Essays in Financial Economics

by

Hayk Hambardzumyan

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Economics)
in The University of Michigan
2023

Doctoral Committee:

Professor John V. Leahy, Chair
Research Professor Vladimir Asriyan, CREi
Assistant Professor Shane H. Miller
Associate Professor Pablo Ottonello

Hayk Hambardzumyan

hayk@umich.edu

ORCID iD: 0000-0002-1928-9727

ACKNOWLEDGMENTS

I am deeply indebted to my dissertation committee, John V. Leahy, Vladimir Asriyan, Shane H. Miller and Pablo Ottonello, for their invaluable advice and continuing discussions. I am also grateful to Toni Whited, Martin Schmalz, Florian Trouvain, John Olson, Dominykas Norgilas, seminar participants at the University of Michigan and Armenian Economics Association Annual Meetings for the helpful comments.

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Abstract

This dissertation comprises three essays in financial economics. The first essay uses a general equilibrium model to explain the cash flow risk and the dynamics of the equity term structure. I propose a novel explanation for cash flow risk that accurately captures the decomposition of forward yields into expected dividend growth and risk premium components. Additionally, it explains the upward sloping term structure during expansions and its inversion during recessions. In the model, asset prices respond instantly to aggregate shocks in the economy (financial effect), but the real effect of the shock is delayed. When a negative shock impacts the economy, cash flows either continue to increase or remain flat due to firms selling their capital and distributing the proceeds as dividends. This phenomenon arises from both the decreased productivity of capital and the high value that households place on dividends during challenging times. Furthermore, the economic mechanism behind the endogenous cash flow process and the discount rate risk is micro-founded using a heterogeneous firm model matched to firm-level data for the US. Additionally, the essay explains the economic mechanism behind the value premium (as the risk premium between long and short duration firms) and the decomposition of forward yields across firms with different durations, taking advantage of the richness of the heterogeneous firm model.

The second essay investigates the relationship between competition, common ownership, and inequality. While previous studies have documented the decrease in competition, the increase in common ownership, and the rise of inequality, no paper has explored the connection between common ownership and inequality. This essay presents empirical evidence, through structural estimation, that common ownership contributes to the increase in income inequality through declining competition *given* the common ownership hypothesis is true. I aim to provide quantitative answers to the following questions: 1) What percentage of the increase in the share of profits (increase in markups) in the last three decades can be attributed to the increase in common ownership? 2) What portion of the increase in income inequality can be attributed to common ownership? According to our model, based on the common ownership hypothesis, the rise of common ownership has contributed to an increase in the total income Gini coefficient from 27.62 to 41.73.

The final essay explores the dynamics of equilibrium asset prices in an economy with wishful thinkers (as defined in Caplin and Leahy, 2019) and rational agents. The optimism of wishful thinkers, their attraction to high-return projectswith low success probabilities, and their occasional wealth accumulation contribute to the formation and bursting os stock market bubbles. The extend of wishful thinking's impact on asset prices depends on the level of wishful thinking, wealth distribution, and the learning pace within the economy.

CHAPTER I.

Equity Term Structure Dynamics in a Heterogeneous Firm Model

1 Introduction

Understanding the factors behind stock market fluctuations is a central focus of asset pricing research. Since an asset's price represents the sum of its discounted future cash flows, it is necessary to explore the factors that drive changes in either the discount rate (nominal bond yield plus risk premium) or cash flows, or both. To gain a better understanding of how cash flow properties are linked to asset prices, we follow the approach of van Binsbergen et al. (2013) and examine forward yields - the difference between equity and bond yields. As forward yields exclude the nominal bond yields, their fluctuations stem from changes in the risk premium and expected dividend growth. According to van Binsbergen et al. (2013), about 70% of the variation in forward yields for the S&P500 is attributed to variation in expected growth rates[1].

While many asset pricing models explain the upward sloping equity term structure in expansions and its inversion in recessions (due to discount rates), they often fail to accurately capture the decomposition of forward yields into expected dividend growth and risk premium components. My model fills this gap. Exploring the economic mechanism that explains why high forward yields in recessions are primarily interpreted as a poor economic outlook in terms of low expected dividend growth, rather than as high risk aversion and high risk premium is crucial for comprehending asset prices. The subsequent paragraph provides a description of the terms used, with their equations defined in the model (section 2.4.1, equations (12)-(14)).

The equity yield at time t with maturity m can be expressed as the difference between the maturity-specific return (discount rate) that investors require for holding dividend risk (nominal bond yield with maturity m plus hold-to-maturity risk premium) and the average expected dividend growth over the next m periods (as defined in van Binsbergen et al., 2013). The equity term structure illustrates how equity yields change across maturities. Without bond yield, the forward

[1] Similar evidence is observed in Giglio et al. (2021), in a bigger sample of 47 years.

equity yield decomposition consists of the difference between the hold-to-maturity dividend risk premium and the m-year average expected dividend growth rate. The term structure of forward equity demonstrates how the spread between equity and bond yields changes across maturities. While bond coupons are fixed and only subject to discount rate risk at the risk-free rate, stock dividends are uncertain and convey both cash flow risk and discount rate risk. In the equity yield decomposition, cash flow risk impacts both the hold-to-maturity risk premium and the expected dividend growth. The challenge lies in the fact that the term structure of discount rates is a complex equilibrium object that depends on the combined dynamics of cash flows and investors' risk preferences, which are interdependent. While the latter has been explored in various asset pricing models, less attention has been given to the cash flow risk.

This paper introduces a general equilibrium model that incorporates firm-side heterogeneity. Similar to consumption asset pricing models (e.g. habit, long run risk, etc.), my model generates forward equity term structure that is upward sloping (forward equity yield increases across maturity) in expansions and inverts in recessions. It also provides explanations for various asset pricing puzzles (including equity premium, lead pro-cyclical variation of stock prices, etc.). The novel aspect of the model lies in its ability to separate the financial effect of aggregate shocks on asset prices and household wealth from the real effect on consumption and dividends. Prices are forward-looking and efficient, as they instantly react to shocks, while dividends respond with a lag. This lag in dividends allows for predictability by yields and that helps to match the data on the forward yield decomposition into risk premium and expected dividend growth components. Moreover, the model derives the economic mechanism behind the cash flow process endogenously from the heterogeneous firm optimization problem, which is matched to firm-level data for the US. Additionally, the model contributes to the cross-sectional understanding by explaining the economic mechanism for the value premium and forward yield decomposition for firms of different durations within the same micro-founded model.

In the model, identical households trade perpetual bonds and stocks and generate a stochastic discount factor (SDF) that shapes the returns and premiums on equities over bonds in an environment of aggregate shocks and business cycles. Meanwhile, firms exhibit heterogeneity in their productivity (high or low) and capital levels, resulting in differences in the expected returns on their stocks. When firms with small capital stock experience a high productivity shock, they have a high retention ratio (reinvest profits into company growth) and pay dividends at a later time. In contrast, firms with small capital stock and low productivity, as well as firms with large capital stock, pay dividends every period. Consequently, stocks have different durations depending on the weighted average time periods in which the majority of dividends are expected to be received. Aggregating a portfolio of stocks with the same duration does not eliminate duration risk (only individual risk disappears). Therefore, the value premium can be measured as the difference between sorted portfolios of short and long duration stocks[2] Short duration stocks are more susceptible to market risk (business cycle fluctuations) compared to long duration stocks: The value premium is formed as a result of the co-movement of the short duration stocks' cash flows with the stochastic discount factor (SDF) of the households while for long duration stocks the change in cash flows as well as the co-movement is negligible. The households determine the SDF of the economy, and the heterogeneous firms generate various cash flow processes that are discounted by the same SDF of the households. This combination results in cross-sectional differences in each firm's price processes and returns in each period, defining the panel of equity term structure dynamics.

While the term structure of bonds is mainly explained by discount rate risk, as captured in the duration of a bond, equities are exposed to both discount rate and cash flow risks. Therefore, the difference between the equity term structure and the bond term structure arises from cash flow uncertainty, and equity duration represents a combination of discount rate and cash flow risks. In the model, the SDF of investors varies depending on the cash flows they receive, and the cash flows of firms depend on the investors' time and risk preferences, as summarized in their SDF. Consumption asset pricing models accurately reproduce the upward-sloping term structure in expansions and its inversion in recessions, as the slope and inversion are driven by the consumers' SDF, which is common in such models (e.g., the habit model, the long-run risk model, etc.). However, these models fail to match the data on the decomposition of equity (forward) yields into the hold-to-maturity return (risk premium) and expected dividend growth components[3]. This discrepancy arises from an incorrect modeling of cash flow risk, where either the expected dividend growth is constant (habit model) or significantly smaller than the risk premium (long-run risk model). Additionally, both habit and long-run risk models lack micro-foundations and rely on exogenous dividend processes and frictions to match the data on equity premium.

The model presented in this paper addresses this gap by accurately capturing both the decomposition of equity yields and the dynamic properties of its term structure. In the available data, the expected dividend growth variation is estimated to account for 70% of the variation in equity yields (see Figure 21 from van Binsbergen et al., 2013, and Figure 22 from Giglio et al., 2021, in the appendix). The improvement in this model lies in the introduction of a lag in the dividend response, allowing for its growth even when the negative shock hits the economy. This lag enables a larger negative expected dividend growth, aligning with the data on equity yield decomposition. In typical asset pricing models, achieving this would require adding more autoregressive (AR) or moving average (MA) terms to their exogenous dividend processes. However, in my model, the dividend process is endogenous. Dividends increase or remain stable when the negative shock hits the economy because firms sell their capital and distribute the proceeds from sale as dividends.

[2]While the conventional definition of the value premium focuses on the return difference between value stocks (high ratio of book-to-market value) and growth stocks (low book-to-market ratio), the intuition behind it aligns with the timing of dividends, and the return difference between short and long duration stocks captures this relationship.

[3]van Binsbergen et al. (2013), and Giglio et al. (2021), van Binsbergen and Koijen (2017)

This occurs because capital becomes less productive, and investors highly value dividends during challenging times. But in the subsequent period, dividend growth turns negative as conditional on already being in a recession, there are two possible scenarios looking forward: (a) tomorrow is a bad day, resulting in decreased consumption as households already depleted their productive wealth today, or (b) tomorrow is a good day, leading to reduced consumption as households invest in rebuilding their productive wealth (reconstruction phase). This generates a more substantial negative expected dividend growth during recessions and produces a lagged consumption response to the shock, aligning with the data (see figures 27-28 in the appendix). Initially, households only experience the financial effect of the shock as their investment portfolios decline, making them feel less wealthy. There is no immediate "real" effect when the negative shock hits the economy because it has not yet impacted household consumption. Only in the later stage, after households have consumed the firms' productive capacity or experienced a positive shock that requires reinvestment, do dividends decline.

An implication of this model is the predictability of dividends by equity yields, as observed in the data (see figures 30-39 in the appendix). Across firms, there is heterogeneity in the decomposition of forward yields. Short duration firms naturally exhibit a more predictable dividend growth process due to their covariation with forward yields, compared to long duration firms (as observed by Maio and Santa-Clara, 2015, in cross-sectional data of stocks).

While Miller (2020), Gormsen and Lazarus (2021), and Giglio, Kelly, and Kozak (2021) find evidence of an upward-sloping or flat average equity term structure, their findings contrast with the downward-sloping term structure reported by Binsbergen, Brandt, and Koijen (2012), Goncalves (2021), Andrews and Goncalves (2020), and Weber (2018), among others[4]. However, the latter view faces challenges due to limited cross-sectional and time series data on traded derivatives, starting only around 2004. Bansal et al. (2021) explain that the negative average slope in the recent short sample of the equity term structure is due to the significant weight of the 2008 recession, and that, in general, the sign in any sample depends on the proportion of recessions and expansions, as the equity term structure is downward-sloping in recessions and upward-sloping in expansions.

The average term structure of bonds exhibits an upward slope (the US economy grows 1.8% on average), and long maturity cash flows offer a premium compared to short-term ones (see Figure 16 in the appendix for dividend strip estimates from Giglio et al., 2021, and bond yield data from Gurkaynak et al., 2006). Both for stocks and bonds the term structure is upward sloping in expansions (reflecting the expected growth in the economy) and inverts in recessions. Additionally, the term structure of equity risk premia follows the same pattern, being upward-sloping in expansions

[4] Kogan et al. (2017) discuss a production-based model with investment-specific shocks, which also supports a negative slope in the term structure of equity returns. Ai et al. (2017) propose another production-based model where individual firms have imperfect information about their productivity and learn about it over time.

and inverting in recessions (see figures 17 and 18 in the appendix for dividend strip estimates from Giglio et al., 2021, and bond yield data from Gurkaynak et al., 2006)[5]. The model presented in this paper successfully captures these observed facts.

According to the leading asset pricing models such as habit formation (Campbell and Cochrane, 1999), long-run risk (Bansal and Yaron, 2004), and the rare disaster model (Baro, 2006), the term structure of equity returns is either upward-sloping or flat (Lettau and Watcher, 2007). Consumption asset pricing models reproduce the observed upward-sloping forward equity term structure during expansions and its inversion in recessions (e.g. habit, long-run risk models, shown in Bansal et al. (2021)).

However, the existing theoretical asset pricing models fail to provide an explanation for the decomposition of equity yield while simultaneously accounting for the dynamics of the equity term structure. In the habits model, that is because of the constant expected dividend growth ($\Delta d_{t+1} = \bar{g} + w_{t+1}$). On the other hand, the long-run risk model performs better in terms of equity yield decomposition due to the inclusion of an AR(1) term in consumption growth. Nonetheless, it primarily attributes high yields to high risk premia, as the effect of the dividend growth shock is mostly contemporaneous (yield decomposition in long-run risk model: 79% risk premia, 21% expected dividend growth). In contrast, empirical data emphasizes the significance of cash flow risk. The decomposition of the forward strip yield reveals that the majority (70%) of the variation is driven by expected growth rates, with only a minor portion attributable to risk premia (see Figure 21 in the appendix, van Binsbergen et al., 2013). Similar findings are reported by Giglio et al. (2021) (see Figure 22 in the appendix).

Throughout this paper, I assume identical households, as household heterogeneity does not impact aggregate variables. However, in section 2.6, I discuss the microeconomic implications of simultaneous heterogeneity in both firm and household aspects. It seems the next reasonable step in economic modeling to have the distribution of firms and households at the same time. As the markets are complete, the challenge of tracking the stock holdings of a potentially infinite number of firms as state variables for each household is resolved by employing a dynamic version of the two-fund separation theorem (Cass and Stiglitz, 1970; Schmedders, 2007). This theorem establishes that every investor holds available risky assets in the same proportions as other investors and adjusts for their risk aversion solely through the weights on bonds relative to the overall risky bundle of stocks. Instead of keeping track of their shares in all individual firms (possibly infinite), the separation theorem reduces the state space of the investor problem to only their holdings in the aggregate stock market and bonds (two state variables) as they diversify away the idiosyncratic shocks of the stocks.

How realistic is the result in the model that investors only hold the market portfolio and a bond (efficient prices)? Over the past quarter-century, there has been a shift away from active

[5] also in Ait-Sahalia, Karaman, and Mancini (2015), Bansal et al (2017).

investment, with recent literature focusing more on the rise of passive investment (e.g., Vanguard, BlackRock, StateStreet) (Backus et al., 2019). According to research firm Morningstar Inc., as of August 2019, passive investment funds in the U.S. managed more assets (\$4.27 trillion) than active investment funds (\$4.25 trillion)[6]. Additionally, several papers debate whether fund managers can add value to the funds they manage (e.g., Carhart, 1997). The ratio of passively managed to actively managed assets serves as a proxy for stock market efficiency, which has changed over time (see Figure 23 in the appendix). These developments reflect the objective processes of increasing market efficiency, as active investors now identify and exploit inefficiencies more swiftly, leaving less room for mispricing. The assumption of general equilibrium, where markets clear in each period and assets are priced efficiently, is supported by the growing number of individuals who prefer to hold the market portfolio rather than attempting to outperform the market.

The remainder of the paper is structured as follows: Section 2 provides a description of the model, its environment, equilibrium, and solution, while Section 3 presents the concluding remarks.

2 Model

2.1 Households

A continuum (unit measure) of infinitely lived identical households, indexed by $i \in \mathbb{I}$, exists in the model. These households possess utility functions with external habits $\alpha_{i,t}$, similar to Campbell and Cochrane (1999). They start with an initial endowments of stocks, $\theta_{ij,-1} = 1$ for each stock j (firm problem defined below), and no initial endowments of bonds, $b_{i,-1} = 0$. They derive utility from a homogeneous consumption good, which is perishable. Additionally, they exhibit impatience, characterized by the parameter β, and can save in a perpetual bond, $b_{i,t}$, with a riskless cash stream CF_f (coupons), or in a risky stock j, $\theta_{ij,t}$, with a risky cash flow $CF_{j,t}$ (dividends) for delayed consumption. The households trade to determine prices, with the consumption good serving as the numeraire.

The problem faced by each household is

$$\max_{\{c_{i,t};\{\theta_{ij,t+1}\}_{j\in\mathbb{J}},b_{i,t+1}\}_{t=0}^{\infty}} \mathbb{E}\sum_{t=0}^{\infty}\beta^t\frac{(c_{i,t}-\alpha_{i,t})^{1-\gamma}}{1-\gamma}$$

subject to the following constraint:

$$c_{i,t} + \int_j \theta_{ij,t+1}p_{j,t}dj + b_{i,t+1}q_t \leq \int_j \theta_{ij,t}(p_{j,t}+CF_{j,t})dj + b_{i,t}(q_t+CF_f)$$

[6]"Index Funds Are the New Kings of Wall Street," Wall Street Journal (Sept. 18, 2019)

Here, $b_{i,t+1}$represents the bond holdings, and $\theta_{ij,t+1}$ represents the stock holdings of household i in firm j that are chosen in period t and carried into period $t+1$. The end-of-period stock and bond prices, after dividends and coupons have been remitted, are denoted as $p_{j,t}$ and q_t, respectively, and households consider these prices as given.

Since the markets are complete, all households share the same stochastic discount factor (SDF) in equilibrium, which is defined as:

$$M_{t+1} = M_{i,t+1} = \beta \frac{u'(c_{i,t+1})}{u'(c_{i,t})} = \beta \left(\frac{c_{i,t+1} - \alpha_i}{c_{i,t} - \alpha_i} \right)^{-\gamma} \tag{1}$$

where the aggregate SDF for the economy, M_{t+1}, is the same as the individual SDF of each household, $M_{i,t+1}$.

2.2 Firms

There is a large population $\mathbb{J}$ (unit measure) of competitive heterogeneous firms that use capital to produce a homogeneous consumption good and maximize the sum of their discounted cash flows, $CF_{j,t}$, but differ in their productivity and capital holdings. The cash flow of each separate firm is allowed to be negative (neoclassical firm model), but the aggregate cash flow will always be positive as it equals the consumption of the households.

For each period, firm j is defined by its predetermined capital stock $k_{j,t} \in \mathbb{R}_+$, its idiosyncratic productivity level $z_{j,t} \in \mathbb{Z}$, and the current aggregate state of the economy $A_t \in \mathbb{A}$. Both shocks follow Markov processes.

Firms produce output using capital according to the production function $A_t z_{j,t} k_{j,t}^{\xi}$, which is increasing with decreasing returns to scale, where $0 < \xi < 1$ represents the share of capital. Firms own capital subject to quadratic adjustment costs of $\frac{\phi}{2} \left(\frac{I_{j,t}}{k_{j,t}} - \delta \right)^2$, where ϕ is the parameter of the quadratic adjustment cost of capital and $I_{j,t}$ is the investment. Capital depreciates at a rate of δ and follows the accumulation equation:

$$k_{j,t+1} = (1 - \delta) k_{j,t} + I_{j,t}$$

The problem for each firm is to maximize the sum of the present discounted value of its cash flows $CF_{j,t}$ (output less investment costs):

$$\max_{\{k_{j,t}\}_{t=0}^{\infty}} E \left[\sum_{t=0}^{\infty} M_{0 \to t} CF_{j,t} \right]$$

$$= \max_{\{k_{j,t}\}_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty} M_{0\to t}\left(A_t z_{j,t} k_{j,t}^{\xi} - I_{j,t} - \frac{\phi}{2}\left(\frac{I_{j,t}}{k_{j,t}} - \delta\right)^2 k_{j,t}\right)\right]$$

subject to:

$$k_{j,t+1} = (1-\delta)k_{j,t} + I_{j,t}$$

$$k_{j,t+1} \geq 0$$

$M_{0\to t}$ represents the stochastic discount factor (SDF) at which households, in aggregate, discount the firm's cash flow from period t to period 0. It can be recursively written as $M_{0\to t} = M_1 \cdot ... \cdot M_t$, where M_t is defined in the household problem above.

2.3 General Equilibrium

2.3.1 Distributional dynamics and recursive problems

I summarize the distribution of firms over (z, k) using the probability measure μ defined on the Borel algebra σ for the product space $\Sigma = \mathbb{Z} \times \mathbb{R}_+$. The aggregate state is (μ, A), where μ represents the current measure (distribution) of firms over capital holdings and productivity status. The part of the law of motion that concerns A is exogenous, and it can be described by A's transition matrix. The part that concerns updating μ is denoted as Γ, meaning that $\mu' = \Gamma(\mu, A, A')$. For the individual firm, the relevant state variables are its capital holdings, productivity status, and the aggregate state: $(z, k; A, \mu)$. The role of the aggregate state is to allow the households and firms to predict future prices.

The recursive formulation of the firm problem is

$$V_F(z, k; A, \mu) = \max_{k'} Azk^{\xi} - k' + (1-\delta)k - \frac{\phi}{2}\left(\frac{k'}{k} - 1\right)^2 k$$

$$+\mathbb{E}\left[M' V_F(z', k'; A', \mu') | z, A\right] \tag{2}$$

The decision rule for updating capital implied by this problem is denoted by the function f^k: $k' = f^k(z, k; A, \mu)$. And the updating rule for the distribution of capital is $\mu' = \Gamma(\mu, A, A')$.

The recursive formulation of the household problem is

$$V_H(\Theta; z, k, A, \mu) =$$

$$\max_{\Theta'} \frac{\left(\int_j \theta_j (p_j + CF_j) dj + b(q + CF_f) - \int_j \theta_j' p_j dj - b'q - \alpha\right)^{1-\gamma} - 1}{1-\gamma}$$

$$+\beta E \left[V_H(\Theta'; z', k', A', \mu') | z, A\right] \tag{3}$$

where Θ is the portfolio of bond and stocks of individual household i.

Although μ summarizes the distribution of firms over (z, k), it only determines bond and aggregate stock market prices, not individual stock prices. However, in the household problem, we require individual stock prices, p_j, as well as individual (z_j, k_j) pairs for each $j \in \mathbb{J}$ (which may be infinitely many). The decision rule for updating the portfolio, as implied by this problem, is denoted by the function $f^\Theta : \Theta' = f^\Theta(\Theta, z, k; A, \mu)$.

2.3.2 Market clearing

Since the homogeneous consumption good is perishable, in each state of the economy, the aggregate production less investment, $CF = \int_{j \in \mathbb{J}} CF_j dj$, is equal to the aggregate consumption, $C = \int_{i \in \mathbb{I}} c_i di$, ensuring goods market clearing. Additionally, the aggregate bond market, $\int_{i \in \mathbb{I}} b_i di$, and the aggregate stock market, $\int_{i \in \mathbb{I}} \theta_{ij} di$, also clear, with bonds having zero net supply and the supply of each firm's stock being one. That determines the prices of stocks and bonds.

2.3.3 Definition of Equilibrium

Given an initial distribution of assets $\Theta_{i,-1}$ with $b_{i,-1} = 0$ and $\theta_{ij,-1} = 1$ for each household i, the equilibrium is a sequence of prices $q(A, \mu)$ and $p_j(z_j, k_j; A, \mu)$ for all $j \in \mathbb{J}$, and an allocation defined such that the following conditions hold:

(i) V_H satisfies the household problem (3), and the utility-maximizing optimal portfolio for household $i \in \mathbb{I}$ is $\Theta' = f^\Theta(\Theta, z, k; A, \mu)$.

(ii) V_f satisfies the firm problem (2), and $k' = f^k(z, k; A, \mu)$ is the optimal capital level of firm j that maximizes its value.

(iii) $C = \int_{i \in \mathbb{I}} c_i di = \int_{j \in \mathbb{J}} CF_j dj = CF$ for each state of the economy.

(iv) $\int_{i \in \mathbb{I}} b_i di = 0$, $\int_{i \in \mathbb{I}} \theta_{ij} di = 1$ for all $j \in \mathbb{J}$ and for each state of the economy.

(v) Γ is generated by f^k, i.e., the appropriate summing up of firms' optimal choices of capital given their current capital and productivity status:

$$\mu' = \Gamma(\mu(z,k), A, A')$$

$$= \int_{A \in \mathbb{A}} \left\{ \int_{(z,k)\in\Sigma} \mathcal{I}_{\{k'=f^k(z,k|A,\mu)\}} d\pi_z(z'|z) d\mu(z,k) \right\} d\pi_A(A'|A)$$

where $\mathcal{I}$ is an indicator function, and $\pi_z(z'|z)$ and $\pi_A(A'|A)$ are the transition probabilities of idiosyncratic and aggregate shocks, respectively.

2.4 Solving the Model

Each firm solves for the optimal capital level $\tilde{k_j}'$ from its first-order condition:

$$\tilde{k_j'} = \left(1 - \frac{1}{\phi}\right) k_j + \frac{k_j}{\phi} \cdot \mathbb{E}\left[M' \left(\xi A' z_j' (k_j')^{\xi-1} + 1 - \delta + \frac{\phi}{2} \left(\frac{k_j''}{k_j'} - 1 \right) \right) \right] \tag{4}$$

Then firms pay dividends

$$CF = \int_j CF_j dj = \int_j \left[A z_j k_j^{\xi} - \tilde{k_j'} + (1-\delta) k_j - \frac{\phi}{2} \left(\frac{\tilde{k_j'}}{k_j} - 1 \right)^2 k_j \right] dj$$

to the households. Each household makes portfolio allocation and consumption decisions by taking the prices as given. The Euler equations of the households price each stock $j \in \mathbb{J}$:

$$p_{j,t} = \mathbb{E}_t \left[M_{t+1} (CF_{j,t+1} + p_{j,t+1}) \right] \tag{5}$$

and the bond:

$$q_t = \mathbb{E}_t \left[M_{t+1} (CF_f + q_{t+1}) \right] \tag{6}$$

using the growth in marginal utilities as a stochastic discount factor given in equation (1).

For the internal consistency of the model, the stochastic discount factor of the households should be the same as the stochastic discount rate used in the firm problem (2).

Note that the value of each firm for their optimal capital levels is equal to the sum of its dividend and price in that period:

$$\underbrace{p_{j,t} + CF_{j,t}}_{V_{F,t}} = \underbrace{A_t z_{j,t} k_{j,t}^{\xi} - \tilde{k}_{j,t+1} + (1-\delta)k_{j,t} - \frac{\phi}{2}\left(\frac{\tilde{k}_{j,t+1}}{k_{j,t}} - 1\right)^2 k_{j,t}}_{CF_{j,t}}$$

$$+\mathbb{E}_t\left[M_{t+1}(\underbrace{p_{j,t+1} + CF_{j,t+1}}_{V_{F,t+1}})\right]$$

We can write down equation (5) as $\mathbb{E}_t[M_{t+1}R_{j,t+1}] = 1$ and equation (6) as $\mathbb{E}_t[M_{t+1}R_{f,t+1}] = 1$, where $R_{j,t+1}$ and $R_{f,t+1}$ are the returns for stock $j \in \mathbb{J}$ and the bond, respectively. By subtracting one from the other, we get the relationship between the excess returns and the SDF:

$$\mathbb{E}_t[M_{t+1}[R_{j,t+1} - R_{f,t+1}]] = \mathbb{E}_t[M_{t+1}R^e_{j,t+1}] = 0 \tag{7}$$

where $R^e_{j,t+1}$ is the excess return on stock j in period $t+1$.

Denote $CF_t = \int_j CF_{j,t}dj$ and $p_t = \int_j p_{j,t}dj$ for each time t. As for each identical household $i \in \mathbb{I}$, the shares are the same for each firm $j \in \mathbb{J}$, $\theta_{i,t} = \theta_{ij,t}$, total dividends and portfolio holdings of all the stocks for household i become $\int_j \theta_{ij,t}CF_{j,t}dj = \theta_{i,t}\int_j CF_{j,t}dj = \theta_{i,t} \cdot CF_t$ and $\int_j \theta_{ij,t}p_{j,t}dj = \theta_{i,t}\int_j p_{j,t}dj = \theta_{i,t} \cdot p_t$, respectively.

Now we can drop the individual state variables (z, k) from the household state-space, as well as replace the portfolio $\Theta_{i,t}$ with holdings in the bond $b_{i,t}$ and a single stock fund $\theta_{i,t}$ that includes all the stocks and represents the market portfolio. The household problem is now computationally tractable.

The recursive formulation of the household problem becomes:

$$\tilde{V}_H(\hat{\Theta}; A, \mu) = \max_{\hat{\Theta}'} \frac{\left(\theta(p + CF) + b(q + CF_f) - \theta^{'}p - b'q - \alpha\right)^{1-\gamma} - 1}{1-\gamma}$$

$$+\beta E\left[\tilde{V}_H(\hat{\Theta}'; A', \mu')|z, A\right] \tag{8}$$

where $\hat{\Theta} = (\theta, b)$ is the portfolio of individual i with the second asset being a risk-free console. The decision rule for updating the portfolio implied by this problem is denoted by the function $f^{\hat{\Theta}}$: $\hat{\Theta}' = f^{\hat{\Theta}}(\hat{\Theta}; A, \mu)$.

Next, to define the market portfolio, we integrate the prices of the stocks. The price of the market portfolio can be obtained by integrating all the stock prices:

$$p_t = \int_{j\in\mathbb{J}} p_{j,t}dj = \int_{j\in\mathbb{J}} \mathbb{E}_t\Bigg[M_{t+1}(CF_{j,t+1} + p_{j,t+1})\Bigg]dj$$

Under some regularity conditions over $G_{j,t} = M_{t+1}(CF_{j,t+1} + p_{j,t+1})$[7], using Fubini's theorem, we can take the integral over i under the expectation operator, and we obtain the price of the market portfolio:

$$p_t = \mathbb{E}_t\Bigg[M_{t+1}(CF_{t+1} + p_{t+1})\Bigg] \tag{9}$$

where CF_t is the aggregate market dividend as defined in equation (iii) in the general equilibrium definition. The households can diversify away all the idiosyncratic risks, so only the systematic risks earn premiums. For households, only the market risk matters as they hold the market portfolio in complete markets. I will also consider portfolios sorted according to the duration, where the idiosyncratic risk will be zero and only the duration-specific risk will be present. But first, we need to define what equity duration is.

2.4.1 Duration and Term Structure of Equity

Note that the SDFs (Stochastic Discount Factors) get multiplied when we insert tomorrow's forward price of stock into today's Euler equation (5):

$$p_{j,t} = \mathbb{E}_t[M_{t+1}(CF_{j,t+1} + \mathbb{E}_{t+1}[M_{t+2}(CF_{j,t+2} + p_{j,t+2})])]$$

$$= \mathbb{E}_t[M_{t+1}CF_{j,t+1}] + \mathbb{E}_t[M_{t+1}M_{t+2}CF_{j,t+2}] + \mathbb{E}_t[M_{t+1}M_{t+2}p_{j,t+2}]$$

$$= \mathbb{E}_t\left[\sum_{m=1}^{T} M_{t\to t+m}CF_{j,t+m}\right] + \mathbb{E}_t[M_{t\to t+T}p_{j,t+T}]$$

where $M_{t\to t+m} = M_{t+1} \cdot ... \cdot M_{t+m}$.

When $T \to \infty$, the term $lim_{T\to\infty}\mathbb{E}_t[M_{t\to t+T}p_{j,t+T}] = 0$ as prices are bounded, and heavy discounting drags the price in further periods towards zero from today's perspective. Therefore, we have:

[7] 1) G is measurable and 2)$\mathbb{E}[|G|] < \infty$ is finite. Measurability holds trivially. Finite expectations are the same as the limited arbitrage (finite Sharpe ratios), which also holds as the markets clear and market forces do not allow prices to drop to zero or jump up to infinity, creating infinite excess returns.

$$p_{j,t} = \mathbb{E}_t\left[\sum_{m=1}^{\infty} M_{t\to t+m} CF_{j,t+m}\right] \tag{10}$$

If a recession occurs at time t, the expected marginal utility in time period s increases relative to the expected marginal utility in future time $s+m$, for all $s \geq t$ and all $m > 0$. This affects all M_{t+m} terms, with a stronger effect for smaller m as the impact of the current shock on far future marginal utility fades out. Consequently, in equation (10), all the cash flows are discounted at a lower rate, but this affects long horizon cash flows differently than short horizon ones. Since the cash flows of different horizons have different risks and a stock is a sum of discounted cash flows, the expected return of stock j depends on the weighted average horizon of cash flows of the firm, which is referred to as its duration. The expected duration of a stock, denoted as $d_{j,t}$, can be defined as:

$$d_{j,t} = \mathbb{E}_t\left[\frac{\sum_{m=1}^{\infty} m\left(M_{t\to t+m} CF_{j,t+m}\right)}{p_{j,t}}\right]$$

Duration weighs the time a payment is received by the present value of the payment.

What creates a long-duration firm in my model? When a firm receives a high idiosyncratic shock, it starts accumulating more capital at the expense of dividends to reach an optimal capital level under the new shock. From an investor's perspective, the firm may not pay dividends currently but is expected to overpay in the future, creating long-duration firms. On the other hand, when a highly productive firm experiences a low productivity shock, it disinvests in its capital slowly because of the capital adjustment costs. This represents short-duration firms.

Since the stock market clearing equation (iv) in general equilibrium holds for each individual stock, it should also hold for any portfolio of stocks with similar duration when aggregated. We can determine the expected return and expected price of a portfolio of stocks with duration in some interval $d_t \subseteq \mathbb{D}_t$ at time t. Here, $\mathbb{D}_t$ represents an interval that includes all the stocks in the market sorted across duration at time t (time is discrete in the model but duration is an average statistic, hence, is continuous), while d_t is a sub-interval that also depends on time. Using portfolio rebalancing, we can map d_t into quantiles of $\mathbb{D}_t$ and use the d-quantile statistic as a fixed point or interval in $[0,1]$ that no longer depends on time. For instance, we can consider the upper 10% of stocks sorted by duration and compare their return with the lower 10%, as the order does not depend on time, although the duration range and stocks in those portfolios change over time. By utilizing Fubini's theorem to exchange the integral and expectation operators, we obtain:

$$p_t^d = \int_{j\in d} p_{j,t}\,dj = \mathbb{E}_t\left[\sum_{m=1}^{\infty} M_{t\to t+m} CF_{t+m}^d\right]$$

where $CF_{t+m}^d = \int_{j\in d} CF_{j,t+m}\,dj$ denotes the aggregate cash flow of portfolio d in period $t+m$.

To find the expected return on the sorted portfolio d in period $t+1$, $\mathbb{E}_t[R^d_{t+1}]$, we integrate over equation (7):

$$\int_{j\in d} \mathbb{E}_t[M_{t+1}R^e_{j,t+1}]dj = \mathbb{E}_t[M_{t+1}R^{e,d}_{t+1}] = 0,$$

Then, using the covariance formula, we have:

$$\mathbb{E}_t[R^d_{t+1}] = \mathbb{E}_t[R_{f,t+1}] - \frac{cov_t(M_{t+1}, R^{e,d}_{t+1})}{\mathbb{E}_t[M_{t+1}]} \tag{11}$$

Now, we turn to computing the dividend strips, equity yields, and forward equity yields (term structure of equity risk premia) used in the simulation of my model.

Dividend Strips: Consider the fully diversified portfolio. The price of a dividend strip m periods ahead is:

$$p_t^{(m)} = \mathbb{E}_t\Bigg[M_{t\to t+m}CF_{t+m}\Bigg] \tag{12}$$

The log return from holding that strip until maturity is $r_{t+m} = log\left[\frac{CF_{t+m}}{p_t^{(m)}}\right]^{\frac{1}{m}}$

Equity yields: The equity yield $e_{t,m}$ at time t with maturity m can be defined as follows (based on van Binsbergen et al., 2013):

$$\underbrace{\mathbb{E}_t[r_{t+m}]}_{\theta_{t,m}+y_{t,m}} = \underbrace{log\left(\frac{CF_t}{p_t^{(m)}}\right)^{\frac{1}{m}}}_{e_{t,m}} + \underbrace{\mathbb{E}_t\left[log\left(\frac{CF_{t+m}}{CF_t}\right)^{\frac{1}{m}}\right]}_{g_{t,m}}$$

$$e_{t,m} = y_{t,m} + \theta_{t,m} - g_{t,m} \tag{13}$$

where $y_{t,m}$ denotes the bond yields, $\theta_{t,m}$ is the risk premium, and $g_{t,m}$ is the expected dividend growth (all at time t with maturity m).

Forward equity yields can be computed as:

$$e^f_{t,m} = e_{t,m} - y_{t,m} = \theta_{t,m} - g_{t,m} \tag{14}$$

In the model, armed with these equations and the duration statistic, we proceed to simulate the equity term structure of those portfolios in the next section.

2.5 Results

2.5.1 Calibration

The inflation-adjusted average annual return on US equities from 1900 to 2017 was 6.5%. Similarly, the inflation-adjusted returns on US 10-year bonds during the same period were 2%. As a result, the risk premium over the 117 years amounted to 4.5% [8].

Between 1871 and 1913, prior to the establishment of the Federal Reserve System, the inflation-adjusted average annual return on US equities was 7.7%, while bonds yielded 4.8%. This resulted in a risk premium of 2.9%. In the past century, bonds were hedged against most of the recessions as the Federal Reserve artificially stimulated demand for bonds during most of the recessions. Consequently, bond returns declined while risk premiums increased.

The impact of the Federal Reserve on bond yields can be observed in Figure 20 of the appendix. The graph compares a period of the Fed's active quantitative easing and low-interest-rate policy during recessions with an inflationary period in which the Fed was unable to do so [9]. When examining agent behavior without significant intervention from the Fed, bond yields are higher during recessions than during economic expansions.

Household				
Param.	Value	Moment it matches	Data	Model
α_{mean}	0.9	risk premium	4.5%	4.3%
$\Delta\alpha$	0.35	risk premium	4.5%	4.3%
β	0.985	risk-free rate	2%	3.4%
γ	2	est. in the data		
Firm				
Param.	Value	Moment it matches	Data	Model
π^{bb}	0.32	recess. duration	1.46y	1.46y
π^{gg}	0.88	expans. duration	8.42y	8.42y
ΔA	0.018	cons. gr. std	3.65%	3.75%

Table 1: Calibration

Table 1 presents the calibration results for two aggregate shocks: a recession and an expansion. The transition probability matrix of these aggregate shocks is calibrated to match an average of 1.46 years of recession (probability of staying in the bad state is $\pi^{bb} = 0.32$) and 8.42 years of

[8] Source: updated monthly stock market data used in Shiller (2000)

[9] it could not fight inflation at the expense of negative growth, hence the flat yield curve in the 1970s and early 1980s

expansion (probability of staying in the good state is $\pi^{gg} = 0.88$)[10]. The calibrated values for the aggregate shocks are set at 0.7 ± 0.018, ensuring that the fluctuations in macroeconomic aggregates align with the magnitude observed in the data for the US during the period 1889-1913.

Furthermore, the time preference parameter β and habit parameters (α_{bad},α_{good}) are chosen to achieve a risk premium of 4.3% while maintaining the risk-free rates at their lowest possible level (3.4%).

	Data	Model 1 (W_1)	Model 2 (opt W_2)	Model 3 (clW_3)
mean(I/k)	0.1114	0.1115	0.1115	0.1115
var(I/k)	0.0086	7.4e-30	6.0e-06	1.6e-04
mean(V/k)	2.3667	2.7254	2.4461	2.6116
var(V/k)	8.5588	2.5339	0.6947	1.7156
mean(CF/k)	0.2041	0.1795	0.1507	0.1670
var(CF/k)	0.0685	0.5781	0.1125	0.3432
Est. param				
ξ		0.1500	0.3000	0.1500
δ		0.1115	0.1115	0.1115
A		0.5000	0.7000	0.6000
ϕ		32.115	14.2141	3.7377
σ_z		1.1000	0.7000	1.0000

Table 2: Simulated Method of Moments for 3 models with different weight matrices

I use the accepted estimates for the risk-aversion parameter γ from the literature and estimate the capital share ξ, the depreciation rate δ, the average aggregate productivity A, the adjustment cost parameter ϕ, and the standard deviation of idiosyncratic shocks σ_z within the model such that the distance between the first two moments of CF/k, V/k, and I/k variables in the data and in the model is minimized (results for the simulated method of moments estimation are in Table 2). The data is obtained from the firm fundamental annual variables in the CRSP/Compustat merged dataset for the period 1970-2022 (annual). To clean the data, I follow the approach of Bazdresch, Kahn, and Whited (2018)[11] with the change that I take PPEGT as the capital variable in the

[10] I matched the contractions and expansions in the consumption data for the 1889-2009 sample, data from Chapter 26 of Shiller (1989).

[11] drop all firms with less than 2 years of data, also those that belong to the financial (SIC 6) or regulated (SIC 49) sectors, quasi-governmental (SIC 9) firms, and the U.S. Postal Service. I also follow them by deleting all the observations where any of the variables used are missing, and by winsorizing at 1% level to minimize the impact of outliers.

denominator of the ratios, while they use total assets due to the leverage in their model.

Let m denote moments, x_i denote actual data, s_{ik} denote the simulated data vector from simulation k, and S denote the number of simulations. The simulated moments $m(s_{ik}(b))$ are functions of the parameter vector $b = (\delta, \xi, A, \phi, \sigma_z)'$ because the moments differ depending on the choice of b.

$$g_n(b) = n^{-1} \sum_{i=1}^{n} \left[m(x_i) - S^{-1} \sum_{k=1}^{S} m(s_{ik}(b)) \right]$$

The estimator for the simulated moments, denoted as $\hat{b}$, is defined as the solution to the minimization problem:

$$\hat{b} = arg \min_b Q(b, n) \equiv g_n(b)' \hat{W}_n g_n(b)$$

For the first model in Table 2, I choose the weight matrix W_1 to assign the highest weight to the mean of CF/k (an identity matrix performed poorly). In the second model, the optimal weight matrix W_2 is selected (the inverse of the variance matrix), and in the third model, the optimally weighted matrix W_3 is the inverse of the variance matrix clustered by firm.

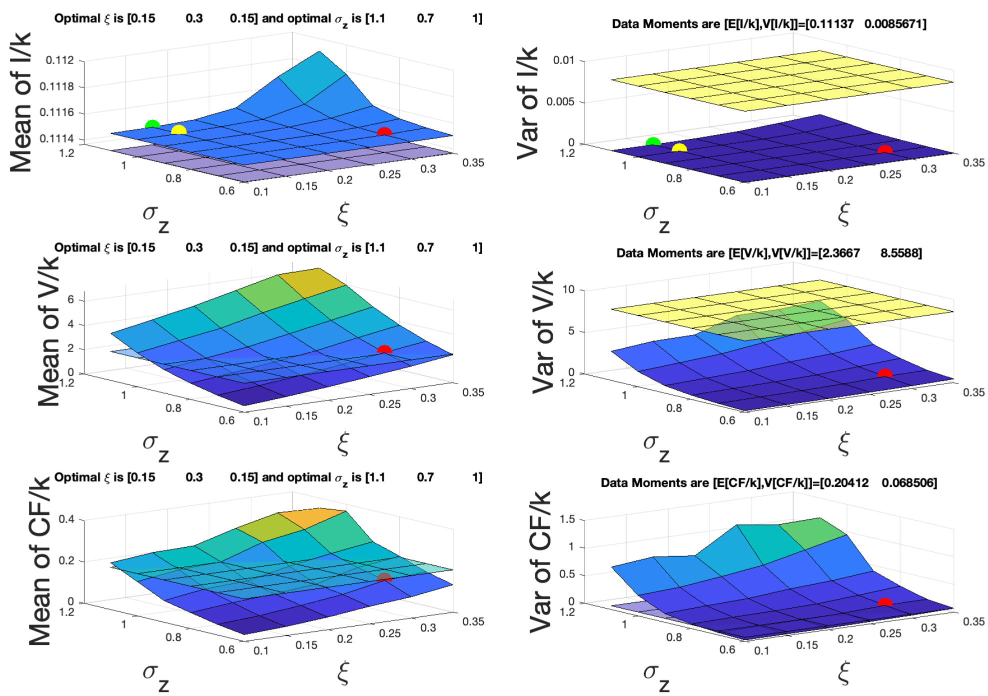


Figure 1: Simulated method of moments: moments' sensitivity to the parameter pair (σ_z, ξ)

Figure 1 illustrates how these moments change with the pair of parameters (σ_z, ξ). Since the average of the variable I/k is 0.11 in the data, the depreciation rate δ is set to 0.11 across models 1-3. The CF/k and V/k variables move in the same direction, making it challenging to precisely match both of them (those change with the parameters A, ξ, σ_z). The adjustment cost parameter ϕ is sufficiently high to slow down the capital accumulation process (it also creates a lag in consumption response, and makes significant distinction between growth and value firms in the model).

A high idiosyncratic productivity shock leads to firms holding less capital and being more flexible, as capital adjustment is costly. Conversely, a small idiosyncratic risk requires fewer capital adjustments, allowing firms to hold a higher level of capital. When individual firm capital holdings are high, firms are more resilient to aggregate shocks, resulting in minimal changes in investments and a more responsive dividend process (with a jump at impact) compared to when firms are small and their capital fluctuates more with shocks. I design the transition probability of the idiosyncratic shock to depend on the aggregate shock state, $\pi(z'|z, A) \neq \pi(z'|z)$ as more firms are unproductive in recessions than in expansions. As a result, high idiosyncratic shocks lead to a higher standard deviation in consumption growth, even though the stock portfolio is fully diversified.

Therefore, firm heterogeneity adds another dimension (a 'free parameter') to control the cash flow reaction to the aggregate shock when our hands might be tightened to change ϕ since it has already been matched to the data.

2.5.2 Simulation

We can strip out the present values of each coupon payment of the perpetuity, effectively creating zero coupon bonds with various maturities. During recessions, bond prices are generally lower compared to booms, as households tend to have lower wealth. To examine how bond returns change over economic cycles, I construct the bond yield curve. By utilizing equation (13) for equity yields, we can substitute the constant coupon payment as the cash flow and generate the yield curve for bonds. From the simulated 1100 periods, I exclude the first 100 observations (need steady-state convergence) and plot the average bond yield curve conditional on the economic phase (recession versus expansion) in Figure 2.

During periods of economic prosperity (booms), investors anticipate economic growth over time. Consequently, rates tend to rise (an upward-sloping bond yield curve). In recessions, the economy is expected to contract, resulting in a decline in bond yields over the horizon. Despite being less wealthy during recessions, households save more over time in anticipation of prolonged recession. Hence, the bond yield curve exhibits a downward slope. The magnitude of the volatility is relatively larger in the model than in the data due to the absence of a monetary authority in the model

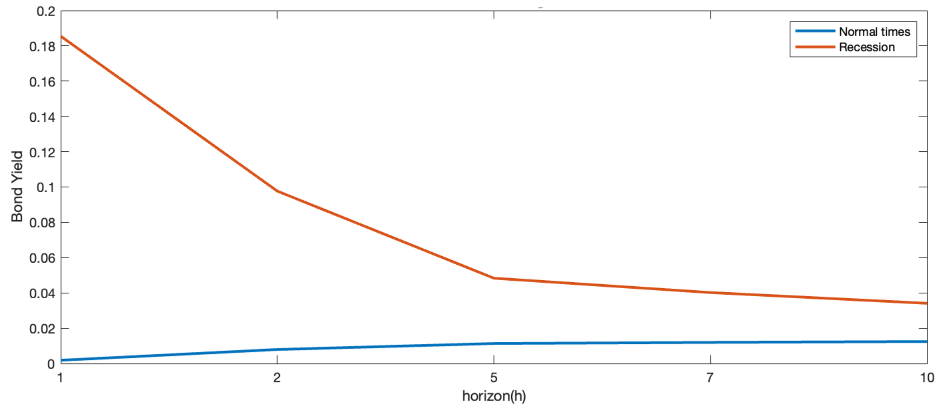


Figure 2: Simulated average term structure of bond yields conditional on the aggregate state

to stimulate the economy during recessions (in the data the flat or increasing bond yield curve in recessions does not solely represent household behavior). In this model, volatility arises from fluctuations in the household Stochastic Discount Factor (SDF). The precautionary saving behavior of households, often observed in typical habit models, is not applied here as the low risk-free rate is not a behavioral phenomenon (see Figure 20 for evidence of the impact of the Fed's intervention on bond yields).

To explore whether the term structure of forward yields is upward or downward-sloping in the model and how it behaves over the business cycle, I simulate the yields on the forward strip for the entire (aggregate) market. Similar to the bond yield curve, I exclude the first 100 observations from the simulated 1100 periods and plot the average term structure of simulated forward yields conditioned on the aggregate state in Figure 3. This qualitative reproduction aligns with the data shown in Figure 17 in the appendix.

Additionally, I plot the forward strip yields for 40 periods for aggregate equity strips with maturities of 1, 3, 5, 7, and 10 years (Figure 4). It is evident that the slope is downward-sloping during recessions and upward-sloping during expansions, similar to Figure 3. The plot also demonstrates how strips of different maturities invert over the business cycle, with longer strips being affected to a lesser extent. The inversion of the term structure in the data has been documented by Bansal et al. (2017), and further confirmed by Giglio, Kelly, and Kozak (2021) using an alternative approach (see Figure 18 in the appendix).

The distinguishing factor of this asset pricing model lies in the modeling of dividends. When the economy experiences a negative aggregate productivity shock in period t,

$$CF_{j,t} = \left[\downarrow A_t z_{j,t} k_{j,t}^{\xi} - \downarrow I_{j,t} - \frac{\phi}{2} \left(\frac{I_{j,t}}{k_{j,t}} - \delta \right)^2 k_{j,t} \right) \right]$$

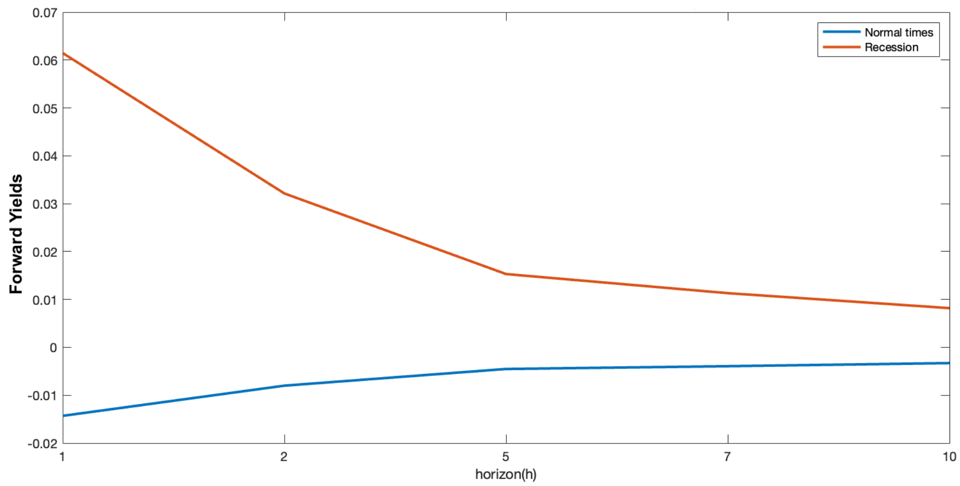


Figure 3: Simulated average term structure of forward yields conditional on the aggregate state

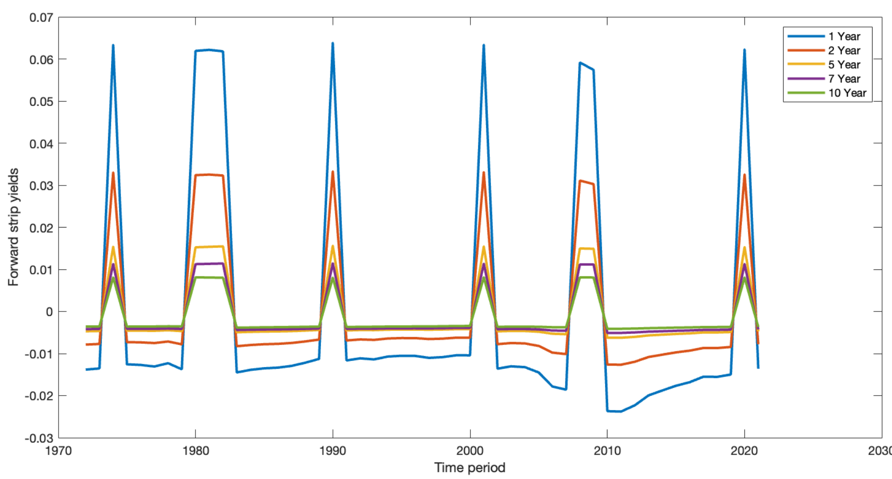


Figure 4: Simulated forward strip yields on aggregate market (high during recessions)

Dividends increase at the moment a negative shock impacts the economy since firms initially sell their nonproductive capital and distribute the proceeds as dividends (Figure 5). In contrast to other asset pricing models where consumption drops immediately upon a negative shock (refer to Figure 24 in the appendix for a shock to the dividend process in the habit model and Figure 25 for a shock to the dividend process in the long-run risk model), the lagged consumption response in this model allows for a more significant negative expected dividend growth. This aligns with the data on the decomposition of equity yields (Table 3) and the data on lagged consumption response (Figures 27 and 28 in the appendix).

In a homogeneous firm model, a similar response can be achieved with a lower value of ϕ to make investment more volatile and dividend (consumption) smoother. However, this disturbs other moments (CF/k, I/k, V/k). A comparison of the same model with homogeneous firms is presented in Figure 29 in the appendix. A similar response is observed when the adjustment cost channel is eliminated.

Maturity	g	θ
5	88%	12%
7	78%	22%
10	70%	30%

Table 3: Simulation: The variance decomposition of forward yields into hold-to-maturity, θ, and expected dividend growth, g, components for maturities 5, 7 and 10 years

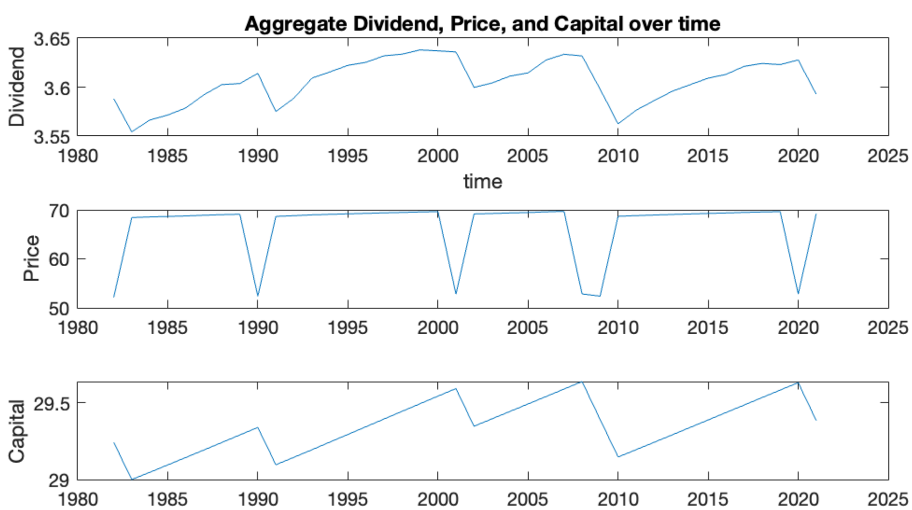


Figure 5: Simulated Dividend, Stock Price, and Capital in a Heterogeneous Firm model

In the data, the forward yield is decomposed into 30% risk premia and 70% expected dividend growth components (van Binsbergen et al., 2013). Figure 6 illustrates the decomposition in my simulated model, which is comparable to the data (Figures 21 and 22 in the appendix). It demonstrates that high forward yields during recessions primarily indicate low expected dividends rather than high risk aversion.

The model implies that equity yields should predict future dividend growth. I test this implication in the data by running regressions of future dividend growths (at different forward horizons) on current dividend strip yields: $g_{t+h,t+h+1} = \beta_0 + \beta_1 e_{t,m} + \varepsilon_t$, where h represents the forward time horizon between dividend growth and equity yields in months, and m is the maturity of the dividend strip. Different versions of the regression, using the annualized 1-year and 2-year forward dividend growth as dependent variables or the stock market yield as an explanatory variable instead of dividend strip yields, yield similar results. Figures 30-39 in the appendix summarize the regression coefficients and their 95% confidence intervals. The figures demonstrate a strong negative relationship between current equity strip yields and future dividend growth, as well as between current stock market yields (dividend-price ratio) and future dividend growth.

For the aggregate US stock market, dividend growth predictability persists but has decreased in recent years. This is attributed to less predictable dividend growth in growth stocks compared to value stocks, combined with the increasing share of growth stocks in the overall market (Rangvid et al., 2014; Maio & Santa-Clara, 2012; Verdickt et al., 2019).

Although the dividend process in the model can be approximated using an exogenous process with many MA or AR terms (see Figure 26 in the appendix for dividend growth following MA(11)), it is preferable to leverage the economic mechanism of the dividend process and explore its interaction with the household SDF in a model that aligns with firm-level data.

In recent data, the bond yield curve does not experience a sharp inversion during recessions due to the intervention of the Federal Reserve, which purchases bonds and controls the bond yield curve to stabilize the market. As a result, the bond yield curve only briefly inverts before the actual recession and becomes upward-sloping again during the recession as a response to the actions of the Federal Reserve. While productivity may decline and stock prices may fall, the artificial demand for bonds created by the Fed makes them a hedge against recessions, leading to a decrease in their risk premia. However, during inflationary periods when the Fed cannot stimulate the economy through low rates and quantitative easing, agents' behavior results in much higher interest rates during recessions compared to expansions (refer to Figure 20(b) and (c) in the appendix).

2.5.3 Cross-sectional result

The heterogeneous firm model is suitable for explaining the value premium observed in the data. The reason why a short duration portfolio earns excess returns over a long duration portfolio is due to the way their cash flows interact with the household SDF (Figure 7).

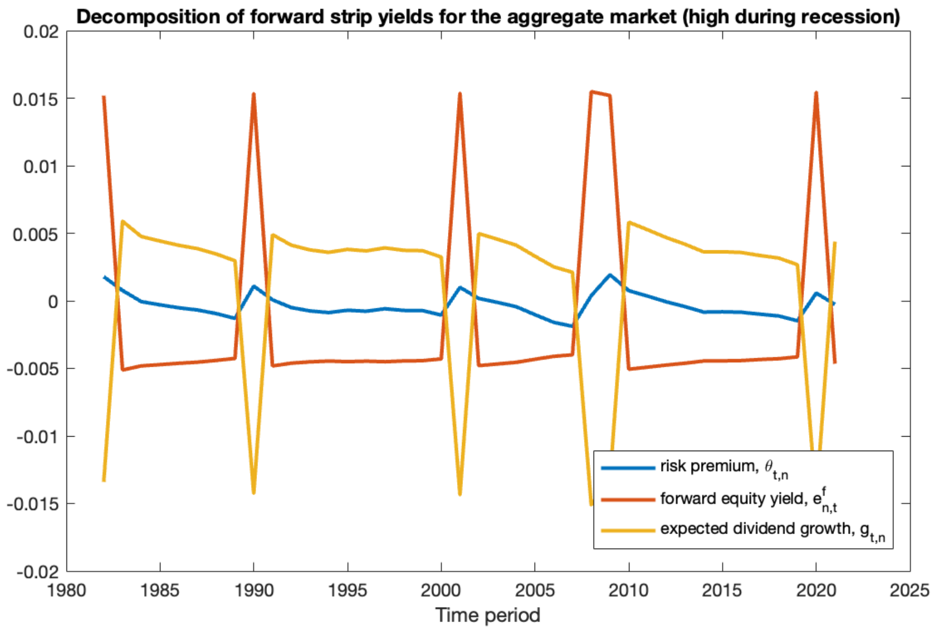


Figure 6: Simulation: Decomposition of 5y Forward Strip Yield

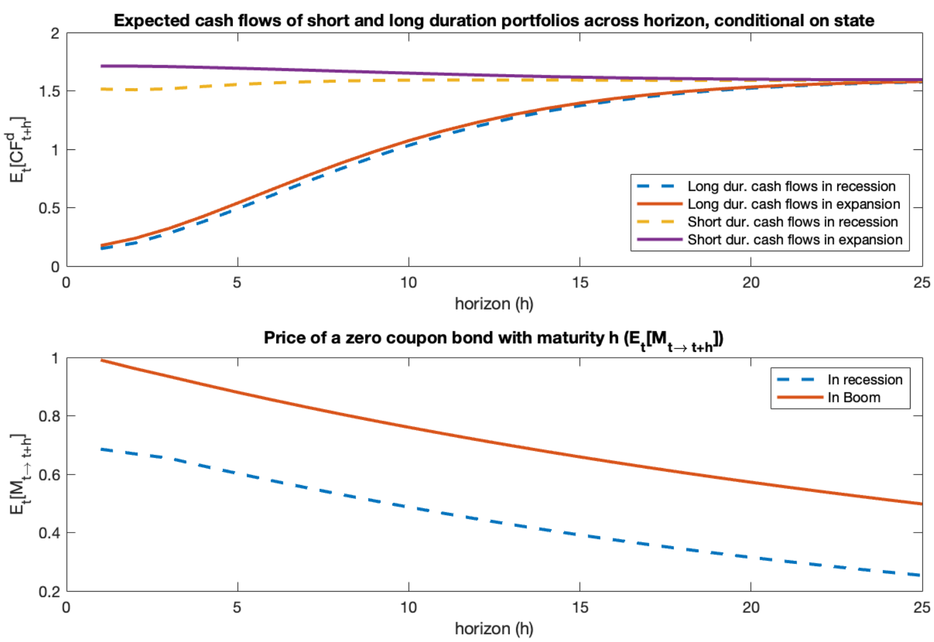


Figure 7: Simulation: Conditional on aggregate state (a) Dividends of Short and Long Duration stocks and (b) SDF

For the short duration portfolio, most of its cash flow weight is in the initial periods where the cash flows are subject to market risk. When buying the short duration stock today, there is a risk that it might pay less in the near future than expected. On the other hand, the market risk has a negligible effect on long duration stocks as they start paying only far in the future. The simple intuition is that relying on stocks to provide dividends in the near future and receiving less is much worse than not relying on them at all.

2.6 Microeconomic Implications of Heterogeneous Households

I introduce the subsistence consumption parameter, inspired by the habit model, using a Stone-Geary type of utility function (or more generally, HARA utility function), $u_i(c_{i,t}) = \frac{(c_{i,t}-\alpha_i)^{1-\gamma}}{1-\gamma}$, where $c_{i,t}$ represents the consumption level of household i in period t, γ is the relative risk aversion parameter common for all households, and $\alpha_i > 0$ represents the subsistence consumption level. The subsistence consumption level ensures that the relative risk aversion decreases with wealth ($\alpha_i = 0$ corresponds to the CRRA case). Therefore, the wealthier an investor, the more they prefer stocks over riskless bonds. The relative risk aversion is given by $rra_{HARA}(c_i) = -\frac{u''(c_i)}{u'(c_i)}c_i = \gamma\frac{c_i}{c_i-\alpha_i}$.

This framework allows for fluctuations in the stochastic discount factor. During recessions, when households lose part of their wealth, they become more risk-averse (decreasing relative risk aversion[12]) and shift their remaining wealth towards bonds more than they would in the CRRA utility case.

As heterogeneity in household labor income has negligible contribution, labor is dropped to simplify the model. Therefore, there is no real effect of household heterogeneity on the aggregate variables or equity structure dynamics. The heterogeneous households can be aggregated into an artificial representative household with weights on each of the heterogeneous households' utility functions.

2.6.1 Heterogeneous Households with HARA utility

I fix parameters α_i according to the households' given initial endowment of assets, where households with high initial wealth have high subsistence consumption α_i. The wealth order of households remains the same, but their wealth levels change over time (see Figure 40 in the appendix).

The household problem transitions to heterogeneous households, where each household has equi-cautious HARA utilities. These utilities have the same risk aversion parameter γ but different subsistence consumption parameters α_i. This configuration represents decreasing relative risk

[12]
$$\frac{\partial rra_{HARA}(c_i)}{\partial c_i} = -\frac{\alpha_i\gamma}{(c_i-\alpha_i)^2} < 0.$$

aversion. The households differ in their initial endowments of stocks $\theta_{ij,-1}$ and subsistence consumption $\alpha_i > 0$, which tends to be higher for wealthier individuals. Initially, no households have any bonds ($b_{i,-1} = 0$). To simplify the analysis, we assume that all households have non-negative initial holdings of each asset and at least a share of stock.

The households trade in period zero, with wealthy households preferring risky stocks and poorer ones favoring bonds.

The household problem is defined as:

$$\max_{\{c_{i,t};\{\theta_{ij,t+1}\}_{j\in\mathbb{J}},b_{i,t+1}\}_{t=0}^{\infty}} \mathbb{E}\sum_{t=0}^{\infty}\beta^t \frac{(c_{i,t}-\alpha_i)^{1-\gamma}}{1-\gamma}$$

subject to:

$$c_{i,t} + \int_j \theta_{ij,t+1} p_{j,t} dj + b_{i,t+1} q_t \leq \int_j \theta_{ij,t}(p_{j,t} + CF_{j,t}) dj + b_{i,t}(q_t + CF_f)$$

Here, the notable change is that α_i is now a fixed habit and differs for each household.

The Stochastic Discount Factor (SDF) for an individual household is defined as:

$$M_{i,t+1} = \beta \frac{u'(c_{i,t+1})}{u'(c_{i,t})} = \beta \left(\frac{c_{i,t+1} - \alpha_i}{c_{i,t} - \alpha_i} \right)^{-\gamma}$$

The aggregate SDF for the economy is the same as the individual SDF for each household.

In complete markets, the marginal utility vectors are collinear. To demonstrate collinearity, I compute the consumption allocations using the Negishi approach (Negishi, 1960), as in Judd et al. (2003). For utility functions of the HARA type, the first-order conditions imply $\lambda_{i'} u'(c_{i',t}) - \lambda_i u'(c_{i,t}) = 0$, where λ_i is the Negishi weight on household i, and $i, i' \in \mathbb{I}$. Since the ratio of marginal utilities for each household is the same, they all have the same stochastic discount factor and agree on asset prices. In other words, the volatility in the marginal utilities of poor and wealthy agents is the same, but it translates into different consumption levels and smoothness due to the subtraction of different individual α_i values from their consumption. From here on, I use the aggregate SDF instead of the individual SDF, as they are equal (i.e., $M_{t+1} = M_{i,t+1}$, $\forall i \in \mathbb{I}$). I look for an equilibrium in which a common SDF exists.

The household problem in my model fits into the setting of Judd et al. (2003). Two results from their work greatly simplify the equilibrium analysis. First, efficient equilibria exhibit time-homogeneous consumption processes and asset prices. This means that consumption allocations and asset prices in date t depend only on the last shock. Second, after one round of initial trading in period 0, each agent's portfolio remains constant across states and time. This implies that for all t and all $i \in \mathbb{I}$

$$c_{i,t} = \int_j CF_{j,t}\theta_{ij,t}dj + CF_f b_{i,t} \tag{15}$$

Households consume only the dividend and interest income, and they hold the same amount of stocks and bonds each period. Wealthy households, being less risk-averse, primarily hold stocks and short bonds (borrowing from the poor and middle class). The main income of poor households comes from bonds. This ensures that not only poor households tend to smooth their consumption but also the middle class mostly holds bonds and does not enjoy the good states of aggregate fruitfulness, as they cannot afford the volatility in their consumption required to buy stocks. Thus, only high net-worth households, who are always further away from their habit level, hold most of the stocks.

Since the households have equi-cautious HARA utilities, the two-fund separation theorem, as described in Schmedders (2007), applies[13]. This theorem states that in an efficient equilibrium, household portfolios exhibit two-fund separation, that is $\theta_{ij} = \theta_{ij'}$ for all risky assets $j, j' \in \mathbb{J}$, and each household $i \in \mathbb{I}$ in every period t holds a weighted average of the two funds: the market portfolio of all stocks and the bond. The theorem and its proof are provided in Appendix A1. From the derivation, we obtain the linear sharing rule for consumption: $c_{i,t} = \theta_{ij}C_t + b_i$, $\forall j$, where θ_{ij}, $\forall j$ depends on the relative weight of the household's wealth in the overall wealth of all households (weights do not change after the initial endowment), whereas b_i also depends on the individual α_i values.

The two-fund separation theorem simplifies the state space of the households by removing the individual state variables of all firms, making the household problem tractable. We denote $CF_t = \int_j CF_{j,t}dj$ and $p_t = \int_j p_{j,t}dj$ for each time t. Since for every household $i \in \mathbb{I}$ the shares are the same for each firm $j \in \mathbb{J}$, $\theta_{i,t} = \theta_{ij,t}$, the total dividends and portfolio holdings of all stocks for household i become $\int_j \theta_{ij,t}CF_{j,t}dj = \theta_{i,t}\int_j CF_{j,t}dj = \theta_{i,t} \cdot CF_t$ and $\int_j \theta_{ij,t}p_{j,t}dj = \theta_{i,t}\int_j p_{j,t}dj = \theta_{i,t} \cdot p_t$, respectively.

Now we can eliminate the individual state variables (z, k) from the household state-space and replace the portfolio $\Theta_{i,t}$ with holdings in the bond $b_{i,t}$ and a single stock fund $\theta_{i,t}$ that includes all the stocks and represents the market portfolio. The household problem becomes computationally tractable.

The households share the aggregate stock market dividend proportional to their wealth levels while taking either long or short positions in bonds depending on their subsistence consumption levels. Without the fixed subsistence consumption parameters α_i, the households hold zero bonds. In the specific case where the linear sharing rule takes the form $c_{i,t} = m_i C_t$, the bond holdings of all households are zero. This corresponds to the habit model of Campbell and Cochrane (1999)

[13] Schmedders added the two-fund separation theorem to the result of Judd et al. (2003)

with heterogeneous agents, which will be discussed next.

2.6.2 Heterogeneous households with state dependant habits

I modify the model to allow changes in the subsistence consumption parameter α_{it}, which now varies across both household i and time t.

Campbell and Cochrane (1999) describe different wealth groups consisting of identical individuals, where each individual's habit depends on the average consumption of their reference group. In this model, I assume that the heterogeneous groups are households with identical members. Following their definition, I define the surplus consumption ratio for household i as $s_{i,t} = \frac{c_{i,t}-\alpha_{i,t}}{c_{i,t}}$. To maintain the heterogeneity of households while preserving the two-fund separation theorem, I model individual consumption as a fixed portion of aggregate consumption: $c_{i,t} = m_i C_t$. In this particular case, we have the same consumption growth among individuals. Moreover, if we start from an initial state where everyone has the same surplus consumption ratio, then all their surplus consumption ratios will be the same and the marginal utilities will be collinear. Since they all agree on asset prices, there is no reason for them to trade away from their initial endowments in any period. Therefore, the linear sharing rule for consumption and the two-fund separation theorem still hold (see Appendix A2 for details). The only drawback compared to the fixed habit case is that now in equilibrium all the investors hold shares of stocks but no bonds.

3 Conclusion

I have constructed a general equilibrium model that matches the firm-level data and accurately captures the decomposition of forward yields into risk premium and expected dividend growth components observed in the data. The model also generates an upward-sloping equity term structure during booms and its inversion during recessions. The mechanism works due to the link between the timing of dividends and expected returns on stocks. Additionally, the model explains the value premium and the forward yield decomposition in the cross-section.

In this model, the expected dividend growth is the main component in the forward yield decomposition. Short-duration firms have a more predictable dividend growth process than long-duration firms, as the market risk does not significantly affect long-duration firms. The value premium emerges due to the co-movement of the cash flows of short-duration stocks with the households' stochastic discount factor, while the effect is negligible for long-duration stocks. Furthermore, the model generates the inversion of the bond yield curve.

CHAPTER II.

Common Ownership and Inequality

1 Introduction

Since the 1980s, markups, inequality, and common ownership have been on the rise in the US. Recent literature on common ownership explores the relationship between firm ownership and their strategic decisions in product markets. The *common ownership hypothesis* suggests that if firms' decision-making is an expression of investors interests, and influential investors have stakes in competing firms, one might expect firms to put a nonzero profit weight on competing firm's profit. Consequently, when large investors own shares in multiple firms within the same industry, one might anticipate reduced competition, higher prices, and other characteristics of a monopolistic market within that industry. This implies significant harm to consumers.

The hypothesis assumes that firms aim to maximize shareholder value, which includes stakes in competing firms. Therefore, firms may strive to maximize a combination of their own profits and their competitors' profits to achieve this objective. However, why do CEOs focus on maximizing their shareholders' portfolio value instead of other factors such as utility? Information regarding shareholders' portfolios is mostly publicly available, and portfolios have a monetary value with corresponding stakes, making them measurable. Conversely, utility is unobservable and heterogeneous among investors. Additionally, CEOs may not possess knowledge about their shareholders' income sources other than portfolio income. Thus, the maximization of investors' portfolios in the common ownership hypothesis represents a simple, intuitive, and convenient formulation. Section 3.3 provides a more detailed explanation, connecting the common ownership hypothesis to the Cournot model.

This concept traces back to Rotemberg's work in 1984, which highlighted how common ownership can lead to monopoly outcomes. Recent research has examined common ownership in the airline industry by Azar, Schmalz, and Tecu (2018), and in the banking sector by Azar, Raina, and Schmalz (2019). Additionally, Backus, Conlon, and Sinkinson (2019) proposed an approach to measure this phenomenon among S&P 500 firms.

Therefore, common ownership enables companies to bypass monopolistic regulations while still operating as part of a larger monopolistic entity. However, what does this have to do with inequality? The idea is straightforward: primarily wealthy individuals have investments in the stock market (indirectly through hedge funds and mutual funds, in other words, indirectly "commonly own" numerous companies or benefit from owning a portion of a monopolized industry through direct stock investments). However, their consumers encompass the entire population, including those who do not benefit from such ownership[14].

This paper aims to quantitatively address the questions surrounding the percentage of the increase in markups and the extent to which the rise in inequality can be attributed to the growth of common ownership *assuming that the common ownership hypothesis is true*, although empirical validation is not the objective of this study.

One crucial aspect of this research is to determine the current position of the US economy on the spectrum from perfect competition (no common ownership) to complete monopoly (high common ownership). As we move towards a world dominated by monopolies, characterized by decreasing efficiency and rising inequality, it is essential to recognize that common ownership is a consequence of passive investment. The shift from active investment to passive investment is evident in the increasing dominance of funds tracking broad US equity indexes, which reached $4.27 trillion in assets as of August 31, 2019, surpassing stock-picking rivals with only $4.25 trillion in assets (Backus et al., 2019)[15]. This rise in indexing and diversification is primarily driven by Vanguard, BlackRock, and StateStreet. Understanding whether this trend has positive or negative implications in terms of productivity, competition, and wealth disparity is equally significant.

The remaining sections of the paper are organized as follows: Section 1.1 provides a brief literature review covering income inequality, common ownership, and the decline in competition. In Section 2, a standard general equilibrium model of monopolistic competition is described. Finally, Section 3 offers concluding remarks.

1.1 Literature Review

There have been various studies conducted on common ownership across different sectors, such as airlines (Azar, Schmalz, and Tecu, 2018) and banking (Azar, Raina, and Schmalz, 2019). However, there is a potential endogeneity issue when estimating the impact of common ownership on competition. When investors anticipate a price increase (resulting in larger industry markups), they might purchase shares of all companies within the industry. This concern has been discussed and addressed in the aforementioned papers.

Barkai (2019) provides empirical evidence regarding the overall decline in competition within the US economy. In his study, Barkai explains the simultaneous decrease in capital and labor shares by

[14] the wealthy top 10% owns 84% of the stocks according to Wolf (2017)

[15] According to research firm Morningstar Inc., during the first-ever monthly reporting period ending on August 31, 2019, funds tracking broad U.S. equity indexes amassed $4.27 trillion in assets. This figure surpassed the assets of stock-picking rivals, who aimed to outperform the market, totaling only $4.25 trillion on the same date.

attributing it to an increase in profit share. He employs a standard model that quantitatively aligns the decline in competition (reflected in markup increases) with the available data. Additionally, Barkai explores the impact of decreased competition on welfare and finds substantial disparities in output, wages, and investment. I will adopt a similar standard model while incorporating the structure of common ownership into it. Another relevant paper alarming reduced competition, authored by Autor et al. (2019), discusses the rise of superstar firms and their increasing influence. However, it focuses on the direct effect of monopolization rather than the indirect effect through common ownership.

On the other hand, extensive literature documents the increase in income inequality since the 1980s, which coincides with the rise of hedge funds (Fichtner, 2013). However, no paper has specifically examined the relationship between income inequality and common ownership.

Alesina, Angeloni, and Etro (2005) present a study on international unions that hold power over the provision of certain public goods and policies. In their model, there exists a trade-off between the benefits of coordination and the loss of independent policymaking. Within their framework, it can be argued that CEOs aim to maximize shareholder utility, leading to a non-linear relationship between shareholders' value and the common ownership measure. Initially, with a small common ownership (CO) measure, CEOs may seek to increase markups. However, at a certain point, further markup increases would start to diminish shareholders' utilities. Nevertheless, in this paper, I adhere to the common ownership hypothesis, which assumes that CEOs are incentivized to maximize shareholders' portfolios rather than their utility. Hence, CEOs aim to maximize their companies' profits along with a linear combination of others' profits.

2 Model

A general equilibrium model of monopolistic competition incorporating common ownership is utilized to examine the relationship between common ownership and inequality. The model incorporates preference and productivity shocks while excluding aggregate shocks.

To calibrate the model, I focus on the US economy, specifically for the year 2016. The underlying idea of the model is as follows: Agents make decisions regarding work, consumption, and saving. While agents are similar in most aspects, their productivity levels differ, resulting in varying labor incomes. Productivity shocks exhibit high persistence, leading highly productive agents to accumulate capital, while less productive agents primarily consume their labor income. Agents have the option to save in either bonds or stocks. Corporate bonds are issued by firms to finance capital purchases required for production. Once production is complete, capital is sold, and the fixed rent for capital, alongside the principal, is paid to bondholders. Stocks, also issued by firms, are riskier in comparison to bonds. Consumer preference for goods (preference shock) impacts a company's profits and stock returns, as the demand for the produced goods fluctuates with the change in preference. Household preference shocks generate differences in stock ownership, with varying risk and return levels for stocks[16]. Owners, therefore, do not completely diversify the risk in stocks; instead, they concentrate their holdings to some extent based on their preferences. Consequently, firms are not entirely commonly owned but only partially. This partial common ownership enables firms to exhibit a more monopolistic behavior compared to a scenario without common ownership.

The model is simulated until a steady-state distribution is achieved, allowing for the examination of inequality in labor income, capital income, and wealth. As this is a model incorporating monopolistic competition, the presence of the common ownership hypothesis intensifies monopolistic behavior, leading to higher markups and profits compared to a standard monopolistic competition model. Since stocks are primarily concentrated in the hands of the wealthiest agents, they receive a significant portion of the profits. However, all agents are consumers of these goods, thereby exacerbating inequality through the disproportionate distribution of monopolistic profits, which are elevated due to the application of the common ownership hypothesis. To precisely quantify the extent to which markups and inequality increase due to common ownership, I eliminate the common ownership term and allow CEOs to maximize their company's profits, similar to a standard monopolistic competition model without common ownership. Then I compare the results of the two scenarios. The subsequent section provides a mathematical description of the model.

2.1 Measure of Common Ownership

Antitrust economics has developed two quantitative methodologies to assess the impact of mergers on pricing and output incentives. The first methodology, known as the Herfindahl-Hirschman Index (HHI), has been employed in the Horizontal Merger Guidelines since 1982. It is based on the Cournot oligopoly model, which assumes quantity competition among firms producing homogeneous products. Another methodology, pioneered by Carl Shapiro (1996) and others, utilizes the diversion ratio and relies on the Bertrand model of price competition among firms offering differentiated products.

The HHI concentration index is commonly used in antitrust as an initial screening tool to evaluate the effects of mergers on competitive incentives. While the HHI is sometimes viewed as an arbitrary measure of concentration, it does have a theoretical foundation in industrial organization

[16] It should be noted that the stock market is considered inefficient in this model, as there exist overvalued and undervalued stocks, primarily due to households of different wealth levels favoring different firms. Less productive households prefer less profitable firms due to their lower income, resulting in lower stock demand (price) for less profitable firms but higher returns compared to more profitable firms. Consequently, the two-fund monetary theorem does not hold in this context, and agents maintain different bundles of risky portfolios.

economics. In the Cournot oligopoly model, where entry barriers protect firms producing homogeneous products, the HHI is related to the margin between market price and cost. This Cournot model can be extended to incorporate partial ownership interests under different assumptions regarding corporate control. Building upon this methodology, Bresnahan and Salop (1986) modified the HHI to account for various alternative financial interest and corporate control scenarios, resulting in the Modified Herfindahl-Hirschman Index (MHHI). The MHHI provides an approximate measure of the impact of partial ownership scenarios on competitive incentives.

In my analysis, I adopt O'Brien and Salop's (2000) version of the MHHI delta. This measure, known as the common ownership concentration, captures the density of the ownership and control network among competitors within a specific market. Its formula allows for evaluating the competitive effects of partial ownership acquisitions in a manner similar to how increases in the HHI are used in merger analysis. The MHHI calculations provide a rough estimate of the impact on the ownership structure. However, it is important to note that both MHHI and conventional HHI calculations make several simplifying assumptions, such as assuming a relevant market without product substitution, prohibitive entry barriers, no other competitive factors, and no efficiency benefits. Despite their limitations, these calculations serve as an initial step, much like the HHI, in merger analysis.

Consider an industry or market with K natural competitors, each owned by I shareholders. The ownership share of shareholder i in firm j is denoted as $\beta_{ij,t}$, and the control share held by shareholder i in firm j is denoted as $\gamma_{ij,t}$. The total portfolio profits of shareholder i are given by $\pi_t^i = \sum_k \beta_{ik,t}\pi_{k,t}$, where $\pi_{k,t}$ represents the profits of portfolio firm k in period t. Firm j maximizes its profits by considering its own profits alongside a linear combination of the profits of other firms in which its controlling shareholders hold ownership stakes. This is represented as

$$max\Pi_{j,t} = max \sum_{i=1}^{I} \gamma_{ij,t} \underbrace{\sum_{k=1}^{K} \beta_{ik,t}\pi_{k,t}}_{\pi_t^i} = max\left[\pi_{j,t} + \sum_{k\neq j} \underbrace{\frac{\sum_i \gamma_{ij,t}\beta_{ik,t}}{\sum_i \gamma_{ij,t}\beta_{ij,t}}}_{\kappa_{jk,t}} \pi_{k,t}\right] \tag{16}$$

While this problem is static for the firm, the timing of shares is essential for the household problem, and thus the t subscript is retained. It should be noted that the value of κ can exceed one, referred to as tunneling in the literature, but in this paper, I only consider cases where κ is less than one, which is approximately 0.7 in the US.

When considering alternative strategies, the manager weighs two effects: the benefit to the firm and the portfolio gains or losses for diversified shareholders. The latter is weighted by their share of control in the firm. Applying this generalized version, instead of the special case of own-firm profit maximization, to a Cournot setting implies that the market share-weighted average markup in the market is given by[17]

$$\epsilon_t \sum_j s_{j,t} \frac{P_t - C'_{j,t}(x_{j,t})}{P_t} = \underbrace{\sum_j \sum_k s_{j,t} s_{k,t} \frac{\sum_i \gamma_{ij,t}\beta_{ik,t}}{\sum_i \gamma_{ij,t}\beta_{ij,t}}}_{MHHI}$$

$$= \underbrace{\sum_j s_{j,t}^2}_{HHI} + \underbrace{\sum_j \sum_{k\neq j} s_{j,t} s_{k,t} \frac{\sum_i \gamma_{ij,t}\beta_{ik,t}}{\sum_i \gamma_{ij,t}\beta_{ij,t}}}_{MHHI\,delta} \tag{17}$$

In this equation, ϵ_t represents the price elasticity of demand, and s_j denotes the market share of firm j[18].

It is important to note that MHHI does not imply tacit or explicit collusion. Rather, it models competition under common ownership, without assuming collusion resulting from common ownership.

Once I match the moments of the simulated model to the data, I can examine the effects of common ownership on inequality and profits by maintaining the same parameters and eliminating the linear combination of the profits term. This will show how inequality and profits change relative to the common ownership case.

2.2 Final Good Producer

Competitive producers combine intermediate inputs, denoted as $y_{j,t}$ where $j = 1, ..., J$, to produce the final good Y_t. The final good is then sold to households at a price P_t^Y. The technology for producing the final consumption good can be represented as:

$$Y_t = \left(\sum_{j=1}^{J} (\xi_{j,t} y_{j,t})^{\frac{\epsilon_t - 1}{\epsilon_t}} \right)^{\frac{\epsilon_t}{\epsilon_t - 1}} \tag{18}$$

In this equation, $y_{j,t}$ represents the quantity of input j used in the production of the final consumption good, ϵ_t is the elasticity of substitution between input variables, and $\xi_{j,t}$ denotes the preference shock. The final good producers purchase these inputs from monopolistically competitive firms at prices $p_{j,t}$, where the firms charge a markup (price over marginal cost) represented by μ_t.

[17] derived from the first order condition of equation (16) with respect to quantity produced

[18] For example, suppose two firms each hold a 50% market share. The HHI would be 5,000 on a scale ranging from 0 (perfect competition) to 10,000 (monopoly). If the firms are separately owned, MHHI delta would be 0, and MHHI would equal HHI. However, if the two shareholders exchange 50% of their shares, both shareholders would now receive 50% of the profits from each firm. Consequently, they would prefer the two firms to operate as if they were two divisions of a monopoly. The HHI would remain at 5,000 since the two firms are still formally independent, but the effective market concentration, reflected by an MHHI of 10,000, would be equivalent to that of a monopoly.

Based on cost minimization, the demand for input variety j can be expressed as:

$$y_{j,t} = \xi_{j,t}^{\epsilon_t - 1} \left(\frac{p_{j,t}}{P_t^Y} \right)^{-\epsilon_t} Y_t \tag{19}$$

and the price of the final good is

$$P_t^Y = \left(\sum_{j=1}^{J} \left(\frac{p_{j,t}}{\xi_{j,t}} \right)^{1-\epsilon_t} \right)^{\frac{1}{1-\epsilon_t}} \tag{20}$$

2.3 Intermediate Goods

Here there is a technology with constant returns to scale in capital, denoted as $k_{i,t}$, and labor, denoted as $l_{i,t}$ inputs: $y_{i,t} = f_t(k_{i,t}, l_{i,t})$.

In period $t-1$, the firm exchanges one-period nominal bonds for dollars and purchases capital at a nominal price P_{t-1}^K. In period t, the firm hires labor in a competitive labor market at a nominal wage w_t and produces $y_{i,t}$, which is sold at price $p_{i,t}(y_{i,t})$. Afterward, the firm sells the undepreciated capital at a nominal price P_t^K and pays the face value of its debt. The firm's nominal profits can be calculated as:

$$\pi_{j,t} = \max_{k_{j,t}, l_{j,t}} p_{j,t}(y_{j,t}) \times y_{j,t} - (1+r_t) P_{t-1}^K k_{j,t} - w_t l_{j,t} + (1-\delta_t) P_t^K k_{j,t}$$

$$= \max_{k_{j,t}, l_{j,t}} p_{j,t}(y_{j,t}) \times y_{j,t} - R_t P_{t-1}^K k_{j,t} - w_t l_{j,t} \tag{21}$$

Here, $R_t = r_t - (1-\delta_t) \frac{P_t^K - P_{t-1}^K}{P_{t-1}^K} + \delta_t$ represents the required rate of return on capital. (For the derivation of the marginal costs MC, refer to appendix C1.)

However, CEOs are not only maximizing firm j's profits but also a linear combination of other firms' profits:

$$\max_{p_{j,t}} \Bigg\{ (p_{j,t} - \xi_{j,t} MC_t) \xi_{j,t}^{\epsilon_t - 1} \left(\frac{p_{j,t}}{P_t^Y} \right)^{-\epsilon_t} Y_t$$

$$+ \sum_{k \neq j} \frac{\sum_i \gamma_{ij,t} \beta_{ik,t}}{\sum_i \gamma_{ij,t} \beta_{ij,t}} (p_{k,t} - \xi_{k,t} MC_t) \xi_{k,t}^{\epsilon_t - 1} \left(\frac{p_{k,t}}{P_t^Y} \right)^{-\epsilon_t} Y_t \Bigg\} \tag{22}$$

These producers consider the nominal wage, nominal interest rate, and aggregate demand as given and determine the demand for labor and capital inputs.

The first-order condition (F.O.C.) for this problem is given by:

$$\frac{1-\epsilon_t}{\epsilon_t} + \left(\frac{p_{j,t}}{\xi_{j,t}P_t^Y}\right)^{1-\epsilon_t} + \frac{\xi_{j,t}MC_t}{P_t^Y}\left(\frac{p_{j,t}}{P_t^Y}\right)^{-1} - \frac{MC_t}{P_t^Y}\left(\frac{p_{j,t}}{\xi_{j,t}P_t^Y}\right)^{-\epsilon_t}$$

$$+ \sum_{k\neq j}\frac{\sum_i \gamma_{ij,t}\beta_{ik,t}}{\sum_i \gamma_{ij,t}\beta_{ij,t}}\frac{p_{k,t}-\xi_{k,t}MC_t}{\xi_{k,t}P_t^Y}\left(\frac{p_{k,t}}{\xi_{k,t}P_t^Y}\right)^{-\epsilon} = 0 \tag{23}$$

Note that I have used the fact that $\frac{\partial P_t^Y}{\partial p_{j,t}} = \left(\frac{p_{j,t}}{\xi_{j,t}P_t^Y}\right)^{-\epsilon_t}\frac{1}{\xi_{j,t}}$, as the effect of each of the n firms is not infinitesimal. In the usual New Keynesian models, this effect through P_t^Y is ignored (because Dixit-Stiglitz aggregator is continuous).

By simplifying the above equation, we obtain

$$(\mu_{j,t}^*)^{-1} = \frac{\epsilon_t - 1}{\epsilon_t} - \frac{\mu_t^{\epsilon_t - 1}}{B_{j,t}}\sum_k\sum_i \gamma_{ij,t}\beta_{ik,t}\left[\mu_{k,t}-1\right]\left[\mu_{k,t}\right]^{-\epsilon_t} \tag{24}$$

In this equation, $\mu_t = \frac{P_t^Y}{MC_t} = \left(\sum_{j=1}^J \mu_{j,t}^{1-\epsilon_t}\right)^{\frac{1}{1-\epsilon_t}}$ represents the aggregate markup, and $\mu_{j,t} = \frac{p_{j,t}}{\xi_{j,t}MC_t}$ represents the individual firm markups. Additionally, $B_{j,t} = \sum_i \gamma_{ij,t}\beta_{ij,t}$ (derivation in appendix C).

Note that the markups will be higher compared to the New Keynesian case, and they are highest when $\gamma_{ik,t}$ and $\beta_{ik,t}$ are the same across k and i - indicating the highest common ownership.

Capital Creation

I assume that all agents in the model have free access to a technology with constant returns to scale, converting output into capital at a ratio of $1 : \varphi_t$. Additionally, I assume this technology is fully reversible. According to no arbitrage, in period t, φ_t units of capital must have the same market value as 1 unit of output. This determines the relative price of capital as

$$\frac{P_t^K}{P_t^Y} = \varphi_t^{-1}$$

2.4 Households

The heterogeneous households in the economy have the same utility function representation but differ in productivity. They derive utility from consumption goods and experience disutility from working. Bonds of all firms are aggregated into one pool as R_t, P_{t-1}^K, and δ_t, hence the return on bonds is the same for all firms. The households can only save in nominal bonds, $a_{i,t+1}$, which pays off $1 + r_{t+1}$ dollars in period $t+1$ for each dollar invested in period t. The households receive dividends from profits in the corporate sector, depending on their ownership of stocks in the intermediate firm. They hold $\gamma_{ij,t}$ shares of the intermediate company j and receive dividends from its profit $\Pi_{j,t}$ for all j. The household's problem is to maximize:

$$\max_{\{c_{i,t},n_{i,t},\gamma_{ij,t+1},a_{i,t+1}\}_{t=0}^{\infty}} \mathbb{E}\sum_{t=0}^{\infty}\beta^t u(c_{i,t},n_{i,t}) \tag{25}$$

subject to the budget constraint:

$$a_{i,t+1} = z_{i,t}w_t n_{i,t} + a_{i,t}(1+r_t) + \sum_{j=1}^{J}\left[\frac{\gamma_{ij,t}\Pi_{j,t}}{\xi_{j,t}} - p^S_{j,t}(\gamma_{ij,t+1}-\gamma_{ij,t})\right] - P^Y_t c_{i,t}$$

I assume above that all control shares $\gamma_{ij,t}$ and ownership shares $\beta_{ij,t}$ are the same and denote them as $\gamma_{ij,t}$ for computational tractability.

Here, $z_{i,t}$ represents the productivity shock for each household and $\xi_{j,t}$ is their preference shock. Both evolve according to AR(1) processes. Specifically,

$$log z_{i,t+1} = (1-\rho_z)log\mu_z + \rho_z log z_{i,t} + \varepsilon^z_{i,t+1}, \tag{26}$$

$$\xi_{j,t+1} = (1-\rho_\xi)\mu_\xi + \rho_\xi\xi_{j,t} + \varepsilon^\xi_{j,t+1}, \tag{27}$$

where $\varepsilon^z_{i,t+1}$ and $\varepsilon^\xi_{j,t+1}$ follow a bivariate normal distribution

$$\begin{pmatrix} \varepsilon^z_{i,t+1} \\ \varepsilon^\xi_{j,t+1} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} \sigma_z^2 & \sigma_{z\xi} \\ \sigma_{z\xi} & \sigma_\xi^2 \end{bmatrix}\right)$$

Hence, households will be heterogeneous in their income (labor and dividends), and firms in their profits. The parameter $\xi_{j,t}$ increases demand for the good with an elasticity of $\epsilon_t - 1$, and also increases total costs with unit elasticity. Households that experience high labor productivity at time t will buy stocks of profitable firms (acquiring most of the shares), and as ρ_ξ is not as high, in $t+1$, high-productivity households will purchase new profitable stocks (again, most of the shares). Thus, I can control the degree of common ownership through ρ_ξ, σ_ξ, and the covariation $\sigma_{z\xi}$ between productivity and preference parameters.

If I additionally assume that household preferences over consumption and labor are represented by the utility function:

$$u(c_{i,t},n_{i,t}) = \left[\frac{c_{i,t}^{1-\gamma}}{1-\gamma} - M\frac{\theta}{\theta+1}n_{i,t}^{\frac{\theta+1}{\theta}}\right]$$

where M represents the disutility from working, θ is the Frish elasticity parameter, and γ is the risk-aversion parameter, I can derive the intra-temporal (MRS) and inter-temporal (Euler equations for bonds and stocks) first-order conditions of the household, respectively.

$$Mn_{i,t}^{\frac{1}{\theta}} = \frac{w_t}{P_t^Y} c_{i,t}^{-\gamma} z_{i,t} \tag{28}$$

$$1 = \mathbb{E}_t \left[\beta \left(\frac{c_{i,t+1}}{c_{i,t}} \right)^{-\gamma} \left(\frac{P_{t+1}^Y}{P_t^Y} \right)^{-1} (1 + r_{t+1}) \right] \tag{29}$$

$$p_{j,t}^S = \mathbb{E}_t \left[\beta \left(\frac{c_{i,t+1}}{c_{i,t}} \right)^{-\gamma} \left(\frac{P_{t+1}^Y}{P_t^Y} \right)^{-1} \left(\frac{\Pi_{j,t+1}}{\xi_{j,t+1}} + p_{j,t+1}^S \right) \right] \tag{30}$$

2.5 Equilibrium

Aggregate **labor market** should be cleared in equilibrium with labor demand of all the firms being equal to the labor supply of all the households in the economy, which can be represented as

$$\sum_{j=1}^{J} l_{j,t} = L_t = \sum_{i=1}^{I} n_{i,t}. \tag{31}$$

Stocks market are cleared each period with total supply of each stock being fixed at 1, expressed as

$$\sum_{i=1}^{I} \gamma_{ij,t} = 1, \ \ \forall j \tag{32}$$

Bond and capital market clearing: As firms use bonds to purchase capital goods, the aggregate bond holdings should be equal to the market value of aggregate capital in the economy. This can be represented as

$$\sum_{i=1}^{I} a_{i,t} = P_t^K K_t = P_t^K \sum_{j=1}^{J} k_{j,t} \tag{33}$$

Definition: The equilibrium is a sequence of prices $(r_t^*, w_t^*, p_{j,t}, p_{j,t}^S, P_t^Y)_{t \in \mathcal{N}}$ for all j and quantities that satisfy the following conditions:

1. Households maximize their utilities.

2. Final good producers minimize their costs.

3. Producers of input variety maximize profits.

4. Labor, capital, asset, and consumption goods markets clear in each period.

5. Exogenous shocks for productivity and preference evolve according to the correlated AR(1) processes.

The **aggregate resource constraint** of the economy, measured in nominal dollars, is given by

$$P_t^Y Y_t = P_t^Y C_t + P_t^K [K_{t+1} - (1-\delta)K_t]$$

which holds due to Walras' law ($\sum_{i=1}^m c_{i,t} = C_t$).

Remark: In the steady state, all variables remain constant. The shares of income for labor services, capital rentals, and profits are defined as follows:

Labor share:

$$s_{L,t} = \frac{w_t L_t}{P_t^Y Y_t} = \frac{1}{\mu_t} \frac{w_t L_t}{w_t L_t + R_t P_{t-1}^K K_t}$$

Capital share:

$$s_{K,t} = \frac{R_t P_{t-1}^K K_t}{P_t^Y Y_t} = \frac{1}{\mu_t} \frac{R_t P_{t-1}^K K_t}{w_t L_t + R_t P_{t-1}^K K_t}$$

Profit share:

$$s_{\Pi,t} = \frac{\Pi_t}{P_t^Y Y_t} = 1 - \frac{1}{\mu_t}$$

These shares sum up to 1. Therefore, for example, if the capital share remains fixed, any decline in the labor share must be offset by an equal increase in markups.

3 Results

One of the main findings of the paper is the derivation of markups in the presence of the common ownership hypothesis within the model.

In a standard monopolistic competition model (e.g., New Keynesian model with fixed prices), markups are solely dependent on the elasticity of demand, represented by

$$\mu_{j,t}^* = \frac{\epsilon_t}{\epsilon_t - 1}$$

This is due to the presence of a continuum of infinitesimal firms, where the influence of an individual good's price on the aggregate price level is negligible. As a result, other terms become inconsequential when taking the derivative.

An extension to this concept is the monopolistic competition model with a limited number of firms, where each firm's price affects the aggregate price level. In such cases, the inverse markups can be expressed as

$$\mu_{j,t}^{*-1} = \frac{\epsilon_t - 1}{\epsilon_t} - \frac{\mu_{j,t} - 1}{\mu_t} \left(\frac{\mu_{j,t}}{\mu_t} \right)^{-\epsilon_t}$$

for all firms j, and $\mu_t = \left(\sum_{j=1}^{J} \mu_{j,t}^{1-\epsilon_t} \right)^{\frac{1}{1-\epsilon_t}}$. The additional term, denoted with a negative sign, indicates that markups are now larger compared to the previous case.

The final extension to the monopolistic competition model with a limited number of firms is the incorporation of a common ownership structure. Consequently, the inverse of the markups can be expressed as:

$$(\mu_{j,t}^{*})^{-1} = \frac{\epsilon_t - 1}{\epsilon_t} - \frac{\mu_t^{\epsilon_t - 1}}{B_{j,t}} \sum_k \sum_i \gamma_{ij,t} \beta_{ik,t} \left[\mu_{k,t} - 1 \right] \left[\mu_{k,t} \right]^{-\epsilon_t}$$

Compared to the previous case, the additional term now subtracts not only firm j's relative markups but also the markups of all other firms, considering their respective weights of common ownership. The higher the level of common ownership among firms, the greater the markups of firm j.

3.1 Data and Calibration

I solve the optimization problems of firms and households to determine the prices that clear the markets. From the steady-state distributions, I calculate the GINI and MHHI indexes, as well as the shares of profits, labor, and capital in the simulated economy. My goal is to estimate the parameters of the model by matching the moments of wealth distribution, labor income distribution, total income distribution, markups, and the shares of labor and capital to the corresponding data. I assume that the economy is in a steady state each year, allowing us to match the data moments for individual years and observe the dynamic evolution of MHHI, GINI, profit shares, labor shares, capital shares, etc..

The primary source of calibration data on common ownership is Backus, Conlon, and Sinkinson (2019)[19]. For the period of 1999-2017, they compile a dataset of 13(f) holdings from publicly available source documents, which can be accessed electronically from the Securities and Exchange Commission (SEC) website. The resulting dataset contains approximately 48 million reported holdings (CIK-CUSIP) for all 76 quarters, with 4,000 to 7,000 CUSIPs and 1,000 to 4,000 investors per quarter.

[19] https://sites.google.com/view/msinkinson/research/common-ownership-data.

Param.	Description	Value	Param.	Description	Value
β	Discount factor	.98	μ_z	Idios. Prod. AR(1)	-3
γ	Utility curv.	2	μ_ξ	Idios. Pref. AR(1)	1
θ	Frisch elast.	.45	ρ_z	Idios. Prod. AR(1)	.99
M	Labor disutility	50	ρ_ξ	Idios. Pref. AR(1)	.7
α	Capital share	.7	$\sigma_{z\xi}$	Cov(Prod.;Pref.)	-0.01
δ	Capital depr.	.03	σ_z	Idios. Prod. AR(1)	.1175
ε	Demand elast.	23.7	σ_ξ	Idios. Pref. AR(1)	.125
φ	Price of capital	2			

Table 4: Model Parameterization

To align the model with US data on stock ownership, I aim for approximately half of the households to own stocks, with the wealthiest top 10% owning 84% of all stocks as of 2014.

Data on inequality is obtained from Piketty, Saez, and Zucman (2018)[20]. I aim to align the model with US data on total income inequality, as well as wealth inequality, capital income, and labor income.

Table 4 presents the calibration of the main parameters. Currently, I have only conducted this exercise for the year 2016, matching the model to the following estimates from the data for that year: a common ownership weight of 0.65 (Backus et al., 2019), markups of 1.6 (De Loecker et al., 2019), and a Gini coefficient of 41.6 for total income according to the World Bank's estimate[21].

3.2 Simulation Results

In the simulated model, I utilize only two firms to minimize computational costs, as each additional firm adds an extra stock as a state variable. However, two firms are sufficient to capture the impact of common ownership. My simulation involves 10 productivity shocks and 2 preference shocks for 2000 agents over 1000 time periods.

In the counterfactual scenario without common ownership, we observe the following results: with a common ownership weight of 0.84, markups decrease from 1.58 to 1.09, and the Gini coefficient decreases from 41.7 to 27.62 in the model without common ownership compared to the model with

[20] http://gabriel-zucman.eu/usdina/
[21] https://data.worldbank.org/indicator/SI.POV.GINI.

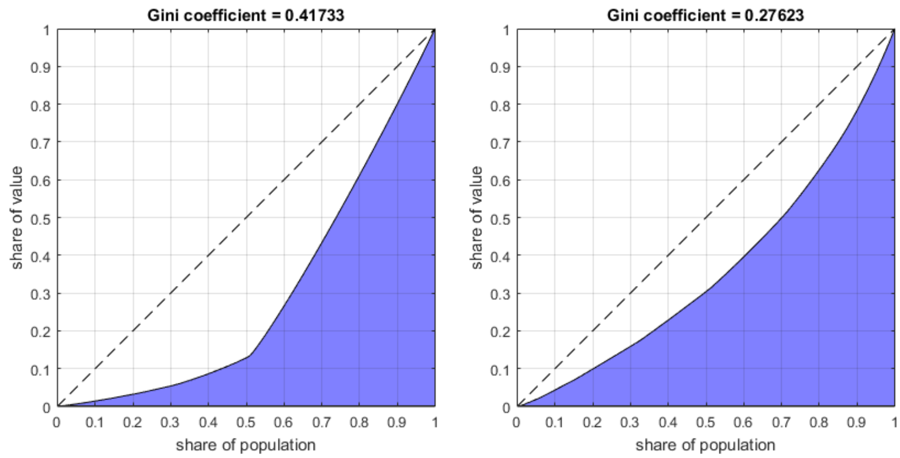


Figure 8: Gini coefficient of total income (41.73) in a model with common ownership (left)
Gini coefficient of total income (27.62) in a model without common ownership (right)

common ownership. The difference in the Gini coefficient is better visualized in the Lorenz curve depicted in Figure 8. The model indicates that removing the common ownership component leads to a significant drop in markups. In response, agents reduce their asset holdings and instead work more hours to compensate for the loss of capital income with labor income. Consequently, the wage Gini decreases (refer to Figure 41 in Appendix B).

The results of the model are presented in Table 5. As the common ownership effect is eliminated, both individual and aggregate markups decrease, indicating a lower profit share. However, this implies that the sum of capital and labor shares should now be larger. This observation is supported by the increase in the wage base from 43.74 to 93.63 and the rise in the interest rate from 0.045 to 0.055. With the higher wage base, individuals choose to work more hours (from 0.064 to 0.133), resulting in a larger average total wage (from 0.121 to 0.598). As interest rates increase, the average expected return on assets also rises, ensuring attractiveness for investors. The prices of goods exhibit a decline: the average price of goods decreases from 1.033 to 1.031, and the aggregate price of goods drops from 0.9987 to 0.9968.

4 Conclusion

There is a substantial body of literature addressing recent developments in the structure of common ownership, with many papers demonstrating how common ownership has led to reduced competition and increased markups. Additionally, numerous studies have explored the relationship between decreased competition, the growth of stock markets, and rising inequality. However, this

	Model with CO	Model without CO
Common Ownership	.8390	.9580
Average Individual Markups	1.5822	1.0921
Aggregate Markups	1.5317	1.0593
Gini of total income	41.73	27.62
Wage base	43.74	93.63
Interest rate	.0450	.0550
Avg. Expected return on assets	.0461	.0720
Max return on assets	.0490	.0795
Min return on assets	.0430	.052
Average output	6.9041	8.5419
Average Wealth	26.76	18.73
Average Wage	.1205	.5976
Average work hour	.0640	.1332
Average Consumption	2.5511	6.3118
Average price of assets	26.75	4.77
Average price of goods	1.0330	1.0310
Aggregate price of goods	.9987	.9968

Table 5: Model Results

paper stands out as the first to establish a connection between inequality and common ownership. Through empirical evidence obtained from structural estimation, I demonstrate that common ownership, *assuming the common ownership hypothesis holds true*, is one of the factors contributing to heightened income inequality. Furthermore, I derive an analytical expression for markups, explicitly demonstrating that increased common ownership leads to higher markups. The quantitative findings derived from the counterfactual analysis in my simulated model suggest that markups would have been 1.09 instead of 1.58 in the absence of the common ownership effect. Moreover, the rise in common ownership contributes to an increase in the total income Gini coefficient from 27.62 to 41.73.

CHAPTER III.

Wishful Thinking, Asset Pricing, and Bubbles

(with John V. Leahy[22])

1 Introduction

In his classic history of asset pricing bubbles, Kindelberger (1978) places optimism at the center of the story. In his telling, bubbles begin with the introduction of a new asset, typically reflecting a new technology such as railroads, information technology, or cryptocurrency. A period of good news then increases interest in the new asset and the price of the new asset begins to rise. Eventually interest evolves into euphoria, and the price rises above what would be expected by an objective observer. Euphoria, however, cannot be sustained indefinitely without evidence. Eventually, reality sets in and prices return to earth.

More recently, several empirical studies of the housing bubble of the 2000's also rely on optimistic beliefs. Optimistic forecasts of future housing demand play a central role in Kaplin, Mitman, and Violante (2020) and Chodorow-Reich, Guren and McQuade (2022). Without optimism, the authors have a hard time fitting the data.

The purpose of this paper is to evaluate the role of optimism as a driver of asset price movements and to create a model in which optimism leads to bubbles that look like the bubbles described by Kindelberger.

We consider a particular model of optimism due to Caplin and Leahy (2019). Caplin and Leahy model belief choice and call it "wishful thinking". To motivate belief choice Caplin and Leahy follow Jevons (1905), Loewenstein (1987) and Caplin and Leahy (2001) and assume that some portion of current well-being depends on the anticipation of future outcomes. In their model, the more likely are positive outcomes, the greater is utility. To constrain belief choice they limit agents to "plausible beliefs" that are not obviously contradicted by the available evidence. To model plausibility they impose an entropy cost in the spirit of Hansen and Sargent (2008). This cost is increasing in the Kullback-Leibler divergence between the chosen beliefs and the beliefs that a neutral observer would

[22]University of Michigan and NBER

hold. This cost is related to the likelihood that the chosen beliefs would be rejected in favor of the objective ones.

The first section of this paper explores the implications of wishful thinking for asset pricing. We consider a simple consumption-savings problem. There are two periods. An agent is endowed with a good that they may consume or invest in an asset with uncertain returns. Agents derive utility in the first period from their beliefs regarding future utility and are able to manipulate these beliefs subject to an entropy constraint. In this setting, wishful thinkers choose to believe that the mean of the asset is greater than its objective mean. This raises current utility. In fact, this effect is so great that if the coefficient of relative risk aversion is less than one, the agent chooses to believe that the asset's return is infinite. The gain to optimistic beliefs rises exponentially in the mean, whereas the costs rise linearly. Similarly, the agent chooses to believe that the variance of the asset is less than its objective variance.

One important result is that the extent to which wishful thinkers manipulate beliefs is increasing in the variance of asset returns. Optimism is increasing in the variance of the asset since optimism is hard to reject in these circumstances.

If the asset is the agent's only source of second period consumption, then optimism only raises the demand for the asset if the coefficient of relative risk aversion is less than one. This is the condition for the substitution effect to dominate the wealth effect. Optimism increases the agent's perceived wealth. If the coefficient of relative risk aversion is greater than one, the agent reacts to the higher return by increasing consumption today rather than investing more.

Hence the condition for the solution to be well behaved is exactly the condition for which optimism reduces demand for the asset. We explore solutions to this problem. If the agent has other sources of period 2 income besides the asset, then the wealth effect is reduced, so that optimism may increase asset demand for coefficients of relative risk aversion above one. If the asset has bounded returns, then optimism is bounded as well, and the agent's problem is well behaved for coefficients of relative risk aversion below one. The implication is that bubbles are more likely to occur the lower is the coefficient of relative risk aversion and in single assets as opposed to the market as a whole.

In the second section we construct a two-period model of consumption and saving in which there are both wishful thinkers and objective agents. Agents can trade both a risk-free bond and a risky stock. The stock return takes on a finite number of values so that the wishful agent's problem is well-defined for all levesls of relative risk aversion. The main result of this section is that the asset price rises in the wealth of wishful thinkers if the coefficient of relative risk aversion is less than one.

In the third section we construct model of bubbles. The model is an overlapping generations model. To keep the analysis manageable and focus on the dynamics introduced by wishful thinkers, we make assumptions such that asset pricing is essentially static. Agents price the dividend in the next period. The dynamics of the model then come from two sources. The first is the movement of wishful thinkers into the asset. We assume that the asset is initially held by objective agents and that wishful thinkers have little wealth. Optimism leads wishful thinkers to save in the asset. This increased saving leads to wealth accumulation. As the wishful thinkers accumulate they begin to influence asset prices, and asset prices rise. This is the bubble phase. The second dynamic element is learning. We assume that the return on the asset is uncertain. This uncertainty allows wishful thinkers freedom to be optimistic. Over time as agent's learn the asset's true return, it gets harder and harder for wishful thinkers to manipulate their beliefs, and the beliefs of wishful thinkers converges to that of objective agents and the bubble disappears.

2 Consumption and Saving and Portfolio Choice with Wishful Thinking

Consider an agent who lives two periods. In the first period, they earn an income y. The agent has the option to consume this income today or invest in an asset that returns x. x is log-normally distributed: $\ln x \sim N(\mu, \sigma^2)$. We assume that the agent has a constant relative risk aversion utility function:

$$\frac{c^{1-\gamma} - 1}{1-\gamma}$$

and discounts future utility by a factor β.

The twist is that the agent derives utility in the first period from their beliefs about second period utility. They prefer to believe that they will be happy in the future. To this end, they can choose subjective beliefs $\tilde{\mu}$ and $\tilde{\sigma}^2$ subject to an entropy constraint. The cost of subjective beliefs $\tilde{P}$ when the objective beliefs are P is the Kullback-Leibler divergence from P to $\tilde{P}$:

$$KL(\tilde{P} \parallel P) = \tilde{E} \ln(\tilde{P}/P)$$

where $\tilde{E}$ is the expectation according to $\tilde{P}$.

We think of this belief choice as reflecting model uncertainty as in Sargent and Hansen (2008). Agents rarely know parameters such as the mean and variance of an asset with certainty. Given that they derive utility from beliefs, they have an incentive to choose optimistic beliefs. The entropy constraint limits agents to plausable beliefs. The Kullback-Leibler divergence is related to the probability that the chosen beliefs would be rejected by data generated by the objective beliefs.

In the case of two log normal variables x and $\tilde{x}$ the Kullback-Leibler divergence is

$$\frac{1}{2}\left[\ln \sigma^2 - \ln \tilde{\sigma}^2 + \frac{\tilde{\sigma}^2 - \sigma^2 + (\mu - \tilde{\mu})^2}{\sigma^2}\right]$$

The agent chooses beliefs $\tilde{P}$ and saving s to maximize

$$V(y, \mu, \sigma) = \max_{s,P} U(y-s) + \tilde{E}U(s\tilde{x}) - \frac{1}{\theta} KL(\tilde{P} \parallel P)$$

μ and σ enter as state variables since they anchor belief choice. θ is a parameter that calibrates the cost of belief choice. The smaller θ the harder it is to manipulate beliefs. Given our functional form assumptions

$$V(y, \mu, \sigma) = \max_{s,\tilde{\mu},\tilde{\sigma}^2} \frac{(y-s)^{1-\gamma} - 1}{1-\gamma} + \beta \frac{s^{1-\gamma} e^{\tilde{\mu}(1-\gamma) + \tilde{\sigma}^2(1-\gamma)^2/2} - 1}{1-\gamma}$$

$$-\frac{1}{2\theta} \left[\ln\left(\frac{\sigma^2}{\tilde{\sigma}^2}\right) + \frac{\tilde{\sigma}^2 - \sigma^2 + (\mu - \tilde{\mu})^2}{\sigma^2} \right]$$

of $\gamma \neq 1$, and

$$V(y, \mu, \sigma) = \max_{s,\tilde{\mu},\tilde{\sigma}^2} \ln(y-s) + \beta \ln s + \beta\tilde{\mu} - \frac{1}{2\theta} \left[\ln\left(\frac{\sigma^2}{\tilde{\sigma}^2}\right) + \frac{\tilde{\sigma}^2 - \sigma^2 + (\mu - \tilde{\mu})^2}{\sigma^2} \right]$$

2.1 Belief Choice

We begin by studying the choice of $\tilde{\mu}$ and $\tilde{\sigma}^2$ given s.

The first order condition for $\tilde{\mu}$ is

$$s^{1-\gamma} \left[e^{\tilde{\mu}(1-\gamma) + \tilde{\sigma}^2(1-\gamma)^2/2} \right] = \frac{1}{\theta} \frac{(\tilde{\mu} - \mu)}{2\sigma^2}$$

The first thing to note is the because the left-hand side is positive, $\tilde{\mu} > \mu$. This is not surprising, optimistic beliefs raise the agent's subjective utility. The second thing to note is that $\tilde{\mu}$ is infinite if $\gamma < 1$. With our functional form assumptions the benefit of optimism rises exponentially, whereas the costs rise linearly. The third thing to note is that when $\gamma = 1$, $\tilde{\mu}$ takes a particularly simple form

$$\tilde{\mu} = \mu + 2\sigma^2\theta\mu$$

Finally, $\gamma \geq 1$ that $s^{1-\gamma}$, θ and σ^2 all have the same qualitative effect on $\tilde{\mu}$. Increases in each work to raise $\tilde{\mu}$.

We now consider the first-order condition for $\tilde{\sigma}^2$

$$(1-\gamma)s^{1-\gamma} \left[e^{\tilde{\mu}(1-\gamma) + \tilde{\sigma}^2(1-\gamma)^2/2} \right] = \frac{1}{2\theta} \left[\frac{1}{\sigma^2} - \frac{1}{\tilde{\sigma}^2} \right]$$

Again the problem blows up if $\gamma < 1$ as the gain to increasing $\tilde{\sigma}^2$ rises exponentially. When $\gamma = 1$, the left-hand side equals zero and $\tilde{\sigma}^2 = \sigma^2$. Intuitively, an increase in $\tilde{\sigma}^2$ raises the expectation of x and increases the riskiness of the asset. When $\gamma = 1$, these two effects balance out. When $\gamma > 1$, the right hand side is negative and so $\tilde{\sigma}^2 < \sigma^2$.

The bottom line is that wishful thinking leads the agent to both over-estimate the mean and under-estimate the variance of the asset. The incentive to overestimate is increasing in the level of uncertainty regarding asset returns. The more uncertain are asset returns, the harder it is to falsify optimistic beliefs.

2.2 Saving

The first order condition for saving is

$$(y-s)^{-\gamma} = e^{\tilde{\mu}(1-\gamma)+\tilde{\sigma}^2(1-\gamma)^2/2}\beta s^{-\gamma}$$

Solving for s

$$s = \frac{y}{1 + e^{-\frac{\tilde{\mu}(1-\gamma)+\tilde{\sigma}^2(1-\gamma)^2/2}{\gamma}}\beta^{-\frac{1}{\gamma}}}$$

With $\gamma > 1$, an increase in $\tilde{\mu}$ reduces saving and hence the demand for the asset. This is because the income effect dominates the substitution effect. The optimistic agent believes that they will be rich in the future and decides to consume some of that wealth in the present. With $\gamma > 1$, an reduction in $\tilde{\sigma}^2$ also reduces saving. $\tilde{\sigma}^2(1-\gamma)$ enters the first order condition in the same way that $\tilde{\mu}$ does, so that when $\gamma > 1$, $\tilde{\sigma}^2$ has the opposite effect of $\tilde{\mu}$.

With log utility

$$s = \frac{\beta y}{1+\beta}$$

saving is independent of beliefs. The agent still manipulates beliefs to raise subjective utility, but these beliefs have no impact on the choice of how much to save.

The bottom line from this analysis is that it is not obvious that wishful thinking increases asset prices. For saving and hence asset demand to rise, we need $\gamma < 1$, but if $\gamma < 1$ the incentive to manipulate beliefs becomes overwhelmingly strong.

There are a couple of possible ways in which we can amend the model so that wishful thinking is compatible with increased demand for the asset. If the agent has another source of second period income then the wealth effect is muted. To see this, consider the case of log utility. Solving the Euler equation for the optimal savings rate

$$s = \frac{\beta y - y_2 e^{\tilde{\mu}}}{1+\beta}(y-s)^{-\gamma} = e^{\tilde{\mu}(1-\gamma)+\tilde{\sigma}^2(1-\gamma)^2/2}\beta s^{-\gamma}$$

where y_2 is the second period endowment. Whereas earlier s was independent of $\tilde{\mu}$, s is now increasing in $\tilde{\mu}$. Intuitively, the additional endowment income, reduces the percent change in consumption and hence magnitude of the wealth effect. Another way is to assume that the asset only takes values in a finite range. This puts an upper bound on the assets return under any beliefs and bounds the level of optimism. Since this second method is mathematically tractable, it is the one we follow in the remainder of the paper. We assume that the asset can take on one of two values.

3 A Two Period Asset Pricing Model

We begin with a simple two-period version of the Lucas Asset Pricing model with optimistic and objective agents and two assets, a stock and a bond. To keep things simple and to illustrate the forces at work, we assume that there are only two second period states. This makes belief choice one dimensional. We also assume that the riskless bond is traded internationally, so that the price of the stock is the only price to be determined in equilibrium.

The two periods are indexed by $t = \{0, 1\}$ and we use a superscript O for objective agents and a superscript W for the wishful thinkers. There is a unit mass of agents. A fraction ϕ are wishful thinkers.

In each period agents receive an endowment of a consumption good. In the first period these endowments are fixed. Let Y_1^O and Y_1^W, denote the endowments of the objective agents and the wishful thinkers respectively. Endowments in the second period are determined by the ownership of two production technologies or Lucas trees. The first tree produces 1 unit of the consumption good in the second period with certainty. Ownership of this technology is therefore equivalent to investing in a riskless bond. This tree is supplied elastically in a large liquid market for international bonds. This market pins down the rate of return. The second tree produces Q^H units of the consumption good in the second period with probability π and Q^L units with probability $1 - \pi$. We assume that $Q^H > Q^L$ so that H indexes the high state. Ownership of this technology is therefore similar to owning a stock. Initially, the objective agents are endowed with S^O units of the stock technology. Similarly wishful thinkers are endowed with S^W. We normalize the number of shares to one so that $(1 - \phi)S^O + \phi S^W = 1$. We assume that the supply of the stock is inelastic and equal to one. Stocks only trade domestically.

In the first period, agents consume, trade endowments and shares of stocks and bonds. In the second, uncertainty is realized and agents consume their remaining wealth. We allow agents to short both the stock and the bond.

Objective agents choose first period consumption $\tilde{C}^O$, stock holdings $\tilde{S}^O$, and bond holdings $\tilde{B}^O$ to maximize a CRRA utility function

$$\max_{\tilde{C}^O,\tilde{S}^O,\tilde{B}^O} \frac{\left(\tilde{C}^O\right)^{1-\sigma}}{1-\sigma} + \pi\frac{(\tilde{S}^O Q^H + \tilde{B}^O)^{1-\sigma}}{1-\sigma} + (1-\pi)\frac{(\tilde{S}^O Q^L + \tilde{B}^O)^{1-\sigma}}{1-\sigma}$$

Here $\tilde{S}^O Q^H + \tilde{B}^O$ is consumption in the high state and $\tilde{S}^O Q^L + \tilde{B}^O$ is consumption in the low state. These choices are subject to the budget constraint

$$C^O + P\tilde{S}^O + \frac{1}{R}\tilde{B}^O = Y_1^O + PS^O$$

where P is the price of the stock and R is the gross return on the bond. The first period consumption good is the numeraire.

Wishful thinkers choose $\tilde{C}^W$, $\tilde{S}^W$, $\tilde{B}^W$, and also the probability of the good state $\tilde{\pi}$ to maximize

$$\begin{aligned}\max_{\tilde{C}^W,\tilde{S}^W,\tilde{B}^W,\tilde{\pi}} & \frac{\left(\tilde{C}^W\right)^{1-\sigma}}{1-\sigma} + \tilde{\pi}\frac{(\tilde{S}^W Q^H + \tilde{B}^W)^{1-\sigma}}{1-\sigma} + (1-\tilde{\pi})\frac{(\tilde{S}^W Q^L + \tilde{B}^W)^{1-\sigma}}{1-\sigma} \\ & -\frac{1}{\theta}\tilde{\pi}\ln\frac{\tilde{\pi}}{\pi} - \frac{1}{\theta}(1-\tilde{\pi})\ln\frac{1-\tilde{\pi}}{1-\pi}\end{aligned}$$

subject to

$$\tilde{C}^W + P\tilde{S}^W + \frac{1}{R}\tilde{B}^W = Y^W + PS^W$$

The last term in the maximand is the cost of belief choice. As in Caplin and Leahy (2019), this cost is increasing in the Kullback-Leibler divergence of the chosen beliefs from the true beliefs. The idea is that it becomes more and more difficult for an agent to recruit information in support of their desired beliefs the further these beliefs are from the objective truth. θ is a parameter that captures the cost of belief choice. The larger is θ the easier it is to fool oneself.

An equilibrium is an allocation $\{\tilde{C}^O, \tilde{S}^O, \tilde{B}^O, \tilde{C}^W, \tilde{S}^W, \tilde{B}^W\}$, a stock price P, and beliefs of the wishful thinkers $\tilde{\pi}$ such that, given P, the allocation and beliefs solve each agent's problem and the stock markets clears

$$(1-\phi)\tilde{S}^O + \phi\tilde{S}^W = 1$$

3.1 Solution

The model is straightforward to solve. Given the CRRA utility we can solve in closed form for the agents' consumption choices as functions of the prices P and R, the beliefs π and $\tilde{\pi}$, and the multipliers on the budget constraint λ^O and λ^W. For $i = O, W$

$$\tilde{C}^i = \left(\frac{1}{\lambda^i}\right)^{1/\sigma} \tag{34}$$

$$\tilde{C}^{i,L} = \tilde{S}^i Q^L + \tilde{B}^i = \left(\frac{(1-\pi^i)(Q^H - Q^L)}{\lambda^i\left(\frac{Q^H}{R} - P\right)}\right)^{1/\sigma} \tag{35}$$

$$\tilde{C}^{i,H} = \tilde{S}^i Q^H + \tilde{B}^i = \left(\frac{\pi^i(Q^H - Q^L)}{\lambda^i\left(P - \frac{Q^L}{R}\right)}\right)^{1/\sigma} \tag{36}$$

Here $\tilde{C}^{i,H}$ is the consumption of agent i in the high state and $\tilde{C}^{i,L}$ is their consumption in the low state. These consumptions implicitly determine the stock and bond holdings, $\tilde{S}^i$ and $\tilde{B}^i$, and we can use these to write the multipliers as:

$$\begin{aligned}(\lambda^i)^{\frac{1}{\sigma}} = \ & \tfrac{1}{Y^i + PS^i} \\ & \times\left(1 + P(Q^H - Q^L)^{(\frac{1}{\sigma}-1)}\left[\left(\tfrac{\pi^i}{\left(P - \frac{Q^L}{R}\right)}\right)^{\frac{1}{\sigma}} - \left(\tfrac{(1-\pi^i)}{\left(\frac{Q^H}{R} - P\right)}\right)^{\frac{1}{\sigma}}\right]\right. \\ & \left. + \tfrac{(Q^H - Q^L)^{\frac{1}{\sigma}-1}}{R}\left[Q^H\left(\tfrac{(1-\pi^i)}{\left(\frac{Q^H}{R} - P\right)}\right)^{\frac{1}{\sigma}} - Q^L\left(\tfrac{\pi^i}{\left(P - \frac{Q^L}{R}\right)}\right)^{\frac{1}{\sigma}}\right]\right)\end{aligned} \tag{37}$$

Finally, the wishful thinker's beliefs satisfy the first order condition

$$\theta\left[\frac{(\tilde{C}^{W,H})^{1-\sigma}}{1-\sigma} - \frac{(\tilde{C}^{W,L})^{1-\sigma}}{1-\sigma}\right] = \ln\frac{\tilde{\pi}}{1-\tilde{\pi}} - \ln\frac{\pi}{1-\pi} \tag{38}$$

and the price P then clears the stock market:

$$(1-\phi)\tilde{S}^O + \phi\tilde{S}^W = 1 \tag{39}$$

Together equations (34) through (39) pin down the solution to the model.

We now discuss the properties of this solution, beginning with the wishful thinkers' choice of $\tilde{\pi}$.

3.2 Belief Choice

There is a feedback between wishful thinking and consumption outcomes. Optimism leads wishful thinkers to demand more stocks. This which raises consumption in the high state (35) and lowers consumption in the low state (36). This in turn feed further optimism. If $\sigma < 1$, this feedback is so strong that $\tilde{\pi}$ will end up at the boundary, equal to either 0 or 1 depending on which state consumption is higher.

According to (35), (36), and (37), consumption is proportionate to wealth $Y^W + PS^W$. Whether an increase in wealth raises or lowers the utility difference between the high and the low states depends on σ. When $\sigma > 1$, marginal utility falls quickly as consumption rises, and a proportionate increase in $\tilde{C}^{W,H}$ and $\tilde{C}$ reduces the gap in utility between the high and the low states. The poor are more likely to be optimistic than the rich. When $\sigma < 1$, this result flips, increases in wealth feed optimism. With log utility the gap in utility depends on the ratio of consumption and proportionate increases in consumption have no effect.

4 An Overlapping Generations Model

In this section we construct a model of bubbles. The model balances wishful thinking with two sources of dynamics. The first is the movement of wishful thinkers into the asset. We assume that the asset is initially held by objective agents and that wishful thinkers have little wealth. Optimism leads wishful thinkers to save in the asset. This increased saving leads to wealth accumulation. As the wishful thinkers accumulate they begin to influence asset prices, and asset prices rise. The second dynamic element is learning. We consider a new asset with uncertain returns. This uncertainty allows wishful thinkers freedom to be optimistic. Over time as agents learn the asset's true return, it gets harder and harder for wishful thinkers to manipulate their beliefs, and the beliefs of wishful thinkers converges to that of objective agents.

4.1 Environment

At any point in time the economy is comprised of one wishful thinker and one rational agent. To simplify the asset pricing and focus on the dynamics of wealth accumulation and learning, we assume that all agents live for two periods and make assumptions such that the price of the asset only depends on dividends in the next period.

Generation t is born at the end of period t. Upon birth they receive an endowment w. This endowment is constant over time. They may consume this endowment or invest it in an asset with a risky returns. The asset generates a dividend d at the beginning of period $t+1$. Agents consume this return and bequeath the asset to their heirs and die. The next generation receives the asset at the end of period $t+1$. Each agent has a single heir.

Since generations do not overlap within the period, there is no one to sell the asset to at the beginning of the period. We also assume that generations do not receive utility from the consumption of their heirs. The value of the asset in period t to generation t is therefore only the expected present value of the dividend. The equilibrium in each period is therefore similar to the equilibrium in the two period economy of the last section with the obvious difference that there is no bond. The additional element is that the wealth of the two types of agent evolves over time and agents learn about the return on the asset from observations of the past. We will discuss dynamics below.

We assume that the dividend takes on one of two values: d_g and d_b. $d_g > d_g$ so that d_g is the dividend in the good state and d_b is the dividend in the bad state. Dividend realizations are iid. The true probability of the good state is p_g. The objective agents know this probability and act accordingly. E^O is the expectation conditional on this probability. In the first period, the wishful thinkers initially twist this probability according. The gain to twisting is the improvement in subjective utility. The cost to twisting is the Kullback-Leibler divergence. In subsequent periods, the future generations of wishful thinkers update these subjective beliefs after observing the realized dividends. We model this updating as follows. We model beliefs as a beta distribution in which the parameters are chosen to yield the first period's subjective beliefs. Wishful thinker sthen update this density after observing based on realized dividends. This updating gives E_t^W.

We now study equilibrium for a single generation t and then discuss dynamics.

4.2 Equilbirum in a given period

The objective agents ($i = O$) born in period t receive their endowment w and are bequeathed a fraction s_t of the stock. They choose consumption $c_{i,t}$ and holdings of the stock $s_{i,t+1}$ to maximize the present value of utility. Let $u(c_t)$ denote utility from consumption c_t in period t. The agent discounts future utility by a factor β. Expectations of future dividends dividends are take with respect to current beliefs $p_{g,t}$. The agent's problem is:

$$V^O(s_{i,t}) = \max_{c_{i,t}, s_{i,t+1}} \left\{ u(c_{i,t}) + \beta E^O u(c_{i,t+1}) \right\} \tag{40}$$

subject to

$$c_{i,t} = w + q_t(s_{i,t} - s_{i,t+1})$$

$$c_{i,t+1} = d_{t+1} s_{i,t+1}$$

q_t is the price of the asset in period t and $s_{i,t} - s_{i,t+1}$ is net purchases of the asset. $c_{i,t+1}$ is equal to their holdings of the stock $s_{i,t+1}$ times dividends in period $t + 1$, d_{t+1}. E^O is the expectation conditional the true probability p_g. available at date t.

In the first period, the wishful thinker solves

$$V^W(0) = \max_{c_{i,t}, s_{i,t+1}} \left\{ u(c_{i,t}) + \frac{\beta}{\theta} \ln E e^{\theta u(c_{i,t+1})} \right\} \tag{41}$$

subject to the same budget constraint ($i = W$). Again E reflects the probability p_g. The log of the expectation and the utility in the exponent reflect the impact of optimal belief choice (see Caplin and Leahy, 2019). Belief choice twist the probabilities toward optimism, and this has the same impact as taking the expectation of a convex transformation of utility.

In subsequent periods the wishful thinkers update their beliefs after observing dividends. The Bayesian updating of the Beta-Bernoulli process takes the following form:

$$Prob(p_g | x_{1:n}) = \frac{p_g^{\sum x_i}(1-p_g)^{n-\sum x_i}}{Prob(x_{1:n})} \mu_g$$

$$= Beta(p_g | a + \sum x_i, b + n - \sum x_i) \tag{42}$$

where the true probability is $0 < p_g < 1$, the subjective probability of the previous period, serves as the conjugate prior of the Beta distribution for Bernoulli binomial, and $x_{1:n} = x_1, ..., x_n$ is the observed dividend state data for the first n periods. $p_{g,t}$ is the expectation of the true p_g. E_t^W is the expectation given $p_{g,t}$.

The wishful thinkers problem in all subsequent periods is similar to the objective agent's problem except that they use the updated probabilities.

$$V^W(s_{i,t}) = \max_{c_{i,t}, s_{i,t+1}} u(c_{i,t}) + \beta E_t u(c_{i,t+1}) \tag{43}$$

Markets clear when asset purchases by the two agents sum to one.

$$s_{W,t+1} + s_{O,t+1} = 1$$

If this condition holds in all periods then $c_{W,t} + c_{O,t} = 2w$ and consumption is equal to the endowment.

Given initial holdings of the asset $s_{W,t}$ and $s_{O,t}$ such that $s_{W,t} + s_{O,t} = 1$, a period-t equilibrium in this economy is a price q_t and asset demands $s_{W,t+1}$ and $s_{O,t+1}$ such that agents choices maximize their utility given prices and markets clear.

4.3 Dynamics

The economy evolves as the period-t equilibrium evolves. Dynamics arise from two sources.

The first is the dynamics of $s_{i,t}$ which reflect maximizing decisions of agents. We assume that the coefficient of relative risk aversion is less than one so that optimism increases the demand for the asset. As $s_{i,t}$ rises, the wishful thinker exerts greater influence on the market price and the market price rises. The second source is the learning of the wishful thinkers. The wishful thinkers begin optimistic and their beliefs converge to those of the rational agents as they observe returns.

The combination leads to hump-shaped dynamics. Initially asset prices rise as the wealth of wishful thinkers rises. Eventually asset prices revert to the objective level as wishful thinkers learn.

4.4 Calibration

We assume constant relative risk aversion with a coefficient of relative risk aversion equal to 1/2. This implies that the substitution effect dominates the income effect with returns rise, so that optimism increases saving and hence asset demand. We assume that the dividend takes on two value 0.5 and 1.5. The finite range of dividends ensures that the wishful thinker's problem is well behaved in spite of the low coefficient of relative risk aversion. We assume that initially the asset is held only by the objective agent. This allows the market share of the wishful thinker to rise over time. We assume that the endowment w is equal to 0.2. This prevents the wishful thinker from purchasing a large share of the asset right from the start. The wishful thinker needs to accumulate wealth before their holdings can rise.

We choose the true value of the p_g to maximize the gap between the wishful thinker's subjective beliefs and the rational agent's beliefs. Given this objective probability we solve the wishful thinkers problem. This gives the initial subjective probability. We choose parameters of the beta distribution to match this probability.

Table 6 summarizes the the calibration of the model.

Parameter	Describtion	Value
γ	rel. risk aversion coef.	1/2
θ	degree of wishfulness	1
β	time preference	0.97
d	dividends	[0.5; 1.5]
w	endowment when young	0.2
p_g	prob. of the good state	0.4286

Table 6: Calibration

4.5 Results

Figure 9 illustrates the dynamics of the model. Initially, wishful thinkers own 0 shares in the

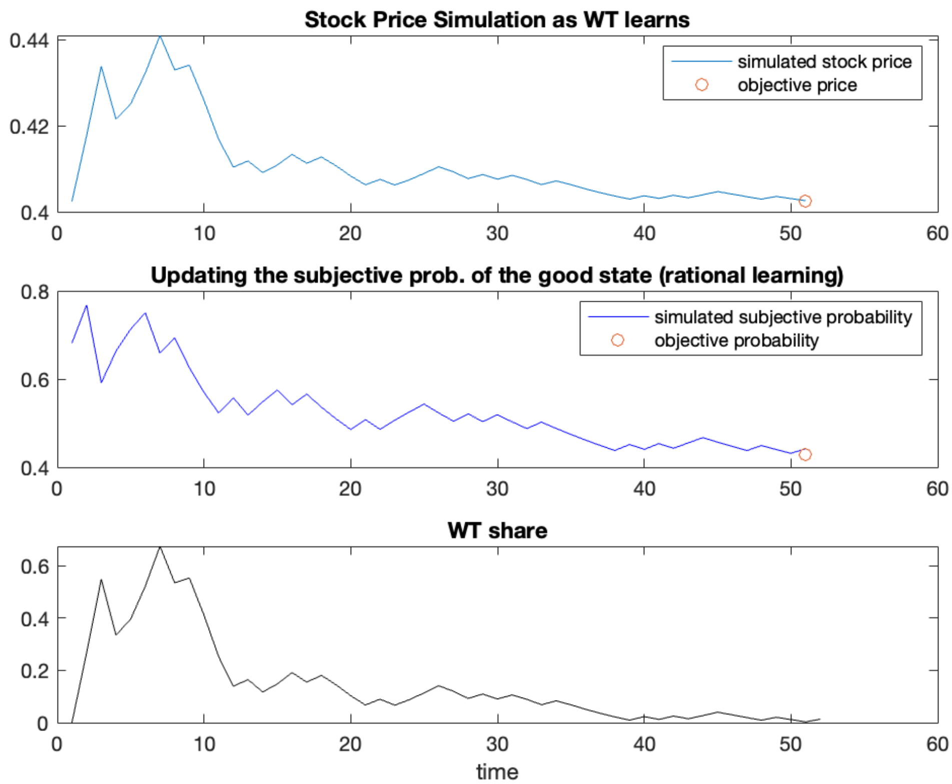


Figure 9: Simulation of (a) the Stock Price, (b) Subjective Probability, (c) Wishful's Share.

stock (figure 9(c)), and the stock is priced objectively. However, their subjective probability (0.68) is significantly higher than the objective probability (0.42). Believing that the stock is undervalued, wishful thinkers begin buying the stock with their savings from the endowment w when they are young (as there is no inheritance from the older generation), thus driving up the price. After six generations, they manage to accumulate more than half of the shares of the stock, reaching their peak holdings as they gradually learn about the actual probability. Figure 9(b) displays the learning process of the probability of the good state, which changes with each new draw of the dividend state. The price and the share of wishful thinkers decrease as their subjective probability converges to the objective one over time. It is important to note that wishful thinkers do not necessarily sell their stocks and exit the market; instead, they retain their shares and gradually become objective as they learn the true probability.

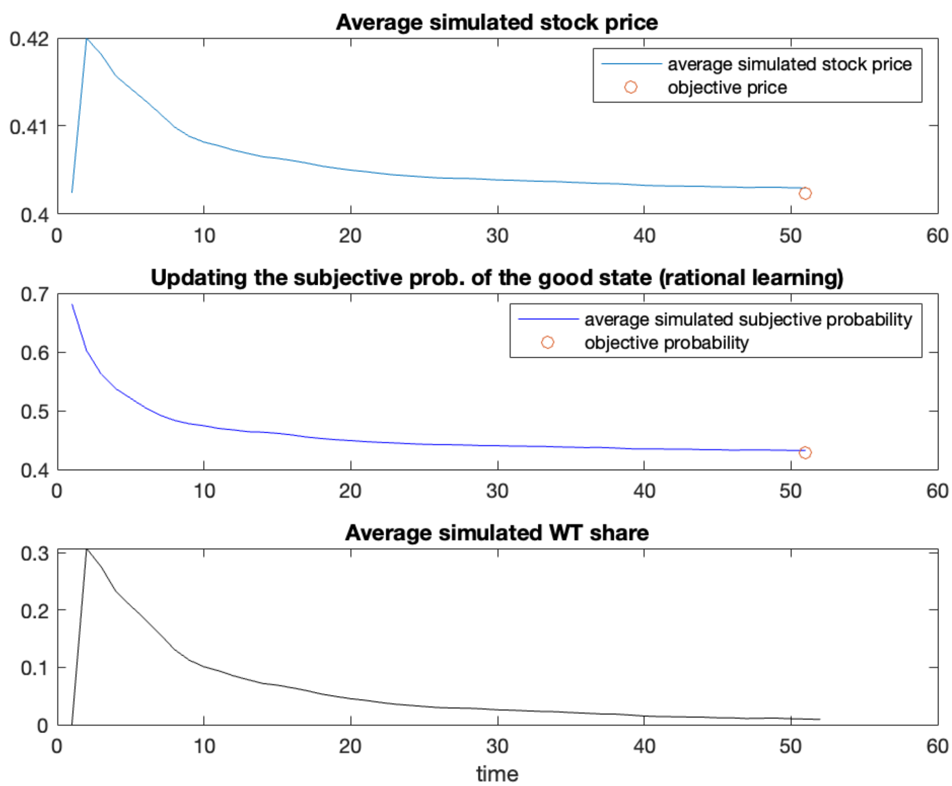


Figure 10: Simulation of (a) the Stock Price, (b) Subjective Probability, (c) Wishful's Share.

Figure 10 shows the average simulated (a) price, (b) subjective probability, and (c) wishful's share from 1000 simulations. The starting prior is the same but now it declines steadily. The price and the wishful's share in stocks start low and jump to their highest level as wishful thinkers have high subjective probability. Then both decrease with the updating of the subjective probability.

5 A Dynasty Model

5.1 Environment

Consider an economy with a continuum of agents indexed by $i \in [0,1]$. Agents indexed by $i \in [0, \phi_t]$ are wishful (W), and the remaining agents are objective (O). Both types want to maximize the quantity

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t U(c_{it})\right], \tag{44}$$

where $E[\cdot]$ is an expectations operator, β is a discount factor, $U(\cdot)$ is a current period utility function, and $c_{i,t}$ is a stochastic process of consumption for $i = \{W, O\}$.

The consumption good is produced on a risky productive unit which we call stock (s). The output in the risky unit, d_t, varies with the state of the world. The process guiding the motion of d_t will be taken to be Markovian with the transition function

$$F(d', d) = P\{d_{t+1} \leq d' | d_t = d\}.$$

Additionally, each period, agents receive a small fixed amount of endowment of a consumption good $\bar{\varepsilon}$.

Output is exogenous and perishable, and it cannot be transferred to the following period. Hence, the feasible consumption level in the economy is limited to

$$0 \leq c_{O,t} + c_{W,t} \leq d_t + 2\bar{\varepsilon}.$$

The two types of agents are different in how they assign probability to different states of the world: wishful thinkers tend to believe that the "good" state of the world is more likely than the "bad" state (following Caplin and Leahy, 2019).

Let $s_{i,t}$ be the beginning-of-period t share holdings of stock for each type $i = \{W, O\}$ (total shares are normalized to 1). The asset price and the ownership of the productive unit are determined each period together by wishful thinkers and objective individuals in a competitive stock market. The unit has one perfectly divisible equity share. In each period t, output is produced and distributed to the shareholders of the productive unit as dividends together with the small fixed endowments $\bar{\varepsilon}$. Then stock shares are traded at a competitively determined price p_t.

Consumers' current consumption, $c_{i,t}$, and saving in stock $s_{i,t+1}$ decisions depend on their current stock holdings of stocks $s_{i,t}$ and the dividend state of the stock d_t. A consumer's behavior is then described by the consumption function $c_{i,t} = c(s_{i,t}, d_t)$ and the policy function $s_{i,t+1} = s(s_{i,t}, d_t)$, so that given the expected behavior of prices, consumers should be able to determine these decision rules optimally. As we have two types of consumers, the equilibrium price functions are determined by their "aggregate" behavior. As their total holdings sum up to 1, knowing the fraction of one agent in the stock market (e.g. knowing $\phi_t = s_{W,t}$) together with the dividend state is enough to pin down the price $p_t = p(s_{W,t}, d_t)$ (from here on we drop the time subscript, and denote the $t+1$ period with a prime).

5.2 Equilibrium

In the economy described above, we make the following assumptions: assume $0 < \beta < 1$, and $U : \mathbb{R}^+ \to \mathbb{R}^+$ is continuously differentiable, bounded, increasing, and strictly concave. For the transition function, assume that $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is continuous, $F(\cdot, d)$ is a distribution function for fixed d, with $F(0, d) = 0$. Also, assume that $\psi(\cdot)$, the unique solution to

$$\psi(d^{'}) = \int F(d^{'}, d) d\psi(d),$$

has a stationary distribution, and for any continuous function $g(d)$, $\int g(d')dF(d', d)$ is a continuous function of d.

An equilibrium will be the price function $p(s_W, d)$ and the optimum value functions $V^O(s_O, d), V^W(s_W, d)$, where the values V^O and V^W are interpreted as the value of the objective (40) for each type of consumer who begins in dividend state d with asset holding s_O for objective agents and s_W for wishful thinkers, and follows an optimum consumption-portfolio policy thereafter.

Definition: An equilibrium is a continuous price function $p(s_W, d) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ and continuous, bounded functions $V^O(s_O, d) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ and $V^W(s_W, d) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that

(i) for wishful thinkers

$$V^W(s_W, d) = max_{s_W^{'}} u(c_W) + \frac{\beta}{\theta} ln\mathbb{E}_d[e^{\theta V^W(s_W^{'}, d^{'})}]$$

subject to

$$c_W \leq ds_W + p(s_W, d)(s_W - s_W^{'}) + \bar{\varepsilon}$$

$$0 \leq s_W^{'} \leq 1$$

$$c_W \geq 0$$

or equivalently the Euler equation

$$p(s_W, d) = \frac{\beta}{\theta} \int \frac{u'(c_W(s_W', d'))}{u'(c_W(s_W, d))} \frac{e^{\theta V(s_W', d')}}{\int e^{\theta V(s_W', d')} dF(d', d)} \left[p(s_W', d') + d'\right] dF(d', d)$$

(ii) for the objective agents

$$V^O(s_O, d) = max_{s_O^{'}} u(c_O) + \beta \mathbb{E}_d[V^O(s_O^{'}, d^{'})]$$

subject to

$$c_O \leq ds_O + p(s_O, d)(s_O - s_O^{'}) + \bar{\varepsilon}$$

$$0 \leq s_O^{'} \leq 1$$

$$c_O \geq 0$$

or equivalently the Euler equation

$$p(s_O, d) = \beta \int \frac{u'(c_O(s_O', d'))}{u'(c_O(s_O, d))} \left[p(s_O', d') + d'\right] dF(d', d)$$

(iii) for each d, $V^W(s_W, d)$ and $V^O(s_O, d)$ are attained by the market clearing conditions

$$c_W(s_W, d) + c_O(s_O, d) = d + 2\bar{\varepsilon}$$

$$s_W^{'}(s_W, d) + s_O^{'}(s_O, d) = 1$$

Condition (i) and (ii) state that both types of consumers optimally allocate their resources among current consumption c and end-of-period share holdings s'. Condition (iii) requires that these consumption and saving decisions clear the market (we only use the stock market clearing condition as the goods market will clear by Walras' law).

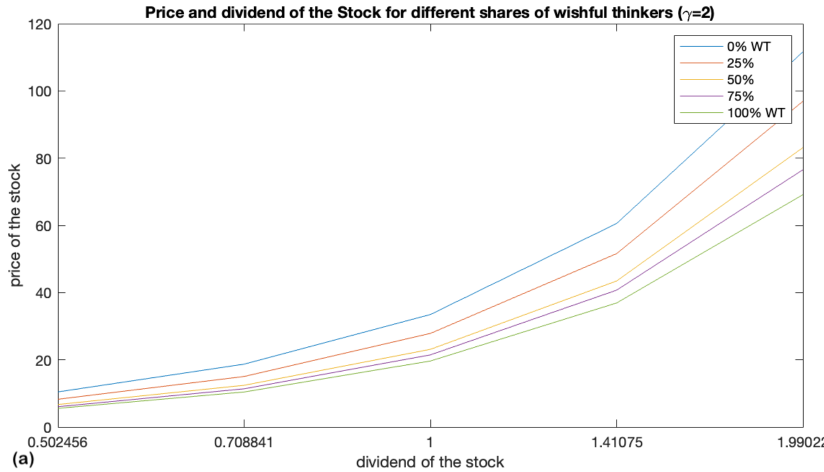
5.3 Solution to the infinite horizon model

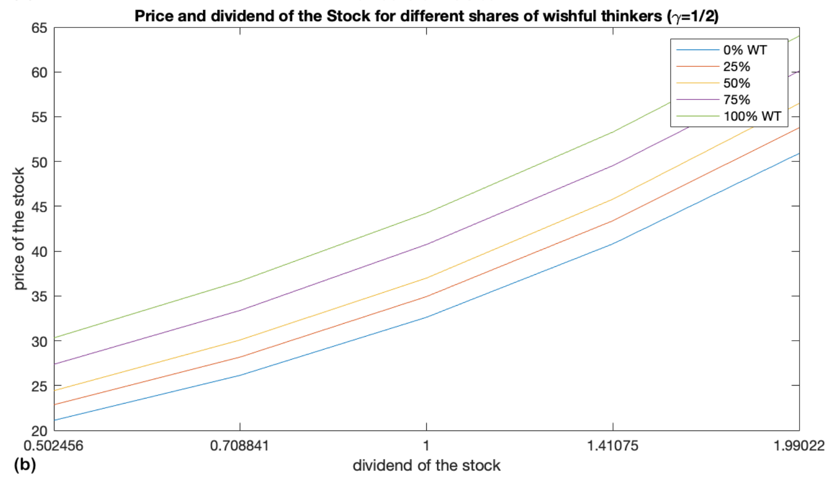
5.3.1 Prices

We take the isoelastic function (CRRA) as the functional form for both agents' utility functions with the same relative risk aversion parameter γ for both agents. The agents consider prices as given when solving their maximization problems, and then prices are adjusted according to market clearing. Figure 11 represents the pricing function of the stock for different dividend states. In all three cases, prices increase with the dividend state as the dividend shock persists. However, the share of wishful thinkers in the market affects the price differently. When the relative risk aversion coefficient γ is equal to 2 for both agents (Figure 11(a)), the price of the stock decreases with the share of wishful thinkers for all dividends. Meanwhile, the price increases with the share of wishful thinkers when $\gamma = 1/2$ (Figure 11(b)), and remains constant when $\gamma = 1$ (Figure 11(c)). This aligns with the results of the two-period asset pricing model in the previous section, where wishful thinking increases with wealth if $\gamma < 1$, decreases with wealth if $\gamma > 1$, and remains constant for $\gamma = 1$. The intuition is that with $\gamma > 1$, the income effect is larger than the substitution effect, and wishful thinkers save less today than objective agents, believing that the probability of high dividends is high for tomorrow. When $\gamma < 1$, the substitution effect is larger than the income effect, and wishful thinkers, believing that the probability of high dividends is higher than it actually is, save more in the stock than objective agents do. With $\gamma = 1$, both effects are equal, resulting in the same price for both agents. The fundamental price of the stock is on the $s_{W,t} = 0$ (or $s_{O,t} = 1$) curve, where only objective agents own stocks. With the participation of wishful thinkers, the price may become overvalued ($\gamma < 1$) or undervalued ($\gamma > 1$), and this deviation from rational pricing generates (and bursts) asset pricing bubbles.

5.3.2 Price and share dynamics

The share dynamics are derived from the policy function of the agents' problem and the market clearing condition. Figure 12 presents the relationship between the price of the stock and the share of wishful thinkers in the market for different dividend states and three cases of γ. It also shows the dynamics of the share of wishful thinkers with small arrows on the curves. In the case of $\gamma = 2$ (Figure 12(a)), regardless of the initial starting share of wishful thinkers, it keeps decreasing until all wishful thinkers are out of the market. This corresponds to Figure 11(a), as the market price drops with the participation of wishful thinkers, resulting in less demand from them compared





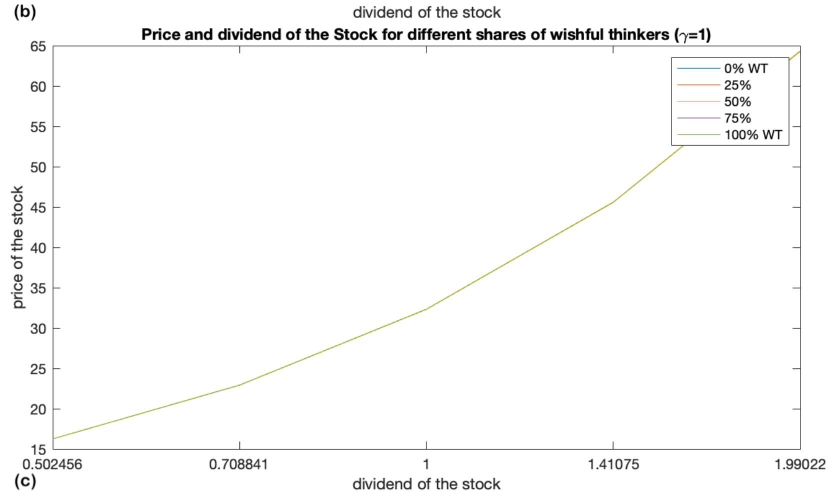


Figure 11: Stock Price when (a) $\gamma = 2$, (b) $\gamma = 1/2$, (c) $\gamma = 1$.

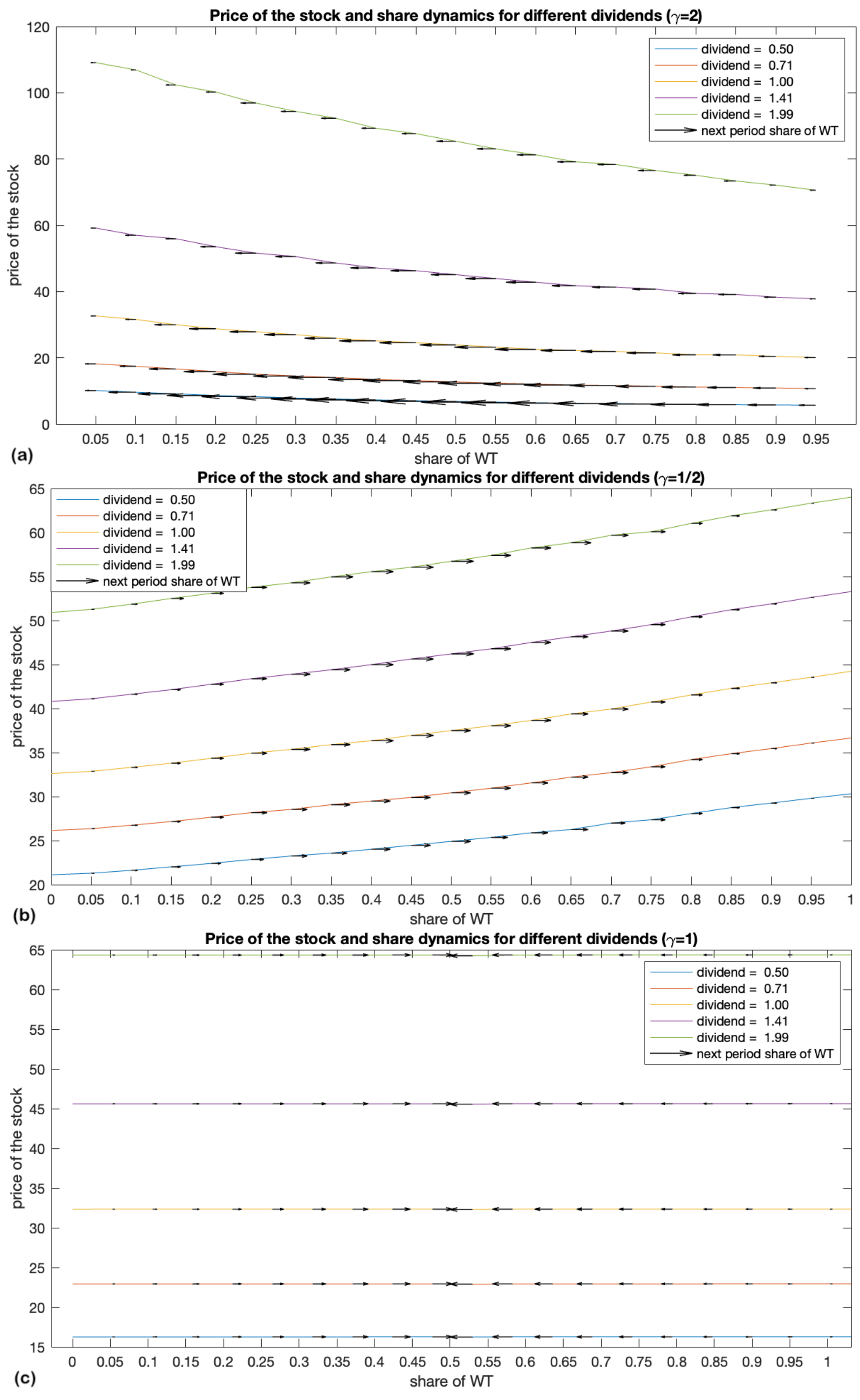


Figure 12: Stock Price and WT share dynamics when (a) $\gamma = 2$, (b) $\gamma = 1/2$, (c) $\gamma = 1$.

to objective agents. Hence, they sell their shares and exit the market (income effect being larger than the substitution effect). In Figure 12(b), the opposite occurs: as $\gamma = 1/2$, wishful thinkers continue to buy all the stocks to benefit from tomorrow's high dividend, as they subjectively assign a high probability to that event. When wishful thinkers start with a small share in the stock market, the arrows indicating their share dynamics are small, as they do not have enough buying power. Starting with a 50% share generates the largest arrows, as they have high buying power and objective agents are not at their highest marginal utility point. Towards the end, the arrows become smaller again as objective agents' marginal utility increases, and they are willing to pay more to save for tomorrow.

In Figure 12(c), when $\gamma = 1$, the price does not change with the share of wishful thinkers. Their share dynamics lead them to the point of exact half of the total shares because, at that point, their marginal utilities are equal.

To generate a bubble where wishful thinkers enter the market with the high dividend, inflate the price, and then leave as the dividend becomes low again, we make the relative risk aversion parameter inversely dependent on the dividend state $\gamma(d)$. When the dividend is low, both agents are more risk-averse than when the dividend state is high. This aligns with the observation that investors' risk preferences change with business cycles, as they are willing to take more risks in booms than in busts. Figure 13(a) illustrates that wishful thinkers are out of the market when the dividend state is low, but they are attracted to the stock and enter the market once the dividend reaches the highest state, driving the price up. They again exit the market once the dividend drops. When the dividend is at the average level (the yellow curve), $\gamma = 1$ but it is expected to drop if tomorrow's dividend is high and expected to increase if tomorrow's dividend is low. This means that if there is a large share of wishful thinkers, they believe that the high dividend state and a small γ are more likely, leading them to buy more shares. The opposite happens when wishful thinkers start with a low share. This is different from the $\gamma = 1$ case in Figure 12(c), where γ is fixed. This aligns with the intuition of a bubble. However, because the entrance of wishful thinkers into the market occurs very slowly (small arrows for the highest state), and because the persistence[23] of the dividend shock is not high enough, this is not enough to significantly inflate the price to higher levels. To address this issue, we also make the wishful thinking parameter θ increase with the dividend state in Figure 13(b). Now wishful thinkers enter the market more rapidly when the dividend state is high, as they become more wishful with the dividend. With the latter version, we simulate the boom-bust dynamics in the next subsection.

[23] it is 0.95 in Marcet and Singleton (1999).

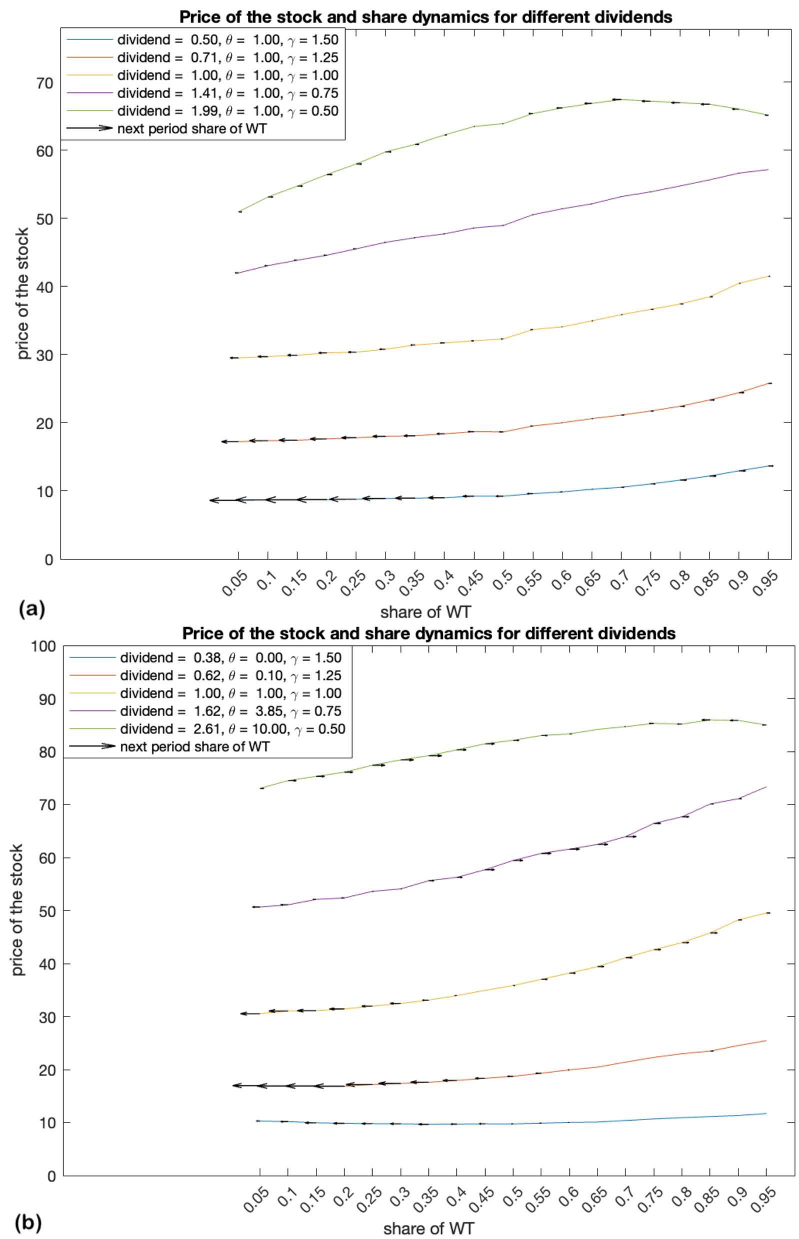


Figure 13: Stock Price and WT share dynamics when (a) γ decreases with dividend state, (b) γ decreases and θ increases with dividend state.

5.3.3 The boom-bust dynamics

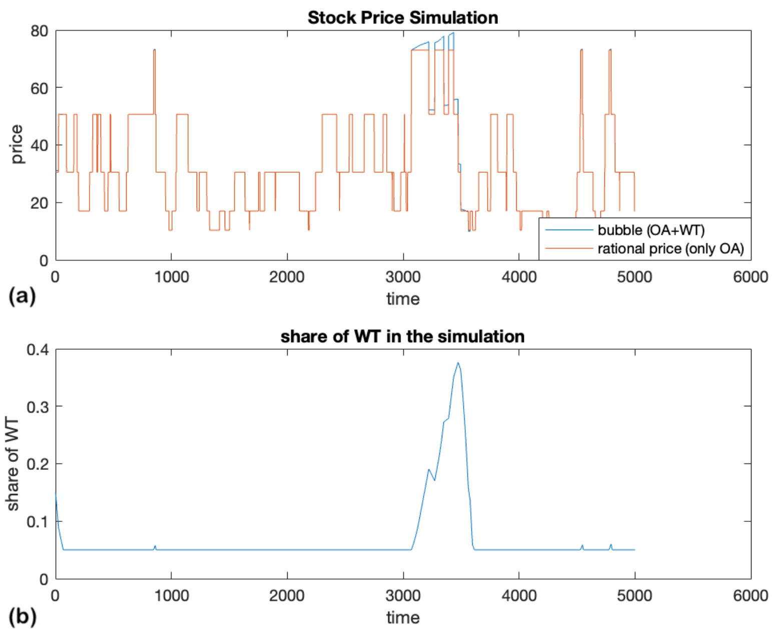


Figure 14: (a) Stock Price Bubble and (b) WT share simulation over time when γ decreases and θ increases with dividend state.

We initiate the simulation from one of the grid points for the share of wishful thinkers in the stock market[24]. We know the next period's share as the share dynamics were solved for each grid point. However, after the first period, the share may end up anywhere between the grid points.

To continue the simulation after the first period, we measure the distance between the new point and its two nearest grid points. To obtain the dynamics function of our new point (where the share moves in the next period), we weight the dynamics functions of both nearest grid points by their respective distances from the new point in between and calculate the dynamics function of the new point as a weighted average of the dynamics of the nearest grid points, with the closer grid point receiving a higher weight.

In Figure 14, initially, the economy is not in the high dividend state, and mostly objective agents own the stock, as wishful thinkers have sold their shares to them. This follows the first stage in the lifecycle of a typical bubble (stages: displacement, boom, euphoria, profit-taking, and panic).

In this stage, the stock has not yet performed to attract the attention of wishful thinkers (public

[24] see "Algorithm for the infinite horizon model" in the appendix for the policy functions

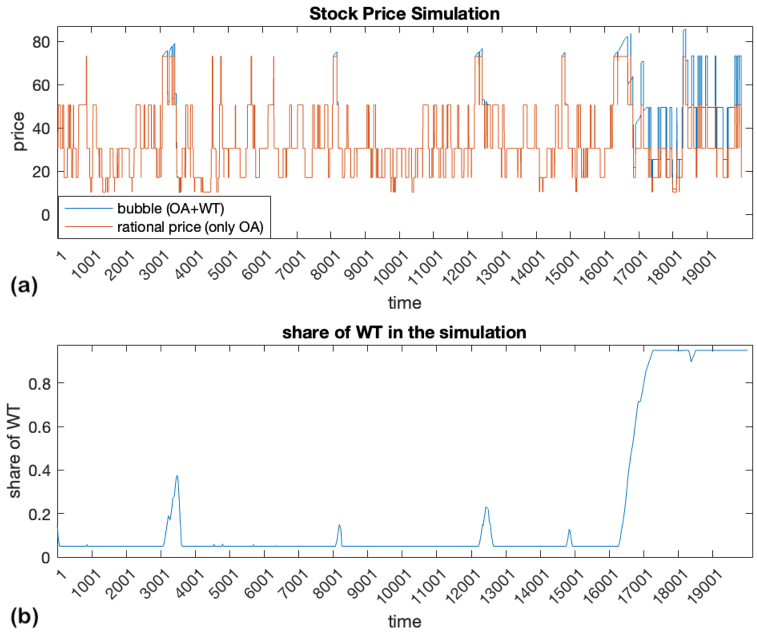


Figure 15: A Longer Simulation for (a) Stock Price Bubble and (b) WT share over time when γ decreases and θ increases with dividend state.

attention). Once the dividend state changes to a high state, it attracts wishful thinkers ("media attention," "enthusiasm"), driving up the price of the asset beyond what can be explained by its fundamentals (blue line in Figure 14(a)). Objective agents sell their stock shares to wishful thinkers as the price cannot be justified by the true dividend generating process (Figure 14(b)). This represents the "boom" phase of the bubble. As wishful thinkers take over the stock market, the objective agents' share in the market decreases ("profit-taking"). The last stage arrives when the dividend changes to a low state again ("panic"). In this stage, the price of the asset drops back to its fundamentals.

If we run the simulation long enough, there will come a period where the price remains high for an extended time, and wishful thinkers take over the market (Figure 15). From that point on, they stay in the market even when shocks are not high anymore, and it may take much longer for the bubble to burst, as wishful thinkers raised the price to unreasonable levels for objective agents to buy.

6 Conclusion

In this paper, we examined the stochastic behavior of the equilibrium stock price in an economy with wishful thinkers, as defined in Caplin, Leahy (2019). We presented a theoretical framework to explain stock market bubbles. Initially, when the dividends are low, the stock price of a productive unit is close to its fundamentals. As soon as the asset starts paying a high dividend, it attracts wishful thinkers who take over the market and drive the price above what can be explained by its fundamentals.

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A Appendix of CHAPTER I

A.1 Mathematical Appendix of CHAPTER I

A.1.1 Derivation of the Two-Fund Separation Theorem

We need to define a financial market equilibrium and derive Proposition 1 and Lemma 1 before proceeding to the proof of the Two-Fund Separation Theorem.

Definition 2.

A financial market equilibrium is a process of portfolio holdings Θ and asset prices (P, q) satisfying the following conditions.

1. $\int_i \theta_{ij,t} di = 1$ and $\int_i b_{i,t} di = 0$ for all $j \in \mathbb{J}$ and all t.
2. For each household $i \in \mathbb{I}$, Θ_i solves their utility maximization problem (1)
3. $\sup |\int_j \theta_{ij,t} p_j dj + b_{i,t} q| < \infty$

Proposition 1.

The economy described in the model has an efficient financial market equilibrium.

It follows from the general equilibrium described in section 1.4 (for the proof see Proposition 2 in Schmedders (2007)).

Lemma 1.

If all the households have equi-cautious HARA utilities, then the consumption allocation of each agent in an efficient equilibrium satisfies a linear sharing rule $c_{i,t} = m_i C_t + n_i$ for all $i \in \mathbb{I}$ and all t, where C_t is the aggregate consumption in period t, and m_i and n_i are household specific constants.

Proof.

Compute the consumption allocations in equilibrium using Negishi approach (Negishi 1960) of Judd et al. (2003). For utility functions of type HARA the first order conditions imply $\lambda_{i'} u'(c_{i',t}) - \lambda_i u'(c_{i,t}) = 0$:

$$\lambda_i (c_{i,t} - \alpha_i)^{-\gamma} - \lambda_{i'} (c_{i',t} - \alpha_{i'})^{-\gamma} = 0, \quad i \in \mathbb{I}, \forall t$$

where λ_i is the Negishi (1960) weight on household i. After some algebra I simplify the consumption to the linear sharing rule

$$c_{i,t} = C_t \frac{\lambda_i^{1/\gamma}}{\int_{i' \in \mathbb{I}} \lambda_{i'}^{1/\gamma} di'} + \left(- \alpha_i + \frac{\lambda_i^{1/\gamma}}{\int_{i' \in \mathbb{I}} \lambda_{i'}^{1/\gamma} di'} \int_{i' \in \mathbb{I}} \alpha_{i'} di' \right)$$

where C_t is the aggregate consumption in period t[25]. $\square$

Remark 1.

Note that for the special case of the CRRA utility function ($\alpha_i = 0$ for all households $i \in \mathbb{I}$), the sharing rule has zero intercept.

Remark 2.

The supplies of all risky assets are each fixed to a unit so market clearing and the requirement from the definition imply that all the households' portfolios exhibit two-fund separation in equilibrium if and only if each household has a constant share of every risky asset in the economy, $\theta_{ij} = \theta_{ij'}$ for all stocks $j, j' \in \mathbb{J}$ and all households $i \in \mathbb{I}$ in all time periods. Furthermore, as the supplies of assets are fixed, for each household i the ratio of wealth invested in any two risky assets $(\theta_{ij}p_{j,t}/\theta_{ij'}p_{j',t})$ equals the ratio of their prices $p_{j,t}/p_{j',t}$ and thus depends on the state of the economy.

Note that in a complete markets setting agents have the same stochastic discount factor and agree on asset prices, hence there is no trade after the initial period, and they consume their asset income each period without rebalancing their portfolios,

$$c_{i,t} = CF_t \Theta_i \tag{45}$$

where CF_t is the vector of all the cash flows of assets (including bond).

Theorem 1. (Two-Fund Separation Theorem[26])

Suppose the economy $\mathbb{H}$ has assets with linearly independent payoff vectors, short sales are permitted, and the returns from any security cannot be dominated by the returns from a linear combination of the other securities using weights (possibly negative) adding to one. The risky assets are in unit net supply and the risk free asset is a consol in zero net supply. If the households have equi-cautious HARA utilities then in an efficient equilibrium their portfolios exhibit two-fund separation, that is $\theta_{ij} = \theta_{ij'}$ for all risky assets $j, j' \in \mathbb{J}$, and each household $i \in \mathbb{I}$ in any period t holds a weighted average of the two funds - the market portfolio of all stocks and the bond.

Proof

Proposition 1 ensures that an efficient equilibrium exists. Lemma 1 implies that sharing rules are linear in aggregate consumption and $c_{i,t} = m_i C_t + n_i$ for all $i \in \mathbb{I}$ and all t. Under the assumptions of the theorem, equation (15) in the model has the unique solution $n_i = b_i$ and $m_i = \theta_{ij}$ $\forall j \in \mathbb{J}$ and all t. $\square$

[25] For more detail on the Negishi approach refer to Negishi (1960), Judd et al. (2003), and Schmedders (2007)

[26] This version is from Schmedders (2007)

A.1.2 Aggregation of the heterogeneous households in the habit model

As the individual consumption levels of the households is already defined to satisfy a linear sharing rule $c_{i,t} = m_i C_t$ for all $i \in \mathbb{I}$ and all t, the two-fund separation theorem above still holds with equation (15) having the unique solution $b_i = 0$ and $m_i = \theta_{ij}$ $\forall j \in \mathbb{J}$ and all t. But first we need to check that the marginal utilities are collinear and that households do not trade after the initial period.

I follow the steps described in the appendix of Campbell and Cochrane (1999a). Despite heterogeneous households have different levels of consumption, the consumption growth is the same across all households,

$$\frac{c_{i,t}}{c_{i,t-1}} = \frac{C_t}{C_{t-1}}$$

Recall that in the habit model the log of the surplus consumption ratio[27] is defined as

$$log(S_{i,t+1}) = s_{i,t+1} = (1-\phi)\bar{s} + \phi s_{i,t} + \lambda(s_{i,t})(c_{i,t+1} - c_{i,t} - g).$$

By choosing to start from an initial state that has the same surplus consumption ratio for all the agents, those will be the same for all households i at any point t in time, $s_{i,t} = s_t$ because consumption growths are the same for each individual. The margianl utility ratios of all the households are the same

$$\frac{MU_{i,t+1}}{MU_{i,t}} = \frac{(c_{i,t+1} - \alpha_{i,t+1})^{-\gamma}}{(c_{i,t} - \alpha_{i,t})^{-\gamma}} = \frac{(m_i C_{t+1} S_{t+1})^{-\gamma}}{(m_i C_t S_t)^{-\gamma}} = \frac{(C_{t+1} S_{t+1})^{-\gamma}}{(C_t S_t)^{-\gamma}}.$$

Hence, they agree on asset prices and have no incentive to trade after the first period. The two-fund separatioon theorem follows from the linear sharing rule with the difference that now households hold zero bonds. The relation between the individual habit, $\alpha_{i,t}$, and the average habit, α_t, will be

$$\alpha_{i,t} = c_{i,t}(1-S_t) = m_i C_t(1-S_t) = m_i C_t \frac{\alpha_t}{C_t} = m_i \alpha_t.$$

Hence, similar to consumption, aggregate habit is the weighted average of individual habits.

[27] $S_t = \frac{C_t - X_t}{C_t}$

A.2 Appendix of Figures of CHAPTER I

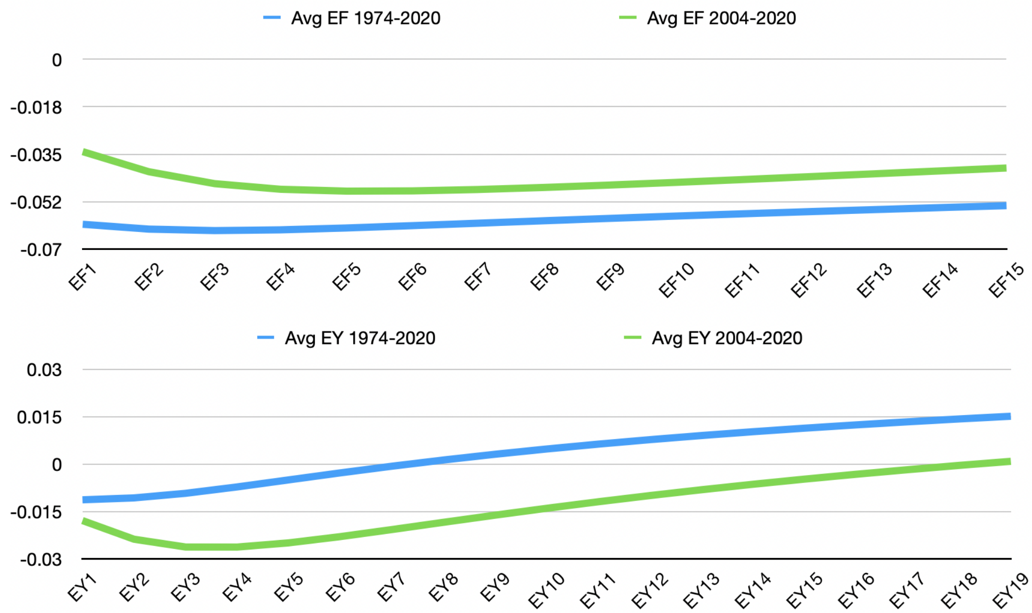

Figure 16: (a) Average equity term structure (EY) and (b) average term structure of forward yields (EF) (dividend strip yield estimates from Giglio et al. (2021), bond yield data from Gurkaynak et al. (2006))

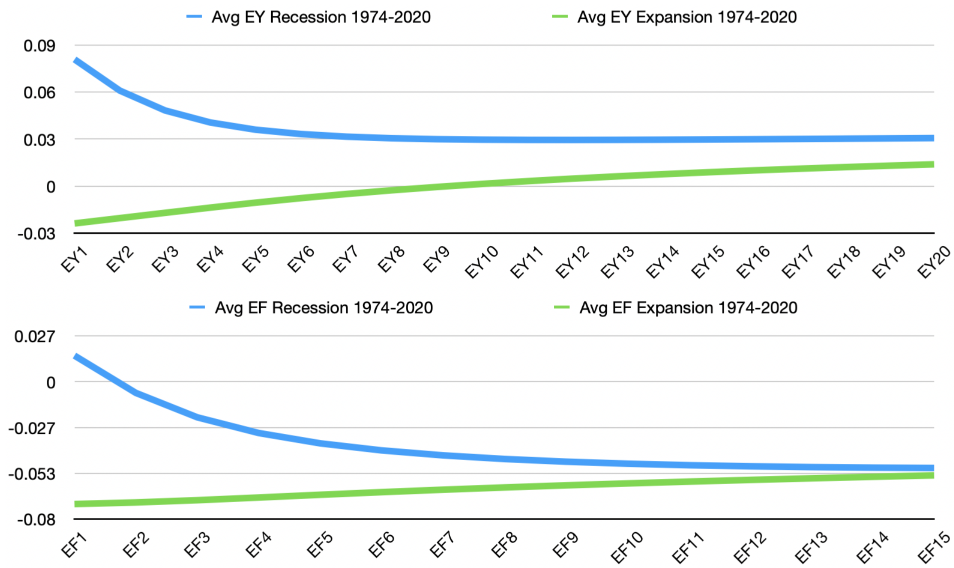

Figure 17: (a) Average equity term structure (EY) and (b) average term structure of forward yields (EF) *conditional* on the phase of the business cycle (dividend strip yield estimates from Giglio et al. (2021), bond yield data from Gurkaynak et al. (2006))

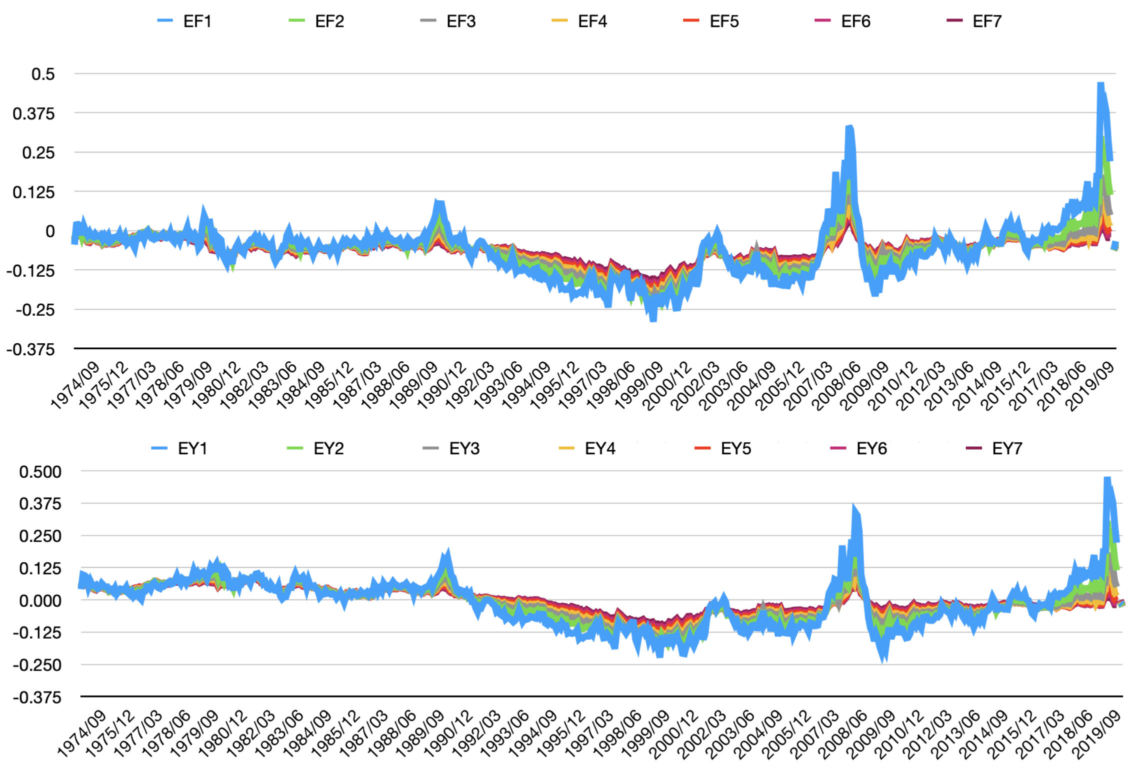

Figure 18: (a) Equity term structure (EY) and (b) term structure of forward yields (EF) (dividend strip yield estimates from Giglio et al. (2021), bond yield data from Gurkaynak et al. (2006))

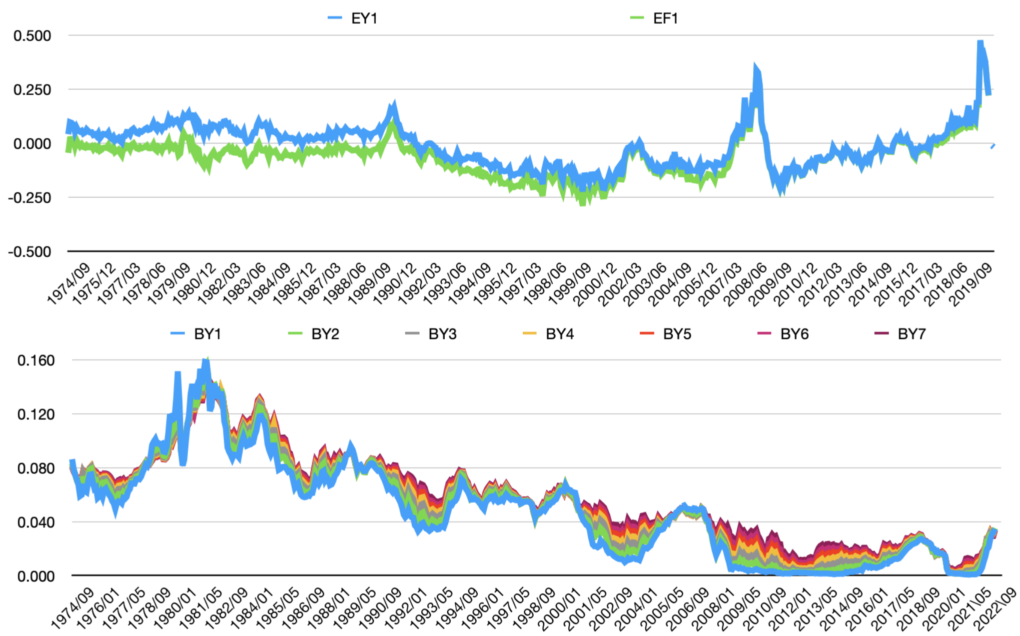

Figure 19: (a) Comparison of the equity yield (EY) & forward yield (EF) both with maturity of 1 year, (b) bond term structure (dividend strip yield estimates from Giglio et al. (2021), bond yield data from Gurkaynak et al. (2006))

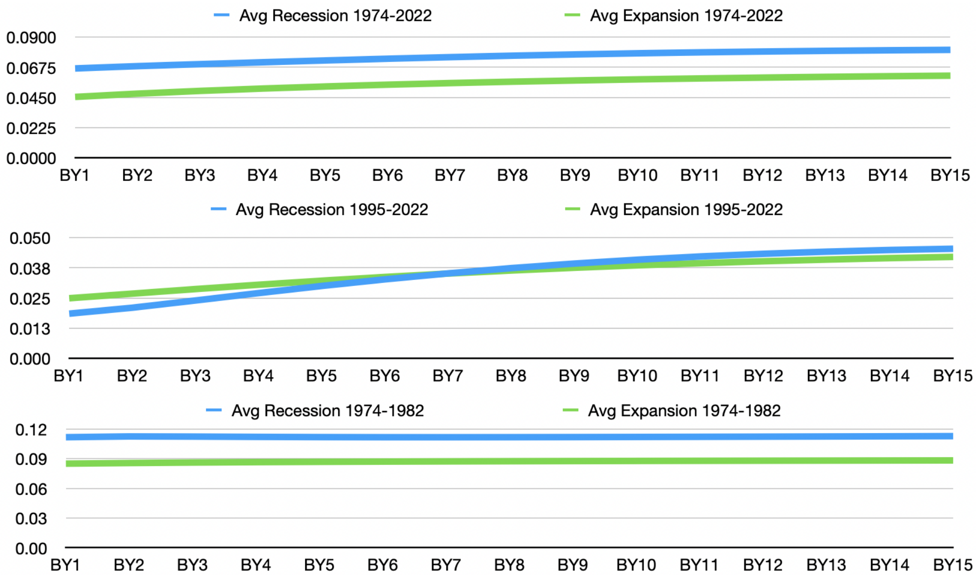

Figure 20: (a) Conditional bond yields in the entire sample, (b) Conditional bond yields for the period when the Fed was more active vs (c) less active in QE and lowering rates during recessions (data from Gurkaynak et al. (2006))

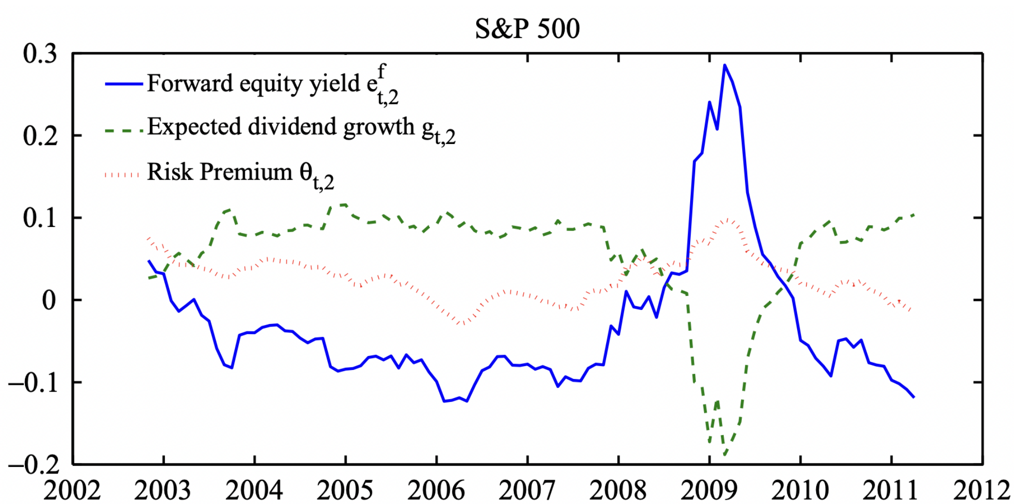

Figure 21: Decomposition of a forward equity yield with maturity of 2 years (figure 7 in van Binsbergen et al. (2013))

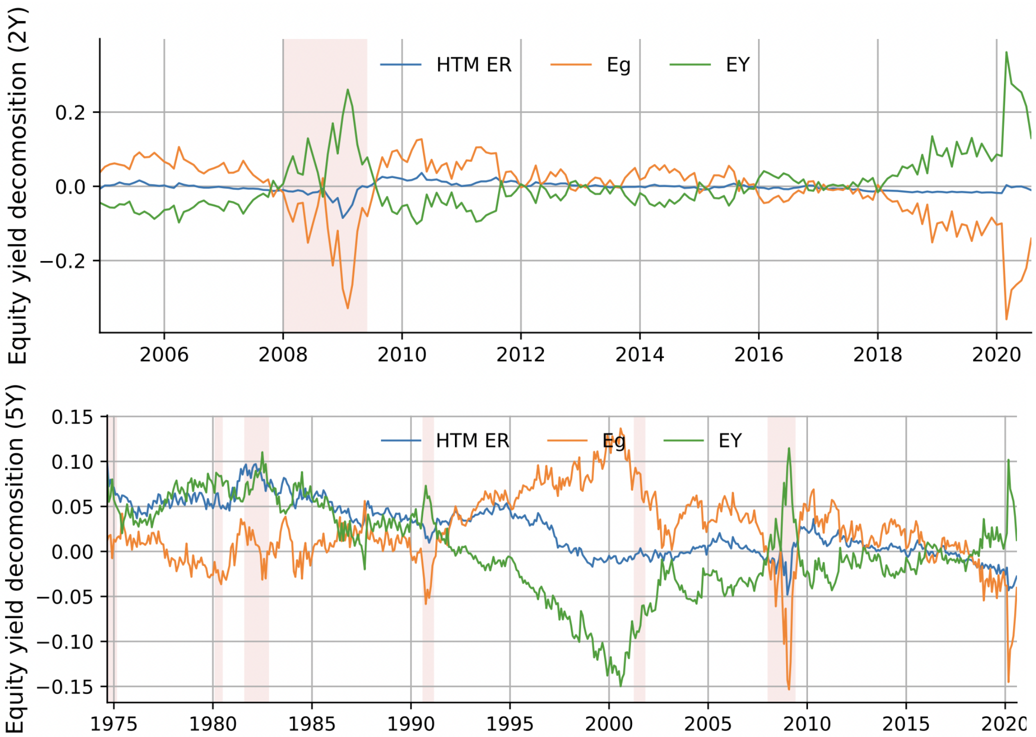

Figure 22: Decomposition of a forward equity yield with maturity of (a) 2 years & (b) 5 years (figures 6 & 9 in Giglio et al. (2021))

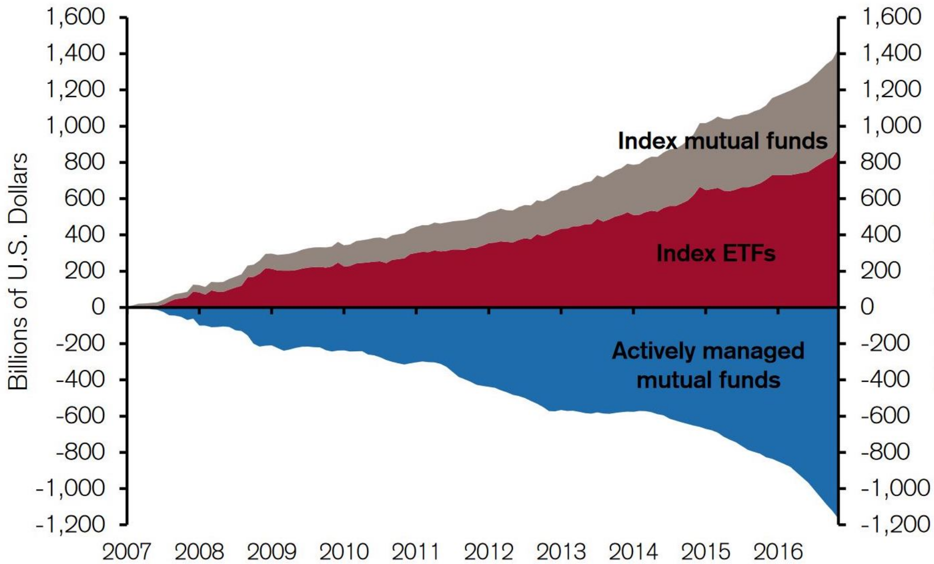

Figure 23: Flows from Active to Passive Funds in US Equities
Source: Investment Company Institute; Simfund; Credit Suisse.

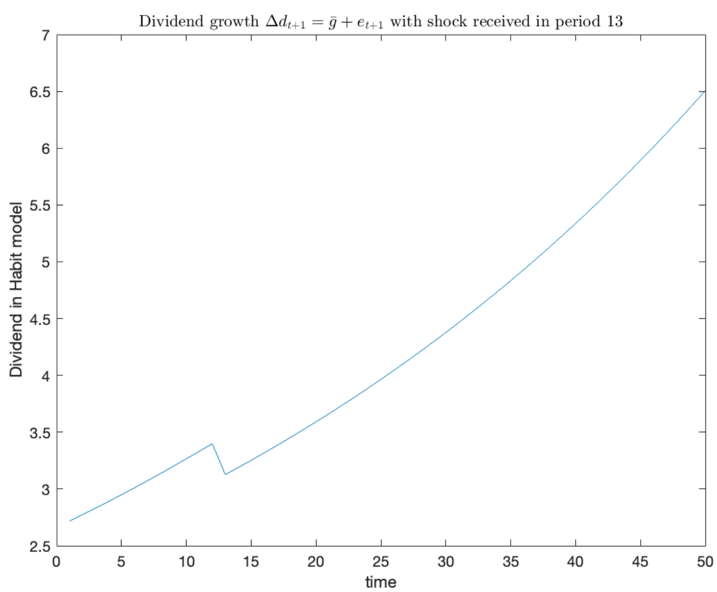

Figure 24: Comparison: Shock (in period 13) to the Consumption in Habit model

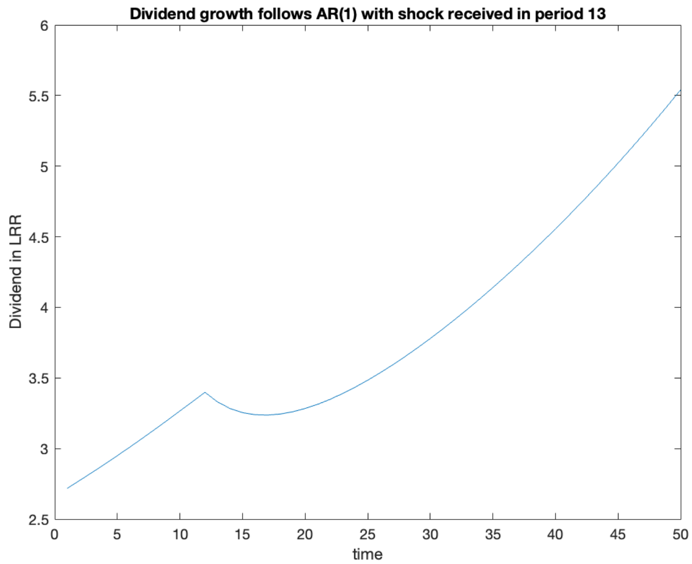

Figure 25: Comparison: Shock (in period 13) to the Consumption in long-run risk model

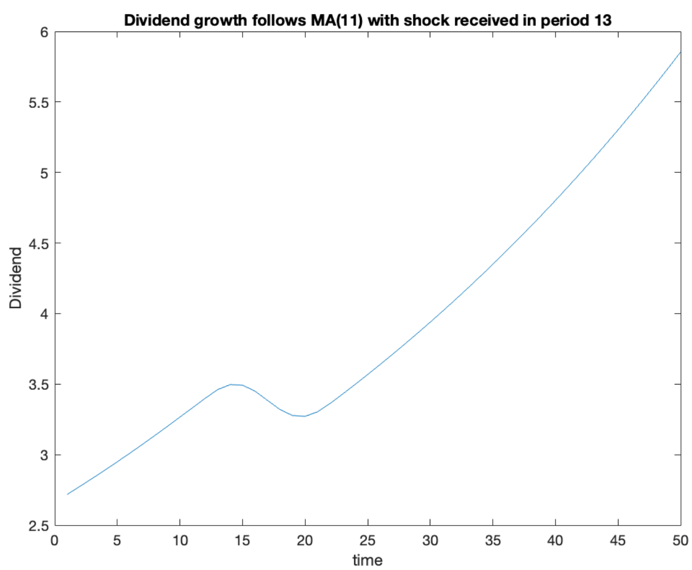

Figure 26: Comparison: Shock (in period 13) to the Consumption in MA(11) model (can do with AR(2))

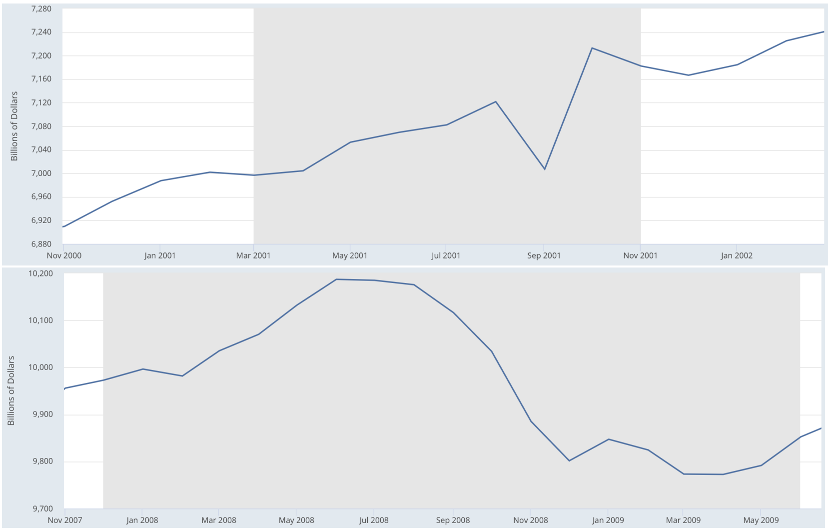

Figure 27: Lagged Consumption response in recessions in the data (blue-PCE, grey-recession, data from Fed)

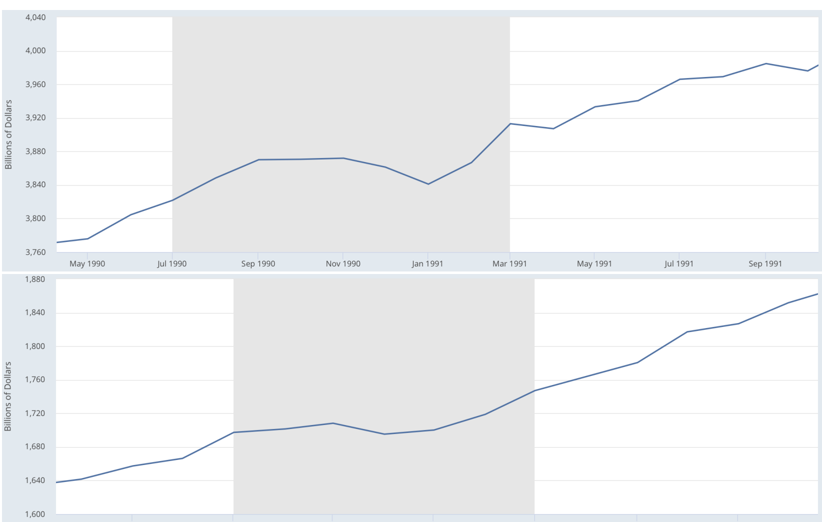

Figure 28: Lagged Consumption response in recessions in the data (blue-PCE, grey-recession, data from Fed)

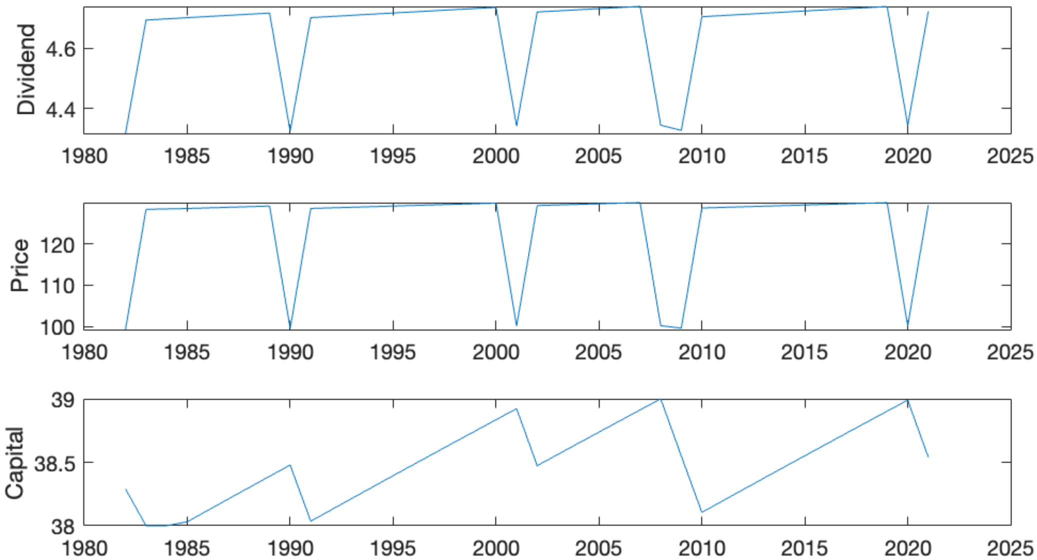

Figure 29: Simulated Dividend, Stock Price, and Capital in a Homogeneous Firm model

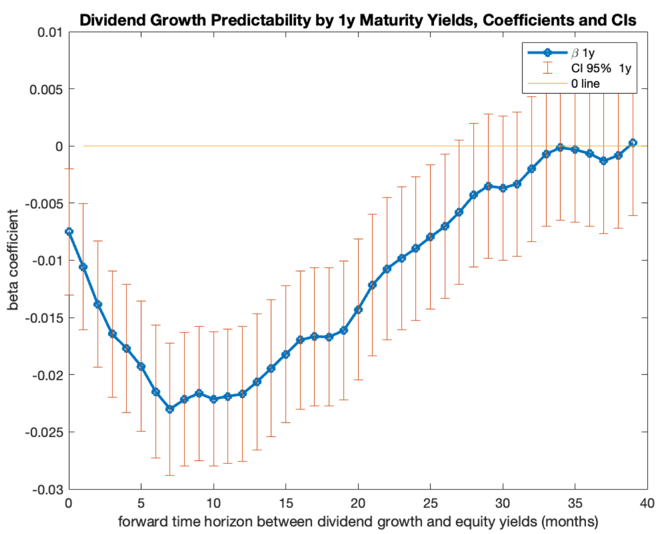

Figure 30: 1 month forward Dividend Growth Predictability by 1 year Maturity Yields

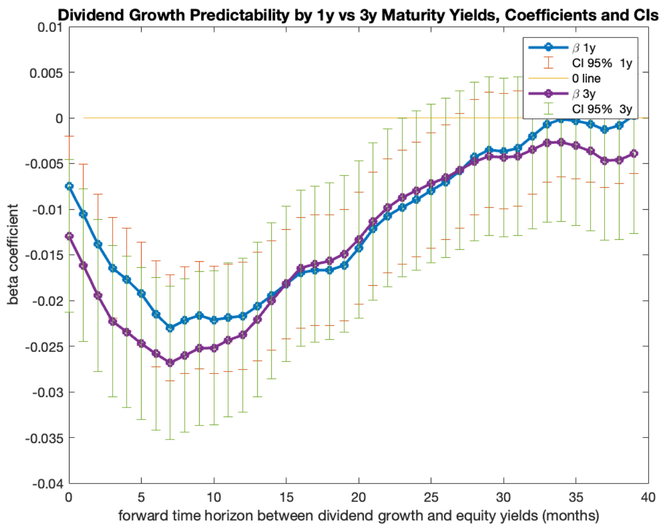

Figure 31: 1 month forward Dividend Growth Predictability by 1 vs 3 year Maturity Yields

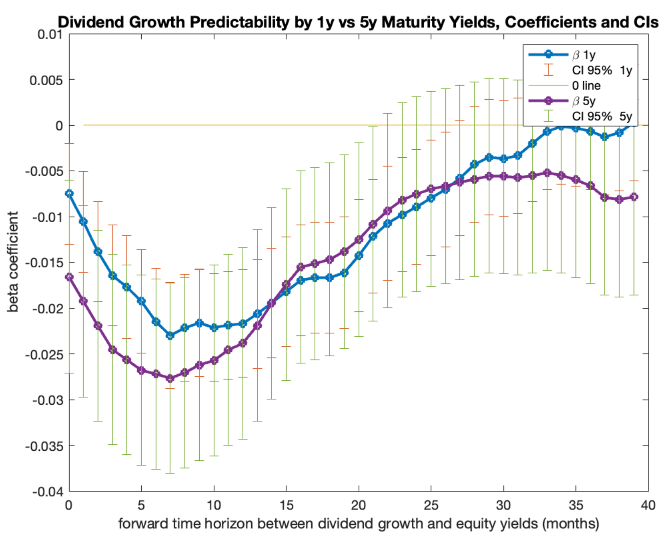

Figure 32: 1 month forward Dividend Growth Predictability by 1 vs 5 year Maturity Yields

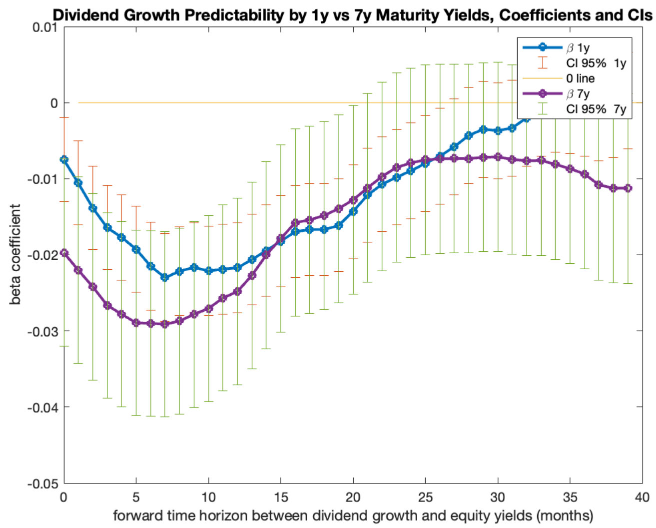

Figure 33: 1 month forward Dividend Growth Predictability by 1 vs 7 year Maturity Yields

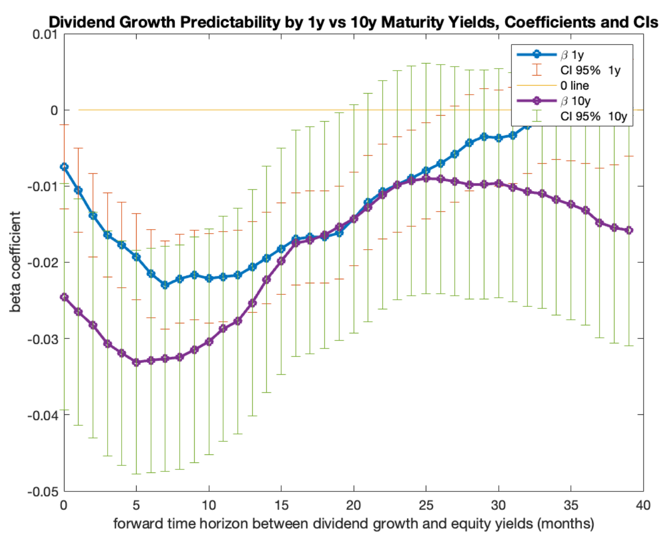

Figure 34: 1 month forward Dividend Growth by 1 vs 10 year Maturity Yields

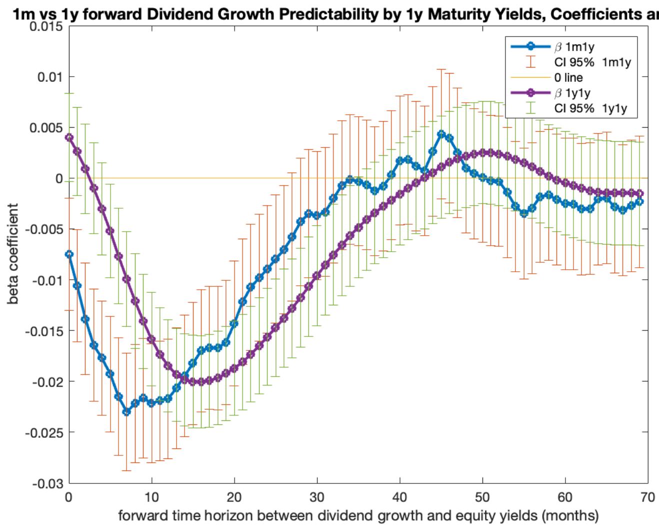

Figure 35: 1 month vs 1 year forward Dividend Growth by 1 year Maturity Yields

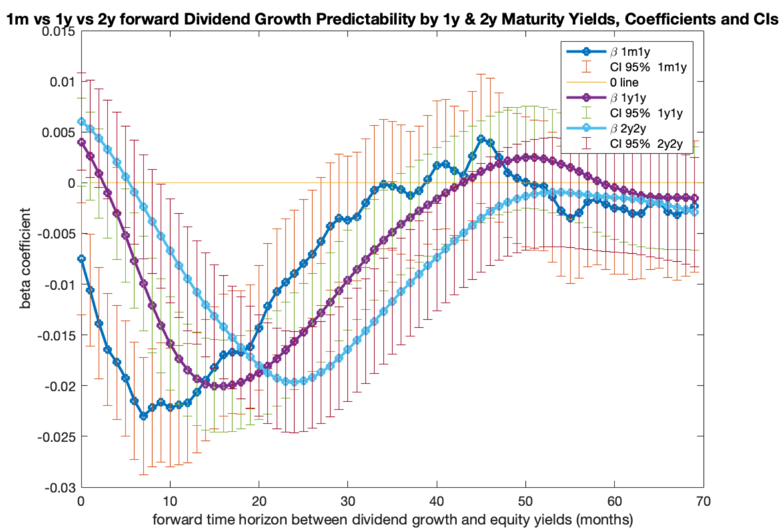

Figure 36: 1 year forward Dividend Growth Predictability by 1 year Maturity Yields and 2 year forward Dividend Growth Predictability by 2 year Maturity Yields

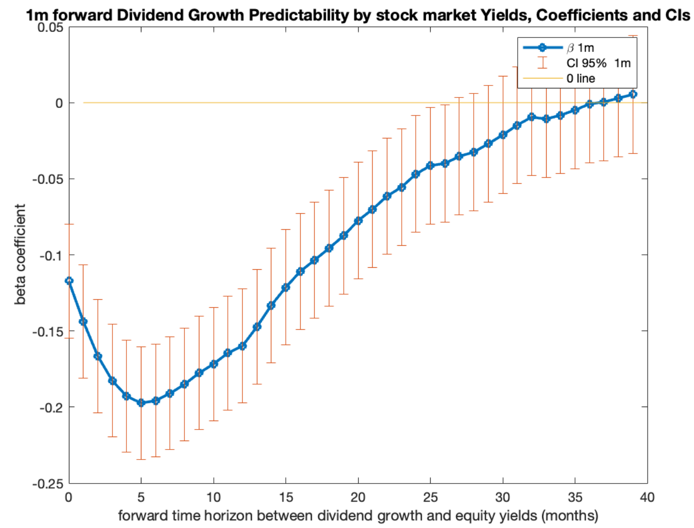

Figure 37: 1 month Dividend Yield Predictability by Stock Market Yields

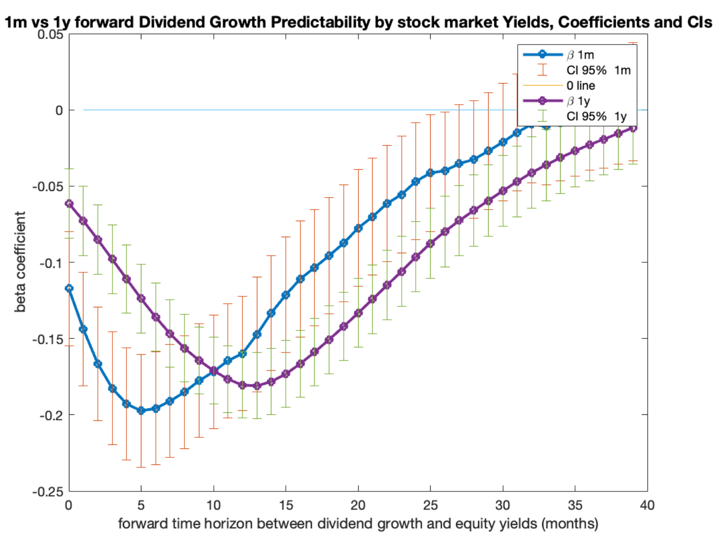

Figure 38: 1 month vs 1 year Dividend Yield Predictability by Stock Market Yields

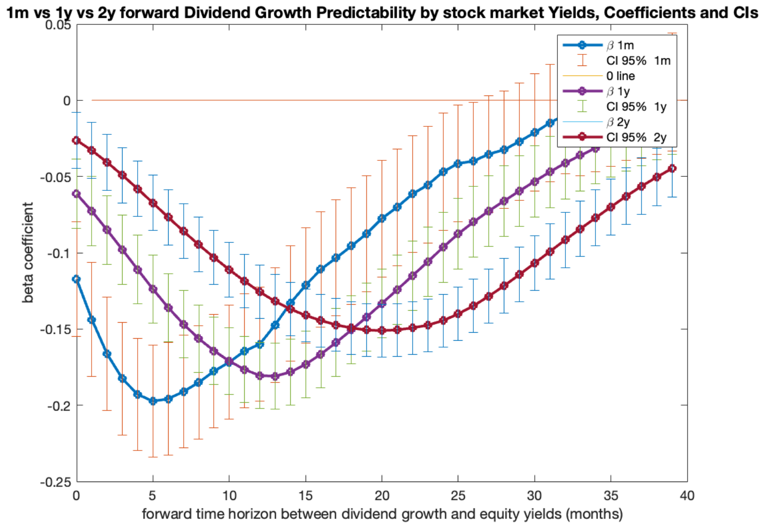

Figure 39: 1 month vs 1 year vs 2 year Dividend Yield Predictability by Stock Market Yields

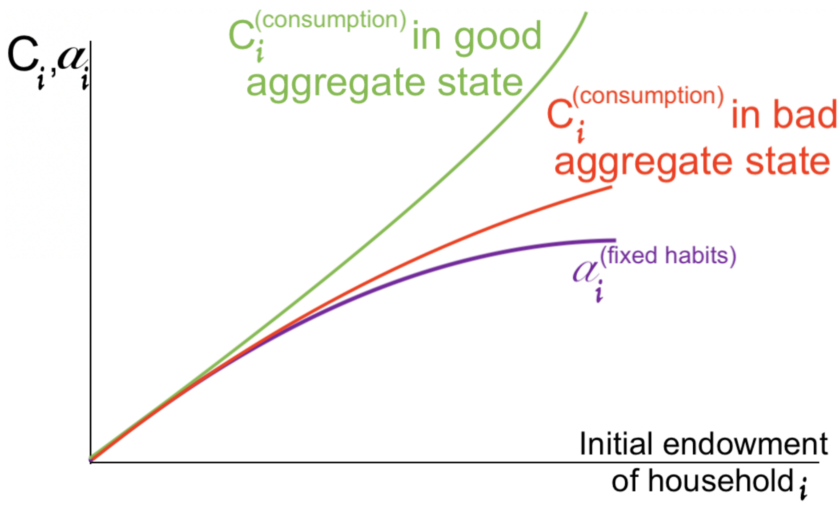

Figure 40: Relationship between consumption, fixed habits and total (asset) income

No two investors neighboring along wealth axis change their order (rank) of wealth over the cycle: Suppose, the difference in their wealth is $\varepsilon > 0$. Then the difference in their portfolio holdings will also be infinitesimal with the investor with ε more wealth holding only slightly more stocks over bonds than her neighbor with less wealth. The price jumps over the cycle are finite in equilibrium, and its product with an infinitesimal number gives an infinitesimal number. Therefore, the two neighbors keep their rank along wealth. Only the consumption volatility of high net-worth households is higher than that of the poor and middle class. Thus there is a trade only at the beggining of the economy in period zero when households allocate their portfolios according to their initial endowments (wealth). But the prices and returns on assets change.

B Appendix of CHAPTER II

B.1 Mathematical Appendix of CHAPTER II

B.1.1 Derivation of the marginal costs for intermediate good producer

The firm's nominal profits are

$$\pi_{j,t} = \max_{k_{j,t},l_{j,t}} p_{j,t}(y_{j,t}) \times y_{j,t} - (1+r_t)P^K_{t-1}k_{j,t} - w_t l_{j,t} + (1-\delta_t)P^K_t k_{j,t} =$$

$$\max_{k_{j,t},l_{j,t}} p_{j,t}(y_{j,t}) \times y_{j,t} - R_t P^K_{t-1}k_{j,t} - w_t l_{j,t}$$

instead solve cost minimization problem

$$J = -w_t l_{j,t} - R_t P^K_{t-1} k_{j,t} + MC_t(k^{\alpha}_{j,t} l^{1-\alpha}_{j,t} - \bar{y})$$

F.O.C. yields:

$$MC_t = \left(\frac{1}{1-\alpha}\right)^{1-\alpha} \left(\frac{1}{\alpha}\right)^{\alpha} w_t^{1-\alpha}(R_t P^K_{t-1})^{\alpha}$$

where $R_t = r_t - (1-\delta_t)\frac{P^K_t - P^K_{t-1}}{P^K_{t-1}} + \delta_t$ is the required rate of return on capital.

B.1.2 Intermediate good producers' optimization problem

CEOs are maximizing not only firm j's profits but also a linear combination of other firms' profits:

$$\max_{p_{j,t}} \left\{ (p_{j,t} - \xi_{j,t}MC_t)\xi_{j,t}^{\epsilon_t - 1}\left(\frac{p_{j,t}}{P^Y_t}\right)^{-\epsilon_t} Y_t + \sum_{k \neq j} \frac{\sum_i \gamma_{ij,t}\beta_{ik,t}}{\sum_i \gamma_{ij,t}\beta_{ij,t}}(p_{k,t} - \xi_{k,t}MC_t)\xi_{k,t}^{\epsilon_t - 1}\left(\frac{p_{k,t}}{P^Y_t}\right)^{-\epsilon_t} Y_t \right\}$$

These producers take nominal wage, nominal interest rate, and aggregate demand as given and determine the demand for labor and capital inputs.

F.O.C.

$$\frac{1-\epsilon_t}{\epsilon_t} + \left(\frac{p_{j,t}}{\xi_{j,t}P^Y_t}\right)^{1-\epsilon_t} + \frac{\xi_{j,t}MC_t}{P^Y_t}\left(\frac{p_{j,t}}{P^Y_t}\right)^{-1} - \frac{MC_t}{P^Y_t}\left(\frac{p_{j,t}}{\xi_{j,t}P^Y_t}\right)^{-\epsilon_t} +$$

$$+\sum_{k\neq j} \frac{\sum_i \gamma_{ij,t}\beta_{ik,t}}{\sum_i \gamma_{ij,t}\beta_{ij,t}} \frac{p_{k,t} - \xi_{k,t}MC_t}{\xi_{k,t}P^Y_t}\left(\frac{p_{k,t}}{\xi_{k,t}P^Y_t}\right)^{-\epsilon} = 0$$

Note that we used the fact that $\frac{\partial P^Y_t}{\partial p_{j,t}} = \left(\frac{p_{j,t}}{\xi_{j,t}P^Y_t}\right)^{-\epsilon_t}\frac{1}{\xi_{j,t}}$ as the effect of each of the n firms is not infinitesimal, whereas in the usual New Keynesian models this effect through P^Y_t is ignored (because Dixit-Stiglitz aggregator is continuous).

Simplifying the above equation yields

$$\sum_i \gamma_{ij,t}\beta_{ij,t}\left[\frac{1-\epsilon_t}{\epsilon_t}+\left[\frac{p_{j,t}-\xi_{j,t}MC_t}{\xi_{j,t}P_t^Y}\right]\left(\frac{p_{j,t}}{\xi_{j,t}P_t^Y}\right)^{-\epsilon_t}+\left(\frac{p_{j,t}}{\xi_{j,t}MC_t}\right)^{-1}\right]+$$

$$+\sum_{k\neq j}\sum_i \gamma_{ij,t}\beta_{ik,t}\frac{p_{k,t}-\xi_{k,t}MC_t}{\xi_{k,t}P_t^Y}\left(\frac{p_{k,t}}{\xi_{k,t}P_t^Y}\right)^{-\epsilon_t}=0$$

$$p_{j,t}=\xi_{j,t}MC_t\underbrace{\left[\frac{\epsilon_t-1}{\epsilon_t}-\frac{\sum_k\sum_i \gamma_{ij,t}\beta_{ik,t}\left[\frac{p_{k,t}-\xi_{k,t}MC_t}{\xi_{k,t}P_t^Y}\right]\left(\frac{p_{k,t}}{\xi_{k,t}P_t^Y}\right)^{-\epsilon_t}}{\sum_i \gamma_{ij,t}\beta_{ij,t}}\right]^{-1}}_{\mu_{j,t}}$$

denoting $\sum_i \gamma_{ij,t}\beta_{ij,t}=B_{j,t}$ and using the markups

$$\left(\frac{p_{j,t}}{\xi_{j,t}P_t^Y}\right)^{-\epsilon_t}=\left[\frac{\mu_{j,t}}{\mu_t}\right]^{-\epsilon_t}\quad and\quad \frac{p_{j,t}-\xi_{j,t}MC_t}{\xi_{j,t}P_t^Y}=\left[\frac{\mu_{j,t}-1}{\mu_t}\right]$$

we can plug these expressions back into the previous equation

$$\mu_{j,t}^{-1}=\frac{\epsilon_t-1}{\epsilon_t}-\frac{\sum_k\sum_i \gamma_{ij,t}\beta_{ik,t}\left[\frac{\mu_{k,t}-1}{\mu_t}\right]\left[\frac{\mu_{k,t}}{\mu_t}\right]^{-\epsilon_t}}{B_{j,t}}$$

where $\mu_t=\frac{P_t^Y}{MC_t}$ and $\mu_{j,t}=\frac{p_{j,t}}{\xi_{j,t}MC_t}$ are the aggregate and individual firm markups, respectively.

$$\mu_{j,t}^{-1}=\frac{\epsilon_t-1}{\epsilon_t}-\frac{\mu_t^{\epsilon_t-1}}{B_{j,t}}\sum_k\sum_i \gamma_{ij,t,t}\beta_{ik,t}\left[\mu_{k,t}-1\right]\left[\mu_{k,t}\right]^{-\epsilon_t}$$

B.2 Appendix of Figures of CHAPTER II

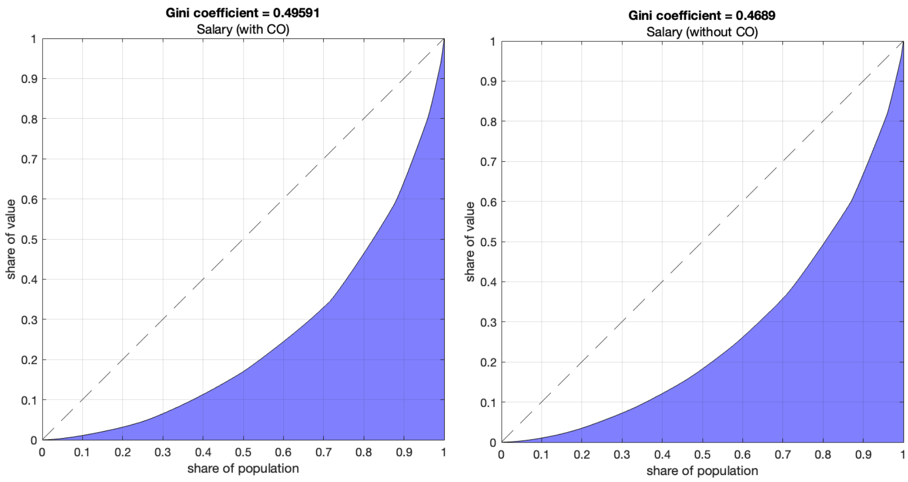

Figure 41: Gini coefficient of wages in a model with common ownership (left with Gini=49.59) Gini coefficient of wages in a model without common ownership (right with Gini=46.89)

C Appendix of CHAPTER III

C.1 Mathematical Appendix of CHAPTER III

C.1.1 Objective agent problem

The first order conditions are

$$
\begin{aligned}
\left(C^O\right)^{-\sigma} &= \lambda^O \\
\pi Q^H(\tilde{S}^O Q^H + \tilde{B}^O)^{-\sigma} + (1-\pi)Q^L(\tilde{S}^O Q^L + \tilde{B}^O)^{-\sigma} &= \lambda^O P \\
\pi(\tilde{S}^O Q^H + \tilde{B}^O)^{-\sigma} + (1-\pi)(\tilde{S}^O Q^L + \tilde{B}^O)^{-\sigma} &= \frac{\lambda^O}{R}
\end{aligned}
$$

We can solve the latter two for consumption in the high and low states

$$
\begin{aligned}
\pi Q^H(\tilde{S}^O Q^H + \tilde{B}^O)^{-\sigma} + (1-\pi)Q^L(\tilde{S}^O Q^L + \tilde{B}^O)^{-\sigma} &= \lambda^O P \\
\pi Q^H(\tilde{S}^O Q^H + \tilde{B}^O)^{-\sigma} + (1-\pi)Q^H(\tilde{S}^O Q^L + \tilde{B}^O)^{-\sigma} &= \frac{\lambda^O}{R} Q^H \\
(1-\pi)(Q^H - Q^L)(\tilde{S}^O Q^L + \tilde{B}^O)^{-\sigma} &= \lambda^O \left(\frac{Q^H}{R} - P\right) \\
C^{O,L} &= \tilde{S}^O Q^L + \tilde{B}^O = \left(\frac{(1-\pi)(Q^H - Q^L)}{\lambda^O \left(\frac{Q^H}{R} - P\right)}\right)^{1/\sigma}
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\pi Q^H(\tilde{S}^O Q^H + \tilde{B}^O)^{-\sigma} + (1-\pi)Q^L(\tilde{S}^O Q^L + \tilde{B}^O)^{-\sigma} &= \lambda^O P \\
\pi Q^L(\tilde{S}^O Q^H + \tilde{B}^O)^{-\sigma} + (1-\pi)Q^L(\tilde{S}^O Q^L + \tilde{B}^O)^{-\sigma} &= \frac{\lambda^O}{R} Q^L \\
\pi(Q^H - Q^L)(\tilde{S}^O Q^H + \tilde{B}^O)^{-\sigma} &= \lambda^O \left(P - \frac{Q^L}{R}\right) \\
C^{O,H} &= \tilde{S}^O Q^H + \tilde{B}^O = \left(\frac{\pi(Q^H - Q^L)}{\lambda^O \left(P - \frac{Q^L}{R}\right)}\right)^{1/\sigma}
\end{aligned}
$$

We can then solve for $\tilde{S}^O$ and $\tilde{B}^O$

$$\begin{aligned}
\tilde{S}^O Q^L + \tilde{B}^O &= \left(\frac{(1-\pi)(Q^H - Q^L)}{\lambda^O\left(\frac{Q^H}{R} - P\right)}\right)^{1/\sigma} \\
\tilde{S}^O Q^H + \tilde{B}^O &= \left(\frac{\pi(Q^H - Q^L)}{\lambda^O\left(P - \frac{Q^L}{R}\right)}\right)^{1/\sigma}
\end{aligned}$$

Differencing

$$\begin{aligned}
\tilde{S}^O Q^H - \tilde{S}^O Q^L &= \left(\frac{\pi(Q^H - Q^L)}{\lambda^O\left(P - \frac{Q^L}{R}\right)}\right)^{1/\sigma} - \left(\frac{(1-\pi)(Q^H - Q^L)}{\lambda^O\left(\frac{Q^H}{R} - P\right)}\right)^{1/\sigma} \\
\tilde{S}^O &= \frac{1}{Q^H - Q^L}(C^{O,H} - C^{O,L}) \\
\tilde{S}^O &= \frac{(Q^H - Q^L)^{(1/\sigma - 1)}}{(\lambda^O)^{1/\sigma}}\left[\left(\frac{\pi}{\left(P - \frac{Q^L}{R}\right)}\right)^{1/\sigma} - \left(\frac{(1-\pi)}{\left(\frac{Q^H}{R} - P\right)}\right)^{1/\sigma}\right]
\end{aligned}$$

$$\begin{aligned}
\frac{\tilde{B}^O}{Q^H} - \frac{\tilde{B}^O}{Q^L} &= \frac{1}{Q^H}\left(\frac{\pi(Q^H - Q^L)}{\lambda^O\left(P - \frac{Q^L}{R}\right)}\right)^{1/\sigma} - \frac{1}{Q^L}\left(\frac{(1-\pi)(Q^H - Q^L)}{\lambda^O\left(\frac{Q^H}{R} - P\right)}\right)^{1/\sigma} \\
\tilde{B}^O &= \frac{1}{\frac{1}{Q^H} - \frac{1}{Q^L}}\left[\frac{1}{Q^H}C^{O,H} - \frac{1}{Q^L}C^{O,L}\right] \\
\tilde{B}^O &= \frac{(Q^H - Q^L)^{1/\sigma - 1}}{(\lambda^O)^{1/\sigma}}\left[Q^H\left(\frac{(1-\pi)}{\left(\frac{Q^H}{R} - P\right)}\right)^{1/\sigma} - Q^L\left(\frac{\pi}{\left(P - \frac{Q^L}{R}\right)}\right)^{1/\sigma}\right]
\end{aligned}$$

Use budget constraint to solve for Lambda

$$\begin{aligned}
Y &= C^O + P\tilde{S}^O + \frac{1}{R}\tilde{B}^O \\
Y &= \left(\frac{1}{\lambda^O}\right)^{1/\sigma} + P\frac{(Q^H - Q^L)^{(1/\sigma - 1)}}{(\lambda^O)^{1/\sigma}}\left[\left(\frac{\pi}{\left(P - \frac{Q^L}{R}\right)}\right)^{1/\sigma} - \left(\frac{(1-\pi)}{\left(\frac{Q^H}{R} - P\right)}\right)^{1/\sigma}\right] \\
&\quad + \frac{1}{R}\frac{(Q^H - Q^L)^{1/\sigma - 1}}{(\lambda^O)^{1/\sigma}}\left[Q^H\left(\frac{(1-\pi)}{\left(\frac{Q^H}{R} - P\right)}\right)^{1/\sigma} - Q^L\left(\frac{\pi}{\left(P - \frac{Q^L}{R}\right)}\right)^{1/\sigma}\right]
\end{aligned}$$

$$\left(\lambda^O\right)^{1/\sigma} = \frac{1}{Y} + \frac{P(Q^H - Q^L)^{(1/\sigma-1)}}{Y}\left[\left(\frac{\pi}{\left(P - \frac{Q^L}{R}\right)}\right)^{1/\sigma} - \left(\frac{(1-\pi)}{\left(\frac{Q^H}{R} - P\right)}\right)^{1/\sigma}\right]$$
$$+\frac{(Q^H - Q^L)^{1/\sigma-1}}{RY}\left[Q^H\left(\frac{(1-\pi)}{\left(\frac{Q^H}{R} - P\right)}\right)^{1/\sigma} - Q^L\left(\frac{\pi}{\left(P - \frac{Q^L}{R}\right)}\right)^{1/\sigma}\right]$$

Note

- Doubling income doubles $\left(\lambda^O\right)^{-1/\sigma}$ which doubles consumption in all states, as well has bond holdings and stock holdings

C.1.2 Wishful Thinker's problem

The wisful thinker's problem is similar except that π is replaced with $\tilde{\pi}$ and $\tilde{\pi}$ satisfies

$$\frac{(\tilde{S}^W Q^H + \tilde{B}^W)^{1-\sigma}}{1-\sigma} - \frac{(\tilde{S}^W Q^L + \tilde{B}^W)^{1-\sigma}}{1-\sigma} = \frac{1}{\theta}\ln\frac{\tilde{\pi}}{\pi} - \frac{1}{\theta}\ln\frac{1-\tilde{\pi}}{1-\pi}$$

- $\tilde{\pi}$ is increasing in the gap between second period utility in the two states
$$\frac{(C_2^{W,H})^{1-\sigma}}{1-\sigma} - \frac{(C_2^{W,L})^{1-\sigma}}{1-\sigma}$$
- A mean preserving spread in utility increases $\tilde{\pi}$
- Income effects lesson wishful thinking...A proportionate increase in consumption in the two states raises $\tilde{\pi}$ if $C_2^{W,H} > C_2^{W,L}$ and $\sigma < 1$, reduces $\tilde{\pi}$ if $\sigma > 1$, and has no effect for log utility. With log utility
$$\frac{\tilde{\pi}}{1-\tilde{\pi}} = \left(\frac{C_2^{W,H}}{C_2^{W,L}}\right)^{\theta}\frac{\pi}{1-\pi}$$
doubling consumption multiplies the gap by
$$2^{1-\sigma}$$

Properties of the solution

1. $\tilde{\pi}$ is increasing in the gap between second period utility in the two states
$$\frac{(C_2^{W,H})^{1-\sigma}}{1-\sigma} - \frac{(C_2^{W,L})^{1-\sigma}}{1-\sigma}$$

- A mean preserving spread in utility increases $\tilde{\pi}$
- An equal increase in consumption in the two states reduces $\tilde{\pi}$ due to diminishing marginal utility.

2. Marshallian demand of rational agents... CRRA utility}

$$
\begin{aligned}
C^{-\sigma} &= \lambda \\
\pi Q^H \left(\tilde{S}^O Q^H + \tilde{B}^O\right)^{-\sigma} + (1-\pi) Q^L \left(\tilde{S}^O Q^L + \tilde{B}^O\right)^{-\sigma} &= \lambda P \\
\pi \left(\tilde{S}^O Q^H + \tilde{B}^O\right)^{-\sigma} + (1-\pi) \left(\tilde{S}^O Q^L + \tilde{B}^O\right)^{-\sigma} &= \frac{\lambda}{R}
\end{aligned}
$$

Solve for consumption in each state

$$
\begin{aligned}
\pi Q^H \left(\tilde{S}^O Q^H + \tilde{B}^O\right)^{-\sigma} + (1-\pi) Q^L \left(\tilde{S}^O Q^L + \tilde{B}^O\right)^{-\sigma} &= \lambda P \\
\pi Q^H \left(\tilde{S}^O Q^H + \tilde{B}^O\right)^{-\sigma} + (1-\pi) Q^H \left(\tilde{S}^O Q^L + \tilde{B}^O\right)^{-\sigma} &= \frac{\lambda}{R} Q^H \\
(1-\pi) Q^H \left(\tilde{S}^O Q^L + \tilde{B}^O\right)^{-\sigma} - (1-\pi) Q^L \left(\tilde{S}^O Q^L + \tilde{B}^O\right)^{-\sigma} &= \frac{\lambda}{R} Q^H - \lambda P \\
C^{O,L} &= \tilde{S}^O Q^L + \tilde{B}^O \\
&= \left(\frac{\lambda \left[\frac{Q^H}{R} - P\right]}{(1-\pi)[Q^H - Q^L]} \right)^{-1/\sigma} \\
&= \left(\frac{(1-\pi)[Q^H - Q^L]}{\lambda \left[\frac{Q^H}{R} - P\right]} \right)^{1/\sigma}
\end{aligned}
$$

Similarly

$$
C^{O,H} = \tilde{S}^O Q^H + \tilde{B}^O = \left(\frac{\pi [Q^H - Q^L]}{\lambda \left[P - \frac{Q^L}{R}\right]} \right)^{1/\sigma}
$$

System become

$$
\begin{aligned}
C^O + P\tilde{S}^O + \frac{1}{R}\tilde{B}^O &= Y \\
C^O &= \left(\frac{1}{\lambda}\right)^{1/\sigma} \\
\tilde{S}^O Q^L + \tilde{B}^O &= \left(\frac{\lambda \left[\frac{Q^H}{R} - P\right]}{(1-\pi)[Q^H - Q^L]} \right)^{-1/\sigma} \\
\tilde{S}^O Q^H + \tilde{B}^O &= \left(\frac{\lambda \left[P - \frac{Q^L}{R}\right]}{\pi [Q^H - Q^L]} \right)^{-1/\sigma}
\end{aligned}
$$

Solve for $\tilde{S}^O$ and $\tilde{B}^O$

$$\begin{aligned}
\tilde{S}^O &= \frac{1}{Q^H - Q^L}\left[\left(\frac{\pi[Q^H - Q^L]}{\lambda\left[P - \frac{Q^L}{R}\right]}\right)^{1/\sigma} - \left(\frac{(1-\pi)[Q^H - Q^L]}{\lambda\left[\frac{Q^H}{R} - P\right]}\right)^{1/\sigma}\right] \\
&= \frac{\left(Q^H - Q^L\right)^{(1-\sigma)/\sigma}}{\lambda^{1/\sigma}}\left[\left(\frac{\pi}{\left[P - \frac{Q^L}{R}\right]}\right)^{1/\sigma} - \left(\frac{(1-\pi)}{\left[\frac{Q^H}{R} - P\right]}\right)^{1/\sigma}\right] \\
\frac{\tilde{B}^O}{Q^H} - \frac{\tilde{B}^O}{Q^L} &= \frac{1}{Q^H}\left(\frac{\pi[Q^H - Q^L]}{\lambda\left[P - \frac{Q^L}{R}\right]}\right)^{1/\sigma} - \frac{1}{Q^L}\left(\frac{(1-\pi)[Q^H - Q^L]}{\lambda\left[\frac{Q^H}{R} - P\right]}\right)^{1/\sigma} \\
\tilde{B}^O &= \frac{1}{\lambda^{1/\sigma}}\frac{[Q^H - Q^L]^{1/\sigma}}{\frac{1}{Q^H} - \frac{1}{Q^L}}\left[\frac{1}{Q^H}\left(\frac{\pi}{\left[P - \frac{Q^L}{R}\right]}\right)^{1/\sigma} - \frac{1}{Q^L}\left(\frac{(1-\pi)}{\left[\frac{Q^H}{R} - P\right]}\right)^{1/\sigma}\right] \\
&= \frac{1}{\lambda^{1/\sigma}}\frac{[Q^H - Q^L]^{1/\sigma}}{\frac{1}{Q^H} - \frac{1}{Q^L}}\left[\frac{1}{Q^H}\left(\frac{\pi}{\left[P - \frac{Q^L}{R}\right]}\right)^{1/\sigma} - \frac{1}{Q^L}\left(\frac{(1-\pi)}{\left[\frac{Q^H}{R} - P\right]}\right)^{1/\sigma}\right]
\end{aligned}$$

Finally

$$\left(\frac{1}{\lambda}\right)^{1/\sigma} + P\tilde{S}^O + \frac{1}{R}\tilde{B}^O = Y$$

Now

$$\begin{aligned}
C_2^{O,H} &= \tilde{S}^O Q^H + \tilde{B}^O = \left(\frac{\pi[Q^H - Q^L]}{\lambda\left[P - \frac{Q^L}{R}\right]}\right)^{1/\sigma} \\
C_2^{O,L} &= \tilde{S}^O Q^L + \tilde{B}^O = \left(\frac{(1-\pi)[Q^H - Q^L]}{\lambda\left[\frac{Q^H}{R} - P\right]}\right)^{1/\sigma}
\end{aligned}$$

and $(\sigma > 0)$

$$\begin{aligned}
\left(\frac{\pi[Q^H - Q^L]}{\lambda\left[P - \frac{Q^L}{R}\right]}\right)^{1/\sigma} &> \left(\frac{(1-\pi)[Q^H - Q^L]}{\lambda\left[\frac{Q^H}{R} - P\right]}\right)^{1/\sigma} \\
(1-\pi)\left[P - \frac{Q^L}{R}\right] &< \pi\left[\frac{Q^H}{R} - P\right]
\end{aligned}$$

Require that gain to stock in high state greater than gain to bond in the low state....suppose that this is the case.

And for the wishful thinkers the determination of $\tilde{\pi}$

$$\begin{aligned}
\frac{1}{\theta}\ln\frac{\tilde{\pi}}{\pi} &= \frac{(C_2^{W,H})^{1-\sigma}}{1-\sigma} - \frac{(C_2^{W,L})^{1-\sigma}}{1-\sigma} + \frac{1}{\theta}\ln\frac{1-\tilde{\pi}}{1-\pi} \\
\ln\frac{\tilde{\pi}}{1-\tilde{\pi}} &= \frac{\theta}{1-\sigma}\left[(C_2^{W,H})^{1-\sigma} - (C_2^{W,L})^{1-\sigma}\right] + \ln\frac{\pi}{1-\pi} \\
\ln\frac{\tilde{\pi}}{1-\tilde{\pi}} &= \frac{\theta}{1-\sigma}\left[\left(\frac{\lambda\left[P-\frac{Q^L}{R}\right]}{\tilde{\pi}[Q^H-Q^L]}\right)^{\frac{\sigma-1}{\sigma}} - \left(\frac{\lambda\left[\frac{Q^H}{R}-P\right]}{(1-\tilde{\pi})[Q^H-Q^L]}\right)^{\frac{\sigma-1}{\sigma}}\right] + \ln\frac{\pi}{1-\pi} \\
\ln\frac{\tilde{\pi}}{1-\tilde{\pi}} &= \frac{\theta}{1-\sigma}\left[\left(\frac{\lambda\left[P-\frac{Q^L}{R}\right]}{\tilde{\pi}[Q^H-Q^L]}\right)^{\frac{\sigma-1}{\sigma}} - \left(\frac{\lambda\left[\frac{Q^H}{R}-P\right]}{(1-\tilde{\pi})[Q^H-Q^L]}\right)^{\frac{\sigma-1}{\sigma}}\right] + \ln\frac{\pi}{1-\pi}
\end{aligned}$$

With $\sigma > 1$, This equaiton holds for $\tilde{\pi} = 1$ and 0, but these are ruled out by the requirement that consumption is greater than $\bar{\varepsilon}$. Is there an interior solution.

$$\frac{\left(\tilde{C}^W\right)^{1-\sigma}}{1-\sigma} + \tilde{\pi}\frac{(\tilde{S}^W Q^H + \tilde{B}^W)^{1-\sigma}}{1-\sigma} + (1-\tilde{\pi})\frac{(\tilde{S}^W Q^L + \tilde{B}^W)^{1-\sigma}}{1-\sigma}$$
$$-\frac{1}{\theta}\tilde{\pi}\ln\frac{\tilde{\pi}}{\pi} - \frac{1}{\theta}(1-\tilde{\pi})\ln\frac{1-\tilde{\pi}}{1-\pi}$$

The first order condition is

$$\frac{(C_2^{W,H})^{1-\sigma}}{1-\sigma} - \frac{(C_2^{W,L})^{1-\sigma}}{1-\sigma} - \frac{1}{\theta}\ln\frac{\tilde{\pi}}{\pi} + \frac{1}{\theta}\ln\frac{1-\tilde{\pi}}{1-\pi}$$

the envelop theorem says do not have to worry about $C_2^{W,H}$ and $C_2^{W,L}$. Now take second derivative ...here need to consider consumption response

C.1.3 Solution with log utility

The maximization problem for wishful thinkers in the CRRA utility case was

$$\frac{\left(\tilde{C}^W\right)^{1-\sigma}}{1-\sigma} + \tilde{\pi}\frac{(\tilde{S}^W Q^H + \tilde{B}^W)^{1-\sigma}}{1-\sigma} + (1-\tilde{\pi})\frac{(\tilde{S}^W Q^L + \tilde{B}^W)^{1-\sigma}}{1-\sigma}$$
$$-\frac{1}{\theta}\tilde{\pi}\ln\frac{\tilde{\pi}}{\pi} - \frac{1}{\theta}(1-\tilde{\pi})\ln\frac{1-\tilde{\pi}}{1-\pi}$$

such that

$$\tilde{C}^W + P\tilde{S}^W + \frac{1}{R}\tilde{B}^W = Y^W + PS^W$$

Know

$$
\begin{aligned}
C_2^{W,H} &= \left(\frac{\tilde{\pi}[Q^H - Q^L]}{\lambda\left[P - \frac{Q^L}{R}\right]}\right)^{1/\sigma} \\
C_2^{W,L} &= \left(\frac{(1-\tilde{\pi})[Q^H - Q^L]}{\lambda\left[\frac{Q^H}{R} - P\right]}\right)^{1/\sigma} \\
C^W &= (1/\lambda)^{1/\sigma}
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{S}^W &= \frac{\left(Q^H - Q^L\right)^{(1-\sigma)/\sigma}}{\lambda^{1/\sigma}}\left[\left(\frac{\tilde{\pi}}{\left[P - \frac{Q^L}{R}\right]}\right)^{1/\sigma} - \left(\frac{(1-\tilde{\pi})}{\left[\frac{Q^H}{R} - P\right]}\right)^{1/\sigma}\right] \\
\tilde{B}^W &= \frac{1}{\lambda^{1/\sigma}}\frac{[Q^H - Q^L]^{1/\sigma}}{\frac{1}{Q^H} - \frac{1}{Q^L}}\left[\frac{1}{Q^H}\left(\frac{\tilde{\pi}}{\left[P - \frac{Q^L}{R}\right]}\right)^{1/\sigma} - \frac{1}{Q^L}\left(\frac{(1-\tilde{\pi})}{\left[\frac{Q^H}{R} - P\right]}\right)^{1/\sigma}\right]
\end{aligned}
$$

So λ is

$$
\begin{aligned}
& 1 + P\left(Q^H - Q^L\right)^{(1-\sigma)/\sigma}\left[\left(\frac{\pi}{\left[P - \frac{Q^L}{R}\right]}\right)^{1/\sigma} - \left(\frac{(1-\pi)}{\left[\frac{Q^H}{R} - P\right]}\right)^{1/\sigma}\right] \\
+ \quad & \frac{1}{R}\frac{[Q^H - Q^L]^{1/\sigma}}{\frac{1}{Q^H} - \frac{1}{Q^L}}\left[\frac{1}{Q^H}\left(\frac{\tilde{\pi}}{\left[P - \frac{Q^L}{R}\right]}\right)^{1/\sigma} - \frac{1}{Q^L}\left(\frac{(1-\tilde{\pi})}{\left[\frac{Q^H}{R} - P\right]}\right)^{1/\sigma}\right] = Y\lambda^{\sigma}
\end{aligned}
$$

Now lets look at log utility

$$
\ln\left(C^O\right) + \pi \ln(\tilde{S}^O Q^H + \tilde{B}^O) + (1-\pi)\ln(\tilde{S}^O Q^L + \tilde{B}^O)
$$

subject to the budget constraint

$$
C^O + P\tilde{S}^O + \frac{1}{R}\tilde{B}^O = Y^O
$$

FOC's

$$
\begin{aligned}
\frac{1}{C^O} &= \lambda^O \\
\frac{\pi Q^H}{\tilde{S}^O Q^H + \tilde{B}^O} + \frac{(1-\pi)Q^L}{\tilde{S}^O Q^L + \tilde{B}^O} &= \lambda^O P \\
\frac{\pi}{\tilde{S}^O Q^H + \tilde{B}^O} + \frac{(1-\pi)}{\tilde{S}^O Q^L + \tilde{B}^O} &= \frac{\lambda^O}{R}
\end{aligned}
$$

These three equations plus the budget constraint may be solved for C_0, $\tilde{S}^O$, and $\tilde{B}^O$

$$\begin{aligned}
\frac{\pi}{\tilde{S}^O Q^H + \tilde{B}^O} + \frac{(1-\pi)}{\tilde{S}^O Q^L + \tilde{B}^O} &= \frac{\lambda^O}{R} \\
\frac{\pi Q^H}{\tilde{S}^O Q^H + \tilde{B}^O} + \frac{(1-\pi)Q^H}{\tilde{S}^O Q^L + \tilde{B}^O} &= \frac{\lambda^O Q^H}{R} \\
\frac{(1-\pi)Q^H}{\tilde{S}^O Q^L + \tilde{B}^O} - \frac{(1-\pi)Q^L}{\tilde{S}^O Q^L + \tilde{B}^O} &= \frac{\lambda^O Q^H}{R} - \lambda^O P \\
\tilde{S}^O Q^L + \tilde{B}^O &= \frac{(1-\pi)[Q^H - Q^L]}{\lambda^O \left[\frac{Q^H}{R} - P\right]}
\end{aligned}$$

Similarly

$$\begin{aligned}
\frac{\pi}{\tilde{S}^O Q^H + \tilde{B}^O} + \frac{(1-\pi)}{\tilde{S}^O Q^L + \tilde{B}^O} &= \frac{\lambda^O}{R} \\
\frac{\pi Q^L}{\tilde{S}^O Q^H + \tilde{B}^O} + \frac{(1-\pi)Q^L}{\tilde{S}^O Q^L + \tilde{B}^O} &= \frac{\lambda^O Q^L}{R} \\
\frac{\pi Q^H}{\tilde{S}^O Q^H + \tilde{B}^O} - \frac{\pi Q^L}{\tilde{S}^O Q^H + \tilde{B}^O} &= \lambda^O P - \frac{\lambda^O Q^L}{R} \\
\tilde{S}^O Q^H + \tilde{B}^O &= \frac{\pi[Q^H - Q^L]}{\lambda^O \left[P - \frac{Q^L}{R}\right]}
\end{aligned}$$

System become

$$\begin{aligned}
C^O + P\tilde{S}^O + \frac{1}{R}\tilde{B}^O &= Y \\
C^O &= \frac{1}{\lambda^O} \\
\tilde{S}^O Q^L + \tilde{B}^O &= \frac{1-\pi}{\lambda^O}\frac{[Q^H - Q^L]}{\left[\frac{Q^H}{R} - P\right]} \\
\tilde{S}^O Q^H + \tilde{B}^O &= \frac{\pi}{\lambda^O}\frac{[Q^H - Q^L]}{\left[P - \frac{Q^L}{R}\right]}
\end{aligned}$$

Solve for $\tilde{S}^O$ and $\tilde{B}^O$

$$\begin{aligned}
\tilde{S}^O &= \frac{1}{Q^H - Q^L}\left[\frac{\pi}{\lambda^O}\frac{[Q^H - Q^L]}{\left[P - \frac{Q^L}{R}\right]} - \frac{1-\pi}{\lambda^O}\frac{[Q^H - Q^L]}{\left[\frac{Q^H}{R} - P\right]}\right] \\
&= \frac{1}{\lambda^O}\left[\frac{\pi}{\left[P - \frac{Q^L}{R}\right]} - \frac{1-\pi}{\left[\frac{Q^H}{R} - P\right]}\right]
\end{aligned}$$

$$
\begin{aligned}
\frac{\tilde{B}^O}{Q^H} - \frac{\tilde{B}^O}{Q^L} &= \frac{\pi}{Q^H\lambda^O}\frac{[Q^H - Q^L]}{\left[P - \frac{Q^L}{R}\right]} - \frac{1-\pi}{Q^L\lambda^O}\frac{[Q^H - Q^L]}{\left[\frac{Q^H}{R} - P\right]} \\
\tilde{B}^O &= \frac{1}{\frac{1}{Q^H} - \frac{1}{Q^L}}\left[\frac{\pi}{Q^H\lambda^O}\frac{[Q^H - Q^L]}{\left[P - \frac{Q^L}{R}\right]} - \frac{1-\pi}{Q^L\lambda^O}\frac{[Q^H - Q^L]}{\left[\frac{Q^H}{R} - P\right]}\right] \\
&= -\frac{Q^H Q^L}{Q^H - Q^L}\left[\frac{\pi}{Q^H\lambda^O}\frac{[Q^H - Q^L]}{\left[P - \frac{Q^L}{R}\right]} - \frac{1-\pi}{Q^L\lambda^O}\frac{[Q^H - Q^L]}{\left[\frac{Q^H}{R} - P\right]}\right] \\
&= \frac{Q^H Q^L}{\lambda^O}\left[\frac{1-\pi}{Q^L\left[\frac{Q^H}{R} - P\right]} - \frac{\pi}{Q^H\left[P - \frac{Q^L}{R}\right]}\right] \\
&= \frac{1}{\lambda^O}\left[\frac{(1-\pi)Q^H}{\left[\frac{Q^H}{R} - P\right]} - \frac{\pi Q^L}{\left[P - \frac{Q^L}{R}\right]}\right]
\end{aligned}
$$

In sum

$$
\begin{aligned}
C^O + P\tilde{S}^O + \frac{1}{R}\tilde{B}^O &= Y \\
C^O &= \frac{1}{\lambda^O} \\
\tilde{S}^O &= \frac{1}{\lambda^O}\left[\frac{\pi}{\left[P - \frac{Q^L}{R}\right]} - \frac{1-\pi}{\left[\frac{Q^H}{R} - P\right]}\right] \\
\tilde{B}^O &= \frac{1}{\lambda^O}\left[\frac{(1-\pi)Q^H}{\left[\frac{Q^H}{R} - P\right]} - \frac{\pi Q^L}{\left[P - \frac{Q^L}{R}\right]}\right]
\end{aligned}
$$

Look at consumption in high and low states

$$
\frac{C_2^{O,H}}{C_2^{O,L}} = \frac{\frac{\pi[Q^H - Q^L]}{\lambda^O\left[P - \frac{Q^L}{R}\right]}}{\frac{(1-\pi)[Q^H - Q^L]}{\lambda^O\left[\frac{Q^H}{R} - P\right]}} = \frac{\pi}{(1-\pi)}\frac{\left[\frac{Q^H}{R} - P\right]}{\left[P - \frac{Q^L}{R}\right]}
$$

depends on probabilities and excess returns....if unlikely...need high payoff. Need $\bar{\varepsilon}$ such that this is interior.

Marshallian demand of the wishful thinkers (log utility) \bigskip

$$
\begin{gathered}
ln(C^W) + \tilde{\pi} ln(\tilde{S}^W Q^H + \tilde{B}^W) + (1-\tilde{\pi}) ln(\tilde{S}^W Q^L + \tilde{B}^W) \\
-\frac{1}{\theta}\tilde{\pi}\ln\frac{\tilde{\pi}}{\pi} - \frac{1}{\theta}(1-\tilde{\pi})\ln\frac{1-\tilde{\pi}}{1-\pi}
\end{gathered}
$$

subject to

$$
\tilde{C}^W + P\tilde{S}^W + \frac{1}{R}\tilde{B}^W = Y^W
$$

Same except that also

$$\frac{\tilde{\pi}}{1-\tilde{\pi}} = \left(\frac{\tilde{C}_2^{W,H}}{\tilde{C}_2^{W,L}}\right)^{\theta} \frac{\pi}{1-\pi}$$

As above

$$\begin{aligned}
\tilde{C}^{W,L} &= \tilde{S}^W Q^L + \tilde{B}^W = \frac{(1-\tilde{\pi})[Q^H - Q^L]}{\lambda^W \left[\frac{Q^H}{R} - P\right]} \\
\tilde{C}^{W,H} &= \tilde{S}^O Q^H + \tilde{B}^O = \frac{\tilde{\pi}[Q^H - Q^L]}{\lambda^W \left[P - \frac{Q^L}{R}\right]}
\end{aligned}$$

So

$$\frac{\tilde{C}_2^{W,H}}{\tilde{C}_2^{W,L}} = \frac{\frac{\tilde{\pi}[Q^H - Q^L]}{\lambda^W \left[P - \frac{Q^L}{R}\right]}}{\frac{(1-\tilde{\pi})[Q^H - Q^L]}{\lambda^W \left[\frac{Q^H}{R} - P\right]}} = \frac{\tilde{\pi}}{(1-\tilde{\pi})} \frac{\left[\frac{Q^H}{R} - P\right]}{\left[P - \frac{Q^L}{R}\right]}$$

Combining FOC w.r.t. $\tilde{\pi}$

$$\frac{\tilde{\pi}}{1-\tilde{\pi}} = \left(\frac{\tilde{C}_2^{W,H}}{}\right.$$

with 2nd period high and low state consumptions ratio

$$\frac{\tilde{C}_2^{W,H}}{\tilde{C}_2^{W,L}} = \frac{\tilde{\pi}}{1-\tilde{\pi}} \cdot \frac{\frac{Q^H}{R} - P}{P - \frac{Q^L}{R}}$$

we get

$$\frac{\tilde{\pi}}{1-\tilde{\pi}} = \frac{\pi}{1-\pi} \left[\frac{\tilde{\pi}}{1-\tilde{\pi}}\right]^{\theta} \left[\frac{\frac{Q^H}{R} - P}{P - \frac{Q^L}{R}}\right]^{\theta}$$

$$\frac{\tilde{\pi}}{1-\tilde{\pi}} = \left[\frac{\pi}{1-\pi}\right]^{\frac{1}{1-\theta}} \left[\frac{\frac{Q^H}{R} - P}{P - \frac{Q^L}{R}}\right]^{\frac{\theta}{1-\theta}}$$

$$\tilde{\pi} = \left[\frac{\pi}{1-\pi}\right]^{\frac{1}{1-\theta}} \left[\frac{\frac{Q^H}{R} - P}{P - \frac{Q^L}{R}}\right]^{\frac{\theta}{1-\theta}} - \left[\frac{\pi}{1-\pi}\right]^{\frac{1}{1-\theta}} \left[\frac{\frac{Q^H}{R} - P}{P - \frac{Q^L}{R}}\right]^{\frac{\theta}{1-\theta}} \cdot \tilde{\pi}$$

$$\tilde{\pi} = \frac{\left[\frac{\pi}{1-\pi}\right]^{\frac{1}{1-\theta}} \left[\frac{\frac{Q^H}{R} - P}{P - \frac{Q^L}{R}}\right]^{\frac{\theta}{1-\theta}}}{1 + \left[\frac{\pi}{1-\pi}\right]^{\frac{1}{1-\theta}} \left[\frac{\frac{Q^H}{R} - P}{P - \frac{Q^L}{R}}\right]^{\frac{\theta}{1-\theta}}}$$

Now using

$$\tilde{C}^W + P\tilde{S}^W + \frac{1}{R}\tilde{B}^W = Y^W$$

$$\tilde{C}^W = \frac{1}{\lambda^W}$$

$$\tilde{S}^W = \frac{1}{\lambda^W}\left[\frac{\tilde{\pi}}{\left[P - \frac{Q^L}{R}\right]} - \frac{1-\tilde{\pi}}{\left[\frac{Q^H}{R} - P\right]}\right]$$

$$\tilde{B}^W = \frac{1}{\lambda^W}\left[Q^H\frac{1-\tilde{\pi}}{\left[\frac{Q^H}{R} - P\right]} - Q^L\frac{\tilde{\pi}}{\left[P - \frac{Q^L}{R}\right]}\right]$$

$$\tilde{\pi} = \frac{\left[\frac{\pi}{1-\pi}\right]^{\frac{1}{1-\theta}}\left[\frac{\frac{Q^H}{R}-P}{P-\frac{Q^L}{R}}\right]^{\frac{\theta}{1-\theta}}}{1+\left[\frac{\pi}{1-\pi}\right]^{\frac{1}{1-\theta}}\left[\frac{\frac{Q^H}{R}-P}{P-\frac{Q^L}{R}}\right]^{\frac{\theta}{1-\theta}}}$$

We solve for $1/\lambda^W$

$$\frac{1}{\lambda^W} + P\cdot\frac{1}{\lambda^W}\left[\frac{\tilde{\pi}}{\left[P - \frac{Q^L}{R}\right]} - \frac{1-\tilde{\pi}}{\left[\frac{Q^H}{R} - P\right]}\right] + \frac{1}{R}\cdot\frac{1}{\lambda^W}\left[Q^H\frac{1-\tilde{\pi}}{\left[\frac{Q^H}{R} - P\right]} - Q^L\frac{\tilde{\pi}}{\left[P - \frac{Q^L}{R}\right]}\right] = Y^W$$

$$\frac{1}{\lambda^W} = \tilde{C}^W = \frac{Y^W}{1+P\left[\frac{\tilde{\pi}}{\left[P-\frac{Q^L}{R}\right]} - \frac{1-\tilde{\pi}}{\left[\frac{Q^H}{R}-P\right]}\right] + \frac{1}{R}\left[Q^H\frac{1-\tilde{\pi}}{\left[\frac{Q^H}{R}-P\right]} - Q^L\frac{\tilde{\pi}}{\left[P-\frac{Q^L}{R}\right]}\right]}$$

where

$$\tilde{\pi} = \frac{\left[\frac{\pi}{1-\pi}\right]^{\frac{1}{1-\theta}}\left[\frac{\frac{Q^H}{R}-P}{P-\frac{Q^L}{R}}\right]^{\frac{\theta}{1-\theta}}}{1+\left[\frac{\pi}{1-\pi}\right]^{\frac{1}{1-\theta}}\left[\frac{\frac{Q^H}{R}-P}{P-\frac{Q^L}{R}}\right]^{\frac{\theta}{1-\theta}}}$$

Simplify the denominators:

$$1+P\left[\frac{\pi}{\left[P-\frac{Q^L}{R}\right]}-\frac{1-\pi}{\left[\frac{Q^H}{R}-P\right]}\right]+\frac{1}{R}\left[Q^H\frac{1-\pi}{\left[\frac{Q^H}{R}-P\right]}-Q^L\frac{\pi}{\left[P-\frac{Q^L}{R}\right]}\right]$$

$$=1+PR\left[\frac{\pi}{PR-Q^L}-\frac{1-\pi}{Q^H-PR}\right]+\left[Q^H\frac{1-\pi}{Q^H-PR}-Q^L\frac{\pi}{PR-Q^L}\right]=$$

$$=1+\left[PR-Q^L\right]\frac{\pi}{PR-Q^L}-\left[PR-Q^H\right]\frac{1-\pi}{Q^H-PR}=1+\pi+1-\pi=2$$

and

$$1+P\left[\frac{\tilde{\pi}}{\left[P-\frac{Q^L}{R}\right]}-\frac{1-\tilde{\pi}}{\left[\frac{Q^H}{R}-P\right]}\right]+\frac{1}{R}\left[Q^H\frac{1-\tilde{\pi}}{\left[\frac{Q^H}{R}-P\right]}-Q^L\frac{\tilde{\pi}}{\left[P-\frac{Q^L}{R}\right]}\right]=2$$

we get

$$\frac{1}{\lambda^W}=\tilde{C}^W=\frac{Y^W}{2}$$

Plug λ^W into $\tilde{S}^W$ and $\tilde{B}^W$

$$\tilde{S}^W=\frac{Y^W}{2}\left[\frac{\tilde{\pi}}{\left[P-\frac{Q^L}{R}\right]}-\frac{1-\tilde{\pi}}{\left[\frac{Q^H}{R}-P\right]}\right]$$

$$\tilde{B}^W=\frac{Y^W}{2}\left[Q^H\frac{1-\tilde{\pi}}{\left[\frac{Q^H}{R}-P\right]}-Q^L\frac{\tilde{\pi}}{\left[P-\frac{Q^L}{R}\right]}\right]$$

And

$$\tilde{C}_2^{W,H}=\tilde{\pi}\frac{Y^W}{2}\left[\frac{Q^H-Q^L}{P-\frac{Q^L}{R}}\right]$$

$$\tilde{C}_2^{W,L}=(1-\tilde{\pi})\frac{Y^W}{2}\left[\frac{Q^H-Q^L}{\frac{Q^H}{R}-P}\right]$$

Need to solve for $\left\{\tilde{S}^O,\tilde{S}^W,\tilde{B}^O,\tilde{B}^W,\tilde{\pi},P,R\right\}$ from the system

$$\tilde{\pi}=\frac{\left[\frac{\pi}{1-\pi}\right]^{\frac{1}{1-\theta}}\left[\frac{\frac{Q^H}{R}-P}{P-\frac{Q^L}{R}}\right]^{\frac{\theta}{1-\theta}}}{1+\left[\frac{\pi}{1-\pi}\right]^{\frac{1}{1-\theta}}\left[\frac{\frac{Q^H}{R}-P}{P-\frac{Q^L}{R}}\right]^{\frac{\theta}{1-\theta}}} \tag{46}$$

$$\tilde{S}^W = \frac{Y^W}{2}\left[\frac{\tilde{\pi}}{\left[P - \frac{Q^L}{R}\right]} - \frac{1-\tilde{\pi}}{\left[\frac{Q^H}{R} - P\right]}\right] \tag{47}$$

$$\tilde{B}^W = \frac{Y^W}{2}\left[Q^H\frac{1-\tilde{\pi}}{\left[\frac{Q^H}{R} - P\right]} - Q^L\frac{\tilde{\pi}}{\left[P - \frac{Q^L}{R}\right]}\right] \tag{48}$$

$$\tilde{S}^O = \frac{Y^O}{2}\left[\frac{\pi}{\left[P - \frac{Q^L}{R}\right]} - \frac{1-\pi}{\left[\frac{Q^H}{R} - P\right]}\right] \tag{49}$$

$$\tilde{B}^O = \frac{Y^O}{2}\left[Q^H\frac{1-\pi}{\left[\frac{Q^H}{R} - P\right]} - Q^L\frac{\pi}{\left[P - \frac{Q^L}{R}\right]}\right] \tag{50}$$

$$(1-\phi)\tilde{S}^O + \phi\tilde{S}^W = 1 \tag{51}$$

$$(1-\phi)\tilde{B}^O + \phi\tilde{B}^W = 0 \tag{52}$$

Now the system becomes:
From (13), (15) and from bond market clearing (17)

$$Q^L\frac{\tilde{\pi}}{PR - Q^L} - Q^H\frac{1-\tilde{\pi}}{Q^H - PR} + \frac{(1-\phi)Y^O}{\phi Y^W}\left[Q^L\frac{\pi}{PR-Q^L} - Q^H\frac{1-\pi}{Q^H-PR}\right] = 0 \tag{53}$$

From (12), (14) and from stock market clearing (16)

$$\frac{\tilde{\pi}}{PR - Q^L} - \frac{1-\tilde{\pi}}{Q^H - PR} + \frac{(1-\phi)Y^O}{\phi Y^W}\left[\frac{\pi}{PR-Q^L} - \frac{1-\pi}{Q^H-PR}\right] = \frac{2}{R\cdot\phi Y^W} \tag{54}$$

and (11) unchanged

$$\tilde{\pi} = \frac{\left[\frac{\pi}{1-\pi}\right]^{\frac{1}{1-\theta}}\left[\frac{Q^H-PR}{PR-Q^L}\right]^{\frac{\theta}{1-\theta}}}{1+\left[\frac{\pi}{1-\pi}\right]^{\frac{1}{1-\theta}}\left[\frac{Q^H-PR}{PR-Q^L}\right]^{\frac{\theta}{1-\theta}}}$$

Multiply the stock market clearing equation (19) with Q^L

$$Q^L\frac{\tilde{\pi}}{PR-Q^L} + Q^L\frac{\pi}{PR-Q^L}\cdot\frac{(1-\phi)Y^O}{\phi Y^W} = \frac{2Q^L}{R\cdot\phi Y^W} + Q^L\frac{1-\tilde{\pi}}{Q^H-PR} + Q^L\frac{1-\pi}{Q^H-PR}\cdot\frac{(1-\phi)Y^O}{\phi Y^W}$$

together with (18) gives

$$Q^H\frac{1-\tilde{\pi}}{Q^H-PR} + Q^H\frac{1-\pi}{Q^H-PR}\cdot\frac{(1-\phi)Y^O}{\phi Y^W} = \frac{2Q^L}{R\cdot\phi Y^W} + Q^L\frac{1-\tilde{\pi}}{Q^H-PR} + Q^L\frac{1-\pi}{Q^H-PR}\cdot\frac{(1-\phi)Y^O}{\phi Y^W}$$

which simplifies to

$$(Q^H - Q^L)\frac{1-\tilde{\pi}}{Q^H - PR} + (Q^H - Q^L)\frac{\frac{(1-\phi)Y^O}{\phi Y^W}(1-\pi)}{Q^H - PR} = \frac{2Q^L}{R \cdot \phi Y^W}$$

$$(Q^H - Q^L)\left[\frac{1-\tilde{\pi} + \frac{(1-\phi)Y^O}{\phi Y^W}(1-\pi)}{Q^H - PR}\right] = \frac{2Q^L}{R \cdot \phi Y^W} \tag{55}$$

Also multiply the stock market clearing equation (19) with Q^H

$$-Q^H\frac{1-\tilde{\pi}}{Q^H - PR} - \frac{(1-\phi)Y^O}{\phi Y^W}Q^H\frac{1-\pi}{Q^H - PR} = \frac{2Q^H}{R \cdot \phi Y^W} - Q^H\frac{\tilde{\pi}}{PR - Q^L} - \frac{(1-\phi)Y^O}{\phi Y^W}Q^H\frac{\pi}{PR - Q^L}$$

together with (18) gives

$$-Q^L\frac{\tilde{\pi}}{PR - Q^L} - \frac{(1-\phi)Y^O}{\phi Y^W}Q^L\frac{\pi}{PR - Q^L} = \frac{2Q^H}{R \cdot \phi Y^W} - Q^H\frac{\tilde{\pi}}{PR - Q^L} - \frac{(1-\phi)Y^O}{\phi Y^W}Q^H\frac{\pi}{PR - Q^L}$$

which simplifies to

$$(Q^H - Q^L)\left[\frac{\tilde{\pi} + \frac{(1-\phi)Y^O}{\phi Y^W}\pi}{PR - Q^L}\right] = \frac{2Q^H}{R \cdot \phi Y^W} \tag{56}$$

(20) and (21) can be represented as (22) and (23), respectively

$$(Q^H - Q^L)(\tilde{\pi} + \frac{(1-\phi)Y^O}{\phi Y^W}\pi) = (1 + \frac{(1-\phi)Y^O}{\phi Y^W})(Q^H - Q^L) - \frac{2Q^L}{R \cdot \phi Y^W}(Q^H - PR) \tag{57}$$

$$(Q^H - Q^L)(\tilde{\pi} + \frac{(1-\phi)Y^O}{\phi Y^W}\pi) = \frac{2Q^H}{R \cdot \phi Y^W}(PR - Q^L) \tag{58}$$

Combine (22) \& (23) to solve for P

$$\frac{\phi Y^W + (1-\phi)Y^O}{2\phi Y^W}(Q^H - Q^L) - \frac{Q^L}{R \cdot \phi Y^W}(Q^H - PR) = \frac{Q^H}{R \cdot \phi Y^W}(PR - Q^L)$$

$$R \cdot \frac{\phi Y^W + (1-\phi)Y^O}{2}(Q^H - Q^L) + Q^L PR = Q^H PR$$

$$P = \phi\frac{Y^W}{2} + (1-\phi)\frac{Y^O}{2} = \bar{Y}$$

Combine (10) and (11) to solve for R (derivation is commented out in the .tex file) for $\theta = 1/2$

$$PR = \frac{(3+\pi\Delta Y)Q^H Q^L\left[\frac{\pi}{1-\pi}\right]^2 - (Q^L)^2\pi\Delta Y \pm Q^L\sqrt{(Q^L\pi\Delta Y + (1-\pi\Delta Y)(Q^H)\left[\frac{\pi}{1-\pi}\right]^2)^2 + 8Q^H Q^L\left[\frac{\pi}{1-\pi}\right]^2}}{\left[\frac{\pi}{1-\pi}\right]^2\left\{(1-\pi\Delta Y)Q^H + 2(1+\pi\Delta Y)Q^L\right\} - Q^L\pi\Delta Y \pm \sqrt{(Q^L\pi\Delta Y + (1-\pi\Delta Y)(Q^H)\left[\frac{\pi}{1-\pi}\right]^2)^2 + 8Q^H Q^L\left[\frac{\pi}{1-\pi}\right]^2}}$$

where $\Delta Y = \frac{(1-\phi)Y^O}{\phi Y^W}$

...

Using $P = \frac{\phi Y^W + (1-\phi)Y^O}{2} = \bar{Y}$ for $\theta = 1/2$ we get

...

$$R = \frac{1}{\bar{Y}} \cdot \frac{(3+\pi\Delta Y)Q^H Q^L\left[\frac{\pi}{1-\pi}\right]^2 - (Q^L)^2\pi\Delta Y \pm Q^L\sqrt{(Q^L\pi\Delta Y + (1-\pi\Delta Y)(Q^H)\left[\frac{\pi}{1-\pi}\right]^2)^2 + 8Q^H Q^L\left[\frac{\pi}{1-\pi}\right]^2}}{\left[\frac{\pi}{1-\pi}\right]^2\left\{(1-\pi\Delta Y)Q^H + 2(1+\pi\Delta Y)Q^L\right\} - Q^L\pi\Delta Y \pm \sqrt{(Q^L\pi\Delta Y + (1-\pi\Delta Y)(Q^H)\left[\frac{\pi}{1-\pi}\right]^2)^2 + 8Q^H Q^L\left[\frac{\pi}{1-\pi}\right]^2}}$$