Parabolic Towers and the Asymptotic Geometry of the Mandelbrot Set

by

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ABSTRACT

Understanding the geometry of the Mandelbrot set has been a central pillar of holomorphic dynamics over the past four decades. Much of its structure is now understood, but a critical question remains unresolved: is the Mandelbrot set locally connected? The first major breakthrough towards this conjecture was achieved by Yoccoz in the nineties, who proved that the Mandelbrot set is locally connected at all parameters which are not infinitely quadratic-like renormalizable. A key ingredient in Yoccoz's work is the PLY-inequality, which bounds the diameter of certain subsets, called limbs, of the Mandelbrot set. These limbs are naturally labeled by the rational numbers, and the PLY-inequality asserts that the p/q-limb of the Mandelbrot set has size $O(1/q)$. Milnor conjectured that $O(1/q^2)$ is the correct scale. For any $N \geq 1$, the main result of this thesis is to verify Milnor's conjecture for all p/q-limbs where a finite continued fraction of p/q has uniformly bounded length. Our strategy relies on careful analysis of the bifurcation of parabolic fixed points; we also further develop some of the classical theory in this area. We introduce parabolic and near-parabolic renormalization operators for maps which have parabolic fixed points of arbitrary multiplier and there perturbations, constructing invariant classes for these operators. We provide an alternative definition to the parabolic towers introduced by Epstein and construct a dynamically natural topology on the space of all parabolic towers. We also study the dynamics of Lavaurs maps, constructing analogues of polynomial external rays for these functions showing that these rays arise as the Hausdorff limits of polynomial external rays.

CHAPTER I
Introduction

A holomorphic function f is said to be *parabolic* if there is some point z_0 such that $f^n(z_0) = z_0$ and the multiplier $\lambda = (f^n)'(z_0)$ is a root of unity; the point z_0 is called a *parabolic periodic point* of f. The local dynamics of f near z_0 are relatively simple; there exists an integer $\nu \geq 1$ such that orbits under iteration of f^n are attracted toward z_0 along νq directions and repelled away from z_0 along νq other directions. The parabolic point z_0 is said to be *simple* when $\lambda = 1$ and *non-degenerate* when $\nu = 1$. Using the dynamics of f near z_0, we can construct a new family of holomorphic functions, called *Lavaurs maps*, by sending points along the attracting directions to points along the repelling directions. While this construction is a priori purely synthetic, it is closely related to the perturbation theory of f; if h is another holomorphic function which approximates f, then high iterates of h may approximate a Lavaurs map. This phenomenon is called *parabolic implosion* and was first developed by Douady and Lavaurs in [Dou94] and [Lav89].

In [Shi98] and [Shi00], Shishikura introduced the *parabolic* and *near-parabolic* renormalization operators, defined by quotienting the dynamics of f and h respectively near z_0, which provide another framework to study parabolic implosion in the simple non-degenerate setting. These operators produce new holomorphic functions which themselves could be parabolic or near-parabolic, so in some cases the renormalization operators can be repeatedly applied. Shishikura produced a class of maps which is invariant under parabolic renormalization in [Shi98], and classes of maps invariant under near-parabolic renormalization have been constructed by Inou and Shishikura in [IS08], Yang in [Yan15], and Chéritat in [Ché22]. These invariant classes have had several remarkable applications in holomorphic dynamics, see for example [BC12], [Che13], [CC15], [CS15], [SY16], [Che17], [AC18], and [Che19].

While the results of Chéritat, Inou, Shishikura, and Yang only apply to parabolic implosion in the simple non-degenerate case, it is natural to ask what can be said in the general case. This question was studied by Oudkerk in [Oud99] and[Oud02], where he gave a comprehsive description of implosion and Lavaurs maps in for degenerate parabolic maps. However, the additional complexity makes it difficult to describe parabolic renormalization operators and construct invariant classes for general parabolic implosion. If we instead consider parabolic maps which are non-degenerate, but allow the multiplier to be any root of unity, then parabolic implosion can be described in analogy to the simple case. In the first chapter of this thesis, we present parabolic implosion in this generality and observe that Chéritat's argument can be applied to construct invariant classes for the corresponding parabolic and near-parabolic renormalizations.

In [Dou94], Douady used parabolic implosion to show that the function from polynomials to their filled Julia sets is discontinuous at parabolic polynomials. The fundamental issue is that for a parabolic polynomial f we can defined an analogue of the filled Julia set for the Lavaurs maps of f, and these sets arise as the limits of filled Julia sets of perturbations of f. If a Lavaurs map L of f is also parabolic, then we can similarly define filled Julia sets of the Lavaurs maps of L; these sets also arise as the limits of filled Julia sets of perturbations of L or perturbations of f. By considering further parabolic Lavaurs maps of L and so on, we can continue enriching the set of limits of filled Julia sets of quadratic polynomials. It is natural to ask, what is the set of *all* possible limits? In [Eps93], Epstein introduced parabolic towers, dynamical systems associated to successively constructed Lavaurs maps, and studied their Julia sets. In the second chapter of this thesis we give an alternative definition of parabolic towers, instead defining them to be sequences of analytic maps constructed by successive parabolic renormalizations. While these two notions or parabolic towers are closely related, indeed they can be directly compared by analytic semi-conjugacies, the upshot of our definition is that we can use the near-parabolic renormalization operators to explicitly construct the basis of a topology on the space of parabolic towers. Using this topology and results in [Eps93], we show that there is a subset $\widehat{\text{Quad}}$ of the space of parabolic towers such that the function from quadratic polynomials to their filled Julia sets extends to the continuous function from towers in $\widehat{\text{Quad}}$ to their associated filled Julia sets. Additionally, we show that the filled Julia sets of towers $\widehat{\text{Quad}}$ give us exactly the set of all possible limits of filled Julia sets of quadratic polynomials.

For a quadratic polynomial, the points which escape under iteration of f, that is the points whose orbit tends to infinity, can be analytically labeled using the Böttcher coordinate for the polynomial. *External rays* for f are constructed from the Böttcher coordinate and can be used to combinatorially describe the dynamics of f. As both the Böttcher coordinate and external rays depend holomorphically on f, similar Böttcher coordinates and external rays can be realized in parameter spaces of polynomials. For a Lavaurs map L of a parabolic map f we can similarly consider the set of points that escape under iteration of L, where here escaping means the orbit of the point eventually leaves the parabolic basin of f. In the third chapter of this thesis, we introduce analogues of Böttcher coordinates and external rays, the latter we call *bubble rays*, for Lavaurs maps of parabolic maps, and study their geometry. The space of all Lavaurs maps associated to a fixed parabolic map provides a natural parameter space of Lavaurs maps, and just as for polynomials we realize coordinates and rays in this parameter space.

As noted above, when h is a perturbation of a parabolic map f, high iterates of h may approximate a Lavaurs map L of f. If both h and f are polynomials, and so have external rays, then we can show that in this case the external rays of h approximate bubble rays of L. Lifting this convergence to parameter space, we can show that external rays in the polynomial parameter space near f approximate the bubble rays in the parameter space of Lavaurs maps for f. But what if f and h are not polynomials? If h has some suitable analogue of external rays near the parabolic fixed point of f, then the same argument above can be used to show convergence of these rays to bubble rays of Lavaurs maps, with similar statement in parameter space. In the final chapter of this thesis, we study the convergence of rays to bubble rays in this generality. In particular, the external rays of polynomials induce suitable rays for their successive near-parabolic renormalizations, allowing us to control the geometry of the external rays of polynomials which are close to a parabolic tower in $\widehat{\mathrm{Quad}}$. This geometric control gives us the main theorem of this thesis:

Theorem I.1. *For any $N \geq 1$, there exists a constant $C_N > 0$ such that if*

$$p/q = \cfrac{\varepsilon_1}{a_1 + \cfrac{\varepsilon_2}{\ddots + \cfrac{\varepsilon_N}{a_N}}},$$

where $\varepsilon_n = \pm 1$ and $a_n \geq 2$ are integers for all $1 \leq n \leq N$, then the diameter of the p/q-limb of the Mandelbrot set has diameter bounded above by C_N/q^2 and below by $1/(C_N q^2)$.

It is natural to ask what we can say about the constant C_N in the theorem above when N tends to infinity. The integer N records exactly how many near-parabolic renormalizations of polynomials must be considered, so to tackle this question we must understand infinite near-parabolic renormalization. At present, the tools developed in this thesis cannot be applied to case of infinitely near-parabolic renormalizable polynomials, but we may hope that the invariant classes constructed Chéritat, Shishikura, and Yang may help resolve this issue in the future.

CHAPTER II
Non-Degenerate Parabolic Implosion

Let us fix some $z_0 \in \hat{\mathbb{C}}$ and a holomorphic function f defined in a neighborhood of z_0. The point z_0 is said to be *k-periodic* for some integer $k \geq 1$, or *fixed* when $k = 1$, if k is the minimal positive integer such that $f^k(z_0) = z_0$. If z_0 is k-periodic, then the *multiplier* of z_0 is the value $\lambda = (f^k)'(z_0)$; z_0 is said to be *attracting*, *repelling*, or *indifferent* if $|\lambda| < 1$, $|\lambda| > 1$, or $|\lambda| = 1$ respectively. We will say that z_0 is p/q-parabolic for some $p/q \in \mathbb{Q}$ if $\lambda = e^{2\pi i p/q}$. If z_0 is k-periodic and p/q-parabolic, then it is *non-degenerate* if

$$f^{kq}(z) = z + a(z - z_0)^{q+1} + O((z - z_0)^{q+2})$$

for z close to z_0 for some $a \in \mathbb{C}^*$. After conjugating by a Möbius transformation and replacing f with f^k, we will usually assume that $0 = z_0$ is a parabolic fixed point of f. The arithmetic properties of p/q will play a role in the local dynamics of f near 0 and its perturbations, so we will briefly recall some facts about continued fractions.

II.1: Modified continued fractions

Let us fix some rational number $x \in [-1/2, 1/2]$. We can associate a (possibly empty) sequence $\omega_x = \langle (a_n, \varepsilon_n) \rangle_{n=1}^{N}$, called the *modified continued fraction* of x, as follows. First, let $x_0 = x$ and assume that $x_n \in [-1/2, 1/2]$ is defined for some $n \geq 0$. If $x_n \neq 0$, then there exists a unique $\varepsilon_{n+1} \in \{\pm 1\}$ and a unique $y_{n+1} \in (0, 1/2]$ such that $x_n = \varepsilon_{n+1} \cdot y_{n+1}$. In this case, there exists a unique integer $a_{n+1} \geq 2$ and a unique $x_{n+1} \in (-1/2, 1/2]$ such that

$$\frac{1}{y_{n+1}} = a_{n+1} + x_{n+1}.$$

We repeat this construction recursively to produce ω_x; as x is rational the sequence is guaranteed to be finite. For $N \geq 0$, we denote by $\mathbb{Q}_N$ the set of all rational numbers in $[-1/2, 1/2]$ whose modified continued fraction has length N. It follows from our construction that if $a_n = 2$ and $\varepsilon_n = -1$, then $n = 1$.

The *approximates* of x are inductively defined as

$$\mathfrak{p}_{-1}(x) = 1, \qquad \mathfrak{p}_0(x) = 0, \qquad \mathfrak{p}_n(x) = a_n \mathfrak{p}_{n-1}(x) + \varepsilon_n \mathfrak{p}_{n-2}(x) \qquad \text{for } 1 \leq n \leq N,$$

$$\mathfrak{q}_{-1}(x) = 0, \qquad \mathfrak{q}_0(x) = 1, \qquad \mathfrak{q}_n(x) = a_n \mathfrak{q}_{n-1}(x) + \varepsilon_n \mathfrak{q}_{n-2}(x) \qquad \text{for } 1 \leq n \leq N.$$

By construction, if $x = p/q \in \mathbb{Q}_N$, then $p = \mathfrak{p}_N(x)$ and $q = \mathfrak{q}_N(x)$. If $N > 0$, then we will call $\mathfrak{p}_{n-1}(p/q)/\mathfrak{q}_{n-1}(p/q)$ the *parent* of p/q.

Proposition II.1. *If $x \in \mathbb{Q}_N$, then $\mathfrak{q}_n(x) \geq \frac{3}{2}\mathfrak{q}_{n-1}(x)$ for all $0 \leq n \leq N$.*

Proof. This holds automatically when $n = 0$, so we assume that $n > 0$ and $\mathfrak{q}_{n-1}(x) \geq \frac{3}{2}\mathfrak{q}_{n-2}(x)$. If $a_n > 2$, then

$$\mathfrak{q}_n(x) \geq 3\mathfrak{q}_{n-1}(x) - \mathfrak{q}_{n-2}(x) \geq \frac{3}{2}\mathfrak{q}_{n-1}(x).$$

If $a_n = 2$, then either $\varepsilon_n = +1$ or $j = 1$. As $\mathfrak{q}_{-1}(x) = 0$, in either case we have

$$\mathfrak{q}_j(x) \geq 2\mathfrak{q}_{n-1}(x) \geq \frac{3}{2}\mathfrak{q}_{n-1}(x).$$

□

Corollary II.2. *If $x \in \mathbb{Q}_N$, then*

$$\left(\frac{2}{3}\right)^N \cdot q < \prod_{n=1}^{N} a_n \leq \left(\frac{4}{3}\right)^N \cdot q$$

Proof. When $x = 0/1$, we have

$$q = 1 = \prod_{n=1}^{0} a_n.$$

So now we assume that $N > 0$ and that the proposition holds for the parent of x. As $a_N \geq 2$, proposition II.1 implies

$$\frac{2}{3} \cdot a_n \mathfrak{q}_{n-1}(x) \leq a_n \mathfrak{q}_{n-1}(x) + \varepsilon_n \mathfrak{q}_{n-2}(x) \leq \frac{4}{3} \cdot a_n \mathfrak{q}_{n-1}(x).$$

The proposition therefore follows by induction on N. □

We define the *signature* of x to be

$$\mathfrak{S}(x) := (-1)^N \prod_{n=1}^{N} \varepsilon_n.$$

Proposition II.3. *If $x \in \mathbb{Q}_N$, then*

$$\mathfrak{S}(x) = \mathfrak{p}_{N-1}(x)\mathfrak{q}_N(x) - \mathfrak{p}_N(x)\mathfrak{q}_{N-1}(x).$$

Proof. If $N = 1$, then $\mathfrak{S}(x) = 1 = 1 \cdot 1 - 0 \cdot 0$. If $N > 1$, then

$$\begin{aligned}\mathfrak{p}_{N-1}(x)\mathfrak{q}_N(x) - \mathfrak{p}_N(x)\mathfrak{q}_{N-1}(x) &= \mathfrak{p}_{N-1}(x)(a_N\mathfrak{q}_{N-1}(x) + \varepsilon_N\mathfrak{q}_{N-2}(x)) - \mathfrak{q}_{N-1}(x)(a_N\mathfrak{p}_{N-1}(x) + \varepsilon_N\mathfrak{p}_{N-2}(x)) \\ &= -\varepsilon_N(\mathfrak{p}_{N-2}(x)\mathfrak{q}_{N-1}(x) - \mathfrak{p}_{N-1}(x)\mathfrak{q}_{N-2}(x))\end{aligned}$$

by induction. □

We define the Möbius transformation $\mu_x : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by

$$\mu_x(z) := \frac{\mathfrak{p}_N(x) + \mathfrak{S}(x) \cdot z \cdot \mathfrak{p}_{N-1}(x)}{\mathfrak{q}_N(x) + \mathfrak{S}(x) \cdot z \cdot \mathfrak{q}_{N-1}(x)} = \cfrac{\varepsilon_1}{a_1 + \cfrac{\varepsilon_2}{\ddots + \cfrac{\varepsilon_N}{a_N + \mathfrak{S}(x) \cdot z}}}. \tag{II.1.1}$$

When $x \neq 0$ and x' is the parent of x, we have

$$\mu_x(z) = \mu_{x'}\left(\frac{\varepsilon_n \cdot \mathfrak{S}(x')}{a_n + \mathfrak{S}(x) \cdot z}\right). \tag{II.1.2}$$

We conclude this section by recording some important properties of μ_x.

Proposition II.4. *If $z \in \overline{\mathbb{D}}$, then $\mu_x(z) \in \overline{\mathbb{D}}$.*

Proof. If $x = 0$, then $\mu_x(z) = z$. So we assume that $x \in \mathbb{Q}_N$ with $N > 0$ and that the proposition holds for the parent x' of x. As $a_N \geq 2$, if $|z| \leq 1$ then

$$\left|\frac{\epsilon_N \cdot \mathfrak{S}(x')}{a_N + \mathfrak{S}(x) \cdot z}\right| \leq \frac{1}{a_N - 1} \leq 1.$$

The inductive hypothesis combined with (II.1.2) implies $|\mu_x(z)| \leq 1$. □

Proposition II.5. *If* $z, w \in \mathbb{D}$ *then*

$$|\mu_x(z) - \mu_x(w)| < \frac{9|z-w|}{\mathfrak{q}(x)^2}.$$

Proof. Using (II.1.1) and proposition II.1, we can directly compute

$$\begin{aligned} |\mu_x(z) - \mu_x(w)| &= \left| \frac{(\mathfrak{p}_{N-1}(x)\mathfrak{q}_N(x) - \mathfrak{p}_N(x)\mathfrak{q}_{N-1}(x)) \cdot \mathfrak{S}(x) \cdot (z-w)}{(\mathfrak{q}_N(x) + \mathfrak{S}(x) \cdot z\mathfrak{q}_{N-1}(x))(\mathfrak{q}_N(x) + \mathfrak{S}(x) \cdot w\mathfrak{q}_{N-1}(x))} \right| \\ &\leq \frac{|z-w|}{(\mathfrak{q}_N(x) - \mathfrak{q}_{N-1}(x))^2} \\ &\leq \frac{9|z-w|}{\mathfrak{q}(x)^2}. \end{aligned}$$

□

Proposition II.6. $\mu_{-x}(z) = -\mu_x(-z)$.

Proof. Straightforward inductions show that $\mathfrak{S}(-x) = -\mathfrak{S}(x)$, $\mathfrak{p}(-x) = -\mathfrak{p}(x)$, and $\mathfrak{q}(-x) = \mathfrak{q}(x)$. It then follows from (II.1.1) that $\mu_{-x}(z) = -\mu_x(-z)$. □

II.2: Parabolic Renormalization

For an analytic map $f : \hat{\mathbb{C}} \dashrightarrow \hat{\mathbb{C}}$, we define a *petal* for f to be a Jordan domain $P \subset Dom(f)$ such that f is univalent on P and there exists a univalent map $\phi : P \to \mathbb{C}$, called a *Fatou coordinate*, which satisfies:

1. For all $z \in P$, the following are equivalent:
   - $f(z) \in P$.
   - $\phi(z) + 1 \in \phi(P)$.
   - $\phi(f(z)) = \phi(z) + 1$.
2. If both w and $w + n$ belong to $\phi(P)$ for some integer $n \geq 0$, then $w + j \in \phi(P)$ for all $0 \leq j \leq n$.
3. For any $w \in \mathbb{C}$, there exists $n \in \mathbb{Z}$ such that $w + n \in \phi(P)$.

A Fatou coordinate ϕ is unique up to post-composition with a translation. If $f(P) \subset P$ or $P \subset f(P)$, then we will say that P is an *attracting* or *repelling* petal respectively. We will call any petal P' of f contained in P a *sub-petal* of P.

Let us now fix a holomorphic function f which has a non-degenerate p/q-parabolic fixed point at 0. The following classical result describes how the local dynamics of f near 0 can be completely characterized by petals.

Theorem II.7. *For any neighborhood V of 0, there exist attracting and repelling petals P^f_{att} and P^f_{rep} respectively for f^q which satisfy (see Figure II.1):*

1. *The following geometric conditions:*
   (a) *The union*
   $$\bigcup_{n=0}^{q-1} f^{nk}(P^f_{att} \cup P^f_{rep})$$
   forms a punctured neighborhood of 0 contained in V.
   (b) *$P^f_{att} \cap P^f_{rep}$ is either a Jordan domain with 0 on its boundary if $q > 1$, or the union of two Jordan domains with 0 on their boundaries if $q = 1$.*
   (c) *There exists some $r_0 > 0$ and $\theta \in \mathbb{R}$ such that*
   $$re^{2\pi i\theta} \in P^f_{att} \setminus \overline{P^f_{rep}} \text{ and } re^{2\pi i(\theta+1/2q)} \in P^f_{rep} \setminus \overline{P^f_{att}}$$
   for all $0 < r \leq r_0$.
2. *The following dynamical conditions:*
   (a) *The sets $P^f_{att}, f(P^f_{att}), \ldots, f^{kq-1}(P^f_{att})$ are all pair-wise disjoint.*
   (b) *Every forward or backward orbit under f which converges asymptotically towards $z = 0$ intersects P^f_{att} or P^f_{rep} respectively.*

Proof. See [Mil06, Chapter 10]. □

Let us fix some P^f_{att} and P^f_{rep} as in theorem II.7. We will also denote $P^f := P^f_{att} \cup P^f_{rep}$. We can analytically extend the attracting Fatou coordinate by defining

$$\rho^f(z) := \phi^f_{att}(f^{nq+m}(z)) - n$$

for all $n \geq 0$, $0 \leq m < q$, and z such that $f^{nq+m}(z) \in P^f_{att}$. We can similarly extend the inverse of the repelling Fatou coordinate by defining

$$\chi^f(w) := f^{nq} \circ (\phi^f_{rep})^{-1}(w - n)$$

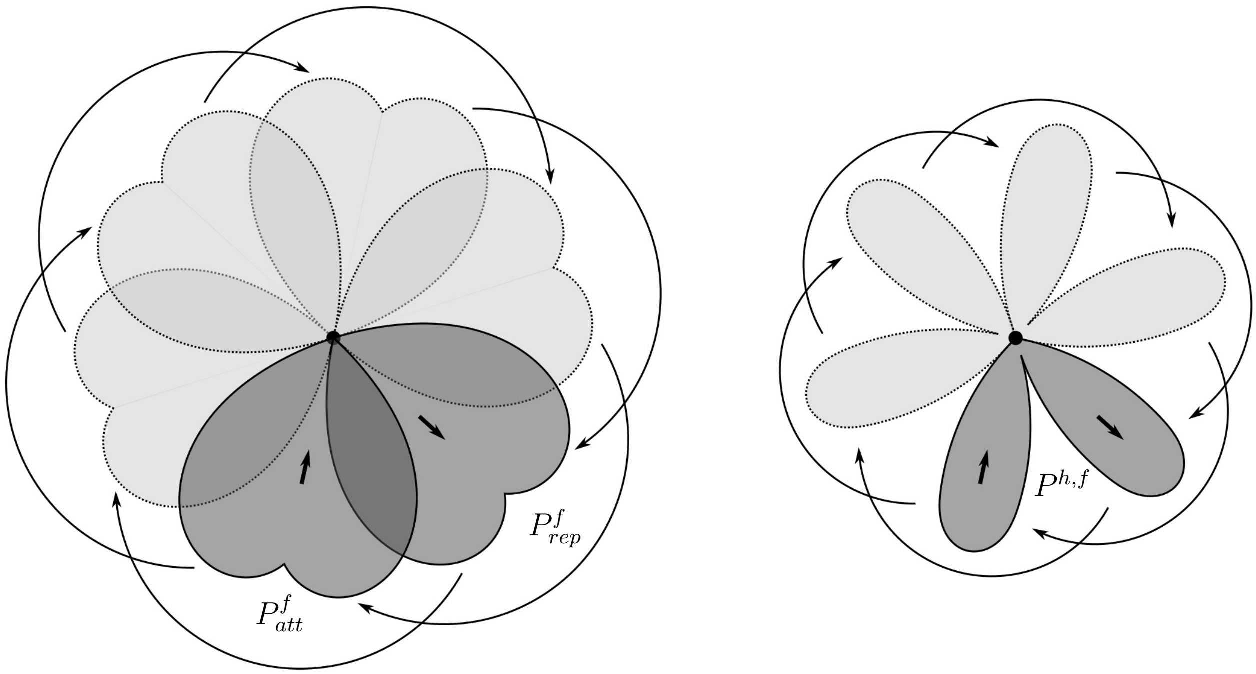


Figure II.1: *Left:* Attracting and repelling petals near a 2/3-parabolic fixed point of f. The long external arrows indicate the action of f, the short internal arrows indicate the action of f^3. *Right:* A petal for a positively implosive perturbation of h of f as in theorem II.22.

for all $n \geq 0$ such that $w - n \in \phi^f_{rep}(P^f_{rep})$ and $(\phi^f_{rep})^{-1}(w-n) \in Dom(f^{nq})$. The *horn map* for f is defined to be

$$H^f := \rho^f \circ \chi^f.$$

Note that the maps ρ^f, χ^f, and H^f depend on our choice of petals and Fatou coordinates for f.

For any integer n, we denote $T_n(w) = w + n$ for all $w \in \mathbb{C}$.

Proposition II.8. *The horn map H^f is defined and univalent on both an upper and a lower half-plane and satisfies:*

$$H^f \circ T_1 = T_1 \circ H^f.$$

There are two constants $c^f_\pm$ such that $H^f(w) - w \to c^f_\pm$ when $\operatorname{Im} w \to \pm\infty$.

Proof. See [Shi00] for the $p/q = 0/1$ case, the general p/q case follows by similar argument. □

Note that changing the attracting or repelling Fatou coordinates post or pre-composes the horn map by a translation. For the rest of this thesis, we will always choose the repelling

Fatou coordinate relative to the attracting Fatou coordinate so that

$$c_+^f = \frac{1 - \mathfrak{S}(p/q)}{2} = \begin{cases} 0 & \text{if } \mathfrak{S}(p/q) = +1, \\ 1 & \text{if } \mathfrak{S}(p/q) = -1, \end{cases}.$$

The horn map H^f is therefore well-defined up to conjugation by a translation.

Let us also record the following fact on the intersection of petals:

Proposition II.9. *For any $x \in \mathbb{R}$, if $\phi_{rep}^f(z) = x - iy$ for some sufficiently large $y > 0$, then $z \in P_{att}^f$. If instead $\phi_{rep}^f(z) = x + iy$ for some sufficiently large $y > 0$, then $f^n(z) \in P_{att}^f$ where $0 \leq n < q$ satisfies $np \equiv -1 \mod q$.*

We will call the component of $P_{att}^f \cap P_{rep}^f$ which contains $\phi_{rep}^f(-t)$ for large $t > 0$ the *lower component*.

Let $\iota : \mathbb{C} \to \mathbb{C}$ be the map which sends each point to its complex conjugate and set $\operatorname{Exp}^+(z) := e^{2\pi i z}$ and $\operatorname{Exp}^-(z) = \operatorname{Exp}^+ \circ \iota(z)$. We define a *parabolic renormalization* of f to be a function of the form

$$\mathcal{R}_\delta^\pm f := \operatorname{Exp}^\pm \circ T_{\delta - c_\pm^f} \circ H^f \circ (\operatorname{Exp}^\pm)^{-1} = \operatorname{Exp}^\pm(\delta) \cdot \mathcal{R}_0^\pm f$$

with $\delta \in \mathbb{C}$. It follows from proposition II.8 that the domain of $\mathcal{R}_\delta^\pm f$ contains punctured neighborhoods of 0 and ∞, and we can analytically extend $\mathcal{R}_\delta^\pm f$ by setting $\mathcal{R}_\delta^\pm f(0) = 0$ and $\mathcal{R}_\delta^\pm f(\infty) = \infty$. Moreover, this extension satisfies

$$(\mathcal{R}_\delta^\pm f)'(0) = \operatorname{Exp}^\pm(\delta) \text{ and } (\mathcal{R}_\delta^\pm f)'(\infty) = \operatorname{Exp}^\pm(\pm(c_+^f - c_-^f) - \delta)$$

While the parabolic renormalization is only well-defined up to linear conjugacy, in the next subsection we will restrict to maps for which the parabolic renormalization can be uniquely defined.

The renormalizations $\mathcal{R}_\delta^+ f$ and $\mathcal{R}_\delta^- f$ are called *top* and *bottom* parabolic renormalizations respectively. We will be primarily interested in top renormalizations and denote $\operatorname{Exp} := \operatorname{Exp}^+$ and $\mathcal{R}_\delta := \mathcal{R}_\delta^+$. We also define the extremal parabolic renormalizations

$$\mathcal{R}_{+i\infty} f : Dom(\mathcal{R}_0 f) \setminus \{\infty\} \to \{0\} \text{ and } \mathcal{R}_{+i\infty} f : Dom(\mathcal{R}_0 f) \setminus \{0\} \to \{\infty\}$$

to be the corresponding constant functions, so $\mathcal{R}_\delta f \to \mathcal{R}_{\pm i\infty} f$ when $\operatorname{Im} \delta \to \pm\infty$.

Let us note that our terminology here differs from other conventions in the literature. Specifically, $\mathcal{R}_0^+ f$ and $\mathcal{R}_0^- f$ are called *the* top and bottom parabolic renormalizations of f respectively in [IS08] as they both have a non-degenerate 0/1-parabolic fixed point at 0.

These specific renormalizations will not play as special of a role in this thesis, so we will not name them.

For any $\delta \in \mathbb{C}$, we define the *attracting δ-elevator of f* to be the map

$$\eta^f_{att,\delta} := \operatorname{Exp} \circ T_{\delta - c^f_+} \circ \rho^f.$$

We also define the *repelling elevator of f* to be the map

$$\eta^f_{rep} := \operatorname{Exp} \circ \phi^f_{rep}.$$

We have the following alternative definition of parabolic renormalizations in terms of elevators:

$$\mathcal{R}_\delta f = \eta^f_{att,\delta} \circ (\eta^f_{rep})^{-1}.$$

II.2.1: Invariant classes

We will now focus on the class $\mathcal{F}$ of holomorphic functions $f : \hat{\mathbb{C}} \dashrightarrow \hat{\mathbb{C}}$ which satisfy the following three conditions:

1. $Dom(f)$ is open and contains both 0 and ∞;
2. $f(0) = 0$ and $f(\infty) = \infty$;
3. the restriction
$$f : f^{-1}(\mathbb{C}^*) \to \mathbb{C}^*$$
is a branched covering map with a unique critical value cv^f and all critical points are of local degree 2.

For example, the class $\mathcal{F}$ contains every quadratic polynomial of the form

$$\mathfrak{f}_\alpha(z) := e^{2\pi i \alpha} z + z^2$$

with $\alpha \in \mathbb{C}$.

Proposition II.10. *Any map in $\mathcal{F}$ can have at most one parabolic cycle, and the cycle is non-degenerate if it exists.*

Proof. This follows from the uniqueness of the critical value and a standard argument, see for example [Shi00, Lemma 4.5.2]. □

We denote by $\mathcal{F}^{\circledast}$ the set of all maps in $\mathcal{F}$ which have a parabolic cycle. Fixing some $f \in \mathcal{F}^{\circledast}$, we denote by

$$U^f := \bigcup_{n\geq 0} f^{-n}(P^f_{att}) = Dom(\rho^f)$$

the parabolic basin of f. The *immediate* parabolic basin U^f_0 is defined to be the component of U^f which contains the critical value cv^f.

Proposition II.11. *There exists a unique choice of P^f_{att} and ϕ^f_{att} such that $cv^f \in \partial P^f_{att}$, $\phi^f_{att}(z) \to 0$ when $z \to cv^f$, and*

$$\phi^f_{att}(P^f_{att}) = \{w \in \mathbb{C} : \operatorname{Re} w > -|\operatorname{Im} w|\}.$$

For any sufficiently large $t > 0$ and $\zeta \in \mathbb{C}$ satisfying $\operatorname{Re}\zeta < |\operatorname{Im}\zeta| - t$, there is a unique choice of P^f_{rep} and ϕ^f_{rep} such that

$$\phi^f_{rep}(P^f_{rep}) = \{w - \zeta \in \mathbb{C} : \operatorname{Re} w < |\operatorname{Im} w|\}.$$

We equip the space of holomorphic functions $\hat{\mathbb{C}} \dashrightarrow \hat{\mathbb{C}}$ with the *compact-open topology with domains*, that is a neighborhood of f is a set of the form

$$\left\{h : \hat{\mathbb{C}} \dashrightarrow \hat{\mathbb{C}} \mid X \subset \operatorname{Dom}(h), \sup_{z\in X} |f(z) - h(z)| < \epsilon\right\}$$

where X is a compact subset of $Dom(f)$ and $\epsilon > 0$. We will write $h \to f$ when h converges to f in this topology. If h is a holomorphic function sufficiently close to f which has a unique critical value cv^h and a a parabolic cycle with the same period and multiplier as f, then proposition II.11 holds for h; we will say that h is a *stable perturbation* of f. We can similarly define stable perturbations of h so that proposition II.11 holds for these maps as well. We define $\operatorname{Comp}^*(\hat{\mathbb{C}})$ to be the set of all non-empty compact subsets of $\hat{\mathbb{C}}$ and equip it with the *Hausdorff metric*, so the distance between $X_1, X_2 \in \operatorname{Comp}^*(\hat{\mathbb{C}})$ is given by

$$\sup_{w\in X_1\cup X_2} \left| \inf_{z_1\in X_1} d_{\hat{\mathbb{C}}}(z_1, w) - \inf_{z_2\in X_2} d_{\hat{\mathbb{C}}}(z_2, w) \right|$$

where $d_{\hat{\mathbb{C}}}$ is the spherical metric. Given non-empty open proper subsets V_1 and V_2 of $\hat{\mathbb{C}}$, we will also say that V_1 converges to V_2 when $\overline{V_1} \to \overline{V_2}$ and $\hat{\mathbb{C}} \setminus V_1 \to \hat{\mathbb{C}} \setminus V_2$ in the Hausdorff metric.

Proposition II.12. *If the same ζ is chosen in proposition II.11, then the compact sets $\overline{P^f_{att}}$, $\overline{P^f_{rep}}$, $\hat{\mathbb{C}} \setminus P^f_{att}$, and $\hat{\mathbb{C}} \setminus P^f_{rep}$ depend continuously on f. Moreover, the Fatou coordinates ϕ^f_{att}*

and ϕ^f_{rep} depend continuously and holomorphically on f.

Proof. See [IS08] or [Oud99]. □

Returning to the dynamics of $f \in \mathcal{F}^{\text{✿}}$, we have the following property of the parabolic basin of f.

Proposition II.13. *Every component of U^f is simply connected, and the first return map to U^f_0 is holomorphically conjugate to the restriction of $\mathfrak{f}_0$ to $U^{\mathfrak{f}_0}$.*

Proof. See [Shi00, Lemma 4.5.2]. □

The parabolic renormalizations $\mathcal{R}_\delta f$ with $\delta \in \mathbb{C}$ are defined exactly on

$$(\mathrm{Exp} \circ (\chi^f)^{-1}(U^f)) \cup \{0, \infty\}.$$

Thus the geometry of U^f descends to the geometry of $Dom(\mathcal{R}_0 f)$.

Corollary II.14. *Every component of $Dom(\mathcal{R}_0 f)$ is simply connected. Moreover, if every component of U^f is a Jordan domain, then every component of $Dom(\mathcal{R}_0 f)$ is a Jordan domain.*

Let us also record here the following properties of the extended Fatou coordinate ρ.

Lemma II.15. *Assume that U^f_0 is a Jordan domain and let $\tilde{X} \subset \mathbb{C}$ be an unbounded simply connected set which avoids the negative integers. There is a unique component of X of $U^f_0 \cap (\rho^f)^{-1}(X)$ which has the parabolic fixed point 0 on its boundary, and ρ^f is univalent on the interior of X. If 0 is in the closure of $\tilde{X}$, then cv^f is in the closure of X. If $\tilde{X}$ is an open set whose boundary is a Jordan arc, then X is a Jordan domain whose boundary intersects ∂U^f_0 only at 0.*

Proof. It follows from the definition that ρ^f is a covering map branched over the negative integers. Thus ρ^f is a covering map over X. Let us assume that X avoids the $\mathbb{R}_{\leq -1}$. There is a unique branch of $(\rho^f)^{-1}$ defined on $\mathbb{C} \setminus \mathbb{R}_{\leq -1}$ which is invariant under f^q and has image in U^f_0; it follows from classical arguments that this component contains cv^f. The boundary of the image of this branch intersects ∂U^f_0 in a connected nonempty f-invariant set, this set is therefore $\{0\}$. The lemma follows. If X does not avoid $\mathbb{R}_{\leq -1}$, then we can apply the same argument to one of the unbounded components of $X \setminus \mathbb{R}_{\leq -1}$ and then perform an analytic continuation. □

One of the main reasons for restricting to the class $\mathcal{F}$ is that it is invariant under parabolic renormalization:

Proposition II.16. *For all $\delta \in \mathbb{C}$, $\mathcal{R}_\delta f \in \mathcal{F}$ and $cv^{\mathcal{R}_\delta f} = \operatorname{Exp}(\delta) = \eta^f_{att,\delta}(cv^f)$.*

Proof. See [Shi00, Lemma 4.5.5] and [Ché22]. □

We denote by $Dom_0(\mathcal{R}_0 f)$ the component of the domain of $\mathcal{R}_0 f$ which contains 0. Let $\varphi^{\mathfrak{f}_0} : \mathbb{D} \to Dom_0(\mathcal{R}_0^{\mathfrak{f}_0})$ be the unique analytic isomorphism with $(\varphi^{\mathfrak{f}_0})'(0) > 0$.

Proposition II.17. *There exists a unique univalent map $\varphi^f : \mathbb{D} \to Dom_0(\mathcal{R}_0 f)$ such that $(\varphi^f)'(0) = (\varphi^{\mathfrak{f}_0})'(0)$ and*

$$(\mathcal{R}_0 f) \circ \varphi^f = (\mathcal{R}_0 \mathfrak{f}_0) \circ \varphi^{\mathfrak{f}_0}.$$

Proof. See [Shi00], [LY14a], or [Ché22]. □

For all $\epsilon > 0$, we define the classes of maps

$$\mathcal{S}_\epsilon := \left\{ \varphi : \mathbb{D}_{1-\epsilon} \to \mathbb{C} \mid \varphi \text{ is univalent}, \varphi(0) = 0, \text{ and } \varphi'(0) = (\varphi^{\mathfrak{f}_0})'(0). \right\}$$

and

$$\mathcal{F}_\epsilon := \left\{ (\mathcal{R}_0 \mathfrak{f}_0) \circ \varphi^{\mathfrak{f}_0} \circ \varphi^{-1} \mid \varphi \in \mathcal{S}_\epsilon \right\}.$$

Equipped with the compact-open topology, it follows from the Koebe distortion theorem that these classes are all compact. Proposition II.17 that $\mathcal{R}_0 f$ has a restriction in $\mathcal{F}_0$. The above constructions could also be recreated for the bottom parabolic renormalization $\mathcal{R}_0^- f$; the following proposition shows that we get the same class of maps:

Proposition II.18. $\mathcal{R}_0^+ \mathfrak{f}_0 = \mathcal{R}_0^- \mathfrak{f}_0$.

Proof. It is shown in [LY14a] that $\iota \circ \mathfrak{f}_0 \circ \iota = \mathfrak{f}_0$ implies that

$$\iota \circ H^{\mathfrak{f}_0} \circ \iota = H^{\mathfrak{f}_0}.$$

Thus $c_-^{\mathfrak{f}_0} = \iota(c_+^{\mathfrak{f}_0})$, so

$$\mathcal{R}_0^- \mathfrak{f}_0 = \operatorname{Exp} \circ \iota \circ T_{-c_-^{\mathfrak{f}_0}} \circ H^{\mathfrak{f}_0} \circ \iota \circ \operatorname{Exp}^{-1} = \operatorname{Exp} \circ T_{-c_+^{\mathfrak{f}_0}} \circ \iota \circ H^{\mathfrak{f}_0} \circ \iota \circ \operatorname{Exp}^{-1} = \mathcal{R}_0^+ \mathfrak{f}_0$$

□

For any $f \in \mathcal{F}_0$ and $p/q \in \mathbb{Q}$, the map $e^{2\pi i p/q} f$ belongs to $\mathcal{F}^{✾}$ and is mapped into $\mathcal{F}_0$ by $\mathcal{R}_0$. We define the *p/q-parabolic fiber renormalization* of f to be

$$\mathcal{R}_{p/q,0} f := \mathcal{R}_0(e^{2\pi i p/q} f),$$

so

$$\mathcal{R}_\delta(e^{2\pi i p/q} f) = e^{2\pi i\delta}\mathcal{R}_{p/q,0} f.$$

Thus the class $\mathcal{F}_0$ is invariant under $\mathcal{R}_{p/q,0}$ for every choice of p/q. For any sequence $\langle p_n/q_n\rangle_{n=1}^\infty$ of rational numbers and $f \in \mathcal{F}_0$, we can define the sequence of maps

$$\langle \mathcal{R}_{p_n/q_n,0} \cdots \mathcal{R}_{p_1/q_1,0} f\rangle_{n=1}^\infty.$$

When each $p_n/q_n = 0/1$, the above sequence has studied in [IS08], [LY14a], and [Ché22]; in particular it is shown that $\mathcal{R}_0 = \mathcal{R}_{0/1,0}$ has a unique attracting fixed point in $\mathcal{F}_0$. We will spend the remainder of this section showing that those results can be extended to more general sequences of rational numbers.

We will first work with $\mathcal{F}_\epsilon$ with $\epsilon > 0$. By restricting the domain, we have a natural inclusion map $\mathcal{F}_\epsilon \hookrightarrow \mathcal{F}_{\epsilon'}$ for any $0 \le \epsilon < \epsilon'$. However, this map is not surjective. For any $\epsilon > 0$ and $\varphi \in \mathcal{S}_0$, the function

$$z \mapsto (1-\epsilon)\varphi\left(\frac{z}{1-\epsilon}\right)$$

belongs to $\mathcal{S}_\epsilon$. This induces homeomorphisms $\mathcal{S}_0 \to \mathcal{S}_\epsilon$ and $\mathcal{F}_0 \to \mathcal{F}_\epsilon$ which depend continuously on ϵ.

Proposition II.19. *For any $p/q \in \mathbb{Q}$ there exists some $\epsilon > 0$ such that $\mathcal{R}_{p/q,0} f$ is defined for all $f \in \mathcal{F}_{\epsilon'}$ and $0 \le \epsilon' \le \epsilon$.*

Proof. This follows immediately from the compactness of $\mathcal{F}_0$. □

Theorem II.20. *For all $p/q \in \mathbb{Q}$, if $\epsilon > 0$ is sufficiently small then there exists $0 < \epsilon' < \epsilon$ such that $\mathcal{R}_{p/q,0} f$ has a restriction in $\mathcal{F}_{\epsilon'}$ for all $f \in \mathcal{F}_\epsilon$.*

Proof. For $p/q = 0/1$, this theorem is the main result in [Ché22]. We will not recreate the full argument here; we will instead examine the two main steps used by Chéritat and observe that the same reasoning applies to the general case. For details, see [Ché22].

Let us fix some $f \in \mathcal{F}_0$, $p/q \in \mathbb{Q}$, and set $h = e^{2\pi i p/q} f$. The first part of the argument in [Ché22] is a contraction: showing that for any $\epsilon > 0$ sufficiently small there exists some $0 < \epsilon' \ll \epsilon$ such that the restriction of $\mathcal{R}_{p/q,0} f = \mathcal{R}_0 h$ in $\mathcal{F}_{\epsilon'}$ depends only on the restriction of f in $\mathcal{F}_\epsilon$. Proving this fact requires showing that U_0^h intersects only finitely many connected components of

$$B := h^{-1}(\{z \in \mathbb{C}^* : |z| \ne |cv^h|\}),$$

and comparing four different metrics:

1. The hyperbolic metric on $Dom_0(\mathcal{R}_0 h)$.

2. The hyperbolic metric on U_0^h.

3. The *box-Euclidean* metric on $Dom_0(h)$, which is the pull-back of the flat metric on $\mathbb{C}^*$ by h.

4. The hyperbolic metric on $Dom_0(h) = Dom_0(f)$.

When we switch to general $p/q \in \mathbb{Q}$, we must consider the structure of $\bigcup_{n=0}^{q-1} U_0$ instead of just U_0^h. In [Ché22], showing that U_0^h intersects only finitely many components of B when $q = 1$ follows from studying the geometry of the set $\rho_h^{-1}(\mathbb{R}_{\geq -1})$. The same analysis can be applied to show that $\bigcup_{n=0}^{q-1} U_0^h$ intersects only finitely many components of B in the general case. The comparison of the four metrics is identical in the $p/q = 0/1$ and general p/q case.

The second part of the argument in [Ché22] is a perturbation: showing that for any ϵ sufficiently small and $f \in \mathcal{F}_0$, when we apply the homeomorphism $\mathcal{F}_0 \to \mathcal{F}_\epsilon$ the orbits of h do not move very far. For this step, the argument for general p/q is identical to the $p/q = 0/1$ case. □

The key feature of theorem II.20 is that we can pick $\epsilon' < \epsilon$. As a consequence, the following proposition shows that every parabolic fiber renormalization is contracting.

Proposition II.21. *For any $0 < \epsilon' < \epsilon$, there is a complete metric $\mathfrak{d}$ on $\mathcal{F}_{\epsilon'}$, such that if $\mathcal{R} : \mathcal{F}_\epsilon \to \mathcal{F}_{\epsilon'}$ is an holomorphic operator, then*

$$\mathfrak{d}(\mathcal{R}(f_1), \mathcal{R}(f_2)) < \frac{1-\epsilon}{1-\epsilon'}\mathfrak{d}(f_1, f_2)$$

for all $f_1, f_2 \in \mathcal{F}_{\epsilon'}$. Convergence in this metric implies convergence in the compact-open topology.

Proof. This is essentially the same as Main Theorem 2 in [IS08]; the same argument can be applied. □

II.3: Near-parabolic renormalization

While the petals, Fatou coordinates, and parabolic renormalizations were defined in the previous section for maps with a parabolic periodic cycle, the fundamental theorem of parabolic implosion is that similar objects can be constructed for maps which have a periodic cycle with multiplier close to a root of unity. The prototypical example is an analytic map h satisfying $h(0) = 0$ and $h'(0) = \lambda$ for some λ close to 1.

II.3.1: Perturbed petals and Fatou coordinates

As in the previous section, we now fix a map f which has a p/q-parabolic k-periodic cycle for some rational p/q and $k \geq 1$. While most of the constructions in this section can be made in this generality, we will also add the assumption that f has a unique critical value cv^f and that proposition II.11 holds; so for example f is a stable perturbation of a map in $\mathcal{F}^{\circledast}$. After conjugating by Möbius transformations, we can assume that 0 is the parabolic periodic point on the boundary of P^f_{att}. Let $h : \hat{\mathbb{C}} \dashrightarrow \hat{\mathbb{C}}$ be another holomorphic function such that

$$h^k(z) = e^{2\pi i \mu_{p/q}(\alpha)} z + O(z^2)$$

for z close to 0 and $\alpha \in \mathbb{C}$. Setting

$$A^+_{1/2} := \{z \in \mathbb{C} \setminus \{0\} : |\alpha| < 1/2, |\arg \alpha| < \pi/5\},$$

we will say that h is a *positively implosive perturbation* of f if $\alpha \in A^+_{1/2}$.

Theorem II.22. *For any $\theta \in [-\pi/4, \pi/4]$ and sufficiently large $M > 0$, if h is a sufficiently close positively implosive perturbation of f, then there exists a unique petal $P^{h,f}$ for h^{kq} satisfying (see Figures II.1 and II.2):*

1. *The following geometric conditions:*
   - *(a) $P^{h,f}$ is bounded by two arcs joining 0 to a non-zero fixed point $\sigma^{h,f}$ of h^{kq}.*
   - *(b) $cv^h \in \partial P^{h,f}$.*
   - *(c) There exists a branch of $\log$ defined on $P^{h,f}$ such that*

     $$\frac{\log P^{h,f}}{2\pi i} \subset \{e^{i\theta'} w \in \mathbb{C} : |\mathrm{Re}\, w| < M'\}$$

     for some constant M' which depends does not depend on h and for some

     $$\theta' = \theta + O(\alpha).$$

2. *The following dynamical conditions:*
   - *(a) There exists a Fatou coordinate $\phi^{h,f}_{att} : P^{h,f} \to \mathbb{C}$ such that*

     $$\phi^{h,f}_{att}(P^{h,f}) = \left\{ e^{i\theta} w \in \mathbb{C} : 0 < \mathrm{Re}\, w < \mathrm{Re}\left(\frac{1/\alpha - M}{e^{i\theta}}\right)\right\}.$$

   - *(b) $z \in P^{h,f}$ tends to 0 or $\sigma^{h,f}$ when $\mathrm{Im}\, \phi^{h,f}_{att}(z)$ tends to $+\infty$ or $-\infty$ respectively.*

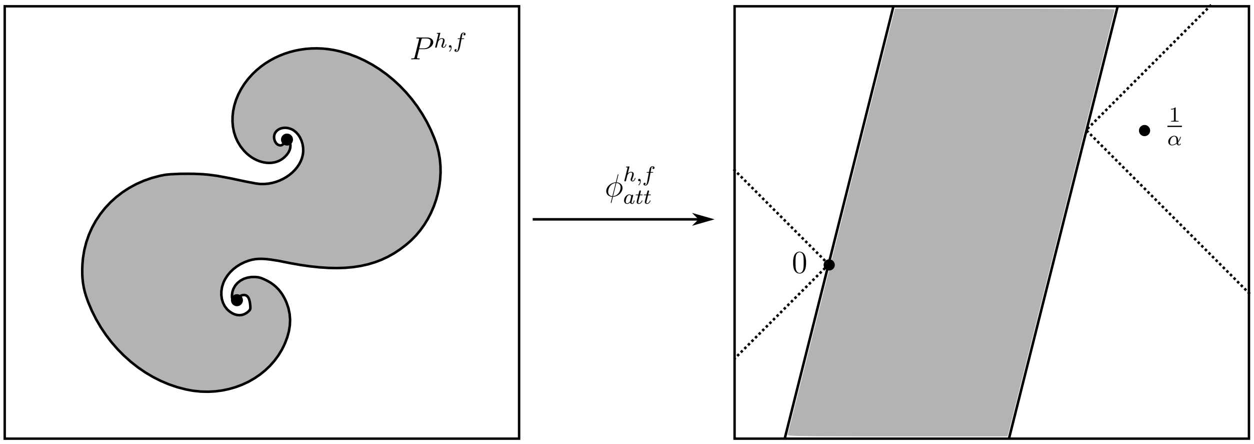


Figure II.2: The image of under the attracting Fatou coordinate of the petal for h relative f as in theorem II.22. The image must be contained in the region bounded by the dashed lines, and changing θ changes the slant.

(c) *The sets* $P^{h,f}, h(P^{h,f}), \ldots, h^{kq-1}(P^{h,f})$ *are all pair-wise disjoint.*

3. *The following continuity conditions:*

(a) *The compact sets* $\overline{P^{h,f}}$ *and* $\widehat{\mathbb{C}} \setminus P^{h,f}$ *depend continuously on* h.

(b) *There are petals* $P_{att}^{f,f} \subset P_{att}^{f}$ *and* $P_{rep}^{f,f} \subset P_{rep}^{f}$ *for* f^{kq} *such that*

$$P^{h,f} \to P^{f,f}$$

when $h \to f$, *where* $P^{f,f} := P_{att}^{f,f} \cup P_{rep}^{f,f}$.

(c) *When* $h \to f$, $\phi_{att}^{h,f} \to \phi_{att}^{f}$ *on* $P_{att}^{f,f}$.

Proof. For the case where $p/q = 0$, almost all of this theorem is proved in [Shi00] and [LY14a], while the general p/q case is proved in [Oud99]. The only parts of this theorem which are not proved in those papers are parts (1c) and (2a), these parts are proved in [BC12, Appendix A] when $\theta = 0$ and $\alpha > 0$; the general case holds by similar argument. □

Note that the petal $P^{h,f}$ depends on the choice of repelling petal P_{rep}^{f} as well as the choice of θ and M in theorem II.22. We will call a consistent choice of P_{rep}^{f}, θ ,and M for h close to f a *choice of petals near* f.

Let us make the following observation which follows from part (2a).

Proposition II.23. *If* $\alpha \in A_{1/2}^{+}$ *is sufficiently close to* 0, *then*

$$\mathrm{Re}\frac{1}{4\alpha} < \sup\{x \in \mathbb{R} : w + x \in \phi^{h,f}(P^{h,f})\} - \inf\{x \in \mathbb{R} : w + x \in \phi^{h,f}(P^{h,f})\} < \mathrm{Re}\frac{7}{4\alpha}$$

for all $w \in \mathbb{C}$.

Proof. First we note that

$$\mathrm{Re}\left(\frac{1/\alpha - M}{e^{i\theta}}\right) = \left(\mathrm{Re}\frac{1}{\alpha} - M\right)\cos\theta + \left(\mathrm{Im}\frac{1}{\alpha}\right)\sin\theta.$$

As $|\theta| < \frac{\pi}{4}$ and $\alpha \in A^+_{1/2}$, $|\sin\theta| < \cos\theta$ and $|\mathrm{Im}\frac{1}{\alpha}| < \tan(\frac{\pi}{5})\mathrm{Re}\frac{1}{\alpha}$. As $\tan(\frac{\pi}{5}) < 0.73 < 3/4$ and M is constant, if $|\alpha|$ is sufficiently small then

$$\left(\mathrm{Re}\frac{1}{4\alpha}\right)\cos\theta < \mathrm{Re}\left(\frac{1/\alpha - M}{e^{i\theta}}\right) < \left(\mathrm{Re}\frac{7}{4\alpha}\right)\cos\theta.$$

It follows from part (2a) of theorem II.22 that we need only prove the proposition for $w = 0$, so

$$\inf\{x \in \mathbb{R} : w + x \in \phi^{h,f}(P^{h,f})\} = 0.$$

For any $x > 0$, $x = e^{i\theta}(\frac{x}{e^{i\theta}})$ and

$$\mathrm{Re}\frac{x}{e^{i\theta}} = x\cos\theta.$$

Thus $x \in \phi^{h,f}(P^{h,f})$ if $x \le \mathrm{Re}\frac{1}{4\alpha}$ and $x \notin \phi^{h,f}(P^{h,f})$ if $x \ge \mathrm{Re}\frac{7}{4\alpha}$. □

Similarly to the Fatou coordinates for f, we can extend the attracting Fatou coordinate. However in this case, the orbit of a point may enter and exit the petal $P^{h,f}$ infinitely often; we must take care to only consider the "first" arrival to $P^{h,f}$. We will say that a point $z \in P^{h,f}$ is *petal-entering* if

$$\mathrm{Re}\,\phi^{h,f}_{att}(z) - \mathrm{Re}\frac{1}{12\alpha} \notin \phi^{h,f}_{att}(P^{h,f}).$$

If $h^{nkq+m}(z)$ is petal-entering for some $0 \le n < \mathrm{Re}\frac{1}{12\alpha}$ and $0 \le m < kq$, then we define

$$\rho^{h,f} := \phi^{h,f}_{att}(h^{nkq+m}(z)) - n.$$

Note that $\rho^{h,f}$ is the extension of a restriction of $\phi^{h,f}_{att}$. Indeed there may be points in $P^{h,f}$ which are not petal-entering and which have different images under $\phi^{h,f}_{att}$ and $\rho^{h,f}$.

Setting

$$\phi^{h,f}_{rep} := T_{-1/\alpha} \circ \phi^{h,f}_{att},$$

we will view $\phi^{h,f}_{rep}$ as the analogue of the repelling Fatou coordinate ϕ^{f}_{rep}. We will say that a point $z \in P^{h,f}$ is *petal-exiting* if

$$\operatorname{Re}\phi^{h,f}_{rep}(z) + \operatorname{Re}\frac{1}{12\alpha} \notin \phi^{h,f}_{rep}(P^{h,f}).$$

If z is petal-exiting and in $Dom(h^{nq})$ for some $0 \le n < \operatorname{Re}\frac{1}{12\alpha}$, then we define

$$\chi^{h,f}(\phi^{h,f}_{rep}(z) + n) := h^{nkq}(z).$$

The *horn map for h relative to f* is defined by

$$H^{h,f} := \rho^{h,f} \circ \chi^{h,f}.$$

Unlike H^f the horn map $H^{h,f}$ is not defined on a T_1-invariant domain; we extend H^f using the equation

$$H^{h,f} \circ T_1 = T_1 \circ H^{h,f}.$$

Proposition II.24. *The horn map $H^{h,f}$ is well-defined and analytic on both an upper and a lower half-plane. There exist constants $c^{h,f}_{\pm}$ such that $H^{h,f}(w) - w \to c^{h,f}_{\pm}$ when $\operatorname{Im} w \to \pm\infty$.*

Proof. See [Shi00] or [Oud99]. □

Our choice $\phi^{h,f}_{rep}$ was made precisely so that we can compute the constants $c^{h,f}_{\pm}$:

Proposition II.25. *If h is sufficiently close to f, then $c^{h,f}_{+} = c^{f}_{+}$ and*

$$c^{h,f}_{-} - 1/\alpha = \frac{2\pi i}{\log(h^q)'(\sigma^{h,f})}.$$

Proof. Let us denote $P_0 = P^{h,f}$ and set $S_{0,att}$ and $S_{0,rep}$ to be the set of all petal-entering and petal-exiting points in P_0 respectively. For all $0 \le j < q$ we set $P_j, S_{j,att}$, and $S_{j,rep}$ to be the components of $h^{-kj}(P_0), h^{-kj}(S_{0,att})$, and $h^{-kj}(S_{0,rep})$ respectively which have 0 on their boundary. Thus the sets P_j are pair-wise disjoint Jordan domains. We define Fatou coordinates on P_j by

$$\phi_{j,att} = \phi^{h,f}_{att} \circ h^{jk} \text{ and } \phi_{j,rep} = \phi^{h,f}_{rep} \circ h^{jk}.$$

As $\alpha \in A^+_r$, so

$$q\operatorname{Re}\mu_{p/q}(\alpha) - p > p/q,$$

the orbit of h^q travels counter-clockwise around 0. In particular, for $0 \le m < q$ satisfying

$$mp \equiv -1 \mod q,$$

and taking the indices modulo q, the orbit of a point in $S_{0,rep}$ close to zero must enter, in order, the sets

$$S_{0,rep}, S_{m,att}, S_{m,rep}, S_{2m,att}, S_{2m,rep}, \dots, S_{qm,att}.$$

Let H be the map in the repelling Fatou coordinate induced by the first non-trivial return to $S_{0,rep}$ under h^{kq}. More precisely, for any $z \in S_{0,rep}$ and minimal integer $n > 0$ such that $h^{nkq}(z) \in S_{0,rep}$ and $h^{n'kq}(z) \notin S_{0,rep}$ for some $0 \leq n' < n$, we set

$$H(\phi_{0,rep}(z)) = \phi_{0,rep}(h^{nkq}(z)) - n.$$

It is shown in [Oud02] that

$$\lim_{\operatorname{Im} w \to +\infty} H(w) - w = -\frac{2\pi i}{\log(h^{kq})'(0)} = \frac{-1}{q\mu_{p/q}(\alpha) - p} = -\frac{q}{\alpha} - \mathfrak{S}(p/q)q'$$

where p'/q' is the parent of p/q. For all $0 \leq j < q$, let H_j be the map in Fatou coordinates induced by the orbit under h^{kq} of points traveling from $S_{j,rep}$ to $S_{j+m,att}$. More precisely, for any $z \in S_{j,rep}$ and minimal integer $n > 0$ such that $h^{nkq}(z) \in S_{j+m,rep}$, we set

$$H_j(\phi_{j,rep}(z)) = \phi_{j+m,att}(h^{nkq}(z)) - n.$$

As $\phi_{j,rep} = T_{-1/\alpha} \circ \phi_{j,att}$ for all j, if $z \in S_{0,rep}$ is sufficiently close to 0 and $w = \phi_{0,rep}(z)$, then

$$H(w) = T_{-1/\alpha} \circ H_{(q-1)m} \circ \cdots \circ T_{-1/\alpha} \circ H_0.$$

Now let us fix some $z_0 \in S_{0,rep}$ and set $w = \phi_{0,rep}(z)$. For all $0 \leq j < q$, we let $z_j \in S_{j,rep}$ be the point such that $h^{jk}(z_j) = z_0$, so $\phi_{j,rep}(z_j) = w$. We assume that $H_0(w)$ is defined, so there exists some minimal integer $n > 0$ such that $z'_m := h^{nkq}(z_0) \in S_{m,att}$. We set $z'_0 = h^{mk}(z'_m) \in S_{0,att}$, and for all $0 \leq j < q$ we let $z'_j \in S_{j,att}$ be the point in $S_{j,att}$ such that $h^{jk}(z'_j) = z'_0$, so $\phi_{j,att}(z'_j) = \phi_{0,att}(z'_0)$. If $m + j < q$, then

$$h^{nkq+jk}(z_j) = h^{nkq}(z_0) = z'_m,$$

so $h^{nkq}(z_j) = z'_{m+j}$ and

$$H_j(w) = \phi_{j+m,att}(z'_{m+j}) - n = \phi_{m,att}(z'_m) - n = H_0(w).$$

If $m+j \geq q$, then

$$h^{(n+1)kq}(z_j) = h^{(n+1)kq-jk}(z_0) = h^{(q-j)k}(z'_m) = z'_{m+j-q},$$

so

$$H_j(w) = \phi_{m+j,att}(z'_{m+j-q}) - n - 1 = \phi_{m,att}(z'_m) - n - 1 = H_0(w) - 1.$$

Note that there are exactly m choices of $0 \leq j < q$ such that $m+j \geq q$.

Now we fix some $z \in S_{0,rep}$ and $w = \phi_{0,rep}(z)$ such that $H^{h,f}(w)$ is defined. Thus there exist integers $0 \leq n < \mathrm{Re}\frac{2}{12\alpha}$ and $0 \leq j < kq$ such that $h^{nkq+j}(z) \in S_{0,att}$ and

$$H^{h,f}(w) = \phi^{h,f}_{0,att}(h^{nkq+j}(z)) - n.$$

If z is close to 0, then we can choose n so that that $h^{nkq}(z) \in S_{m,att}$ and $j = mk$. Hence

$$H^{h,f}(w) = \phi^{h,f}_{0,att}(h^{nkq+mk}(z)) - n = \phi_{m,att}(h^{nkq}z) - n = H_0(w).$$

Thus

$$\begin{aligned}
\lim_{\mathrm{Im}\, w \to +\infty} H(w) - w &= -\frac{q}{\alpha} + \sum_{j=0}^{q-1} \lim_{\mathrm{Im}\, w \to +\infty} H_j(w) - w \\
&= -\frac{q}{\alpha} - m + q\left(\lim_{\mathrm{Im}\, w \to +\infty} H_0(w) - w\right) \\
&= -\frac{q}{\alpha} - m + qc^{h,f}_+,
\end{aligned}$$

so

$$c^{h,f}_+ = \frac{m - \mathfrak{S}(p/q)q'}{q}.$$

Setting p'/q' to be the parent of p/q, or $p'/q' = 1/0$ if $p/q = 0/1$, we have

$$p'q - pq' = \mathfrak{S}(p/q) = \pm 1.$$

We can therefore directly compute that

$$m = \frac{1 - \mathfrak{S}(p/q)}{2} q + \mathfrak{S}(p/q)q' = \begin{cases} q' & \text{if } \mathfrak{S}(p/q) = +1, \\ q - q' & \text{if } \mathfrak{S}(p/q) = -1 \end{cases}.$$

Hence

$$c^{h,f}_+ = \frac{1 - \mathfrak{S}(p/q)}{2} = c^f_+.$$

The computation of $c_-^{h,f}$ is similar. As $\sigma^{h,f}$ is not on the boundary of $h^n(P^{h,f})$ for an $0 < n < q$, it follows that

$$H(w) = T_{-1/\alpha} \circ H^{h,f}(w)$$

for z close to $\sigma^{h,f}$ and $w = \phi_{rep}^{h,f}(w)$. It is similarly shown in [Oud99] that

$$\lim_{\operatorname{Im} w \to -\infty} H(w) - w = \frac{2\pi i}{\log(h^{kq})'(\sigma^{h,f})}.$$

□

Corollary II.26. *When $h \to f$, $\phi_{rep}^{h,f}$ converges to the restriction of ϕ_{rep}^f to $P_{rep}^{f,f}$. Additionally, $\chi^{h,f} \to \chi^f$ and $H^{h,f} \to H^f$.*

Proof. It suffices to just show that $\phi_{rep}^{h,f}$ converges to a restriction of ϕ_{rep}^f, the other convergences then follow immediately from the definitions. It follows from Montel's theorem that up to a subsequence $\phi_{rep}^{h,f}$ converges locally uniformly to a Fatou coordinate ϕ on $P_{rep}^{f,f}$. It follows from proposition II.25 that

$$\lim_{\operatorname{Im} w \to +\infty} \rho^f \circ \phi^{-1}(w) = \lim_{\operatorname{Im} w \to +\infty} \lim_{h \to f} \rho^{h,f} \circ (\phi_{rep}^{h,f})^{-1}(w) = c_+^f,$$

so $\phi = \phi_{rep}^f$ on $P_{rep}^{f,f}$ by the uniqueness of the Fatou coordinate. □

The convergence of Fatou coordinates gives us the following classical result:

Theorem II.27 (Douady, Lavaurs, Oudkerk)**.** *For any $\delta \in \mathbb{C}$, if $h \to f$ and $n - 1/\alpha \to \delta$ when $n \to +\infty$, then for any $z \in U_0^f$,*

$$h^{nq} \to \chi^f \circ T_\delta \circ \rho^f$$

locally uniformly near z if the latter function is defined at z_0.

Proof. For any $z_0 \in U_0^f$, there exists some $m \geq 0$ such that $f^m(z) \in P_{att}^{f,f}$ for all z close to z_0. Additionally, there exists some $k \geq 0$ such that $T_{\delta-k} \circ \rho^f(z) \in \phi_{rep}^f(P_{rep}^{f,f})$ for all z close to z. Thus

$$\begin{aligned}
h^q &= h^{(n-m)q} \circ (\phi_{att}^{h,f})^{-1} \circ \phi_{att}^{h,f} \circ h^{mq} \\
&= h^{kq} \circ h^{(n-k-m)q} \circ (\phi_{rep}^{h,f})^{-1} \circ T_{-m-1/\alpha} \circ \rho^{h,f} \\
&= h^{kq} \circ (\phi_{rep}^{h,f})^{-1} \circ T_{n-k-1/\alpha} \circ \rho^{h,f} \\
&= \chi^{h,f} \circ T_{n-1/\alpha} \circ \rho^{h,f} \\
&\to \chi^f \circ T_\delta \circ \rho^f
\end{aligned}$$

locally uniformly near z_0 when $h \to f$ and $n - 1/\alpha \to \delta$. □

Let us note that the maps $\rho^{h,f}$, $\chi^{h,f}$, and $H^{h,f}$ all depend on our choice of petals near f. However, the following proposition shows that this dependence evaporates as h tends to f:

Proposition II.28. *For any other choice of petals $\tilde{P}^{h,f}$ near f and any compact set X contained in the domain of either ρ^f or χ^f, the restriction of $\rho^{h,f}$ or $\chi^{h,f}$ respectively to X does not change if we replace $P^{h,f}$ with $\tilde{P}^{h,f}$ when h is sufficiently close to f.*

Proof. Let $\tilde{\phi}^{h,f}_{att}$ and $\tilde{\phi}^{h,f}_{rep}$ be the corresponding Fatou coordinates for $\tilde{P}^{h,f}$. As cv^h lies on boundary of both $P^{h,f}$ and $\tilde{P}^{h,f}$, it follows from the uniqueness of Fatou coordinates that $\phi^{h,f}_{att} = \tilde{\phi}^{h,f}_{att}$ on $P^{h,f} \cap \tilde{P}^{h,f}$. It then follows from the definition that $\phi^{h,f}_{rep} = \tilde{\phi}^{h,f}_{rep}$ on $P^{h,f} \cap \tilde{P}^{h,f}$. It follows from part (2a) of theorem II.22 that for any compact set $X \subset \mathbb{C}$, there exists a constant $n \geq 0$ such that $X + n \subset \phi^{h,f}_{att}(P^{h,f} \cap \tilde{P}^{h,f})$ and $X - n \subset \phi^{h,f}_{rep}(P^{h,f} \cap \tilde{P}^{h,f})$ when h is sufficiently close to f. Thus

$$\chi^{h,f} = f^{nkq} \circ (\phi^{h,f}_{rep})^{-1} \circ T_{-n}$$

on X does not depend on the choice of $P^{h,f}$ or $\tilde{P}^{h,f}$. By similar reasoning, the analogous statement holds for $\rho^{h,f}$. □

A *near-parabolic renormalization of h relative to f* is defined to be a map of the form

$$\mathcal{R}^{\pm}_f h := \mathrm{Exp}^{\pm} \circ T_{-1/\alpha} \circ H^{h,f} \circ (\mathrm{Exp}^{\pm})^{-1}.$$

It follows from proposition II.24 that $\mathcal{R}^{\pm}_f h$ is defined on punctured neighborhoods of 0 and ∞ and we can analytically extend by setting $\mathcal{R}^{\pm}_f h(0) = 0$ and $\mathcal{R}^{\pm}_f h(\infty) = \infty$. Additionally proposition II.28 implies that for any two choices of petals near f and any compact subset X of $Dom(\mathcal{R}^{\pm}_0 f)$, if h is sufficiently close to f then the restriction of $\mathcal{R}^{\pm}_f h$ to X does not depend on the choice of petals. So while $\mathcal{R}^{\pm}_f h$ is not canonically defined as a function which depends only on f and h, this is a technicality which we can ignore asymptotically when $h \to f$. As in the parabolic case, $\mathcal{R}^{+}_f h$ and $\mathcal{R}^{-}_f h$ are called the *top* and *bottom* renormalizations of h relative to f respectively, we will focus on the top renormalization and denote $\mathcal{R}_f = \mathcal{R}^{+}_f$. Corollary II.26 implies that $\mathcal{R}_f h \to \mathcal{R}_\delta f$ when $h \to f$ and $-1/\alpha \to \delta \mod 1$. For any integer $j \geq 0$ which depends on h, we will say that h converges to $\langle f, \mathcal{R}_\delta f \rangle$ with *combinatorics* j if $j - 1/\alpha \to \delta - c^f_+$ when $h \to f$.

We define the *attracting elevator of h relative to f* to be the map

$$\eta^{h,f}_{att} := \mathrm{Exp} \circ T_{-1/\alpha} \circ \rho^{h,f}.$$

We define the *repelling elevator of h relative to f* to be the map

$$\eta_{rep}^{h,f}(z) := \mathrm{Exp} \circ \phi_{rep}^{h,f}(z),$$

defined only when $z \in P^{h,f}$ is petal-exiting. Similarly to the parabolic case, it follows immediately from the definitions that $\eta_{att}^{h,f} \circ (\eta_{rep}^{h,f})^{-1}$ is a restriction of $\mathcal{R}_f h$. It follows from corollary II.26 that $\eta_{rep}^{h,f} \to \eta_{rep}^{f}$ when $h \to f$, and it follows from part (3c) of theorem II.22 that $\eta_{att}^{h,f} \to \eta_{att,\delta}^{f}$ when $h \to f$ and $-1/\alpha \to \delta \mod 1$.

Proposition II.29. *For any z and z', if*

$$\mathcal{R}_f h \circ \eta_{rep}^{h,f}(z) = \eta_{rep}^{h,f}(z'),$$

then there exist integers $\mathrm{Re}\frac{1}{12\alpha} < n < \mathrm{Re}\frac{23}{12\alpha}$ and $0 \le m < q$ which depend continuously on z and z' such that $h^{nkq+m}(z) = z'$.

Proof. Set $w = \phi_{rep}^{h,f}(z), w' = \phi_{rep}^{h,f}(z')$, $\zeta = \mathrm{Exp}(w)$, and $\zeta' = \mathrm{Exp}(w')$. If $\mathcal{R}_f h(\zeta) = \zeta'$, then there exists an integer n so that $T_{-1/\alpha} \circ H^{h,f}(w) = w' - n$. As $H^{h,f}(w)$ is defined and z is petal-exiting, there exists some integers $0 \le j < \mathrm{Re}\frac{1}{6\alpha}$ and $0 \le m < q$ so that $h^{jkq+m}(z) \in P^{h,f}$ is petal-entering and

$$H^{h,f}(w) = \phi_{att}^{h,f}(h^{jkq+m}(z)) - j.$$

Hence

$$\begin{aligned}
\phi_{rep}^{h,f}(z') &= w' \\
&= H^{h,f}(w) + n - 1/\alpha \\
&= \phi_{att}^{h,f}(h^{jkq+m}(z)) + n - j - 1/\alpha \\
&= \phi_{rep}^{h,f}(h^{jkq+m}(z)) + n - j.
\end{aligned}$$

As $\phi_{rep}^{h,f}$ is a Fatou coordinate, z' is petal-exiting, and $h^{jkq+m}(z)$ is petal-entering, it follows from proposition II.23 that we can conclude that $\mathrm{Re}\frac{1}{12\alpha} < n - j < \mathrm{Re}\frac{7}{4\alpha}$ and $z' = h^{nkq+m}(z)$. It follows immediately from the above argument that the integers n and j both depend continuously on w and w'. □

Proposition II.30. *For any z, z', if*

$$\eta_{att}^{h,f}(z) = \eta_{rep}^{h,f}(z),$$

then there exist integers $\mathrm{Re}\frac{1}{12\alpha} < n < \mathrm{Re}\frac{11}{6\alpha}$ *and* $0 \le k < q$ *which depend continuously on* z *and* z' *such that* $h^{nq+k}(z) = z'$.

Proof. This follows directly from the same argument used to prove proposition II.29 above. □

Proposition II.31. *Fix some* $\delta \in \mathbb{C}$, $z \in Dom(\eta^f_{att,\delta})$, *and set* $\zeta = \eta^f_{att,\delta}(z)$. *For any* $\theta \in [-\pi/4, \pi/4]$ *and* $M > 0$, *if* h *is sufficiently close to* f *and* $-1/\alpha$ *is sufficiently close to* δ *modulo* 1, *then for any simply connected set* $Y \subset \mathbb{C}$ *which avoids* $\mathrm{Exp}(-1/\alpha)$, *contains* ζ, *and on which there is a continuous branch of* Exp^{-1} *satisfying*

$$\mathrm{Exp}^{-1}(Y) \subset \{e^{i\theta}w : |\mathrm{Re}\, w| < M\},$$

the petals near f *can be chosen so that there exists a unique continuous branch of* $(\eta^{h,f}_{att})^{-1}$ *defined on* Y *which sends* ζ *close to* z. *If* $\tilde{Y}$ *has* cv^h *on its boundary and* Y *has* 0 *on its boundary, then* 0 *is on the boundary of* $\tilde{Y}$.

Moreover, using the above branch and setting $\tilde{Y} := (\eta^{h,f}_{att})^{-1}$, *there exists a branch of* Exp^{-1} *defined on* $\tilde{Y}$ *satisfying*

$$\mathrm{Exp}^{-1}(\tilde{Y}) \subset \{e^{i\tilde{\theta}}w : |\mathrm{Re}\, w| < \tilde{M}\},$$

where

$$\tilde{\theta} = \theta + O(\alpha)$$

when $\alpha \to 0$ *and* $\tilde{M}$ *depends only on* M *and* z_0.

Proof. Set $w_0 = \rho^f(z)$. If h is sufficiently close to f, then we can choose the petals near f so that

$$\{e^{i\tilde{\theta}}w + w_0 : |\mathrm{Re}\, w| < \tilde{M}\}$$

is contained in the image of ρ^f. If h is close to f and $n - 1/\alpha$ is close to δ for some $n \in \mathbb{Z}$, then there is a unique branch of $(\mathrm{Exp} \circ T_{n-1/\alpha})^{-1}$ defined on Y which sends ζ close to w_0. It follows from the definition that $\rho^{h,f}$ is a covering map branched over the negative integers; hence if h is sufficiently close to f then there exists a unique branch $(\eta^{h,f}_{att})^{-1}$ defined on Y which sends ζ close to z. If cv^h is on the boundary of $\tilde{Y}$, and 0 is on the boundary of Y, then the same argument used in the proof of lemma II.15 allows us to conclude that 0 is on the boundary of $\tilde{Y}$.

If $\tilde{Y} \subset P^{h,f}_{rep}$, then the second part of the proposition follows from part (1c) of theorem

II.22. Otherwise, the same conclusion follows from the fact that

$$\tilde{Y} \subset \bigcup_{m=0}^{\tilde{M}} h^{-mkq}(P^{h,f})$$

for some integer $\tilde{M}$ which does not depend on α. □

Let us observe that in the above constructions we never actually use the function f, just that h is sufficiently close to f. Thus if g is a stable perturbation of h which is close enough to f, then $P^{h,f} = P^{h,g}$, $H^{h,f} = H^{h,g}$, $\mathcal{R}_f h = \mathcal{R}_g h$, etc.

II.3.2: Negatively implosive perturbations

Let us now assume that instead $h'(0) = e^{2\pi i \mu_{p/q}(\alpha)}$ for some $\alpha \in A^-_{1/2} := -A^+_{1/2}$. We set $f^* = \iota \circ f \circ \iota$ and $h^* = \iota \circ h \circ \iota$, so

$$(f^*)'(0) = \mathrm{Exp}(-p/q) \text{ and } (h^*)'(0) = \mathrm{Exp}(-\iota(\mu_{p/q}(\alpha))) = \mathrm{Exp}(\mu_{-p/q}(-\iota(\alpha)))$$

by proposition II.6. Thus h^* is a positively implosive perturbation of f^*; in this case we will say that h is a *negatively implosive perturbation of f*. We define

$$P^{h,f}_{att} := \iota(P^{h^*,f^*}_{att}) \text{ and } P^{h,f}_{rep} := f^n \circ \iota(P^{h^*,f^*}_{rep}),$$

where $0 \le n < q$ satisfies $np \equiv 1 \mod q$. We also define

$$\begin{aligned}
\eta^{h,f}_{att} &:= \mathrm{Exp} \circ \iota \circ T_{c^{h^*,f^*}_- - c^f_+} \circ \mathrm{Exp}^{-1} \circ \eta^{h^*,f^*}_{att} \circ \iota, \\
\eta^{h,f}_{rep} &:= \mathrm{Exp} \circ \iota \circ T_{c^{h^*,f^*}_- - c^f_+} \circ \mathrm{Exp}^{-1} \circ \eta^{h^*,f^*}_{rep} \circ \iota \circ h^{-n}, \text{ and} \\
\mathcal{R}_f h &:= \eta^{h,f}_{att} \circ (\eta^{h,f}_{rep})^{-1}.
\end{aligned}$$

using the branch of h^{-n} which sends $P^{h,f}_{rep}$ to $\iota(P^{h^*,f^*}_{rep})$. It follows from the definitions that on their respective domains, we can alternatively write these maps as

$$\begin{aligned}
\eta^{h,f}_{att} &= \mathrm{Exp} \circ \iota \circ T_{c^{h^*,f^*}_- - c^f_+ + 1/\iota(\alpha)} \circ \rho^{h^*,f^*} \circ \iota, \\
\eta^{h,f}_{rep} &= \mathrm{Exp} \circ \iota \circ T_{c^{h^*,f^*}_- - c^f_+} \circ \phi^{h^*,f^*}_{rep} \circ \iota \circ f^{-n}, \text{ and} \\
\mathcal{R}_f h(z) &= \mathrm{Exp}^-(c^{h^*,f^*}_- - c^f_+) \cdot \mathcal{R}^-_{f^*} h^* \left(\frac{z}{\mathrm{Exp}^-(c^{h^*,f^*}_- - c^f_+)} \right).
\end{aligned}$$

So in particular, we can compute the multiplier of $\mathcal{R}_f h$ at 0 and ∞ as

$$(\mathcal{R}_f h)'(0) = (\mathcal{R}^-_{f^*} h^*)'(0) = \mathrm{Exp}^-(c_-^{h^*,f^*} + 1/\iota(\alpha)) = \mathrm{Exp}(\iota(c_-^{h^*,f^*}) + 1/\alpha), \text{ and}$$
$$(\mathcal{R}_f h)'(\infty) = (\mathcal{R}^-_{f^*} h^*)'(\infty) = \mathrm{Exp}^-(-(c_+^{h^*,f^*} + 1/\iota(\alpha))) = \mathrm{Exp}(-1/\alpha).$$

Proposition II.32. *The petals near f and f^* can be chosen so that*

$$\eta_{att}^{h,f} \to \eta_{att,\delta}^{f},$$
$$\eta_{rep}^{h,f} \to \eta_{rep}^{f}, \text{ and}$$
$$\mathcal{R}_f h \to \mathcal{R}_\delta f$$

when $h \to f$ and $\iota(c_-^{f^*}) + 1/\alpha \to \delta \mod 1$.

Proof. First we observe that $\iota(P_{att}^{f^*})$ is an attracting petal for f with cv^f on its boundary and with Fatou coordinate $\iota \circ \phi_{att}^{f^*} \circ \iota$. By the uniqueness of P_{att}^f and ϕ_{att}^f, we therefore have

$$P_{att}^f = \iota(P_{att}^{f^*}) \text{ and } \phi_{att}^f = \iota \circ \phi_{att}^{f^*} \circ \iota.$$

As $c_+^f \in \mathbb{Z}$, it follows that

$$\eta_{att}^{h,f} \to \mathrm{Exp} \circ \iota \circ T_{\iota(\delta) - c_+^f} \circ \rho^{f^*} \circ \iota = \mathrm{Exp} \circ T_{\delta - c_+^f} \circ \rho^f = \eta_{att,\delta}^f$$

when $h \to f$ and $\iota(c_-^{f^*}) + 1/\alpha \to \delta \mod 1$. Similarly, the uniqueness of ϕ_{rep}^f combined with part (1c) of theorem II.7 implies that we can choose the petals near f and f^* so that $P_{rep}^f = f^n \circ \iota(P_{rep}^{f^*})$. By the uniqueness of Fatou coordinates, it follows that there is some $\lambda \in \mathbb{C}$ so that

$$\phi_{rep}^f = T_\lambda \circ \iota \circ \phi_{rep}^{f^*} \circ \iota \circ f^{-n}$$

on P_{rep}^f, using the branch of f^{-n} which sends P_{rep}^f to $\iota(P_{rep}^{f^*})$. Thus for any fixed $x \in \mathbb{R}$, if y is sufficiently large then proposition II.9 implies that

$$\begin{aligned} H^f(x+iy) &= \rho^f \circ (\phi_{rep}^f)^{-1}(x+iy) \\ &= \phi_{att}^{h,f} \circ (\phi_{rep}^f)^{-1}(x+iy) \\ &= \iota \circ \phi_{att}^{f^*} \circ (f^*)^n \circ (\phi_{rep}^{f^*})^{-1} \circ \iota \circ T_{-\lambda}(x+iy) \\ &= \iota \circ \rho^{f^*} \circ (\phi_{rep}^{f^*})^{-1} \circ \iota \circ T_{-\lambda}(x+iy) \\ &= \iota \circ H^{f^*} \circ \iota \circ T_{-\lambda}(x-iy). \end{aligned}$$

Thus

$$\begin{aligned}
\iota(c_+^f) &= \lim_{y\to+\infty} \iota(H^f(x+iy)-(x+iy)) \\
&= \lim_{y\to+\infty} H^{f^*}(x-iy-\iota(\lambda))-(x-iy-\iota(\lambda))-\iota(\lambda) \\
&= c_-^{f^*}-\iota(\lambda),
\end{aligned}$$

so

$$\eta_{rep}^{h,f} \to \operatorname{Exp}\circ\iota\circ T_{\iota(\lambda)}\circ\phi_{rep}^{f^*}\circ\iota\circ f^{-n} = \operatorname{Exp}\circ\phi_{rep}^{f} = \eta_{rep}^{f}$$

on the domain of η_{rep}^f. The convergence $\mathcal{R}_f h \to \mathcal{R}_\delta f$ immediately follows. $\square$

In this setting, for any integer $j \leq 0$ which depends on h we will say that h converges to $\langle f, \mathcal{R}_\delta f\rangle$ with combinatorics j if $j - 1/\alpha \to \iota(c_-^{f^*}) - \delta$ when $h \to f$.

Proposition II.32 ensures that the top parabolic renormalizations of f are exactly the limits of the top near-parabolic renormalizations of h for both positively and negatively implosive perturbations. We will say that h is an *implosive perturbation* of f if it is either a positively or negatively implosive perturbation of f. We will say that h is a *non-implosive perturbation* of f if it is neither an implosive perturbation of f nor a stable perturbation of f.

II.3.3: Comparing renormalizations

So far in this chapter, we have introduced several different renormalization operators. We will now focus on exactly how we can relate these renormalizations.

We consider a map f_0 and $\delta \in \mathbb{C}$ such that both f_0 and f_1 are stable perturbations of maps in $\mathcal{F}^{\text{❀}}$. Let g_0 be an implosive perturbation of f_0 such that $g_1 = \mathcal{R}_{f_0} g_0$ is a stable perturbation of f_1. In particular, g_1 has a parabolic cycle with the same multiplier and period as the parabolic cycle of f_1. If h_0 is another map close to g_0, then h_0 is also an implosive perturbation of f_0; we set $h_1 = \mathcal{R}_{f_0} h_0$.

Proposition II.33. *In the situation above, g_0 also has a parabolic cycle. Moreover, if h_0 is sufficiently close to g_0 and h_1 is sufficiently close to g_1, then h_0 is a stable or implosive perturbation of g_0 if and only if h_1 is the same for g_1 respectively.*

Proof. If g_1 has a parabolic periodic cycle contained in $\mathbb{C}^*$, then it follows immediately from proposition II.29 that g_0 does as well. Moreover, the dynamics of h_1 near the parabolic cycle of g_1 are locally analytically conjugate by $\eta_{rep}^{h_0,f_0}$ to the dynamics of h_0 near the parabolic cycle of g_0, which completes the proof in this case.

Now that us assume that 0 is a p_1/q_1 parabolic fixed point of g_1 for some rational $p/q \in [-1/2, 1/2]$, so $g_1'(0) = \operatorname{Exp} \circ \mu_{p_1/q_1}(0)$. As f_0 has a p_0/q_0-parabolic k-periodic point, after conjugating by Möbius maps we can assume that 0 is the fixed point of g_0^{kq} on the upper end of P^{g_0,f_0}, that is $\operatorname{Im} \phi^{g_0,f_0}(z) \to +\infty$ when $z \to 0$, and that $(g_0^{kq})'(0) = \operatorname{Exp} \circ \mu_{p/q}(\alpha)$, where

$$c_+^{f_0} - 1/\alpha \equiv p_1/q_1 \mod 1.$$

Let us assume that g_0 is a positively implosive perturbation of f_0, so there exists some integer $j \gg 0$ satisfying

$$\alpha = \frac{1}{j + c_+^{f_0} - \mu_{p_1/q_1}(0)},$$

in particular there exists some rational $p_0'/q_0' \in [-1/2, 1/2]$ such that

$$\mu_{p_0'/q_0'}(z) = \frac{1}{j + c_+^{f_0} - \mu_{p_1/q_1}(z)}.$$

Thus 0 is a p_0'/q_0'-parabolic periodic point of g_0. Recall that h_1 is a stable or implosive perturbation of g_1 if and only if h_1 is close to g_1 and

$$h_1'(0) = \operatorname{Exp} \circ \mu_{p_1/q_1}(\alpha')$$

for some $\alpha' \in A_{1/2} \cup \{0\}$. Again conjugating by a Möbius transformation, if h_0 is sufficiently close to g_0 then 0 is a fixed point of h_0^{kq} on the boundary of P^{h_0,f_0} and

$$(h_0^{kq})'(0) = \operatorname{Exp} \circ \mu_{p/q}\left(\frac{1}{j + c_+^{f_0} - \mu_{p_1/q_1}(\alpha')}\right) = \operatorname{Exp} \circ \mu_{p_0'/q_0'}(\alpha').$$

Thus h_0 is a stable or implosive perturbation of g_0 if and only if h_1 is a stable or implosive perturbation of g_1 respectively.

The argument when g_0 is an negatively implosive perturbation is similar. If instead ∞ is a parabolic fixed point of f_0, then the desired results follow from the same argument as above and proposition II.25. □

Proposition II.34. *Assume that h_1 is an implosive perturbation of g_1. If g_0 is sufficiently close to f_0, then there exist choices of petals near f_0, g_0, and g_1 such that*

$$\eta_{att}^{h_0,f_0}(P^{h_0,g_0}) = P^{h_1,g_1}$$

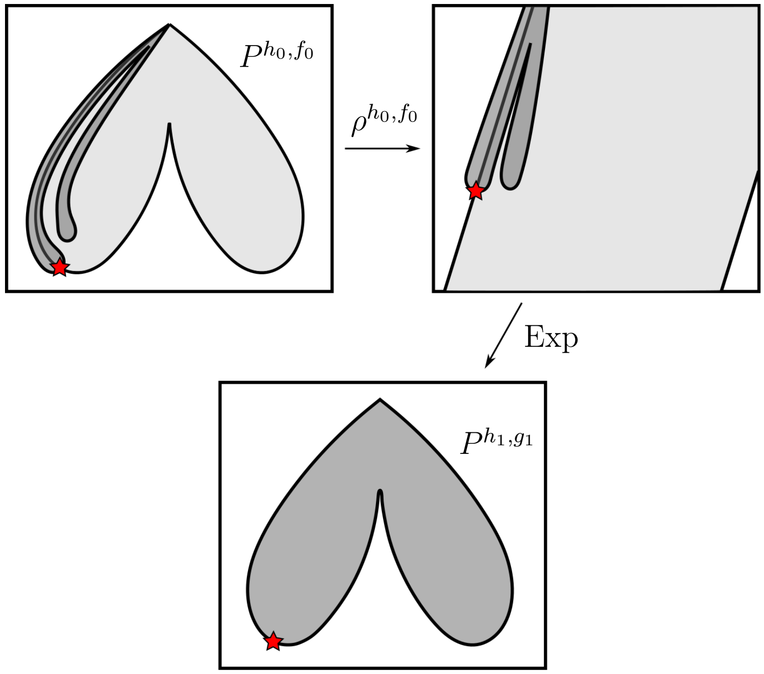


Figure II.3: The lift of P^{h_1,g_1} to P^{h_0,g_0} as in proposition II.34. The critical value cv^{h_0} and its corresponding images are given by the red starts.

and

$$\eta_{rep}^{h_0,g_0} = \eta_{rep}^{h_1,g_1} \circ \eta_{att}^{h_0,f_0}$$

when h_0 and h_1 are sufficiently close to g_0 and g_1 respectively. For any compact $X \subset (\eta_{att}^{g_0,f_0})^{-1}(Dom(\rho^{g_1}))$, if h_0 is sufficiently close to g_0 and if h_1 is a sufficiently close to g_1, then

$$\eta_{att}^{h_0,g_0} = \eta_{att}^{h_1,g_1} \circ \eta_{att}^{h_0,f_0}$$

on X.

Proof. Let us assume that f_0 has a p_0/q_0-parabolic k_0-periodic cycle and that g_1 has a p_1/q_1-parabolic k_1-periodic cycle. We set α to be the complex number satisfying

$$\phi_{rep}^{h_0,f_0} = T_{-1/\alpha} \circ \phi_{att}^{h_0,f_0}.$$

Lemma II.35. *For any $\epsilon > 0$ and $|\theta| < \frac{\pi}{4} - \epsilon$, there exists a choice of petals near g_1 and a constant $M > 0$ such that if h_1 is sufficiently close to g_1 then there exists a branch of* log *defined on P^{h_1,g_1} satisfying*

$$\log P^{h_1,g_1} \subset \{e^{i\theta} w \in \mathbb{C} : |\mathrm{Re}\, w| < M\}.$$

Proof. If the parabolic cycle of g_1 avoids $\mathbb{C}^*$ then the petals P^{g_1,g_1} are compactly contained in $\mathbb{C}^*$. The convergence of P^{h_1,g_1} to P^{g_1,g_1} therefore implies the lemma. If instead 0 or ∞ is the parabolic fixed point of g_1, then the lemma follows from part (1c) of theorem II.22. $\square$

Let us fix some $|\theta| < \frac{\pi}{4}$ and continuous choice of petals near g_1 such that there exists a branch of log defined on P^{h_1,g_1} satisfying

$$\log P^{h_1,g_1} \subset \{e^{i\theta}w \in \mathbb{C} : |\mathrm{Re}\, w| < M\}$$

for some constant M as in the above lemma. It therefore follows from part (2a) of theorem II.22 that there is a choice of petals near f_0 for which there exists a connected component P' of $(\eta_{rep}^{h_0,f_0})^{-1}(P^{h_1,g_1})$ on which $\eta_{rep}^{h_0,f_0}$ is univalent. It similarly follows from lemma II.35 and proposition II.31 that if g_0 is sufficiently close to f_0 and if h_0 is sufficiently close to g_0, then there exists a unique connected component P of $(\eta_{att}^{h_0,f_0})^{-1}(P^{h_1,g_1})$ which has cv^{h_0} on its boundary. As P is connected, proposition II.30 implies that there exists an integer

$$q_0 k_0 \mathrm{Re}\frac{1}{12\alpha} < m < q_0\left(1 + k_0\mathrm{Re}\frac{11}{6\alpha}\right)$$

such that $h_0^m(P) = P'$ and $\eta_{att}^{h_0,f_0} = \eta_{rep}^{h_0,f_0} \circ h_0^m$ on P.

As $\tilde{P}$ is connected, it follows from proposition II.29 that there is an integer

$$q_0 k_0 \mathrm{Re}\frac{1}{12\alpha} < n < q_0\left(1 + k_0\mathrm{Re}\frac{23}{12\alpha}\right)$$

so that $\eta_{rep}^{h_0,f_0}$ conjugates h_0^n on $\tilde{P}$ to $h_1^{k_1 q_1}$ on P^{h_1,g_1}. In particular, $\tilde{P}$ is a petal for h_0^n. Moreover, $\tilde{P}$ converges to the union of attracting and repelling petals for g_0^n when $h_0 \to g_0$ and $h_1 \to g_1$. It follows from the proofs of proposition II.29 and II.30 that $m < n$. As h_0^n is univalent near 0, it follows that P is also a petal for h_0^n and converges to the union of attracting and repelling petals for g_0^n when $h_0 \to g_0$ and $h_1 \to g_1$. It follows from the uniqueness of petals that there exists a choice of petals near g_0 so that $P = P^{h_0,g_0}$.

It follows automatically from the above that

$$\eta_{att}^{h_0,f_0}(P^{h_0,g_0}) = P^{h_1,g_1}.$$

If X is a connected compact subset of $(\eta_{att}^{g_0,f_0})^{-1}(Dom(\rho^{g_1}))$, then $X_1^{g_1} := \eta_{att}^{g_0,f_0}(X)$ is a compact subset of $Dom(\rho^{g_1})$ and there exists some integer $j \geq 0$ so that $g_1^j(x_1^{g_1}) \subset P_{att}^{g_1}$. If h_0 is sufficiently close to g_0, then X is a compact subset of $Dom(\eta_{att}^{h_0,f_0})$. If h_1 is also close to g_1, then $X_1^{h_1} := \eta_{att}^{h_0,f_0}(X)$ is a compact subset of ρ^{h_1,g_1} and $h_1^j(X_1^{h_1}) \subset P^{h_1,g_1}$. Proposition

II.29 implies that for any connected compact subset of $\tilde{X}$ of $(\eta_{rep}^{h_0,f_0})^{-1}(X_1^{h_1})$ there exists an integer J such that $h_0^J(\tilde{X}) \subset \tilde{P}$ and

$$J < jq_0\left(1 + k_0\mathrm{Re}\frac{23}{12\alpha}\right) \leq 24jq_0k_0\mathrm{Re}\frac{1}{12\alpha} < 24jn.$$

Thus $h_0^{n+J}(X) \subset P$, moreover if h_0 and h_1 are sufficiently close to g_0 and g_1 respectively we can ensure that $h_0^{n+J}(X)$ is a subset of the petal-entering points in P. As $J \leq 12jn$, if h_0 is sufficiently close to g_0 then $X \subset Dom(\rho^{h_0,g_0})$, and

$$\eta_{att}^{h_0,g_0} = \eta_{att}^{h_1,g_1} \circ \eta^{h_0,f_0}$$

on X. We can extend from connected X to arbitrary X by including X in a larger connected compact subset of $(\eta_{att}^{g_0,f_0})^{-1}(Dom(\rho^{g_1}))$. □

Proposition II.36. *Assume that h_1 is a stable perturbation of g_1. If g_0 is sufficiently close to f_0, then there exist choices of petals near f_0, g_0, and g_1 such that*

$$\eta_{att}^{h_0,f_0}(P_{att}^{h_0}) = P_{att}^{h_1} \text{ and } \eta_{att}^{h_0,f_0}(P_{rep}^{h_0}) = P_{rep}^{h_1},$$

and

$$\phi_{att}^{h_0} = \phi_{att}^{h_1} \circ \eta_{att}^{h_0,f_0} \text{ and } \phi_{rep}^{h_0} = \phi_{rep}^{h_1} \circ \eta_{att}^{h_0,f_0}$$

when h_0 and h_1 are sufficiently close to g_0 and g_1 respectively.

Proof. The argument is the same as that in the proof of proposition II.34 above. □

Corollary II.37. *If g_0 is sufficiently close to f_0 and if h_1 is a stable perturbation of g_1, then*

$$\mathcal{R}_\delta h_0 \text{ and } \mathcal{R}_\delta h_1$$

eventually agree on every compact subset of $Dom(\mathcal{R}_0 g_1)$ for any $\delta \in \mathbb{C}$ when $h_0 \to g_0$ and $h_1 \to g_1$. If h_1 is an implosive perturbation of g_1, then

$$\mathcal{R}_{g_1} h_1 \text{ and } \mathcal{R}_{g_0} h_0$$

eventually agree on every compact subset of $Dom(\mathcal{R}_0 g_1)$ when $h_0 \to g_0$ and $h_1 \to g_1$.

II.3.4: Invariant classes

Let us fix some $f_0 \in \mathcal{F}_0$, and rational $p/q \in [-1/2, 1/2]$. Setting $A_r := A_r^+ \cup A_r^-$ for all $r > 0$, for any sufficiently small $\alpha \in A_{1/2}$ we can define the $(p/q, \alpha)$-*fiber renormalization* of f_0 to be

$$\mathcal{R}_{p/q,\alpha} f_0 := \operatorname{Exp} \circ H^{f,h} \circ \operatorname{Exp}^{-1}$$

where $f = e^{2\pi i p/q} f_0$ and $h = e^{2\pi i \mu_{p/q}(\alpha)} f_0$. The fiber renormalization is closely related to near-parabolic renormalization, indeed

$$\mathcal{R}_f h = \begin{cases} \operatorname{Exp}(-1/\alpha) \cdot \mathcal{R}_{p/q,\alpha} f_0 & \text{if } \alpha \in A^+_{1/2} \\ \operatorname{Exp}(\iota(c_-^{h^*,f^*} - c_+^f) + 1/\alpha) \cdot \mathcal{R}_{p/q,\alpha} f_0 & \text{if } \alpha \in A^-_{1/2}. \end{cases}$$

It follows from the compactness of $\mathcal{F}_0$ that for any rational p/q there exists $\epsilon > 0$ and $r > 0$ such that $\mathcal{R}_{p/q,\alpha} f_0$ can be similarly defined for all $f_0 \in \mathcal{F}_\epsilon$ and $\alpha \in A_r$.

Proposition II.38. *For all $p/q \in \mathbb{Q}$, if $\epsilon > 0$ is sufficiently small then there exists $r > 0$ and $0 < \epsilon' < \epsilon$ such that $\mathcal{R}_{p/q,\alpha} f$ has a restriction in $\mathcal{F}_{\epsilon'}$ for all $f \in \mathcal{F}_\epsilon$ and $\alpha \in A_r$.*

Proof. Theorem II.20 implies that for all $\epsilon > 0$, there exists some $0 < \epsilon' < \epsilon$ such that $\mathcal{R}_{p/q,0} f$ has a restriction in $\mathcal{F}_{\epsilon'}$ for all $f \in \mathcal{F}_\epsilon$. The local uniform convergence of $\mathcal{R}_{p/q,\alpha} f$ to $\mathcal{R}_{p/q,0} f$ when $\alpha \to 0$ implies that if $r > 0$ is sufficiently small, then $\mathcal{R}_{p/q,\alpha} f$ has a restriction in $\mathcal{F}_{(\epsilon'+\epsilon)/2}$ for all $\alpha \in A_r$. As $\mathcal{F}_\epsilon$ is compact, r can be chosen uniformly. □

II.4: Orbit correspondences

We end this chapter by generalizing a classical result in parabolic implosion, the *parabolic orbit correspondence* introduced in [Lei00], or similarly the *Tour de Valse* introduced in [DH85].

Let us fix a map f and let P_{att} and P_{rep} be attracting and repelling petals for f with Fatou coordinates ϕ_{att} and ϕ_{rep} respectively. Let us fix some $\epsilon, r > 0$ and compact sets $X \subset P_{att}$, $Y \subset P_{rep}$. Let h_α be a holomorphic family of maps parameterized by $\alpha \in \mathbb{D}$ such that for any $\alpha \in A^+_r$ sufficiently close to 0:

1. There exists a petal P_α for h_α with Fatou coordinates $\phi_{att,\alpha}$ and $\phi_{rep,\alpha}$ which depend holomorphically on α and satisfy

$$\phi_{rep,\alpha} = T_{-1/\alpha} \circ \phi_{att,\alpha}.$$

2. Both X and Y are subsets of P_α and

$$\sup_{z \in X} |\phi_{att}(z) - \phi_{att,\alpha}(z)| < \epsilon \text{ and } \sup_{z \in Y} |\phi_{rep}(z) - \phi_{rep,\alpha}(z)| < \epsilon$$

for all $\alpha \in A_r^+$.

Proposition II.39. *There exists an integer $n_0 > 0$ and constant $C > 0$ which depend only on r, ϵ, X, and Y such that if $x : A_r^+ \to X$ and $y : A_r^+ \to Y$ are holomorphic functions, then for any $n \geq n_0$ the equation*

$$\phi_{att}(x(\alpha)) + n - \frac{1}{\alpha} = \phi_{rep,\alpha}(y(\alpha))$$

has a unique solution $\alpha_n \in A_r^+$ which satisfies

$$\left|\alpha_n - \frac{1}{n}\right| < \frac{C}{n^2}$$

when $n \geq n_0$. Moreover, if $y = y_t$ depends holomorphically or univalently on $t \in \mathbb{D}$, then $\alpha_n = \alpha_{n,t}$ also depends holomorphically or univalently on t respectively.

Proof. Let us fix some large $M > 0$ such that

$$\operatorname{Diam} \phi_{att}(X) + \operatorname{Diam} \phi_{rep}(Y) + 2\epsilon < M.$$

Thus

$$|\phi_{att,\alpha}(a) - \phi_{att}(a')| + |\phi_{rep,\alpha}(b) - \phi_{rep}(b')| < M$$

for all $a, a' \in X$, $b, b' \in Y$, and $\alpha \in A_r^+$. We fix some points $a_0 \in X$, $b_0 \in Y$, and set

$$\zeta = \phi_{att}(a_0) - \phi_{rep}(b_0).$$

For all integers $n > 0$, we define the holomorphic function $F_n(\alpha) = \zeta + n + \frac{1}{\alpha}$ on A_r^+. Let n_0 be sufficiently large so that

$$D_n := \{1/z : |z - \zeta - n| < 2M\} \subset A_r^+$$

for all $n \geq n_0$. Thus the equation $F_n(\alpha) = 0$ has a unique solution in D_n for all large n.

Now let us fix some $n \geq n_0$, and define the holomorphic functions

$$G_n(\alpha) = \phi_{att,\alpha}(x(\alpha)) + n - \frac{1}{\alpha} - \phi_{rep,\alpha}(y(\alpha)),$$
$$E(\alpha) = \phi_{att,\alpha}(x(\alpha)) - \phi_{rep,\alpha}(y(\alpha)) - \zeta$$

on A_r^+. Thus $F_n(\alpha) + E(\alpha) = G_n(\alpha)$ and $|E(\alpha)| < M$. As

$$\sup_{\alpha \in \partial D_n} |F_n(\alpha) - G_n(\alpha)| = \sup_{\alpha \in \partial D_n} |E(\alpha)| < M < 2M = \inf_{\alpha \in \partial D_n} |F_n(\alpha)|,$$

Rouché's theorem implies that the equation $G_n(\alpha) = 0$ has a unique solution α_n inside D_n. Thus, increasing n_0 if necessary, we have

$$\left|\alpha_n - \frac{1}{n}\right| = \frac{|\alpha_n|}{n} \cdot \left|n - \frac{1}{\alpha_n}\right| < \frac{2M + |\zeta|}{n(n - 2M - |\zeta|)} < \frac{4M + 2|\zeta|}{n^2}$$

when $n \geq n_0$.

Now we suppose that $y = y_t$ depends on $t \in \mathbb{D}$, so $G_n = G_{n,t}$ and $\alpha_n = \alpha_{n,t}$ also depend on t. If y_t depends holomorphically on $t \in \mathbb{D}$, then for any $t_0 \in \mathbb{D}$ we can repeat the above argument to conclude that G_{n,t_0} is univalent in a neighborhood of α_{n,t_0}. Thus $\alpha_{n,t}$ depends holomorphically on t near t_0 by the implicit function theorem. Moreover, if y_t depends univalently on t, then $\alpha_{n,t}$ also depends univalently on t. $\square$

CHAPTER III
Parabolic Towers

We will call any sequence $\mathcal{T} = \langle f_n \rangle_{n=1}^{N}$ of holomorphic functions, with $1 \leq N \leq \infty$, a *tower* of height N. For any $1 \leq M < N$, we will call $\langle f_n \rangle_{n=1}^{M}$ the *height M sub-tower* of $\mathcal{T}$. We will not distinguish between a sequence of length 1 and its sole entry, so for example we will write $\langle f_1 \rangle = f_1$. If $\mathcal{T} = \langle f_n \rangle_{n=1}^{N}$ is a sequence of length $1 < N < \infty$, then we will denote

$$\lfloor \mathcal{T} \rfloor := \langle f_n \rangle_{n=1}^{N-1} \text{ and } \lceil \mathcal{T} \rceil := \langle f_n \rangle_{n=2}^{N}.$$

We define a *parabolic tower* to be a tower $\mathcal{T} = \langle f_n \rangle_{n=1}^{N}$ such that for all $1 \leq n < N$, $f_n \in \mathcal{F}^{\text{❀}}$ and $f_{n+1} = \mathcal{R}_{\delta_n} f_n$ for some $\delta_n \in \mathbb{C}/\mathbb{Z}$. We will say that the $\mathcal{T}$ is *strictly* parabolic if $N < \infty$ and $f_N \in \mathcal{F}^{\text{❀}}$. We will call the function f_1 the *base* of the tower and the sequence $\langle \delta_n \rangle_{n=1}^{N-1}$ the *data* of the tower. Thus any parabolic tower is uniquely determined by its base and data, and every sub-tower of a parabolic tower is a strictly parabolic tower.

We denote by $\mathfrak{T}$ the set of all parabolic towers, and for all $N \geq 1$ we denote by $\mathfrak{T}_N$ the set of all parabolic towers with height less than or equal to N. We define $\pi_N : \mathfrak{T} \to \mathfrak{T}_N$ to be the map which is the identity on $\mathfrak{T}_N$ and sends

$$\langle f_n \rangle_{n=1}^{N'} \mapsto \langle f_n \rangle_{n=1}^{N}$$

when $N' > N$.

III.1: The space of parabolic towers

Let us now assume that $\mathcal{T} = \langle f_n \rangle_{n=1}^{N}$ is a finite height parabolic tower. For any other parabolic tower $\mathcal{T}' = \langle g_n \rangle_{n=1}^{N'}$, we define a *renormalization tower of $\mathcal{T}'$ relative to $\mathcal{T}$* to be a tower

$$\mathcal{R}_{\mathcal{T}} \mathcal{T}' := \langle h_n \rangle_{n=1}^{N},$$

$$\begin{array}{cccccccccc}
\mathcal{T} & f_1 & & f_2 & & f_3 & f_4 & & f_5 \\
\mathcal{R}_{\mathcal{T}}\mathcal{T}' & h_1 & \xrightarrow{\mathcal{R}_{f_1}h_1} & h_2 & \xrightarrow{\mathcal{R}_{f_2}h_2} & h_3 & h_4 & \xrightarrow{\mathcal{R}_{f_4}h_4} & h_5 \\
 & \Updownarrow & & & & & \Updownarrow & & \\
\mathcal{T}' & g_1 & & & & & g_2 & &
\end{array}$$

Figure III.1: A renormalization tower of $\mathcal{T}' = \langle g_n \rangle_{n=1}^{2}$ relative to $\mathcal{T} = \langle f_n \rangle_{n=1}^{5}$ with jump-heights $\langle 1, 4, 6 \rangle$. The double arrow $\Leftrightarrow$ indicates equality.

if it exists, such that there exists some $1 \le M \le \min(N, N')$ and integers

$$1 = s_1 < \cdots < s_{M+1} = N + 1$$

such that for all $1 \le j \le M$:

1. $h_{s_j} = g_j$,
2. for all $s_j \le n < s_{j+1} - 1$, h_n is an implosive perturbation of f_n and $h_{n+1} = \mathcal{R}_{f_n} h_n$,
3. if $j > 1$ and $n = s_j - 1$, then h_n is a stable perturbation of f_n.

We will call the sequence $\langle s_j \rangle_{j=1}^{M+1}$ the *jump-heights* of $\mathcal{R}_{\mathcal{T}}\mathcal{T}'$. The renormalization tower $\mathcal{R}_{\mathcal{T}}\mathcal{T}'$ is uniquely defined when we fix a choice of petals near each f_n. We note that the renormalization tower $\mathcal{R}_{\mathcal{T}}\mathcal{T}'$ depends only on $\pi_M(\mathcal{T}')$.

For any sequence $\langle \mathcal{N}_n \rangle_{n=1}^{N}$ where $\mathcal{N}_n$ is a neighborhood of f_n for all $1 \le n \le N$ and any choice of petals near f_n for all $1 \le n < N$, we will call the set of all parabolic towers $\mathcal{T}' = \langle g_n \rangle_{n=1}^{N'}$ which satisfy the following two conditions an *unbounded neighborhood* of $\mathcal{T}$:

1. The renormalization tower $\mathcal{R}_{\mathcal{T}}\mathcal{T}' = \langle h_n \rangle_{n=1}^{N}$ is defined for the choices of petals.
2. For all $1 \le n \le N$, $h_n \in \mathcal{N}_n$.

If $\mathcal{T}$ is strictly parabolic, then for any set $\mathcal{N}_{N+1}$ which is a union of neighborhoods of $\mathcal{R}_{\pm i\infty} f_N$ and any choice of petals near f_N, we will call the set of all such $\mathcal{T}'$ which also satisfy the following additional conditions a *bounded neighborhood* of $\mathcal{T}$:

3. If $M < N'$ and h_N is a stable perturbation of f_N, then $g_{M+1} \in \mathcal{N}_{N+1}$;
4. If h_N is an implosive perturbation of f_N, then $\mathcal{R}_{f_N} g_N \in \mathcal{N}_{N+1}$.

For any $N \geq 1$, we will say that a subset X of $\mathfrak{T}$ is a bounded or unbounded neighborhood of height N if X is a bounded or unbounded neighborhood of a parabolic tower of height N respectively. Note that these bounded and unbounded neighborhoods are partially ordered by inclusion.

Let us recall that a topology on a space X is *first-countable* if for every $x \in X$, there exists a a sequence of $\langle V_n \rangle_{n=1}^{\infty}$ of neighborhoods of x such that if V is a neighborhood of x, then there exists some $n \geq 1$ such that $V_n \subset V$.

Proposition III.1. *The set of all sufficiently small bounded and unbounded neighborhoods forms the basis of a first-countable topology on $\mathfrak{T}$.*

Proof. Every parabolic tower is automatically in every unbounded neighborhood of any of its sub-towers, so the union of all bounded and unbounded neighborhoods covers $\mathfrak{T}$. Fix some finite height tower $\mathcal{T} = \langle f_n \rangle_{n=1}^{N} \in \mathfrak{T}$ and let $\mathcal{N}$ be a either a bounded or unbounded neighborhood of $\mathcal{T}$. Fix some $\tilde{\mathcal{T}} = \langle \tilde{f}_n \rangle_{n=1}^{\tilde{N}} \in \mathcal{N}$, so the renormalization tower

$$\mathcal{R}_{\mathcal{T}} \tilde{\mathcal{T}} = \langle \tilde{g}_n \rangle_{n=1}^{N}$$

is defined with jump-heights $\langle \tilde{s}_j \rangle_{j=1}^{\tilde{M}+1}$ for some $1 \leq \tilde{M} \leq \min(N, \tilde{N})$. Let $\mathcal{T}' = \langle h_n \rangle_{n=1}^{N'}$ be another parabolic tower such that

$$\mathcal{R}_{\tilde{\mathcal{T}}} \mathcal{T}' = \langle \tilde{h}_n \rangle_{n=1}^{\tilde{N}}$$

is defined with jump-heights $\langle t_j \rangle_{j=1}^{M+1}$ for some $1 \leq M \leq \min(\tilde{N}, N')$. Let us observe that when $\mathcal{T}'$ belongs to a sufficiently small neighborhood of $\tilde{\mathcal{T}}$, we can define a tower $\langle g_n \rangle_{n=1}^{N}$ such that

1. $g_{\tilde{s}_j} = \tilde{h}_j$, and
2. for all $\tilde{s}_j \leq n < \tilde{s}_{j+1} - 1$, $g_{n+1} = \mathcal{R}_{f_n} g_n$,

for all $1 \leq j \leq \tilde{M}$. Indeed for any $1 \leq j \leq \tilde{M}$ and $\tilde{s}_j \leq n < \tilde{s}_{j+1} - 2$, $\tilde{g}_n$ is an implosive perturbation of f_n. Thus if g_n is defined and sufficiently close to $\tilde{g}_n$, then g_n is also an implosive perturbation of f_n and $\mathcal{R}_{f_n} g_n$ is close to $\mathcal{R}_{f_n} \tilde{g}_n$. By taking $\mathcal{T}'$ in a sufficiently small neighborhood of $\tilde{\mathcal{T}}$, we can force $g_{\tilde{s}_j} = \tilde{h}_j$ to be arbitrarily close to $\tilde{f}_j = \tilde{g}_{\tilde{s}_j}$, combined with the above this guarantees that the tower $\langle g_n \rangle_{n=1}^{N}$ is defined. Moreover, by taking $\mathcal{T}'$ in a sufficiently small neighborhood of $\tilde{\mathcal{T}}$, we can force each g_n to be arbitrarily close to $\tilde{g}_n$.

Setting $s_j = \tilde{s}_{t_j}$ for all $1 \leq j \leq M + 1$, let us now show that $\langle g_n \rangle_{n=1}^{N} = \mathcal{R}_{\mathcal{T}} \mathcal{T}'$ and has jump-heights $\langle s_j \rangle_{n=1}^{M+1}$. As $\tilde{h}_{t_j} = h_j$, we first need to check that for any $1 \leq j \leq M$ and all

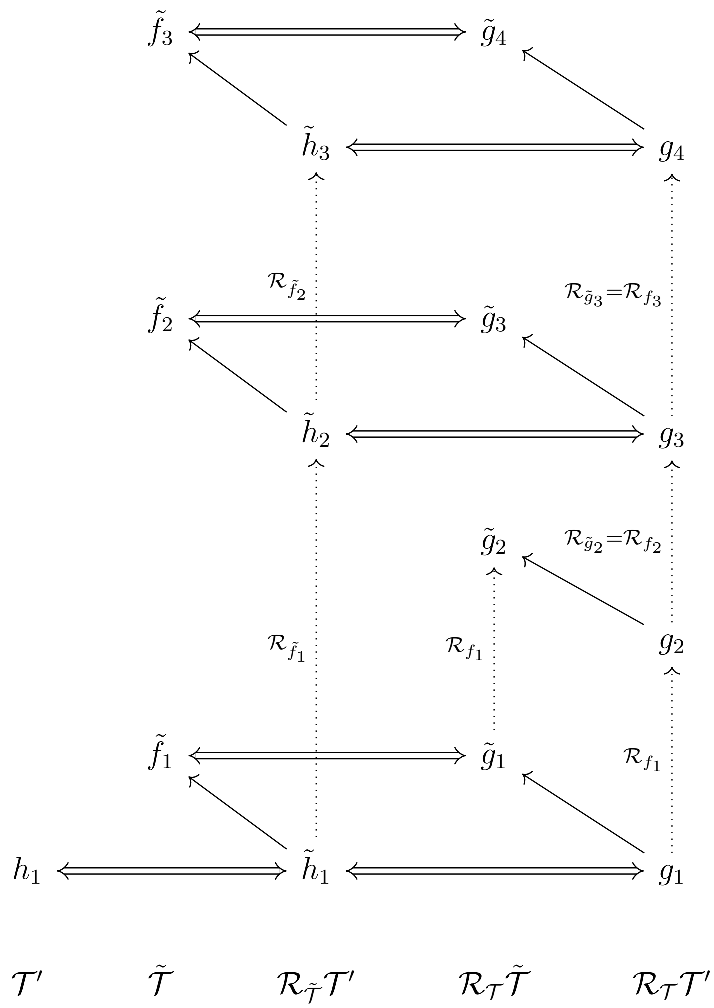


Figure III.2: An example of the renormalization towers in the proof of proposition III.1. Here the double arrow $\Leftrightarrow$ indicates equality, the arrow $\rightarrow$ indicates convergence when $\mathcal{T}'$ is taken in successively smaller bounded neighborhoods of $\tilde{\mathcal{T}}$, and the dashed arrow $\dashrightarrow$ indicates the action of the labeled operator.

$t_j < k < t_{j+1}$,

$$g_{\tilde{s}_k} = \mathcal{R}_{f_{\tilde{s}_k-1}} g_{\tilde{s}_k-1}.$$

By definition, $\tilde{g}_{s_k-1}$ is a stable perturbation of f_{s_k-1}, so

$$\mathcal{R}_{f_{\tilde{s}_k-1}} g_{\tilde{s}_k-1} = \mathcal{R}_{\tilde{g}_{\tilde{s}_k-1}} g_{\tilde{s}_k-1}.$$

Repeated application of corollary II.37 implies that, by taking $\mathcal{T}'$ in a sufficiently small unbounded neighborhood of $\tilde{\mathcal{T}}$ and requiring that $\mathcal{N}$ is sufficiently small, we can ensure that

$$\mathcal{R}_{\tilde{g}_{\tilde{s}_k-1}} g_{\tilde{s}_k-1} = \mathcal{R}_{\tilde{g}_{\tilde{s}_{k-1}}} g_{\tilde{s}_{k-1}} = \mathcal{R}_{\tilde{f}_{k-1}} \tilde{h}_{k-1} = \tilde{h}_k = g_{\tilde{s}_k}$$

on any compact subset of the domain of $\mathcal{R}_0 \tilde{g}_{\tilde{s}_k-1}$. It follows from the definition that for any $1 < k \leq \tilde{M}+1$,

$$g_{\tilde{s}_k-1} = \left(\mathcal{R}_{f_{\tilde{s}_k-2}} \cdots \mathcal{R}_{f_{\tilde{s}_{k-1}}}\right) \tilde{h}_{k-1}.$$

As

$$\tilde{g}_{\tilde{s}_k-1} = \left(\mathcal{R}_{f_{\tilde{s}_k-2}} \cdots \mathcal{R}_{f_{\tilde{s}_{k-1}}}\right) \tilde{f}_{k-1}$$

is a stable perturbation of $f_{\tilde{s}_k-1}$ for all $1 \leq k \leq \tilde{M}+1$ and $\tilde{h}_{t_j-1}$ is a stable perturbation of $\tilde{f}_{t_j-1}$ for all $1 < j \leq M$, repeated application of corollary II.37 implies that $g_{s_j-1} = g_{\tilde{s}_{t_j}-1}$ is a stable perturbation of f_{s_j-1} for all $1 < j \leq M$ if $\mathcal{N}$ was chosen sufficiently small. This verifies that $\langle g_n \rangle_{n=1}^N = \mathcal{R}_{\mathcal{T}} \mathcal{T}'$ when $\mathcal{T}$ is in a sufficiently small unbounded neighborhood of $\tilde{\mathcal{T}}$.

Thus if $\mathcal{N}$ is an unbounded neighborhood of $\mathcal{T}$, then there exists an unbounded neighborhood of $\tilde{\mathcal{T}}$ contained in $\mathcal{N}$. If instead $\mathcal{N}$ is a bounded neighborhood, then $f_N \in \mathcal{F}^{\circledast}$. If $\tilde{g}_N$ is an implosive perturbation of f_N, then g_N is also an implosive perturbation and $\mathcal{R}_{f_N} g_N$ can be made arbitrarily close to $\mathcal{R}_{f_N} \tilde{g}_N$ by forcing g_N to be close to $\tilde{g}_N$. If $\tilde{g}_N$ is a stable perturbation of f_N, then repeated application of corollary II.37 implies that g_N is a stable or implosive perturbation of $\tilde{g}_N$ if and only if $\tilde{h}_M$ is a stable perturbation of $\tilde{f}_N$. In the stable case, $\mathcal{R}_0 g_N$ and $\mathcal{R}_0 \tilde{h}_M$ agree on any compact subset of the domain of $\mathcal{R}_0 \tilde{g}_N$ when $\tilde{h}_M$ is close to $\tilde{f}_M$ and g_N is close to $\tilde{g}_N$. In the implosive case, $\mathcal{R}_{\tilde{g}_N} g_N$ and $\mathcal{R}_{\tilde{f}_M} \tilde{h}_M$ agree on any compact subset of the domain of $\mathcal{R}_0 \tilde{g}_N$ when $\tilde{h}_M$ and g_N are sufficiently close to $\tilde{f}_M$ and $\tilde{g}_N$ respectively. Thus there exists either a bounded or unbounded neighborhood $\tilde{\mathcal{N}}$ of $\tilde{\mathcal{T}}$ contained in $\mathcal{N}$. It follows from the definition that any non-empty intersection of two bounded or unbounded neighborhoods of $\mathcal{T}$ contains a bounded neighborhood of $\mathcal{T}$. Thus if N_1 and N_2 are sufficiently small bounded or unbounded neighborhoods of $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively and $\mathcal{T} \subset \mathcal{N}_1 \cap \mathcal{N}_2$, then we can find a bounded neighborhood of $\mathcal{T}$ contained in $\mathcal{N}_1 \cap \mathcal{N}_2$. So we can conclude that the bounded and unbounded neighborhoods form the basis of a topology. The fact that this topology is first-countable follows immediately from the fact that the compact-open topology is first-countable. □

For the rest of this thesis, we equip $\mathfrak{T}$ with the topology described by proposition III.1. While we have a natural inclusion $\mathfrak{T}_N \subset \mathfrak{T}$, we will not equip $\mathfrak{T}_N$ with the subspace topology. To define an alternative topology on $\mathfrak{T}_N$, we first make the following observation.

Proposition III.2. *Let $\mathcal{N}$ be a bounded or unbounded neighborhood of height N and set $M = N + 1$ or $M = N$ respectively. For any $\mathcal{T} \in \mathfrak{T}$, $\mathcal{T} \in \mathcal{N}$ if and only if $\pi_M(\mathcal{T}) \in \mathcal{N}$.*

Proof. This follows immediately from the fact that $\mathcal{R}_{\mathcal{T}}\mathcal{T}'$ depends only on $\pi_N(\mathcal{T}')$ for any $\mathcal{T} \in \mathfrak{T}_N$. □

For all $N \geq 1$, we endow $\mathfrak{T}_N$ with the topology generated by the sets $\pi_N(X)$ where X is a bounded or unbounded neighborhood of height no more that $N-1$ or N respectively. We can repeat the proof of proposition III.1 to verify that these sets do indeed form the basis of a first-countable topology on $\mathfrak{T}_N$. It follows from proposition III.2 that with this choice of topology the map π_N is continuous for any N. Note that given two finite height parabolic towers $\mathcal{T}$ and $\mathcal{T}'$, if we write $\mathcal{T}' \to \mathcal{T}$ then there is some ambiguity to whether we are considering the convergence in $\mathfrak{T}$ or $\mathfrak{T}_N$ for some $N \geq 1$. To avoid possible confusion, we will always assume that the convergence is in $\mathfrak{T}$ unless explicitly stated.

Let $\mathcal{T} = \langle f_n \rangle_{n=1}^{N}$ be a parabolic tower with data $\langle \delta_n \rangle_{n=1}^{N-1}$, let $\mathcal{T}'$ be another parabolic tower with $\mathcal{R}_{\mathcal{T}}\mathcal{T}' = \langle h_n \rangle_{n=1}^{N}$, and let $\langle k_n \rangle_{n=1}^{N-1}$ be a sequence of integers which depends on $\mathcal{T}'$. When $\mathcal{T}'$ converges to $\mathcal{T}$ in $\mathfrak{T}$ or $\mathfrak{T}_N$, we will say that the convergence has *combinatorics* $\langle k_n \rangle_{n=1}^{N-1}$ if h_n converges to $\langle f_n, \mathcal{R}_{\delta_n} f_n \rangle$ with combinatorics k_n for all $1 \leq n \leq N-1$ such that h_n is an implosive perturbation of f_n. Note that while we normally consider the entries δ_n of the data of $\mathcal{T}$ in $\mathbb{C}/\mathbb{Z}$, in this context we instead take δ inside $\mathbb{C}$.

It will be important to understand how convergence of towers induces convergence of their sub-towers. Let $\mathcal{T} = \langle f_n \rangle_{n=1}^{N}$ and $\mathcal{T}' = \langle g_n \rangle_{n=1}^{N'}$ be parabolic towers with N finite and such that the renormalization tower $\mathcal{R}_{\mathcal{T}}\mathcal{T}'$ is defined with jump heights $\langle s_n \rangle_{n=1}^{M'+1}$. Setting $M = s_{M'} \leq N$, let us denote $\mathcal{T}_M = \pi_M(\mathcal{T})$ and $\mathcal{T}'_{M'} = \pi_{M'}(\mathcal{T}')$.

Proposition III.3. *If the jump heights remain constant when $\mathcal{T}' \to \mathcal{T}$, then $\mathcal{T}'_{M'} \to \mathcal{T}$. If $M' > 1$, then additionally $\lfloor \mathcal{T}'_{M'} \rfloor \to \lfloor \mathcal{T}_M \rfloor$.*

Proof. It follows from the definition that $\mathcal{R}_{\mathcal{T}}\mathcal{T}'_{M'} = \mathcal{R}_{\mathcal{T}}\mathcal{T}' = \langle h_n \rangle_{n=1}^{N}$. When $\mathcal{T}' \to \mathcal{T}$ we have that $h_n \to f_n$ for all n. Moreover, if h_N is an implosive perturbation of f_N, then $\mathcal{R}_{f_N} h_N$

is eventually contained in every union of neighborhoods of $\mathcal{R}_{\pm i\infty} f_N$, so $\mathcal{T}'_{M'} \to \mathcal{T}$. If $M' > 1$, so $M > 1$, then a h_{M-1} is a stable perturbation of f_{M-1} and

$$\mathcal{R}_{\lfloor \mathcal{T}_M \rfloor} \lfloor \mathcal{T}'_{M'} \rfloor = \langle h_n \rangle_{n=1}^{M-1},$$

hence $\lfloor \mathcal{T}'_{M'} \rfloor \to \lfloor \mathcal{T}_M \rfloor$. □

III.1.1: Inverse limits

Let $\langle X_n, f_n \rangle_{n=1}^{\infty}$ be a sequence such that X_n is a topological space and $f_n : X_{n+1} \to X_n$ is continuous for all n. Such a sequence is called an *inverse system*. The *inverse limit* of the inverse system, which is denoted by $\varprojlim \langle X_n, f_n \rangle_{n=1}^{\infty}$, is defined to be the set of all sequences $\langle x_n \rangle_{n=1}^{\infty}$ such that $x_n \in X_n$ and $f_n(x_{n+1}) = x_n$ for all $n \geq 1$. The inverse limit is a subset of the infinite product $\prod_{n\geq 1} X_n$ and is equipped with the subspace topology.

Proposition III.4. *$\mathfrak{T}$ is homeomorphic to $\varprojlim \langle \mathfrak{T}_n, \pi_n \rangle_{n=1}^{\infty}$.*

Proof. Given any parabolic tower $\mathcal{T}$, we can define the sequence $\langle \mathcal{T}_n \rangle_{n=1}^{\infty}$ by $\mathcal{T}_n = \pi_n(\mathcal{T})$. It is easy to check that the resulting map $\varphi : \mathfrak{T} \to \varprojlim \langle \mathfrak{T}_n, \pi_n \rangle_{n=1}^{\infty}$ is a bijection. Given any bounded or unbounded neighborhood X of height N and setting $M = N + 1$ or $M = N$ respectively, proposition III.2 implies that

$$\varphi(X) = \left(\varprojlim \langle \mathfrak{T}_n, \pi_n \rangle_{n=1}^{\infty}\right) \cap \left(\left(\prod_{1\leq n<M} \mathfrak{T}_n\right) \times \pi_M(X) \times \left(\prod_{n>M} \mathfrak{T}_n\right)\right).$$

In particular, φ maps the basis of the topology on $\mathfrak{T}$ to a basis of the topology on $\varprojlim \langle \mathfrak{T}_n, \pi_n \rangle_{n=1}^{\infty}$. Thus φ is a homeomorphism. □

III.1.2: Elevators

For any finite height parabolic tower $\mathcal{T} = \langle f_n \rangle_{n=1}^{N}$ with data $\langle \delta_N \rangle_{n=1}^{N-1}$, we define the *attracting elevator* of $\mathcal{T}$ inductively by

$$\eta_{att}^{\mathcal{T}}(z) := \begin{cases} z & \text{if } N = 1, \\ \eta_{att,\delta_{N-1}}^{f_{N-1}} \circ \eta_{att}^{\lfloor \mathcal{T} \rfloor}(z) & \text{if } N > 1. \end{cases}$$

We denote the domain of $\eta_{att}^{\mathcal{T}}$ by $U^{\lfloor \mathcal{T} \rfloor}$, so $U^{\lfloor \mathcal{T} \rfloor} = \mathbb{C}^*$ when $N = 1$ and

$$U^{\lfloor \mathcal{T} \rfloor} = (\eta_{att}^{\lfloor \mathcal{T} \rfloor})^{-1}(U^{f_{N-1}})$$

when $N > 1$. Thus if $\mathcal{T}$ is a strictly parabolic tower, then $U^{\mathcal{T}}$ is defined.

Proposition III.5. *The map $\eta_{att}^{\mathcal{T}} : U^{\lfloor \mathcal{T} \rfloor} \to \mathbb{C}^*$ is an analytic branched covering map whose unique critical value, if it exists, is $cv^{f_N} = \eta_{att}^{\mathcal{T}}(cv^{f_1})$. Moreover, cv^{f_1} is not a critical point of $\eta_{att}^{\mathcal{T}}$.*

Proof. The proposition clearly holds when $N = 1$, so assume that $N > 1$ and that the proposition holds for $\eta_{att}^{\lfloor \mathcal{T} \rfloor}$. It follows from the definition of $\rho^{f_{N-1}}$ that $cv^{f_{N-1}} = \eta_{att}^{\lfloor \mathcal{T} \rfloor}(cv^{f_1})$ is not a critical point of $\rho^{f_{N-1}}$, so cv^{f_1} is not a critical point of $\eta_{att}^{\mathcal{T}}$. Additionally, the critical values of $\rho^{f_{N-1}}$ are all integers, so $\mathrm{Exp}(\delta_{N-1}) = cv^{f_N}$ is the unique critical value of $\eta_{att}^{\mathcal{T}}$. □

Proposition III.6. *For any open set V which contains a point in $\partial U^{\lfloor \mathcal{T} \rfloor}$ and any $y \in \mathbb{C}^*$, there exists infinitely many points $x \in V$ such that $\eta_{att}^{\mathcal{T}}(x) = y$.*

Proof. This is a special case of proposition III.18 below. □

Corollary III.7. *If $N > 1$ and $\mathcal{T}$ is a strictly parabolic tower, then*

$$\partial U^{\mathcal{T}} = \partial U^{\lfloor \mathcal{T} \rfloor} \cup (\eta_{att}^{\mathcal{T}})^{-1}(\partial U^{f_N}).$$

Proof. It follows immediately from the definition that

$$\partial U^{\mathcal{T}} \cap U^{\lfloor \mathcal{T} \rfloor} = (\eta_{att}^{\mathcal{T}})^{-1}(\partial U^{f_N}).$$

Moreover, $\partial U^{\mathcal{T}} \subset \overline{U^{\lfloor \mathcal{T} \rfloor}}$ as $U^{\mathcal{T}} \subset U^{\lfloor \mathcal{T} \rfloor}$. Proposition III.6 implies that any point in $\partial U^{\lfloor \mathcal{T} \rfloor}$ can be approximated by points inside or outside of $(\eta_{att}^{\mathcal{T}})^{-1}(U^{f_N})$, so $\partial U^{\lfloor \mathcal{T} \rfloor} \subset \partial U^{\mathcal{T}}$. □

We also define the *repelling elevator of $\mathcal{T}$* inductively by

$$\eta_{rep}^{\mathcal{T}}(z) := \begin{cases} z & \text{if } N = 1, \\ \eta_{rep}^{f_{N-1}} \circ \eta_{rep}^{\lfloor \mathcal{T} \rfloor}(z) & \text{if } N > 1. \end{cases}.$$

Let $\mathcal{T}' = \langle g_n \rangle_{n=1}^{N'}$ be another parabolic tower of possibly infinite height whose renormalization tower $\mathcal{R}_{\mathcal{T}}\mathcal{T}' = \langle h_n \rangle_{n=1}^{N}$ is defined with jump heights $\langle s_n \rangle_{n=1}^{M'+1}$. Setting $M = s_{M'} \leq N$, so $h_M = g_{M'}$, let us denote $\mathcal{T}_M = \pi_M(\mathcal{T})$ and $\mathcal{T}'_{M'} = \pi_{M'}(\mathcal{T}')$. We define the *attracting* and *repelling elevators of $\mathcal{T}'$ relative to $\mathcal{T}$* to be

$$\eta_{att}^{\mathcal{T}',\mathcal{T}} := \eta_{att}^{h_{N-1},f_{N-1}} \circ \cdots \circ \eta_{att}^{h_M,f_M} \circ \eta_{att}^{\mathcal{T}'_{M'}}, \text{ and}$$
$$\eta_{rep}^{\mathcal{T}',\mathcal{T}} := \eta_{rep}^{h_{N-1},f_{N-1}} \circ \cdots \circ \eta_{rep}^{h_M,f_M} \circ \eta_{rep}^{\mathcal{T}'_{M'}}.$$

Proposition III.8. *If $\pi_N(\mathcal{T}') \to \mathcal{T}$ in $\mathfrak{T}_N$, then $\eta_{att}^{\mathcal{T}',\mathcal{T}} \to \eta_{att}^{\mathcal{T}}$ and $\eta_{rep}^{\mathcal{T}',\mathcal{T}} \to \eta_{rep}^{\mathcal{T}}$.*

Proof. We will only prove the proposition for the repelling elevators, the same argument can be used for the attracting elevators. Up to a subsequence we can assume that the jump heights $\langle s_n \rangle_{n=1}^{M'+1}$ do not change when $\mathcal{T}' \to \mathcal{T}$.

If $N = 1$, then $M = M' = 1$ and the proposition holds by definition. So we assume that $N > 1$ and that the proposition holds for smaller values of N. If $M < N$, then

$$\eta_{rep}^{\mathcal{T}',\mathcal{T}} = \eta_{rep}^{h_{N-1},f_{N-1}} \circ \eta_{rep}^{\mathcal{T}',\lfloor \mathcal{T} \rfloor},$$

so the proposition holds by the inductive hypothesis and the convergence of $\eta_{rep}^{h_{N-1},f_{N-1}}$ to $\eta_{rep}^{f_{N-1}}$. If $N = M = s_{M'}$, then $N > 1$ and $s_1 = 1$ implies that $M' > 1$. We set $M_0 = s_{M'-1}$, so $g_{M'-1} = h_{M_0}$. As

$$h_{M-1} = \mathcal{R}_{f_{N-2}} \circ \mathcal{R}_{f_{N-3}} \circ \cdots \circ \mathcal{R}_{f_{N_0}} h_{M_0},$$

and h_{M-1} is a stable perturbation of f_{M-1}, repeated application of proposition II.34 implies that

$$\eta_{rep}^{g_{M'-1}} = \eta_{rep}^{h_{M_0}} = \eta_{rep}^{h_{N-1}} \circ \eta_{rep}^{h_{N-2},f_{N-2}} \circ \cdots \circ \eta_{rep}^{h_{M_0},f_{M_0}}.$$

The inductive hypothesis and proposition III.3 therefore imply that

$$\eta_{rep}^{\mathcal{T}',\mathcal{T}} = \eta_{rep}^{\mathcal{T}'_{M'}} = \eta_{rep}^{g_{M'-1}} \circ \eta_{rep}^{\lfloor \mathcal{T}'_{M'} \rfloor} = \eta_{rep}^{g_{M'-1}} \circ \eta_{rep}^{\lfloor \mathcal{T}'_{M'} \rfloor,\lfloor \mathcal{T}_M \rfloor} \to \eta_{rep}^{f_{N-1}} \circ \cdots \circ \eta_{rep}^{f_{M_0}} \circ \eta_{rep}^{\lfloor \mathcal{T}_M \rfloor} = \eta_{rep}^{\mathcal{T}}.$$

□

Proposition III.9. *Fix some $z \in Dom(\eta_{att}^{\mathcal{T}})$, and set $\zeta = \eta_{att}^{\mathcal{T}}(z)$. For any $|\theta| < \pi/4$ and $M > 0$, if $\mathcal{T}'$ is sufficiently close to $\mathcal{T}$, then for any simply connected set $Y \subset \mathbb{C}$ which avoids cv^{h_N}, contains ζ, and on which there is a continuous branch of Exp^{-1} satisfying*

$$\operatorname{Exp}^{-1}(Y) \subset \{e^{i\theta} w : |\operatorname{Re} w| < M\},$$

the petals near $\mathcal{T}$ can be chosen so that there exists a unique continuous branch of $(\eta_{att}^{\mathcal{T}',\mathcal{T}})^{-1}$ defined on Y which sends ζ close to z.

Proof. If $N = 1$, then both $\eta_{att}^{\mathcal{T}}$ and $\eta_{att}^{\mathcal{T}',\mathcal{T}}$ are the identity, so the proposition holds automatically. So we assume that $N > 1$ and that the proposition holds for towers of lesser height. It follows from the definition that either

$$\eta_{att}^{\mathcal{T}',\mathcal{T}} = \eta_{att}^{h_{N-1},f_{N-1}} \circ \eta_{rep}^{\mathcal{T}',\lfloor \mathcal{T} \rfloor},$$

or $\eta_{att}^{\mathcal{T}',\mathcal{T}} = \eta_{att}^{\mathcal{T}''}$ where $\mathcal{T}''$ is a sub-tower of $\mathcal{T}'$. In the former case, the proposition holds by proposition II.31 and the inductive hypothesis. In the latter case, the proposition holds as $\eta_{att}^{\mathcal{T}''}$ is a covering map onto $\mathbb{C}^*$ branched at cv^{h_N}. □

III.2: Quadratic parabolic towers

For all $\alpha \in \mathbb{C}$, we define the quadratic polynomial

$$\mathfrak{f}_\alpha(z) = e^{2\pi i\alpha} z + z^2.$$

We also define $\mathfrak{f}_{+i\infty}(z) := z^2$, and define $\mathfrak{f}_{-i\infty}$ to be the constant function from $\hat{\mathbb{C}} \setminus \{0\}$ to $\{\infty\}$.

Proposition III.10. *When* $\operatorname{Im}\alpha \to \pm\infty$, $\mathfrak{f}_\alpha \to \mathfrak{f}_{\pm i\infty}$.

Proof. When $\operatorname{Im}\alpha \to +i\infty$, $e^{2\pi i\alpha} \to 0$. For any $0 < \epsilon < 1$ and $z \in \mathbb{C}$ satisfying $\epsilon < |z| < 1/\epsilon$,

$$|\mathfrak{f}_\alpha(z)| = |z||e^{2\pi i\alpha} + z| > \epsilon(|e^{2\pi i\alpha}| - 1/\epsilon) \to \infty$$

when $\operatorname{Im}\alpha \to -i\infty$. □

We denote by Quad the space of all $\mathfrak{f}_\alpha$ with $\alpha \in \mathbb{C} \cup \{\pm i\infty\}$ with the compact-open topology. it follows from proposition III.10 that Quad is homeomorphic to $\mathbb{C}/\mathbb{Z} \cup \{\pm i\infty\}$ with the natural topology; in particular Quad is compact and Hausdorff.

We will say that a parabolic tower is *quadratic* if its base is in Quad, and define $\widehat{\text{Quad}} \subset \mathfrak{T}$ to be the space of all quadratic parabolic towers, and for all $N \geq 1$ we denote

$$\widehat{\text{Quad}}_N := \pi_N(\widehat{\text{Quad}}) \subset \mathfrak{T}_N.$$

Proposition III.11. *For all* $N \geq 1$, $\widehat{\text{Quad}}_N$ *is Hausdorff.*

Proof. Fix some $1 \leq N_1 \leq N_2 \leq N$ and let $\mathcal{T}_1 = \langle f_{n,1}\rangle_{n=1}^{N_1}$ and $\mathcal{T}_2 = \langle f_{n,2}\rangle_{n=1}^{N_2}$ be quadratic parabolic towers. For all $1 \leq n \leq N_1$ let $\mathcal{N}_{n,1}$ be a neighborhood of $f_{n,1}$ and for all $1 \leq n \leq N_2$ let $\mathcal{N}_{n,2}$ be a neighborhood of $f_{n,2}$. Together with choices of petals, these neighborhoods define unbounded neighborhoods $\mathcal{N}_1$ and $\mathcal{N}_2$ of $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively. Let us assume that $\mathcal{T}_1 \neq \mathcal{T}_2$, so either there exists some minimal $1 \leq M \leq N_1$ so that $\pi_M(\mathcal{T}_1) \neq \pi_M(\mathcal{T}_2)$ or $N_1 < N_2$ and $\mathcal{T}_1 = \pi_{N_1}(\mathcal{T}_2)$.

First we consider the case where $\pi_M(\mathcal{T}_1) \neq \pi_M(\mathcal{T}_2)$ for some minimal $1 \leq M \leq N_1$. Thus $f_{M,1} \neq f_{M,2}$. If $M = 1$, then we can pick $\mathcal{N}_{M,1}$ and $\mathcal{N}_{M,2}$ so that the intersection $\mathcal{N}_{M,1} \cap \mathcal{N}_{M,2}$ is empty as Quad is Hausdorff. If $M > 1$, then the same conclusion holds from the fact that $f_{M,1} = \mathcal{R}_{\delta_1} f_{M-1,1}$ and $f_{M,2} = \mathcal{R}_{\delta_1} f_{M-1,1}$ for some $\delta_1 \neq \delta_2$. If $\mathcal{T}$ is a quadratic parabolic tower such that the renormalization towers $\mathcal{R}_{\mathcal{T}_1}\mathcal{T} := \langle h_{n,1}\rangle_{n=1}^{N_1}$ and $\mathcal{R}_{\mathcal{T}_2}\mathcal{T} := \langle h_{n,2}\rangle_{n=1}^{N_2}$, then it follows from the minimality of M that $h_{M,1} = h_{M_2}$. Hence the intersection $\mathcal{N}_1 \cap \mathcal{N}_2$ is empty.

Now we consider the case where $N_1 < N_2$ and $\mathcal{T}_1 = \pi_{N_1}(\mathcal{T}_2)$. Thus $f_{N_1+1,2} = \mathcal{R}_\delta f_{N_1,1}$ for some $\delta \in \mathbb{C}$. Let $\mathcal{N}_{N_1+1,1}$ be a union of neighborhoods of $\mathcal{R}_{\pm i\infty} f_{N_1,1}$, this defined a bounded neighborhood $\tilde{N}_1$ of $\mathcal{T}_1$. As $\delta \in \mathbb{C}$, we can choose $\mathcal{N}_{N_1+1,1}$ and $\mathcal{N}_{N+1,2}$ so that the intersection $\mathcal{N}_{N_1+1,1} \cap \mathcal{N}_{N_1+1,2}$ is empty. Repeating the argument above, we can conclude that the intersection $\tilde{\mathcal{N}_1} \cap \mathcal{N}_2$ is empty. □

Recall that a topological space X is *sequentially compact* if every sequence in X has a convergent subsequence.

Proposition III.12. *For all $N \geq 1$, $\widehat{\mathrm{Quad}}_N$ is sequentially compact.*

Proof. Let

$$\langle \mathcal{T}_m = \langle f_{n,m}\rangle_{n=1}^{N_m}\rangle_{m=1}^{\infty}$$

be a sequence in $\widehat{\mathrm{Quad}}_N$. Up to a subsequence, we can assume that N_m does not depend on m. As Quad is a compact, up to a subsequence there exists some $f_1 \in \mathrm{Quad}$ such that $f_{1,m} \to f_1$ when $m \to \infty$. Setting $\mathcal{T} = \langle f_1 \rangle$, the renormalization towers $\mathcal{R}_{\mathcal{T}}\mathcal{T}_m$ are therefore defined for all large m and have jump-heights $\langle 1, 2\rangle$. Let us now assume more generally that there is some quadratic parabolic tower $\mathcal{T} = \langle f_n \rangle_{n=1}^{M}$ with height $1 \leq M \leq N$ such that up to a subsequence the renormalization towers

$$\mathcal{R}_{\mathcal{T}}\mathcal{T}_m = \langle h_{n,m}\rangle_{n=1}^{M}$$

are defined for all large $m \geq 0$ and have jump-heights $\langle s_j \rangle_{n=1}^{M'+1}$ for some $1 \leq M' \leq \min(M, N_m)$ which do not depend on m. Additionally, let us assume that $h_{n,m} \to f_n$ for all $1 \leq n \leq M$ when $m \to \infty$. If $f_M \notin \mathcal{F}^{\text{\ding{96}}}$ or $M = N$, then by definition $\mathcal{T}_m \to \mathcal{T}$ in $\widehat{\mathrm{Quad}}_N$ when $m \to \infty$. If $f_M \in \mathcal{F}^{\text{\ding{96}}}$ and $M < N$, then up to a further subsequence we can assume that $h_{M,m}$ is either a non-implosive, implosive, or stable perturbation of f_M for all large m. In the non-implosive case, it follows from the definition that $\mathcal{T}_m \to \mathcal{T}$ in $\widehat{\mathrm{Quad}}_N$. In the implosive case, up to a further subsequence there exists some $\delta_M \in \mathbb{C} \cup \{\pm i\infty\}$ such that $h_M \to \mathcal{R}_{\delta_M} f_M$ when $m \to \infty$. If $\delta_M = \pm i\infty$ then by definition $\mathcal{T}_m \to \mathcal{T}$ in $\widehat{\mathrm{Quad}}_N$, otherwise we can set $f_{M+1} := \mathcal{R}_{\delta_M} f_M$. In the stable case, if $M = N_m$ then by definition $\mathcal{T}_m \to \mathcal{T}$. Otherwise, by definition there exists some sequence $\langle \delta_{M',m}\rangle_{m=1}^{\infty} \in \mathbb{C}/\mathbb{Z}$ so that

$$h_{M+1,m} = f_{M'+1,m} = \mathcal{R}_{\delta_{M',m}} f_{M',m} = \mathcal{R}_{\delta_{M',m}} h_{M',m}$$

by proposition II.37. Thus up to a subsequence there exists some $\delta_M \in \mathbb{C} \cup \{\pm i\infty\}$ so that $\delta_{M',m} \to \delta_M$ when $m \to \infty$. If $\delta_M = \pm i\infty$ then by definition $\mathcal{T}_m \to \mathcal{T}$ in $\widehat{\text{Quad}}_N$; otherwise we can set $f_{M+1} := \mathcal{R}_{\delta_M} f_M$ and $h_{M+1,m} \to f_{M+1}$. The proposition therefore follows by induction on M. □

We have the following classical result on inverse limits:

Proposition III.13. *If $\langle X_n, f_n \rangle_{n=1}^{\infty}$ is an inverse system of sequentially compact Hausdorff spaces, then $\varprojlim \langle X_n, f_n \rangle_{n=1}^{\infty}$ is sequentially compact and Hausdorff.*

Corollary III.14. $\widehat{\text{Quad}}$ *is sequentially compact and Hausdorff.*

III.3: Continuity of filled Julia sets

For any non-constant analytic function $f : \hat{\mathbb{C}} \dashrightarrow \hat{\mathbb{C}}$ defined on an open subset of $\hat{\mathbb{C}}$, the *Fatou set* $\Omega(f)$ of f is defined to be the set of all $z \in \hat{\mathbb{C}}$ which have a neighborhood U satisfying either

1. $U \subset Dom(f^n) \setminus Dom(f^{n+1})$ for some $n \geq 0$, or

2. $U \subset Dom(f^n)$ for all $n \geq 0$ and the family $\{f^n_{|U}\}$ is normal.

The *Julia set* of f is defined to be $\hat{\mathbb{C}} \setminus \Omega(f)$.

An analytic map $f : \hat{\mathbb{C}} \dashrightarrow \hat{\mathbb{C}}$ defined on an open subset of $\hat{\mathbb{C}}$ is said to be *finite-type* if f has finitely many singular values; that is there exists a finite set $X \subset \hat{\mathbb{C}}$ such that the restriction

$$f : Dom(f) \setminus f^{-1}(X) \to \hat{\mathbb{C}} \setminus X$$

is a covering map. Finite-type maps were introduced by Epstein [Eps93], where he proved the following two theorems.

Theorem III.15 (Epstein)**.** *For any finite type map f,*

$$J(f) = \overline{\{\text{repelling periodic points of f}\}}.$$

Theorem III.16 (Epstein)**.** *For any finite type map f, every component of $\Omega(f)$ which belongs to $\bigcap_{n\geq 0} Dom(f^n)$ is eventually periodic under f. Moreover, any periodic component of $\Omega(f)$ is either either an attracting basin, a parabolic basin, a Siegel disk, or a Herman ring.*

Note that every map in $\mathcal{F}$ is finite-type. For any quadratic polynomial f, the *filled Julia set* $K(f)$ is defined to be the complement of the unique component of $\Omega(f)$ which contains ∞. The filled Julia sets of polynomials have been well studied, see for example [DH84]. In particular it is known that $K(f)$ is non-empty and compact,

$$K(f) = \{z \in \mathbb{C} : f^n(z) \not\to \infty\},$$

and that $\partial K(f) = J(f)$. We also define

$$J(\mathfrak{f}_{-i\infty}) := K(\mathfrak{f}_{-i\infty}) = \{0, -1, \infty\},$$

so that the $J(f)$ and $K(f)$ are defined for all $f \in \mathrm{Quad}$.

Proposition III.17. *When* $\operatorname{Im} \alpha \to -\infty$,

$$\lim K(\mathfrak{f}_\alpha) = \lim J(\mathfrak{f}_\alpha) = K(\mathfrak{f}_{-i\infty}).$$

Proof. Set $\mu = e^{2\pi i\alpha}$ and $f_\mu(z) = \mathfrak{f}_\alpha(z) = \mu z + z^2$. If $|z| > |\mu| + 2$, then

$$|f_\mu(z)| \geq |z|^2 - |\mu||z| > 2|z|.$$

For any $\epsilon > 0$, if $|z| > \epsilon$ and $|1 + z| > \epsilon$, then

$$|f_\mu^2(z)| = |z(\mu + z)(\mu(1 + z) + z^2)| > \frac{\epsilon^2}{4}|\mu|^2 > |\mu| + 2$$

when $|\mu|$ is sufficiently large. Hence $\lim J(f_\mu) \subset \{0, -1, \infty\}$ when $\mu \to \infty$. As 0 and $1 - \mu$ are repelling fixed points of f_μ and $f_\mu(-1) = 1 - \mu$, it follows that $\{0, -1, \infty\} \subset \lim J(f_\mu)$ when $\mu \to \infty$. $\square$

We will say that an analytic map $f : \hat{\mathbb{C}} \dashrightarrow \hat{\mathbb{C}}$ has *hyperbolic domain* if $Dom(f)$ is a hyperbolic subset of $\hat{\mathbb{C}}$; that is $\mathbb{C} \setminus Dom(f)$ contains at least three points. It is shown in [Eps93] that for such maps, every point in the boundary acts as an essential singularity:

Proposition III.18. *If f is a finite-type map with hyperbolic domain which is a branched covering over $\mathbb{C}^*$, then for $z \in \partial U$, open set U containing z, and any $y \in \mathbb{C}^*$, there are infinitely many $x \in U$ satisfying $f(x) = y$.*

For an analytic map f with hyperbolic domain, we define the filled Julia set by

$$K(f) = \{z \in \hat{\mathbb{C}} : f^n(z) \notin \hat{\mathbb{C}} \setminus \overline{Dom(f)} \text{ for any } n \geq 0\}.$$

It follows from the definition that $K(f)$ is non-empty and compact.

Proposition III.19. *If f has hyperbolic domain, then*

$$J(f) = \partial K(f) = \overline{\bigcup_{n\geq 1} \partial Dom(f^n)}.$$

Proof. For any z which is not in $Dom(f^n)$ for some $n \geq 0$, it follows from the definitions that both $z \in J(f)$ and $z \in K(f)$ hold if and only if there exists some $0 \leq m \leq n$ such that $z \in \partial Dom(f^m)$.

Let z be a point which is in $Dom(f^n)$ for all $n \geq 0$. As $Dom(f)$ is a hyperbolic, by Montel's theorem it follows from the definitions that both $z \in J(f)$ and $z \in \partial K(f)$ hold if and only if there is a sequence of points $\langle z_k \rangle_{k=1}^{\infty}$ such that

$$z_k \notin \bigcap_{n\geq 1} Dom(f^n)$$

and $z_k \to z$ when $z \to \infty$. As both $Dom(f^n)$ and $\hat{\mathbb{C}} \setminus \overline{Dom(f^n)}$ are open for any $n \geq 1$, for any connected neighborhood V of z it follows that there exists a point belonging to

$$V \cap \left(\bigcup_{n\geq 1} \partial Dom(f^n) \right).$$

□

The filled Julia set of a map with hyperbolic domain is in many ways analogous to the filled Julia set of a polynomial; indeed both are the set of all points which do not "escape" for some notion of escaping. Given any $f \in \mathcal{F}$ we will say that an analytic map h is an *appropriate* perturbation of h if f and h either both belong to Quad or both have hyperbolic domain.

Let us recall that for the Hausdorff metric the lim inf and lim sup of a sequence $\langle X_n \rangle_{n=1}^{\infty}$ in $\text{Comp}^*(\hat{\mathbb{C}})$ are defined by

$$\liminf X_n := \left\{ z \in \hat{\mathbb{C}} : \lim_{n\to\infty} \inf_{x\in X_n} d_{\hat{\mathbb{C}}}(z,x) = 0 \right\}, \text{ and}$$
$$\limsup X_n := \left\{ z \in \hat{\mathbb{C}} : \lim_{n\to\infty} \inf_{x\in \bigcup_{m\geq n} X_m} d_{\hat{\mathbb{C}}}(z,x) = 0 \right\}.$$

Moreover, if $\liminf X_n = \limsup X_n = X$, then $X = \lim X_n$ when $n \to \infty$.

Theorem III.20. *Fix some $f \in \mathcal{F}$ which is either a polynomial or has hyperbolic domain. If $\langle f_n \rangle_{n=1}^{\infty}$ is a sequence of appropriate perturbations of f which converge to f when $n \to \infty$, then*

$$J(f) \subset \liminf J(f_n) \subset \limsup K(f_n) \subset K(f).$$

Moreover, every Siegel disk and Herman ring of f is contained in $\liminf J(f_n)$ and every attracting basin of f is contained in $\liminf K(f_n)$. If f has a parabolic cycle and either $\langle f_n \rangle_{n=1}^{\infty}$ is a sequence of either

1. *stable perturbations of f,*
2. *non-implosive perturbations of f, or*
3. *implosive perturbations of f such that $\mathcal{R}_f f_n \to \mathcal{R}_{\pm i\infty} f$,*

then every parabolic basin of f is contained in $\liminf K(f_n)$.

Proof. When f is a polynomial, this theorem follows from results in [Dou94] and [McM00]. The argument in [Dou94] relies on studying the perturbation of periodic components of $\Omega(f)$. When instead f has hyperbolic domain, by proposition III.16 we have the same classification of periodic Fatou components, and we can apply the same argument used in [Dou94]. Let us note that while the perturbation of Herman rings is not treated in [Dou94], the same analysis used to study Siegel disks can be used in that case. □

III.3.1: Filled Julia sets of towers

For a finite height quadratic parabolic tower $\mathcal{T} = \langle f_N \rangle_{n=1}^{N}$, we define the *filled Julia set* of $\mathcal{T}$ to be

$$K(\mathcal{T}) := \overline{(\eta_{att}^{\mathcal{T}})^{-1}(K(f_N))}$$

if $N > 1$ is finite, and

$$K(\mathcal{T}) := \bigcap_{M \geq 1} K(\pi_M(\mathcal{T}))$$

if $N = \infty$. It follows from the definition that if $\mathcal{T}$ is a strictly parabolic tower then

$$U^{\mathcal{T}} \subset K(\mathcal{T}) \subset \overline{U^{\lfloor \mathcal{T} \rfloor}}.$$

We define the *Julia set* of the tower to be

$$J(\mathcal{T}) := \overline{(\eta_{att}^{\mathcal{T}})^{-1}(J(f_N))}$$

if N is finite and

$$J(\mathcal{T}) = \overline{\bigcup_{M\geq 1} J(\pi_M(\mathcal{T}))}$$

if $M = \infty$. If $\mathcal{T}$ has height 1, then we define $K(\lfloor\mathcal{T}\rfloor)$ and $J(\lfloor\mathcal{T}\rfloor)$ to be $\hat{\mathbb{C}}$ and the empty set respectively.

Proposition III.21. *If $\mathcal{T}$ has finite height $N \geq 1$, then*

$$K(\mathcal{T}) = J(\lfloor\mathcal{T}\rfloor) \cup (\eta_{att}^{\mathcal{T}})^{-1}(K(f_N)) \text{ and}$$
$$J(\mathcal{T}) = J(\lfloor\mathcal{T}\rfloor) \cup (\eta_{att}^{\mathcal{T}})^{-1}(J(f_N)).$$

Proof. It follows immediately from the definition that any point $z \in K(\mathcal{T})$ either belongs to $\partial U^{\lfloor\mathcal{T}\rfloor}$ or is mapped into $K(f_N)$ by $\eta_{att}^{\mathcal{T}}$. Conversely, proposition III.6 implies that any point in $\partial U^{\lfloor\mathcal{T}\rfloor}$ can be approximated by points which project by $\eta_{att}^{\mathcal{T}}$ into $K(f_N)$. With a similar argument for $J(\mathcal{T})$, the proposition follows. □

Corollary III.22. *For any $\mathcal{T} \in \widehat{\mathrm{Quad}}$, $J(\mathcal{T}) = \partial K(\mathcal{T})$. If $\mathcal{T}'$ is a subtower of $\mathcal{T}$, then*

$$J(\mathcal{T}') \subsetneq J(\mathcal{T}) \subset K(\mathcal{T}) \subsetneq K(\mathcal{T}').$$

Thus as we increase the height of a parabolic tower, the Julia set grows and the filled Julia set shrinks. In the limit, for infinite height towers, Epstein showed that these two sets equalize.

Theorem III.23 (Epstein)**.** *If $\mathcal{T}$ has infinite height, then $K(\mathcal{T}) = J(\mathcal{T})$.*

Proof. This is one of the main results in [Eps93]. The formalism used by Epstein differs from ours, so we will briefly comment on how to translate from our towers to the the objects in [Eps93].

For any parabolic tower $\mathcal{T} = \langle f_n\rangle_{n=1}^N$, we can define the pre-sheaf $\mathcal{O}_{\mathcal{T}}$ of holomorphic functions induced by all possible compositions of local continuous branches of

$$(\eta_{rep}^{\mathcal{T}_M})^{-1} \circ f_M^j \circ \eta_{att}^{\mathcal{T}_M},$$

where $1 \leq M \leq N$, $\mathcal{T}_M = \langle f_n\rangle_{n=1}^M$, and $j \geq 0$. The pre-sheaf $\mathcal{O}_{\mathcal{T}}$ is a *holomorphic dynamical system* in the sense of Epstein. In [Eps93], Epstein proved that if $\mathcal{T} \in \widehat{\mathrm{Quad}}$, or more generally if the base of $\mathcal{T}$ is finite-type, then the holomorphic dynamical system $\mathcal{O}_{\mathcal{T}}$ has no wandering domains. If $\mathcal{T}$ has infinite height, then any component X of the interior of $K(\mathcal{T})$ is a wandering domain of of $\mathcal{O}_{\mathcal{T}}$ as the orbit under f_M of $\eta_{att}^{\mathcal{T}_M}(X)$ converges towards the parabolic cycle of f_M. □

Just as for polynomials, we have semi-continuity of filled Julia sets of towers.

Proposition III.24. *For any integer $N \geq 1$ and towers $\mathcal{T}_1, \mathcal{T}_2 \in \widehat{\mathrm{Quad}}_N$,*

$$J(\mathcal{T}_1) \subset \liminf J(\mathcal{T}_2) \subset \limsup K(\mathcal{T}_2) \subset K(\mathcal{T}_1)$$

when $\mathcal{T}_2 \to \mathcal{T}_1$ in $\widehat{\mathrm{Quad}}_N$. Moreover, if the height of $\mathcal{T}_1$ is strictly less than N, then

$$\lim K(\mathcal{T}_2') = K(\mathcal{T}_1)$$

when $\mathcal{T}_2 \to \mathcal{T}_1$ in $\widehat{\mathrm{Quad}}_N$ for any quadratic parabolic tower $\mathcal{T}_2'$ which has $\mathcal{T}_2$ as a subtower.

Proof. Let $\mathcal{T}_1 = \langle f_n \rangle_{n=1}^{N_1}$ and $\mathcal{T}_2 = \langle g_n \rangle_{n=1}^{N_2}$ be quadratic parabolic towers in $\widehat{\mathrm{Quad}}_N$. Setting $\mathcal{R}_{\mathcal{T}_1}\mathcal{T}_2 = \langle h_n \rangle_{n=1}^{N_1}$, there exists some maximal M_1 such that $h_{M_1} = g_{M_2}$ for some $1 \leq M_2 \leq M_1$. Up to a subsequence we can assume that M_1 and M_2 remain constant when $\mathcal{T}_2 \to \mathcal{T}_1$ in $\widehat{\mathrm{Quad}}$. We set $\mathcal{T}_1' = \pi_{M_1}(\mathcal{T}_1)$ and $\mathcal{T}_2' = \pi_{M_2}(\mathcal{T}_2)$. Proposition III.3 implies that $\mathcal{T}_2' \to \mathcal{T}_1$ when $\mathcal{T}_2 \to \mathcal{T}_1$, and $\lfloor \mathcal{T}_2' \rfloor \to \lfloor \mathcal{T}_1' \rfloor$ if $M_2 > 1$. We also define $\tilde{\eta}_{att}^{\mathcal{T}_1}$ and $\tilde{\eta}_{rep}^{\mathcal{T}_1}$ to be the functions so that

$$\eta_{att}^{\mathcal{T}_1} = \tilde{\eta}_{att}^{\mathcal{T}_1} \circ \eta_{att}^{\mathcal{T}_1'} \text{ and } \eta_{rep}^{\mathcal{T}_1} = \tilde{\eta}_{rep}^{\mathcal{T}_1} \circ \eta_{rep}^{\mathcal{T}_1'}.$$

We similarly define $\tilde{\eta}_{att}^{\mathcal{T}_2}$ and $\tilde{\eta}_{rep}^{\mathcal{T}_2}$ to be the functions so that

$$\eta_{att}^{\mathcal{T}_2,\mathcal{T}_1} = \tilde{\eta}_{att}^{\mathcal{T}_2} \circ \eta_{att}^{\mathcal{T}_2'} \text{ and } \eta_{rep}^{\mathcal{T}_2,\mathcal{T}_1} = \tilde{\eta}_{rep}^{\mathcal{T}_2} \circ \eta_{rep}^{\mathcal{T}_2'}.$$

As $\eta_{att}^{\mathcal{T}_2'} = \eta_{att}^{\mathcal{T}_2',\mathcal{T}_1'}$ and $\eta_{rep}^{\mathcal{T}_2'} = \eta_{rep}^{\mathcal{T}_2',\mathcal{T}_1'}$, it follows from proposition III.8 that $\tilde{\eta}_{att}^{\mathcal{T}_2} \to \tilde{\eta}_{att}^{\mathcal{T}_1}$ and $\tilde{\eta}_{rep}^{\mathcal{T}_2} \to \tilde{\eta}_{rep}^{\mathcal{T}_1}$ when $\mathcal{T}_2 \to \mathcal{T}_1$. Moreover, it follows from proposition III.21 that

$$K(\mathcal{T}_1) = J(\lfloor \mathcal{T}_1 \rfloor) \cup \left((\eta_{att}^{\mathcal{T}_1'})^{-1} \circ (\tilde{\eta}_{att}^{\mathcal{T}_1})^{-1}(K(f_{N_1})) \right)$$

and

$$J(\mathcal{T}_1) = J(\lfloor \mathcal{T}_1 \rfloor) \cup \left((\eta_{att}^{\mathcal{T}_1'})^{-1} \circ (\tilde{\eta}_{att}^{\mathcal{T}_1})^{-1}(J(f_{N_1})) \right).$$

It follows from corollary III.22 that if $M_2 > 1$ then

$$K(\mathcal{T}_2) \subset K(\mathcal{T}_2') = J(\lfloor \mathcal{T}_2' \rfloor) \cup (\eta_{att}^{\mathcal{T}_2'})^{-1}(K(g_{M_2}))$$

and

$$J(\lfloor \mathcal{T}_2' \rfloor) \cup (\eta_{att}^{\mathcal{T}_2'})^{-1}(J(g_{M_2})) = J(\mathcal{T}_2') \subset J(\mathcal{T}_2).$$

We proceed by induction on N_1. To prove the proposition, it suffices to show

(III.3.1) $$J(\lfloor\mathcal{T}_1\rfloor) \subset \liminf J(\mathcal{T}_2') \subset \limsup K(\mathcal{T}_2') \subset K(\lfloor\mathcal{T}_1\rfloor)$$

and

(III.3.2) $$(\tilde{\eta}_{att}^{\mathcal{T}_1})^{-1}(J(f_{N_1})) \subset U^{f_{M_1}} \cap \liminf J(g_{M_2}) \subset U^{f_{M_1}} \cap \limsup K(g_{M_2}) \subset (\tilde{\eta}_{att}^{\mathcal{T}_1})^{-1}(K(f_{N_1})).$$

The inclusions (III.3.1) hold automatically when $N_1 = 1$ and by the inductive hypothesis when $N_1 > 1$, so we only need to prove (III.3.2). If $N_1 = M_1$ then $\tilde{\eta}_{att}^{\mathcal{T}_1}$ and $\tilde{\eta}_{att}^{\mathcal{T}_2}$ are both the identity, so the desired inclusions follow from theorem III.20. So we assume that $N_1 > M_1$. Fix some connected compact set

$$X \subset U^{f_{M_1}} \setminus (\tilde{\eta}_{att}^{\mathcal{T}_1})^{-1}(K(f_{N_1})).$$

Thus there exists some integer j so that

$$f_N^j \circ \tilde{\eta}_{att}^{\mathcal{T}_1}(X) \subset \hat{\mathbb{C}} \setminus \overline{Dom(f_N)}.$$

Thus there exists a connected compact set $X' \subset \hat{\mathbb{C}} \setminus \overline{U^{f_{M_1}}} = \hat{\mathbb{C}} \setminus K(f_{M_1})$ whose interior compactly contains a component of

$$(\tilde{\eta}_{rep}^{\mathcal{T}_1})^{-1}(f_N^j \circ \tilde{\eta}_{att}^{\mathcal{T}_1}(X)).$$

Thus when $\mathcal{T}_2$ is close enough to $\mathcal{T}_1$, theorem III.20 and $g_{M_2} = h_{M_1} \to f_{M_1}$ implies that $X' \subset \hat{\mathbb{C}} \setminus K(g_{M_2})$. Moreover, there exists a compact set $X_2' \subset X'$ such that

$$\tilde{\eta}_{rep}^{\mathcal{T}_2}(X_2') = h_N^j \circ \tilde{\eta}_{att}^{\mathcal{T}_2}(X).$$

Repeated application of propositions II.29 and II.30 therefore implies that X is mapped into X' by an iterate of g_{M_2}, hence $X \subset \hat{\mathbb{C}} \setminus K(g_{M_2})$ as $K(g_{M_2})$ is invariant under $g_{M_2}^{-1}$. Thus we have shown that

$$U^{f_{M_1}} \cap \limsup K(g_{M_2}) \subset (\tilde{\eta}_{att}^{\mathcal{T}_1})^{-1}(K(f_{N_1})).$$

Now fix some point

$$x \in (\tilde{\eta}_{att}^{\mathcal{T}_1})^{-1}(J(f_{N_1}))$$

and let $X \subset U^{f_{M_1}}$ be any open neighborhood of x. It follows from theorem III.15 that there exists a point $x_1 \in X$ such that $\tilde{\eta}_{att}^{\mathcal{T}_1}(x_1)$ is a repelling periodic point of f_{N_1}. As repelling periodic points are stable under perturbation, when $\mathcal{T}_2$ is sufficiently close to $\mathcal{T}_1$ there exists a point $x_2 \in X$ such that $\tilde{\eta}^{\mathcal{T}_1}_{att}(x_2)$ is a repelling periodic point of h_{N_1}. Repeated application of propositions II.29 and II.30 therefore implies that x_2 is mapped onto a repelling periodic point of g_{M_2} by an iterate of g_{M_2}, theorem III.15 therefore implies that $x_2 \in J(g_{M_2})$ as $J(g_{M_2})$ is invariant under $g^{-1}_{M_2}$. Thus we have shown that

$$(\tilde{\eta}^{\mathcal{T}_1}_{att})^{-1}(J(f_{N_1})) \subset U^{f_{M_1}} \cap \liminf J(g_{M_2}),$$

completing our verification of (III.3.2).

To show the second statement in the proposition, let us observe that if $\mathcal{T}_1$ is not a strictly parabolic tower, then theorem III.20 implies that $\lim K(h_{N_1}) = K(f_{N_1})$, so the above implies that $K(\mathcal{T}_1) = \liminf J(\pi_{M_2}(\mathcal{T}_2))$. Thus $K(\mathcal{T}_1) = \lim K(\mathcal{T}_2')$ for any quadratic parabolic tower $\mathcal{T}_2'$ which has $\mathcal{T}_2$ as a sub-tower. If $\mathcal{T}_1$ is a strictly parabolic tower and $N_1 < N$, then our definition of bounded neighborhoods of $\mathcal{T}_1$ combined with theorem III.20 implies similarly that $K(\mathcal{T}_1) = \lim K(\pi_{M_2}(\mathcal{T}_2))$. Up to a subsequence one of the following cases holds:

1. h_{N_1} is a stable perturbation of f_{N_1} and g_{M_1+1} converges to $\mathcal{R}_{\pm i\infty} f_{N_1}$,

2. h_{N_1} is an implosive perturbation of f_{N_1} and $\mathcal{R}_{f_{N_1}} h_{N_1} \to \pm$, or

3. h_{N_1} is a non-implosive perturbation of f_{N_1}.

The first case implies that $g_{M_2+1} \notin \mathcal{F}^{\circledast}$, and we can recreate the earlier arguments in this proof to conclude that $K(\mathcal{T}_1) = \lim K(\mathcal{T}_2)$. The second and third cases above imply that h_{N_1} has an attracting cycle, so $h_{M_1} = g_{M_2} \notin \mathcal{F}^{\circledast}$. Thus $\pi_{M_2}(\mathcal{T}_2) = \mathcal{T}_2$. Note that in all three cases, the same holds for any parabolic tower which has $\mathcal{T}_2$ as a sub-tower. □

As K is continuous and $\mathrm{Comp}^*(\hat{\mathbb{C}})$ is Hausdorff, $K(\widehat{\mathrm{Quad}})$ is sequentially compact and hence closed. As we have the following commutative diagram for the embedding $f \mapsto \langle f \rangle$ of Quad into $\widehat{\mathrm{Quad}}$

$$\begin{array}{ccc} \widehat{\mathrm{Quad}} & & \\ \uparrow & \searrow^{K} & \\ \mathrm{Quad} & \xrightarrow{K} & \mathrm{Comp}^*(\hat{\mathbb{C}}), \end{array}$$

it follows that

$$\overline{\{K(f) : f \in \mathrm{Quad}\}} \subset K(\widehat{\mathrm{Quad}}).$$

The following proposition implies that the above inclusion is actually an equality.

Proposition III.25. *For any* $\mathcal{T} \in \widehat{\mathrm{Quad}}$*, there exists a sequence* $\langle g_m \rangle_{m=1}^{\infty}$ *with* $g_m \in \mathrm{Quad}$ *such that* $\langle g_m \rangle \to \mathcal{T}$*.*

Proof. For any $\mathcal{T} = \langle f_1 \rangle$, we can set $g_m = f_1$ for all $m \geq 1$. If f_1 has a p/q-parabolic k-periodic cycle for some p/q and k, after conjugating by a Möbius transformation we can assume that 0 belongs to the parabolic cycle. It follows from [MnSS83] that Quad has no persistently parabolic cycles, that is for any sufficiently small $\alpha \in \mathbb{C}$ we can find some $g_{m,\alpha}$ such that, after conjugating by a Möbius transformation, 0 is k-periodic for $g_{m,\alpha}$,

$$(g_{m,\alpha})'(0) = e^{2\pi i \mu_{p/q}(\alpha)},$$

and $g_{m,\alpha} \to g_m$ when $\alpha \to 0$. For any $\delta \in \mathbb{C}$, we set

$$\alpha_k := \frac{1}{k + c_+^{f_1} - \delta}.$$

If $\mathcal{R}_\delta f_1 \in \mathcal{F}^{\text{✿}}$, then we can modify α_k slightly so that g_{m,α_k} is a stable perturbation of $\mathcal{R}_\delta f_1$ when m and k are sufficiently large. Thus $g_{m,\alpha_k} \to \langle f_1, \mathcal{R}_\delta f_1 \rangle$ when $m \to \infty$ and $k \to \infty$.

Let us fix a tower $\mathcal{T} = \langle f_n \rangle_{n=1}^{N}$ for some integer $N \geq 1$ and assume that there is a sequence $\langle g_m \rangle_{n=1}^{\infty}$ such that $g_m \to \mathcal{T}$. We additionally assume that for $\mathcal{R}_{\mathcal{T}} g_m = \langle h_{m,n} \rangle_{n=1}^{N}$, $h_{m,N}$ is a stable perturbation of f_N for all large m. Thus $h_{m,N}$ has a parabolic cycle with the same period and multiplier as f_N, and proposition II.33 implies that g_m also has a parabolic periodic cycle. It follows from the above that for any $\delta \in \mathbb{C}$ we can produce a sequence $\langle \tilde{g}_{m,k} \rangle_{k=1}^{\infty}$ such that $\tilde{g}_{m,k} \to \langle g_m, \mathcal{R}_\delta g_m \rangle$ when $k \to \infty$. Corollary II.37 implies that

$$\mathcal{R}_\delta g_m = \mathcal{R}_\delta h_{m,N} \to \mathcal{R}_\delta f_N$$

when $m \to \infty$, hence

$$\tilde{g}_{m,k} \to \mathcal{T} \oplus \langle \mathcal{R}_\delta f_N \rangle := \mathcal{T}'$$

when both m and k tend to ∞. Moreover, we can modify $\tilde{g}_{m,k}$ slightly so that $\tilde{g}_{m,k}$ is a stable perturbation of $\mathcal{T}'$.

By induction, this proves the proposition for all finite height towers. We can extend to infinite height towers by a diagonalization argument. ☐

It is well known that the function $K : \mathrm{Quad} \to \mathrm{Comp}^*(\hat{\mathbb{C}})$ is injective, see for example [Fer89]. We can ask if K remains injective on $\widehat{\mathrm{Quad}}$. If it is injective, then we can conclude that

$$K : \widehat{\mathrm{Quad}} \to \overline{\{K(f) : f \in \mathrm{Quad}\}}$$

is a homeomorphism as every continuous bijection from a sequentially compact first-countable space to a Hausdorff space is a homeomorphism.

CHAPTER IV
The Geometry and Dynamics of Lavaurs Maps

Let us fix a map f which is Möbius conjugate to a map in $\mathcal{F}^{\text{✿}}$. Without loss of generality we will assume that f has a p/q-parabolic fixed point at 0 for some rational $p/q \in [-1/2, 1/2]$; the general periodic case can be studied similarly by considering the appropriate iterate of f and conjugating by a Möbius transformation.

In this chapter, we restrict our attention to the case where every connected component of U^f is a Jordan domain, in this case we will say that f has *Jordan basin*. Fixing some $\delta \in \mathbb{C}$, we will only consider the dynamics of the restriction of $\mathcal{R}_\delta f$ to $Dom_0(\mathcal{R}_\delta f)$; let us denote this restriction by h. Corollary II.14 implies that $Dom(h)$ is a Jordan domain. Note that proposition II.18 implies that we could similarly study instead the restriction of $\mathcal{R}_\delta^- f$ to the component of its domain containing 0.

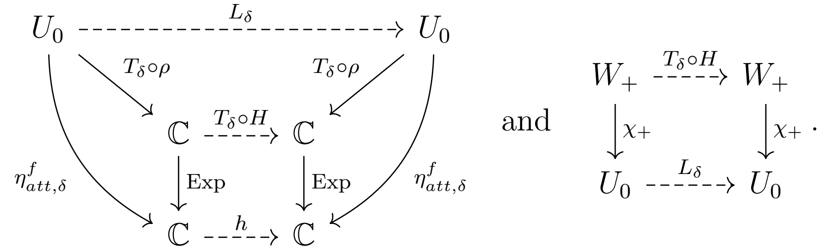
Let $0 \le k_+^f \le q$ be the integer such that $k_+^f p \equiv -1 \mod q$. Proposition II.9 implies that

$$f^{k_+^f} \circ \chi^f(W_+^f) = U_0^f.$$

Denoting $\chi_+^f := f^{k_+^f} \circ \chi^f$, we define the *(upper) δ-Lavaurs map* for f to be

$$L_\delta^f := \chi_+^f \circ T_\delta \circ \rho^f.$$

We will suppress the dependence on f and the dependence on δ in the notation when the choices are clear, so for example $\chi_+ = \chi_+^f$ and $L = L_\delta^f$. As $0 \le k_+^f < q$, it follows from the definition of ρ that we have the following commutative diagrams:



It follows from the definition that

$$L_\delta \circ f^q = L_{\delta+1} = f^q \circ L_\delta.$$

For any (possibly negative) integer m we define

$$L_\delta \circ f^{qm} := L_{\delta+m}.$$

We equip $\mathbb{Z}^2$ with the lexicographic ordering; so $(m,n) > (m',n')$ if and only if either $m > m'$ or $m = m'$ and $n > n'$. The function $L_\delta^d \circ f^{qm}$ is therefore defined if and only $(d,m) \geq (0,0)$. While L_δ is not necessarily defined on all of U_0^f, any $z \in U_0^f$ lies in the domain of $L_\delta \circ f^{-qm}$ when $m \geq 0$ is sufficiently large.

IV.1: Escaping sets

We denote $\mathcal{E}_1^f := \mathbb{C} \setminus \overline{W_+^f}$. As ∂W_+ is a Jordan arc and $\chi_+(W_+) = U_0$, we can continuously extend χ_+ to $\partial W_+ = \partial \mathcal{E}_1$ such that

$$\chi_+(\partial \mathcal{E}_1) = \partial U_0,$$

where all the boundaries above are taken in $\mathbb{C}$.

Proposition IV.1. *There exists a unique homeomorphism*

$$\Upsilon_1^f : \overline{\mathcal{E}_1} \to \overline{\mathbb{H}}$$

which satisfies:

1. *Υ_1 is univalent on $\mathcal{E}_1$.*
2. *$\Upsilon_1 \circ T_1 = T_1 \circ \Upsilon_1$.*
3. *$t = 0$ is the minimal real number satisfying $\chi_+ \circ \Upsilon_1^{-1}(t) = 0$.*

Proof. As $\mathcal{E}_1$ is simply connected, contains a lower half-plane, avoids an upper half-plane, and is invariant under T_1, there exists an analytic isomorphism $\Upsilon_1 : \mathcal{E}_1 \to \mathbb{H}$, unique up to post-composition by a translation, which commutes with T_1. As the boundary of $\mathcal{E}_1$ in $\hat{\mathbb{C}}$ is a Jordan curve in, Υ_1 extends continuously to a homeomorphism from $\overline{\mathcal{E}_1} \to \overline{\mathbb{H}}$. As χ_+ maps a left half-plane to a punctured neighborhood of 0 and $\chi_+(\partial \mathcal{E}_1) = \partial U_0$, there exists some minimal $t_0 \in \mathbb{R}$ such that $\chi_+ \circ \Upsilon_1^{-1}(t_0) = 0$. We can uniquely post-compose Υ_1 by a translation so that $t_0 = 0$. □

We will say that a point in $\hat{\mathbb{C}}$ is 0-*nonescaping*, 0-*Julia*, or 0-*nonescaping* for L_δ if it belongs to U_0, ∂U_0, or $\mathbb{C} \setminus U_0$ respectively. For any $d \geq 1$, we will similarly say that a point z is d-nonescaping, d-Julia, or d-nonescaping for L if it is $(d-1)$-nonescaping and

$$T_\delta \circ \rho \circ L^{d-1}(z)$$

belongs to $W_+, \partial W_+$, or $\mathcal{E}_1$ respectively. For all $d \geq 0$, we denote by K_d^L, J_d^L, and E_d^L the set of all d-nonescaping, d-Julia, and d-escaping points respectively. We will say that L is d-nonescaping, d-Julia, or d-escaping when cv^f is the same respectively for L. Given a component B of E_d for $d \geq 1$, we will say that $z \in \partial B$ is a *parent* of B if z is not d-Julia.

We will say that a point $z \in \overline{U_0}$ is *pre-critical* for L if there is some $(d, m) > (0, 0)$ such that z is d-nonescaping and $L^d \circ f^{mq}(z) = cv^f$. Otherwise, we will say that z is *off-critical* for L. If L is d-escaping or d-Julia for any $d \geq 1$, then the critical value cv^f is automatically off-critical. We have the following alternative characterization of off-critical points:

Proposition IV.2. *For all $d \geq 0$, $z \in K_d$ is pre-critical for L if and only there is some integer m with $(d, m) > (0, 0)$ such that z is a critical point of $L^d \circ f^{mq}$.*

Proof. Let us fix some point $z \in U_0$; z is a critical point of f^{mq} for some $m > 0$ if and only if there exists some $0 < m' \leq m$ such that $f^{m'q}(z) = cv^f$. It follows from the definition of ρ that z is a critical point of ρ if and only if $f^{mq}(z) = cv^f$ for some $m > 0$. It follows from the definition of χ_+ that any point $w \in W_+$ is a critical point of χ_+ if and only if there is some integer $m \geq 0$ such that $\chi_+ \circ T_{-m}(w) = cv^f$. Thus $z \in K_1$ is a critical point of $L \circ f^{mq}$ for any integer m if and only if either $f^{m'q}(z) = cv^f$ for some integer $m' > 0$ or $L \circ f^{(m-m')q}(z) = cv^f$ for some $m' \geq 0$. The proposition then follows by a straightforward induction on d. $\square$

For all $d \geq 1$, we define the function

$$\upsilon_d^L := \Upsilon_1 \circ T_\delta \circ \rho \circ L^{d-1} : E_d \cup J_d \to \overline{\mathbb{H}}.$$

Let $\mathcal{A} = \mathcal{A}^f$ denote the set of all $x \in \mathbb{R}$ such that

$$\chi_+ \circ \Upsilon_1^{-1}(x + m) = 0$$

for some integer m. We will say that a point z is *d-asymptotic* for L if $\upsilon_d(z) \in \mathcal{A}$. We will also say that every point in

$$\chi_+ \circ \Upsilon_1^{-1}(\mathcal{A}) \subset J_0$$

is 0-asymptotic. We have the following description of the escaping and Julia sets:

Proposition IV.3. *For all $d \geq 1$, J_d has countably many connected components. Each component is either an open Jordan arc or the union of two open Jordan arcs which intersect at a pre-critical point of L. Conversely, any pre-critical point in J_d lies in the intersection of two distinct curves in J_d.*

For any continuous branch γ of υ_d^{-1} defined on $\mathbb{R}$, both limits $\lim_{t\to\pm\infty}\gamma(t)$ exist and are $(d-1)$-asymptotic points for L. For any $(d-1)$-asymptotic off-critical point z and large $M > 0$, there exists a unique continuous branch γ of υ_d^{-1} defined on $\{t \in \mathbb{R} : |t| > M\}$ such that

$$\lim_{t\to\pm\infty}\gamma(t) = z.$$

Proof. It follows from the definitions and proposition IV.2 that the restriction $\upsilon_{d|_{J_d}} : J_d \to \mathbb{R}$ is an infinite degree branched covering map whose critical points are exactly the pre-critical points of L inside J_d. Moreover, if $z \in J_d$ is a critical point of υ_d, we can repeat the proof of proposition IV.2 to conclude that either $\upsilon_{d|_{J_d}}$ has local degree two at z or there exists $(0,0) < (d_1, m_1) < (d_2, m_2) \leq (d, +\infty)$ such that

$$L^{d_1} \circ f^{m_1 q}(z) = L^{d_2} \circ f^{m_2 q}(z) = cv^f.$$

The latter case implies that

$$L^{d-d_1} \circ f^{mq}(cv^f) \in J_0$$

for some $(d,m) > 0$, and

$$L^{d_2-d_1} \circ f^{(m_2-m_1)q}(cv^f) = cv^f,$$

which is a contradiction. This completes the proof of the first paragraph in the proposition.

It follows from proposition II.15 that for any large $M > 0$ there is a unique branch γ_0 of υ_1^{-1} defined on $\{t \in \mathbb{R} : |t| > M\}$ such that $\gamma_0(t) \to 0$ when $|t| \to +\infty$. Every other branch of υ_1^{-1} must correspond to pre-images of γ_0 by iterates of f, which completes the proof of the second paragraph in the proposition when $d = 1$. It then follows from the definition of χ_+ that for any continuous branch of $\chi_+^{-1} \circ \upsilon_1^{-1}$ defined on $\mathbb{R}$ with image in W_+, both limits $\lim_{t\to\pm\infty}\gamma(t)$ exist and are pre-images under χ_+ of 0-asymptotic points in J_0. Pulling back by $(T_\delta \circ \rho)^{-1}$, the second paragraph in the proposition when $d = 2$ follows. We can repeat this argument inductively to complete the proof. □

Proposition IV.4. *If L is 1-escaping, then E_1 is has a single connected component and that component is simply connected. Otherwise, E_d is a countable union of disjoint Jordan domains for all d. In either case, every parent of a component of E_d is $(d-1)$-asymptotic.*

Proof. This follows from a similar argument to the proof of proposition IV.3. □

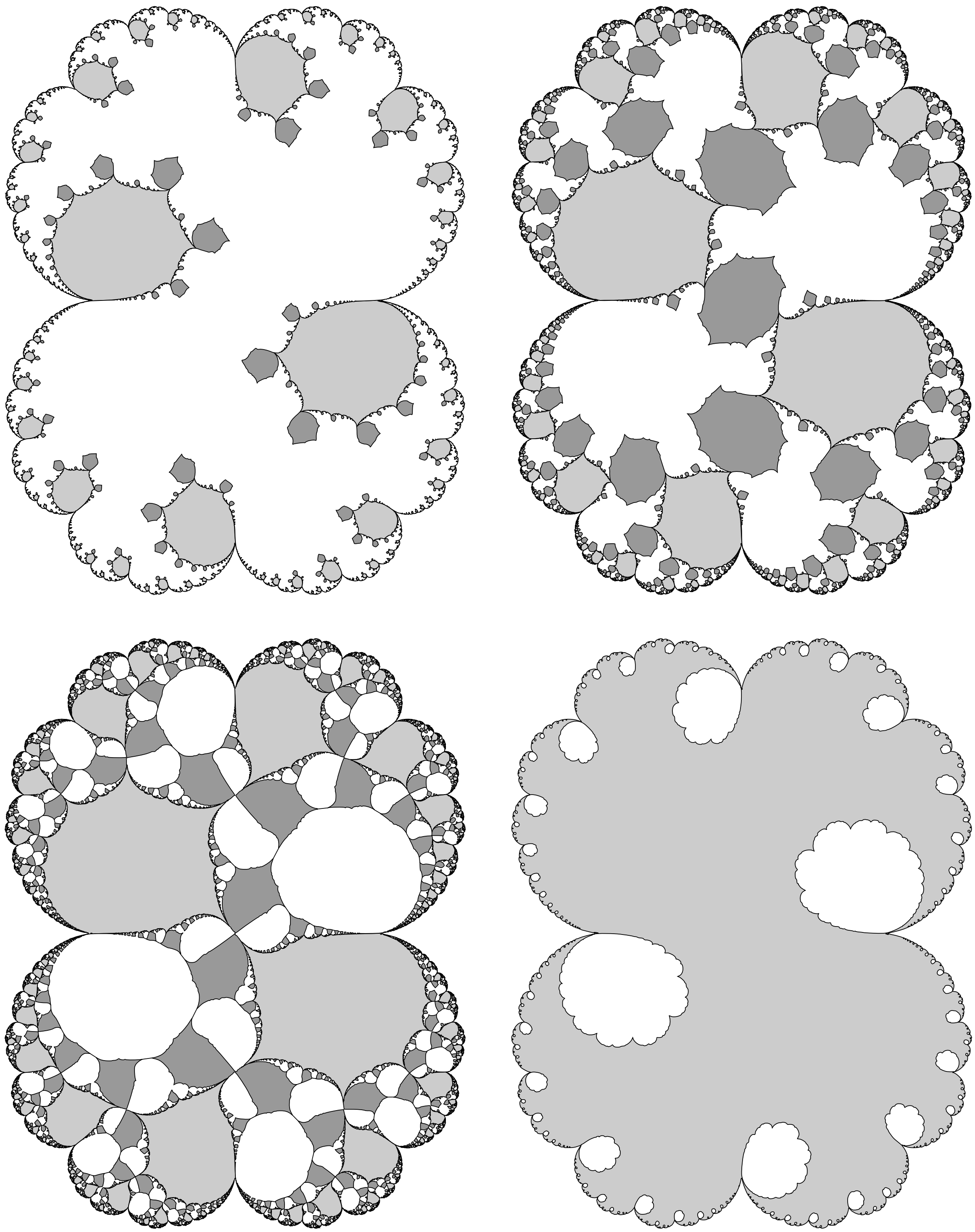


Figure IV.1: The 1-escaping set (in light gray) and the 2-escaping set (in dark gray) for $L = L_{\delta}^{\mathfrak{f}_1}$ and different choices of δ. *Upper left:* L is 2-nonescaping. *Upper right:* L is 2-escaping. *Bottom left:* L is weakly 1-escaping. *Bottom right:* L is 1-escaping.

For any $d \geq 1$ and component B of E_d, we inductively define a component B' of $E_{d'}$ with $d' \geq d$ to a *descendant* of B if either $d' = d$ and $B' = B$ or $d' > d$ and a parent of B' belongs to the closure of a descendant of B.

For $d \geq 1$ we define an *enriched angle of depth* d for f to be a sequence $\Theta = \langle \theta_n \rangle_{n=1}^d$ with $\theta_n \in \mathcal{A}$ for all n. We will also denote $\Theta + 1 = \langle \theta_n + 1 \rangle_{n=1}^d$. When $d = 1$, we define the point

$$z_\Theta = z_\Theta^L := \chi_+ \circ \Upsilon_1^{-1}(\theta_1) \in J_0.$$

For general Θ of depth $d \geq 1$, if z_Θ is defined and off-critical for L, then by propositions IV.4 and IV.3 we can define B_Θ^L to be the unique component of E_d which has z_Θ as a parent. If L is d-nonescaping or d-escaping and z_Θ is defined, then it follows from proposition IV.3 that there is a unique branch γ_Θ of υ_d^{-1} defined on $\mathbb{R}$ such that $\gamma_\Theta(t) \to z_\Theta$ when $t \to +\infty$. If instead L is d-Julia, then it follows from propositions IV.3 and IV.4 that there is a unique continuous branch γ_Θ of υ_d^{-1} defined on $\mathbb{R}$ whose image is contained in ∂B_Θ and which satisfies $\gamma_\Theta(t) \to z_\Theta$ when $t \to +\infty$. In either case above, for any $\theta \in \mathcal{A}$ we define

$$z_{\Theta \oplus \theta} := \gamma_\Theta^{-1}(\theta) \in J_d.$$

Thus we have inductively defined z_Θ for all enriched angles Θ of depth $d \geq 2$ when L is $(d-2)$-active and $z_{\lfloor \Theta \rfloor}$ is off-critical. In particular, if L is $(d-1)$-active then z_Θ is defined for all enriched angles Θ of depth d and every $(d-1)$-asymptotic point is labeled by an enriched angle. We will say that the enriched angle Θ is off-critical for L if z_Θ is defined and off-critical for L. Note that by definition, if Θ has depth $d > 1$ and is off-critical for L, then $\lfloor \Theta \rfloor$ is off-critical.

We will say that two enriched angles $\Theta = \langle \theta_n \rangle_{n=1}^d$ and $\Theta' = \langle \theta'_n \rangle_{n=1}^{d'}$ are *equivalent*, and write $\Theta \sim \Theta'$, if and only if

1. $d = d'$,
2. $\theta_n = \theta'_n$ for all $1 < n \leq d$, and
3. $z_{\langle \theta_1 \rangle} = z_{\langle \theta'_1 \rangle}$.

Up to equivalence, off-critical asymptotic points are uniquely determined by enriched angles.

Proposition IV.5. *For any two off-critical enriched angles* Θ, Θ' *of depth* $d \geq 1$*,* $z_\Theta = z_{\Theta'}$ *if and only if* $\Theta \sim \Theta'$.

Proof. If $d = 1$, then by definition $\Theta \sim \Theta'$ if and only if $z_\Theta = z_{\Theta'}$. If $d > 1$, then z_Θ is uniquely determined by $\upsilon_d(z_\Theta)$ and $z_{\lfloor \Theta \rfloor}$, so the proposition follows by induction. □

Our labeling also respects the dynamics of f^q and L:

Proposition IV.6. *For any $d \geq 1$, $f^q(z_\Theta) = z_{\Theta+1}$ for any enriched angle Θ of depth d such that z_Θ is defined. If additionally $d > 1$, then $L(z_\Theta) = z_{\lceil\Theta\rceil}$.*

Proof. Set $\Theta = \langle\theta_n\rangle_{n=1}^d$. If $d = 1$, then

$$f^q(z_\Theta) = f^q(\chi_+ \circ \Upsilon(\theta_1)) = \chi_+ \circ \Upsilon(\theta_1 + 1) = z_{\Theta+1}.$$

If $d > 1$, then we observe that the map $\xi := f^q \circ \gamma_{\lfloor\Theta\rfloor} \circ T_{-1}$ is a continuous branch of υ_d^{-1}. Moreover if $f^q(z_{\lfloor\Theta\rfloor}) = z_{\lfloor\Theta\rfloor+1}$, then

$$\lim_{t\to+\infty} \xi(t) = \lim_{t\to+\infty} f^q \circ \gamma_{\lfloor\Theta\rfloor}(t-1) = f^q\left(\lim_{t\to+\infty} \gamma_{\lfloor\Theta\rfloor}(t-1)\right) = f^q(z_{\lfloor\Theta\rfloor}) = z_{\lfloor\Theta\rfloor+1}.$$

Hence $\xi(t) = \gamma_{\lfloor\Theta\rfloor+1}(t)$ for all $t > 0$ sufficiently large, so

$$f^q(z_\Theta) = f^q \circ \gamma_{\lfloor\Theta\rfloor}(\theta_d) = \gamma_{\lfloor\Theta\rfloor+1}(\theta_d + 1) = z_{\Theta+1}.$$

If $d = 2$, then

$$\begin{aligned} L(z_\Theta) &= L \circ \gamma_{\lfloor\Theta\rfloor}(\theta_2) \\ &= L \circ \upsilon_1^{-1}(\theta_2) \\ &= (\chi_+ \circ T_\delta \circ \rho) \circ \left(\rho^{-1} \circ T_{-\delta} \circ \Upsilon\right)(\theta_2) \\ &= \chi_+ \circ \Upsilon(\theta_2) \\ &= z_{\lceil\Theta\rceil}. \end{aligned}$$

If $d > 2$, then we observe that $L \circ \gamma_{\lfloor\Theta\rfloor}$ is a continuous branch of υ_{d-1}^{-1}. Moreover, if $L(z_{\lfloor\Theta\rfloor}) = z_{\lceil\lfloor\Theta\rfloor\rceil}$ then

$$\lim_{t\to+\infty} L \circ \gamma_{\lfloor\Theta\rfloor}(t) = L\left(\lim_{t\to+\infty} \gamma_{\lfloor\Theta\rfloor}(t)\right) = L(z_{\lfloor\Theta\rfloor}) = z_{\lceil\lfloor\Theta\rfloor\rceil}.$$

As L is automatically $(d-1)$-active, hence $(d-2)$-nonescaping, $B_{\lceil\lfloor\Theta\rfloor\rceil}$ is the unique component of E_{d-1} which has $z_{\lceil\lfloor\Theta\rfloor\rceil}$ as a parent. Thus $L \circ \gamma_{\lfloor\Theta\rfloor} = \gamma_{\lceil\lfloor\Theta\rfloor\rceil}$, so

$$L(z_\Theta) = L \circ \gamma_{\lfloor\Theta\rfloor}(\theta_d) = \gamma_{\lceil\lfloor\Theta\rfloor\rceil}(\theta_d) = z_{\lceil\Theta\rceil}.$$

By induction on d, the proof is complete. ☐

We will say that an enriched angle is $\Theta = \langle\theta_n\rangle_{n=1}^d$ is *basic* if $\langle\theta_1\rangle \sim \langle 0\rangle$. The basic off-critical enriched angles always capture the critical value when it escapes:

Proposition IV.7. *For any $d \geq 1$ and component B of E_d, if $cv^f \in \overline{B}$, then there exists a basic off-critical enriched angle Θ so that $B = B_\Theta$. Moreover, if L is not 1-escaping then Θ is unique up to equivalence.*

Proof. As $cv^f \in \overline{B}$, the Lavaurs map L is either d-escaping, d-Julia, or $(d-1)$-Julia. Thus L is automatically $(d-1)$-active, so there is an enriched angle Θ of depth d such that z_Θ is a parent of B. The point z_Θ will be the unique component unless either L is 1-escaping or B contains a pre-critical point of L. If B contains a pre-critical point of L then L is d-escaping and $cv^f \in B$ implies that there is an integer $m > 0$ such that $f^{mq}(B) = B$. This is only possible when B has the fixed point 0 on its boundary, so L is 1-escaping. Thus if L is not 1-escaping, then z_Θ is the unique parent of B and Θ. If z_Θ is pre-critical for L, then L is $(d-1)$-Julia hence $z_\Theta = cv^f$. Moreover, as z_Θ is pre-critical for L there exists an integer $m > 0$ such that $f^{mq}(cv^f)$ which is a contradiction. Hence Θ is off-critical for L, so $B = B_\Theta$ and Θ is unique up to equivalence if L is not 1-escaping by proposition IV.5.

Let us now show that Θ is always basic. If L is 1-escaping then we can just pick $\Theta = 0$, so we assume that L is not 1-escaping. Set $\Theta_d = \Theta$ and for all $1 \leq n < d$ set $\Theta_n = \lfloor \Theta_{n+1} \rfloor$. Let

$$\gamma \subset \bigcup_{n=1}^{d} \overline{B_{\Theta_n}}$$

be a simple curve connecting cv to z_{Θ_1}. As L is $(d-1)$-nonescaping, $\overline{B_{\Theta_j}}$ also avoids the critical points of ρ for all $1 \leq j < d$. Our argument above that Θ is unique up to equivalence implies that additionally B_Θ avoids all the critical points of ρ. Thus we can choose γ so that it avoids the critical points of ρ. Thus we can choose γ so that $\rho(\gamma \cap U_0)$ is a simple curve connected 0 to ∞ which avoids the non-negative integers. Lemma II.15 therefore implies that $0 \in \gamma$, so $z_{\Theta_1} = 0$ and Θ is basic. $\square$

IV.1.1: Bubble rays

If L is ∞-nonescaping, then we define a *bubble ray* of L to be a sequence $\langle B_d \rangle_{d=1}^{\infty}$, where B_d is a connected component of E_d for all d, such that B_d is a descendant of B_{d-1} for all $d > 1$. We will say that the bubble ray is at enriched angle $\Theta = \langle \theta_d \rangle_{d=1}^{\infty}$ if for any $n \geq 1$, $B_n = B_{\langle \theta_d \rangle_{d=1}^{n}}$. We will say that two bubble rays are equivalent if all of the corresponding bubbles are equivalent.

IV.1.2: Pre-petals

As U_0 is a Jordan domain, for any $\theta \in \mathcal{A}$ there exists a unique minimal integer $m \geq 0$ such that $f^{mq}(z_{\langle \theta \rangle}) = 0$, using the continuous extension of f^{mq} to ∂U_0. We define the *attracting* *pre-petal* for f at enriched angle $\langle\theta\rangle$ to be the unique component $P^f_{\langle\theta\rangle,att}$ of $f^{-mq}(P^f_{att})$ contained in U_0 and which has $z_{\langle\theta\rangle}$ on its boundary. As f^{mq} is a covering map over P^f_{rep}, we can define the *repelling pre-petal* for f at enriched angle $\langle\theta\rangle$ to be the unique component $P^f_{\langle\theta\rangle,att}$ of $f^{-mq}(P^f_{rep})$ which contains the corresponding pre-image of the lower component of $P^f_{att}\cap P^f_{rep}$ contained in $P^f_{\langle\theta\rangle,att}$.

If Θ is an enriched angle of depth $d\geq 1$ which is off-critical for L, then proposition IV.2 implies that there is some $(d,m)\geq(0,0)$ and $\theta\in\mathcal{A}$ such that $L^d\circ f^{mq}$ is univalent in a neighborhood of z_Θ and maps z_Θ to $z_{\langle\theta\rangle}$. We can therefore define the attracting and repelling pre-petals for L at enriched angle Θ to be the corresponding pre-images $P^L_{att,\Theta}$ and $P^L_{rep,\Theta}$ of sub-petals of $P^f_{att,\langle\theta\rangle}$ and $P^f_{rep,\langle\theta\rangle}$. Thus there exists some $m'\geq m$ such that

$$L^d\circ f^{m'q}(P^L_{att,\Theta})\subset P^f_{att} \text{ and } L^d\circ f^{m'q}(P^L_{rep,\Theta})\subset P^f_{rep},$$

so we can define

$$\begin{aligned}\phi^L_{\Theta,att} &:= \phi^f_{att}\circ L^d\circ f^{m'q}: P_{\Theta,att}\to\mathbb{C}, \text{ and}\\ \phi^L_{\Theta,rep} &:= \phi^f_{rep}\circ L^d\circ f^{m'q}: P_{\Theta,rep}\to\mathbb{C}.\end{aligned}$$

We can define the pre-petals uniquely, up to our choice of P^f_{rep}, by requiring that

$$\phi^L_{\Theta,att}(P^L_{att,\Theta}) = T_M\circ\phi^f_{att}(P^f_{att}) \text{ and } \phi^L_{\Theta,rep}(P^L_{rep,\Theta}) = T_{-M}\circ\phi^f_{rep}(P^f_{rep})$$

for some maximal $M\geq 0$.

IV.1.3: Virtually parabolic Lavaurs maps

If h has a parabolic periodic cycle, then that cycle lifts by $\eta^f_{att,\delta}$ to a parabolic periodic cycle of L. If h has instead a parabolic fixed point at 0, we cannot similarly lift the parabolic fixed point but we can lift the parabolic dynamics; in this case L is said to have a *virtually parabolic* fixed point.

Let us fix some rational $p/q\in[-1/2,1/2]$ and assume that $h'(0)=e^{2\pi ip/q}$. As both P^h_{att} and P^h_{rep} are Jordan domains with 0 on their boundaries and avoid cv^h, $\eta^f_{att,\delta}$ is a covering map over both of these sets. As $cv^h=\eta^f_{att,\delta}(cv^f)$ lies on the boundary of P^h_{att}, we can define $\tilde{P}^L_{att}$ to be the component of $(\eta^h_{att,\delta})^{-1}(P^h_{att})$ which has 0 on its boundary. We can then define $\tilde{P}^h_{rep}$ to be the component of $(\eta^h_{att,\delta})^{-1}(P^h_{rep})$ which contains the lift of the lower component of $P^h_{att}\cap P^h_{rep}$ contained in $\tilde{P}^L_{att}$.

Proposition IV.8. *Both $\tilde{P}^L_{att}$ and $\tilde{P}^L_{rep}$ are Jordan domains which have* 0 *on their boundaries.*

There exists some integer m such that $\eta^f_{att,\delta}$ conjugates $L^{q'} \circ f^{mq}$ on $\tilde{P}^L_{att} \cup \tilde{P}^L_{rep}$ to $h^{q'}$ on $P^h_{att} \cup P^h_{rep}$.

Proof. Set $X = T_{\delta - c^f_+} \circ \rho(\tilde{P}^L_{att})$. So by construction, X is a component of $\mathrm{Exp}^{-1}(P^h_{att})$. Thus X is simply connected, contains points with arbitrarily large positive imaginary part, and has boundary which is an open Jordan arc. It therefore follows from lemma II.15 that $\tilde{P}^L_{att}$ is a Jordan domain, moreover the fact that cv^f lies on its boundary implies that 0 also lies on its boundary. The semi-conjugacy between the horn map and h implies that there is an integer m so that

$$T_m \circ (T_{\delta - c^f_+} \circ H^f)^{q'}(\tilde{P}^L_{att}) \subset X.$$

The semi-conjugacy between the horn map and the Lavaurs map implies that there is a component $\tilde{X}$ of $(T_{\delta - c^f_+} \circ \rho)^{-1}(X)$ such that $L^{q'} \circ f^{mq}(\tilde{P}^L_{att}) \subset \tilde{X}$. As X contains points with arbitrarily large imaginary part, it follows from the definition of χ_+ that $\tilde{X}$ has 0 on its boundary. The uniqueness in lemma II.15 therefore implies that $\tilde{X} = \tilde{P}$. The argument for $\tilde{P}^L_{rep}$ is similar; the non-empty intersection of $\tilde{P}^L_{att}$ and $\tilde{P}^L_{rep}$ ensures that we can use the same integer m. □

We denote by $\tilde{U}^L_0$ the unique connected component of $(\eta^f_{att,\delta})^{-1}(U^h_0)$ which contains $\tilde{P}^L_{att}$.

Proposition IV.9. *Every component of U^h is a Jordan domain.*

Proof. When $p/q = 0/1$, this fact is proved in [LY14a]. The same argument there can be applied for the general p/q case. □

Corollary IV.10. *The set $\tilde{U}^L_0$ is a Jordan domain on which $\eta^f_{att,\delta}$ is injective.*

Proof. We use the same argument as for proposition IV.8. The injectivity follows from lemma II.15 and the fact that $cv^f \in \tilde{U}^L_0$. □

For all basic $\theta \in \mathcal{A}^h$, we denote $\tilde{z}^L_\theta = 0$. If $\theta \in \mathcal{A}^h$ is not basic, then $z^h_\theta \neq 0$ and we denote by $\tilde{z}^L_\theta$ the unique lift by $(\eta^f_{att,\delta})^{-1}$ of z^h_θ which lies on the boundary of $\tilde{U}^L_0$. We similarly define $\tilde{P}^L_{\theta,att}$ to be the unique lift by $(\eta^f_{att,\delta})^{-1}$ of $P^h_{\theta,att}$ contained in $\tilde{U}^L_0$, and define $\tilde{P}^L_{\theta,rep}$ to be the unique lift by $(\eta^f_{att,\delta})^{-1}$ of $P^h_{\theta,rep}$ which contains the lift of the lower component of $P^h_{\theta,att} \cap P^h_{\theta,rep}$ contained in $\tilde{P}^L_{\theta,att}$. Thus $L^{q'} \circ f^{mq}$ maps $\tilde{P}^L_{\theta,att}$ and $\tilde{P}^L_{\theta,rep}$ to $\tilde{P}^L_{\theta+1,att}$ and $\tilde{P}^L_{\theta+1,rep}$ for any nonbasic $\theta \in \mathcal{A}^h$, where m is the integer in proposition IV.8.

Proposition IV.11. *If $h'(0) = e^{2\pi i p'/q'}$ for some $p/q \in \mathbb{Q}$, then there is a unique bubble ray $\langle B_d \rangle_{d=1}^{\infty}$ up to equivalence and some $d_0 \geq 1$ such that:*

1. *If B is a descendant of B_{d_0}, then $\overline{B} \subset \tilde{P}^L_{rep}$.*

2. *If B is a component of E_d for some $d \geq 1$ and there is a point in $\overline{B} \cap \tilde{P}^L_{rep}$ which is sufficiently close to 0, then B is a descendant of B_{d_0}.*

3. *For all $d > q'$,*
$$L^{q'} \circ f^{mq}(B_d) = B_{d-q'},$$
where m is the integer in proposition IV.8.

Proof. For all $0 \leq n < q'$ we set $U^h_n := h^n(U^h_0)$. It follows from proposition II.14 that U^h_n is a Jordan domain for all n. Proposition II.13 implies that there is a unique analytic isomorphism $\xi : U^{\mathfrak{f}_0}_0 \to U^h_0$ which conjugates $\mathfrak{f}_0$ to h and which extends continuously to a homeomorphism between the closures. For all $0 \leq n < q'$ we define γ_n to be the arc $h^n \circ \xi([-1, 0])$. As
$$\lim_{t \to -1} h(h^{q-1} \circ \xi(t)) = \lim_{t \to -1} \xi \circ \mathfrak{f}_0(t) = \xi(0) = 0,$$
the curve $\gamma_{q'-1}$ contains a point on the boundary of $Dom_0(h)$. As $Dom_0(h)$ is a Jordan domain, we can define Γ to be the union of all γ_n and a simple curve which connects the point in $\gamma_{q'-1} \cap \partial Dom_0(h)$ to ∞, avoids $Dom_0(h)$, and which is contained in $\mathbb{R}$ near ∞. It follows from the construction that the the image of $\Gamma \cap Dom_0(h)$ under h is contained in h. We define $\mathcal{V} := \mathbb{C} \setminus \Gamma$.

Lemma IV.12. *$\mathcal{V}$ is simply connected.*

Proof. For all $z \in \overline{U^h_0}$, $0 \leq n < q$, and $t_0 \in [-1, 0)$,
$$\lim_{t \to t_0} h^n \circ \xi(t) = h^n \circ \xi(t_0),$$
using the continuous extension of h^n to $\overline{U}_0$. As the only asymptotic values of h are 0 and ∞, it follows that $\xi(t_0)$ is contained in $Dom(h^{q-1})$. By similar argument we can conclude that as long as $(n, t_0) \neq (q' - 1, -1)$, the point $h^n \circ \xi(t_0)$ is in the domain of h. If $\mathcal{V}$ is not simply connected, then there exists some $0 \leq n_1 < n_2 < q'$ and $t_1, t_2 \in [-1, 0)$ such that
$$\lim_{t \to t_1} h^{n_1} \circ \xi(t) = \lim_{t \to t_2} h^{n_2} \circ \xi(t) := z.$$
If $t_2 \neq -1$, then it follows from the above that $h^{q'-n_2}(z') \in U^h_0$, so $n_1 = n_2$ which is a contradiction. If $t_2 = -1$, then $h^{q'-1-n_2}(z') \notin Dom(h)$ and the above implies that $n_1 = n_2$ which is again a contradiction. □

As $\mathcal{V}$ avoids all the critical values of $h^{q'}$ and $h^{q'}$ is injective in a neighborhood of 0, there are exactly q' components of $h^{-q'}(\mathcal{V})$ which have 0 on their boundary. Let us fix one such component $\mathcal{U}$ and let $\zeta : \mathcal{V} \to \mathcal{U}$ be the corresponding branch of $h^{-q'}$. As 0 is on the boundary of $\mathcal{U}$, the Denjoy-Wolff theorem implies that the iterates of ζ converge locally uniformly to 0. As $\mathcal{V}$ is connected, there exists some integer $0 \leq n < q$ such that for any $z \in \mathcal{V}$, $\zeta^j(z) \in h^n(P^h_{rep})$ for all large $j > 0$. Replacing $\mathcal{U}$ with another component of $h^{-q'}(\mathcal{V})$, we can assume that $n = 0$; in particular $P^h_{rep} \subset \mathcal{V}$. We define $\tilde{\mathcal{V}}$ to be the component of $(\eta^f_{att,\delta})^{-1}(\mathcal{V})$ which contains $\tilde{P}^L_{rep}$, and define $\tilde{U}$ to be the component of $(\eta^f_{att,\delta})^{-1}(\mathcal{U})$ contained in $\tilde{U}$. Thus $\eta^f_{att,\delta}$ conjugates the restriction $L^{q'} \circ f^{mq} : \tilde{\mathcal{U}} \to \tilde{\mathcal{V}}$ to the restriction $h^{q'} : \mathcal{U} \to \mathcal{V}$, where m is the integer in proposition IV.8. Let $\tilde{\zeta} : \tilde{\mathcal{V}} \to \tilde{\mathcal{U}}$ be the corresponding inverse map, so

$$\eta^f_{att,\delta} \circ \tilde{\zeta} = \zeta \circ \eta^f_{att,\delta}.$$

As $\mathcal{V}$ intersects the compliment of $Dom_0(h)$ in a single component, it follows that $\tilde{\mathcal{V}}$ intersects E^L_1 in a single connected component. For ever integer $s \geq 0$, let $B_{1+sq'}$ be the unique component of $E_{1+sq'}$ which contains $\tilde{\zeta}^s(\tilde{\mathcal{V}} \cap E^L_1)$.

Lemma IV.13. *Fix some integers $s, s' \geq 0$ and a component B of $E_{s'}$ with parent z_B. If $\overline{B} \cap \tilde{\zeta}^s(\tilde{\mathcal{V}})$ is non-empty, the one of the following holds:*

1. *$s' = 1 + sq'$ and $B = B_{1+sq'}$.*

2. *$s' > 2 + sq'$ and $\overline{B} \subset \tilde{\zeta}^s(\tilde{\mathcal{V}})$.*

3. *$s' = 2 + sq'$, $\overline{B} \setminus \{z_B\} \subset \tilde{\zeta}^s(\tilde{\mathcal{V}})$, and $z_B \in \partial B_{1+sq'}$.*

Proof. As B and $\tilde{\zeta}^s(\tilde{V})$ are Jordan domains, if $\overline{B} \cap \tilde{\zeta}^s(\tilde{\mathcal{V}})$ is non-empty, then $B \cap \tilde{\zeta}^s(\tilde{\mathcal{V}})$ is also non-empty. As $L^{q'} \circ f^{mq} = \tilde{\zeta}^{-1}$, we must have $s' \geq 1 + sq'$. Moreover, the uniqueness in the definition of $B_{1+sq'}$ implies that $B = B_{1+sq'}$ if $s' = 1 + sq'$. Thus we can restrict to the case $s' > 1 + sq'$.

Assume that there is some point

$$x' \in \overline{B} \subset J_{s'} \cup E_{s'} \cup J_{s'-1}$$

which belongs to the boundary of $\tilde{\zeta}^s(\tilde{\mathcal{V}})$. As $s' > 1 + sq'$, the point $x := (L^{q'} \circ f^{mq})^s(x')$ is defined and belongs to $\partial\tilde{\mathcal{V}}$. We have four possible cases:

1. $x \notin U^f_0$,

2. $\eta^f_{att,\delta}(x) \notin \overline{Dom_0(h)}$,

3. $\eta^f_{att,\delta}(x) \in \Gamma \cap \partial Dom_0(h)$, or

4. $\eta^f_{att,\delta}(x) \in \Gamma \cap Dom_0(h)$.

The first case above implies that $x \in J^L_{sq'}$, and the second case implies that $x' \in E_{1+sq'}$. Both are impossible as $s' > 1 + sq'$. The third case above implies that $x \in J_1 \cap \partial B_1$, so $x' \in J_{1+sq'} \cap B_{1+sq'}$. Hence $x' = z_B$ and $s' = 2 + sq'$. The fourth case above is impossible as every point in $\Gamma \cap Dom_0(h)$ has infinite forward orbit under h.

Thus we have shown that if $s > 1 + sq'$, then $\overline{B}$ can intersect the boundary of $\tilde{\zeta}^s(\tilde{\mathcal{V}})$ only at the parent of B and only when $s = 2 + sq'$, which completes the proof. $\square$

For all integers $d \geq 1$ not equivalent to 1 modulo q', we inductively define B_d to be the unique component of E_d which has the parent of B_{d+1} on its closure. Lemma IV.13 implies that $\langle B_d \rangle_{d=1}^{\infty}$ is a bubble ray for all n. It follows from the construction of $\tilde{\zeta}$ and lemma IV.13 that this bubble ray has the desired properties. $\square$

We will call the $\langle B_d \rangle_{d=1}^{\infty}$ as in proposition IV.11 the *parabolic bubble ray* for L. We can associate to this bubble ray an infinite depth enriched angle $\langle \theta_d \rangle_{d=1}^{\infty}$, which we will call the *parabolic enriched angle* for L, so that $B_d = B_{\langle \theta_n \rangle_{n=1}^{d}}$ for all $d \geq 1$.

We end this section with the following observation, which follows from our analysis in the proof of proposition IV.11.

Corollary IV.14. *The only point in $\overline{U^h_0}$ which is not in the domain of $h^{q'}$ is z^h_{-1}.*

IV.2: Parameter spaces

Let us fix some f as in the previous section. For any $d \geq 1$, we define the *d-nonescaping parameter set*, *d-Julia parameter set*, and the *d-escaping parameter set* to be

$$\mathcal{K}^f_d := \left\{ \delta \in \mathbb{C} : L^f_\delta \text{ is } d\text{-nonescaping} \right\},$$
$$\mathcal{J}^f_d := \left\{ \delta \in \mathbb{C} : L^f_\delta \text{ is } d\text{-Julia} \right\}, \text{ and}$$
$$\mathcal{E}^f_d := \left\{ \delta \in \mathbb{C} : L^f_\delta \text{ is } d\text{-escaping} \right\}$$

respectively. We will say that a parameter $\delta \in \mathcal{J}^f_d$ is d-asymptotic if cv^f is d-asymptotic for L^f_δ. For any basic enriched angle Θ for f, we similarly define

$$\mathcal{B}^f_\Theta := \left\{ \delta \in \mathbb{C} : cv^f \in B^{L^f_\delta}_\Theta \right\}.$$

For all $d \geq 0$, we define the function $\Upsilon^f_d : \mathcal{E}^f_d \cup \mathcal{J}^f_d \to \overline{\mathbb{H}}$ by

$$\Upsilon^f_d(\delta) = \upsilon^{L_\delta}_d(cv^{\mathcal{T}}).$$

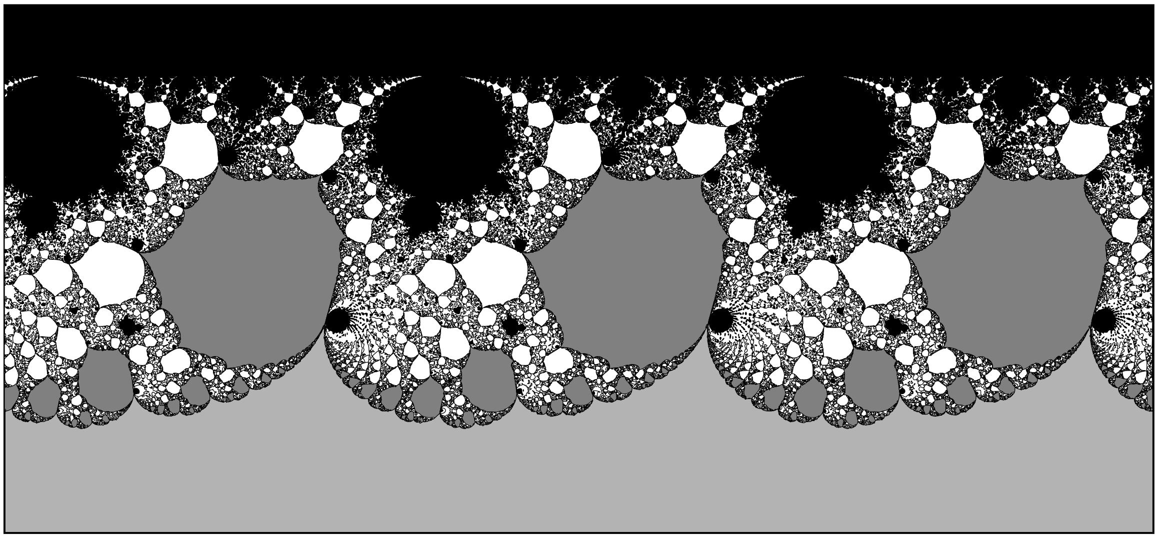


Figure IV.2: The parameter space of the tower $\mathcal{T} = \langle \mathfrak{f}_1 \rangle$. The d-escaping parameter set is shown in light gray for $d = 1$, dark gray for $d = 2$, and white for $d > 2$.

Proposition IV.15. *Both $\mathcal{E}_1^f$ and Υ_1^f agree with our definitions in the previous section.*

Proof. We observe that by definition, L_δ is 1-escaping if and only if

$$\delta = T_\delta \circ \rho(cv^f) \in \mathcal{E}_1^f.$$

Moreover,

$$v_1^L(cv^f) = \Upsilon_1(\delta)$$

by definition. □

We can generalize proposition IV.15 to arbitrary basic enriched angles.

Proposition IV.16. *For any basic enriched angle Θ of depth $d > 1$, $\mathcal{B}_\Theta$ is a Jordan domain and Υ_d restricts to an analytic isomorphism from $\mathcal{B}_\Theta$ to $\mathbb{H}$.*

Proof. Let us fix some enriched angle Θ of depth $d > 1$ and assume that there exists some $\delta_0 \in \mathcal{B}_\Theta$. We set $w_0 = v_d^{L_{\delta_0}}(cv^f)$. First we will show that in this case, Υ_d restricts to an analytic isomorphism from $\mathcal{B}_\Theta$ to $\mathbb{H}$.

For all $w \in i\mathbb{H}$, we define the quasiconformal map $\xi_w : \mathcal{E}_1 \to \mathcal{E}_1$ by

$$\xi_w(z) = \Upsilon_1^{-1}\left(\mathrm{Re}\,\Upsilon_1(z) + iw\mathrm{Im}\,\Upsilon_1(z)\right).$$

It follows from the definition that ξ_w commutes with T_1 and is equal to the identity on $\partial\mathcal{E}_1$. Let us denote by $\tilde{\lambda}_w$ the Beltrami differential on $\mathcal{E}_1$ defined by pulling back the standard complex structure by ξ_w. Pulling back by iterates of the restriction of $T_{\delta_0} \circ H$ to W_+, we extend $\tilde{\lambda}_w$ to a T_1 and $T_{\delta_0} \circ H$ invariant Beltrami differential on $\mathbb{C}$. Pulling $\tilde{\lambda}_w$ back by $T_{\delta_0} \circ \rho$, we can define a Beltrami differential λ_w on U_0 which is invariant under both f^q and L_{δ_0}. As χ_+ semi-conjugates $T_{\delta_0} \circ H$ to $L_\delta = \chi_+ \circ T_{\delta_0} \circ \rho$, the pullback of λ_w by the restriction of χ_+ to W_+ agrees with $\tilde{\lambda}_w$ on W_+. By the measurable Riemann mapping theorem, there exists a quasiconformal homeomorphism $\varphi_w : U_0 \to U_0$ depends continuously on w and which pulls back the standard complex structure to λ_w. The map

$$\varphi_w \circ f^q \circ \varphi_w^{-1} : U_0 \to U_0$$

is therefore holomorphic and quasiconformally conjugate to f^q. After post-composing φ_w with an analytic automorphism of U_0 if necessary, we can ensure that $\varphi_w \circ f^q = f^q \circ \varphi_w$ and $\varphi_w(cv^f) = cv^f$. The continuous extension of φ_w to $\overline{U_0}$ must therefore satisfy $\varphi_w(z) = z$ for all $z \in \partial U_0$.

On the intersection of $\overline{W_+}$ and left half-plane, the map

$$\tilde{\varphi}_w := \phi_{rep} \circ \varphi_w \circ \phi_{rep}^{-1}$$

is defined an commutes with T_1. We can therefore extend $\tilde{\varphi}_w$ to a quasiconformal map from W_+ to $\mathbb{C}$ which extends continuously to ∂W_+ as the identity. We can continuously extend $\tilde{\varphi}_w$ to all of $\mathbb{C}$ by defining

$$\tilde{\varphi}_w(z) = \begin{cases} \tilde{\varphi}_w(z) & \text{if } z \in \overline{W_+} \\ \xi_w(z) & \text{if } z \in \overline{\mathcal{E}_1} \end{cases}.$$

It follows from this construction that $\tilde{\varphi}_w$ pulls back the standard complex structure to $\tilde{\lambda}_w$.

The set $\varphi_w(P_{att})$ is an attracting petal for f^q with Fatou coordinate

$$\tilde{\varphi}_w \circ T_{\delta_0} \circ \phi_{att}^f \circ \varphi_w^{-1} : \varphi_w(P_{att}) \to \mathbb{C}.$$

The uniqueness of Fatou coordinates therefore implies that there is some $\delta_w \in \mathbb{C}$ so that

$$\tilde{\varphi}_w \circ T_{\delta_0} \circ \phi_{att}^f \circ \varphi_w^{-1} = T_{\delta_w} \circ \phi_{att}^f.$$

The function $\tilde{\chi}_w : \tilde{\varphi}_w(W_+ \cup \phi_{rep}^f(P_{rep}^f)) \to \mathbb{C}$ defined by

$$\tilde{\chi}_w(z) := \begin{cases} \varphi_w \circ \chi_+ \circ \tilde{\varphi}_w^{-1}(z) & \text{if } z \in \overline{\tilde{\varphi}_w(W_+)} \\ \chi_+(z) & \text{if } z \in \overline{\tilde{\varphi}_w(\mathcal{E}_1)} \end{cases}$$

is holomorphic and the image under $\tilde{\chi}_w$ of $\tilde{\varphi}_w(\phi^f_{rep}(P^f_{rep}))$ is a repelling petal for f^q with Fatou coordinate $\tilde{\chi}_w^{-1}$, the uniqueness of Fatou coordinates therefore implies that there is some $\delta' \in \mathbb{C}$ so that

$$\tilde{\chi}_w = \chi_+ \circ T_{\delta'}.$$

Post-composing $\tilde{\varphi}_w$ with a translation if necessary, we can assume that $\delta' = 0$. Thus

$$\varphi_w \circ L_{\delta_0} \circ \varphi_w^{-1} = \varphi_w \circ \chi_+ \circ \tilde{\varphi}_w^{-1} \circ \tilde{\varphi}_w \circ T_{\delta_0} \circ \rho \circ \varphi_w^{-1} = \chi_+ \circ T_{\delta_w} \circ \rho = L_{\delta_w}$$

on $\varphi_w(K_1^{L_{\delta_0}})$. Moreover,

$$\begin{aligned}
\upsilon_d^{L_{\delta_w}}(cv^f) &= T_{\delta_w} \circ \rho \circ L_{\delta_w}^{d-1}(cv^f) \\
&= \Upsilon_1 \circ \tilde{\varphi}_w \circ T_{\delta_0} \circ \rho \circ \varphi_w^{-1} \circ L_{\delta_w}^{d-1}(cv^f) \\
&= \Upsilon_1 \circ \xi_w \circ T_{\delta_0} \circ \rho \circ L_{\delta_0}^{d-1} \circ \varphi_w^{-1}(cv^f) \\
&= \Upsilon_1 \circ \xi_w \circ \Upsilon_1^{-1} \circ \upsilon_d^{L_{\delta_0}}(cv^f) \\
&= \operatorname{Re} w_0 + iw \operatorname{Im} w_0.
\end{aligned}$$

Thus $\delta_w \in \mathcal{E}_d$ and $\Upsilon_d(\delta_w) = \operatorname{Re} w_0 + iw\operatorname{Im} w_0$. As φ_w depends continuously on w, the function $w \mapsto \delta_w$ is continuous. Moreover, this function induces a continuous branch of Υ_d^{-1} defined on $\mathbb{H}$ whose image lies in $\mathcal{B}_\Theta$, the image must therefore be all of $\mathcal{B}_\Theta$. Hence Υ_d restricts to an analytic isomorphism from the connected component of $\mathcal{B}_\Theta$ containing δ_0 to $\mathbb{H}$.

To show that a δ_0 as above exists, and that $\mathcal{B}_\Theta$ is a Jordan domain, we observe that if cv^f belongs to $\overline{B_\Theta^{L_\delta}}$ for some δ, then it follows from proposition IV.7 that Θ is off-critical. Thus, a neighborhood of cv^f inside $\overline{B_\Theta^{L_\delta}}$ moves holomorphically near δ, so it follows from standard arguments that the geometry of $\overline{\mathcal{B}_\Theta}$ near δ mirrors the geometry of $\overline{B_\Theta^{L_\delta}}$ near cv^f. Thus if a component of $\mathcal{B}_\Theta$ is non-empty, then it will be a Jordan domain so long as it is compactly contained in $\mathbb{C}$. If such a component of $\mathcal{B}_\Theta$ is not compactly contained in $\mathbb{C}$, then it is contained in a horizontal strip as it must avoid $\mathcal{E}_1$ and if $\operatorname{Im}\delta > 0$ is sufficiently large then $\mathcal{R}_\delta f$ has an attracting fixed point at 0 so L_δ cannot be d-nonescaping. The parameter bubbles $\mathcal{B}_\Theta + j = \mathcal{B}_{\Theta+j}$ with $j \in \mathbb{Z}$ must therefore have an accumulation point $\delta \in \mathbb{C}$, more precisely for any $\epsilon > 0$ there exists some j and $\delta' \in \mathcal{B}_{\Theta+j}$ such that $|\delta - \delta'| < \epsilon$. But this would imply the same property for the bubbles $B_{\Theta+j}^{L_\delta}$ near cv^f which is a contradiction as either cv^f is d-Julia for L_δ, so $J_d^{L_\delta}$ is near cv^f is an arc, or cv^f is d-non-escaping for L_δ, so $J_d^{L_\delta}$ avoids cv^f. Thus every non-empty component of $\mathcal{B}_\Theta$ is a Jordan domain.

Let us now suppose that either $d = 2$ or $d > 2$ and that the proposition holds for smaller values of d. Thus for there is a unique parameter $\mathcal{Z}_\Theta \in \partial\mathcal{B}_{\lfloor\Theta\rfloor}$ such that $cv^f = z_\Theta^{L_{\mathcal{Z}_\Theta}}$. Thus

$\overline{\mathcal{B}_\Theta}$ is non-empty, and by the above there is a unique component of $\mathcal{B}_\Theta$ which has $\mathcal{Z}_\Theta$ on its boundary. Additionally, any non-empty component of $\mathcal{B}_\Theta$ must have $\mathcal{Z}_\Theta$ on its boundary, so it follows by induction on d that $\mathcal{B}_\Theta$ is non-empty and has a unique connected component. $\square$

For any basic enriched angle Θ of depth larger than 1, we denote by $\mathcal{Z}_\Theta^f = \mathcal{Z}_\Theta$ the unique parameter in $\partial\mathcal{B}_\Theta$ such that $cv^f = z_\Theta^{L_{\mathcal{Z}_\Theta}}$; we call $\mathcal{Z}_\Theta$ the *parent* of $\mathcal{B}_\Theta$. We define a *parameter bubble ray* of $\mathcal{T}$ to be a sequence $\langle \mathcal{B}_d \rangle_{d=1}^{\infty}$, where $\mathcal{B}_d$ is a component of $\mathcal{E}_d$ and the parent of $\mathcal{B}_{d+1}$ is on the boundary of $\mathcal{B}_d$ for all $d \geq 1$.

CHAPTER V
The Near-Parabolic Geometry of the Mandelbrot Set

V.1: Quadratic polynomials

Let us now focus our attention to the family of quadratic polynomials. Most of the objects and statements in this section are classical results in holomorphic dynamics, for more details and proofs we refer the reader to [DH84], [DH85], or [Hub16].

For any

$$\mathfrak{f}_\alpha(z) = e^{2\pi i\alpha} z + z^2$$

with $\alpha \in \mathbb{C}$, $\mathfrak{f}_\alpha$ has a fixed point at $z = 0$ with multiplier $\mathrm{Exp}(\alpha)$, and a unique critical value $cv^{\mathfrak{f}_\alpha} := -\mathrm{Exp}(2\alpha)/4$. The *Green's function* for $\mathfrak{f}_\alpha$ is defined by

$$G^{\mathfrak{f}_\alpha}(z) = \lim_{n\to\infty} \frac{1}{2^n} \log^+ \mathfrak{f}_\alpha^n(z).$$

We can alternatively define the filled Julia set of $\mathfrak{f}_\alpha$ by

$$K(\mathfrak{f}_\alpha) := (G^{\mathfrak{f}_\alpha})^{-1}(0),$$

We also define

$$\tilde{K}(\mathfrak{f}_\alpha) := \{z \in \mathbb{C} : G^{\mathfrak{f}_\alpha}(z) \leq G^{\mathfrak{f}_\alpha}(cv^{\mathfrak{f}_\alpha})/2\}.$$

The Green's function is continuous, harmonic on $\mathbb{C} \setminus K(\mathfrak{f}_\alpha)$, depends continuously on λ, and satisfies

$$G^{\mathfrak{f}_\alpha}(\mathfrak{f}_\alpha(z)) = 2G^{\mathfrak{f}_\alpha}(z).$$

For all $g \geq 0$, let us denote

$$\mathbb{H}_g := \{x - iy \in \mathbb{C} : y > g\}.$$

Proposition V.1. *For any α with $G^{\mathfrak{f}_\alpha}(cv^{\mathfrak{f}_\alpha})/2 = 2\pi g \geq 0$, there exists a unique analytic covering map*

$$\psi^{\mathfrak{f}_\alpha} : \mathbb{H}_g \to \mathbb{C} \setminus \tilde{K}(\mathfrak{f}_\alpha)$$

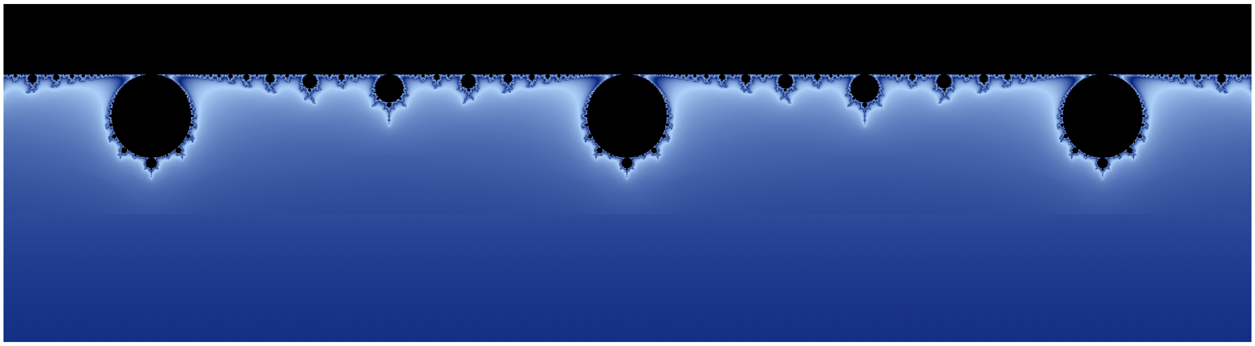


Figure V.1: The log-Mandelbrot set $\mathcal{M}$.

such that

1. $\psi^{\mathfrak{f}_\alpha}(w) = \psi^{\mathfrak{f}_\alpha}(w')$ *if and only if* $w \equiv w' \mod 1$,
2. $\psi^{\mathfrak{f}_\alpha}(2w) = \mathfrak{f}_\alpha \circ \psi^{\mathfrak{f}_\alpha}(w)$,
3. $G^{\mathfrak{f}_\alpha} \circ \psi^{\mathfrak{f}_\alpha}(w) = -2\pi \operatorname{Im} w$, *and*
4. $\frac{\psi^{\mathfrak{f}_\alpha}(w)}{\operatorname{Exp}(w)} \to 1$ *when* $\operatorname{Im} w \to -\infty$.

We will call $\psi^{\mathfrak{f}_a}$ the *Böttcher parameter* for $\mathfrak{f}_\alpha$. For any $\theta \in \mathbb{R}$, we will call the image under $\psi^{\mathfrak{f}_\alpha}$ of the vertical line $\operatorname{Re} w = \theta$ the *external ray* of $\mathfrak{f}_\alpha$ at angle θ; we will say that the ray *lands* at a point z if

$$\psi^{\mathfrak{f}_\alpha}(w) \to z$$

when $w \in \mathbb{H}$ tends to θ. We will say that a parameter $\alpha \in \mathbb{C}$ is *parabolic* if $\mathfrak{f}_\alpha$ has a parabolic periodic cycle.

Proposition V.2. *If α is parabolic, then every external ray of $\mathfrak{f}_\alpha$ lands. Moreover, there is rational angle θ such*

$$\psi^{\mathfrak{f}_\alpha}(w) \in P_{rep}^{\mathfrak{f}_\alpha}$$

for all $w \in \mathbb{H}$ sufficiently close to θ.

The *log-Mandelbrot set* $\mathcal{M}$ is defined to be the set of all $\alpha \in \mathbb{C}$ such that $G^{\mathfrak{f}_\alpha}(cv^{\mathfrak{f}_\alpha}) = 0$, or alternatively the set of all α so that $K(\mathfrak{f}_\alpha)$ is connected.

Proposition V.3. *There is a unique analytic isomorphism*

$$\Psi : \mathbb{H} \to \mathbb{C} \setminus \mathcal{M},$$

satisfying $2\Psi(w) - w \to \frac{1}{2} + \frac{\log 4}{2\pi i}$ *when* $\operatorname{Im} w \to -\infty$ *and such that if* $\Psi(w) = \alpha$*, then* $cv^{f_\alpha} = \psi^{f_\alpha}(w)$*.*

For any $\theta \in \mathbb{R}$ we define the *parameter ray* at angle θ to be the image under Ψ of the vertical line $\operatorname{Re} w = \theta$. We will say that the parameter ray at angle θ *lands* at a parameter α if

$$\Psi(w) \to \alpha$$

when $w \in \mathbb{H}$ tends to θ.

Proposition V.4. *For any parabolic* $\alpha \in \mathbb{C}$*, there are exactly two angles* $\theta_\alpha^- < \theta_\alpha^+$ *such that the corresponding parameter rays land at* α*.*

For any parabolic parameter α, it follows from proposition V.4 that the closure of the two parameter rays which land at α cuts the plane into two connected components, the *parabolic wake* of α, which we denote by $\operatorname{Wake}(\alpha)$, is the connected component which avoids the upper half-plane $-\mathbb{H}$. The α*-limb* of $\mathcal{M}$ is defined to be

$$\mathcal{L}_\alpha := \mathcal{M} \cap \operatorname{Wake}(\alpha),$$

or equivalently $\mathcal{L}_\alpha$ is the unique component of $\mathcal{M} \setminus \{\alpha\}$ which is compactly contained in $\mathbb{C}$.

The *PLY-inequality*, developed independently by Pommerenke [Pom86], Levin [Lev91], and Yoccoz [Hub93], bounds the geometry of the limbs of $\mathcal{M}$. For any set $X \subset \mathbb{C}$, we denote by $\operatorname{Diam} X$ the Euclidean diameter of X.

Theorem V.5 (PLY inequality)**.** *For any* $p/q \in \mathbb{Q}$*,*

$$\operatorname{Diam} \mathcal{L}_{p/q} < \frac{\log 2}{\pi q}.$$

Actually, the PLY inequality is even more powerful: it gives a bound on the diameter of every limb in $\mathcal{M}$, but it is difficult to directly relate the diameter to combinatorial data in general.

In [Mil94], Milnor conjectured that the $O(1/q)$ bound in the PLY-inequality could be improved to $O(1/q^2)$. Such a bound was first verified in [Kap21] for the limbs $\mathcal{L}_{1/q}$ with $q \geq 2$. For the rest of this chapter, we will generalize the argument in [Kap21] and prove the following theorem:

Theorem V.6. *For any integer* $M \geq 0$*, there exists a constant* $C > 0$ *such that for all* $p/q \in \mathbb{Q}_M$*,*

$$\operatorname{Diam} \mathcal{L}_{p/q} < \frac{C}{q^2}.$$

Similarly to the PLY-inequality, our argument will allow us to control the diameter of many other limbs of $\mathcal{M}$, but it is difficult to explicitly relate the bounds to combinatorial data. The precise statement of the bound we prove is given in theorem V.7 below.

V.2: Satellite towers

We define a *satellite tower* to be a strictly parabolic tower $\mathcal{T} = \langle f_n \rangle_{n=1}^{N}$ such that $f_1 \in \text{Quad}$ and f_n has a parabolic fixed point at either 0 or ∞ for all $1 \leq n \leq N$. Our goal for the remainder of this chapter is to prove the following:

Theorem V.7. *For any satellite tower $\mathcal{T}$ of height $N \geq 1$ there exists a constant $C > 0$ such that if $\mathfrak{f}_\alpha$ converges to $\mathcal{T}$ in $\widehat{\text{Quad}}_N$ with combinatorics $\langle k_n \rangle_{n=1}^{N-1}$, then*

$$\text{Diam}\, \mathcal{L}_\alpha < \frac{C}{\prod_{n=1}^{N-1} k_n^2}$$

when $\mathfrak{f}_\alpha$ is sufficiently close to $\mathcal{T}$.

First let us show that theorem V.7 implies theorem V.6.

Proof of theorem V.6 assuming theorem V.7. Let us fix some $M \geq 0$ and assume that theorem V.6 does not hold. Thus there is a sequence $\langle p_j/q_j \rangle_{n=1}^{\infty}$ in $\mathbb{Q}_M$ such that

$$\text{Diam}\, \mathcal{L}_{p_j/q_j} \geq \frac{j}{q_j^2}. \tag{V.2.1}$$

For all j, let $\langle a_{m,j}, \varepsilon_{m,j} \rangle_{m=1}^{M}$ be the modified continued fraction for p_j/q_j. Up to a subsequence, for every $1 \leq m \leq M$ there exists some $a_m \in \mathbb{Z}_{>1} \cup \{\infty\}$ and $\varepsilon_m \in \{\pm 1\}$ such that $a_{m,j} \to a_m$ and $\varepsilon_{m,j} \to \varepsilon_m$ when $j \to \infty$. Thus there exists some $0 \leq k_0 \leq M$ and integers $0 = m_0 < m_1 < \cdots < m_{k_0+1} = M + 1$ such that $a_m = \infty$ if and only if $m = m_k$ for some $1 \leq k \leq k_0$. For any $1 \leq m \leq M$ which is not equal to m_k for some $1 \leq k \leq k_0$, we have $a_{m,j} = a_m$ and $\varepsilon_{m,j} = \varepsilon_j$ for all j sufficiently large.

For all $0 \leq k \leq k_0$, let $\delta_k := \tilde{p}_k/\tilde{q}_k$ be the rational number whose modified continued fraction is given by $\langle a_m, \varepsilon_m \rangle_{m=m_k+1}^{m_{k+1}-1}$, recalling that $0/1$ has empty continued fraction expansion. We also define $\delta_{k,j} := p_{k,j}/q_{k,j}$ to be the rational number whose modified continued fraction is given by $\langle a_{m,j}, \varepsilon_{m,j} \rangle_{m=m_k+1}^{M}$. As $a_{m,j} = a_m$ for all $m_k + 1 \leq m < m_{k+1}$ and $a_{m_{k+1},j} \to +\infty$ for $1 \leq k \leq k_0$, it follows that $\delta_{k,j} \to \delta_k$ for all $1 \leq k \leq k_0$.. Moreover,

$$\delta_{k,j} = \mu_{\delta_k} \left(\frac{\varepsilon_{m_{k+1}} \cdot \mathfrak{S}(\delta_k)}{a_{m_{k+1},j} + \delta_{k+1,j}} \right)$$

for all $0 \le k < k_0$ by (II.1.2).

By theorem V.7 and corollary II.2, to reach a contradiction it suffices to show that $h_{1,j} := \mathfrak{f}_{p_j/q_j}$ converges in $\widehat{\mathrm{Quad}}_{k_0}$ with combinatorics $\langle a_{m_k,j}\rangle_{k=1}^{k_0}$ to some height k_0+1 satellite tower when $j \to \infty$. Indeed, combined with corollary II.2 this would imply that there is a constant $C > 0$ such that

$$\begin{aligned}
\operatorname{Diam} \mathcal{L}_{p_j/q_j} &< \frac{C}{\prod_{k=1}^{k_0} a_{m_k,j}^2} \\
&\le \left(\frac{C}{\prod_{m=1}^{M} a_{m,j}^2}\right)\left(\prod_{k=0}^{k_0}\prod_{m=m_k+1}^{m_{k+1}-1} a_{m,j}^2\right) \\
&\le \left(\frac{C}{(2/3)^{2M} q_j^2}\right)\left(\prod_{k=0}^{k_0}(4/3)^{m_{k+1}-m_k-1}\tilde{q}_k\right)^2 \\
&\le \frac{4^M C \prod_{k=0}^{k_0}\tilde{q}_k^2}{q_j^2}
\end{aligned}$$

for all large j, which contradicts (V.2.1).

Setting $f_1 = \mathfrak{f}_{\delta_0}$, the convergence of $p_j/q_j = \delta_{0,j}$ to δ_0 implies that $h_{1,j} := \mathfrak{f}_j$ converges to f_1 in $\widehat{\mathrm{Quad}}_1$ when $j \to \infty$. If $k_0 = 0$ then we are done, so we assume that $k_0 > 0$. As $h_{1,j}'(0) = \mathrm{Exp}(\delta_{0,j})$ and $a_{m_1,j} \to +\infty$, $h_{1,j}$ is a positively or negatively implosive perturbation of f_1 depending on whether $\varepsilon_{m_1} \cdot \mathfrak{S}(\delta_0)$ is positive or negative respectively. Setting $h_{2,j} := \mathcal{R}_{f_1} h_{1,j}$, it follows that

$$\mathrm{Exp}(-\delta_{1,j}) = \begin{cases} h_{2,j}'(0) & \text{if } \varepsilon_{m_1} \cdot \mathfrak{S}(\delta_0) = +1 \text{ or} \\ h_{2,j}'(\infty) & \text{if } \varepsilon_{m_1} \cdot \mathfrak{S}(\delta_0) = -1. \end{cases}$$

Setting f_2 to be the top parabolic renormalization of f_1 with a $-\delta_1$-parabolic fixed point at 0 or ∞ depending on whether $\varepsilon_{m_1} \cdot \mathfrak{S}(\delta_0)$ is positive or negative respectively, it follows that $h_{1,j}$ converges to $\langle f_1, f_2\rangle$ in $\widehat{\mathrm{Quad}}_2$ with combinatorics $a_{m_1,j}$ when $j \to \infty$. Repeating this argument inductively completes the proof. □

For any sequence of integers $\kappa = \langle k_n\rangle_{n=1}^{N}$, we denote

$$|\kappa| := \min_{1 \le n \le N} |k_n|$$

and

$$\|\kappa\| := \prod_{n=1}^{N} |k_n|.$$

If κ is the empty sequence, then we will instead define $|\kappa| = \infty$ and $\|\kappa\| = 1$. To begin proving theorem V.7, we first make the following observation.

Proposition V.8. *Fix $N \geq 1$ and let $\mathcal{T} = \langle f_n \rangle_{n=1}^{N}$ be a finite height satellite tower. If $\kappa = \langle k_n \rangle_{n=1}^{N-1}$ is a (possibly empty) sequence of integers with $|\kappa|$ sufficiently large, then there exists a univalent function $\mu_\kappa^{\mathcal{T}} : \mathbb{D} \to \mathbb{C}$ satisfying:*

1. *For all $\alpha \in \mathbb{D}$, the function*

$$h_{\kappa,\alpha}^{\mathcal{T}} := \left(\mathcal{R}_{f_{N-1}} \circ \cdots \circ \mathcal{R}_{f_1}\right) \mathfrak{f}_{\mu_\kappa^{\mathcal{T}}(\alpha)}$$

is defined.

2. *If f_N has a p_N/q_N-parabolic fixed point at 0 or ∞ for some $p_N/q_N \in (-1/2, 1/2]$, then either*

$$(h_{\kappa,\alpha}^{\mathcal{T}})'(0) = e^{2\pi i \mu_{p_N/q_N}(\alpha)} \text{ or } (h_{\kappa,\alpha}^{\mathcal{T}})'(\infty) = e^{2\pi i \mu_{p_N/q_N}(\alpha)}$$

respectively.

3. *There exists a constant $C > 0$ which does not depend on κ such that*

$$\left|\mu_\kappa^{\mathcal{T}}(s) - \mu_\kappa^{\mathcal{T}}(t)\right| < \frac{C|s-t|}{\|\kappa\|^2}$$

for all $s, t \in \mathbb{D}$.

For any $k \geq 0$, if $\mathfrak{f}_t$ is sufficiently close to $\mathcal{T}$ in $\widehat{\mathrm{Quad}}_N$, then there exists some κ with $|\kappa| > k$ and $\alpha \in \mathbb{D}$ such that $t = \mathrm{Exp} \circ \mu_\kappa^{\mathcal{T}}(\alpha)$.

Proof. In the case where $N = 1$, κ is automatically the empty sequence and we can define $\mu_\kappa^{\mathcal{T}} := \mu_{p_1/q_1}$, where f_1 has a p_1/q_1-parabolic fixed point at 0 and $p_1/q_1 \in (-1/2, 1/2]$.

So now we assume that $N > 1$, $\mu_{\lfloor\kappa\rfloor}^{\lfloor\mathcal{T}\rfloor}$ is defined on $\mathbb{D}$, and $k_N > 0$. If f_N has a p_N/q_N-parabolic fixed point at 0 for some $p_N/q_N \in (-1/2, 1/2]$, then we can define

$$\mu_\kappa^{\mathcal{T}}(\alpha) = \mu_{\lfloor\kappa\rfloor}^{\lfloor\mathcal{T}\rfloor}\left(\frac{1}{k_N + c_+^{f_N} - \mu_{p_N/q_N}(\alpha)}\right)$$

for all $\alpha \in \mathbb{D}$ when $|\kappa|$ is sufficiently large. In this case, all of the desired properties follow automatically. If instead f_N has a p_N/q_N-parabolic fixed point at ∞ for some $p_N/q_N \in (-1/2, 1/2]$, then we have some increased difficulty as we want to define $\mu_\kappa^{\mathcal{T}}$ so that

$$\mu_\kappa^{\mathcal{T}}(\alpha) = \mu_{\lfloor\kappa\rfloor}^{\lfloor\mathcal{T}\rfloor}(\alpha'),$$

where

$$\alpha' = \frac{1}{k_N + c_-^{h^{\lfloor \mathcal{T} \rfloor}_{\lfloor \kappa \rfloor, \alpha'}, f_{N-1}} + \mu_{p_N/q_N}(\alpha)}.$$

As $c_-^{h,f_{N-1}}$ converges to $c_-^{f_{N-1}}$ when $h \to f_{N-1}$ and the equation

$$\alpha' = \frac{1}{k_N + c_-^{f_{N-1}} + \mu_{p_N/q_N}(\alpha)}$$

has a unique solution α' for any $\alpha \in \mathbb{D}$ when $k_N > 0$ is large, we can produce the desired function $\mu_\kappa^{\mathcal{T}}$ by Rouché's theorem. If instead $-k_N > 0$ is large, the we can similarly produce $\mu_\kappa^{\mathcal{T}}$ by considering negatively implosive perturbations. The proposition therefore follows by induction on N. □

We will call the collection of maps $h_{\kappa,\alpha}^{\mathcal{T}}$ the *renormalized quadratic perturbations* of $\mathcal{T}$. Let us now fix some finite height satellite tower $\mathcal{T} = \langle f_n \rangle_{n=1}^N$ and sequence of integers $\kappa = \langle k_n \rangle_{n=1}^{N-1}$. Without loss of generality we assume that f_N has a p/q-parabolic fixed point at 0 for some rational $p/q \in (-1/2, 1/2]$ and $k_N \geq 0$, the other cases can be handled similarly. Fixing some $\alpha \in \mathbb{D}$, to simplify notation, we assume that $|\kappa|$ is large and denote

$$f = f_N, \tilde{h} = \mathfrak{f}_{\mu_\kappa^{\mathcal{T}}(\alpha)}, \text{ and } h = h_{\kappa,\alpha}^{\mathcal{T}}.$$

Thus $\tilde{h}$ is semi-conjugate to h by $\eta_{att}^{\tilde{h},\mathcal{T}}$. It is important to remember that both h and $\tilde{h}$ depend on κ and α even though we suppress this fact in our notation; we similarly consider several other objects in this section which are assumed to depend on κ and α unless stated otherwise.

We will say that a map $\xi : \mathbb{C} \to \mathbb{C}$ is $\mathbb{Q}$-*linear* if

$$\xi(z) = az + b$$

for some rational $a > 0$ and rational b; so ξ preserves both $\mathbb{H}$ and $\mathbb{Q}$ and sends vertical or horizontal lines to vertical or horizontal lines respectively.

We will now assume that there exists a $\mathbb{Q}$-linear map $\xi_0 = \xi_{0,\kappa}^{\mathcal{T}}$ and rational $x_0 > 0$, $\lambda \geq 2$ which all do not depend on α such that

1. Setting
$$X_0 = X_{\kappa,0}^{\mathcal{T}} := \{x - iy : |x| < x_0, \lambda^{-1} < |y| < \lambda\}$$
and $X_m = \lambda^{-m} X_0$ for all integers m, if $|\kappa|$ is sufficiently large and α is sufficiently small

then the map

$$\psi_0 := \eta_{att}^{h,\mathcal{T}} \circ \psi^{\tilde{h}} \circ \xi_0 \tag{V.2.2}$$

is defined on $\bigcup_{m=0}^{M} X_m$ and satisfies

$$h^q \circ \psi_0(w) = \psi_0(\lambda w) \tag{V.2.3}$$

wherever both sides of the equation are defined.

2. There exists a compact subset of $P_{rep}^{f,f}$ which does not depend on κ or α and which avoids U^f such that any neighborhood of that compact set contains the image under ψ_0 of $\bigcup_{m=0}^{M}(X_m)$ when $|\kappa|$ is sufficiently large and α is sufficiently small.

We will say that the renormalized quadratic perturbations of $\mathcal{T}$ have *external rays* when the above conditions are satisfied. We will call any image under ψ_0 of a vertical line an *external ray* of h. If $\mathcal{T}$ has height 1, so $h = \tilde{h}$ and f are polynomials, then these external rays are exactly external rays of of h. Our strategy in proving theorem V.7 is to use the geometry of bubble rays for Lavaurs maps of f to control the geometry of the external rays of h and lift this control to parameter space. By propositions II.14 and IV.9, the fact that $\mathcal{T}$ is a satellite tower implies that f has Jordan basin; so we can apply all the results in the previous chapter to the Lavaurs maps of f.

Given a sequence $\langle x_j \rangle_{j=1}^{\infty}$, we will call any limit of a subsequence of x_j when $j \to \infty$ a *subsequential limit* of the sequence.

Proposition V.9. *Set $d = 1$ and $\Theta = \langle 0 \rangle$. For any $\delta \in \mathbb{C}$, $M \in \mathbb{Z}$, and sufficiently large $s > 0$, if h is sufficiently close to f and if $n - 1/\alpha$ is sufficiently close to δ, then either we can univalently extend ψ_Θ to $\bigcup_{m=s}^{n-M} X_m$ or there exists an integer $M' > M$ and $k \geq 0$ such that*

$$cv^h \in h^k \circ \psi_\Theta(X_{n-M'}).$$

The former case holds if $M > 0$ is sufficiently large.

If ψ_Θ is defined on $\bigcup_{m=s}^{n-M} X_m$ and $s \leq m_n \leq n - M$ is an integer which depends on n, then any subsequential limit X of $\overline{\psi_\Theta(X_{m_n})}$ when $h \to f$, $n - 1/\alpha \to \delta$, and $n \to \infty$ satisfies

$$X \subset \begin{cases} E_d^{L_\delta} \cup J_d^{L_\delta} & \text{if } n - m_n < O(1), \\ P_{\Theta,rep}^{L_\delta} \setminus K_{d-1}^{L_\delta} & \text{if } m_n < O(1), \\ \{z_\Theta^{L_\delta}\} & \text{if } m_n \to +\infty \text{ and } n - m_n \to -\infty. \end{cases}$$

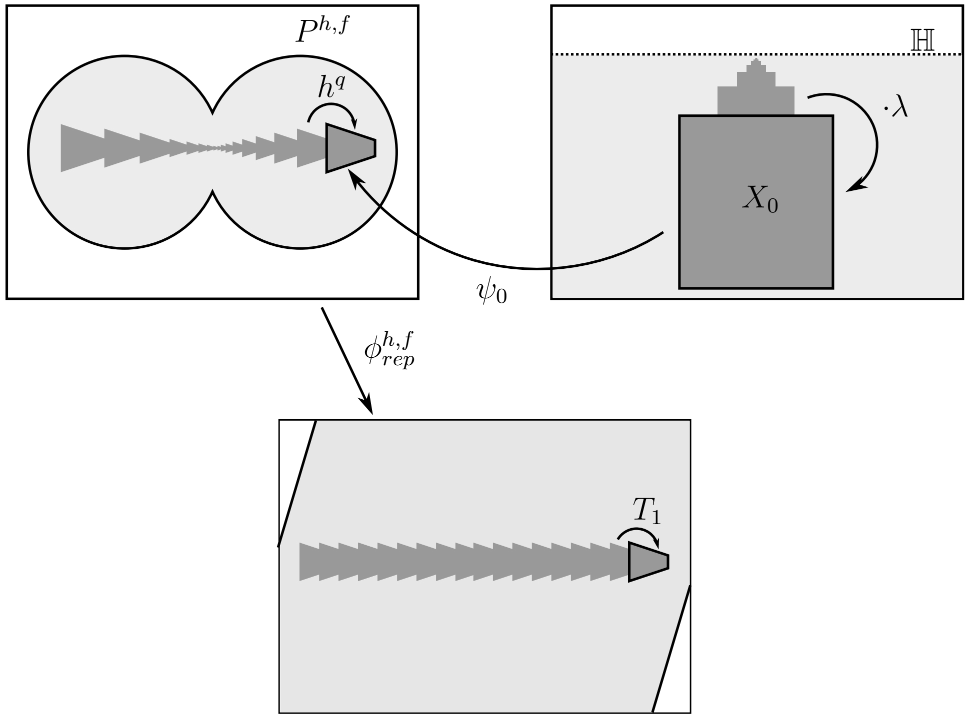


Figure V.2: The extension of ψ_0 in proposition V.9

If $M > 0$ is sufficiently large or if L_δ is d-nonescaping, then additionally

$$X \subset \overline{B_\Theta^{L_\delta}}$$

if $n - m_n < O(1)$.

Proof. For simplicity, we assume without loss of generality that $s = 0$. Recall that $h'(0) = e^{2\pi i \mu_{p/q}(\alpha)}$, so if h is sufficiently close to f and if $n - 1/\alpha$ is sufficiently close to δ for some integer $n > 0$ and $\delta \in \mathbb{C}$, then h is a positive implosive perturbation of f. By definition, there is a compact set $Y_0 \subset P^f_{rep}$ which avoids U^f such that any neighborhood of Y_0 contains $\psi_0(X_0) \subset P^{h,f}$ when h is sufficiently close to f. Let $M_0 \geq 0$ be sufficiently large so that

$$T_{M_0 - \delta} \circ \phi^f_{rep}(Y_0) \subset \phi^f_{att}(P^{f,f}_{att}).$$

Thus if h is sufficiently close to f and if $n - 1/\alpha$ is sufficiently close to δ, then

$$T_{-m} \circ \phi^{h,f}_{att} \circ \psi_0(X_0) = T_{n-m-n+1/\alpha} \circ \phi^{h,f}_{rep} \circ \psi_0(X_0) \subset \phi^{h,f}_{att}(P^{h,f})$$

for all $0 \leq m \leq n - M_0$. We can therefore define

$$\psi_0(w) := (\phi_{att}^{h,f})^{-1} \circ T_{-m} \circ \phi_{att}^{h,f} \circ \psi_0(\lambda^m w)$$

on X_m for all $0 \leq m \leq n - M_0$, which gives an analytic extension of ϕ_0. If $\psi_0(X_m)$ is defined and does not contain cv^h for some integer m, then there exists a unique branch of h^{-q} defined on $\psi_0(X_m)$ whose image contains $\psi_0(\lambda^{-m} \cdot (-3i/4))$; using this branch we can define ψ_0 on X_{m+1}. By induction on m, this completes the proof of the first part of the proposition.

Now let us suppose that ψ_0 is defined on $\bigcup_{m=0}^{n-M} X_m$ for some M. For all $m \geq 0$, let Y_m be any subsequential limit of $\overline{\psi_0(X_m)}$ when $h \to f$. By the definition of the external rays, $Y_m \subset P_{rep}^{f,f} \setminus U_0^f$ for all integers $m \geq 0$. For all integers $m \geq M$, let $Y_{\infty,m}$ be the subsequential limit of $\psi_0(X_{n-m})$. If $m \geq M_0$, then

$$\psi_0(X_{n-m}) = (\phi_{rep}^{h,f})^{-1} \circ T_{n-m-n+1/\alpha} \circ \phi_{att}^{h,f}(X_0),$$

so $Y \subset P_{att}^{f,f}$ and $L_\delta(Y_\infty) = f^{k_+^f}(Y_0)$, so $Y_{\infty,m} \subset \overline{B_0^{L_\delta}} \cap P_{att}^{f,f}$. If $m < M_0$, then we can choose the subsequential limits so that

$$f^{M_0 - m}(Y_{\infty,m}) = Y_{\infty,M_0},$$

so $Y_{\infty,m} \subset E_1^{L_\delta} \cup J_1^{L_\delta}$. If L_δ is 1-nonescaping, then the components of $E_1^{L_\delta}$ have pairwise disjoint closures; as any subsequential limit of $\overline{\psi_0(\bigcup_{j=n-M_0}^{n-m} X_j)}$ is connected it follows that $Y_{\infty,m} \subset \overline{B_0^{L_\delta}}$. If m_n is an integer which depends on n such that $m_n \to +\infty$ and $n - m_n \to -\infty$, then it follows from the above that any subsequential limit of $\psi_0(X_{m_n})$ is contained in $\overline{P^f}$ but does not contain any points in the domain of ϕ_{att}^f or ϕ_{rep}^f, hence the limit must be contained in $\{0\}$. □

We now assume that for any choice of petals near f and any $\theta \in \mathcal{A}^f$, when $\alpha \in A_{1/2}^+$ is sufficiently close to 0 and h is sufficiently close to f there exists a Jordan domain $P_\theta^{h,f}$ such that:

1. $P_0^{h,f} = P^{h,f}$;
2. if θ is not basic, then h^q restricts to an analytic isomorphism from $P_\theta^{h,f}$ to $P_{\theta+1}^{h,f}$;
3. if $\theta \sim \theta'$, then $P_\theta^{h,f} = P_{\theta'}^{h,f}$;
4. $P_\theta^{h,f} \to P_\theta^{f,f}$ when $\alpha \to 0$ and $h \to f$.

We will say that the renormalized quadratic perturbations of $\mathcal{T}$ have *pre-petals* when the above conditions are satisfied. If h and f are polynomials, then every pre-petal of f is compactly contained in $Dom(f)$ and the local uniform convergence of h to f implies the existence of $P_\theta^{h,f}$ for any $\theta \in \mathcal{A}^f$ when h is sufficiently close to f. For general f, we are guaranteed that P_{-1}^f is compactly contained in $\mathbb{C}^*$, so P_θ^f is compactly contained in $Dom(f)$ for any $\theta \in \mathcal{A}^f$ non equivalent to 0 or -1. Thus to ensure that the renormalized quadratic approximations of $\mathcal{T}$ have pre-petals, it suffices to show that $P_{-1}^{h,f}$ exists when h is close to f.

We recall that for any enriched angle Θ of depth $d \geq 1$ which is off-critical for $L = L_\delta^f$ for some $\delta \in \mathbb{C}$, there is some $\theta \in \mathcal{A}$ and integer k such that $L_\delta^d \circ f^{kq}$ is univalent on a neighborhood of z_Θ^L and maps z_Θ^L to z_θ^f. Thus if h is sufficiently close to f and $n - 1/\alpha$ is sufficiently close to $\delta - c_+^f$, then there exists a Jordan domain $P_\Theta^{h,L}$ on which $h^{d(nq+k_+^f)+kq}$ restricts to an analytic isomorphism onto a sub-petal of $P_\Theta^{h,L}$ to $P_\theta^{h,f}$. Moreover, we have the convergence

$$\overline{P_\Theta^{h,L}} \to \overline{P_\Theta^L} \text{ and } \hat{\mathbb{C}} \setminus P_\Theta^{h,L} \to \hat{\mathbb{C}} \setminus P_\Theta^L$$

when $h \to f$ and $n - 1/\alpha \to \delta - c_+^f$. Using the corresponding branch of $h^{-d(nq+m)-k}$, for any large $s \geq 0$ we can define the analytic function $\psi_\Theta^{h,L} : X_s \to \hat{\mathbb{C}}$ by

$$\psi_\Theta^{h,L} := h^{-d(nq+k)-m} \circ \psi_0^{h,f}$$

when h is sufficiently close to f and $n-1/\alpha$ is sufficiently close to $\delta - c_+^f$. We will call $\psi_\Theta^{h,L}$ the *external coordinate* for h relative to L at enriched angle Θ. The following two propositions follow immediately from these definitions.

Proposition V.10. *Fix an enriched angle Θ which is off-critical for L_δ for some $\delta \in \mathbb{C}$. For any sufficiently large $s \geq 0$, if h is sufficiently close to f and $n-1/\alpha$ is sufficiently close to $\delta - c_+^f$, then*

$$h^q \circ \psi_\Theta^{h,L} = \psi_{\Theta+1}^{h,L}$$

on X_s. If Θ has height $d > 1$, then for any sufficiently large integer $t \geq 0$

$$h^{(n-t)q+m} \circ \psi_\Theta^{h,f} = \psi_{\lceil\Theta\rceil - t}^{h,f}$$

on X_s.

Proposition V.11. *For any $\delta \in \mathbb{C}$, proposition V.9 holds for any $d \geq 1$ and enriched angle Θ of depth d which is off-critical for L_δ.*

For any negative $\theta \in \mathcal{A}$ with $|\theta|$ sufficiently large, proposition II.9 implies that

$$P_\theta^f \subset f^{k_+^f}(P_{rep}^f).$$

Hence $P_\theta^{h,f} \subset h^{k_+^f}(P^{h,f})$ when h_0 is sufficiently close to f and α is sufficiently close to 0. We will say that $X \subset \mathbb{H}$ is *above* or *below* $X' \subset \mathbb{H}$ if for all $w \in X$ there exists some $t \geq 0$ so that $w - it \in X'$ or $w + it \in X'$ respectively. For the rest of this section we will assume that for any integer $s \geq 0$, if $\theta < 0$ lies in $\mathcal{A}^f$ with $|\theta|$ sufficiently large, then there exists a $\mathbb{Q}$-linear map $\xi_\theta = \xi_\theta^{\kappa,\mathcal{T}}$ which does not depend on α or s such that if $\alpha \in A_{1/2}^+$ is sufficiently close to 0 and $|\kappa|$ is sufficiently large, then:

1. $\xi_\theta(X_s)$ is above X_s.
2. $\psi_0^{h,f}$ can be univalently extended to the set

   (V.2.4) $$\{w \in \mathbb{H} : w \text{ is above } X_s \text{ and below } \xi_\theta(X_s)\}.$$

3. On X_s,
   $$h^{q-m} \circ \psi_\theta^{h,f} = \psi_0^{h,f} \circ \xi_\theta.$$
4. there is a compact subset of P_{rep}^f which does not depend on κ or α and avoids U^f such that any neighborhood of that compact set contains the image under $\psi_0^{h,f}$ of the set (V.2.4) whenever $\alpha \in A_{1/2}^+$ is sufficiently small and h is sufficiently close to f.

We will say that the renormalized quadratic perturbations of $\mathcal{T}$ have *compatible* external rays and pre-petals when the above conditions hold. If $\mathcal{T}$ has height 1, then these conditions are automatically satisfied.

Proposition V.12. *For every enriched angle Θ of depth $d > 1$, there exists a $\mathbb{Q}$-linear map $\xi_\Theta = \xi_\Theta^{\kappa,\mathcal{T}}$ which does not depend on α such that if Θ is off-critical for L_δ for some $\delta \in \mathbb{C}$, $s \geq 0$ is sufficiently large, h is sufficiently close to f, and if $n - 1/\alpha$ is sufficiently close to δ, then*

1. *$\xi_\Theta(X_s)/\lambda^n$ is above X_s*
2. *There exists some minimal $g \geq 0$ which depends on h such that $\psi_{\lfloor\Theta\rfloor}$ can be univalently extended to*

   (V.2.5) $$\{w \in \mathbb{H} : w \text{ is above } X_s \text{ and } w + ig \text{ is below } \xi_\Theta(X_s)/\lambda^n\}.$$

   Either $g = 0$ or cv^h belongs to the forward orbit under h of the image of the set (V.2.5) under $\psi_{\lfloor\Theta\rfloor}$.

3. If $g = 0$, then there exists some enriched angle Θ' of depth d such that

$$\psi_{\Theta'}(w) = \psi_{\lfloor\Theta\rfloor}(\xi_\Theta(w)/\lambda^n)$$

on X_s. If L_δ is $(d-1)$-nonescaping, then $\Theta' \sim \Theta$.

Proof. The argument is identical to the proof of propositions V.9 and V.11. □

V.2.1: Virtually parabolic Lavaurs maps

Let us now fix some $p'/q' \in \mathbb{Q}$ so that the Lavaurs map $L = L_{p'/q'}$ has a virtually parabolic fixed point. To simplify our notation we will assume that $c_+^f = 0$, the case where $c_+^f = 1$ can be handled similarly. Let $\Theta_\infty = \langle\theta_d\rangle_{d=1}^\infty$ be the parabolic enriched angle for L, and for all $D \geq 1$ set $\Theta_D = \langle\theta_d\rangle_{d=1}^D$. We also denote $\kappa' := \kappa \oplus \langle n\rangle$ for some positive integer n, and denote

$$n' = (nq + k_+^f)q' + mq$$

where m is the integer in proposition IV.8. So

$$h^{n'} \to L^{q'} \circ f^{mq}$$

locally uniformly on $K_{q'-1}^L$ when $h \to f$ and $n - 1/\alpha \to p'/q'$ by theorem II.27.

By proposition IV.11, there exists some $d_0 \gg 0$ such that $\overline{B_{\Theta_d}} \subset P_{rep}^L$ for all $d \geq 1 + (d_0 - 1)q'$. Additionally,

$$L^{q'} \circ f^{mq}(\overline{B_{\Theta_d}}) = \overline{B_{\Theta_{d-q'}}}$$

for all $d \geq q'$. Proposition V.10 therefore implies that there is an integer $s \geq 0$ so that

$$h^{n'} \circ \psi_{\Theta_{d+q'}} = \psi_{\Theta_d}(w)$$

on X_s for any $1 \leq d \leq d_0q' + 1$ when h is sufficiently close to f and $n - 1/\alpha$ is sufficiently close to p'/q'. For all $1 < d \leq (d_0 + 1)q'$, we denote

$$\xi_{\Theta_d}^*(w) := \xi_{\Theta_d}(w/\lambda^n),$$

so

$$\psi_{\Theta_{d+1}} = \psi_{\Theta_d} \circ \xi_{\Theta_{d+1}}^*$$

on X_s by proposition V.12. Repeated application of proposition V.12 implies that for all

$0 \le d \le d_0$, the set

$$\xi^*_{\Theta_{2+dq'}} \circ \cdots \circ \xi^*_{\Theta_{1+(d+1)q'}}(X_s)$$

is above X_0, $\psi_{\Theta_{1+dq'}}$ can be vertically extended between $\xi^*_{\Theta_{2+dq'}} \circ \cdots \circ \xi^*_{\Theta_{1+(d+1)q'}}(X_s)$ and X_0, and

$$\psi_{\Theta_{1+(d+1)q'}} = \psi_{\Theta_{1+dq'}} \circ \xi^*_{\Theta_{2+dq'}} \circ \cdots \circ \xi^*_{\Theta_{1+(d+1)q'}}$$

on X_s when h is sufficiently close to f and $n - 1/\alpha$ is sufficiently close to p'/q'.

Proposition V.13. *For all* $0 \le d \le d_0$,

$$\xi^*_{\Theta_{2+dq'}} \circ \cdots \circ \xi^*_{\Theta_{1+(d+1)q'}} = \xi^*_{\Theta_2} \circ \cdots \circ \xi^*_{\Theta_{1+q'}}.$$

Proof. When $d = 0$ the proposition holds automatically, so we assume that $d > 0$ and that the proposition holds for smaller values of d. As

$$\begin{aligned}
\psi_{\Theta_{1+(d-1)q'}} \circ \xi^*_{\Theta_{2+(d-1)q'}} \circ \cdots \circ \xi^*_{\Theta_{1+dq'}} &= \psi_{\Theta_{1+dq'}} \\
&= h^{n'} \circ \psi_{\Theta_{1+(d+1)q'}} \\
&= h^{n'} \circ \psi_{\Theta_{1+dq'}} \circ \xi^*_{\Theta_{dq'+2}} \circ \cdots \circ \xi^*_{\Theta_{1+(d+1)q'}} \\
&= \psi_{\Theta_{1+(d-1)q'}} \circ \xi^*_{\Theta_{dq'+2}} \circ \cdots \circ \xi^*_{\Theta_{1+(d+1)q'}},
\end{aligned}$$

the inductive hypothesis implies that

$$\xi^*_{\Theta_{dq'+2}} \circ \cdots \circ \xi^*_{\Theta_{1+(d+1)q'}} = \xi^*_{\Theta_{(d-1)q'+2}} \circ \cdots \circ \xi^*_{\Theta_{1+dq'}} = \xi^*_{\Theta_2} \circ \cdots \circ \xi^*_{\Theta_{1+q'}}.$$

□

Let $a > 0$ and b be the rational numbers which depend only on κ' and which satisfy

$$\xi^*_{\Theta_2} \circ \cdots \circ \xi^*_{\Theta_{1+q'}}(w) = aw + b,$$

so

$$\begin{aligned}
\psi_{\Theta_{1+dq'}}\left(\frac{w-b}{a}\right) &= h^{n'} \circ \psi_{\Theta_{1+(d+1)q'}}\left(\frac{w-b}{a}\right) \\
&= h^{n'} \circ \psi_{\Theta_{1+dq'}} \circ \xi^*_{\Theta_2} \circ \cdots \circ \xi^*_{\Theta_{1+q'}}\left(\frac{w-b}{a}\right) \\
&= h^{n'} \circ \psi_{\Theta_{1+dq'}}(w)
\end{aligned}$$

on $aX_s + b$ for all $0 \le d \le d_0$. We set $\lambda'_{\kappa'} = \lambda' := a^{-1}$ and $\xi_{\Theta_\infty,\kappa'} = \xi_{\Theta_\infty}(w) := w + \frac{b}{1-a}$, so

$$\frac{\xi_{\Theta_\infty}(w) - b}{a} = \frac{w + \frac{b}{1-a} - b}{a} = a^{-1}w + \frac{b}{1-a} = \xi_{\Theta_\infty}(\lambda' w)$$

and

$$a\xi_{\Theta_\infty} + b = a\left(w + \frac{b}{1-a}\right) + b = aw + \frac{b}{1-a} = \xi_{\Theta_\infty}(w/\lambda').$$

Proposition V.14. *If $|\kappa'|$ is sufficiently large and $n - 1/\alpha$ is sufficiently close to p'/q', then there exists some $x'_0 > 0$ which depends only on κ' such that:*

1. *Setting*
$$X'_j := \{(\lambda')^j(x - iy) : |x| < x'_0, (\lambda')^{-1} < y < \lambda'\}$$
for all integers j, $\xi_{\Theta_\infty}(X'_1)$ is above X_0 and $\psi^h_{\Theta_{dq'}}$ can be univalently extended to the set
$$\{w \in \mathbb{H} : w \text{ is above } X_0 \text{ and below } \xi_{\Theta_d}\}$$
for all $1 \le d \le d_0 + 1$.

2. *For all $0 \le d \le d_0$,*
$$h^{n'} \circ \psi_{\Theta_{1+dq'}} \circ \xi_{\Theta_\infty}(w) = \psi_{\Theta_{1+dq'}} \circ \xi_{\Theta_\infty}(\lambda' \cdot w)$$
and
$$\psi_{\Theta_{1+(d+1)q'}} \circ \xi_{\Theta_\infty}(w) = \psi_{\Theta_{1+dq'}} \circ \xi_{\Theta_\infty}(w/\lambda')$$
wherever both sides of the equations are defined.

3. *There is a compact subset of $\tilde{P}^L_{rep}$ which avoids $\tilde{U}^L$ such that any neighborhood of that compact set contains $\psi_{\Theta_{1+d_0q'}} \circ \xi_{\Theta_\infty}(X'_0)$ whenever $|\kappa'|$ is sufficiently large and $n - 1/\alpha$ is sufficiently close to p'/q'.*

Proof. By construction,

$$h^{n'} \circ \psi_{\Theta_{1+dq'}} \circ \xi_{\Theta_\infty}(w) = \psi_{\Theta_{1+dq'}}\left(\frac{\xi_{\Theta_\infty}(w) - b}{a}\right) = \psi_{\Theta_{1+dq'}} \circ \xi_{\Theta_\infty}(\lambda' w)$$

on $\xi^{-1}_{\Theta_\infty}(aX_s + b) =$ for all $0 \le d \le d_0$ Similarly,

$$\psi_{\Theta_{1+(d+1)q}} \circ \xi_{\Theta_\infty}(w) = \psi_{\Theta_{1+dq'}} \circ \xi^*_{\Theta_2} \circ \cdot \circ \xi^*_{\Theta_{1+q'}} \circ \xi_{\Theta_\infty}(w) = \psi_{\Theta_{1+dq'}} \circ \xi_{\Theta_\infty}(w/\lambda')$$

on $\xi^{-1}_{\Theta_\infty}(X_s)$.

Lemma V.15. *When $n \to \infty$,*

$$|\log a + nq' \log \lambda| < O(1) \text{ and } |\lambda^n b| < O(1).$$

Proof. For all $2 \le d \le q' + 1$, set

$$\xi^*_{\Theta_2} \circ \cdots \circ \xi^*_{\Theta_d}(w) = a_d w + b_d,$$

so $a = a_{q'+1}$ and $b = b_{q'+1}$. Setting $a_1 = 1$ and $b_1 = 0$, it follows that

$$a_d w + b_d = \frac{\xi_{\Theta_d}(a_{d-1}w + b_{d-1})}{\lambda^n}$$

for all $2 \le d \le q' + 1$. As ξ_{Θ_d} does not depend on n, it follows from a straightforward induction on d that

$$|\log a_d + nq' \log \lambda| < O(1) \text{ and } |\lambda^n b_d| < O(1)$$

for all $2 \le d \le q' + 1$. □

As $s \ge 0$ and the above lemma implies that we can pick $x'_0 > 0$ which depends on n and does not depend on α and which is small enough so that X'_1 is below

$$\xi^{-1}_{\Theta_\infty}(a(aX_s + b) + b) = \left((\lambda')^{-2}\left(X_s - \frac{b}{1-a}\right)\right) \approx (\lambda')^{-2}(X_s)$$

and above $\xi^{-1}_{\Theta_\infty}(X_0) \approx X_0$. Thus $\psi_{\Theta_{1+dq'}}$ can be vertically extended between $\xi_{\Theta_\infty}(X'_0)$ and X_0 for all $0 \le d \le d_0$. It follows from proposition V.11 that the image under $\psi_{\Theta_{1+d_0 q'}} \circ \xi_{\Theta_\infty}$ of X'_0 is contained in any neighborhood of

$$\bigcup_{d=1+(d_0-1)q'}^{1+(d_0+1)q'} \overline{B_{\Theta_d}} \subset \tilde{P}^L_{rep}$$

when κ' is sufficiently large and $n - 1/\alpha$ is sufficiently close to p'/q', which completes the proof. □

Let us now set $f_{N+1} := R_{p'/q'} f_N$ and $\mathcal{T}' = \mathcal{T} \oplus \langle f_{N+1} \rangle$, so $\mathcal{T}'$ is also a satellite tower. As an immediate consequence of proposition V.14, we have the following:

Corollary V.16. *The renormalized quadratic perturbations of $\mathcal{T}'$ have external rays.*

Proof. When $|\kappa|$ and n is sufficiently large, for all $\alpha' \in \mathbb{D}$ sufficiently close to 0 and

$$\alpha = \frac{1}{n - \mu_{p'/q'}(\alpha')}$$

we can set

$$\psi_0^{h^{\mathcal{T}'}_{\kappa',\alpha'},\mathcal{T}'} := \eta_{att}^{h^{\mathcal{T}}_{\kappa,\alpha},\mathcal{T}} \circ \psi_{\Theta_{1+d_0 q'}} \circ \xi_{\Theta_\infty}.$$

□

Additionally, we can show the following:

Proposition V.17. *The renormalized quadratic perturbations of $\mathcal{T}'$ have pre-petals.*

Proof. It follows from corollary IV.14 that $z_{-1}^{f_{N+1}}$ is the unique point in $\overline{U_0^{f_{N+1}}}$ not contained in the domain of f_{N+1}. Thus $\tilde{z}_{-1}^L$ is the only point in $\overline{\tilde{U}_0^L}$ which belongs to U_0^f and which is not q'-nonescaping. Moreover, $\tilde{z}_{-1}^L$ is q'-asymptotic as

$$f_{N+1}^{q'}(z_{-1}^{f_{N+1}}) = z_0^{f_{N+1}} = 0$$

implies that

$$L^{q'} \circ f^{mq}(\tilde{z}_{-1}^L) = \tilde{z}_0^L = 0.$$

Thus there exists an enriched angle Θ of depth $q'+1$ so that $\tilde{z}_{-1}^L = z_\Theta^L$.

Let us recall that $\tilde{P}_{-1,att}^L$ is by definition the unique component of $(L^{q'} \circ f^{mq})^{-1}(\tilde{P}_{att}^L)$ which has $\tilde{z}_{-1}^L$ on its boundary and which is contained in $\tilde{U}_0^L$. For simplicity, let us assume that $\tilde{P}_{att}^L$ is contained in P_{att}^f and $L^{q'} \circ f^{mq}$ restricts to any analytic isomorphism from $P_{\Theta,att}^L$ to P_{att}^f; the general case will hold by similar argument and applying a homotopy. Thus there is a unique component of $(L^{q'} \circ f^{mq})^{-1}(\tilde{P}_{att}^L)$ which is contained in $P_{\Theta,att}^L$, and this component has z_Θ^L on its boundary. A straightforward induction on q' implies that the restriction of $L^{q'} \circ f^{mq}$ to the intersection of $K_{q'}^L$ and a neighborhood of z_Θ^L is injective, so the above component must be $\tilde{P}_{-1,att}^L$. If we similarly assume that $\tilde{P}_{rep}^L$ is contained in P_{rep}^L, then we can conclude by similar argument that $\tilde{P}_{-1,rep}^L$ is the unique component of $(L^{q'} \circ f^{mq})^{-1}(\tilde{P}_{rep}^L)$ which is contained in $P_{\Theta,att}^L$.

Let us denote $g = \mathcal{R}_f h$. Repeating the argument in the proof of proposition II.34, we can find petals $\tilde{P}^{h,f} \subset P^{h,f}$ such that

$$\eta_{att}^{h,f}(\tilde{P}^{h,f}) = P^{g,f_{N+1}}.$$

Setting $\tilde{P}^{h,f}_{-1}$ to be the unique component of $h^{-n'}(\tilde{P}^{h,f})$ contained in $P^{h,f}_{\Theta}$, we can set

$$P^{g,f_{N+1}}_{-1} := \eta^{h,f}_{att}(\tilde{P}^{h,f}_{-1}).$$

Thus $g^{q'}(P^{g,f_{N+1}}_{-1}) = P^{g,f_{N+1}}$, and it follows from the construction that $P^{g,f_{N+1}}$ converges to $P^{f_{N+1},f_{N+1}}$ when $|\kappa'| \to \infty$ and $g \to f_{N+1}$. □

Corollary V.18. *The renormalized quadratic perturbations of $\mathcal{T}'$ have compatible external rays and pre-petals.*

Proof. This follows from corollary V.16 and proposition V.17 as we can pull-back ψ_{Θ_∞} by iterates of L and f^q and project down by $\eta^{h,f}_{att}$. Part (2) of proposition IV.11 ensures compatibility. □

It follows from corollary V.18 and induction on height that the renormalized quadratic perturbations of any satellite tower have compatible external rays and pre-petals.

V.2.2: Parameter rays

Using the external rays for quadratic perturbations of $\mathcal{T}$ above, we will now produce rays in parameter space.

Proposition V.19. *For any sufficiently large $C > 0$ there exists an integer $n_0 \geq 0$ such that if $|\kappa|$ is sufficiently large, then there exists a $\mathbb{Q}$-linear map Ξ_κ satisfying:*

1. *The map*

$$\Psi_\kappa := (\mu^{\mathcal{T}}_\kappa)^{-1} \circ \Psi \circ \Xi_\kappa$$

   is defined on $\bigcup_{n=n_0}^{\infty} X_n$.

2. *For all $n \geq n_0$ and $w \in X_n$, $\alpha = \Psi_\kappa(w)$ is the unique choice of α such that*

   (a) $\psi^h_0(w)$ can be univalently extended to a neighborhood of the set

$$\{z \in \mathbb{H} : z \text{ is above } X_{n_0} \text{ and below } w\},$$

   (b) $cv^h = \psi^h_0(w)$, and

   (c)

$$\left|\alpha - \frac{1}{n}\right| < \frac{C}{n^2}.$$

Proof. It follows from the definition of the external rays for h that there is a compact set $Y \subset P^{f}_{rep}$ which does not depend on κ and which contains $\psi^h_0(X_0)$ when $|\kappa|$ is sufficiently large and α is sufficiently close to 0. As cv^h and ψ^h_0 depend holomorphically on α, proposition II.39 implies that there exists some $n_0 \geq 0$ and $C > 0$ which do not depend on κ such that if $|\kappa|$, $w_0 \in X_0$, and if $n \geq n_0$, then there exists $\alpha \in \mathbb{D}$ such that

$$\phi^{h,f}_{att}(cv^h) + n = \phi^{h,f}_{att} \circ \psi^h_0(w_0)$$

and

$$\left|\alpha - \frac{1}{n}\right| < \frac{C}{n^2}.$$

Repeating the argument used in the proof of proposition V.9, we can analytically extend ψ^h_0 to a neighborhood V of $[w_0/\lambda^n, w_0]$ and $cv^h = \psi^h_0(w_0/\lambda^n)$. Moreover, as the image under ψ^h_0 of V can only get close to 0 inside P^h_{att}, it follows from proposition III.9 that there exists a map $\tilde{\psi}$ such that

$$\eta^{h,\mathcal{T}}_{att} \circ \tilde{\psi} = \psi^h_0$$

on V and $\tilde{\psi}(w_0/\lambda^n) = cv^{\tilde{h}}$. As

$$\eta^{h,\mathcal{T}}_{att} \circ \tilde{\psi} = \psi^h_0 = \eta^{h,\mathcal{T}}_{att} \circ \psi^{\tilde{h}} \circ \xi_0$$

on X_0, it follows from the definition of the attracting elevators that there exist integers $j, \tilde{j} \geq 0$ such that

$$\tilde{h}^{\tilde{j}} \circ \tilde{\psi} = \tilde{h}^j \circ \psi^{\tilde{h}} \circ \xi_0.$$

Thus there exists a $\mathbb{Q}$-linear map $\tilde{\xi}$ such that

$$\tilde{\psi} = \psi^{\tilde{h}} \circ \tilde{\xi}.$$

Hence $\mu^{\mathcal{T}}_{\kappa}(\alpha) = \Psi \circ \tilde{\xi}(w_0/\lambda^n)$. It follows from the above construction that $\tilde{\xi}$ depend continuously on α, which in turn depends continuously on w_0. As the set of all $\mathbb{Q}$-linear maps is discrete the map $\tilde{\xi}$ does not actually depend on w_0. The map $\Xi_\kappa := \tilde{\xi}$ has the desired properties. □

As Ψ is defined on all of $\mathbb{H}$ and $\mu^{\mathcal{T}}_{\kappa}$ is defined on $\mathbb{D}$, if Ψ_κ is defined at some $w_0 \in \mathbb{H}$ then we can analytically extend Ψ_κ to a neighborhood of w_0. Moreover, if $\alpha = \Psi_\kappa(w)$ for some w close to w_0, then the uniqueness of analytic continuation guarantees that $\psi^h_0(w)$ is defined and equal to cv^h.

Proposition V.20. *For any $\theta \in \mathcal{A}^f$ with $-\theta$ sufficiently large, integer $s \geq 0$, and sufficiently large $C > 0$, there exists some $n_{s,\theta} \geq 0$ such that if $|\kappa|$ is sufficiently large and $n \geq n_{s,\theta}$, then Ψ_κ can be analytically extended to the set*

$$\{\lambda^n w : w \text{ is above } X_0 \text{ and below } \xi_\theta(X_s)\}.$$

For any w in the above set, $\alpha = \Psi_\kappa(w)$ is the unique choice of α such that ψ_0^h can be analytically extended to the set

$$\{w \in \mathbb{H} : w \text{ is above } X_0 \text{ and below } w\},$$

$cv^h = \psi_0^h(w)$, and

$$\left|\alpha - \frac{1}{n}\right| < \frac{C}{n^2}.$$

Proof. The argument is identical to that of proposition V.19, using the compatibility of external rays and pre-petals. □

A key part of our proof of theorem V.7 will be understanding extensions of Ψ_κ, specifically how some parameter rays land. Instead of extending Ψ_κ and showing that some rays land, we will work backwards: we will produce alternative rays in parameter space which land and then show that these are extensions of Ψ_κ.

Proposition V.21. *There exists $d_1 \geq 0$ and $C' > 0$ such that if $|\kappa'|$ is sufficiently large, then there exists a $\mathbb{Q}$-linear map $\Xi_{\kappa'}$ satisfying:*

1. *The map*

   $$\Psi_{\kappa'} := (\mu_\kappa^{\mathcal{T}})^{-1} \circ \Psi \circ \Xi_{\kappa'}$$

   is defined on $\bigcup_{d=d_1}^{\infty} X'_d$.

2. *If $\alpha = \Psi_{\kappa'}(w)$ for some w, then ψ_0 can be univalently extended to a neighborhood the set*

   $$\{z \in \mathbb{H} : z \text{ is above } X_0 \text{ and below } w\}$$

   and $cv^h = \psi_0^h \circ \xi_{\Theta_\infty}(w)$.

3. *For all $d \geq d_1$ and $w \in X'_d$,*

   $$\left|\Psi_{\kappa'}(w) - \frac{1}{n - p'/q'}\right| < \frac{C'}{dn^2}.$$

4. Any subsequential limit of

$$n - \frac{1}{\Psi_{\kappa'}(-(\lambda')^{-d_1} i)}$$

when $|\kappa'| \to \infty$ is contained in

$$\overline{\mathcal{B}_{\Theta_{d_1 q'}}} \setminus \mathcal{Z}_{\Theta_{d_1 q'}}.$$

Proof. Our argument is similar to the proof of proposition V.19. As Θ_∞ is basic, it follows from proposition V.14 that there exists a compact set $Y \subset \tilde{P}^L_{rep}$ which contains

$$\psi_{\Theta_{d_0}} \circ \xi_{\Theta_\infty}(X'_0) = \psi_0 \circ \xi_{\Theta_\infty}(X'_0/(\lambda')^{d_0})$$

when $|\kappa'|$ large and $n - 1/\alpha$ is close to p'/q'. We can apply proposition II.39 and conclude that there exists an integer $d_1 \geq 0$ and constant $C > 0$ such that if $|\kappa'|$ is sufficiently large then for any $w_0 \in X'_0$ and $d \geq d_1$ there exists some $\alpha' \in \mathbb{D}$ such that

$$\tilde{\phi}^{h,f}_{att}(cv^h) + d - d_0 = \tilde{\phi}^{h,f}_{att} \circ \psi^h_0 \circ \xi_{\Theta_\infty}(w_0/(\lambda')^{d_0})$$

and

$$\left|\alpha' - \frac{1}{d}\right| < \frac{C}{d^2}$$

when $d \geq d_1$, where

$$\alpha = \frac{1}{n - \mu_{p'/q'}(\alpha')}.$$

Thus

$$\left|\alpha - \frac{1}{n - p'/q'}\right| = \left|\frac{\mu_{p'/q'}(\alpha') - p'/q'}{(n - \mu_{p'/q'}(\alpha'))(n - p'/q'))}\right| < \frac{9|\alpha'|}{(q')^2(n-1)^2} < \frac{C'}{dn^2}$$

for some constant C' which does not depend on κ' or d by proposition II.5. By the same argument used in the proof of proposition V.9, we can vertically extend ψ^h_0 between a neighborhood of $w_0/(\lambda')^d$ and X_0, and

$$cv^h = \psi^h_0 \circ \xi_{\Theta_\infty}(w_0/(\lambda')^d).$$

Similarly to the proof of proposition V.19, the image under $\psi^h_{\Theta_{d_0}} \circ \xi_{\Theta_\infty}$ of this neighborhood can get close to 0 only inside $P^{h,f}$ or $\tilde{P}^{h,L}$, which implies by proposition III.9 that there is some $\mathbb{Q}$-linear map ξ which does not depend on w_0 and which satisfies

$$cv^{\tilde{h}} = \psi^{\tilde{h}} \circ \xi(w_0/(\lambda')^d),$$

hence

$$\alpha = (\mu_\kappa^{\mathcal{T}})^{-1} \circ \Psi \circ \xi(w_0/(\lambda')^d).$$

We define $\Xi_{\kappa'} := \xi$.

For any $w_0 \in X'_{d_1}$ and $\alpha = \Xi_{\kappa'}(w_0)$, it follows from the above that

$$\left| n - \frac{1}{\alpha} \right| = \left| \alpha - \frac{1}{n} \right| \cdot \frac{n}{|\alpha|} < C''$$

for some $C'' > 0$ which does not depend on κ', so every subsequential limit δ of $n - 1/\alpha$ belongs to $\overline{\mathbb{D}_{C''}}$. As

$$cv^h = \psi_0^h \circ \xi_{\Theta_\infty}(w_0) = \psi_{\Theta_{d_1}} \circ \xi_{\Theta_\infty}((\lambda')^{d_1} w_0),$$

if $w_0 = -(\lambda')^{-d_1} i$, then proposition V.11 implies that

$$cv^f \in \overline{B_{\Theta_{d_1}}^{L_\delta}} \setminus z_{\Theta_{d_1}}^{L_\delta},$$

so $\delta \in \overline{\mathcal{B}_{\Theta_{d_1}}} \setminus \mathcal{Z}_{\Theta_{d_1}}$. □

Just as for Ψ_κ, we can analytically extend $\Psi_{\kappa'}$ to a larger domain when the corresponding image is contained in $\mathbb{D}$.

Proposition V.22. *For any sufficiently large integer $s \geq 0$ and any integer $1 < d \leq d_1 q$, if $|\kappa'|$ is sufficiently large then $\Psi_{\kappa'}$ can be univalently extended to*

$$\{w \in \mathbb{H} : w \text{ is below } X'_{d_1} \text{ and above } X_{(d-1)n+s}\}. \tag{V.2.6}$$

If $m_n \geq s$ is an integer and if w_n belongs to the intersection of $X_{(d-1)n+m_n}$ with (V.2.6), then any subsequential limit δ of $n - \frac{1}{\Psi_{\kappa'}(w_n)}$ satisfies

$$\delta \in \begin{cases} \overline{\mathcal{B}_{\Theta_d}} \setminus \{\mathcal{Z}_{\Theta_d}\} & \text{if } |n - m_n| < O(1) \\ \overline{\mathcal{B}_{\Theta_{d-1}}} \setminus \{\mathcal{Z}_{\Theta_{d-1}}\} & \text{if } |m_n| < O(1) \\ \{\mathcal{Z}_{\Theta_d}\} & \text{if } m_n \to +\infty \text{ and } m_n - n \to -\infty. \end{cases}$$

Consequently, if $s \geq 0$ is sufficiently large then there exists a constant $C > 0$ such that

$$\left| \Psi_{\kappa'}(w) - \frac{1}{n} \right| < \frac{C}{n^2} \tag{V.2.7}$$

for all w belonging to (V.2.6).

Proof. Fixing some d and s, we denote by V the set (V.2.6) and assume that $\Psi_{\kappa'}$ is defined on V. We will further assume that there exists some integer s_n, $w_0 \in V \cap X_{dn+s_n}$, and $\delta_0 \in \overline{\mathcal{B}_{\Theta_d}} \setminus \{\mathcal{Z}_{\Theta_d}\}$ such that $|s_n| < O(1)$ and $n - \frac{1}{\Psi_{\kappa'}(w_0)} \to \delta_0$ when $|\kappa'| \to \infty$. We note that the latter assumption follows from proposition V.21 when $d = d_1 q'$.

Let $m_n \geq 0$ be an integer which depends on n and choose some $w \in V \cap X_{(d-1)n+m_n}$. We set $d = d'q' - r'$ for integers $d' \geq 1$ and $0 \leq r' < q'$. Setting $\alpha = \Psi_{\kappa'}(w)$, it follows from proposition V.21 that

$$cv^h = \psi_0 \circ \xi_{\Theta_\infty}(w) = \psi_{\Theta_{d'q'}} \circ \xi_{\Theta_\infty}((\lambda')^{d'} w) = \psi_{\Theta_{d-1}} \circ \xi^*_{\Theta_d} \circ \cdots \circ \xi^*_{\Theta_{d'q'}} \circ \xi_{\Theta_\infty}((\lambda')^{d'} w).$$

As ξ_{Θ_j} does not depend on n and $|\log \lambda'/\lambda^{q'}| < O(1)$ when $n \to \infty$, it follows that

$$\xi^*_{\Theta_d} \circ \cdots \circ \xi^*_{\Theta_{d'q'}} \circ \xi_{\Theta_\infty}((\lambda')^{d'} w) \approx \lambda^{d'q'-r'-1} w = \lambda^{d-1} w.$$

In particular, our definition of V and X'_0 ensures that there is some integer m'_n with $|m'_n| < O(1)$ so that

$$w' := \xi^*_{\Theta_d} \circ \cdots \circ \xi^*_{\Theta_{d'q'}} \circ \xi_{\Theta_\infty}((\lambda')^{d'} w) \in X_{m_n+m'_n}.$$

Let Ω be a subsequential limit of the set $n - \frac{1}{\Psi_{\kappa'}([w_0,w])}$ and let $\delta \in \mathbb{C}$ be a subsequential limit of $n - \frac{1}{\Psi_{\kappa'}(w)}$ when $|\kappa'| \to \infty$.

First we consider the case where $|m_n - n| < O(1)$. If $\delta \in \mathcal{K}_{d-1}$, then proposition V.11 implies that cv^f, which is the limit of $cv^h = \psi_{\Theta_{d-1}}(w')$, is contained $E_d^{L_\delta} \cup J_d^{L_\delta}$, so $\delta \in \mathcal{E}_d \cup \mathcal{J}_d$. If instead $\delta = \mathcal{Z}_{\Theta_d}$, so Θ_d is off-critical for L_δ, then the same argument implies that $cv^f \in E_d^{L_\delta} \cup J_d^{L_\delta}$ which is a contradiction. Thus Ω must avoid $\mathcal{Z}_{\Theta_d}$ and can intersect $\mathcal{K}_{d-1}$ only in $\mathcal{E}_d \cup \mathcal{J}_d$. As $\delta_0 \in \mathcal{B}_{\Theta_d}$, Ω is connected, and the components of $\mathcal{E}_d$ have pairwise disjoint closures, we can conclude that

$$\delta\Omega \subset \overline{\mathcal{B}_{\Theta_d}} \setminus \{\mathcal{Z}_{\Theta_d}\}.$$

Now let us consider the case where $m_n \to +\infty$ and $m_n - n \to -\infty$. If δ is sufficiently close to $\mathcal{Z}_{\Theta_d}$, so Θ_d is off-critical for L_δ, then proposition V.11 implies that $cv^f = z^{\mathcal{T}}_{\Theta_d}$. So in particular $\delta = \mathcal{Z}_{\Theta_d}$. A similar argument shows that δ cannot be contained in $\mathcal{K}_{d-1}$. As Ω is connected, it follows from the above that $\Omega \subset \overline{\mathcal{B}_{\Theta_d}}$, so in particular

$$\delta = \mathcal{Z}_{\Theta_d}.$$

Now let us consider the case where $|m_n| < O(1)$. It follows from the above that for any neighborhood $\mathcal{U}$ of $\mathcal{Z}_{\Theta_d}$, there exists some $S > 0$ such that if $m_n > S$ then $\delta \in \mathcal{U}$; indeed otherwise we could choose m_n so that $m_n \to +\infty$, $m_n - n \to -\infty$, and $\delta \neq \mathcal{Z}_{\Theta_d}$. If $\mathcal{U}$ is sufficiently small, then Θ_d is off-critical for L_δ and we can again apply proposition V.11 and similarly conclude that $\delta \in \overline{\mathcal{B}_{\Theta_{d-1}}} \setminus \mathcal{Z}_{\Theta_{d-1}}$.

The proposition therefore follows by induction on d, with one technical detail. We had to assume that $\Psi_{\kappa'}$ is defined on V, which is not immediately implied by induction. However, if $\Psi_{\kappa'}$ is not defined on all of V, then there must be some $w \in V$ which is mapped to $\partial\mathbb{D}$. But if n is sufficiently large then this contradicts (V.2.7). □

Corollary V.23. *If $|\kappa'|$ is sufficiently large, then $\Psi_{\kappa'} = \Psi_\kappa \circ \xi_{\Theta_\infty}$.*

Proof. Fix some large $s_0 \geq 0$, $w \in X_{n+s_0} \cap \xi_{\Theta_\infty}^{-1}(X_{n+s_0})$, and set $\alpha = \Psi_{\kappa'}(w)$. Thus

$$cv^h = \psi_0^h \circ \xi_{\Theta_\infty}(w)$$

and $\left|\alpha - \frac{1}{n}\right| < \frac{C}{n}^2$ for some constant $C > 0$ by proposition V.21. The uniqueness in V.20 implies that if $|\kappa'|$ is sufficiently large, then $\alpha = \Psi_\kappa \circ \xi_{\Theta_\infty}(w)$. As this holds for an open set, the analytic extension holds over the whole connected domain. □

Let us set $\kappa'_+ := \kappa \oplus \langle n+2 \rangle$ and $\kappa'_- := \kappa \oplus \langle n-2 \rangle$. We define $\Omega_{\kappa'}$ to be the set formed by the union of the interval

$$\left[\frac{1}{n+2-p'/q'}, \frac{1}{n-2-p'/q'}\right],$$

the curves

$$\Psi_{\kappa'_+}(-i(0, \lambda^{-n-s})) \text{ and } \Psi_{\kappa'_-}(-i(0, \lambda^{-n-s}))$$

for some large $s \geq 0$, and the curve

$$\Psi_\kappa([\xi_{\Theta_\infty, \kappa'_+}(-i\lambda^{-n-s}), \xi_{\Theta_\infty, \kappa'_-}(-i\lambda^{-n-s})]).$$

Proposition V.24. *If κ' is sufficiently large, then $\Omega_{\kappa'}$ is a Jordan curve. There exists a constant $C > 0$ which does not depend on κ' such that*

$$\sup_{\alpha \in \Omega_{\kappa'}} \left|\alpha - \frac{1}{n - p'/q'}\right| < \frac{C}{n^2}.$$

Proof. It follows from proposition V.21 and corollary V.23 that $\Omega_{\kappa'}$ is a Jordan curve. The existence of a constant $C > 0$ follows from propositions V.19, V.20, and V.21. □

Equipped with proposition V.24, we are now ready to prove theorem V.7.

Proof of theorem V.7. Taking $\mathcal{T}$ and $\mathcal{T}'$ as described in this section, we note that if $\mathfrak{f}_t$ converges to $\mathcal{T}$ in $\widehat{\text{Quad}}_N$ with combinatorics κ, then proposition V.8 implies that that there exists $\alpha_t \in \mathbb{D}$ such that $t = \mu_\kappa^{\mathcal{T}}(\alpha_t)$. If $\mathfrak{f}_t$ additionally converges to $\mathcal{T}'$ in $\widehat{\text{Quad}}_N$ with combinatorics κ', then $n + c_+^f - 1/\alpha_t \to \delta$. If $\alpha_t \in -\mathbb{H}$, so

$$\left|(h_{\kappa,\alpha_t}^{\mathcal{T}})'(0)\right| = \left|e^{2\pi i \mu_{p_N/q_N}(\alpha_t)}\right| < 1,$$

then it follows from the same argument as in proposition II.33 that $\mathfrak{f}_t$ has an attracting periodic cycle. As $c_+^f \in \{0, 1\}$, it follows from the definition of $\Omega_{\kappa'}^{\mathcal{T}'}$ that if $n + c_+^f - 1/\alpha$ is sufficiently close to p'/q', then α belongs to either $-\mathbb{H}$ or $\overline{\Omega_{\kappa'}^{\mathcal{T}'}}$. Hence $\alpha_t \in \overline{\Omega_{\kappa'}^{\mathcal{T}'}}$. Thus

$$\tilde{\Omega} := \mu_\kappa^{\mathcal{T}}(\Omega_{\kappa'}^{\mathcal{T}'})$$

is a Jordan curve by proposition V.24 with t on its boundary. As there exists parameters in $\mathcal{L}_t$ close to t which do not have an attracting periodic cycle, it follows from the above that if $\mathfrak{f}_t$ is sufficiently close to $\mathcal{T}$ then $(\mu_\kappa^{\mathcal{T}})^{-1}(\mathcal{L}_t)$ must contain points not in $-\mathbb{H}$. Thus $\mathcal{L}_t$ is contained in the bounded component of the complement of $\tilde{\Omega}$. Thus

$$\begin{aligned}
\sup_{\lambda \in \mathcal{L}_t} |\lambda - t| &\leq \sup_{\lambda \in \tilde{\Omega}} |\lambda - t| \\
&\leq \sup_{\alpha \in \Omega_\kappa^{\mathcal{T}}} \left|\mu_\kappa^{\mathcal{T}}(\alpha) - \mu_\kappa^{\mathcal{T}}(\alpha_t)\right| \\
&\leq \sup_{\alpha \in \Omega_\kappa^{\mathcal{T}}} \frac{C|\alpha - \alpha_t|}{\|\kappa\|^2} \\
&\leq \sup_{\alpha \in \Omega_\kappa^{\mathcal{T}}} \frac{C'|\alpha - \frac{1}{n - p'/q'}|}{\|\kappa\|^2} \\
&\leq \frac{C''}{n^2\|\kappa\|^2} \\
&\leq \frac{C''}{\|\kappa'\|^2}
\end{aligned}$$

for some constants $C, C', C'' > 0$ which do not depend on κ' by propositions V.8 and V.24, provided $|\kappa'|$ is sufficiently large. □

V.3: Generalizing theorem V.7

While theorem V.7 is limited to the satellite towers, this restriction is only mildly required in the proof. The same argument can be used to prove a version of theorem V.7 for any quadratic strictly parabolic tower $\mathcal{T} = \langle f_n \rangle_{n=1}^N$ of finite height with data $\langle \delta_n \rangle_{n=1}^{N-1}$ which satisfies the following three conditions:

1. For all $1 \leq n < N$, there exists δ arbitrarily close to δ_n so that $\mathcal{R}_\delta f_n \notin \mathcal{F}^{\text{❀}}$.
2. For all $1 \leq n \leq N$, f_n has Jordan basin.
3. For all $1 \leq n < N$, there exists a bubble ray of $L_{\delta_n}^{f_n}$ whose image under $\eta_{att,\delta_n}^{f_n}$ lands at a parabolic periodic point of f_{n+1} insider $P_{att}^{f_{n+1}}$.

These conditions are automatically satisfied for satellite towers, and we can ask what other quadratic parabolic towers also satisfy these conditions. We conjecture that the first two conditions above are satisfied by all quadratic strictly parabolic tower. More care needs to be taken in the third case; if the parabolic cycle of f_{n+1} is not inside $Dom_0(f_{n+1})$, so $\mathcal{L}_{\delta_n}^{f_n}$ is not infinitely non-escaping, then we would need to introduce an alternative definition of escaping sets and bubble rays for the Lavaurs maps of f_n. Nonetheless, the same type of analysis should be possible, so we conjecture that the restriction to satellite towers in theorem V.7 can be removed:

Conjecture V.25. *For any quadratic parabolic tower $\mathcal{T}$ of height $N \geq 1$, there exists a constant $C > 0$ such that if $\mathfrak{f}_\alpha$ converges to $\mathcal{T}$ in $\widehat{\text{Quad}}_N$ with combinatorics $\langle k_n \rangle_{n=1}^{N-1}$, then*

$$\operatorname{Diam} \mathcal{L}_\alpha < \frac{C}{\prod_{n=1}^{N-1} k_n^2}$$

when $\mathfrak{f}_\alpha$ is sufficiently close to $\mathcal{T}$.

The above conjecture still requires that the height of the parabolic tower is finite, we can similarly ask what happens when the height is infinite. As shown by propositions II.20 and II.21, for some infinite height quadratic parabolic towers $\langle f_n \rangle_{n=1}^{\infty}$ the sequence of maps may converge towards a limit function f_∞. As the constant C in proposition V.7 is determined by the geometry of parameter spaces of Lavaurs maps, the convergence of f_n to f_∞ may allow some control of the diameter of limbs in this infinite tower case.

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