Classification Results of Orbit Closures of Translation Surfaces

by

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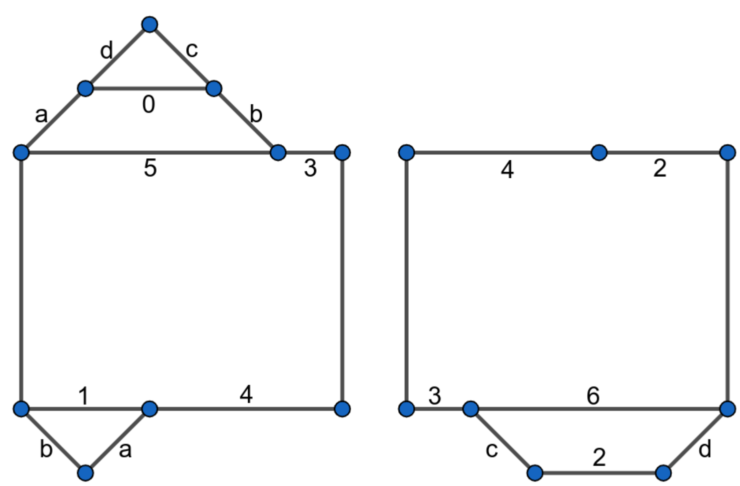
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M	A translation surface.
X	The underlying Riemann surface of a translation surface M.
ω	The underlying holomorphic 1-form of a translation surface M.
Σ	The set of singularities of a translation surface M.
$\mathcal{M}$	An invariant subvariety in a stratum of translation surfaces see Definition 1.2.1.
p	The forgetful map $H^1(M, \Sigma) \to H^1(M)$ see Definition 2.1.1.
$g(\mathcal{M})$	The genus of the surfaces in $\mathcal{M}$ see Definition 2.1.1
$\operatorname{rk} \mathcal{M}$	The rank of an invariant subvariety see Definition 2.1.1
$\operatorname{rel} \mathcal{M}$	The rel of an invariant subvariety see Definition 2.1.1
$T_M^{\mathbb{R}}\mathcal{M}$	The real tangent space see Definition 2.1.2
$[\gamma]$	The class of γ in $H_1(M)$ see Definition 2.2.1.
γ^*	Poincare dual of γ when viewed as an element of $H_1(M - \Sigma)$ see Definition 2.2.1.
σ_C	The standard twist of C see Definition 2.2.7
$\operatorname{Tw} M$	Twist space of M see Definition 2.3.1.

ABSTRACT

Translation surfaces are a type of flat surface that generalizes the dynamics on flat tori to higher genera. This has applications to billiards and the finite blocking problem. Studying dynamics on individual translation surfaces is often done by studying a different dynamical system on the moduli space of translation surfaces. This thesis covers three classification results of orbit closures in these moduli spaces. First, we use the transfer principle to classify periodic points on a certain family of Veech surfaces. The next result is classifying orbit closures in a product of two components of strata. This is done with an induction argument and investigating the boundary of orbit closures. Finally, we reprove the classification of rank 2 orbit closures in some genus 3 strata. The main contribution of this new proof is providing code that can automatically check certain conditions to significantly simplify the work needed for those proofs. This code would also be useful in classifying orbit closures in other strata.

CHAPTER 1

Introduction

A **translation surface** M is a closed surface obtained by gluing together polygons in the plane by translation. These polygons give coorindate charts for M with transitions maps in the group of translations of $\mathbb{R}^2$. These transition maps preserve the complex structure of the plane as well as the holomorphic one form dz, so M has an underlying compact Riemann surface X and a holomorphic 1-form ω. In fact, the data (X, ω) also determines M although we do not like to think of a translation surface in this way. Let Σ be the set of zeros of ω. $|dz|$ also defines a singular flat Riemannian metric on M with singularities at Σ. Thus, we call Σ the **singularities** of M. Each singularity p will have cone angle $(d_p + 1) \cdot 2\pi$, where d_p is the degree of vanishing of ω at p.

The following is an example of how differential and algebraic geometry are related on a translation surface. Let $\mathcal{X}$ be the upward unit vector field on $M - \Sigma$. Around each $p \in \Sigma$, $\mathcal{X}$ has index $-d_p$ if p has cone angle $(d + 1) \cdot 2\pi$. By Poincare-Hopf, the Euler characteristic of M is $\chi = -\sum_{p \in \Sigma} d_p$ or

$$\sum_{p \in \Sigma} d_p = 2g - 2.$$

We also know that ω is a section of the cotangent bundle which has degree $2g - 2$, so this gives another derivation of this equation.

Remark 1.0.1. This this not intended as an introduction to translation surfaces. See [Wri15b] for an introduction geared at similar topics as this thesis.

1.1 Motivating Problems

There are two dynamical systems relating to translation surfaces. On a single translation surface, we can study the straight line flow or geodesic flow, while $\mathrm{GL}^+(2, \mathbb{R})$ acts on the moduli space of translation surfaces by actings on the polygons that make up a translation surface. We will see that studying the $\mathrm{GL}^+(2, \mathbb{R})$-action on moduli space leads to many insights about the dynamics of straight line flow.

1.1.1 Billiards

The dynamics of billiards is a classical problem in mathematics. We start with a table that is a rectangle or maybe some other polygon. Inside we have a single billiard ball that we model as a point. The point travels in straight lines and reflects off the walls of the billiard table so that the angle of incidence equals the angle of reflection. This billiard trajectory can be extended infinitely into the past and the future. We can see that a rectangular billiard table has many closed billiard trajectories, but the following question is open in general.

Question 1.1.1. Does every triangular billiard table have a closed billiard trajectory?

We say that a polygon has **rational angles** if the measure of every angle is a ration number of degree or equivalently 2π times a rational number of radians. For rational polygons, there is a famous unfolding construction that obtains a translation surface see for example [Wri15b, Section 1.2]. Billiard trajectories on the billiard table correspond to geodesics on the translation surface up to the action of the deck group. (To make this notion more precise, a billiard table can be thought of as an orbifold and the translation surface a finite cover.) If the polygon does not have rational angles, the construction is still possible but gives a translation surface of infinite type see [DHV, Section 1.2.1].

Theorem 1.1.2. *Every translation surface has a dense set of directions with a closed geodesic in that direction.*

See [MT02, Theorem 4.1] for a proof. Proofs of these results all use the structure of strata of translation surfaces. As Question 1.1.1 remains open, its seems that no one has been able to make this work for translation surfaces of infinite type. The unfolding construction gives us this corollary.

Corollary 1.1.3. *Every polygonal billiard table with rational angles has a dense set of directions with a periodic billiard trajectory in that direction.*

1.1.2 Optimal Dynamics

Let $\mathbb{T}$ be the square torus $\mathbb{R}^2/\mathbb{Z}^2$ with the flat metric. Every line in $\mathbb{R}^2$ projects to a geodesic in $\mathbb{T}$, and every geodesic on $\mathbb{T}$ can be lifted to a line in $\mathbb{R}^2$ unique up to the action of the deck group. The following is a classical theorem.

Theorem 1.1.4. *A geodesic on $\mathbb{T}$ is closed if and only if it lifts to a line with rational slope and dense if and only if it lifts to a line with irrational slope. In the latter case, the flow is uniquely ergodic.*

Definition 1.1.5. For a fixed translation surface, a direction where every geodesic that doesn't go through a singularity is closed is called a **periodic direction**. A translation surface such that the straight-line flow in every direction is either periodic or uniquely ergodic is said to have **optimal dynamics**.

The following generalization of Theorem 1.1.4 exemplifies the theme that the $\mathrm{GL}^+(2,\mathbb{R})$ dynamics gives information about individual translation surfaces.

Definition 1.1.6. A **Veech surface** is a translation surface with closed $\mathrm{GL}^+(2,\mathbb{R})$-orbit.

Theorem 1.1.7 (Veech Dichotomy)**.** *If M is a Veech surface, then it has optimal dynamics.*

Although no full converse is known, Wright proved a partial converse in [Wri15a, Theorem 1.5]. We study Veech surfaces more in Chapter 3.

1.1.3 Finite Blocking Problem

The following is another classical result.

Theorem 1.1.8. *For any fixed points $p, q \in \mathbb{T}$, there is a finite set B such that any geodesic from p to q crosses at least one point of B.*

Proof. Choose $\hat{p}, \hat{q}$ to be lifts of p, q to $\mathbb{R}^2$. Every geodesic from p to q on $\mathbb{T}$ can be lifted to a straight line segment from $\hat{p}$ to a point of $\hat{q} + \mathbb{Z}^2$ on $\mathbb{R}^2$. Consider the set of midpoints of all such segments. Each of these points lies on one of the four lattices $\frac{\hat{p}+\hat{q}}{2} + \mathbb{Z}^2$, $\frac{\hat{p}+\hat{q}+(1,0)}{2} + \mathbb{Z}^2$, $\frac{\hat{p}+\hat{q}+(0,1)}{2} + \mathbb{Z}^2$, $\frac{\hat{p}+\hat{q}+(1,1)}{2} + \mathbb{Z}^2$. Thus, $B = \{\frac{p+q}{2}, \frac{p+q+(1,0)}{2}, \frac{p+q+(0,1)}{2}, \frac{p+q+(1,1)}{2}\}$ is a finite blocking set. $\square$

Using the unfolding trick, we can also block any billiard trajectory between p, q usually a larger blocking set. Initially, this result may seem surprising that a finite number of points can block an infinite number of geodesics that pass between p and q. One may then try to generalize this result.

Question 1.1.9. Is there a finite blocking set for any two points in an arbitrary translation surface?

Although we will not discuss the connection in detail in this work, such a blocking set will be a periodic point defined below. By studying periodic points, [LMW16] and [AW21a] show that for the "majority" of translation surfaces M, there are only a finite number of pairs (p, q) of points on M with a finite blocking set.

1.2 Prior Work

This section gives a brief overview of existing work in the field as well as how the results of this work fit in. Formal statements of the results can be found in the introductions of Chapter 3, Chapter 4, and Chapter 5.

The focus of this work is studying various classification problems of orbit closures of the $\mathrm{GL}^+(2,\mathbb{R})$ action on the moduli space of translation surfaces. We start by surveying previous work on these types of problems. In genus 2, McMullen classifies closed $\mathrm{GL}^+(2,\mathbb{R})$-invariant sets of translation surfaces in [McM07]. The classification is almost complete except for a classification of orbits of square-tiled surfaces in $\mathcal{H}(1^2)$. Each orbit closure turned out to be a special type of orbifold that was defined by linear equations with real coefficients in periodic cooridinates. Then in [EM18] and [EMM15], closed $\mathrm{GL}^+(2,\mathbb{R})$-invariant sets of translation surfaces in any genera were proven to have this linear orbifold structure. Although we have not given a precise statement, this theorem is often called the Magic Wand Theorem, and it led to rapid advancements in the field. Filip in [Fil16] also proved that these orbit closures are algebraic varieties. Thus, orbit closures are invariant subvarieties defined as follows:

Definition 1.2.1. Let $\mathcal{H}$ be a component of a stratum of translation surfaces. An **invariant subvariety** is a closed $\mathrm{GL}^+(2,\mathbb{R})$-invariant irreducible algebraic variety $\mathcal{M} \subset \mathcal{H}$ that is cut out by linear equations with real coefficients in period corrdinate charts. We will extend this definition to other moduli spaces with periodic coordinate charts namely strata of translation surfaces with marked points and strata of multi-component translation surfaces.

One plentiful source of invariant subvarieties comes from **square-tiled surfaces**. Square-tiled surfaces, also called origamis, are covers of the square torus. The orbits of each square-tiled surface are closed and these orbits are dense in each stratum. [HL06] and [McM05] classify these invariant subvarieties in $\mathcal{H}(2)$ but the problem is open in all other strata. A different approach is to classify invariant subvarieties that do not come from covering constructions, which are called **primitive**. [McM03] and [Cal04] independently discovered infinite families of primitive invariant subvarieties in genus 2.Wright in [Wri15a] defines the rank of an invariant subvariety, and these invariant subvarieties turn out to be rank 1. In each genus, while there are infinitely many rank 1 invarieties, [EFW18, Theorem 1.5] states that there are finitely many invariant subvarieties of rank at least 2. Mirzakhani conjectured that the only primitive invariant subvarieties were components of strata of translation or half translation surfaces. [NW14], [ANW16], [AN16], and [AN20] proved the conjecture in genus 3, and Apisa proved it in hyperelliptic components in [Api18], but counterexamples were found in [MMW17] and [EMMW20]. It remains open whether there are invariant subvarieties of rank at least 3 that are not components of strata. Towards this end, Apisa and Wright proved the lack of invariant varieties for rank at least $\frac{g}{2}+1$ in [AW21b]. In this thesis, we add to this disucussion; Chapter 5 of this work simplifies the proof of the classification of rank 2 invariant subvarieties in $\mathcal{H}(3,1)$,$\mathcal{H}^{odd}(2,2)$, and $\mathcal{H}(2,1,1)$ strata that was already done in [AN16] and [AN20]. Along with [Api18, Main Theorem 1] and [AW21b, Corollary 7.3], this reproves the classification of rank 2 invariant subvarieties in genus 3 in every stratum except $\mathcal{H}(1,1,1,1)$.

The results of Chapter 3 can also be formulated in the language of classifying invariant subvarieties. If M is a Veech surface, the stabilizer $\mathrm{SL}(M)$ of the action of $\mathrm{GL}^+(2,\mathbb{R})$ is a lattice in $\mathrm{SL}(2,\mathbb{R})$. A point $p \in M$ with finite orbit under $\mathrm{SL}(M)$ is a **periodic point**. Chapter 3 classifies periodic points on two infinite families of Veech surfaces: the regular $2n$-gon surfaces and the double $2n+1$-gon surfaces. Now we explain how this relates to classifying invariant subvarieties. We can consider a translation surface M and a set Σ, where Σ contains the singular points of M along with a fixed number of marked points. The moduli space of such (M, Σ) is a **stratum with marked points**. If $\mathcal{H}$ is a stratum without marked points, we will use $\mathcal{H}^{*n}$ to denote a stratum with n marked points. $\mathcal{F} : \mathcal{H}^{*n} \to \mathcal{H}$ is the map forgetting marked points.

If $\mathcal{N} \subset \mathcal{H}^{*n}$ is an invariant subvariety, then $\overline{\mathcal{F}(\mathcal{N})}$ is an invariant subvariety $\mathcal{M} \subset \mathcal{H}$ by the Magic Wand Theorem.

Definition 1.2.2. $\mathcal{N}$ is called an ***n*-point marking** of $\mathcal{M}$. If $n = 1$ and $\dim \mathcal{N} = \dim \mathcal{M}$, then $\mathcal{N}$ is a **periodic point** of $\mathcal{M}$.

This definition generalizes the previous definition of periodic point and shows that it is an invariant subvariety. Types of point markings were classified by Apisa and Wright in [Api18] and [AW21a]. In addition, we can see that a blocking set from Section 1.1.3 is an n-point marking i.e. the set $\mathcal{B}$ of (M, B) for M a translation surface and B a blocking set of size n between two points on M. For any $A \in \mathrm{GL}^+(2,\mathbb{R})$, AB is a blocking set between two points on AM, so $\mathcal{B}$ is an invariant set. Thus, it makes sense that classification results of invariant subvarieties were used to prove theorems about the finite blocking problem, see [LMW16] and [AW21a].

Finally, Chapter 4 classifies certain types of invariant subvarieties in strata of multi-component translation surfaces. While Section 4.1 goes into more depth on the motivation, the main one is that multi-component translation surfaces appear in the WYSIWYG boundary in strata of single-component translation surfaces and are useful for induction arguments using this boundary. Chapter 3, Chapter 4, and Chapter 5 are all independ of each other and can be read in any order.

CHAPTER 2

Background

In this chapter we review some facts about translation surfaces, and in some cases extend them to multi-component translation surfaces. The multi-component versions of these results is only needed for Chapter 4.

2.1 Invariant Subvarieties

Here we will cover some notation. Recall the definition of invariant subvariety from Definition 1.2.1.

Definition 2.1.1. Let $\mathcal{M}$ be an invariant subvariety. Note that $\mathcal{M}$ is also an orbifold that has a finite cover which is a manifold. Let $M \in \mathcal{M}$ be a manifold point of $\mathcal{M}$. Then, there is a tangent space $T_M\mathcal{M} \subset H^1(M, \Sigma; \mathbb{C})$. Let $p : H^1(M, \Sigma; \mathbb{C}) \to H^1(M; \mathbb{C})$ be the forgetful map. The **rank** of $\mathcal{M}$ is denoted $\operatorname{rk} \mathcal{M} := \frac{1}{2} \dim p(T_M\mathcal{M})$, which is an integer since $p(T_M\mathcal{M})$ is symplectic [AEM17, Theorem 1.4]. Let $g(\mathcal{M})$ be the genus of the surfaces in $\mathcal{M}$. $\mathcal{M}$ is **full rank** if $\operatorname{rk} \mathcal{M} = g(\mathcal{M})$. We define **rel** $\mathcal{M} := \dim T_M\mathcal{M} \cap \ker p$. For orbitfold points, we can make the same definitions after choosing a marking.

Definition 2.1.2. Let $\mathcal{M}$ be an invariant subvariety and $M \in \mathcal{M}$. Define the **real tangent space** $T_M^{\mathbb{R}}\mathcal{M} := T_M\mathcal{M} \cap H^1(M, \Sigma; \mathbb{R})$. Since $\mathcal{M}$ is linear in period coordinates, there is some domain $U \subset T_M\mathcal{M}$, where U can be viewed as the domain for the period coordinate chart $\Phi : U \to \mathcal{M}$. Then for $v \in U$, $M + v$ is the translation surface $\Phi(\Phi^{-1}(M) + v)$. If $v \in T_M^{\mathbb{R}}\mathcal{M}$, then $M + v$ is a **real deformation**.

Lemma 2.1.3. *Let M be horizontally periodic. After a real deformation, $M + v$ has the same cylinder diagram as M. The horizontal $\mathcal{M}$-parallel classes of $M + v$ will also be the same as M.*

Proof. A real deformation does not change the imaginary parts of periods of saddle connections, so it does not change whether a saddle is horizontal. □

2.1.1 Prime Invariant Subvarieties

Prime invariant subvarieties first defined in [CW21].

Definition 2.1.4. An invariant subvariety of multi-component surfaces $\mathcal{M}$ is called a **prime invariant subvariety** if it cannot be written as a product $\mathcal{M}_1 \times \mathcal{M}_2$ of two other multi-component invariant subvarieties.

Theorem 2.1.5 (Chen-Wright)**.** *[CW21, Theorem 1.3] For a prime invariant subvariety $\mathcal{P} \subset \mathcal{H}_1 \times \cdots \times \mathcal{H}_n$, the absolute periods locally determine each other. In particular, the rank of each component $\overline{\pi_i(\mathcal{P})}$ is the same, so we can define this to be the rank of $\mathcal{P}$.*

Proposition 2.1.6 (Chen-Wright)**.** *[CW21, Corollary 7.4] In a prime invariant subvariety, the g_t action is ergodic on the unit area locus. Thus, the ratio of areas of the components is constant.*

Remark 2.1.7. For any quasidiagonal Δ we get an infinite number of quasidiagonals $\Delta_r = \{(M_1, rM_2) : (M_1, M_2) \in \Delta\}$. By Proposition 2.1.6, we can scale Δ so that both components have the same area.

2.2 Cylinder Deformations

Although cylinders of translation surfaces had been studied before, the Cylinder Deformation Theorem (Theorem 2.2.9) led the way in many advances in the understanding of invariant subvarieties through the investigation of cylinders. These cylinder-based methods are the basis of most arguments in this paper.

Let γ be a closed geodesic on a translation surface M. Then, any geodesic parallel to and close enough to γ will also be closed. Thus, we have a tubular neighborhood of γ that is foliated by parallel closed geodesics. This neighborhood can be extended on both sides of the closed geodesic until you hit a singularity of M.

Definition 2.2.1. A **cylinder** C is such a maximal open neighborhood foliated by parallel closed geodesics. Note that this definition does not require a choice of γ, but we will make an arbitary choice. The **core curve** of C will be the closed geodesic γ in the middle of C. While the core cure of C can be thought of as a curve, we will primarily consider the homology class $[\gamma] \in H_1(M; \mathbb{Z})$. We may also consider the class of γ in the group $H_1(M - \Sigma)$. By Poincare duality, this gives an element of $H^1(M, \Sigma)$, which we call γ^*. We remark that $[\gamma]$ and γ^* would not change if we chose a different geodesic parallel to γ. A cylinder C has two **boundary components**, which are closed geodesics in the closures of C that contain singularities. A **cross curve** is a saddle connection contained in the closure of C that goes from one boundary component to the other. For a translation surface M'

close enough to M, there is a natural isomorphism $H_1(M,\mathbb{Z}) \cong H_1(M',\mathbb{Z})$, so there is a homology element $[\gamma']$ associated to $[\gamma]$. There is a closed geodesic γ' on M' that has homology class $[\gamma']$. Thus M' contains a cylinder C' corresponding to C. A cylinder is **horizontal** if its core curve is horizontal. A translation surface is **horizontally periodic** if it is a union of horizontal cylinders, horizontal saddle connections, and singularities.

Let $\mathcal{M} \subset \mathcal{H} = \mathcal{H}_1 \times \cdots \times \mathcal{H}_n$ be an invariety subvariety of multi-component surfaces. We reproduce a theorem by Smillie-Weiss [SW04, Theorem 5]. The original theorem was only proven in the single-component case. The proof of the multicomponent case has been deferred to the appendix. Let $H_t = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\} \subset \mathrm{SL}(2,\mathbb{R})$.

Theorem 2.2.2 (Smillie-Weiss 2004)**.** *Every H_t orbit closure in a stratum of multicomponent quadratic differentials contains a surface q such that every component of q is horizontally periodic.*

Lemma 2.2.3. *Let $M = (M_1, \ldots, M_n) \in \Delta$ be a surface in a prime invariant subvariety. Choose a period coordinate chart U around M and let $M^t = (M_1^t, \ldots, M_n^t)$, $t \in [0,1]$, be a path in U with $M^0 = M$. If for some i, for each t the imaginary parts of the absolute periods of M_i^t are the same, then the same is true for all i.*

Proof. By Theorem 2.1.5, the absolute periods of M_i determine each other and $\mathcal{M}$ is cut out by equations with real coefficients in period coordinate charts. □

Lemma 2.2.4. *Let $M = (M_1, \ldots, M_n) \in \mathcal{M}$ be a prime invariant subvariety. If M_i is horizontally periodic, then M_j must be horizontally periodic for every $1 \le j \le n$. In this case, we say that M is* ***horizontally periodic****.*

Proof. Assume by contradiction and with loss of generality that M_1 is horizontally periodic and M_2 is not. By Theorem 2.2.2, there is a sequence $t_n \to \infty$ such that

$$\begin{pmatrix} 1 & t_n \\ 0 & 1 \end{pmatrix} M_2 \to M_2^\infty$$

where M_2^∞ is horizontally periodic. Since M_1 is horizontally periodic, its orbit closure under the action of H_t is an n-dimensional torus T, which contains surfaces with locally the same imaginary periods. Since the H_t-orbit closure of M is compact, after passing to a subsequence, we also have that $\begin{pmatrix} 1 & t_n \\ 0 & 1 \end{pmatrix} M_1 \to M_1^\infty$ and M_1^∞ must also be horizontally periodic. We choose a coordinate chart $U \times V$ centered around $M^\infty = (M_1^\infty, \ldots, M_n^\infty)$, where $U \subset \mathcal{M}_1$ and $V \subset \mathcal{M}_2 \times \cdots \times \mathcal{M}_n$. For large enough t_n, $M_1^{t_n} := \begin{pmatrix} 1 & t_n \\ 0 & 1 \end{pmatrix} M_1 \in U$ and since $M_1^{t_n}$ and M_1^{∞} all lie on T they all have the same real periods. However, for large enough n, there is a surface $M_2^{t_n} := \begin{pmatrix} 1 & t_n \\ 0 & 1 \end{pmatrix} M_2$ such that there is some cylinder C of M_2^{∞} that persists on $M_2^{t_n}$ but it is not horizontal. Thus, the core curve of C is an absolute period that changes in imaginary part on a path from $M_2^{t_n}$ to M_2^{∞}. This contradicts Lemma 2.2.3. □

Let $\mathcal{C} = \{C_1, \ldots, C_r\}$ be a collection of cylinders on $M = (M_1, \ldots, M_n) \in \mathcal{M}$ and let γ_i be a core curve of C_i. The tangent space $T_M\mathcal{M}$ is a subspace of $T_M\mathcal{H} = H^1(M, \Sigma; \mathbb{C}) := H^1(M_1, \Sigma; \mathbb{C}) \times \cdots \times H^1(M_n, \Sigma; \mathbb{C})$. Thus, there is a projection $\pi : H_1(M, \Sigma; \mathbb{C}) \to (T_M\mathcal{M})^*$. We view γ_i as elements of $H_1(M, \Sigma; \mathbb{C})$ by setting the $H_1(M_j, \Sigma; \mathbb{C})$ to be zero on the components M_j that do not contain C_i.

Definition 2.2.5. $\mathcal{C}$ is called **$\mathcal{M}$-parallel** if all $\pi(\gamma_i)$ are colinear in $(T_M\mathcal{M})^*$. Being $\mathcal{M}$-parallel is an equivalence relation on cylinders, so we call $\mathcal{C}$ an **$\mathcal{M}$-parallel class** if it is an equivalence class of $\mathcal{M}$-parallel cylinders.

Remark 2.2.6. Intuitively, $\mathcal{M}$-parallel means that there is a neighborhood $M \in U \subset \mathcal{M}$ such that all cylinders in $\mathcal{C}$ remain parallel in this neighborhood. See [Wri15a] for a more detailed discussion on $\mathcal{M}$-parallel cylinders.

Definition 2.2.7. Let $\mathcal{C}$ consist of cylinders $C_1, \ldots, C_r$. Let h_i and γ_i be the height and core curve of C_i respectively. Define the **standard twist** of $\mathcal{C}$:

$$\sigma_{\mathcal{C}} = \sum_{i=1}^{r} h_i \gamma_i^*$$

See also [Wri15a, Section 2] for a more detailed discussion of $\sigma_{\mathcal{C}}$.

Remark 2.2.8. If $\mathcal{C}$ is horizontal, moving in the direction of $i\sigma_{\mathcal{C}}$ in period coordinates stretches all the cylinders in $\mathcal{C}$ in proportion to the height of the cylinder and doesn't change the rest of the surface. Moving in the direction of $\sigma_{\mathcal{C}}$ shears all the cylinders of $\mathcal{C}$ in proportion to their heights and doesn't change the rest of the surface. We are now able to state the theorem.

Theorem 2.2.9 (Cylinder Deformation Theorem, Wright 2015)**.** *Let $\mathcal{M} \subset \mathcal{H}$ be an invariant subvariety of a multi-component stratum. Let $\mathcal{C}$ be a $\mathcal{M}$-parallel class of cylinders on M. Then $\sigma_{\mathcal{C}} \in T_M\mathcal{M}$. (Here $T_M\mathcal{M}$ denotes the tangent space to $\mathcal{M}$ at M, which is a subspace of $H^1(M, \Sigma; \mathbb{C})$).*

We will sketch a proof of Wright's Cylinder Deformation Theorem for multi-component surfaces because the original theorem was only stated for single-component surfaces. The proof is identical to the original proof but we omit many details, which can be found in the original paper [Wri15a]. The following lemma is [Wri15a, Lemma 3.1].

Lemma 2.2.10. *Let M be horizontally periodic and C be an $\mathcal{M}$-parallel class of cylinders. Let the moduli of the cylinders of C be independent over $\mathbb{Q}$ of the moduli of the remaining horizontal cylinders. Then $\sigma_C \in T_M\mathcal{M}$.*

Proof. The H_t-flow of a horizontally periodic surface is the same as the flow on a r-dimensional torus whose slope is determined by the moduli of the cylinders. □

The following lemma can be found in [Wri15a, Lemma 4.9].

Lemma 2.2.11. *Let V be a finite-dimensional vector space and $F \subset V^*$ a finite collections of linear functionals on V, no two of which are colinear. The collection of functions $1/w$ for $w \in F$ are linearly independent over $\mathbb{R}$. This remains true when the functions are restricted to any nonempty open set of V.*

Proof of Theorem 2.2.9. Let C be an $\mathcal{M}$-parallel class of cylinders on M. By Theorem 2.2.2, there is a horizontally periodic surface M' in the H_t orbit closure of M. The corresponding set of cylinders, which we still call C, is an $\mathcal{M}$-parallel class on M'.

Claim 2.2.12. There is a surface M'' that is a real deformation of M' such that the moduli of the cylinders in C are independent over $\mathbb{Q}$ of the moduli of the cylinders not in C. [Wri15a, Lemma 4.10]

By the claim we have a surface M'' where the moduli of the cylinders in C are independent of the rest of the cylinders. Lemma 2.2.10 finishes the proof. Thus, it suffices to prove the claim. Let $C_{r+1}, \ldots, C_l$ be the cylinders of M' not in C, and let m_i be the moduli of the cylinder C_i. Assume by contradiction that the claim is false. Because $\mathcal{M}$ is cut out by real linear equations in period coordinates, there is some rational relation that holds for small real deformations of M'

$$\sum_{i=1}^{r} q_i m_i = \sum_{j=r+1}^{l} q_j m_j$$

for some $q_i \in \mathbb{Q}$, where neither the right hand nor the left hand side is identically zero. Recall that $m_i = h_i/c_i$ is the modulus of a cylinder. Since the cylinders in C are all $\mathcal{M}$-parallel, the c_i are all multiples of each other in a small neighborhood. Allowing the coefficients to be real numbers, we can remove all but one representative from each $\mathcal{M}$-parallel class. Thus, we have an equation of the form

$$r_1 m_1 = \sum_{\substack{j \in J, \\ J \subset \{r+1, \dots, l\}}} q_j m_j$$

where neither side is zero, no two of the cylinders C_j are $\mathcal{M}$-parallel. However, $1/m_{i_j}$ are linear functional (over an open set in the space of real deformations) that are not colinear, so this relation cannot hold by Lemma 2.2.11. This is a contradiction, so the claim and the theorem are proven. $\square$

The following corollary, which is [NW14, Proposition 3.2], is useful for classifying invariant subvarieties. It immediately generalizes to the multi-component setting.

Corollary 2.2.13 (Cylinder Proportionality)**.** *Let M be a surface in an invariety subvariety of multi-component surfaces $\mathcal{M}$. Let C, C' be $\mathcal{M}$-parallel classes of cylinders on M, and let C, D be cylinders in C'. Then,*

$$\frac{\text{area}(C \cap C)}{\text{area}(C)} = \frac{\text{area}(D \cap C)}{\text{area}(D)}.$$

As another application of the Cylinder Deformation Theorem, we have a lemma about the relation between $\mathcal{M}$-parallel classes on a prime invariant subvariety and its components.

Lemma 2.2.14. *Let $\mathcal{M}$ be a prime invariant subvariety and $\mathcal{C}$ be a $\mathcal{M}$-parallel class of cylinders on a surface $M = (M_1, \dots, M_n) \in \mathcal{M}$. Define $\mathcal{M}_i := \overline{\text{proj}_i(\mathcal{M})}$ to be the closure of the i-th projection of $\mathcal{M}$. Let $\mathcal{C}_i$ be the cylinders of $\mathcal{C}$ on M_i. Then, $\mathcal{C}_i$ is a nonempty $\mathcal{M}_i$-parallel class of cylinders. Furthermore, if $\mathcal{C}_i, \mathcal{C}_i'$ are two distinct $\mathcal{M}_i$-parallel classes, then there are distinct Δ-parallel classes C, C' that contain $\mathcal{C}_i, \mathcal{C}_i'$ respectively.*

Proof. First we show each $\mathcal{C}_i$ is nonempty. Assume by contradiction that M_i does not have a cylinder in C, but M_j does. By the Cylinder Deformation Theorem, we can perform standard cylinder dilation on C while remaining in Δ. This causes the absolute periods of M_j to change without changing the absolute periods of M_i, which contradicts Theorem 2.1.5.

Let $p : H^1(M, \Sigma; \mathbb{C}) \to H^1(M; \mathbb{C})$ be the projection from relative to absolute cohomology. Then, $(pT_M\mathcal{M})^* \subset (T_M\mathcal{M})^*$. Since the core curves of cylinders γ_i are elements of absolute homology $H_1(M; \mathbb{C})$, we have $\pi(\gamma_i) \in (pT_M\mathcal{M})^*$ (where π is defined in the discussion before Definition 2.2.5). Now, let $\{\gamma_j\}$ be the core curves of cylinders of M_i. By [CW21, Theorem 1.3], $(pT_M\Delta)^* \cong (pT_{M_i}\mathcal{M}_i)^*$, so $\pi(\gamma_i)$ are colinear in $(pT_{M_i}\mathcal{M}_i)^*$ if and only if they are colinear in $(pT_M\Delta)^*$. Thus, the γ_i they are $\mathcal{M}_i$-parallel if and only if they are Δ-parallel. This proves the rest of the lemma. $\square$

2.3 Cylindrically Stable Surfaces

A lot of the proof techniques in this work involves moving around an invariant subvariety $\mathcal{M}$ by changing the shape of cylinders. The Cylinder Deformation Theorem guarantees that the standard twist always stays in $\mathcal{M}$. If $\operatorname{rel}\mathcal{M} = 0$, these are all the twists, but otherwise the space of these twists is even larger.

Definition 2.3.1. Let M be a horizontally periodic translation surface contained in a fixed invariant subvariety $\mathcal{M}$. Let $C_1, \ldots, C_k$ be the horizontal cylinders on M and γ_i the core curve of C_i, $\gamma_i^* \in H^1(M, \Sigma; \mathbb{R})$. The **twist space** of M, $\operatorname{Tw} M = T_M^{\mathbb{R}}\mathcal{M} \cap \operatorname{Span}_{i=1}^k \gamma_i^*$. Although the definition depends on $\mathcal{M}$, we leave it out of the notation as it will be clear from context.

Lemma 2.3.2. *[Wri15a, Lemma 8.10] Let $M \in \mathcal{M}$ be a horizontally periodic surface. It's twist space is isotropic. Thus,* $\dim \operatorname{Tw} M \le \operatorname{rk}\mathcal{M} + \operatorname{rel}\mathcal{M}$ *and* $\dim p(\operatorname{Tw} M) \le \operatorname{rk}\mathcal{M}$.

Lemma 2.3.3. *Let $M \in \mathcal{M}$ be horizontally periodic. If* $\dim \operatorname{Tw} M < \operatorname{rk}\mathcal{M} + \operatorname{rel}\mathcal{M}$*, any neighborhood containing M contains a horizontally periodic surface M' with more horizontal cylinders than M.*

Proof. By [Wri15a, Lemma 8.8] and Lemma 2.3.2, $\dim \operatorname{Pres}(M, \mathcal{M}) \ge \dim \mathcal{M} - p(\dim \operatorname{Tw} M) \ge \dim \mathcal{M} - \operatorname{rk}\mathcal{M} = \operatorname{rk}\mathcal{M} + \operatorname{rel}\mathcal{M}$. The proof follows from [Wri15a, Lemma 8.6] noting that the proof of [Wri15a, Lemma 8.6] can find a surface in any neighborhood around M. □

Definition 2.3.4. Let $M \in \mathcal{M}$ be a horizontally periodic translation surface. M is **cylindrically stable** if $\dim \operatorname{Tw} M = \operatorname{rk}\mathcal{M} + \operatorname{rel}\mathcal{M}$.

Lemma 2.3.5. *Let $\mathcal{H}$ be a stratum in genus g with s singular points. If $M \in \mathcal{H}$ is a horizontally period surface with $g + s - 1$ cylinders, then M is cylindrically stable.*

Proof. Let $C_1, \ldots, C_k$ be the horizontal cylinders of M. Then $\gamma_1^*, \ldots, \gamma_k^*$ are independent vectors in $H^1(M, \Sigma; \mathbb{C})$, and $p(\operatorname{Span}_{i=1}^k \gamma_i^*)$ is isotropic. Thus, M must have at most $g+s-1$ cylinders. Now assume that M has $g + s - 1$ cylinders. By Lemma 2.3.3, if M were not cylindrically stable, then there exists M' with more cylinders, but $g + s - 1$ is the maximum number of cylinders in $\mathcal{H}$. □

Remark 2.3.6. Because $p(\operatorname{Tw} M)$ is an isotropic subspace of $p(T_M\mathcal{M})$, if M is cylindrically stable, $\operatorname{Tw} M \cap \ker p = T_M\mathcal{M} \cap \ker p$.

Lemma 2.3.7. *If M is a horizontally periodic surface with less than $g + s - 1$ cylinders such that every cylinder of M is free, then M is not cylindrically stable.*

The ideas used to prove Lemma 2.3.7 are from [MW18, Section 5], but it was not written in a way that is easy to cite. Thus, a proof is needed. Define the **cylinder digraph** Γ of a horizontally periodic surface M to be a digraph with one vertex for each horizontal cylinder of M and a directed edge for each horizontal saddle connection on M, where the saddle goes from the cylinder below the saddle to the cylinder above the saddle. Let V, E be the set of vertices, edges of Γ. Define $\mathbb{Q}^E$ to be the free vector space generated by the elements of E. A loop in Γ gives an element of $\mathbb{Q}^E$ by taking the some of the edges in the loop. Define the **loop space** $L(\Gamma)$ as the subspace of $\mathbb{Q}^E$ generated by the loops in Γ that pass through each edge at most once.

Proof of Lemma 2.3.7. By [MW18, Lemma 5.1], $\dim L(\Gamma) = |E| - |V| + 1 = 2g + s - 1 - |V|$. Since every cylinder is free $\dim \mathrm{Tw}(M) = |V| = 2g + s - 1 - \dim L(\Gamma) < g + s - 1$ by assumption. Thus, $\dim L(\Gamma) > g \geq \operatorname{rk} \mathcal{M}$. By [MW18, Lemma 5.3], $\dim \mathrm{Tw}(M) < \dim \mathrm{CP}(M)$, so M is not cylindrically stable. □

Remark 2.3.8. [MW18, Lemma 5.4] states that, if $\mathcal{M}$ contains a surface with $g + s - 1$ horizontal free cylinders, then $\mathcal{M}$ is a connected component of a stratum.

2.4 WYSIWYG Compactification

We give a short overview of the WYSIWYG compactification. See [MW17] and [CW21] for a more formal treatment.

Definition 2.4.1. Let $\mathcal{H}, \mathcal{H}'$ be strata of multi-component translation surfaces potentially having marked points. Let $M_n = (X_n, \omega_n) \in \mathcal{H}$ and Σ_n be its set of singularities and marked points, and let $M = (X, \omega) \in \mathcal{H}'$ and Σ its set of singularities and marked points. We say that M_n converges to M if there are decreasing neighborhoods $\Sigma \subset U_i \subset M$ such that there are $g_i : X - U_i \to X_i$ that are diffeomorphisms onto their images satisfying

1. $g_i^*(\omega_i) \to \omega$ in the compact-open topology on $M - \Sigma$.

2. The injectivity radius of points not in the image of g_i goes to zero uniformly in i.

See [MW17, Definition 2.2].

Thus, we can construct $\partial\mathcal{H}$ from $\mathcal{H}$ by including all $\mathcal{H}'$ such that a sequence of surfaces in $\mathcal{H}$ converges to a surface in $\mathcal{H}'$. Multiple copies of a stratum can be included if there are two sequences that converge to the same surface in $\mathcal{H}'$ but are not close in $\mathcal{H}$. We call the union $\overline{\mathcal{H}} = \mathcal{H} \cup \partial\mathcal{H}$ (with the topology given by the above convergence of sequences) the WYSIWYG partial compactification of $\mathcal{H}$. For any invariant subvariety $\mathcal{M} \subset \mathcal{H}$, we define ∂M to be $\overline{\mathcal{M}} - \mathcal{M}$, where the closure $\overline{\mathcal{M}}$ is taken in $\overline{\mathcal{H}}$.

Remark 2.4.2. Even if M_n is a convergent sequence of surfaces without marked points, its limit may have marked points.

Let $M_n = (X_n, \omega_n) \in \mathcal{M}$ be a sequence of multi-component translation surfaces that has a limit $M = (X, \omega) \in \partial\mathcal{M}$. Let $\mathcal{H}'$ be the stratum with marked points that contains M. Let $\mathcal{N}$ be the connected component of $\mathcal{H}' \cap \partial\mathcal{M}$ that contains M. We call $\mathcal{N}$ the **component of the boundary of** $\mathcal{M}$ that contains M. The sequence X_n will approach a limit X' in the Deligne-Mumford compactification. For large enough n there is a map $f_n : X_n \to X'$ called the collapse map. There is also a map $g : X \to X'$ identifying together marked points of X. Define $(f_n)_* : H_1(X_n, \Sigma_n) \to H_1(X, \Sigma)$ and $V_n = \ker((f_n)_*)$.

Proposition 2.4.3. *After identifying $H_1(X_n, \Sigma_n)$ for different n, V_n eventually becomes constant which we call V. For large enough n, $T_M\mathcal{H}'$ can be identified with* $\mathrm{Ann}(V)$.

This proposition was proven for multi-component surfaces in [MW17, Proposition 2.5 and Proposition 2.6].

Theorem 2.4.4 (Mirzakhani-Wright 2017, Chen-Wright 2021)**.** *Let $\mathcal{M}$ be an invariant variety in a stratum $\mathcal{H}$ of connected translation surfaces with marked points. Let $M_n \in \mathcal{M}$ be a sequence that converges to $M \in \partial M$. Let $\mathcal{H}'$ be the stratum that contains M and $\mathcal{M}'$ be the component of the boundary of $\mathcal{M}$ that contains M. By Proposition 2.4.3 we identify $T_M\mathcal{H}'$ with* $\mathrm{Ann}(V)$. *Then, $T_M\mathcal{M}'$ can be identified with $T_{M_n}\mathcal{M} \cap$* $\mathrm{Ann}(V)$.

This theorem was proven in [MW17, Theorem 1.1] when M is connected, and in [CW21, Theorem 1.2] when M is disconnected.

Theorem 2.4.5. *[MW18, Theorem 1.1] An invariant subvariety is full rank if and only if it is a full stratum or a hyperelliptic locus.*

We clarify our definition of hyperelliptic locus in the case of marked points.

Definition 2.4.6. Let $\mathcal{H}$ be a stratum of genus ≥ 2 with marked points. $M \in \mathcal{H}$ is called **hyperelliptic** if there is an involution $J : M \to M$ that is the hyperelliptic involution on the underlying Riemann surface of M such that J maps marked points to marked points. $\mathcal{H}$ is **hyperelliptic** if every surface in $\mathcal{H}$ is hyperelliptic. An invariant subvariety $\mathcal{M} \subset \mathcal{H}$ is a **hyperelliptic locus** if it contains exactly the hyperelliptic surfaces of $\mathcal{H}$.

Corollary 2.4.7. *If $\mathcal{M}$ is a full rank invariant subvariety, then any component $\mathcal{N}$ of $\partial\mathcal{M}$ is a full rank invariant subvariety with marked points.*

Proof. By Theorem 2.4.5, a full rank invariant subvariety is a stratum or hyperelliptic locus. A stratum is cut out by no equations in period coordinates, so by Theorem 2.4.4 $\mathcal{N}$ is cut out by no equations, so it is a stratum. A hyperelliptic locus with hyperelliptic involution J is cut out by the equations $\int_\gamma \omega + \int_{J(\gamma)} \omega = 0$ for all saddles γ. J restricts to the hyperelliptic involution on $\mathcal{N}$ and the equations that cut out $\mathcal{M}$ restrict to the corresponding equations that cut out $\mathcal{N}$. Thus, $\mathcal{N}$ is a hyperelliptic locus. $\square$

CHAPTER 3

Classifying Periodic Points of Regular n-gon Veech Surfaces (Joint with Paul Apisa and Rafael Saavedra)

3.1 Introduction

The group $\mathrm{GL}^+(2,\mathbb{R})$ acts on the moduli space of translation surfaces, which is stratified by specifying the number of singularities of the flat metric and their cone angles. This action, which is generated by complex scalar multiplication and Teichmüller geodesic flow, preserves these strata. In the sequel, an element of a stratum will be denoted (X,ω) where X is a Riemann surface and ω is a holomorphic 1-form that induces the translation surface structure. Given such a point, its stabilizer $\mathrm{SL}(X,\omega)$ in $\mathrm{SL}(2,\mathbb{R})$ is called the **Veech group**, and (X,ω) is called a **Veech surface** when this group is a lattice.

Definition 3.1.1. A point p on a Veech surface (X,ω) is called *periodic* if p is not a zero of ω and if its orbit under $\mathrm{Aff}(X,\omega)$–the affine diffeomorphism group of (X,ω)–is finite.

Remark 3.1.2. Our definition is equivalent to the one used in Apisa [Api20]. A version of this definition which includes the zeros of ω first appeared in Gutkin-Hubert-Schmidt [GHS03]. Under this definition, an equivalent notion of a periodic point is a point marked by a holomorphic multisection of the universal curve over a suitable finite cover of the Teichmüller curve associated to (X,ω). See Möller [Mö06, Lemma 1.2] for details.

Consider the following translation surfaces. For n even, the **regular n-gon surface** is the regular n-gon with opposite sides identified, and for n odd, the **double n-gon surface** is two copies of a regular n-gon differing by a rotation by π with parallel sides glued together. The regular 10-gon and the double 7-gon surfaces are depicted in Figure 3.2. Both the regular and double n-gon surfaces are hyperelliptic with their hyperelliptic involutions being affine diffeomorphisms of derivative $-\mathrm{Id}$. The **Weierstrass points** are the fixed points of this involution. In [Vee89], Veech proved that the regular n-gon and double n-gon surfaces are Veech surfaces for all $n \geq 3$. Our main result is a classification of periodic points on these surfaces.

Theorem 3.1.3. *When $n \geq 5$ and $n \neq 6$, the periodic points of the regular n-gon and double n-gon surfaces are exactly the Weierstrass points that are not singularities of the flat metric.*

Remark 3.1.4. For the regular n-gon surface, the Weierstrass points are the center, midpoints of the sides, and, when $4 \mid n$, the vertices of the regular n-gon that comprises the surface. For the double n-gon surface, the Weierstrass points are the midpoints of the sides and vertices of the two regular n-gons that comprise the surface.

Remark 3.1.5. When $n = 5, 8$, and 10 this result was shown by Möller [Mö06, Theorems 5.1, 5.2]. When $n = 3, 4$ or 6 the surfaces are tori and have infinitely many periodic points coming from torsion points.

The proof can be divided into two steps. First, we use the transfer principle[1], recalled in the first sentence of the proof of Proposition 3.3.2, to reduce the problem to classifying periodic points on an explicit set of saddle connections[2]. Second, we classify the periodic points on these saddle connections by covering them with two collections of non-parallel cylinders and using the "rational height lemma", recalled as Lemma 3.3.5 below.

The regular n-gon and double n-gon surfaces belong to a larger infinite family of Veech surfaces called the Veech-Ward-Bouw-Möller surfaces [BM10]. It would be interesting to know whether our methods could be used to classify the periodic points on these surfaces, which, by Hooper [Hoo13] and Wright [Wri13], also admit a presentation as a disjoint union of regular polygons with side identifications.

Theorem 3.1.3 has consequences for the finite blocking problem.

Definition 3.1.6. Two points P, Q on a billiard table (resp. translation surface) M are **finitely blocked** if there is a finite set of points $S \subset M - \{P, Q\}$ such that all billiard trajectories (resp. straight line segments that do not contain singularities in their interior) from P to Q pass through a point in S.

The following corollaries will be proven after the proof of the main theorem.

Corollary 3.1.7. *When $n \geq 5$ and $n \neq 6$, the pairs of finitely blocked points on the regular n-gon and double n-gon surfaces consist precisely of any point that is not a singularity and its image under the hyperelliptic involution.*

Via the unfolding construction of Katok-Zemlyakov [ZK75], the $\left(\frac{\pi}{2}, \frac{\pi}{n}, \frac{(n-2)\pi}{2n}\right)$ triangle unfolds to the regular n-gon or double n-gon surface when n is even or odd respectively. Therefore, a consequence of the previous corollary is the following.

[1]Our use of the phrase "transfer principle" is inspired by and analogous to the usage in Calsamiglia-Deroin-Francaviglia [CDF23]

[2]A saddle connection is a closed geodesic arc on a translation surface whose endpoints are singular points and which has no singular points in its interior.

Corollary 3.1.8. *When $n \geq 5$ and $n \neq 6$, the $\left(\frac{\pi}{2}, \frac{\pi}{n}, \frac{(n-2)\pi}{2n}\right)$ triangle admits a pair of finitely blocked points if and only if n is even, in which case the only such pair is the vertex of angle $\frac{\pi}{n}$ and itself.*

Remark 3.1.9. Similar statements can be deduced for the $\left(\frac{\pi}{n}, \frac{\pi}{n}, \frac{(n-2)\pi}{n}\right)$ and $\left(\frac{2\pi}{n}, \frac{(n-2)\pi}{2n}, \frac{(n-2)\pi}{2n}\right)$ triangles, which unfold to the regular n-gon surface, the double n-gon surface, or a double cover of one of those surfaces.

The remainder of the paper is divided into three sections. In Section 3.2 we establish some facts about the flat geometry of the regular and double n-gon surfaces. This will require a lemma on the rationality of ratios of sines that will be proven in Section 3.4. Section 3.3 contains the proof of Theorem 3.1.3 and its corollaries.

Context. Some of the earliest results on periodic points and the finite blocking problem, especially for Veech surfaces, are due to Gutkin-Hubert-Schmidt [GHS03], Hubert-Schmoll-Troubetzkoy [HST08], and Monteil [Mon05, Mon09]; see especially [Mon05, Theorem 1] for related work on the finite blocking problem in the regular n-gon.

Periodic points in strata of Abelian and quadratic differentials were classified by Apisa [Api20] and Apisa-Wright [AW21a] respectively. Periodic points for every genus two translation surface were classified by Möller [Mö06] and Apisa [Api17]. These works use the classification of $\mathrm{GL}(2, \mathbb{R})$ orbit closures in genus two strata of Abelian differentials by McMullen [McM07]. The only periodic point on a genus two translation surface that is not a Weierstrass point was discovered by Kumar-Mukamel [KM17] and relates to orbit closures discovered by Eskin-McMullen-Mukamel-Wright [EMMW20].

For recent work on the finite blocking and illumination problems that was inspired by work of Eskin, Mirzakhani, and Mohammadi ([EM18, EMM15]), see Lelièvre-Monteil-Weiss [LMW16], Apisa-Wright [AW21a], and Wolecki [Wol19].

3.2 Preliminaries

Fix an integer n so that either $n = 5$ or $n \geq 7$. Let R_1 denote a regular n-gon circumscribed in the unit circle in $\mathbb{C}$ centered at the origin and so that one of its vertices lies at the point i. When n is even, the regular n-gon surface is R_1 with opposite sides identified. To form the double n-gon surface when n is odd we take a copy of R_1 rotated by $\frac{\pi}{n}$, which we call R_2, and identify parallel sides between R_1 and R_2. By triangulating these polygons and computing Euler characteristic, it is easy to see that the genus of the regular n-gon surface (resp. double n-gon surface) is $\lfloor \frac{n}{4} \rfloor$ (resp. $\frac{n-1}{2}$).

Let Γ_n denote the Veech group of the regular n-gon surface when n is even and the double n-gon surface when n is odd. Make the following definitions,

$$r_n = \begin{pmatrix} \cos\frac{\pi}{n} & -\sin\frac{\pi}{n} \\ \sin\frac{\pi}{n} & \cos\frac{\pi}{n} \end{pmatrix}, \qquad s_n = \begin{pmatrix} 1 & 2\cot\frac{\pi}{n} \\ 0 & 1 \end{pmatrix}.$$

Theorem 3.2.1 ([Vee89] (Definition 5.7, Theorem 5.8); see also [MT02] (Theorem 5.4))**.** *When n is even, Γ_n is generated by $\{r_n^2, s_n, r_n s_n r_n^{-1}\}$ and is isomorphic to the $(n/2, \infty, \infty)$ triangle group. In particular, $\mathbb{H}/\Gamma_n$ has two cusps.*

When n is odd, Γ_n is generated by $\{r_n, s_n\}$ and is isomorphic to the $(2, n, \infty)$ triangle group. In particular, $\mathbb{H}/\Gamma_n$ has one cusp.

Remark 3.2.2. In fact when n is even, Veech considered a double cover of the regular n-gon surface; however, it is well-known that the Veech group of the two surfaces are identical. Nevertheless, we will only ever use that, when n is even, Γ_n is contained in the Veech group of the regular n-gon surface, which is clear since each generator has that property.

Remark 3.2.3. It is well known (see for instance Veech [Vee89, Section 3] or Hubert-Schmidt [HS06, Lemma 4]) that for Veech surfaces, the maximal parabolic subgroups of the Veech group are in one-to-one correspondence with cylinder directions. The correspondence is given by associating to each cylinder direction, its stabilizer in the Veech group. Under this correspondence, the action of the Veech group on cylinder directions corresponds to its action by conjugation on maximal parabolic subgroups. Since conjugacy classes of maximal parabolic subgroups correspond to cusps of the quotient of the upper half plane by the Veech group, we see that each cusp corresponds to the orbit of a cylinder direction under the Veech group. Thus, every cylinder direction can be moved to one of these prescribed directions (described below) by an element of the Veech group.

In light of this observation, by Theorem 3.2.1, on the double n-gon surface any cylinder direction may be sent to any other by an affine diffeomorphism. Similarly, there are two orbits of cylinders under the action of the affine diffeomorphism group on the regular n-gon surface. We will now describe these two cylinder directions.

The first is the horizontal direction, which is covered by $\lceil \frac{n}{4} \rceil$ (resp. $\frac{n-1}{2}$) cylinders when n is even (resp. odd), as seen on the left in Figure 3.1. Let g_n' denote the number of horizontal cylinders. Notice that g_n' is greater than or equal to the genus of the surface. Since the vertices of R_1 lie at the points $\left\{ i\exp\left(\frac{2j\pi i}{n}\right)\right\}_{j=0}^{n-1}$, it is easy to see that the heights of the horizontal cylinders are

$$h_j := \operatorname{Im}\left(i\exp\left(\frac{2j\pi i}{n}\right) - i\exp\left(\frac{(2j+2)\pi i}{n}\right)\right)$$

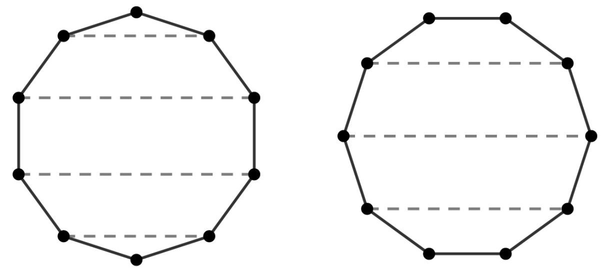

Figure 3.1: Two cylinder directions for the regular decagon.

for $j \in \{0, \ldots, g_n' - 1\}$. We can simplify the expression for the heights as follows,

$$h_j = \operatorname{Im}\left(i\exp\left(\frac{(2j+1)\pi i}{n}\right)\left(\exp\left(\frac{-\pi i}{n}\right) - \exp\left(\frac{\pi i}{n}\right)\right)\right) = 2\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{(2j+1)\pi}{n}\right). \tag{3.1}$$

When n is odd, notice that since $\sin(x) = \sin(\pi - x)$ for any x, for $j > \left\lfloor \frac{n-3}{4} \right\rfloor$ we can write $\sin\left(\frac{(2j+1)\pi}{n}\right) = \sin\left(\frac{(n-2j-1)\pi}{n}\right)$, and so

$$\{h_j\}_{j=0}^{g_n'-1} = \left\{2\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{k\pi}{n}\right)\right\}_{k=1}^{g_n'}. \tag{3.2}$$

When n is even, there are two orbits of cylinder directions under the action of the Veech group. Under the equivalences explained above, these two cylinder directions can be chosen to be stabilized by the maximal parabolic subgroup of Γ_n generated by s_n and $r_n s_n r_n^{-1}$. Using this observation (which is explained more fully in [Vee89, Section 5]), a cylinder direction that cannot be sent to the horizontal one by the Veech group can be described as follows. Rotate the regular n-gon surface so that R_1 remains circumscribed in a unit circle but with one of its edges being horizontal. Let R_1' denote this rotated copy of R_1. The horizontal direction is now covered by $g_n := \left\lfloor \frac{n}{4} \right\rfloor$ cylinders, as seen on the right in Figure 3.1; the notation g_n is chosen since it is equal to the genus of the surface. Since the vertices of R_1' lie at the points $\left\{i\exp\left(\frac{(2j+1)\pi i}{n}\right)\right\}_{j=0}^{n-1}$, it is easy to see that the heights of the horizontal cylinders are

$$h_j' := \operatorname{Im}\left(i\exp\left(\frac{(2j+1)\pi i}{n}\right) - i\exp\left(\frac{(2j+3)\pi i}{n}\right)\right)$$

for $j \in \{0, \ldots, g_n - 1\}$. We can simplify the expression for the heights as follows,

$$h_j' = \operatorname{Im}\left(i\exp\left(\frac{(2j+2)\pi i}{n}\right)\left(\exp\left(\frac{-\pi i}{n}\right) - \exp\left(\frac{\pi i}{n}\right)\right)\right) = 2\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{(2j+2)\pi}{n}\right). \tag{3.3}$$

The following result was stated in McMullen [McM06, page 7] where it is indicated that its proof follows from an application of the bounds in the proof of [McM06, Theorem 2.1]. Since the deduction is not entirely trivial, we offer a proof in Section 3.4. In our deduction, we will not use the full strength of [McM06, Theorem 2.1], which shows that the number of sine ratios of any fixed degree over $\mathbb{Q}$ is finite and which can be used to find all such ratios.

Lemma 3.2.4. *For rational numbers* $0 < \alpha < \beta \leq \frac{1}{2}$, $\frac{\sin(\pi\alpha)}{\sin(\pi\beta)}$ *is rational if and only if* $\alpha = \frac{1}{6}$ *and* $\beta = \frac{1}{2}$.

Lemma 3.2.5. *On the regular n-gon and double n-gon surfaces, at least one cylinder direction has the property that the ratio of heights and circumferences of distinct cylinders in this direction have irrational ratio; every cylinder direction has this property whenever n is not congruent to* 0 *or* 6 *mod* 12. *Moreover, for any two parallel cylinders sharing a boundary saddle connection, their heights and circumferences have an irrational ratio.*

Proof. By Remark 3.2.3, any cylinder direction can be sent to one of the two directions specified in Remark 3.2.3 by an element of the Veech group. In particular, the ratio of heights of two distinct parallel cylinders is either $\frac{h_j}{h_k}$ or $\frac{h'_j}{h'_k}$. By Equations (3.1), (3.2), and (3.3), up to inverting the ratio, these ratios always have the form $\frac{\sin(\pi\alpha)}{\sin(\pi\beta)}$ for rational numbers $0 < \alpha < \beta \leq \frac{1}{2}$ where $n\alpha$ and $n\beta$ are integers. By Lemma 3.2.4 such a ratio is rational if and only if $\alpha = \frac{1}{6}$ and $\beta = \frac{1}{2}$. In particular by Equation (3.2) the ratio of heights of distinct parallel cylinders is irrational when n is odd.

Suppose therefore that n is even. By Equation (3.1), $\frac{h_j}{h_k} = \frac{\sin\left(\frac{(2j+1)\pi}{n}\right)}{\sin\left(\frac{(2k+1)\pi}{n}\right)}$. If $j < k$ then this ratio is rational if and only if $\frac{2j+1}{n} = \frac{1}{6}$ and $\frac{2k+1}{n} = \frac{1}{2}$. This implies that $n = 12j + 6$ and $k = 3j + 1$. Therefore, the ratio of heights of distinct cylinders in this cylinder direction is irrational if and only if n is not congruent to 6 mod 12. Moreover, since j and k correspond to cylinders that share a boundary saddle connection if and only if $|j - k| = 1$ we have that cylinders that share a boundary saddle connection have an irrational ratio of height as long as $j > 0$, which is the case since we have assumed that $n \neq 6$.

By Equation (3.3), $\frac{h'_j}{h'_k} = \frac{\sin\left(\frac{(2j+2)\pi}{n}\right)}{\sin\left(\frac{(2k+2)\pi}{n}\right)}$. If $j < k$ then this ratio is rational if and only if $\frac{2j+2}{n} = \frac{1}{6}$ and $\frac{2k+2}{n} = \frac{1}{2}$. This implies that $n = 12j + 12$ and $k = 3j + 2$. Therefore, the ratio of heights of distinct cylinders in this cylinder direction is irrational if and only if n is not congruent to 0 mod 12. As before cylinders that share a boundary saddle connection have an irrational ratio of height.

By Remark 3.2.3 any cylinder direction can be moved by an element of the Veech group to one of the two cylinder directions analyzed in the preceding paragraphs, so the result follows. The claims for circumferences hold since the ratio of moduli [3] of parallel cylinders is rational for any Veech surface. $\square$

[3]The modulus of a cylinder of height h and circumference c on a translation surface is $\frac{h}{c}$.

Definition 3.2.6. Given translation surfaces (X, ω) and (X', ω') a *translation cover* $f : (X, \omega) \to (X', \omega')$ is a holomorphic map $f : X \to X'$ such that $f^*\omega' = \omega$. Similarly, if (Y, q) is a quadratic differential that is not the square of a holomorphic one-form, then we say that $f : (X, \omega) \to (Y, q)$ is a **half-translation cover** if $f : X \to Y$ is holomorphic and $f^*q = \omega^2$.

Lemma 3.2.7. *Let (X, ω) be the regular n-gon or double n-gon surface. If (X', ω') is a translation surface so that $f : (X, \omega) \to (X', \omega')$ is a translation cover then $(X, \omega) = (X', \omega')$.*

Proof. Suppose to a contradiction that there is a translation cover $f : (X, \omega) \to (X', \omega')$ where the genus g' of X' is less than the genus g of X. For each cylinder C of circumference c on (X, ω), there is an integer m so that $f(C)$ is a cylinder of circumference c/m. By Lemma 3.2.5, there is a cylinder direction on (X, ω) in which all distinct pairs of cylinders have an irrational ratio of circumferences and hence map to distinct cylinders on (X', ω') under f. Since every cylinder direction on (X, ω) has at least g cylinders (see the description of cylinder directions in Remark 3.2.3), it follows that there is a cylinder direction on (X', ω') with g distinct cylinders. Since ω has at most two zeros, the number of zeros s' of ω' is also at most two. Therefore, $g \leq g' + 1$, since the number of cylinders on (X', ω') is bounded above by $g' + s' - 1$. Since we have assumed that $g' < g$ we see that $g' = g - 1$ and $s' = 2$.

By the Riemann-Hurwitz formula, this implies that $g \leq 3$ and hence that $n \in \{5, 7, 8, 10, 12, 14\}$. The condition that ω has two singularities reduces the possibilities to just $n \in \{10, 14\}$. Since 10 and 14 are not congruent to 0 or 6 mod 12 it follows from Lemma 3.2.5 that these surfaces possess cylinder directions with $g + 1$ cylinders so that the ratio of circumferences of distinct cylinders is irrational. As argued above, this implies that (X', ω') also has such a cylinder direction and hence that $g = g'$, which is a contradiction. $\square$

Corollary 3.2.8. *For the regular n-gon and double n-gon, the affine diffeomorphism group is isomorphic to the Veech group.*

Proof. Let (X, ω) be the regular n-gon or double n-gon surface. Letting $\mathrm{Aut}(X, \omega)$ be the group of affine diffeomorphisms of derivative Id, we have the following short exact sequence (see [Vee89, Equation (2.6)]).

$$0 \longrightarrow \mathrm{Aut}(X, \omega) \longrightarrow \mathrm{Aff}(X, \omega) \longrightarrow \mathrm{SL}(X, \omega) \longrightarrow 0.$$

It suffices to show that $\mathrm{Aut}(X, \omega)$ is trivial. This follows from Lemma 3.2.7 since the cover $(X, \omega) \to (X, \omega)/\mathrm{Aut}(X, \omega)$ must be the identity. $\square$

From now on, we consider the Veech group acting on the regular or double n-gon by the above isomorphism.

Remark 3.2.9. It is standard that the Weierstrass points that do not coincide with singularities of the flat metric are periodic points for any translation surface in a hyperelliptic locus of a stratum. However, in our case, this is particularly easy to see since the isomorphism between $\mathrm{Aff}(X, \omega)$ and $\mathrm{SL}(X, \omega)$ shows that the hyperelliptic involution is in the center of $\mathrm{Aff}(X, \omega)$–since it is sent to $-\mathrm{Id}$–and hence that $\mathrm{Aff}(X, \omega)$ permutes its fixed points.

Remark 3.2.10. Let n be even and let p provisionally denote the center of the regular n-gon R_1 whose opposite sides are identified to form the regular n-gon surface. We will show that p is fixed by every element of the affine diffeomorphism group. It is obvious that it is fixed by the rotation r_n. The remaining generators of the affine diffeomorphism group are shears in the cylinder directions identified in Remark 3.2.3. In cylinder direction of slope $\frac{\pi}{n}$, p lies on a boundary saddle connection of a cylinder and is trivially fixed. In the horizontal cylinder direction, p lies on the core curve of a cylinder C of modulus $\tan(\pi/n)$. The corresponding generator of the maximal parabolic subgroup is s_n, which performs two Dehn twists in C and hence fixes p. Since p is fixed by the generators of the affine diffeomorphism group, it is fixed by every element in it.

3.3 Proof of Theorem 3.1.3 and its corollaries

We now begin our study of periodic points using the transfer principle. Our goal is to reduce the main theorem to identifying the periodic points on finitely many saddle connections. We start with the following definition.

Definition 3.3.1. When n is even let P_n denote the point in Remark 3.2.10. When n is odd, let P_n denote the unique cone point of the flat metric of the double n-gon surface. Both points are fixed by every element of Γ_n.

Proposition 3.3.2. *The Γ_n orbit of any periodic point on the regular n-gon or double n-gon surface contains a point lying on the leaf of the horizontal foliation passing through P_n or, when n is even, a point lying on the leaf of the foliation passing through P_n that makes an angle of $\frac{\pi}{n}$ with the horizontal (see Figure 3.2). These leaves are saddle connections or possibly, when n is even, the core curve of a cylinder.*

Proof. The transfer principle states that if G and H are topological groups acting continuously, from the left and right respectively, on a topological space $\mathcal{X}$ then the following are in bijective correspondence:

1. Closed (resp. dense) G orbits on $\mathcal{X}/H$.

2. Closed (resp. dense) $G \times H$ orbits on $\mathcal{X}$.

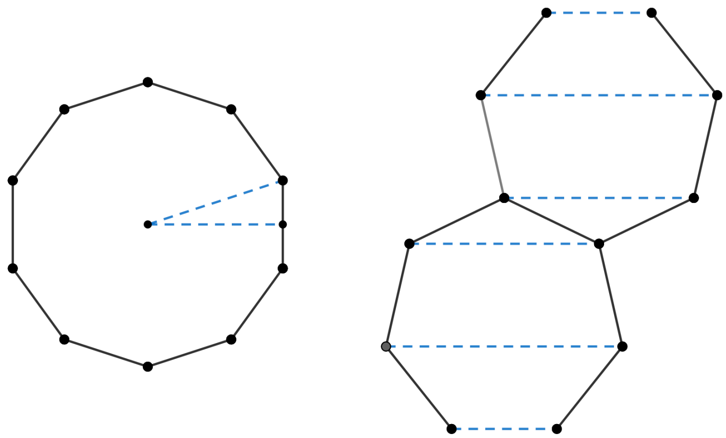

Figure 3.2: After applying Proposition 3.3.2, any periodic point can be assumed to lie on one of the dashed lines or its image under the hyperelliptic involution.

3. Closed (resp. dense) H orbits on $G\backslash\mathcal{X}$

We will briefly sketch a proof of this claim for closed orbits (the case of dense orbits is similar). Let $\pi : \mathcal{X} \to \mathcal{X}/H$ be the quotient map. If $C \subseteq \mathcal{X}/H$ is a closed G-invariant set, then its preimage $\pi^{-1}(C)$ is a closed $G \times H$-invariant set. Conversely, if $D \subseteq \mathcal{X}$ is a closed $G \times H$ invariant set, then, since it is H-invariant, $\pi(D)$ is the complement of $\pi(\mathcal{X} - D)$. Since π is an open map, $\pi(\mathcal{X} - D)$ is open and hence $\pi(D)$ is a closed G-invariant set. Therefore, taking images and preimages under π establishes a bijection between closed G-invariant subsets of $\mathcal{X}/H$ and closed $G \times H$ invariant subsets of $\mathcal{X}$. This bijection restricts to the desired bijection between closed orbits.

Under these correspondences, a $G \times H$ orbit of $x \in \mathcal{X}$ will be sent to the G orbit of xH or the H orbit of Gx. In our context, G is the Veech group Γ_n, $\mathcal{X}$ is $\mathrm{SL}(2,\mathbb{R})$, and H is the unipotent subgroup $U := \left\{\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} : s \in \mathbb{R}\right\}$. The quotient $\mathrm{SL}(2,\mathbb{R})/U$ can be identified with $\mathbb{R}^2 - \{0\}$ by sending $g \in \mathrm{SL}(2,\mathbb{R})$ to $g \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Under this identification, the action of Γ_n on $\mathrm{SL}(2,\mathbb{R})/U$ is given by the standard linear action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}^2 - \{0\}$. Define $a_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ where t is any real number.

It is a foundational result of Dani [Dan78, Theorem A] that the only U orbits of $\Gamma_n\backslash\mathrm{SL}(2,\mathbb{R})$ are closed or dense, and the closed orbits are horocycles around the cusps. Recall from Theorem 3.2.1 that cusps of $\Gamma_n\backslash\mathrm{SL}(2,\mathbb{R})$ correspond to conjugacy classes of maximal parabolic subgroups, and these are generated by s_n and, when n is even, $r_n s_n r_n^{-1}$. In particular, the closed horocycles corresponding to the cusps of $\Gamma_n\backslash\mathrm{SL}(2,\mathbb{R})$ are given by $\Gamma_n a_t U$ and also, when n is even, $\Gamma_n r_n a_t U$ where t is any real number. By the transfer principle, the only vectors in $\mathbb{R}^2 - \{0\}$ that do not have dense Γ_n orbit are vectors parallel to a vector in $\Gamma_n \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or, when n is even, parallel to a vector in $\Gamma_n \cdot \begin{pmatrix} \cos \frac{\pi}{n} \\ \sin \frac{\pi}{n} \end{pmatrix}$.

Now let p be any periodic point that is distinct from P_n. By definition, the orbit of p under Γ_n is finite. In particular, $\Gamma_n \cdot p$ remains a bounded distance away from P_n. Since the regular n-gon surface is comprised of a convex polygon with opposite sides identified and since the double n-gon surface is comprised of two regular n-gons whose vertices correspond to P_n, it follows that there is a straight line segment γ from P_n to p, the holonomy of which we will denote by v. Since p remains a bounded distance away from P_n we have that $\Gamma_n \cdot v$ is not dense in $\mathbb{R}^2$ (in particular, the orbit is not dense in a neighborhood of 0) and hence there is a vector in the Γ_n orbit of v that is parallel to a vector in $\Gamma_n \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or, when n is even, one in $\Gamma_n \cdot \begin{pmatrix} \cos \frac{\pi}{n} \\ \sin \frac{\pi}{n} \end{pmatrix}$. This shows that $\Gamma_n \cdot p$ contains a point on either the horizontal leaf through P_n or, in the case of n even, the leaf that makes an angle of $\frac{\pi}{n}$ with the horizontal. $\square$

Definition 3.3.3. When n is odd, call the saddle connections identified in Proposition 3.3.2 *candidate line segments*. When n is even, notice that the hyperelliptic involution fixes both line segments identified in Proposition 3.3.2. Each line segment can be partitioned into two subsegments, which are exchanged by the hyperelliptic involution and have one endpoint at P_n. For each line segment identified in Proposition 3.3.2 we choose one of these subsegments and call them *candidate line segments* (see Figure 3.2). Since the hyperelliptic involution preserves the collection of periodic points, any periodic point can be moved by an element of the Veech group to a candidate line segment. (Recall that, as observed in Remark 3.2.9, the hyperelliptic involution can be identified with the element $-\mathrm{Id}$ of the Veech group.)

In the sequel, we will adopt the convention that all cylinders are assumed to be closed; that is, they contain their boundary.

Definition 3.3.4. A point contained in a cylinder C is said to have *rational height in C* if its distance from the boundary of C is a rational multiple of the height of C.

The following observation is well-known and a version of it appears in [HS00, Lemma 4] and [Api20, Lemma 5.4].

Lemma 3.3.5. *A periodic point on a Veech surface has rational height in any cylinder containing it.*

Proof. Let p be the periodic point and suppose that it is contained in a cylinder C. After perhaps rotating the surface we may suppose without loss of generality that C is horizontal. Denote its height and circumference by h and c respectively. Since the surface is Veech, the Veech group contains an element $g := \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ where $s = kc/h$ for some nonzero integer k. Choosing flat coordinates so that the bottom boundary of C lies on the x-axis, we see that g sends a point $(x, y) \in C$ to $(x + ykc/h, y)$ where the x coordinate is taken modulo c. Thus, if (x, y) has finite orbit under the Veech group, then y is a rational multiple of h. □

In the following lemma, it will be useful to use the notation $\overline{PQ}$ to refer to a straight line segment on a flat surface between points P and Q. In general on a flat surface there are infinitely many straight line segments between any two points, so we emphasize that this notation presupposes a choice of a line segment between P and Q and that the line segment is not uniquely determined by its endpoints.

Lemma 3.3.6. *Let C_2 and C_3 be two adjacent parallel cylinders whose ratio of heights is irrational. Let C_1 be another cylinder. Suppose that $\overline{PQ}$ is a line segment satisfying the following:*

1. *$\overline{PQ}$ is neither parallel nor perpendicular to the core curves of C_1, C_2, and C_3.*
2. *$\overline{PQ}$ is contained in C_1 and its interior does not intersect the boundary of C_1.*
3. *$\overline{PQ}$ is contained in $C_2 \cup C_3$ and its interior intersects the boundary of C_2 and C_3 in a unique point R. $\overline{PR}$ (resp. $\overline{RQ}$) is contained in C_2 (resp. C_3) and has nonzero length, see Figure 3.3.*
4. *The orthogonal projection of $\overline{PQ}$ (resp. $\overline{PR}$, $\overline{RQ}$) to the core curve of C_1 (resp. C_2, C_3) is a proper subset of the core curve.*
5. *P (resp. Q) has rational height in C_1 and C_2 (resp. C_1 and C_3).*

Then the only point on $\overline{PR}$ (resp. $\overline{RQ}$) that has rational height in both C_1 and C_2 (resp. C_1 and C_3) is P (resp. Q).

Proof. Assume for the sake of contradiction that there is a point S on $\overline{PR}$ other than P with rational height inside both C_1 and C_2. Without loss of generality, perhaps after rotating the surface, we may suppose that C_1 is horizontal.

When P is contained in the interior of C_1, let ℓ denote the leaf of the horizontal foliation passing through P. When P is contained in the boundary of C_1, let ℓ denote the boundary of C_1 containing P. Since the interior of $\overline{PQ}$ is contained in the interior of C_1 (by (2)), we may think of C_1 as a Euclidean cylinder and orthogonally project $\overline{PQ}$ onto ℓ. By (1) and (4), this projection is a line segment $\overline{PU}$ where $U \neq P$. Let $\overline{QU}$ denote the vertical line segment contained in C_1 from Q to U.

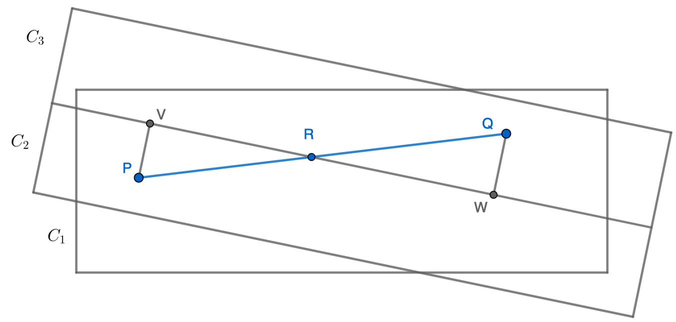

Figure 3.3: The three cylinders in Lemma 3.3.6.

The triangle, which we will denote ΔPQU, formed by $\overline{PQ}$, $\overline{PU}$, and $\overline{QU}$ is, by (2), a right triangle contained in C_1 as shown in Figure 3.4. Let T be the point on $\overline{QU}$ so that ΔSQT is similar to ΔPQU (see Figure 3.4).

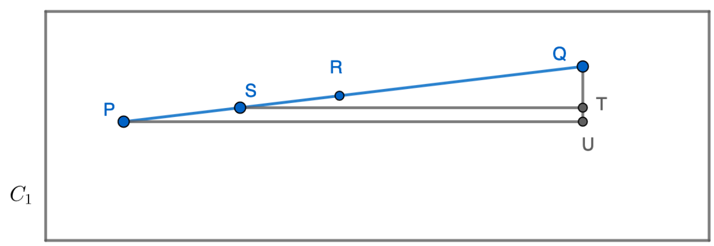

Figure 3.4: The triangles SQT and PQU are similar.

Since, by (5), P, Q, and S (and hence also T and U) have rational height in C_1, it follows that the length of $\overline{UT}$ is a rational multiple of the length of $\overline{TQ}$. Since ΔSQT and ΔPQU are similar, it follows that the length of $\overline{PS}$ is a rational multiple of the length of $\overline{SQ}$.

The preceding argument (with the role of $\overline{PQ}$ being played by $\overline{PR}$ and that of C_1 by C_2) shows that the length of $\overline{PS}$ is a rational multiple of the length of $\overline{SR}$ (notice that (3) is the analogue of (2) here). It follow that the lengths of $\overline{PR}$ and $\overline{RQ}$ have a rational ratio.

Finally, let V (resp. W) denote the orthogonal projection of P (resp. Q) to the component of the boundary of C_2 (resp. C_3) containing R (see Figure 3.3). We note that the triangles ΔPVR and ΔQWR, which are formed in the same way we formed ΔPQU, are similar. Since the lengths of $\overline{PR}$ and $\overline{RQ}$ have a rational ratio, so do the lengths of $\overline{PV}$ and $\overline{RW}$. However, by (5), the lengths of $\overline{PV}$ and $\overline{RW}$ are rational multiples of the height of C_2 and C_3 respectively. Therefore, we have shown that C_2 and C_3 have a rational ratio of heights, which is a contradiction.

$\square$

Proof of Theorem 3.1.3. Suppose first that n is even. By Proposition 3.3.2, any periodic point can be moved by an element of the Veech group to one of the two candidate line segments (see Definition 3.3.3 and Figure 3.5). The endpoints of the candidate line segments are singularities of the flat metric or Weierstrass points. It suffices to show that these endpoints are the only periodic points on a candidate line segment. Choosing a candidate line segment, let P denote P_n, which is one endpoint, and let Q denote the other endpoint. We will let $\overline{PQ}$ denote the candidate line segment. Notice that $\overline{PQ}$ is contained in a single cylinder C_1 that makes an angle of $-\frac{\pi}{n}$ with the horizontal and to which $\overline{PQ}$ is not parallel. This is the dotted cylinder in Figure 3.5.

The line segment $\overline{PQ}$ is also contained in the union of two parallel cylinders C_2 and C_3 in the cylinder direction that makes an angle of $-\frac{2\pi}{n}$ with the horizontal. The cylinders C_2 and C_3 share a boundary saddle connection and so by Lemma 3.2.5 they have an irrational ratio of heights. These cylinders are the dashed cylinders in Figure 3.5.

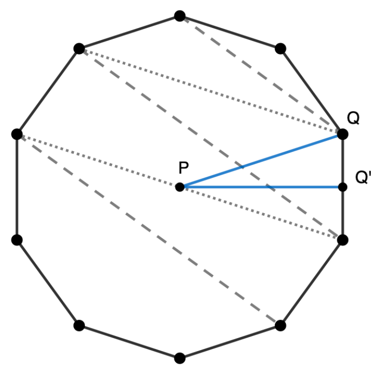

Figure 3.5: The cylinders dividing candidate lines segments in the even case. Both Q and Q' are endpoints of candidate line segments.

Since the endpoints of the candidate line segments are either singularities of the flat metric or periodic points they have rational height in any cylinder containing them by Lemma 3.3.5. By Lemma 3.3.6 it follows that any point lying in the interior of $\overline{PQ}$ has irrational height in at least one of C_1, C_2, and C_3. Therefore, any point lying in the interior of a candidate line segment cannot be periodic by Lemma 3.3.5 as desired.

Now suppose that n is odd. Recall that the double n-gon surface is comprised of two regular n-gons, which we denote by R_1 and R_2, that differ from each other by a rotation of π/n and so that parallel sides are identified. Recall too that the hyperelliptic involution exchanges R_1 and R_2. Since every candidate line segment is contained in either R_1 or R_2 it suffices to classify the periodic points on the candidate line segments in just R_2. Recall that we have supposed that R_2 is circumscribed in a circle centered at the origin in $\mathbb{C}$ and has a vertex lying at the point $-i$.

We will begin by showing that if a candidate line segment passes through the interior of R_2 then its interior contains no periodic points. Notice that such a candidate line segment is contained in the interior of a cylinder C_1 that makes an angle of $-\frac{\pi}{n}$ with the horizontal and in the union of two

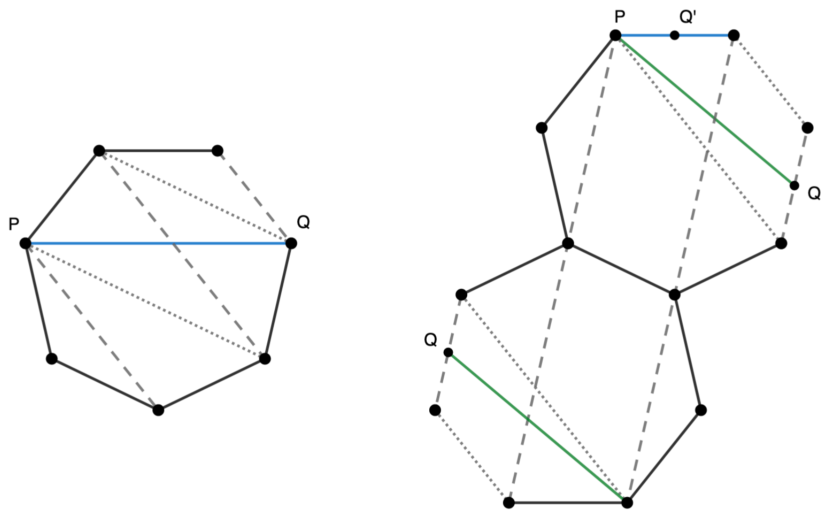

Figure 3.6: The cylinders dividing candidate line segments in the odd case. The boundary of C_1 (resp. C_2 and C_3) are the dotted (resp. dashed) lines.

cylinders C_2 and C_3 that share a boundary saddle connection and make an angle of $-\frac{2\pi}{n}$ with the horizontal (see the left subfigure in Figure 3.6). The claim that this candidate line segment contains no periodic points in its interior is now identical to the argument in the case of n even.

It remains to consider the candidate line segment ℓ that is an edge of R_2 (see the right subfigure in Figure 3.6). Let Q' denote the midpoint of this candidate line segment, which is a Weierstrass point. We will show that Q' is the only periodic point on the interior of ℓ. As before, let P denote P_n and let $\overline{PQ'}$ denote the line segment contained in ℓ that begins at P, travels in the positive horizontal direction and ends at Q'.

Notice that ℓ is entirely contained in a cylinder C_1 that makes an angle of $-\frac{\pi}{n}$ with the horizontal. Apply the element $r_n^{-1}s_nr_n$ of the Veech group, which shears the cylinders parallel to C_1. When n is odd, any two parallel cylinders have equal moduli (Veech showed this for one specific cylinder direction in [Vee89, Equation 5.2] and so it holds for every cylinder direction by Theorem 3.2.1, see Remark 3.2.3). Therefore, $r_n^{-1}s_nr_n$ acts on C_1 by performing a Dehn twist. In particular, the image $\overline{PQ'}$ under $r_n^{-1}s_nr_n$ remains in C_1 and becomes a line segment from P to a new point Q (see the right subfigure of Figure 3.6). Let $\overline{PQ}$ denote this line segment.

It is easy to see that the line segment $\overline{PQ}$ is contained in two cylinders C_2 and C_3 that share a boundary saddle connection and make an angle of $-\frac{\pi}{n}$ with the vertical. Proceeding as in the case of n even yields that $\overline{PQ}$ contains no periodic points in its interior and so the same must hold for $\overline{PQ'}$. Since ℓ is the union of $\overline{PQ'}$ and its image under the hyperelliptic involution, we have that Q' is the only periodic point contained in the interior of ℓ as desired. □

Proof of Corollary 3.1.7. By [Mö06, Theorem 2.6] whenever a translation surface (X, ω) is not a translation cover of a torus there is a translation cover $\pi_{X_{min}} : (X, \omega) \to (X_{min}, \omega_{min})$ so that any translation cover with domain (X, ω) is a factor of $\pi_{X_{min}}$. Similarly, by [AW21a, Lemma 3.3], there is a quadratic differential (Q_{min}, q_{min}) and a degree one or two (half)-translation cover π : $(X_{min}, \omega_{min}) \to (Q_{min}, q_{min})$ so that any half-translation cover is a factor of $\pi_{Q_{min}} := \pi \circ \pi_{X_{min}}$. By [AW21a, Theorem 3.6 and Lemma 3.8], if (p, q) are finitely blocked on (X, ω) then one of the following occurs:

1. p and q are periodic points or zeros of ω and the blocking set may be taken to be the collection of all other distinct periodic points.
2. Neither p nor q are periodic points or zeros of ω, but $\pi_{Q_{min}}(p) = \pi_{Q_{min}}(q)$ and the blocking set may be taken to be the union of the periodic points with $\pi^{-1}_{Q_{min}}\left(\pi_{Q_{min}}(p)\right)$.

Let (X, ω) now denote the regular n-gon or double n-gon surface. By Lemma 3.2.7, (X, ω) is not a translation cover of a torus and $\pi_{X_{min}}$ is the identity. Since $(X, \omega) = (X_{min}, \omega_{min})$, the discussion above shows that $\pi_{Q_{min}}$ must be degree one or two. By uniqueness of $\pi_{Q_{min}}$, if (X, ω) admits any degree two map to a quadratic differential this map must be $\pi_{Q_{min}}$. Since (X, ω) is hyperelliptic, the quotient by the hyperelliptic involution is such a map and hence must be $\pi_{Q_{min}}$.

By [AW21a, Lemma 3.1], all pairs (p, q) where p is not a zero of ω and q is its image under the hyperelliptic involution are finitely blocked. (Note that the statement of [AW21a, Lemma 3.1] does not include the case when $p = q$ is a Weierstrass point that is not a zero of ω. However, the proof is identical.) By Remark 3.1.4, it is clear that a zero of ω is never blocked from itself by the collection of Weierstrass points and hence is never finitely blocked from itself.

It remains to show that if p and q are distinct points that are Weierstrass points or zeros of ω, then they are not finitely blocked from each other. Since the blocking set would have to consist of the other Weierstrass points, convexity of the n-gons comprising (X, ω) and the explicit description of the Weierstrass points in the preceding paragraph shows that this is never possible. Thus, the only finitely blocked points are the ones listed in the statement of the corollary. □

Proof of Corollary 3.1.8. Let T be a billiard table that unfolds to a translation surface (X, ω). Two points p and q on T are finitely blocked if and only if every preimage of p is finitely blocked from every preimage of q on (X, ω). When (X, ω) is the regular n-gon or double n-gon surface, each point is finitely blocked from at most one other by Corollary 3.1.7. When T is the $\left(\frac{\pi}{2}, \frac{\pi}{n}, \frac{(n-2)\pi}{2n}\right)$ triangle the only points on T that have two or fewer preimages on (X, ω) are the vertices of angle $\frac{(n-2)\pi}{2n}$ and $\frac{\pi}{n}$, which, in the first case, unfolds to the zeros of ω and, in the second, to either the Weierstrass point P_n when n is even or to two points exchanged by the hyperelliptic involution when n is odd. Since the zeros of ω are not finitely blocked from any other point, we see that the only possible pair of finitely blocked points on T is the vertex of angle $\frac{\pi}{n}$ from itself. When n is even, the preimage of this vertex on (X, ω) is P_n, which is finitely blocked from itself. When n is odd, the preimage of this vertex consists of two points swapped by the hyperelliptic involution, and each point is not finitely blocked from itself.

□

3.4 Proof of Lemma 3.2.4

In this section we will show that for rational numbers $0 < \alpha < \beta \leq \frac{1}{2}$, $\frac{\sin(\pi\alpha)}{\sin(\pi\beta)}$ is rational if and only if $\alpha = \frac{1}{6}$ and $\beta = \frac{1}{2}$. McMullen stated this result in [McM06, page 7] and indicated that its proof follows from an application of the bounds in the proof of [McM06, Theorem 2.1]. Since we were unable to find an explicit proof in the literature, we provide one here.

For any positive integer m, let $\zeta_m := \exp\left(\frac{2\pi i}{m}\right)$ and

$$g(m) := \begin{cases} m & m \equiv 2 \bmod 4 \\ 2m & 4 \mid m \\ 4m & m \text{ odd.} \end{cases}$$

We begin with the following simple lemma, which is well known. Throughout this section, if p and q are positive integers we will let (p, q) denote their greatest common divisor.

Lemma 3.4.1. *For positive integers k and n with $(k, n) = 1$, $\mathbb{Q}\left(\sin\left(\frac{\pi k}{n}\right)\right)$ is the maximal real subfield of $\mathbb{Q}(\zeta_{g(n)})$.*

Proof. It is well known that for positive integers ℓ and m with $(\ell, m) = 1$, $\cos\left(\frac{2\pi\ell}{m}\right)$ generates the maximal real subfield of $\mathbb{Q}(\zeta_m)$. Notice that

$$\sin\left(\frac{\pi k}{n}\right) = \cos\left(\frac{\pi}{2} - \frac{\pi k}{n}\right) = \cos\left(\frac{2\pi(n-2k)}{4n}\right).$$

Since $(k, n) = 1$, the only prime that might divide both $n - 2k$ and $4n$ is 2. When n is odd, we see that the numerator is odd, so $(n - 2k, 4n) = 1$ and the claim holds. Similarly, when $4 \mid n$, k is odd and so $\frac{n-2k}{4n} = \frac{\frac{n}{2}-k}{2n}$, where $\left(\frac{n}{2} - k, 2n\right) = 1$ since the numerator is odd. Finally, suppose that $n \equiv 2 \bmod 4$, which in particular implies that $\mathbb{Q}(\zeta_n) = \mathbb{Q}(\zeta_{n/2})$. We see that $4 \mid n - 2k$ and that 8 is the largest power of 2 that divides $4n$. Therefore, when $\frac{n-2k}{4n}$ is put into lowest terms the denominator is either n or $\frac{n}{2}$ and the result follows. □

Now let $\alpha = \frac{k_1}{n_1}$ and $\beta = \frac{k_2}{n_2}$ be rational numbers expressed in lowest terms where $0 < \alpha < \beta \leq \frac{1}{2}$ and so that $\frac{\sin(\pi\alpha)}{\sin(\pi\beta)}$ is rational. If $n_2 = 2$, then it is well-known that $\alpha = \frac{1}{6}$ and $\beta = \frac{1}{2}$, see Niven

[Niv56, Cor. 3.12]. So suppose in order to deduce a contradiction that n_2 and hence also n_1 are greater than 2. Let N be the least common multiple of n_1 and n_2.

Lemma 3.4.2. *If n_1 and n_2 are as in the preceding paragraph, then $n_1 = n_2$*

Proof. Suppose in order to deduce a contradiction that $n_1 \neq n_2$. Since g is an injection onto the even integers, $\mathbb{Q}(\zeta_{g(n_1)}) \neq \mathbb{Q}(\zeta_{g(n_2)})$. The compositum of these two fields is $\mathbb{Q}(\zeta_M)$ where M is the least common multiple of $g(n_1)$ and $g(n_2)$. Since, by Lemma 3.4.1, $\mathbb{Q}\left(\sin\left(\frac{\pi k_i}{n_i}\right)\right)$ is the maximal real subfield of $\mathbb{Q}(\zeta_{g(n_i)})$ for $i \in \{1, 2\}$ and since $\frac{\sin(\pi\alpha)}{\sin(\pi\beta)}$ is rational, we see that the maximal real subfields of $\mathbb{Q}(\zeta_{g(n_1)})$ and $\mathbb{Q}(\zeta_{g(n_2)})$ coincide. In general, if E and F are subfields of $\mathbb{C}$ so that, letting $K := E \cap F$, we have that E/K and F/K are Galois, then the compositum EF/K is Galois and $\mathrm{Gal}(EF/K) \cong \mathrm{Gal}(E/K) \times \mathrm{Gal}(F/K)$. In our case, since we assumed that $n_i > 2$ for $i \in \{1, 2\}$, $\mathbb{Q}(\zeta_{g(n_i)})$ is a degree two extension of its maximal real subfield K. This shows that $\mathbb{Q}(\zeta_M)$ is a degree four extension of K and so $\phi(M) = 2\phi\left(g(n_1)\right) = 2\phi\left(g(n_2)\right)$ where ϕ denotes the Euler phi function. Using that g is injective and that $n_1 \neq n_2$, this implies that $M = 12k$ where $3 \nmid k$ for a positive integer k and where, up to exchanging n_1 with n_2, $g(n_1) = 6k$ and $g(n_2) = 4k$. Since $\mathrm{Gal}\left(\mathbb{Q}(\zeta_M)/\mathbb{Q}\right)$ is isomorphic to $(\mathbb{Z}/M\mathbb{Z})^\times$, we see that under the Galois correspondence $\mathbb{Q}(\zeta_{g(n_1)}) = \mathbb{Q}(\zeta_M^2)$ corresponds to the subgroup generated by $6k+1$ and the maximal real subfield of $\mathbb{Q}(\zeta_M)$ corresponds to the subgroup generated by -1. Again by the Galois correspondence, $\mathbb{Q}(\zeta_{g(n_2)}) = \mathbb{Q}(\zeta_M^3)$ must correspond to the subgroup generated by $-(6k+1)$. This implies that $-3(6k+1) \equiv 3 \bmod 12k$, equivalently $6k \equiv 6 \bmod 12k$, which implies that $k = 1$. However, in this case $g(n_2) = 4$ which implies that $n_2 = 1$, a contradiction to the assumption that $n_2 > 2$. □

Following McMullen [McM06, Proof of Theorem 2.1], there is a constant C_1 such that $\frac{N}{n_1} \leq C_1$. By Lemma 3.4.2 we may set $C_1 = 1$. By [McM06, Proof of Theorem 2.1], $\frac{1}{2} \leq \frac{5\log(2N)}{N}$, which implies that $N < 45$. By Lemma 3.4.2 we have the following

$$\frac{\sin(\pi\alpha)}{\sin(\pi\beta)} = \frac{\zeta_{2N}^{k_2} - \zeta_{2N}^{-k_2}}{\zeta_{2N}^{k_1} - \zeta_{2N}^{-k_1}} = q$$

for some positive rational number $q \in \mathbb{Q}$. Since $k_2 > k_1$, this implies that ζ_{2N} is a root of the polynomial

$$F(x) := x^{2k_2} - qx^{k_1+k_2} + qx^{k_2-k_1} - 1.$$

The minimal polynomial for ζ_{2N}, which is the $(2N)$th cyclotomic polynomial, has degree $\phi(2N)$ and divides F.

Lemma 3.4.3. *F is the $(2N)$th cyclotomic polynomial.*

Proof. If not, then because $\phi(2N) \mid 2k_2$, it would follow that $2\phi(2N) \leq 2k_2$. Since $\frac{k_2}{N} = \beta < \frac{1}{2}$ we have, $\frac{\phi(2N)}{2N} < \frac{1}{4}$. Let Π be the set of primes that divide $2N$, then we have

$$\frac{\phi(2N)}{2N} = \prod_{p \in \Pi} \left(\frac{p-1}{p} \right) < \frac{1}{4}$$

For this inequality to hold, N would need to have at least three prime factors aside from 2. The smallest such number is 105, which is greater than 45. □

Since k_2 is coprime to N and since $2k_2 = \phi(2N)$, we have that $(N, \phi(N)) = 1$. This implies that N is squarefree and since $N > 2$, that N is odd. Since $N < 45$, N is prime or $N \in \{15, 21, 33, 35, 39\}$. We can discard the cases of $N \in \{21, 39\}$ since $(N, \phi(N)) \neq 1$. When N is prime, the $(2N)$th cyclotomic polynomial is $\sum_{k=0}^{N-1} (-1)^k x^k$, which is never the same as F. The $(2N)$th cyclotomic polynomials for $N \in \{15, 33, 35\}$ all have more than four nonzero coefficients of monomial terms, so again they cannot be F and we have a contradiction.

CHAPTER 4

Classifying Quasidiagonals of Full Rank Invariant Subvarieties

4.1 Introduction

For $\mathcal{H}$ a component of a stratum of the moduli space of translation surfaces, there is an $\mathrm{GL}^+(2,\mathbb{R})$-action on $\mathcal{H}$. The breakthrough work of Eskin, Mirzakhani, and Mohammadi in [EM18] and [EMM15] showed that $\mathrm{GL}^+(2,\mathbb{R})$ orbit closures of translation surfaces are immersed submanifolds that are cut out by linear equations in period coordinates, and furthermore Filip [Fil16] showed that they are also subvarieties. For a multi-component stratum $\mathcal{H}_1 \times \cdots \times \mathcal{H}_n$, we have a diagonal $\mathrm{GL}^+(2,\mathbb{R})$ action.

Definition 4.1.1. Let $\mathcal{H} = \mathcal{H}_1 \times \cdots \times \mathcal{H}_n$ be a product of connected components of strata of translation surfaces. An **invariant subvariety** of multi-component surfaces is a closed $\mathrm{GL}^+(2,\mathbb{R})$-invariant irreducible variety $\mathcal{L} \subset \mathcal{H}$ that is cut out by linear equations with real coefficients in period coordinate charts. Invariant subvarieties should be assumed to be single-component unless otherwise specified. The term invariant subvariety includes whole strata.

Definition 4.1.2. A invariant subvariety of multi-component surfaces $\mathcal{M}$ is called a **prime invariant subvariety** if it cannot be written as a product $\mathcal{M}_1 \times \mathcal{M}_2$ of two other multi-component invariant subvarieties.

One source of these multi-component invariant subvarieties comes from the WYSIWYG boundary of Mirzakhani and Wright [MW17]. Starting with a stratum of single-component translation surfaces, going to the boundary may produce multi-component surfaces. Chen and Wright proved in [CW21, Theorem 1.2] that the boundary of invariant subvarieties are multi-component invariant subvarieties.

Lemma 4.1.3. *Let $\mathcal{M}$ be an invariant subvariety. Then the **diagonal** in $\mathcal{M} \times \mathcal{M}$ defined as*

$$\mathcal{D} := \{(M, M) \in \mathcal{M} \times \mathcal{M} : M \in \mathcal{M}\}$$

*is an invariant subvariety. The **antidiagonal** defined as*

$$\overline{\mathcal{D}} := \{(M, -\operatorname{Id}(M)) \in \mathcal{M} \times \mathcal{M} : M \in \mathcal{M}\}$$

is also an invariant subvariety.

Proof. $\mathcal{D}$ is $GL^+(2, \mathbb{R})$-invariant since the action is the diagonal action. Let $a_1, \ldots, a_n$ be periods of the first surface while $b_1, \ldots, b_n$ be the periods of the same saddles on the second surface. Then, $a_i = b_i$ are equations that cut out $\mathcal{D}$. The proof for $\overline{\mathcal{D}}$ is similar. □

Remark 4.1.4. In hyperelliptic strata, the diagonal and antidiagonal are the same.

We generalize the above examples in the following definition.

Definition 4.1.5. Let $\mathcal{M}_1, \mathcal{M}_2$ be invariant subvarieties. A prime invariant subvariety Δ is a **quasidiagonal** in $\mathcal{M}_1 \times \mathcal{M}_2$ if the projection maps $\operatorname{proj}_i : \Delta \to \mathcal{M}_i$ are dominant. As a shorthand, we will write "$\Delta \subset \mathcal{M}_1 \times \mathcal{M}_2$ is a quasidiagonal". By Proposition 2.1.6 and Remark 2.1.7 below, every quasidiagonal has a corresponding quasidiagonal where both sides have equal area. Thus, we will assume throughout the paper that both components have equal area unless otherwise stated.

Remark 4.1.6. Any prime invariant subvariety Δ is a quasidiagonal in $\overline{\operatorname{proj}_1(\Delta)} \times \overline{\operatorname{proj}_2(\Delta)}$.

Definition 4.1.7. Let (X, ω) is a **hyperelliptic translation surface** if there exists a hyperelliptic involution $j : X \to X$ such that $j^*(\omega) = -\omega$. Let $\mathcal{H}$ be a component of a stratum of translation surfaces. The **hyperelliptic locus** in $\mathcal{H}$ is an invariant subvariety that consists of all hyperelliptic translation surfaces of $\mathcal{H}$.

The following is the main theorem of the paper and proves a conjecture by Apisa and Wright [AW21c, Conjecture 8.35] in the case of Abelian differentials.

Theorem 4.1.8. *Let $\mathcal{M}_i$ be either a connected component of a stratum of translation surface in genus at least 2 (without marked points) or a hyperelliptic locus in such a stratum for $i = 1, 2$. There exists (equal area) quasidiagonals $\Delta \subset \mathcal{M}_1 \times \mathcal{M}_2$ only if $\mathcal{M}_1 = \mathcal{M}_2$. In this case, Δ must be the the diagonal or antidiagonal.*

Example 4.1.9. We do not allow marked points because this will give rise to many uninteresting examples of quasidiagonals. For example, let $\Delta \subset \mathcal{M}_1 \times \mathcal{M}_2$ be a quasidiagonal. Then $\{((M_1, p), M_2) : (M_1, M_2) \in \Delta, p \in M_1\}$ is a quasidiagonal in $\mathcal{M}_1^* \times \mathcal{M}_2$. (Here $\mathcal{M}_1^*$ denotes the invariant subvariety which is $\mathcal{M}_1$ along with a free marked point.)

Classifying quasidiagonals is helpful with inductive arguments that use the WYSIWYG boundary. In addition, this classification is interesting since quasidiagonals show relationships between $\mathcal{M}_1$ and $\mathcal{M}_2$.

Definition 4.1.10. A continuous, $SL(2,\mathbb{R})$-invariant map between invariant subvarieties $\phi : \mathcal{M} \to \mathcal{N}$ is called a **morphism** if it is linear in period coordinates.

A morphism $\phi : \mathcal{M} \to \mathcal{N}$ between invariant subvarieties gives a quasidiagonal $\{(M, \phi(M)) : M \in \mathcal{M}\} \subset \mathcal{M} \times \overline{\phi(\mathcal{M})}$. For example, when $\mathcal{H}$ is not hyperelliptic, $-\,\mathrm{Id}$ gives rise to a nontrivial automorphism of $\mathcal{H}$. This corresponds to the antidiagonal.

Corollary 4.1.11. *Let $\mathcal{H}, \mathcal{H}'$ be strata. There are no dominant morphisms $\phi : \mathcal{H} \to \mathcal{H}'$ other than* $\mathrm{Id}, -\mathrm{Id} : \mathcal{H} \to \mathcal{H}$.

The above support the following heuristic: a quasidiagonal $\Delta \subset \mathcal{M}_1 \times \mathcal{M}_2$ exists if and only if $\mathcal{M}_1$ and $\mathcal{M}_2$ are "related".

Example 4.1.12. Let $\widetilde{\mathcal{H}}(2, 0^2) \subset \mathcal{H}(2^2, 1^2)$ be the space of all double covers of surfaces in $\mathcal{H}(2)$ branched at two marked points. There is a quasidiagonal $\Delta \subset \mathcal{H}(2) \times \widetilde{\mathcal{H}}(2, 0^2)$ consisting of all $(M, \widetilde{M})$ where $\widetilde{M}$ is a branched double cover of M.

Lemma 4.1.13. *Given quasidiagonals $\Delta_L \subset \mathcal{M}_1 \times \mathcal{M}_2$ and $\Delta_R \subset \mathcal{M}_2 \times \mathcal{M}_3$, then $\Delta_L * \Delta_R \subset \mathcal{M}_1 \times \mathcal{M}_3$ defined as the closure of $\{(M_1, M_3) \in \mathcal{M}_1 \times \mathcal{M}_3 : \exists M_2$ such that $(M_1, M_2) \in \Delta_L, (M_2, M_3) \in \Delta_R\}$ is a quasidiagonal.*

Proof. By Theorem 2.1.5, for $(M_1, M_2) \in \Delta_L$ the absolute periods of M_1 locally determine the absolute periods of M_2, and similarly for $(M_2, M_3) \in \Delta_R$, the absolute periods of M_2 locally determine the absolute periods of M_3. Thus, the absolute periods of each side of $\Delta_L * \Delta_R$ locally determines the absolute periods of the other. Thus, $\Delta_L * \Delta_R$ cannot be the product of invariant subvarieties of multi-component surfaces, so it must be prime. $\mathrm{proj}_2(\Delta_L) \cap \mathrm{proj}_1(\Delta_R)$ is a Zariski open subset of $\mathcal{M}_2$. $\mathrm{proj}_1 : \Delta_L * \Delta_R \to \mathcal{M}_1$ is dominant since $\mathrm{proj}_1(\Delta_L * \Delta_R) \supset \mathrm{proj}_1(\mathrm{proj}_2^{-1}(\mathrm{proj}_2(\Delta_L) \cap \mathrm{proj}_1(\Delta_R)))$. □

Corollary 4.1.14. *Let $\mathcal{M}_1, \mathcal{M}_2$ be invariant subvarieties. We define $\mathcal{M}_1 \sim \mathcal{M}_2$ if there exists a quasidiagonal $\Delta \in \mathcal{M}_1 \times \mathcal{M}_2$. Then, $\sim$ is an equivalent relation.*

Theorem 4.1.8 implies that there are many distinct $\sim$ equivalence classes. It would be interesting to classify these equivalence classes. The only ways the author is aware of to make related invariant subvarieties is through adding marked points as in Example 4.1.9 or branched covering constructions as in Example 4.1.12.

Question 4.1.15. Let $\mathcal{R}$ be an equivalence class of related invariant subvarieties with marked points in rank ≥ 2. Does $\mathcal{R}$ contain a minimal element through which all other related invariant subvarieties can be produced through adding marked points and branched covering constructions?

Another application of this work is to measurable joinings of Masur-Veech measures.

Definition 4.1.16. Let (X_1, μ_1, T_1) and (X_2, μ_2, T_2), where μ_i is a measure on a space X_i and $T_i : X_i \to X_i$ is a measure preserving transformation. A **joining** is a measure on $X_1 \times X_2$ invariant under the product transformation $T_1 \times T_2$, whose marginals on X_i are μ_i. A measure μ on a space X is **prime** if it cannot be written as a product $\mu = \mu_1 \times \mu_2$, $X = X_1 \times X_2$, where μ_i is a measure on X_i.

Assuming a multi-component version of Eskin-Mirzakhani's measure classification result, Theorem 4.1.8 classifies ergodic measurable joinings of Masur-Veech measures on strata.

Assumption 4.1.17. (See [MW17, Conjecture 2.10]) We define an affine measure as in [EM18, Definition 1.1]. For a multi-component stratum under the diagonal action of $\mathrm{SL}(2, \mathbb{R})$, the only ergodic invariant measures are $\mathrm{SL}(2, \mathbb{R})$-invariant and affine.

Corollary 4.1.18. *Let $\mathcal{M}_i$ be either a stratum of translation surfaces or the hyperelliptic locus in such a stratum and μ_i be the Masur-Veech measure on the unit area locus of $\mathcal{M}_i$, for $i = 1, 2$. Under Assumption 4.1.17, the only prime ergodic joinings of μ_1, μ_2 (under the diagonal action of* $\mathrm{SL}(2, \mathbb{R})$*) are the Masur-Veech measure on the diagonal or antidiagonal.*

Proof. By Assumption 4.1.17, the only ergodic $\mathrm{SL}(2, \mathbb{R})$-invariant measures are affine, so they are supported on an invariant subvariety $\mathcal{M}$. Since the joining is prime, $\mathcal{M}$ must be prime. The condition on the marginals implies that $\mathcal{M}$ must be a quasidiagonal in $\mathcal{M}_1 \times \mathcal{M}_2$. Our assumptions also guarantee equal area on both sides. We may now apply Theorem 4.1.8. □

Organization: Section 2 is the bulk of the paper and contains a summary of the techniques used in the proof of the main theorem. The proof will use induction. Section 3, which is slightly technical, is the base case of the proof. The heart of the proof is in Section 4. Section 5 is an appendix.

4.2 Background

While some of the background can be found in Chapter 2, we leave some background more specific to this chapter here.

4.2.1 Cylinder Collapse

We has different sections on cylinder collapses here and in Chapter 5 because although similar, we use slightly different notation and types of cylinder collapse. Let $M \in \mathcal{M}$ and C be an $\mathcal{M}$-parallel class on M. Let σ_C be the standard twist. Fixing a period coordinate chart, we can travel in the direction of $e^{i\theta}\sigma_C$ as long as we remain in the coordinate chart. Traveling in this way we may reach the boundary of $\mathcal{M}$ as two singularities come together.

We will define cylinders collapse and diamonds in a similar fashion to [AW21b, Section 4.2]. Let $\mathcal{M}$ be an invariant subvariety of multi-component surfaces with marked points. Let $M \in \mathcal{M}$ and $\mathcal{C}$ an $\mathcal{M}$-parallel class of horizontal cylinders on M.

$$\mathrm{Col}_{\mathcal{C}} M := \lim_{t \to 1^-} M - ti\sigma_{\mathcal{C}}.$$

We call $\mathrm{Col}_{\mathcal{C}} M$ the **collapse of** M **along** $\mathcal{C}$ or simply the collapse of $\mathcal{C}$. If there is no vertical saddle connection, $\mathrm{Col}_{\mathcal{C}} M$ will remain in $\mathcal{M}$ but all cylinders of $\mathcal{C}$ will have zero height. Otherwise, $\mathrm{Col}_{\mathcal{C}} M$ will be on ∂M by [MW17, Lemma 3.1] or [AW21b, Lemma 4.9].

In this chapter, we want cylinder collapses to always degenerate to ∂M. Thus, we must ensure that a cross curve will always degenerate. Let γ be a cross curve of some cylinder $C \in \mathcal{C}$ with height h. Let $z \in \mathbb{C}$ such that z points in the direction of γ with positive imaginary part and $|z|h = |\gamma|$. Then

$$\mathrm{Col}_{\mathcal{C},\gamma} M := \lim_{t \to 1^-} M - z\sigma_{\mathcal{C}}.$$

This operation has been defined so that γ degenerates, so it will always be in ∂M unlike the previous definitions. If M is a multicomponent surface, this means that the component that contains γ degenerates, while the other components of M may or may not degenerate. We also define

$$\mathrm{Col}_{\mathcal{C},\gamma} \mathcal{M}$$

to be the component of $\partial\mathcal{M}$ that contains $\mathrm{Col}_{\mathcal{C},\gamma} M$. We define $\mathrm{Col}_{\mathcal{C},\gamma} \mathcal{C}$ to be the $\mathcal{M}$-parallel collection of saddles on $\mathrm{Col}_{\mathcal{C},\gamma} M$ from the collapsed cylinders in $\mathcal{C}$.

Let $\mathcal{C}_1, \mathcal{C}_2$ be disjoint equivalence classes of cylinders on M, and let γ_1, γ_2 be cross curves of cylinders of $\mathcal{C}_1, \mathcal{C}_2$ respectively. As an abuse of notation, for $j \neq i$, we also use the notation $\mathcal{C}_j, \gamma_j$ to refer to the equivalence class of cylinders and saddle connection on $\mathrm{Col}_{\mathcal{C}_i,\gamma_i} M$ that correspond to $\mathcal{C}_j, \gamma_j$ respectively. We define

$$\mathrm{Col}_{\mathcal{C}_1,\mathcal{C}_2,\gamma_1,\gamma_2} M = \mathrm{Col}_{\mathcal{C}_1,\gamma_1} \mathrm{Col}_{\mathcal{C}_2,\gamma_2} M = \mathrm{Col}_{\mathcal{C}_2,\gamma_2} \mathrm{Col}_{\mathcal{C}_1,\gamma_1} M.$$

Definition 4.2.1. Let $\mathcal{M}$ be an invariety subvariety of multi-component surfaces. A surface $M \in \mathcal{M}$ is called $\mathcal{M}$**-generic** if two saddles on the same component of M are parallel only if they are $\mathcal{M}$-parallel. If $\mathcal{M}$ is clear from context, we will just call M generic.

Lemma 4.2.2. *For any multi-component invariant subvarity, a dense G_δ set of surfaces are generic.*

Proof. The condition that two saddles that are not generically parallel are parallel defines a linear subspace in period coordinates. There are countably many saddles on a surface, and the coefficients of the equation must be in a the field of definition of the component that the two saddles are on. The field of definition is a finite extension of $\mathbb{Q}$ by [Wri14, Theorem 1.1]. $\square$

Lemma 4.2.3. *Let $\mathcal{M}$ be a full rank invariant subvariety, and let $M \in \mathcal{M}$ be generic. Then, every cylinder C on $\mathcal{M}$ is simple.*

Proof. By Theorem 2.4.5, $\mathcal{M}$ is a stratum or hyperelliptic locus. Assume by contradiction there was a cylinder C that was not simple. Then, there would be a boundary component with more than one saddle. These saddles would be $\mathcal{M}$-parallel because M is generic. That is not possible in a stratum, so $\mathcal{M}$ must be a hyperelliptic locus. Let γ_1, γ_2 be parallel saddle on the boundary of C. Every cylinder on M is either fixed or swapped with another cylinder. In either case, the quotient surface N has a corresponding cylinder, which we still call C, with two saddles, which we still call γ_1 and γ_2. Now we appeal to [MZ08], and we will use the formulation and terminology of [AW21a, Proposition 4.4]. By this theorem, removing γ_1, γ_2 from N disconnects the surface into two components $\mathcal{A}, \mathcal{B}$, and the component $\mathcal{B}$ not containing C has trivial holonomy. Since $\mathcal{M}$ is hyperelliptic, N is genus 0, so $\mathcal{B}$ is topologically a cylinder. However, gluing this cylinder back along γ_1, γ_2 would create genus. This is a contradiction since N is genus 0. Thus, the cylinder $C \subset M$ must have been simple. $\square$

Lemma 4.2.4. *Let $\Delta \subset \mathcal{M}_1 \times \mathcal{M}_2$ be a quasidiagonal, $M \in \Delta$ a generic surface, $\mathcal{C} \subset M$ a Δ-parallel class of cylinders on M, and γ a cross curve of a cylinder in $\mathcal{C}$. Let $\Delta := \mathrm{Col}_{\mathcal{C},\gamma}\Delta$. Then* $\dim \Delta' = \dim \Delta - 1$ *and* $\dim \mathcal{M}_i - 1 \le \overline{\mathrm{proj}_i(\Delta')} \le \dim \mathcal{M}_i$ *for $i = 1, 2$.*

Proof. Let $M' := \mathrm{Col}_{\mathcal{C},\gamma} M$. Since M is generic, the space of vanishing cycles is one-dimensional, so by Proposition 2.4.3, $\dim \Delta' = \dim \Delta - 1$, and $T_{M'}\Delta'$ can be viewed as a one-dimensional subspace of $T_M\Delta$. Then, $\dim(\mathrm{proj}_i T_M\Delta) - 1 \le \dim(\mathrm{proj}_i T_{M'}\Delta') \le \dim(\mathrm{proj}_i T_M\Delta)$. The result follows. $\square$

Lemma 4.2.5. *Let $\mathcal{M}$ be an invariant subvariety and $\mathcal{N} \subset \mathcal{M}$ be a codimension* 1 *subvariety. Then $\mathcal{M}, \mathcal{N}$ must have the same rank.*

Proof. $p(T\mathcal{N}) \subset p(T\mathcal{M})$ is codimension at most 1 and the symplectic form on $p(T\mathcal{M})$ restricts to a symplectic form on $p(T\mathcal{N})$, so in fact $p(T\mathcal{M}) \cong p(T\mathcal{N})$. $\square$

Lemma 4.2.6. *Let $\Delta \subset \mathcal{M}_1 \times \mathcal{M}_2$ be a quasidiagonal, where $\mathcal{M}_i$ is a full rank invariant subvariety, $M = (M_1, M_2) \in \Delta$ is generic, $\mathcal{C}$ a Δ-equivalence class of cylinders on M, and γ a cross curve of a cylinder $C \in \mathcal{C}$ on M_1. If* $\mathrm{Col}_{C,\gamma}\mathcal{M}_1$ *is lower rank than $\mathcal{M}_1$, then there must be a $\gamma' \subset M_2$ generically parallel to γ that is a cross curve of a cylinder in $\mathcal{C}$.*

Proof. Assume by contradiction no saddle of $\mathcal{M}_2$ collapses in $\mathrm{Col}_{\mathcal{C},\gamma}\Delta$, so $\mathcal{M}_2' := \overline{\mathrm{proj}_2(\mathrm{Col}_{\mathcal{C},\gamma}\Delta)} \subset \mathcal{M}_2$. However, $\mathcal{M}_2'$ is dimension at most one less than $\mathcal{M}_2$ by Lemma 4.2.4 and has lower rank than $\mathcal{M}_2$ by Theorem 2.1.5. This contradicts Lemma 4.2.5. Thus, some $\gamma' \subset M_2$

collapses in $\mathrm{Col}_{C,\gamma}\Delta$. A priori γ' may cross multiple adjacent cylinders. Since we assumed M is generic, by Lemma 4.2.3 all cylinders must be simple. Adjacent simples cylinders meet at marked points, but we assumed that there are no marked points, so there are no adjacent cylinders. $\square$

Lemma 4.2.7. *Let $\Delta \subset \mathcal{M}_1\times\mathcal{M}_2$ be a quasidiagonal, where $\mathcal{M}_i$ is a full rank invariant subavariety, $M = (M_1, M_2) \in \Delta$ is generic, $\mathcal{C}$ a Δ-equivalence class of cylinders on M, and γ a cross curve a cylinder $C \in \mathcal{C}$. Let $\mathcal{M}'_i := \overline{\mathrm{proj}_i(\mathrm{Col}_{\mathcal{C},\gamma}\Delta)}$. Then, $\mathrm{Col}_{\mathcal{C},\gamma}\Delta \subset \mathcal{M}'_1 \times \mathcal{M}'_2$ is a quasidiagonal, and $\mathcal{M}'_i$ is a full rank invariant subvariety with marked points.*

Proof. By [AW21b, Lemma 9.1], $\Delta' := \mathrm{Col}_{\mathcal{C},\gamma}\Delta$ is a prime invariant subvariety, so it is a quasidiagonal in $\mathcal{M}'_1 \times \mathcal{M}'_2$. It remains to show that $\mathcal{M}'_i$ is full rank. Let $M = (M_1, M_2)$ and $\mathcal{C}_i$ be the cylinders of $\mathcal{C}$ on M_i. Let $M' = (M'_1, M'_2) = \mathrm{Col}_{\mathcal{C},\gamma} M$. Without loss of generality let $\gamma \subset M_1$. By Corollary 2.4.7, $\mathcal{M}'_1 = \mathrm{Col}_{\mathcal{C}_1,\gamma}\mathcal{M}_1$ is full rank invariant subavariety with marked points. By Lemma 4.2.4, $\mathcal{M}'_2$ is codimension at most 1. If $\mathcal{M}'_2 \subset \mathcal{M}_2$, $\mathcal{M}'_2$ is full rank by Lemma 4.2.5. Otherwise, some saddle connection $\gamma' \subset M_2$ collapses, so $\mathcal{M}'_2 = \mathrm{Col}_{\mathcal{C}_2,\gamma'}\mathcal{M}_2$, so by Corollary 2.4.7, $\mathcal{M}'_2$ is full rank. $\square$

Definition 4.2.8. Let $\mathcal{M}$ be an invariant subvariety in a potentially multi-component stratum. Let $M \in \mathcal{M}$ be generic and $\mathcal{C}_1, \mathcal{C}_2$ be two distinct $\mathcal{M}$-parallel classes of cylinders. Let γ_i be a cross curve on a cylinder in $\mathcal{C}_i$. Assume furthermore that the components of $\mathrm{Col}_{\mathcal{C}_1,\gamma_1,\mathcal{C}_2,\gamma_2} M$ have no translation surface automorphisms other than the identity. Then, $(\mathcal{M}, M, \mathcal{C}_1, \mathcal{C}_2, \gamma_1, \gamma_2)$ is called a **good diamond**. Note that this definition is a special case of a skew diamond defined in [AW22, Section 5].

The following is similar to [AW21c, Lemma 3.31].

Lemma 4.2.9. *Let $\mathcal{M}$ be a prime invariant subvariety of rank at least 2. Then, there is a G_δ set $U \subset \mathcal{M}$ such that for $M \in U$ there is a good diamond that contains M. Furthermore, if $\mathcal{M}$ is rank at least 3, there is a G_δ set of M such M contains three cylinders C_1, C_2, C_3 such that there is a good diamond containing any two C_i.*

Proof. By Lemma 4.2.2, a G_δ set U' of surfaces in $\mathcal{M}$ are generic. Now we show that we can find k cylinder classes on any surface M in a dense open set V. By definition, $\mathcal{M}_i = \overline{\mathrm{proj}_i(\mathcal{M})}$ has rank k. By [Wri15a, Theorem 1.10], a dense subset of the set of horizontally periodic surfaces in $\mathcal{M}_i$ has at least k disjoint cylinder equivalence classes. There is an open subset around each of these points, where these cylinders persist and remain disjoint, so a dense set of surfaces in $\mathcal{M}_i$ has at least k disjoint cylinder equivalence classes. By Lemma 2.2.14, if $\mathrm{proj}_i M$ contains k distinct $\mathcal{M}_i$-parallel classes, then M contains k distinct Δ-parallel classes. Since proj_i is a submersion, there is a dense set of surfaces $\mathrm{proj}_i^{-1}(M) \subset \mathcal{M}$ that have k disjoint cylinder equivalence classes. Thus, the set U'' of generic surfaces in Δ that contain at least k distinct Δ-parallel classes is a G_δ set.

For each $M \in U''$ there is an open set V_M around M where the k cylinder classes persist. For any two of the cylinder classes C_1, C_2, the set of surfaces in $\mathrm{Col}_{C_1,\gamma_1,C_2,\gamma_2} \mathcal{M}$ that have nontrivial translation surface isomorphisms is a set of isolated points. Thus, there is a countable union of two dimensional spaces of surfaces in Δ where C_1, C_2 do not belong to a good diamond. Thus, after removing these sets for all possible pairs of Δ-parallel classes from V_M we are left with an open dense set of V_M that satisfy the lemma. Taking a union of all of these sets gives the G_δ set U that is the conclusion of the lemma. $\square$

Definition 4.2.10. Let $\mathcal{M}$ be an invariant subvariety with marked points. The **diagonal** $\mathcal{D} \subset \mathcal{M} \times \mathcal{M}$ is the set of surfaces (M, N) such that there is a translation surface isomorphism $M \to N$ that takes marked points to marked points. The **antidiagonal** $\overline{\mathcal{D}} \subset \mathcal{M} \times \mathcal{M}$ is the set of surfaces (M, N) such that the $N = -\mathrm{Id}(M)$ and $-\mathrm{Id}$ takes marked points to marked points.

Lemma 4.2.11. *Let $\Delta \subset \mathcal{M}_1 \times \mathcal{M}_2$ be an (equal area) quasidiagonal. If for any good diamond $(\Delta, M, C_1, C_2, \gamma_1, \gamma_2)$, both $\mathrm{Col}_{C_i,\gamma_i} \Delta$, $i = 1, 2$ are diagonals (resp. antidiagonals) up to rescaling (see Remark 2.1.7), then Δ is a diagonal (resp. antidiagonal).*

Proof. We prove the statement for diagonals as the statement for antidiagonals is similar. Let $(\Delta, M, C_1, C_2, \gamma_1, \gamma_2)$ be any good diamond, and let $M = (M_1, M_2)$. Assume both $\mathrm{Col}_{C_i,\gamma_i} \Delta$, $i = 1, 2$ are diagonals up to rescaling. Then, there are constants $r_i > 0$ and isomorphisms $f_i : \mathrm{Col}_{C_i,\gamma_i} M_1 \to r_i \, \mathrm{Col}_{C_i,\gamma_i} M_2$ that restrict to isomorphisms

$$\begin{aligned}
\mathrm{Col}_{C_2,\gamma_2} f_1 &: \mathrm{Col}_{C_1,\gamma_1,C_2,\gamma_2} M_1 \to r_1 \, \mathrm{Col}_{C_1,\gamma_1,C_2,\gamma_2} M_2 \\
\mathrm{Col}_{C_1,\gamma_1} f_2 &: \mathrm{Col}_{C_1,\gamma_1,C_2,\gamma_2} M_1 \to r_2 \, \mathrm{Col}_{C_1,\gamma_1,C_2,\gamma_2} M_2
\end{aligned}$$

And these must be the same isomorphism since we chose a good diamond. We also get that $r_1 = r_2$ because isomorphic translation surfaces must have the same area. We can now define an isomorphism $f : M_1 \to r_1 M_2$ as follows. $M_i - \overline{C_j}$ can be identified with $\mathrm{Col}_{C_j,\gamma_j} M_i - \mathrm{Col}_{C_j,\gamma_j} C_j$ for $i, j = 1, 2$. Thus, f can be defined to be f_i on $M_1 - \overline{C_i}$. f is defined twice on $M_1 - \overline{C_1} - \overline{C_2}$, but these definitions agree. This defines f as an isomorphism of punctured translation surfaces $M_1 - \overline{C_1} \cap \overline{C_2} \to r_1 M_2 - r_1 \overline{C_1} \cap \overline{C_2}$. $\overline{C_1} \cap \overline{C_2}$ consists of a finite set of points, so we can extend f to an isomorphism from M_1 to M_2.

We note that $r_1 = 1$ because we assumed M_1, M_2 have the same area. By Lemma 4.2.9 for a dense set of M, there exists a good diamond that contains M, and for these M both components M_i are isomorphic, so Δ is a diagonal. $\square$

4.2.2 Marked Points

See [AW21a] for a more in depth discussion of marked points.

Definition 4.2.12. Let $\mathcal{H}$ be a stratum and $\mathcal{H}^{*n}$ be the set of surfaces in $\mathcal{H}$ with n distinct marked points. Let $\mathcal{F} : \mathcal{H}^{*n} \to \mathcal{H}$ be the map that forgets marked points. Let $\mathcal{M}$ be an invariant subvariety of $\mathcal{H}$. An n**-point marking** is an invariant subvariety of $\mathcal{H}^{*n}$ that maps to a dense subset of $\mathcal{M}$ under $\mathcal{F}$. We use the term point marking to refer to an n-point marking for any value of n. A point marking is **reducible** if it is a fiberwise union of two other point markings, otherwise it is irreducible. A marked point p on $M \in \mathcal{M}$ is $\mathcal{M}$**-free** if there are no relations in $\mathcal{M}$ between p and any of the other marked points on M. When $\mathcal{M}$ is understood, we call p a free marked point. Similarly when $\mathcal{M}$ is understood, we say a set of marked points on M is irreducible if $\mathcal{M}$ is irreducible.

Lemma 4.2.13. *Let $\mathcal{M}$ be an invariant subvariety in a genus g stratum. Let $M \in \mathcal{M}$ and $\mathbf{C}$ be an $\mathcal{M}$-parallel class that contains only a single simple cylinder C. Let γ be a cross curve of C. Let $M' = \mathrm{Col}_{C,\gamma} M$ and $\mathcal{M}' = \mathrm{Col}_{C,\gamma} \mathcal{M}$. If the endpoints of $\mathrm{Col}_{C,\gamma} C$ are distinct from each other, then the genus $g(\mathcal{M}) = g(\mathcal{M}') + 1$.*

Similarly, assume $\mathbf{C}$ contains two simple cylinders and assume the four endpoints of $\mathrm{Col}_{C,\gamma} \mathbf{C}$ contains at least three distinct points. Then $g(\mathcal{M}) = g(\mathcal{M}') + 2$.

Proof. Let $\mathcal{M}'$ be contained in a stratum with marked points $\mathcal{H} = \mathcal{H}(\kappa)$, where κ is a tuple containing the orders of the singularities of the surfaces in $\mathcal{H}$. Let $|\kappa|$ be the sum of the entries of κ and s be the number of elements in κ. Note that $2g - 2 = |\kappa|$ for any stratum $\mathcal{H}(\kappa)$. The sum of the angles around all singularities of M' is $2\pi(s + |\kappa|)$. After gluing in a cylinder, the sum of the angles around the singularities of M is $2\pi(s + |\kappa| + 1)$. But the two distinct endpoints of $\mathrm{Col}_{C,\gamma} C$ fuse into one, so M has $s - 1$ singularities, so the sum of the singularities is $|\kappa| + 2 = 2g(\mathcal{M}) - 2$. Thus, $g(\mathcal{M}) = g(\mathcal{M}') + 1$. To prove the second statement, we can glue in the cylinders one at a time. Each time we increase the genus by one. $\square$

Theorem 4.2.14. *Let $\mathcal{P}$ be a nonempty irreducible point marking on an invariant subvariety $\mathcal{M}$. If $\mathcal{M}$ is a full rank, then either $\mathcal{P} = \mathcal{M}^*$ or $\mathcal{M}$ is a hyperelliptic locus with hyperelliptic involution J and $\mathcal{P}$ is one of the following*

$$\{(M, p) : M \in \mathcal{M}, p \text{ is a Weierstrass point}\}$$
$$\{(M, p, J(p)) : M \in \mathcal{M}, p \in M\}$$

This theorem was proven for strata in [Api20, Theorem 1.5] and [AW21a, Theorem 1.4] in general.

Lemma 4.2.15. *Let $\mathcal{M}$ be an invariant subvariety with marked points. Let $M \in \mathcal{M}$, and let Γ be a set of $\mathcal{M}$-parallel saddle connections on M. Assume that no saddle connects a marked point to itself. Let $\gamma \in \Gamma$ be a saddle such that at least one endpoint of γ is a marked point but not a periodic* *point. Then, for each endpoint p of γ that is a marked point but not a periodic point, there exists an irreducible set $P \ni p$ of marked points such that for every $\gamma_i \in \Gamma$, P contains an endpoint of γ_i.*

Proof. Let P be a maximal irreducible set of marked points of M containing p. We may move the points of P without moving the rest of the surface. No saddle $\gamma' \in \Gamma$ can exist that does not have an endpoint in P otherwise we can make γ not parallel to γ', which is a contradiction. □

Lemma 4.2.16. *Let $\mathcal{M}$ be an invariant subvariety (without marked points), $M \in \mathcal{M}$, C an $\mathcal{M}$-parallel class of simple cylinders on M, and γ a cross curve of C. Assume there is at least one marked point on $M' := \mathrm{Col}_{C,\gamma} M$ and $\Gamma := \mathrm{Col}_{C,\gamma} C$.*

1. *If $\mathcal{M}$ is a stratum, each marked point of M' is free. Furthermore, Γ contains a single saddle connection.*

2. *If $\mathcal{M}$ is a hyperelliptic locus, and let J be the induced hyperelliptic involution on $\mathcal{M}'$. Then, M' contains no periodic points and Γ either consists of one saddle connection fixed by J or two saddle connections swapped by J.*

Proof. First assume $\mathcal{M}$ is a stratum. By Theorem 2.4.4, every marked point is free. Let $\Gamma := \mathrm{Col}_{C,\gamma} C$. Let $\gamma \in \Gamma$ be a saddle such that at least one endpoint of γ is a marked point. By Lemma 4.2.15 and since all marked points are free, γ cannot be $\mathrm{Col}_{C,\gamma} \mathcal{M}$-parallel to any other saddle connection on M'. Thus, C only consisted of a single cylinder, so there is only one saddle in Γ. Now assume $\mathcal{M}$ is a hyperelliptic locus.

Claim 4.2.17. None of the marked points on M' are Weierstrass points.

Proof. Assume by contradiction that there exists a saddle $\gamma_1 \in \Gamma$ such that one of the endpoints is a Weierstrass point p. Let $\gamma_2 = J'(\gamma_1)$. We note that $\gamma_2 \neq \gamma_1$ otherwise M has marked points, which contradicts our assumptions. Then, there are two cylinders $C_1, C_2 \in C$, which collapse to become γ_1, γ_2 respectively. Let q_1 be the other endpoint of γ, so $q_2 = J'(q_1)$ is the other endpoint of γ_2. First assume $q_1 \neq q_2$. Note that M is topologically M' plus two handles, so it has genus two more than M'. Thus, the hyperelliptic involution on M must have four more fixed points than the hyperelliptic involution on M'. By our assumptions, J swaps C_1 and C_2. The fixed point of J are the fixed points of J' minus p plus the singularity that comes out of the fusion of q_1, q_2. Thus, J same number of fixed points as J'. Thus, this is not possible. The remaining case is that $q_1 = q_2$. In this case, M has genus one more than M'. However, there are two fewer fixed points of J than there are fixed points of J', which is also a contradiction. Thus, M' cannot have Weierstrass points. □

By the Claim and Theorem 4.2.14, M' has no periodic points. C consists of at most two cylinders. If C is one cylinder, it must be fixed by the hyperelliptic involution. In this case, Γ is one saddle γ,

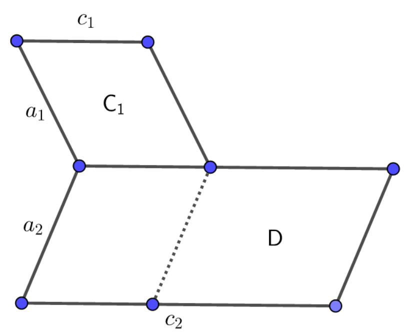

Figure 4.1: Horizontally periodic surface is $\mathcal{H}(2)$.

and γ is fixed by J. Now assume that C contains two cylinders. These cylinders must be swapped by the hyperelliptic involution, so Γ consists of two saddles swapped by J. This concludes the proof. $\square$

4.3 Genus 2

As a base case for the induction, we must prove Theorem 4.1.8 for quasidiagonals in $\mathcal{H}(2)\times\mathcal{H}(2)$.

Theorem 4.3.1. *The only (equal area) quasidiagonal* $\Delta \in \mathcal{H}(2)\times\mathcal{H}(2)$ *is the diagonal* $\{(M, M) : M \in \mathcal{H}(2)\}$.

Let $\Delta \subset \mathcal{H}(2)\times\mathcal{H}(2)$ be a quasidiagonal. Let $(M, M') \in \Delta$ be any generic surface. To prove the theorem, it suffices to show that M' is equal to M, which is Lemma 4.3.5 below.

Lemma 4.3.2. *Let* $\mathcal{H}_1, \mathcal{H}_2$ *be hyperelliptic components, and* $\Delta \subset \mathcal{H}_1, \times\mathcal{H}_2$ *a quasidiagonal. Let* C *be a cylinder equivalence class on a surface* $M \in \Delta$*. Then,* C *consists of one cylinder on each component.*

Proof. By Lemma 2.2.14, C consists of one $\mathcal{H}_1$-parallel class and one $\mathcal{H}_2$-parallel class. For any stratum $\mathcal{H}$, a $\mathcal{H}$ parallel class consists of an equivalence class of homologous cylinders, and on a hyperelliptic component no two cylinders can be homologous. $\square$

Choose two disjoint cylinders C_1, D on M. By Lemma 4.3.2, there are cylinders C_1', D' on M' such that $\mathcal{C}_1 = \{C_1, C_1'\}$ and $\mathcal{D} = \{D, D'\}$ are Δ-parallel classes of cylinders. By Corollary 2.2.13, C_1', D' must be disjoint. We rotate (M, M') to make C horizontal and perform a cylinder shear on $\mathcal{D}$ until M is horizontally periodic. By Lemma 2.2.4, M' is also horizontally periodic. Let C_2 be the other horizontal cylinder on M. We fix a cross curve γ_i of C_i, where γ_2 is a boundary curve of D. Let a_i be the period of γ_i. Let c_i be the period of the core curve of C_i. See Figure 4.1.

Because M is horizonally periodic, we have that $\operatorname{Im} c_i = 0$. Label the corresponding cylinders, saddle connections, and periods of M' with primes.

Since Δ is cut out by linear equations in period coordinates, we have that

$$T' \cdot \begin{pmatrix} a'_1 \\ a'_2 \\ c'_1 \\ c'_2 \end{pmatrix} = T \cdot \begin{pmatrix} a_1 \\ a_2 \\ c_1 \\ c_2 \end{pmatrix}$$

for some real matrices T, T'. By Theorem 2.1.5, the absolute periods determine each other, so the above matrices are invertible. Thus we can assume $T' = \mathrm{Id}$. By Lemma 4.2.6, we may choose a_1, a'_1 so that they are Δ-parallel. In addition, a_2, a'_2 are Δ-parallel since they are boundary curves of Δ-parallel cylinders. This means that a'_i doesn't depend on any period except a_i. Changing $\operatorname{Im} a_1, \operatorname{Im} a_2$ does not affect $\operatorname{Im} c'_1, \operatorname{Im} c'_2$ since the surface must remain horizontally periodic by Lemma 2.2.4. Thus, we can simplify the matrix

$$T = \begin{pmatrix} f_{11} & 0 & 0 & 0 \\ 0 & f_{22} & 0 & 0 \\ 0 & 0 & g_{11} & g_{12} \\ 0 & 0 & g_{21} & g_{22} \end{pmatrix}.$$

Lemma 4.3.3. *With the above notation,* $g_{12} = g_{21} = 0$.

Proof. We can dilate $\mathcal{D}$ while keeping the surface horizontally periodic. This changes the circumferences c_2 without changing c_1, so $g_{12} = 0$. The following combination of shears will change c_1 without changing c'_2 or a'_1. Dilate $\mathcal{D}$ while keeping the surface horizontally periodic. Now shrink the whole surface in the real direction (i.e. by a matrix of the form $\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$) so that c_2 is its original size. This changes c_1 without changing c_2, so $g_{21} = 0$. □

Lemma 4.3.4. *Let* (M, M') *be a generic surface in* Δ. *Let* C, C' *be* Δ*-parallel simple cylinders on* M, M' *respectively. Then, they have the same modulus.*

Proof. Let $\mathcal{C} = \{C, C'\}$. Shear $\mathcal{C}$ so that a cross curve γ of C is vertical. By Lemma 4.2.6, C' also has a vertical saddle connection γ'. Now continue to shear $\mathcal{C}$ until the first time M once again has a vertical saddle connection. We notice that both must have been sheared one full rotation. Thus, the moduli must be equal. □

Lemma 4.3.5. *Let* $\Delta \subset \mathcal{H}(2) \times \mathcal{H}(2)$ *be a quasidiagonal and* $(M, M') \in \Delta$ *be a generic surface. Then* M, M' *are isomorphic translation surfaces.*

Proof. We label the surfaces using the notation above. Because the combinatorics of the surfaces are the same, it suffices to show that $T = \mathrm{Id}$. We first show $g_{22} = f_{22}$ and $f_{11} = g_{11}$ is similar. We take a sequence of generic surfaces in Δ that converge to (M, M'). For surfaces close enough to (M, M') there are cylinders corresponding to C_i, C_i'. By Lemma 4.3.4, C_i, C_i' have the same modulus along this sequence, so they have the same moduli on (M, M'). Let h_i, h_i' be the heights of C_i, C_i' respectively. Then, $\frac{c_i}{h_i} = \frac{c_i'}{h_i'} = \frac{g_{ii}c_i}{f_{ii}h_i}$, so $f_{ii} = g_{ii}$.

Now we show $g_{11} = f_{22}$. We shear C_1, C_2 so that γ_1, γ_2 are vertical, so that M, M' are both vertically and horizontally periodic. By the same argument as above, D, D', must have the same modulus. Thus

$$\frac{c_2 - c_1}{a_2} = \frac{c_2' - c_1'}{a_2'} = \frac{g_{22}c_2 - g_{11}c_1}{f_{22}a_2},$$

so $f_{11} = f_{22} = g_{11} = g_{22}$. Now, we see that

$$\begin{pmatrix} a_1' \\ a_2' \\ c_1' \\ c_2' \end{pmatrix} = f_{11} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ c_1 \\ c_2 \end{pmatrix}.$$

Since we assumed both sides have the same area, we have that $f_{11} = 1$. Thus, $T = \mathrm{Id}$. This finishes the proof of Theorem 4.3.1. □

4.4 Proof of Main Theorem

By Theorem 2.4.5, Theorem 4.1.8 is equivalent to the following:

Theorem 4.4.1. *Let $\mathcal{M}_1, \mathcal{M}_2$ be full rank invariant subvarieties (without marked points) in a stratum of translation surfaces in genus $g \geq 2$. There exists (equal area) quasidiagonals $\Delta \subset \mathcal{M}_1 \times \mathcal{M}_2$ only if $\mathcal{M}_1 = \mathcal{M}_2$. In this case, Δ must be the diagonal and antidiagonal.*

Proof. We use induction. Theorem 4.3.1 is the theorem for a quasidiagonal $\Delta \subset \mathcal{H}(2) \times \mathcal{H}(2)$, which is the base case. Let r, s be given such that the theorem holds for any $\Delta \subset \mathcal{M}_1 \times \mathcal{M}_2$ if either $\operatorname{rk}\Delta < r$ or $\operatorname{rk}\Delta = r$ and $\operatorname{rel}\mathcal{M}_1 + \operatorname{rel}\mathcal{M}_2 < s$. Now, we will prove the theorem for $\operatorname{rk}\Delta = r$ and $\operatorname{rel}\mathcal{M}_1 + \operatorname{rel}\mathcal{M}_2 = s$. Let $M \in \Delta$ be a generic surface and C a Δ-parallel class on M. Let γ be a cross curve of a cylinder of C. Let $\Delta' := \operatorname{Col}_{C,\gamma}\Delta$ and $\mathcal{M}_j' := \overline{\operatorname{proj}_j \Delta'}$. By Lemma 4.2.7, $\Delta' \subset \mathcal{M}_1' \times \mathcal{M}_2'$ is a quasidiagonal, where $\mathcal{M}_j'$ are full rank invariant subvarieties with marked points.

Claim 4.4.2. $\Delta' \subset \mathcal{M}_1' \times \mathcal{M}_2'$ is a diagonal or antidiagonal. Furthermore, if either $\mathcal{M}_1$ or $\mathcal{M}_2$ is a hyperelliptic locus, then Δ' is both a diagonal and antidiagonal. Recall that the diagonal and antidiagonal were defined for invariant subvarieties with marked points in Definition 4.2.10.

Proof. When $\mathcal{M}_1'$ and $\mathcal{M}_2'$ both do not have marked points, then the claim is true by the induction hypothesis. Let $\mathcal{F}$ be the functor that forgets marked points. When marked points are present, the induction hypothesis can still be used on $\mathcal{F}\Delta' \subset \mathcal{F}\mathcal{M}_1' \times \mathcal{F}\mathcal{M}_2'$. Define $\mathcal{N} := \mathcal{F}\mathcal{M}_1' = \mathcal{F}\mathcal{M}_2'$, which is full rank. Let $(M_1', M_2') = \mathrm{Col}_{C,\gamma}(M_1, M_2)$. Since $\mathcal{F}\Delta'$ is a diagonal or quasidiagonal, there is a map between the surfaces $I : \mathcal{F}M_1' \to \mathcal{F}M_2'$, which is either the identity map if $\mathcal{F}\Delta'$ is the diagonal or $-\mathrm{Id}$ if $\mathcal{F}\Delta'$ is the antidiagonal. When $\mathcal{N}$ is hyperelliptic, Δ' is a diagonal and an antidiagonal, so we let I denote the identity and I' denote $-\mathrm{Id}$. We may construct a translation surface with marked points $\widehat{M}$ by taking the underlying surface $\mathcal{F}M_2'$, the marked points of M_2', and the image of the marked points of of M_1' under I. By taking the set of all $\widehat{M}$, we get $\widehat{\mathcal{N}}$ an invariant subvariety with marked points. Let $\Gamma = \mathrm{Col}_{C,\gamma}\, C$, defined in the beginning of Section 4.2.1. As an abuse of notation, we will use to term saddle to include geodesic segments between marked points or singularities allowing marked points but not singularities on the interior of the segment. Let $\Gamma_1 = I(\Gamma \cap M_1')$ and $\Gamma_2 = \Gamma \cap M_2'$. Under our definition of saddle, Γ_1 and Γ_2 are sets of saddles on $\widehat{M}$.

By Lemma 4.2.16, none of the marked points are periodic points. In addition, no saddle of Γ_1 or Γ_2 can connect a marked point to itself because otherwise it would come from collapsing two marked points on M, but we assumed M does not have marked points. Since Γ is a Δ'-parallel set of saddle connections on M', then $\Gamma_1 \cup \Gamma_2$ is a $\widehat{\mathcal{N}}$-parallel set of saddle connections on $\widehat{M}$. Thus, by Lemma 4.2.15 there is an irreducible set of marked points $P \subset \widehat{M}$ containing an endpoint of every saddle in $\Gamma_1 \cup \Gamma_2$. By Theorem 4.2.14, P consists of either one free marked point or two points that are swapped by the hyperelliptic involution.

Subclaim 4.4.3. If $\gamma_1, \gamma_2 \in \Gamma_1 \cup \Gamma_2$ are $\widehat{\mathcal{N}}$-parallel saddles that share an endpoint that is also a marked point, then $\gamma_1 = \gamma_2$.

Proof. Since γ_1, γ_2 don't contain any singularities (other than marked points) in the interior, there is a flat coordinate chart of $\widehat{M}$ that contains γ_1 and γ_2. Let p be a marked point on γ_1 and γ_2. By Lemma 4.2.16, no marked points of $\widehat{M}$ are Weierstrass point. Let P be the maximal irreducible set of marked points containing p. $\mathcal{F}\widehat{\mathcal{N}}$ is a stratum or a hyperelliptic locus, so by Theorem 4.2.14, P is either $\{p\}$ or $p, J(p)$, where J is the hyperelliptic involution on $\widehat{M}$. We may perturb P without changing the rest of the surface. If $P = \{p\}$, it is clear that if $\gamma_1 \neq \gamma_2$, there is some perturbation that makes them not parallel. Since γ_1, γ_2 are $\widehat{\mathcal{N}}$-parallel, we must have $\gamma_1 = \gamma_2$. Now assume $P = \{p, J(p)\}$. If $J(p)$ is not among the four endpoints of γ_1, γ_2, this is equivalent to the previous case. We say that γ_1, γ_2 are on opposite sides of p if the vectors from p pointing towards the interior of γ_i are 180 degrees from each other. If γ_1 and γ_2 are on opposite sides of p then a small perturbation of p will make them not parallel. If γ_1, γ_2 both connect p and $J(p)$ and $\gamma_1 \neq \gamma_2$, then γ_1 and γ_2 are on opposite sides of p, so perturbing will also cause γ_1 and γ_2 to not be parallel. The last case is that γ_1 connects p and $J(p)$, and γ_2 connects p to another point q. We perturb p be a small vector δ, so $J(p)$ is changed by $-\delta$. Thus, we change the period of γ_1 by 2δ and the period of γ_2 by δ. γ_1 and γ_2 are still parallel under any such perturbation, so the period of γ_2 must be half the period of γ_1. In addition, γ_1 and γ_2 are on the same side of p. Thus, q is the midpoint of γ_1. However, this is a Weierstrass point since γ_1 is fixed by J, and no marked points of $\widehat{M}$ are Weierstrass points. This rules out the final case, so the subclaim is proven. $\square$

By Subclaim 4.4.3, each point in P is adjacent to exactly one saddle. Thus, $\Gamma_1 \cup \Gamma_2$ consists of either one saddle connection or, in the case that $\mathcal{N}$ is a hyperelliptic locus two saddles. In the latter case by applying Subclaim 4.4.3 again, we get that the two saddles must be swapped by the hyperelliptic involution J. Now, using this understanding of $\Gamma_1 \cup \Gamma_2$ and Lemma 4.2.16, we can understand Γ_1 and Γ_2.

Case 1: $\mathcal{M}_1$ and $\mathcal{M}_2$ are strata.

By Lemma 4.2.16, Γ_1, Γ_2 each contains a single saddle γ_1, γ_2 respectively. Thus, γ_1 and γ_2 either must be the same saddle connection, or they are swapped by the hyperelliptic involution. If $\gamma_1 = \gamma_2$, I takes marked points to marked points, so Δ' is a diagonal or antidiagonal. Otherwise $\mathcal{N}$ is hyperelliptic and I' takes marked points to marked points, so Δ' is an antidiagonal.

Case 2: $\mathcal{M}_1$ and $\mathcal{M}_2$ are a hyperelliptic loci.

If $\Gamma_1 \cup \Gamma_2$ consists of a single saddle connection, then $\Gamma_1 = \Gamma_2$ is a single saddle and by Lemma 4.2.16 this saddle is fixed by the hyperelliptic involution. Thus both I, I' take marked points to marked points, so Δ' is both a diagonal and antidiagonal. Now assume $\Gamma_1 \cup \Gamma_2 = \{\gamma, J(\gamma)\}$, where $\gamma \neq J(\gamma)$. By Lemma 4.2.16, Γ_i contains one saddle that is fixed by J or contains two saddles that are swapped by J. Since none of the saddles in $\Gamma_1 \cup \Gamma_2$ are fixed by J, $\Gamma_1 = \Gamma_2 = \{\gamma, J(\gamma)\}$, and Δ' is both a diagonal and antidiagonal.

Case 3: $\mathcal{M}_1$ is a stratum and $\mathcal{M}_2$ is a hyperelliptic locus.

By Lemma 4.2.16, Γ_1 is a single saddle connection, and Γ_2 can be either one saddle fixed by J or two saddle connections swapped by J. We will show be contradiction that Γ_2 cannot consists of two distinct saddle connections. Assume $\Gamma_2 = \{\gamma, J(\gamma)\}$, $\gamma \neq J(\gamma)$. First we will show that among the four endpoints of $\gamma, J(\gamma)$, there are at least three distinct points. Using Lemma 4.2.15 on $\widehat{M}$, γ and $J(\gamma)$ must each have at least one marked point as an endpoint. For each saddle, its two endpoints cannot be the same marked point because we assumed $\mathcal{M}_1$ and $\mathcal{M}_2$ do not have marked points. In addition, the marked points of γ and $J(\gamma)$ are distinct by the Subclaim. Thus, among the four endpoints there are at least two distinct marked points along with at least one more distinct endpoint. Thus by Lemma 4.2.13, the genus $g(\mathcal{M}_1) \leq g(\mathcal{N}) + 1$ but $g(\mathcal{M}_2) = g(\mathcal{N}) + 2$, which is a contradiction since we assumed $\mathcal{M}_1$ and $\mathcal{M}_2$ are full rank invariant subvarieties with the same rank. Thus, Γ_2 can only have a single saddle which is fixed by J.

Recall $\Gamma_1 \cup \Gamma_2$ is either a single saddle fixed by J or two saddles swapped by J. But $\Gamma_1 \cup \Gamma_2$

cannot be two saddles swapped by the hyperelliptic involutions because Γ_2 contains a saddle fixed by J. Thus, $\Gamma_1 \cup \Gamma_2$ must consist of only a single saddle connection, so $\Gamma_1 = \Gamma_2$, and we get that Δ' is both a diagonal and antidiagonal. The case where $\mathcal{M}_1$ is a hyperelliptic locus and $\mathcal{M}_2$ is a stratum is equivalent to this case, so we have proven the claim. $\square$

Let $(\mathcal{M}, M, C_1, C_2, \gamma_1, \gamma_2)$ be a good diamond, which exists by Lemma 4.2.9. By the claim, if $\mathcal{M}_1$ or $\mathcal{M}_2$ is hyperelliptic, then $\mathrm{Col}_{C_i,\gamma_i} \Delta$ is a diagonal for $i = 1, 2$. Thus by Lemma 4.2.11, Δ must be a diagonal. Now it remains to consider when $\mathcal{M}_1$ and $\mathcal{M}_2$ are not hyperelliptic. In this case, the rank is at least 3, so by Lemma 4.2.9 we can find M and three disjoint equivalence classes of cylinders C_1, C_2, C_3 such that for any two $C_i \neq C_j$, we can find a good diamond $(\mathcal{M}, M, C_i, C_j, \gamma_i, \gamma_j)$. By the Claim for $i = 1, 2, 3$, $\mathrm{Col}_{C_i,\gamma_i} \Delta$ is a diagonal or quasidiagonal. By the Pigeon Hole Principle, there are two (C_i, γ_i) such that $\mathrm{Col}_{C_i,\gamma_i} \Delta$ are either both diagonals or both quasidiagonals. Now Lemma 4.2.11 finishes the proof. $\square$

CHAPTER 5

Classifying Rank 2 Invariant Subvarieties in Genus 3

The classification of rank 2 invariant subvarieties in genus 3 was first done by Aulicino, Nguyen, and Wright in [NW14], [ANW16], [AN16], and [AN20]. They prove:

Theorem 5.0.1 (Aulicino, Nguyen, Wright)**.** *All rank 2 invariant subvarieties in genus 3 are Abelian or quadratic doubles.*

Since then, many new techniques have been developed that lead to shorter proofs. In particular, [Api20, Main Theorem 1] proves a similar statement for all hyperelliptic loci and [AW21b, Corollary 7.3] proves such a statement for $\mathcal{H}(4)$. We redo the proof for all remaining genus 3 strata except $\mathcal{H}(1^4)$.

Theorem 5.0.2. *All rank 2 invariant subvarieties in* $\mathcal{H}^{odd}(2^2)$, $\mathcal{H}(3,1)$, *and* $\mathcal{H}(2,1^2)$ *are Abelian and quadratic doubles.*

This theorem will be proven in each component separately in Theorem 5.3.2, Theorem 5.3.3, and Theorem 5.3.9.

5.1 Abelian and Quadratic Doubles

In this section, we define Abelian and quadratic doubles and find all rank 2 ones in genus 3. We know that these are all rank 2 invariant subvarieties by Theorem 5.0.1, but later in this chapter we will give differents proofs of this in select strata. We include the list at the top of this section for easy reference, and definitions will come below.

Proposition 5.1.1. *The only rank 2 Abelian or quadratic doubles in genus 3 are the following*

1. *Dimension 4*

    (a) $\widetilde{Q}(3,-1^3) \subset \mathcal{H}^{odd}(4)$

(b) $\widetilde{\mathcal{H}}^{hyp}(2) = \widetilde{Q}(1^2, -1^2) \subset \mathcal{H}^{hyp}(2^2)$

(c) $\widetilde{\mathcal{H}}^{odd}(2) \subset \mathcal{H}^{odd}(2^2)$

2. *Dimension 5*

   (a) $\widetilde{Q}(4, -1^4) \subset \mathcal{H}^{odd}(2^2)$

   (b) $\widetilde{Q}(2, 1, -1^3) \subset \mathcal{H}(2, 1^2)$

   (c) $\widetilde{\mathcal{H}}(1^2) \subset \mathcal{H}(1^4)$

3. *Dimension 6*

   (a) $\widetilde{Q}(2^2, -1^4) \subset \mathcal{H}(1^4)$

Definition 5.1.2. If $\mathcal{H}$ is a component of a stratum of translation surfaces, the space of double covers of $M \in \mathcal{H}$ is an invariant subvariety $\widetilde{\mathcal{H}}$ called an **Abelian double**. The rank and rel of $\widetilde{\mathcal{H}}$ is the same as $\mathcal{H}$.

Lemma 5.1.3. *$\widetilde{\mathcal{H}}(2)$ has two components. One component is contained in $\mathcal{H}^{hyp}(2^2)$ and is called $\widetilde{\mathcal{H}}^{hyp}(2)$, and the other component is contained in $\mathcal{H}^{odd}(2^2)$ and is called $\widetilde{\mathcal{H}}^{odd}(2)$. $\widetilde{\mathcal{H}}(1^2)$ is connected.*

Proof Sketch. This proof is based on the proof of [AW22, Lemma 7.2]. The facts used in this proof all come from [AW22, Lemma 7.2]. Let $M \in \mathcal{H}(2)$. Double covers of M are isomorphic to surjective maps from $H_1(M, \mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$, which correspond to W, the set of unordered pairs of distinct Weierstrass points on M. The orbits of W under $\pi_1^{orb}(\mathcal{H}(2))$ thus correspond to connected components of $\widetilde{\mathcal{H}}(2)$. Let w_0 be the singularity of M and $w_1, \ldots, w_5$ be the other Weierstrass points. $\pi_1^{orb}(\mathcal{H}(2))$ fixes w_0 and acts by the symmetric group S_5 on $w_1, \ldots, w_5$. Thus, there are two orbits of W under $\pi_1^{orb}(\mathcal{H}(2))$. One orbit of pairs $\{w_0, w_i\}$, which corresponds to $\widetilde{\mathcal{H}}^{hyp}(2)$ and another of pairs $\{w_i, w_j\}$, $i, j \neq 0$, which corresponds to $\widetilde{\mathcal{H}}^{odd}(2)$. We can fix an element of $\mathcal{H}(2)$ can see how many of each of the 15 double covers lie in each component of $\mathcal{H}(2^2)$. Then, one sees which connect component corresponds to which orbit based on the size of the orbit. An alternate proof can be found in [AN16, Lemma 6.19]. A similar argument shows that $\widetilde{\mathcal{H}}(1^2)$ is connected. □

Let Q be a half translation surface and Σ its set of singularities. The holonomy map $\pi_1(Q-\Sigma) \to \mathbb{Z}/2\mathbb{Z}$ is nontrivial when Q is not a translation surface. This defines a double cover $\widehat{X} \to Q - \Sigma$ that can be extended to a branched double cover $X \to Q$, where X is a compact Riemann surface. The quadratic differential q on Q pulls back to ω^2, where ω is a holomorphic 1-form on X. The translation surface $M := (X, \omega)$ is the **holonomy double cover** of Q. M contains an involution J such that $J^*\omega = -\omega$, and $Q = M/J$.

Definition 5.1.4. If $\mathcal{Q}$ is a component of a stratum of half translation surfaces, then the **quadratic double** $\widetilde{\mathcal{Q}}$ is the space of holonomy double covers of $Q \in \mathcal{Q}$. It is a connected invariant subvariety in a stratum of translation surfaces.

Lemma 5.1.5. *Let $\widetilde{\mathcal{Q}}$ be a quadratic double. Then,*

$$\operatorname{rk} \widetilde{\mathcal{Q}} = g(\widetilde{\mathcal{Q}}) - g(\mathcal{Q})$$

Proof. $\widetilde{\mathcal{Q}}$ is defined by the equations

$$\int_\gamma \omega + \int_{J_*\gamma} \omega = 0.$$

The dimension of the subspace $V \subset H_1(M)$ of γ such that $J_*\gamma = \gamma$ is $2(g(\widetilde{\mathcal{Q}}) - g(\mathcal{Q}))$ since $H_1(M)/V \cong H_1(Q)$. For $\gamma \in V$, this equation is vacuous while otherwise this is an equation cutting out $\widetilde{\mathcal{Q}}$. □

In the next lemma, we write out in more detail the proof sketch in [AW21a, Lemma 4.2].

Lemma 5.1.6. *The number of odd degree singularities of a quadratic differential Q are in bijection with the fixed points of J on its holonomy double cover.*

$$\operatorname{rk} \widetilde{\mathcal{Q}} = \frac{\text{\# of odd degree singularities of } \mathcal{Q}}{2} + g(\mathcal{Q}) - 1,$$
$$\operatorname{rel} \widetilde{\mathcal{Q}} = \text{\# of even degree singularities of } \mathcal{Q}.$$

Proof. An odd degree singularity cannot be a fixed point of J since $J^*\omega = -\omega$ and an odd degree singularity is locally $z^k dz$ for k odd. Thus, every fixed point of J can cone angle $(2k+1) \cdot 2\pi$, so quotient is an odd degree singularity. Thus, # fixed points of J = # odd degree singularities of Q.

By Riemann Hurwicz,

$$2 - 2g(\widetilde{\mathcal{Q}}) + \text{\# fixed points of } J = 2(2 - 2g(\mathcal{Q})).$$

Rearranging we get:

$$g(\widetilde{\mathcal{Q}}) - g(\mathcal{Q}) = \frac{\text{\# fixed points of } J}{2} + g(\mathcal{Q}) - 1.$$

The equation for rank follows from # fixed points of J = # odd degree singularities of Q and Lemma 5.1.5. The equation for rel comes from the equation for the dimension of $\mathcal{Q}$ see [KZ03, Section 2.1]. □

Proof of Proposition 5.1.1. The only rank 2 Abelian doubles are covers of genus 2 Abelian differentials, so they are $\widetilde{\mathcal{H}}(2) \subset \mathcal{H}(2^2)$ and $\widetilde{\mathcal{H}}(1^2) \subset \mathcal{H}(1^4)$. The components of each are classified in Lemma 5.1.3.

By Lemma 5.1.5, a rank 2 quadratic double in genus 3 is a double cover of genus 1 quadratic differentials $\mathcal{Q}(\kappa)$. By Lemma 5.1.6, since there are four fixed points of such a covering map, κ must have four odd integers. Thus, we must enumerate all tuples of integers in $\{-1\} \cup \mathbb{N}$ with exactly four odd integers that sum to 0. All such tuples are: $(1, 1, -1, -1)$, $(2, 1, -1, -1, -1)$, $(2, 2, -1, -1, -1, -1)$, $(3, -1, -1, -1)$, $(4, -1, -1, -1, -1)$. □

As we mentioned all holonomy double covers must have an involution J. We provide some more discussion about such involutions.

Definition 5.1.7. Let M be a translation surface and ω its holomorphic 1-form. An affine diffeomorphism $J : M \to M$ such that $J^*\omega = -\omega$ is called a **180 degree involution**. We may view M/J as a Riemann surface. If M/J is genus 0 then J is the **hyperelliptic involution** of M. If M/J is genus 1, then J is the **Prym involution** of M.

Lemma 5.1.8. *If M is a horizontally periodic surface with a* 180 *degree involution J, then every cylinder C of M is either fixed by J or swapped with another cylinder. If C is fixed, it must have the same number of horizontal saddles along the top and bottom of C. If C is swapped with C', the number of saddles on the top of C equals the number of saddles on the bottom of C' and the number of saddles on the bottom of C equals the number of saddles on the top of C'.*

Proof. J must take horizontal saddles to horizontal saddles. □

We close this section by stating some results from [AW21b] that they used to classify rank 2 invariant subvarieties in $\mathcal{H}(4)$ and are helpful to other genus 3 strata.

Theorem 5.1.9. *[AW21b, Theorem 7.2] Let $\mathcal{M}$ be an invariant subvariety with no rel. If there is a surface $(X, \Phi) \in \mathcal{M}$ that contains a nested free cylinder, then $\mathcal{M}$ is a stratum of translation surfaces or a quadratic double.*

Proposition 5.1.10. *[AW21b, Proposition 7.13] Suppose that $\mathbf{k}(\mathcal{M}) = \mathbb{Q}$ and $\mathcal{M}$ has no rel. Then there is a cylindrically stable surface $M \in \mathcal{M}$ such that either:*

1. *There is a free horizontal cylinder on M that contains a nested free cylinder.*

2. *None of the horizontal cylinders on M are free.*

Corollary 5.1.11. *If $\mathcal{M}$ is an invariant subvariety with no rel such that every cylindrically stable surface in $\mathcal{M}$ has a free cylinder, then $\mathcal{M}$ is a stratum of translation surfaces or a quadratic double.*

Proof. This follows from Theorem 5.1.9 and Proposition 5.1.10. □

5.2 Background

5.2.1 Homology Relations

Definition 5.2.1. An n-cylinder **pair of pants** P is a set of cylinders $\{C_1, \ldots, C_n\}$ with the following property: There is a cylinder C_1 that is the **waist** of the pants such that the bottom (resp. top) of C_1 is adjacent to only $C_2, \ldots, C_n$ and the tops (resp. bottoms) of C_i is only adjacent to C_1 for $i > 1$. The rel curve γ_P of P is the homology element $\gamma_1 - \sum_{i=2}^{n} \gamma_i$.

Lemma 5.2.2. *Let $M \in \mathcal{M}$. Let $C_1, \ldots, C_n$ be the cylinders on M in an n-cylinder pair of pants. Then the cylinders $C_1, \ldots, C_n$ cannot be contained in exactly two cylinder equivalence classes.*

Proof. The core curves of the cylinders satisfy $\gamma_1 = \sum_{i=2}^{n} \gamma_i$ in absolute homology. Without loss of generality assume γ_1 and γ_2 are in different equivalence classes. Then, any other γ_i can be written as a positive multiple of either γ_1 or γ_2. Thus, the equation in homology becomes $k_1\gamma_1 = k_2\gamma_2$, where k_2 is not zero. If $k_1 = 0$, this is a contradiction since k_2 cannot be zero. If $k_1 \neq 0$, then γ_1 and γ_2 are $\mathcal{M}$-parallel, which is also a contradiction of our assumptions. $\square$

Lemma 5.2.3. *Let $\mathcal{M}$ be an invariant subvariety and $M \in \mathcal{M}$. Let C_1, C_2, C_3, C_4 be four cylinders on M whose core curves $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ satisfy the relation $\gamma_1 + \gamma_2 = \gamma_3 + \gamma_4$ in homology. Assume the four cylinders are contained into two distinct $\mathcal{M}$-parallel classes $\mathcal{C}_1, \mathcal{C}_2$. Then there are two possibilities up to relabeling $\mathcal{C}_1, \mathcal{C}_2$:*

1. *$\{C_1, C_3\} \subset \mathcal{C}_1, \{C_2, C_4\} \subset \mathcal{C}_2$. In this case, γ_1, γ_3 have the same period and γ_2, γ_4 have the same period in a neighborhood of M.*
2. *$\{C_1, C_4\} \subset \mathcal{C}_1, \{C_2, C_3\} \subset \mathcal{C}_2$. In this case, γ_1, γ_4 have the same period and γ_2, γ_3 have the same period in a neighborhood of M.*

Proof. Let $\pi : H_1(M, \Sigma) \to (T_M\mathcal{M})^*$ be the natural projection. Assume three cylinders are in one class and one cylinder is in the other class. We consider the case that $\{C_1, C_2, C_3\} \subset \mathcal{C}_1$ and $C_4 \in \mathcal{C}_2$ but the other cases are similar. By the definition of $\mathcal{M}$-parallel, there are some $k_1, k_2 > 0$ such that $k_1\pi(\gamma_1) = \pi(\gamma_3)$ and $k_2\pi(\gamma_2) = \pi(\gamma_3)$. Thus $0 = (1 - k_1 - k_2)\pi(\gamma_3) + \pi(\gamma_4)$, so $\pi(\gamma_3)$ and $\pi(\gamma_4)$ are collinear, which contradicts the assumption that C_4 is in a different $\mathcal{M}$-parallel class than C_3. Assume $\{C_1, C_2\} \subset \mathcal{C}_1$ and $\{C_3, C_4\} \subset \mathcal{C}_2$. Then, $\pi(\gamma_1) = k_1\pi(\gamma_2)$ and $\pi(\gamma_3) = k_2\pi(\gamma_4)$ for some $k_1, k_2 > 0$. But then $(1 + k_1)\pi(\gamma_1) = (1 + k_2)\pi(\gamma_3)$, so C_1 and C_3 are also $\mathcal{M}$-parallel, which is a contradiction.

Now assume $\{C_1, C_3\} \subset \mathcal{C}_1$ and $\{C_2, C_4\} \subset \mathcal{C}_2$. Then for a neighborhood around M, $\pi(\gamma_3) = k_1\pi(\gamma_1)$ and $\pi(\gamma_4) = k_2\pi(\gamma_2)$, so $\pi(\gamma_1) + \pi(\gamma_2) = k_1\pi(\gamma_1) + k_2\pi(\gamma_2)$. In order for C_1, C_2, C_3, C_4

to not be all contained in the same $\mathcal{M}$-parallel class this equation must be trivial, so $k_1 = k_2 = 1$. Thus, γ_1 and γ_3 must have the same period in a neighborhood of M. Similarly, γ_2, γ_4 have the same period in a neighborhood of M. The case that $\{C_1, C_4\} \subset \mathcal{C}_1$ and $\{C_2, C_3\} \subset \mathcal{C}_2$ is similar. $\square$

Lemma 5.2.4. *Let $\mathcal{M}$ be a rank 2 invariant subvariety and $M \in \mathcal{M}$ be a horizontally periodic surface with at least three 3 $\mathcal{M}$-parallel classes of horizontal cylinders. For any three of these classes $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$, let $\sigma_{\mathcal{C}_i}$ be the standard twist of $\mathcal{C}_i$. Then, up to relabeling there exist $a_1, a_2, a_3 > 0$ such that*

$$a_1 p(\sigma_{\mathcal{C}_1}) = a_2 p(\sigma_{\mathcal{C}_2}) + a_3 p(\sigma_{\mathcal{C}_3}) \tag{5.1}$$

Note that the core curve of every horizontal cylinder of M appears in the equation with a nonzero coefficient.

Proof. Let $\sigma_{\mathcal{C}_i}$ be the standard twist of $\mathcal{C}_i$. The $p(\sigma_{\mathcal{C}_i})$ span a Lagrangian subspace of $T_M\mathcal{M}$, which is dimensional at most 2 since $\mathcal{M}$ is rank 2. There must be some relation $\sum_i a_i p(\sigma_{\mathcal{C}_i}) = 0$. Now, we will show that the a_i's are all nonzero. Assume by contradiction that $a_3 = 0$. Note that the same argument works if a_1 or $a_2 = 0$. Each can be written as a sum of twists $\sigma_{\mathcal{C}_i} = \sum_{C_j \in \mathcal{C}_i} w_j \gamma_j^*$ for $w_j > 0$. Writing out the $p(\sigma(\mathcal{C}_i))$ this way and taking the dual, the equation $a_1 p(\sigma_{\mathcal{C}_1}) = a_2 p(\sigma_{\mathcal{C}_2})$ is equivalent to

$$\sum_{C_j \in \mathcal{C}_1} y_j [\gamma_j] = \sum_{C_k \in \mathcal{C}_2} z_k [\gamma_k]$$

for some $y_j, z_k > 0$. Let π be the projecting from $H_1(M, \Sigma)$ to $(T_M\mathcal{M})^*$. By the definition of $\mathcal{M}$-parallel all the $\pi[\gamma_i]$ are colinear for $C_i \in \mathcal{C}_1$ and similarly for $\pi[\gamma_j]$. Thus the equation gives that the $\pi(\gamma_i)$ and $\pi(\gamma_j)$ are colinear with each other, so $\mathcal{C}_1$ and $\mathcal{C}_2$ are actually the same $\mathcal{M}$-parallel class. This is a contradiction. Thus, $a_1, a_2, a_3 \neq 0$. In addition, up to relabeling we get that $a_1 p(\sigma_{\mathcal{C}_1}) = a_2 p(\sigma_{\mathcal{C}_2}) + a_3 p(\sigma_{\mathcal{C}_2})$, for $a_1, a_2, a_3 > 0$. $\square$

5.2.2 Cylinder Collapse

We will define a similar cylinder collapse as in Section 4.2.1. For this chapter, we will extend the definition to elements of the twist space other than standard twists of $\mathcal{M}$-parallel classes. Let $\alpha \in \operatorname{Tw} M$. We can similarly define

$$\operatorname{Col}_\alpha M := \lim_{t \to t_0^-} M + ti\alpha,$$

where t_0 is chosen to be the first time when at least one cylinder has zero height. If γ_i are the core curves of the cylinders in C, then $\alpha = \sum c_i \gamma_i^* - \sum c_j \gamma_j^*$, for some $c_i, c_j > 0$. The cylinders C_j will collapse to height zero and the cylinders C_i will increase in height. The cylinders C_j will be called the **collapsing cylinders** of α. $\mathrm{Col}_\alpha M \in \partial \mathcal{M}$ if and only if one of the collapsing cylinders has a vertical cross curve. We define $\mathrm{Col}_\alpha \mathcal{M}$ to be the invariant subvariety in $\overline{M}$ that contains $\mathrm{Col}_\alpha M$ i.e. $\mathrm{Col}_\alpha \mathcal{M} = \mathcal{M}$ if $\mathrm{Col}_\alpha M \in \mathcal{M}$, otherwise $\mathrm{Col}_\alpha \mathcal{M}$ is the component of ∂M that contains $\mathrm{Col}_\alpha M$.

Remark 5.2.5. The special case when C is an $\mathcal{M}$-parallel class of cylinders and $\alpha = -\sigma_C$ was briefly discussed in Section 4.2.1. Then $\mathrm{Col}_\alpha M = \mathrm{Col}_C M$ by definition. When C contains only a single cylinder C, we define $\mathrm{Col}_C M := \mathrm{Col}_C M$. We use similar definitions for $\mathrm{Col}_C \mathcal{M}$ and $\mathrm{Col}_C \mathcal{M}$,

Lemma 5.2.6. *Let $\mathcal{M} \subset \mathcal{H}(\kappa)$ be an invariant subvariety and $M \in \mathcal{M}$. Let $\alpha \in \mathrm{Tw}\, M$.*

1. *Assume there is exactly one collapsing cylinder C of α, and it is simple and contains a vertical cross curve β. Assume that the endpoints of β are distinct points p, q. Then, $\mathrm{Col}_\alpha \mathcal{M} \subset \mathcal{H}(\kappa')$, where κ' is obtained from κ by merging p and q.*

2. *Assume there is exactly one collapsing cylinder C of α, and it is simple and contains a vertical cross curve β. Assume that the endpoints of β are the same point. Then, $\mathrm{Col}_\alpha \mathcal{M}$ has lower genus than $\mathcal{M}$.*

3. *Assume there are exactly two collapsing cylinders C_1, C_2 of α, and each is simple and contains a vertical cross curve β_1, β_2 respectively. Assume the endpoints of β_1 are p, q and the endpoints of β_2 are p, r, where p, q, r are distinct. Then, $\mathrm{Col}_\alpha \mathcal{M} \subset \mathcal{H}(\kappa')$, where κ' is obtained from κ by merging p,q, and r.*

Proof. First we assume that there is exactly one collapsing cylinder $C \subset M$ of α that is simple and it contains a vertical cross curve β. Let C be the cylinder on M that contains β. C collapses to a saddle connection $\mathrm{Col}_\alpha C$ on $M' := \mathrm{Col}_\alpha M$. By gluing a cylinder into $\mathrm{Col}_\alpha C$, we get M. If the endpoints of $\mathrm{Col}_\alpha C$ are the distinct, then adding a cylinder is topologically adding a handle, so it increases the genus. This proves 2. Now assume both endpoints of $\mathrm{Col}_\alpha C$ are the same point p. Then, adding a cylinder to $\mathrm{Col}_\alpha C$ separates the singularity into two. The sum of the degrees of the two singularities give the degree of p. This proves 1. To prove 3, we can separate Col_α into two individual collapses of C_1 and C_2. Since we only care about the stratum of the resulting surface, it doesn't matter if we leave $\mathcal{M}$. 3 followings from applying 1 twice. □

Lemma 5.2.7. *Let $\mathcal{M}$ be an invariant subvariety and $M \in \mathcal{M}$. Let $\alpha \in \mathrm{Tw}\, M$, and let the collapsing cylinders of α be semisimple. If the cross curves of the collapsing cylinders are all $\mathcal{M}$-parallel, and $M' := \mathrm{Col}_\alpha M$ is the same genus as M, then $\mathcal{M}' := \mathrm{Col}_\alpha M$ is the same rank as $\mathcal{M}$.*

Proof. There is a one-dimensional space of vanishing cycles, so $\mathcal{M}'$ is dimension 1 less than $\mathcal{M}$. Now we assume that they have same genus. The only way for $\mathcal{M}'$ to be dimension one less than $\mathcal{M}$ and have lower rank is to have higher rel than $\mathcal{M}$, but we will show that is not possible.

There is an inclusion $\iota : H^1(M', \Sigma_{M'}) \hookrightarrow H^1(M, \Sigma_M)$. Let K be the kernel of the projection map $p_M : H^1(M, \Sigma_M) \to H^1(M)$ and K' is the kernel of $p_{M'} : H^1(M', \Sigma_{M'}) \to H^1(M')$. We claim that ι takes K' to K. An element of K' is an element $\Phi \in H^1(M', \Sigma_{M'})$ such that for every $\alpha \in H^1(M')$, $\Phi(\alpha) = 0$. Since M, M' have the same genus, there is a natural isomorphism $H^1(M') \cong H^1(M)$, so for every element $\alpha \in H^1(M)$, $\iota(\Phi)(\alpha) = 0$, so ι takes K' to K. Thus $K' \subset K$ and $T_{M'}\mathcal{M}' \subset T_M\mathcal{M}$, so $\operatorname{rel}\mathcal{M} = \dim(T_M\mathcal{M} \cap K)$ is at least $\operatorname{rel}\mathcal{M}' = \dim(T_{M'}\mathcal{M}' \cap K')$. Thus, $\mathcal{M}'$ has the same rank as $\mathcal{M}$. □

Lemma 5.2.8. *Let $M' := \operatorname{Col}_\alpha M$ and $\mathcal{M}' = \operatorname{Col}_\alpha \mathcal{M}$. Let C_1, C_2 be $\mathcal{M}$-parallel cylinders on M that persist on M'. Let C_i' be the cylinder on M' corresponding to C_i. Then, C_1', C_2' are $\mathcal{M}'$-parallel on M'.*

Proof. We have the following commutative diagram.

$$\begin{array}{ccc} H_1(M,\Sigma) & \longrightarrow & (T_M\mathcal{M})^* \\ \downarrow & & \downarrow \\ H_1(M',\Sigma) & \longrightarrow & (T_{M'}\mathcal{M}')^* \end{array}$$

Let $[\gamma_i] \in H_1(M, \Sigma)$ be the core curve of C_i and $[\gamma_i'] \in H_1(M', \Sigma)$ be the core curve of C_i'. By the definition of $\mathcal{M}$-parallel cylinders, $[\gamma_1], [\gamma_2]$ project to collinear vectors in $(T_M\mathcal{M})^*$. $[\gamma_i']$ is the projection of $[\gamma_i]$ to $H_1(M', \Sigma)$. Thus, $[\gamma_1'], [\gamma_2']$ project to collinear vectors in $(T_{M'}\mathcal{M}')^*$, so C_1', C_2' are $\mathcal{M}'$-parallel. □

5.2.3 Overcollapse

Let M be a horizontally periodic surface, and $\alpha \in \operatorname{Tw} M$. Then, $M(t) = M + ti\alpha^*$ is a linear path in period coordinates. The height of the collapsing cylinders of α shrink as t increases. In Section 5.2.2, we stopped when a single cylinder reached height 0. This was called a cylinder collapse. If we furthermore assume that the collapsing cylinders of α have no vertical cross curves, no saddles will degenerate along this path. Thus, we can continue it slightly past the point of cylinder collapse. We call this an **overcollapse of M along α** or the overcollapse of M along C, when $\alpha = -\sigma(C)$. The resulting surface will still be a horizontally periodic surface in $\mathcal{M}$ but will usually have a different cylinder diagram than M. Thus, this is a way to find horizontally periodic surfaces in $\mathcal{M}$ with different cyinder diagrams. To visualize these surfacese, we can always find a period coordinate chart as in Figure 5.1 that contains the entire path of the overcollapse.

Let C be a horizontal semisimple cylinder such that the bottom of C has a single saddle connection and the top has more than one saddle connection. Let p be the singularity on the bottom of

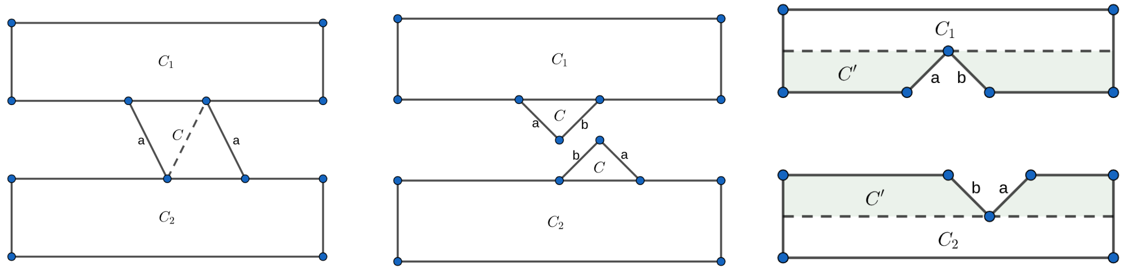

Figure 5.1: Assume that C is a free cylinder on $M \in \mathcal{M}$. The whole overcollapse is contained in a properly chosen period coordinate chart. Here we are ignoring the cylinders that aren't relavent to the overcollapse. The cylinder C' shaded green replaces C. We have the following relation between core curves $\gamma' = \gamma_1 + \gamma_2 - \gamma$.

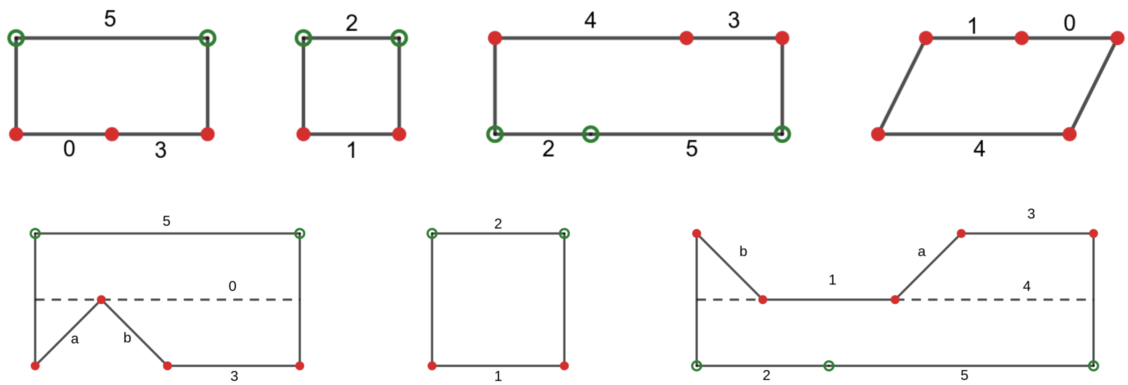

Figure 5.2: Top: Cylinder diagram (0,3)-(5) (1)-(2) (2,5)-(3,4) (4)-(0,1). The straight line in C_3 that starts at the vertex on the bottom of the cylinder intersects s_0. Bottom: Computing the overcollapse of C_3 in cylinder diagram (0,3)-(5) (1)-(2) (2,5)-(3,4) (4)-(0,1) through s_0. The new cylinder diagram is (0)-(5) (1)-(2) (2,5)-(4,1) (4,3)-(3,0). Up to relabeling this is cylinder diagram 3 in B.1.1

C and draw a staight line up from p until it hits the top boundary of C. Since we assumed C does not have a vertical saddle connection, this line will hit the interior of a saddle connection s. To be more specific, we will call this overcollapse, the overcollapse of M along C **through** s. By twisting C, we may overcollapse through any saddle along the top of C, and the resulting surfaces may have different cylinder diagrams. We will also use this terminology when the top of C is a single saddle connection, and the bottom has more than one saddle connection. An example is show in Figure 5.2.

We say two horizontal cylinders C, C' are **adjacent on the same side** of a horizontal cylinder D if C, C' are both adjacent to the top or both adjacent to the bottom of D. A collection of horizontal cylinders $\mathcal{C}$ on a horizontally periodic surface M is called **overcollapsible** if

1. Each cylinder $C \in \mathcal{C}$ is semisimple

2. No two cylinders of $\mathcal{C}$ are adjacent

3. No two cylinders $C, C' \in \mathcal{C}$ are adjacent on the same side to a horizontal cylinder D of M'.

The above construction is also valid if we replace C by an overcollapsible collection of horizontal cylinders $\mathcal{C} = \{C_1, \dots, C_k\}$. Let h_i be the height of C_i and $\gamma^* = \sum_i h_i \gamma_i^*$. Now, $M - ti\gamma^*$ for t slightly larger than 1 is an overcollapse of M along $\mathcal{C}$.

Lemma 5.2.9. *Let $\mathcal{M}$ be a rank 2 invariant variety, and $M \in \mathcal{M}$ is cylindrically stable. Assume M has two horizontal $\mathcal{M}$-parallel classes of cylinders $\mathcal{C}_0, \mathcal{C}_1$ and $\mathcal{C}_1$ consists of a single semisimple cylinder C. Then, the overcollapse of M along C is cylindrically stable.*

Proof. Since M is cylindrically stable, $\mathrm{Tw}\, M$ has dimension $\mathrm{rk}\, \mathcal{M} + \mathrm{rel}\, \mathcal{M} = 2 + \mathrm{rel}\, \mathcal{M}$. Let $\mathrm{Tw}\, \mathcal{C}_i$ be the subspace of $\mathrm{Tw}\, M$ generated by the core curves of cylinders of $\mathcal{C}_i$. By assumption $\dim \mathrm{Tw}\, \mathcal{C}_0 = 1 + \mathrm{rel}\, \mathcal{M}$ and $\dim \mathrm{Tw}\, \mathcal{C}_1 = 1$. Overcollapsing $\mathcal{C}_1$ gives a horizontally periodic surface M' with the same number of cylinders as M. All cylinders of $\mathcal{C}_0$ persist on M' and remain $\mathcal{M}$-parallel, so we call this $\mathcal{M}$-parallel class on M' $\mathcal{C}_0$ as well. The remaining cylinder C on M' correspond to the cylinder C' on M. Let C_i be the cylinders in $\mathcal{C}_0$ and γ_i be the core curve of C_i. Let γ, γ' be the core curves of C, C' respectively. Then, $\gamma' = \gamma_1 + \gamma_2 - \gamma$. Thus, C' is not in $\mathcal{C}_0$. Thus, there is a horizontal $\mathcal{M}$-parallel class $\mathcal{C}_1' \neq \mathcal{C}_0$, and $\dim \mathrm{Tw}\, \mathcal{C}_1' = 1$, so $\dim \mathrm{Tw}\, M' = \dim \mathcal{C}_0 + \dim \mathcal{C}_1' = \mathrm{rk}\, \mathcal{M} + \mathrm{rel}\, \mathcal{M}$, so M' must be cylindrically stable. □

5.2.4 Automation

Using Sage and the flat-surface package, we wrote a program that can automatically check certain conditions on cylinder diagrams described below. The code can be found here: `https://github.com/chriszhang3/cylinders`.

We first explain how to describe a horizontally periodic translation surfaces as a cylinder diagram, see Figure 5.3 for an example. The horizontal saddle connections are numbered starting from 0. Two tuples connected by a dash corresponds to a cylinder. The first tuple is the horizontal saddles on the bottom of the cylinder, which go from left to right.The second tuple is the saddles on the top of the cylinder, and these go from right to left.

The surface-dynamics package in Sage has functions that investigate cylinder diagrams, see documentation. There is also a database of which cylinder diagrams can be found in certain components of strata, see `https://flatsurf.github.io/surface-dynamics/database.html`. The contents of this database for components of strata relavant to this paper is reproduced in Appendix B.1.

Given a cylinder diagram with n cylinders and an integer m, our program lists all ways to partition the horizontal cylinders into m $\mathcal{M}$-parallel classes, which we call a **partition**. Our program then

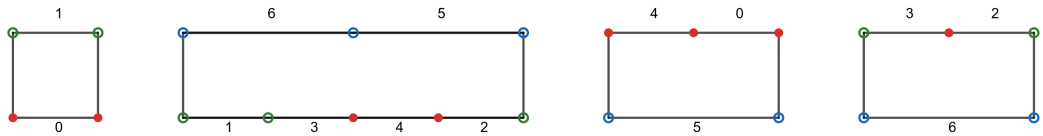

Figure 5.3: A surface with the cylinder diagram (0)-(1) (1,3,4,2)-(5,6) (5)-(0,4) (6)-(2,3) in $\mathcal{H}(2, 1^2)$. The degree 2 singularity is the red dots and the degree 1 singularities are the green and blue dots.

filters out partitions that do not satisfy certain conditions. In particular, our program will do the following:

1. Finds all n-cylinder pants. If the cylinder diagram has at least one n-cylinder pants, filters out the partitions that do not satisfy Lemma 5.2.2.

   Definition 5.2.10. The **cylinder graph** G of a horizontally periodic translation surface M is constructed as follows. Each cylinder of M is a node of G. For each horizontal saddle s of M, add a directed edge from the cylinder underneath s to the cylinder above s. We say a node n' is a **successor** of a node n if there is a directed edge from n to n'. n' is a **predecessor** of n if there is a directed edge from n' to n.

   The code finds n-cylinder pants by creating the cylinder graph and checking each node n. If every successor of n only has n a predecessor, then we have found a pair of pants. Similarly, if every predecessor of n only has n as a successor, then we have found a pair of pants. After finding all pairs of pants, its straightforward to check Lemma 5.2.2.

2. Finds all homologous cylinders. If the cylinder diagram has at least one set of homologous cylinders, filters out the partitions where the cylinders are in different $\mathcal{M}$-parallel class.

   Lemma 5.2.11. *Let M be a horizontally periodic translation surface. Let $\gamma_0, \ldots, \gamma_k$ be the horizontal saddle connections of M, and choose a cross curve α_i of each horizontal cylinder C_i. $H^1(M, \Sigma; \mathbb{Z})$ is isomorphic to the $\mathbb{Z}$-module generated by γ_j, α_i with the relations*

   $$\sum_{\gamma_i \text{ on the bottom of } C} \gamma_i = \sum_{\gamma_j \text{ on the top of } C} \gamma_j.$$

   Proof. A 1-skeleton of M is given by the γ_j and α_i, which gives us the generators. We create a 2-skeleton by adding each cylinder, which gives us the relations. □

Thus, the $\mathbb{R}$ vector space generated by the γ_j with the relations from Lemma 5.2.11 is a subspace of $H_1(M, \Sigma; \mathbb{R})$ that contains the core curves of each cylinder. We then check if any two core curves are the same element of this space.

Remark 5.2.12. Using Lemma 5.2.11, we can also compute the dimension of the span of the core curves of all horizontal cylinders.

3. Filters out partitions that do not satisfy Lemma 5.2.13, which is a restatement of [AN16, Lemma 2.11].

   Lemma 5.2.13. *Let $\mathcal{M}$ be an invariant subvariety defined over $\mathbb{Q}$ of rank at least 2. Let M be a horizontally periodic surface with horizontal cylinders $C_1, \ldots, C_k$. Assume the core curves of $\gamma_1, \ldots, \gamma_k$ span a subspace of $(T_M\mathcal{M})^*$ of dimension at least two. If C_1 is a simple cylinder that is only adjacent to C_2, then C_1 and C_2 cannot be in the same equivalence class.*

   Definition 5.2.14. For a directed graph G and a node n, we define a **neighbor** of n as any node that has an incoming edge from n or an outgoing edge to n. A **leaf** is a node with only one neighbor.

   The assumptions of Lemma 5.2.13 are satisfied in our situation because we consider $\mathcal{M}$ that is rank 2 in genus 3, so $\mathcal{M}$ is defined over $\mathbb{Q}$. In addition, M is a cylindrically stable surface, so the core curves of horizontal cylinders span a subspace of $(T_M\mathcal{M})^*$ of dimension at least two. The condition that C_1 is only adjacent to C_2 is equivalent to the following: the node c_1 in the cylinder graph corresponding to C_1 is a leaf node, and c_2 is the unique neighbor of c_1. We must also check that C_1 is simple.

4. Filters out partitions that do not satisfy Lemma 5.2.4.

   By Lemma 5.2.4, if M has three $\mathcal{M}$-parallel classes of cylinders C_1, C_2, C_3, one of the following must hold:

$$a_1 p(\sigma_1) + a_2 p(\sigma_2) - a_3 p(\sigma_3) = 0,$$
$$a_2 p(\sigma_2) + a_3 p(\sigma_3) - a_1 p(\sigma_1) = 0,$$
$$a_3 p(\sigma_3) + a_1 p(\sigma_1) - a_2 p(\sigma_2) = 0,$$

   where $a_1, a_2, a_3 > 0$. If M has more than 3 $\mathcal{M}$-parallel classes, then for each 3-element subset, one such equation must hold, although in that case, the equations will not all be independent. We will show that finding a positive solution to $a_1 p(\sigma_1) + a_2 p(\sigma_2) - a_3 p(\sigma_3) = 0$ is equivalent to determining whether there exists a nonnegative solution to a linear system of equations

$$Ax = b,$$

$x_i \geq 0$ for all i, which can be determined using a computer. In our case, we use the Sage MixedIntegerLinearProgram class: see documentation. Although all x_i are real and not integers, this class can still be used.

Now, we show how we will reduce our problem to the above form. The horizontal saddle connections are elements of $H_1(M, \Sigma; \mathbb{Z}) \subset H_1(M, \Sigma; \mathbb{R})$ and span a $g + s - 1$ dimensional space U. Notice that $\operatorname{Tw} M \subset U$. The program chooses a linearly independent subset of these saddle connections, so this gives us a fixes isomorphism $U \cong \mathbb{R}^{g+s-1}$ by sending these saddle connections to standard basis vectors. The program treats core curves of cylinders as elements of $\mathbb{R}^{g+s-1}$ through this isomorphism. Up to relabeling, we want to check whether $a_1 p(\sigma_1) + a_2 p(\sigma_2) - a_3 p(\sigma_3) = 0$ has a solution, where a_1, a_2, a_3.

Each $p(\sigma_i)$ is some sum $\sum_j h_j \gamma_j$, where the sum is over all cylinders C_j in the $\mathcal{M}$-parallel class $\mathcal{C}_i$ and $h_j > 0$ is the height of C_j. Thus

$$a_1 p(\sigma_1) + a_2 p(\sigma_2) - a_3 p(\sigma_3) = a_1 \sum_{C_i \in \mathcal{C}_1} h_i \gamma_i + a_2 \sum_{C_j \in \mathcal{C}_2} h_j \gamma_j + a_3 \sum_{C_k \in \mathcal{C}_3} h_k \gamma_k$$

Let A be a matrix whose columns are the $\gamma_i, \gamma_j, \gamma_k$ above and y be a vector of coefficients, so that the above expression equals Ay. Then, it suffices to check whether $Ay = 0$ has a solution, where $y_i > 0$ for all i. A solution $y = (y_1, \ldots, y_k)$ can be scaled so that $y_i \geq 1$ for all i. Thus, it suffices to check whether:

$$A(y - \underline{1}) + A\underline{1} = 0,$$

where $\underline{1}$ is the vector of all 1's. Letting $b = -A\underline{1}$, $x_i = y_i - 1$, and $x = (x_1, \ldots, x_k)$, we have to check whether

$$Ax - b = 0,$$

has a solution, where $x_i \geq 0$ for all i. Thus, we have reduced our problem to the desired linear programming problem.

Output of the program when relevant to this paper is reproduced in Appendix B.2.

5.3 Strata

Lemma 5.3.1. *Assume $\mathcal{M}$ is a rank 2 invariant subvariety such that every horizontally periodic surface has at most three cylinders. Then $\mathcal{M}$ is a stratum or a quadratic double.*

Proof. Let M be a cylindrically stable surface on $\mathcal{M}$. First assume that $\mathcal{M}$ has no rel. Then, M has two horizontal $\mathcal{M}$-parallel classes and at most three cylinders, so at least one horizontal cylinder is free. By Corollary 5.1.11, $\mathcal{M}$ is a stratum or quadratic double. Now assume $\mathcal{M}$ is rel 1. Then, Tw M has dimension 3, so every cylinder must be free. Then Lemma 2.3.7 contradicts the assumption that M is cylindrically stable. If rel $\mathcal{M} > 1$, then $\mathcal{M}$ must contain a cylindrically stable surface with at least 4 cylinders. $\square$

5.3.1 $\mathcal{H}(3, 1)$

In this section we prove:

Theorem 5.3.2. *There are no rank* 2 *invariant subvarieties in* $\mathcal{H}(3, 1)$.

Proof. Assume $\mathcal{M}$ is a rank two invariant subvariety in $\mathcal{H}(3, 1)$. Note that there are no rank 2 Abelian or quadratic doubles in $\mathcal{H}(3, 1)$. By Lemma 5.3.1, we can choose M to be a cylindrically stable surface in $\mathcal{M}$ with four horizontal cylinders. Using the computer program (Section 5.2.4), the only possible cylinder diagrams for M and partitions of horizontal cylinders into $\mathcal{M}$-parallel classes are listed in Appendix B.2.1.

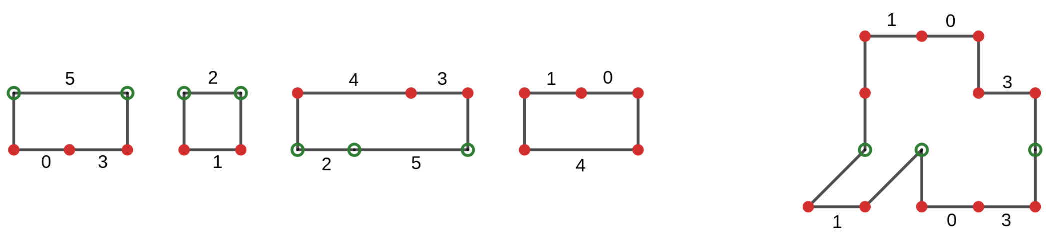

Figure 5.4: Two ways to view cylinder diagram (0,3)-(5) (1)-(2) (2,5)-(3,4) (4)-(0,1). We label the cylinders C_0 to C_3 from left to right. C_0, C_1, C_2 make a pants.

2. (0,3)-(5) (1)-(2) (2,5)-(3,4) (4)-(0,1)

   Assume M has this cylinder diagram. Then, {{0, 1, 2}, {3}} is the only possible partition of the horizontal cylinders into $\mathcal{M}$-parallel classes. We may overcollapse C_3 to get a surface $M' \in \mathcal{M}$ with cylinder diagram 3: (0,1)-(0,2) (2)-(3) (3,4)-(1,5) (5)-(4) (see Figure 5.2). By Lemma 2.3.5, M' is cylindrically stable, but no cylindrically stable surface in $\mathcal{M}$ has cylinder diagram 3, so M cannot have cylinder diagram 2 either.

4. (0,1)-(0,4,5) (2,3)-(1) (4)-(2) (5)-(3)

   The remaining case is that every cylindrically stable surface in $\mathcal{M}$ has this cylinder diagram. In addition, {{0}, {1, 2, 3}} is the only possible partition of the horizontal cylinders into $\mathcal{M}$-parallel classes. If $\mathcal{M}$ is rel 0, then every cylindrically stable surface contains a free cylinder, so $\mathcal{M}$ is a quadratic double by Corollary 5.1.11, but quadratic doubles do not exist in $\mathcal{H}(3,1)$ by Proposition 5.1.1. Thus, $\mathcal{M}$ is rel 1. $\gamma_1^* - \gamma_2^* - \gamma_3^*$ generates the $\ker p \cap T_M\mathcal{M}$, so $\gamma_1^* - \gamma_2^* - \gamma_3^* \in \operatorname{Tw} M$. Thus, we may overcollapse C_1 by increasing the heights of C_2 and C_3. Overcollaping C_1 through s_2 gives a surface with cylinder diagram 1, but we no that $\mathcal{M}$ cannot contain such a surface.

We have shown that a rank 2 invariant variety cannot contain any cylindrically stable surface, so no rank 2 invariant variety exists. □

5.3.2 $\mathcal{H}^{odd}(2,2)$

In this section we prove:

Theorem 5.3.3. *The only rank 2 invariant subvarieties in $\mathcal{H}^{odd}(2^2)$ are $\widetilde{\mathcal{H}}^{odd}(2)$ and $\widetilde{Q}(4,-1^4)$.*

Lemma 5.3.4. *Let $\mathcal{M} \subset \mathcal{H}^{odd}(2^2)$ be a rank 2 invariant subvariety and $M \in \mathcal{M}$ be cylindrically stable. In the following cases, $\mathcal{M} = \widetilde{Q}(4,-1^4)$.*

1. *M has cylinder diagram (0,3)-(5) (1)-(0) (2,5)-(3,4) (4)-(1,2) and C_1 is free.*

4. *M has cylinder diagram (0,3)-(0,5) (1,2)-(1,4) (4)-(3) (5)-(2) and C_2 or C_3 is free.*

5. *M has cylinder diagram (0,2,1)-(3,4,5) (3)-(1) (4)-(2) (5)-(0) and C_1, C_2, or C_3 is free.*

Proof. Let C be the free cylinder. We can check in each case that C is a simple cylinder that has distinct zeros on the top and bottom boundaries. By Lemma 5.2.7, $\mathcal{M}' := \operatorname{Col}_C \mathcal{M}$ is a rank 2 invariant subvariety, and by Lemma 5.2.6, $\mathcal{M}' \subset \mathcal{H}(4)$. Thus, $\mathcal{M}'$ must be $\widetilde{Q}(3,-1^3)$, so M' has a Prym involution. Let γ be the saddle on $M' := \operatorname{Col}_C M$ resulting from collapsing C. Using Lemma 5.1.8, we check in each case that γ is fixed by the involution. Thus, the involution extends to M by fixing C_2. We see that M must be contained in $\widetilde{Q}(4,-1^4)$. By Lemma 2.1.3, any real deformation of M has the same cylinder diagram and the same $\mathcal{M}$-parallel classes, so it is also in $\widetilde{Q}(4,-1^4)$. Thus, an open set around M in $\mathcal{M}$ is contained in $\widetilde{Q}(4,-1^4)$. Since $\mathcal{M}$ and $\widetilde{Q}(4,-1^4)$ are $\mathrm{GL}^+(2,\mathbb{R})$ invariant and the action is ergodic on $\mathcal{M}$, then $\mathcal{M} \subset \widetilde{Q}(4,-1^4)$. Since $\mathcal{M}$ contains a rank 2 invariant subvariety on its boundary, $\mathcal{M}$ must be rank 2 rel 1, so $\mathcal{M} = \widetilde{Q}(4,-1^4)$. □

Lemma 5.3.5. *Let $\mathcal{M} \subset \mathcal{H}^{odd}(2^2)$ be a rank 2 invariant subvariety, and assume $M \in \mathcal{M}$ is a cylindrically stable surface. If M has one of the following cylinder diagrams:*

1. *(0,3)-(5) (1)-(0) (2,5)-(3,4) (4)-(1,2)*

5. *(0,2,1)-(3,4,5) (3)-(1) (4)-(2) (5)-(0)*

then $\mathcal{M}$ is $\widetilde{\mathcal{Q}}(4,-1^4)$. If M has one of the following cylinder diagrams then

3. *(0,5)-(3,4) (1,4)-(2,5) (2)-(0) (3)-(1)*

4. *(0,3)-(0,5) (1,2)-(1,4) (4)-(3) (5)-(2)*

then $\mathcal{M}$ is $\widetilde{\mathcal{H}}^{odd}(2)$ or $\widetilde{\mathcal{Q}}(4,-1^4)$.

Proof. First the case that M has cylinder diagram 1. If M has two equivalence classes of cylinders, then by Lemma 5.2.3 either $\mathrm{len}(\gamma_0) = \mathrm{len}(\gamma_1)$ or $\mathrm{len}(\gamma_1) = \mathrm{len}(\gamma_3)$ both of which are not possible. Thus, M has three distinct equivalence classes. Assume by contradiction that C_0 is a free cylinder of M. Then by Lemma 5.2.6, $\mathcal{M}' := \mathrm{Col}_{C_0} \mathcal{M}$ is contained in $\mathcal{H}(4)$, and $\mathcal{M}'$ is rank 2 by Lemma 5.2.7. Thus, $\mathcal{M}' = \widetilde{\mathcal{Q}}(3,-1^3)$. We can collapse C_0 in a way so that the resulting cylinder diagram is (1)-(0) (2,0,3)-(3,4) (4)-(1,2). Every surface in $\mathcal{M}$ has a Prym involution J, but this contradicts Lemma 5.1.8. Thus, C_0 cannot be a free cylinder of M. By Appendix B.2.2, we see that $\{\{0,3\}, \{1\}, \{2\}\}$ must be the horizontal $\mathcal{M}$-parallel classes of M. In this case, C_1 is free so by Lemma 5.3.4, $\mathcal{M}$ must be $\widetilde{\mathcal{Q}}(4,-1^4)$.

Now assume M has cylinder diagram 5. M cannot have exactly two equivalence classes of cylinders by Lemma 5.2.2, so we assume there are three distinct cylinder equivalence classes. At least one of C_1, C_2, C_3 must be a free cylinder, so by Lemma 5.3.4, $\mathcal{M}$ must be $\widetilde{\mathcal{Q}}(4,-1^4)$.

Now, consider cylinder diagram 3. If C_2 or C_3 is free, we may overcollapse it to get a horizontally periodic surface with cylinder diagram 1. Similarly, it C_2 and C_3 are in one $\mathcal{M}$-parallel class, we can also overcollapse to get a surface with cylinder diagram 4. Checking Appendix B.2.2, we see that we covered all possible partitions for this cylinder diagram.

It remains to check cylinder diagram 4. Our argument is similar to [AN16, Lemma 6.19]. By Appendix B.2.2, the only possible partitions are $\{\{0\},\{1,2,3\}\}$, $\{\{1\},\{0,2,3\}\}$, and $\{\{0,1\},\{2,3\}\}$. First assume M has partition $\{\{0\},\{1,2,3\}\}$. C_1 contains a nested free cylinder D that is contained in a $\mathcal{M}$-parallel class $\mathcal{D}$. By Corollary 2.2.13, $\mathcal{D}$ is completely contained in the $\mathcal{M}$-parallel class $\mathcal{C}_1$ containing C_1. By another application of Corollary 2.2.13, there must be another cylinder in $\mathcal{D}$ that intersects C_2 and C_3, but this is not possible. Thus, M cannot have partition $\{\{0\},\{1,2,3\}\}$. Similarly, M cannot have partition $\{\{1\},\{0,2,3\}\}$. The only remaining possibility is the partition $\{\{0,1\},\{2,3\}\}$. If $\mathcal{M}$ is rel 1 and since M is cylindrically stable, then $\gamma_2^* - \gamma_3^* \in \mathrm{Tw}\, M$ and $h_2\gamma_2^* + h_3\gamma_3^* \in \mathrm{Tw}\, M$, so $\gamma_2^*, \gamma_3^* \in \mathrm{Tw}\, M$, so C_2 and C_3 are free, so $\mathcal{M} = \widetilde{\mathcal{Q}}(4,-1^4)$ by Lemma 5.3.4. The remaining case is when $\mathcal{M}$ is rel 0.

Let $\mathcal{C} = \{C_0, C_1\}$. Twist $\mathcal{C}$ so that the nested cylinder D_0 in C_0 is vertical. By Corollary 2.2.13, D_0 must be $\mathcal{M}$-parallel to the nested cylinder D_1 in C_1, so D_1 is also vertical.

Claim 5.3.6. If either C_2 or C_3 has a vertical cross curve, then both do.

Proof. Let $C' = \{C_2, C_3\}$. If one of C_2 or C_3 contains a vertical saddle and the other does not then, $\mathrm{Col}_{C'} \mathcal{M}$ is rank 2 by Lemma 5.2.7, but this is not possible since $\mathcal{M}$ is rank 2 rel 0. $\square$

Thus, M is both horizontally and vertically periodic.

Claim 5.3.7. C_0 and C_1 are isometric.

Proof. The closure of C_0 and C_1 can be viewed as two slit tori. By Corollary 2.2.13, we may apply [ANW16, Lemma 8.1]. $\square$

Claim 5.3.8. C_2 and C_3 are isometric.

Proof. Let $C' = \{C_2, C_3\}$. Twist C' until the first time that either C_2 or C_3 has a vertical saddle again. By Claim 5.3.6, both do. Thus, C_2 and C_3 have the same modulus. They have the same circumference, so they must be isometric. $\square$

We see that $M \in \widetilde{\mathcal{H}}(2)$. Any real deformation of M has the same cylinder diagram, so is also an element of $\widetilde{\mathcal{H}}(2)$ and $\mathcal{M}$ is also the same dimension, so $\mathcal{M} = \widetilde{\mathcal{H}}(2)$. $\square$

Proof of Theorem 5.3.3. Assume $\mathcal{M}$ is a rank two invariant subvariety in $\mathcal{H}^{odd}(2^2)$. Let M be a cylindrically stable surface in $\mathcal{M}$. If M had the cylinder diagrams in Lemma 5.3.5, the lemma shows that $\mathcal{M}$ is $\widetilde{Q}(4, -1^4)$, which is a quadratic double, or $\widetilde{\mathcal{H}}^{odd}(2^2)$, which is an Abelian double. Now, we will show that no other $\mathcal{M}$ exist.

Assume that $\mathcal{M}$ is a rank two invariant subvariety that does not contain a cylindrically stable surface with a cylinder diagrams in Lemma 5.3.5. The only remaining cylinder diagram is (0,1)-(0,5) (2)-(4) (3,4)-(1) (5)-(2,3). By Appendix B.2.2, the only possible partitions of the horizontal cylinders of M into $\mathcal{M}$-parallel classes are

1. $\{\{0\}, \{1, 2, 3\}\}$
2. $\{\{0, 2, 3\}, \{1\}\}$
3. $\{\{0, 1\}, \{2, 3\}\}$

Let M be a cylindrically stable surface. By Corollary 2.2.13, $\{0, 1\}$ cannot be a cylinder equivalence class of M since C_0 contains a nested cylinder and C_1 does not. Thus, M cannot have partition 3. Assume furthermore than $\mathcal{M}$ has no rel. By our assumptions, for any cylindrically stable surface, M must have cylinder diagram (0,1)-(0,5) (2)-(4) (3,4)-(1) (5)-(2,3) and partitions 1 or 2. Thus by Corollary 5.1.11, $\mathcal{M}$ is a quadratic double.

Our remaining case is that $\mathcal{M}$ is rank 2 rel 1. Label the cylinders of (0,1)-(0,5) (2)-(4) (3,4)-(1) (5)-(2,3) C_0 to C_3 from left to right, and let γ_i be the core curve of the cylinder $C_i \subset M$. $\mathrm{rel}\, \mathcal{H}^{odd}(2^2) = 1$, so $\gamma_2^* - \gamma_3^*$ spans $\ker p$, and $\mathrm{rel}\, \mathcal{M} = 1$, so $\gamma_2^* - \gamma_3^* \in \mathrm{Tw}\, M$. Let $M' := \mathrm{Col}_{\gamma_2^* - \gamma_3^*} M$

and $\mathcal{M}' := \mathrm{Col}_{\gamma_2^*-\gamma_3^*}\mathcal{M}$. By Lemma 5.2.6, $\mathcal{M}' \subset \mathcal{H}(4)$ and by Lemma 5.2.7 $\mathcal{M}'$ is rank 2, so it must be $\widetilde{Q}(3,-1^3)$, so it must have a Prym involution. M' has the cylinder diagram (0,1)-(0,2,3) (2)-(4) (3,4)-(1). We can see that that the above cylinder diagram cannot have a Prym involution by Lemma 5.1.8. □

5.3.3 $\mathcal{H}(2,1,1)$

In this section, we will prove the following theorem.

Theorem 5.3.9. *The only rank 2 invariant subvariety in $\mathcal{H}(2,1^2)$ is $\widetilde{Q}(2,1,-1^3)$.*

We start by finding which cylinder diagrams are possible for $\widetilde{Q}(2,1,-1^3)$, see Proposition 5.3.12. The majority of the work is ruling out four cylinder diagrams first in rel 0 in Proposition 5.3.16, and then in rel 1 in Proposition 5.3.19.

Lemma 5.3.10. *Let $\mathcal{M}$ be a rank 2 invariant subvariety in $\mathcal{H}(2,1^2)$, and $M \in \mathcal{M}$. If C is a semisimple cylinder cylinder of M such that one boundary of C contains a degree 2 singularity and the other boundary contains a degree 1 singularity, then C cannot be free.*

Proof. Assume by contradiction that C was free. Twist C so that there is a vertical saddle between the rank 2 and a rank 1 singularity. By Lemma 5.2.7, collapsing C would create a rank 2 invariant subvariety $\mathcal{M}' \in \mathcal{H}(3,1)$, which does not exist. □

Lemma 5.3.11. *Let $M \in \mathcal{H}(2,1^2)$ contained in a rank 2 invariant subvariety $\mathcal{M}$. Let $\mathcal{C} = \{C, C'\}$ be an $\mathcal{M}$-parallel class of cylinders. Assume that both C, C' are both simple and have one degree one and one degree two singularity on its boundaries. Then, if either of C, C' has a vertical saddle then both do, and both have the same modulus.*

Proof. Twist $\mathcal{C}$ so that C has a vertical saddle connection. If C' has no vertical saddle and we collapse $\mathcal{C}$, then we'd get a rank 2 invariant subvariety of $\mathcal{H}(3,1)$, which does not exist. Thus, C' must have a vertical saddle connection. Similarly, if C' has a vertical saddle connection then C must have one as well. Starting from when C, C' have a vertical saddle connection, we can twist $\mathcal{C}$ until the first time they have a vertical saddle connection again. Since C, C' are simple, this is one full twist around each cylinder. The standard twist of $\mathcal{C}$ twists each cylinder in proportion to its height, so C, C' have the same modulus. □

Proposition 5.3.12. *Let $\mathcal{M}$ be a rank 2 invariant subvariety in $\mathcal{H}(2,1^2)$, and $M \in \mathcal{M}$ be a cylindrically stable surface. If M has one of the following five-cylinder diagrams, then $\mathcal{M}$ is $\widetilde{Q}(2,1,-1^3)$:*

3. *(0,1)-(0,6) (2)-(5) (3)-(4) (4,5)-(1) (6)-(2,3)*

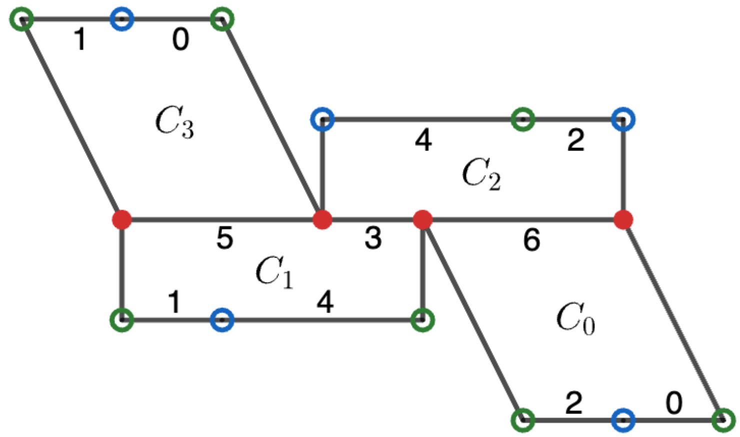

Figure 5.5: Four cylinder diagram 9 in $\mathcal{H}(2, 1^2)$. Cylinders are labeled $C_0, \ldots, C_3$ and saddles are labeled $s_0, \ldots, s_6$. Singularities p_0, p_1, p_2 are blue, green, and red respectively. In this picture, there is a vertical saddle between p_1 and p_3 in both C_3 and C_0. The points q, q' are defined in the proof of Lemma 5.3.15

5. *(0,2)-(6) (1)-(3) (3,6)-(4,5) (4)-(0) (5)-(1,2)*

7. *(0,6)-(4,5) (1,2)-(3,6) (3)-(2) (4)-(1) (5)-(0)*

8. *(0,6)-(4,5) (1,2)-(3,6) (3)-(0) (4)-(2) (5)-(1)*

Proof. Our proof for cylinder diagrams 5 and 7 will be based on [AN20, Proof of Proposition 6.2]. Assume M has cylinder diagram 7. By Appendix B.2.3, the only possible partitions are $\{\{0, 1\}, \{2, 4\}, \{3\}\}$ and $\{\{0, 2\}, \{1, 4\}, \{3\}\}$. Since $[\gamma_1] + [\gamma_4] = [\gamma_0] + [\gamma_2]$, Lemma 5.2.3 gives that $\{\{0, 2\}, \{1, 4\}, \{3\}\}$ is not possible. Thus, the simple cylinders C_2 and C_4 are in the same $\mathcal{M}$-parallel class $\mathcal{C}$. By Lemma 5.3.11, C_2, C_4 have the same modulus, and we can twist $\mathcal{C}$ until both C_2 and C_4 have vertical saddle connections. $M' := \mathrm{Col}_{\mathcal{C}}\, M$ has cylinder diagram (0,6)-(4,0) (1,2)-(2,6) (4)-(1), and it is contained in a rank 2 invariant subvariety $\mathcal{M}' := \mathrm{Col}_{\mathcal{C}}\, \mathcal{M} \subset \mathcal{H}(4)$, so $\mathcal{M}' = \mathcal{Q}(3, -1^3)$ and M' has a Prym involution J. By Lemma 5.1.8, the cylinder (4)-(1) on M' corresponding to C_3 must be fixed by J, so the other cylinders must be swapped. There are two saddles s_0, s_2 on M' corresponding to the collapsed cylinders C_2 and C_4 and these must be swapped by J. Thus, they must be the same length, and since C_2 and C_4 have the same modulus and twist, they must be isometric. Thus, J extends to a Prym involution on M, so $M \in \widetilde{\mathcal{Q}}(2, 1, -1^3)$. Since for any real deformation of M, it has the same cylinder diagram and thus a Prym involution, $\mathcal{M} \subset \widetilde{\mathcal{Q}}(2, 1, -1^3)$. $\mathcal{M}$ must have rel at least 1 since M is a cylindrically stable surface with 5 cylinders, so $\mathcal{M} = \widetilde{\mathcal{Q}}(2, 1, -1^3)$.

Now assume M has cylinder diagram 5. By Lemma 5.3.10, C_1 and C_3 cannot by free. In addition $\gamma_0 + \gamma_1 = \gamma_3 + \gamma_4$, so $\{\{0,3\},\{1,4\},\{2\}\}$ is not possible by Lemma 5.2.3. By Appendix B.2.3, the remaining partition is $\{\{0,4\},\{1,3\},\{2\}\}$. Using a similar argument as for cylinder diagram 7 on the $\mathcal{M}$-parallel class of simple cylinders $\{C_1, C_3\}$, $\mathcal{M}$ must be $\widetilde{Q}(2,1,-1^3)$.

For cylinder diagrams 3 and 8, we defer to the proof of [AN20, Proposition 6.11]. In both of these cases, [AN20, Lemma 6.12], [AN20, Lemma 6.13], and [AN20, Lemma 6.14] will be clear as we will show below, so only [AN20, Section 6.2.1] remains of the proof. For cylinder diagram 8, what we call C_0, C_1 are called C_1, C_2 by Aulicino and Nguyen. Our C_3, C_4 are also their C_3, C_4, and our C_2 is their C_5, which is simple. Appendix B.2.3 gives that the only possible partition is $\{\{0,1,3,4\},\{2\}\}$, which is [AN20, Lemma 6.12] and [AN20, Lemma 6.13], and [AN20, Lemma 6.14] is clear in our setting. For cylinder diagram 8, what we call C_3, C_4 are called C_1, C_2 by Aulicino and Nguyen. Our C_1, C_2 are their C_3, C_4, and our C_0 is their C_5, which is not simple. Appendix B.2.3 gives that the only possible partition is $\{\{0\},\{1,2,3,4\}\}$, which is [AN20, Lemma 6.12] and [AN20, Lemma 6.13], and [AN20, Lemma 6.14] is clear in our setting. □

Lemma 5.3.13. *The following cylinder diagrams each have one pair of homologous cylinders listed below. Cylinders are labeled C_0 to C_3 from left to right.*

7. *(0,1)-(0,3) (2,5)-(1,6) (3,6)-(4,5) (4)-(2)* $\{C_1, C_2\}$
8. *(0,2)-(0,5) (1,3)-(1,6) (4,5)-(3) (6)-(2,4)* $\{C_2, C_3\}$
12. *(0,1,2)-(0,1,6) (3)-(5) (4,5)-(2) (6)-(3,4)* $\{C_2, C_3\}$
19. *(0,2,1)-(5,6) (3,6)-(0,4,1) (4)-(2) (5)-(3)* $\{C_0, C_1\}$
23. *(0,1,4)-(0,1,6) (2,3)-(2,5) (5)-(4) (6)-(3)* $\{C_2, C_3\}$
24. *(0,5,2)-(3) (1,3)-(1,6) (4)-(5) (6)-(0,4,2)* $\{C_0, C_3\}$

Proof. Each cylinder diagram can be manually checked. □

Lemma 5.3.14. *Let $\mathcal{M} \subset \mathcal{H}(2,1^2)$ be a rank 2 invariant subvariety. Let M be a horizontally periodic surface of $\mathcal{M}$. If M has one of the cylinder diagrams 7, 12, 23, or 24, it cannot have two horizontal $\mathcal{M}$-parallel classes that each have 2 cylinders.*

Proof. By Lemma 5.3.13, each of the cylinder diagrams contains a pair of homologous cylinders, so those must be in the same $\mathcal{M}$-parallel class. The remaining two cylinders C, C' cannot be in the same cylinder equivalence class since they have a different number of nested cylinders, but by Corollary 2.2.13 each nested cylinder in C (or C') must be $\mathcal{M}$-parallel to a nested cylinder in C' (or C). □

Lemma 5.3.15. *Let $\mathcal{M}$ be a rank 2 invariant subvariety in $\mathcal{H}(2, 1^2)$. If M is a cylindrically stable surface in $\mathcal{M}$, it cannot have cylinder diagrams:*

9. *(0,2)-(6) (1,4)-(3,5) (3,6)-(2,4) (5)-(0,1)*
15. *(0,3,1)-(6) (2)-(3) (4,6)-(0,5,1) (5)-(2,4)*
18. *(0,2,1)-(5,6) (3,4)-(0,2,1) (5)-(4) (6)-(3)*
20. *(0,2,1)-(5,6) (3,6)-(0,4,1) (4)-(3) (5)-(2)*
22. *(0,2,1)-(6) (3,6)-(4,5) (4)-(3) (5)-(0,2,1)*

Proof. We check each cylinder diagram separately.

9. Assume by contradiction that M has cylinder diagram 9. By Lemma 5.3.10, C_0, C_3 cannot be free. All of the partitions with three classes in Appendix B.2.3 have that C_0 or C_3 is free, so M cannot have 3 horizontal $\mathcal{M}$-parallel classes. By Lemma 5.2.3, $\mathcal{C}_1 = \{C_0, C_3\}$ and $\mathcal{C}_2 = \{C_1, C_2\}$ must be the horizontal $\mathcal{M}$-parallel classes of M. $\mathcal{M}$ cannot be rel 2 since all cylinders would be free, so by Lemma 2.3.7, M would not be cylindrically stable. First assume $\mathcal{M}$ is rel 1. $\ker p \cap T_M\mathcal{M}$ is generated by $\gamma_0^* + \gamma_1^* - \gamma_2^* - \gamma_3^* \in \mathrm{Tw}\, M$. Let $\sigma_{\mathcal{C}_2} = h_1\gamma_1^* + h_2\gamma_2^* \in \mathrm{Tw}\, M$, $h_1, h_2 > 0$. A linear combination of these vectors is $\gamma_0^* + (1 + \frac{h_1}{h_2})\gamma_1^* - \gamma_3^* \in \mathrm{Tw}\, M$. Thus, we can collapse a cross curve of $\mathcal{C}_2$, which gives a rank 2 invariant subvariety in $\mathcal{H}(3, 1)$, which is not possible.

   Thus, $\mathcal{M}$ can only be rel 0. Thus, any collapse of $\mathcal{M}$ must be rank 1. Label the saddle connections $s_0, \dots, s_6$ based on the labeling from the cylinder diagram. Let p_0 be the singularity on the left of s_0 and p_1 the singularity on the right of s_0. p_2 is the degree 2 singularity, see Figure 5.5. Assume there is a vertical saddle α between p_0 and p_2 inside C_0. If there is no vertical saddle in C_3 generically parallel to α, then we can find a nearby surface $\tilde{M}$ where α is the only vertical saddle in $\mathcal{C}_2$. Then $\mathrm{Col}_{\mathcal{C}_2} \tilde{M}$ is a rank 2 invariant subvariety by Lemma 5.2.7 and has lower dimension than $\mathcal{M}$, which is not possible. There is a similar contradiction if there is a vertical saddle between p_1 and p_2 in C_3 that is generically parallel to α. Thus, there must be a vertical saddle between p_0 and p_2 generically parallel to α. Now we twist $\mathcal{C}_2$ slightly by $\sigma_{\mathcal{C}_2}$ so there are no vertical saddle connections in C_0 or C_3. s_1 lies directly over the singularity in C_3 and s_0 lies directly under the singularity in C_0. We would like to overcollapse $\mathcal{C}_2$, but C_0 and C_3 are adjacent, so we must be careful. As in Figure 5.6, we can find a period coordinate chart that contains the whole overcollapse and use it to compute the cylinder diagram of the resulting surface M', which is cylinder diagram 7. C_1, C_2 persist on M' and remain $\mathcal{M}$-parallel. There are cylinders C_0', C_3' that correspond to C_0, C_3. Let γ_i, γ_i'

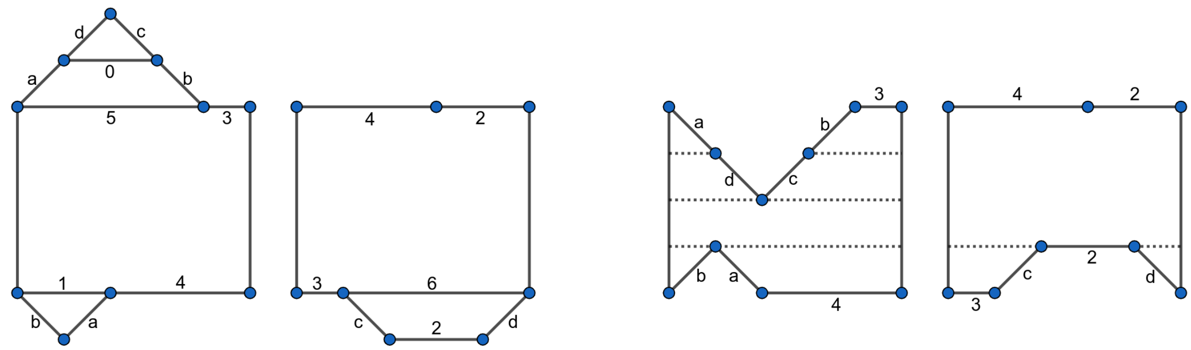

Figure 5.6: Left: A period coordinate chart that contains the whole cylinder overcollapse. Right: After the overcollapse. We now compute the cylinder diagram of the new surface.

be the core curves of C_i, C_i' respectively. Then, $\gamma_3' = 2\gamma_1 - \gamma_3$ and $\gamma_0' = \gamma_1 + \gamma_2 - \gamma_0$, so C_0', C_3' are not $\mathcal{M}$-parallel to C_1, C_2. Thus, C_0', C_3' are in an $\mathcal{M}$-parallel class on M'. We also see that M' is cylindrically stable since it has two $\mathcal{M}$-parallel classes. By Lemma 5.3.14, a surface with cylinder diagram 7 and this partition cannot exist in $\mathcal{M}$.

15. Assume by contradiction M has cylinder diagram 15. We have the relation $\gamma_0 + \gamma_3 = \gamma_1 + \gamma_2$, so we can apply Lemma 5.2.3. We see that $\Phi(\gamma_0) \neq \Phi(\gamma_1)$ and $\Phi(\gamma_1) \neq \Phi(\gamma_3)$, so no partition with two $\mathcal{M}$-parallel classes is possible. Checking Appendix B.2.3, we see that $\{0\}, \{1,2\}, \{3\}$ and $\{0,3\}, \{1\}, \{2\}$ are the only possible partitions with three $\mathcal{M}$-parallel classes. However, this is a contradiction since both of these are not possible by Lemma 5.3.10.

18. Assume by contradiction that M has cylinder diagram 18. By Appendix B.2.3, the $\mathcal{M}$-parallel classes must be $\{\{0,1\}, \{2\}, \{3\}\}$. Collapsing C_2 we get M' with cylinder diagram (0,2,1)-(6,4) (3,4)-(1,2,0) (6,3) in a rank 2 invariant variety in $\mathcal{M}' \subset \mathcal{H}(2^2)$. $\mathcal{M}' = \widetilde{\mathcal{Q}}(1^2, -1^2)$ or $\mathcal{M}' \subset \widetilde{\mathcal{Q}}(4, -1^4)$, so M' must have a Prym involution. However, the only 180 involution on M' is a hyperelliptic involution by Lemma 5.1.8.

20. Assume by contradiction that M has cylinder diagram 20. Neither C_2 nor C_3 is free by Lemma 5.3.10. By Appendix B.2.3, M cannot have three cylinder equivalence classes. By Lemma 5.2.3, $\mathcal{C}_1 = \{C_0, C_1\}$ and $\mathcal{C}_2 = \{C_2, C_3\}$ must be the $\mathcal{M}$-parallel classes. Twist $\mathcal{C}_2$ so that C_2 has a vertical saddle connection α. Now we perturb M, so that α is only parallel to saddles that it is $\mathcal{M}$-parallel to. Then $\mathcal{M}' := \mathrm{Col}_{\mathcal{C}_2} \mathcal{M}$, is a rank 2 invariant subvariety by Lemma 5.2.7 that has one lower dimension than $\mathcal{M}$. Thus, $\mathcal{M}$ must have rel ≥ 1. $\mathcal{M}$ cannot be rel 2 by Lemma 2.3.7, so $\mathcal{M}$ is rank 2 rel 1. $\ker p \cap T_M\mathcal{M}$ is generated by $\gamma_0^* + \gamma_2^* - \gamma_1^* - \gamma_3^* \in \mathrm{Tw}\, M$. Let $\sigma_{\mathcal{C}_2} = h_0\gamma_0^* + h_1\gamma_1^* \in \mathrm{Tw}\, M$, $h_0, h_1 > 0$. A linear combination of these vectors is $(1 + \frac{h_0}{h_1})\gamma_0^* + \gamma_2^* - \gamma_3^* \in \mathrm{Tw}\, M$. Thus, we can collapse a cross curve of C_3,

which gives a rank 2 invariant subvariety in $\mathcal{H}(3,1)$. This is a contradiction, so M does not have cylinder diagram 20.

22. Now assume by contradiction M has cylinder diagram 22. Checking Appendix B.2.3, we see that the only possible partition is $\{0,3\},\{1\},\{2\}$. Collapsing C_2, we get M' with cylinder diagram (0,2,1)-(6) (3,6)-(3,5) (5)-(0,2,1) in a rank 2 invariant subvariety $\mathcal{M}' \subset \mathcal{H}(2^2)$. M' has a Prym involution that swaps the singularities of M', so it is contained in $\widetilde{\mathcal{Q}}(4,-1^4)$. Thus, $\mathcal{M}' = \widetilde{\mathcal{Q}}(4,-1^4)$ or $\widetilde{\mathcal{H}}^{odd}(2)$. All surfaces in $\widetilde{\mathcal{H}}^{odd}(2)$ are hyperelliptic, but by Lemma 5.1.8 M' has a unique 180 degree involution, which is the Prym involution, so $M' \notin \widetilde{\mathcal{H}}^{odd}(2)$. Thus, $\mathcal{M}' = \widetilde{\mathcal{Q}}(4,-1^4)$ and $\mathcal{M}$ must be rel 2, but a horizontally periodic surface with four horizontal cylinders cannot be cylindrically stable in an rank 2 rel 2 invariant subvariety by Lemma 2.3.7. This is a contradiction.

□

Proposition 5.3.16. *There does not exist a rank* 2 *rel* 0 *invariant subvariety* $\mathcal{M} \subset \mathcal{H}(2,1^2)$ *such that every cylindrically stable* $M \in \mathcal{M}$ *has four cylinders.*

Proof. Assume by contradiction that $\mathcal{M}$ is a rank 2 rel 0 invariant subvariety in $\mathcal{H}(2,1,-1^2)$ such that every cylindrically stable $M \in \mathcal{M}$ has four cylinders. If all cylindrically stable surfaces $M \in \mathcal{M}$ have a free cylinder, then $\mathcal{M}$ is a quadratic double by Corollary 5.1.11, but the only quadratic double in $\mathcal{H}(2,1^2)$ is $\widetilde{\mathcal{Q}}(2,1,-1^3)$, which is rel 1. Thus, we consider M such that there does not exist a cylinder in an $\mathcal{M}$-parallel class by itself. Looking at the output of the program in Appendix B.2.3.5, as well as Proposition 5.3.12, Lemma 5.3.15, and Lemma 5.3.14, the remaining cases are cylinder diagrams 8 and 19.

First assume that M has cylinder diagram 8. We are assuming no horizontal cylinder is in an $\mathcal{M}$-parallel class by itself, and C_2, C_3 are homologous, so they are in an $\mathcal{M}$-parallel class. Thus, C_0, C_1 are the other $\mathcal{M}$-parallel class. C_0 contains a nested cylinder D_0 so by Corollary 2.2.13 it must be $\mathcal{M}$-parallel to the nested cylinder D_1 in C_1. We may collapse $\{D_0, D_1\}$ to get M' in $\mathcal{M}' \subset \mathcal{H}(2,0^2)$. M' has cylinder diagram (2)-(5) (3)-(6) (4,5)-(3) (6)-(2,4). Let C_i' be the cylinder on M' corresponding to C_i on M, and let γ_i' be the core curve of C_i'. There are twists of the form $\beta_1 = a_0\gamma_0' + a_1\gamma_1'$ for $a_0, a_1 > 0$ and $\beta_2 = a_2\gamma_2' + a_3\gamma_3'$ for $a_2, a_3 > 0$ in $\operatorname{Tw} M'$. Since $\mathcal{M}$ is rank 2 rel 0, $\mathcal{M}'$ must be rank 1. Thus, $\operatorname{Tw} M'$ is generated by the standard twist σ and $\ker p$. Let $\operatorname{Tw}_i M$ be the imaginary part of $\operatorname{Tw} M$ i.e. $\operatorname{Tw}_i M = iH^1(M,\Sigma;\mathbb{R}) \cap \operatorname{Tw} M$. Perturbing the periods of M' by a small multiple of $i\sigma$ changes the area of M' while $\operatorname{rel} p \cap \operatorname{Tw}_i M$ preserves area. Since these generate $\operatorname{Tw}_i M$, an element $\alpha \in \operatorname{Tw}_i M$ is in $\operatorname{rel} p$ if and only if α preserves the area of M'. However, there is a linear combination $\beta_1 + b\beta_2$ that preserves the area of M' but cannot be in $\operatorname{rel} p$ since it changes the period of a cross curve of C_0', which is in absolute homology. This is a contradiction. Thus, M cannot have cylinder diagram 8.

The remaining case is that M has cylinder diagram 19 such that $\mathcal{C}_1 = \{C_0, C_1\}$ and $\mathcal{C}_2 = \{C_2, C_3\}$ are the two $\mathcal{M}$-parallel classes. Overcollaping M along $\mathcal{C}_2$ gives a surface M' with cylinder diagram 23. C_0, C_1 are still present on M' and are $\mathcal{M}$-parallel. There are there are horizontal cylinders C_2', C_3' on M' corresponding to C_2, C_3 respectively. Let γ_i, γ_i' be the core curve of C_i, C_i respectively. We have that $\gamma_2' = \gamma_0 + \gamma_1 - \gamma_2$ and $\gamma_3' = \gamma_0 + \gamma_1 - \gamma_3$, so C_2', C_3' are not $\mathcal{M}$-parallel to C_0, C_1. Since $\mathcal{M}$ is rank 2 rel 0, there are two $\mathcal{M}$-parallel classes $\{C_0, C_1\}$ and $\{C_2, C_3\}$ on M', and M' is cylindrically stable. This is not possible by Lemma 5.3.14. □

Lemma 5.3.17. *Each of the following cylinder diagrams has a generalized pair of pants.*

1. *(0)-(1) (1,3,4,2)-(5,6) (5)-(0,4) (6)-(2,3)**
2. *(0)-(1,2) (1,4,2,3)-(5,6) (5)-(4) (6)-(0,3)*
3. *(0)-(3) (1,3,2,4)-(5,6) (5)-(4) (6)-(0,2,1)*
4. *(0)-(3) (1,4,2,3)-(5,6) (5)-(4) (6)-(0,2,1)*
5. *(0,1)-(0,2,5) (2)-(3) (3,6)-(1,4) (4,5)-(6)**
6. *(0,2)-(5) (1,3)-(6) (4)-(0,1) (5,6)-(2,4,3)**

10. *(0,1)-(0,2) (2,4)-(6) (3,6)-(1,5,4) (5)-(3)*
11. *(0,1)-(0,3,5,6) (2,4)-(1) (3,6)-(4) (5)-(2)**
13. *(0,4,1)-(6) (2)-(3) (3,6)-(4,5) (5)-(0,2,1)**
14. *(0,3,1)-(6) (2)-(4) (4,6)-(0,5,1) (5)-(2,3)**
16. *(0,1,2)-(3,6,4,5) (3,4)-(2) (5)-(0) (6)-(1)*
17. *(0,1,2)-(0,1,6) (3,6)-(4,5) (4)-(3) (5)-(2)*
21. *(0,4,2)-(5,6) (1,3)-(0,1,2) (5)-(4) (6)-(3)*
25. *(0,1,2)-(0,1,5,6) (3,4)-(2) (5)-(4) (6)-(3)**
26. *(0,1,2)-(0,5,1,6) (3,4)-(2) (5)-(3) (6)-(4)**
27. *(0,2,1)-(6) (3,6)-(0,5,4,1) (4)-(3) (5)-(2)*

For all cylinder diagrams not marked with an asterisk, the program tells us that this cylinder diagram is not possible.

Proof. The pair of pants of each cylinder diagram can be found manually. We can check Appendix B.2.3 for the output of the program. □

Lemma 5.3.18. *Let $\mathcal{M}$ be a rel 1 invariant subvariety in $\mathcal{H}(2,1^2)$. For the cylinder diagrams in Lemma 5.3.13,* $\ker p \cap T_M\mathcal{M}$ *is spanned by $\gamma_i^* - \gamma_j^*$, where C_i and C_j are homologous cylinders. For the cylinder diagrams in Lemma 5.3.17,* $\ker p \cap T_M\mathcal{M}$ *is spanned by $\gamma_i^* - \gamma_j^* - \gamma_k^*$, where C_i, C_j, C_k form a pants.*

Proof. Checking Appendix B.1.3, for all cylinder diagrams in Lemma 5.3.13 and Lemma 5.3.17, $p(\operatorname{Span}\gamma_i)$ is three dimensional, so $\ker p \cap \operatorname{Span}\gamma_i$ is one dimensional. Since M is cylindrically stable, $\ker p \cap T_M\mathcal{M} \subset \operatorname{Tw} M \subset \operatorname{Span}\gamma_i$, so $\ker p \cap T_M\mathcal{M} \subset \ker p \cap \operatorname{Span}\gamma_i$, and $\ker p \cap T_M\mathcal{M}$ is one dimensional since $\mathcal{M}$ is rel 1. Thus, $\ker p \cap T_M\mathcal{M} = \ker p \cap \operatorname{Span}\gamma_i$. Lemma 5.3.13 gives that $p(\gamma_i^*) - p(\gamma_j^*) = 0$ if C_i and C_j are homologous cylinders, so $\gamma_i^* - \gamma_j^*$ generates $\ker p \cap T_M\mathcal{M}$. The argument is similar for Lemma 5.3.17. □

Proposition 5.3.19. *There does not exist a rank* 2 *invariant subvariety in $\mathcal{H}(2,1^2)$ such that every cylindrically stable $M \in \mathcal{M}$ has four cylinders.*

Proof. By Proposition 5.3.16, it suffices to prove no rank 2 rel 1 invariant subvarieties in $\mathcal{H}(2,1^2)$ exist. Assume by contradiction $\mathcal{M}$ is a rank 2 rel 1 invariant subvariety in $\mathcal{H}(2,1^2)$. By Lemma 5.3.15, it remains to consider the cases when M has a cylinder diagram in Lemma 5.3.13 or Lemma 5.3.17.

Case 1: M has a cylinder diagram from Lemma 5.3.17.
The starred cylinder diagrams have been already been ruled out by the program. If M has cylinder diagram 11, $\gamma_1 - \gamma_2 - \gamma_3 \in \operatorname{Tw} M$ by Lemma 5.3.18, so we can collapse C_1. This gives a rank 2 invariant subvariety in $\mathcal{H}(3,1)$, which does not exist. Similarly, if M has cylinder diagram 26, we can also collapse C_1 to get a rank 2 invariant subvariety in $\mathcal{H}(3,1)$, which does not exist. If M has cylinder diagram 5 or 25, M has a cylinder nested in a free cylinder, so it must also be free. The singularities on each side of the cylinder have different degree, so this contradicts Lemma 5.3.10.

Now we have the following overcollapse computations:

1. If M has cylinder diagram 1, we overcollapse C_0 to get a surface with cylinder diagram 11.
2. If M has cylinder diagram 6, we overcollapse C_2 through s_0 to get a surface with cylinder diagram 5.
3. If M has cylinder diagram 13, we overcollapse C_3 through s_0 to get a surface with cylinder diagram 21.

4. If M has cylinder diagram 14, we overcollapse C_3 through s_2 to get a surface with cylinder diagram 13.

By Lemma 5.2.9, each of these collapses produces a cylindrically stable surface. $\mathcal{M}$ does not have a cylindrically stable surface with cylinder diagrams 5, 11, or 21, so it cannot have a cylindrically stable surface with cylinder diagrams 1, 6, or 13. Thus, it also cannot have a cylindrically stable surface with cylinder diagram 14. This covers all cylinder diagrams of Lemma 5.3.17.

<u>Case 2:</u> M has a cylinder diagram from Lemma 5.3.13.
We now claim that M cannot have one of the following cylinder diagrams.

8. (0,2)-(0,5) (1,3)-(1,6) (4,5)-(3) (6)-(2,4)

12. (0,1,2)-(0,1,6) (3)-(5) (4,5)-(2) (6)-(3,4)

23. (0,1,4)-(0,1,6) (2,3)-(2,5) (5)-(4) (6)-(3)

24. (0,5,2)-(3) (1,3)-(1,6) (4)-(5) (6)-(0,4,2)

The above cylinder diagrams use a similar argument, so we will only give the proof for cylinder diagram 8. Assume by contradiction M has cylinder diagram 8. C_2 and C_3 are homologous cylinders, so by Lemma 5.3.18, $\gamma_2^* - \gamma_3^* \in T_M\mathcal{M}$. Thus, we may collapse C_3 by increasing the height of C_2. The singularities on the top of C_3 are degree 1 and the singularity on the bottom of C_3 is degree 2, so this would create a rank 2 invariant subvariety in $\mathcal{H}(3,1)$ which is a contradiction.

Now assume M has cylinder diagram 7: (0,1)-(0,3) (2,5)-(1,6) (3,6)-(4,5) (4)-(2). The two possible partitions are $\{\{0\},\{1,2,3\}\}$ and $\{\{0,1,2\},\{3\}\}$. In the first case, C_0 has a nested free cylinder. Collapsing the nested free cylinder, we get a surface M' that is either a rank 2 rel 0 invariant subvariety or a rank 1 rel 2 invariant subvariety $\mathcal{M}'$ in $\mathcal{H}(2^2)$. Since $\mathcal{H}(2^2)$ has at most one rel, $\mathcal{M}'$ must be rank 2 rel 0. The rank 2 rel 0 invariant subvarieties in $\mathcal{H}(2^2)$ are $\widetilde{\mathcal{H}}^{odd}(2) \subset \widetilde{\mathcal{Q}}(4,-1^4)$ and $\widetilde{\mathcal{H}}^{even}(2) = \widetilde{\mathcal{Q}}(1^2,-1^2)$, which both have a Prym involution with four fixed points. The Prym involution on M' swaps the two singularities, so $\mathcal{M} = \widetilde{\mathcal{H}}^{odd}(2)$. Let C_i' be the cylinder of M' corresponding to C_i. The Prym involution swaps C_0' and C_3' so they must be in one $\mathcal{M}'$-parallel class and C_1' and C_2' are the other. This contradicts Lemma 5.2.8. The remaining case is M has partition $\{\{0,1,2\},\{3\}\}$. Then, we overcollapse C_3 to get a surface M' with cylinder diagram 8. M' is cylindrically stable by Lemma 5.2.9, but we have already shown that such a surface cannot exist on $\mathcal{M}$.

Finally, assume M has cylinder diagram 19: (0,2,1)-(5,6) (3,6)-(0,4,1) (4)-(2) (5)-(3). The only possible partitions are $\{\{0,1,3\},\{2\}\}$ and $\{\{0,1,2\},\{3\}\}$. These cases we can overcollapse the free cylinder to get a cylinder diagram 12 and 24 respectively, and the resulting surface is cylindrically stable by Lemma 5.2.9. We already determined that $\mathcal{M}$ cannot have a cylindrically stable surface with either of these cylinder diagrams. We have now covered all four cylinder diagrams in $\mathcal{H}(2, 1^2)$.

$\square$

Proof of Theorem 5.3.9. Let $\mathcal{M}$ be a rank 2 invariant subvariety in $\widetilde{\mathcal{H}}(2, 1^2)$. By Proposition 5.3.19, $\mathcal{M}$ must contain at least one five-cylinder diagram, M. By Proposition 5.3.12, if M has a cylinder diagram listed in that lemma, then $\mathcal{M}$ is $\widetilde{Q}(2, 1, -1^3)$. Now we go over the other cylinder diagrams.

1. (0)-(2) (1)-(3) (2,4,3)-(5,6) (5)-(4) (6)-(0,1)

   By Appendix B.2.3, the only possible partition is $\{\{0, 1, 4\}, \{2\}, \{3\}\}$, but C_3 satisfies the hypothesis of Lemma 5.3.10, so it cannot be free.

2. (0,4)-(6) (1)-(0) (2)-(3) (3,6)-(4,5) (5)-(1,2)

   By Appendix B.2.3, the only possible partition are $\{\{0, 1\}, \{2\}, \{3, 4\}\}$ and $\{\{0, 4\}, \{1, 3\}, \{2\}\}$. We have the homology relation $[\gamma_0] + [\gamma_4] = [\gamma_1] + [\gamma_3]$, so by Lemma 5.2.3, either $\Phi(\gamma_1) = \Phi(\gamma_0)$ or $\Phi(\gamma_1) = \Phi(\gamma_4)$, which are both not possible.

4. (0,2)-(6) (1)-(0) (3,6)-(4,5) (4)-(3) (5)-(1,2)

   By Appendix B.2.3, there are no possible partitions.

6. (0,1)-(0,2) (2)-(3) (3,4)-(5,6) (5)-(4) (6)-(1)

   By Appendix B.2.3, there are no possible partitions.

$\square$

APPENDIX A

Minimal Sets

In this section, we prove Theorem 2.2.2, which is a multicomponent version of [SW04, Corollary 6]. Most of the work will be adapting [MW02, Corollary 2.7] to the multicomponent setting. Let $\mathcal{Q}$ be a stratum of multicomponent quadratic differentials and $\mathcal{Q}_1$ the unit area locus of $\mathcal{Q}$. Define $h_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$, and $H_t = \{h_t, t \in \mathbb{R}\}$.

Theorem A.0.1. *Every closed H_t-invariant set of $\mathcal{Q}_1$ contains a minimal closed H_t-invariant set. A minimal closed H_t-invariant set is compact.*

The proof of Theorem A.0.1 will be broken down into Proposition A.0.4 and Proposition A.0.7, which depend on the following nondivergence result. Let

$$\mathrm{Avg}_{T,q}(K) := \frac{|\{t \in [0,T] : u_t q \in K\}|}{T}$$

Theorem A.0.2. *For any $\epsilon > 0$ and $\eta > 0$, there is a compact subset K such that for any $q \in \mathcal{Q}_1$, one of the following statements holds:*

1. $\liminf_{T\to\infty} \mathrm{Avg}_{T,q}(K) \geq 1 - \epsilon$.
2. *q contains a horizontal saddle connection of length less than η.*

Theorem A.0.2 is very similar to but more general than Theorem H2 from [MW02]. Theorem A.0.2 is not stated in this way in [MW02], but this formulation is more coninvient for our purposes. The proof of this theorem follows exactly the proof of Theorem H2 in the single component case, and we copy it here for the coninvience of the reader. It mainly uses Theorem 6.3 of [MW02], which we prove for the multicomponent case, assuming the single component case.

Let $\widetilde{\mathcal{Q}}$ be the marked quadratic differentials that cover $\mathcal{Q}$. Let $\widetilde{\mathcal{Q}}_1$ be the unit area locus of $\widetilde{\mathcal{Q}}$. For $q \in \widetilde{\mathcal{Q}}$, let $\mathcal{L}_q$ be the set of saddle connections on q. Let $l_{q,\delta}(t)$ be the length of the saddle δ on $u_t q$, and $\|\cdot\|_I$ the L^∞-norm on an interval $I \subset \mathbb{R}$. Let $\alpha_q(t)$ be the length of the shortest saddle connection on $u_t q$.

Theorem A.0.3. *There are positive constants C, α, ρ_0, depending only on the genus, such that if $q \in \widetilde{Q}_1$, an interval $I \subset \mathbb{R}$, and $0 < \rho' \leq \rho_0$, satisfy:*

$$\text{for any } \delta \in \mathcal{L}_q, \|l_{q,\delta}\|_I \geq \rho',$$

then for any $0 < \epsilon < \rho'$ we have:

$$|\{t \in I : \alpha_q(t) < \epsilon\}| \leq C \left(\frac{\epsilon}{\rho'}\right)^\alpha |I|$$

Proof. Let $q = (q_1, \ldots, q_n)$ be a multicomponent quadratic differential. The condition $\forall\delta \in \mathcal{L}_q, \|l_{q,\delta}\|_I \geq \rho'$ implies this condition holds on each component. By the single component version of this theorem [MW02, Theorem 6.3], $|\{t \in I : \alpha_{q_i}(t) < \epsilon\}| \leq C_i \left(\frac{\epsilon}{\rho'}\right)^{\alpha_i} |I|$ and for some $\rho' < \min(\rho_{0,1}, \ldots, \rho_{0,n})$ for all i. Thus,

$$|\{t \in I : \alpha_q(t) < \epsilon\}| = |\bigcup_i \{t \in I : \alpha_{q_i}(t) < \epsilon\}| \leq n \max_i C_i \left(\frac{\epsilon}{\rho'}\right)^{\min_i \alpha_i} |I|$$

□

Proof of Theorem A.0.2. Let C, α, ρ_0 be the constants from Theorem A.0.3. For fixed ϵ and η, choose ϵ' small enough so that

$$C \left(\frac{\epsilon'}{\rho_0}\right)^\alpha < \epsilon, \epsilon' < \rho_0, \text{ and } \epsilon' < \eta.$$

Let K be the sets of surfaces in Q_1 with no saddle of length less than ϵ'. This is compact by Masur's Compactness Criterion. Let $q \in \widetilde{Q}$, and suppose q does not contain a horizontal saddle of length less than η. Let $\rho = \min(\rho_0, \eta)$. The set

$$\mathcal{L}_0 = \{\delta \in \mathcal{L}_q : l_{q,\delta}(0) < \rho\}$$

is finite by [MW02, Proposition 4.8]. Since we are assuming none of the functions $t \mapsto l_{q,\delta}(t)$ are constant for $\delta \in \mathcal{L}_0$, they diverge by [MW02, Lemma 4.4]. Thus there is some T_0 such that for all $\delta \in \mathcal{L}_0, l_\delta(T_0) \geq \rho$. For any $T \geq T_0$ we can apply Theorem A.0.3 with $I = [0, T]$ and $\rho' = \rho$, and obtain that

$$\mathrm{Avg}_{T,q}(Q_1 - K) < \epsilon.$$

□

Proposition A.0.4. *Let Q_1 be a stratum of unit-area multi-component quadratic differentials. Every* *closed H_t-invariant set $X \subset \mathcal{Q}_1$ contains a minimal closed H_t-invariant set.*

Lemma A.0.5. *Let X be a closed invariant set such that*

$$\rho := \inf\{l_{q,\delta}(0) : q \in X, \delta \text{ is a horizontal saddle connection on } q\} > 0.$$

Then X contains a minimal closed invariant set.

Proof. Choose any $0 < \epsilon < 1$ and $0 < \eta < \rho$ and let K be the compact set obtained from Theorem A.0.2. Then K intersects $U_t q$ for all $q \in X$. Let $\{X_\alpha\}$ be any totally ordered family of closed invariant subsets of X. Any finite intersection is nonempty $K \cap X_{\alpha_1} \cap \cdots \cap X_{\alpha_i}$ since every H_t trajectory meets K. Thus, $K \cap \bigcap_\alpha X_\alpha$ is nonempty by compactness. By Zorn's lemma, X contains a minimal closed invariant set. □

Proof of Proposition A.0.4. We must prove that X contains a set X_0 that satisfies the hypotheses of Lemma A.0.5. Let $q \in X$ have the maximal number of horizontal saddle connections, and let $X_0 = \overline{U_t q}$. For any $q' \in X_0$, the set of horizontal saddles on q is isometric to a set of horizontal saddles on q'. Since q has the maximal number of horizontal saddles, it cannot have more horizontal saddles. Thus, every surface in X_0 has the same horizontal saddles, so they have the same value of ρ. □

We do not include a proof of the following lemma which is Lemma 7.2 in [MW02].

Lemma A.0.6. *Let $\{T_t\}$ be an action of $\mathbb{R}$ by homeomorphisms on a locally compact space Z. Suppose there is a compact $K \subset Z$ such that for every $z \in Z$, the subsets $\{t \geq 0 : T_t z \in K\}$ and $\{t \leq 0 : T_t z \in K\}$ are unbounded. Then Z is compact.*

Proposition A.0.7. *If a minimal H_t-invariant set exists, then it is compact.*

Proof. By Lemma A.0.6 is suffices to prove that $\{t \geq 0 : u_t q \in X\}$ and $\{t \leq 0 : u_t q \in X\}$ are dense in X. We prove this for $\{t \geq 0 : u_t q \in X\}$ as $\{t \leq 0 : u_t q \in X\}$ is similar. Define $X^+(q)$ to be the limit points of $\{t \geq 0 : u_t q \in X\}$, so it is a closed H_t-invariant set. It is nonempty by Theorem A.0.2. Since X is minimal, $X^+(q) = X$. Thus, $\{t \geq 0 : u_t q \in X\}$ is dense in X. □

Proof of Theorem 2.2.2. By Theorem A.0.1, a H_t-orbit closure contains a minimal closed H_t-invariant set X, and X is compact. Let $q = (q_1, \ldots, q_n) \in X$, so $\overline{U_t q} \subset X$. Let proj_i be the projection onto the i-th component. $\overline{U_t q_i}$ is contained in a compact set, namely $\text{proj}_i(X)$, so by [SW04, Theorem 5], q_i is horizontally periodic. This argument does not depend on i, so every component of q is horizontally periodic. □

APPENDIX B

Computer Output

This appendix contains all relevant computer output for Chapter 5.

B.1 All Cylinder Diagrams

For the convinience of the reader, this section lists all cylinder diagrams of horizontally periodic surfaces with at least four cylinder diagrams in the strata $\mathcal{H}(3,1), \mathcal{H}^{odd}(2^2), \mathcal{H}(2,1^2)$.

B.1.1 $\mathcal{H}(3,1)$

B.1.1.1 Four Cylinders

```
1. (0)-(2) (1,2,3)-(4,5) (4)-(3) (5)-(0,1)
2. (0,3)-(5) (1)-(2) (2,5)-(3,4) (4)-(0,1)
3. (0,1)-(0,2) (2)-(3) (3,4)-(1,5) (5)-(4)
4. (0,1)-(0,4,5) (2,3)-(1) (4)-(2) (5)-(3)
```

B.1.2 $\mathcal{H}^{odd}(2,2)$

B.1.2.1 Four Cylinders

```
1. (0,3)-(5) (1)-(0) (2,5)-(3,4) (4)-(1,2)
2. (0,1)-(0,5) (2)-(4) (3,4)-(1) (5)-(2,3)
3. (0,5)-(3,4) (1,4)-(2,5) (2)-(0) (3)-(1)
4. (0,3)-(0,5) (1,2)-(1,4) (4)-(3) (5)-(2)
5. (0,2,1)-(3,4,5) (3)-(1) (4)-(2) (5)-(0)
```

B.1.3 $\mathcal{H}(2,1,1)$

B.1.3.1 Five Cylinders

1. (0)−(2) (1)−(3) (2,4,3)−(5,6) (5)−(4) (6)−(0,1)
2. (0,4)−(6) (1)−(0) (2)−(3) (3,6)−(4,5) (5)−(1,2)
3. (0,1)−(0,6) (2)−(5) (3)−(4) (4,5)−(1) (6)−(2,3)
4. (0,2)−(6) (1)−(0) (3,6)−(4,5) (4)−(3) (5)−(1,2)
5. (0,2)−(6) (1)−(3) (3,6)−(4,5) (4)−(0) (5)−(1,2)
6. (0,1)−(0,2) (2)−(3) (3,4)−(5,6) (5)−(4) (6)−(1)
7. (0,6)−(4,5) (1,2)−(3,6) (3)−(2) (4)−(1) (5)−(0)
8. (0,6)−(4,5) (1,2)−(3,6) (3)−(0) (4)−(2) (5)−(1)

B.1.3.2 Four Cylinders

For four cylinder diagrams in $\mathcal{H}(2, 1^2)$ only, next to each cylinder diagram, we also include the dimension of the span of the core curves of all horizontal cylinders see Remark 5.2.12.

1. (0)−(1) (1,3,4,2)−(5,6) (5)−(0,4) (6)−(2,3) 3
2. (0)−(1,2) (1,4,2,3)−(5,6) (5)−(4) (6)−(0,3) 3
3. (0)−(3) (1,3,2,4)−(5,6) (5)−(4) (6)−(0,2,1) 3
4. (0)−(3) (1,4,2,3)−(5,6) (5)−(4) (6)−(0,2,1) 3
5. (0,1)−(0,2,5) (2)−(3) (3,6)−(1,4) (4,5)−(6) 3
6. (0,2)−(5) (1,3)−(6) (4)−(0,1) (5,6)−(2,4,3) 3
7. (0,1)−(0,3) (2,5)−(1,6) (3,6)−(4,5) (4)−(2) 3
8. (0,2)−(0,5) (1,3)−(1,6) (4,5)−(3) (6)−(2,4) 3
9. (0,2)−(6) (1,4)−(3,5) (3,6)−(2,4) (5)−(0,1) 3
10. (0,1)−(0,2) (2,4)−(6) (3,6)−(1,5,4) (5)−(3) 3
11. (0,1)−(0,3,5,6) (2,4)−(1) (3,6)−(4) (5)−(2) 3
12. (0,1,2)−(0,1,6) (3)−(5) (4,5)−(2) (6)−(3,4) 3
13. (0,4,1)−(6) (2)−(3) (3,6)−(4,5) (5)−(0,2,1) 3
14. (0,3,1)−(6) (2)−(4) (4,6)−(0,5,1) (5)−(2,3) 3
15. (0,3,1)−(6) (2)−(3) (4,6)−(0,5,1) (5)−(2,4) 3
16. (0,1,2)−(3,6,4,5) (3,4)−(2) (5)−(0) (6)−(1) 3
17. (0,1,2)−(0,1,6) (3,6)−(4,5) (4)−(3) (5)−(2) 3
18. (0,2,1)−(5,6) (3,4)−(0,2,1) (5)−(4) (6)−(3) 2
19. (0,2,1)−(5,6) (3,6)−(0,4,1) (4)−(2) (5)−(3) 3
20. (0,2,1)−(5,6) (3,6)−(0,4,1) (4)−(3) (5)−(2) 3
21. (0,4,2)−(5,6) (1,3)−(0,1,2) (5)−(4) (6)−(3) 3
22. (0,2,1)−(6) (3,6)−(4,5) (4)−(3) (5)−(0,2,1) 2
23. (0,1,4)−(0,1,6) (2,3)−(2,5) (5)−(4) (6)−(3) 3
24. (0,5,2)−(3) (1,3)−(1,6) (4)−(5) (6)−(0,4,2) 3

```
25. (0,1,2)-(0,1,5,6) (3,4)-(2) (5)-(4) (6)-(3) 3
26. (0,1,2)-(0,5,1,6) (3,4)-(2) (5)-(3) (6)-(4) 3
27. (0,2,1)-(6) (3,6)-(0,5,4,1) (4)-(3) (5)-(2) 3
```

B.2 Valid Equivalence Classes

This section contains the output of the computations from Section 5.2.4. We number all cylinder diagrams as in Appendix B.1. Under each cyliner diagram, we list the sets of $\mathcal{M}$-parallel classes of cylinders that have not been ruled out by the program. All cylinders are labeled from left to right starting from 0.

B.2.1 $\mathcal{H}(3,1)$

B.2.1.1 4 Cylinders, 2 Equivalence Classes

```
1. (0)-(2) (1,2,3)-(4,5) (4)-(3) (5)-(0,1)
[]
2. (0,3)-(5) (1)-(2) (2,5)-(3,4) (4)-(0,1)
[{{0, 1, 2}, {3}}]
3. (0,1)-(0,2) (2)-(3) (3,4)-(1,5) (5)-(4)
[]
4. (0,1)-(0,4,5) (2,3)-(1) (4)-(2) (5)-(3)
[{{0}, {1, 2, 3}}]
```

B.2.1.2 4 Cylinders, 3 Equivalence Classes

```
1. (0)-(2) (1,2,3)-(4,5) (4)-(3) (5)-(0,1)
[]
2. (0,3)-(5) (1)-(2) (2,5)-(3,4) (4)-(0,1)
[]
3. (0,1)-(0,2) (2)-(3) (3,4)-(1,5) (5)-(4)
[]
4. (0,1)-(0,4,5) (2,3)-(1) (4)-(2) (5)-(3)
[]
```

B.2.2 $\mathcal{H}^{odd}(2,2)$

B.2.2.1 4 Cylinders, 3 Equivalence Classes

```
1. (0,3)-(5) (1)-(0) (2,5)-(3,4) (4)-(1,2)
[{{0, 3}, {1}, {2}}, {{0}, {1, 2}, {3}}]
2. (0,1)-(0,5) (2)-(4) (3,4)-(1) (5)-(2,3)
[]
3. (0,5)-(3,4) (1,4)-(2,5) (2)-(0) (3)-(1)
[]
4. (0,3)-(0,5) (1,2)-(1,4) (4)-(3) (5)-(2)
[]
5. (0,2,1)-(3,4,5) (3)-(1) (4)-(2) (5)-(0)
[{{0}, {1, 3}, {2}}, {{0}, {1}, {2, 3}}, {{0}, {1, 2}, {3}}]
```

B.2.2.2 4 Cylinders, 2 Equivalence Classes

```
1. (0,3)-(5) (1)-(0) (2,5)-(3,4) (4)-(1,2)
[{{0}, {1, 2, 3}}, {{0, 1, 2}, {3}}, {{0, 1, 3}, {2}}, {{0, 2,
   3}, {1}}, {{0, 1}, {2, 3}}, {{0, 3}, {1, 2}}, {{0, 2}, {1,
   3}}]
2. (0,1)-(0,5) (2)-(4) (3,4)-(1) (5)-(2,3)
[{{0}, {1, 2, 3}}, {{0, 2, 3}, {1}}, {{0, 1}, {2, 3}}]
3. (0,5)-(3,4) (1,4)-(2,5) (2)-(0) (3)-(1)
[{{0, 1, 2}, {3}}, {{0, 1, 3}, {2}}, {{0, 1}, {2, 3}}]
4. (0,3)-(0,5) (1,2)-(1,4) (4)-(3) (5)-(2)
[{{0}, {1, 2, 3}}, {{0, 2, 3}, {1}}, {{0, 1}, {2, 3}}]
5. (0,2,1)-(3,4,5) (3)-(1) (4)-(2) (5)-(0)
[]
```

B.2.3 $\mathcal{H}(2,1,1)$

B.2.3.1 5 Cylinders, 4 Equivalence Classes

```
1. (0)-(2) (1)-(3) (2,4,3)-(5,6) (5)-(4) (6)-(0,1)
[]
2. (0,4)-(6) (1)-(0) (2)-(3) (3,6)-(4,5) (5)-(1,2)
[]
3. (0,1)-(0,6) (2)-(5) (3)-(4) (4,5)-(1) (6)-(2,3)
[]
4. (0,2)-(6) (1)-(0) (3,6)-(4,5) (4)-(3) (5)-(1,2)
[]
```

```
5. (0,2)-(6) (1)-(3) (3,6)-(4,5) (4)-(0) (5)-(1,2)
[]
6. (0,1)-(0,2) (2)-(3) (3,4)-(5,6) (5)-(4) (6)-(1)
[]
7. (0,6)-(4,5) (1,2)-(3,6) (3)-(2) (4)-(1) (5)-(0)
[]
8. (0,6)-(4,5) (1,2)-(3,6) (3)-(0) (4)-(2) (5)-(1)
[]
```

B.2.3.2 5 Cylinders, 3 Equivalence Classes

```
1. (0)-(2) (1)-(3) (2,4,3)-(5,6) (5)-(4) (6)-(0,1)
[{{0, 1, 4}, {2}, {3}}]
2. (0,4)-(6) (1)-(0) (2)-(3) (3,6)-(4,5) (5)-(1,2)
[{{0, 1}, {2}, {3, 4}}, {{0, 4}, {1, 3}, {2}}]
3. (0,1)-(0,6) (2)-(5) (3)-(4) (4,5)-(1) (6)-(2,3)
[]
4. (0,2)-(6) (1)-(0) (3,6)-(4,5) (4)-(3) (5)-(1,2)
[]
5. (0,2)-(6) (1)-(3) (3,6)-(4,5) (4)-(0) (5)-(1,2)
[{{0}, {1}, {2, 3, 4}}, {{0, 1, 2}, {3}, {4}}, {{0, 3}, {1, 4},
   {2}}, {{0, 4}, {1, 3}, {2}}]
6. (0,1)-(0,2) (2)-(3) (3,4)-(5,6) (5)-(4) (6)-(1)
[]
7. (0,6)-(4,5) (1,2)-(3,6) (3)-(2) (4)-(1) (5)-(0)
[{{0, 1}, {2, 4}, {3}}, {{0, 2}, {1, 4}, {3}}]
8. (0,6)-(4,5) (1,2)-(3,6) (3)-(0) (4)-(2) (5)-(1)
[]
```

B.2.3.3 5 Cylinders, 2 Equivalence Classes

```
1. (0)-(2) (1)-(3) (2,4,3)-(5,6) (5)-(4) (6)-(0,1)
[]
2. (0,4)-(6) (1)-(0) (2)-(3) (3,6)-(4,5) (5)-(1,2)
[]
3. (0,1)-(0,6) (2)-(5) (3)-(4) (4,5)-(1) (6)-(2,3)
[{{0}, {1, 2, 3, 4}}]
4. (0,2)-(6) (1)-(0) (3,6)-(4,5) (4)-(3) (5)-(1,2)
```

```
[]
5. (0,2)-(6) (1)-(3) (3,6)-(4,5) (4)-(0) (5)-(1,2)
[]
6. (0,1)-(0,2) (2)-(3) (3,4)-(5,6) (5)-(4) (6)-(1)
[]
7. (0,6)-(4,5) (1,2)-(3,6) (3)-(2) (4)-(1) (5)-(0)
[]
8. (0,6)-(4,5) (1,2)-(3,6) (3)-(0) (4)-(2) (5)-(1)
[{{0, 1, 3, 4}, {2}}]
```

B.2.3.4 4 Cylinders, 3 Equivalence Classes

```
1. (0)-(1) (1,3,4,2)-(5,6) (5)-(0,4) (6)-(2,3)
[]
2. (0)-(1,2) (1,4,2,3)-(5,6) (5)-(4) (6)-(0,3)
[]
3. (0)-(3) (1,3,2,4)-(5,6) (5)-(4) (6)-(0,2,1)
[]
4. (0)-(3) (1,4,2,3)-(5,6) (5)-(4) (6)-(0,2,1)
[]
5. (0,1)-(0,2,5) (2)-(3) (3,6)-(1,4) (4,5)-(6)
[]
6. (0,2)-(5) (1,3)-(6) (4)-(0,1) (5,6)-(2,4,3)
[]
7. (0,1)-(0,3) (2,5)-(1,6) (3,6)-(4,5) (4)-(2)
[]
8. (0,2)-(0,5) (1,3)-(1,6) (4,5)-(3) (6)-(2,4)
[]
9. (0,2)-(6) (1,4)-(3,5) (3,6)-(2,4) (5)-(0,1)
[{{0}, {1}, {2, 3}}, {{0, 1}, {2}, {3}}]
10. (0,1)-(0,2) (2,4)-(6) (3,6)-(1,5,4) (5)-(3)
[]
11. (0,1)-(0,3,5,6) (2,4)-(1) (3,6)-(4) (5)-(2)
[]
12. (0,1,2)-(0,1,6) (3)-(5) (4,5)-(2) (6)-(3,4)
[]
13. (0,4,1)-(6) (2)-(3) (3,6)-(4,5) (5)-(0,2,1)
```

```
[]
14. (0,3,1)-(6) (2)-(4) (4,6)-(0,5,1) (5)-(2,3)
[]
15. (0,3,1)-(6) (2)-(3) (4,6)-(0,5,1) (5)-(2,4)
[{{0, 3}, {1}, {2}}, {{0}, {1, 2}, {3}}]
16. (0,1,2)-(3,6,4,5) (3,4)-(2) (5)-(0) (6)-(1)
[{{0}, {1, 3}, {2}}, {{0}, {1}, {2, 3}}, {{0}, {1, 2}, {3}}]
17. (0,1,2)-(0,1,6) (3,6)-(4,5) (4)-(3) (5)-(2)
[]
18. (0,2,1)-(5,6) (3,4)-(0,2,1) (5)-(4) (6)-(3)
[{{0, 1}, {2}, {3}}]
19. (0,2,1)-(5,6) (3,6)-(0,4,1) (4)-(2) (5)-(3)
[]
20. (0,2,1)-(5,6) (3,6)-(0,4,1) (4)-(3) (5)-(2)
[{{0}, {1, 3}, {2}}, {{0, 2}, {1}, {3}}]
21. (0,4,2)-(5,6) (1,3)-(0,1,2) (5)-(4) (6)-(3)
[]
22. (0,2,1)-(6) (3,6)-(4,5) (4)-(3) (5)-(0,2,1)
[{{0, 3}, {1}, {2}}]
23. (0,1,4)-(0,1,6) (2,3)-(2,5) (5)-(4) (6)-(3)
[]
24. (0,5,2)-(3) (1,3)-(1,6) (4)-(5) (6)-(0,4,2)
[]
25. (0,1,2)-(0,1,5,6) (3,4)-(2) (5)-(4) (6)-(3)
[]
26. (0,1,2)-(0,5,1,6) (3,4)-(2) (5)-(3) (6)-(4)
[]
27. (0,2,1)-(6) (3,6)-(0,5,4,1) (4)-(3) (5)-(2)
[]
```

B.2.3.5 4 Cylinders, 2 Equivalence Classes

```
1. (0)-(1) (1,3,4,2)-(5,6) (5)-(0,4) (6)-(2,3)
[{{0}, {1, 2, 3}}]
2. (0)-(1,2) (1,4,2,3)-(5,6) (5)-(4) (6)-(0,3)
[]
3. (0)-(3) (1,3,2,4)-(5,6) (5)-(4) (6)-(0,2,1)
```

```
[]
4. (0)-(3) (1,4,2,3)-(5,6) (5)-(4) (6)-(0,2,1)
[]
5. (0,1)-(0,2,5) (2)-(3) (3,6)-(1,4) (4,5)-(6)
[{{0}, {1, 2, 3}}]
6. (0,2)-(5) (1,3)-(6) (4)-(0,1) (5,6)-(2,4,3)
[{{0, 1, 3}, {2}}]
7. (0,1)-(0,3) (2,5)-(1,6) (3,6)-(4,5) (4)-(2)
[{{0}, {1, 2, 3}}, {{0, 1, 2}, {3}}, {{0, 3}, {1, 2}}]
8. (0,2)-(0,5) (1,3)-(1,6) (4,5)-(3) (6)-(2,4)
[{{0}, {1, 2, 3}}, {{0, 2, 3}, {1}}, {{0, 1}, {2, 3}}]
9. (0,2)-(6) (1,4)-(3,5) (3,6)-(2,4) (5)-(0,1)
[{{0}, {1, 2, 3}}, {{0, 1, 2}, {3}}, {{0, 1, 3}, {2}}, {{0, 2,
   3}, {1}}, {{0, 1}, {2, 3}}, {{0, 3}, {1, 2}}, {{0, 2}, {1,
   3}}]
10. (0,1)-(0,2) (2,4)-(6) (3,6)-(1,5,4) (5)-(3)
[]
11. (0,1)-(0,3,5,6) (2,4)-(1) (3,6)-(4) (5)-(2)
[{{0}, {1, 2, 3}}]
12. (0,1,2)-(0,1,6) (3)-(5) (4,5)-(2) (6)-(3,4)
[{{0}, {1, 2, 3}}, {{0, 2, 3}, {1}}, {{0, 1}, {2, 3}}]
13. (0,4,1)-(6) (2)-(3) (3,6)-(4,5) (5)-(0,2,1)
[{{0, 1, 2}, {3}}]
14. (0,3,1)-(6) (2)-(4) (4,6)-(0,5,1) (5)-(2,3)
[{{0, 1, 2}, {3}}]
15. (0,3,1)-(6) (2)-(3) (4,6)-(0,5,1) (5)-(2,4)
[{{0}, {1, 2, 3}}, {{0, 1, 2}, {3}}, {{0, 1, 3}, {2}}, {{0, 2,
   3}, {1}}, {{0, 1}, {2, 3}}, {{0, 3}, {1, 2}}, {{0, 2}, {1,
   3}}]
16. (0,1,2)-(3,6,4,5) (3,4)-(2) (5)-(0) (6)-(1)
[]
17. (0,1,2)-(0,1,6) (3,6)-(4,5) (4)-(3) (5)-(2)
[]
18. (0,2,1)-(5,6) (3,4)-(0,2,1) (5)-(4) (6)-(3)
[]
19. (0,2,1)-(5,6) (3,6)-(0,4,1) (4)-(2) (5)-(3)
```

```
[{{0, 1, 2}, {3}}, {{0, 1, 3}, {2}}, {{0, 1}, {2, 3}}]
20. (0,2,1)-(5,6) (3,6)-(0,4,1) (4)-(3) (5)-(2)
[{{0, 1}, {2, 3}}, {{0, 2}, {1, 3}}]
21. (0,4,2)-(5,6) (1,3)-(0,1,2) (5)-(4) (6)-(3)
[]
22. (0,2,1)-(6) (3,6)-(4,5) (4)-(3) (5)-(0,2,1)
[]
23. (0,1,4)-(0,1,6) (2,3)-(2,5) (5)-(4) (6)-(3)
[{{0}, {1, 2, 3}}, {{0, 2, 3}, {1}}, {{0, 1}, {2, 3}}]
24. (0,5,2)-(3) (1,3)-(1,6) (4)-(5) (6)-(0,4,2)
[{{0, 1, 3}, {2}}, {{0, 2, 3}, {1}}, {{0, 3}, {1, 2}}]
25. (0,1,2)-(0,1,5,6) (3,4)-(2) (5)-(4) (6)-(3)
[{{0}, {1, 2, 3}}]
26. (0,1,2)-(0,5,1,6) (3,4)-(2) (5)-(3) (6)-(4)
[{{0}, {1, 2, 3}}]
27. (0,2,1)-(6) (3,6)-(0,5,4,1) (4)-(3) (5)-(2)
[]
```

B.2.4 $\mathcal{H}(1, 1, 1, 1)$

Although we do not use it in our analysis, we included all partitions for 4 or more cylinders in $\mathcal{H}(1^4)$ for the interested reader.

B.2.4.1 6 Cylinders, 5 Equivalence Classes

```
1. (0,1)-(7) (2)-(1) (3)-(0) (4,7)-(5,6) (5)-(4) (6)-(2,3)
[]
2. (0,1)-(7) (2)-(0) (3)-(4) (4,7)-(5,6) (5)-(1) (6)-(2,3)
[]
3. (0,3)-(6,7) (1,2)-(4,5) (4)-(3) (5)-(0) (6)-(2) (7)-(1)
[]
4. (0,3)-(6,7) (1,2)-(4,5) (4)-(1) (5)-(3) (6)-(2) (7)-(0)
[]
```

B.2.4.2 6 Cylinders, 4 Equivalence Classes

```
1. (0,1)-(7) (2)-(1) (3)-(0) (4,7)-(5,6) (5)-(4) (6)-(2,3)
[]
```

```
2. (0,1)-(7) (2)-(0) (3)-(4) (4,7)-(5,6) (5)-(1) (6)-(2,3)
[{{0, 5}, {1}, {2, 4}, {3}}]
3. (0,3)-(6,7) (1,2)-(4,5) (4)-(3) (5)-(0) (6)-(2) (7)-(1)
[]
4. (0,3)-(6,7) (1,2)-(4,5) (4)-(1) (5)-(3) (6)-(2) (7)-(0)
[]
```

B.2.4.3 6 Cylinders, 3 Equivalence Classes

```
1. (0,1)-(7) (2)-(1) (3)-(0) (4,7)-(5,6) (5)-(4) (6)-(2,3)
[{{0, 1, 2, 5}, {3}, {4}}]
2. (0,1)-(7) (2)-(0) (3)-(4) (4,7)-(5,6) (5)-(1) (6)-(2,3)
[{{0, 5}, {1, 3}, {2, 4}}]
3. (0,3)-(6,7) (1,2)-(4,5) (4)-(3) (5)-(0) (6)-(2) (7)-(1)
[{{0, 1, 4, 5}, {2}, {3}}, {{0, 1, 2, 3}, {4}, {5}}, {{0, 1}, {2,
    5}, {3, 4}}, {{0, 1}, {2, 4}, {3, 5}}]
4. (0,3)-(6,7) (1,2)-(4,5) (4)-(1) (5)-(3) (6)-(2) (7)-(0)
[{{0, 2}, {1, 5}, {3, 4}}, {{0, 1}, {2, 5}, {3, 4}}]
```

B.2.4.4 6 Cylinders, 2 Equivalence Classes

```
1. (0,1)-(7) (2)-(1) (3)-(0) (4,7)-(5,6) (5)-(4) (6)-(2,3)
[]
2. (0,1)-(7) (2)-(0) (3)-(4) (4,7)-(5,6) (5)-(1) (6)-(2,3)
[]
3. (0,3)-(6,7) (1,2)-(4,5) (4)-(3) (5)-(0) (6)-(2) (7)-(1)
[]
4. (0,3)-(6,7) (1,2)-(4,5) (4)-(1) (5)-(3) (6)-(2) (7)-(0)
[]
```

B.2.4.5 5 Cylinders, 4 Equivalence Classes

```
1. (0)-(3) (1)-(2) (2,4,3,5)-(6,7) (6)-(0,1) (7)-(4,5)
[]
2. (0)-(4) (1)-(2,3) (2,5,3,4)-(6,7) (6)-(5) (7)-(0,1)
[]
3. (0,3)-(7) (1,2)-(6) (4)-(2,3) (5)-(0,1) (6,7)-(4,5)
[]
4. (0,4)-(6) (1,6)-(4,7) (2,7)-(3,5) (3)-(1) (5)-(0,2)
```

```
[]
5. (0,1)-(0,4) (2,3)-(1,7) (4,7)-(5,6) (5)-(3) (6)-(2)
[]
6. (0,1)-(0,3) (2,3)-(6) (4,6)-(5,7) (5)-(4) (7)-(1,2)
[]
7. (0,1,2)-(0,1,7) (3)-(6) (4)-(5) (5,6)-(2) (7)-(3,4)
[]
8. (0,4,1)-(7) (2)-(4) (3)-(5) (5,7)-(0,6,1) (6)-(2,3)
[]
9. (0,3,1)-(7) (2)-(4) (4,7)-(5,6) (5)-(3) (6)-(0,2,1)
[]
10. (0,3,1)-(7) (2)-(3) (4,7)-(5,6) (5)-(4) (6)-(0,2,1)
[]
11. (0,1,2)-(0,1,7) (3,4)-(5,6) (5)-(4) (6)-(2) (7)-(3)
[]
12. (0,2,1)-(6,7) (3,4)-(0,5,1) (5)-(4) (6)-(2) (7)-(3)
[]
13. (0,2,1)-(6,7) (3,4)-(0,5,1) (5)-(2) (6)-(4) (7)-(3)
[]
```

B.2.4.6 5 Cylinders, 3 Equivalence Classes

```
1. (0)-(3) (1)-(2) (2,4,3,5)-(6,7) (6)-(0,1) (7)-(4,5)
[{{0, 1, 3}, {2}, {4}}]
2. (0)-(4) (1)-(2,3) (2,5,3,4)-(6,7) (6)-(5) (7)-(0,1)
[{{0, 1, 4}, {2}, {3}}]
3. (0,3)-(7) (1,2)-(6) (4)-(2,3) (5)-(0,1) (6,7)-(4,5)
[{{0}, {1}, {2, 3, 4}}, {{0, 1, 4}, {2}, {3}}, {{0, 2}, {1, 3},
   {4}}, {{0, 3}, {1, 2}, {4}}]
4. (0,4)-(6) (1,6)-(4,7) (2,7)-(3,5) (3)-(1) (5)-(0,2)
[{{0, 2}, {1, 4}, {3}}, {{0, 4}, {1, 2}, {3}}]
5. (0,1)-(0,4) (2,3)-(1,7) (4,7)-(5,6) (5)-(3) (6)-(2)
[]
6. (0,1)-(0,3) (2,3)-(6) (4,6)-(5,7) (5)-(4) (7)-(1,2)
[]
7. (0,1,2)-(0,1,7) (3)-(6) (4)-(5) (5,6)-(2) (7)-(3,4)
[]
```

```
8. (0,4,1)-(7) (2)-(4) (3)-(5) (5,7)-(0,6,1) (6)-(2,3)
[{{0, 1}, {2}, {3, 4}}, {{0, 4}, {1, 3}, {2}}]
9. (0,3,1)-(7) (2)-(4) (4,7)-(5,6) (5)-(3) (6)-(0,2,1)
[{{0}, {1}, {2, 3, 4}}, {{0, 1, 2}, {3}, {4}}, {{0, 3}, {1, 4},
   {2}}, {{0, 4}, {1, 3}, {2}}]
10. (0,3,1)-(7) (2)-(3) (4,7)-(5,6) (5)-(4) (6)-(0,2,1)
[]
11. (0,1,2)-(0,1,7) (3,4)-(5,6) (5)-(4) (6)-(2) (7)-(3)
[]
12. (0,2,1)-(6,7) (3,4)-(0,5,1) (5)-(4) (6)-(2) (7)-(3)
[{{0, 1}, {2, 3}, {4}}, {{0, 2}, {1, 3}, {4}}]
13. (0,2,1)-(6,7) (3,4)-(0,5,1) (5)-(2) (6)-(4) (7)-(3)
[]
```

B.2.4.7 5 Cylinders, 2 Equivalence Classes

```
1. (0)-(3) (1)-(2) (2,4,3,5)-(6,7) (6)-(0,1) (7)-(4,5)
[]
2. (0)-(4) (1)-(2,3) (2,5,3,4)-(6,7) (6)-(5) (7)-(0,1)
[]
3. (0,3)-(7) (1,2)-(6) (4)-(2,3) (5)-(0,1) (6,7)-(4,5)
[]
4. (0,4)-(6) (1,6)-(4,7) (2,7)-(3,5) (3)-(1) (5)-(0,2)
[]
5. (0,1)-(0,4) (2,3)-(1,7) (4,7)-(5,6) (5)-(3) (6)-(2)
[{{0}, {1, 2, 3, 4}}]
6. (0,1)-(0,3) (2,3)-(6) (4,6)-(5,7) (5)-(4) (7)-(1,2)
[]
7. (0,1,2)-(0,1,7) (3)-(6) (4)-(5) (5,6)-(2) (7)-(3,4)
[{{0}, {1, 2, 3, 4}}]
8. (0,4,1)-(7) (2)-(4) (3)-(5) (5,7)-(0,6,1) (6)-(2,3)
[]
9. (0,3,1)-(7) (2)-(4) (4,7)-(5,6) (5)-(3) (6)-(0,2,1)
[]
10. (0,3,1)-(7) (2)-(3) (4,7)-(5,6) (5)-(4) (6)-(0,2,1)
[]
11. (0,1,2)-(0,1,7) (3,4)-(5,6) (5)-(4) (6)-(2) (7)-(3)
```

[]
12. (0,2,1)-(6,7) (3,4)-(0,5,1) (5)-(4) (6)-(2) (7)-(3)
[]
13. (0,2,1)-(6,7) (3,4)-(0,5,1) (5)-(2) (6)-(4) (7)-(3)
[{{0, 1, 3, 4}, {2}}]

B.2.4.8 4 Cylinders, 3 Equivalence Classes

1. (0)-(4) (1,4,3,2,5)-(6,7) (6)-(5) (7)-(0,3,1,2)
[]
2. (0,3)-(0,5) (1,2)-(1,4) (4,6)-(3,7) (5,7)-(2,6)
[]
3. (0,7)-(5,6) (1,5)-(2,7) (2,4)-(1,3) (3,6)-(0,4)
[{{0}, {1, 3}, {2}}, {{0, 2}, {1}, {3}}]
4. (0,1)-(0,2,5) (2,3)-(7) (4,5)-(6) (6,7)-(1,4,3)
[]
5. (0,1)-(0,2) (2,5,6)-(7) (3,7)-(1,6,4,5) (4)-(3)
[]
6. (0,1)-(0,2,5,6) (2,4,6)-(7) (3,7)-(1,4) (5)-(3)
[]
7. (0,1)-(5) (2,3,5,4)-(6,7) (6)-(0,2) (7)-(1,4,3)
[]
8. (0,1)-(0,4,5) (2,4,3,5)-(6,7) (6)-(2) (7)-(1,3)
[]
9. (0,2,1)-(7) (3)-(2) (4,5)-(6) (6,7)-(0,5,3,4,1)
[]
10. (0,6,1)-(7) (2,7)-(0,5,3,4,1) (3)-(2) (4,5)-(6)
[]
11. (0,5,1)-(6,7) (2,6)-(5) (3,7)-(0,4,1) (4)-(2,3)
[{{0, 3}, {1}, {2}}, {{0}, {1, 2}, {3}}]
12. (0,1,4)-(0,1,7) (2,3)-(2,6) (5,6)-(4) (7)-(3,5)
[]
13. (0,4,2)-(3,7) (1,3)-(1,5) (5,7)-(0,6,2) (6)-(4)
[]
14. (0,1,2)-(0,5,1,6) (3,7)-(2,4) (4,5)-(7) (6)-(3)
[]
15. (0,1,2)-(0,1,5,6,7) (3,4)-(2) (5,7)-(4) (6)-(3)

```
[]
16. (0,2,4)-(0,2,7) (1,6,3)-(4) (5)-(6) (7)-(1,5,3)
[]
17. (0,3,5)-(0,3,7) (1,2,4)-(1,2,6) (6)-(5) (7)-(4)
[]
18. (0,5,2)-(1,7,3) (1,4,3)-(0,6,2) (6)-(5) (7)-(4)
[]
19. (0,5,2)-(1,7,3) (1,4,3)-(0,6,2) (6)-(4) (7)-(5)
[{{0}, {1, 3}, {2}}, {{0, 2}, {1}, {3}}]
20. (0,7,2)-(1,6,3) (1,4,3)-(7) (5)-(4) (6)-(0,5,2)
[{{0}, {1, 3}, {2}}, {{0, 2}, {1}, {3}}]
21. (0,2,4)-(0,2,3,1) (1,5,3)-(6,7) (6)-(5) (7)-(4)
[]
22. (0,4,2)-(6) (1,6,3)-(5,7) (5)-(4) (7)-(0,3,1,2)
[]
23. (0,3,1,2)-(4,6,5,7) (4)-(1) (5)-(0) (6,7)-(2,3)
[{{0}, {1, 3}, {2}}, {{0}, {1}, {2, 3}}, {{0}, {1, 2}, {3}}]
24. (0,2,1,3)-(4,5,7,6) (4)-(2) (5,6)-(3) (7)-(0,1)
[{{0}, {1, 3}, {2}}, {{0}, {1}, {2, 3}}, {{0}, {1, 2}, {3}}]
25. (0,3,1,2)-(6,7) (4,5)-(0,3,1,2) (6)-(5) (7)-(4)
[{{0, 1}, {2}, {3}}]
26. (0,3,1,2)-(7) (4,7)-(5,6) (5)-(4) (6)-(0,3,1,2)
[{{0, 3}, {1}, {2}}]
27. (0,2,3,1)-(0,6,1,2,7) (4,5)-(3) (6)-(4) (7)-(5)
[]
```

B.2.4.9 4 Cylinders, 2 Equivalence Classes

```
1. (0)-(4) (1,4,3,2,5)-(6,7) (6)-(5) (7)-(0,3,1,2)
[]
2. (0,3)-(0,5) (1,2)-(1,4) (4,6)-(3,7) (5,7)-(2,6)
[{{0}, {1, 2, 3}}, {{0, 2, 3}, {1}}, {{0, 1}, {2, 3}}]
3. (0,7)-(5,6) (1,5)-(2,7) (2,4)-(1,3) (3,6)-(0,4)
[{{0}, {1, 2, 3}}, {{0, 1, 2}, {3}}, {{0, 1, 3}, {2}}, {{0, 2,
   3}, {1}}, {{0, 1}, {2, 3}}, {{0, 3}, {1, 2}}, {{0, 2}, {1,
   3}}]
4. (0,1)-(0,2,5) (2,3)-(7) (4,5)-(6) (6,7)-(1,4,3)
```

```
[{{0}, {1, 2, 3}}]
5. (0,1)-(0,2) (2,5,6)-(7) (3,7)-(1,6,4,5) (4)-(3)
[]
6. (0,1)-(0,2,5,6) (2,4,6)-(7) (3,7)-(1,4) (5)-(3)
[{{0}, {1, 2, 3}}]
7. (0,1)-(5) (2,3,5,4)-(6,7) (6)-(0,2) (7)-(1,4,3)
[{{0}, {1, 2, 3}}]
8. (0,1)-(0,4,5) (2,4,3,5)-(6,7) (6)-(2) (7)-(1,3)
[]
9. (0,2,1)-(7) (3)-(2) (4,5)-(6) (6,7)-(0,5,3,4,1)
[{{0, 2, 3}, {1}}]
10. (0,6,1)-(7) (2,7)-(0,5,3,4,1) (3)-(2) (4,5)-(6)
[]
11. (0,5,1)-(6,7) (2,6)-(5) (3,7)-(0,4,1) (4)-(2,3)
[{{0}, {1, 2, 3}}, {{0, 1, 2}, {3}}, {{0, 1, 3}, {2}}, {{0, 2,
   3}, {1}}, {{0, 1}, {2, 3}}, {{0, 3}, {1, 2}}, {{0, 2}, {1,
   3}}]
12. (0,1,4)-(0,1,7) (2,3)-(2,6) (5,6)-(4) (7)-(3,5)
[{{0}, {1, 2, 3}}, {{0, 2, 3}, {1}}, {{0, 1}, {2, 3}}]
13. (0,4,2)-(3,7) (1,3)-(1,5) (5,7)-(0,6,2) (6)-(4)
[{{0, 1, 2}, {3}}, {{0, 2, 3}, {1}}, {{0, 2}, {1, 3}}]
14. (0,1,2)-(0,5,1,6) (3,7)-(2,4) (4,5)-(7) (6)-(3)
[{{0}, {1, 2, 3}}]
15. (0,1,2)-(0,1,5,6,7) (3,4)-(2) (5,7)-(4) (6)-(3)
[{{0}, {1, 2, 3}}]
16. (0,2,4)-(0,2,7) (1,6,3)-(4) (5)-(6) (7)-(1,5,3)
[{{0}, {1, 2, 3}}, {{0, 1, 3}, {2}}, {{0, 2}, {1, 3}}]
17. (0,3,5)-(0,3,7) (1,2,4)-(1,2,6) (6)-(5) (7)-(4)
[{{0}, {1, 2, 3}}, {{0, 2, 3}, {1}}, {{0, 1}, {2, 3}}]
18. (0,5,2)-(1,7,3) (1,4,3)-(0,6,2) (6)-(5) (7)-(4)
[{{0, 1, 2}, {3}}, {{0, 1, 3}, {2}}, {{0, 1}, {2, 3}}]
19. (0,5,2)-(1,7,3) (1,4,3)-(0,6,2) (6)-(4) (7)-(5)
[{{0, 1}, {2, 3}}, {{0, 2}, {1, 3}}]
20. (0,7,2)-(1,6,3) (1,4,3)-(7) (5)-(4) (6)-(0,5,2)
[{{0}, {1, 2, 3}}, {{0, 1, 2}, {3}}, {{0, 1, 3}, {2}}, {{0, 2,
   3}, {1}}, {{0, 1}, {2, 3}}, {{0, 3}, {1, 2}}, {{0, 2}, {1,
```

3}}]
21. (0,2,4)-(0,2,3,1) (1,5,3)-(6,7) (6)-(5) (7)-(4)
[]
22. (0,4,2)-(6) (1,6,3)-(5,7) (5)-(4) (7)-(0,3,1,2)
[{{0}, {1, 2, 3}}]
23. (0,3,1,2)-(4,6,5,7) (4)-(1) (5)-(0) (6,7)-(2,3)
[]
24. (0,2,1,3)-(4,5,7,6) (4)-(2) (5,6)-(3) (7)-(0,1)
[]
25. (0,3,1,2)-(6,7) (4,5)-(0,3,1,2) (6)-(5) (7)-(4)
[]
26. (0,3,1,2)-(7) (4,7)-(5,6) (5)-(4) (6)-(0,3,1,2)
[]
27. (0,2,3,1)-(0,6,1,2,7) (4,5)-(3) (6)-(4) (7)-(5)
[{{0}, {1, 2, 3}}]

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