Repeated Binary Action Coordination Games with Uniform Noise and Prior

by

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For my family

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ABSTRACT

I explore a game where two players simultaneously choose between two actions in finitely many stages. Players have incomplete information about their payoffs. In each stage, they receive a noisy signal about a parameter that determines their payoffs; the signal's noise is a random variable that is uniformly distributed over the closed interval from 0 to 1 and independent of the payoff parameter. Players share a common prior belief that the payoff parameter is uniformly distributed over the closed interval from 0 to 1 and they commonly update this belief given the full history of actions. Players are restricted to "cutoff" strategies: They play action "1" if they see that the average of their signals up to that stage is strictly above a cutoff value, and they play action "0" otherwise. I describe how players obtain cutoffs that are optimal given the publicly known posterior beliefs about the payoff parameter, their privately observed average of signals in each stage, and the opposing player's cutoff strategy. Each player is further restricted to choosing the maximum, "pessimistic," cutoff if multiple exist. I give the exact pessimistic equilibrium cutoffs in Stages 1 and 2 for all possible average of signals. Using Monte Carlo integration methods, I simulate this game and approximate the pessimistic cutoffs for each player in Stages 3 and beyond. I give a lower bound on the possible values of any nonzero pessimistic cutoffs for each player. I show that there is a range of average signals where each player almost always plays action "1." With certain assumptions, I give a range of signal averages for any stage such that players will always choose action "0" when a nonzero cutoff exists and I provide numerical evidence for this property. Finally, I look at two alternative equilibria—one where each player always chooses

a cutoff of 0 and one where each player chooses the smallest nonzero cutoff when multiple nonzero cutoffs exist.

CHAPTER I

Introduction

I explore a class of games inspired by the *global game* framework introduced by Carlsson and van Damme (CVD) [CvD93] and expanded by Morris and Shin (MS) [MS01]. Two players simultaneously choose between two possible actions. Player $i \in \{1, 2\}$ receives a payoff depending on each players' actions and a parameter θ. Under this framework, θ is determined by Nature, where it is drawn from an underlying distribution unknown to both players. θ is unknown to both players until they receive a payoff for their choice of actions.

Table I.1 shows the payoffs for Players 1 and 2 in the game considered by MS. The rows of the payoff matrix correspond to Player 1's choice of actions and the columns correspond to Player 2's choice of actions; the ith entry in a cell indicates Player i's payoff for the corresponding pair of actions.

If both players know θ, then there exist pure Nash equilibria when $\theta < 0$ and $\theta > 1$, where both play action "0" and both play action "1," respectively. If both players know the value of $\theta \in [0, 1]$, then both players simultaneously choosing action "0" and simultaneously choosing action "1" are pure Nash equilibria. A mixed Nash equilibrium exists where both players choose action "0" with probability θ.

Per MS, players believe that θ is "uniformly distributed on the real line" (players share an uninformative prior distribution on θ). Player i privately observes a signal $x_i = \theta + \epsilon_i$, where ϵ_i is distributed normally with mean 0 and variance σ^2. Players (privately) obtain a posterior probability distribution for θ given their signal following Bayes' Theorem, and in

		Player 2	
		0	1
Player 1	0	$0, 0$	$0, \theta - 1$
	1	$\theta - 1, 0$	θ, θ

Table I.1: Payoff matrix for Players 1 and 2 considered by Morris and Shin [MS01].

turn Player i believes that the alternative player's signal is distributed normally with mean x_i and standard deviation $\sqrt{2}\sigma$.

MS restrict players to *switching strategies*: Players select action "1" if their signals are strictly above a threshold value. With the posterior beliefs above, Player i is indifferent between playing action "0" or action "1" when

$$x_i = \Phi\left(\frac{k - x_i}{\sqrt{2}\sigma}\right),$$

where the difference of the left-hand side and right-hand side of the equality is the expected payoff for Player i when the alternative player uses a switching strategy about k; x_i is Player i's expected value of θ. Φ indicates the normal cumulative distribution function, and the right-hand side gives Player i the probability that the alternative player has a signal less than or equal to k conditional on x_i.

Player i's best-response switching strategy to the alternative player's switch strategy about k is the right-hand side of the equation above. If both players utilize cutoff strategies and know that each player uses a cutoff strategy, then each player's optimal threshold is $1/2$. This *Bayesian Nash equilibrium strategy* is the unique signal value of x_i for each player that solves that equation above, given that Player $-i$ also uses a cutoff value of $1/2$.

In this paper, I consider the game described above repeated in finitely many stages. I will assume that both players start off with a standard uniform prior on the game parameter θ. The noise terms that each player sees in the each stage of the game will be drawn from a standard uniform distribution. I show how repetition of the game above introduces the possibility of multiple equilibria in as early as the second stage.

CVD show that all two-action, two-player global games have a unique equilibrium. Frankel, Morris, and Pauzner (FMP) [FMP03] generalize this uniqueness result to include games with many players and many actions. The simultaneous game played in each stage of the game I discuss here exhibits qualities considered by FMP. Namely, the payoff function governing each stage shown in Table I.1 is *supermodular*—a player choosing action "1" incentivizes the opposing player to also choose action "1"; the payoffs in each stage are *state-monotonic*—a higher θ makes that action "1" more desirable for a player than action "0". Yet this "limit uniqueness" does not arise in the second stage of this game. Angeletos, Hellwig, and Pavan (AHP) [GMA07] demonstrate that their game repeated in a discrete-number of stages initially exhibits the qualities of the game considered by FMP. Without a history of actions and information of the global game parameter in Stage 1, there is a unique equilibrium in Stage 2. However, introducing this history can allow for one or many equilibria to exist depending on the prior belief players share over the global parameter.

In Chapter II, I define the game that is played in Stages $1, 2, \ldots, T < \infty$. Given a player's privately known average of signals up to Stage t and the publicly known history of actions, I describe the posterior belief about θ given this information and their expected payoff if they play action "1". In each stage, players choose cutoff strategies that maximize their expected payoffs in that stage alone. When multiple cutoffs exist, I further assume that players choose the highest amongst these cutoffs—what I call the *pessimistic* cutoff in each stage. Theorem II.2 gives the pessimistic cutoff strategy of 0 for each player in Stage 1—a result similar to the one-shot games considered by CVD and MS. However, Theorem II.3 gives a regime of average signal values where three possible cutoffs arise given the history of actions in Stage 1 and the average of players' private signals up to Stage 2. Finally, Theorem II.7 provides a range of signal averages in Stage $t \geq 3$ where both players have a unique cutoff equal to 0.

In Chapter III, I perform Monte Carlo simulations of the game described in Chapter II. There are four possible action profile histories that can arise from Stage 2 (players match at the high action, players match at the low action, or players mismatch actions) and three possible posterior beliefs players have about θ in Stage 3. The aim of these simulations is to approximate the pessimistic cutoff strategies for each player in each stage over a finite time horizon. I start with crude estimators of integrals defined in Chapter II; here the components of sampled vectors are drawn uniformly and independently. I then define a modified sampling scheme where the components are drawn from a different distribution that gives estimations with smaller probabilistic variance. The simulations with the modified scheme indicate that a pessimistic cutoff selection leads to a history-independent choice of actions from each player; I provide a simplified sampling scheme that simulates such a game. Assuming this history-independence, and provided a nonzero pessimistic cutoff exists, I show that Player i always chooses $a_i^{(t)} = 0$ when $X^{(t)} \leq 1$.

Finally, Chapter IV explores two alternate equilibrium selection criteria and how these criteria can affect the cutoff choices for each player in all stages. I first consider the zero-cutoff strategy in each stage, where players choose a cutoff of 0 for their signal averages. I then explore an alternate *optimistic* choice of cutoffs, where players choose the lowest nonzero cutoff when multiple exist. I employ the modified Monte Carlo sampling scheme from Chapter III to visualize the effect of this optimistic choice on each players' expected value of θ.

I.1: Applications and Further Reading

Global games can model a myriad of economic and social situations. The global game parameter takes on different roles in different models; it captures economic fundamentals that are unknown to any number of players in the game. If players know said fundamentals, they would have perfect information about their respective payoff functions.

Morris and Shin [MS04] consider a model of debt pricing: A continuum of creditors finance a project and must decide to hold or liquidate debt prior to project's completion (be that a successful project or otherwise). A creditor who holds onto debt up to completion receives either a high payoff or low payoff depending on the game parameter; a creditor who liquidates receives a payoff somewhere in between the high and low payoffs. θ captures a borrower's ability to succeed in the project (when creditors get the high payoff). Similarly, Morris and Shin [MS98] discuss a model where a continuum of currency speculators decide to attack (short) or not attack a currency. θ represents economic fundamentals that affect the government-unregulated exchange rate of said currency. Heinemann, Nagel, and Ockenfels [FHO04] perform an experiment to test conclusions from this speculative attack game, where they report that agent behavior closely follows the global game solution. Goldstein and Pauzner [GP05] develop a continuum-of-agents model concerning bank runs (here, they consider a continuum of depositors); the global state represents fundamentals that affect the total return on money left deposited in a bank.

Shadmehr and Bernhardt (SB) [SB11] present a model of the "calculus of protest," where two citizens choose between submitting to the status quo of a regime or revolting against it; θ represents the allure of revolt. The formulation given by SB is similar to the setup of Table I.1 given by MS. SB investigate cutoff strategies for this game and there exists a critical cutoff where information received by both citizens can cause their actions to be either strategic complements or substitutes. AHP explore a model of regime change, where θ has the opposite interpretation as the strength of the regime. Edmond [Edm13] incorporates a propaganda component, where a regime can secretly affect the citizens' beliefs about regime strength.

Whilst some of the models I mention are derived from one-stage simultaneous play from a number of players, one can consider a repetition of the one-stage game where players receive new information about the underlying game parameter in multiple stages. The AHP model on regime change allows the citizens to attack the regime in multiple stages until the status quo is finally overcome. Chassang [Cha10] looks at infinite horizon exit games, where two players simultaneously choose between actions "Stay" and "Exit" in many stages, until at least one player chooses "Exit". Frankel [Fra12] considers a speculative attack game à la MS where a continuum of players receive signals over a countably infinite number of stages.

Lastly, I draw attention to some literature concerning supermodular games with strategic complementarities and *global game selection*. Heinemann [Hei15] defines this latter concept as the unique limit equilibrium that a global game "chooses" as the signal noise for all players vanishes. Bergemann and Morris [BM16] characterize Bayesian Nash equilibria corresponding to what they call "Bayes correlated equilibria". Morris, Oyama, and Takahashi [SMT20] give characterizations of binary action global game selections (Bayesian Nash equilibria) with respect to "sequential obedience" criterion.

CHAPTER II

The Game

Let "Player $-i$" denote the alternative to Player i. Consider the game described in Table I.1 repeated for T stages where, in Stage t, Players i and $-i$ simultaneously choose actions $a_i^{(t)}, a_{-i}^{(t)} \in \{0,1\}$, respectively. Neither player knows θ. Instead, both players initially share a common prior belief that θ is distributed uniformly over the interval $[0,1]$. In Stage t, Player i privately observes a signal $x_i^{(t)} = \theta + \epsilon_i^{(t)}$. For all i, t, $\epsilon_i^{(t)}$ are independently and identically, uniformly distributed over $[0,1]$; each of these noise terms are independent of θ. The total payoff for Player i is the sum of the undiscounted payoffs from each individual stage; neither player sees their payoff from any stage until the end of the game. Assume that players are myopic: Each player is only interested in maximizing their expected payoff in the current stage.

Let $X^{(\tau)}$ be the average of Player i's signals $x_i^{(1)}, \dots x_i^{(\tau)}$; in Stage t, Player i remembers $X^{(1)}, \dots, X^{(t-1)}$. Actions are publicly observed and remembered: In Stage t, Player i and $-i$ know the history of actions played in Stages $1, \dots, t-1$, $A^{(t-1)} = \{a_i^{(1)}, a_{-i}^{(1)}, \dots, a_i^{(t-1)}, a_{-i}^{(t-1)}\}$. At the beginning of each stage, both players obtain a common posterior distribution of θ given this full history of action profiles. In each stage, both players are restricted to cutoff strategies defined as follows: Let $c_i^{(t)}$ be a cutoff Player i chooses in Stage t; if $X^{(t)} \le c_i^{(t)}$, then Player i will play $a_i^{(t)} = 0$; otherwise, Player i chooses action $a_i^{(t)} = 1$. A history of actions $A^{(t-1)}$ encodes cutoff information for each player. For example, in Stage 2, if Player 1 played action $a_1^{(1)} = 1$ with a cutoff of $c_1^{(1)}$ and Player 2 played action $a_2^{(1)} = 0$ with a cutoff of $c_2^{(1)}$, then both players know the history

$$A^{(1)} = \{\{a_1^{(1)} = 1\}, \{a_2^{(1)} = 0\}\} = \{\{x_1^{(1)} > c_1^{(1)}\}, \{x_2^{(1)} \leq c_2^{(1)}\}\}.$$

For this game, I consider symmetric Bayesian Nash equilibria in cutoff strategies:

Definition II.1. A collection of cutoff strategies for both players, $\{c_1^{(t)}, c_2^{(t)}\}_{t=1}^{T}$, is a **Bayesian Nash equilibrium** if for all i, t, the cutoff strategy defined by $c_i^{(t)}$ maximizes Player i's expected payoff in Stage t given their average of signals $X^{(t)}$, the action history $A^{(t-1)}$, and Player $-i$'s cutoff.

Any such equilibrium cutoff-strategy profile is a *pure-strategy equilibrium* in cutoff strategies; the cutoff strategies in each stage describe what action Player i chooses for any average of signals in that stage.

Finally, assume that players choose *pessimistic* cutoffs: In Stage t, if Player i has multiple optimal cutoff strategies (in the sense that a strategy maximizes Player i's expected payoff) given Player $-i$'s cutoff strategy, then Player i selects the maximum cutoff value amongst the optimal cutoffs.

II.1: Stage 1

To begin, the sum of n independent random variables that are all uniformly distributed over $[0, 1]$ is *Irwin-Hall* [Irw27, Hal27] distributed with parameter n and with probability density function given by

$$f(x) = \frac{1}{(n-1)!} \sum_{k=0}^{\lfloor x \rfloor} (-1)^k \binom{n}{k} (x-k)^{n-1}, \ 0 < x < n.$$

Let $\mathbf{1}_A(\cdot)$ denote the indicator function over a set A. Both players share a common prior belief that the distribution of θ is $f_\theta(\omega) = \mathbf{1}_{[0,1]}(\omega)$, $0 \leq \omega \leq 1$. Player i privately observes signal $x_i^{(1)}$; Player i's private posterior probability density of θ given $x_i^{(1)}$ is

$$f_{\theta:x_i^{(1)}}(\omega : x) = \frac{\mathbf{1}_{(x-1,x)}(\omega)\,\mathbf{1}_{[0,1]}(\omega)}{f_{x_i^{(1)}}(x)},$$

where $x_i^{(1)}$ is an Irwin-Hall distributed with probability density function

$$f_{x_i^{(1)}}(x) = x\mathbf{1}_{(0,1]}(x) + (2-x)\mathbf{1}_{(1,2)}(x).$$

Player i is indifferent between choosing the high action "1" and the low action "0" when their expected payoffs of selecting either action are equal. Player i always receives a payoff of 0 if they select $a_i^{(1)} = 0$. Therefore, Player i is indifferent between both actions when

$$\text{(II.1.1)} \qquad \mathbf{E}[\theta : x_i^{(1)}] - \mathbf{P}(a_{-i}^{(1)} = 0 : x_i^{(1)}) = 0.$$

$\mathbf{E}[\theta : x_i^{(1)}]$ denotes the expected value of θ given Player i's private signal and $\mathbf{P}(a_{-i}^{(1)} = 0 : x_i^{(1)})$ denotes the probability that Player i assigns to the event that Player $-i$ plays action "0," conditional on Player i's private signal. The left-hand side of Equation (II.1.1) is the calculation of Player i's expected payoff of selecting action "1," given their private signal information. If the realization of Player i's signal is $x_i^{(1)} = x$, then

$$\text{(II.1.2)} \qquad \begin{aligned} \mathbf{E}[\theta : x_i^{(1)} = x] &= \frac{1}{f_{x_i^{(1)}}(x)} \int_0^1 \omega \cdot \mathbf{1}_{(x-1,x)}(\omega)\, d\omega \\ &= \begin{cases} \dfrac{x^2}{2x}, & 0 < x \leq 1, \\ \\ \dfrac{2x - x^2}{2(2-x)}, & 1 < x < 2, \end{cases} \\ &= \frac{x}{2},\ 0 < x < 2. \end{aligned}$$

Note that $f_{x_i^{(1)}}(x)$ vanishes on $(-\infty, 0] \cup [2, \infty)$. However, one can define the expectation above to be 0 when $x = 0$ and 1 when $x = 2$. $x_i^{(1)} = 0$ only when $\theta = 0$; $x_i^{(1)} = 2$ only when $\theta = 1$. If Player $-i$'s cutoff value is $c_{-i}^{(1)} = c$, then

$$
\begin{aligned}
\text{(II.1.3)} \qquad \mathbf{P}(a_{-i}^{(1)} = 0 : x_i^{(1)} = x) &= \mathbf{P}(x_{-i}^{(1)} \leq c_{-i}^{(1)} = c : x_i^{(1)} = x) \\
&= \frac{1}{f_{x_i^{(1)}}(x)} \int_0^c f_{x_i^{(1)}, x_{-i}^{(1)}}(x, y)\, dy \\
&= \frac{1}{f_{x_i^{(1)}}(x)} \int_0^c \int_{\mathbb{R}} f_{x_i^{(1)}, x_{-i}^{(1)}:\theta}(x, y : \omega) \cdot \mathbf{1}_{[0,1]}(\omega)\, d\omega\, dy \\
&= \frac{1}{f_{x_i^{(1)}}(x)} \int_0^1 \int_0^c \mathbf{1}_{(x-1,x)}(\omega) \cdot \mathbf{1}_{(\omega,\omega+1)}(y)\, dy\, d\omega, \\
&= \begin{cases}
\dfrac{c^2}{2x}, & 0 < x \leq 1,\ 0 \leq c \leq x, \\
c - \dfrac{x}{2}, & 0 < x \leq 1,\ x < c \leq 1, \\
c - \dfrac{x}{2} - \dfrac{(c-1)^2}{2x}, & 0 < x \leq 1,\ 1 < c \leq x + 1, \\
1, & 0 < x \leq 1,\ x + 1 < c \leq 2, \\
\dfrac{(c - x + 1)^2}{2(2 - x)}, & 1 < x < 2,\ x - 1 < c \leq 1, \\
c - \dfrac{x}{2}, & 1 < x < 2,\ 1 < c \leq x, \\
\dfrac{c^2 - 4c + 2x}{2(x - 2)}, & 1 < x < 2,\ x < c \leq 2, \\
0, & 1 < x < 2,\ 0 \leq c \leq x - 1.
\end{cases}
\end{aligned}
$$

where $f_{x_i^{(1)}, x_{-i}^{(1)}}$ denotes the joint probability density of Player i and Player i's signals. The integrand of Equation (II.1.3) is due to the independence of $x_i^{(1)}$ and $x_{-i}^{(1)}$ conditional on θ. The full derivation is shown in APPENDIX B.

II.1.1: Stage 1 Equilibrium Cutoff Strategy

Player i's *best response* to Player $-i$'s cutoff of $c_{-i}^{(1)} = c$ is given by the conditional probability in Equation (II.1.1). Specifically, given $x_i^{(1)} = x$, if c^* is a fixed point of

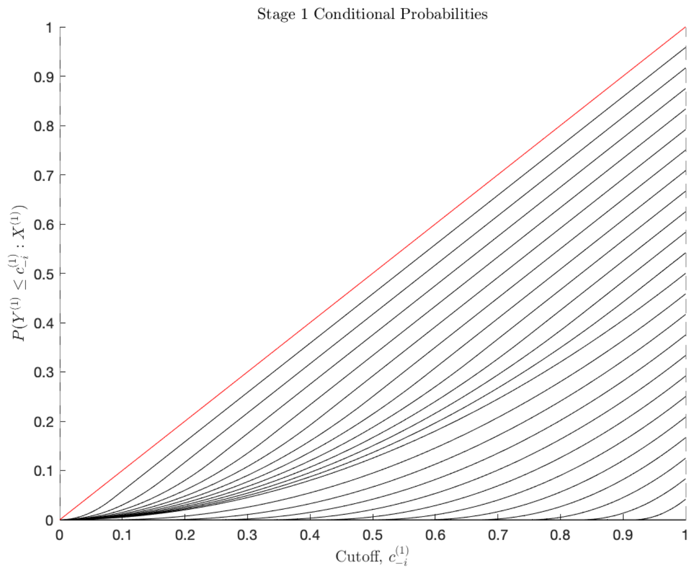

Figure II.1: Stage 1 Conditional Probabilities for twenty-five values of Player i's signal in $(0, 2)$ and cutoffs $c_{-i}^{(1)} \in [0, 1]$.

$P(c; x) = \mathbf{P}(x_{-i}^{(1)} \le c_{-i}^{(1)} = c : x_i^{(1)} = x)$, then the signal $c_i^{(1)*}$ which solves

$$\mathbf{E}[\theta : x_i^{(1)} = c_i^{(1)*}] - c^* = 0$$

is Player i's equilibrium cutoff in Stage 1.

Theorem II.2. *In Stage 1, the pessimistic equilibrium cutoff for both players is* $c_1^{(1)*} = 0 = c_2^{(1)*}$.

See APPENDIX A for the proof of this theorem. Given $0 < x \le 1$, the unique fixed point solving $P(c^*; x) = c^*$ is $c^* = 0$. The same is true for when $1 \le x < 2$. This is illustrated in MATLAB-generated Figure II.1, where the conditional probabilities are plotted for $c \in [0, 1]$; each black curve represents this conditional probability for a value of $x \in (0, 2)$. For a fixed value of c, $\mathbf{P}(x_{-i}^{(1)} \le c : x_i^{(1)} = x)$ decreases as x increases. The red curve indicates the forty-five degree line; each black curve is strictly below this red line, except when $c = 0$. This is in-line with the result from CVD and MS. Player i's equilibrium cutoff for Stage 1 solves the equation $c_i^{(1)*}/2 = \mathbf{E}[\theta : x_i^{(1)} = c_i^{(1)*}] = 0$. Player i chooses action $a_i^{(1)} = 1$ with probability 1.

II.2: Stage 2

In Stage 2, players know the action profile from Stage 1, $A^{(1)} = \{a_i^{(1)}, a_{-i}^{(1)}\}$, and the associated cutoff strategy used by each player. Players commonly update their beliefs about the parameter θ based on this publicly observed action profile; the Stage 2 posterior probability density of θ given $A^{(1)}$ is

$$\text{(II.2.1)} \qquad f_\theta^{(2)}(\omega) = \mathbf{1}_{g(A^{(1)})}(\omega) \cdot \frac{\mathbf{P}(A^{(1)} : \theta = \omega)}{\mathbf{P}(A^{(1)})},$$

where

(II.2.2)

$$g(A^{(1)}) = \bigcap_{i=1,2} \{\omega \in [0,1] : \text{Given } \theta = \omega,\ \exists\, \epsilon_i^{(1)} \in [0,1] \text{ such that Player } i \text{ chose } a_i^{(1)}\}.$$

Restricting the support of Equation (II.2.1) ensures that $\mathbf{P}(A^{(1)}) > 0$, except when either player receives a Stage 1 signal $x_i^{(1)} = 0$. Equation (II.2.2) can be expressed more specifically with the equilibrium cutoffs in Stage 1. The pessimistic equilibrium cutoff for Player i in Stage 1 is $c_i^{(1)*} = 0$; with probability 1, the actions played in Stage 1 are given by

$$A^{(1)} = \{a_1^{(1)} = 1,\ a_2^{(1)} = 1\} = \{x_1^{(1)} > 0\} \cap \{x_2^{(1)} > 0\}.$$

Thus,

$$\begin{aligned} \text{(II.2.3)} \qquad g(A^{(1)}) &= \bigcap_{i=1,2} \{\omega \in [0,1] : \exists\, \epsilon_i^{(1)} \in [0,1] \text{ such that } x_i^{(1)} = \omega + \epsilon_i^{(1)} > 0\} \\ &= \bigcap_{i=1,2} [0,1] \\ &= [0,1], \end{aligned}$$

and

$$f_\theta^{(2)}(\omega) = \mathbf{1}_{[0,1]}(\omega) \cdot \frac{\mathbf{P}(x_1^{(1)} > 0, x_2^{(1)} > 0 : \theta = \omega)}{\mathbf{P}(x_1^{(1)} > 0, x_2^{(1)} > 0)}$$
$$= \mathbf{1}_{[0,1]}(\omega) \cdot \frac{1}{\mathbf{P}(x_1^{(1)} > 0, x_2^{(1)} > 0)} \int_0^2 \int_0^2 f_{x_1^{(1)}, x_2^{(1)} : \omega}(x, y : \theta)\, dx\, dy$$
$$= \mathbf{1}_{[0,1]}(\omega) \cdot \frac{1}{\mathbf{P}(x_1^{(1)} > 0, x_2^{(1)} > 0)} \int_0^2 \int_0^2 \mathbf{1}_{(\omega,\omega+1)}(x) \cdot \mathbf{1}_{(\omega,\omega+1)}(y)\, dx\, dy$$
$$= \mathbf{1}_{[0,1]}(\omega) \cdot \frac{1}{\mathbf{P}(x_1^{(1)} > 0, x_2^{(1)} > 0)} \left(\int_\omega^{\omega+1} dx \right)^2$$
$$= \mathbf{1}_{[0,1]}(\omega). \tag{II.2.4}$$

In Stage 2, Player i remembers their signal from Stage 1; they privately observes another signal $x_i^{(2)}$ and averages both signals. Player i is indifferent between actions "1" and "0" if the expected payoff of choosing the high action equals the expected payoff of choosing the low action. Similar to Equation (II.1.1), Player i is indifferent between both actions if

$$\mathbf{E}[\theta : A^{(1)}, X^{(2)}] - \mathbf{P}(a_{-i}^{(1)} = 0 : A^{(1)}, X^{(2)}) = 0, \tag{II.2.5}$$

where Player i believes θ has probability density $f_\theta^{(2)}$. The following probability distribution is relevant to the equation above and subsequent stages: The average of n uniform-$[0,1]$ random variables is *Bates*-distributed [Bat55] with parameter n, with probability density function

$$B_n(x) = \frac{n}{(n-1)!} \sum_{k=0}^{\lfloor nx \rfloor} (-1)^k \binom{n}{k} (nx - k)^{n-1},\ 0 < x < 1.$$

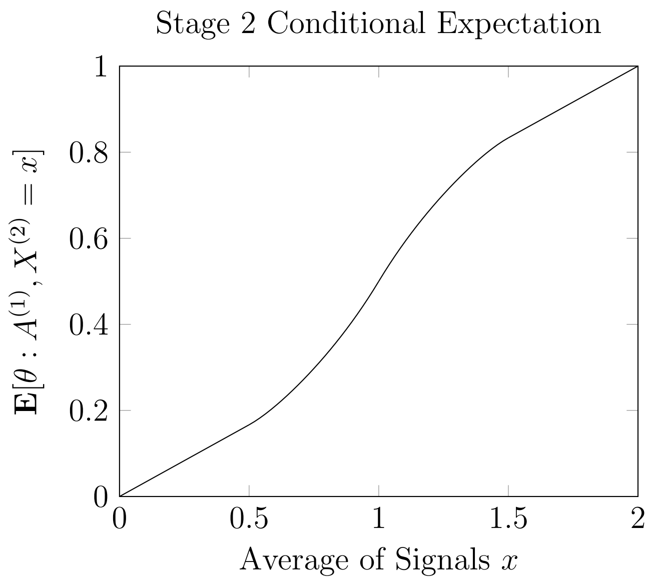

Figure II.2: Stage 2 conditional expectation of θ given Player i's average of signals $X^{(2)} = x$ and history of actions $A^{(1)}$.

If the realization of Player i's average of signals in Stage 2 is $X^{(2)} = x$, then

$$\text{(II.2.6)} \qquad \mathbf{E}[\theta : A^{(1)}, X^{(2)} = x] = \frac{1}{f_{X^{(2)}}(x)} \int_{\mathbb{R}} \omega \cdot f_{X^{(2)};\theta}(x : \omega) \cdot f_\theta^{(2)}(\omega)\, d\omega$$
$$= \frac{\displaystyle\int_0^1 \omega \cdot B_2(x-\omega) \cdot \mathbf{1}_{(x-1,x)}(\omega)\, d\omega}{\displaystyle\int_0^1 B_2(x-\omega) \cdot \mathbf{1}_{(x-1,x)}(\omega)\, d\omega}$$
$$= \begin{cases} \dfrac{x}{3}, & 0 < x < \dfrac{1}{2}, \\ \dfrac{(2x)^3 - 2(2x-1)^3}{6(2x)^2 - 12(2x-1)^2}, & \dfrac{1}{2} \leq x < 1, \\ \dfrac{12(x-1) - 6(2x-2)^2 + 6 - (2x-2)^3}{12 - 6(2x-2)^2}, & 1 \leq x < \dfrac{3}{2}, \\ \dfrac{x+1}{3}, & \dfrac{3}{2} \leq x < 2. \end{cases}$$

The full derivation of Equation (II.2.6) is shown in APPENDIX C. Figure II.2 shows a plot of $\mathbf{E}[\theta : A^{(1)}, X^{(2)} = x]$ for $0 < x < 2$. Note that $f_{X^{(2)}}(x) = 0$ when $x = 0, 2$, but the conditional expectation can be defined for these average of signal values as was done in Stage 1. The marginal density of $X^{(2)}$ is likewise derived in APPENDIX C alongside the

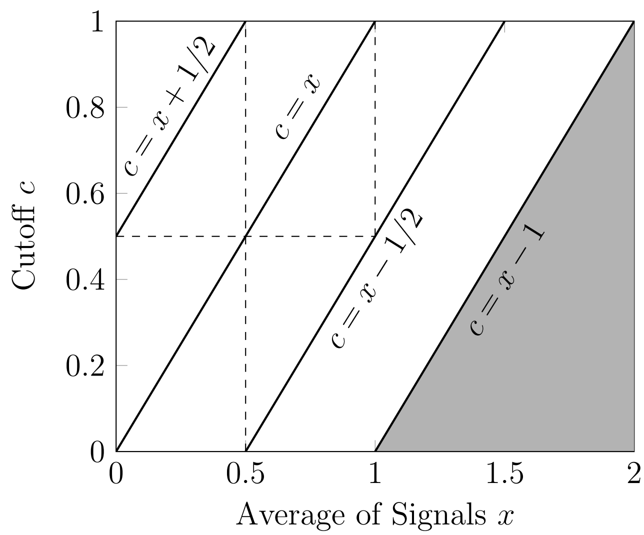

Figure II.3: Subregions of $\{(x,c) \in \mathbb{R}^2 : 0 < x < 2, 0 \leq c \leq 2\}$ relevant to Equation (II.2.8).

expectation of θ given $A^{(1)}$ and $X^{(2)}$:

$$\text{(II.2.7)} \qquad f_{X^{(2)}}(x) = \begin{cases} 2x^2, & 0 < x < \dfrac{1}{2}, \\ -2x^2 + 4x - 1, & \dfrac{1}{2} \leq x < \dfrac{3}{2}, \\ 2x^2 - 8x + 8, & \dfrac{3}{2} \leq x < 2. \end{cases}$$

Let $Y^{(2)}$ denote the average of Player $-i$'s signals up to Stage 2. If Player $-i$'s cutoff value in Stage 2 is $c_{-i}^{(2)} = c$, then

$$\begin{aligned} \text{(II.2.8)} \qquad \mathbf{P}(a_{-i}^{(2)} = 0 : A^{(1)}, X^{(2)} = x) &= \mathbf{P}(Y^{(2)} \leq c_{-i}^{(2)} = c : A^{(1)}, X^{(2)} = x) \\ &= \frac{1}{f_{X^{(2)}}(x)} \int_0^c f_{X^{(2)},Y^{(2)}}(x,y)\,dy \\ &= \frac{1}{f_{X^{(2)}}(x)} \int_0^c \int_{\mathbb{R}} f_{X^{(2)},Y^{(2)}:\theta}(x,y:\omega) \cdot f_\theta^{(2)}(\omega)\,d\omega\,dy \\ &= \frac{1}{f_{X^{(2)}}(x)} \int_0^1 \int_0^c B_2(x-\omega) \cdot B_2(y-\omega)\,dy\,d\omega. \end{aligned}$$

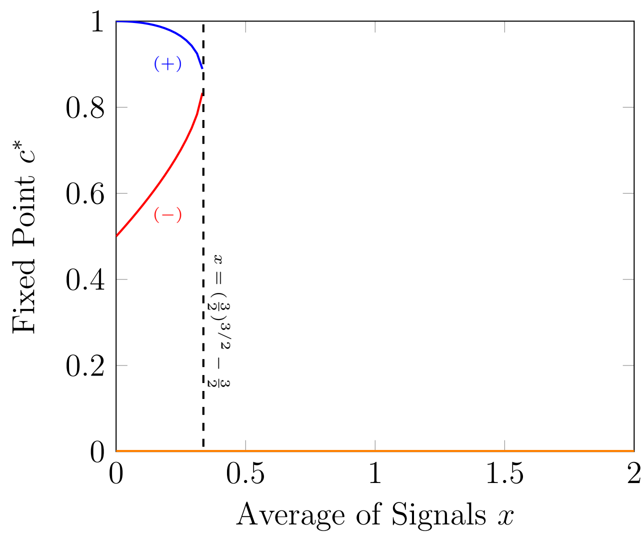

Figure II.4: Fixed points of $\mathbf{P}(Y^{(2)} \leq c : A^{(1)}, X^{(2)} = x),\ 0 < x < 2.$

II.2.1: Stage 2 Pessimistic Equilibrium Cutoff Strategy

Like in Stage 1, the derivation of this probability involves comparing x and c in the region $\{(x, c) \in \mathbb{R}^2 : 0 < x < 2, 0 \leq c \leq 2\}$. However, one need only consider cutoff values $0 \leq c \leq 1$ to determine Player i's pessimistic equilibrium cutoff. Even then, consider the ten subregions of $\{(x, c) \in \mathbb{R}^2 : 0 < x < 2, 0 \leq c \leq 1\}$ in order to fully express Player i's best response cutoff to Player $-i$'s cutoff. APPENDIX C details the derivation of Equation (II.2.8); Figure II.3 illustrates the subregions. Note that the conditional probability is 0 for (x, c) in the gray subregion.

Like in Stage 1, given the actualization of Player i's average of signals, $X^{(2)} = x$, and history $A^{(1)}$, Player i's pessimistic equilibrium cutoff value in Stage 2 is the maximum average of signals, $c_i^{(2)*}$, that satisfy

$$\text{(II.2.9)} \qquad \mathbf{E}[\theta : A^{(1)}, X^{(2)} = c_i^{(2)*}] - c^* = 0,$$

where c^* is a cutoff value for Player $-i$ satisfying $c^* = \mathbf{P}(Y^{(2)} \leq c^* : A^{(1)}, X^{(2)} = x)$. The MATLAB-generated Figure II.5 shows $\mathbf{P}(Y^{(2)} \leq c : A^{(1)}, X^{(2)})$ for various values of Player i's average of signals within $(0, 2)$ and values of Player $-i$'s cutoff, c. The red curve represents the forty-five degree line. Multiple black curves intersect this line at, at most, three different points. There is a regime of $X^{(2)}$ values where multiple fixed points exist. $c^* = 0$ is always a fixed point for all $X^{(2)}$. Specifically, $\mathbf{P}(Y^{(2)} \leq c : A^{(1)}, X^{(2)} = x)$ has multiple fixed points for some regime of Player i's signal averages in $(0, 1]$; these distinct nonzero fixed points are within $[1/2, 1]$. Solving the equation

$$c^* = -2(c^*)^2 + \frac{4\,c^*\,x}{3} + 4\,c^* - \frac{x^2}{3} - \frac{4\,x}{3} - 1 \tag{II.2.10}$$

for c^*, where $0 < x < \dfrac{1}{2}$, yields at least one nonzero fixed point.

Theorem II.3. *Let Player i's Stage 2 average of signals be $X^{(2)} = x$.*

- *If $0 < x \leq (3/2)^{3/2} - 3/2$, then the maximum nonzero solution to the equation $c^* = \boldsymbol{P}(Y^{(2)} \leq c^* : A^{(1)}, X^{(2)} = x)$ is*

$$c^* = \frac{x}{3} + \frac{\sqrt{-8\,x^2 - 24\,x + 9}}{12} + \frac{3}{4}.$$

- *If $(3/2)^{3/2} - 3/2 < x < 2$, then $c^* = 0$ is the unique solution to the equation $c^* = \boldsymbol{P}(Y^{(2)} \leq c^* : A^{(1)}, X^{(2)} = x)$.*

See APPENDIX A for the proof. Figure II.4 shows the two regimes of Player i's average of signals and the corresponding fixed points for those values. The dashed vertical line that runs along $x = (3/2)^{3/2} - 3/2$ separates the two regimes. The orange line that runs along $c^* = 0$ signifies that 0 is always a fixed point for any x in $(0, 2)$. The blue and red curves represent the nonzero fixed point branches.

If $(3/2)^{3/2} - 3/2 < x < 2$, Player i's pessimistic equilibrium cutoff is $c_i^{(2)*} = 0$. If $0 < x \leq (3/2)^{3/2} - 3/2$, then Player i's pessimistic equilibrium cutoff corresponds to the

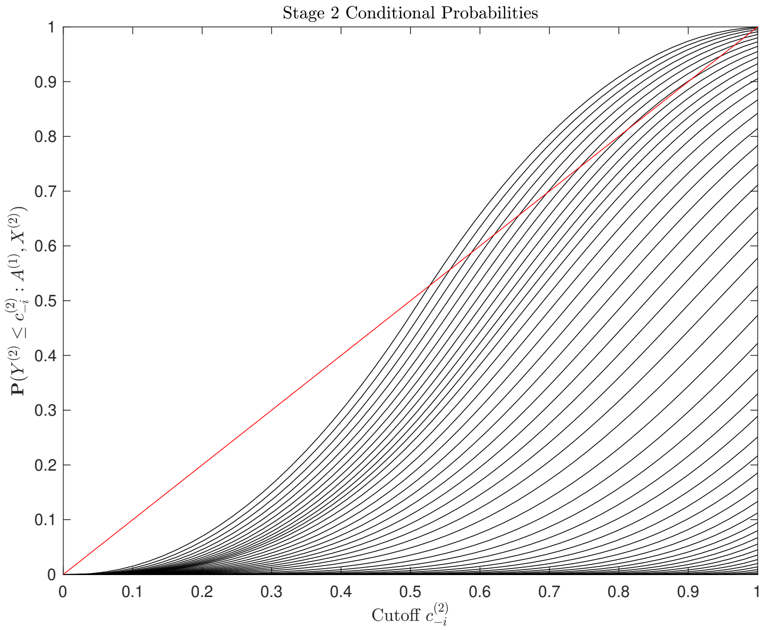

Figure II.5: Stage 2 Conditional Probabilities for twenty-five different values of $X^{(2)}$, for all cutoffs $c_{-i}^{(2)} \in [0, 1]$.

maximal fixed point for that average signal value; from Equation (II.2.9),

$$\mathbf{E}[\theta : A^{(1)}, X^{(2)} = c_i^{(2)*}] - c^* = \frac{1}{3}c_i^{(2)*} + \frac{1}{3} - \frac{x}{3} - \frac{\sqrt{-8\,x^2 - 24\,x + 9}}{12} - \frac{3}{4} = 0$$

$$\Rightarrow c_i^{(2)*} = x + \frac{\sqrt{-8\,x^2 - 24\,x + 9}}{4} + \frac{5}{4}. \tag{II.2.11}$$

Note that the upper branch of fixed points ranges from $(\sqrt{6} + 1)/4$ to 1; the portion of Equation (II.2.6) defined on $3/2 \leq x < 2$ ranges from 5/6 to 1. Figure II.6 shows Player i's pessimistics cutoffs corresponding to the high nonzero branch (blue curve) and 0 (orange line) versus their average of signal values (black line). The vertical dashed line runs through the bifurcation point $x = (3/2)^{3/2} - 3/2$. When Player i selects a high-branch cutoff, their average of signals will be strictly less than this cutoff and they will play the low action $a_i^{(2)} = 0$. Evidently, if Player i selects the low nonzero branch of cutoffs, then they would still play the low action. If Player i sees an average of signals strictly greater than $(3/2)^{3/2} - 3/2$, then they select $c_i^{(t)*} = 0$ and will play the high action $a_i^{(2)} = 1$ with probability 1.

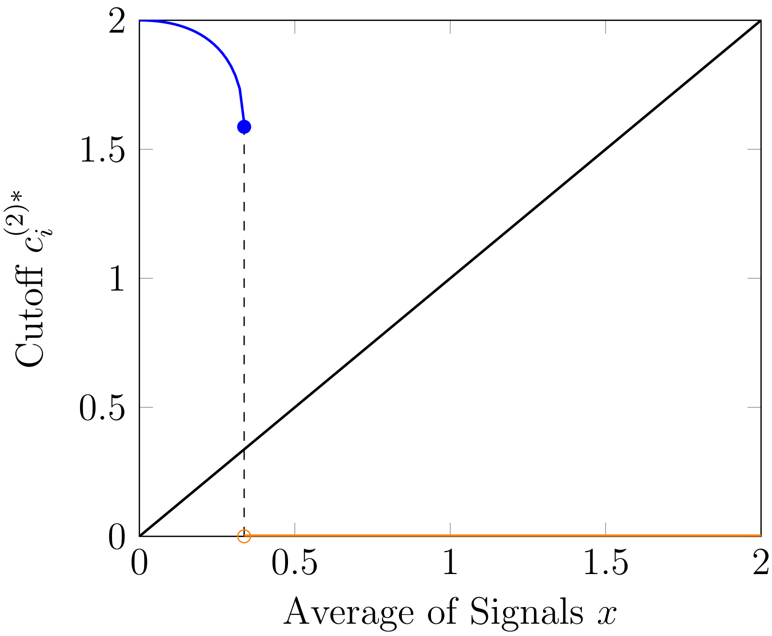

Figure II.6: Player i's Stage 2 equilibrium cutoffs versus their corresponding average of signal values.

II.3: Stage $3 \leq t \leq T$

In Stage t, players commonly remember the action profile history,

$$A^{(t-1)} = \{a_i^{(1)}, a_{-i}^{(1)}, \ldots, a_i^{(t-1)}, a_{-i}^{(t-1)}\}$$

. For $1 \leq \tau \leq t-1$, $a_i^{(\tau)}$ encodes information about Player i's average of signals, $X^{(\tau)}$ and their pessimistic equillibrium cutoff $c_i^{(\tau)*}$ in Stage τ. Let $I_i^{(\tau)} \subseteq (0,2)$ be the interval that $X^{(\tau)}$ lies within such that Player i chose action $a_i^{(\tau)}$. For example, if $a_i^{(\tau)} = 0$, then $X^{(\tau)} \leq c_i^{(\tau)*}$ and $I_i^{(\tau)} = (0, c_i^{(\tau)*}]$.

Both players commonly update their beliefs about θ given the publicly observed history of action profiles. The Stage t posterior probability distribution of θ given $A^{(t-1)}$ is

$$f_\theta^{(t)}(\omega) = \mathbf{1}_{g(A^{(t-1)})}(\omega) \cdot \frac{\mathbf{P}(A^{(t-1)} : \theta = \omega)}{\mathbf{P}(A^{(t-1)})}, \tag{II.3.1}$$

where

$$\text{(II.3.2)} \quad g(A^{(t-1)}) = \bigcap_{i=1,2} \bigcap_{\tau=1}^{t-1} \left\{ \omega \in [0,1] : \text{Given } \theta = \omega, \exists \{\epsilon_i^{(s)}\}_{s=1}^{\tau} \text{ such that } X^{(\tau)} \in I_i^{(\tau)} \right\}.$$

There are two situations where $g(A^{(t-1)}) \subsetneq [0,1]$:

- For at least one player, if there is at least one stage $1 \leq \tau \leq t-1$ where $c_i^{(\tau)*} \geq 1$ and $a_i^{(\tau)} = 1$, then $g(A^{(t-1)}) \subseteq (c_i^{(\tau)*} - 1, 1]$.
- For at least one player, if there is at least one stage $1 \leq \tau \leq t-1$ where $c_i^{(\tau)*} < 1$ and $a_i^{(\tau)} = 0$, then $g(A^{(t-1)}) \subseteq [0, c_i^{(\tau)*}]$.

By Theorem II.3, $g(A^{(2)}) = [0,1]$.

The collection of Player i's and Player $-i$'s average of signals up to Stage $1, \ldots, t-1$, $X^{(1)}, \ldots, X^{(t-1)}, Y^{(1)}, \ldots, Y^{(t-1)}$ given $\theta = \omega$ are transforms of the collection of noise terms $\epsilon_i^{(1)}, \ldots, \epsilon_i^{(t-1)}, \epsilon_{-i}^{(1)}, \ldots, \epsilon_{-i}^{(t-1)}$, such that for $2 \leq \tau \leq t-1$

$$\begin{aligned} \epsilon_i^{(\tau)} &= \tau X^{(\tau)} - (\tau - 1) X^{(\tau-1)} - \omega \\ \epsilon_{-i}^{(\tau)} &= \tau Y^{(\tau)} - (\tau - 1) Y^{(\tau-1)} - \omega, \end{aligned}$$

and $\epsilon_i^{(1)} = X^{(1)} - \omega, \epsilon_{-i}^{(1)} = Y^{(1)} - \omega$. Recall that for $i = 1,2$ and for all $1 \leq \tau \leq t-1$, $\epsilon_i^{(\tau)}$ is uniformly distributed over $[0,1]$. Therefore the joint probability distribution of all of Player i's and Player $-i$'s average of signals is

$$f(\vec{x}, \vec{y} : \omega) = ((t-1)!)^2 \cdot \prod_{\tau=1}^{t-1} \mathbf{1}_{(0,1)}(\tau x_\tau - (\tau-1)x_{\tau-1} - \omega) \cdot \mathbf{1}_{(0,1)}(\tau y_\tau - (\tau-1)y_{\tau-1} - \omega),$$

where $\vec{x} = (x_1, \ldots, x_{t-1}), \vec{y} = (y_1, \ldots, y_{t-1})$, and $((t-1)!)^2$ is the Jacobian of the linear transformation which takes the collection of the average of signals to the collection of noise terms. One can express the probability that $A^{(t-1)}$ occurs given $\theta = \omega$ as

$$\text{(II.3.3)} \quad \mathbf{P}(A^{(t-1)} : \theta = \omega) = ((t-1)!)^2 \cdot \int_{I_i^{(1)} \times \cdots \times I_i^{(t-1)}} \prod_{\tau=1}^{t-1} \mathbf{1}_{(0,1)}(\tau x_\tau - (\tau - 1)x_{\tau-1} - \omega)\, d\vec{x} \cdot \int_{I_{-i}^{(1)} \times \cdots \times I_{-i}^{(t-1)}} \prod_{\tau=1}^{t-1} \mathbf{1}_{(0,1)}(\tau y_\tau - (\tau - 1)y_{\tau-1} - \omega)\, d\vec{y}.$$

In Stage t, Player i remembers their Stage $t-1$ average of signals. Player i privately observes a new signal $x_i^{(t)}$ and obtains a new average of signals, $X^{(t)}$. Player i is indifferent between actions “0” and action “1” if

$$\text{(II.3.4)} \qquad \mathbf{E}[\theta : A^{(t-1)}, X^{(t)}] - \mathbf{P}(a_{-i}^{(t)} = 0 : A^{(t-1)}, X^{(t)}) = 0,$$

where Player i has the posterior belief that θ is distributed with probability density function $f_\theta^{(t)}$. The expected value of θ given history $A^{(t-1)}$ and $X^{(t)} = x$ is

$$\text{(II.3.5)} \qquad \mathbf{E}[\theta : A^{(t-1)}, X^{(t)} = x] = \frac{\displaystyle\int_{g(A^{(t-1)})} \omega \cdot B_t(x-\omega) \cdot \mathbf{P}(A^{(t-1)} : \theta = \omega)\, d\omega}{\displaystyle\int_{g(A^{(t-1)})} B_t(x-\omega) \cdot \mathbf{P}(A^{(t-1)} : \theta = \omega)\, d\omega}.$$

If Player $-i$ chooses a cutoff value of $c_{-i}^{(t)} = c$, then the probability that Player i assigns to the event that Player $-i$ chooses the low action, given $A^{(t-1)}$ and their new average of signals, is

(II.3.6)

$$\mathbf{P}(Y^{(t)} \le c : A^{(t-1)}, X^{(t)} = x) = \frac{\displaystyle\int_0^c \int_{g(A^{(t-1)})} f_{X^{(t)},Y^{(t)}:\theta}(x, y : \omega) \cdot \mathbf{P}(A^{(t-1)} : \theta = \omega)\, d\omega\, dy}{\displaystyle\int_{g(A^{(t-1)})} B_t(x-\omega) \cdot \mathbf{P}(A^{(t-1)} : \theta = \omega)\, d\omega},$$

where $f_{X^{(t)},Y^{(t)}:\theta}(x, y : \omega) = B_t(x-\omega) \cdot B_t(y-\omega)$.

II.3.1: Stage t Pessimistic Equilibrium Cutoff Strategy

In Stage $t \geq 3$, Player i selects their pessimistic equilibrium cutoff as they do in Stage 2. Let $X^{(t)} = x$ be the average of Player i's signals up to Stage t. $c_i^{(t)*}$ is the average of signal values that satisfies $\mathbf{E}[\theta : A^{(t-1)}, X^{(t)} = c_i^{(t)*}] = c^*$, where c^* satisfies $c^* = \mathbf{P}(Y^{(t)} \leq c^* : A^{(t-1)}, X^{(t)} = x)$. If multiple c^*'s exist, then Player i selects the cutoff $c_i^{(t)*}$ corresponding to the maximal fixed point of $\mathbf{P}(Y^{(t)} \leq c : A^{(t-1)}, X^{(t)} = x)$.

Finding the corresponding cutoff for a given fixed point satisfying $c^* = \mathbf{P}(Y^{(t)} \leq c^* : A^{(t-1)}, X^{(t)})$ requires that the Stage t expected value of θ conditional on $X^{(t)} = x$ be an invertible function of x.

Claim II.4. For $1 \leq t \leq T$, $\mathbf{E}[\theta : A^{(t-1)}, X^{(t)} = x]$ is a strictly increasing function of x.

To prove this claim, it is sufficient to show that for all $t \geq 1$ and for all $u \in [0, 1]$, $\mathbf{P}(\theta > u : A^{(t-1)}, X^{(t)} = x)$ is increasing in x. This is because,

$$\begin{aligned}
\mathbf{E}[\theta : A^{(t-1)}, X^{(t)} = x] &= \int_0^1 \omega f_{\theta:X^{(t)}}(\omega : x)\, d\omega \\
&= \int_0^1 \left(\int_0^1 \mathbf{1}_{\omega > u}(u)\, du \right) f_{\theta:A^{(t-1)},X^{(t)}}(\omega : x)\, d\omega \\
&= \int_0^1 \left(\int_0^1 \mathbf{1}_{\omega > u}(\omega) f_{\theta:A^{(t-1)},X^{(t)}}(\omega : x)\, d\omega \right) du \\
&= \int_0^1 \mathbf{P}(\theta > u : A^{(t-1)}, X^{(t)} = x)\, du,
\end{aligned}$$

where $f_{\theta:A^{(t-1)},X^{(t)}}$ is the conditional density of θ given a history $A^{(t-1)}$ and average of signals $X^{(t)}$. This is equivalent to showing the underlying distribution of θ given $A^{(t-1)}$ and $X^{(t)} = x$ is *stochastically increasing* for all $x \in (0, 2)$, as defined by Topkis [Top98]. Intuitively, if Player i sees a high average of signals, they should expect θ to be high as well. $\mathbf{E}[\theta : A^{(t-1)}, X^{(t)} = x]$ is strictly increasing for Stages $t = 1, 2$ (see Equations (II.1.2) and (II.2.6)). In Stages 3 and beyond, the property above allows one to determine the cutoff corresponding to a fixed point of Equation II.3.6 and the average signal regimes where Player i always plays the high action or always plays the low action. In Chapter III, the Monte Carlo simulations of the game provide numerical evidence in support of this claim.

The following lemma informs the value of nonzero cutoffs, provided that they exist. To begin, the Bates cumulative distribution function with parameter n is

$$\bar{B}_n(x) = \begin{cases} 0, & x \leq 0, \\ \dfrac{1}{n!} \displaystyle\sum_{k=0}^{\lfloor nx \rfloor} (-1)^k \binom{n}{k} (nx - k)^n, & 0 < x < 1, \\ 1, & x \geq 1. \end{cases} \tag{II.3.7}$$

A key feature of $\bar{B}_n(x)$ is that it is strictly convex on $(0, 1/2)$ for all $n \geq 3$. This is because $B_n(x)$ is strictly increasing on the same interval. Similarly, $B_n(x)$ is symmetric about $x = 1/2$ and strictly decreases on $(1/2, 1)$ for all $n \geq 3$. Thus $\bar{B}_n(x)$ is strictly concave on this same interval, for the same parameter n.

Lemma II.5. *Let $A^{(t-1)}$ be a history of actions up to Stage $t-1$. For all possible values of Player i's average of signals, $X^{(t)}$, in Stage $t \geq 2$, if $0 < c < \frac{1}{2}$, then $\boldsymbol{P}(Y^{(t)} \leq c : A^{(t-1)}, X^{(t)} = x) < c$.*

Proof. (Lemma II.5) For $t = 2$, for fixed $c \in (0, 1/2)$, $2(c - \omega)^2 < c$ for all $\omega \in [0, c]$. Therefore, for fixed $c \in (0, 1/2)$,

$$\begin{aligned}
\mathbf{P}(Y^{(2)} \leq c : A^{(1)}, X^{(2)} = x) &= \frac{\displaystyle\int_{g(A^{(1)})} B_2(x-\omega) \cdot \bar{B}_2(c-\omega)\mathbf{P}(A^{(1)} : \theta = \omega)\, d\omega}{\displaystyle\int_{g(A^{(1)})} B_2(x-\omega)\mathbf{P}(A^{(1)} : \theta = \omega)\, d\omega} \\
&= \frac{\displaystyle\int_0^1 B_2(x-\omega) \cdot 2(c-\omega)^2 \mathbf{1}_{(0,c]}(\omega)\, d\omega}{\displaystyle\int_0^1 B_2(x-\omega)\, d\omega} \\
&< c \cdot \frac{\displaystyle\int_0^1 B_2(x-\omega)\, d\omega}{\displaystyle\int_0^1 B_2(x-\omega)\, d\omega} \\
&= c.
\end{aligned}$$

For $t \geq 3$, consider the multivariate function $g(c, \omega) = \bar{B}_t(c - \omega)$ on $[0, 1/2] \times [0, 1]$. For fixed $c \in (0, 1/2)$, $g(c, \omega)$ is strictly decreasing on the interval $\omega \in [0, c]$. For fixed $c \in (0, 1/2)$, $g(c, \omega) \leq g(c, 0) = \bar{B}_t(c)$ for all $\omega \in (0, c]$. Therefore, for a fixed $c \in (0, 1/2)$,

$$\begin{aligned}
\mathbf{P}(Y^{(t)} \leq c : A^{(t-1)}, X^{(t)} = x) &= \frac{\displaystyle\int_{g(A^{(t-1)})} B_t(x-\omega) \cdot \bar{B}_t(c-\omega)\mathbf{P}(A^{(t-1)} : \theta = \omega)\, d\omega}{\displaystyle\int_{g(A^{(t-1)})} B_t(x-\omega)\mathbf{P}(A^{(t-1)} : \theta = \omega)\, d\omega} \\
&\leq \frac{\displaystyle\int_{g(A^{(t-1)})} B_t(x-\omega) \cdot \bar{B}_t(c)\mathbf{P}(A^{(t-1)} : \theta = \omega)\, d\omega}{\displaystyle\int_{g(A^{(t-1)})} B_t(x-\omega)\mathbf{P}(A^{(t-1)} : \theta = \omega)\, d\omega} \\
&= \bar{B}_t(c) \\
&< c.
\end{aligned}$$

□

In other words, nonzero fixed points of the conditional probability in Equation (II.3.6) can only lie within the interval $[1/2, 1]$ for Stages 2 and onward. Furthermore, if Player i sees a signal that is sufficiently high, then their pessimistic equilibrium cutoff is 0. The lemma below bounds the probability in Equation (II.3.6) in the case where Player $-i$ chooses a cutoff in the interval $[1/2, 1]$.

Lemma II.6. *Suppose $1/2 \leq c \leq 1$ and $t \geq 3$. Let $A^{(t-1)}$ be a history of actions up to Stage t. If Player i's average of signals in Stage t, $X^{(t)} = x$, lies within the interval $[\underline{x}^{(t)}(c) + 1, 2)$, where $\bar{B}_t(c - \underline{x}^{(t)}(c)) = c$, then $\boldsymbol{P}(Y^{(t)} \leq c : A^{(t-1)}, X^{(t)} = x) < c$.*

Proof. (Lemma II.6) For $t \geq 3$, fix $1/2 \leq c \leq 1$, and let $g(c, \omega) = \bar{B}_t(c - \omega)$. Because $\bar{B}_t(\cdot)$ is strictly concave on $(1/2, 1)$, it follows that $c < g(c, 0) = \bar{B}_t(c)$ (note: $g(1/2, 0) = 1/2$ and $g(1, 0) = 1$). $g(c, \omega)$ takes on values in the interval $[0, \bar{B}_t(c)]$ and is continuous for all $\omega \in [0, c]$. By the intermediate value theorem, there must be a $\underline{\mathrm{x}}^{(t)}(c) \in [0, c]$ such that $g(c, \underline{\mathrm{x}}^{(t)}(c)) = c$.

Let $1 < x < 2$. For all $\omega \in (\underline{\mathrm{x}}^{(t)}(c), c]$, $g(c, \omega) < c$. Recall that the support of $B_t(x)$ is $(x - 1, 1)$. The upper integrand of Equation (II.3.6) is nonzero for all $\omega \in (x-1, x) \cap [0, c] = (x-1, c]$. Therefore, if $x \geq \underline{\mathrm{x}}^{(t)}(c) + 1$, $g(c, \omega) < c$ for all $\omega \in (x-1, c]$ and the conditional probability in Equation (II.3.6) will be strictly less than c.

□

Theorem II.7. *Let $t \geq 3$ and $\underline{x}^{(t)} = \max\limits_{\frac{1}{2} \leq c \leq 1} \{\underline{x}^{(t)}(c)\}$. Let $A^{(t-1)}$ be as history of actions up to Stage $t-1$. If Player i's average of signals in Stage t, $X^{(t)} = x$, is within $[\underline{x}^{(t)} + 1, 2)$, then $\boldsymbol{P}(Y^{(t)} \leq c : X^{(t)} = x) < c$ for all $0 < c \leq 1$.*

Proof. (Theorem II.7) Let $t \geq 3$. By Lemma II.6, for all $c \in [1/2, 1]$, there exists an $\underline{\mathrm{x}}^{(t)}(c)$ such that if Player i's average of signals in Stage t is in $[\underline{\mathrm{x}}^{(t)}(c) + 1, 2)$, then $\mathbf{P}(Y^{(t)} \leq c : A^{(t-1)}, X^{(t)} = x) < c$. Let $\underline{\mathrm{x}}^{(t)}$ be the maximum over the collection of zeros, $\{\underline{\mathrm{x}}^{(t)}(c)\}_{1/2\leq c\leq 1}$, for Stage t only. Therefore, by Lemma II.6,

$$
\begin{aligned}
x-1 &\geq \underline{\mathrm{x}}^{(t)} \\
\Rightarrow x-1 &\geq \underline{\mathrm{x}}^{(t)}(c),\ \forall\, c \in [1/2,1] \\
\Rightarrow \mathbf{P}(Y^{(t)} \leq c : A^{(t-1)}, X^{(t)} = x) &< c,\ \forall\, c \in [1/2,1].
\end{aligned}
$$

By Lemma II.5, for all $x \in (0,2)$ and $c \in (0,1/2)$, $\mathbf{P}(Y^{(t)} \leq c : A^{(t-1)}, X^{(t)} = x) < c$. $\square$

This theorem does not exclude the possibility that 0 is Player i's pessimistic equilibrium cutoff when $0 < x < \underline{x}^{(t)} + 1$. However, if one can determine the value of $\underline{x}^{(t)}$, then one need only consider this range of average of signal values in each stage when searching for nonzero fixed points and corresponding cutoffs.

CHAPTER III

Monte Carlo Integration Methods

Here I construct Monte Carlo estimators for the relevant integrals in Equations (II.3.5) and (II.3.6). The choice of a probabilistic integration method over a quadrature rule is motivated by the construction of the Stage t public posterior probability distribution of θ, $f_{\theta}^{(t)}(\omega)$, in Equation (II.3.1). The dimension of the integrating region for $\mathbf{P}(A^{(t-1)} : \theta = \omega)$ increases by 2 each subsequent stage; furthermore, a deterministic numerical integration method would have to account for the supports of probability densities $f_{X^{(1)},\ldots,X^{(t-1)}:\theta}(\vec{x} : \omega)$ and $f_{Y^{(1)},\ldots,Y^{(t-1)}:\theta}(\vec{y} : \omega)$. In order to approximate the pessimistic equilibrium cutoff in Stage t, I employ three integral estimators. Let $N \geq 1$ be the number of samples used in the estimators; the number of samples will be the same for each of the estimators. For fixed $0 < x < 2$

$$\text{(III.0.1)} \qquad \int_{g(A^{(t-1)})\times\mathcal{I}} f(\vec{v}; x)\, d\vec{v} \approx \frac{1}{N}\sum_{n=1}^{N} \frac{f(\vec{v}_n; x)}{p(\vec{v}_n; x)},$$

where $\mathcal{I} = I_i^{(1)} \times \cdots \times I_i^{(t-1)} \times I_{-i}^{(1)} \times \cdots \times I_{-i}^{(t-1)}$, $\vec{v} = (\omega, x_1, \cdots, x_{t-1}, y_1, \cdots, y_{t-1})$, and

$$f(\vec{v}; x) = B_t(x-\omega)\prod_{\tau=1}^{t-1} \mathbf{1}_{(0,1)}(\tau x_\tau - (\tau-1)x_{\tau-1} - \omega) \cdot \mathbf{1}_{(0,1)}(\tau y_\tau - (\tau-1)y_{\tau-1} - \omega).$$

$\vec{v}_n$ is drawn from the probability distribution $p(\vec{v}; x)$. The sum above approximates the denominators of Equations (II.3.5) and (II.3.6). For the numerator of (II.3.5), the same distribution is used to sample $\vec{v}_n$:

$$\text{(III.0.2)} \qquad \int_{g(A^{(t-1)})\times\mathcal{I}} \omega \cdot f(\vec{v};x)\,d\vec{v} \approx \frac{1}{N}\sum_{n=1}^{N} \frac{\omega_n \cdot f(\vec{v}_n;x)}{p(\vec{v}_n;x)}.$$

For the numerator of (II.3.6), and for fixed $0 < x < 2, 0 \leq c \leq 1$, and $N \geq 1$,

$$\text{(III.0.3)} \qquad \int_{g(A^{(t-1)})\times\mathcal{I}\times[0,c]} h(\vec{z};x)\,d\vec{z} \approx \frac{1}{N}\sum_{n=1}^{N} \frac{h(\vec{z}_n;x)}{q(\vec{z}_n;x,c)},$$

where $\vec{z} = (\omega, x_1, \cdots, x_{t-1}, y_1, \cdots, y_{t-1}, y)$ and

$$\begin{aligned} h(\vec{z};x) = B_t(x-\omega)B_t(y-\omega)\prod_{\tau=1}^{t-1}\big(&\mathbf{1}_{(0,1)}(\tau x_\tau - (\tau-1)x_{\tau-1} - \omega) \\ &\cdot\mathbf{1}_{(0,1)}(\tau y_\tau - (\tau-1)y_{\tau-1} - \omega)\big). \end{aligned}$$

$\vec{z}_n$ is drawn from the probability distribution $q(\vec{z}_n;x,c)$. As the number of samples N approaches ∞, each of the sums converges almost surely to the integral they approximate.

III.1: Multivariate Uniform Sampling

Probability distributions p, q are chosen in order to minimize the variance in each of the sums in Equations (III.0.1), (III.0.2), and (III.0.3). I use MATLAB to generate visualizations of approximations to Player i's Stage 2 conditional expectation and conditional probability. Firstly, consider when $\vec{v}, \vec{z}$ are chosen uniformly such that each coordinate is sampled independent of all others. Generate the sets of points $\{\xi_j\}_{j=1}^{50}, \{c_k\}_{k=1}^{50}$ that are linearly spaced on $(0,2)$ and $[0,1]$, respectively. Fix j; for all k I obtain $N = 10000$ summand samples for

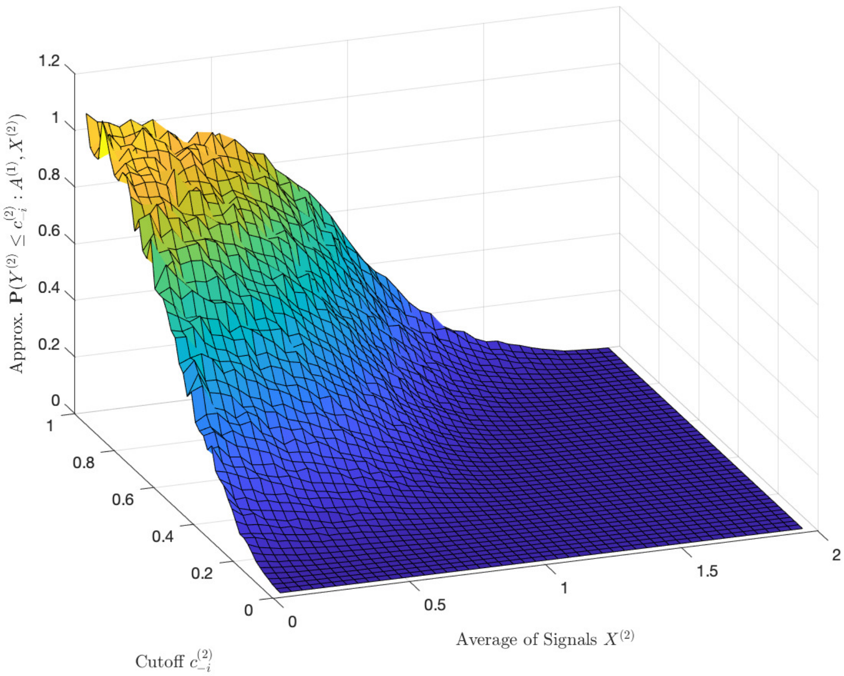

Figure III.1: Monte Carlo approximation to Stage 2 conditional probabilities for fifty values of $X^{(2)}$ and fifty values of $c_{-i}^{(2)}$, $N = 10000$ uniform samples.

the estimators described above using the following functions:

$$
\begin{aligned}
f(\omega, x_1, y_1; x) &= B_2(x-\omega) \cdot \mathbf{1}_{(0,1)}(x_1 - \omega) \cdot \mathbf{1}_{(0,1)}(y_1 - \omega) \\
p(\omega, x_1, y_1; x) &= \frac{1}{2 \cdot 2} \mathbf{1}_{[0,1]}(\omega) \cdot \mathbf{1}_{(0,2)}(x_1) \cdot \mathbf{1}_{(0,2)}(y_1) \\
h(\omega, x_1, y_1, y; x) &= B_2(x-\omega) \cdot B_2(y-\omega) \cdot \mathbf{1}_{(0,1)}(x_1 - \omega) \cdot \mathbf{1}_{(0,1)}(y_1 - \omega) \\
q(\omega, x_1, y_1, y; x, c) &= \frac{1}{2 \cdot 2 \cdot c} \mathbf{1}_{[0,1]}(\omega) \cdot \mathbf{1}_{(0,2)}(x_1) \cdot \mathbf{1}_{(0,2)}(y_1) \cdot \mathbf{1}_{[0,c]}(y).
\end{aligned}
$$

Figure III.1 shows the resulting Monte Carlo approximations of $\mathbf{P}(Y^{(2)} \leq c_{-i}^{(2)} = c : A^{(1)}, X^{(2)} = x)$ represented as a surface plot over $\{(x, c) \in (0,2) \times [0,1]\}$. Figure III.2 gives the surface generated using the piecewise-defined Stage 2 conditional probability over the same region (detailed in APPENDIX C). Any Monte Carlo method used will have variance in the approximated integral values. I quantify this by finding the mean squared error for each ξ_j; Figure III.4 shows the mean squared errors for each method. I

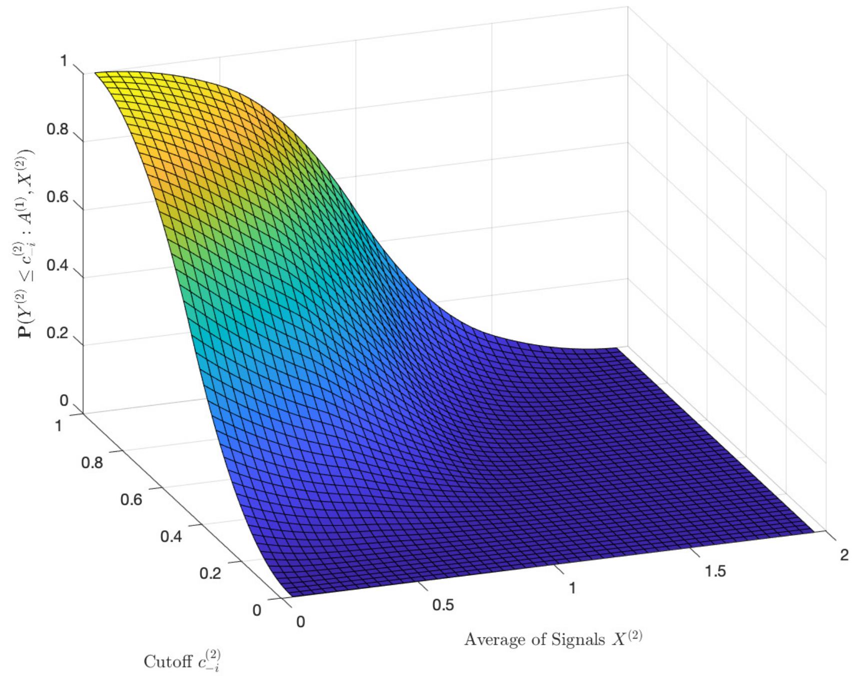

Figure III.2: Stage 2 conditional probabilities for fifty average signal values and fifty cutoffs for Player $-i$.

take the "actual" value of the conditional probability to be the evaluation of expressions in APPENDIX C at (ξ_j, c_k). The red data points correspond to the mean squared error using independent uniform coordinate sampling; the blue data points correspond to a modified sampling scheme detailed below. Compared to this modified method, independent uniform coordinate sampling leads to higher variance for the lower average of signal values. The primary reason for approximating the conditional probabilities as functions of c in Stage t is to approximate the fixed points of this function. Recall that in Stage 2, two nonzero fixed point branches exist for a low-valued regime of $X^{(2)}$ values.

III.2: Modified Sampling

The main idea of the modified sampling scheme below is to choose p, q that are similar to the integrand of $\mathbf{P}(A^{(t-1)} : \theta = \omega)$. For a fixed value of x, ω is sampled uniformly but on an interval where $B_t(x - \omega) > 0$. For all $1 \le \tau \le t - 1$, Algorithm 1 then checks if it is possible

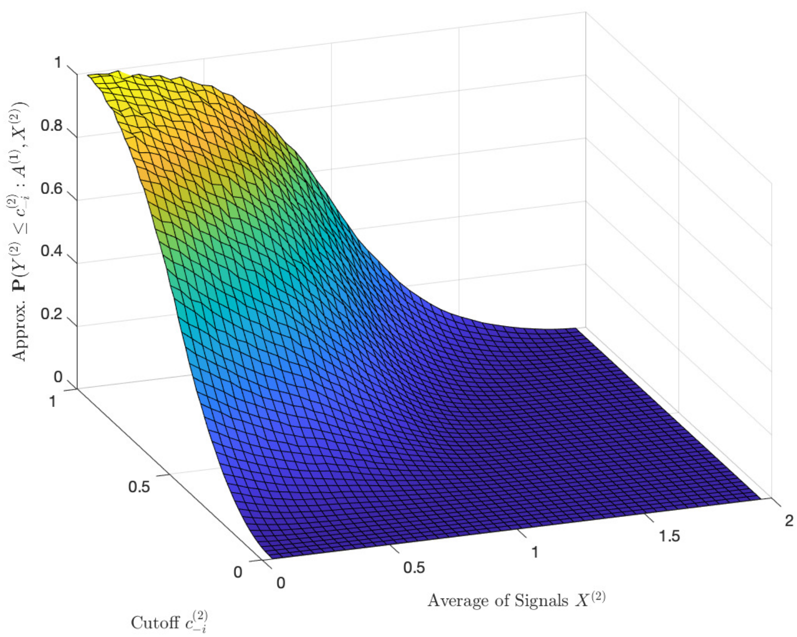

Figure III.3: Monte Carlo approximation to Stage 2 conditional probabilities for 50 values of $X^{(2)}$ and 50 values of $c_{-i}^{(2)}$, $N = 10000$ modified samples.

to sample x_τ, y_τ such that $\mathbf{1}_{(0,1)}(\tau x_\tau - (\tau - 1)x_{\tau-1} - \omega)$ and $\mathbf{1}_{(0,1)}(\tau y_\tau - (\tau - 1)y_{\tau-1} - \omega)$ are both 1. If it is not possible to sample $\omega, \{x_\tau\}, \{y_\tau\}$ in such a way, the algorithm returns a sample for Equation (III.0.1) equal to 0. Otherwise, x_τ is sampled uniformly on an interval dependent on a linear combination of $x_{\tau-1}$ and θ; y_τ is sampled in a similar way. This process generates a sample for the denominators in Equations (III.0.1), (III.0.2), (III.0.3), and the numerator in Equation (III.0.2). Finally, in order to generate a sample for the numerator in Equation (III.0.3), the algorithm checks if y sampled uniformly on an interval dependent on c_k and the previously sampled ω gives $B_t(y - \omega) > 0$. The following algorithm describes how the nth sample for all the estimators are generated.

Algorithm 1 Modified Monte Carlo Sampling

Require: $\xi_j, c_k, t, g(A^{(t-1)}), I_i^{(1)}, \dots, I_i^{(t-1)}, I_{-i}^{(1)}, \dots, I_{-i}^{(t-1)}$
Ensure: ω_n, f_n, h_n
  $f_n \leftarrow 0$
  $p_n^{-1} \leftarrow 0.$ ▷ p_n^{-1} corresponds to the sampling distribution for Equations (III.0.1) and (III.0.2).
  $\Omega_n \leftarrow (\xi_j - 1, \xi_j) \cap g(A^{(t-1)})$.
  if $|\Omega_n| \neq 0$ **then** ▷ $|I|$ indicates the length of the interval I.
    Sample ω_n uniformly on Ω_n.
    $f_n \leftarrow B_t(\xi_j - \omega_n)$. ▷ f_n will be 0 if $(\xi_j - \omega_n) \notin (0, 1)$.
    $p_n^{-1} \leftarrow |\Omega_n|$.
  end if
  $\tau \leftarrow 1$.
  while $f_n \neq 0$ & $\tau \leq t-1$ **do**
    $I_i \leftarrow I_i^{(\tau)} \cap (\frac{\tau-1}{\tau}(x_{\tau-1})_n + \frac{\tau-1}{\tau}\omega_n, \frac{\tau-1}{\tau}(x_{\tau-1})_n + \frac{\tau-1}{\tau}\omega_n + \frac{1}{\tau})$. ▷ $(x_0)_n = 0$ for all n.
    $I_{-i} \leftarrow I_{-i}^{(\tau)} \cap (\frac{\tau-1}{\tau}(y_{\tau-1})_n + \frac{\tau-1}{\tau}\omega_n, \frac{\tau-1}{\tau}(y_{\tau-1})_n + \frac{\tau-1}{\tau}\omega_n + \frac{1}{\tau})$. ▷ $(y_0)_n = 0$ for all n.
    if $|I_i| \neq 0$ & $|I_{-i}| \neq 0$ **then**
      Sample x_τ uniformly on I_i.
      Sample y_τ uniformly on I_{-i}.
      $p_n^{-1} \leftarrow |I_i| \cdot |I_{-i}|$.
    else
      $f_n \leftarrow 0$.
    end if
    $\tau \leftarrow \tau + 1$.
  end while
  $f_n \leftarrow f_n \cdot p_n^{-1}$. ▷ This is the nth sample for Equation (III.0.1).
  if $|[0, c_k] \cap (\omega, \omega+1)| \neq 0$ **then**
    Sample y_n uniformly on $[0, c_k] \cap (\omega, \omega+1)$.
    $h_n \leftarrow |[0, c_k] \cap (\omega, \omega+1)| \cdot B_t(y_n - \omega_n) \cdot f_n$. ▷ This is the nth sample for Equation (III.0.3).
  else
    $h_n \leftarrow 0$.
  end if

In short, the modified sampling distributions for the estimators are given by

$$
\begin{aligned}
p(\vec{v};x) &= \frac{1}{|\Omega(x)|}\mathbf{1}_{\Omega(x)}(\omega)\cdot\prod_{\tau=1}^{t-1}\frac{1}{|I_i(x_{\tau-1},\omega)|}\mathbf{1}_{I_i(x_{\tau-1},\omega)}(x_\tau)\cdot\frac{1}{|I_{-i}(y_{\tau-1},\omega)|}\mathbf{1}_{I_{-i}(y_{\tau-1},\omega)}(y_\tau)\\
q(\vec{z};x,c) &= \frac{1}{|[0,c]\cap(\omega,\omega+1)|}\mathbf{1}_{[0,c]\cap(\omega,\omega+1)}(y)\cdot p(\vec{v};x),
\end{aligned}
$$

where

$$
\begin{aligned}
\Omega(x) &= (x-1,x)\cap g(A^{(t-1)}),\\
I_i(x_{\tau-1},\omega) &= I_i^{(\tau)}\cap\left(\frac{\tau-1}{\tau}(x_{\tau-1}+\omega),\frac{\tau-1}{\tau}(x_{\tau-1}+\omega)+\frac{1}{\tau}\right),\\
I_{-i}(y_{\tau-1},\omega) &= I_{-i}^{(\tau)}\cap\left(\frac{\tau-1}{\tau}(y_{\tau-1}+\omega),\frac{\tau-1}{\tau}(y_{\tau-1}+\omega)+\frac{1}{\tau}\right),
\end{aligned}
$$

provided that none of the sets above are empty. The algorithm above checks that these sets are nonempty and constructs the normalizing factors such that the p, q are probability densities. Figure III.3 shows the resulting approximation of $\mathbf{P}(Y^{(2)} \le c : A^{(1)}, X^{(2)} = x)$, represented as a surface over a 50-by-50 grid of (ξ_j, c_k) points in $(0,2)\times[0,1]$. The approximated surface appears more like the true surface represented in Figure III.2 when compared to the uniformly sampled approximation. Switching to this alternative method reduces the error for low ξ_j values by about a factor of 10^{-2}. Figure III.5 shows the approximated values of $\mathbf{E}[\theta : A^{(1)}, X^{(2)}]$ using both sampling methods with $N = 10000$ samples and 50 equally spaced values of ξ_j. The blue dots represent the modified sampling results and more closely follow the true black conditional expectation curve (generated with MATLAB's symbolic plotting tool); the mean squared error is about 5.6011×10^{-5}. In contrast, the red circles from the uniform sampling method shows that there is a noticeable difference from the true values of the conditional expectation, with a corresponding mean squared error of about 6.1902×10^{-4}.

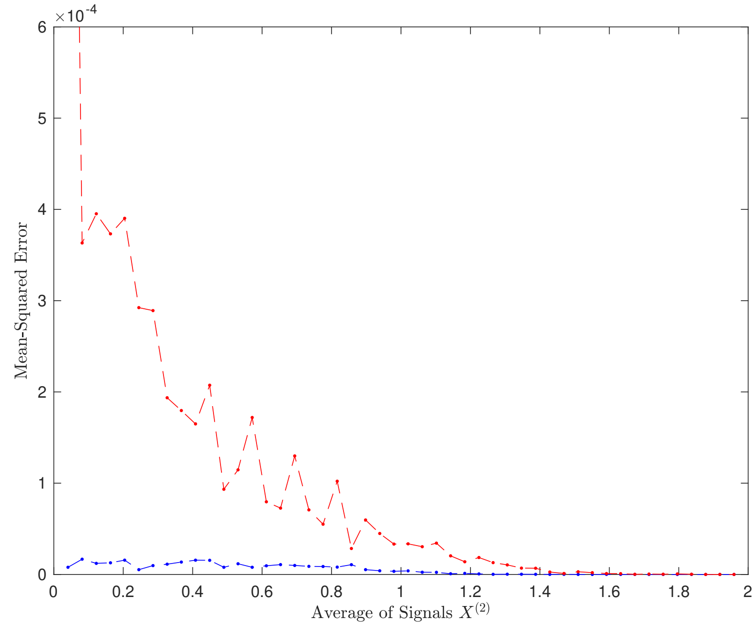

Figure III.4: Mean squared errors associated with the uniform-sampling (red) and modified-sampling (blue) Monte Carlo integral estimator methods, with $N = 10000$ samples for each method, for $\mathbf{P}(Y^{(2)} \leq c : A^{(1)}, X^{(2)} = x)$.

III.2.1: Stage 2 Approximate Pessimistic Equilibrium Cutoffs

For each ξ_j I fit a polynomial to the data approximating $\mathbf{P}(Y^{(t)} \leq c : A^{(t-1)}, X^{(t)})$. Using this polynomial, $\bar{p}(c; \xi_j)$, I obtain approximations to the relevant fixed points by numerically solving $\bar{p}(c; \xi_j) = c$; the maximum of the set of fixed points obtained, $\bar{c}^*(\xi_j)$, is the desired approximation. I perform a similar fitting process to the data approximating $\mathbf{E}[\theta : A^{(t-1)}, X^{(t)} = x]$, where x is variable. Using this polynomial, $\bar{e}(x)$, I finally obtain an approximation to the pessimistic cutoff when Player i sees $X^{(2)} = \xi_j$ by solving $\bar{e}(x) = \bar{c}^*(\xi_j)$.

Figure III.6 shows the approximate Stage 2, using $N = 10000$ modified samples and 50 values of ξ_j and c_k. For each ξ_j, MATLAB functions `polyfit` and `roots` are used on a tenth-degree polynomial, $\bar{p}(c; \xi_j) = c$, fitting the Stage 2 conditional probability. Like in Figure II.4, this process generates a high and low branch of nonzero fixed points; appropriately, 0 is always a fixed point for each ξ_j. The black dashed curve is the actual high branch of fixed points obtained from Equation (II.2.10). Figure III.7 shows Player i's corresponding

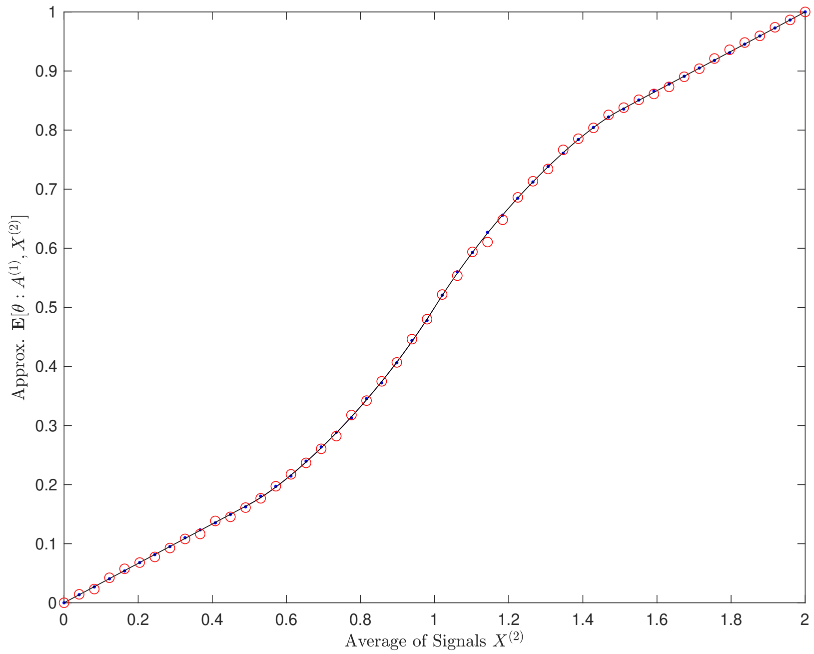

Figure III.5: Approximate conditional expectation of θ given $X^{(2)}$ and $A^{(1)}$ generated with uniform (red) and modified (blue) sampling Monte Carlo estimators, with $N = 10000$ samples for each method, plotted against the exact conditional expectation curve.

approximate pessimistic cutoffs. Again, the black curve plots the actual cutoffs as in Figure II.6. The stray point near $x = 0$ corresponds to the low branch fixed point for that value of ξ_j; in reality, the pessimistic cutoff for this ξ_j should lie closer to the black curve.

III.3: Simulating Stage Game

All approximations and curve fitting processes are performed in MATLAB. Let $T \geq 1$ be the number of stages Players $i, -i$ play the game. Begin by randomly generating θ uniformly on $[0, 1]$ and a collection of T signals for each player, $\{x_i^{(\tau)}\}_{\tau=1}^T, \{x_{-i}^{(\tau)}\}_{\tau=1}^T$. For all Monte Carlo estimators, fix the number of samples N; I set $N = 100000$ for the results that follow. With probability 1, the Stage 1 actions will be $a_i^{(1)} = 1 = a_{-i}^{(1)}$ with equilibrium cutoffs $c_i^{(1)*} = 0 = c_{-i}^{(1)*}$.

At the beginning of the simulated Stage $t \in \{2, 3, \ldots, T\}$, generate the average of Player

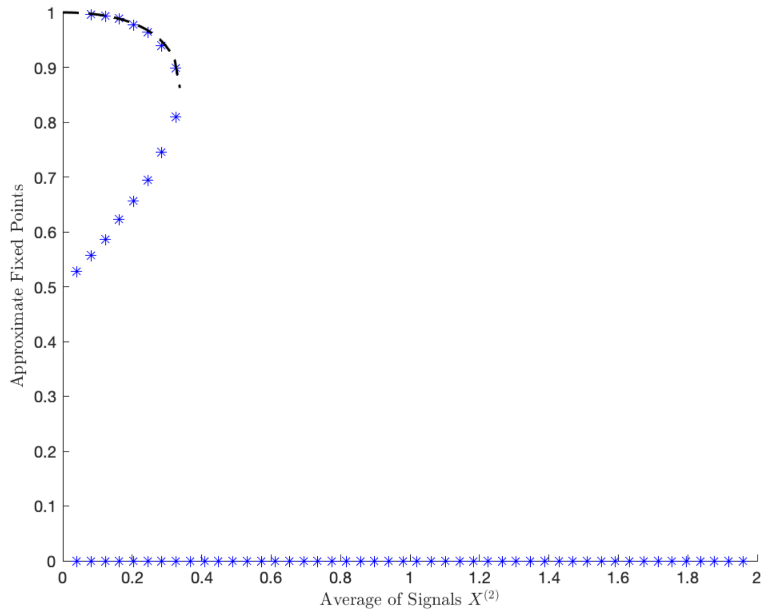

Figure III.6: Approximate fixed points of $\mathbf{P}(Y^{(2)} \leq c : A^{(1)}, X^{(2)} = x)$, generated with a tenth-degree polynomial fit, with $N = 10000$ modified samples.

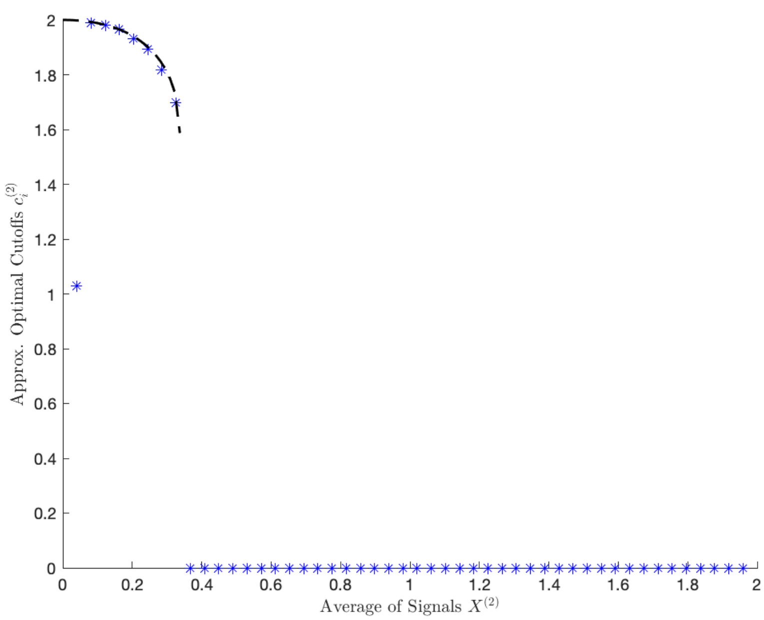

Figure III.7: Approximations to Player i's pessimistic cutoffs in Stage 2, generated with a tenth-degree polynomial fit, $N = 10000$ modified samples.

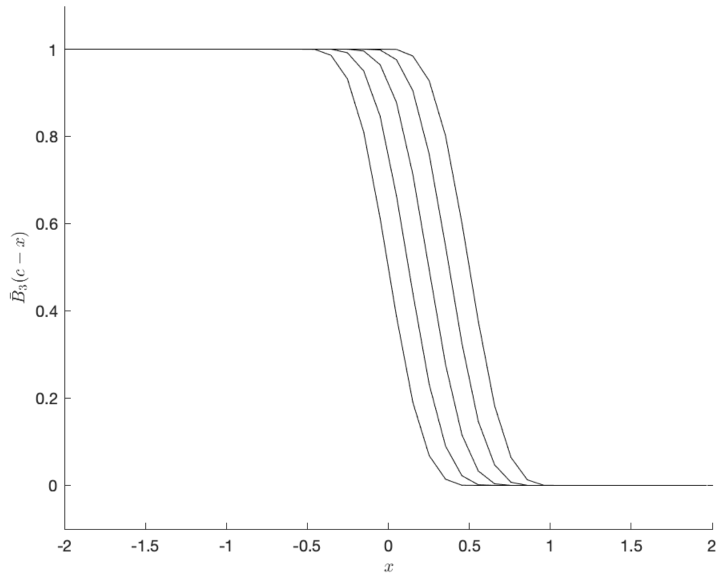

Figure III.8: Plots of $\bar{B}_3(c-x)$ for five different values of $c \in [1/2, 1]$.

$i, -i$'s signals up to Stage t, $X^{(t)}$ and $Y^{(t)}$. Furthermore, update $g(A^{(t-1)})$ based on the prior history of actions and optimal cutoffs. Recall Theorem II.7: If $X^{(t)} \geq \underline{x}^{(t)} + 1$, then Player i's action in Stage t will be $a_i^{(t)} = 1$ (almost surely) with cutoff $c_i^{(t)*} = 0$; the same condition holds for Player $-i$ and $Y^{(t)}$. Thus, before implementing Algorithm 1, one can approximate $\underline{x}^{(t)}$; if the inequality above holds for this approximation, one can bypass the algorithm and set the approximated cutoff to 0. Figure III.8 shows plots for $\bar{B}_3(c-x)$ for five different values of $c \in [1/2, 1]$. As c increases, $\bar{B}_3(c-x)$ shifts to the right for all x. Note the curve has a sigmoidal shape and maintains this shape as t increases.

Fix c_k, chosen from a twenty-five-point discretization of $[1/2, 1]$. Since $\bar{B}_t(c_k - x)$ is a piecewise continuous polynomial for $x \in (c_k - 1, c_k)$, fit a sigmoid-shaped curve whose function has the form

$$\bar{B}_t(c_k - x) \approx \frac{\alpha}{1 + \beta e^{\gamma x}},$$

to values of $\bar{B}_t(c_k - x)$ evaluated at points $\{\xi_j\}_{j=1}^{25}$, $\xi_j \in (0, 2)$. α, β, γ are all positive parameters generated with MATLAB's `fit` function via a nonlinear least squares regression. Next, solve the equation $\dfrac{\alpha}{1 + \beta e^{\gamma x}} = c_k$; the solution, $\underline{\xi}^{(t)}(c_k)$, approximates $\underline{x}^{(t)}(c_k)$. Repeat

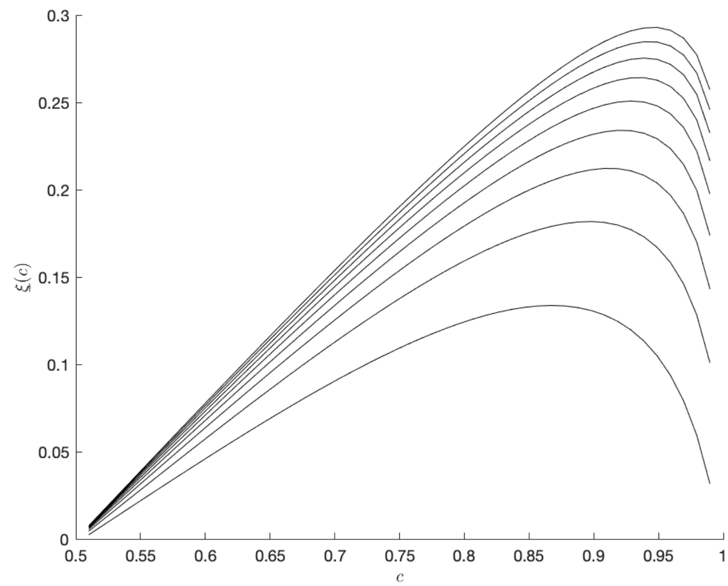

Figure III.9: Approximations to $\underline{x}(c)$ for twenty-five values of $c \in [1/2, 1]$ in Stage 2 through 10.

the fitting and solving processes above for all c_k and determine the maximum value of the collection of solutions, $\{\underline{\xi}^{(t)}(c_k)\}_{k=1}^{25}$; this maximum value approximates $\underline{x}^{(t)}$ for Stage t. Figure III.9 shows approximations to $\underline{x}^{(t)}(c)$ in Stages 2 through 10; for a fixed c_k, $\underline{\xi}^{(t)}(c_k)$ increases as t increases. Finally, I use this approximation to $\underline{x}^{(t)}$ in lieu of the actual value, as shown in Theorem II.7, for each player's average of signals in Stage t.

If Player i's average of signals, $X^{(t)} = x$, is strictly below the approximate value of $\underline{x}^{(t)}+1$, then the simulation proceeds with modified Monte Carlo sampling to determine Player i's approximate pessimistic equilibrium cutoff. In order to generate data for $\mathbf{P}(Y^{(t)} \leq c : A^{(t-1)}, X^{(t)})$, I implement Algorithm 1 for all c_k in a twenty-five-point discretization of the interval $[1/2, 1]$. Then, I fit a cubic polynomial, $\bar{p}(c; x)$, to data generated from the quotient of the estimator averages. Finally, the simulator selects the maximal solution to $\bar{p}(c; x) = c$ from solutions within $[1/2, 1]$.

If $\bar{c}^*(x) \neq 0$, then approximate the signal value, $c_i^{(t)*}$, such that the expected value of θ given $X^{(t)} = c_i^{(t)*}$ and a history $A^{(t-1)}$ is $\bar{c}^*(x)$. For this, I implement the modified sampling algorithm for $\{\xi_j\}_{j=1}^{25}$ taken from a discretization of $(0, 2)$. Note that c_k in Algorithm 1 can be any value since the Monte Carlo estimators for the integral components of $\mathbf{E}[\theta : A^{(t-1)}, X^{(t)}]$ do not depend on it. Next, I fit a tenth-degree polynomial curve, $\bar{e}$, to the Monte Carlo approximations of $\{\mathbf{E}[\theta : A^{(t-1)}, X^{(t)} = \xi_j]\}_{j=1}^{25}$. Then, solve the equation $\bar{e}(\bar{c}_i^{(t)*}) = \bar{c}^*(x)$; this gives an approximation to Player i's pessimistic equilibrium cutoff in Stage t. In order to obtain Player i's action in Stage t, compare $\bar{c}_i^{(t)*}$ and $X^{(t)}$.

III.3.1: Results

Starting with the approximations to $\mathbf{E}[\theta : A^{(t-1)}, X^{(t)}]$, the Monte Carlo simulations illustrate Claim II.4: The expected value of θ strictly increases as Player i's average of signals in Stage t increases; for every cutoff strategy employed by Player $-i$ in Stage t, there is a unique cutoff response for Player i. Figure III.10 shows two different approximations to $\mathbf{E}[\theta : A^{(t-1)}, X^{(t)}]$ for Stages 1 through 10. In both plots, for a fixed $X^{(t)}$, the expected value of θ increases as t increases, except when $X^{(t)}$ is fixed at 0 and 2; the plots also indicate that $\mathbf{E}[\theta : X^{(t)} = 1] = 1/2$ for all $1 \leq t \leq T$. In the simulation which generated the left plot, both players chose the low action in each stage; in the simulation reflected in the right plot, players chose combinations of both the high and low action throughout all the stages. Numerically, the absolute difference between a point in one plot and the corresponding point in the other is at most 2×10^{-3} (exactly zero if the points were generate for Stage 1 or if they are the endpoints of plot).

This suggests that when players both implement their pessimistic equilibrium strategies up to Stage t, a player's fixed points and corresponding cutoffs depend only on the average of signals up to that stage and $g(A^{(t-1)})$.

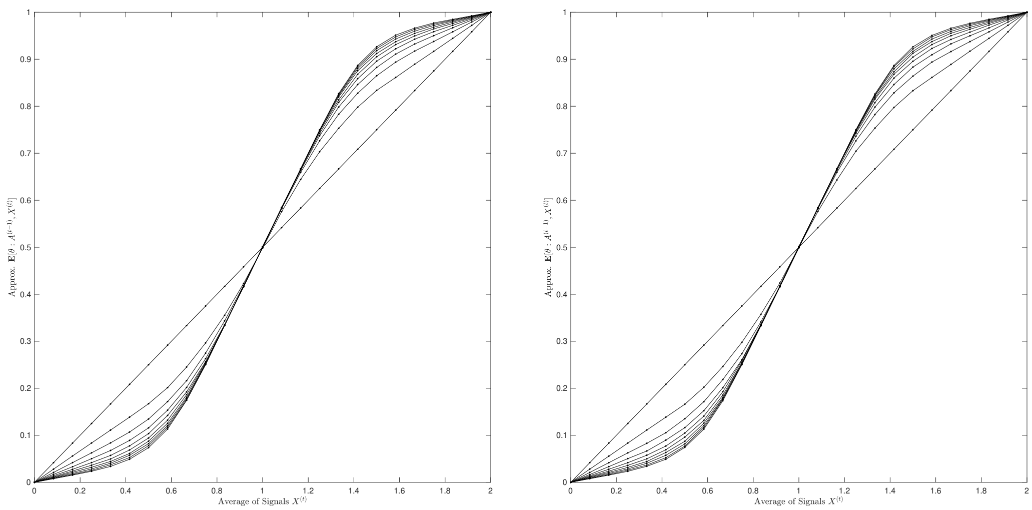

Figure III.10: Two different approximations of $\mathbf{E}[\theta : A^{(t-1)}, X^{(t)}]$ plotted against twenty-five different values of $X^{(t)}$ for Stages 1 through 10, $N = 100000$ samples.

Claim III.1. If Player i and $-i$ choose pessimistic cutoffs strategies up to Stage t, then

$$\mathbf{E}[\theta : A^{(t-1)}, X^{(t)}] = \frac{\displaystyle\int_{g(A^{(t-1)})} \omega \cdot B_t(x-\omega)\, d\omega}{\displaystyle\int_{g(A^{(t-1)})} B_t(x-\omega)\, d\omega},$$

$$\mathbf{P}(Y^{(t)} \le c_{-i}^{(t)} : A^{(t-1)}, X^{(t)}) = \frac{\displaystyle\int_0^{c_{-i}^{(t)}} \int_{g(A^{(t-1)})} B_t(x-\omega) B_t(y-\omega)\, d\omega\, dy}{\displaystyle\int_{g(A^{(t-1)})} B_t(x-\omega)\, d\omega}.$$

The claim above holds for Stage 2 (in fact, if players both play the high action with cutoff 0 in every stage before Stage t, then $\mathbf{P}(A^{(t-1)} : \theta = \omega) = 1$). The proof of Claim III.1 for a general stage hinges on showing the subintegrals of $\mathbf{P}(A^{(t-1)} : \theta = \omega)$ are constant for $\omega \in g(A^{(t-1)}) \cap (x-1, x)$, where x is the value of Player i's average of signals in Stage t. Recall that in Algorithm 1, ω_n is sampled uniformly on this interval. Consider a simplification to Algorithm 1 that excludes the `while`-loop representing a Monte Carlo estimator for $\mathbf{P}(A^{(t-1)} : \theta = \omega)$, where $\omega \in g(A^{(t-1)}) \cap (x-1, x)$. The implementation of the simplified algo-

t	$X^{(t)}$	$c_i^{(t)*}$	$Y^{(t)}$	$c_{-i}^{(t)*}$
1	0.8019	0	0.6665	0
2	0.7023	0	0.7049	0
3	0.6871	0	0.5778	0
4	0.6667	0	0.6360	0
5	0.7493	0	0.6561	1.6148
6	0.6435	1.8337	0.5772	1.8876
7	0.6024	1.8826	0.5145	1.9100
8	0.6254	1.8745	0.5517	1.8733
9	0.6749	1.8536	0.5808	1.8812
10	0.6417	1.8578	0.6036	1.8615

Table III.1: Approximate pessimistic cutoffs and average of signals for Players $i, -i$ in a simulated game with modified Monte Carlo sampling, $N = 100000$, $\theta \approx 0.1054$.

rithm for Stage t gives the same approximations to $\mathbf{E}[\theta : A^{(t-1)}, X^{(t)}]$ and $\mathbf{P}(Y^{(t)} \leq c_{-i}^{(t)} : A^{(t-1)}, X^{(t)})$ that the original algorithm generates if both Players choose the high action with cutoff 0 in all preceding stages.

If the claim above holds, then it follows that the support of $f_\theta^{(t)}$ in Equation (II.3.1) is never a proper subset of $[0, 1]$:

Claim III.2. If players choose pessimistic cutoff strategies in all stages, then $g(A^{(t)})$ is $[0, 1]$ for all $2 \leq t \leq T$.

Claim III.2 is further supported by the original modified Monte Carlo sampling implementation. The threshold of averages that generate nonzero cutoffs appears to be within the interval $(0, 1)$. By Lemma II.5, nonzero fixed points can only lie within the interval $[1/2, 1]$. As shown by Figure III.10, the corresponding cutoffs appear to lie within $[1, 2)$. The original Monte Carlo simulations with modified sampling suggest Player i behaves the same as in Stage 2: When they see a sufficiently low signal, their pessimistic cutoff will be high enough such that they play the low action; otherwise, they play the high action with equilibrium cutoff 0. Table III.1 lists the results of such a game where both Players choose the high action and low action throughout the course of the $T = 10$ stages. If Claim III.1 is true, then each players' choice of cutoff in Stage t is dependent only on their average of signals up to that stage; the average of signal thresholds, similar to the illustration in Figure II.6, would be the same for both players in each stage.

The simulations above indicate that when a player chooses a pessimistic cutoff in a particular stage, then they will play the low action (their average of signals in the stage will be less than or equal to their chosen cutoff).

Claim III.3. For every Stage $t \geq 1$, if $X^{(t)} \leq 1$ and if there exists a nonzero fixed point of Equation (II.3.6), then Player i will play action $a_i^{(t)} = 0$.

If Claim III.1 is true, then

$$\begin{aligned}
\mathbf{E}[\theta : A^{(t-1)}, X^{(t)} = 1] &= \frac{\displaystyle\int_0^1 \omega B_t(1-\omega)\,d\omega}{\displaystyle\int_0^1 B_t(1-\omega)\,d\omega} \\
&= \frac{\displaystyle\int_0^1 (1-\omega) B_t(\omega)\,d\omega}{\displaystyle\int_0^1 B_t(1-\omega)\,d\omega} \\
&= \frac{\displaystyle\int_0^1 B_t(\omega)\,d\omega}{\displaystyle\int_0^1 B_t(1-\omega)\,d\omega} - \frac{\displaystyle\int_0^1 \omega B_t(\omega)\,d\omega}{\displaystyle\int_0^1 B_t(1-\omega)\,d\omega} \\
&= \frac{\displaystyle\int_0^1 B_t(\omega)\,d\omega}{\displaystyle\int_0^1 B_t(\omega)\,d\omega} - \frac{\displaystyle\int_0^1 \omega B_t(1-\omega)\,d\omega}{\displaystyle\int_0^1 B_t(1-\omega)\,d\omega} \\
&= 1 - \mathbf{E}[\theta : A^{(t-1)}, X^{(t)} = 1] \\
\mathbf{E}[\theta : A^{(t-1)}, X^{(t)} = 1] &= \frac{1}{2}.
\end{aligned}$$

Note that $B_t(1-\omega) = B_t(\omega)$ for all $\omega \in [0, 1]$. If Claim II.4 holds for all Stages t, it must be that

$$\mathbf{E}[\theta : A^{(t-1)}, X^{(t)} = x] \leq \mathbf{E}[\theta : A^{(t-1)}, X^{(t)} = 1] = 1/2, \ \forall\, 0 < x \leq 1.$$

Let $0 < x \leq 1$. By Lemma II.5, any nonzero c that satisfies

$$P(Y^{(t)} \leq c : A^{(t-1)}, X^{(t)} = x) = c$$

must be within $[1/2, 1]$. Suppose there exists a c^* that satisfies this equation. Then

$$\mathbf{E}[\theta : A^{(t-1)}, X^{(t)} = x] \leq 1/2 \leq c^*.$$

Let $c_i^{(t)*}$ be Player i's cutoff in Stage t that corresponds to c^*. Again, if Claim II.4 is true for all Stages t, it must be that

$$\mathbf{E}[\theta : A^{(t-1)}, X^{(t)} = x] \leq c^* = \mathbf{E}[\theta : A^{(t-1)}, X^{(t)} = c_i^{(t)*}] \Rightarrow x \leq c_i^{(t)*}.$$

III.4: Simplified Simulations

The application of Algorithm 1 suggests that players that choose pessimistic cutoffs make choices of actions that are unaffected by histories, as stated in Claim III.1. This independence from history reduces the variance in the Monte Carlo sampling method. Firstly, $\mathbf{E}[\theta : A^{(t-1)}, X^{(t)} = x]$ can be calculated using only the definition of $\bar{B}_t(\cdot)$:

$$\text{(III.4.1)} \qquad \mathbf{E}[\theta : A^{(t-1)}, X^{(t)} = x] = \begin{cases} \dfrac{\int_0^x \bar{B}_t(u)\,du}{\bar{B}_t(x)}, & 0 < x < 1, \\ \dfrac{(x-1) - \bar{B}_t(x-1) + \int_{x-1}^1 \bar{B}_t(u)\,du}{1 - \bar{B}_t(x-1)}, & 1 \leq x < 2. \end{cases}$$

In fact, the equation above gives the Stage 1 and Stage 2 conditional expectations (see APPENDIX C). Figure III.11 compares the mean squared errors between modified and simplified approximations of $\mathbf{E}[\theta : A^{(t-1)}, X^{(t)}]$ and values of $\mathbf{E}[\theta : A^{(t-1)}, X^{(t)}]$ from the equation above in $T = 10$ stages. Each simulation generates an approximation for twenty-five linearly

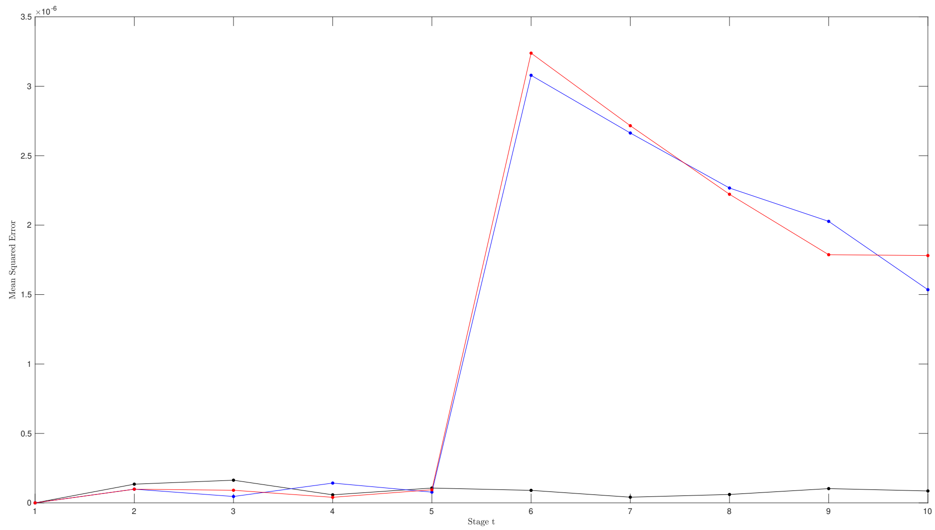

Figure III.11: Mean squared errors between values from Equation (III.4.1) and approximated conditional expectation values from modified and simplified sampling schemes, $N = 100000$ samples for each method.

spaced points on the interval $(0, 2)$ and for each point, the approximations are generated using $N = 100000$ points. The black curve represents the errors common between Player i and $-i$ using the simplified sampling scheme (both players share the same conditional expectation curve in Stage t). The blue and red curves represent Player i and $-i$'s errors, respectively, using the modified sampling scheme. The figure illustrates how close the modified approximations are to the values generated from Equation (III.4.1); this lends further credence to Claim III.1, where the black curve demonstrates the reduction in the variance of approximations.

Algorithm 2 outlines the following simplifications to the modified sampling algorithm: Firstly, there is no need for the estimators in Equations (III.0.1) and (III.0.2) since one can generate data for $\mathbf{E}[\theta : A^{(t-1)}, X^{(t)} = x]$ using $\bar{B}_t(x)$ and $B_t(x)$. Secondly, note that that if $0 \leq c \leq 1$, then

$$\int_0^c B_t(y - \omega)\, dy = \bar{B}_t(c - \omega) \cdot \mathbf{1}_{(c-1,c)}(\omega). \tag{III.4.2}$$

Algorithm 2 Simplified Monte Carlo Sampling

Require: ξ_j, c_k, t
Ensure: H_n
  $H_n \leftarrow 0$
  $\Omega_n \leftarrow (\xi_j - 1, \xi_j) \cap [c_k - 1, c_k] \cap [0, 1]$.
  if $|\Omega_n| \neq 0$ **then**
    Sample ω_n uniformly on Ω_n.
    $H_n \leftarrow |\Omega_n| \cdot B_t(\xi_j - \omega_n) \cdot \bar{B}_t(c_k - \omega_n)$.
  end if

Thus, the simplified algorithm outlines a sampling procedure to approximate the numerator in Equation (II.3.6); the integrand is $B_t(\xi_j - \omega) \cdot \bar{B}_t(c_k - \omega)$ and ω is sampled on $(\xi_j - 1, \xi_j) \cap [c_k - 1, c_k] \cap [0, 1]$, where $\xi_j \in (0, 2)$, $c_k \in [1/2, 1]$.

Figure III.12 shows the approximated fixed points for Player i using the simplified sampling algorithm for Stages 2 through 20. The fixed points are overlaid over one another to illustrate the change in the fixed point branches as t increases. The orange asterisks correspond to fixed point, and cutoff, values of 0. The red asterisks correspond to the lower branch of nonzero fixed points for a particular stage; they arise due to errors in fitting a cubic polynomial to the approximation data of $\mathbf{P}(Y^{(t)} \leq c : A^{(t-1)}, X^{(t)} = x)$ and errors in the subsequent procedure to find fixed points. In reality, these $X^{(t)}$ where red points reside should appear as blue asterisks. The blue asterisks are the fixed point values corresponding to pessimistic cutoffs that Player i will select. As t increases, the pessimistic fixed points coalesce closer to the line $c = 1$; this effect is better illustrated in Figure III.13. Furthermore, the regime of average of signal values where Player i encounters nonzero fixed points appears to expand as t increases. This suggests that the probability that Player i plays the low action—that they select a high cutoff—increases as t increases.

However, this does not preclude the possibility where Player i has a high cutoff in one stage and a cutoff of 0 in the next. As an example, suppose in Stage 2 that Player i sees an average of signals equal to $X^{(2)} = (3/2)^{3/2} - 3/2$—their bifurcation point in Stage 2. The possible average of signals that Player i could see in Stage 3 would be within the interval $[2X^{(2)}/3, 2X^{(2)}/3 + 2/3]$. Figure III.14 shows the approximate values of

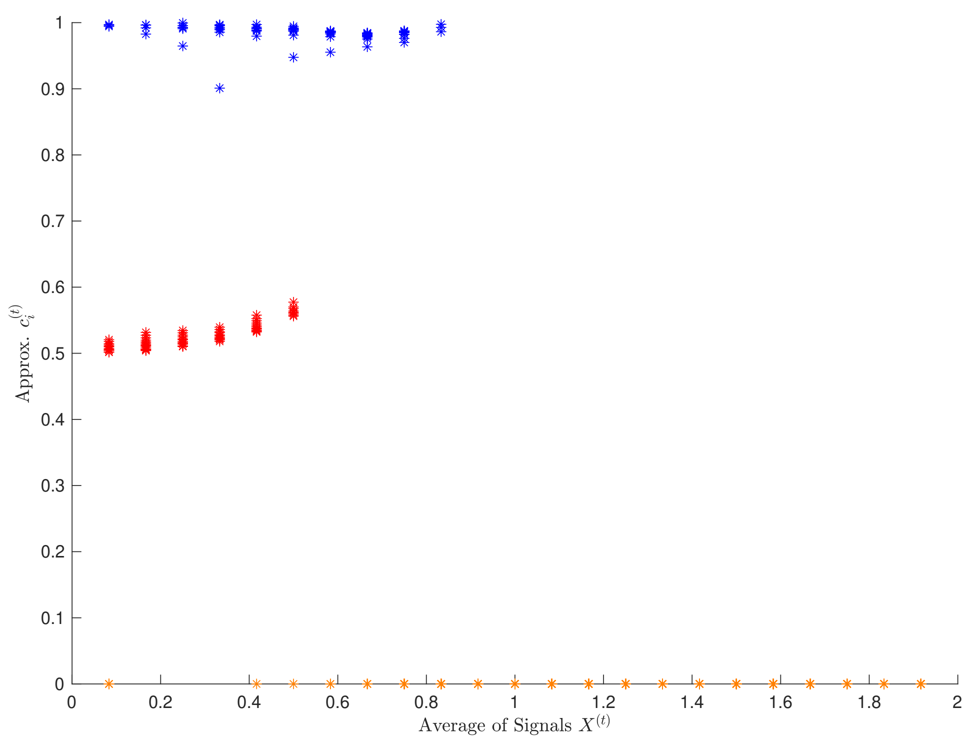

Figure III.12: Approximate fixed points generated from simplified Monte Carlo sampling for Stages 2 through 20 overlaid on top of one another, $N = 100000$ samples.

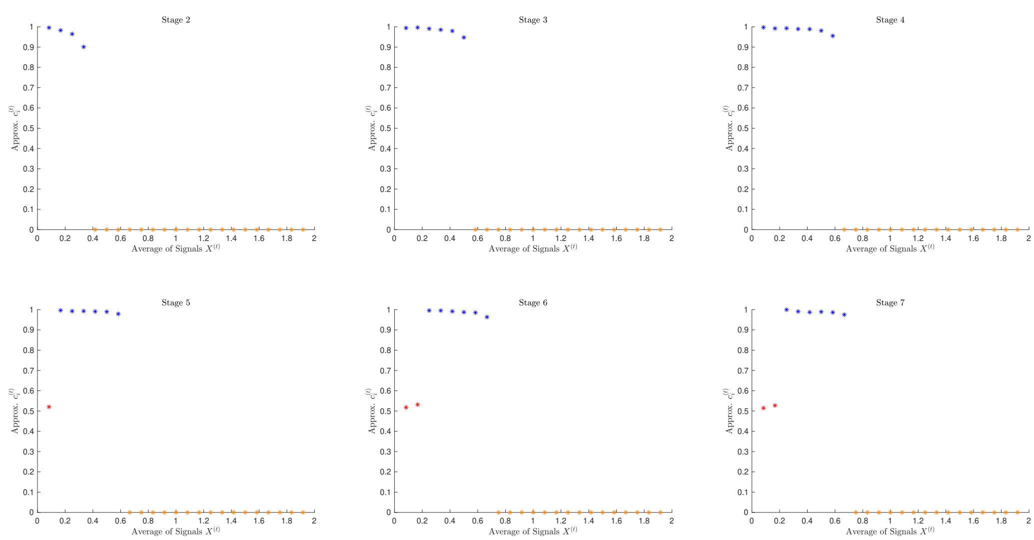

Figure III.13: Stage 2-7 approximate fixed points generated from simplified Monte Carlo sampling, $N = 100000$.

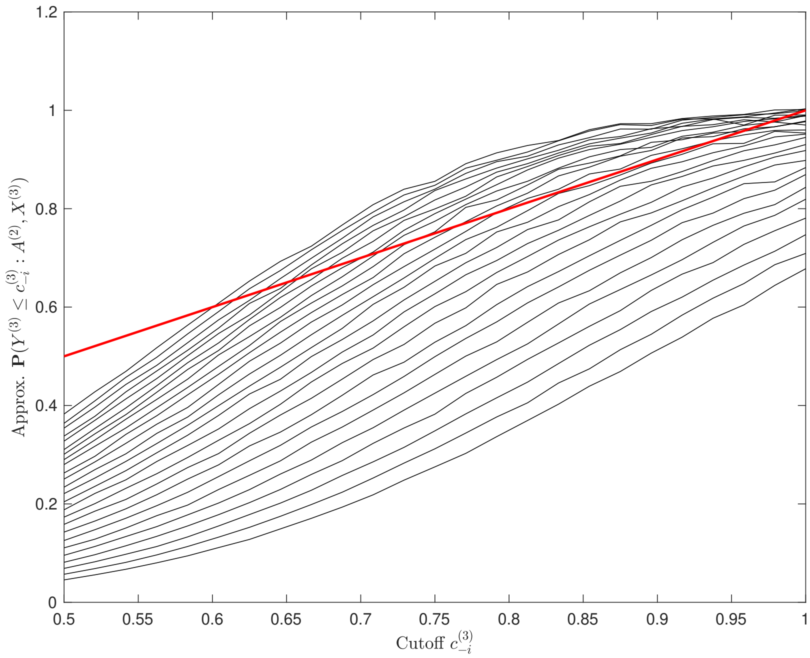

Figure III.14: Player i's approximate conditional probabilities in Stage 3 when they see an average of signals $X^{(2)} = (3/2)^{(3/2)} - 3/2$, $N = 30000$ samples.

$\mathbf{P}(Y^{(3)} \leq c_{-i}^{(3)} : A^{(2)}, X^{(3)})$ for twenty-five average of signal values in this interval. The red curve is the forty-five degree line. From these twenty-five values of $X^{(3)}$, there are more corresponding black curves that are strictly below the red line than there are curves that cross this line; Player i is less likely to observe signals of the latter kind that would lead them to choose a nonzero cutoff.

CHAPTER IV

Alternative Equilibria

Monte Carlo estimation of pessimistic Player i's best responses and cutoffs in Stage t suggests that each players' aggregate signal is the only piece of information which informs their action in a given stage. Here I analyze two different cutoff choices Player i can make instead of the maximal cutoff value for a given a history of actions and an average of signals.

IV.1: Zero-Cutoff Strategies

In Stage t, given a history $A^{(t-1)}$ and for all of Player i's average of signals $X^{(t)}$, $c^* = 0$ satisfies the equation $\mathbf{P}(Y^{(t)} \leq c^* : A^{(t-1)}, X^{(t)}) = c^*$, where the corresponding cutoff for both players is 0. Suppose both players always choose this cutoff. With probability 1, Player i's resulting action in Stage t is $a_i^{(t)} = 1$. Claim III.1 is true if players choose this zero-cutoff strategy in all stages: For all $\omega \in [0, 1]$,

$$\mathbf{P}(A^{(t-1)} : \theta = \omega) = \mathbf{P}\left(\bigcap_{\tau=1}^{t-1} \{X^{(\tau)} > 0\} \cap \{Y^{(\tau)} > 0\} : \theta = \omega\right) = 1$$

since, from Equation (II.3.3), $I_i^\tau = (0,2)$ and

$$\begin{aligned}\tau \cdot \int_{I_i^{(\tau)}} \mathbf{1}_{(0,1)}(\tau x_\tau - (\tau-1)x_{\tau-1} - \omega)\, dx_\tau &= \tau \cdot \int_0^2 \mathbf{1}_{(0,1)}(\tau x_\tau - (\tau-1)x_{\tau-1} - \omega)\, dx_\tau \\ &= \tau \cdot \frac{1}{\tau} \\ &= 1\end{aligned}$$

for all $1 \leq \tau \leq t-1$. Consequently, $g(A^{(t-1)}) = [0,1]$ and $f_\theta^{(t)}(\omega) = \mathbf{1}_{[0,1]}(\omega)$ for all t (Claim III.2, where players choose zero-cutoff strategies instead of pessimistic ones).

IV.1.1: Optimistic Cutoff Strategies

Now consider the game described in Chapter II where Player i chooses an *optimistic* cutoff strategy in each Stage t: Given a history of actions and Player i's average of signals in Stage t, if multiple nonzero cutoffs exist, then Player i selects the minimum amongst them. Other than this modification, the game proceeds as when players chose pessimistic equilibrium cutoffs.

Recall Figure II.4 and the two branches of fixed points that satisfy the equation $\mathbf{P}(Y^{(2)} \leq c^* : A^{(1)}, X^{(2)} = x) = c^*$. A portion of the red-colored branch corresponds to the optimistic fixed points Player i chooses if they see an average of signals within the interval $[0, (3/2)^{3/2} - 3/2]$. The branch given by

$$\tilde{c}^*- = \frac{x}{3} - \frac{\sqrt{-8\,x^2 - 24\,x + 9}}{12} + \frac{3}{4}$$

is strictly greater than $x + 1/2$ when $1/3 \leq x \leq (3/2)^{3/2} - 3/2$.

For signal averages in $(0, 1/3)$, I consider $c^* \in [1/2, x+1/2]$, such that

$$\begin{aligned}c^* = \frac{1}{2x^2}\bigg(&\frac{4}{3}(c^*)^4 - \frac{16}{3}(c^*)^3 x - \frac{8}{3}(c^*)^3 + 4(c^*)^2x^2 + 8(c^*)^2x + 2(c^*)^2 \\ &- \frac{8}{3}(c^*)x^3 - 4(c^*)x - \frac{2}{3}c^* + \frac{2}{3}x^4 + \frac{2}{3}x + \frac{1}{12}.\bigg)\end{aligned} \tag{IV.1.1}$$

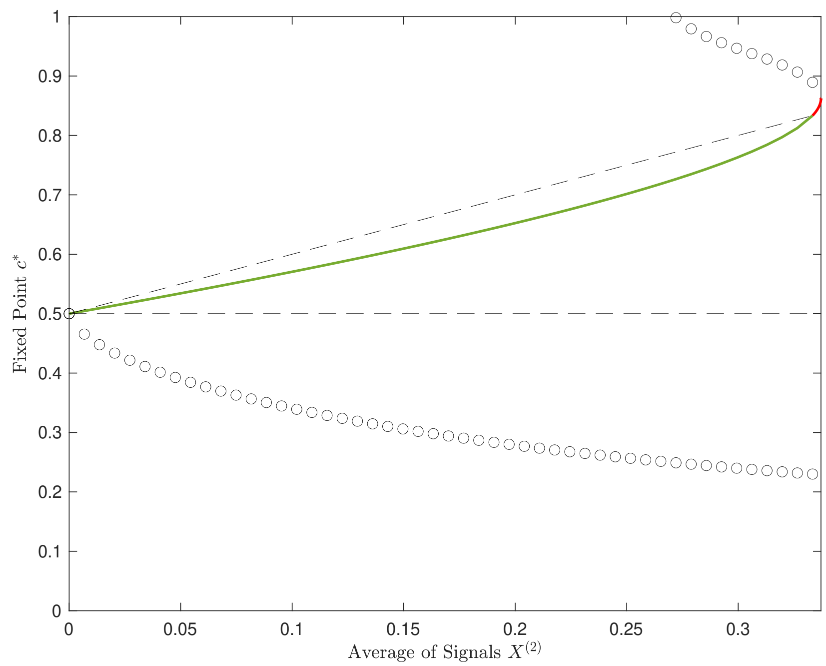

Figure IV.1: Optimistic fixed points for Player i in Stage 2 generated via evaluations of symbolic expressions at fifty values of $X^{(2)} \in (0, 1/3]$. The green curve represents c^* that satisfy $1/2 \leq c^* \leq x + 1/2$ (dashed lines). The red curve is $\tilde{c}^*$ on $(1/3, (3/2)^{3/2} - 3/2]$.

(See APPENDIX C). Using the MATLAB symbolic `solve` function, I generate symbolic solutions to Equation (IV.1.1) represented via the `root` function; this does not return the exact form of c^*. Instead, I evaluate the four symbolic roots for fifty different values of $x \in [0, 1/3]$. Figure IV.1 shows these evaluations for roots within $[0, 1]$. The desired root that corresponds to Player i's optimistic cutoff is highlighted in green. For comparison, the portion of $\tilde{c}^*$ that satisfies $x + 1/2 < \tilde{c}^*$ is highlighted in red. The circle markers indicate the alternative solutions to Equation (IV.1.1) that do not satisfy $1/2 \leq c^* \leq x + 1/2$.

The optimistic c^* that lie between $[1/2, x + 1/2]$ range from 1/2 to 5/6. Thus, Player i's optimistic equilibrium cutoffs in Stage t are the signals $c_i^{(2)*}$ that solve $c_i^{(2)*}/3 + 1/3 = \tilde{c}^*$ for $x \in (1/3, (3/2)^{3/2} - 3/2]$, such that

(IV.1.2) $$c_i^{(2)*} = x - \frac{\sqrt{-8\,x^2 - 24\,x + 9}}{4} + \frac{5}{4}.$$

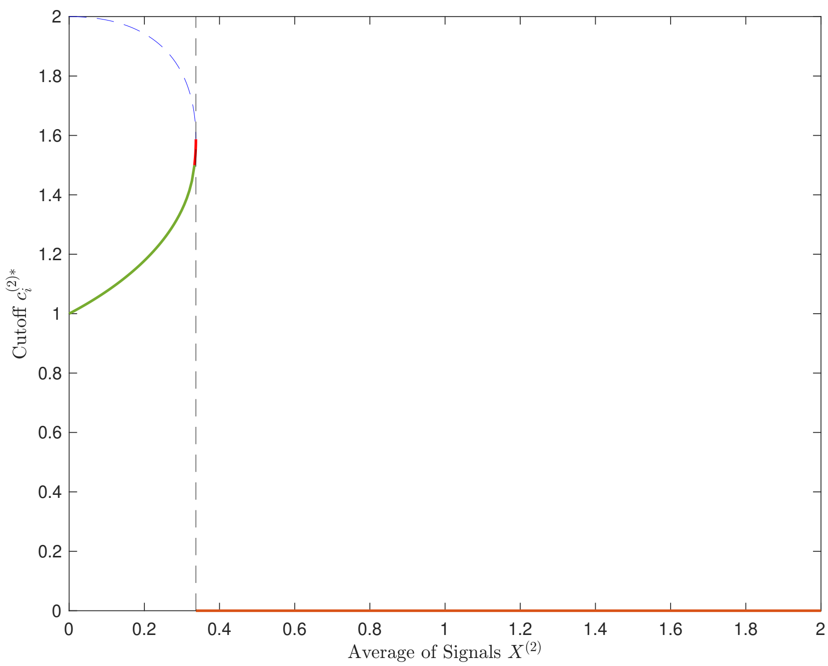

Figure IV.2: Optimistic Cutoffs for Player i in Stage 2 corresponding to the fixed points in Figure IV.1 for averages in $[0, (3/2)^{3/2} - 3/2]$. The Stage 2 pessimistic cutoffs are plotted for comparison (dashed blue curve).

For $x \in (0, 1/3]$, $c_i^{(2)*}$ solves

$$\text{(IV.1.3)} \qquad \frac{12(c_i^{(2)*} - 1) - 6(2c_i^{(2)*} - 2)^2 + 6 - (2c_i^{(2)*} - 2)^3}{12 - 6(2c_i^{(2)*} - 2)^2} = c^*.$$

Figure IV.2 shows the evaluation of the symbolic expressions of optimistic $c_i^{(2)*}$ for all $x \in (0, 2)$. The green curve shows optimistic cutoffs from Equation (IV.1.3) for fifty values of $x \in (0, 1/3]$; the red curve corresponds to the symbolic plot of Equation (IV.1.2) on $(1/3, (3/2)^{3/2} - 3/2]$. The dashed blue curve corresponds to Player i's pessimistic cutoffs given by Equation (II.2.11). As in the pessimistic game setup, when Player i sees an average of signals greater than $(3/2)^{3/2} - 3/2$, their optimistic cutoff is 0 (orange curve). This follows Claim III.3: Player i chooses $a_i^{(2)} = 0$ when they select optimistic cutoff strategies if their average of signals is within $(0, (3/2)^{3/2} - 3/2]$.

The cutoffs for an optimistic Player i seem to depend on the players' history of actions.

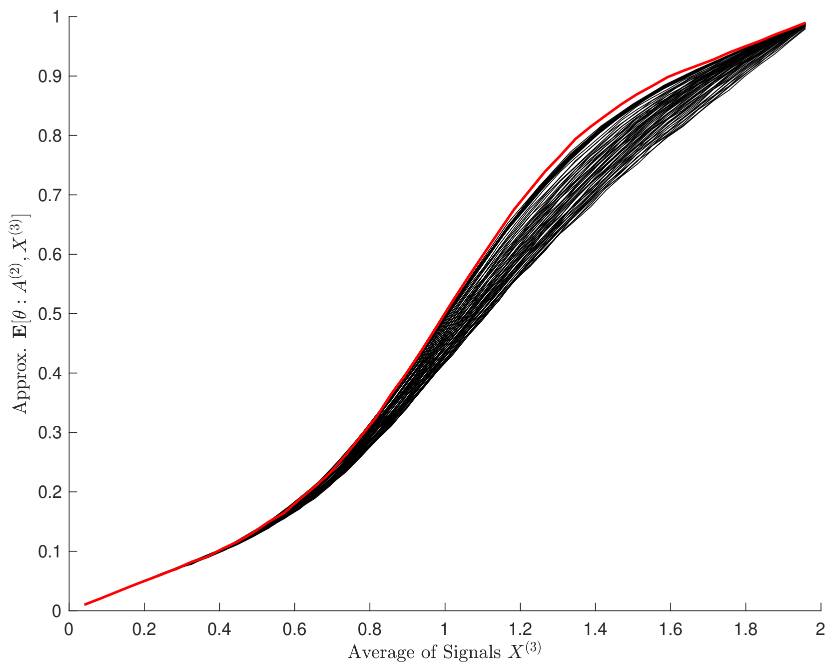

Figure IV.3: Player i's approximate conditional expected value of θ in Stage 3 when they select optimistic cutoffs in Stage 2. The curves are generated using the modified Monte Carlo scheme for fifty $X^{(2)} \in (0, 1/3]$ and twenty-five $X^{(2)} \in (1/3, (3/2)^{3/2} - 3/2]$, $N = 10000$ samples. The red curve corresponds to the action profile $a_i^{(2)} = 1 = a_{-i}^{(2)}$, where both players use a cutoff of 0 in Stage 2.

Consider Player i's Stage 3 approximate conditional expectation curve generated via the modified Monte Carlo sampling scheme, such that Player i, or Player $-i$, or both players chose a nonzero optimistic cutoff in Stage 2. Figure IV.3 (where Player i chooses the low action in Stage 2) shows that Player i's conditional expected value of θ is nondecreasing for all $X^{(3)}$ as the Stage 2 optimistic cutoff increases. The red curve corresponds to both players' conditional expected value given $X^{(3)}$ when $a_i^{(2)} = 1 = a_{-i}^{(2)}$; this is the history-independent expected value in Stage 3 for both players. The key feature is that this history-independent expected value of θ is greater than or equal to the history-dependent expected values for all $X^{(3)}$. This suggests that Claim III.3 holds even when both players select optimistic cutoffs in this stage. This is also shown in Figure IV.4 (where Player $-i$ chooses the low action in Stage 2) and Figure IV.5 (where both players choose the low action in Stage 2).

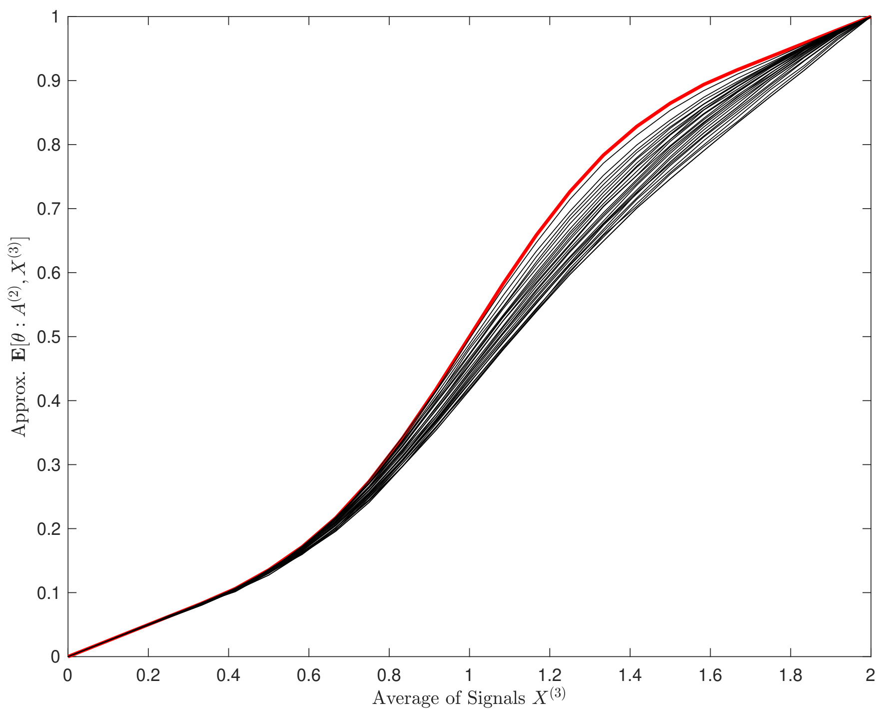

Figure IV.4: Player i's approximate conditional expected value of θ in Stage 3 when they select optimistic cutoffs in Stage 2. The curves are generated using the modified Monte Carlo scheme for twenty-five $Y^{(2)} \in (0, (3/2)^{3/2} - 3/2]$, $N = 10000$ samples. The red curve corresponds to the action profile $a_i^{(2)} = 1 = a_{-i}^{(2)}$, where both players use a cutoff of 0 in Stage 2.

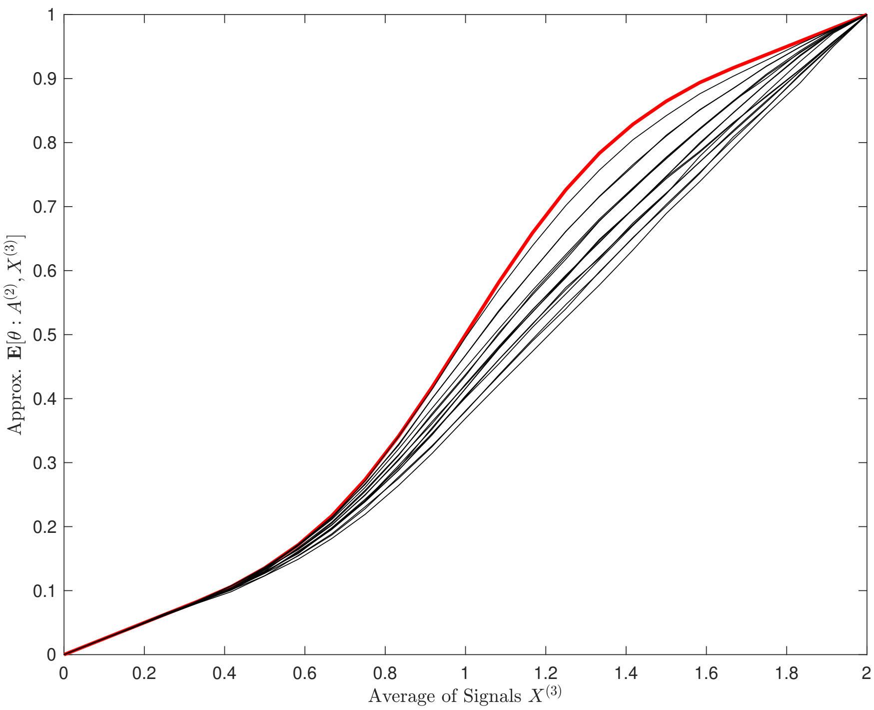

Figure IV.5: Player i's approximate conditional expected value of θ in Stage 3 when they select optimistic cutoffs in Stage 2. The curves are generated using the modified Monte Carlo scheme for 625 combinations of $(X^{(2)}, Y^{(2)}) \in (0, (3/2)^{3/2} - 3/2]^2$, $N = 10000$ samples. The red curve corresponds to the action profile $a_i^{(2)} = 1 = a_{-i}^{(2)}$, where both players use a cutoff of 0 in Stage 2.

CHAPTER V

Conclusions

In the stage-game described by Table I.1, both players receive higher payoffs if they coordinate to the high action "1." However, given each players' average of signals and the history of actions in each stage, the possibility of miscoordination appears to be ever-present in Stages $t \geq 2$. If Player i's average of signals is sufficiently low, and if Player i believes Player $-i$ will choose the low action "0" with positive probability (that is, if Player i's pessimistic equilibrium cutoff is nonzero), then Player i will choose the low action and avoid a negative payoff for miscoordinating. If Claim III.3 is true for the pessimistic game, an average of signals less than or equal to 1 is enough to ensure that Player i chooses the low action when a nonzero cutoff exists.

When players choose pessimistic cutoffs in the game described in Chapter II, the history of actions and cutoffs up to Stage $t-1$ does not seem to affect the actions played in Stage t. Rather, the Monte Carlo simulations of this game suggest that in Stage t, Player i's pessimistic cutoff is determined only by their average of signals in that stage (see Claim III.1). Via the simplified, history-independent Monte Carlo sampling scheme detailed in Algorithm 2, it appears the set of average-signal values for which Player i assigns positive probability to the event $\{a_{-i}^{(t)} = 0\}$ increases to $(0, 1]$ as t increases (see Figure III.12).

Theorem II.7 states that Player i will choose a cutoff of 0 in Stage t when their average of signals is in $[\underline{x}^{(t)} + 1, 2)$. With a sufficiently high average signal, Player i always finds that Player $-i$ chooses action "0" with probability 0. It remains to show what Player i's pessimistic equilibrium cutoff and action will be when their average of signals in Stage t is within $(1, \underline{\mathrm{x}}^{(t)} + 1)$.

The history of actions up to Stage $t-1$ in the optimistic equilibrium appears to affect the optimistic cutoff values for each player in Stage t. Nonetheless, Monte Carlo simulations of this game and optimistic equilibrium suggest that Player i's optimistic nonzero cutoffs will be greater than $X^{(t)}$ for any Stage t. It remains to show that the distribution of θ conditional on $X^{(t)} = x$ is stochastically increasing in $x \in (0,2)$ (Claim II.4); this would show that expected value of θ given $X^{(t)} = x$ is increasing in x, regardless of the choice of cutoff. Furthermore, Claim III.3 would extend to players in the optimistic equilibrium. It remains to show exactly how the history of actions and optimistic cutoffs in previous stages affect Player i's expected value of θ and the probability they assign to Player $-i$ choosing the low action.

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APPENDIX A

Proofs

Proof. (Theorem II.2) Consider $c^* \in [0,1]$ that satisfies

$$c^* = \mathbf{P}(x_{-i}^{(1)} \leq c_{-i}^{(1)} = c^* : x_i^{(t)} = x) \tag{A}$$

for fixed x. Let $P(c;x) = \mathbf{P}(x_{-i}^{(1)} \leq c_{-i}^{(1)} = c : x_i^{(t)} = x)$.

Fix $0 < x_1 \leq 1$. On the interval $0 \leq c \leq x_1$, $P(c;x_1) = c^2/2x_1$, so

$$c^* = \frac{(c^*)^2}{2x_1}$$
$$\Rightarrow c^* = 0,\ 2x_1.$$

However, $2x_1 \nleq x_1$, so $c^* = 0$ is the unique fixed point for $P(c;x_1)$ on the interval $[0,x_1]$. On the interval $x_1 < c \leq 1$, $P(c;x_1) = c - x_1/2$; there are no c^* that satisfy Equation (A) on $(x_1, 1]$.

Now fix $1 < x_2 < 2$. On the interval $x_2 - 1 < c \leq 1$,

$$P(c;x_2) = \frac{(c - x_2 + 1)^2}{2(2 - x_2)},$$

so

$$
\begin{aligned}
c^* &= \frac{(c^* - x_2 + 1)^2}{2(2 - x_2)} \\
\Rightarrow\ c^* &= 1 \pm \sqrt{x_2(2 - x_2)}.
\end{aligned}
$$

$1 + \sqrt{x_2(2 - x_2)}$ is real and is strictly greater than 1. Furthermore, $x_2 > 2 - x_2 > 0$. Thus,

$$
\begin{aligned}
x_2 &> 2 - x_2 \\
\sqrt{x_2} &> \sqrt{2 - x_2} \\
\sqrt{x_2}\sqrt{2 - x_2} &> \sqrt{2 - x_2}\sqrt{2 - x_2} = 2 - x_2 \\
1 - \sqrt{x_2(2 - x_2)} &< 1 - (2 - x_2) = x_2 - 1.
\end{aligned}
$$

Therefore, there are no c^* in $(x_2 - 1, 1]$ that satisfy Equation (A). Finally, on the interval $0 \leq c \leq x_2 - 1$, $P(c; x_2) = 0$; $c^* = 0$ is the unique value that satisfies Equation (A) on $[0, x_2 - 1]$. $\square$

Proof. (Theorem II.3) Consider $c^* \in [0,1]$ that satisfy

$$c^* = \mathbf{P}(Y^{(2)} \leq c_{-i}^{(2)} = c^* : X^{(2)} = x). \tag{B}$$

for fixed x. Let $P(c;x) = \mathbf{P}(Y^{(2)} \leq c_{-i}^{(2)} = c : X^{(2)} = x)$. By Lemma II.5, any nonzero c^* that satisfies Equation (B) must be in the interval $[1/2, 1]$.

Fix $0 < x_1 < 1/2$. On $x + 1/2 < c \leq 1$,

$$\begin{aligned} P(c;x_1) &= \frac{1}{2x_1^2}\left(-4c^2x_1^2 + \frac{8}{3}cx_1^3 + 8cx_1^2 - \frac{2}{3}x_1^4 - \frac{8}{3}x_1^3 - 2x_1^2\right) \\ &= -2c^2 + \frac{4}{3}cx_1 + 4c - \frac{1}{3}x_1^2 - \frac{4}{3}x_1 - 1. \end{aligned}$$

(See APPENDIX C for details on how to obtain $P(c;x)$). Thus,

$$\begin{aligned} c^* &= P(c^*;x_1) \\ \Rightarrow c^* &= \frac{x_1}{3} \pm \frac{\sqrt{-8\,x_1^2 - 24\,x_1 + 9}}{12} + \frac{3}{4}. \end{aligned}$$

Let $c_+^* = \dfrac{x_1}{3} + \dfrac{\sqrt{-8\,x_1^2 - 24\,x_1 + 9}}{12} + \dfrac{3}{4}$. c_+^* (and the alternate root above) is real and nonzero only if $-(3/2)^{3/2} - 3/2 \leq x_1 \leq (3/2)^{3/2} - 3/2 \approx 0.3371$. Using Mathematica, one can verify that $x_1 + 1/2 < c_+^*$ for any $0 < x_1 \leq (3/2)^{3/2} - 3/2$; furthermore, $c_+^* < 1$ for any $0 < x_1 \leq (3/2)^{3/2} - 3/2$.

Suppose x_1 is a fixed number within $[0, (3/2)^{3/2} - 3/2]$. Because $\mathbf{E}[\theta : X^{(2)} = x]$ is a strictly increasing function of x on $(0,2)$ (see Equation (II.2.6)), if Player i sees an average of signals equal to x_1, then their pessimistic cutoff for Stage 2 corresponds to c_+^*.

Suppose x_1 is a fixed number within $((3/2)^{3/2} - 3/2, 1/2)$. Then there are no real c^* in $(x_1 + 1/2, 1]$ that satisfy Equation (B) when $X^{(2)} = x_1$. On $1/2 \leq c \leq x_1 + 1/2$,

$$\frac{d}{dc}[P](c;x_1) = \frac{1}{2x_1^2}\left(\frac{16}{3}c^3 - 16c^2x_1 - 8c^2 + 8cx_1^2 + 16cx_1 + 4c - \frac{8}{3}x_1^3 - 4x_1 - \frac{2}{3}\right).$$

Using Mathematica, one can verify that $\frac{d}{dc}[P](c;x_1) > 0$ for all $1/2 \le c \le x_1 + 1/2$. As such,

$$P(c;x_1) \le P(c;x_1)\Big|_{c=x_1+\frac{1}{2}} = -x_1^2 + \frac{4}{3}x_1 + \frac{1}{2} < c, \text{ for all } 1/2 \le c \le x_1 + 1/2.$$

Therefore, for $x \in ((3/2)^{3/2} - 3/2, 1/2)$, there are no nonzero c^* that satisfy Equation (B).

Now fix $1/2 \le x_2 < 1$. On $1/2 \le c \le x_2$,

$$\frac{d}{dc}[P](c;x_2) = \frac{-8c^3 + 24c^2x_2 - 8c^2 - 16cx_2^2 + 8c + \frac{16}{3}x_2^3 - 8x_2^2 + 8x_2 - 4}{-2x_2^2 + 4x_2 - 1}.$$

Using Mathematica, one can verify that $\frac{d}{dc}[P](c;x_2) > 0$ for all $1/2 \le c \le x_2$. As such,

$$P(c;x_2) \le P(c;x_2)\Big|_{c=x_2} = -x_2^2 + 2x_2 - \frac{1}{2} < c, \text{ for all } 1/2 \le c < x_2.$$

On $x_2 < c \le 1$,

$$\begin{aligned}\frac{d}{dc}[P](c;x_2) = \frac{1}{-2x_2^2 + 4x_2 - 1}\bigg(&\frac{4}{3}c^4 - \frac{16}{3}c^3x_2 - \frac{8}{3}c^3 + 12c^2x_2^2 + 4c^2 - 8cx_2^3 \\ &- 8cx_2^2 + 8cx_2 - 4c + 2x_2^4 + \frac{8}{3}x_2^3 - 2x_2^2 + \frac{1}{2}\bigg).\end{aligned}$$

Using Mathematica, one can verify that $\frac{d}{dc}[P](c;x_2) > 0$ for all $x_2 < c \le 1$. As such,

$$P(c;x_2) \le P(c;x_2)\Big|_{c=x_2} = -x_2^2 + \frac{2}{3}x_2 + \frac{5}{6} < c, \text{ for all } x_2 < c \le 1.$$

Therefore, for $x \in [1/2, 1)$, there are no nonzero c^* that satisfy Equation (B).

Now fix $1 \le x_3 < 3/2$. On $x_3 - 1 < c \le x_3 - 1/2$,

$$\frac{d^2}{dc^2}[P](c;x_3) = \frac{1}{-2x_3^2 + 4x_3 - 1}8(c - x_3 + 1)^2 > 0.$$

$P(c; x_3)$ is convex on $x_3 - 1 < c \leq x_3 - 1/2$, such that

$$\begin{aligned} P(c; x_3) &\leq \frac{2}{3(-2x_3^2 + 4x_3 - 1)} \left(\frac{1}{2}\right)^3 (c - x_3 + 1) \\ &< \frac{2}{3(-2x_3^2 + 4x_3 - 1)} \left(\frac{1}{2}\right)^3 c \\ &< c. \end{aligned}$$

On $x_3 - 1/2 < c \leq 1$,

$$\frac{d^2}{dc^2}[P](c; x_3) = \frac{1}{-2x_3^2 + 4x_3 - 1} \left(8c^2 - 16cx_3 + 16c + 8x_3^2 - 16x_3 + 8\right).$$

Using Mathematica, one can verify that $\frac{d^2}{dc^2}[P](c; x_3) > 0$ for all $x_3 - 1/2 < c \leq 1$. Thus, $P(c; x_3)$ is convex on this interval, such that

$$\begin{aligned} P(c; x_3) &\leq \frac{-24x_3^3 + 92x_3^2 - 102x_3 + 23}{24x_3^2 - 48x_3 + 12}(c - 1) + \left(x_3^2 - \frac{10}{3}x_3 + \frac{17}{6}\right) \\ &< \frac{-24x_3^3 + 92x_3^2 - 102x_3 + 23}{24x_3^2 - 48x_3 + 12} c \\ &< c. \end{aligned}$$

Therefore, for $x \in [1, 3/2)$, there are no nonzero c^* that satisfy Equation (B).

Finally, fix $3/2 \leq x_4 < 2$. On $x_4 - 1 < c \leq 1$,

$$\frac{d^2}{dc^2}[P](c; x_4) = \frac{1}{2x_4^2 - 8x_4 + 8} 8(c - x_4 + 1)^2 > 0.$$

$P(c;x_4)$ is convex on $x_4 - 1 < c \leq 1$, such that

$$\begin{aligned} P(c;x_4) &\leq \frac{2-x_4}{3}(c - x_4 + 1) \\ &< \frac{2-x_4}{3}c \\ &< c. \end{aligned}$$

Therefore, for $x \in [3/2, 2)$, there are no nonzero c^* that satisfy Equation (B).

$\square$

APPENDIX B

Stage 1 Conditional Probability

Here I derive $P = \mathbf{P}(a_{-i}^{(1)} = 0 : x_i^{(1)})$ shown in Equation (II.1.3). Suppose the actualization of Player i's signal in Stage 1 is $x_i^{(1)} = x$ and that Player $-i$'s cutoff value in Stage 1 is $c_{-i}^{(1)} = c$. Firstly,

$$\int_0^c \mathbf{1}_{(\omega,\omega+1)}(y)\,dy = (c-\omega)\mathbf{1}_{[c-1,c)}(\omega) + \mathbf{1}_{(-\infty,c-1)}(\omega).$$

Thus,

$$P = \frac{1}{f_{x_i^{(1)}}(x)} \int_0^1 \mathbf{1}_{(x-1,x)}(\omega)((c-\omega)\mathbf{1}_{[c-1,c)}(\omega) + \mathbf{1}_{(-\infty,c-1)}(\omega))\,d\omega$$

Note that $f_{x_i^{(1)}}(x) = x$ on $0 < x \leq 1$ and $f_{x_i^{(1)}}(x) = 2 - x$ on $1 < x \leq 2$.

If $0 < x \leq 1$, $0 \leq c \leq x$, then

$$P = \frac{1}{x}\int_0^c (c-\omega)\,d\omega = \frac{c^2}{2x}.$$

If $0 < x \leq 1$, $x < c \leq 1$, then

$$P = \frac{1}{x}\int_0^x (c-\omega)\,d\omega = c - \frac{x}{2}.$$

If $0 < x \leq 1$, $1 < c \leq x + 1$, then

$$P = \frac{1}{x}\int_{c-1}^{x}(c-\omega)\,d\omega + \frac{1}{x}\int_{0}^{c-1}d\omega = c - \frac{x}{2} - \frac{(c-1)^2}{2x}$$

If $0 < x \leq 1$, $x + 1 < c \leq 2$, then

$$P = \frac{1}{x}\int_{0}^{x}d\omega = 1.$$

If $1 < x < 2$, $x - 1 < c \leq 1$, then

$$P = \frac{1}{2-x}\int_{x-1}^{c}(c-\omega)d\omega = \frac{(c-x+1)^2}{2(2-x)}.$$

If $1 < x < 2$, $1 < c \leq x$, then

$$P = \frac{1}{2-x}\int_{x-1}^{1}(c-\omega)\,d\omega = c - \frac{x}{2}.$$

If $1 < x < 2$, $x < c \leq 2$, then

$$P = \frac{1}{2-x}\int_{c-1}^{1}(c-\omega)\,d\omega + \frac{1}{2-x}\int_{x-1}^{c-1}d\omega = \frac{c^2-4c+2x}{2(x-2)}.$$

Otherwise, if $1 < x < 2$, $0 \leq c \leq x - 1$, the $P = 0$.

APPENDIX C

Stage 2 Conditional Expectation and Probability

Here I derive the terms shown in Equation (II.2.5).

Conditional Expectation

Below is the expected value of θ given Player i's average of signals up to Stage 2, $\mathbf{E}[\theta : X^{(2)}]$, which is summarized in Equation (II.2.6). Suppose the actualization of Player i's average of signals up to Stage 2 is $X^{(2)} = x$.

$$
\begin{aligned}
\mathbf{E}[\theta : A^{(1)}, X^{(2)} = x] &= \frac{1}{f_{X^{(2)}}(x)} \int_{\mathbb{R}} \omega \cdot f_{X^{(2)}:\theta}(x : \omega) \cdot f_{\theta}^{(2)}(\omega)\, d\omega \\
&= \frac{\displaystyle\int_0^1 \omega \cdot B_2(x-\omega) \cdot \mathbf{1}_{(x-1,x)}(\omega)\, d\omega}{\displaystyle\int_0^1 B_2(x-\omega) \cdot \mathbf{1}_{(x-1,x)}(\omega)\, d\omega} \\
&= \frac{\displaystyle\int_0^1 \omega \cdot 2 \sum_{k=0}^{\lfloor 2(x-\omega) \rfloor} (-1)^k \binom{2}{k} (2(x-\omega)-k) \cdot \mathbf{1}_{(x-1,x)}(\omega)\, d\omega}{\displaystyle\int_0^1 2 \sum_{k=0}^{\lfloor 2(x-\omega) \rfloor} (-1)^k \binom{2}{k} (2(x-\omega)-k) \cdot \mathbf{1}_{(x-1,x)}(\omega)\, d\omega}
\end{aligned}
$$

If $0 < x < 1$, then

$$\mathbf{E}[\theta : A^{(1)}, X^{(2)} = x] = \frac{\displaystyle\int_0^x \omega \cdot 2 \sum_{k=0}^{\lfloor 2(x-\omega) \rfloor} (-1)^k \binom{2}{k} (2(x-\omega)-k)\, d\omega}{\displaystyle\int_0^x 2 \sum_{k=0}^{\lfloor 2(x-\omega) \rfloor} (-1)^k \binom{2}{k} (2(x-\omega)-k)\, d\omega}$$

$$= \frac{\displaystyle\int_0^x (x-u) \cdot 2 \sum_{k=0}^{\lfloor 2u \rfloor} (-1)^k \binom{2}{k} (2u-k)\, du}{\displaystyle\int_0^x 2 \sum_{k=0}^{\lfloor 2u \rfloor} (-1)^k \binom{2}{k} (2u-k)\, d\omega}$$

$$= \frac{\displaystyle\frac{1}{12} \sum_{k=0}^{\lfloor 2u \rfloor} (-1)^k \binom{2}{k} (2u-k)^3 \Big|_0^x}{\displaystyle\frac{1}{2} \sum_{k=0}^{\lfloor 2u \rfloor} (-1)^k \binom{2}{k} (2u-k)^2 \Big|_0^x}$$

$$= \frac{\displaystyle\frac{1}{12} \sum_{k=0}^{\lfloor 2x \rfloor} (-1)^k \binom{2}{k} (2x-k)^3}{\displaystyle\frac{1}{2} \sum_{k=0}^{\lfloor 2x \rfloor} (-1)^k \binom{2}{k} (2x-k)^2}.$$

If $1 \leq x < 2$, then

$$\mathbf{E}[\theta : X^{(2)} = x] = \frac{\displaystyle\int_{x-1}^1 \omega \cdot 2 \sum_{k=0}^{\lfloor 2(x-\omega) \rfloor} (-1)^k \binom{2}{k} (2(x-\omega)-k)\, d\omega}{\displaystyle\int_{x-1}^1 2 \sum_{k=0}^{\lfloor 2(x-\omega) \rfloor} (-1)^k \binom{2}{k} (2(x-\omega)-k)\, d\omega}$$

$$= \frac{\displaystyle x - \frac{1}{2} - \sum_{k=1}^{\lfloor 2x \rfloor - 2} (-1)^k \binom{2}{k} \left(\frac{1}{2}(2x-2-k)^2 + \frac{1}{12}(2x-2-k)^3 \right)}{\displaystyle 1 - \frac{1}{2} \sum_{k=1}^{\lfloor 2x \rfloor - 2} (-1)^k \binom{2}{k} (2x-2-k)^2}.$$

I retrieve the polynomials in the numerator and denominator by evaluating $\lfloor 2x \rfloor$.

If $0 < x < \dfrac{1}{2}$, then $\lfloor 2x \rfloor = 0$ and

$$\mathbf{E}[\theta : X^{(2)} = x] = \frac{\frac{1}{12}(2x)^3}{\frac{1}{2}(2x)^2} = \frac{x}{3}.$$

If $\dfrac{1}{2} \le x < 1$, then $\lfloor 2x \rfloor = 1$ and

$$\mathbf{E}[\theta : X^{(2)} = x] = \frac{\frac{1}{12}(2x)^3 - \frac{1}{6}(2x-1)^3}{\frac{1}{2}(2x)^2 - (2x-1)^2} = \frac{(2x)^3 - 2(2x-1)^3}{6(2x)^2 - 12(2x-1)^2}.$$

If $1 \le x < \dfrac{3}{2}$, then $\lfloor 2x \rfloor = 2$ and

$$\begin{aligned}\mathbf{E}[\theta : X^{(2)} = x] &= \frac{(x-1) - \frac{1}{2}(2x-2)^2 + \frac{1}{2} - \frac{1}{12}(2x-2)^3}{1 - \frac{1}{2}(2x-2)^2} \\ &= \frac{12(x-1) - 6(2x-2)^2 + 6 - (2x-2)^3}{12 - 6(2x-2)^2}.\end{aligned}$$

If $\dfrac{3}{2} \le x < 2$, then $\lfloor 2x \rfloor = 3$ and

$$\begin{aligned}\mathbf{E}[\theta : X^{(2)} = x] &= \frac{(x-1) - \frac{1}{2}(2x-2)^2 + (2x-3)^3 + \frac{1}{2} - \frac{1}{12}(2x-2)^3 + \frac{1}{6}(2x-3)^3}{1 - \frac{1}{2}(2x-2)^2 + (2x-3)^2} \\ &= \frac{x+1}{3}.\end{aligned}$$

Conditional Probability

Here I derive the integral in Equation (II.2.8) For $0 \le c \le 1$. Let

$$
\begin{aligned}
I &= \int_0^1 \int_0^c B_2(x-\omega) \cdot B_2(y-\omega) \mathbf{1}_{(x-1,\, x)}(\omega) \mathbf{1}_{(\omega,\, \omega+1)}(y)\, dy\, d\omega \\
&= \int_0^1 B_2(x-\omega) \cdot \bar{B}_2(c-\omega) \mathbf{1}_{(x-1,\, x) \cap (c-1,\, c)}(\omega)\, d\omega.
\end{aligned}
$$

For given x, let $b_0 = 4(x-\omega)$ and $b_1 = -4(2x - 2\omega - 1)$. Then,

$$
B_2(x-\omega) = b_0 \mathbf{1}_{(x-1/2,\, x]}(\omega) + (b_0 + b_1) \mathbf{1}_{(x-1,\, x-1/2]}(\omega).
$$

For given c, let $p_0 = 2(c-\omega)^2$ and $p_1 = -(2c - 2\omega - 1)^2$. Then,

$$
\bar{B}_2(c-\omega) = p_0 \mathbf{1}_{(c-1/2,\, c]}(\omega) + (p_0 + p_1) \mathbf{1}_{(c-1,\, c-1/2]}(\omega).
$$

If $0 < x < 1/2$, $0 \le c \le x$, then

$$
\begin{aligned}
I &= \int_0^c b_0 p_0\, d\omega \\
&= -\frac{2}{3} c^4 + \frac{8}{3} c^3 x.
\end{aligned}
$$

If $0 < x < 1/2$, $x < c < 1/2$, then

$$
\begin{aligned}
I &= \int_0^c b_0 p_0\, d\omega \\
&= 4c^2 x^2 - \frac{8}{3} c x^3 + \frac{2}{3} x^4.
\end{aligned}
$$

If $0 < x < 1/2$, $1/2 \le c \le x + 1/2$, then

$$\begin{aligned} I &= \int_{c-1/2}^{x} b_0 p_0 \, d\omega + \int_0^{c-1/2} b_0(p_0 + p_1) \, d\omega \\ &= \frac{4}{3}c^4 - \frac{16}{3}c^3 x - \frac{8}{3}c^3 + 4c^2x^2 + 8c^2x + 2c^2 - \frac{8}{3}cx^3 - 4cx - \frac{2}{3}c + \frac{2}{3}x^4 + \frac{2}{3}x + \frac{1}{12}. \end{aligned}$$

If $0 < x < 1/2$, $x + 1/2 < c \le 1$, then

$$\begin{aligned} I &= \int_0^x b_0(p_0 + p_1) \, d\omega \\ &= -4c^2x^2 + \frac{8}{3}cx^3 + 8cx^2 - \frac{2}{3}x^4 - \frac{8}{3}x^3 - 2x^2. \end{aligned}$$

If $1/2 \le x < 1$, $0 \le c \le x - 1/2$, then

$$\begin{aligned} I &= \int_0^c (b_0 + b_1) p_0 \, d\omega \\ &= \frac{2}{3}c^4 - \frac{8}{3}c^3 x + \frac{8}{3}c^3. \end{aligned}$$

If $1/2 \le x < 1$, $x - 1/2 < c < 1/2$, then

$$\begin{aligned} I &= \int_{x-1/2}^{c} b_0 p_0 \, d\omega + \int_0^{x-1/2} (b_0 + b_1) p_0 \, d\omega \\ &= -\frac{2}{3}c^4 + \frac{8}{3}c^3 x - 8c^2x^2 + 8c^2x - 2c^2 + \frac{16}{3}cx^3 - 8cx^2 \\ &\quad + 4cx - \frac{2}{3}c - \frac{4}{3}x^4 + \frac{8}{3}x^3 - 2x^2 + \frac{2}{3}x - \frac{1}{12}. \end{aligned}$$

If $1/2 \le x < 1$, $1/2 \le c \le x$, then

$$\begin{aligned} I &= \int_{x-1/2}^{c} b_0 p_0 \, d\omega + \int_{c-1/2}^{x-1/2} (b_0 + b_1) p_0 \, d\omega + \int_0^{c-1/2} (b_0 + b_1)(p_0 + p_1) \, d\omega \\ &= -2c^4 + 8c^3x - \frac{8}{3}c^3 - 8c^2x^2 + 4c^2 + \frac{16}{3}cx^3 - 8cx^2 + 8cx - 4c - \frac{4}{3}x^4 + \frac{8}{3}x^3 - 2x^2 + \frac{1}{2}. \end{aligned}$$

If $1/2 \le x < 1$, $x < c \le 1$, then

$$\begin{aligned} I &= \int_{c-1/2}^{x} b_0 p_0 \, d\omega + \int_{x-1/2}^{c-1/2} b_0(p_0+p_1)\, d\omega + \int_{0}^{x-1/2} (b_0+b_1)(p_0+p_1)\, d\omega \\ &= \frac{4}{3}c^4 - \frac{16}{3}c^3x - \frac{8}{3}c^3 + 12c^2x^2 + 4c^2 - 8cx^3 - 8cx^2 + 8cx - 4c + 2x^4 + \frac{8}{3}x^3 - 2x^2 + \frac{1}{2}. \end{aligned}$$

If $1 \le x < 3/2$, $x-1 < c \le x-1/2$, then

$$\begin{aligned} I &= \int_{x-1}^{c} (b_0+b_1)p_0 \, d\omega \\ &= \frac{2}{3}(c-x+1)^4. \end{aligned}$$

If $1 \le x < 3/2$, $x-1/2 < c \le 1$, then

$$\begin{aligned} I &= \int_{x-1/2}^{c} b_0 p_0 \, d\omega + \int_{c-1/2}^{x-1/2} (b_0+b_1)p_0 \, d\omega + \int_{x-1}^{c-1/2} (b_0+b_1)(p_0+p_1)\, d\omega \\ &= -2c^4 + 8c^3x - \frac{8}{3}c^3 - 12c^2x^2 + 8c^2x + 8cx^3 - 8cx^2 + \frac{4}{3}c - 2x^4 + \frac{8}{3}x^3 - \frac{4}{3}x + \frac{1}{2}. \end{aligned}$$

If $3/2 \le x < 2$, $x-1 < c \le 1$, then

$$\begin{aligned} I &= \int_{x-1}^{c} (b_0+b_1)p_0 \, d\omega \\ &= \frac{2}{3}(c-x+1)^4. \end{aligned}$$

Otherwise, if $1 \le x < 2$, $0 \le c \le x-1$, then $I = 0$.