_{n+1}$ that sends every $g \in \widehat{A}_{n+1}$ to

$$\widetilde{\phi}(g) = \sum_{i \geq 0} \phi(f_i) t^i$$

where f_i are such that $g = \sum_{i \geq 0} f_i t^i$ for $f_i \in \widehat{A}_n$. This gives us a natural inclusion of rings

$$\operatorname{End}_{\Bbbk}(\widehat{A}_n) \lhook\joinrel\longrightarrow \operatorname{End}_{\Bbbk}(\widehat{A}_{n+1})$$

$$\phi \longmapsto \widetilde{\phi}.$$

It is easy to check that this inclusion maps $\widehat{A}_n$ to $\widehat{A}_{n+1}$ and $\widehat{D}_n$ into $\widehat{D}_{n+1}$ by sending x_i to x_i and ∂_{x_i} to ∂_{x_i} for all $1 \leq i \leq n$, giving us a ring homomorphism

$$\widehat{D}_n \hookrightarrow \widehat{D}_{n+1}.$$

Let M be a left $\widehat{D}_{n+1}$-module, not necessarily coherent. Then via the ring homomorphism of rings of differential operators above, we get a structure of a $\widehat{D}_n$-module on M. Since all the ∂_{x_i} commute with t, the $\widehat{A}_{n+1}$-linear map

$$M \xrightarrow{t\cdot} M$$

is in fact $\widehat{D}_n$-linear. Thus $\ker(M \xrightarrow{x_n\cdot} M)$ and $M/x_n M$ naturally have the structure of a

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$\widehat{D}_n$-module induced by the $\widehat{D}_n$-module structure on M (which, in turn, was induced by the $\widehat{D}_{n+1}$-module structure on M). We make the following observations.

Example 2.48. $\frac{\widehat{A}_{n+1}}{t\widehat{A}_{n+1}} \cong \widehat{A}_n$ as $\widehat{D}_n$-modules.

Example 2.49. Let $0 \le k \le n$ and $i_1, \ldots, i_k \ge 0$. We have short exact sequences of $\widehat{A}_{n+1}$-modules

$$0 \longrightarrow (\widehat{A}_{n+1})_{x_{i_1}\ldots x_{i_k}} \xrightarrow{\ t\cdot\ } (\widehat{A}_{n+1})_{x_{i_1}\ldots x_{i_k}} \longrightarrow (\widehat{A}_n)_{x_{i_1}\ldots x_{i_k}} \longrightarrow 0$$

and isomorphisms $(\widehat{A}_{n+1})_{x_{i_1}\ldots x_{i_k}t} \xrightarrow{\ t\cdot\ } (\widehat{A}_{n+1})_{x_{i_1}\ldots x_{i_k}t}$. Hence we get a short exact sequence of Čech complexes

$$\check{C}^\bullet(x_{n-k+1},\ldots,x_n,t;\widehat{A}_{n+1}) \rightarrowtail \xrightarrow{\ t\cdot\ } \check{C}^\bullet(x_{n-k+1},\ldots,x_n,t;\widehat{A}_{n+1}) \twoheadrightarrow \check{C}^\bullet(x_{n-k+1},\ldots,x_n;\widehat{A}_n).$$

As observed above, the $\widehat{D}_{n+1}$-module structure on the left two objects in each of the above short exact sequences of $\widehat{A}_{n+1}$-modules (resp. the short exact sequence of Čech complexes) induces $\widehat{D}_n$-module structures on them. With these induced $\widehat{D}_n$-module structures, they are short exact sequences of $\widehat{D}_n$-modules (resp. a short exact sequence of complexes of $\widehat{D}_n$-modules).

(a) The induced $\widehat{D}_n$-module structure on $(\widehat{A}_n)_{x_{i_1}\ldots x_{i_k}}$ from the short exact sequences co-incides with the natural $\widehat{D}_n$-module structure on $(\widehat{A}_n)_{x_{i_1}\ldots x_{i_k}}$ from localizing $\widehat{A}_n$ as in Example 2.32.

(b) Let $I = (x_{n-k+1},\ldots,x_n,t)$. Taking cohomology on the short exact sequence of Čech complexes, we get the exact sequence of $\widehat{D}_n$-modules

$$H_I^k(\widehat{A}_{n+1}) = 0 \longrightarrow H_{I\widehat{A}_n}^k(\widehat{A}_{n-1}) \longrightarrow H_I^{k+1}(\widehat{A}_{n+1}) \xrightarrow{\ t\cdot\ } H_I^{k+1}(\widehat{A}_{n+1}) \longrightarrow H_{I\widehat{A}_n}^{k+1}(\widehat{A}_n) = 0.$$

As a result of the previous observations, we see that the induced $\widehat{D}_n$-module structure

on

$$H^k_{I\widehat{A}_n}(\widehat{A}_n) \cong \ker\left(H^{k+1}_I(\widehat{A}_{n+1}) \xrightarrow{t\cdot} H^{k+1}_I(\widehat{A}_{n+1})\right)$$

coincides with the one defined in Example 2.33(a).

Definition 2.50 (Shifted Inverse Image)**.** Let $\pi : \widehat{A}_n \twoheadrightarrow A_{n-k}$ as in $(\star)$. If $k = 1$, we define the *shifted inverse image* functor

$$\pi^! := \ker((-) \xrightarrow{x_n\cdot} (-))$$

from left $\widehat{D}_n$-modules to left $\widehat{D}_{n-k}$-modules. For $k > 1$, we can factor $\pi = \pi_k \circ \ldots \circ \pi_1$ where, for $1 \leq i \leq k$,

$$\pi_i : \widehat{A}_{n-i+1} \twoheadrightarrow \widehat{A}_{n-i}$$

are ring homomorphisms as in $(\star)$. We define the *shifted inverse image* functor

$$\pi^! = \pi^!_k \circ \ldots \circ \pi^!_1.$$

Lemma 2.51. Let $\pi : \widehat{A}_n \twoheadrightarrow \widehat{A}_{n-1}$ as in $(\star)$. Then we have a natural isomorphism of functors $\Phi : \Gamma_{x_n} \cong \pi_! \pi^!$.

Proof. Let M be a left $\widehat{D}_n -$ module. Define the increasing chain of $\widehat{A}_n$-submodules,

$$M_k := \{m \in M, x_n^{k+1}m = 0\} \subseteq \Gamma_{x_n}(M)$$

for all $k \geq -1$. Then $M_0 = \ker(M \xrightarrow{x_n\cdot} M)$. Observe the following.

Observation 2.52. Suppose $u \in M$ is such that $u \in M_k \setminus M_{k-1}$, i.e $x_n^i u = 0$ if and only if $i \geq k + 1$. Then we have $x_n^i \partial_n \cdot u = \partial_n x_n^i \cdot u - ix_n^{i-1} \cdot u = 0$ if $i > k + 1$ and $\neq 0$ if $i = k + 1$ (and hence $\neq 0$ if $i < k + 1$). In other words,

$$\partial_n \cdot (M_k \setminus M_{k-1}) \subseteq (M_{k+1} \setminus M_k).$$

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Observation 2.53. Applying Observation 2.52 iteratively, we can see that if $u \in M_0$ and $u \neq 0$, then $\partial_n^k u \in M_k \setminus M_{k-1}$ for all $k \geq 1$. Thus, $x_n^i \partial_n^k \cdot u = 0$ for all $i > k$ and $x_n^k \partial_n^k \cdot u \neq 0$.

Observation 2.54. Note that for every $l \geq k \geq 0$,

$$[x_n^k, \partial_n^l] = (-1)^k \frac{l!}{(l-k)!} \partial_n^{l-k}$$

in $\widehat{D}_n$. On the other hand,

$$x_n^k \cdot \partial_n^l = (-1)^k \frac{l!}{(l-k)!} \partial_n^{l-k}$$

in $k[\partial_n] \cong \frac{\widehat{D}_{k[[x_n]]/k}}{\widehat{D}_{k[[x_n]]/k} \cdot x_n}$.

By Observation 2.53 above, we can define the map

$$\pi_! \pi^! M = M_0 \otimes_k k[\partial_n] \xrightarrow{\quad \Phi_M \quad} \Gamma_{x_n}(M)$$

$$m \otimes f, \text{ where } m \in M_0, f \in k[\partial_n] \longmapsto f \cdot m.$$

It is straightforward to check that this map is natural in M and additive. Since

- Φ_M is additive,

- For $k > \alpha_n + l$, the left hand side is 0 by definition of the $\widehat{D}_n$-module structure on $\pi_! M_0$, and the right hand side is 0 by Observation 2.53, and

- every element $g \in \widehat{D}_n$ is a finite sum of elements of the form

$$\left(\sum_{k \geq 0} f_k x_n^k \right) \partial^\alpha \text{ for } f_k \in \widehat{A}_{n-1},$$

to check that Φ_M is in fact $\widehat{D}_n$-linear, it is enough to check that for every $f \in \widehat{A}_{n-1}$, $\alpha \in \mathbb{N}$, $l + \alpha_n \geq k \geq 0$, and $m \in M_0$,

$$\Phi_M \left(f x_n^k \partial^\alpha \cdot (m \otimes_k \partial_n^l) \right) = \left(f \partial_1^{\alpha_1} \dots \partial_{n-1}^{\alpha_{n-1}} \right) \left(x_n^k \partial_n^{\alpha_n + l} \right) \cdot m.$$

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In the case $k = 0$, this is clear. For $k > 0$, starting on the right hand side, we compute

$$\left(f\partial_1^{\alpha_1}\ldots\partial_{n-1}^{\alpha_{n-1}}\right)\left(x_n^k\partial_n^{\alpha_n+l}\right)\cdot m \stackrel{x_n^k\cdot m=0}{=} \left(f\partial_1^{\alpha_1}\ldots\partial_{n-1}^{\alpha_{n-1}}\right)\left(x_n^k\partial_n^{\alpha_n+l}-\partial_n^{\alpha_n+l}x_n^k\right)\cdot m$$

$$= \left(f\partial_1^{\alpha_1}\ldots\partial_{n-1}^{\alpha_{n-1}}\right)[x_n^k,\partial_n^{\alpha_n+l}]\cdot m \stackrel{\text{Observation } 2.54}{=} \Phi_m(f\partial_1^{\alpha_1}\ldots\partial_{n-1}^{\alpha_{n-1}}\cdot m \otimes_{\Bbbk} x_n^k\cdot\partial_n^{\alpha_n+l})$$

$$= \Phi_M\left(fx_n^k\partial^\alpha\cdot(m\otimes_{\Bbbk}\partial_n^l)\right).$$

Therefore, the natural maps Φ_M give rise to a natural transformation of functors on $\widehat{D}_n$-modules

$$\Phi : \pi_!\pi^! \to \Gamma_{x_n}.$$

We need to show that Φ is an isomorphism. To do so, we proceed as in [Cou95, Thm 17.2.4].

Φ_M is injective: Let $x \in M_0 \otimes_{\Bbbk} \Bbbk[\partial_n]$ be nonzero. We write $x = \sum_{i=0}^k u_i \otimes_{\Bbbk} \partial_n^i$ for $u_i \in M_0$ such that $u_k \neq 0$. Suppose $\Phi_M(x) = \sum_{i=0}^k \partial_n^i \cdot u_i = 0$. Then $0 = x_n^k \cdot \Phi_M(x) = \sum_{i=0}^k x_n^k\partial_n^i \cdot u_i \stackrel{\text{Observation } 2.53}{=} x_n^k\partial_n^k u_k \stackrel{\text{Observation } 2.53}{\neq} 0$, which is a contradiction.

Φ_M is surjective: We have $\bigcup_{k\geq 0} M_k = \Gamma_{x_n}(M)$, and by Observation 2.52, we have $\partial_n \cdot M_k \subseteq M_{k+1}$, so it is enough to show that in fact $M_k + \partial_n \cdot M_k = M_{k+1}$.

Let $m \in M_{k+1}$ and $m' = x_n\partial_n m + (k+2)m$. We have that

$$x_n^{k+2}\partial_n \cdot m = \partial_n x_n^{k+2}\cdot m - (k+2)x_n^{k+1}m = -(k+2)x_n^{k+1}m.$$

Hence we have $x_n^{k+1}\cdot m' = 0$ so $m' \in M_k$. On the other hand, $x_n m \in M_k$ and $\partial_n x_n m = x_n\partial_n \cdot m + m$. Hence we have that

$$m = \frac{1}{k+1}\left(m' - \partial_n x_n m\right),$$

so $m \in M_k + \partial_n \cdot M_k$.

Hence Φ_M is an isomorphism for every $\widehat{D}_n -$ module M, so Φ is an isomoprhism of functors. $\square$

Let $I \subset \widehat{A}_n$ be an ideal. The full subcategory of $\widehat{D}_n$-modules with support in $V(I)$ is an abelian category. As observed in Definition 2.44, if $\pi : \widehat{A}_n \to \widehat{A}_{n-k}$ is a ring homomorphism as in $(\star)$ and M is an $\widehat{D}_{n-k}$-module, then $\pi_! M$ has support in $V(x_{n-k+1}, \ldots, x_n)$. Thus the functor $\pi_!$ factors through the subcategory of $\widehat{D}_n$-modules with support in $V(x_{n-k+1}, \ldots, x_n)$. We have the following equivalence.

Proposition 2.55 (Kashiwara's Equivalence)**.** Let $\pi : \widehat{A}_n \twoheadrightarrow A_{n-k}$ as in $(\star)$, and denote $I = (x_{n-k+1}, \ldots, x_n)$. Then $\pi_!$ induces an equivalences of categories between left $\widehat{D}_{n-k} -$ modules and left $\widehat{D}_n -$ modules with support in $V(I)$. Its inverse functor is given by the restriction of $\pi^!$ to the full abelian subcategory of left $\widehat{D}_n -$ modules with support in $V(I)$.

Proof. We will show the case $k = 1$ first. Suppose that M is a left $\widehat{D}_n -$ module with support in $V(x_n)$. By Lemma 2.51, $\pi_! \pi^! M \cong \Gamma_{x_n}(M) \overset{\text{Proposition } 2.2}{=} M$. Hence $\pi_! \pi^! \cong id$ on the category of left $\widehat{D}_n$-modules with support in $V(x_n)$.

On the other hand, suppose that M is a left $\widehat{D}_{n-1} -$ module. Observe that we naturally have an inclusion of $\widehat{D}_{n-1}$-modules

$$M \subset \ker \left(M \otimes_{\mathscr{k}} \mathscr{k}[\partial_n] \xrightarrow{x_n \cdot} M \otimes_{\mathscr{k}} \mathscr{k}[\partial_n] \right),$$

thus defining a natural transformation of functors $id \to \pi^! \pi_!$. To show that this natural transformation is an isomorphism, we need to show that the inclusion above is in fact an equality for every M. By the same observation in the proof of Lemma 2.51, we have that $x_n^i \cdot (m \otimes_{\mathscr{k}} \partial_n^k) = x_n^i \partial_n^k \cdot (m \otimes_{\mathscr{k}} 1) = 0$ if and only if $i > k$ for all $m \in M$. Let $m = \sum_{i=0}^k m_i \otimes_{\mathscr{k}} \partial_n^i$ such that $m_k \neq 0$ be an arbitrary nonzero element of $\pi_! M$. Then $x_n^k m = x^k(m_k \otimes_{\mathscr{k}} \partial_n^k) \neq 0$. Hence $x_n m = 0$ if and only if $k = 0$, so $m \in M$. Thus the inclusion is an equality, and we have $id \cong \pi^! \pi_!$. This completes the proof for $k = 1$.

Note that if M is a left $\widehat{D}_n$-module with support in $V(J)$ where $J \subset \widehat{A}_n$ is an ideal, then $\pi^! M$ has support in $V(J\widehat{A}_{n-1})$ since it is a $\widehat{A}_n$-submodule of M. Since $\pi_!$ and $\pi^!$ give an equivalence of categories between $\widehat{D}_{n-1}$-modules and $\widehat{D}_n$-modules with support in $V(x_n)$,

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and $\pi^!$ takes $\widehat{D}_n$-modules with support in $V(J)$ to $\widehat{D}_{n-1}$-modules with support in $V(J\widehat{A}_{n-1})$ for every ideal $J \subset \widehat{A}_n$, $\pi_!$ and $\pi^!$ restrict to an equivalence of categories between $\widehat{D}_n$-modules with support in $V(J)$ and $\widehat{D}_{n-1}$-modules with support in $V(J\widehat{A}_{n-1})$ for every ideal $J \subset \widehat{A}_n$ such that $x_n \in J$.

Now let $k > 1$, and assume that the proposition is true for $k - 1$. Denote the ring homomorphisms

$$\pi_1 : \widehat{A}_{n-1} \twoheadrightarrow \widehat{A}_{n-k}$$

and

$$\pi_2 : \widehat{A}_n \twoheadrightarrow \widehat{A}_{n-1}$$

as in ($\star$). Consider the following diagram of functors,

$$\widehat{D}_n\text{-modules with support in } V(I) \underset{\pi_2^!}{\overset{\pi_{2,!}}{\rightleftarrows}} \widehat{D}_{n-1}\text{-modules with support in } V(I\widehat{A}_{n-1})$$

$$\pi_! \searrow \qquad \pi_1^! \downarrow \uparrow \pi_{1,!}$$

$$\pi^! \searrow \widehat{D}_{n-k}\text{-modules},$$

where we are restricting the shifted inverse image functors and the direct image functors to the subcategories above. We have $\pi = \pi_1 \circ \pi_2$, so $\pi_! \cong \pi_{2,!} \circ \pi_{1,!}$ and $\pi^! \cong \pi_1^! \circ \pi_2^!$. Furthermore, by the previous remark for the case $k = 1$, we have that the horizontal functors give an equivalence, and by the induction hypothesis, the vertical functors also give an equivalence since $I\widehat{A}_{n-1} = (x_{n-k+1}, \ldots, x_{n-1}) = \ker(\pi_1)$. Thus we get natural isomorphism of functors

$$\pi_! \pi^! \cong \pi_{2,!} \circ \pi_{1,!} \circ \pi_1^! \circ \pi_2^! \cong id$$

and

$$\pi^! \pi_! \cong \pi_1^! \circ \pi_2^! \circ \pi_{2,!} \circ \pi_{1,!} \cong id.$$

Hence we are done. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 2.56. Given Kashiwara's Equivalence and the fact that $\pi_!$ preserves coherence of

$\widehat{D}_n$-modules, $\pi_!$ restricts to an equivalence of categories between coherent left $\widehat{D}_{n-k}-$modules and coherent left $\widehat{D}_n-$modules with support in $V(x_{n-k+1}, \ldots, x_n)$. Its inverse is given by the restriction of $\pi^!$ to the full abelian subcategory of coherent left $\widehat{D}_n-$modules with support in $V(x_{n-k+1}, \ldots, x_n)$.

2.2.2.3 Holonomic $\widehat{D}_n$-modules

Coherent $\widehat{D}_n$-modules have many good properties, such as the ones we saw above. However, they are unfortunately not preserved under localization. Here is an example to illustrate this:

Example 2.57. Consider $\widehat{A}_1 = \mathcal{k}[[x]]$ and its ring of differential operators $\widehat{D}_1$. We will show that $\widehat{D}_1$-module $M = \widehat{D}_1[x^{-1}]$ is not finitely generated as a left $\widehat{D}_1$-module.

First note that if $f, g \in M$ such that

$$f = \sum_{i=0}^{m} f_i \partial^i,$$

where $f_m \neq 0$, and

$$g = \sum_{i=0}^{m-1} g_i \partial^i,$$

then $f \neq g$: take $k >> 0$ such that $x^k f_i, x^k g_j \in \widehat{D}_1$ for all $m \geq i \geq 0$, $m - 1 \geq j \geq 0$, and denote $\overline{f_i} := x^k f_i$, $\overline{g_j} := x^k g_j$ in $\widehat{A}_1$ and

$$\overline{f} := x^k f = \sum_{i=0}^{m} \overline{f_i} \partial^i$$

and

$$\overline{g} := x^k g = \sum_{i=0}^{m-1} \overline{g_i} \partial^i.$$

Then $\overline{g} \in D_{\mathcal{k}}^{m-1}(\widehat{A}_1)$, but we saw in the proof of injectivity for Theorem 2.27 that $\overline{f} \notin D_{\mathcal{k}}^{m-1}(\widehat{A}_1)$. Hence we cannot have $f = g$.

Now consider the submodules $M_i = \widehat{D}_1 \cdot \frac{1}{x^i} \subset \widehat{D}_1[x^{-1}]$ for $i \geq 0$. We have $M_i \subset M_{i+1}$ for all $i \geq 0$ and every element $m \in M_i$ is of the form $m = P \cdot \frac{1}{x^i}$ for some $P \in \widehat{D}_1$. Now, for any $k \geq 0$ and $i > 0$,

$$\partial \cdot \frac{1}{x^i} \partial^k = \frac{1}{x^i} \partial^{k+1} - i \frac{1}{x^{i+1}} \partial^k.$$

This means that for any $m \geq 0$, $P \cdot \frac{1}{x^i}$ has top degree of ∂ being m if and only if $P = \sum_{i=0}^{m} f_i \partial^i$ such that $f_m \neq 0$. Thus if $P \cdot \frac{1}{x^i} = \frac{1}{x^{i+1}}$, then $P = f_0$, but then we cannot have $P \cdot \frac{1}{x^i} = \frac{1}{x^{i+1}}$. Hence we must have $M_i \neq M_{i+1}$ for all $i \geq 0$. Thus, M is not finitely generated.

This means that even if M is a coherent $\widehat{D}_n$-module, it is not clear if its local cohomology modules are coherent. There is a smaller category of $\widehat{D}_n$-modules called *holonomic* $\widehat{D}_n$-modules which possesses all the properties that coherent $\widehat{D}_n$-modules have and are also preserved under localizations. In this section, we recall the definition of this category. We first begin by giving a property of coherent (left) $\widehat{D}_n$-modules..

Theorem 2.58 (Bernstein's Inequality). [Bj1, Cor 2.7.2] For any coherent nonzero left $\widehat{D}_n - $ module M, $d(M) \geq n$.

This inequality leads us to the following definition.

Definition 2.59. Let M be a coherent left (resp. right) $\widehat{D}_n - $ module. M is called *holonomic* if $M = 0$ or $d(M) = n$.

Proposition 2.60. [Bj1, Thm 2.7.13] Holonomic $\widehat{D}_n$-modules have finite length as $\widehat{D}_n$-modules.

Definition 2.61. A holonomic $\widehat{D}_n$-module of length 1 is called a *simple* $\widehat{D}_n$-module.

Example 2.62. (a) Just as with coherent $\widehat{D}_n$-modules, quotients and submodules of holonomic $\widehat{D}_n$-modules are holonomic.

(b) $\widehat{A}_n$ is simple as a $\widehat{D}_n - $ module.

Holonomic $\widehat{D}_n$-modules have many good properties. We list a few in what follows.

Proposition 2.63 (Localization). [Bj1, Thm 3.4.1] Let M be a holonomic left $\widehat{D}_n$-module and $f \in \widehat{A}_n$. Then M_f is a holonomic $\widehat{D}_n$-module.

Proposition 2.64. Let M be a holonomic left $\widehat{D}_n$-module and $I \subset \widehat{A}_n$. Then the local cohomology modules $H_I^i(M)$ are holonomic.

Proof. Using the Čech complex description of $R\Gamma_I(M)$, this proposition follows from Proposition 2.63 since submodules and quotient modules of holonomic are holonomic. $\square$

Proposition 2.65. Let $\pi : \widehat{A}_n \twoheadrightarrow \widehat{A}_{n-k}$ as in ($\star$). Let M be a holonomic $\widehat{D}_{n-k}$-module. Then $\pi_! M$ is holonomic. In particular, $\pi_!$ restricts to an equivalence of categories between holonomic left $\widehat{D}_{n-k}$-modules and holonomic left $\widehat{D}_n$-modules with support in $V(x_{n-k+1)}, \ldots, x_n)$.

Proof. By Kashiwara's equivalence, $\pi_! M$ is a coherent $\widehat{D}_n$-module. Hence, we can apply Lemma 2.47 and get that $d(\pi_! M) = d(M) + k = n - k + k = n$. Thus $\pi_! M$ is a holonomic $\widehat{D}_n$-module. $\square$

Remark 2.66. Adding to the previous proposition, Kashiwara's Equivalence furthermore implies that M is a holonomic $\widehat{D}_{n-k}$-module of length d if and only if $\pi_! M$ is a $\widehat{D}_n$-module of length d.

Proposition 2.67. Let $\pi : \widehat{A}_n \twoheadrightarrow \widehat{A}_{n-k}$ as in ($\star$). Let M be a holonomic $\widehat{D}_n$-module. Then $\pi^! M$ is holonomic.

Proof. By Proposition 2.51 and 2.64, $\Gamma_I(M) = \pi_! \pi^! M$ is holonomic. By Kashiwara's equivalence, we therefore have that $\pi^! M$ is holonomic. $\square$

Example 2.68. (a) Let $\pi : \widehat{A}_n \twoheadrightarrow \widehat{A}_{n-1}$ as in ($\star$). Let $1 \leq k \leq n$ and $I = (x_{n-k+1}, \ldots, x_n)$. By Example 2.49, we have $H_{I\widehat{A}_{n-1}}^{k-1}(\widehat{A}_{n-1}) \cong \pi^! H_I^k(\widehat{A}_n)$ as $\widehat{D}_{n-1}$-modules. Since $H_I^k(\widehat{A}_n)$ has support in $V(x_n)$, by Kashiwara's Equivalence, we therefore have

$$\pi_! H_{I\widehat{A}_{n-1}}^{k-1}(\widehat{A}_{n-1}) \cong H_I^k(\widehat{A}_n)$$

as $\widehat{D}_n$-modules. In particular, if $k = 1$, since $\widehat{A}_{n-1} = \Gamma_{x_n \widehat{A}_{n-1}}(\widehat{A}_{n-1})$, we have $\pi_! \widehat{A}_{n-1} \cong H^1_{x_n}(\widehat{A}_n)$.

(b) Let $\pi : \widehat{A}_n \twoheadrightarrow \widehat{A}_{n-k}$ as in $(\star)$ and $I = (x_{n-k+1}, \ldots, x_n)$. For $1 \leq i \leq k$, denote the ring homomorphisms

$$\pi_i : \widehat{A}_{n-i+1} \twoheadrightarrow \widehat{A}_{n-i}$$

as in $(\star)$. We have $\pi = \pi_k \circ \ldots \circ \pi_1$ and $I\widehat{A}_{n-i} = (x_{n-k+1}, \ldots, x_{n-i})\widehat{A}_{n-i}$ for all i. Then we get

$$\pi_! \widehat{A}_{n-k} \cong \pi_{1,!} \ldots \pi_{k,!} \widehat{A}_{n-k} \cong \pi_{1,!} \ldots \pi_{k-1,!} H^1_{x_{n-k+1} \widehat{A}_{n-k+1}}(\widehat{A}_{n-k+1}) \cong \ldots \cong H^k_I(\widehat{A}_n).$$

Recall from Corollary 2.36 that $\mathrm{Supp}(M)$ is closed in $\mathrm{Supp}(\widehat{A}_n)$. We have the following lemma.

Lemma 2.69. Let $M \neq 0$ be a simple $\widehat{D}_n$ − module. Then $\mathrm{Supp}(M)$ is irreducible and the only associated prime of M in $\widehat{A}_n$ is the unique minimal element of $\mathrm{Supp}(M)$.

Proof. Let $I \subset \widehat{A}_n$ be a radical ideal such that $V(I) = \mathrm{Supp}(M)$. Since $\widehat{A}_n$ is noetherian and $M \neq 0$, M has an associated prime. Thus, it is enough to show that if $\mathfrak{p}$ is an associated prime of M, then $I = \mathfrak{p}$.

Let $\mathfrak{p} \subset \widehat{A}_n$ be an associated prime of M. By definition, $\mathfrak{p} = \mathrm{Ann}(m)$ for some $m \in M$. If $M_{\mathfrak{p}} = 0$, then there is $f \notin \mathfrak{p}$ such that $f \cdot m = 0$, which is a contradiction. Hence we must have $M_{\mathfrak{p}} \neq 0$, so $\mathfrak{p} \in \mathrm{Supp}(M)$. Thus $\mathfrak{p} \supseteq I$.

Furthermore, $\Gamma_{\mathfrak{p}}(M) \neq 0$. Since M is simple, we must therefore have $\Gamma_{\mathfrak{p}}(M) = M$. Let $\mathfrak{q} \subseteq \widehat{A}_n$ be another prime ideal such that $\mathfrak{q} \in \mathrm{Supp}(M)$. If $\mathfrak{q} \not\supseteq \mathfrak{p}$, then $\exists f \in \mathfrak{p} \setminus \mathfrak{q}$. For all $m \in M$, $\exists n > 0$ such that $f^n m = 0$ since $\Gamma_{\mathfrak{p}}(M) = M$. Then $M_f = 0$, which implies $M_{\mathfrak{q}} = 0$. This is a contradiction. Hence we must have that $\mathfrak{q} \supseteq \mathfrak{p}$, and hence $I = \bigcap_{\mathfrak{q} \supseteq I} \mathfrak{q} \supseteq \mathfrak{p}$. This implies that $\mathfrak{p} = I$. $\qquad\square$

Corollary 2.70. Let $I \subset \widehat{A}_n$ an ideal such that $\widehat{A}_n/I$ is regular. Let $d = \operatorname{ht}(I)$. Then $H_I^d(\widehat{A}_n)$ is a simple $\widehat{D}_n$-module with support in $V(I)$.

Proof. The surjection $\widehat{A}_n \twoheadrightarrow \widehat{A}_n/I$ is isomorphic to a ring homomorphism

$$\pi : \widehat{A}_n \twoheadrightarrow \widehat{A}_{n-d} \cong \widehat{A}_n/I$$

as in $(\star)$. By Example 2.68, $H_I^d(\widehat{A}_n) \cong \pi_! \widehat{A}_{n-d}$ as $\widehat{D}_n$-modules. By Remark 2.66, $H_I^d(\widehat{A}_n) = \pi_! \widehat{A}_{n-d}$ is simple as a $\widehat{D}_n - \text{module}$ since $\widehat{A}_{n-d}$ is simple as a $\widehat{D}_{n-d} - \text{module}$. $\qquad\square$

2.3: A Result on a Conjecture of Lyubeznik

Recall from Definition 2.14 that for any ideals $I \subset J \subset R$ we have natural transformations of functors on $R - \text{modules}$

$$H_J^i(-) \to H_I^i(-)$$

for all i. Furthermore, in the case when R is a power series ring, these natural transformations induce natural transformations of functors on $D_R - \text{modules}$ (see Example 2.33(b)).

We begin this section by proving the following result concerning these natural transformations, which will be the key tool in proving the main results Theorem 2.87 and Theorem 2.89. The result is identical to a result of Lewis [Lew22, Cor 4.5], which generalizes a result of Hellus [Hel01, Cor 2]. However, we use different methods from her proof and the result is slightly more precise.

Theorem 2.71. Let R be a noetherian ring, and let $I \subset R$ be an ideal with $\operatorname{ht}(I) = d$. Then there is $f \in R$ such that $\operatorname{ht}(I + (f)) > d$ and the natural transformation

$$H_{I+(f)}^i(-) \longrightarrow H_I^i(-)$$

is an isomorphism for $i > d + 1$ and gives an exact sequence

$$0 \longrightarrow H^1_{I+(f)} H^d_I(-) \longrightarrow H^{d+1}_{I+(f)}(-) \longrightarrow H^{d+1}_I(-) \longrightarrow 0.$$

We will need the following lemmas for the proof of Theorem 2.71.

Lemma 2.72. [Eis95, Cor 10.7] Let $(R, \mathfrak{m})$ be a local noetherian ring of dimension n. Then there exist $x_1, \ldots, x_n \in \mathfrak{m}$ such that $\sqrt{(x_1, \ldots, x_n)} = \mathfrak{m}$.

Lemma 2.73. Let R be a noetherian ring and $I \subset R$ an ideal with $\mathrm{ht}(I) = d$ such that $\mathfrak{p} = \sqrt{I}$ is a prime ideal. Then there exists $f \in R \setminus \mathfrak{p}$ and an ideal $J \subset R_f$ generated by d elements such that $\sqrt{J} = \sqrt{I} \cdot R_f$.

Proof. By Lemma 2.72, we have $x_1, \ldots, x_d \in \mathfrak{p}$ such that $\mathfrak{p} R_\mathfrak{p} = \sqrt{(x_1, \ldots, x_d) R_\mathfrak{p}}$. Let $M = \frac{\mathfrak{p}}{\sqrt{(x_1, \ldots, x_d)}}$. Then $M_\mathfrak{p} = \frac{\mathfrak{p} R_\mathfrak{p}}{\sqrt{(x_1, \ldots, x_d) R_\mathfrak{p}}} = 0$, so we can find $f \in R \setminus \mathfrak{p}$ such that $M_f = 0$. Hence, $\sqrt{(x_1, \ldots, x_d) R_f} = \mathfrak{p} R_f$. Then this f and $J = (x_1, \ldots, x_d) R_f$ satisfy the conclusion of the lemma. $\square$

Lemma 2.74. Let R be a noetherian ring and $I \subset R$ an ideal with $\mathrm{ht}(I) = d$. Then there is a radical ideal $J \supset \sqrt{I}$ such that $\mathrm{ht}(J) > d$ and for all $\mathfrak{p} \notin V(J)$ we have that

$$H^i_{IR_\mathfrak{p}}(R_\mathfrak{p}) = 0$$

for all $i > d$.

Proof. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ be the minimal primes containing I. By Proposition 2.3 and Lemma 2.73, for every $i = 1, \ldots, m$, we have $f_i \in R \setminus \mathfrak{p}_i$ and ideal $J_i \subset R_{f_i}$ such that $\mathfrak{p}_i R_{f_i} = \sqrt{J_i}$, so $R\Gamma_{\mathfrak{p}_i R_{f_i}}(R_{f_i}) = R\Gamma_{J_i}(R_{f_i})$ and J_i is generated by $\mathrm{ht}(\mathfrak{p}_i)$ elements.

For all $i = 1, \ldots, m$, define the open set

$$U_i := D(f_i) \setminus \left(\bigcup_{j \neq i} V(\mathfrak{p}_j) \right) \ni \mathfrak{p}_i.$$

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We have $U_i \cap V(\mathfrak{p}_j) = \emptyset$ for all $j \neq i$ and $U_i \cap V(\mathfrak{p}_i)$ is a nonempty subset of $D(f_i)$. Now consider the open set

$$U = \bigcup_{\mathrm{ht}(\mathfrak{p}_i)=d} U_i.$$

We deduce that

$$U \cap V(\mathfrak{p}_i) = \begin{cases} U_i \cap V(\mathfrak{p}_i) \neq \emptyset & \text{if } \mathrm{ht}(\mathfrak{p}_i) = d \\ \\ \emptyset & \text{otherwise} \end{cases}$$

is contained in $D(f_i)$ for all i, and

$$U \cap V(I) = \bigcup_{i=1}^{m} (U \cap V(\mathfrak{p}_i)) = \bigcup_{\mathrm{ht}(\mathfrak{p}_i)=d} (U_i \cap V(\mathfrak{p}_i)) \overset{U_i \cap V(\mathfrak{p}_j)=\emptyset \text{ for } i \neq j}{=} \bigsqcup_{\mathrm{ht}(\mathfrak{p}_i)=d} U_i \cap V(\mathfrak{p}_i)$$

is a disjoint union. Consequently we get the following:

1. For every prime ideal $\mathfrak{p} \in V(I)$ such that $\mathrm{ht}(\mathfrak{p}) = d = \mathrm{ht}(I)$, we must have that $\mathfrak{p} = \mathfrak{p}_i$ for some i. Hence $\mathfrak{p} \in U \cap V(\mathfrak{p}_i) \subseteq U \cap V(I)$. Therefore for every $\mathfrak{p} \in V(I) \setminus U$, we must have $\mathrm{ht}(\mathfrak{p}) > d$.

2. If $\mathfrak{p} \notin V(I)$, then $H^q_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}}) = 0$ for all q.

3. Let $\mathfrak{p} \in V(I) \cap U$. Then there is i with $\mathrm{ht}(\mathfrak{p}_i) = d$ such that $\mathfrak{p} \in V(\mathfrak{p}_i)$, and for all $j \neq i$, $\mathfrak{p} \notin V(\mathfrak{p}_j)$. Hence $\sqrt{I}R_{\mathfrak{p}} = \mathfrak{p}_i R_{\mathfrak{p}}$, so

$$H^q_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}}) \overset{\text{Proposition 2.3}}{=} H^q_{\sqrt{I}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) = H^q_{\mathfrak{p}_i R_{\mathfrak{p}}}(R_{\mathfrak{p}})$$

for all q. Since $\mathfrak{p} \in U \cap V(\mathfrak{p}_i) \subseteq D(f_i)$, we get $\mathfrak{p}_i R_{\mathfrak{p}} = \sqrt{J_i}R_{\mathfrak{p}}$ and $J_i R_{\mathfrak{p}}$ is generated by $\mathrm{ht}(\mathfrak{p}_i) = d$ elements. Thus we have

$$H^q_{\mathfrak{p}_i R_{\mathfrak{p}}}(R_{\mathfrak{p}}) = H^q_{J_i R_{\mathfrak{p}}}(R_{\mathfrak{p}})$$

and by Proposition 2.9, for all $q > d$, $H^q_{J_i R_{\mathfrak{p}}}(R_{\mathfrak{p}}) = 0$.

By taking $J \subset R$ to be the radical ideal such that $V(J) = V(I) \setminus U$, we get the conclusion of the lemma. $\qquad \square$

Now we prove Theorem 2.71.

Proof. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ denote the minimal primes containing I. By Lemma 2.74, we get an ideal $J \supseteq \sqrt{I}$ such that $\mathrm{ht}(J) > d$ and for all $\mathfrak{p} \notin V(J)$, we have $H^i_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}}) = 0$ for all $i > d$. Since $\mathrm{ht}(J) > d$, $J \not\subset \mathfrak{p}_i$ for any i such that $\mathrm{ht}(\mathfrak{p}_i) = d$. Hence we get that $J \not\subset \bigcup_{\mathrm{ht}(\mathfrak{p}_i)=d} \mathfrak{p}_i$ by the Prime Avoidance Lemma. Let $f \in J \setminus \bigcup_{\mathrm{ht}(\mathfrak{p}_i)=d} \mathfrak{p}_i$. Then we have the following:

(a) For every $\mathfrak{p} \notin V(f)$, we have $H^i_I(R) \otimes_R R_{\mathfrak{p}} \overset{\mathrm{Lemma\ 2.12}}{=} H^i_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}}) = 0$ for all $i > d$ since $V(f) \supseteq V(J)$.

(b) For all i with $\mathrm{ht}(\mathfrak{p}_i) = d$, $I + (f) \not\subseteq \mathfrak{p}_i$ since $f \notin \mathfrak{p}_i$. If $\mathfrak{p} \supseteq I + (f)$ is a prime ideal, then $\mathfrak{p} \supseteq \mathfrak{p}_i$ for some i, and we either have $\mathrm{ht}(\mathfrak{p}_i) > d$ or $\mathrm{ht}(\mathfrak{p}_i) = d$ with strict inclusion $\mathfrak{p} \supsetneq \mathfrak{p}_i$. In other words, $\mathrm{ht}(I + (f)) > d$.

Since we have found $f \in R$ such that $\mathrm{ht}(I+(f)) > d$, we are left with showing the assertions on the natural transformations of functors $H^i_{I+(f)}(-) \to H^i_I(-)$ for $i \geq d+1$.

Let M be any $R-$module. Recall that we have a local cohomology spectral sequence for the ideals I, (f), and the R-module M:

$$E_2^{p,q} = H^p_f H^q_I(M) \Rightarrow_p H^{p+q}_{(f)+I}(M).$$

By Proposition 2.9, $H^q_f(M) = 0$ for all $q > 1$. Furthermore, by (a), we get that for all $q > d$, $H^q_I(M)$ has support in $V(f)$. Thus Lemma 2.24 tells us that for all $q > d$, we have $H^p_f H^q_I(M) = 0$ for $p > 0$, and $H^0_f H^q_I(M) = H^q_I(M)$. Hence we have $E_2^{p,q} = H^p_f H^q_I(M) = 0$ if $p \neq 0,1$ OR $p = 1$ and $q \neq d$. Then for all $r > 2$, we inductively get

(a) For $p < 0$, $E_r^{p,q} = 0$.

(b) For $p - r + 1 \geq 0$: $p > 1$ in this case since $r > 2$, so $E_{r-1}^{p,q} = 0$. Hence $E_r^{p,q} = E_2^{p,q} = 0$.

(c) For $p > 0$ and $p - r + 1 < 0$: The differential $d_{r-1}^{p-r+1,q+r-2}$ maps from $E_{r-1}^{p-r+1,q+r-2} = 0$ to $E^{p,q}$. On the other hand, since $p > 0$, $p + r > 2$, so the differential $d_{r-1}^{p,q}$ maps from $E_{r-1}^{p,q}$ to $E_{r-1}^{p+r-1,q-(r-1)+1} = 0$. Hence $E_r^{p,q} = E_2^{p,q} = H_f^p H_I^q(M)$ in this case.

(d) (a) and (b) imply $E_r^{p,q} = E_2^{p,q} = 0$ if $p \neq 0, 1$ OR $p = 1$ and $q > d$.

This means that the spectral sequence degenerates at E_2. Thus we get canonical isomorphisms

$$H_{I+(f)}^q(M) \xrightarrow{\sim} E_\infty^{q,0} = H_I^q(M)$$

for all $q > d + 1$ and we have a canonical short exact sequence

$$0 \longrightarrow E_\infty^{1,d} = \left(H_f^1 \circ H_I^d\right)(M) \longrightarrow H_{I+(f)}^{d+1}(M) \longrightarrow E_\infty^{0,d+1} = H_I^{d+1}(M) \longrightarrow 0 \ .$$

Example 2.22 shows the canonical isomorphisms and the surjective map in the short exact sequence above are the same as those arising from the natural transformation of functors $H_{I+(f)}^i(-) \to H_I^i(-)$ in Definition 2.14. This completes the proof of the theorem. $\qquad\square$

We next discuss two theorems which follow as a consequence of Theorem 2.71. They both hold for a large class of noetherian rings containing a field of characteristic 0. The first theorem describes the dimension of the closure of the support of the local cohomology modules for ideals of pure height. We will need the following facts and lemmas.

Definition 2.75 (Going Down Property)**.** Let $\phi : A \to B$ be a ring homomoprhisms. It satisfies the *going down property* if for any prime ideal $\mathfrak{q}'$ in B and prime ideal $\mathfrak{p} \subseteq \phi^{-1}(\mathfrak{q}')$ in A, there is a prime ideal $\mathfrak{q} \subseteq \mathfrak{q}'$ in B such that $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$.

Lemma 2.76. [Eis95, Lemma 10.11] Flat ring homomorphisms satisfy the going down property.

Lemma 2.77. Let $\phi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local ring homomorphism of local rings. Then it is faithfully flat.

Proof. It is a well known fact that ϕ is faithfully flat if and only if it is flat and every prime ideal in R lifts to a prime ideal in S. Since ϕ is flat, it has the going down property. Since every prime ideal in R is contained in $\mathfrak{m}$, the going down property implies that every prime ideal in R lifts to a prime ideal in S. $\qquad\square$

Lemma 2.78. Let $(R, \mathfrak{m})$ be a local noetherian ring. Then $R \to \widehat{R}$ is faithfully flat.

Proof. By [Eis95, Theorem 7.2(b)], $\widehat{R}$ is flat over R. Since $R \to \widehat{R}$ is a flat local ring homomorphism of local rings, it is faithfully flat. $\qquad\square$

Lemma 2.79. [Eis95, Corollary 10.14] Regular local rings are integral domains.

Lemma 2.80. Let $(R, \mathfrak{m})$ be a local noetherian ring, and let $I \subset R$ be an ideal. Then $\widehat{R/I} \cong \widehat{R}/I\widehat{R}$.

Proof. By [Eis95, Theorem 7.2(a)], we get

$$\widehat{R}/I\widehat{R} = R/I \otimes_R \widehat{R} \cong \widehat{R/I}.$$

$\qquad\square$

Lemma 2.81. [Gro64, Lemma 17.3.8.1] Let $(R, \mathfrak{m})$ be a regular local ring. Then $\widehat{R}$ is regular.

Corollary 2.82. Let $(R, \mathfrak{m})$ be a local noetherian ring, and let $J \subset R$ be an ideal such that R/J is a regular ring. Then $J\widehat{R} \subset \widehat{R}$ is an ideal such that $\widehat{R}/J\widehat{R}$ is regular.

Proof. By Lemma 2.80, $\widehat{R}/J\widehat{R} = \widehat{R/J}$. By Lemma 2.81, $\widehat{R/J}$ is regular since R/J is regular. $\qquad\square$

Lemma 2.83. Let $(R, \mathfrak{m})$ be a local noetherian ring and $I \subset R$ be an ideal. Let

$$\phi : R \to \widehat{R}$$

be the canonical homomorphism. Let $I \subset \mathfrak{p} \subset R$ be a prime ideal and $\mathfrak{q} \subset \widehat{R}$ be a minimal prime ideal containing $\mathfrak{p}\widehat{R}$. Then $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$, and if $\mathfrak{p}$ is not a minimal prime ideal containing I, then $\mathfrak{q}$ is not a minimal prime ideal containing $I\widehat{R}$.

Proof. Let $\mathfrak{p}' = \phi^{-1}(\mathfrak{q}) \supseteq \mathfrak{p}$. Since ϕ is flat, by Lemma 2.76, it satisfies the going down property. Hence, we have a prime ideal $\mathfrak{q}' \subseteq \mathfrak{q}$ in $\widehat{R}$ such that $\phi^{-1}(\mathfrak{q}') = \mathfrak{p}$. In particular, we have $\mathfrak{p}\widehat{R} \subseteq \mathfrak{q}' \subseteq \mathfrak{q}$. Then we must have $\mathfrak{q}' = \mathfrak{q}$. This shows the first part of the lemma.

Suppose $\mathfrak{p}$ is not a minimal prime ideal containing I. Arguing by contradiction, suppose that $\mathfrak{q}$ is a minimal prime ideal containing $I\widehat{R}$. Since $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$ is not a minimal prime ideal containing I, we have a prime ideal $\mathfrak{p}' \subsetneq \mathfrak{p}$ in R such that $I \subseteq \mathfrak{p}'$. By the going down property again, we have $\mathfrak{q}' \subsetneq \mathfrak{q}$ such that $\phi^{-1}(\mathfrak{q}') = \mathfrak{p}'$. In particular, $\mathfrak{q}' \supseteq \mathfrak{p}'\widehat{R} \supseteq I\widehat{R}$. This is a contradiction to the assumption that $\mathfrak{q}$ is a minimal prime containing $I\widehat{R}$. Hence $\mathfrak{q}'$ is not a minimal prime ideal containing $I\widehat{R}$. $\qquad\square$

Lemma 2.84. Let R be a catenary ring of dimension n, $I \subset R$ be an ideal of pure height $d < n$ (i.e all the minimal primes containing I have height d), and $f \in R$ such that $I+(f) \neq R$ and $\operatorname{ht}(I + (f)) > d$. Then $I + (f)$ is of pure height $d + 1$.

Proof. Let $\mathfrak{p}$ be a minimal prime containing $I + (f)$. Since $\operatorname{ht}(I + (f)) > d$ and I has pure height d, f is not contained in any of the minimal prime ideals containing I. Then there is a minimal prime ideal $\mathfrak{q}$ containing I such that $\mathfrak{p} \supseteq \mathfrak{q} + (f) \supsetneq \mathfrak{q}$. The length of a maximal chain of prime ideals between $\mathfrak{q}$ and $\mathfrak{p}$ is $\operatorname{ht}(\mathfrak{p}/\mathfrak{q})$. Hence by the property of being catenary, $\operatorname{ht}(\mathfrak{p}) = \operatorname{ht}(\mathfrak{q}) + \operatorname{ht}(\mathfrak{p}/\mathfrak{q})$. Note that $\mathfrak{p}/\mathfrak{q}$ is a minimal prime ideal containing the principal ideal $f \cdot R/\mathfrak{q}$ and that $R/\mathfrak{q}$ is a domain. Thus by Krull's Height Theorem, $\operatorname{ht}(\mathfrak{p}/\mathfrak{q}) = 1$. Therefore we have $\operatorname{ht}(\mathfrak{p}) = \operatorname{ht}(\mathfrak{q}) + 1 = d + 1$. $\qquad\square$

Definition 2.85. Let S be a noetherian ring. We say that it is *J-0* if there is a non-empty open set $U \subset \operatorname{Spec}(S)$ such that $S_{\mathfrak{p}}$ is regular for all $\mathfrak{p} \in U$.

Remark 2.86. Suppose that S is reduced. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ be the minimal prime ideals of S, and suppose that for every i, $S/\mathfrak{p}_i$ is J-0. That means that for every i, there is an open

subset U_i such that $U_i \cap V(\mathfrak{p}_i) \neq \emptyset$ and $S/\mathfrak{p}_i$ is regular at every point in U_i. Note that for every $\mathfrak{p} \in U_i$ such that $\mathfrak{p} \notin V(\mathfrak{p}_j)$ for all $j \neq i$, we have $\mathfrak{p}_i S_\mathfrak{p} = 0$ and so $(S/\mathfrak{p}_i)_\mathfrak{p} = S_\mathfrak{p}$. Take

$$U = \bigsqcup_{i=1}^{m} \left(U_i \setminus \bigcup_{j \neq i} V(\mathfrak{p}_j) \right).$$

Then U is a *dense* open subset of $\mathrm{Spec}(S)$ such that S is regular at every point in U.

Theorem 2.87. Let R be a regular ring of dimension n containing $\mathbb{Q}$ such that for every prime ideal $\mathfrak{p} \subset R$, the quotient ring $R/\mathfrak{p}$ is J-0. Let $I \subset R$ be an ideal of pure height d. Then $\mathrm{codim}\left(\overline{\mathrm{Supp}\left(H_I^i(R) \right)} \right) \geq i+1$ for all $i > d$.

Proof. We prove this by descending induction on d. In the base case where $d = n$, $H_I^i(R) = 0$ for all $i > n$ by Proposition 2.4, so its support is the empty set, and the result follows trivially.

Now let $d < n$. Assume the theorem holds for every ideal $J \subset R$ of pure height with $\mathrm{ht}(J) > d$. By Theorem 2.71, there is $f \in R$ such that f is not in any minimal prime containing I and if $J = I + (f)$, then,

$$H_J^i(R) \cong H_I^i(R) \text{ for all } i > d+1$$

and

$$H_J^{d+1}(R) \longrightarrow H_I^{d+1}(R) \qquad\qquad (\star)$$

is surjective. If $J = R$, then $H_J^i(R) = 0$ for all i so $H_I^i(R) = 0$ for all $i > d$. Then $\mathrm{Supp}(H_I^i(R)) = \emptyset$. Now assume $J \neq R$. Since R is regular, it is catenary. By Lemma 2.84, J is of pure height $d+1$. By the induction hypothesis, we have that

$$\mathrm{codim}\left(\overline{\mathrm{Supp}\left(H_I^i(R) \right)} \right) = \mathrm{codim}\left(\overline{\mathrm{Supp}\left(H_J^i(R) \right)} \right) \geq i+1$$

for $i > d+1$. Thus we only need to show that $\overline{\mathrm{Supp}\left(H_I^{d+1}(R) \right)}$ has codimension $\geq d+2$.

Note that

$$\overline{\mathrm{Supp}\left(H_I^{d+1}(R)\right)} \subseteq \overline{\mathrm{Supp}\left(H_J^{d+1}(R)\right)} \subseteq V(J)$$

and $\mathrm{codim}\,(V(J)) = \mathrm{ht}(J) = d+1$. By Lemma 2.3, we may replace J with its radical. Then by the assumption in the hypothesis and Remark 2.86, we can find an open set U such that $U \cap V(J)$ is dense in $V(J)$ and $\forall \mathfrak{p} \in U \cap V(J)$, $R_{\mathfrak{p}}/J_{\mathfrak{p}}R_{\mathfrak{p}}$ is regular. If we can show that for every $\mathfrak{p} \in U \cap V(J)$, $H_I^{d+1}(R) \otimes_R R_{\mathfrak{p}} \overset{\text{Lemma 2.12}}{=} H_{IR_{\mathfrak{p}}}^{d+1}(R_{\mathfrak{p}}) = 0$, then we get that

$$\overline{\mathrm{Supp}\left(H_I^{d+1}(R)\right)} \subseteq V(J) \setminus U.$$

Since $\mathrm{codim}\,(V(J) \setminus U) > d+1$, this gives the assertion that we need.

Let $\mathfrak{p} \in U \cap V(J)$. By Lemma 2.79, since $R_{\mathfrak{p}}/JR_{\mathfrak{p}}$ is regular, it is a domain, so $JR_{\mathfrak{p}}$ is a prime ideal in $R_{\mathfrak{p}}$. Let $A := \widehat{R_{\mathfrak{p}}}$. Since R contains $\mathbb{Q}$, we have $A \cong \Bbbk[[x_1, \ldots, x_{\mathrm{ht}(\mathfrak{p})}]]$ where $\Bbbk$ is the residue field of A. By Corollary 2.82 and Lemma 2.79, JA is a prime ideal in A such that A/JA is regular.

Since $R_{\mathfrak{p}} \hookrightarrow A$ is faithfully flat, $H_{IR_{\mathfrak{p}}}^{d+1}(R_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}} A \overset{\text{Lemma 2.12}}{\cong} H_{IA}^{d+1}(A)$ and $H_{IR_{\mathfrak{p}}}^{d+1}(R_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}} A = 0$ if and only if $H_{IR_{\mathfrak{p}}}^{d+1}(R_{\mathfrak{p}}) = 0$. Hence, it is enough to show that $H_{IA}^{d+1}(A) = 0$.

By Lemma 2.15, we have a commutative diagram of A-modules

$$
\begin{array}{ccc}
H_J^{d+1}(R) \otimes_R A & \longrightarrow\!\!\!\!\!\rightarrow & H_I^{d+1}(R) \otimes_R A \\
\| \wr & & \| \wr \\
H_{JA}^{d+1}(A) & \longrightarrow & H_{IA}^{d+1}(A).
\end{array}
$$

Since the top map is surjective, the bottom map must also be surjective. Furthermore, by Example 2.33(b), the bottom map is $D_{A/\Bbbk}$−linear.

If $\mathrm{ht}(JA) \neq d+1$, then $H_{JA}^{d+1}(A) = 0$ by Corollary 2.6 and Proposition 2.9. Thus we get $H_{IA}^{d+1}(A) = 0$.

Now assume that $\mathrm{ht}(JA) = d+1$. By Corollary 2.70, $H_{JA}^{d+1}(A)$ is a simple $D_{A/\Bbbk}$−module with support in $V(JA)$. Thus either $H_{IA}^{d+1}(A) = 0$ or $H_{IA}^{d+1}(A) = H_{JA}^{d+1}(A)$.

Arguing by contradiction, suppose that the latter is true. We consider $B := \widehat{A_{JA}} \cong$ $K[[x_1, \ldots, x_{d+1}]]$, where $K = \Bbbk(JA)$ is the residue field of A at JA. Since $JR_{\mathfrak{p}}$ has height $d + 1$ and $IR_{\mathfrak{p}} \subset JR_{\mathfrak{p}}$ has pure height d, $JR_{\mathfrak{p}}$ cannot be a minimal prime ideal containing $IR_{\mathfrak{p}}$. Since JA is a prime ideal and $JA \supset IA$, we have that by Lemma 2.83, JA cannot be a minimal prime ideal containing IA and $JA_{JA} \supseteq IA_{JA}$ cannot be a minimal prime ideal containing IA_{JA}. Since JB is a prime ideal (in fact, it is the maximal ideal of B) and $JB \supset IB$, we have that by Lemma 2.83 again, JB cannot be minimal prime ideal containing IB. Then we have that $\dim(B/IB) > 0$. By Theorem 2.18, $H_{IB}^{d+1}(B) = 0$. Then we get $H_{JA_{JA}}^{d+1}(A_{JA}) \otimes_{A_{JA}} B \overset{\text{Lemma } 2.3}{=} H_{JB}^{d+1}(B) = H_{IB}^{d+1}(B) = 0$. Since $A_{JA} \longrightarrow B$ is faithfully flat, this means that $H_{JA_{JA}}^{d+1}(A_{JA}) = 0$. However, by our assumption, $\text{ht}(JA) = d + 1$. Then by Proposition 2.6, we have a contradiction. Therefore, we must have $H_{IA}^{d+1}(A) = 0$. $\qquad\square$

The next theorem is a special case to Lyubeznik's conjecture. A slightly weaker result is shown by Puthenpurakal [Put16] using different methods. Our proof is simpler, as it follows almost immediately from Theorem 2.71 using the same argument as in the proof of Theorem 2.87. First we need another lemma, which relates the associated primes of a module over a noetherian ring with the associated primes in a suitable completion.

Lemma 2.88. Let R be a noetherian ring and M be an $R-$ module. If $\mathfrak{p} \subset R$ is an associated prime of M and $\mathfrak{q} \subset R$ is a prime ideal containing $\mathfrak{p}$, then there is an associated prime ideal $\mathfrak{p}' \subset \widehat{R_{\mathfrak{q}}}$ of $M \otimes_R \widehat{R_{\mathfrak{q}}}$ lying over $\mathfrak{p}$.

Proof. We have the following homomorphisms:

$$R \overset{\phi_1}{\longrightarrow} R_{\mathfrak{q}} \overset{\phi_2}{\longrightarrow} \widehat{R_{\mathfrak{q}}} \ .$$

with ϕ spanning from R to $\widehat{R_{\mathfrak{q}}}$.

$\mathfrak{p}$ is an associated prime of M if and only if we have an injective homomorphism

$$R/\mathfrak{p} \hookrightarrow M.$$

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Since $\mathfrak{p} \subset \mathfrak{q}$ and $\widehat{R_{\mathfrak{q}}}$ is flat over R, we get an injective homomoprhism

$$\widehat{R_{\mathfrak{q}}}/\mathfrak{p}\widehat{R_{\mathfrak{q}}} \cong R/\mathfrak{p} \otimes_R \widehat{R_{\mathfrak{q}}} \hookrightarrow M \otimes_R \widehat{R_{\mathfrak{q}}}.$$

Let $\mathfrak{p}' \subset \widehat{R_{\mathfrak{q}}}$ be a minimal prime ideal containing $\mathfrak{p}\widehat{R_{\mathfrak{q}}}$. By Lemma 2.83, $\phi_2^{-1}(\mathfrak{p}')$ is a minimal prime ideal of $\mathfrak{p}R_{\mathfrak{q}}$. Since $\mathfrak{p}R_{\mathfrak{q}}$ is a prime ideal, we must have $\phi_2^{-1}(\mathfrak{p}') = \mathfrak{p}R_{\mathfrak{q}}$. Thus we have

$$\phi^{-1}(\mathfrak{p}') = \phi_1^{-1}(\phi_2^{-1}(\mathfrak{p}')) = \phi_1^{-1}(\mathfrak{p}R_{\mathfrak{q}}) = \mathfrak{p}.$$

In other words, $\mathfrak{p}'$ lies over $\mathfrak{p}$.

On the other hand, since $\mathfrak{p}'$ is a minimal prime containing $\mathfrak{p}\widehat{R_{\mathfrak{q}}}$, it is an associated prime of $\widehat{R_{\mathfrak{q}}}/\mathfrak{p}\widehat{R_{\mathfrak{q}}}$. Thus we get an injective homomorphism

$$\widehat{R_{\mathfrak{q}}}/\mathfrak{p}' \hookrightarrow \widehat{R_{\mathfrak{q}}}/\mathfrak{p}\widehat{R_{\mathfrak{q}}}.$$

This gives us an injective homomorphims $\widehat{R_{\mathfrak{q}}}/\mathfrak{p}\widehat{R_{\mathfrak{q}}} \hookrightarrow M \otimes_R \widehat{R_{\mathfrak{q}}}$, completing the proof. $\qquad\square$

Theorem 2.89. Let R be a regular ring of dimension n containing $\mathbb{Q}$ with the property that for every prime ideal $\mathfrak{p} \subset R$ of height $\geq n-1$, the quotient ring $R/\mathfrak{p}$ is J-0 (note that this is automatic for prime ideals of height n). Let $I \subset R$ be an ideal. Then $H_I^{n-1}(R)$ has finitely many associated primes.

Proof. Recall from Corollary 2.6 that if $\mathrm{ht}(I) = n$, then $H_I^{n-1}(R) = 0$. Furthermore, it is already known (see [Mar01, Proposition 1.1(b)]) that $H_I^{\mathrm{ht}(I)}(R)$ has finitely many associated primes. Thus the assertion in the theorem is known if $\mathrm{ht}(I) = n$ and $\mathrm{ht}(I) = n-1$. Let us suppose first that $\mathrm{ht}(I) = n-2$. By Theorem 2.71, there is an ideal $J \supset I$ with $\mathrm{ht}(J) = n-1$ such that the natural morphism

$$H_J^{n-1}(R) \longrightarrow H_I^{n-1}(R) \qquad\qquad (\star)$$

is surjective. By Lemma 2.3, we may replace J with its radical. Then by the assumption in the hypothesis and Remark 2.86, we can find an open set U such that $U \cap V(J)$ is dense in $V(J)$ and $\forall \mathfrak{p} \in U \cap V(J)$, $R_\mathfrak{p}/JR_\mathfrak{p}$ is regular.

Let J' be an ideal such that $V(J') = V(J) \setminus U$. Then we must have $J' = R$ or $\mathrm{ht}\,(J') > \mathrm{ht}(J) \geq n-1$, so $V(J')$ must be a set of finitely many closed points. Since every associated prime of $H_I^{n-1}(R)$ lies either in $V(J')$ or in U, it is enough to show that the only associated primes of $H_I^{n-1}(R)$ in U are those which are minimal primes containing J. We proceed as in the proof of Theorem 2.71 to show this.

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ be the minimal primes containing J. Then $U \cap V(J) = \bigsqcup_{i=1}^m U \cap V(\mathfrak{p}_i)$ is a disjoint union since $U \cap V(J)$ is regular, and $U \cap V(\mathfrak{p}_i) \neq \emptyset$ since $U \cap V(J)$ is dense in $V(J)$. Let $\mathfrak{p} \in U \cap V(\mathfrak{p}_i)$. By Lemma 2.79, since $R_\mathfrak{p}/JR_\mathfrak{p}$ is regular, it is a domain, so $JR_\mathfrak{p}$ is a prime ideal in $R_\mathfrak{p}$ and we have $JR_\mathfrak{p} = \mathfrak{p}_i R_\mathfrak{p}$.

Let $A := \widehat{R_\mathfrak{p}} \cong \Bbbk[[x_1, \ldots, x_{\mathrm{ht}(\mathfrak{p})}]]$, where $\Bbbk = \Bbbk(\mathfrak{p})$ is the residue field of R at $\mathfrak{p}$. By Corollary 2.82 and Lemma 2.79, $JA = \mathfrak{p}_i A$ is a prime ideal in A such that A/JA is regular.

By Lemma 2.15, we have a commutative diagram of A-modules

$$
\begin{array}{ccc}
H_J^{n-1}(R) \otimes_R A & \longrightarrow\!\!\!\!\!\rightarrow & H_I^{n-1}(R) \otimes_R A \\
\| \wr & & \| \wr \\
H_{JA}^{n-1}(A) & \longrightarrow & H_{IA}^{n-1}(A).
\end{array}
$$

Since the top map is surjective, the bottom map must also be surjective. Furthermore, by Example 2.33(b), the bottom map is $D_{A/\Bbbk} - $ linear.

If $\mathrm{ht}(JA) \neq n-1$, then $H_{JA}^{n-1}(A) = 0$ by Corollary 2.6 and Proposition 2.9. Thus we get $H_{IA}^{n-1}(A) = 0$, hence $\mathfrak{p}A$ is not an associated prime of $H_{IA}^{n-1}(A)$. Since the only minimal prime ideal in A containing $\mathfrak{p}A$ is $\mathfrak{p}A$ itself, by Lemma 2.88, $\mathfrak{p}$ is not an associated prime of $H_I^{n-1}(R)$.

Now assume that $\mathrm{ht}(JA) = n-1$. By Proposition 2.70, we have that $H_{JA}^{n-1}(A) = H_{\mathfrak{p}_i A}^{n-1}(A)$ is a simple $D_{A/\Bbbk} - $ module with support in $V(\mathfrak{p}_i)$. Thus either $H_{IA}^{n-1}(A) = 0$

or $H_{IA}^{n-1}(A) = H_{\mathfrak{p}_i A}^{n-1}(A)$. In the latter case, this means that the only associated prime of $H_{IA}^{n-1}(A)$ is $JA = \mathfrak{p}_i A$. By Lemma 2.88, $\mathfrak{p}$ cannot be an associated prime of $H_I^{n-1}(R)$ unless $\mathfrak{p} = \mathfrak{p}_i$.

Therefore we must have

$$\mathbf{Ass}(H_I^{n-1}(R)) \cap U \subseteq \{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}.$$

This completes the proof in the case $\mathrm{ht}(I) = n - 2$.

Now let I be any ideal with height $\mathrm{ht}(I) = d < n - 2$. Assume the theorem holds for any ideal $J \subset R$ with $\mathrm{ht}(J) > d$. Applying Theorem 2.71 again, we have an ideal $J \supset I$ with height $\mathrm{ht}(J) > d$ such that

$$H_J^i(R) \cong H_I^i(R)$$

for all $i > d + 1$. In particular, $d + 1 < n - 1$ so $H_J^{n-1}(R) \cong H_I^{n-1}(R)$ has finitely many associated primes by the induction hypothesis. $\square$

Remark 2.90.   (a) The J-0 conditions in Theorem 2.87 and in Theorem 2.89 are satisfied for most rings one encounters. All excellent rings satisfy these conditions (see [Mat80, (34.A)] for a reference on excellent rings).

(b) The hypothesis in [Put16, Theorem 1.3] is slightly stronger than in Theorem 2.89, since it assumes that R is excellent. It seems that the main uses of excellence in Puthenpurakal's proof are to (1) reduce to the case that R contains an uncountable field by taking a flat extension $R \hookrightarrow R[[X]]_X$ and using a result of [Rot80] to say that $R[[X]]$ (and therefore $R[[X]]_X$) is excellent if R is excellent, and (2) use that $R/\sqrt{I}R$ is J-0 for every ideal $I \subset R$ (not just those of height $\geq n - 1$).

CHAPTER 3

Motivic Chern Classes

3.1: Background on Motivic Chern Classes

We begin this chapter with the relevant background on equivariant motivic Chern classes. Our main references are [CG10] and [FRW21]. Throughout this chapter, all varieties will be over $\mathbb{C}$.

Our set-up is as follows. Let G be a linear algebraic group. Denote its multiplication map

$$m : G \times G \longrightarrow G.$$

Let X be a G-variety of dimension n, i.e we have an algebraic morphism

$$\sigma : G \times X \longrightarrow X$$

which gives an action of G on X. If Y is another G-variety with G-action given by σ', then an algebraic morphism $f : X \to Y$ is G-equivariant if we have commutativity

$$
\begin{array}{ccc}
G \times X & \xrightarrow{id_G \times f} & G \times Y \\
\downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma'} \\
X & \xrightarrow{\quad f \quad} & Y.
\end{array}
$$

3.1.1: Grothendieck Group of G-Equivariant Coherent Sheaves

By associativity of the G-action on X, we have the commutative diagram

$$
\begin{array}{ccc}
G \times X & \xrightarrow{\ \sigma\ } & X \\
\text{pr}_2 \downarrow & \text{pr}_{23} & \nwarrow \text{pr}_2 \\
X & G \times G \times X & \xrightarrow{id_G \times \sigma} G \times X \\
& \text{pr}_2 \quad \downarrow m \times id_X & \downarrow \sigma \\
& G \times X & \xrightarrow{\ \sigma\ } X.
\end{array}
$$

Definition 3.1 (*G-equivariant Sheaf*)**.** Let $\mathscr{F}$ be a coherent sheaf on X. We say $\mathscr{F}$ is a G-equivariant sheaf if there is an isomorphism of $\mathcal{O}_X$ − modules

$$
\Psi : \text{pr}_2^\star \mathscr{F} \xrightarrow{\ \sim\ } \sigma^\star \mathscr{F}
$$

such that it is compatible with the associativity diagram 3.1.1, i.e the resulting isomorphisms of $\mathcal{O}_{G \times G \times X}$ − modules

$$
\text{pr}_{23}^\star \Psi : \text{pr}_{23}^\star \text{pr}_2^\star \mathscr{F} \xrightarrow{\ \sim\ } \text{pr}_{23}^\star \sigma^\star \mathscr{F} = (id_G \times \sigma)^\star \text{pr}_2^\star \mathscr{F} \ ,
$$

$$
(id_G \times \sigma)^\star \Psi : (id_G \times \sigma)^\star \text{pr}_2^\star \mathscr{F} \xrightarrow{\ \sim\ } (id_G \times \sigma)^\star \sigma^\star \mathscr{F} = (m \times id_X)^\star \sigma^\star F \ ,
$$

and

$$
(m \times id_X)^\star \Psi : \text{pr}_{23}^\star \text{pr}_2^\star \mathscr{F} = (m \times id_X)^\star \text{pr}_2^\star F \xrightarrow{\ \sim\ } (m \times id_X)^\star \sigma^\star \mathscr{F}
$$

satisfy the co-cyle condition $(id_G \times \sigma)^\star \Psi \circ \text{pr}_{23}^\star \Psi = (m \times id_X)^\star \Psi$.

A G-equivariant homomorphism between G-equivariant sheaves $\phi : (\mathscr{F}', \Psi') \to (\mathscr{F}, \Psi)$ on X is a homomorphism of sheaves such that the natural diagram

$$
\begin{array}{ccc}
\text{pr}_2^\star \mathscr{F}' & \xrightarrow{\text{pr}_2^\star \phi} & \text{pr}_2^\star \mathscr{F} \\
\Psi' \downarrow & & \downarrow \Psi \\
\sigma^\star \mathscr{F}' & \xrightarrow{\sigma^\star \phi} & \sigma^\star \mathscr{F}
\end{array}
$$

commutes.

Example 3.2. (a) $\mathcal{O}_X$ has the natural structure of a G-equivariant sheaf given by

$$id : \mathrm{pr}_2^\star \mathcal{O}_X = \mathcal{O}_{G \times X} \longrightarrow \sigma^\star \mathcal{O}_X = \mathcal{O}_{G \times X}$$

(b) Let $Y \subset X$ be a closed subvariety invariant under the G-action. Then from the previous example, $\mathcal{O}_Y$ has a natural structure of a G-equivariant sheaf such that

$$\mathcal{O}_X \twoheadrightarrow \mathcal{O}_Y$$

is a G-equivariant map. Since pr_2 and σ are smooth morphisms, $\mathrm{pr}_2^\star$ and $\sigma^\star$ are exact. Thus $\mathcal{I}_Y := \ker (\mathcal{O}_X \twoheadrightarrow \mathcal{O}_Y)$ has an induced natural structure of a G-equivariant sheaf.

(c) Let Ω_X denote the cotangent sheaf on X. For $g \in G$, denote $\sigma_g := \sigma(g, -)$ the multiplication by g map on X. Then we have a natural isomorphism

$$\psi_g : \sigma_g^\star \Omega_X \xrightarrow{\sim} \Omega_X.$$

These natural isomorphisms satisfy

$$\psi_{gh} = \sigma_h^\star \psi_g \circ \psi_h \qquad (\star)$$

for any $g, h \in G$. On $G \times X$, we have natural maps

$$\sigma^\star \Omega_X \longrightarrow \Omega_{G \times X} \quad \cong \quad \mathrm{pr}_1^\star \Omega_G \oplus \mathrm{pr}_2^\star \Omega_X \longrightarrow \mathrm{pr}_2^\star \Omega_X \ .$$

with a dotted curved arrow labeled Ψ from $\sigma^\star \Omega_X$ to $\mathrm{pr}_2^\star \Omega_X$.

Over point $g \in G$, the induced map on fibers

$$\Psi|_{\{g\} \times G} : (\sigma^\star \Omega_X)|_{\{g\} \times X} \cong \sigma_g^\star \Omega_X \longrightarrow (\mathrm{pr}_2^\star \Omega_X)|_{\{g\} \times X} \cong \Omega_X$$

coincides with ψ_g above. For any $(g, h) \in G \times G$, we get

$$((m \times id_X)^\star \Psi)|_{\{(g,h)\} \times X} = \psi_{gh}$$

$$(\mathrm{pr}_{23}^\star \Psi)|_{\{(g,h)\} \times X} = \psi_h$$

$$((id_G \times \sigma)^\star \Psi)|_{\{(g,h)\} \times X} = \sigma_h^\star \psi_g$$

so by $(\star)$ they satisfy the cocyle condition on fibers and hence do so globally.

(d) Let $\mathcal{E}$ be a locally free sheaf on X and $\pi : E = \mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_X}^\bullet \mathcal{E}^\vee) \to X$ the associated vector bundle over X. Equipping $\mathcal{E}$ with the structure of a G-equivariant sheaf is equivalent to extending the G-action on X to an action on E such that π is a G-equivariant morphism of algebraic varieties, and for every $x \in X$, the induced map

$$\pi^{-1}(x) \longrightarrow \pi^{-1}(g \cdot x)$$

given by multiplication by g is $\mathbb{C}$-linear.

Definition 3.3 (Grothendieck Group). We define the *Grothendieck group of G-eqvuiariant coherent sheaves $K_G(X)$* to be the quotient of the free abelian group on isomorphisms classes of G-equivariant coherent sheaves by the relations $[\mathcal{F}] - [\mathcal{F}'] - [\mathcal{F}'']$ for any G-equivariant short exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0.$$

The sum operation is given by direct sum.

Definition 3.4 (Grothendieck Ring)**.** We can similarly define the *Grothendieck ring of G-equivariant vector bundles* $K^G(X)$ to be the quotient of the free abelian group on isomorphisms classes of G-equivariant vector bundles by the relations $[\mathcal{E}] - [\mathcal{E}'] - [\mathcal{E}'']$ for any G-equivariant short exact sequence

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0.$$

The sum operation is given by direct sum. The multiplication operation is given by the tensor product, and the unit is $1 := [\mathcal{O}_X]$.

Example 3.5. Suppose $G = \{0\}$.

(a) Let $X = \mathbb{A}^n$. Then
$$K^0(X) \xrightarrow{\ \sim\ } \mathbb{Z}$$

$$[\mathcal{E}] \longmapsto \mathrm{rk}(\mathcal{E})$$

(b) Let $X = \mathbb{P}^n$. Then $K^0(X) = \mathbb{Z}[h]/(1-h)^{n+1}$ where $h = [\mathcal{O}_{\mathbb{P}^n}(-1)]$.

Example 3.6. Suppose $G = \mathbb{T}^n$. Then $K^G(pt) = \mathbb{Z}[\xi_1^{\pm 1}, \ldots, \xi_n^{\pm 1}]$ where ξ_i^k is the class of vector space $\mathbb{C}$ with the action of $\mathbb{T}^n$ given by $(\zeta_1, \ldots, \zeta_n) \cdot x = \zeta_i^k x$.

Definition 3.7. We have the *cap product* action

$$K^G(X) \times K_G(X) \xrightarrow{\ \cap\ } K_G(X)$$

$$([\mathcal{E}], [\mathcal{F}]) \longmapsto [\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}].$$

Proposition 3.8. Let $f : X \to Y$ be a G-equivariant morphism of G-equivariant smooth varieties. Then we have the following properties:

(a) [Pullback] [CG10, 5.2.5] If $\mathcal{E}$ is a G-equivariant locally free sheaf on Y, then $f^{\star}\mathcal{E}$ is a

G-equivariant locally free sheaf on X. We have a ring homomorphism

$$f^\star : K^G(Y) \longrightarrow K^G(X)$$

$$[\mathcal{E}] \longmapsto [f^\star \mathcal{E}].$$

(b) [Proper Pushforward] [CG10, 5.2.13] Suppose f is proper. If $\mathcal{F}$ is a coherent G-equivariant sheaf on X, then for all i, $R^i f_\star \mathcal{F}$ is a coherent G-equivariant sheaf on Y. We have a group homomorphism

$$f_\star : K_G(X) \longrightarrow K_G(Y)$$

$$[\mathcal{E}] \longmapsto \sum_{i \geq 0} (-1)^i [R^i f_\star \mathcal{E}].$$

For any other proper morphism $g : Y \to Z$, we have an isomorphism $(g \circ f)_\star \cong g_\star \circ f_\star$.

(c) [Projection Formula][CG10, 5.3.12] For any $\alpha \in K^G(Y)$ and $\beta \in K_G(X)$ we have

$$f_\star(f^\star \alpha \cap \beta) = \alpha \cap f_\star f^\star \beta.$$

Example 3.9. Let Y be a G-variety. If $\pi : X \to Y$ is a G-equivariant vector bundle over Y, then $f^\star : K^0(Y) \simeq K^0(X)$ is an isomorphism. Let $s_0 : Y \hookrightarrow X$ be the zero section map. Then $s_0^\star$ is the inverse map of $f^\star$.

Example 3.10. Let V be a vector space of dimension n and $\mathbb{C}^\star$ act on V by scaling, i.e $\zeta \cdot x = (\zeta x_1, \ldots, \zeta x_n)$ for any $\zeta \in \mathbb{C}^\star$ and $x = (x_1, \ldots, x_n)$. Denote $t = [\mathcal{O}_V(-H)]$ in $K^{\mathbb{C}^\star}(V)$ where $H \subset V$ is any hyperplane and $\mathcal{O}_V(-H) = \ker(\mathcal{O}_V \twoheadrightarrow \mathcal{O}_H)$.

Let k with $0 \leq k \leq n$ and $\iota_W : W \subset V$ be a linear subspace of dimension k. Then

$$\iota_{W,\star}[\mathcal{O}_W] = (1-t)^{n-k}.$$

To compute this, we proceed by decreasing induction on k, since the case of $k = n$ is when $W = V$. Assume that for any linear subspace W of dimension k, the above claim holds. Let H be a hyperplane not containing W, then $\dim(W \cap H) = k - 1$ and we have the short exact sequence

$$0 \longrightarrow \mathcal{O}_W(-H) \longrightarrow \mathcal{O}_W \longrightarrow \mathcal{O}_{W \cap H} \longrightarrow 0 \ .$$

So

$$\iota_{W \cap H, \star}[\mathcal{O}_{W \cap H}] = \iota_{W, \star}[\mathcal{O}_W] - \iota_{W, \star}(t|_W) \overset{\text{Proj Formula}}{=} \iota_{W, \star}[\mathcal{O}_W] - t \cdot \iota_{W, \star}[\mathcal{O}_W]$$

$$= (1-t)\iota_{W, \star}[\mathcal{O}_W] \overset{\text{Ind hyp}}{=} (1-t)^{n-(k-1)}.$$

Proposition 3.11. [CG10, Prop 5.1.28] If X is smooth and quasiprojective, then the map $(-) \cap [\mathcal{O}_X]$ induces an isomorphism

$$K^G(X) \simeq K_G(X).$$

Example 3.12. Suppose X is smooth and quasiprojective of dimension n. Let $\mathcal{F}_1, \mathcal{F}_2$ be G-equivariant sheaves. Then in the Grothendieck ring $K_G(X) \simeq K^G(X)$, we have

$$[\mathcal{F}_1] \cdot [\mathcal{F}_2] = \sum_{i=0}^{n} (-1)^i \mathcal{T}or^i_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{F}_2)$$

Definition 3.13 (Group pullback)**.** Let $\phi : H \to G$ be a homomorphism of linear algebraic groups. Then ϕ induces a canonical ring homomorphism

$$K^G(X) \xrightarrow{\phi^\star} K^H(X)$$

which commutes with proper pushforwards and pullbacks. In the case we have a group automorphism $\phi : G \xrightarrow{\sim} G$, we will denote by ϕX the variety X with action given by

$$g \cdot x = \phi(g) \cdot x \text{ for } g \in G, x \in X.$$

Remark 3.14. Note that $R(G) := K^G(pt)$ is the ring of G-representations.

Definition 3.15. If V is a G-equivariant vector space, we will consider

$$|_0 : K^G(V) \xrightarrow{\sim} R(G)$$

to be the pullback morphism along $\{0\} \hookrightarrow V$ (see Example 3.9).

Definition 3.16. Let G be a simply connected linear algebraic group and $\mathbb{T} \subset G$ be a maximal torus. Denote by $N_G(\mathbb{T})$ the normalizer of $\mathbb{T}$ in G, and define $W_G(\mathbb{T}) = N_G(\mathbb{T})/\mathbb{T}$ to be the *Weyl group* of $\mathbb{T}$ in G.

For any $\mathbb{T} -$ representation $\phi : \mathbb{T} \to \mathrm{End}_{\mathbb{C}}(V)$, where V is a vector space, and any $g \in N_G(\mathbb{T})$ we can define a new $\mathbb{T} -$ representation $\phi_g : \mathbb{T} \to \mathrm{End}_{\mathbb{C}}(V)$ given by

$$\phi_g(\zeta) = \phi(g\zeta g^{-1}).$$

Since $\phi = \phi_\zeta$ for any $\zeta \in \mathbb{T}$, we get an induced action of the quotient group $W_G(\mathbb{T})$ on $R(\mathbb{T})$.

Proposition 3.17. [CG10, Theorem 6.1.22] The restriction homomorphism defined in 3.13 induces an isomorphism

$$R(G) \simeq R(\mathbb{T})^{W_G(\mathbb{T})}.$$

Example 3.18. (a) Let $\mathbb{T}^n \subset \mathrm{GL}_n$ be the diagonal $n \times n$ matrices with nonzero entries on the diagonal. We have that the normalizer group $N_{\mathrm{GL}_n}(\mathbb{T}^n)$ consists of the generalized permutation matrices. Hence

$$W_{\mathrm{GL}_n}(\mathbb{T}^n) = \mathbf{S}_n$$

is the symmetric group on n elements, acting on $R(\mathbb{T}^n)$ by permuting the variables ξ_i. Thus we get

$$R(\mathrm{GL}_n) \cong \mathbb{Z}[\xi_1^{\pm 1}, \ldots, \xi_n^{\pm 1}]^{\mathbf{S}_n}.$$

(b) Let $\mathbb{T}^n \subset \mathrm{B}_n$ (resp. B_n^{-1}) where B_n (resp. B_n^{-1}) $\subset \mathrm{GL}_n$ is the Borel subgroup of upper (resp. lower) triangular invertible $n \times n$ matrices. The only generalized permutation matrices which are upper (resp. lower) triangular are the diagonal matrices. Hence $W_{\mathrm{B}_n}(\mathbb{T}^n)$ (resp. $W_{\mathrm{B}_n^{-1}}(\mathbb{T}^n)$) $= \{1\}$. Thus we get

$$R(\mathrm{B}_n) \text{ (resp. } R(\mathrm{B}_n^{-1})) \cong R(\mathbb{T}^n) = \mathbb{Z}[\xi_1^{\pm 1}, \ldots, \xi_n^{\pm 1}].$$

(c) Let $\mathbb{T}^n \subset P$ where P is a parabolic subgroup of GL_n containing B_n^{-1}, i.e

$$P = \left\{ \begin{bmatrix} S_{r_1} & & & 0 \\ & \ddots & & \\ \star & & S_{r_{k-1}} & \\ & & & S_{r_k} \end{bmatrix} \text{ s.t } S_{r_i} \in \mathrm{GL}_{r_i} \text{ for all } i \right\}$$

where $n \geq r_1, \ldots, r_k \geq 0$ are such that $\sum r_i = n$. Then $W_P(\mathbb{T}^n) \cong \prod_{i=1}^{k} \mathbf{S}_{r_i} \subset \mathbf{S}_n$.

Definition 3.19. Let G be a simply connected linear algebraic group and $\mathbb{T} \subset G$ a maximal torus. Suppose $H \subset G$ is a simply connected closed algebraic subgroup and such that $\mathbb{T} \subset H$. Then $W_H(\mathbb{T}) \subset W_G(\mathbb{T})$ and we put $W^H(\mathbb{T}) := W_G(\mathbb{T})/W_H(\mathbb{T})$ to be the coset.

3.1.2: Motivic Chern Classes

In this section our ground field will be $k = \mathbb{C}$ and G will be a complex linear algebraic group. All varieties will be complex G-varieties. Furthermore, M will denote a smooth quasi-projective complex G-equivariant variety. We denote the ring $K_y^G(M) := K^G(M)[y]$.

Definition 3.20 (Lambda class)**.** If $\mathcal{E}$ is a G-equivariant locally free sheaf on M, then define

$$\lambda_y(\mathcal{E}) = \sum_{i=0}^{\mathrm{rk}(\mathcal{E})} \left[\bigwedge^i \mathcal{E} \right] y^i$$

in the Grothendieck ring $K_y^G(M)$.

Proposition 3.21. Suppose we have a short exact sequence of G-equivariant locally free sheaves on M

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0.$$

Then we have

$$\lambda_y(\mathcal{E}) = \lambda_y(\mathcal{E}')\lambda_y(\mathcal{E}'')$$

in $K_y^G(M)$.

Theorem 3.22 (Motivic Chern Class). [FRW21, 2.3] To any G-equivariant morphism of G − varieties, $f : X \to M$, we can assign the unique class $\mathrm{mC}_y^G(X \xrightarrow{f} M) \in K_y^G(M)$ in such a way as to satisfy the following properties.

1. **Normalization:** $\mathrm{mC}_y^G(id_M) = \lambda_y(\Omega_M)$.

2. **Localness:** If $U \subset M$ is open then

$$\mathrm{mC}_y^G(X \to M)|_U = \mathrm{mC}_y^G(f^{-1}(U) \to U).$$

3. **Additivity:** If $X = Z_1 \sqcup Z_2$ where $Z_1, Z_2 \subset X$ are locally closed subsets, then

$$\mathrm{mC}_y^G(X \to M) = \mathrm{mC}_y^G(Z_1 \to M) + \mathrm{mC}_y^G(Z_2 \to M).$$

4. **Functoriality:** If $\rho : M \to M'$ is a proper G-equivariant morphism of smooth quasi-projective complex G-varieties, then

$$\mathrm{mC}_y^G(X \xrightarrow{\rho \circ f} M') = \rho_\star \mathrm{mC}_y^G(X \xrightarrow{f} M).$$

Example 3.23. Let $f : X \to M$ be a G-equivariant morphism. If X is smooth, of dimension

m, and f is proper, then

$$\mathrm{mC}_y^G(X \xrightarrow{f} M) = f_\star \lambda_y(\Omega_X) = \sum_{i=0}^{m} \sum_{j \geq 0} (-1)^j [R^j f_\star \Omega_X^i] y^i.$$

Definition 3.24. For the case of $f : X \to pt$ where X is smooth, we denote

$$R\Gamma(X, \alpha) := f_\star \alpha$$

for $\alpha \in K_y^G(X)$.

Example 3.25. [FRW21, 2.7] Let $G = \mathbb{T}^n$ act linearly on $\mathbb{C}$. This gives $\mathbb{C}$ the algebraic action of $\mathbb{T}^n$ on it, and we will denote the $\mathbb{T}^n$-equivariant vector space $\mathbb{C}$ with this action as L. Let $\alpha \in R(\mathbb{T}^n)$ be the class associated to this $\mathbb{T}^n$-equivariant vector space. Denote $\iota : \{0\} \hookrightarrow \mathbb{C}$ and $\pi : \mathbb{C} \to \{0\}$. Recall from Definition 3.15 that we have an isomorphism

$$|_0 : K_y^{\mathbb{T}^n}(\mathbb{C}) \simeq R(\mathbb{T}^n)[y] \cong \mathbb{Z}[\xi_1^{\pm 1}, \ldots, \xi_n^{\pm 1}, y].$$

Under this isomorphism, we have $[\pi^\star L]|_0 = (\pi^\star \alpha)|_0 = \alpha$.

(a) We get that $\mathrm{mC}_y^{\mathbb{T}^n}(\mathbb{C} \subset \mathbb{C})|_0 = 1 + y\alpha$ in $R(\mathbb{T}^n)[y]$. For example, if the action of $\mathbb{T}^n$ on $\mathbb{C}$ is given by

$$(\zeta_1, \ldots, \zeta_n) \cdot x = \zeta_1^{k_1} \ldots \zeta_n^{k_n} x,$$

then $\alpha = \xi_1^{k_1} \ldots \xi_n^{k_n}$ and $\mathrm{mC}_y^{\mathbb{T}^n}(\mathbb{C} \subset \mathbb{C})|_0 = 1 + y\xi_1^{k_1} \ldots \xi_n^{k_n}$.

(b) Then $\pi^\star L$ is a $\mathbb{T}^n -$ equivariant rank one locally free sheaf on $\mathbb{C}$. We have the $\mathbb{T}^n$-equivariant short exact sequence

$$0 \longrightarrow \pi^\star L \xrightarrow{\cdot x} \mathcal{O} \longrightarrow \iota_\star \mathcal{O}_0 \longrightarrow 0.$$

Thus we get $\mathrm{mC}_y^{\mathbb{T}^n}(\{0\} \subset \mathbb{C})_0 = \left(\iota_\star \mathrm{mC}_y^{\mathbb{T}^n}(\{0\} \subset \{0\})\right)|_0 = [\iota_\star \mathcal{O}_0]|_0 = (1 - \pi^\star[L])|_0 =$

$1 - \alpha$.

(c) By additivity, we get $\mathrm{mC}_y^{\mathbb{T}^n}(\mathbb{C} - \{0\} \subset \mathbb{C}) = (1 + y\alpha) - (1 - \alpha) = (1 + y)\alpha$.

Remark 3.26. Let $f : X \to M$ a G-equivariant morphism. Let $\phi : H \to G$ be the homomorphism of linear algebraic groups. Then $\phi^\star \mathrm{mC}_y^G(X \xrightarrow{f} M) = \mathrm{mC}_y^H \left(X \xrightarrow{f} M \right)$.

Lemma 3.27. [AMSS22, Thm 4.2(3)] The motivic Chern class $\mathrm{mC}_y^{\mathbb{T}^n}$ commutes with external products. More precisely, let $f_1 : X_1 \to M_1$ and $f_2 : X_2 \to M_2$ be $\mathbb{T}^n$-equivariant morphisms with M_1 and M_2 smooth quasi-projective. Then

$$\mathrm{mC}_y^{\mathbb{T}^n}\left(X_1 \times X_2 \xrightarrow{f_1 \times f_2} M_1 \times M_2\right) = \mathrm{pr}_1^\star\left(\mathrm{mC}_y^{\mathbb{T}^n}(X_1 \xrightarrow{f_1} M_1)\right) \cdot \mathrm{pr}_2^\star\left(\mathrm{mC}_y^{\mathbb{T}^n}(X_2 \xrightarrow{f_2} M_2)\right)$$

in $K^{\mathbb{T}}(M_1 \times M_2)[y]$.

Example 3.28. Suppose that $\mathbb{T}^m$ acts on $\mathbb{C}^n$ by $\zeta \cdot x = (\zeta_1^{i_{11}} \ldots \zeta_m^{i_{m1}} x_1, \ldots, \zeta_1^{i_{1n}} \ldots \zeta_n^{i_{mn}} x_n)$ for $\zeta \in \mathbb{T}^m$ and $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$. Denote $\pi : \mathbb{A}^n \to pt$. For $1 \leq k \leq m$, let

- L_k be the one dimensional vector space such that $\mathbb{T}^m$ acts by $\zeta \cdot x = \zeta_k x$ for $\zeta \in \mathbb{T}^m$ and $x \in L_k$,

- $t_k = \pi^\star L_k$, and

- L_k' be the one dimensional vector space such that $\mathbb{T}^m$ acts by $\zeta \cdot x = \zeta_1^{i_{1k}} \ldots \zeta_m^{i_{nk}} x$ for $\zeta \in \mathbb{T}^m$ and $x \in L_k'$.

Then

$$\mathrm{mC}_y^{\mathbb{T}^m}(\mathbb{C}^n \to \mathbb{C}^n) \stackrel{\text{Lemma 3.27}}{=} \prod_{k=1}^n \mathrm{mC}_y^{\mathbb{T}^m}(L_i' \subseteq L_i') \stackrel{\text{Example 3.25}}{=} \prod_{k=1}^n (1 + t_1^{i_{1k}} \ldots t_m^{mk} y)$$

in $K_y^{\mathbb{T}^m}(\mathbb{C}^n)$.

Example 3.29. Let M_n be the vector space of $n \times n$ matrices. Let $\mathbb{T}^n$ be the diagonal $(n \times n) - $ matrices with nonzero entries acting on M_n by conjugation. Let $\mathbb{T} := \mathbb{T}^n \times \mathbb{C}^\star$

act on M_n by $(D, \zeta) \cdot M = \zeta D M D^{-1}$ for $D \in \mathbb{T}^n$, $\zeta \in \mathbb{C}^\star$, and $M \in M_n$. For $1 \leq i, j \leq n$, $r \leq n - i + 1, k \leq n - j + 1$, denote the closed embedding

$$M_{r,k} \xhookleftarrow{\qquad \iota_{i,j,r,k} \qquad} M_n$$

$$A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,k} \\ \vdots & & \vdots \\ a_{r,1} & \cdots & a_{r,k} \end{bmatrix} \longmapsto \begin{bmatrix} 0 & \overbrace{}^{\}i-1} & & \cdots & & 0 \\ \underbrace{\cdots}_{j-1} & a_{1,1} & \cdots & a_{1,k} & \vdots \\ & \vdots & & \vdots & \\ \vdots & a_{r,1} & \cdots & a_{r,k} & \vdots \\ 0 & \cdots & & \cdots & 0 \\ \vdots & & & & \vdots \end{bmatrix}$$

which sends the $r \times k$ matrix A to the $n \times n$ matrix whose $r \times k$ submatrix with top left corner at position (i, j) is A, and all other entries are 0.

We have that for any matrix $M \in M_n$, $D = Diag(a_1, \ldots, a_n)$, and $\zeta \in \mathbb{C}^\star$

$$(\zeta D M D^{-1})_{u,v} = \zeta a_u M_{u,v} a_v^{-1}$$

This means that we get an induced algebraic action of $\mathbb{T}$ on $M_{r,k}$ such that the closed embedding $\iota_{i,j,r,k}$ is $\mathbb{T}$-equivariant.

We write $M_{i,j,r,k}$ to be $M_{r,k}$ with this induced action. Hence, for any set of 4-tuples

$$S := \{(i_1, j_1, r_1, k_1), \ldots, (i_s, j_s, r_s, k_s)\}$$

such that the rectangles $[i_1, i_1 + r_1] \times [j_1, j_1 + k_1], \ldots, [i_s, i_s + r_s] \times [j_s, j_s + k_s]$ partition the square $[1, n] \times [1, n]$, we get

$$M \cong \prod_{l=1}^{s} M_{i_l, j_l, r_l, k_l}$$

as $\mathbb{T}^n$-varieties.

Let $Z \subseteq M_n$ be a locally closed subset invariant under the action of $\mathbb{T}$. Suppose we have locally closed subsets Z_{i_l,j_l,r_l,k_l} in M_{i_l,j_l,r_l,k_l} such that

$$Z \cong \prod_{l=1}^{s} Z_{i_l,j_l,r_l,k_l} \subseteq M_n \cong \prod_{l=1}^{s} M_{i_l,j_l,r_l,k_l}$$

as product varieties. If $Z_{i_l,j_l,r_l,k_l} \subseteq M_{i_l,j_l,r_l,k_l}$ are $\mathbb{T}$-invariant for all $l = 1, \ldots, s$, then by Lemma 3.27

$$\mathrm{mC}_y^{\mathbb{T}}(Z \subseteq M_n) = \prod_{(i,j,r,k) \in S} \mathrm{mC}_y^{\mathbb{T}}(Z_{i,j,r,k} \subset M_{i,j,r,k}).$$

For example, the identity matrix I_n is invariant under the $\mathbb{T}$ action. Each component $M_{i,j,1,1}$ of M_n, for $i,j = 1, \ldots n$, is a copy of $\mathbb{C}$ but with the action of $\mathbb{T}$ given by

$$(\zeta_1, \ldots, \zeta_n, \zeta) \cdot x = \zeta \zeta_i \zeta_j^{-1} x.$$

Hence, combining the above remarks with Example 3.25, we get

$$\mathrm{mC}_y^{\mathbb{T}}(\{I_n\} \subset M_n)|_{I_n} = \prod_{u=1}^{n} \prod_{v=1}^{n} \left(1 + y\xi \frac{\xi_i}{\xi_j} \right)$$

in $K_y^{\mathbb{T}}(M_n) \cong \mathbb{Z}[\xi_1^{\pm 1}, \ldots, \xi_n^{\pm 1}][\xi^{\pm 1}, y]$.

Remark 3.30. We make the following remark regarding D-modules and motivic Chern classes, refering to [BSY10] for the details. For every complex algebraic variety X, Saito constructs in [Sai90] an Abelian category $MHM(X)$ of objects called mixed Hodge modules which satisfies the *six functor formalism*. Futhermore, every complex variety X has an object $\mathbf{Q}_X^H$ in $D^b(MHM(X))$. When X is smooth, an object $M \in MHM(X)$ has an underlying filtered D-module $(M, F_\bullet M)$ where $F_\bullet M$ is a good filtration on M. Then we can define

$$mH_y(M) := \sum_{i,p} (-1)^i \left[\mathcal{H}^i \mathrm{GR}_{-p}^F DR_X(M) \right] (-y)^p$$

where $DR_X(M)$ is the *de Rham complex* of M with an induced filtration F from $F_\bullet M$.

Suppose X is smooth and quasiprojective. Recall that we have a natural isomorphpism $K^0(X) \cong K_0(X)$ as abelian groups. Then we get that for every morphism $f : V \to X$, where V is an arbitrary complex variety and X is smooth,

$$\mathrm{mC}_y(V \xrightarrow{f} X) = mH_y(f_! \mathbf{Q}_V^H)$$

in the Grothendieck group of coherent sheaves $K_0(X)$.

3.2: The Equivariant Motivic Chern Class of a Nilpotent Cone

Let M_n be the vector space of $n \times n$ matrices. Let $\mathrm{GL}_n \subset M_n$ be the general linear group. Let B_n (resp. B_n^-) $\subset \mathrm{GL}_n$ be the Borel group of all upper (resp. lower) triangular matrices with nonzero diagonal entries. The standard maximal torus $\mathbb{T}^n \subset \mathrm{B}_n$ (resp. B_n^-) is the subgroup of diagonal matrices with nonzero diagonal entries.

Throughout this section, we will use the following notation for matrices.

- I_k is the $k \times k$ identity matrix.

- Z_k is the $k \times k$ zero matrix.

- $\S_{k,l}$ is any $k \times l$ matrix of full rank.

- If M is an $n \times n$ matrix, and $1 \leq i_1 \leq i_2 \leq n$, $1 \leq j_1 \leq j_2 \leq n$ $M[i_1 : i_2, j_1 : j_2]$ denotes the $(i_2 - i_1 + 1) \times (j_2 - j_1 + 1)$ submatrix of M with rows i_1 to i_2 and columns j_1 to j_2.

In addition, we denote $G := \mathrm{GL}_n \times \mathbb{C}^\star$ and the standard maximal torus in it $\mathbb{T} := \mathbb{T}^n \times \mathbb{C}^\star$. We fix the action of G on GL_n to be

$$(M, \zeta) \cdot N = \zeta M N M^{-1}$$

for $M, N \in \mathrm{GL}_n$, $\zeta \in \mathbb{C}^\star$. $\mathbb{T}$ acts on GL_n by restricting the action from G.

3.2.1: Flag Varieties and the Nilpotent Cone

Fix n and $0 \leq k \leq n$. Let $G(k,n)$ denote the Grassmannian parametrizing $k-$dimensional subspaces of $\mathbb{C}^n$. We can represent an element of $V \in G(k,n)$ by a $n \times k$ matrix such that V is spanned by the columns. Then we have an algebraic action of GL_n on $G(k,n)$ given by left multiplication on the $n \times k$ matrices.

Definition 3.31. A sequence of integers $(x_1 \leq \ldots \leq x_k)$ is a partition of n if $x_1 + \ldots + x_k = n$. Let $a_1, \ldots, a_m, b_0, \ldots, b_{m-1} > 0$ be integers. We will denote the partition of n,

$$\left(\underbrace{\lambda_1 \leq \ldots \leq \lambda_1}_{a_1} \leq \underbrace{\lambda_2 \leq \ldots \leq \lambda_2}_{a_2} \leq \ldots \leq \underbrace{\lambda_m \leq \ldots \leq \lambda_m}_{a_m} \right),$$

where $\lambda_i = b_0 + \ldots + b_{i-1}$, as $\lambda := (a_1, \ldots, a_m, b_0, \ldots, b_{m-1})$ and call λ a *Young diagram* partitioning n.

Definition 3.32. Let $\lambda = (a_1, \ldots, a_m, b_0, \ldots, b_{m-1})$ be a Young diagram partitioning n. We will fix the following notation for the rest of this section. Define for $i = 0, \ldots, \lambda_m$,

$$r_i := \begin{cases} i a_m & i \leq b_{m-1} \\ i a_m + (i - b_{m-1})a_{m-1} & b_{m-1} < i \leq b_{m-1} + b_{m-2} \\ i a_m + (i - b_{m-1})a_{m-1} + (i - (b_{m-1} + b_{m-2})) a_{m-2} & b_{m-1} + b_{m-2} < i \leq b_{m-1} + b_{m-2} + b_{m-3} \\ \vdots \\ \sum_{k=1}^m (i - (b_{m-1} + \ldots + b_k)) a_k & b_{m-1} + \ldots + b_1 \leq i \leq b_{m-1} + \ldots + b_0 = \lambda_m \end{cases}$$

Note that $r_0 = 0$ and $r_{\lambda_m} = n$ and r_i is an increasing sequence.

Definition 3.33. Let λ be a Young diagram. The *parabolic subgroup P_λ associated to λ* is the subgroup $\mathrm{B}_n^- \subset P_\lambda \subset \mathrm{GL}_n$ of lower triangular matrices with invertible diagonal blocks of sizes $n - r_{\lambda_m - 1}, \ldots, r_2 - r_1, r_1$.

Definition 3.34. Let λ be a Young diagram. Denote by Fl_λ to be the partial flag variety parametrizing chains of subspaces

$$0 = V^0 \subset V^1 \subset \ldots \subset V^{\lambda_m - 1} \subset V^{\lambda_m} = \mathbb{C}^n, \qquad \dim(V^i) = r_i.$$

As a variety, Fl_λ is defined as a closed subset of a product of Grassmanians:

$$\mathrm{Fl}_\lambda = \{(V_1, \ldots, V_{\lambda_m}) \,|\, V_i \subset V_{i+1} \text{ for all } i = 1, \ldots, \lambda_m - 1\} \subset G(r_1, n) \times \ldots G(r_{\lambda_m}, n).$$

Then we can represent an element $V^\bullet = (V^1, \ldots, V^{\lambda_m})$ in Fl_λ by an $n \times n$ matrix such that the *last r_i* columns span V^i.

Definition 3.35. Denote the standard flag in Fl_λ to be $E^\bullet$ where $E^i \in G(r_i, n)$ is the vector space represented by the identity matrix I_{r_i}.

The general linear group action on Grassmanians induces an algebraic group actions on Fl_λ. The algebraic action of GL_n on $G(r_i, n)$ for all i via multiplication on the left gives us the algebraic action of GL_n on the product $G(r_1, n) \times \ldots \times G(r_{\lambda_m}, n)$. Fl_λ is invariant under this action, so we obtain an action of the algebraic group GL_n on Fl_λ. Explicitly, if we represent a flag $V^\bullet$ in Fl_λ by an $n \times n$ matrix A, then $M \cdot V^\bullet$ for $M \in \mathrm{GL}_n$ is the flag represented by the matrix MA. From now on, we will fix the action of G on Fl_λ to be GL_n acting on Fl_λ via this action and $\mathbb{C}^\star$ to be acting trivially.

Now let $P_\lambda \subset \mathrm{GL}_n$ be the parabolic subgroup associated to λ. The GL_n-action by left multiplication is transitive, making Fl_λ a homogeneous space, and the stabilizer for the standard flag $E^\bullet \in \mathrm{Fl}_\lambda$ is P_λ. Hence Fl_λ is smooth, and we get a surjective morphism of varieties

$$\mathrm{GL}_n \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Fl}_\lambda$$

$$M \longmapsto M \cdot E^\bullet$$

identifying Fl_λ as the set of left cosets GL_n/P_λ. This morphism maps a matrix $M \in \mathrm{GL}_n$ to

the flag it represents, and it is $\mathbb{T}$-equivariant.

Remark 3.36. Note that if $M \in M_n$ is diagonalizable and fixes a vector space $V \subset \mathbb{C}^n$ of dimension k, then V must be generated by k eigenvectors of M. Let D be a diagonal matrix with nonzero diagonal entries. Then D fixes a flag $V^\bullet \in \mathrm{Fl}_\lambda$ if and only if V^k is generated by a subset $e_{i_1}, \ldots, e_{i_{r_k}}$ of the standard basis $e_1, \ldots, e_n$. Thus every fixed point of Fl_λ is of the form $\omega \cdot E^\bullet$ for some permutation $\omega \in \mathbf{S}_n = W_{\mathrm{GL}_n}(\mathbb{T}) =: W_{\mathrm{GL}_n}$. On the other hand, a matrix M fixes $E^\bullet$ if and only if $M \in P_\lambda$. Hence ω fixes $E^\bullet$ if and only if $\omega \in W_{P_\lambda} := W_{P_\lambda}(\mathbb{T})$. Then $W^{P_\lambda} := W_{\mathrm{GL}_n}/W_{P_\lambda}$ indexes the $\mathbb{T}-$fixed points of Fl_λ, i.e the $\mathbb{T}-$fixed points of Fl_λ are precisely $\omega \cdot E^\bullet$ for a unique coset $\omega W_{P_\lambda} \in W^{P_\lambda}$, $\omega \in \mathbf{S}_n$.

Proposition 3.37. The dimenson of Fl_λ is $\sum_{i=1}^{\lambda_m - 1} r_i(r_{i+1} - r_i)$.

Proof. The surjective morphism $\pi : \mathrm{GL}_n \twoheadrightarrow \mathrm{Fl}_\lambda$ has all fibers isomorphic to P_λ. Thus we get

$$\dim(\mathrm{Fl}_\lambda) = \dim(\mathrm{GL}_n) - \dim(P_\lambda) = n^2 - \dim(P_\lambda) = n^2 - \left(n^2 - \sum_{i=1}^{\lambda_m - 1} r_i(r_{i+1} - r_i) \right) = \sum_{i=1}^{\lambda_m - 1} r_i(r_{i+1} - r_i).$$

$\square$

Recall that B_n is a connected and closed subset of GL_n consisting of upper triangular matrices with nonzero diagonal entries. Denote the set

$$U = \mathrm{B}_n \cdot E^\bullet \subset \mathrm{Fl}_\lambda.$$

Since U is the orbit of an algebraic group acting morphically on Fl_λ, it is a locally closed subset of Fl_λ. Hence, we can endow U with the structure of a locally closed algebraic subvariety of Fl_λ, and it is smooth since B_n acts transitively on it. We have the following

diagram of $\mathbb{T}$-equivariant morphisms

$$
Z := \left\{ \begin{bmatrix} I_{n-r_{\lambda_{m-1}}} & & & \star \\ & \ddots & & \\ 0 & & I_{r_2-r_1} & \\ & & & I_{r_1} \end{bmatrix} \in \mathrm{GL}_n \right\} \quad \subset \quad \mathrm{B}_n \quad \subset \quad \mathrm{GL}_n \qquad M
$$

where $f : Z \to U$ is the restriction to the closed subset $Z \subset \mathrm{B}_n$ of the morphism $\mathrm{B}_n \twoheadrightarrow U$.

Proposition 3.38. $U = \mathrm{B}_n \cdot E^\bullet \subset \mathrm{Fl}_\lambda$ is in fact an open subset of Fl_λ, and the morphism $f : Z \to U$ is an isomorphism of algebraic varieties.

Proof. First note that Z is irreducible, since it is isomorphic to an affine space, and U is connected, since B_n is connected and maps surjectively onto U. Because U is smooth, this means U is irreducible as well. Therefore, by lemma 3.39, if f is bijective, then it is an isomorphism. Hence, we will need to show that f is bijective to complete the proof.

We will first show injectivity of f. Let $M \in Z$, and let $V^\bullet \in \mathrm{Fl}_\lambda$ be the flag it represents, i.e V^i is the span of the last r_i columns of M for every $1 \leq i \leq \lambda_m$. We want to show that M is the unique matrix in Z representing the flag $V^\bullet$.

For each $0 \leq i \leq \lambda_m$, denote

$$
M_i := M[1 : n, (n - r_i + 1) : n],
$$

the $n \times r_i$ submatrix of the last r_i columns of M. Our goal is to show that for all $0 \leq i \leq \lambda_m$,

M_i is the unique matrix of the form

$$
\begin{bmatrix}
\star & & \cdots & \star \\
I_{r_i - r_{i-1}} & & & \vdots \\
& \ddots & & \\
\vdots & & I_{r_2 - r_1} & \\
0 & \cdots & & I_{r_1}
\end{bmatrix} \qquad (\star)_i
$$

such that for all $j \leq i$, its last r_j columns span V^i. The injectivity of f is equivalent to showing this holds for $i = \lambda_m$.

We proceed by induction on i. For the case of $i = 0$, we have $M_0 = 0$, so it is trivial. Now assume that M_{i-1} is the unique matrix of the form $(\star)_{i-1}$ such that for all $j \leq i - 1$, its last r_j columns span V^j. Note that M_i has form

$$
M_i = \begin{bmatrix}
\star \\
I_{r_i - r_{i-1}} \quad M_{i-1} \\
0
\end{bmatrix}.
$$

Suppose we have another $n \times r_i$ matrix M_i' of the form $(\star)_i$ such that for all $0 \leq j \leq i$, its last r_j columns span V^j. Then the submatrix $M_i'[1 : n, (r_i - r_{i-1} + 1) : n]$ of the last r_{i-1} columns of M_i' is a matrix of the form $(\star)_{i-1}$ such that for all $j \leq i - 1$, its last r_j columns span V^i. By the induction hypothesis, $M_i'[1 : n, (r_i - r_{i-1} + 1) : n] = M_{i-1}$. Hence, M_i' has form

$$
M_i' = \begin{bmatrix}
\star \\
I_{r_i - r_{i-1}} \quad M_{i-1} \\
0
\end{bmatrix}.
$$

The bottom $r_{i-1} \times r_{i-1}$ submatrix of M_{i-1} is of full rank r_{i-1} since it is an upper triangular matrix with 1's along the diagonal. Since M_i and M_i' are both of full rank r_i, there is $A \in \mathrm{GL}_{r_i}$ such that $M_i = M_i' A$. By Lemma 3.40, we must have $A = I_{r_i}$, so $M_i = M_i'$.

Therefore, by induction, we have shown that f is injective.

We now show surjectivity of f. Suppose $V^\bullet \in U$ and $M \in \mathrm{B}_n$ is an $n \times n$ matrix representing $V^\bullet$. To show that there is a matrix in Z which represent $V^\bullet$, it is enough to show that there is an $n \times n$ matrix A of the form

$$A = \begin{bmatrix} S_{n-r_{\lambda_m-1}} & & & 0 \\ & \ddots & & \\ 0 & & S_{r_2-r_1} & \\ & & & S_{r_1} \end{bmatrix}$$

such that $MA \in Z$.

For each i, again denote the $n \times r_i$ submatrix $M_i := M[1:n, (n-r_i+1):n]$. We proceed by induction on i to show that there is a matrix A_j of the form

$$A_i = \begin{bmatrix} S_{r_i-r_{i-1}} & & & 0 \\ & \ddots & & \\ 0 & & S_{r_2-r_1} & \\ & & & S_{r_1} \end{bmatrix}, \qquad (\star\star)_i$$

such that

$$M_i A_i = \begin{bmatrix} \star & & \cdots & & \star \\ I_{r_i-r_{i-1}} & & & & \vdots \\ & & \ddots & & \\ \vdots & & & I_{r_2-r_1} & \\ 0 & & \cdots & & I_{r_1} \end{bmatrix}. \qquad (\star\star\star)_i$$

For $i = 1$, We have

$$M_1 = \begin{bmatrix} M_{1,T} \\ M_{i,B} \end{bmatrix}$$

where $M_{1,T}$ is the top $(n - r_1) \times (n - r_1)$ submatrix of M_i and $M_{i,B}$ is the bottom $r_1 \times r_1$

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submatrix of M_1. Then M_1 is an upper triangular matrix, so there is $A_1 \in \mathrm{GL}_{r_1}$ such that $M_{1,B} A_1 = I_{r_1}$. Then $M_1 A_1$ has form $(\star\star\star)_1$.

For $i > 1$, assume that there is a matrix A_{i-1} of form $(\star\star)_{i-1}$ such that $M_{i-1} A_{i-1}$ has form $(\star\star\star)_{i-1}$. We can write M_i in the following form

$$M_i = \begin{bmatrix} M_{i,T} \\ M_{i,B} \quad M_{i-1} \\ 0 \end{bmatrix}$$

where $M_{i,T}$ is the top left $(n - r_i) \times (r_i - r_{i-1})$ submatrix of M_i and $M_{i,B}$ is the $r_i - r_{i-1} \times r_i - r_{i-1}$ submatrix of M_i directly below $M_{i,T}$. Since $M \in \mathrm{B}_n$, $M_{i,B}$ is an upper triangular matrix with nonzero diagonals, so it is of full rank $r_i - r_{i-1}$. Hence, there is a matrix $A \in \mathrm{GL}_{r_i - r_{i-1}}$ such that $M_{i,B} A = I_{r_i - r_{i-1}}$. Let

$$A_i = \begin{bmatrix} A & 0 \\ 0 & A_{i-1} \end{bmatrix}.$$

Then $M_i A_i$ is of form

$$M_i A_i = \begin{bmatrix} M_{i,T} A \\ M_{i,B} A \quad M_{i-1} A_{i-1} \\ 0 \end{bmatrix} = (\star\star\star)_i.$$

Hence, we are done by induction. Therefore, f is surjective.

So far, we have shown that f is an isomorphism. Since it is an isomorphism, we see that

$$\dim(U) = \sum_{i=1}^{\lambda_m - 1} r_i(r_{i+1} - r_i) = \dim(\mathrm{Fl}_\lambda).$$

Hence $\overline{U} = \mathrm{Fl}_\lambda$, so U is in fact an *open* neighborhood of $E^\bullet$ in Fl_λ. $\qquad\square$

Lemma 3.39. Suppose $f : X \to S$ is a morphism of irreducible algebraic varieties. Suppose S is furthermore normal. If f is bijective, then it is an isomorphism of algebraic varieties.

Proof. The proof is based on [hs]. By Zariski's Main Theorem [Gro67, Corollary 18.12.13], there is a factorization of f

$$X \xrightarrow{\ \ j\ \ } T$$

such that j is an open immersion and π is a finite morphism. Identifying X with its image, we treat it as an open set in T. By replacing T with $\overline{X} \subset T$, we may assume that T is irreducible. Furthermore, since f is surjective, π is surjective.

Consider $Z = T \setminus X$. Since π is a finite morphism, it is a closed morphism and $\dim(S) = \dim(T)$. Then $Z' := f(Z)$ is a proper closed set of S, since $\dim(Z') \leq \dim(Z) < \dim(T) = \dim(S)$. Then we have $V := S \setminus Z'$ is a nonempty *open* set (hence, a dense open set since S is irreducible) such that $f^{-1}(V) \subseteq X$. Since f is bijective, we have that for any point $x \in V$, $|f^{-1}(x)| = 1$. Then π is a finite morphism of degree 1. Since S is normal, π must therefore be an isomorphism. But then we must have $X = T$. $\qquad\square$

Lemma 3.40. Let $0 < l < r \leq n$. Let M be an $n \times r$ matrix. We write

$$M = \begin{bmatrix} M_L & M_R \end{bmatrix}$$

where M_L is the left $n \times (r - l)$ submatrix and M_R is the right $n \times l$ submatrix. Suppose that

$$M_L = \begin{bmatrix} M_{L,T} \\ I_{r-l} \\ 0 \end{bmatrix}$$

where $M_{L,T}$ is its top $(n-r) \times (r-l)$ submatrix, and that

$$M_R = \begin{bmatrix} M_{R,T} \\ M_{R,B} \end{bmatrix}$$

where $M_{R,T}$ is its top $(n-l) \times l$ submatrix and $M_{R,B}$ is of full rank l. If $A \in M_r$ is such that

$$MA = \begin{bmatrix} \star \\ I_{r-l} \quad M_R \\ 0 \end{bmatrix},$$

then $A = I_r$.

Proof. Write

$$A = \begin{bmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{bmatrix}$$

where A_{TL} is the top left $(r-l) \times (r-l)$ submatrix, A_{BL} is the bottom left $l \times (r-l)$ submatrix, A_{TR} is the top right $(r-l) \times l$ submatrix, and A_{BR} is the bottom right $l \times l$ submatrix. Then we have the following.

- We have

$$\begin{bmatrix} M_{L,T}A_{TL} \\ A_{TL} \\ 0 \end{bmatrix} + \begin{bmatrix} M_{R,T}A_{BL} \\ M_{R,B}A_{BL} \end{bmatrix} = M_L A_{TL} + M_R A_{BL} = \begin{bmatrix} \star \\ I_{r-l} \\ 0 \end{bmatrix}.$$

So $M_{R,B}A_{BL} = 0$. However $M_{R,B}$ is of full rank l, so we must have that $A_{BL} = 0$.

- Since $A_{BL} = 0$, we get $A_{TL} = I_{r-l}$.

- Next, we have

$$\begin{bmatrix} M_{L,T} A_{TR} \\ A_{TR} \\ 0 \end{bmatrix} + \begin{bmatrix} M_{R,T} A_{BR} \\ M_{R,B} A_{BR} \end{bmatrix} = M_L A_{TR} + M_R A_{BR} = M_R = \begin{bmatrix} M_{R,T} \\ M_{R,B} \end{bmatrix}.$$

So $M_{R,B} A_{BR} = M_{R,B}$. Again, since $M_{R,B}$ is of full rank l and hence invertible, $A_{BR} = I_l$.

- Continuing the previous point, we get that since $A_{BR} = I_l$,

$$\begin{bmatrix} M_{L,T} A_{TR} \\ A_{TR} \end{bmatrix} + M_{R,T} = M_{R,T}.$$

This implies that $A_{TR} = 0$.

Thus we get that $A = I_r$. $\qquad \square$

Definition 3.41. The Nilpotent Cone $N \subset M_n$ is the closed set of nilpotent matrices, i.e

$$N = \{ M \in M_n, M^n = 0 \}.$$

Let G act on M_n by $(D, \varsigma) \cdot M = \varsigma D M D^{-1}$ for $(D, \varsigma) \in G$ and $M \in M_n$. N is invariant under this action, so we induce an action of G on N.

Let $J_{k,p}$ denote the matrix of p Jordan blocks of size k with 0 along the diagonal. For

example, we have

$$
J_{3,2} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
$$

Definition 3.42. Let λ be a Young diagram partitioning n. Denote

$$
\sigma_\lambda := GL_n \cdot \begin{bmatrix} J_{\lambda_1,a_1} & & 0 \\ & \ddots & \\ 0 & & J_{\lambda_m,a_m} \end{bmatrix} = \left\{ M \begin{bmatrix} J_{\lambda_1,a_1} & & 0 \\ & \ddots & \\ 0 & & J_{\lambda_m,a_m} \end{bmatrix} M^{-1}, \quad M \in \mathrm{GL}_n \right\}.
$$

Let $N_\lambda := \overline{\sigma_\lambda} \subset N \subset M_n$. Then N_λ is invariant under the GL_n-action and σ_λ is open in N_λ.

Definition 3.43. Let $M_n \times \mathrm{Fl}_\lambda$ have the product G-action. Define the sets

$$
\widetilde{N}_\lambda := \left\{ (M, V^\bullet) \in M_n \times \mathrm{Fl}_\lambda \mid MV^i \subseteq V^{i-1}, \ \forall i = 1, \ldots, \lambda_m \right\}
$$

and

$$
\widetilde{\sigma}_\lambda := \left\{ (M, V^\bullet) \in M_n \times \mathrm{Fl}_\lambda \mid MV^i = V^{i-1}, \ \forall i = 1, \ldots, \lambda_m \right\}.
$$

These sets are G-equivariant.

Lemma 3.44. [BM83] $\widetilde{N}_\lambda$ is closed in $M_n \times \mathrm{Fl}_\lambda$ and $\widetilde{\sigma}_\lambda$ is open in $\widetilde{N}_\lambda$. Furthermore, we have the following diagram:

$$
\begin{array}{ccccccc}
\widetilde{\sigma}_\lambda = \rho^{-1}(\sigma_\lambda) & \subset & \widetilde{N}_\lambda & \subset & M_n \times \mathrm{Fl}_\lambda & & \\
\shortparallel & & \downarrow & & \downarrow{\scriptstyle \rho} & \searrow{\scriptstyle \pi} & \\
\sigma_\lambda & \subset & N_\lambda & \subset & M_n & & \mathrm{Fl}_\lambda
\end{array}
$$

where ρ and π are the projection onto the first and second coordinates respectively.

Note that ρ, π, and the inclusions in the top row of the diagram in Lemma 3.44 are all G-equivariant morphisms. Since $\widetilde{\sigma}_\lambda$ and $\widetilde{N}_\lambda$ are invariant under G, we get that σ_λ and N_λ are invariant under G. Then the diagram in Lemma 3.44 is in fact a diagram of G-equivariant morphisms, and σ_λ is in fact a G-orbit.

Any complex matrix has a Jordan decomposition. Furthermore, a Jordan normal matrix is nilpotent if and only if its diagonal entries are zero. Hence, we have the following.

Proposition 3.45. The G-orbits of N are precisely the σ_λ where λ runs over the Young diagrams partitioning n.

We are now ready to show the main goal of section §3.2.

3.2.2: Motivic Chern Class of the Nilpotent Cone

Our goal in this section is to compute the G-equivariant motivic Chern classes

$$\mathrm{mC}_y^G(\sigma_\lambda \subset M_n)$$

of the orbits σ_λ of the Nilpotent Cone for every Young diagram λ, thereby computing the motivic Chern class of the Nilpotent cone in M_n. The computation will follow the ideas of [FRW21] where motivic Chern class of matrix Schubert cells are computed. The following are some of the results in [FRW21] that we will be using in our computation.

Definition 3.46 (Weight function). [FRW21, Def 7.1] For $k \leq m$ and $I = \{i_1 < \ldots < i_d\} \subseteq \{1, \ldots, m\}$ such that $|I| = d \leq k$, let $\gamma = (\gamma_1, \ldots, \gamma_k)$ and $\delta = (\delta_1, \ldots, \delta_m)$ be two sets of variables. Define

$$W_{k,m,I}(\gamma, \delta) := \frac{1}{(k-d)!} \sum_{\sigma \in \mathbf{S}_k} U_{k,m,I}(\sigma(\gamma), \delta),$$

where

$$U_{k,m,I}(\gamma, \delta) = \prod_{u=1}^{k} \prod_{v=1}^{m} \psi_{I,u,v}(\gamma_u/\delta_v) \cdot \prod_{u=1}^{d} \prod_{v=u+1}^{k} \frac{1 + y\frac{\gamma_v}{\gamma_u}}{1 - \frac{\gamma_v}{\gamma_u}}$$

and

$$\psi_{I,u,v}(\xi) = \begin{cases} 1 - \xi & u > d \text{ or } v < i_u \\ (1+y)\xi & u \leq d \text{ and } v = i_u \\ 1 + y\xi & u \leq d \text{ and } v > i_u. \end{cases}$$

For $k \leq m$, consider the action of the torus $\mathbb{T}^k \times \mathbb{T}^m$ on $M_{m,k}$ given by

$$(A, B) \cdot M = BMA^{-1}, \quad \text{for } A \in \mathbb{T}^k, B \in \mathbb{T}^m, M \in M_{m,k}.$$

Definition 3.47. For $r \leq k$, define

$$\Sigma_{k,m}^r := \{M \in M_{m,k} | \mathrm{rk}M = k - r\}$$

This locally closed subvariety is invariant under the action of $\mathbb{T}^k \times \mathbb{T}^m$.

Definition 3.48. For $I \subseteq \{1, \ldots, m\}$ such that $|I| \leq k$, define

$$\Omega_{k,m,I} := \{M \in M_{m,k} | \forall 0 \leq r \leq n, \ \mathrm{rk}(\text{top r rows of M}) = |I \cap (1, \ldots, r)|\}.$$

This is locally closed subvariety invariant under the action of $\mathbb{T}^k \times \mathbb{T}^m$.

Theorem 3.49. [FRW21, Theorem 7.4] For every $0 \leq k \leq m$ and $I \subseteq \{1, \ldots, m\}$ such that $|I| \leq k$, we have

$$\mathrm{mC}_y^{\mathbb{T}^k \times \mathbb{T}^m} (\Omega_{k,m,I} \subset M_{m,k}) = W_{k,m,I}$$

in $\mathbb{Z}[\gamma_1^{\pm 1}, \ldots, \gamma_k^{\pm 1}, \delta_1^{\pm 1}, \ldots, \delta_m^{\pm 1}, y]$.

Theorem 3.50. Let $\mathbb{T}^k \times \mathbb{T}^m$ act on $M_{m,k}$ as defined above. Define the action $\mathbb{T}^k \times \mathbb{T}^m \times \mathbb{C}^\star$ on $M_{m,k}$ by $(A, B, \xi) \cdot M = (A, B) \cdot \xi M$. Then for $I \subset \{1, \ldots, m\}$ such that $|I| \leq k$, we have that $\Omega_{k,m,I}$ is invariant under the action of $\mathbb{T}^k \times \mathbb{T}^m \times \mathbb{C}^\star$ and

$$\mathrm{mC}_y^{\mathbb{T}^k \times \mathbb{T}^m \times \mathbb{C}^\star}(\Omega_{k,m,I} \subset M_{m,k}) = W_{k,m,I}(\beta\gamma, \delta)$$

in $\mathbb{Z}[\gamma_1^{\pm 1}, \ldots, \gamma_k^{\pm 1}, \delta_1^{\pm 1}, \ldots, \delta_m^{\pm 1}, \beta^{\pm 1}, y]$.

Proof. Consider the morphism of algebraic groups

$$\phi : \mathbb{T}^k \times \mathbb{T}^m \times \mathbb{C}^\star \to \mathbb{T}^k \times \mathbb{T}^m$$

$$(A, B, \xi) \mapsto (\xi A, B)$$

This induces

$$\phi^\star : R(\mathbb{T}^k \times \mathbb{T}^m) = \mathbb{Z}[\gamma^{\pm 1}, \delta^{\pm 1}] \longrightarrow R(\mathbb{T}^k \times \mathbb{T}^m \times \mathbb{C}^\star) = \mathbb{Z}[\gamma^{\pm 1}, \delta^{\pm 1}, \beta^{\pm 1}]$$

$$\gamma, \delta \longmapsto \beta\gamma, \delta$$

where $\gamma = (\gamma_1, \ldots, \gamma_k)$ and $\delta = (\delta_1, \ldots, \delta_m)$. We see that

$$\phi^\star \mathrm{mC}_y^{\mathbb{T}^k \times \mathbb{T}^m}(\Omega_{k,m,I} \subset M_{m,k}) = \mathrm{mC}_y^{\mathbb{T}^k \times \mathbb{T}^m \times \mathbb{C}^\star}(\Omega_{k,m,I} \subset M_{m,k})$$

in their fraction rings, since the action of $\mathbb{T}^k \times \mathbb{T}^m \times \mathbb{C}^\star$ on $M_{m,k}$ acts via the group homomorphism ϕ. Hence Theorem 3.49 gives the proposition. $\square$

Theorem 3.51. In the same set-up as Thm 3.50, we have that for every $0 \le k \le m$ and $r \le k$, $\Sigma_{k,m}^r$ is invariant under the action of $\mathbb{T}^k \times \mathbb{T}^m \times \mathbb{C}^\star$ on $M_{m,k}$ and

$$\mathrm{mC}_y^{\mathbb{T}^k \times \mathbb{T}^m \times \mathbb{C}^\star}(\Sigma_{k,m}^r \subset M_{m,k}) = \sum_{\substack{I \subset \{1,\ldots,m\} \\ |I| = k-r}} \mathrm{mC}_y^{\mathbb{T}^k \times \mathbb{T}^m \times \mathbb{C}^\star}(\Omega_{k,m,I} \subset M_{m,k})$$

$$= \sum_{\substack{I \subset \{1,\ldots,m\} \\ |I| = k-r}} W_{k,m,I}(\beta\gamma, \delta)$$

in $\mathbb{Z}[\gamma_1^{\pm 1}, \ldots, \gamma_k^{\pm 1}, \delta_1^{\pm 1}, \ldots, \delta_m^{\pm 1}, \beta^{\pm 1}, y]$.

Proof. This follows from the same argument for [FRW21, Theorem 8.2] using Thm 3.50. $\square$

Proposition 3.52 ([FRW21], Prop 7.5)**.** Let $\mathbb{T}$ be a torus. Let K be a complete smooth $\mathbb{T}$-variety with finitely many fixed points and V a $\mathbb{T}$-vector space with no zero weight. Denote $M := V \times K$. Then for every $\mathfrak{c} \in K^{\mathbb{T}}(M)$, we have

$$(\pi_V)_\star(\mathfrak{c})|_0 = \sum_{z \in M^{\mathbb{T}}} \mathfrak{c} \cdot \frac{\lambda_{-1}(T_0^\star V)}{\lambda_{-1}(T_z^\star M)}$$

in $R(\mathbb{T})$, where $\pi_V : M \to V$ is the projection,

$$(\pi_V)_\star : K^{\mathbb{T}}(M) \longrightarrow K^{\mathbb{T}}(V)$$

is the proper pushforward defined in Proposition 3.8,

$$|_0 : K^{\mathbb{T}}(M) \to R(\mathbb{T})$$

is the restriction to 0 defined in Definition 3.15, and $M^{\mathbb{T}}$ is the set of torus-fixed points of M.

The main result of this section is the following.

Theorem 3.53. For every Young diagram λ partitioning n,

$$\mathrm{mC}_y^G(\sigma_\lambda \subset M_n) = \frac{1}{\prod_{i=1}^{\lambda_m}(r_i - r_{i-1})!} \sum_{\omega \in \mathbf{S}_n} V(\omega(\alpha), \beta)$$

in $\mathbb{Z}[\alpha_1^{\pm 1}, \ldots, \alpha_n^{\pm 1}, \beta^{\pm 1}, y]$, where

$$V(\alpha, \beta) = \left(\prod_{i=1}^{\lambda_m-1} \prod_{u=n-r_{i+1}+1}^{n-r_i} \prod_{v=n-r_{i+1}}^{n} \frac{1 + y\frac{\alpha_u}{\alpha_v}}{1 - \frac{\alpha_u}{\alpha_v}} \right) \cdot$$

$$\left(\prod_{i=2}^{\lambda_m-1} \prod_{u=n-r_{i+1}+1}^{n-r_i} \prod_{v=n-r_{i+1}}^{n} 1 + y\frac{\beta\alpha_u}{\alpha_v} \right) \left(\prod_{i=0}^{\lambda_m-1} \prod_{u=n-r_{i+1}+1}^{n-r_i} \prod_{v=1}^{n-r_i-1} 1 - \frac{\beta\alpha_u}{\alpha_v} \right) \cdot$$

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$$
\left(\prod_{i=1}^{\lambda_m - 1} \sum_{\substack{I \subset \{1,\ldots,r_{i+1}-r_i\} \\ |I| = r_i - r_{i-1}}} W_{r_{i+1}-r_i, r_{i+1}-r_i, I}((\beta \alpha_{n-r_i+1}, \ldots, \beta \alpha_{n-r_{i-1}}), (\alpha_{n-r_{i+1}+1}, \ldots, \alpha_{n-r_i})) \right) (1 - \beta)^n
$$

Proof. Recall from Lemma 3.44, we have the following diagram:

$$
\begin{array}{ccccc}
\widetilde{\sigma}_\lambda = \rho^{-1}(\sigma_\lambda) & \subset & \widetilde{N}_\lambda & \subset & M := M_n \times \mathrm{Fl}_\lambda \\
\| \wr & & \downarrow & & \downarrow \rho \quad \searrow \pi \\
\sigma_\lambda & \subset & N_\lambda & \subset & M_n \qquad\qquad \mathrm{Fl}_\lambda
\end{array}
$$

where ρ and π are the projection onto the first and second coordinates respectively. By functoriality of motivic Chern classes, we get

$$
\mathrm{mC}_y^G(\sigma_\lambda \subset M_n) = \rho_\star \left(\mathrm{mC}_y^G(\widetilde{\sigma}_\lambda \subset M) \right).
$$

We will now compute $\rho_\star \left(\mathrm{mC}_y^G(\widetilde{\sigma}_\lambda \subset M) \right)$ in the following steps.

Denote $\mathbb{T} := \mathbb{T}^n \times \mathbb{C}^\star$.

Step 1: Consider the following commutative diagram of ring homomorphisms

$$
\begin{array}{ccc}
K^G(\mathrm{M}_n)[y] & \xrightarrow[\sim]{|_0} & R(G)[y] = R(\mathbb{T})^{\mathrm{W_G}}[y] = \mathbb{Z}[\alpha_1^{\pm 1}, \ldots, \alpha_n^{\pm 1}]^{\mathbf{S}_n}[\beta^{\pm 1}, y] \\
\downarrow & & \downarrow \\
K^{\mathbb{T}}(\mathrm{M}_n)[y] & \xrightarrow[\sim]{|_0} & R(\mathbb{T})[y] = R(\mathbb{T}^n)[\beta^{\pm 1}, y] = \mathbb{Z}[\alpha_1^{\pm 1}, \ldots, \alpha_n^{\pm 1}, \beta^{\pm 1}, y]
\end{array}
$$

where the vertical maps are injective from Proposition 3.17. To determine $\rho_\star \left(\mathrm{mC}_y^G(\widetilde{\sigma}_\lambda \subset M) \right)$, it is enough to look at its restriction to the action of $\mathbb{T} \subset G$ and then its restriction to 0. Note that $y \in M$ is a $\mathbb{T}-$ fixed point if and only if $y = 0$. This is clear because if $M \neq 0$, then $\zeta I_n M I_n^{-1} = \zeta M \neq M$ for $\zeta \neq 1$. Now, applying Proposition 3.52 to Fl_λ, M_n, and M with the $\mathbb{T}$ action defined above, we have

$$
\rho_\star \left(\mathrm{mC}_y^{\mathbb{T}}(\widetilde{\sigma}_\lambda \subset M) \right)\big|_0 = \sum_{z \in M^{\mathbb{T}}} \mathrm{mC}_y^{\mathbb{T}}(\widetilde{\sigma}_\lambda \subset M)\big|_z \frac{\lambda_{-1}^{\mathbb{T}}(T_0^\star M_n)}{\lambda_{-1}^{\mathbb{T}}(T_x^\star M)}
$$

$$= \sum_{x \in \mathrm{Fl}_\lambda^{\mathbb{T}}} \mathrm{mC}_y^{\mathbb{T}}(\widetilde{\sigma}_\lambda \subset M)\big|_{(0,x)} \frac{\lambda_{-1}^{\mathbb{T}}(T_0^\star M_n)}{\lambda_{-1}^{\mathbb{T}}\left(T_{(0,x)}^\star M\right)}$$

$$= \sum_{x \in \mathrm{Fl}_\lambda^{\mathbb{T}}} \frac{\mathrm{mC}_y^{\mathbb{T}}(\widetilde{\sigma}_\lambda \subset M)\big|_{(0,x)}}{\lambda_{-1}^{\mathbb{T}}(T_x^\star \mathrm{Fl}_\lambda)}$$

$$\overset{\mathrm{Rmk}\ 3.36}{=} \sum_{\omega W_{P_\lambda} \in W^P} \frac{\mathrm{mC}_y^{\mathbb{T}}(\widetilde{\sigma}_\lambda \subset M)\big|_{(0,\omega \cdot E^\bullet)}}{\lambda_{-1}^{\mathbb{T}}(T_{\omega \cdot E^\bullet}^\star \mathrm{Fl}_\lambda)}. \tag{$\star$}$$

Step 2: We next simplify $(\star)$.

For each permutation $\omega \in W_{\mathrm{GL}_n} = \mathbf{S}_n \subset \mathrm{GL}_n$, consider the group isomorphism

$$\omega : \mathbb{T} \overset{\sim}{\longrightarrow} \mathbb{T}$$

$$\left(D = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}, \xi \right) \longmapsto \left(\omega D \omega^{-1} = \begin{bmatrix} a_{\omega^{-1}(1)} & & 0 \\ & \ddots & \\ 0 & & a_{\omega^{-1}(n)} \end{bmatrix}, \xi \right)$$

which induces the isomorphism of rings

$$R(\mathbb{T}) \overset{\sim}{\longleftarrow} R(\mathbb{T}) : \omega^\star$$

$$f(\omega(\alpha), \beta) = f\left(\alpha_{\omega^{-1}(1)}, \ldots \alpha_{\omega^{-1}(n)}, \beta\right) \longleftarrow\!\!\shortmid f(\alpha_1, \ldots, \alpha_n, \beta).$$

We now refer to the notation and remarks in Definition 3.13. We have a commutative square of $\mathbb{T}$-equivariant isomorphisms

$$\begin{array}{ccccc} \omega\widetilde{\sigma}_\lambda & \subset & \omega M & & (\omega M \omega^{-1}, \omega V^\bullet) \\ {\scriptstyle\wr}\big\uparrow & & {\scriptstyle\wr}\big\uparrow{\scriptstyle\omega} & & \big\uparrow \\ \widetilde{\sigma}_\lambda & \subset & M & & (M, V^\bullet) \end{array}$$

which gives

$$\mathrm{mC}_y^{\mathbb{T}}(\omega\widetilde{\sigma}_\lambda \subset \omega M)\big|_{(0,\omega \cdot E^\bullet)} = \mathrm{mC}_y^{\mathbb{T}}(\widetilde{\sigma}_\lambda \subset M)\big|_{(0,E^\bullet)}.$$

We also have

$$\omega^\star\left(\left.\mathrm{mC}_y^{\mathbb{T}}(\widetilde{\sigma}_\lambda \subset M))\right|_{(0,\omega\cdot E^\bullet)}\right) = \left.\left(\omega^\star\,\mathrm{mC}_y^{\mathbb{T}}(\widetilde{\sigma}_\lambda \subset M))\right)\right|_{(0,\omega\cdot E^\bullet)} = \left.\mathrm{mC}_y^{\mathbb{T}}(\omega\widetilde{\sigma}_\lambda \subset \omega M)\right|_{(0,\omega\cdot E^\bullet)}.$$

Therefore $(\star)$ becomes

$$\left.\rho_\star\left(\mathrm{mC}_y^{\mathbb{T}}(\widetilde{\sigma}_\lambda \subset M))\right|_0 = \sum_{\omega\dot{W}_{P_\lambda}\in W^{P_\lambda}} \frac{(\omega^\star)^{-1}\,\left.\mathrm{mC}_y^{\mathbb{T}}(\widetilde{\sigma}_\lambda \subset M)\right|_{(0,E^\bullet)}}{\lambda_{-1}^{\mathbb{T}}(T_{E^\bullet}^\star \mathrm{Fl}_\lambda)}$$

$$= \frac{1}{|W_{P_\lambda}|}\sum_{\omega\in\mathbf{S}_n} \frac{(\omega^\star)^{-1}\,\left.\mathrm{mC}_y^{\mathbb{T}}(\widetilde{\sigma}_\lambda \subset M)\right|_{(0,E^\bullet)}}{\lambda_{-1}^{\mathbb{T}}(T_{E^\bullet}^\star \mathrm{Fl}_\lambda)}$$

$$= \frac{1}{\prod_{i=1}^{\lambda_m}(r_i - r_{i-1})!}\sum_{\omega\in\mathbf{S}_n} \frac{(\omega^\star)^{-1}\,\left.\mathrm{mC}_y^{\mathbb{T}}(\widetilde{\sigma}_\lambda \subset M)\right|_{(0,E^\bullet)}}{\lambda_{-1}^{\mathbb{T}}(T_{E^\bullet}^\star \mathrm{Fl}_\lambda)}.$$

It remains to determine

$$\frac{\left.\mathrm{mC}_y^{\mathbb{T}}(\widetilde{\sigma}_\lambda \subset M)\right|_{(0,E^\bullet)}}{\lambda_{-1}^{\mathbb{T}}(T_{E^\bullet}^\star \mathrm{Fl}_\lambda)}. \qquad\qquad (\star\star)$$

Step 3: We now compute $(\star\star)$. Take the Schubert cell

$$U = \mathrm{B}_n \cdot E^\bullet \subset \mathrm{Fl}_\lambda.$$

By Prop 3.38, $U \subset \mathrm{Fl}_\lambda$ is an open $\mathbb{T}$-invariant neighborhood of $E^\bullet$ and we have the $\mathbb{T}$-

equivariant isomorphism

$$U \simeq \left\{ \begin{bmatrix} I_{n-r_{\lambda_m-1}} & & & \star \\ & \ddots & & \\ 0 & & I_{r_2-r_1} & \\ & & & I_{r_1} \end{bmatrix} \in \mathrm{GL}_n \right\} \subset \mathrm{B}_n \subset \mathrm{GL}_n$$

where $\mathbb{T}$ acts on GL_n by $(D,\zeta) \cdot M = DMD^{-1}$. By the localization property of motivic Chern classes in Theorem 3.22, we get

$$\mathrm{mC}_y^{\mathbb{T}}(U \subset U)|_{E^\bullet} = \mathrm{mC}_y^{\mathbb{T}}(\mathrm{Fl}_\lambda \subset \mathrm{Fl}_\lambda)|_{E^\bullet} = \lambda_y^{\mathbb{T}}(T_{E^\bullet}^\star \mathrm{Fl}_\lambda).$$

From Example 3.28, we compute

$$\mathrm{mC}_y^{\mathbb{T}}(U \subset U)|_{E^\bullet} = \prod_{i=1}^{\lambda_m-1} \prod_{u=n-r_{i+1}+1}^{n-r_i} \prod_{v=n-r_{i+1}}^{n} \left(1 + y\frac{\alpha_u}{\alpha_v}\right).$$

Hence $(\star\star)$ becomes

$$\frac{\mathrm{mC}_y^{\mathbb{T}}(\widetilde{\sigma}_\lambda \subset M)\big|_{(0,E^\bullet)}}{\prod_{i=1}^{\lambda_m-1} \prod_{u=n-r_{i+1}+1}^{n-r_i} \prod_{v=n-r_{i+1}}^{n} \left(1 - \frac{\alpha_u}{\alpha_v}\right)}.$$

Step 4: We refer to the diagram and notation in Lemma 3.44. Let

$$Z := \pi^{-1}(\{E^\bullet\}) \cap \widetilde{N}_\lambda = \{M \in M_n | ME^i \subseteq E^{i-1}, \forall i = 1, \ldots, \lambda_m\} \subset M_n.$$

be the fiber over $E^\bullet \in \mathrm{Fl}_\lambda$ of the map $\pi : \widetilde{N}_\lambda \to \mathrm{Fl}_\lambda$. Define the open set in Z

$$Z^o := \pi^{-1}(\{E^\bullet\}) \cap \widetilde{\sigma}_\lambda := \{M \in M_n | ME^i = E^{i-1}, \forall i = 1, \ldots, \lambda_m\}.$$

We can describe $\widetilde{N}_\lambda \cap \pi^{-1}(U)$ and $\widetilde{\sigma}_\lambda \cap \pi^{-1}(U)$ in terms of Z, Z^o as follows:

$$\widetilde{N}_\lambda \cap \pi^{-1}(U) = \{(bMb^{-1}, b \cdot E^\bullet) \,\big|\, b \cdot E^\bullet \in U, M \in T^\star_{E^\bullet}\mathrm{Fl}_\lambda\}$$

and

$$\widetilde{\sigma}_\lambda \cap \pi^{-1}(U) = \{(bMb^{-1}, b \cdot E^\bullet) \,\big|\, b \cdot E^\bullet \in U, M \in Z^o\}.$$

Thus restricting the diagram in Lemma 3.44 to U, we get the commutative diagram

$$
\begin{array}{ccccccccc}
Z^o \times U & \subset & Z \times U & \subset & M_n \times U & \subset & M_n \times \mathrm{B}_n & & (b^{-1}Mb, b) \\
\uparrow\wr & & \uparrow\wr & & \uparrow\wr & & \uparrow\wr & & \Big\updownarrow \\
\widetilde{\sigma}_\lambda \cap \pi^{-1}(U) & \subset & \widetilde{N}_\lambda \cap \pi^{-1}(U) & \subset & \pi^{-1}(U) = M_n \times U & \subset & M_n \times \mathrm{B}_n & & (M, b) \\
\downarrow\wr & & \downarrow\rho & & \downarrow\rho & \searrow^{\pi} & & & \\
\sigma_\lambda & \subset & N_\lambda & \subset & M_n & & U & \subset & \mathrm{B}_n
\end{array}
$$

.

$$b \cdot E^\bullet \longmapsto b.$$

The vertical and diagonal maps in the bottom half of the diagram and all the horizontal maps in the diagram are the $\mathbb{T}$-equivariant maps which have already appeared earlier in this section. The only new map are the vertical isomorphisms in the top half. We see that for every $(D, \xi) \in \mathbb{T}$ and $M \in M_n$, $b \in \mathrm{B}_n$, this map sends

$$(D, \xi) \cdot (M, b) = (\xi DMD^{-1}, DbD^{-1})$$

to

$$((DbD^{-1})^{-1}(\zeta DMD^{-1})(DbD^{-1}), DbD^{-1}) = (\zeta Db^{-1}MbD^{-1}, DbD^{-1}) = (D, \xi) \cdot (b^{-1}Mb, b).$$

Hence it is $\mathbb{T}$-equivariant.

Then the numerator in $(\star\star)$ becomes

$$\mathrm{mC}_y^{\mathbb{T}}(\widetilde{\sigma}_\lambda \subset M)\big|_{(0,E^\bullet)} \overset{\text{Theorem 3.22, Localness}}{=\!=\!=} \mathrm{mC}_y^{\mathbb{T}}(\widetilde{\sigma}_\lambda \cap \pi^{-1}(U) \subset M_n \times U)\big|_{(0,E^\bullet)}$$

$$= \mathrm{mC}_y^{\mathbb{T}}(Z^o \times U \subset M_n \times U)\big|_{(0,E^\bullet)}$$

$$\overset{\text{Lemma 3.27}}{=} \left(\mathrm{mC}_y^{\mathbb{T}}(Z^o \subset M_n)\big|_0\right)\left(\mathrm{mC}_y^{\mathbb{T}}(U \subset U)\big|_{E^\bullet}\right)$$

$$= \left(\mathrm{mC}_y^{\mathbb{T}}(Z^o \subset M_n)\big|_0\right)\left(\mathrm{mC}_y^{\mathbb{T}}(\mathrm{Fl}_\lambda \subset \mathrm{Fl}_\lambda)\big|_{E^\bullet}\right)$$

$$= \left(\mathrm{mC}_y^{\mathbb{T}}(Z^o \subset M_n)\big|_0\right)\prod_{i=1}^{\lambda_m-1}\prod_{u=n-r_{i+1}+1}^{n-r_i}\prod_{v=n-r_{i+1}}^{n}\left(1+y\frac{\alpha_u}{\alpha_v}\right).$$

So we are left with computing

$$\mathrm{mC}_y^{\mathbb{T}}(Z^o \subset M_n)\big|_0. \qquad\qquad (\star\star\star)$$

Step 5: We now compute $(\star\star\star)$. First, we can explicitly describe

$$Z^o = \left\{\begin{bmatrix} Z_{n-r_{\lambda_m-1}} & S_{n-r_{\lambda_m-1},r_{\lambda_m-1}-r_{\lambda_m-2}} & & \star & & \star \\ & Z_{r_{\lambda_m-1}-r_{\lambda_m-2},r_{\lambda_m-2}-r_{\lambda_m-3}} & \ddots & & & \\ & & \ddots & S_{r_3-r_2,r_2-r-1} & & \star \\ & & & Z_{r_2-r_1} & S_{r_2-r_1,r_1} & \\ 0 & & & & & Z_{r_1} \end{bmatrix}\right\} \subset M_n,$$

where the action on Z^o comes from the action on M_n given by

$$(D,\zeta)\cdot M = \zeta D M D^{-1} \text{ for } D \in \mathbb{T}^n, \zeta \in \mathbb{C}^\star, M \in M_n.$$

Recall that

$$M_{n-r_{i+1}+1,n-r_i+1,r_{i+1}-r_i,r_i-r_{i-1}}$$

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denotes the affine space $M_{r_{i+1}-r_i,r_i-r_{i-1}}$ considered as a closed $\mathbb{T}$-subspace of M_n via the closed embedding

$$\iota_{n-r_{i+1}+1,n-r_i+1,r_{i+1}-r_i,r_i-r_{i-1}} : M_{r_{i+1}-r_i,r_i-r_{i-1}} \hookrightarrow M_n$$

described in Example 3.29. Following the computation in Example 3.29, if we can show that for every $i = 1, \ldots, \lambda_m - 1$, the open subset

$$S_{r_{i+1}-r_i,r_i-r_{i-1}} \subset M_{n-r_{i+1}+1,n-r_i+1,r_{i+1}-r_i,r_i-r_{i-1}}$$

is preserved under the induced $\mathbb{T}$-action, then we can obtain the following

$$\mathrm{mC}_y^{\mathbb{T}}(Z^o \subset M_n)\big|_0 = \underbrace{\left(\prod_{i=2}^{\lambda_m-1} \prod_{u=n-r_{i+1}+1}^{n-r_i} \prod_{v=n-r_i+1}^{n} 1 + y\frac{\beta\alpha_u}{\alpha_v} \right)}_{\star \text{ components}} \underbrace{\left(\prod_{i=0}^{\lambda_m-1} \prod_{u=n-r_{i+1}+1}^{n-r_i} \prod_{v=1}^{n-r_i-1} 1 - \frac{\beta\alpha_u}{\alpha_v} \right)}_{\text{non-diagonal zero components}} .$$

$$\underbrace{\left(\prod_{i=1}^{\lambda_m-1} \mathrm{mC}_y^{\mathbb{T}}(S_{r_{i+1}-r_i,r_i-r_{i-1}} \subset M_{n-r_{i+1}+1,n-r_i+1,r_{i+1}-r_i,r_i-r_{i-1}}) \right)}_{\text{nonsingular block components}} \underbrace{(1-\beta)^n}_{\text{diagonal components}} .$$

We need to check that $S_{r_{i+1}-r_i,r_i-r_{i-1}} \subset M_{n-r_{i+1}+1,n-r_i+1,r_{i+1}-r_i,r_i-r_{i-1}}$ is indeed preserved under the induced $\mathbb{T}$-action and then compute

$$\mathrm{mC}_y^{\mathbb{T}}(S_{r_{i+1}-r_i,r_i-r_{i-1}} \subset M_{n-r_{i+1}+1,n-r_i+1,r_{i+1}-r_i,r_i-r_{i-1}}).$$

To do these tasks, we look at how $\mathbb{T}$ acts on the portion

$$
\begin{array}{ccc}
\overset{(n-r_{i+1}+1,n-r_i+1)}{\bullet} & \cdots & \overset{(n-r_{i+1}+1,n-r_{i-1})}{\bullet} \\[6pt]
\vdots & u < v & \vdots \\[6pt]
\underset{(n-r_i,n-r_i+1)}{\bullet} & \cdots & \underset{(n-r_i,n-r_{i-1})}{\bullet}
\end{array}
$$

of M_n (where u indexes the row and v indexes the column). For $r \leq s$, consider

$$\mathbb{T}_{r,s} := \{(a_r, \ldots, a_s) | (a_1, \ldots, a_n) \in \mathbb{T}^n\}.$$

Thus, for every $M \in M_{n-r_{i+1}+1,n-r_i+1,r_{i+1}-r_i,r_i-r_{i-1}}$, $D = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n i+1 \end{bmatrix} \in \mathbb{T}^n$, and $\zeta \in \mathbb{C}^\star$, we have

$$(D, \zeta) \cdot M = \zeta \begin{bmatrix} a_{n-r_{i+1}+1} & & 0 \\ & \ddots & \\ 0 & & a_{n-r_i} \end{bmatrix} M \begin{bmatrix} a_{n-r_i+1} & & 0 \\ & \ddots & \\ 0 & & a_{n-r_{i-1}} \end{bmatrix}^{-1}.$$

This action preserves the rank of a matrix $M \in M_{n-r_{i+1}+1,n-r_i+1,r_{i+1}-r_i,r_i-r_{i-1}}$. Hence $S_{r_{i+1}-r_i,r_i-r_{i-1}} \subset M_{n-r_{i+1}+1,n-r_i+1,r_{i+1}-r_i,r_i-r_{i-1}}$ is preserved under the $\mathbb{T}$-action on $M_{n-r_{i+1}+1,n-r_i+1,r_{i+1}-r_i,r_i-r_{i-1}}$. Furthermore, since the intersection of the set of indices

$$\{n - r_i + 1, \ldots, n - r_{i-1}\} \cap \{n - r_{i+1} + 1, \ldots, n - r_i\} = \emptyset,$$

we have that $\mathbb{T}$ acts on $M_{n-r_{i+1}+1,n-r_i+1,r_{i+1}-r_i,r_i-r_{i-1}}$ via $\mathbb{T}_{n-r_i+1,n-r_{i-1}} \times \mathbb{T}_{n-r_{i+1}+1,n-r_i} \times \mathbb{C}^\star$ as in Thm 3.50. Then $S_{r_{i+1}-r_i,r_i-r_{i-1}} = \sum_{r_i-r_{i-1},r_{i+1}-r_i}^0$ with this action. Thus we get

$$\mathrm{mC}_y^{\mathbb{T}}(S_{r_{i+1}-r_i,r_i-r_{i-1}} \subset M_{n-r_{i+1}+1,n-r_i+1,r_{i+1}-r_i,r_i-r_{i-1}})$$

$$= \mathrm{mC}_y^{\mathbb{T}^{r_i-r_{i-1}} \times \mathbb{T}^{r_{i+1}-r_i} \times \mathbb{C}^\star}(\Sigma_{r_i-r_{i-1},r_{i+1}-r_i}^0 \subset M_{n-r_{i+1}+1,n-r_i+1,r_{i+1}-r_i,r_i-r_{i-1}})$$

$$\overset{\text{Thm } 3.51}{=} \sum_{\substack{I \subset \{1,\ldots,r_{i+1}-r_i\} \\ |I|=r_i-r_{i-1}}} W_{r_{i+1}-r_i,r_{i+1}-r_i,I}\left((\beta\alpha_{n-r_i+1}, \ldots, \beta\alpha_{n-r_{i-1}}), (\alpha_{n-r_{i+1}+1}, \ldots, \alpha_{n-r_i})\right).$$

Therefore we have

$$
\mathrm{mC}_y^{\mathbb{T}}(Z^o \subset M_n)\big|_0 = \left(\prod_{i=2}^{\lambda_m-1} \prod_{u=n-r_{i+1}+1}^{n-r_i} \prod_{v=n-r_i+1}^{n} 1 + y\frac{\beta\alpha_u}{\alpha_v} \right) \left(\prod_{i=0}^{\lambda_m-1} \prod_{u=n-r_{i+1}+1}^{n-r_i} \prod_{v=1}^{n-r_i-1} 1 - \frac{\beta\alpha_u}{\alpha_v} \right) \cdot
$$

$$
\left(\prod_{i=1}^{\lambda_m-1} \sum_{\substack{I\subset\{1,\dots,r_{i+1}-r_i\} \\ |I|=r_i-r_{i-1}}} W_{r_{i+1}-r_i,r_{i+1}-r_i,I} \left((\beta\alpha_{n-r_i+1},\dots,\beta\alpha_{n-r_{i-1}}), (\alpha_{n-r_{i+1}+1},\dots,\alpha_{n-r_i}) \right) \right) (1-\beta)^n \quad .
$$

Assembling everything together, we get the result. $\qquad\square$

3.3: The Equivariant Motivic Chern Class of the Affine cone over a Smooth Hypersurface

For any $n \geq 0$, consider $\mathbb{P}^n$ the projective space of dimension n. For any $d \geq 0$, let $D \subset \mathbb{P}^n$ be a hypersurface of degree d. In this section, our goal is to compute the $\mathbb{C}^\star$-equivariant motivic Chern class of the affine cone over D. This is joint work with Sridhar Venkatesh.

3.3.1: Smooth Hypersurface in Projective Space

We begin by making some computations regarding the motivic Chern class of hypersurfaces in the projective space $\mathbb{P}^n$. These computations will be used in the last section of the chapter when we compute the motivic Chern class of their affine cones.

Recall that $K^0(\mathbb{P}^n) = \mathbb{Z}[h]/(1-h)^{n+1}$ where $h = [\mathcal{O}_{\mathbb{P}^n}(-1)]$. Let $\mathbb{T}^k$ act trivially on $\mathbb{P}^n$, i.e $\mathbb{T}^k$ acts on $\mathbb{P}^n$ via the group homomorphism $\mathbb{T}^k \to 1$. Then we have a ring homomorphism

$$
K^0(\mathbb{P}^n) \hookrightarrow K^{\mathbb{T}^k}(\mathbb{P}^n).
$$

We will consider the class h in $K^{\mathbb{T}^k}(\mathbb{P}^n)$.

By Proposition 3.21, the Euler short exact sequence on $\mathbb{P}^n$

$$0 \longrightarrow \Omega_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0$$

gives

$$\lambda_y\left(\Omega_{\mathbb{P}^n}\right) = \frac{(1 + hy)^{n+1}}{1 + y}.$$

Let $D \subset \mathbb{P}^n$ be a smooth hypersurface of degree d. Denote $\overline{h} := h|_D = [\mathcal{O}_D(-1)]$ in $K_y^{\mathbb{T}^k}(D)$. We have the short exact sequence

$$0 \longrightarrow O_D(-d) \longrightarrow \Omega_{\mathbb{P}^n}|_D \longrightarrow \Omega_D \longrightarrow 0$$

on D. Hence by Proposition 3.21 we have

$$\lambda_y\left(\Omega_D\right) = \frac{\lambda_y\left(\Omega_{\mathbb{P}^n}|_D\right)}{1 + \overline{h}^d y} = \frac{(1 + \overline{h}y)^{n+1}}{(1 + \overline{h}^d y)(1 + y)}$$

$$= \frac{1}{1 + y} \left(\sum_{j=0}^{n+1} \binom{n+1}{j} (\overline{h}y)^j\right) \left(\sum_{i=0}^{\infty} (-\overline{h}^d y)^i\right)$$

$$= \frac{1}{1 + y} \sum_{i=0}^{\infty} \sum_{j=0}^{n+1} (-1)^i \binom{n+1}{j} \overline{h}^{j+di} y^{i+j}$$

$$= \frac{1}{1 + y} \sum_{m=0}^{\infty} y^m \sum_{j=0}^{\min(m,n+1)} (-1)^{m-j} \binom{n+1}{j} \overline{h}^{j+d(m-j)}$$

$$= \left(\sum_{p=0}^{\infty} (-y)^p\right) \left(\sum_{q=0}^{\infty} y^q \sum_{j=0}^{\min(q,n+1)} (-1)^{q-j} \binom{n+1}{j} \overline{h}^{j+d(q-j)}\right)$$

$$= \sum_{m=0}^{\infty} y^m \sum_{q=0}^{m} \sum_{j=0}^{\min(q,n+1)} (-1)^{m-j} \binom{n+1}{j} \overline{h}^{j+d(q-j)}$$

in $K^{\mathbb{T}^k}(D)[[y]]$. Since $\bigwedge^m \Omega_D = 0$ for $m > n - 1$, we have

Equation 3.54.

$$\lambda_y\left(\Omega_D\right) = \sum_{m=0}^{n-1} y^m \sum_{q=0}^{m} \sum_{j=0}^{q} (-1)^{m-j} \binom{n+1}{j} \overline{h}^{j+d(q-j)}$$

in $K_y^{\mathbb{T}^k}(D)$.

We also have

$$R\Gamma(D, \lambda_y(\Omega_D)) = \sum_{p=0}^{n-1} \sum_{q=0}^{n-1} (-1)^q \left[H^q(D, \Omega_D^p)\right] y^p = \sum_{p=0}^{n-1} \chi(D, \Omega_D^p) y^p.$$

Then, by [Ara12], Corollary 17.2.2 and Proposition 17.3.2, we get

Equation 3.55.

$$R\Gamma(D, \lambda_y(\Omega_D)) = \sum_{p=0}^{n-1} y^p \sum_{j=0}^{p} (-1)^j \binom{n+1}{p-j} \binom{n+j-p}{n}.$$

3.3.2: Blow-up of Affine Space at the Origin

In this section, we explain the set-up for the rest of the chapter and make a couple additional computations which will also be needed in the final section. Let $\mathbb{C}^\star$ act by scaling on $\mathbb{A}^{n+1}$, i.e for $x = (x_0, \ldots, x_n) \in \mathbb{A}^{n+1}$ and $\zeta \in \mathbb{C}^\star$,

$$\zeta \cdot x = (\zeta x_0, \ldots, \zeta x_n).$$

We will consider the $\mathbb{C}^\star$-equivariant locally free sheaf

$$\mathcal{O}_{\mathbb{A}^{n+1}}(-H) := \ker(\mathcal{O}_{\mathbb{A}^{n+1}} \twoheadrightarrow \mathcal{O}_H)$$

where H is a $\mathbb{C}^\star$-invariant hyperplane (i.e $H \ni 0$). By Example 3.9(a) and Example 3.6, we have that

$$K_y^{\mathbb{C}^\star}(\mathbb{A}^{n+1}) = \mathbb{Z}[t^{\pm 1}][y]$$

where $t = [\mathcal{O}_{\mathbb{A}^{n+1}}(-H)]$. Let $p : Z = \mathfrak{Bl}_0 \mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$ be the blow-up at $0 \in \mathbb{A}^{n+1}$. Let $\mathbb{C}^\star$ act trivially on $\mathbb{P}^n$.

Denote $A := \mathbb{C}[x_0, \ldots, x_n]$ and $\mathfrak{m} = (x_0, \ldots, x_n)$. We have a closed embedding

$$Z = \operatorname{Proj}\left(\bigoplus_{m \geq 0} \mathfrak{m}^m t^m \right) \subset \operatorname{Proj}\left(A[y_0, \ldots, y_n] \right) = \mathbb{A}^{n+1} \times \mathbb{P}^n$$

via the surjective graded ring homomorphism

$$A[y_0, \ldots, y_n] \xrightarrow{\phi} \bigoplus_{m \geq 0} \mathfrak{m}^m t^m$$

$$y_i \longmapsto x_i t.$$

Let $q : Z \to \mathbb{P}^n$ be the restriction to Z of the projection map from $\mathbb{A}^{n+1} \times \mathbb{P}^n$ on to $\mathbb{P}^n$. Then we can identify Z with the total space $\mathbb{V}(\mathcal{O}_{\mathbb{P}^n}(-1)) = \operatorname{Spec}\left(\bigoplus_{m \geq 0} \mathcal{O}_{\mathbb{P}^n}(m) \right)$ over $\mathbb{P}^n$ via the map

$$\mathbb{C}^{n+1} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^n} \twoheadrightarrow \mathcal{O}_{\mathbb{P}^n}(1)$$

on $\mathbb{P}^n$.

$\mathbb{C}^\star$ acts on $\mathbb{A}^{n+1} \times \mathbb{P}^n$ via the actions on $\mathbb{A}^{n+1}$ and $\mathbb{P}^n$. One can check that Z is invariant under this $\mathbb{C}^\star$ action from checking how ϕ behaves with the induced action of $\mathbb{C}^\star$ on the coordinate rings of the affine charts $D^+(y_i)$. Thus, we get an algebraic action of $\mathbb{C}^\star$ on Z such that the maps p and q are $\mathbb{C}^\star$-equivariant. Let $E = p^{-1}(0) = \{0\} \times \mathbb{P}^n \subset Z$ be the exceptional divisor. Then $\mathbb{C}^\star$ acts trivially on E. We denote $e = [\mathcal{O}_Z(E)]$ in $K^{\mathbb{C}^\star}(Z)$.

We summarize the $\mathbb{C}^\star$-equivariant maps in the following diagram.

$$
\begin{array}{c}
E \\
\cap \\
Z = \mathfrak{Bl}_0\,\mathbb{A}^{n+1} = \mathbb{V}(O(-1)) \\
\{0\} \xrightarrow{\iota_0} \mathbb{A}^{n+1} \qquad\qquad \mathbb{P}^n
\end{array}
$$

with maps p and q from Z, ι_0 from $\{0\}$ to $\mathbb{A}^{n+1}$.

Using additivity of motivic Chern classes, Example 3.28, and Example 3.10 we get

Equation 3.56.

$$
\mathrm{mC}_y^{\mathbb{C}^\star}(Z \to \mathbb{A}^{n+1}) = \mathrm{mC}_y^{\mathbb{C}^\star}(E \to \mathbb{A}^{n+1}) + \mathrm{mC}_y^{\mathbb{C}^\star}(Z \setminus E \to \mathbb{A}^{n+1})
$$

$$
= \mathrm{mC}_y^{\mathbb{C}^\star}(E \to \mathbb{A}^{n+1}) + \mathrm{mC}_y^{\mathbb{C}^\star}(\mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{A}^{n+1})
$$

$$
= \iota_{0,\star}\mathrm{mC}_y^{\mathbb{C}^\star}(\mathbb{P}^n \to \{0\}) + \mathrm{mC}_y^{\mathbb{C}^\star}(\mathbb{A}^{n+1} \to \mathbb{A}^{n+1}) - \mathrm{mC}_y^{\mathbb{C}^\star}(\{0\} \to \mathbb{A}^{n+1})
$$

$$
= (1-t)^{n+1} \sum_{p=0}^{n} \sum_{q=0}^{n} (-1)^q H^q(\mathbb{P}^n, \Omega^q) y^p + (1+ty)^{n+1} - (1-t)^{n+1}
$$

$$
= (1-t)^{n+1} \sum_{p=0}^{n} (-y)^p + (1+ty)^{n+1} - (1-t)^{n+1}.
$$

Let $D \subset \mathbb{P}^n$ be a closed subvariety, $X = C(D) \subset \mathbb{A}^{n+1}$ the affine cone over D, and $Y = \mathfrak{Bl}_0\,X$ the strict transform of X. Then X and Y are $\mathbb{C}^\star$-invariant in $\mathbb{A}^{n+1}$ and Z, respectively. Furthermore, we have that $Y = q^{-1}D = \mathbb{V}(\mathcal{O}_D(-1))$.

Example 3.57. Let $H \subset \mathbb{A}^{n+1}$ be a hyperplane and $D \subset \mathbb{P}^n$ be the projective hyperplane it lies over. Then we have $\mathcal{O}_Z(\mathfrak{Bl}_0\,H) = \mathcal{O}_Z(p^\star H - E)$ as $\mathbb{C}^\star$-equivariant lines bundles. Pulling back the $\mathbb{C}^\star$-equivariant short exact sequence on $\mathbb{P}^n$

$$
0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_D \longrightarrow 0
$$

to Z, we get the $\mathbb{C}^\star$-equivariant short exact sequence

$$0 \longrightarrow O_Z(-1) \longrightarrow \mathcal{O}_Z \longrightarrow q^\star \mathcal{O}_D = \mathcal{O}_{\mathfrak{Bl}_0 H} \longrightarrow 0$$

so $\mathcal{O}_Z(-1) = \mathcal{O}_Z(E - p^\star H)$.

Recall the notations $e = [\mathcal{O}_Z(E)]$ and $t = \mathcal{O}_{\mathbb{A}^{n+1}}(-H)$. By an abuse of notation, we will also denote $\mathcal{O}_Z(-p^\star H)$ by t. Denote $a := [\mathcal{O}_Z(-1)] = [\mathcal{O}_Z(E - p^\star H)] = et$ in $K^{\mathbb{C}^\star}(Z)$. Our goal for the remainder of this section is to compute $p_\star a^i$ in $K^{\mathbb{C}^\star}(\mathbb{A}^{n+1})$ for all $i \geq 0$. Let

$$V_0 \subset V_1 \subset \ldots V_{n-1} \subset V_n = \mathbb{A}^{n+1}$$

be a flag of linear subspaces such that $\dim(V_k) = k + 1$. For every $k = 0, \ldots, n$, denote by $\widetilde{\iota}_k : V_k \subset \mathbb{A}^{n+1}$ the inclusion map. Let $\iota_k : Z_k = \mathfrak{Bl}_0 V_k \subset Z$ be the strict transform of V_k (so $Z_{n+1} = Z$), $p_k : Z_k \to V_k$ be the restriction of p to Z_k, and $D_k \subset \mathbb{P}^n$ be the projective linear subspace which is the image of V_k in $\mathbb{P}^n$ (so $D_n = \mathbb{P}^n$ and D_0 is a point in $\mathbb{P}^n$). Then for all $k > 0$, we have $\mathbb{C}^\star$-equivariant short exact sequences

$$0 \longrightarrow \mathcal{O}_{Z_k}(-1) \longrightarrow \mathcal{O}_{Z_k} \longrightarrow \mathcal{O}_{Z_{k-1}} \longrightarrow 0$$

by pulling back the $\mathbb{C}^\star$-equivariant short exact sequences

$$0 \longrightarrow \mathcal{O}_{D_k}(-1) \longrightarrow \mathcal{O}_{D_k} \longrightarrow \mathcal{O}_{D_{k-1}} \longrightarrow 0$$

along $q : Z \to \mathbb{P}^n$. Tensoring the above $\mathbb{C}^\star$-equivariant short exact sequences by $\mathcal{O}_Z(i)$, for $i > 0$, gives us the following $\mathbb{C}^\star$-equivariant short exact sequences

$$0 \longrightarrow \mathcal{O}_{Z_k}(i - 1) \longrightarrow \mathcal{O}_{Z_k}(i) \longrightarrow \mathcal{O}_{Z_{k-1}}(i) \longrightarrow 0 \qquad (1).$$

We put $a_{k,i} := \iota_{k,\star}[\mathcal{O}_{Z_k}(i)]$ in $K^{\mathbb{C}^\star}(Z)$ (so $a_{n,i} = a^i$). The $\mathbb{C}^\star$-equivariant short exact sequence

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(1) gives us

$$a_{k,i} = a_{k,i-1} + a_{k-1,i} = a_{k,0} + \sum_{l=1}^{i} a_{k-1,l}$$

for $k > 0$, $i > 0$. Note that Z_0 is the cone over a point in $\mathbb{P}^n$ and hence we have $\mathcal{O}_{Z_0}(i) = q^\star \mathcal{O}_{pt}(i) = q^\star \mathcal{O}_{pt} = \mathcal{O}_{Z_0}$. Thus

Equation 3.58.

$$a_{0,i} = a_{0,0}$$

for all i.

Define

$$C_{k,i,j} := \sum_{l_1=1}^{i} \cdots \sum_{l_{k-j}=1}^{l_{k-j-1}} 1.$$

for $i, k > 0$, $j < k$. Notice that $C_{k,i,j}$ can be inductively defined as

$$C_{k,i,j} = \sum_{l=1}^{i} C_{k-1,l,j}.$$

Furthermore, $C_{k,i,j}$ are generalized tetrahedral numbers whose formulas are given by

Equation 3.59.

$$C_{k,i,j} = \binom{i+k-j-1}{k-j}$$

(see [Bau19]).

The claim now is that

Equation 3.60.

$$a_{k,i} = a_{k,0} + \sum_{j=0}^{k-1} C_{k,i,j} a_{j,0}$$

for $k > 0$.

We will check this fact by induction. First, for $k = 1$, we indeed have

$$a_{1,i} = a_{1,0} + \sum_{l=1}^{i} a_{0,i} \overset{\text{Equation 3.58}}{=} a_{1,0} + a_{0,0} \sum_{l=1}^{i} 1 = a_{1,0} + C_{1,i,0} a_{0,0}.$$

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Assume, for $k > 1$, that we have

$$a_{k,i} = a_{k,0} + \sum_{j=0}^{k-1} C_{k,i,j} a_{j,0}.$$

Then we check

$$a_{k+1,i} = a_{k+1,0} + \sum_{l=1}^{i} a_{k,l} \overset{\text{Inductive Assumption}}{=} a_{k+1,0} + \sum_{l=1}^{i} \left(a_{k,0} + \sum_{j=0}^{k-1} C_{k,l,j} a_{j,0} \right)$$

$$= a_{k+1,0} + \sum_{l=1}^{i} a_{k,0} + \sum_{l=1}^{i} \sum_{j=0}^{k-1} C_{k,l,j} a_{j,0} = a_{k+1,0} + C_{k+1,i,k} a_{k,0} + \sum_{j=0}^{k-1} a_{j,0} \sum_{l=1}^{i} C_{k,l,j}$$

$$= a_{k+1,0} + C_{k+1,i,k} a_{k,0} + \sum_{j=0}^{k-1} C_{k+1,i,j} a_{j,0} = a_{k+1,0} + \sum_{j=0}^{k} C_{k+1,i,j} a_{j,0}.$$

This completes the proof of the claim.

Note that since V_k are smooth, we have $Rp_\star \mathcal{O}_{Z_k} = \mathcal{O}_{V_k}$. Now consider the commutative diagram

$$
\begin{array}{ccc}
Z_k & \overset{\iota_k}{\subset} & Z \\
\downarrow{\scriptstyle p_k} & & \downarrow{\scriptstyle p} \\
V_k & \underset{\tilde{\iota}_k}{\subset} & \mathbb{A}^{n+1}
\end{array}
$$

for $k \geq 0$. We see that for all $k \geq 0$, we have

Equation 3.61.

$$p_\star a_{k,0} = p_\star \iota_{k,\star}[\mathcal{O}_{Z_k}] = \tilde{\iota}_{k,\star} p_\star [\mathcal{O}_{Z_k}] = \tilde{\iota}_{k,\star}[\mathcal{O}_{V_k}] \overset{\text{Example 3.10}}{=} (1-t)^{n-k}.$$

Thus we have by Equations 3.60, 3.61, and 3.59,

Equation 3.62.

$$p_\star a^i = p_\star a_{n,i} = p_\star a_{n,0} + \sum_{j=0}^{n-1} C_{k,i,j} p_\star a_{j,0} = 1 + \sum_{j=0}^{n-1} C_{n,i,j} (1-t)^{n-j} = 1 + \sum_{j=0}^{n-1} \binom{i+n-j-1}{n-j}(1-t)^{n-j}.$$

3.3.3: Affine Cone over a Smooth Projective Hypersurface

Fix $D \subset \mathbb{P}$ to be a smooth hypersurface of degree d with $\mathbb{C}^\star$ acting trivially on it. Let $X = C(D) \subset \mathbb{A}^{n+1}$ be the affine cone over D. Define the integers

$$A_{m,k} = \sum_{q=0}^{m} \sum_{j=0}^{q} (-1)^{m-j} \binom{n+1}{j} \left(\binom{j + d(m-j) + n - k}{n+1-k} - \binom{j + d(m-j+1) + n - k}{n+1-k} \right).$$
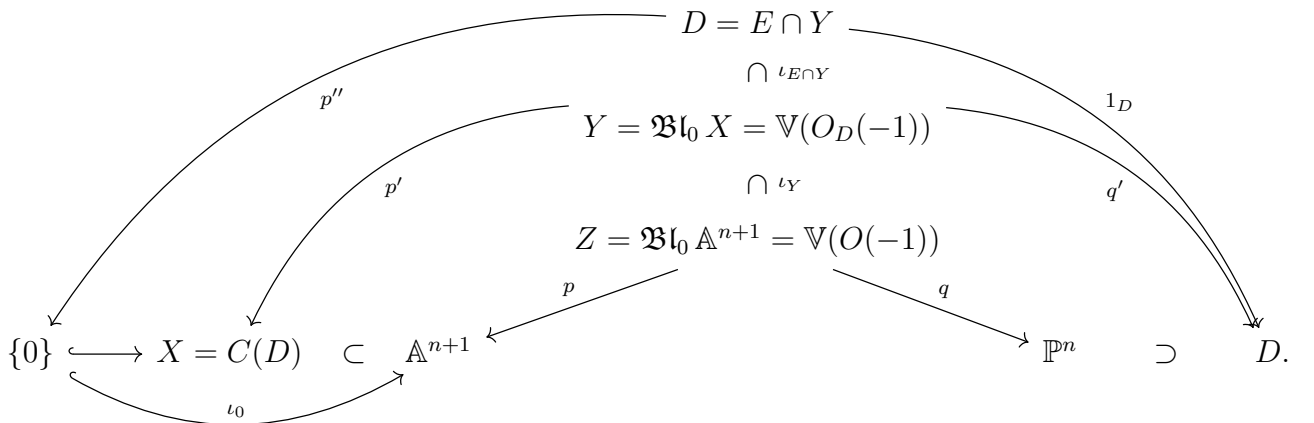
for all $n - 1 \geq m \geq 0, \geq k \geq 0$. The entire section will be dedicated to showing the following result.

Theorem 3.63. With the setup as above,

$$\mathrm{mC}_y^{\mathbb{C}^\star} \left(X \subset \mathbb{A}^{n+1} \right) = 2(1+ty)^{n+1} +$$

$$(1-t)^{n+1} \sum_{p=1}^{n} y^p \left((-1)^p + t^{-1} \sum_{j=0}^{p-1} (-1)^j \binom{n+1}{p-1-j} \binom{n+j-(p-1)}{n} \right) -$$

$$\sum_{k=0}^{n} (1-t)^{n-k} (1 + t^{-1} y) \sum_{p=0}^{n-1} A_{p,k} y^p$$

in $K_y^{\mathbb{C}^\star}(\mathbb{A}^{n+1})$.

Let $Y = \mathfrak{Bl}_0 X \subset Z$ be the strict transform of X. Then Y is smooth and we have the following diagram of $\mathbb{C}^\star$-equivariant varieties

Consider $\bar{a} = a|_Y, \bar{e} = e|_Y$ in $K^{\mathbb{C}^\star}(Y)$. The following short exact sequence on Y

$$0 \longrightarrow q'^\star \Omega_D \longrightarrow \Omega_Y \longrightarrow \mathcal{O}_Y(1) \longrightarrow 0$$

gives us

$$\lambda_y(\Omega_Y) = q'^\star \lambda_y(\Omega_D) \cdot (1 + a^{-1}y) = q'^\star \lambda_y(\Omega_D) \cdot (1 + \bar{e}^{-1}t^{-1}y).$$

Since $-t^{-1}y(1 - \bar{e}^{-1}) + 1 + t^{-1}y = 1 + \bar{e}^{-1}t^{-1}y$, we write

$$\lambda_y(\Omega_Y) = q'^\star \lambda_y(\Omega_D) \cdot \left(-t^{-1}y(1 - \bar{e}^{-1}) + 1 + t^{-1}y\right) =$$

$$\underbrace{-t^{-1}y \cdot q'^\star \lambda_y(\Omega_D)(1 - \bar{e}^{-1})}_{(1)} + \underbrace{q'^\star \lambda_y(\Omega_D) \cdot (1 + t^{-1}y)}_{(2)} \qquad (\star)$$

We now need to simplify (1) and then (2). Note that $q'^\star \lambda_y(\Omega_D)|_{E \cap Y} = \lambda_y(\Omega_D)$. Furthermore, from the short exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-E) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_{E \cap Y} \longrightarrow 0,$$

we have

$$\iota_{E \cap Y, \star}\left(q'^\star \lambda_y(\Omega_D)|_{E \cap Y}\right) \overset{\text{Proj Formula}}{=\!=\!=\!=} q'^\star \lambda_y(\Omega_D)(1 - \bar{e}^{-1}).$$

Thus we get

Equation 3.64.

$$p_\star \iota_{Y,\star}\left(-t^{-1}y \cdot q'^\star \lambda_y(\Omega_D)(1 - \bar{e}^{-1})\right) \overset{\text{Proj Formula}}{=\!=\!=\!=} -t^{-1}y \cdot p_\star \iota_{Y,\star}\left(\lambda_y(\Omega_D)(1 - \bar{e}^{-1})\right)$$

$$= -t^{-1}y \cdot p_\star \iota_{Y,\star} \iota_{E \cap Y,\star}\left(\lambda_y(\Omega_D)|_{E \cap Y}\right) = -t^{-1}y \cdot p_\star \iota_{Y,\star} \iota_{E \cap Y,\star} \lambda_y(\Omega_D)$$

$$= -t^{-1}y \cdot \iota_{0,\star} p''_\star \lambda_y(\Omega_D) = -t^{-1}y \cdot \iota_{0,\star} R\Gamma(D, \lambda_y(\Omega_D))$$

$$= -t^{-1}y \cdot R\Gamma(D, \lambda_y(\Omega_D)) \cdot \iota_{0,\star}[\mathcal{O}_0] \overset{\text{Example } 3.10}{=\!=\!=\!=} -t^{-1}y(1 - t)^{n+1} \cdot R\Gamma(D, \lambda_y(\Omega_D))$$

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$$\overset{\text{Equation 3.55}}{=} -t^{-1}y(1-t)^{n+1} \sum_{p=0}^{n-1} y^p \sum_{j=0}^{p} (-1)^j \binom{n+1}{p-j} \binom{n+j-p}{n}$$

in $K_y^{\mathbb{C}^\star}(\mathbb{A}^{n+1})$.

For simplifying (2), we use the short exact sequence

$$0 \longrightarrow \mathcal{O}_Z(-d) \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_Y = q^\star \mathcal{O}_D \longrightarrow 0 \ .$$

to get $\iota_{Y,\star}[\mathcal{O}_Y] = 1 - a^d$. Hence

$$p_\star \iota_{Y,\star} \left(q'^\star \lambda_y(\Omega_D)(1+t^{-1}y) \right) \overset{\text{Equation 3.54}}{=} p_\star \iota_{Y,\star} \left((1+t^{-1}y) \sum_{m=0}^{n-1} y^m \sum_{q=0}^{m} \sum_{j=0}^{q} (-1)^{m-j} \binom{n+1}{j} \overline{a}^{j+d(q-j)} \right)$$

$$\overset{\text{Proj Formula}}{=} (1+t^{-1}y) p_\star \left((1-a^d) \sum_{m=0}^{n-1} y^m \sum_{q=0}^{m} \sum_{j=0}^{q} (-1)^{m-j} \binom{n+1}{j} a^{j+d(q-j)} \right)$$

$$= (1+t^{-1}y) \sum_{m=0}^{n-1} y^m \sum_{q=0}^{m} \sum_{j=0}^{q} (-1)^{m-j} \binom{n+1}{j} p_\star \left(a^{j+d(m-j)} - a^{j+d(m-j+1)} \right)$$

in $K_y^{\mathbb{C}^\star}(\mathbb{A}^{n+1})$. We get

$$p_\star \left(a^{j+d(m-j)} - a^{j+d(m-j+1)} \right) \overset{\text{Equation 3.60}}{=}$$

$$p_\star \left(\left(a_{n+1,0} + \sum_{k=0}^{n} C_{n+1,j+d(m-j),k} a_{k,0} \right) - \left(a_{n+1,0} + \sum_{k=0}^{n} C_{n+1,j+d(m-j+1),k} a_{k,0} \right) \right) =$$

$$\sum_{k=0}^{n} \left(C_{n+1,j+d(m-j),k} - C_{n+1,j+d(m-j+1),k} \right) p_\star a_{k,0} \overset{\text{Equations 3.61 and 3.59}}{=}$$

$$\sum_{k=0}^{n} \left(\binom{j+d(m-j)+n-k}{n+1-k} - \binom{j+d(m-j+1)+n-k}{n+1-k} \right) (1-t)^{n-k}.$$

Thus we get

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Equation 3.65.

$$p_\star \iota_{Y,\star}\left(q'^{\star}\lambda_y(\Omega_D)(1+t^{-1}y)\right) = (1+t^{-1}y)\sum_{m=0}^{n-1} y^m \sum_{q=0}^{m}\sum_{j=0}^{q}(-1)^{m-j}\binom{n+1}{j}\cdot$$

$$\sum_{k=0}^{n}\left(\binom{j+d(m-j)+n-k}{n+1-k} - \binom{j+d(m-j+1)+n-k}{n+1-k}\right)(1-t)^{n-k} =$$

$$(1+t^{-1}y)\sum_{m=0}^{n-1} y^m \sum_{k=0}^{n}(1-t)^{n-k}.$$

$$\sum_{q=0}^{m}\sum_{j=0}^{q}(-1)^{m-j}\binom{n+1}{j}\left(\binom{j+d(m-j)+n-k}{n+1-k} - \binom{j+d(m-j+1)+n-k}{n+1-k}\right) =$$

$$(1+t^{-1}y)\sum_{m=0}^{n-1} y^m \sum_{k=0}^{n} A_{m,k}(1-t)^{n-k}$$

in $K_y^{\mathbb{C}^\star}(\mathbb{A}^{n+1})$.

Putting Equations 3.64 and 3.65 into $(\star)$, we get

Equation 3.66.

$$\mathrm{mC}_y^{\mathbb{C}^\star}(Y \to \mathbb{A}^{n+1}) = p_\star\iota_{Y,\star}\lambda_y(\Omega_Y) =$$

$$-t^{-1}y(1-t)^{n+1}\sum_{p=0}^{n-1} y^p \sum_{j=0}^{p}(-1)^j\binom{n+1}{p-j}\binom{n+j-p}{n} + (1+t^{-1}y)\sum_{m=0}^{n-1} y^m \sum_{k=0}^{n} A_{m,k}(1-t)^{n-k}.$$

Using additivity of Motivic Chern classes, Example 3.28, and Equations 3.56 and 3.66, we get

$$\mathrm{mC}_y^{\mathbb{C}^\star}(X \to \mathbb{A}^{n+1}) = \mathrm{mC}_y^{\mathbb{C}^\star}(\mathbb{A}^{n+1} \to \mathbb{A}^{n+1}) - \mathrm{mC}_y^{\mathbb{C}^\star}(\mathbb{A}^{n+1} \setminus X \to \mathbb{A}^{n+1})$$

$$\overset{\mathbb{A}^{n+1}\setminus X=Z\setminus Y}{=} \mathrm{mC}_y^{\mathbb{C}^\star}(\mathbb{A}^{n+1} \to \mathbb{A}^{n+1}) + \mathrm{mC}_y^{\mathbb{C}^\star}(Z \setminus Y \to \mathbb{A}^{n+1})$$

$$= \mathrm{mC}_y^{\mathbb{C}^\star}(\mathbb{A}^{n+1} \to \mathbb{A}^{n+1}) + \mathrm{mC}_y^{\mathbb{C}^\star}(Z \to \mathbb{A}^{n+1}) - \mathrm{mC}_y^{\mathbb{C}^\star}(Y \to \mathbb{A}^{n+1})$$

$$= (1+ty)^{n+1} + (1-t)^{n+1}\sum_{p=0}^{n}(-y)^p + (1+ty)^{n+1} - (1-t)^{n+1}+$$

$$t^{-1}y(1-t)^{n+1}\sum_{p=0}^{n-1}y^p\sum_{j=0}^{p}(-1)^j\binom{n+1}{p-j}\binom{n+j-p}{n} - (1+t^{-1}y)\sum_{m=0}^{n-1}y^m\sum_{k=0}^{n}A_{m,k}(1-t)^{n-k}$$

$$= 2(1+ty)^{n+1} + (1-t)^{n+1}\sum_{p=1}^{n}y^p\left((-1)^p + t^{-1}\sum_{j=0}^{p-1}(-1)^j\binom{n+1}{p-1-j}\binom{n+j-(p-1)}{n}\right) -$$

$$\sum_{k=0}^{n}(1-t)^{n-k}(1+t^{-1}y)\sum_{p=0}^{n-1}A_{p,k}y^p.$$

This completes the proof.

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