Convergence of Measures on Non-Archimedean Hybrid Spaces

by

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ABSTRACT

We study the convergence of certain classes of complex geometric measures to certain non-Archimedean measures. This convergence takes place on the non-Archimedean hybrid space introduced by Boucksom and Jonsson.

Given a family X of complex analytic spaces parametrized by the punctured unit complex disk, the hybrid space associated to this family is a partial compactification of this family obtained by filling in the puncture with the Berkovich analytification of X. The topology of the hybrid space is given by local logarithmic convergence. Furthermore, if each of the complex analytic spaces in the family carry a natural measure, we can think of these measures as being supported on the hybrid space, then their weak limit is a measure supported on the Berkovich space.

First, we study the convergence of volume forms on a degenerating holomorphic family of log Calabi–Yau varieties, extending a result of Boucksom and Jonsson. More precisely, let (X, B) be a holomorphic family of sub log canonical, log Calabi–Yau complex varieties parameterized by the punctured unit disk. Let η be a meromorphic form on X with poles along B such that the restriction of η is a top-dimensional form on each of the fibers. We show that the (possibly infinite) measures induced by the restriction of $|\eta \wedge \bar{\eta}|$ to a fiber weakly converge to a measure on the Berkovich analytification of $X \setminus B$ as we approach the puncture. The limit measure is a sum of suitably normalized Lebesgue measures supported on certain skeletal subsets of the Berkovich space.

Secondly, we prove a folklore conjecture that the Bergman measures along a holomorphic

family of curves parametrized by the punctured unit disk weakly converge to the Zhang measure on the associated Berkovich space. We also study the convergence of the Bergman measures to a measure on a metrized curve complex in the sense of Amini and Baker.

CHAPTER I

Introduction

I.1: Non-archimedean geometry

I.1.1: Valued fields

Let k be a field. Algebraic geometry is the study of *varieties* over k, which are spaces defined by the zero sets polynomials in k. In this section, we briefly discuss how additional structures on k gives rise to a richer theory.

A multiplicative norm on k is a function $|\cdot| : k \to \mathbb{R}_{\geq 0}$ that satisfies

- $|x| = 0$ if and only if $x = 0$,
- $|xy| = |x||y|$, and
- (Triangle inequality) $|x + y| \leq |x| + |y|$.

A valued field is a field k equipped with a multiplicative norm. Some examples of valued fields are as follows

- $(\mathbb{R}, |\cdot|)$ Real numbers equipped with the usual absolute value

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

- $(\mathbb{C}, |\cdot|)$ Complex numbers equipped with the usual absolute value

$$|x+iy| = \sqrt{x^2+y^2}$$

- $(k, |\cdot|_0)$ Any field equipped with the trivial norm.

$$|x|_0 = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

- $(\mathbb{Q}, |\cdot|_p)$: the rational numbers equipped with the p-adic norm for some prime number p,

$$\left| p^m \frac{a}{b} \right|_p = p^{-m},$$

where we write a rational number in the form $p^m \frac{a}{b}$, where $m \in \mathbb{Z}$ and a, b are integers that are not divisible by p.

- $k((t))$, the field of *formal Laurent series* over k equipped with the norm given by the *vanishing order* at 0. The field $k((t))$ is the fraction field of the ring of formal power series over k and is given by

$$k((t)) = \left\{ f = \sum_{i=-\infty}^{\infty} a_i t^i \;\middle|\; a_i = 0 \text{ for } i \ll 0 \right\}.$$

The vanishing order norm is given by $|f| = e^{-m}$, where $m \in \mathbb{Z}$ is such that $a_i = 0$ for all $i < m$ and $a_m \neq 0$.

I.1.2: Archimedean fields and complex analytification

A valued field is said to satisfy the *Archimedean* property if for any non-zero $x \in k$, there exists an $n \in \mathbb{N}$ such that

$$|\underbrace{x + \cdots + x}_{n \text{ times}}| \geq 1.$$

Among the examples above, we see that only $(\mathbb{R}, |\cdot|)$ and $(\mathbb{C}, |\cdot|)$ satisfy the Archimedean property. We have a characterization of all fields that satisfy the Archimedean property.

Theorem I.1. (Ostrowski) Any Archimedean field is a sub-field of $\mathbb{C}$ i.e. if $(k, |\cdot|)$ is an Archimedean field, then there exists an embedding $k \subset \mathbb{C}$ such that the norm on k is the restriction of the usual absolute value on $\mathbb{C}$.

Thus, we see that $\mathbb{C}$ is essentially the only Archimedean field. We briefly discuss the geometry over $\mathbb{C}$. Serre defined the *analytification functor* which associates a *complex analytic space* to a variety over $\mathbb{C}$ [Ser56]. The complex analytic space captures the "finer" details of the geometry (for example, we can look at all holomorphic functions instead of just polynomial functions). The analytification functor acts as a bridge between algebraic geometry and complex analysis, and opens up the door to using analytic techniques in algebraic geometry. For example, we get comparisons between algebraic and analytic cohomologies.

I.1.3: Non-Archimedean analytification theories

A valued field that does not satisfy the Archimedean property is called as a *non-Archimedean field*. Some examples of non-Archimedean fields are trivially valued fields, p-adic numbers, and the Laurent series.

It is easy to see that non-Archimedean fields satisfy a stronger version of triangle inequality, called the *ultrametric triangle inequality*:

$$|x + y| \leq \max(|x|, |y|) \text{ for all } x, y \in k.$$

The ultrametric triangle inequality leads to interesting non-intuitive geometric properties of non-Archimedean fields – for example, all triangles are isosceles and any interior point of a disk is also a center of the disk.

One could ask if there is an analytification functor over non-Archimedean fields. However unlike in the case of complex analytic spaces, there is no pre-existing notion of non-

Archimedean analytic spaces. So such a theory would also have to describe what these non-Archimedean analytic spaces look like. The exist several such theories, see [Con08] for a detailed comparison.

The first theory of non-Archimedean theory was due to Tate, which was based on his work on uniformizing p-adic elliptic curves [Tat71]. Tate's theory of *rigid analytic spaces* was constructed by looking at *maximal ideals* of *affinoid algebras*. One drawback of Tate's theory is that the underlying topology of a rigid analytic space is totally disconnected. Tate's theory also did not work over trivially valued fields.

Raynaud's theory of *formal models* described an equivalence class of generic fibers of formal schemes as an analytic space [Ray74]. However, the drawback here is that the analytic spaces don't have a single underlying topological space, but instead have a collection of underlying topological spaces.

Berkovich's theory of analytic spaces was introduced to counter some of these drawbacks, which were constructed by looking at the space of *valuations* on the affinoid algebras [Ber99]. Berkovich analytic spaces contain more points than the rigid analytic spaces, which help the analytifications be connected and Hausdorff when equivalent properties hold for the underlying variety. We discuss Berkovich analytic spaces in more detail in the following section.

Huber's theory of adic spaces is another theory of analytic spaces obtained by considering *higher rank valuations* [Hub96]. Scholze's theory of *perfectoid spaces*, which have been of recent interest in algebraic geometry due to their applications in the mixed characteristic, are special types of adic spaces [Sch12]. However Huber's adic spaces have "too many points" in the sense that the underlying topological spaces are no longer Hausdorff.

I.2: Berkovich spaces

Berkovich constructed an analytification functor which associates an *analytic space*, X^{an}, to a variety X over a non-Archimedean field k. The analytic space X^{an} is a locally ringed space i.e. it carries the data of a topological space as well as the data of sheaf of *analytic functions* on the space.

This functor satisfies the following topological properties:

- X^{an} is Hausdorff if and only if X is separated.
- X^{an} is compact if and only if X is proper.
- X^{an} is connected if and only if X is connected.

These topological properties of the analytification functor make the Berkovich analytic spaces ideal in our setting.

We briefly explain the construction of the topological space underlying X^{an}. As a set, X^{an} consists of all pairs $x = (\xi_x, |\cdot|_x)$, where ξ_x is a (scheme-theoretic) point in X and $|\cdot|_x$ is a multiplicative norm on the residue field $k(\xi_x)$ whose restriction to k is the underlying non-Archimedean norm on k. Given an open set $U \subset X$, a regular function $f \in \mathcal{O}(X)$, and a non-negative real number a, open sets in X^{an} are generated by sets of the form

$$\{x = (\xi_x, |\cdot|_x) \in X^{\text{an}} \mid \xi_x \in U \text{ and } |f|_x \leq a\}.$$

The structure sheaf of analytic functions is given by the affinoid algebras on the *affinoid subdomains* of X^{an} – we refer the reader to [Ber93] for more details. For the purposes of this dissertation, we only need to consider the underlying topological structure of X^{an}. We give an alternate construction of the analytic space over the non-Archimedean field $\mathbb{C}(\!(t)\!)$ in Section I.3. We also explain the Berkovich affine line in Section I.2.1.

Berkovich spaces have found numerous applications in various areas of mathematics. For example:

- The étale cohomology of Berkovich spaces are used in the Langland's program to produce Galois representations over local fields [HT01].

- Thullier proved using Berkovich spaces that the homotopy type of the dual complex of a log resolution of a pair over a perfect field is independent of the chosen resolution [Thu07].

- Berkovich spaces are closely related to tropical spaces, and a Berkovich space can be seen as an inverse limit of tropical spaces [Pay09].

- Berkovich spaces over $(\mathbb{C}, |\cdot|_0)$ have applications to the field of K-stability. The various functionals from Kähler geometry have non-Archimedean analogues and these non-Archimedean functionals correspond to slope at infinity of these functionals along geodesic rays. For example, a version of Yau-Tian-Donaldson conjecture can be proven using these non-Archimedean functionals [BHJ19, BHJ17, BHJ22, BBJ21, BJ22].

- Berkovich spaces over $k((t))$ are closely related to degenerations of varieties over k. We expand on this in Section I.3.

I.2.1: Berkovich affine line

Let $(k, |\cdot|)$ be an algebraically closed non-Archimedean field. We describe the Berkovich affine line $\mathbb{A}_k^{1,\mathrm{an}}$. The points in $\mathbb{A}_k^{1,\mathrm{an}}$ correspond to multiplicative seminorms on the algebra $k[T]$, which restrict to the given norm on k.

Pick $a \in k$ and $r \in \mathbb{R}_{\geq 0}$. Then, associated to the disk $D(a, r) := \{x \in k \mid |x - a| \leq r\}$, we have a seminorm $|\cdot|_{D(a,r)}$ given by the following.

$$\left|\sum_i a_i T^i\right|_{D(a,r)} = \max_i |a_i| r^i.$$

These seminorms correspond to three different types of points in the Berkovich affine line:

- Type 1: When $r = 0$, these seminorms are given by $f(T) \mapsto |f(a)|$. These points correspond to the points of the rigid analytic line.

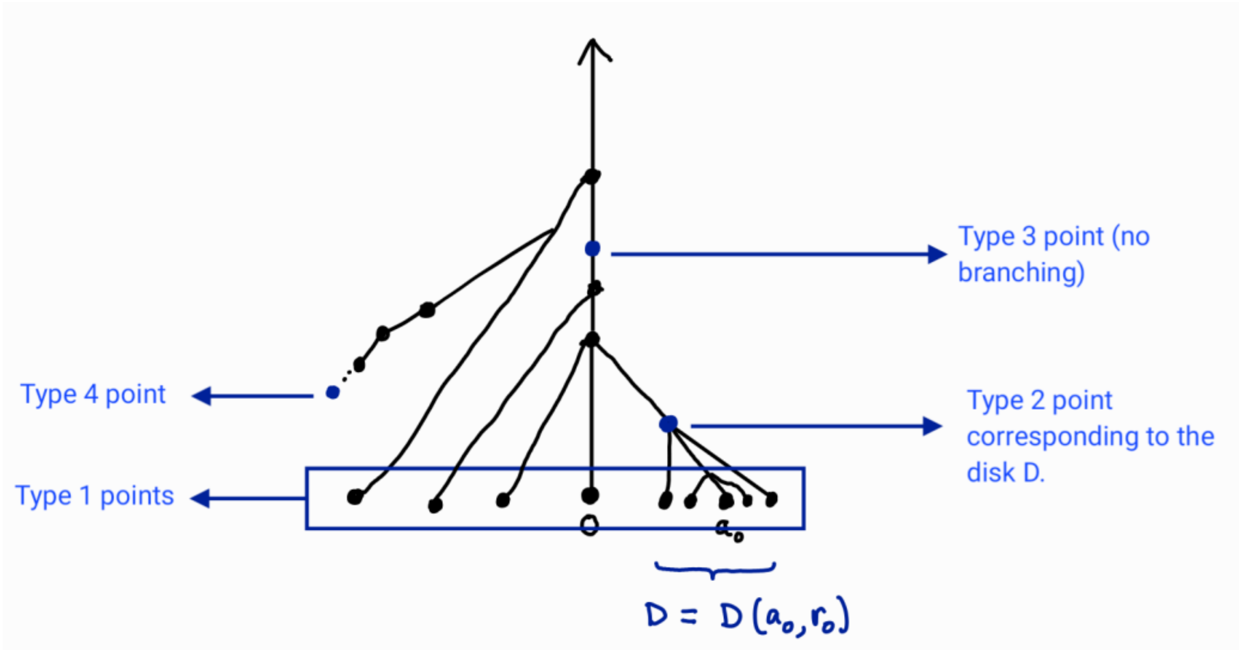


Figure I.1: Berkovich Affine Line

- Type 2: When r lies in the value group $\Gamma = \{|a| \mid a \in k^\times\} \subset \mathbb{R}_{>0}$, then these seminorms correspond to the points in the Berkovich affine line with branching.

- Type 3: When r does not lie in the value group, these seminorms correspond to points in the Berkovich affine line with no branching.

When k is *spherically complete*, these are all the types of points in the Berkovich affine line. If not, there are Type 4 points in the Berkovich space which occur as limits of points of Type 2 and 3. These points are given by

$$|f(T)| = \inf_{D(a_i,r_i)\in\mathcal{F}} |f(T)|_{D(a_i,r_i)},$$

where $\mathcal{F}$ is a family of nested disks $D(a_1, r_1) \supset D(a_2, r_2) \supset \dots$ with empty intersection. See Figure I.1.

As we see from the figure, the Berkovich affine line is a type of infinite graph [FJ04]. It has various graph subsets with vertices given by Type 2 points and edges being the line segments joining these Type 2 points. Similarly, in higher dimensions, there will be various polytope-like subsets of the Berkovich space. These "skeletal" subsets will play an important role in the later sections.

I.3: Berkovich analytification of a degeneration

Let $\mathbb{D} = \{t \in \mathbb{C} \mid |t| < 1\}$ denote the complex unit disk, $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$ and t denote a coordinate on $\mathbb{D}^*$. Consider a family of compact complex manifolds parametrized by $\mathbb{D}^*$ i.e. consider a complex manifold X with a proper smooth map $\pi : X \to \mathbb{D}^*$. Such a family is called a degeneration of complex manifolds. Degenerations play a central role in algebraic geometry, particularly in the study of moduli spaces.

We would like to think of the family X as a variety over $\mathbb{C}((t))$. To do this, we further assume that X is a projective family i.e. X is a closed subset of $\mathbb{P}^N \times \mathbb{D}^*$ cut out by homogeneous polynomials whose coefficients are functions that are holomorphic on $\mathbb{D}^*$ and meromorphic on $\mathbb{D}$. If we denote t as the coordinate on $\mathbb{D}^*$, then we can view these coefficients as elements of $\mathbb{C}((t))$. Then, we can think of X as a variety over $\mathbb{C}((t))$, which we denote as $X_{\mathbb{C}((t))}$. We denote the Berkovich analytification of $X_{\mathbb{C}((t))}$ as $X^{\mathrm{an}}_{\mathbb{C}((t))}$. We can understand the topology of $X^{\mathrm{an}}_{\mathbb{C}((t))}$ as an inverse limit of dual complexes of snc models – we explain this below.

A *model* of X is a normal complex analytic space $\mathscr{X}$ with a map $\pi : \mathscr{X} \to \mathbb{D}$ such that $\pi^{-1}(\mathbb{D}^*)$ is biholomorphic to X as spaces over $\mathbb{D}^*$. Note that existence of a model is guaranteed if we assume that X is a projective family. The *central fiber* $\mathscr{X}_0$ of a model $\mathscr{X}$ is the divisor given by $\pi^{-1}(0)$. A model $\mathscr{X}$ is said to *simple normal crossing (snc)* if $\mathscr{X}$ is regular and the central fiber is a simple normal crossing divisor in $\mathscr{X}$. Given two models $\mathscr{X}$ and $\mathscr{X}'$ of X, there is always a bimeromorphic map $\mathscr{X}' \dashrightarrow \mathscr{X}$ that commutes with the projection to $\mathbb{D}$. We say that $\mathscr{X}'$ dominates $\mathscr{X}$ if this bimeromorphic map is holomorphic. Given two snc models, we can always find a third one that dominates the two. Given a model $\mathscr{X}$, we can construct a new model dominating it by performing a *blow up* along a smooth subvariety contained in the central fiber.

Let $\mathscr{X}$ be an snc model and let $\mathscr{X}_0 = \sum_i b_i E_i$ be the central fiber, where E_i are the irreducible components of $\mathscr{X}_0$ and b_i are their respective multiplicities in $\mathscr{X}_0$. Then, the *dual complex* of $\mathscr{X}$, denoted $\Delta(\mathscr{X})$ is a cell complex with vertices v_{E_i} in bijection with irreducible components E_i. Each *stratum* i.e. an irreducible component Y of an intersection $E_{i_0} \cap \cdots \cap E_{i_m}$ corresponds to a face

$$\sigma_Y = \{(x_0, \ldots, x_m) \mid \sum_{k=0}^{m} b_{i_k} x_k = 1\}.$$

Two faces $\sigma_{Y_1} \subset \sigma_{Y_2}$ are glued if an only if $Y_2 \subset Y_1$, and the gluing map is given by setting the coordinates of the irreducible components not involved in defining Y_1 to 0.

Given two snc models $\mathscr{X}$ and $\mathscr{X}'$ of X such that $\mathscr{X}'$ dominates $\mathscr{X}$, there is a surjective map of cell complexes $r_{\mathscr{X}',\mathscr{X}} : \Delta(\mathscr{X}') \to \Delta(\mathscr{X})$ (see Section II.4.2 for more details). We explain this map in the case when $\mathscr{X}'$ is obtained by blowing up $\mathscr{X}$ along smooth subvarieties contained in the central fiber. If we write $\mathscr{X}_0 = \sum_i b_i E_i$, then $\mathscr{X}_0' = \sum_i b_i \widetilde{E_i} + b_{\mathrm{exc}} E_{\mathrm{exc}}$, where $\widetilde{E_i}$ is the strict transforms of E_i, E_{exc} is the exceptional divisor of the blowup and its multiplicity b_{exc} depends on the center of the blowup. Thus, we see that $\Delta(\mathscr{X}')$ contains a new vertex $v_{E_{\mathrm{exc}}}$ corresponding to E_{exc}. Now there are two cases:

- If the blowup is along a stratum Y of $E_{i_0} \cap \cdots \cap E_{i_m}$, then $\Delta(\mathscr{X}')$ is obtained from $\Delta(\mathscr{X})$ by subdividing the face σ_Y using the vertex $v_{E_{\mathrm{exc}}}$ and the map $r_{\mathscr{X}',\mathscr{X}}$ is in fact a homeomorphism in this case.

- If the blowup is along a subvariety Z which is not the irreducible component of the previous form, let Y be the smallest such intersection containing Z. Then, we obtain $\Delta(\mathscr{X}')$ by forming a cone over the face σ_Y. The apex of the cone corresponds to the new vertex $v_{E_{\mathrm{exc}}}$ and the map $r_{\mathscr{X}',\mathscr{X}}$ is obtained by collapsing the cone to σ_Y. The image of the cone point is determined by the multiplicities of the irreducible component defining Y.

For example, consider the dual complex of a model $\mathscr{X}$ whose central fiber is given by $\mathscr{X}_0 = E_0 + E_1 + E_2$, where the pairwise intersections $E_i \cap E_j$ and the intersection $E_0 \cap E_1 \cap E_2$

are non-empty and irreducible. Then, $\Delta(\mathscr{X})$ is a triangle an the intersection $E_0 \cap E_1 \cap E_2$ corresponds to a triangular face, pairwise intersections $E_i \cap E_j$ correspond to the boundary line segments, and the irreducible components correspond to the vertices. Let $Y = E_0 \cap E_1$. Then, blowing up $\mathscr{X}$ at Y corresponds to subdividing the line segment joining v_{E_0} and v_{E_1}. Blowing up $\mathscr{X}$ along a smooth center $Z \subsetneq Y$ not containing $E_0 \cap E_1 \cap E_2$ corresponds to adding a new triangle with vertices $v_{E_{\text{exc}}}, v_{E_0}$, and v_{E_1}. See Figure I.2.

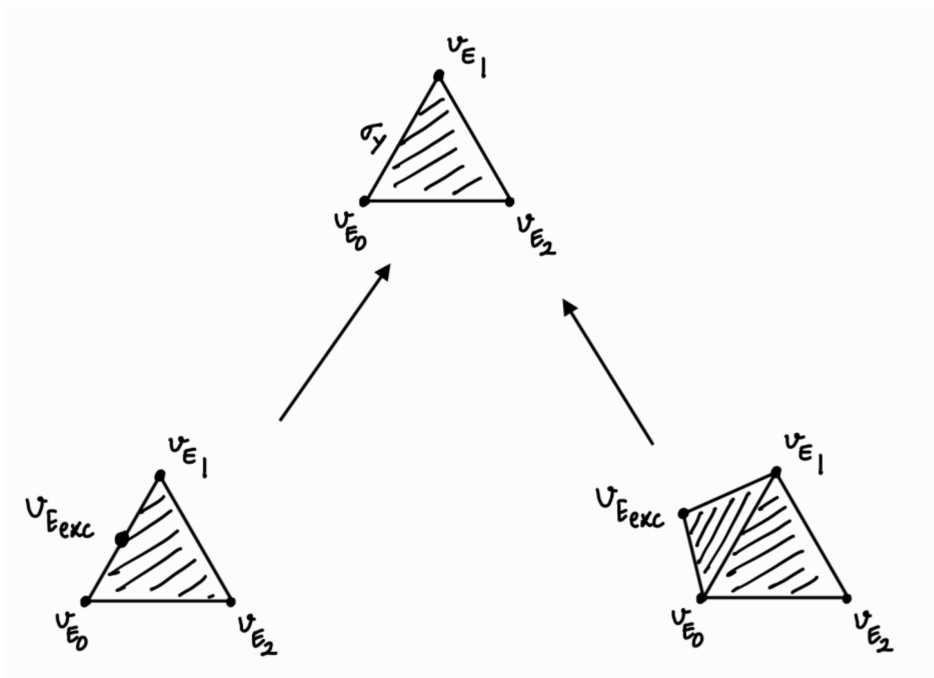


Figure I.2: (Top) Dual complex of a model $\mathscr{X}$ with central fiber given by $\mathscr{X}_0 = E_0 + E_1 + E_2$ where the pairwise intersections $E_i \cap E_j$ and the intersection $E_0 \cap E_1 \cap E_2$ are non-empty and irreducible. (Bottom left) Dual complex of the model obtained by blowing up $\mathscr{X}$ along $Y = E_0 \cap E_1$. (Bottom Right) Dual complex of the model obtained by blowing up $\mathscr{X}$ along a smooth center $E_0 \cap E_1 \cap E_2 \not\subset Z \subsetneq Y$.

We have the following result relating the dual complexes $\Delta(\mathscr{X})$ and the Berkovich analytification $X_{\mathbb{C}((t))}$.

Theorem I.2. There exits a homeomorphism of topological spaces

$$X^{\mathrm{an}}_{\mathbb{C}((t))} \simeq \varprojlim_{\mathscr{X}'} \Delta(\mathscr{X}')$$

where the projective limit is taken over all possible snc models of X.

I.4: Hybrid Space

The *hybrid space* is a topological space that is "mixture" of a complex analytic and a non-Archimedean space. As a set,

$$X^{\mathrm{hyb}} = X \sqcup X^{\mathrm{an}}$$

One of the ways to construct X^{hyb} is by first constructing a "smaller" hybrid space $\mathscr{X}^{\mathrm{hyb}}$ for an snc model $\mathscr{X}$ of X, which as a set is given by

$$\mathscr{X}^{\mathrm{hyb}} = X \sqcup \Delta(\mathscr{X}).$$

The topology on $\mathscr{X}^{\mathrm{hyb}}$ can be described locally by using a local "logarithm" map. Let Y be an irreducible component of $E_{i_0} \cap \cdots \cap E_{i_m}$ and let $U \subset \mathscr{X}$ be an open set intersecting Y such that

- $U \cap \mathscr{X}_0 = U \cap (\bigcup_{k=0}^{m} E_{i_k})$.
- U has coordinates $(z_0, \ldots, z_n)$ such that $\{z_k = 0\} = E_{i_k} \cap U$ for $k = 0, \ldots, m$, and
- the projection to $\mathbb{D}$ is given by $t = hz_0^{b_{i_0}} \ldots z_m^{b_{i_m}}$ where h is a bounded non-vanishing function on U.

Then, we define a local logarithm function $\mathrm{Log}_U : U \cap X \to \sigma_Y$ given by

$$\mathrm{Log}_U(z_0, \ldots, z_n) = \left(\frac{\log|z_0|}{\log|z_0^{b_{i_0}} \ldots z_m^{b_{i_m}}|}, \ldots, \frac{\log|z_m|}{\log|z_0^{b_{i_0}} \ldots z_m^{b_{i_m}}|} \right).$$

The topology of $\mathscr{X}^{\mathrm{hyb}}$ is defined is such as way as to ensure that the function $(U \cap X) \sqcup \sigma_Y \to \sigma_Y$ given by $\mathrm{Log}_U \sqcup \mathrm{Id}_{\sigma_Y}$ is continuous. We can the define $X^{\mathrm{hyb}} := \varprojlim_{\mathscr{X}} \mathscr{X}^{\mathrm{hyb}}$. For more details, see Section II.2.

Berkovich introduced a variant of the hybrid space in [Ber09] as the analytification of X over a certain Banach ring. Boucksom and Jonsson first introduced this version of hybrid space in [BJ17] inspired by the work of Morgan and Shalen [MS84].

A few applications of the hybrid spaces include

- Berkovich used a variant of the hybrid space to describe the weight zero subspace of the mixed Hodge structure of a degeneration $X \to \mathbb{D}^*$ [Ber09].

- Boucksom and Jonsson introduced the hybrid space in [BJ17] and they used it to study the limits of Calabi-Yau measures. Pille-Schneider used the hybrid spaces to study the limits of Kähler-Einstein volume forms [PS22]. In this dissertation, we study the limits of log Calabi-Yau measures and of Bergman measures on Riemann surfaces.

- Kontsevich and Soibelman gave insight into the possible use of Berkovich spaces to study the Strominger-Yau-Zaslow (SYZ) conjecture in mirror symmetry [KS06]. They conjectured that the Gromov-Hausdorff limit of a maximally degenerate Calabi-Yau family is a skeletal subset of the non-Archimedean space. The Berkovich hybrid spaces have found applications in the construction of SYZ fibrations and in understanding the Gromov-Hausdorff limits of degenerations [Oda18, Sus18, Li22].

- The hybrid spaces have found applications in both complex and arithmetic dynamics [Fav17, DKY20].

- Chambert-Loir and Ducros have developed a theory of differential forms over Berkovich spaces [CLD12]. Ducros, Hrushovski and Loeser showed that non-Archimedean integrals arise as asymptotics of complex integrals [DHL23]. While their approach does not directly use hybrid spaces, their techniques can be used to show such a convergence using a hybrid space.

I.5: Convergence of measures

Now also suppose that for each $t \in \mathbb{D}^*$, there is a natural measures μ_t on X_t. We would like to understand the limit of these measures μ_t as $t \to 0$. To do this, we consider μ_t as a measure on X^{hyb} (using the pushforward of μ_t along the inclusion $X_t \hookrightarrow X^{\mathrm{hyb}}$). By considering the weak limit of the measures μ_t on the space X^{hyb}, we can make precise the notion of convergence of the measures μ_t to a measure that lives on the Berkovich space $X^{\mathrm{an}}_{\mathbb{C}((t))}$. We consider two such families of measures in this dissertation

- We extend Boucksom and Jonsson's result regarding limits of Calabi-Yau measures to the case of log Calabi-Yau measures.
- We prove that Bergman measures on a family of Riemann surfaces converges to the Zhang measure.

We describe these two results in the following two subsections.

I.5.1: Log Calabi-Yau measures [1]

Let Y be an irreducible, normal and compact complex analytic space. Let η be a top-dimensional meromorphic form on the smooth locus, Y^{reg} of Y, and let $D \subset Y$ be a (possibly not reduced or not effective) divisor such that η is holomorphic and does not vanish on $Y^{\mathrm{reg}} \setminus D$, and has poles (and zeroes) given exactly by D. Then the pair (Y, D) is called log Calabi-Yau. Any two such forms η and η' on Y^{reg} which have poles given by D will be equal up to a scalar factor. Let $|D|$ denote the support of D. The form η gives rise to a volume form $|\eta \wedge \overline{\eta}| = i^{(\dim Y)^2} \eta \wedge \overline{\eta}$ on $Y^{\mathrm{reg}} \setminus |D|$, and thus a positive Radon measure on $Y^{\mathrm{reg}} \setminus |D|$. For a log Calabi-Yau pair (Y, D), this measure is unique up to scaling. Note that locally near $|D|$ and Y^{sing}, it is possible for the mass of this measure to be infinite. When $D = 0$ and Y is smooth, Y is said to be Calabi-Yau. More generally, we get such a canonical measure if we assume that $K_Y + D$ is $\mathbb{Q}$-Cartier and $K_Y + D \sim_{\mathbb{Q}} 0$. See Section II.3 for details.

[1] This section has been largely reproduced from [Shi22, Section 1] with the publisher's permission.

Let $X \to \mathbb{D}^*$ be a proper flat family of irreducible normal complex analytic spaces. Let B be a horizontal $\mathbb{Q}$-Weil divisor on X such that $K_{X/\mathbb{D}^*} + B$ is $\mathbb{Q}$-Cartier and is $\mathbb{Q}$-linearly equivalent to 0. We don't need to assume that B is effective. Then, (X_t, B_t) is log Calabi-Yau, where $B_t := B|_{X_t}$. As above, using a trivializing section η of $\mathcal{O}_X(m(K_{X/\mathbb{D}^*} + B))$ for some sufficiently divisible integer m, we can obtain Radon measures μ_t on each of the fibers $X_t^{\mathrm{reg}} \setminus |B_t|$ for $|t| \ll 1$ (see Section II.3 for more details on how to handle the $\mathbb{Q}$-divisor case). This measure μ_t remains unchanged if we replace m by $m_1 m$ and η by $\eta^{\otimes m_1}$. Two such families μ_t and μ'_t obtained by picking $\eta, \eta' \in H^0(X, m(K_{X/\mathbb{D}^*} + B))$ differ by a factor of $|h(t)|^{2/m}$, where h is a holomorphic function on $\mathbb{D}^*$. Our goal is to understand if the measures μ_t converge as $t \to 0$.

We also assume that (X, B) is projective i.e. X is a closed subset of $\mathbb{P}^N \times \mathbb{D}^*$ for some $N \in \mathbb{N}$ and X and B are cut out by homogeneous polynomials whose coefficients are holomorphic functions on $\mathbb{D}^*$ and meromorphic on $\mathbb{D}$. This guarantees that there exists a proper flat family $\mathscr{X}$ over $\mathbb{D}$ with $\mathscr{X}$ normal and $\mathscr{X}|_{\mathbb{D}^*} \simeq X$, and a $\mathbb{Q}$-Cartier divisor $\mathscr{D}$ on $\mathscr{X}$ extending $K_{X/\mathbb{D}^*} + B$ such that $\mathscr{D} \sim_{\mathbb{Q}} 0$. (Such an $\mathscr{X}$ is called a model of X).

Secondly, we assume that there exists a section ψ of $\mathcal{O}_{\mathscr{X}}(m\mathscr{D})$ which extends the section η of $m(K_{X/\mathbb{D}^*} + B)$. In this case, we say that η admits a meromorphic extension. Recall that two families of measures μ_t and μ'_t obtained by picking two trivializing sections $\eta, \eta' \in H^0(X, m(K_{X/\mathbb{D}^*} + B))$ differ by a factor of $|h(t)|^{2/m}$, where h is a holomorphic function on $\mathbb{D}^*$. If we also assume that η and η' admit meromorphic extensions, then we further get that h is meromorphic at 0, and thus $h(t) \sim Ct^{\alpha}$ as $t \to 0$ for some $\alpha \in \mathbb{Z}$.

In a manner similar to the Boucksom-Jonsson hybrid space, we construct a locally compact hybrid topological space $(X, |B|)^{\mathrm{hyb}}$, which as a set is a disjoint union of $X^{\mathrm{reg}} \setminus |B|$ and $(X^{\mathrm{reg}}_{\mathbb{C}((t))})^{\mathrm{an}} \setminus |B|^{\mathrm{an}}_{\mathbb{C}((t))}$ (see Section II.2 for more details). We have the following convergence theorem for the measures μ_t on $(X, |B|)^{\mathrm{hyb}}$.

Theorem A. Suppose (X, B) is a projective log Calabi-Yau pair over $\mathbb{D}^*$ and let $\eta \in H^0(X, m(K_{X/\mathbb{D}^*} + B))$ be a generating section that admits a meromorphic extension. Let

μ_t be the Radon measure on $X^{\mathrm{reg}} \setminus |B_t|$ defined by $i^{(\dim(X_t))^2}(\eta|_{X_t} \wedge \overline{\eta|_{X_t}})^{1/m}$. In addition, assume that the pair (X, B) is sub log canonical in the sense of the minimal model program. Then, there exists a non-zero measure μ_0 on $(X^{\mathrm{reg}}_{\mathbb{C}((t))})^{\mathrm{an}} \setminus |B|^{\mathrm{an}}_{\mathbb{C}((t))}$ and constants $d \in \mathbb{N}$ and $\kappa \in \mathbb{Q}$ such that the measures $\frac{\mu_t}{|t|^{2\kappa}(2\pi \log|t|^{-1})^d}$ converge weakly to μ_0, when viewed measures on $(X, |B|)^{\mathrm{hyb}}$.

In the above theorem, sub log canonical (sometimes just called log canonical in the literature) is in the sense of the minimal model program i.e. $\mathrm{discrep}(X, B) \geq -1$ (see [KM98, Section 2.3]) and we don't assume that B is effective.

The measure μ_0 is easy to describe when (X, B) is log-smooth i.e. when X is smooth and B has snc support. In this case, the support of μ_0 is the locus where a certain *weight function* associated to (X, B, η), constructed in [MN15] and [BM19], is minimized (see also [Tem16]). The minimizing locus of the weight function is called the *essential skeleton* in the literature, and thus we have that the measure μ_0 is the Lebesgue measure on the top-dimensional faces of the essential skeleton of the triple (X, B, η). In general, the support of μ_0 is the image of a skeleton under a birational map $(X', B') \to (X, B)$.

If the pair (X, B) is not sub log canonical, then there is no reasonable convergence in this non-Archimedean setting (see Example II.9). This is consistent with the observation that the essential skeleton of (X, B, η) is empty when (X, B) is not sub log canonical.

As an example of Theorem A, we get a convergence result for a torus $T = (\mathbb{C}^*)^n$. We have a canonical embedding $\mathbb{R}^n \hookrightarrow T^{\mathrm{an}}_{\mathbb{C}((t))}$ given by sending $r \in \mathbb{R}^n$ to the valuation $\sum_{m \in \mathbb{Z}^n} a_m z^m \mapsto \max_m\{|a_m| e^{-\langle r, m\rangle}\}$. Consider the constant family $T \times \mathbb{D}^*$ and the associated hybrid space $(T \times \mathbb{D}^*) \cup T^{\mathrm{an}}_{\mathbb{C}((t))}$. Then by applying Theorem A to a smooth projective toric compactification of T we get that as $t \to 0$, the Haar measure on $T \times \{t\}$ scaled by a factor of $\frac{1}{(2\pi \log|t|^{-1})^n}$ converges weakly to the Lebesgue measure on $\mathbb{R}^n$. See Examples II.15 and II.28 for more details.

It is enough to prove Theorem A in the case when the pair (X, B) is log-smooth i.e. when X is regular and $B \subset X$ is an snc divisor. Indeed, we can then prove Theorem A for a general pair (X,B), by taking a log resolution $(X',B')\to(X,B)$, and using Theorem A for (X',B') (see Section II.4.5 for details). So, for the remainder of this section, we will assume that the pair (X,B) is log-smooth.

We employ an approach similar to [BJ17]. To prove Theorem A, we first prove Theorem B below, which shows the convergence on certain skeletal subsets of $X^{\mathrm{an}}_{\mathbb{C}((t))}\setminus|B|^{\mathrm{an}}_{\mathbb{C}((t))}$. Since our measures are no longer finite, we would have to allow for the limit measures to be infinite and this would not be possible if we use Lebesgue measure on a compact simplicial complex. The solution is to allow our simplices to have unbounded faces. Pick a model $\mathscr{X}$ such that $\mathscr{X}_0+\overline{B}$ is an snc divisor, where $\overline{B}$ denotes the component-wise closure of B in $\mathscr{X}$. A good candidate for this is $\Delta(\mathscr{X},|B|)$, the dual intersection complex of a pair, introduced in [Tyo12] [BPR13] [BPR16] in the one-dimensional case and in [GRW16] [BM19] for higher dimensions.

We briefly explain the construction of $\Delta(\mathscr{X},|B|)$ here. Let E_i denote the irreducible components of $\mathscr{X}_0$ and let $\mathscr{X}_0=\sum_{i=1}^N b_iE_i$. A connected component Y of $\left(\bigcap_{i\in I}E_i\right)\cap\left(\bigcap_{j\in J}\overline{B}_j\right)$ is called a *stratum* where $\{E_i\mid i\in I\}$ denotes a non-empty collection of irreducible components of $\mathscr{X}_0$ and $\{B_j\mid j\in J\}$ denotes a possibly empty collection of irreducible components of the support of B. Associated to every such stratum Y is a face $\sigma_Y=\{(\underline{r},\underline{s})\in\mathbb{R}^{|I|+|J|}_{\geq0}\mid\sum_{i\in I}b_ir_i=1\}$ of $\Delta(\mathscr{X},|B|)$. These faces are then glued together via some attaching maps to get $\Delta(\mathscr{X},|B|)$. See Section II.1 for more details.

Associated to such a model $\mathscr{X}$, we construct a hybrid space $(\mathscr{X},|B|)^{\mathrm{hyb}}=(X\setminus|B|)\cup\Delta(\mathscr{X},|B|)$, where the topology is given by logarithmic rate of convergence. We prove the following convergence theorem on this hybrid space.

Theorem B. Let $X\to\mathbb{D}^*$ be a holomorphic family of proper complex manifolds. Let B be an snc $\mathbb{Q}$-divisor such that $K_{X/\mathbb{D}^*}+B\sim_{\mathbb{Q}}0$ and the pair (X,B) is sub log canonical. Let $\mathscr{X}$ be a regular model of X such that $\mathscr{X}_0+\overline{B}$ is snc and $\mathscr{D}$ be an snc divisor on $\mathscr{X}$ extending $K_{X/\mathbb{D}^*}+B$ such that $\mathscr{D}\sim_{\mathbb{Q}}0$. Let $\psi\in H^0(\mathscr{X},m\mathscr{D})$ be a generating section for sufficiently divisible m and $\mu_t=i^{(\dim X_t)^2}(\psi|_{X_t}\wedge\overline{\psi|_{X_t}})^{1/m}$ be the Radon measure induced on

$X_t \setminus |B_t|$ by ψ. Then, there exists a non-zero measure μ_0 supported on a subcomplex $\Delta(\mathscr{D})$ of $\Delta(\mathscr{X}, |B|)$ and explicit constants $d \in \mathbb{N}$, $\kappa_{\min} \in \mathbb{Q}$ such that

$$\frac{\mu_t}{|t|^{2\kappa_{\min}}(2\pi\log|t|^{-1})^d} \to \mu_0$$

converges weakly as measures on $(\mathscr{X}, |B|)^{\text{hyb}}$.

On each top-dimensional faces of $\Delta(\mathscr{D})$, μ_0 is a suitably normalized Lebesgue measure while on all other faces of $\Delta(\mathscr{X}, |B|)$, μ_0 is zero. For a precise description of μ_0, see Section II.3.

Note that for Theorem B, we don't need to assume that (X, B) is projective. The projectivity assumption is needed in Theorem A in order to define $X^{\text{an}}_{\mathbb{C}((t))}$ and $|B|^{\text{an}}_{\mathbb{C}((t))}$. In this case, we can view $\Delta(\mathscr{X})$ and $\Delta(\mathscr{X}, |B|)$ as subsets of the Berkovich analytification, $X^{\text{an}}_{\mathbb{C}((t))}$. Moreover, $\Delta(\mathscr{X})$ is a strong deformation retract of $X^{\text{an}}_{\mathbb{C}((t))}$.

In the case when $B = 0$, Theorem A follows from Theorem B by using the following result. The collection of $\Delta(\mathscr{X})$ for all snc models $\mathscr{X}$ is a directed system and $X^{\text{an}}_{\mathbb{C}((t))} \simeq \varprojlim_{\mathscr{X}} \Delta(\mathscr{X})$ (See [KS06, Theorem 10], [BFJ16, Corollary 3.2]).

We prove a similar result (see Theorem II.16) that

Theorem C. Let (X, B) be a log-smooth pair. Then there exists a canonical homeomorphism.

$$X^{\text{an}}_{\mathbb{C}((t))} \setminus |B|^{\text{an}}_{\mathbb{C}((t))} \simeq \varprojlim_{\mathscr{X}} \Delta(\mathscr{X}, |B|).$$

Theorem B and C together prove Theorem A.

As a corollary of Theorem C, we also get the following result. Suppose that $U \to \mathbb{D}^*$ is a smooth meromorphic family of quasi-projective complex manifolds. Consider an snc compactification of $U \subset X$ with snc boundary divisor B. Then, $U^{\text{hyb}} = (X, |B|)^{\text{hyb}} = \varprojlim_{\mathscr{X}} (\mathscr{X}, |B|)^{\text{hyb}}$ is independent of the choice of (X, B). The space U^{hyb} can also be seen as the Berkovich analytification of U with respect to a suitable Banach ring [Ber09] [BJ17, Appendix]. The hybrid spaces $(\mathscr{X}, |B|)^{\mathrm{hyb}}$ can be thought of as approximations of U^{hyb} and we hope that they will be useful in future applications.

I.5.2: Bergman and Zhang measures

Any compact Riemann surface Y of genus $g \geq 1$ carries a canonical measure called the *Bergman measure*, defined as follows. Note that there is a positive definite Hermitian metric on $H^0(Y, \Omega_Y)$, the g-dimensional complex vector space of holomorphic 1-forms on Y, given by

$$\langle \phi, \psi \rangle = \frac{i}{2} \int_Y \phi \wedge \overline{\psi}.$$

Pick an orthonormal basis $\phi_1, \ldots, \phi_g$ with respect to the above pairing. Then, the positive $(1,1)$-form defined by $\frac{i}{2} \sum_{i=1}^{g} \phi_i \wedge \overline{\phi_i}$ does not depend on the choice of the orthonormal basis and the associated measure on Y is called as the Bergman measure on Y. We can also define the Bergman measure on Y using the pullback of the flat metric from the Jacobian of Y along the Abel-Jacobi map.

The Bergman measure has many applications. For example, the variation of the Bergman measure gives rise to a metric on the Teichmüller space of genus g curves for $g \geq 2$ that is invariant under the action of the mapping class group [HJ98].

Let X be a complex surface with a holomorphic submersion $X \to \mathbb{D}^*$ with fibers being compact complex curves of genus at least 1. For $t \in \mathbb{D}^*$, let μ_t denote the Bergman measure on the fiber X_t. We would like to understand the convergence of the measures μ_t on the hybrid space X^{hyb} as $t \to 0$. Since we are working in the case of dimension 1, we call the dual complex of an snc model $\mathscr{X}$ as the *dual graph* and denote it as $\Gamma_{\mathscr{X}}$. The associated Berkovich space $X^{\mathrm{an}}_{\mathbb{C}((t))}$ is now an inverse limit of graphs.

The *Zhang measure* on a metric graph is a weighted sum of Lebesgue measures on edges and Dirac masses on vertices. It was introduced by Zhang in [Zha93] to define a non-Archimedean analogue of the Bergman pairing on a Riemann surface. The Zhang measure has been used in the study of potential theory on the Berkovich projective line [BR10]. The weight of the Zhang measure on an edge is a function involving the length of the edge and the resistance across the endpoints after removing the the edge from the graph. The weight of the Zhang measure on a vertex is the genus of the irreducible component associated to it.

The Zhang measures on the dual graphs of all normal crossing models of X are compatible and thus give rise to a measure on $X^{\mathrm{an}}_{\mathbb{C}((t))}$.

There are several reasons to believe that the Zhang measure is the non-Archimedean analogue of the Bergman measure. Firstly, the Weierstrass points on a Riemann surface are equidistributed with respect to the Bergman measure [Nee84]. It is possible to define Weierstrass points on a Berkovich curve or on a metric graph and it turns out that they are equidistributed with respect to the Zhang measure [Ami14],[Ric18]. Secondly, recall that the Bergman measure can be obtained as a pullback of the flat metric from the Jacobian under the Abel-Jacobi map. Similarly, the Zhang measure can be realized as the pullback of a certain canonical metric on the tropical Jacobian under the tropical Abel-Jacobi map [BF11]. Thirdly, a version of Kazhdan's theorem for the Bergman measure on a Riemann surface is true for the Zhang measure on a metric graph [SW19].

Indeed, it is a well-known conjecture that the Bergman measure converges to the Zhang measure in the hybrid space setting. See [SW19, Section 1.1] for an explicit statement. This conjecture was communicated to us by M. Jonsson, who was informed of it by M. Baker. We give a positive answer to this conjecture.

Theorem D. The Bergman measure μ_t on the fiber X_t converges weakly to a measure μ_0 on the Berkovich space $X^{\mathrm{an}}_{\mathbb{C}((t))}$, where the convergence takes place on the hybrid space X^{hyb}. The measure μ_0 is supported on a subspace of $X^{\mathrm{an}}_{\mathbb{C}((t))}$ that is isomorphic to a metric graph, and is a weighted sum of Lebesgue measures on edges and Dirac masses on points.

Moreover, if we assume that X has a semistable model, then μ_0 is the Zhang measure on the Berkovich space $X^{\mathrm{an}}_{\mathbb{C}((t))}$

In the above theorem, the existence of a semistable model is asking for a normal crossing model $\mathscr{X}$ of X such that $\mathscr{X}_0$ is reduced. Such a model always exists after performing a finite base change $\mathbb{D}^* \to \mathbb{D}^*$ given by $t \mapsto t^n$.

Amini and Nicolussi have been able to obtain a version of Theorem D using techniques from Hodge theory [AN20].

A key step involved in the proof of Theorem D is to prove the convergence on the "smaller" hybrid space $\mathscr{X}^{\text{hyb}} = X \sqcup \Gamma_{\mathscr{X}}$, associated to a fixed normal crossing model $\mathscr{X}$ of X. See Section III.3 for details on the topology of the space $\mathscr{X}^{\text{hyb}}$.

Theorem D′. Suppose that X has semistable reduction and let $\mathscr{X}$ be a normal crossing model of X. On the space $\mathscr{X}^{\text{hyb}}$, the measures μ_t converge weakly to the Zhang measure on $\Gamma_{\mathscr{X}}$.

We are also able to prove a convergence statement on a hybrid space which has the *metrized curve complex* in the sense of Amini and Baker [AB15] as the central fiber. The metrized curve complex associated to a normal crossing model $\mathscr{X}$ of X is a topological space obtained by replacing each nodal point in $\mathscr{X}_0$ by a line segment. We get $\mathscr{X}_0$ from the associated metrized curve complex by collapsing the line segments. We also get the dual graph $\Gamma_{\mathscr{X}}$ by collapsing the Riemann surfaces in the metrized curve complex to points. We construct a hybrid space $\mathscr{X}^{\text{hyb}}_{\text{CC}}$ which is a partial compactification of X with the central fiber the metrized curve complex associated to $\mathscr{X}$.

Theorem E. Assume that X has semistable reduction and let $\mathscr{X}$ be a normal crossing model of X. Then, there exists a measure μ_{CC} on the metrized curve complex associated to $\mathscr{X}$ such that μ_t converges weakly to μ_{CC} as $t \to 0$, when seen as measures on $\mathscr{X}^{\text{hyb}}_{\text{CC}}$.

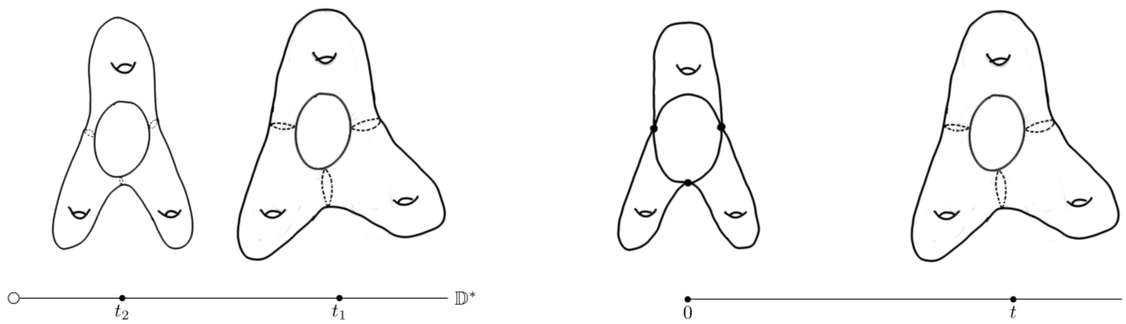
The measure μ_{CC} restricted to each Riemann surface of positive genus in the metrized curve complex is exactly the Bergman measure on that Riemann surface. The measure μ_{CC} places no mass on any genus zero Riemann surface in the metrized curve complex. The restriction of μ_{CC} on an edge is exactly the Zhang measure restricted to the edge. This shows us that the Dirac masses that show up in the Zhang measure correspond to collapsed Bergman measures.

Theorem D′ is closely related to [dJ19, Remark 16.4]. The main difference between the two results is that [dJ19] does not involve any Berkovich spaces and the limiting measure lives on the singular curve $\mathscr{X}_0$ while in our case the limiting measure is on the metric graph $\Gamma_{\mathscr{X}}$. Another difference is that de Jong's result only applies to semistable models of X while we also deal with the case when the central fiber is not necessarily reduced. The limiting measure in [dJ19, Remark 16.4] is the sum of the Bergman measures on the normalization of positive genus irreducible components of $\mathscr{X}_0$ and some Dirac masses on nodal points. The mass at a nodal point is equal to the total mass of the corresponding edge in the Zhang measure. Theorem E serves as a concrete link between the two: the pushforward of μ_{CC} to $\mathscr{X}_0$ gives the limiting measure in [dJ19, Remark 16.4] while its pushforward to $\Gamma_{\mathscr{X}}$ gives the Zhang measure. So we recover both Theorem D′ and [dJ19, Remark 16.4] from Theorem E by considering the continuous maps $\mathscr{X}^{\mathrm{hyb}}_{\mathrm{CC}} \to \mathscr{X}^{\mathrm{hyb}}$ and $\mathscr{X}^{\mathrm{hyb}}_{\mathrm{CC}} \to \mathscr{X}$.

To prove Theorem D using Theorem D′, we just need to show that the convergence given by Theorem D′ for different models are compatible i.e. if $\mathscr{X}, \mathscr{X}'$ are models of X such that we have a proper map $\mathscr{X}' \to \mathscr{X}$ which restricts to identity on X, then the limiting measures seen as measures on $\Gamma_{\mathscr{X}'}$ using $\Gamma_{\mathscr{X}} \hookrightarrow \Gamma_{\mathscr{X}'}$ are the same. Now, using the fact that $X^{\mathrm{hyb}} = \varprojlim_{\mathscr{X}} \mathscr{X}^{\mathrm{hyb}}$, we get Theorem D in the case when X has semistable reduction. Since a semistable reduction always exists after a base change, to prove Theorem D in general, we only need to understand what happens after a base change.

To prove Theorem D′ on $\mathscr{X}^{\mathrm{hyb}}$ for a normal crossing model $\mathscr{X}$ of X, we make a careful choice of elements of $H^0(\mathscr{X}, \Omega_{\mathscr{X}/\mathbb{D}})$ that restrict to a basis of $H^0(X_t, \Omega_{X_t})$ for all t and also to good basis of $H^0(\mathscr{X}_{0,\mathrm{red}}, \omega_{\mathscr{X}_{0,\mathrm{red}}})$. We also work with $\mathscr{X}_{0,\mathrm{red}}$ instead of $\mathscr{X}_0$ because the dualizing sheaf, $\omega_{\mathscr{X}_{0,\mathrm{red}}}$, is better behaved. We express the Bergman measure in terms of this basis and compute some asymptotics. Our analysis strongly uses the analogy between one-forms on Riemann surfaces and on metric graphs.

To prove Theorem E, we first construct the metrized curve complex hybrid space $\mathscr{X}^{\mathrm{hyb}}_{\mathrm{CC}}$ for a normal crossing model $\mathscr{X}$ of X. We then analyze the convergence in a small enough



(a) A family of genus 4 curves obtained by pinching the dotted lines.

(b) The minimal normal crossing model of the family

Figure I.3: A family of genus 4 curves and its minimal normal crossing model.

neighborhood of each point in the central fiber. For non-nodal points that lie on an irreducible component of $\mathscr{X}_0$ or points in the interior of a line segment, this computation is a minor modification of the computations done to prove Theorem D′. So, we only need to study the convergence in a neighborhood of a point that is the intersection of an irreducible component of $\mathscr{X}_0$ and a line segment. The proof of this part uses the same kind of analysis, just a more careful one.

A major difference between the results of [BJ17] and this paper is that the limiting measure in [BJ17] is either always Lebesgue or always atomic, but never a sum of both. For $g = 1$, Theorem D recovers the one-dimensional case of the convergence theorem in [BJ17]. See also [CLT10, Corollary 4.8] for a related statement.

We would also like to point out that some of the asymptotics that we use to prove Theorem D′ are similar to the ones used by de Jong to prove [dJ19, Remark 16.4]. For example, compare Lemma III.9 and [dJ19, Equation (16.7)]. De Jong's asymptotics are more versatile as they involve families $\mathscr{X} \to \mathbb{D}^m$ and are proved using the theory of variation of mixed Hodge structures. We don't use any variation of mixed Hodge structures and prove these asymptotics for $m = 1$ by explicit computations.

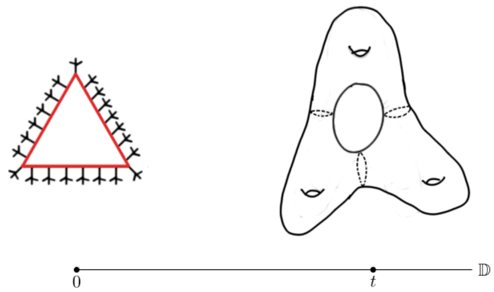


Figure I.4: The hybrid space associated to the family in Figure I.3a. The support of the Zhang measure is shown in red.

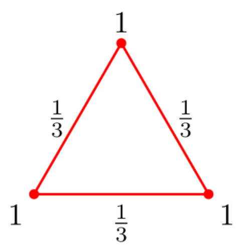


Figure I.5: The Zhang measure on the Berkovich space associated to the family in Figure I.3a

I.5.3: An example

Let $X \to \mathbb{D}^*$ be a family of compact genus 4 Riemann surfaces given by pinching the dotted simple closed curves in Figure I.3a. Then, the central fiber of the minimal normal crossing model, $\mathscr{X}$, has three irreducible components each of genus one intersecting at 3 nodal points (see Figure I.3b). The associated hybrid space is shown in Figure I.4.

In this case, the dual graph, $\Gamma_{\mathscr{X}}$, is a triangle with all three vertices of genus one. The Zhang measure is a sum of a Lebesgue measure on each of edge of mass $\frac{1}{3}$ and a Dirac mass on each vertex of mass 1. The central fiber of the hybrid space has a subspace homeomorphic to $\Gamma_{\mathscr{X}}$.

The curve complex hybrid space associated to the minimal normal crossing model is shown in Figure I.6. The measure μ_{CC} on the metrized curve complex in the sum of the Bergman (Haar) measures on each of the genus 1 curves and Lebesgue measure of mass $\frac{1}{3}$ on each of the edges.

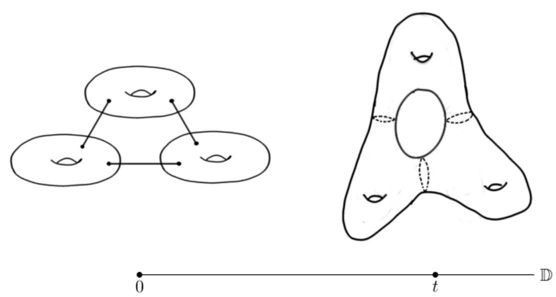


Figure I.6: The curve complex hybrid space associated to the family in Figure I.3b.

CHAPTER II

Convergence of Log Calabi-Yau Measures [1]

Structure of the chapter

The chapter is structured as follows. In Section II.1, we recall the construction of the dual complex $\Delta(\mathscr{X}, |B|)$ associated to an snc model $\mathscr{X}$ of a log smooth pair (X, B) and in Section II.2, we construct the hybrid space $(\mathscr{X}, |B|)^{\text{hyb}}$, associated to a model $\mathscr{X}$. In Section II.3, we prove Theorem B. In Section II.4, we construct the space $(X, |B|)^{\text{hyb}}$, realize it and its the central fiber as a non-Archimedean space and prove Theorem A.

Notations and Conventions

We use $\mathbb{D}$ to denote the (open) unit complex disc, $\{t \in \mathbb{C} \mid |t| < 1\}$, and $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$.

We will use X, Y etc to denote families of complex analytic spaces parametrized by $\mathbb{D}^*$ and use $\mathscr{X}, \mathscr{Y}$ etc to denote extensions of these families to $\mathbb{D}$. We will use B to denote horizontal divisors in X and $\overline{B}$ will denote its component-wise closure in $\mathscr{X}$. We will denote the irreducible components of $\mathscr{X}_0$ by E_i's and their multiplicities by b_i's. We will denote the irreducible components of the support of B as B_j's and their multiplicities by β_j's. The support of a divisor D will be denoted by $|D|$.

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II.1: The dual simplicial complex associated to an snc model

In this section, we recall the definition of a model and the construction of the dual intersection complex associated to an snc model of a log-smooth pair (X, B). Let X be a holomorphic flat family of compact complex manifolds parametrized by $\mathbb{D}^*$ i.e. X is a smooth complex manifold with a proper smooth map $X \to \mathbb{D}^*$. Let B be a horizontal snc $\mathbb{Q}$-divisor in X. Write $B = \sum_j \beta_j B_j$, where $\beta_j \in \mathbb{Q}$ and B_j are prime divisors. A *model* $\mathscr{X}$ of X is a a normal complex analytic space $\mathscr{X}$ which is proper and flat over $\mathbb{D}$ such that $\mathscr{X}|_{\mathbb{D}^*} = X$. Let $\mathscr{X}_0$ denote the *central fiber* of $\mathscr{X}$ i.e. the fiber over $0 \in \mathbb{D}$.

We say that (X, B) has *removable singularities* at the origin if there exists a model $\mathscr{X}$ of X such that the topological closure of $|B|$ in $\mathscr{X}$ is a divisor in $\mathscr{X}$. Throughout this chapter, we will assume that all pairs have removable singularities at the origin. This is automatic if we assume that (X, B) is projective.

In this section, we don't need to assume that (X, B) is projective or that the pair (X, B) is log Calabi-Yau.

II.1.1: Snc models of (X, B)

We say that a model $\mathscr{X}$ of X is an *snc model* of (X, B) if $\mathscr{X}$ is regular and $(\mathscr{X}_0 + \overline{B})_{\text{red}}$ is an snc divisor in $\mathscr{X}$, where $\overline{B} = \sum_j \beta_j \overline{B}_j$ denotes the component-wise closure of B in $\mathscr{X}$. Let $\mathscr{X}_0 = \sum_i b_i E_i$, where E_i are the irreducible components of the central fiber.

Since (X, B) has removable singularities at the origin, using Hironaka's theorem, we can always find an snc model of (X, B). Thus, the existence of an snc model of (X, B) is equivalent to (X, B) having removable singularities at the origin.

Given an snc model $\mathscr{X}$ of (X, B), we can obtain new snc models of (X, B) by blowing up $\mathscr{X}$ at any smooth subvariety of $\mathscr{X}_0$.

Example II.1. If $X = \mathbb{P}^1 \times \mathbb{D}^*$, then $\mathscr{X} = \mathbb{P}^1 \times \mathbb{D}$ is an snc model of X. The blowup of $\mathscr{X}$ at the point $(0, 0)$ is also an snc model of X.

II.1.2: The dual complex

Let $\mathscr{X}$ be an snc model of (X, B). Let $E_{i_0}, \ldots, E_{i_p}$ for $p \geq 0$ denote a non-empty collection of irreducible components of $\mathscr{X}_0$. Let $B_{j_0}, \ldots, B_{j_q}$ for $q \geq 0$ denote a (possibly empty) collection of irreducible components of $|B|$.

A non-empty connected component Y of $E_{i_0} \cap \cdots \cap E_{i_p} \cap \overline{B}_{j_1} \cap \cdots \cap \overline{B}_{j_q}$ is called a *stratum* in $\mathscr{X}_0 + \overline{B}$ and we write $Y \subset_{\text{conn. comp.}} E_{i_0} \cap \cdots \cap E_{i_p} \cap \overline{B}_{j_1} \cap \cdots \cap \overline{B}_{j_q}$.

Note that in our definition of a stratum, we insist that we start with a non-empty collection of irreducible components of $\mathscr{X}_0$. Thus, we do not consider $\overline{B}_j$ or $\cap_j \overline{B}_j$ to be strata of $\mathscr{X}_0 + \overline{B}$, while E_i is a stratum of $\mathscr{X}_0 + \overline{B}$. All strata are smooth subvarieties of $\mathscr{X}_0$.

Given a stratum $Y \subset_{\text{conn. comp.}} E_{i_0} \cap \cdots \cap E_{i_p} \cap \overline{B}_{j_1} \cap \cdots \cap \overline{B}_{j_q}$, let b_{i_k} denote the multiplicity of E_{i_k} in $\mathscr{X}_0$. Then the *face* associated to Y is the (open) simplex σ_Y defined as follows.

$$\sigma_Y = \left\{ (x_0, \ldots, x_p, y_1 \ldots, y_q) \in \mathbb{R}^{p+1}_{\geq 0} \times \mathbb{R}^{q}_{\geq 0} \;\middle|\; \sum_k b_{i_k} x_k = 1 \right\}.$$

We define the *dual complex* $\Delta(\mathscr{X}, |B|)$ associated to the snc model $\mathscr{X}$ of (X, B) to be the CW complex (with possibly open faces) obtained by gluing the faces σ_Y for all possible strata Y.

More precisely,

$$\Delta(\mathscr{X}, |B|) = \left(\bigcup_{Y \text{strata}} \sigma_Y \right) / \sim$$

where $\sim$ is an equivalence relation generated by the following identification. If Y' and Y are strata with $Y \subset Y'$, then without loss of generality we can write $Y' \subset_{\text{conn. comp.}} E_{i_0} \cap \cdots \cap E_{i_{p'}} \cap \overline{B}_{j_1} \cap \cdots \cap \overline{B}_{j_{q'}}$ and $Y \subset_{\text{conn. comp.}} E_{i_0} \cap \cdots \cap E_{i_p} \cap \overline{B}_{j_1} \cap \cdots \cap \overline{B}_{j_q}$ for some $p' \leq p$ and $q' \leq q$, and we can identify $\sigma_{Y'}$ as a subset of σ_Y via

$$(x_0, \ldots, x_{p'}, y_1, \ldots, y_{q'}) \mapsto (x_0, \ldots, x_{p'}, \underbrace{0, \ldots, 0}_{p-p' \text{ times}}, y_1, \ldots, y_{q'}, \underbrace{0, \ldots, 0}_{q-q' \text{ times}}).$$

For example, if $\dim(X_t) = 1$, then $\Delta(\mathscr{X}, |B|)$ is the dual graph of $\mathscr{X}_0 + \overline{B}$ with the vertices associated to $|\overline{B}|$ as well as the edges with both endpoints in $|\overline{B}|$ removed. The dual complex of a pair was introduced in [GRW16] [BM19].

Note that $\Delta(\mathscr{X}, |B|)$ is independent of β_j's and thus only depends on $|B|$.

Given a face σ of $\Delta(\mathscr{X}, |B|)$, we denote Y_σ to be the stratum associated to the face σ.

The complex $\Delta(\mathscr{X}) := \Delta(\mathscr{X}, 0)$ is just the subcomplex of $\Delta(\mathscr{X}, |B|)$ consisting of all the bounded faces.

Example II.2 (The dual complex associated to $\mathbb{P}^1 \times \mathbb{D}$)**.** Let $X = \mathbb{P}^1 \times \mathbb{D}^*$, with the projection map to $\mathbb{D}^*$, and $B = \{0\} \times \mathbb{D}^* + \{\infty\} \times \mathbb{D}^*$ is a horizontal divisor in X. Consider the model $\mathscr{X} = \mathbb{P}^1 \times \mathbb{D}$. Then, the dual complex $\Delta(\mathscr{X}, |B|)$ is homeomorphic to $\mathbb{R}$, with 0 being the vertex $\sigma_{\mathbb{P}^1 \times \{0\}}$, the positive axis being identified with $\sigma_{(0,0)}$ and the negative axis with $\sigma_{(\infty,0)}$. See Figure II.1.

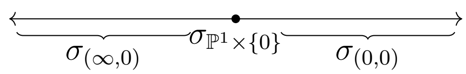

Figure II.1: The dual complex $\Delta(\mathscr{X}, |B|)$ for $\mathscr{X} = \mathbb{P}^1 \times \mathbb{D}$ and $\overline{B} = \{0\} \times \mathbb{D} + \{\infty\} \times \mathbb{D}$

II.1.3: Integral piecewise affine structure on the dual intersection complex

We briefly discuss some results related to the natural integral piecewise affine structure on $\Delta(\mathscr{X}, |B|)$. The reader can take a look at [Ber99], [Ber04], [CLD12] and [BJ17, Section 1.3] for more details. Let $\sigma = \{(x_0, \ldots, x_p) | \sum_{i=0}^{p} b_i x_i = 1\} \times \mathbb{R}^q_{\geq 0}$ be a simplex. Let M_σ denote the abelian group of integral affine functions on $\mathbb{R}^{p+1+q}$ restricted to σ (two such functions are identified if they are equal on σ). Let $(M_\sigma)_\mathbb{R} := M_\sigma \otimes_\mathbb{Z} \mathbb{R}$ and let $(M_\sigma)^\vee_\mathbb{R}$ be its $\mathbb{R}$-dual. Denote $b_\sigma := \gcd(b_0, \ldots, b_p)$.

Let $\mathbf{1}_\sigma \in M_\sigma$ denote the constant function 1 on σ. The evaluation map $\sigma \to (M_\sigma)^\vee_\mathbb{R}$ realizes σ as a simplex of codimension one in $(M_\sigma)^\vee_\mathbb{R}$ contained in the affine plane $\{\nu | \nu(\mathbf{1}_\sigma) =$

$1\}$. So, the tangent space of σ in $(M_\sigma)^\vee_\mathbb{R}$ can be realized as $(\overrightarrow{M_\sigma})^\vee_\mathbb{R}$, where

$$\overrightarrow{M_\sigma} = M_\sigma/(\mathbb{Q}\mathbf{1}_\sigma \cap M_\sigma)$$

and $(\overrightarrow{M_\sigma})^\vee_\mathbb{R}$ is the $\mathbb{R}$-dual of $\overrightarrow{M_\sigma} \otimes_\mathbb{Z} \mathbb{R}$.

Consider the Lebesgue measure on $(\overrightarrow{M_\sigma})^\vee_\mathbb{R}$ for which the lattice $\mathrm{Hom}_\mathbb{Z}(\overrightarrow{M_\sigma}, \mathbb{Z})$ has covolume 1. This gives rise to a measure on σ. This is called the normalized Lebesgue measure λ_σ of σ. The following remark, stated with a typo in [BJ17, Remark 1.3], gives an explicit description of the normalized Lebesgue measure, which will be useful for computations. We provide a quick proof here for the convenience of the reader.

Proposition II.3 ([BJ17, Remark 1.3])**.** Let $b_0, \dots, b_p \in \mathbb{N}_+$ and let

$$\sigma = \{(x_0, \dots, x_p, y_1, \dots, y_q) \in \mathbb{R}^{p+q+1}_{\geq 0} | \sum_{i=0}^{p} b_i x_i = 1\}$$

be a simplex. Then, we have a homeomorphism

$$\sigma \to \{(x_1, \dots, x_p, y_1, \dots, y_q) \in \mathbb{R}^{p+q}_{\geq 0} | \sum_{i=1}^{p} b_i x_i \leq 1\},$$

where we can recover x_0 by $x_0 = b_0^{-1}(1 - \sum_{i=1}^{p} b_i x_i)$. Under this homeomorphism, the normalized Lebesgue measure is given by

$$\lambda_\sigma = b_\sigma b_0^{-1} |dx_1 \wedge \dots \wedge dx_p \wedge dy_1 \wedge \dots \wedge dy_q|$$

Proof. Note that $1_\sigma, X_1, \dots, X_p, Y_1, \dots, Y_q$ is an $\mathbb{R}$-basis for $(M_\sigma)_\mathbb{R}$, where X_i and Y_j denote projection to the x_i and y_j coordinates. Let $1^*_\sigma, X^*_1, \dots, Y^*_q$ denote its dual basis. Then, $X^*_1, \dots, Y^*_q$ is a $\mathbb{R}$-basis for the $(\overrightarrow{M_\sigma})^\vee_\mathbb{R}$ and $\mathrm{Hom}_\mathbb{Z}(\overrightarrow{M_\sigma}, \mathbb{Z})$ is a sub lattice of $\Lambda = \mathbb{Z}X^*_1 + \dots + \mathbb{Z}Y^*_q$.

Note that we can view $\mathrm{Hom}_\mathbb{Z}(\overrightarrow{M_\sigma}, \mathbb{Z})$ as the kernel of the map $\phi : \Lambda \to \mathbb{Z}/b_0\mathbb{Z}$ given by

$\alpha_1 X_1^* + \cdots + \alpha_p X_p^* + \gamma_1 Y_1^* + \cdots + \gamma_q Y_q^* \to b_1\alpha_1 + \cdots + b_p\alpha_p + b_0\mathbb{Z}$. The image of ϕ is generated by b_σ and the size of the image is $\frac{b_0}{b_\sigma}$. Thus, the index of $\mathrm{Hom}_{\mathbb{Z}}(\overrightarrow{M_\sigma}, \mathbb{Z})$ in Λ is $\frac{b_0}{b_\sigma}$, and thus $b_\sigma b_0^{-1}|dx_1 \wedge \cdots \wedge dx_p \wedge dy_1 \wedge \cdots \wedge dy_q|$ is the normalized Lebesgue measure on σ. $\square$

II.2: The hybrid space associated to a dual complex

Let X be a holomorphic flat family of compact complex manifolds parametrized by $\mathbb{D}^*$ i.e. X is a smooth complex manifold with a proper smooth map $X \to \mathbb{D}^*$. Let B be a horizontal snc $\mathbb{Q}$-divisor in X. Write $B = \sum_j \beta_j B_j$, where $\beta_j \in \mathbb{Q}$ and B_j are prime divisors. We don't need to assume that (X, B) is projective or that (X, B) is log Calabi-Yau. The constructions in this section only depend on $|B|$ i.e. they are independent of β_j's. Let $\mathscr{X}$ be an snc model of (X, B) and write $\mathscr{X}_0 = \sum_i b_i E_i$.

In this section, we construct the hybrid space $(\mathscr{X}, |B|)^{\mathrm{hyb}}$, associated to the snc model $\mathscr{X}$ of (X, B); this is a topological space over $\mathbb{D}$ such that the fiber over $\mathbb{D}^*$ is isomorphic to $X \setminus |B|$ and the central fiber is isomorphic to $\Delta(\mathscr{X}, |B|)$. This construction closely follows [BJ17, Section 2.2], where the construction for $B = 0$ was done.

II.2.1: Local Log function

To construct the hybrid space, we will first construct a Log function on $\mathscr{X}$ and glue $X \setminus |B|$ and $\Delta(\mathscr{X}, |B|)$ using this Log function. To do this, we first construct a local version of the Log function. Let $Y \subset_{\text{conn. comp.}} E_0 \cap \cdots \cap E_p \cap \overline{B}_1 \cap \cdots \cap \overline{B}_q$ denote a stratum of $\mathscr{X}_0 + \overline{B}$ (see Section II.1.2 for the definition of a stratum) and let b_i denote the multiplicity of E_i in $\mathscr{X}_0$. For an open set $U \subset \mathscr{X}$ and for local coordinates (z, w, y) on U where $z = (z_0, \ldots, z_p)$, $w = (w_1, \ldots, w_q)$ and $y = (y_1, \ldots, y_r)$, we say that $(U, (z, w, y))$ is *adapted* to the stratum Y if

- The only irreducible components of $\mathscr{X}_0 + \overline{B}$ intersecting U are $E_0, \ldots, E_p$, and $\overline{B}_1, \ldots, \overline{B}_q$
- $U \cap (E_0 \cap \cdots \cap E_p \cap \overline{B}_1 \cap \cdots \cap \overline{B}_q) = U \cap Y$.

- We have $|z_i|, |w_j|, |y_k| < 1$ on U and $E_i \cap U = \{z_i = 0\}$ and $\overline{B}_j \cap U = \{w_j = 0\}$.

If there exists a choice of coordinates that make U adapted to some stratum of $\mathscr{X}_0 + \overline{B}$, we say that U is an adapted coordinate chart in $\mathscr{X}$.

Suppose that $(U, (z, w, y))$ is a coordinate chart adapted to a stratum $Y \subset_{\text{conn. comp.}} E_0 \cap \cdots \cap E_p \cap \overline{B}_1 \cap \cdots \cap \overline{B}_q$. Then, we can define $\mathrm{Log}_U : U \setminus |\mathscr{X}_0 + \overline{B}| \to \sigma_Y$. Let $f_U := z_0^{b_0} \ldots z_p^{b_p}$. Then there exists a bounded invertible holomorphic function u on U such that the projection $U \to \mathbb{D}$ is given by $t = u \cdot f_U$. Define

$$\mathrm{Log}_U(z, w, y) := \left(\frac{\log|z_0|}{\log|f_U|}, \ldots, \frac{\log|z_p|}{\log|f_U|}, \frac{\log|w_1|}{\log|f_U|}, \ldots, \frac{\log|w_q|}{\log|f_U|} \right).$$

Note that Log_U depends on the choice of the coordinates on an adapted coordinate chart U, however the following lemma tells us that the difference between two such maps goes off to 0 for a different choice of coordinates on U.

Remark II.4 ([BJ17, Prop 2.1])**.** If $(U, (z, w, y))$ and $(U', (z', w', y'))$ are adapted to a stratum Y, then

$$\mathrm{Log}_U - \mathrm{Log}_{U'} = O\left(\frac{1}{\log|t|^{-1}} \right)$$

as $t \to 0$ uniformly on compact subsets of $U \cap U'$ where t denotes the coordinate on $\mathbb{D}$.

Here, we view Log_U and $\mathrm{Log}_{U'}$ as maps with image $\sigma_Y \subset \mathbb{R}^{p+1+q}$ and $\mathrm{Log}_U - \mathrm{Log}_{U'} = O\left(\frac{1}{\log|t|^{-1}}\right)$ just means that the equality is true coordinate-wise on $\mathbb{R}^{p+1+q}$.

II.2.2: Constructing the global Log function

Here, we globalize the Log construction by patching up the local Log functions and to do so, we will have to find a 'nice' open covering of $\mathscr{X}_0 + \overline{B}$. The following construction, as well as Proposition II.5 is similar to [BJ17, Proposition 2.1], but we provide some more details.

For a non-empty collection $\{E_i \mid i \in I\}$ of the irreducible components of $\mathscr{X}_0$, denote $E_I = \cap_{i \in I} E_i$. Similarly, for a (possibly empty) collection $\{B_j \mid j \in J\}$ of irreducible components of B, denote $\overline{B}_J = \cap_{j \in J} \overline{B}_j$.

Following [Cle77, Theorem 5.7], we can find tubular neighborhoods $U_{I,J}$ of $D_{I,J} := E_I \cap \overline{B}_J$ and a smooth projection $\pi_{I,J} : U_{I,J} \to D_{I,J}$ satisfying $U_{I,J} \cap U_{I',J'} = U_{I\cup I', J\cup J'}$. In particular, if $U_{I,J}$ and $U_{I',J'}$ intersect, then $D_{I,J} \cap D_{I',J'} \neq \emptyset$. Also, note that $U_{I,J}$ has as many connected components as $D_{I,J}$ and each connected component U_Y of $U_{I,J}$ corresponds to a stratum $Y \subset_{\text{conn. comp.}} E_I \cap \overline{B}_J$.

Pick $x \in \mathscr{X}_0$. Suppose that Y_x is the smallest stratum containing x. Around x, pick an open neighborhood U_x that is adapted to Y_x and lies in U_{Y_x}. The union of all such U_x for $x \in \mathscr{X}_0$ covers $\mathscr{X}_0$. Since $\mathscr{X}_0$ is compact, we only need finitely many of these to cover $\mathscr{X}_0$. Call these open sets $U_1, \dots, U_l$ and let their corresponding strata be $Y_1, \dots, Y_l$ respectively. Let $\chi_1, \dots, \chi_l$ be a partition of unity with respect to $U_1, \dots, U_l$ and let $V = \bigcup_{\lambda=1}^{l} U_\lambda$. Then, V is a neighborhood of $\mathscr{X}_0$.

Proposition II.5. The function $\mathrm{Log}_V : V \setminus |\mathscr{X}_0 + \overline{B}| \to \Delta(\mathscr{X}, |B|)$ given by $\mathrm{Log}_V = \sum_{\lambda=1}^{l} \chi_\lambda \mathrm{Log}_{U_\lambda}$ is well defined. Here, addition in $\Delta(\mathscr{X}, |B|)$ means that the sum makes sense in a face of $\Delta(\mathscr{X}, |B|)$.

Proof. Note that $\mathrm{Log}_{U_{\lambda_1}}$ and $\mathrm{Log}_{U_{\lambda_2}}$ are maps with image $\sigma_{Y_{\lambda_1}}$ and $\sigma_{Y_{\lambda_2}}$ respectively. A priori, there might not be a face of $\Delta(\mathscr{X}, |B|)$ that contains both $\sigma_{Y_{\lambda_1}}$ and $\sigma_{Y_{\lambda_2}}$, in which case there is no way for us to make sense of the sum $\sum_{\lambda=1}^{l} \chi_\lambda \mathrm{Log}_{U_\lambda}$ in $\Delta(\mathscr{X}, |B|)$ at a point $x \in V \setminus |\mathscr{X}_0 + \overline{B}|$ where $\chi_{\lambda_1}(x), \chi_{\lambda_2}(x) \neq 0$. To show the well-definedness of the map, we need to show that such a scenario does not happen.

Pick a point $x \in V \setminus |\mathscr{X}_0 + \overline{B}|$. After a possible re-indexing, suppose $x \in (U_1 \cap \cdots \cap U_a) \setminus (U_{a+1} \cup \cdots \cup U_l)$. Then, we would like to define $\mathrm{Log}_V(x) = \chi_1(x)\mathrm{Log}_{U_1}(x) + \cdots + \chi_a(x)\mathrm{Log}_{U_a}(x)$. For this to make sense, it is enough to find a face σ' of $\Delta(\mathscr{X}, |B|)$ such that $\sigma_{Y_1}, \dots, \sigma_{Y_a} \subset \sigma'$. Note that $U_1 \cap \cdots \cap U_a \subset U_{Y_1} \cap \cdots \cap U_{Y_a}$. Each connected component of $\bigcap_{\lambda=1}^{a} U_{Y_\lambda}$ corresponds to a stratum of $\bigcap_{\lambda=1}^{a} Y_\lambda$. Let Y' be the stratum corresponding to the connected component of $\bigcap_{\lambda=1}^{a} U_{Y_\lambda}$ containing x. Then, $\sigma' := \sigma_{Y'}$ contains σ_{Y_λ} for all $\lambda = 1, \dots, a$ and $\mathrm{Log}_V(x) = \chi_1(x)\mathrm{Log}_{U_1}(x) + \cdots + \chi_a(x)\mathrm{Log}_{U_a}(x)$ makes sense in σ'. $\square$

Proposition II.6. Let U be an open set adapted to a stratum Y. Then, $\mathrm{Log}_V - \mathrm{Log}_U = O(\frac{1}{\log|t|^{-1}})$ locally uniformly as $t \to 0$, where the equality is interpreted as being true cooridinate-wise on some faces of $\Delta(\mathscr{X}, |B|)$ containing σ_Y.

Proof. We may replace U by $U \cap U_Y$ and assume that $U \subset U_Y$. It is enough to prove the result in a small neighborhood of every point $x \in U \cap \mathscr{X}_0$.

Suppose $x \in U \cap \mathscr{X}_0$ such that $x \in (U_1 \cap \cdots \cap U_a) \setminus (U_{a+1} \cup \cdots \cup U_l)$. Then, from the previous proof, we know that there exists a stratum Y' such that $x \in U_{Y'}$ and $\mathrm{Log}_V(x) = \chi_1(x)\mathrm{Log}_{U_1}(x) + \cdots + \chi_a(x)\mathrm{Log}_{U_a}(x)$ makes sense in $\sigma_{Y'}$. Since $x \in U_Y \cap U_{Y'}$, we get that $Y \cap Y' \neq \emptyset$. Let Z be the stratum corresponding to the connected component of $U_Y \cap U_{Y'}$ containing x. Then, $\sigma_Y, \sigma_{Y'} \subset \sigma_Z$ and we can think of Log_U and Log_V as maps with image contained in $\sigma_Z \subset \mathbb{R}^N$. We now need to show that $\mathrm{Log}_U - \mathrm{Log}_V = O(\frac{1}{\log|t|^{-1}})$ coordinate-wise on σ_Z.

Suppose $x \in E_i$, $z_i = 0$ defines E_i in U, and $z'_i = 0$ defines E_i in U_1. Then, z_i and z'_i differ by the factor of a unit in a neighborhood of x and we get that $\frac{\log|z_i|}{\log|f_U|} - \frac{\log|z'_i|}{\log|f_{U_1}|} = O(\frac{1}{\log|t|^{-1}})$ in a neighborhood of x.

Suppose $x \notin E_i$ and $z'_i = 0$ defines E_i in U_1. Then, $\log|z'_i|$ is bounded near x and we get that $\frac{\log|z'_i|}{\log|t|} = O(\frac{1}{\log|t|^{-1}})$ in a neighborhood of x.

Using a similar argument for B_j's as well gives us that $\mathrm{Log}_U - \mathrm{Log}_{U_1} = O(\frac{1}{\log|t|^{-1}})$ in a neighborhood of x. Repeating the argument for all U_k for $k = 1, \ldots, a$, we get that $\mathrm{Log}_U - \mathrm{Log}_V = O(\frac{1}{\log|t|})$ in a neighborhood of x.

□

II.2.3: The hybrid space

The hybrid space of an snc model $\mathscr{X}$ of (X, B), as a set, is defined as $(\mathscr{X}, |B|)^{\mathrm{hyb}} := (X \setminus |B|) \cup \Delta(\mathscr{X}, |B|)$. The topology on the hybrid space is defined to be the coarsest one satisfying the following.

- $X \setminus |B| \hookrightarrow (\mathscr{X}, |B|)^{\mathrm{hyb}}$ is an open immersion.

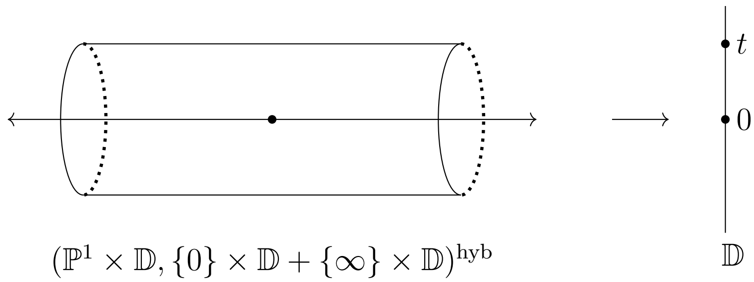

Figure II.2: The hybrid space $(\mathbb{P}^1 \times \mathbb{D}, \{0\} \times \mathbb{D} + \{\infty\} \times \mathbb{D})^{\mathrm{hyb}}$ with the projection to $\mathbb{D}$

- The projection map $\pi : (\mathscr{X}, |B|)^{\mathrm{hyb}} \to \mathbb{D}$, given by the projection $X \setminus |B| \to \mathbb{D}^*$ and by sending $\Delta(\mathscr{X}, |B|)$ to the origin, is continuous.
- $\mathrm{Log}_{V^{\mathrm{hyb}}} : (V \setminus |\mathscr{X}_0 + \overline{B}|) \cup \Delta(\mathscr{X}, |B|) \to \Delta(\mathscr{X}, |B|)$ defined by Log_V on $V \setminus |\mathscr{X}_0 + \overline{B}|$ and identity on $\Delta(\mathscr{X}, |B|)$ is continuous.

Note that the hybrid space does not contain $|B|$. It follows from Proposition II.6 that the topology of the hybrid space does not depend on the global log function we pick. Also note that the fiber of $\pi : (X, |B|)^{\mathrm{hyb}} \to \mathbb{D}$ over $t \in \mathbb{D}^*$ is $X_t \setminus |B_t|$.

Example II.7 (Hybrid space of $\mathbb{P}^1 \times \mathbb{D}$)**.** The hybrid space $(\mathscr{X}, |B|)^{\mathrm{hyb}}$ for Example II.2 is given by $\mathbb{C}^* \times \mathbb{D}$ with the identification $(re^{i\theta_1}, 0) \sim (re^{i\theta_2}, 0)$ for all $r \in \mathbb{R}, \theta_i \in [0, 2\pi]$. Over any $t \in \mathbb{D}^*$ the fiber is $\mathbb{P}^1 \setminus \{0, \infty\}$, which is topologically a cylinder. Over $t = 0$, the fiber is homeomorphic to $\mathbb{R}$. See Figure II.2.

The hybrid space $\mathscr{X}^{\mathrm{hyb}} := (\mathscr{X}, 0)^{\mathrm{hyb}}$, constructed in [BJ17], is compact over a compact neighborhood of the origin. But the hybrid space $(\mathscr{X}, |B|)^{\mathrm{hyb}}$ that we construct is not always compact over a compact neighborhood of the origin, as can be seen from Example II.7. However, the following proposition tells us that it is not too bad. In particular, it implies that the hybrid space is locally compact.

Proposition II.8. The map $\mathrm{Log}_{V^{\mathrm{hyb}}} : (V \setminus |\mathscr{X}_0 + \overline{B}|) \cup \Delta(\mathscr{X}, |B|)) \to \Delta(\mathscr{X}, |B|)$ is proper near the central fiber, in the sense that there exists an $r > 0$ such that for any compact set $K \subset \Delta(\mathscr{X}, |B|)$, $\mathrm{Log}^{-1}_{V^{\mathrm{hyb}}}(K) \cap \pi^{-1}(r\overline{\mathbb{D}})$ is a compact subset of $(\mathscr{X}, |B|)^{\mathrm{hyb}}$.

Proof. By rescaling the coordinate t, we may without loss of generality assume that $V = \mathscr{X}$. Let $V = \bigcup_\alpha U_\alpha$ such that $\mathrm{Log}_V = \sum_\alpha \chi_\alpha \mathrm{Log}_{U_\alpha}$ for adapted coordinate charts U_α.

Pick a compact set $K \subset \Delta(\mathscr{X}, |B|)$. It is enough to show that $L = \mathrm{Log}_{V^{\mathrm{hyb}}}^{-1}(K) \cap \pi^{-1}(\frac{1}{2}\overline{\mathbb{D}})$ is compact. Let $\bigcup_{\lambda \in \Lambda} V_\lambda$ be an open cover of L. Since $K \subset L$ is compact, there exists a finite subset $\Lambda' \subset \Lambda$ such that $K \subset \bigcup_{\lambda \in \Lambda'} V_\lambda$. For a point $P \in \Delta(\mathscr{X}, |B|) \subset (\mathscr{X}, |B|)^{\mathrm{hyb}}$, the sets of the form $\mathrm{Log}_{V^{\mathrm{hyb}}}^{-1}(W) \cap \pi^{-1}(\frac{1}{N}\mathbb{D})$, where $W \subset \Delta(\mathscr{X}, |B|)$ is an open neighborhood of P in $\Delta(\mathscr{X}, |B|)$ and $N \in \mathbb{N}$, form basic open neighborhoods of P in $(\mathscr{X}, |B|)^{\mathrm{hyb}}$. For every point $P \in K$, we pick such a neighborhood contained in $\bigcup_{\lambda \in \Lambda'} V_\lambda$. Since finitely many such neighborhoods cover K, we can pick an $r > 0$ such that $K \subset \bigcup_{j=1}^m \mathrm{Log}_{V^{\mathrm{hyb}}}^{-1}(W_j) \cap \pi^{-1}(r\overline{\mathbb{D}}) \subset \bigcup_{\lambda \in \Lambda'} V_\lambda$. Thus, $\mathrm{Log}_{V^{\mathrm{hyb}}}^{-1}(K) \cap \pi^{-1}(r\overline{\mathbb{D}}) \subset \bigcup_{\lambda \in \Lambda'} V_\lambda$.

Now it is enough to show that we need finitely many V_λ's to cover $L \cap \pi^{-1}(\{r \leq |t| \leq \frac{1}{2}\})$. To do this, it is enough to show that $L \cap \pi^{-1}(\{r \leq |t| \leq \frac{1}{2}\})$ is relatively compact in $X \setminus |B|$. Suppose to the contrary that the closure of L in $\mathscr{X}$ intersects B. Then, there exists a sequence $b_n \in L$ with limit $b \in B$. By the assumption $V = \mathscr{X}$, b lies in an adapted coordinate chart U_α for some α. Let U_α be adapted to the stratum $Y_\alpha \subset_{\text{conn. comp.}} E_{I_\alpha} \cap \overline{B}_{J_\alpha}$. As $b \in U_\alpha$, $U_\alpha \cap \overline{B} \neq \emptyset$ and since U_α is an adapted coordinate chart, it follows that J_α is non-empty. But this implies that $\mathrm{Log}_{U_\alpha}(b_n)$ is an unbounded sequence in σ_{Y_α}, which is a contradiction.

□

II.3: Convergence of measure

In this section, we prove Theorem B by imitating the proof of [BJ17, Theorem A]. Since $(\mathscr{X}, |B|)^{\mathrm{hyb}}$ is not compact, we can no longer use Stone-Weierstrass as done in [BJ17]. Instead, we use Lemma II.11. Let (X, B) be as in the previous section. Further assume that $K_{X/\mathbb{D}^*} + B \sim_{\mathbb{Q}} 0$ and that (X, B) is sub log canonical. Here, sub log canonical means that (X, B) is log canonical in the sense of the minimal model program (i.e. $\mathrm{discrep}(X, B) \geq -1$) but we are not necessarily assuming that B is effective.

Write $B = \sum_j \beta_j B_j$ for irreducible components B_j of B. Since (X, B) is log-smooth, being sub log canonical is equivalent to requiring that $\beta_j \leq 1$ for all j. Fix an snc model $\mathscr{X}$ of the pair (X, B). Note that we don't yet need to assume that X is projective. Let $n+1$ denote the complex dimension of X i.e. each of the fibers X_t has dimension n.

II.3.1: The subcomplex $\Delta(\mathscr{D})$ of $\Delta(\mathscr{X}, |B|)$

Suppose $\mathscr{D}$ is a $\mathbb{Q}$-divisor on $\mathscr{X}$ that extends $K_{X/\mathbb{D}^*} + B$ such that $\mathscr{D} \sim_{\mathbb{Q}} 0$. Denote $K^{\log}_{\mathscr{X}/\mathbb{D}} := K_{\mathscr{X}/\mathbb{D}} - \mathscr{X}_0 + (\mathscr{X}_0)_{\mathrm{red}}$. Then, $\mathscr{D}$ differs from $K^{\log}_{\mathscr{X}/\mathbb{D}} + B$ by only divisors supported in $\mathscr{X}_0$. Recall that we write $\mathscr{X}_0 = \sum_i b_i E_i$, where E_i are the irreducible components of $\mathscr{X}_0$. Thus, we can write $\mathscr{D} = K^{\log}_{\mathscr{X}/\mathbb{D}} - \sum_i a_i E_i + \sum_j \beta_j \overline{B}_j$ for some $a_i \in \mathbb{Q}$. Let $\kappa_i = \frac{a_i}{b_i}$ and $\kappa_{\min} = \min_i \kappa_i$.

Define the subcomplex $\Delta(\mathscr{D}) \subset \Delta(\mathscr{X}, |B|)$ as follows. If $Y \subset_{\text{conn. comp.}} E_I \cap \overline{B}_J$ is a stratum, then $\sigma_Y \in \Delta(\mathscr{D})$ if $\kappa_i = \kappa_{\min}$ for all $i \in I$ and if $\beta_j = 1$ for all $j \in J$. In the case when $\dim(X_t) = 1$, this just means that we pick the subgraph generated by vertices corresponding to irreducible components with minimal κ-value and the rays corresponding to intersections $E_i \cap \overline{B}_j$ with $\kappa_i = \min_k \kappa_k$ and $\beta_j = 1$.

For a stratum $Y \subset_{\text{conn. comp.}} E_I \cap \overline{B}_J$, define $b_{\sigma_Y} = \gcd(b_i)_{i \in I}$ and let λ_{σ_Y} be the normalized Lebesgue measure on σ_Y (see Section II.1.3). Define $d := \dim(\Delta(\mathscr{D}))$, the maximum of the dimensions of the faces of $\Delta(\mathscr{D})$.

II.3.2: The residual measure

Let m be a sufficiently divisible integer. Given a section $\psi \in H^0(\mathscr{X}, m\mathscr{D})$ and a stratum $Y \subset_{\text{conn. comp.}} E_I \cap \overline{B}_J$, we can get a section $\mathrm{Res}_Y(\psi) \in H^0(Y, m(\mathscr{D} - \sum_{j \in J} \overline{B}_j - \sum_{i \in I} E_i)|_Y)$. Suppose that $z_0 = 0, \dots, z_p = 0, w_1 = 0, \dots, w_q = 0$ define Y locally. Thinking of ψ as a relative m-canonical section, we can write $\psi = f\left(\frac{dz_0}{z_0} \wedge \cdots \wedge \frac{dz_p}{z_p} \wedge \frac{dw_1}{w_1} \wedge \dots \frac{dw_q}{w_q} \wedge \phi\right)^{\otimes m}$ locally for some local meromorphic function f. Then, $\mathrm{Res}_Y(\psi) := f \cdot \phi|_Y^{\otimes m}$.

Note that $\dim(Y) = n - p - q$ and $|\mathrm{Res}_Y(\psi)|^{2/m}$ gives rise to a $(n-p-q, n-p-q)$-form on $Y \setminus (\bigcup_{i|Y\not\subset E_i} E_i|_Y \cup \bigcup_{j|Y\not\subset B_j} \overline{B}_j|_Y)$. Thus, $|\mathrm{Res}_Y(\psi)|^{2/m}$ gives rise to a positive measure on Y.

II.3.3: The Convergence Theorem

Let m be sufficiently divisible integer. Let $\eta \in H^0(X, m(K_{X/\mathbb{D}^*}+B))$ be a generating section and suppose there exists a generating section $\psi \in H^0(\mathscr{X}, m\mathscr{D})$ that extends η. Let ψ_t denote the restriction $\psi|_{X_t}$ for $t \neq 0$. If $\psi_t = \alpha \cdot (dx_1 \wedge \cdots \wedge dx_n)^{\otimes m}$ on a local chart, then $i^{n^2}(\psi_t \wedge \overline{\psi_t})^{1/m}$ given locally by

$$i^{n^2}(\psi_t \wedge \overline{\psi_t})^{1/m} = |\alpha|^{2/m}(idx_1 \wedge d\overline{x}_1) \wedge \ldots (idx_n \wedge d\overline{x}_n)$$

is a well-defined positive continuous volume form on $X_t \setminus |B_t|$.

Define a measure

$$\mu_t = \frac{i^{n^2}}{|t|^{2\kappa_{\min}}(2\pi \log|t|^{-1})^d}(\psi_t \wedge \overline{\psi_t})^{1/m}$$

on $X_t \setminus |B_t|$, and a measure

$$\mu_0 := \sum_{\sigma \subset_{\text{face}} \Delta(\mathscr{D}), \dim(\sigma)=d} \left(\int_{Y_\sigma} |\mathrm{Res}_{Y_\sigma}(\psi)|^{2/m} \right) b_\sigma^{-1} \lambda_\sigma$$

on $\Delta(\mathscr{X}, |B|)$, where Y_σ denotes the stratum associated to the face σ. We will see in the proof of Lemma II.10 that the measure $|\mathrm{Res}_{Y_\sigma}(\psi)|^{2/m}$ is a finite measure on Y_σ and thus $\int_{Y_\sigma} |\mathrm{Res}_{Y_\sigma}(\psi)|^{2/m}$ is well defined .

Example II.9. This example illustrates the importance of the sub log canonical assumption. For simplicity, assume that X has relative dimension 1. Let E_0 be an irreducible component of $\mathscr{X}_0$ and let B_0 be an irreducible component of $|B|$ occurring with multiplicity $\beta_0 > 1$. Let $\sigma \simeq \mathbb{R}_{\geq 0}$ be the face corresponding to $E_0 \cap \overline{B}_0$. Let z and w denote the functions that define E_0 and B_0 in an open neighborhood U of $E_0 \cap \overline{B}_0$ such that $|z|, |w| < 1$ on U. We may assume that $t = z^{b_0}$

We have $\mathrm{Log}_U : (U \setminus (E_0 + \overline{B}_0)) \to \mathbb{R}_{\geq 0}$ given by $(z, w) \mapsto \frac{\log|w|}{\log|t|}$. Suppose we had that $(\mathrm{Log}_U)_*(\alpha(t)\mu_t)$ weakly converged to a measure μ_0 on $\mathbb{R}_{\geq 0}$ for some positive scaling function $\alpha(t)$. By scaling by a suitable power of $|t|$, we may assume that $\mu_t = i|w|^{-2\beta_0} dw \wedge d\overline{w}$. Pick a compactly supported continuous function f on $\mathbb{R}_{\geq 0}$. Then,

$$\int_{U_t} (f \circ \mathrm{Log}_U) d\mu_t = \int_{U_t} f\left(\frac{\log|w|}{\log|t|}\right) i|w|^{-2\beta_0} dw \wedge d\overline{w}.$$

Making a change of variable $w = |t|^u e^{i\theta}$, we get

$$\int_{U_t} (f \circ \mathrm{Log}_U) d\mu_t = \frac{2\pi}{(\log|t|^{-1})} \int_0^\infty f(u)|t|^{-2(\beta_0 - 1)u} du$$

If we pick a function f that is close to the indicator function of $[0, N]$, then $\alpha(t) \int_{U_t} (f \circ \mathrm{Log}_U) d\mu_t = O(\frac{\alpha(t)}{\log|t|^{-1}} \frac{|t|^{-2(\beta_0-1)N}}{\log|t|^{-1}})$ as $t \to 0$. If we require that this expression converge for all values of N as $t \to 0$, then it is easy to see that this is only possible if μ_0 is the zero measure and $\frac{1}{\alpha(t)}$ is growing super-polynomially as $t \to 0$. Thus, we see that the convergence in this hybrid space setting is not very interesting if don't assume that (X, B) is sub log canonical.

To prove Theorem B, we first prove a local version for functions that are pulled-back from a face σ_Y via a local Log map.

Lemma II.10. Let $(U, (z, w, y))$ be a coordinate chart adapted to a stratum Y of $\mathscr{X}_0$. Let f be a compactly-supported continuous (real-valued) function on σ_Y and let χ be a compactly supported continuous function on U. If a maximal face of $\Delta(\mathscr{D})$ is contained in σ_Y, let $\sigma_{Y'}$ denote this (unique) maximal face and let Y' be the stratum associated to $\sigma_{Y'}$.

If a maximal face of $\Delta(\mathscr{D})$ is contained in σ_Y, then

$$\int_{U \cap X_t} (f \circ \mathrm{Log}_U) \chi d\mu_t \to \left(\int_{Y'} \chi |\mathrm{Res}_{Y'}(\psi)|^{2/m}\right) \int_{\sigma_{Y'}} f b_{\sigma_{Y'}}^{-1} \lambda_{\sigma_{Y'}}$$

as $t \to 0$. If σ_Y does not contain a maximal face of $\Delta(\mathscr{D})$, then the above limit is 0.

Proof. By replacing $\mathscr{D}$ by $\mathscr{D} - \kappa_{\min}\mathscr{X}_0$ and ψ by $t^{m\kappa_{\min}}\psi$, we may assume that $\kappa_{\min} = 0$.

Suppose $Y = E_0 \cap \cdots \cap E_p \cap \overline{B}_1 \cap \cdots \cap \overline{B}_q$ locally. The proof for the case $q = 0$ can be found in [BJ17, Lemma 3.5]. The new estimate we need is Equation (II.3.2). Let (z, w, y) be coordinates on U such that $E_i = \{z_i = 0\}$ and $\overline{B}_j = \{w_j = 0\}$ on U. To simplify notation, denote $\underline{z}^{\underline{a}} := z_0^{a_0} \dots z_p^{a_p}$ and $\underline{w}^{\underline{\beta}} := w_1^{\beta_1} \dots w_q^{\beta_q}$. Then, we can write ψ locally in U as

$$\begin{aligned}\psi \otimes \left(\frac{dt}{t}\right)^{\otimes m} &= u \cdot \underline{z}^{m\underline{a}} \cdot \underline{w}^{-m\underline{\beta}} \cdot \left(\frac{dz_0}{z_0} \wedge \cdots \wedge \frac{dz_p}{z_p} \wedge dw_1 \wedge \cdots \wedge dw_q \wedge d\underline{y}\right)^{\otimes m} \\ &= \frac{u \cdot \underline{z}^{m\underline{a}} \cdot \underline{w}^{m(1-\underline{\beta})}}{b_0^m} \left(\frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_p}{z_p} \wedge \frac{dw_1}{w_1} \cdots \wedge \frac{dw_q}{w_q} \wedge d\underline{y}\right)\Bigg|_{X_t}^{\otimes m} \otimes \left(\frac{dt}{t}\right)^{\otimes m}\end{aligned}$$

for some invertible function u on U. Here, the second equality follows from $\frac{dt}{t} = \sum_{i=0}^p b_i \frac{dz_i}{z_i}$. Thus, we have the following expression for ψ_t.

$$\psi_t = \frac{u \cdot \underline{z}^{m\underline{a}} \cdot \underline{w}^{m(1-\underline{\beta})}}{b_0^m} \left(\frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_p}{z_p} \wedge \frac{dw_1}{w_1} \cdots \wedge \frac{dw_q}{w_q} \wedge d\underline{y}\right)\Bigg|_{X_t}^{\otimes m} \tag{II.3.1}$$

and

$$i^{n^2}(\psi_t \wedge \overline{\psi_t})^{1/m} = \frac{|u|^{2/m}|z|^{2a}|w|^{2(1-\beta)}}{b_0^2} \left(\frac{|dz_1 \wedge d\overline{z}_1| \wedge \cdots \wedge |d\underline{y} \wedge d\underline{\overline{y}}|}{|z_1|^2 \dots |z_p|^2 |w_1|^2 \dots |w_q|^2}\right)$$

.

For $t \in \mathbb{D}^*$, denote by $\mathrm{Log}_t : ((X_t \cap U) \setminus |B|) \to \sigma_Y \times Y$ the map given by $\mathrm{Log}_t(z, w, y) = (\mathrm{Log}_U(z, w), y)$.

We now apply a change of variables using log-polar coordinates. Let $u_i = b_i \frac{\log|z_i|}{\log|t|}$ and $v_j = \frac{\log|w_j|}{\log|t|}$, $\langle \underline{\kappa}, \underline{u} \rangle := \sum_{i=0}^p \kappa_i u_i$, $\langle \underline{v}, -\underline{\beta} + 1 \rangle := \sum_{j=1}^q v_j(-\beta_j + 1)$. Then, we can write

$$\begin{aligned}&\int_{X_t \cap U} (f \circ \mathrm{Log}_U) d\mu_t = \\ &\qquad C(\log|t|^{-1})^{p+q-d} \int_{\sigma_p \times \mathbb{R}^q_{\geq 0} \times Y} |t|^{2(\langle \underline{\kappa}, \underline{u} \rangle + \langle \underline{v}, -\underline{\beta}+1 \rangle)} \left(\int_{\mathrm{Log}_t^{-1}(u,v,y)} \phi \rho_{t,u,v,y}\right) dudv|dy|^2,\end{aligned}$$

where $\phi = f \circ \mathrm{Log}_U$, $\rho_{t,u,v,y}$ is the Haar measure on the torsor $\mathrm{Log}_t^{-1}(u, v, y)$ for the (possibly disconnected) Lie-group $\{(\theta_0,\dots,\theta_p)\in(S^1)^{p+1}|e^{i\theta_0}\dots e^{i\theta_p}=1\}\times(S^1)^q$ and C is a constant.

First, let us try to figure out the order of magnitude of the expression on the right-hand side. After re-indexing, assume that $\kappa_0=\min_{i=1}^p\kappa_i$. Note that

$$\int_\sigma |t|^{2\langle\kappa,u\rangle}du = O\left(\frac{|t|^{2\kappa_0}}{(\log|t|^{-1})^{\#\{i|\kappa_i>\kappa_0\}}}\right),$$

and for a fixed N such that $\mathrm{supp}(f)\subset\{\sum_{i=0}^p u_i=1\}\times[0,N]^q$,

$$\text{(II.3.2)}\qquad \int_{[0,N]^q}|t|^{\sum_{j=1}^q(-2\beta_j+2)v_j}dv = O\left(\frac{1}{(\log|t|^{-1})^{\#\{j|\beta_j<1\}}}\right).$$

Thus, we see that

$$\int_{U\cap X_t}(f\circ \mathrm{Log}_U)\chi d\mu_t = O\left(\frac{|t|^{2\kappa_0}}{(\log|t|^{-1})^{d-p-q+\#\{i|\kappa_i>\kappa_0\}+\#\{j|\beta_j<1\}}}\right).$$

Note that the right hand side in the above expression goes off to 0, unless $\kappa_0=0$ and $d=\#\{i|\kappa_i=0\}+\#\{j|\beta_j=1\}$. This corresponds exactly to the case when there exists a face $\sigma_{Y'}\subset\sigma_Y$ such that $\sigma_{Y'}\subset\Delta(\mathscr{D})$ and $\sigma_{Y'}$ has dimension d.

After a possible re-indexing, assume that $\kappa_0=\dots=\kappa_{p'}=0$ and $\kappa_i>0$ for all $i>p'$, and $\beta_1=\dots=\beta_q=1$ and $\beta_j<1$ for all $j>q'$, and $p'+q'=d$. Then, $Y'\subset_{\text{conn. comp.}} E_0\cap\dots\cap E_{p'}\cap\overline{B}_1\cap\dots\cap\overline{B}_{q'}$.

In this case, the Poincaré residue of ψ at Y' is given by

$$|\mathrm{Res}_{Y'}(\psi)|^{2/m} = |u|^{2/m}|z_{p'+1}|^{2a_{p'+1}}\dots|z_p|^{2a_p}|w_{q'+1}|^{2(1-\beta_{q'+1})}\dots|w_q|^{2(1-\beta_q)}\cdot$$
$$\cdot\left|\frac{dz_{p'+1}}{z_{p'+1}}\wedge\dots\wedge\frac{dz_p}{z_p}\wedge\frac{dw_{q'+1}}{w_{q'+1}}\wedge\dots\wedge\frac{dw_q}{w_q}\wedge d\underline{y}\right|^2.$$

Note that $|\mathrm{Res}_{Y'}(\psi)|^{2/m}$ is a finite measure on Y' as $a_i,(1-\beta_j)>0$ for all $i>p'$ and

$j > q'$. Using the expression of ψ_t in Equation (II.3.1), we can write

$$i^{n^2}(\psi_t \wedge \overline{\psi_t})^{1/m} = \left|\frac{1}{b_0}\frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_{p'}}{z_{p'}} \wedge \frac{dw_1}{w_1} \wedge \cdots \wedge \frac{dw_{q'}}{w_{q'}}\right|^2 \wedge |\mathrm{Res}_{Y'}(\psi)|^{2/m}.$$

Make a change of variables $z_i = |t|^{u_i}e^{i\theta_i}$ for $1 \le i \le p'$ and $w_j = |t|^{v_j}e^{i\vartheta_j}$ for $1 \le j \le q'$. Writing $z' = (z_{p'+1}, \ldots, z_p)$ and $w' = (w_{q'+1}, \ldots, w_q)$, we can view (z', w', y) as coordinates on $Y' \cap U$. Let $\widetilde{\sigma} = \{(u,v) \in \mathbb{R}^{p+q}_{\ge 0} | \sum_{i=1}^{p} b_i u_i \le 1\}$. Write

$$S := \left\{(u, v, z', w', y) \in \widetilde{\sigma} \times (Y' \cap U) \left| \sum_{i=1}^{p'} b_i u_i + \sum_{i=p'+1}^{p} \frac{b_i \log|z_i|}{\log|t|} \le 1\right.\right\}$$

and let $\mathbf{1}_S$ denote its indicator function. Applying the change of variables, we get

$$\begin{aligned}\frac{1}{(2\pi\log|t|)^d}\int_{U\cap X_t}(f\circ \mathrm{Log}_U)\chi\left|\frac{1}{b_0}\frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_{p'}}{z_{p'}} \wedge \frac{dw_1}{w_1} \wedge \cdots \wedge \frac{dw_{q'}}{w_{q'}}\right|^2 \wedge |\mathrm{Res}_{Y'}(\psi)|^{2/m} \\ = \frac{1}{b_0^2(2\pi)^d}\int_{\widetilde{\sigma}\times[0,2\pi]^{p'+q'}\times(Y'\cap U)} \sum_{\{z_0 | z_0^{b_0} = t/\Pi_{i=1}^{p'} z_i^{b_i}\}} f\cdot\chi\cdot\mathbf{1}_S du\ dv\ d\theta\ d\vartheta\ |\mathrm{Res}_{Y'}(\psi)|^{2/m}.\end{aligned}$$

The integral on the right hand side is taken over $\widetilde{\sigma} \times [0, 2\pi]^{p'+q'} \times (Y' \cap U)$, where we view $(u, v) \in \widetilde{\sigma}$, $\theta_i \in [0, 2\pi]$ for $1 \le i \le p'$, $\vartheta_j \in [0, 2\pi]$ for $1 \le j \le q'$ and $(z', w', y) \in Y' \cap U$.

Let us analyze the pointwise limit of each of the factors appearing in the right hand side of the previous expression. We have that

$$f\left(1 - \sum_{i=1}^{p'} b_i u_i - \sum_{i=p'+1}^{p}\frac{b_i\log|z_i|}{\log|t|}, u, \frac{\log|z'|}{\log|t|}, v, \frac{\log|w'|}{\log|t|}\right) \to f\left(1 - \sum_{i=1}^{p'} b_i u_i, u, \underline{0}, v, \underline{0}\right)$$

pointwise on $\widetilde{\sigma} \times (Y' \cap U)$ as $t \to 0$.

As for χ, note that $z_0 \to 0$ as $t \to 0$ for a fixed $(u, v, z', w', y) \in \widetilde{\sigma} \times (Y' \cap U)$. So,

$$\chi(z_0, |t|^u e^{i\theta}, z', |t|^v e^{i\vartheta}, w', y) \to \chi(0, z', 0, w', y)$$

as $t \to 0$.

It is easy to check that $\mathbf{1}_S \to 1$ a.e on $\widetilde{\sigma} \times (Y' \cap U)$ as $t \to 0$, and from our analysis in Proposition II.3, we have that $b_{\sigma_{Y'}}^{-1} \lambda_{\sigma_{Y'}} = \frac{1}{b_0} dudv$ under the homeomorphism $\sigma_{Y'} \xrightarrow{\simeq} \widetilde{\sigma}$ given by $(u_0, \ldots, u_{p'}, v_1, \ldots, v_{q'}) \to (u_1, \ldots, u_{p'}, v_1, \ldots, v_{q'})$. The remaining factor of $\frac{1}{b_0}$ is taken care of by the fact that the number of solutions z_0 to the equation $z_0^{b_0} = \frac{t}{\Pi_{i=1}^{p} z_i^{b_i}}$ is exactly b_0.

Using Lebesgue's dominated convergence theorem, we have the result. $\square$

The following lemma helps to 'glue' to the result of the previous lemma to obtain a global version.

Lemma II.11. Let L be a compact subset of $(\mathscr{X}, |B|)^{\text{hyb}}$. Then, $\limsup_{t \to 0} \int_{X_t \cap L} d\mu_t < \infty$.

Proof. Consider a global log function Log_V on $\mathscr{X}$. Without loss of generality, assume that $V = \mathscr{X}$. We may further rescale t to assume that $\{\text{Log}_{V^{\text{hyb}}}^{-1}(K) \cap \pi^{-1}(\frac{1}{2}\overline{\mathbb{D}})\}_{K \subset_{cpt} \Delta(\mathscr{X}, |B|)}$ forms a compact exhaustion of $(\mathscr{X}, |B|)^{\text{hyb}} \cap \pi^{-1}(\frac{1}{2}\overline{\mathbb{D}})$ (see Proposition II.8). So, we may enlarge L to assume that $L = \text{Log}_{V^{\text{hyb}}}^{-1}(K) \cap \pi^{-1}(\frac{1}{2}\overline{\mathbb{D}})$ for some compact $K \subset \Delta(\mathscr{X}, |B|)$.

We wish to show that $\limsup_{t \to 0} \int_{X_t} \mathbf{1}_K \circ \text{Log}_V d\mu_t < \infty$. Let $V = \bigcup_{i \in I} U_i$ for adapted coordinate charts U_i and let $\{\chi_i\}_{i \in I}$ be a partition of unity on $\{U_i\}_{i \in I}$ such that $\text{Log}_V = \sum_i \chi_i \text{Log}_{U_i}$. It is enough to show that $\limsup_{t \to 0} \int_{U_i \cap X_t} \chi_i (\mathbf{1}_K \circ \text{Log}_V) d\mu_t < \infty$ for all i.

Since $\text{Log}_V - \text{Log}_{U_i} = O(\frac{1}{\log |t|^{-1}})$ on the support of χ_i, we can find a compactly supported continuous function f on $\Delta(\mathscr{X}, |B|)$ such that $f \circ \text{Log}_{U_i} \geq \mathbf{1}_K \circ \text{Log}_V$ on $(U_i \setminus (\mathscr{X}_0 + |\overline{B}|)) \cap \text{supp}(\chi_i)$.

Then,

$$\limsup_{t \to 0} \int_{U_i \cap X_t} \chi_i (\mathbf{1}_K \circ \text{Log}_V) d\mu_t \leq \lim_{t \to 0} \int_{U_i \cap X_t} \chi_i (f \circ \text{Log}_{U_i}) d\mu_t,$$

and the right hand side exists and is finite by Lemma II.10. $\square$

We now prove the statement of Theorem B for functions that are pulled back from compactly-supported continuous functions on $\Delta(\mathscr{X}, |B|)$ via a global Log map.

Lemma II.12. Let f be a continuous compactly supported function on $\Delta(\mathscr{X}, |B|)$ and let Log_V be a global log function on $\mathscr{X}$. Then, $\int_{X_t} (f \circ \text{Log}_V) d\mu_t \to \int_{\Delta(\mathscr{X}, |B|)} f d\mu_0$ as $t \to 0$.

Proof. Let $V = \bigcup_{i \in I} U_i$ and let χ_i be a partition of unity on U_i so that $\mathrm{Log}_V = \sum_i \chi_i \mathrm{Log}_{U_i}$. Let Y_i be the stratum associated to U_i.

Then, we can write $\int_{X_t} (f \circ \mathrm{Log}_V) d\mu_t = \sum_i \int_{U_i \cap X_t} \chi_i (f \circ \mathrm{Log}_V) d\mu_t$. It follows from Lemma II.11 and from Proposition II.6 that

$$\lim_{t \to 0} \left| \int_{U_i \cap X_t} \chi_i (f \circ \mathrm{Log}_V) d\mu_t - \int_{U_i \cap X_t} \chi_i (f \circ \mathrm{Log}_{U_i}) d\mu_t \right| = 0. \tag{II.3.3}$$

If σ_{Y_i} contains a maximal face $\sigma_{Y'}$ of $\Delta(\mathscr{D})$, it follows from Lemma II.10 that

$$\lim_{t \to 0} \int_{U_i \cap X_t} \chi_i (f \circ \mathrm{Log}_{U_i}) d\mu_t = \left(\int_{Y'} \chi_i |\mathrm{Res}_{Y'}(\psi)|^{2/m} \right) \int_{\sigma_{Y'}} f b_{\sigma_{Y'}}^{-1} \lambda_{\sigma_{Y'}}$$

.

If σ_{Y_i} does not contain a maximal face of $\Delta(\mathscr{D})$, then the above limit is 0. Note that any σ_{Y_i} contains at most one maximal face $\sigma_{Y'}$ of $\Delta(\mathscr{D})$ and this happens if and only if Y' intersects U_i. Thus, for all $i \in I$, we have

$$\lim_{t \to 0} \int_{U_i \cap X_t} \chi_i (f \circ \mathrm{Log}_{U_i}) d\mu_t = \sum_{\sigma \subset \Delta(\mathscr{D}), \dim(\sigma) = d} \left(\int_{Y_\sigma} \chi_i |\mathrm{Res}_{Y_\sigma}(\psi)|^{2/m} \right) \left(\int_\sigma f b_\sigma^{-1} \lambda_\sigma \right). \tag{II.3.4}$$

Combining Equations (II.3.3) and (II.3.4), we are done. □

Now, we are ready to prove Theorem B.

Proof of Theorem B. Let f be a continuous compactly supported function on $(\mathscr{X}, |B|)^{\mathrm{hyb}}$. Fix a global log function Log_V and let χ be a continuous function on $(\mathscr{X}, |B|)^{\mathrm{hyb}}$ that is 1 in a neighborhood of $\Delta(\mathscr{X}_0, B)$ and is supported in $\pi^{-1}(\frac{1}{2}\overline{\mathbb{D}})$. By replacing f by $(f|_{\Delta(\mathscr{X}, |B|)} \circ \mathrm{Log}_{V^{\mathrm{hyb}}}) \cdot \chi - f$, we may assume that $f|_{\Delta(\mathscr{X}, |B|)} = 0$.

Let $K = \mathrm{supp}(f)$ and pick $\epsilon > 0$. Since f is continuous and compactly supported, there exists $t_0 \ll 1$ such that $|f| \leq \epsilon$ on $\pi^{-1}(t_0 \overline{\mathbb{D}})$. Then, $\limsup_{t \to 0} |\int_{X_t} f d\mu_t| \leq \epsilon \limsup_{t \to 0} \int_{K \cap X_t} d\mu_t$, which goes to 0 as $\epsilon \to 0$ by Lemma II.11. □

Remark II.13 (Independence of m)**.** Note that Theorem B seems to involve the choice of a sufficiently divisible integer m. However, it is easy to see that both the measures μ_t and μ_0 remain invariant if we replace m by km for some positive integer k.

Example II.14 (Convergence of Haar measure on $(\mathbb{P}^1, 0+\infty)^{\text{hyb}}$)**.** In the setting of Example II.7, let μ_t denote the Haar measure on $(\mathbb{P}^1 \setminus \{0,\infty\}) \times \{t\}$. Then, $\frac{1}{2\pi \log|t|^{-1}}\mu_t$ weakly converges to the Lebesgue measure on $\mathbb{R} \simeq \Delta(\mathscr{X}, |B|)$ as measures on the hybrid space $(\mathbb{P}^1, 0+\infty)^{\text{hyb}}$.

More generally, we can prove a similar result for toric varieties.

Example II.15 (Convergence for the Haar measure on a torus)**.** Let N be a free abelian group of rank n. Let $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ and $T = \text{Spec}\ (\mathbb{C}[M])$ be the associated torus. Let Y be a smooth projective toric compactification of T i.e. a smooth projective toric variety associated to a regular fan in $N_{\mathbb{R}}$ (For example, $Y = \mathbb{P}^n$). Let ω be a torus invariant meromorphic n-form on Y. Note that there is a canonical choice of such an ω up to a sign and ω has poles of order one along all boundary divisors. Let D be the reduced divisor given by the sum of the boundary divisors. Then, $K_Y + D$ is linearly equivalent to 0 and $\omega \in H^0(K_Y + D)$ is a trivializing section.

Consider the constant family $Y \times \mathbb{D}^*$ over $\mathbb{D}$. Then $(Y \times \mathbb{D}^*, D \times \mathbb{D}^*)$ is log smooth and consider the projective snc model $\mathscr{Y} = Y \times \mathbb{D}$ of $(Y \times \mathbb{D}^*, D \times \mathbb{D}^*)$. Then, $\Delta(\mathscr{Y}, D \times \mathbb{D}^*)$ is canonically isomorphic to $N_{\mathbb{R}}$, with the faces given by the cones in the fan defining Y. Thus we have a hybrid space given by $(\mathscr{Y}, D \times \mathbb{D}^*)^{\text{hyb}} = (T \times \mathbb{D}^*) \cup N_{\mathbb{R}}$. We also get a top-dimensional meromorphic form η on $Y \times \mathbb{D}^*$ whose restriction to each fiber gives the measure ω. Let μ_t denote the measure given by $\frac{i^{n^2} \omega \wedge \overline{\omega}}{(2\pi \log|t|^{-1})^n}$ on the fiber $T \times \{t\}$.

Applying Theorem B to this setting, we get that the measures μ_t converge to the Lebesgue measure on each of the cones. The Lebesgue measures on each of the cones is exactly the Lebesgue measure on $N_{\mathbb{R}}$ (normalized such that the lattice N has unit covolume) restricted to that cone. Thus, μ_t converges weakly to the Lebesgue measure on $N_{\mathbb{R}}$ as $t \to 0$.

II.4: Convergence on the limit hybrid model

Let (X, B) be a log-smooth pair over $\mathbb{D}^*$ with removable singularities at the origin (i.e. there exists an snc model of (X, B)). The choice of a hybrid space $(\mathscr{X}, |B|)^{\text{hyb}}$ depends on the choice of an snc model $\mathscr{X}$ of (X, B). We construct a canonical hybrid space $(X, |B|)^{\text{hyb}}$ that does not depend on a choice of a model. Such a space is obtained by an inverse limit $(X, |B|)^{\text{hyb}} = \varprojlim_{\mathscr{X}} (\mathscr{X}, |B|)^{\text{hyb}}$. Theorem II.16 implies that this definition matches with the definition in the introduction when (X, B) is a projective over $\mathbb{D}^*$. We also explain how the space $(X, |B|)^{\text{hyb}}$ can itself be viewed as a Berkovich analytic space when (X, B) is projective over $\mathbb{D}^*$.

II.4.1: The limit hybrid model

Given two models $\mathscr{X}', \mathscr{X}$ of (X, B), there is always a bimeromorphic map $\mathscr{X}' \dashrightarrow \mathscr{X}$ induced by the given isomorphism with X over $\mathbb{D}^*$. We say that $\mathscr{X}'$ dominates $\mathscr{X}$ when this bimeromorphic map extends to a morphism. More precisely, we say that $\mathscr{X}'$ dominates $\mathscr{X}$ if we have a proper holomorphic map $\mathscr{X}' \to \mathscr{X}$ which is compatible with the isomorphisms $\mathscr{X}'|_{\mathbb{D}^*} \simeq X \simeq \mathscr{X}|_{\mathbb{D}^*}$.

When $\mathscr{X}$ and $\mathscr{X}'$ are snc models of (X, B) such that $\mathscr{X}'$ dominates $\mathscr{X}$ via a map $\pi : \mathscr{X}' \to \mathscr{X}$, we also have an integral affine map $\pi_* : \Delta(\mathscr{X}', |B|) \to \Delta(\mathscr{X}, |B|)$ and also a continuous surjective map $(\mathscr{X}', |B|)^{\text{hyb}} \to (\mathscr{X}, |B|)^{\text{hyb}}$ as in Section 4.2 and Section 4.8 of [BJ17]. If $\sigma_{Y'}$ is a face of $\Delta(\mathscr{X}', |B|)$, associated to a stratum Y' of $\mathscr{X}'_0$, let Y be the smallest stratum that contains $\pi(Y')$. Then, $\pi_*(\sigma_{Y'}) \subset \sigma_Y$. We describe these maps in detail in the projective case in the following subsection.

The collection of all snc models of (X, B) is a directed system. See [BJ17, Lemma 4.1] for more details. We can then define $(X, |B|)^{\text{hyb}} := \varprojlim_{\mathscr{X}} (\mathscr{X}, |B|)^{\text{hyb}}$. It is easy to see that we have a projection map $(X, |B|)^{\text{hyb}} \to \mathbb{D}$ such that $\pi^{-1}(\mathbb{D}^*) \simeq X \setminus |B|$, and the central fiber, ${(X, |B|)^{\text{hyb}}}_0$, is $\varprojlim_{\mathscr{X}} \Delta(\mathscr{X}, |B|)$. Here the inverse limit runs over all snc models $\mathscr{X}$ of

(X, B), and the inverse limit is taken in the category of topological spaces. Theorem II.16 tells us why this definition of $(X, |B|)^{\mathrm{hyb}}$ matches with the one in the introduction when (X, B) is projective.

Suppose now that (X, B) is projective over $\mathbb{D}^*$, i.e. we can view X as a closed subset of $\mathbb{P}^N \times \mathbb{D}^*$ for some N such that X and B are cut out by polynomials whose coefficients are holomorphic on $\mathbb{D}^*$ and meromorphic on $\mathbb{D}$. Thus, we can view the coefficients of the defining equations as elements of $\mathbb{C}(\!(t)\!)$. Using the same defining equations in $\mathbb{P}^N_{\mathbb{C}(\!(t)\!)}$, we get varieties $X_{\mathbb{C}(\!(t)\!)}$ and $B_{\mathbb{C}(\!(t)\!)}$ over Spec $\mathbb{C}(\!(t)\!)$. A projective snc model $\mathscr{X}$ of (X, B) gives rise to an snc model $\mathscr{X}_{\mathbb{C}[[t]]}$ over Spec $\mathbb{C}[[t]]$ whose generic fiber is $X_{\mathbb{C}(\!(t)\!)}$ and special fiber is $\mathscr{X}_0$, and $\mathscr{X}_0 + \overline{B_{\mathbb{C}(\!(t)\!)}}$ is an snc divisor in $\mathscr{X}_{\mathbb{C}[[t]]}$. Then, we can define $\Delta(\mathscr{X}_{\mathbb{C}[[t]]}, B_{\mathbb{C}(\!(t)\!)})$ similar to the construction in Section II.1.2, and we have a canonical identification $\Delta(\mathscr{X}_{\mathbb{C}[[t]]}, B_{\mathbb{C}(\!(t)\!)}) \simeq \Delta(\mathscr{X}, |B|)$.

The following theorem, analogous to [KS06, Theorem 10] [BFJ16, Cor 3.2], realizes the central fiber ${(X, |B|)^{\mathrm{hyb}}}_0$ as a non-Archimedean space.

Theorem II.16. Let (X, B) be a projective log-smooth pair over $\mathbb{D}^*$. We have an isomorphism $X^{\mathrm{an}}_{\mathbb{C}(\!(t)\!)} \setminus |B|^{\mathrm{an}}_{\mathbb{C}(\!(t)\!)} \simeq \varprojlim_{\mathscr{X}} \Delta(\mathscr{X}, |B|)$ where $(\_)^{\mathrm{an}}$ denotes the Berkovich analytification with respect to the t-adic norm on $\mathbb{C}(\!(t)\!)$ and, the inverse limit is taken over all projective snc models $(\mathscr{X}, B)$ of (X, B).

We will prove the above theorem in the following subsection, after setting up some preliminaries.

II.4.2: The central fiber of the limit hybrid model as a non-Archimedean space

In this section, we will work over the field $\mathbb{C}(\!(t)\!)$ instead of $\mathbb{D}^*$. So, let X be a smooth projective variety over the discretely valued field $K = \mathbb{C}(\!(t)\!)$, B be an snc divisor on X and, $\mathscr{X}$ be a smooth projective integral scheme over $R = \mathbb{C}[[t]]$ along with a specified isomorphism $\mathscr{X}_K \simeq X$ such that $\mathscr{X}$ is an snc model of (X, B) (that is, $\mathscr{X}_0 + \overline{B}$ is a snc divisor in $\mathscr{X}$). Then, $\Delta(\mathscr{X}, |B|)$ is the dual intersection complex defined similar to the construction in

Section II.1.2. We also have a CW complex $\Delta(\mathscr{X}) := \Delta(\mathscr{X}, 0)$, which can be viewed as a subcomplex of $\Delta(\mathscr{X}, |B|)$.

Let X^{an} and $|B|^{\text{an}}$ denote the Berkovich analytification of X and $|B|$, respectively, with respect to the t-adic norm on K. We recall a few definitions related to the Berkovich analytification that will be useful in this section. Recall that points in X^{an} correspond to valuations on the residue field at (scheme) points of X that extend the valuation on $\mathbb{C}(\!(t)\!)$. This gives us a continuous map $\ker : X^{\text{an}} \to X$ which sends a valuation to the underlying point i.e a point $x \in X^{\text{an}}$ is a valuation on the residue field of its *kernel* $\ker(x) \in X$. We also have a map $\text{red}_{\mathscr{X}} : X^{\text{an}} \to \mathscr{X}_0$ defined as follows.

Let $x \in X^{\text{an}}$ and let $k(\ker(x))$ denote the residue field at $\ker(x)$. Then x is a valuation on $k(\ker(x))$. Let $k(\ker(x))^{\circ}$ denote the valuation ring in $k(\ker(x))$ with respect to the valuation x. By the valuative criteria of properness, the map Spec $(k(\ker(x))) \to X$ lifts to a unique map Spec $(k(\ker(x))^{\circ}) \to \mathscr{X}$. The image of the closed point of Spec $(k(\ker(x))^{\circ})$ is denoted as $\text{red}_{\mathscr{X}}(x)$ and is called the *center* of the valuation x. The map $\text{red}_{\mathscr{X}} : X^{\text{an}} \to \mathscr{X}_0$ is anti-continuous in the sense that the inverse image of an open set is closed.

We have an inclusion $i_{\mathscr{X}} : \Delta(\mathscr{X}) \to X^{\text{an}}$ and a retraction $r_{\mathscr{X}} : X^{\text{an}} \to \Delta(\mathscr{X})$ as constructed in [MN15]. We would like to do a similar construction for $\Delta(\mathscr{X}, |B|)$ and $X^{\text{an}} \setminus |B|^{\text{an}}$.

Let $\mathscr{X}$ be a snc model of (X, B). Then, we have an inclusion map $i_{(\mathscr{X},|B|)} : \Delta(\mathscr{X}, |B|) \to X^{\text{an}} \setminus |B|^{\text{an}}$, which is given as follows. Let $Y \subset_{\text{conn. comp.}} E_0 \cap \dots \cap E_p \cap \overline{B}_1 \cap \dots \cap \overline{B}_q$ denote a stratum of $\mathscr{X}_0$. Pick a point $(r_0, \dots, r_p, s_1, \dots, s_q) \in \sigma_Y$. Let z_i and w_j locally define E_i and $\overline{B}_j$ near Y for $0 \leq i \leq p$ and $1 \leq j \leq q$. Then, we have an isomorphism $\widehat{\mathcal{O}_{\mathscr{X}_0,Y}} \simeq \mathbb{C}[[z_0, \dots, z_p, w_1, \dots, w_q]]$. Pulling back the valuation defined by $\nu(\sum_{\alpha \in \mathbb{N}^{p+1}, \beta \in \mathbb{N}^q} c_{\alpha,\beta} z^{\alpha} w^{\beta}) = \min_{c_{\alpha,\beta} \neq 0} \{\alpha \cdot \underline{r} + \beta \cdot \underline{s}\}$, we get an element of $X^{\text{an}} \setminus |B|^{\text{an}}$. It is clear that $i_{(\mathscr{X},|B|)}$ is injective, and it follows from [MN15, Proposition 3.1.4] that $i_{(\mathscr{X},|B|)}$ is continuous. We will often identify $\Delta(\mathscr{X}, |B|)$ with its image under $i_{(\mathscr{X},|B|)}$.

We also have a continuous retraction map $r_{(\mathscr{X},|B|)} : X^{\text{an}} \setminus |B|^{\text{an}} \to \Delta(\mathscr{X}, |B|)$, which is a left inverse to the map $i_{(\mathscr{X},|B|)}$, defined as follows. Pick $x \in X^{\mathrm{an}} \setminus |B|^{\mathrm{an}}$ and let $\mathrm{red}_{\mathscr{X}}(x)$ be the center of the valuation x. Pick the smallest stratum $Y \subset_{\text{conn. comp.}} E_0 \cap \cdots \cap E_p \cap \overline{B}_1 \cap \cdots \cap \overline{B}_q$ containing $\mathrm{red}_{\mathscr{X}}(x)$. Then, we define

$$r_{(\mathscr{X},|B|)}(x) = (\nu_x(E_0), \ldots, \nu_x(E_p), \nu_x(\overline{B}_1), \ldots, \nu_x(\overline{B}_q))$$

in σ_Y.

To see why $r_{(\mathscr{X},|B|)}$ is continuous, recall that the map $X^{\mathrm{an}} \to \mathscr{X}_0$ taking any valuation to its center is anti-continuous (i.e. the inverse image of a closed set is open). For any stratum $Y \subset_{\text{conn. comp.}} E_0 \cap \cdots \cap E_p \cap \overline{B}_1 \cap \cdots \cap \overline{B}_q$ of $\mathscr{X}_0$, the subset $r^{-1}_{(\mathscr{X},|B|)}(\sigma_Y) \subset X^{\mathrm{an}} \setminus |B|^{\mathrm{an}}$ is a closed set as it corresponds to a subset of X^{an} whose center lies on an open set of $\mathscr{X}_0$. Therefore, it is enough to prove that $r_{(\mathscr{X},|B|)}|_{r^{-1}_{(\mathscr{X},|B|)}(\sigma_Y)} : r^{-1}_{(\mathscr{X},|B|)}(\sigma_Y) \to \sigma_Y$ is continuous for all possible strata Y. But this is clear from the description of the map above.

We also have a continuous retraction map $\phi_{\mathscr{X}} : \Delta(\mathscr{X}, |B|) \to \Delta(\mathscr{X})$, which we obtain from the composition.

$$\Delta(\mathscr{X}, |B|) \xrightarrow{i_{(\mathscr{X},|B|)}} X^{\mathrm{an}} \setminus |B|^{\mathrm{an}} \hookrightarrow X^{\mathrm{an}} \xrightarrow{r_{\mathscr{X}}} \Delta(\mathscr{X}).$$

Explicitly, if $Y \subset_{\text{conn. comp.}} E_0 \cap \cdots \cap E_p \cap \overline{B}_1 \cap \cdots \cap \overline{B}_q$, let $Y' \subset_{\text{conn. comp.}} E_0 \cap \cdots \cap E_p$ be the stratum containing Y. Then, $\phi_{\mathscr{X}}(\sigma_Y) \subset \sigma_{Y'}$ and

$$\phi_{\mathscr{X}}(r_0, \ldots, r_p, s_1, \ldots, s_q) = (r_0, \ldots, r_p).$$

If $\mathscr{X}$ and $\mathscr{X}'$ are two snc models of (X, B) such that $\mathscr{X}'$ dominates $\mathscr{X}$, then there is a surjective map $r_{\mathscr{X}',\mathscr{X},|B|} : \Delta(\mathscr{X}', |B|) \to \Delta(\mathscr{X}, |B|)$ given by

$$\Delta(\mathscr{X}', |B|) \xhookrightarrow{i_{(\mathscr{X}',|B|)}} X^{\mathrm{an}} \setminus |B|^{\mathrm{an}} \xrightarrow{r_{(\mathscr{X},|B|)}} \Delta(\mathscr{X}, |B|).$$

The surjectivity of the map follows from [MN15, Proposition 3.17].

We have an explicit description of $r_{\mathscr{X}',\mathscr{X},|B|}$ similar to [BJ17, Section 4.2] as follows. Let $\rho : \mathscr{X}' \to \mathscr{X}$ denote the proper map between $\mathscr{X}'$ and $\mathscr{X}$, let $Y' \subset_{\text{conn. comp.}} E'_0 \cap \cdots \cap E'_{p'} \cap \overline{B}_1 \cap \cdots \cap \overline{B}_{q'}$ be a stratum of $\mathscr{X}'_0$, and let $Y \subset_{\text{conn. comp.}} E_0 \cap \cdots \cap E_p \cap \overline{B}_1 \cdots \cap \overline{B}_q$ be the stratum of $\mathscr{X}_0$ containing the image of Y'. Note that $q' \leq q$. Let $E'_i, \overline{B}_j$ be locally defined by $z'_i = 0$ and $w'_j = 0$ near Y' and let E_i and $\overline{B}_j$ be locally defined by $z_i = 0$ and $w_j = 0$ near Y. Then, we can write $\rho^*(z_i) = u_i \cdot \prod_{k=0}^{p'} (z'_k)^{c_{i,k}}$ and $\rho^* w_j = v_j \cdot w'_j \cdot \prod_{k=0}^{p'} (z'_k)^{d_{j,k}}$ for units $u_i, v_j \in \mathcal{O}_{\mathscr{X}',Y'}$ and for some $c_{i,k}, d_{j,k} \in \mathbb{N}$. Then, $r_{\mathscr{X}',\mathscr{X},|B|}(\sigma_{Y'}) \subset \sigma_Y$ and is given by

$$r_i = \sum_{k=0}^{p'} c_{i,k} r'_k$$

and

$$s_j = s'_j + \sum_{k=0}^{p'} d_{j,k} r'_k.$$

Proposition II.17. We have a commutative diagram

$$\begin{tikzcd}
\Delta(\mathscr{X}', |B|) \arrow[r, "\phi_{\mathscr{X}'}"] \arrow[d, "r_{\mathscr{X}',\mathscr{X},|B|}"] & \Delta(\mathscr{X}') \arrow[d, "r_{\mathscr{X}',\mathscr{X}}"] \\
\Delta(\mathscr{X}, |B|) \arrow[r, "\phi_{\mathscr{X}}"] & \Delta(\mathscr{X})
\end{tikzcd}$$

which gives rise to a continuous map $\phi : \varprojlim_{\mathscr{X}} \Delta(\mathscr{X}, |B|) \to \varprojlim_{\mathscr{X}} \Delta(\mathscr{X})$.

Proof. To see that the diagram commutes, it enough to use the fact that $r_{\mathscr{X}',\mathscr{X}} \circ r_{\mathscr{X}'} = r_{\mathscr{X}}$ [MN15, Proposition 3.1.7] and show that $\phi_{\mathscr{X}} \circ r_{(\mathscr{X},|B|)} = r_{\mathscr{X}}$ on $X^{\text{an}} \setminus |B|^{\text{an}}$. Pick $\nu \in X^{\text{an}} \setminus |B|^{\text{an}}$. Let $Y \subset_{\text{conn. comp.}} E_0 \cap \cdots \cap E_p \cap \overline{B}_1 \cap \cdots \cap \overline{B}_q$ be the minimal stratum of $\mathscr{X}_0 + \overline{B}$ containing the center of ν. Then,

$$r_{(\mathscr{X},|B|)}(\nu) = (\nu(E_0), \ldots, \nu(E_p), \nu(\overline{B}_1), \ldots, \nu(\overline{B}_q))$$

in σ_Y.

Let $Y' \subset_{\text{conn. comp.}} E_0 \cap \cdots \cap E_p$ be the stratum containing Y. Then, Y' is the minimal stratum in $\mathscr{X}_0$ containing the center of ν and $r_{\mathscr{X}}(\nu) = (\nu(E_0), \ldots, \nu(E_p))$ in $\sigma_{Y'}$. It follows from the description of $\phi_{\mathscr{X}}$ that $\phi_{\mathscr{X}}(r_{(\mathscr{X},|B|)}(\nu)) = r_{\mathscr{X}}(\nu)$ □

Proposition II.18. If $\mathscr{X}'$ is a blowup of $\mathscr{X}$ along a stratum $Y \subset_{\text{conn. comp.}} E_0 \cap \cdots \cap E_p \cap \overline{B}_1 \cap \cdots \cap \overline{B}_q$, then $r_{\mathscr{X}',\mathscr{X},|B|} : \Delta(\mathscr{X}', |B|) \to \Delta(\mathscr{X}, |B|)$ is a homeomorphism obtained by a subdivision.

Proof. This follows from a local blowup computation. Let E' denote the exceptional divisor in $\mathscr{X}'$. Then, the maximal strata of $\mathscr{X}'$ that map down to Y are of the form $E' \cap \widetilde{E}_I \cap \widetilde{\overline{B}}_1 \cap \cdots \cap \widetilde{\overline{B}}_q$ and $E' \cap \widetilde{E}_0 \cap \cdots \cap \widetilde{E}_p \cap \widetilde{\overline{B}}_J$, where I and J denote subsets of $\{0, \ldots, p\}$ and $\{1, \ldots, q\}$ of size p and $(q-1)$ respectively and $\widetilde{E}_i$ and $\widetilde{\overline{B}}_j$ denote the strict transforms of E_i and $\overline{B}_j$.

First, let's compute the image of $\sigma_{E'}$ in $\Delta(\mathscr{X}, |B|)$. Note that $\operatorname{div}_{\mathscr{X}'}(t) = \sum_i b_i \widetilde{E}_i + (\sum_{i=0}^p b_i)E'$. Let $\nu_{E'}$ denote the divisorial valuation corresponding to $\sigma_{E'}$. Then,

$$\nu_{E'}(E_i) = \nu_{E'}(\widetilde{E}_i + E') = \nu_{E'}(E') = \frac{1}{\operatorname{ord}_{E'}(t)} = \frac{1}{\sum_{i=0}^p b_i}$$

for all $i = 0, \ldots, p$. Similarly, $\nu_{E'}(B_j) = \frac{1}{\sum_{i=0}^p b_i}$ for all $j = 1, \ldots, q$. Thus, the image of $\sigma_{E'}$ in $\Delta(\mathscr{X}, |B|)$ is $\frac{1}{\sum_{i=0}^p b_i}(1, \ldots, 1) \in \sigma_Y$.

It is easy to check that the $\Delta(\mathscr{X}', |B|) \to \Delta(\mathscr{X}, |B|)$ is a subdivision obtained by adding the vertex $\sigma_{E'}$ to σ_Y. For example, let's compute the image of $\sigma_{Y'}$ for $Y' = E' \cap \widetilde{E}_1 \cap \cdots \cap \widetilde{E}_p \cap \widetilde{\overline{B}}_1 \cap \cdots \cap \widetilde{\overline{B}}_q$. Note that

$$\sigma_{Y'} = \left\{ (x_0, \ldots, x_p, y_1, \ldots, y_q) \;\middle|\; \left(\sum_{i=0}^p b_i\right) x_0 + \sum_{i=1}^p b_i x_i = 1 \right\}.$$

Suppose ν is a valuation represented by $(x_0, \ldots, x_p, y_1, \ldots, y_q) \in \sigma_{Y'}$. Then, $\nu(E_0) = \nu(\widetilde{E}_0 + E') = \nu(E') = x_0$ and $\nu(E_i) = \nu(\widetilde{E}_i + E') = x_i + x_0$ for $i = 1, \ldots, p$. Similarly, $\nu(\overline{B}_j) = y_j + x_0$ for $j = 1, \ldots, q$.

Thus, we see that $r_{\mathscr{X}',\mathscr{X},|B|}|_{\sigma_{Y'}}$ is given by

$$(x_0, \dots, x_p, y_1, \dots, y_q) \mapsto (x_0, x_1 + x_0, \dots, x_p + x_0, y_1 + x_0, \dots, y_q + x_0)$$

$\square$

In general, the map $\Delta(\mathscr{X}') \to \Delta(\mathscr{X})$ is not a homeomorphism, as illustrated by the following example.

Example II.19 (Blowup of $\mathbb{P}^1 \times \mathbb{D}$)**.** Let the notation be the same as in Example II.2. Let $E_0 = \mathbb{P}^1 \times \{0\}$, $\overline{B}_1 = \{0\} \times \mathbb{D}$, $\overline{B}_2 = \{\infty\} \times \mathbb{D}$. Let $\mathscr{X}'$ denote the blowup of $\mathscr{X}$ at $E_0 \cap \overline{B}_1$ and let $\mathscr{X}''$ denote the blowup of $\mathscr{X}$ at some point in E_0 that is different from 0 and ∞. Then $\Delta(\mathscr{X}', |B|)$ is obtained from $\Delta(\mathscr{X}, |B|)$ by adding a vertex along the ray $E_0 \cap \overline{B}_1$ and $\Delta(\mathscr{X}'', |B|)$ is obtained from $\Delta(\mathscr{X}, |B|)$ by adding an extra vertex and joining it to σ_{E_0}. The retraction $r_{\mathscr{X}',\mathscr{X},|B|} : \Delta(\mathscr{X}', |B|) \to \Delta(\mathscr{X}, |B|)$ is an isomorphism, while $r_{\mathscr{X}'',\mathscr{X},B} : \Delta(\mathscr{X}'', |B|) \to \Delta(\mathscr{X}, |B|)$ is given by collapsing the newly added edge and vertex to σ_{E_0}.

(a) (b)

Figure II.3: The dual complex of the $(\mathbb{P}^1 \times \mathbb{D}, \{0\} \times \mathbb{D} + \{\infty\} \times \mathbb{D})$ after blowing up after (a) blowing up at $(0, 0)$, and (b) blowing up at (1,0)

Lemma II.20. Let $\mathscr{X}$ be a snc model of (X, B) and let $K \subset \Delta(\mathscr{X}, |B|)$ be a compact set. Then there exists a snc model $\mathscr{X}'$ of (X, B) dominating $\mathscr{X}$ such that $r^{-1}_{\mathscr{X}',\mathscr{X},|B|}(K) \subset \Delta(\mathscr{X}')$.

Proof. For a valuation $\nu \in X^{\mathrm{an}}$ and a divisor $D \subset \mathscr{X}$ not contained in $\overline{\{\ker \nu\}}$, set $\nu(D) := \nu(f)$, where f defines D in an open neighborhood of the $\mathrm{red}_{\mathscr{X}}(\nu)$. We identify $\Delta(\mathscr{X}, |B|)$ with its image under $i_{(\mathscr{X},|B|)}$ and think of points in $\Delta(\mathscr{X}, |B|)$ as valuations.

Since it is enough to prove the result for some small enough compact neighborhoods of all points in K, we may assume without loss of generality that there exists an irreducible component E of $\mathscr{X}_0$ and an $\epsilon > 0$ such that $\nu(E) \geq \epsilon$ for all $\nu \in K$. Let $\overline{B}_1, \ldots, \overline{B}_q$ be the irreducible components of $|\overline{B}|$ containing the centers of all $\nu \in K$. It is enough to show that there exists an snc model $\mathscr{X}'$ of (X, B) such that $\mathrm{red}_{\mathscr{X}'}(\nu')$ is not contained in the closures of $\overline{B}_1, \ldots, \overline{B}_q$ in $\mathscr{X}'$ for all $\nu' \in \Delta(\mathscr{X}, |B|)$ such that $r_{\mathscr{X}', \mathscr{X}, |B|}(\nu') \in K$. Note that if $q = 0$, we are done. We will prove the result by induction on q.

Pick $N > 0$ large enough so that $N\nu(E) \geq \nu(\overline{B}_1)$ for all $\nu \in K$. Let $\mathcal{I}_E$ and $\mathcal{I}_{\overline{B}_1}$ be the ideal sheaf defining E and $\overline{B}_1$ respectively. Let $\widetilde{\mathscr{X}}$ be the blowup of $\mathscr{X}$ along the ideal sheaf $\mathcal{I}_E^N + \mathcal{I}_{\overline{B}_1}$. Then, $\widetilde{\mathscr{X}}$ is a model of X although it may not necessarily be regular. Pick $\nu \in K$ and let U be an affine open neighborhood of $\mathrm{red}_{\mathscr{X}}(\nu)$. If E is defined by $z = 0$ and $\overline{B}_1$ is defined by $w_1 = 0$ on U, then $\widetilde{U} = \mathrm{Spec}\ \mathcal{O}_{\mathscr{X}}(U)[\frac{z^N}{w_1}]$ is a chart of the blowup. Let $\mathscr{X}'$ be a resolution of singularities of $\widetilde{\mathscr{X}}$ such that $\mathscr{X}'$ is a snc model for (X, B). Pick $\nu' \in \Delta(\mathscr{X}', |B|)$ such that $r_{\mathscr{X}', \mathscr{X}, |B|}(\nu') = \nu$. Then, $\nu'(\frac{z^N}{w_1}) = \nu(\frac{z^N}{w_1}) \geq 0$. Thus, the center of ν' in $\widetilde{\mathscr{X}}$ is contained in $\widetilde{U}$. But $\widetilde{U}$ misses the strict transform of $\overline{B}_1$, and thus the center of ν' in $\mathscr{X}'$ is not contained in $\overline{B}_1$. Thus, the irreducible components of $\overline{B}$ in $\mathscr{X}'$ containing the centers of any valuations $\nu' \in r^{-1}_{\mathscr{X}', \mathscr{X}, |B|}(K)$ can only be $\overline{B}_2, \ldots, \overline{B}_q$. Thus we are done by induction. □

To simplify the discussion, for the remainder of this subsection we will identify $\Delta(\mathscr{X}, |B|)$ with its image under $i_{(\mathscr{X}, |B|)}$.

Corollary II.21. Let $\nu \in \varprojlim_{\mathscr{X}} \Delta(\mathscr{X}, |B|)$ be defined by a sequence of valuations $\nu_{\mathscr{X}} \in \Delta(\mathscr{X}, |B|)$ for each snc model $\mathscr{X}$ of (X, B). Then, given a snc model $\mathscr{X}$ of (X, B), there exists a snc model $\mathscr{X}'$ of (X, B) dominating $\mathscr{X}$ such that the center of $\nu_{\mathscr{X}'}$ in $\mathscr{X}'$ does not intersect $|B|$.

Proof. This easily follows Lemma II.20. Once we find a model $\mathscr{X}'$ of $\mathscr{X}$ such that $\mathrm{red}_{\mathscr{X}'}(\nu_{\mathscr{X}'})$ is not contained in the closure of $|B|$, we can further blowup to assume that the two become disjoint. □

Proposition II.22. The map $\phi : \varprojlim_{\mathscr{X}} \Delta(\mathscr{X}, |B|) \to \varprojlim_{\mathscr{X}} \Delta(\mathscr{X})$ is open and injective, where $\mathscr{X}$ ranges over all snc models $\mathscr{X}$ of (X, B).

Proof. Let ν, ν' be two distinct elements in $\varprojlim_{\mathscr{X}} \Delta(\mathscr{X}, |B|)$ defined by sequences $\nu_{\mathscr{X}}, \nu'_{\mathscr{X}} \in \Delta(\mathscr{X}, |B|)$ respectively. Once again, we identify the elements of $\Delta(\mathscr{X}, |B|)$ with its image under $i_{(\mathscr{X},|B|)}$ and think of them as valuations. Let $\mathscr{X}$ be an snc model of (X, B) such that $\nu_{\mathscr{X}} \neq \nu'_{\mathscr{X}}$ in $\Delta(\mathscr{X}, |B|)$. From Corollary II.21, we can find a model $\mathscr{X}'$ such that $\phi_{\mathscr{X}'}(\nu_{\mathscr{X}'}) = \nu_{\mathscr{X}'}$ and $\phi_{\mathscr{X}'}(\nu'_{\mathscr{X}'}) = \nu'_{\mathscr{X}'}$. Note that $\nu_{\mathscr{X}'} \neq \nu'_{\mathscr{X}'}$ as $r_{\mathscr{X}',\mathscr{X},|B|}(\nu_{\mathscr{X}'}) \neq r_{\mathscr{X}',\mathscr{X},|B|}(\nu'_{\mathscr{X}'})$. Thus, ϕ is injective.

To see that ϕ is open, it is enough to show that given an snc model $\mathscr{Y}$ of (X, B) and an open set $U \subset \Delta(\mathscr{Y}, B)$

$$\phi(\{\nu \in \varprojlim_{\mathscr{X}} \Delta(\mathscr{X}, |B|) | \nu_{\mathscr{Y}} \in U\})$$

is an open set. We may further assume that U is small enough and has compact closure. Using Lemma II.21, we can find a model $\mathscr{Y}'$ such that $U' := r^{-1}_{\mathscr{Y}',\mathscr{Y},B}(U) \subset \Delta(\mathscr{Y}')$. Then, it is easy to check that

$$\phi(\{\nu \in \varprojlim_{\mathscr{X}} \Delta(\mathscr{X}, |B|) | \nu_{\mathscr{Y}} \in U\}) = \{\nu \in \varprojlim_{\mathscr{X}} \Delta(\mathscr{X}) | \nu_{\mathscr{Y}'} \in U'\}.$$

□

To prove Theorem II.16, we exploit the isomorphism $X^{\mathrm{an}} \xrightarrow{\simeq} \varprojlim_{\mathscr{X}} \Delta(\mathscr{X})$ (see [KS06, Theorem 10], [BFJ16, Cor. 3.2]).

Remark II.23. The homeomorphism $X^{\mathrm{an}} \xrightarrow{\simeq} \varprojlim_{\mathscr{X}} \Delta(\mathscr{X})$ in [KS06, Theorem 10] is stated when the inverse limit runs over all snc models $\mathscr{X}$ of X. However, we may as well take the inverse limit over all snc models $\mathscr{X}$ of (X, B) because such models form a cofinal system.

Proof of Theorem II.16. We obtain a map $r_{(X,|B|)} : X^{\mathrm{an}} \setminus |B|^{\mathrm{an}} \to \varprojlim_{\mathscr{X}} \Delta(\mathscr{X}, |B|)$ by considering the inverse limit over the retraction map $r_{(\mathscr{X},|B|)} : X^{\mathrm{an}} \setminus |B|^{\mathrm{an}} \to \Delta(\mathscr{X}, |B|)$.

Observe that we have the following commutative diagram where the bottom map is a homeomorphism.

$$\begin{tikzcd} X^{\mathrm{an}} \setminus |B|^{\mathrm{an}} \arrow[r, "r_{(X,|B|)}"] \arrow[d, hook, "i"] & \varprojlim_{\mathscr{X}} \Delta(\mathscr{X}, |B|) \arrow[d, hook, "\phi"] \\ X^{\mathrm{an}} \arrow[r, "r_X", "\simeq"'] & \varprojlim_{\mathscr{X}} \Delta(\mathscr{X}) \end{tikzcd}$$

Therefore, it is enough to show that the image of $|B|^{\mathrm{an}}$ in $\varprojlim_{\mathscr{X}} \Delta(\mathscr{X})$ does not intersect with the image of ϕ. Let ν be an element of $\varprojlim_{\mathscr{X}} \Delta(\mathscr{X}, |B|)$ defined by a sequence $\nu_{\mathscr{X}} \in \Delta(\mathscr{X}, |B|)$. Let $\nu_1 := r_X^{-1}(\phi(\nu))$. Without loss of generality, assume to the contrary that $\nu_1 \in B_1^{\mathrm{an}}$.

Using Corollary II.21, we can find a model $\mathscr{X}$ such that the center of $\nu_{\mathscr{X}}$ in $\mathscr{X}$ does not intersect $|B|$. Then, $\phi_{\mathscr{X}}(\nu_{\mathscr{X}}) = \nu_{\mathscr{X}}$. We also have that $r_{\mathscr{X}}(\nu_1) = \phi_{\mathscr{X}}(\nu_{\mathscr{X}}) = \nu_{\mathscr{X}}$ and the center of ν_1 in $\mathscr{X}$ is contained in the center of $\nu_{\mathscr{X}}$ in $\mathscr{X}$. But the center of ν_1 is contained in the closure of B_1, which is a contradiction. $\square$

II.4.3: The limit hybrid space as a Berkovich analytic space

Let (X, B) be a log-smooth pair of projective varieties over $\mathbb{D}^*$. In this section, for any $0 < r < 1$, we realize $(X, |B|)^{\mathrm{hyb}}{}_r := (X, |B|)^{\mathrm{hyb}}|_{\overline{r\mathbb{D}}}$ as the analytification of a scheme over a Banach ring, A_r.

As in [Ber09], consider the Banach ring

$$A_r = \left\{ \sum_i c_i t^i \in \mathbb{C}((t)) \;\middle|\; c_i \in \mathbb{C} \text{ and } \sum_{i \in \mathbb{Z}} ||c_i||_{\mathrm{hyb}} r^i < \infty \right\},$$

where $||c_i||_{\mathrm{hyb}} = \max\{|c_i|, 1\}$ if $c_i \neq 0$ and $||0||_{\mathrm{hyb}} = 0$. Then, its Berkovich spectrum $\mathcal{M}(A_r)$ is homeomorphic to $r\overline{\mathbb{D}}$. For more details, see [Ber09] [BJ17, Appendix 1]. Note that any function that is holomorphic in open neighborhood of $r\overline{\mathbb{D}} \setminus \{0\}$ and meromorphic at 0 gives an element of A_r.

Given a projective family $X \to \mathbb{D}^*$, we can think of X as a finite type scheme over

Spec A_r because the coefficients of the homogeneous equations cutting out X in $\mathbb{P}^N \times \mathbb{D}^*$ can be viewed as elements of A_r. We denote this scheme as X_{A_r}. Similarly, we get $|B|_{A_r} \subset X_{A_r}$. Let $(\_)^{\mathrm{An}}$ denote the Berkovich analytification functor on the category of finite type schemes over Spec A_r. The map $X_{A_r} \setminus |B|_{A_r} \to \operatorname{Spec} A_r$ gives rise to the canonical map $X_{A_r}^{\mathrm{An}} \setminus |B|_{A_r}^{\mathrm{An}} \to \mathcal{M}(A_r) \simeq r\overline{\mathbb{D}}$. The following proposition tells us how this analytic space is related to $(X, |B|)^{\mathrm{hyb}}$.

Proposition II.24. We have a homeomorphism $X_{A_r}^{\mathrm{An}} \setminus |B|_{A_r}^{\mathrm{An}} \xrightarrow{\simeq} (X, |B|)_r^{\mathrm{hyb}}$ as spaces over $r\overline{\mathbb{D}}$.

Proof. Let $\pi_r : (X_{A_r} \setminus |B|_{A_r}) \to r\overline{\mathbb{D}} \simeq \mathcal{M}(A_r)$ be the canonical projection map. From [BJ17, Lemma A.6] we have the following homeomorphisms:

$$\pi_r^{-1}(r\overline{\mathbb{D}}^*) \simeq (X \setminus |B|)|_{r\overline{\mathbb{D}}^*} \text{ and } \pi_r^{-1}(0) \simeq (X_{\mathbb{C}((t))}^{\mathrm{an}} \setminus |B|_{\mathbb{C}((t))}^{\mathrm{an}}).$$

Moreover, the first homeomorphism is compatible with the projections to $r\overline{\mathbb{D}}^*$.

The above homeomorphisms let us define a bijection $X_{A_r}^{\mathrm{An}} \setminus |B|_{A_r}^{\mathrm{An}} \to (X, |B|)_r^{\mathrm{hyb}}$. It remains to check that this map is continuous. To do this, first note that we have an embedding $(X, B)^{\mathrm{hyb}} \hookrightarrow X^{\mathrm{hyb}}$, where $X^{\mathrm{hyb}} := \varprojlim_{\mathscr{X}} \mathscr{X}^{\mathrm{hyb}}$, given by the canonical inclusion over $\mathbb{D}^*$ and by Proposition II.17 over the central fiber. We also have a homeomorphism $X_{A_r}^{\mathrm{An}} \to X_r^{\mathrm{hyb}}$ as topological spaces over $r\overline{\mathbb{D}}$ [BJ17, Proposition 4.12]. It is straightforward to check that the following diagram of topological spaces over $r\overline{\mathbb{D}}$ commutes.

$$\begin{tikzcd}
X_{A_r}^{\mathrm{An}} \setminus |B|_{A_r}^{\mathrm{An}} \arrow[r] \arrow[d, hook] & (X, |B|)_r^{\mathrm{hyb}} \arrow[d, hook] \\
X_{A_r}^{\mathrm{An}} \arrow[r, "\simeq"] & X_r^{\mathrm{hyb}}
\end{tikzcd}$$

Since the map at the bottom is a homeomorphism, the vertical maps are open immersions, and the top map is a bijection, the top map is also a homeomorphism. ☐

Now, we can define the hybrid space associated to a (not necessarily log-smooth) projective pair (X,B) over $\mathbb{D}^*$ as $(X,|B|)^{\mathrm{hyb}} := \varinjlim_{0<r<1}(X^{\mathrm{reg}} \setminus |B|)^{\mathrm{An}}_{A_r}$. Proposition II.24 tells us that this matches with our previous definition when (X,B) is log-smooth.

II.4.4: Convergence on the limit hybrid model

Let (X,B) be a projective log-smooth pair of varieties over $\mathbb{D}^*$. The convergence described in Theorem B depends on the choice of a model $(\mathscr{X},B)$ of (X,B). We would like to remedy this by describing the convergence on $(X,|B|)^{\mathrm{hyb}} = \varprojlim_{\mathscr{X}}(\mathscr{X},|B|)^{\mathrm{hyb}}$, which is independent of the choice of a model.

Suppose we have two models $\mathscr{X}$ and $\mathscr{X}'$ of (X,B) with $\mathscr{X}'$ dominating $\mathscr{X}$ via $\rho : \mathscr{X}' \to \mathscr{X}$. Suppose that we have a $\mathbb{Q}$-Cartier divisor $\mathscr{D}$ on $\mathscr{X}$ extending $K_{X/\mathbb{D}^*}+B$ and a generating section $\psi \in H^0(\mathscr{X}, m\mathscr{D})$ extending $\eta \in H^0(X, m(K_{X/\mathbb{D}^*}+B))$. Then, we can get a $\mathbb{Q}$-Cartier divisor $\mathscr{D}' = \rho^*\mathscr{D}$ on $\mathscr{X}'$ extending $K_{X/\mathbb{D}^*}+B$ and a section $\psi' = \rho^*\psi$ extending η. Applying Theorem *B* to both $\mathscr{X}$ and $\mathscr{X}'$, we get measures $\mu_0^{\mathscr{X}}$ and $\mu_0^{\mathscr{X}'}$ on $\Delta(\mathscr{X},|B|)$ and $\Delta(\mathscr{X}',|B|)$ respectively which are the limits of μ_t on $(\mathscr{X},|B|)^{\mathrm{hyb}}$ and $(\mathscr{X}',|B|)^{\mathrm{hyb}}$ respectively. Since the pushforward of Radon measures commutes with weak limits, we have that $\mu_0^{\mathscr{X}}$ is just the push-forward of the measure $\mu_0^{\mathscr{X}'}$ under the map $r_{\mathscr{X}',\mathscr{X},|B|}$.

Thus, we get a compatible system of measure $\mu_0^{\mathscr{X}'}$ on all models $\mathscr{X}'$ dominating a fixed model $\mathscr{X}$. This gives rise to a measure on μ_0 on $(X,|B|)_0^{\mathrm{hyb}}$, and thus we get the following convergence theorem.

Theorem II.25. Let (X,B) be a projective log-smooth pair over $\mathbb{D}^*$. Suppose that $K_{X/\mathbb{D}^*} + B \sim_{\mathbb{Q}} 0$ and let $\eta \in H^0(X, m(K_{X/\mathbb{D}^*}+B))$ admit a meromorphic extension (i.e. there exists a model $\mathscr{X}$ of (X,B), a $\mathbb{Q}$-Cartier divisor $\mathscr{D}$ extending $K_{X/\mathbb{D}^*}+B$ and $\psi \in H^0(\mathscr{X}, m\mathscr{D})$ extending η). Then, there exists $\kappa_{\min} \in \mathbb{Q}$ and $d \in \mathbb{N}$ such that the measure $\mu_t = \frac{i^{n^2}(\eta_t \wedge \overline{\eta_t})^{1/m}}{|t|^{2\kappa_{\min}}(2\pi \log|t|^{-1})^d}$ converges weakly to a measure μ_0 on $(X,|B|)^{\mathrm{hyb}}$.

Moreover if we fix an snc model $\mathscr{X}$, a $\mathbb{Q}$-Cartier divisor $\mathscr{D}$ and a section $\psi \in H^0(\mathscr{X}, m\mathscr{D})$ extending η, then μ_0 is supported on $\Delta(\mathscr{D}) \subset \Delta(\mathscr{X},|B|) \subset X^{\mathrm{an}}_{\mathbb{C}((t))} \setminus |B|^{\mathrm{an}}_{\mathbb{C}((t))}$, and $d, \kappa_{\min}$ and μ_0 have the same description as in Section II.3.3.

Example II.26. Following up Example II.14, we see that the Haar measures on $\mathbb{P}^1$ converges to the Lebesgue measure on $\mathbb{R}$, which can be thought of as the unique line joining the type 1 points corresponding 0 and ∞ in $(\mathbb{P}^1_{\mathbb{C}((t))})^{\mathrm{an}}$. More generally, we could take B_t to be given by $p(t) + q(t)$ for distinct functions p, q which are meromorphic on $\mathbb{D}$ and holomorphic on $\mathbb{D}^*$. Then, there exists an isomorphism of pairs $(\mathbb{P}^1 \times \mathbb{D}^*, \{(p(t), t)\} + \{(q(t), t)\}) \simeq (\mathbb{P}^1 \times \mathbb{D}^*, [0] \times \mathbb{D}^* + [\infty] \times \mathbb{D}^*)$. This extends to an isomorphism $(\mathbb{P}^1_{\mathbb{C}((t))})^{\mathrm{an}} \setminus \{p, q\} \simeq (\mathbb{P}^1_{\mathbb{C}((t))})^{\mathrm{an}} \setminus \{0, \infty\}$, where p, q denote the type 1 points corresponding to $p(t)$ and $q(t)$. Thus, as $t \to 0$, the Haar measure on $\mathbb{P}^1 \setminus \{p(t), q(t)\}$ converges to the Lebesgue measure on the unique line joining the points p and q in $(P^1_{\mathbb{C}((t))})^{\mathrm{an}} \setminus \{\widetilde{p}, \widetilde{q}\}$.

Example II.27. Similar to the above example, let $X = \mathbb{P}^1 \times \mathbb{D}^*$ denote the constant family. Let $B = \{z^2 + a_1 z + a_2 = 0\} \subset \mathbb{P}^1 \times \mathbb{D}^*$, where z denotes the coordinate on $\mathbb{P}^1$ and a_1, a_2 are functions that are meromorphic on $\mathbb{D}$ and holomorphic on $\mathbb{D}$. Then, (X, B) is log Calabi-Yau. Also assume that the polynomial $z^2 + a_1 z + a_2 \in \mathbb{C}((t))[z]$ is irreducible.

Fix a square root $u = \sqrt{t}$ and consider the field extension $\mathbb{C}((t)) \to \mathbb{C}((u))$. This corresponds to a degree two map $\mathbb{D}^* \to \mathbb{D}^*$ given by $u \mapsto u^2$.

The polynomial $z^2 + a_1 z + a_2 \in \mathbb{C}((t))[z]$ splits into factors $(z - p)(z - q)$ in $\mathbb{C}((u))[z]$. By the previous example, as $u \to 0$, the Haar measure on $\mathbb{P}^1 \setminus \{p(u), q(u)\}$ converges to the Lebesgue measure on the line joining p and q in $(\mathbb{P}^1_{\mathbb{C}}((u)))^{\mathrm{an}} \setminus \{p, q\}$. Call this measure $\widetilde{\mu_0}$. We have a map $(\mathbb{P}^1_{\mathbb{C}}((u)))^{\mathrm{an}} \setminus \{p, q\} \to (\mathbb{P}^1_{\mathbb{C}((t))})^{\mathrm{an}} \setminus |\overline{B}|^{\mathrm{an}}$.

To understand the convergence of the Haar measure on $\mathbb{P}^1 \setminus |B_t|$, note that $\mathbb{P}^1 \setminus |B_t| \simeq \mathbb{P}^1 \setminus \{p(u), q(u)\}$. Thus, as $t \to 0$ the Haar measure on $\mathbb{P}^1 \setminus |B_t|$ converges to the pushforward of $\widetilde{\mu_0}$ to $(\mathbb{P}^1_{\mathbb{C}((t))})^{\mathrm{an}} \setminus |\overline{B}|^{\mathrm{an}}$.

Example II.28. Following up Example II.15, we get that the (scaled) Haar measure on the constant family of tori $T = N \otimes \mathbb{C}^*$ converges to the Lebesgue measure on $\mathbb{R}^n$. For any smooth projective toric compactification Y of T with boundary divisor D, the image of $\Delta(Y, D) \subset T^{\mathrm{an}}_{\mathbb{C}((t))}$ coincides with the image of $N_{\mathbb{R}} \hookrightarrow T^{\mathrm{an}}_{\mathbb{C}((t))}$ given by sending $\sum n_i \otimes r_i \in N_{\mathbb{R}}$ to the seminorm $|\sum_j a_j \chi^{m_j}| = \max_j \{|a_j| e^{-\sum_i r_i \langle m_j, n_i \rangle}\}$.

II.4.5: Convergence for general sub log canonical pairs (X, B)

In this subsection, we drop the assumption that (X, B) is log-smooth and prove Theorem A in general.

Suppose that (X, B) is a projective pair over $\mathbb{D}^*$ such that (X, B) is a sub log canonical and log Calabi-Yau pair. Here, sub log canonical means that (X, B) is log canonical in the sense of the minimal model program (i.e. $\operatorname{discrep}(X, B) \geq -1$) but we are not necessarily assuming that B is effective. Let $\eta \in H^0(X, m(K_{X/\mathbb{D}^*} + B))$ be a generating section that admits a meromorphic extension.

Let $\pi : (Y, B') \to (X, B)$ be a log resolution of singularities. Here, B' is the divisor supported on the exceptional locus and the preimage of B such that $K_{Y/\mathbb{D}^*} + B' = \pi^*(K_{X/\mathbb{D}^*} + B) \sim_{\mathbb{Q}} 0$. Moreover, π gives an isomorphism $Y \setminus |B'| \simeq X^{\mathrm{reg}} \setminus |B|$.

Since (X, B) is sub log canonical, all the coefficients that show up in B' are at most 1. Thus, the pair (Y, B') is log-smooth, sub log canonical and log Calabi-Yau. Let $\eta' \in H^0(Y, m(K_{Y/\mathbb{D}^*} + B'))$ denote the section $\eta' = \pi^*(\eta)$. Applying Theorem B to Y, we get that there exist $\kappa_{\min} \in \mathbb{Q}, d \in \mathbb{N}_+$ such the measures $\mu'_t = \frac{i^{n^2}(\eta'_t \wedge \overline{\eta'_t})^{1/m}}{(2\pi \log|t|^{-1})^d |t|^{2\kappa_{\min}}}$ converge weakly to a measure μ'_0 on the space $(Y, B')^{\mathrm{hyb}}_r$ for any $0 < r < 1$.

Note that the map $\pi^{\mathrm{An}}_{A_r} : (Y \setminus |B'|)^{\mathrm{An}}_{A_r} \to (X^{\mathrm{reg}} \setminus |B|)^{\mathrm{An}}_{A_r}$ is a homeomorphism as the restriction of π to $Y \setminus |B'|$ is an isomorphism. Taking $\varinjlim_{0 \leq r \leq 1}$, we get a homeomorphism $(Y, |B'|)^{\mathrm{hyb}} \simeq (X, |B|)^{\mathrm{hyb}}$. Then, it follows from the change of variables formula that $\mu_t := (\pi^{\mathrm{An}}_{A_r})_*(\mu'_t) = \frac{i^{n^2}(\eta_t \wedge \overline{\eta_t})^{1/m}}{(2\pi \log|t|^{-1})^d |t|^{2\kappa_{\min}}}$. Since the pushforward of Radon measures under a continuous map commutes with weak limits, it follows that $\mu_t \to (\pi^{\mathrm{An}}_{A_r})_*(\mu'_0)$, which finishes the proof of Theorem A.

CHAPTER III

Convergence of Bergman Measures Towards the Zhang Measure

III.1: Introduction

Structure of the chapter

In Section III.2, we discuss some preliminaries. In Section III.3, we recall the construction of the hybrid space. In Section III.4, we recall some properties of the dualizing sheaf of curves with at worst simple nodal singularities. In Section III.5, we compute some asymptotics related to the Bergman measure. In Section III.6, we prove Theorems D′ and D. The key technical result in this section is Lemma III.12. In Section III.7, we work out the convergence on the metrized curve complex hybrid space, proving Theorem E.

III.2: Preliminaries

III.2.1: Curves and models

Throughout this chapter, a *family of curves* X over $\mathbb{D}^*$ of genus $g \geq 1$ refers to a complex manifold X of dimension 2 such that we have a smooth projective holomorphic map $X \to \mathbb{D}^*$ with fibers being connected smooth complex projective curves of genus g. We also assume that the family is meromorphic at 0 i.e. there exists a projective flat family $\mathscr{X} \to \mathbb{D}$ extending $X \to \mathbb{D}^*$ with $\mathscr{X}$ normal and having a non-empty fiber over 0.

A *model* $\mathscr{X}$ of X is a flat projective holomorphic family $\mathscr{X} \to \mathbb{D}$ such that $\mathscr{X}|_{\mathbb{D}^*}$ is biholomorphic to X as spaces over $\mathbb{D}^*$. We say that $\mathscr{X}$ is a *regular model* of X if $\mathscr{X}$ is regular. The fiber over 0, $\mathscr{X}_0$, is called the special fiber. Let $\mathscr{X}_{0,\mathrm{red}}$ denote the reduced induced structure on $\mathscr{X}_0$.

We say that $\mathscr{X}$ is a *normal crossing model* (abbreviated as *nc model*) of X if $\mathscr{X}$ is regular and $\mathscr{X}_{0,\mathrm{red}}$ is a normal crossing divisor.

III.2.2: Semistable reduction and minimal nc models

We refer the reader to [Rom13] for a detailed introduction to models over a DVR. We summarize some of the results that we will use.

For any model $\mathscr{X}$ of X, we always have that $\mathscr{X}_0$ is connected [Liu02, Corollary 8.3.6].

A family of curves X is said to have *semistable reduction* if there exists an nc model $\mathscr{X}$ of X with reduced special fiber i.e. $\mathscr{X}_0 = \mathscr{X}_{0,\mathrm{red}}$ and such an $\mathscr{X}$ is called a *semistable model* of X.

A family of curves X of genus $g \geq 1$ always has semistable reduction after performing a finite base change $\mathbb{D}^* \to \mathbb{D}^*$ given by $u \mapsto t^n$. This follows from [DM69, Corollary 2.7] in the case when $g \geq 2$. See [Sta20, Tag 0CDN] for a general statement.

A family of curves X of genus $g \geq 1$ always has a *minimal nc model* i.e. there exists an nc model $\mathscr{X}_{\min}$ of X such that for any nc model $\mathscr{X}$ of X, there is a proper morphism $\mathscr{X} \to \mathscr{X}_{\min}$. Such a model is unique up to a unique isomorphism. See [Rom13, Theorem 2.5.1] or [Sta20, Tag 0C6B] for details.

When X has semistable reduction, the minimal nc model is also semistable [Sta20, Tag 0CDG]. In addition, the special fiber of the minimal nc model has no non-singular rational component that meets the rest of the component in only one point.

III.2.3: Blowups and getting new models from old ones

Given two models $\mathscr{X}$ and $\mathscr{X}'$ of X, we say that $\mathscr{X}'$ *dominates* $\mathscr{X}$ and write $\mathscr{X}' \geq \mathscr{X}$ if we have a proper holomorphic map $\mathscr{X}' \to \mathscr{X}$ such that its restriction to $\mathscr{X}'|_{\mathbb{D}^*}$ commutes with the isomorphism to X.

If $\mathscr{X}, \mathscr{X}'$ are two nc models of X such that $\mathscr{X}' \geq \mathscr{X}$, then $\mathscr{X}'$ is obtained from $\mathscr{X}$ by a sequence of blowups at closed points in the special fiber [Lic68, Theorem 1.15].

If $\mathscr{X}$ is an nc model of X, we can get a new nc model $\mathscr{X}'$ dominating $\mathscr{X}$ by blowing up $\mathscr{X}$ at a closed point in $\mathscr{X}_0$. Given two models $\mathscr{X}$ and $\mathscr{X}'$ of X, there always exists a model $\mathscr{X}''$ such that $\mathscr{X}'' \geq \mathscr{X}$, $\mathscr{X}'' \geq \mathscr{X}'$, and $\mathscr{X}''$ is obtained from both $\mathscr{X}$ and $\mathscr{X}'$ by a sequence of blowups in the special fiber [Lic68, Proposition 4.2].

III.2.4: Dual graph associated to a model

Let $\mathscr{X} \to \mathbb{D}$ be an nc model of X. The *dual graph* $\Gamma_{\mathscr{X}}$ associated to $\mathscr{X}$ a connected metric graph . The vertices of $\Gamma_{\mathscr{X}}$ correspond to the irreducible components of $\mathscr{X}_0$. If P is a node in $\mathscr{X}_0$ that lies in the intersection of the components E_0 and E_1, then we add an edge e_p between the vertices v_{E_0} and v_{E_1}. Let $V(\Gamma_{\mathscr{X}})$ and $E(\Gamma_{\mathscr{X}})$ denote the vertex and edge set of the dual graph respectively. Note that $\Gamma_{\mathscr{X}}$ might have loop edges and multiple edges between a pair of vertices.

We define a length on each edge i.e. we have a function $l : E(\Gamma_{\mathscr{X}}) \to \mathbb{Q}_{\geq 0}$ defined as follows. Let z, w be the (analytic) local equations defining the irreducible components containing a node P. Then, locally near P, the map $\mathscr{X} \to \mathbb{D}$ is given by $(z, w) \mapsto z^a w^b$, where a and b are the respective multiplicities of the irreducible components. We define the length of e_P to be $\frac{1}{ab}$.

It is also useful to keep track of the genus of the irreducible components. So our metric graph also comes with the data of a genus function $g : V(\Gamma_{\mathscr{X}}) \to \mathbb{N}$ given by taking the value of the genus of the normalization of an irreducible component at every vertex. We also define the genus of $\Gamma_{\mathscr{X}}$ to be its first Betti number i.e.

$$g(\Gamma_{\mathscr{X}}) = |E(\Gamma_{\mathscr{X}})| - |V(\Gamma_{\mathscr{X}})| + 1.$$

Note that if $\mathscr{X}$ is a semistable model of X, all the edges in the dual graph $\Gamma_{\mathscr{X}}$ would have length 1. For more details about the dual metric graphs, refer to [BF11], [BPR13] and [BPR16].

Let $\mathscr{X}'$ be obtained by blowing up $\mathscr{X}$ at a closed point in $\mathscr{X}_0$. Then, $\Gamma_{\mathscr{X}}$ and $\Gamma_{\mathscr{X}'}$ are related as follows.

- If $\mathscr{X}'$ is obtained by blowing up a smooth point on an irreducible component $E_0 \subset \mathscr{X}_0$ of multiplicity a, then $\Gamma_{\mathscr{X}'}$ is obtained from $\Gamma_{\mathscr{X}}$ by adding a new vertex v_E corresponding to the exceptional divisor of the blowup and adding an edge of length $\frac{1}{a^2}$ between v_E and v_{E_0}. The genus function is extended to one on $\Gamma_{\mathscr{X}'}$ by defining it to be 0 on v_E.

- If $\mathscr{X}'$ is obtained by blowing up a node $P = E_1 \cap E_2$ for (possibly same) irreducible components $E_1, E_2 \subset \mathscr{X}_0$, then $\Gamma_{\mathscr{X}'}$ is obtained from $\Gamma_{\mathscr{X}}$ by subdividing the edge e_P into edges of lengths $\frac{1}{a(a+b)}$ and $\frac{1}{(a+b)b}$ by adding a vertex v_E corresponding to the exceptional divisor. This makes sense as

  $$\frac{1}{ab} = \frac{1}{a(a+b)} + \frac{1}{(a+b)b}.$$

  The genus function is extended to $\Gamma_{\mathscr{X}'}$ by defining it to be 0 on v_E.

In both the cases, we see that we have an inclusion $\Gamma_{\mathscr{X}} \hookrightarrow \Gamma_{\mathscr{X}'}$ as well as a retraction $\Gamma_{\mathscr{X}'} \to \Gamma_{\mathscr{X}}$, and thus $\Gamma_{\mathscr{X}'}$ is a deformation retract of $\Gamma_{\mathscr{X}}$. They both also have the 'same' genus function.

More generally, given two nc models $\mathscr{X}$ and $\mathscr{X}'$, they can both be dominated by a common model $\mathscr{X}''$ obtained by a sequence of blowups from both $\mathscr{X}$ and $\mathscr{X}'$. Thus, we see that $g(\Gamma_{\mathscr{X}}) = g(\Gamma_{\mathscr{X}'})$ and $\sum_{v \in V(\Gamma_{\mathscr{X}})} g(v) = \sum_{v \in V(\Gamma_{\mathscr{X}'})} g(v)$. Let $g' = g(\Gamma_{\mathscr{X}})$.

The following remark is a consequence of the invariance of the genus functions under blowups.

Remark III.1. Suppose that X has a semistable model and let $\mathscr{X}$ be an nc model of X. Then, any irreducible component $E \subset \mathscr{X}_0$ whose normalization has positive genus, has multiplicity 1.

Remark III.2 (The two choices of the length function)**.** There are two possible ways of assigning lengths that we can assign to a node P given by the intersection of two irreducible components of $\mathscr{X}_0$ with multiplicities a and b respectively. One way is to define the lengths as above, by setting

$$l_1(e_P) = \frac{1}{ab}.$$

Yet another way is to define the length by

$$l_2(e_P) = \frac{1}{\operatorname{lcm}(a, b)}.$$

Both these lengths are compatible with respect to blowups. This follows from the fact that

$$\frac{1}{\operatorname{lcm}(a, b)} = \frac{1}{\operatorname{lcm}(a, a + b)} + \frac{1}{\operatorname{lcm}(a + b, b)}.$$

See [BN16] for comparisons between the two metrics. The advantage of using the first length function is that it makes our computations easier and the advantage of using the second one is that it is well-behaved with respect to ground field extensions.

In our case, it turns out that we could have chosen either one of the above metrics and it would not matter. The reason for this is that if we assume that X has a semistable model, the two notions of length can only differ on bridge edges of the dual graphs associated to any model. Since our aim is to compute the Zhang measure on the dual graph using the length function, it is enough to realize that Zhang measure remains invariant under change of length of any bridge edge.

III.2.5: The Zhang measure on the dual graph

Let Γ be a metric graph along with a genus function $g : V(\Gamma) \to \mathbb{N}$. Let l_e denote the length of an edge e. Let δ_v denote the unit Dirac measure at a vertex v and let $dx|_e$ denote the Lebesgue measure on an edge e normalized such that $\int_e dx|_e = l_e$. We can also view Γ as an resistor network with the resistance of each edge being l_e. Let r_e denote the resistance between the endpoints of the edge e in the in the resistor network obtained by removing the interior of the edge e from the graph Γ. For the precise definition of r_e, see [Zha93, Section 3] (The precise definition of r_e is not very important for this chapter as we will use a different characterization of the Zhang measure in the proof; see Proposition III.5).

The *Zhang measure* on Γ is a measure and is given as follows.

$$\mu_{Zh} = \sum_{v \in V(\Gamma)} g(v)\delta_v + \sum_{e \in E(\Gamma)} \frac{dx|_e}{l_e + r_e}$$

When e is a bridge edge i.e. removing e from Γ disconnects Γ, then $r_e := \infty$ and $\frac{1}{l_e + r_e} = 0$. Thus, the Zhang measure places no mass on bridge edges. For more details, see [Zha93]. Note that our definition differs from Zhang's original definition by a factor of g. This is done so that so that the total mass of Zhang measure is now equal to $g = \sum_{v \in V(\Gamma)} g(v) + g(\Gamma)$. For an interpretation of $\frac{1}{l_e + r_e}$ in terms of spanning trees and electrical networks, refer to [BF11, Section 6].

Remark III.3. Note that the Zhang measure is invariant under the following operations.

- If we subdivide an edge of length l into two edges of lengths $l_1, l - l_1$, the Zhang measure does not change.
- If we introduce a new vertex v' and add a new edge e between v' and an existing vertex v, the Zhang measure on the new graph is the same as the one on the old graph as the edge e would be a bridge and would not alter any of the resistances in the old graph.

- If we multiply all the lengths by a fixed factor $N \in \mathbb{R}_+$, the Zhang measure does not change. This is because the resistance is linear as a function of edge lengths and thus the quantity $\frac{l_e}{l_e+r_e}$ remains unchanged.

The first two operations correspond to altering an nc model by blowups and the third operation corresponds to ground field extension.

III.2.6: Bergman measure on a complex curve

Let Y be a complex curve of genus $g \geq 1$. Then there exists a natural Hermitian metric on $H^0(Y, \Omega_Y)$ given by

$$(\vartheta, \vartheta') \mapsto \frac{i}{2} \int_Y \vartheta \wedge \overline{\vartheta'}. \tag{III.2.1}$$

Let $\vartheta_1, \ldots, \vartheta_g$ be an orthonormal basis of $H^0(Y, \Omega_Y)$ with respect to this pairing. Then, we get a positive $(1,1)$-form $\frac{i}{2}\sum_i \vartheta_i \wedge \overline{\vartheta_i}$ on Y. It is easy to verify that this $(1,1)$-form does not depend on the choice of the orthonormal basis. This $(1,1)$-form gives rise to a measure on Y which is known as the *Bergman measure*. Note that the total mass of the Bergman measure is g. For more details regarding the Bergman measure, see [Ber10] and [BSW19, Section 3.3].

III.2.7: Associated Berkovich space

Let $X_{\mathbb{C}((t))}$ be the projective variety cut out by the defining equations of X, where we view the coefficients of the defining polynomial as elements of $\mathbb{C}((t))$ by looking at the power series expansion around 0.

The collection of all nc models of X forms a directed system. Given a proper morphism $\mathscr{X}' \to \mathscr{X}$, we get a retraction map $\Gamma_{\mathscr{X}'} \to \Gamma_{\mathscr{X}}$. For example, if $\mathscr{X}'$ is obtained by blowing up $\mathscr{X}$ at a node in $\mathscr{X}_0$, then this map is an isometry and if $\mathscr{X}'$ is obtained by blowing up $\mathscr{X}$ at a smooth point $P \subset \mathscr{X}_0$, then this map is obtained by collapsing the vertex and edge associated to the exceptional divisor and the new node respectively to the vertex on $\Gamma_{\mathscr{X}}$ associated to the irreducible component containing P. More generally, see [MN15] for a description of this map.

Then, we have an homeomorphism [KS06, Theorem 10] [BFJ16, Corollary 3.2]

$$X^{\mathrm{an}}_{\mathbb{C}((t))} \simeq \varprojlim_{\text{nc models } \mathscr{X}} \Gamma_{\mathscr{X}}.$$

III.3: The hybrid space

Given an nc model $\mathscr{X}$ of X, we can construct the hybrid space $\mathscr{X}^{\mathrm{hyb}}$ given set theoretically as $\mathscr{X}^{\mathrm{hyb}} = X \sqcup \Gamma_{\mathscr{X}}$. We recall the topology of the hybrid space in the one dimensional case.

Consider a chart given by an open subset $U \subset \mathscr{X}$ such that $U \cap \mathscr{X}_0 = U \cap E$, where E is an irreducible component of $\mathscr{X}_0$ of multiplicity a and there exist coordinates (z, w) on U with $|z|, |w| < 1$ such that the projection to $\mathbb{D}$ is given by $(z, w) \mapsto z^a$. Following the terminology of [BJ17], we call such a coordinate chart as being *adapted* to E. In this case, we define $\mathrm{Log}_U : U \setminus E \to v_E$ to be the constant function, where $v_E \in \Gamma_{\mathscr{X}}$ is the vertex corresponding to E.

Now, let $P = E_1 \cap E_2$ be a node where E_1 and E_2 are either two distinct irreducible components of $\mathscr{X}_0$, or correspond to two different local analytic branches of the same irreducible component. Let the multiplicities of E_1, E_2 in $\mathscr{X}_0$ be a, b respectively. Now consider a coordinate chart given by an open set $U \subset \mathscr{X}$ such that $U \cap \mathscr{X}_0 = U \cap (E_1 \cup E_2)$ and there exist coordinates (z, w) on E with $|z|, |w| < 1$, $U \cap E_1 = \{z = 0\}$, $U \cap E_2 = \{w = 0\}$ and the projection to the disk is given by $(z, w) \mapsto t = z^a w^b$. Such a coordinate chart is said to be *adapted* to the node $P = E_1 \cap E_2$. In this case, we define $\mathrm{Log}_U : U \setminus \mathscr{X}_0 \to e_P$ by $(z, w) \mapsto \frac{\log|z|}{b\log|t|}$, where we identify e_P with $[0, \frac{1}{ab}]$ with v_{E_2} corresponding to 0 and v_{E_1} corresponding to $\frac{1}{ab}$.

A coordinate chart adapted to either an irreducible component of $\mathscr{X}_0$, or to a node in

$\mathscr{X}_0$ is called such a coordinate chart as an adapted coordinate chart.

Let $V = \bigcup_i U_i$ be a finite cover of an open neighborhood of $\mathscr{X}_0$ by adapted coordinate charts and let χ_i be a partition of unity with respect to the cover U_i. Then the function $\mathrm{Log}_V : V \setminus \mathscr{X}_0 \to \Gamma_{\mathscr{X}}$ defined by $\mathrm{Log}_V = \sum_i \chi_i \mathrm{Log}_{U_i}$ is well-defined (note that addition in $\Gamma_{\mathscr{X}}$ is not well-defined, but it makes sense on an edge using the identification $e_P \simeq [0, l_{e_P}]$). Such a function is called a global log function. The following remark is very useful and is proved using [Cle77, Theorem 5.7].

Remark III.4 (Proposition 2.1, [BJ17])**.** If V and W are open neighborhoods of $\mathscr{X}_0$ with global log functions Log_V and Log_W, then as $t \to 0$

$$\mathrm{Log}_V - \mathrm{Log}_W = O\left(\frac{1}{\log|t|^{-1}}\right)$$

uniformly on compact sets of $V \cap W$.

We define the topology on $\mathscr{X}^{\mathrm{hyb}}$ to be the coarsest topology satisfying

- The map $X \to \mathscr{X}^{\mathrm{hyb}}$ is an open immersion.
- The map $\mathscr{X}^{\mathrm{hyb}} \to \mathbb{D}$ given by extending $\pi : X \to \mathbb{D}^*$ and sending $\Gamma_{\mathscr{X}}$ to the origin is continuous.
- Given a global log function Log_V, the map $V \cup \Gamma_{\mathscr{X}} \to \Gamma_{\mathscr{X}}$ given by Log_V on V and identity on $\Gamma_{\mathscr{X}}$ is continuous.

It follows from Remark III.4 that the topology induced on $\mathscr{X}^{\mathrm{hyb}}$ does not depend on the choice of the global log function.

We can define X^{hyb} to be $\varprojlim_{\mathscr{X}} \mathscr{X}^{\mathrm{hyb}}$, where $\mathscr{X}$ runs over all normal crossing models. Since we have that $X^{\mathrm{an}}_{\mathbb{C}((t))} = \varprojlim \Gamma_{\mathscr{X}}$, we get that the central fiber of X^{hyb} is homeomorphic to $X^{\mathrm{an}}_{\mathbb{C}((t))}$. In fact, it is possible to see the space X^{hyb} as the Berkovich analytification of X seen as a scheme over a certain Banach ring [Ber09], [BJ17, Appendix]. See also [Poi13].

III.4: The canonical sheaf on $\mathscr{X}_{0,\mathrm{red}}$

If Y is a a smooth projective complex curve, then we define its *dualizing sheaf* ω_Y as the sheaf of holomorphic de-Rham differentials Ω_Y i.e. $\omega_Y = \Omega_Y$. This sheaf satisfies Serre duality i.e. for any line bundle $\mathcal{L}$ and for $i = 0, 1$

$$H^i(Y, \mathcal{L}) \simeq H^{1-i}(Y, \omega_Y \otimes_{\mathcal{O}_Y} \mathcal{L}^\vee)^\vee.$$

In certain more general situations, it is possible to define a sheaf that satisfies similar duality properties. For example, if Y is a Cohen-Macaulay variety, one can define a dualizing sheaf ω_Y. [Har77, Section III.7]

Let $\mathscr{X}$ be an nc model of X. A simple computation shows that $\mathscr{X}_{0,\mathrm{red}}$ is a Cohen-Macaulay variety and thus it is possible to define $\omega_{\mathscr{X}_{0,\mathrm{red}}}$. The sheaf $\omega_{\mathscr{X}_{0,\mathrm{red}}}$ is in fact a line bundle. We give a more explicit description of it later in this section.

Let us first calculate $\dim_{\mathbb{C}} H^0(\mathscr{X}, \omega_{\mathscr{X}_{0,\mathrm{red}}})$. Since $\omega_{\mathscr{X}_{0,\mathrm{red}}}$ is the dualizing sheaf of $\mathscr{X}_{0,\mathrm{red}}$, by applying Serre duality we get that

$$H^0(\mathscr{X}, \omega_{\mathscr{X}_{0,\mathrm{red}}}) \simeq H^1(\mathscr{X}_{0,\mathrm{red}}, \mathcal{O}_{\mathscr{X}_{0,\mathrm{red}}})^\vee.$$

Let $\widetilde{\mathscr{X}_{0,\mathrm{red}}}$ denote the normalization of $\mathscr{X}_{0,\mathrm{red}}$ and let $p : \widetilde{\mathscr{X}_{0,\mathrm{red}}} \to \mathscr{X}_{0,\mathrm{red}}$ denote the normalization map. Then, $\widetilde{\mathscr{X}_{0,\mathrm{red}}}$ is a possibly disconnected union of curves. By looking at the long exact sequence induced in cohomology by

$$0 \to \mathcal{O}_{\mathscr{X}_{0,\mathrm{red}}} \to p_*(\mathcal{O}_{\widetilde{\mathscr{X}_{0,\mathrm{red}}}}) \to \sum_{P \in \mathscr{X}_{0,\mathrm{red}} \text{ node}} \mathbb{C}(P) \to 0,$$

it follows that

$$\dim_{\mathbb{C}} H^1(\mathscr{X}_{0,\mathrm{red}}, \mathcal{O}_{\mathscr{X}_{0,\mathrm{red}}}) = g(\Gamma_{\mathscr{X}}) + \sum_{v \in V(G)} g(v).$$

If $\mathscr{X}$ is a semistable model of X, then $\mathscr{X}_0 = \mathscr{X}_{0,\mathrm{red}}$ and the invariance of the arithmetic genus in flat families guarantees that $\dim_{\mathbb{C}} H^1(\mathscr{X}_0, \mathcal{O}_{\mathscr{X}_0}) = g$. Thus, in this case, we see that $g = g(\Gamma_{\mathscr{X}}) + \sum_{v \in V(\Gamma_{\mathscr{X}})} g(v)$.

More generally, the same holds true for any model $\mathscr{X}$ as long as we assume that X has a semistable model. This follows from the fact that $g(\Gamma_{\mathscr{X}}) + \sum_{v \in V(\Gamma_{\mathscr{X}})} g(v)$ does not depend on the choice of the nc model. (See Section III.2.4). In this case, it also follows that $\dim_{\mathbb{C}} H^0(\mathscr{X}_{0,\mathrm{red}}, \omega_{\mathscr{X}_{0,\mathrm{red}}}) = g$.

III.4.1: An explicit description of $\omega_{\mathscr{X}_{0,\mathrm{red}}}$

It is possible to give an explicit description of the elements of $H^0(\mathscr{X}_{0,\mathrm{red}}, \omega_{\mathscr{X}_{0,\mathrm{red}}})$: they correspond to meromorphic 1-forms ψ on $\widetilde{\mathscr{X}_{0,\mathrm{red}}}$, the normalization of $\mathscr{X}_{0,\mathrm{red}}$, with at worst simple poles at the points that lie above the nodes in $\mathscr{X}_0$ such that if P' and P'' lie above the node P, then the residues of ψ at P' and P'' add up to 0 [DM69, Section I].

Let $E_1, \ldots, E_m$ denote the irreducible components of $\mathscr{X}_{0,\mathrm{red}}$. Then, note that $\widetilde{\mathscr{X}_{0,\mathrm{red}}} = \bigsqcup_i \widetilde{E_i}$, where $\widetilde{E_i}$ is the normalization of E_i. When E_i does not have a self-node, then $E_i = \widetilde{E_i}$.

Let $P_1^{(i)}, \ldots, P_{r_i}^{(i)}$ denote the points in $\widetilde{E_i}$ that lie over nodal points in $\mathscr{X}_0$. The above description gives rise to the following short exact sequence of sheaves on $\mathscr{X}_{0,\mathrm{red}}$:

$$0 \to \omega_{\mathscr{X}_{0,\mathrm{red}}} \to \bigoplus_i \omega_{\widetilde{E}_i}(P_1^{(i)} + \cdots + P_{r_i}^{(i)}) \to \bigoplus_{P \in \mathscr{X}_0 \text{ node}} \mathbb{C}(P) \to 0,$$

where the first map is given by the restrictions $\psi \mapsto (\psi|_{\widetilde{E}_1}, \ldots, \psi|_{\widetilde{E}_m})$ and the second map is given by taking the sum of residues.

We also have a natural inclusion $\omega_{\widetilde{\mathscr{X}_{0,\mathrm{red}}}} = \bigoplus_i \omega_{\widetilde{E}_i} \hookrightarrow \omega_{\mathscr{X}_{0,\mathrm{red}}}$ as the sections of $\omega_{\widetilde{\mathscr{X}_{0,\mathrm{red}}}}$ have zero residue at all points. Since $H^0(\widetilde{\mathscr{X}_{0,\mathrm{red}}}, \omega_{\widetilde{\mathscr{X}_{0,\mathrm{red}}}})$, the vector space of holomorphic 1-forms on $\widetilde{\mathscr{X}_{0,\mathrm{red}}}$, has dimension $\sum_{v \in V(\Gamma)} g(v)$, it follows that the subspace of $H^0(\mathscr{X}_{0,\mathrm{red}}, \omega_{\mathscr{X}_{0,\mathrm{red}}})$ spanned by 1-forms that have no poles has dimension $\sum_{v \in V(\Gamma)} g(v) = g - g(\Gamma_{\mathscr{X}})$.

III.4.2: One-forms on metric graphs

We refer the reader to [BF11, Section 2.1] for a detailed introduction to one-forms on metric graphs. Let Γ be a connected metric graph of genus g'. Assume that Γ is oriented i.e. a choice of an orientation for each edge of Γ. Then we define the space of *one-forms* on Γ as:

$$\Omega(\Gamma) = \left\{ \omega : E(\Gamma) \to \mathbb{C} \;\middle|\; \sum_{e|e^+=v} \omega(e) = \sum_{e|e^-=v} \omega(e) \text{ for all } v \in V(\Gamma) \right\}.$$

It is easy to see that $\dim_{\mathbb{C}} \Omega(\Gamma) = g'$. There exists a positive definite Hermitian pairing on $\Omega(\Gamma)$ given by

$$\langle \omega, \omega' \rangle = \sum_e \omega(e) \overline{\omega'(e)} l_e. \tag{III.4.1}$$

This Hermitian pairing should be thought of as the analogue of (III.2.1) for metric graphs.

Proposition III.5 (Theorem 5.10, Theorem 6.4 in [BF11])**.** Let $\omega_1, \ldots, \omega_{g'}$ be an orthonormal basis of $\Omega(\Gamma)$ with respect to the Hermitian pairing (III.4.1). Let r_e denote the resistance between e^- and e^+ in the graph obtained by removing the interior of the edge e from Γ. Then,

$$\sum_{i=1}^{g'} |\omega_i(e)|^2 = \frac{1}{l_e + r_e}.$$

Proof. Translating to the notation used by [BF11], we have that

$$\sum_{i=1}^{g'} |\omega_i(e)|^2 = \left\| \frac{1}{l_e} \int_e \right\|_{\mathrm{L}^2}^2 .$$

Using [BF11, Theorem 5.10], we get that

$$\left\| \frac{1}{l_e} \int_e \right\|_{\mathrm{L}^2}^2 = \frac{F(e)}{l_e},$$

where $F(e)$ is the Foster coefficient defined by Baker and Faber. Now, [BF11, Theorem 6.4] tells us that

$$\frac{F(e)}{l_e} = \frac{1}{l_e + r_e}.$$

□

III.4.3: Relation between the residues and the dual graph

Let $\psi_1, \ldots, \psi_g$ be a basis of $H^0(\mathscr{X}_{0,\mathrm{red}}, \omega_{\mathscr{X}_{0,\mathrm{red}}})$. Let $g' = g(\Gamma_{\mathscr{X}})$. Following the discussion in Section III.4.1, we may assume that $\psi_{g'+1}, \ldots, \psi_g$ are holomorphic i.e. have zero residues at all nodal points in $\mathscr{X}_{0,\mathrm{red}}$.

Note that the residues of $\psi_1, \ldots, \psi_{g'}$ at the points in $\widetilde{\mathscr{X}_{0,\mathrm{red}}}$ that lie over nodes in $\mathscr{X}_{0,\mathrm{red}}$ cannot be arbitrary; they must satisfy the following constraints

- The residue theorem ensures that the sum of the residues of ψ_i is zero on every irreducible component of $\widetilde{\mathscr{X}_{0,\mathrm{red}}}$ for all $1 \leq i \leq g'$.
- If P' and P'' are points in $\widetilde{\mathscr{X}_{0,\mathrm{red}}}$ that map to a node P in $\mathscr{X}_0$, then the residues of ψ_i at P' and P'' sum to zero for all $1 \leq i \leq g'$.

Now pick an arbitrary orientation for each of the edges of $\Gamma_{\mathscr{X}}$. For an edge e, let e^- and e^+ denote the initial and the final vertex respectively. For each ψ_i and a node $P \in \mathscr{X}_{0,\mathrm{red}}$, let C_i^P denote the residue of ψ_i at the point that lies over P in the irreducible component associated to e_P^-. The data of the residues of ψ_i defines an element $\omega_i \in \Omega(\Gamma_{\mathscr{X}})$ by $\psi_i \mapsto (e_P \mapsto C_i^P)$. Conversely, given $\omega \in \Omega(\Gamma_{\mathscr{X}})$, we can get a $\psi \in H^0(\mathscr{X}_{0,\mathrm{red}}, \omega_{\mathscr{X}_{0,\mathrm{red}}})$ using the residue theorem. Such an element is uniquely determined up to an element that has no poles on $\widetilde{\mathscr{X}_0}$ i.e. up to an element in the linear span of $\psi_{g'+1}, \ldots, \psi_g$.

Summarizing, we have the following short exact sequence of complex vector spaces.

$$0 \to H^0(\widetilde{\mathscr{X}_{0,\mathrm{red}}}, \omega_{\widetilde{\mathscr{X}_{0,\mathrm{red}}}}) \to H^0(\mathscr{X}_{0,\mathrm{red}}, \omega_{\mathscr{X}_{0,\mathrm{red}}}) \to \Omega(\Gamma_{\mathscr{X}}) \to 0 \tag{III.4.2}$$

Lemma III.6. We can pick a basis $\psi_1, \ldots, \psi_g$ of $H^0(\mathscr{X}_{0,\mathrm{red}}, \omega_{\mathscr{X}_{0,\mathrm{red}}})$ such that

$$\sum_P C_j^P \overline{C_k^P} l_{e_P} = \delta_{jk}$$

for all $1 \leq j, k \leq g'$ and

$$\int_{\widetilde{\mathscr{X}_{0,\mathrm{red}}}} \psi_j \wedge \overline{\psi_k} = \delta_{jk}$$

for $g' + 1 \leq j, k \leq g$.

Proof. We have a positive definite Hermitian pairing on $H^0(\widetilde{\mathscr{X}_{0,\mathrm{red}}}, \omega_{\widetilde{\mathscr{X}_{0,\mathrm{red}}}}) = \bigoplus_i H^0(E_i, \omega_{E_i})$ given by the Hermitian pairing on each direct summand. We pick $\psi_{g'+1}, \ldots, \psi_g$ to be orthonormal with respect to this pairing.

We pick $\psi_1, \ldots, \psi_{g'}$ so that the induced $\omega_1, \ldots, \omega_{g'} \in \Omega(\Gamma)$ form an orthonormal basis with respect to the pairing (III.4.1). $\square$

It follows immediately from Proposition III.5 that for a node $P \in \mathscr{X}_0$ and for the choice of ψ_i's in Lemma III.6,

$$\sum_{i=1}^{g'} |C_i^P|^2 = \frac{1}{l_{e_P} + r_{e_P}},$$

which is the coefficient of $dx|_{e_P}$ that shows up in the Zhang measure.

III.4.4: Relation between $\omega_{\mathscr{X}_{0,\mathrm{red}}}$ and the canonical bundle on $\mathscr{X}$

Let $\mathscr{X}$ be an nc model of X. Let $\omega_{\mathscr{X}} \simeq \Omega^2_{\mathscr{X}}$ denote the canonical line bundle of $\mathscr{X}$ i.e. the sheaf of 2-forms on $\mathscr{X}$. Note that we have an isomorphism $\Omega_{\mathscr{X}/\mathbb{D}} \simeq \omega_{\mathscr{X}}$ between the sheaf of relative holomorphic 1-forms and the canonical line bundle. This isomorphism is given by 'unwedging dt', where t is the coordinate on $\mathbb{D}$.

Note that $\mathscr{X}_0, \mathscr{X}_{0,\mathrm{red}}$ are Cartier divisors on $\mathscr{X}$ and we can consider the line bundle

$$\mathcal{L} := \omega_{\mathscr{X}}(-\mathscr{X}_0 + \mathscr{X}_{0,\mathrm{red}}) := \omega_{\mathscr{X}} \otimes_{\mathcal{O}_{\mathscr{X}}} \mathcal{O}_{\mathscr{X}}(-\mathscr{X}_0 + \mathscr{X}_{0,\mathrm{red}}).$$

Since $\mathscr{X}_0 = \mathrm{div}(t)$ is a principal divisor, we have a canonical isomorphism $\mathcal{L} \simeq \omega_{\mathscr{X}}(\mathscr{X}_{0,\mathrm{red}})$ For $t \in \mathbb{D}^*$, note that

$$\mathcal{L}|_{X_t} \simeq \omega_{\mathscr{X}}(\mathscr{X}_{0,\mathrm{red}})|_{X_t} \simeq \omega_{\mathscr{X}}|_{X_t} \simeq \Omega_{\mathscr{X}/\mathbb{D}}|_{X_t} \simeq \omega_{X_t}.$$

For the central fiber, we can use adjunction formula [Liu02, 9.1.37] to conclude that

$$\mathcal{L}|_{\mathscr{X}_{0,\mathrm{red}}} \simeq \omega_{\mathscr{X}_0}(\mathscr{X}_{0,\mathrm{red}})|_{\mathscr{X}_{0,\mathrm{red}}} \simeq \omega_{\mathscr{X}_{0,\mathrm{red}}}.$$

Lemma III.7. Let $\mathscr{X}'$ be a nc model obtained from $\mathscr{X}$ by a single blowup at a closed point in $\mathscr{X}_0$. Let $q_0 : \mathscr{X}'_{0,\mathrm{red}} \to \mathscr{X}_{0,\mathrm{red}}$ be the map induced by the blowup map $q : \mathscr{X}' \to \mathscr{X}$. Then, we have an isomorphism obtained by "pulling back differential forms".

$$q_0^* : H^0(\mathscr{X}_{0,\mathrm{red}}, \omega_{\mathscr{X}_{0,\mathrm{red}}}) \xrightarrow{\sim} H^0(\mathscr{X}'_{0,\mathrm{red}}, \omega_{\mathscr{X}'_{0,\mathrm{red}}}).$$

Proof. We first describe the map q_0^*. To do this, we use a few elementary facts regarding blowups. Let E denote the exceptional divisor and let b denote its multiplicity in $q^*(\mathscr{X}_{0,\mathrm{red}})$. Note that b is either 1 or 2, depending on whether we blowup at a smooth or at a nodal point in $\mathscr{X}_0$.

$$q^*(\omega_{\mathscr{X}}) \otimes \mathcal{O}_{\mathscr{X}'}(E) = \omega_{\mathscr{X}'}$$
$$q^*(\mathscr{X}_{0,\mathrm{red}}) = \mathscr{X}'_{0,\mathrm{red}} + (b-1)E$$

Using the above facts, we conclude that

$$q^*(\omega_{\mathscr{X}}(\mathscr{X}_{0,\mathrm{red}})) = \omega_{\mathscr{X}'}(\mathscr{X}'_{0,\mathrm{red}} - E + (b-1)E).$$

Restricting the above equation to $\mathscr{X}'_{0,\mathrm{red}}$, we get

$$q_0^*(\omega_{\mathscr{X}_{0,\mathrm{red}}}) = \omega_{\mathscr{X}'_{0,\mathrm{red}}}((b-2)E).$$

The following composition is the map q_0^* and we claim that it is an isomorphism.

$$H^0(\mathscr{X}_{0,\mathrm{red}}, \omega_{\mathscr{X}_{0,\mathrm{red}}}) \to H^0(\mathscr{X}'_{0,\mathrm{red}}, q^*(\omega_{\mathscr{X}_{0,\mathrm{red}}})) \to H^0(\mathscr{X}'_{0,\mathrm{red}}, \omega_{\mathscr{X}'_{0,\mathrm{red}}}).$$

Note that both $H^0(\mathscr{X}_{0,\mathrm{red}}, \omega_{\mathscr{X}_{0,\mathrm{red}}})$ and $H^0(\mathscr{X}'_{0,\mathrm{red}}, \omega_{\mathscr{X}'_{0,\mathrm{red}}})$ are vector spaces of dimension g. So, it is enough to show that q_0^* is injective. Note that any section $\psi \in H^0(\mathscr{X}_{0,\mathrm{red}}, \omega_{\mathscr{X}_{0,\mathrm{red}}})$ is determined by all restrictions $\psi|_{E_i}$ for all the irreducible components E_i of $\widetilde{\mathscr{X}_{0,\mathrm{red}}}$. Also, $q_0^*\psi|_{E_i'} = \psi|_{E_i}$, where E_i' is the strict transform of E_i. Thus, ψ is determined by $q_0^*\psi$ and the map q_0^* is injective. □

Lemma III.8. Suppose that X has semistable reduction. Let $\mathscr{X}$ be an nc model of X. Then there exist an $r \in (0,1)$ and 2-forms $\theta_1, \dots, \theta_g \in H^0(r\mathbb{D}, \omega_{\mathscr{X}}(-\mathscr{X}_0 + \mathscr{X}_{0,\mathrm{red}}))$ such that $\theta_1|_{X_t}, \dots, \theta_g|_{X_t}$ is a basis of $H^0(X_t, \omega_{X_t})$ for all $t \in r\mathbb{D}^*$ and $\theta_1|_{\mathscr{X}_{0,\mathrm{red}}}, \dots, \theta_g|_{\mathscr{X}_{0,\mathrm{red}}}$ is a basis of $H^0(\mathscr{X}_0, \omega_{\mathscr{X}_{0,\mathrm{red}}})$.

Proof. As above, let $\mathcal{L}$ denote the line bundle $\omega_{\mathscr{X}}(-\mathscr{X}_0 + \mathscr{X}_{0,\mathrm{red}})$.

First suppose that $\mathscr{X}$ is a semistable model of X. Then $\mathscr{X}_{0,\mathrm{red}} = \mathscr{X}_0$ and we have that

$$\dim_{\mathbb{C}} H^0(\mathscr{X}_0, \omega_{\mathscr{X}_0}) = \dim_{\mathbb{C}} H^0(\mathscr{X}_{0,}, \omega_{\mathscr{X}_{0,\mathrm{red}}}) = g = \dim_{\mathbb{C}} H^0(X_t, \omega_{X_t})$$

for all $t \in \mathbb{D}^*$ and thus the dimension of $H^0(\mathscr{X}_t, \mathcal{L}|_{\mathscr{X}_t})$ for all $t \in \mathbb{D}$ remains constant. By a theorem of Grauert [Gra60] (see [Har77, Cor 3.12.19] for an algebraic version), we get that $\pi_*(\mathcal{L})$ is a locally free sheaf and its fiber over 0, $\pi_*(\mathcal{L})_{(0)}$, is isomorphic to $H^0(\mathscr{X}_0, \omega_{\mathscr{X}_0})$. Now we pick $\theta_1, \dots, \theta_g \in \pi_*(\mathcal{L})_{(0)}$ that map to a basis in $H^0(\mathscr{X}_0, \omega_{\mathscr{X}_0})$. Then there exists an $0 < r < 1$ and $\widetilde{\theta}_1, \dots, \widetilde{\theta}_g \in H^0(r\mathbb{D}, \mathcal{L})$ which restrict to a basis $\psi_1, \dots, \psi_g$ of $H^0(\mathscr{X}_{0,\mathrm{red}}, \omega_{\mathscr{X}_{0,\mathrm{red}}})$. Since being linearly independent is an open condition, we may pick a smaller r so that

$\theta_1, \ldots, \theta_g$ remain linearly independent (and hence form a basis) after restricting to X_t for $|t| \ll r$. This completes the proof when $\mathscr{X}$ is a semistable model of X.

Any nc model can be obtained from the minimal nc model by a sequence of blowups at closed points in the central fiber. By induction, we may reduce to the proof to the case of a single blowup.

Suppose now that the result is true for a nc model $\mathscr{X}$; we would like to prove the result for a nc model $\mathscr{X}'$ obtained by a single blowup $q : \mathscr{X}' \to \mathscr{X}$ at a closed point in $\mathscr{X}_0$. Let E denote the exceptional divisor of the blowup.

Let $\theta_1, \ldots, \theta_g$ be sections of $\omega_{\mathscr{X}_0}(-\mathscr{X}_0 + \mathscr{X}_{0,\text{red}})$ in a neighborhood of $\mathscr{X}_{0,\text{red}}$ that satisfy the required conditions for the model $\mathscr{X}$. We claim that $q^*\theta_1, \ldots, q^*\theta_g$ satisfy the required conditions for $\mathscr{X}'$, where q^* denotes the usual pullback of differential forms.

Since $q|_X$ is an isomorphism, it is clear that $(q^*\theta_1)|_{X_t}, \ldots, (q^*\theta_g)|_{X_t}$ form a basis of $H^0(X_t, \omega_{X_t})$. The fact that $(q^*\theta_1)|_{\mathscr{X}'_{0,\text{red}}}, \ldots, (q^*\theta_g)|_{\mathscr{X}'_{0,\text{red}}}$ is also a basis follows by Lemma III.7. $\square$

III.5: Asymptotics

In this section, we compute some asymptotics to describe the Bergman measure in terms on $\theta_1, \ldots, \theta_g$. Suppose that X has semistable reduction. We pick an nc model $\mathscr{X}$ of X. Let $\Gamma = \Gamma_{\mathscr{X}}$ denote its dual graph and let $g' = g(\Gamma)$. Then, we have that $g = g' + \sum_{v \in V(\Gamma)} g(v)$. By Lemma III.8, we can find two-forms $\theta_1, \ldots, \theta_g$ defined in a neighborhood of $\mathscr{X}_0$ such that their restrictions form a basis of $H^0(X_t, \omega_{X_t})$ and $H^0(\mathscr{X}_{0,\text{red}}, \omega_{\mathscr{X}_{0,\text{red}}})$ for all $t \in \mathbb{D}^*$. Let $\psi_i = \theta_i|_{\mathscr{X}_{0,\text{red}}}$. After applying a (complex) linear transformation to θ_i's, we may assume that the ψ_i satisfy the conditions in Lemma III.6.

III.5.1: Relating θ_i, $\theta_{i,t}$ and ψ_i

For doing computations, we would like to express $\theta_{i,t} := \theta_i|_{X_t}$ and ψ_i explicitly in terms of θ_i in a local coordinate chart.

Let U be a coordinate chart adapted to an irreducible component $E \subset \mathscr{X}_0$ of multiplicity a. Let (z, w) be coordinates on U such that $E = \{z = 0\}$ and $t = z^a$. Then, θ_i must vanish along E to the order $a - 1$ and has a power series expansion of the form

$$\theta_i = \sum_{\alpha \geq a-1, \beta \in \mathbb{N}} c^{(i)}_{\alpha\beta} z^\alpha w^\beta dw \wedge dz.$$

Then, $\theta_{i,t}$ is just obtained by 'unwedging dt'. To do this, note that $dt = az^{a-1}dz$ and thus

$$\theta_{i,t} = \sum_{\alpha \geq a-1, \beta \in \mathbb{N}} \frac{c^{(i)}_{\alpha\beta}}{a} t^{\frac{\alpha - a + 1}{a}} w^\beta dw.$$

Here we think of the coordinates on X_t as being given by $|w| < 1$. Taking the a-th root of t corresponds to the fact that $U \cap X_t$ is disconnected and has a connected components. Choosing a connected component corresponds to choosing an a-th root of t. Note that we will be somewhat sloppy while writing fractional powers of t. This should be interpreted as being true in a small enough chart where the roots are well defined.

Tracing through the isomorphism in Section III.4.4, we see that ψ_i is obtained from θ_i by getting rid of $z^{a-1}dz$ and then setting $z = 0$. Thus,

$$\psi_i = \sum_{\beta \in \mathbb{N}} c^{(i)}_{a-1,\beta} w^\beta dw$$

and we see that $\lim_{t \to 0} \theta_{i,t} = \frac{1}{a}\psi_i$, where the limit is taken pointwise as a function of w.

Now consider a coordinate chart U adapted to a node $P = E_1 \cap E_2$. We allow the possibility that E_1 and E_2 correspond to two local branches of the same irreducible component. Let the coordinates on U be (z, w) such that $|z|, |w| < 1$, $E_1 = \{z = 0\}$, $E_2 = \{w = 0\}$ and $t = z^a w^b$. On $X_t \cap U$, we can either use the local coordinate z with $|t|^{1/a} < |z| < 1$ or the coordinate w with $|t|^{1/b} < |w| < 1$. We also have coordinates w on $E_1 \cap U$ for $|w| < 1$ and coordinates z on $E_2 \cap U$ for $|z| < 1$. Also note that $X_t \cap U \to \{w \in \mathbb{D}^* \mid |t|^{1/b} < |w| < 1\}$ is a a-sheeted cover with the fibers corresponding to choosing an a-th root to determine

$z = (\frac{t}{w^b})^{1/a}$. Since θ_i must vanish along E_1 to order $a-1$ and along E_2 to order $b-1$, we can write

$$\theta_i = \sum_{\alpha \geq a-1, \beta \geq b-1} c^{(i)}_{\alpha,\beta} z^\alpha w^\beta dw \wedge dz$$

on U. Let us compute $\theta_{i,t}$ and ψ_i in the w-coordinates. Using $dt = az^{a-1}w^b dz + bz^a w^{b-1} dw$, we have that

$$\theta_{i,t} = \sum_{\alpha \geq a-1, \beta \geq b-1} \frac{c^{(i)}_{\alpha,\beta}}{a} \left(\frac{t}{w^b}\right)^{\frac{\alpha-a+1}{a}} w^{\beta-b} dw.$$

To obtain ψ_i on E_1, we need to get rid of $z^{a-1}w^{b-1}d(zw)$ and set $z = 0$. This gives us

$$\psi_i = \sum_{\beta \geq b-1} c^{(i)}_{a-1,\beta} w^{\beta-b} dw.$$

Similarly, we can compute $\theta_{i,t}$ in the z coordinates and can obtain ψ_i on E_2.

Once again we see that $\lim_{t\to 0} \theta_{i,t} = \frac{1}{a}\psi_i$ for fixed w and $\lim_{t\to 0} \theta_{i,t} = \frac{1}{b}\psi_i$ for fixed z.

From the local description, we also see that $\psi_i|_{E_1}$ has a simple pole at P with residue $c^{(i)}_{a-1,b-1}$ and $\psi_i|_{E_2}$ has a simple pole at P with residue $-c^{(i)}_{a-1,b-1}$. Set $C^P_i = c^{(i)}_{a-1,b-1}$ for ease of notation.

III.5.2: Bergman measure in terms of $\theta_1, \ldots, \theta_g$

For $t \in \mathbb{D}^*$, let $A(t)$ be the complex $g \times g$ skew-Hermitian matrix with (j,k)-th entries

$$A(t)_{j,k} = (\theta_{j,t}, \theta_{k,t}) = \frac{i}{2}\int_{X_t} \theta_{j,t} \wedge \overline{\theta_{k,t}}.$$

Then the Bergman measure (as a $(1,1)$-form) on X_t is given by

$$\mu_t = \frac{i}{2}\sum_{j,k} (\overline{A(t)})^{-1}_{j,k} \theta_{j,t} \wedge \overline{\theta_{k,t}}.$$

To understand the asymptotics of the Bergman measure, we need to understand the entries of the matrix $A(t)^{-1}$ as $t \to 0$. We start by understanding the entries of the matrix

$A(t)$. Similar asymptotics can be found in [HJ96, Proposition 4.1] and [dJ19, Equation (16.7)].

Lemma III.9. For a suitable choice of $\theta_1, \ldots, \theta_g$, the matrix A is of the form

$$A = \begin{pmatrix} B & C^* \\ C & F \end{pmatrix},$$

where $B = 2\pi \log|t|^{-1}\mathcal{I}_{g'} + O(1)$ is a $g' \times g'$ matrix, $C = O(1)$ and $F = \mathbf{I}_{g-g'} + o(1)$ is a $(g-g') \times (g-g')$ matrix as $t \to 0$.

Proof. We pick $\theta_1, \ldots, \theta_g$ such that $\psi_1, \ldots, \psi_g$ satisfy the conditions of Lemma III.6.

By using partitions of unity, to understand the asymptotics of $\int_{X_t} \theta_{j,t} \wedge \overline{\theta_{k,t}}$ it is enough to understand the asymptotics of $\int_{U\cap X_t} \theta_{j,t} \wedge \overline{\theta_{k,t}}$ for an adapted coordinate chart U.

Let U be a coordinate chart adapted to an irreducible component $E \subset \mathscr{X}_0$ occurring with multiplicity a. For all i, we have that $\theta_{i,t} \to \frac{\psi_i}{a}$ as $t \to 0$. By shrinking U if needed, we may further assume that $\theta_{i,t} \to \frac{\psi_i}{a}$ uniformly. Since all the ψ_i's are bounded on $U \cap E$, by the dominated convergence theorem, we have that $\int_{U\cap X_t} \theta_{j,t} \wedge \overline{\theta_{k,t}} \to \int_{U\cap E} \psi_j \wedge \overline{\psi_k}$ as $t \to 0$ for all $1 \leq j, k \leq g$.

If U is a coordinate chart adapted to $P = E_1 \cap E_2$, then we break up the integral as

$$\int_{U\cap X_t} \theta_{j,t} \wedge \overline{\theta_{k,t}} = \int_{|t|^{1/2a}<|z|<1} \theta_{j,t} \wedge \overline{\theta_{k,t}} + \int_{|t|^{1/2b}<|w|<1} \theta_{j,t} \wedge \overline{\theta_{k,t}}.$$

On the set $|t|^{1/2b} < |w| < 1$, using the discussion in Section III.5.1, we can write

$$\theta_{i,t}(w) = \frac{C_i^P}{aw}dw + \sum_{\substack{\alpha\geq a-1,\beta\geq b-1,\\ (\alpha,\beta)\neq(a-1,b-1)}} \frac{c_{\alpha,\beta}^{(i)}}{a}\left(\frac{t}{w^b}\right)^{\frac{\alpha-a+1}{a}} w^{\beta-b}dw.$$

Since $|t| < |w|^b$ in the region $|t|^{1/2b} < |w| < 1$, we get

$$\theta_{i,t}(w) = \frac{C_i^P}{aw}dw + O(|w|^{-1+\frac{1}{a}})dw.$$

where the $O(|w|^{-1+\frac{1}{a}})$ is with respect to $|w|$ as $|w| \to 0$ uniformly in t. Thus,

$$\begin{aligned}\frac{i}{2}\int_{U\cap X_t}\theta_{j,t}\wedge\overline{\theta_{k,t}} &= \frac{i}{2}\int_{|t|^{1/2b}<|w|<1}\left(\frac{C_j^P\overline{C_k^P}}{a^2|w|^2}+O(|w|^{-2+\frac{1}{a}})\right)dw\wedge d\overline{w}\\ &= a\int_0^{2\pi}\int_{|t|^{1/2b}}^1\left(\frac{C_j^P\overline{C_k^P}}{a^2 r}+O(r^{-1+\frac{1}{a}})\right)dr,\end{aligned}$$

where the second $O(1)$ is with respect to r as $r \to 0$ and the factor a appears on the right-hand side because $X_t \cap U \to \{w \in \mathbb{D} \mid |t|^{1/b} < |w| < 1\}$ is an a-sheeted cover. So,

$$\frac{i}{2}\int_{|t|^{1/2b}<|w|<1}\theta_{j,t}\wedge\overline{\theta_{k,t}} = \pi\frac{C_j^P\overline{C_k^P}}{ab}\log|t|^{-1}+O(1).$$

Using a similar computation in the z coordinates and using $l_{e_P} = \frac{1}{ab}$ shows that

$$\frac{i}{2}\int_{X_t\cap U}\theta_{j,t}\wedge\overline{\theta_{k,t}} = 2\pi\frac{C_j^P\overline{C_k^P}}{ab}\log|t|^{-1}+O(1) = 2\pi C_j^P\overline{C_k^P}l_{e_P}\log|t|^{-1}+O(1).$$

Summing up, we see that

$$\frac{i}{2}\int_{X_t}\theta_{j,t}\wedge\overline{\theta_{k,t}} = 2\pi\sum_{\text{nodes}P\in\mathscr{X}_0}C_j^P\overline{C_k^P}l_{e_P}\log|t|^{-1}+O(1). \tag{III.5.1}$$

By the choice of θ_i's, $C_i^P = 0$ for all P and for all $i > g'$ giving the required asymptotics for the matrix C. The asymptotics for B follows from $\sum_P C_j^P\overline{C_k^P}l_{e_P} = \delta_{jk}$.

To get the asymptotics for the matrix F, we need to analyze the $O(1)$-term in Equation (III.5.1). Recall that $\theta_{i,t} \to \frac{1}{a}\psi_i$ as $t \to 0$ for a fixed w in the set $\{w \in \mathbb{D}^* \mid |t|^{1/2b} < |w| < 1\}$, and ψ_i is bounded on $U \cap E_1$ for all $g' \le i \le g$. Thus, as $t \to 0$ we have that

$$\int_{|t|^{1/2b}<|w|<1}\theta_{j,t}\wedge\overline{\theta_{k,t}} \to \frac{1}{a}\int_{U\cap E_1}\psi_j\wedge\overline{\psi_k}.$$

Thus, after applying a partition of unity argument, we get that

$$\int_{X_t} \theta_{j,t} \wedge \overline{\theta_{k,t}} \to \sum_E \frac{1}{\mathrm{mult}_{\mathscr{X}_0} E} \int_E \psi_j \wedge \overline{\psi_k}.$$

For $g' + 1 \leq i \leq g$, ψ_i is holomorphic on $\widetilde{\mathscr{X}_0}$ on any irreducible component and therefore must be zero on any irreducible component with genus 0. Since X has a semistable reduction, all positive genus irreducible components must occur with multiplicity 1 (see Remark III.1). Thus, we further get that

$$\int_{X_t} \theta_{j,t} \wedge \overline{\theta_{k,t}} \to \int_{\mathscr{X}_{0,\mathrm{red}}} \psi_j \wedge \overline{\psi_k}.$$

By the choice of θ_i's, we have that $\frac{i}{2} \int_{\mathscr{X}_{0,\mathrm{red}}} \psi_j \wedge \overline{\psi_k} = \delta_{j,k}$ and thus we get the asymptotics for F. $\square$

Since $F \to \mathbf{I}_{g-g'}$ as $t \to 0$, F is invertible for $|t|$ small enough. We now apply elementary row reduction operations to $(A, \mathcal{I}_g)$ to obtain the following result.

Corollary III.10. For a suitable choice of $\theta_1, \ldots, \theta_g$, the matrix A^{-1} is of the form

$$A^{-1} = \begin{pmatrix} B' & (C')^* \\ C' & F' \end{pmatrix}$$

where

$$B' = \frac{1}{2\pi \log|t|^{-1}} \mathcal{I}_{g'} + O\left(\frac{1}{(\log|t|^{-1})^2}\right),$$

$$C' = O\left(\frac{1}{\log|t|^{-1}}\right),$$

and

$$F' = \mathbf{I}_{g-g'} + o(1).$$

$\square$

III.6: Convergence Theorem

III.6.1: Convergence on $\mathscr{X}^{\mathrm{hyb}}$

In this section, we prove Theorems D′ and D.

Suppose that X has semistable reduction and let $\mathscr{X}$ is an nc model of X. Let μ_t denote the Bergman measure on X_t.

III.6.2: Bergman measure on $\widetilde{\mathscr{X}_{0,\mathrm{red}}}$

By the Bergman measure on $\widetilde{\mathscr{X}_{0,\mathrm{red}}}$, we mean the sum of the Bergman measures on all positive genus connected components of $\widetilde{\mathscr{X}_{0,\mathrm{red}}}$. The Bergman measure on $\widetilde{\mathscr{X}_{0,\mathrm{red}}}$ is given by the two-form $\frac{i}{2}\sum_{i=g'+1}^{g}\psi_i \wedge \overline{\psi_i}$ Let $\widetilde{\mu}_0$ denote the pushforward of the Bergman measure on $\widetilde{\mathscr{X}_{0,\mathrm{red}}}$ to $\mathscr{X}_{0,\mathrm{red}}$.

The following lemma gives the contribution of the Dirac mass on the vertices of $\Gamma_{\mathscr{X}}$ in the limiting measure.

Lemma III.11. Consider an open set $U \subset \mathscr{X}$ adapted to an irreducible component E of $\mathscr{X}_0$ of multiplicity a, i.e. $U \cap \mathscr{X}_0 = E \cap U$ and there exist coordinates z, w on U with $|z|, |w| < 1$ such that $E \cap U = \{z = 0\}$ and the projection $U \to \mathbb{D}$ is given by $(z, w) \mapsto z^a$ and $|z|, |w| < 1$ on U. Let χ be a compactly supported continuous function on U. Then, as $t \to 0$,

$$\int_{U\cap X_t} \chi\mu_t \to \int_{U\cap E} \chi\widetilde{\mu}_0.$$

Proof. Recall that $\mu_t = \frac{i}{2}\sum_{j,k}(\overline{A(t)})^{-1}_{j,k}\theta_{j,t} \wedge \overline{\theta_{k,t}}$.

If either $j \leq g'$ or $k \leq g'$, then $\overline{A(t)}^{-1}_{j,k} = O\left(\frac{1}{\log|t|^{-1}}\right)$, $\theta_{i,t}$ is bounded on U and using Corollary III.10,

$$\int_{U\cap X_t} \chi \cdot (\overline{A(t)})^{-1}_{j,k}\theta_{j,t} \wedge \theta_{k,t} = O\left(\frac{1}{\log|t|^{-1}}\right)$$

and hence goes to 0 as $t \to 0$.

Therefore, we only need to worry about the terms for which $j,k > g'$. Recall that $\theta_{i,t} \to \frac{1}{a}\psi_i$ as $t \to 0$. Using a similar computation as in the proof of Lemma III.9, we get that

$$\lim_{t\to 0}\int_{U\cap X_t}\chi\mu_t = \frac{1}{a}\int_{U\cap E}\chi\cdot\left(\sum_{j,k=g'+1}^{g}(\lim_{t\to 0}\overline{F'(t)_{j,k}})\cdot\psi_j\wedge\overline{\psi_k}\right),$$

where F' is the matrix from Corollary III.10. Since $\lim_{t\to 0}F'(t) = \mathbf{I}_{g-g'}$, we get that

$$\lim_{t\to 0}\int_{U\cap X_t}\chi\mu_t = \int_{U\cap E}\chi\cdot\left(\frac{1}{a}\sum_{i=g'+1}^{g}\psi_i\wedge\overline{\psi_i}\right),$$

Using Remark III.1, we get that $\psi_i = 0$ unless $a = 1$ for $g - g' + 1 \leq i \leq g$. Thus,

$$\lim_{t\to 0}\int_{U\cap X_t}\chi\mu_t = \int_{U\cap E}\chi\cdot\left(\sum_{i=g'+1}^{g}\psi_i\wedge\overline{\psi_i}\right),$$

The right-hand side is exactly $\int_{U\cap E}\chi\widetilde{\mu}_0$. $\square$

In the following lemma, the first term on the right-hand side contributes to the Lebesgue measure in μ_0 while the second term contributes to the Dirac mass in μ_0.

Lemma III.12. Let $U \subset \mathscr{X}$ be an open set adapted to a node $P = E_1 \cap E_2$ in $\mathscr{X}_{0,\text{red}}$, where E_1, E_2 are irreducible components of $\mathscr{X}_0$ with multiplicities a, b respectively. Let χ be a compactly supported function on U and let f be a continuous function on $[0, \frac{1}{ab}]$. Write the coordinates in U as z, w with $|z|, |w| < 1$, $E_1 = \{z = 0\}$, $E_2 = \{w = 0\}$ and the projection to $\mathbb{D}$ given by $(z,w) \mapsto t = z^a w^b$. Let the coordinate on $X_t \cap U$ be w with $|t|^{1/2b} < |w| < 1$. Then, as $t \to 0$ we have that

$$\int_{|t|^{1/2b}<|w|<1}\chi\cdot\left(f\circ\mathrm{Log}_U\left(\left(\frac{t}{w^b}\right)^{1/a}, w\right)\right)\mu_t \to$$
$$\chi(P)\cdot\frac{1}{l_{e_P}+r_{e_P}}\cdot\int_0^{1/2ab}f(u)du + f(0)\cdot\int_{U\cap E_1}\chi\widetilde{\mu}_0,$$

where $\frac{1}{l_{e_P}+r_{e_P}}$ is the coefficient of $dx|_{e_P}$ in the Zhang measure. (See Section III.2.5 for details.)

Proof. To analyze the integral in the left-hand side of the lemma, we first note that

$$\int_{|t|^{1/2b}<|w|<1} \chi(f \circ \mathrm{Log}_U)\mu_t = \frac{i}{2}\sum_{j,k=1}^{g} (\overline{A(t)})^{-1}_{j,k} \int_{|t|^{1/2b}<|w|<1} \chi(f \circ \mathrm{Log}_U)\theta_{j,t} \wedge \overline{\theta_{k,t}}$$

and then analyze each of the terms. To do this, we break them up into three cases.

- ($j \leq g'$ and $k > g'$) or ($j > g'$ and $k < g'$)
- $j, k \leq g'$
- $j, k > g'$

We will prove that the first case does not contribute at all in the limit, the second case contributes the first term in right-hand side of the Lemma and the third case contributes the second term.

For the first case, note that if $j \leq g'$ and $k > g'$, then $(\overline{A(t)})^{-1}_{j,k} = O\left(\frac{1}{\log|t|^{-1}}\right)$. Since $|t| < |w|^{2b}$ on the region that we are integrating on, we see from the power series expansion that

$$\begin{aligned} \theta_{j,t} =& \quad \left(\tfrac{C_j}{aw} + O(|w|^{-1+\frac{1}{a}})\right) dw \\ \theta_{k,t} =& \qquad O(|w|^{-1+\frac{1}{a}}) dw \end{aligned}$$

where the $O(|w|^{-1+\frac{1}{a}})$ above are with respect to $|w|$ as $|w| \to 0$ uniformly in t. If we do a change of coordinates $w = re^{i\theta}$, we have that

$$\theta_{j,t} \wedge \overline{\theta_{k,t}} = O(|w|^{-2+\frac{1}{a}}) dw \wedge d\overline{w} = O(r^{-1+\frac{1}{a}}) dr d\theta$$

where the last $O(r^{-1+\frac{1}{a}})$ is with respect to r as $r \to 0$.

Thus we see that

$$(\overline{A(t)})^{-1}_{j,k} \int_{|t|^{1/2b}<|w|<1} \chi(f \circ \mathrm{Log}_U)\theta_{j,t} \wedge \overline{\theta_{k,t}} = O\left(\frac{1}{\log|t|^{-1}}\right).$$

By symmetry, the same holds when $j > g'$ and $k \leq g'$.

Now consider the second case when $j, k \leq g'$. Then,

$$\overline{A(t)}^{-1}_{j,k} = \frac{\delta_{jk}}{2\pi \log|t|^{-1}} + O\left(\frac{1}{(\log|t|^{-1})^2}\right)$$

and

$$\frac{i}{2}\theta_{j,t} \wedge \overline{\theta_{k,t}} = \frac{i}{2}\left(\frac{C_j\overline{C_k}}{a^2|w|^2} + O(|w|^{-2+\frac{1}{a}})\right) dw \wedge d\overline{w} = \left(\frac{C_j\overline{C_k}}{a^2 r} + O(r^{-1+\frac{1}{a}})\right) dr d\theta,$$

First note that if $j \neq k$ and $j, k \leq g'$, then,

$$\begin{aligned}(\overline{A(t)})^{-1}_{j,k} \int_{|t|^{1/2b}<|w|<1} \chi(f \circ \mathrm{Log}_U)\theta_{j,t} \wedge \overline{\theta_{k,t}} \quad &= O\left(\tfrac{1}{(\log|t|^{-1})^2} \int_{|t|^{1/2b}<r<1} \tfrac{dr}{r}\right) \\ &= O\left(\tfrac{1}{\log|t|^{-1}}\right) \to 0 \quad \text{ as } t \to 0.\end{aligned}$$

If $j \leq g'$, then,

$$\begin{aligned}&\frac{i}{2}(\overline{A(t)})^{-1}_{j,j} \int_{|t|^{1/2b}<|w|<1} \chi(f \circ \mathrm{Log}_U)\theta_{j,t} \wedge \overline{\theta_{j,t}} \\ &\qquad = \frac{1}{2a\pi \log|t|^{-1}} \int_{|t|^{1/2b}<r<1} \chi f\left(\frac{\log r}{a \log|t|}\right) \frac{|C_j|^2 dr d\theta}{r} + O\left(\frac{1}{\log|t|^{-1}}\right).\end{aligned}$$

It is enough to figure out the limit of the integral on the right-hand side. To do this, consider a change of variable $u = \frac{\log r}{a\log|t|}$. Then, the integral on the right-hand side becomes

$$|C_j|^2 \int_0^{1/2ab} \int_0^{2\pi} \chi \cdot f(u) d\theta du.$$

The integrand converges to $\chi(0,0)f(u)$ pointwise almost everywhere as $t \to 0$. Since the integrand is bounded, by the dominated convergence theorem, we have that

$$\lim_{t\to 0} \frac{i}{2}(\overline{A(t)})^{-1}_{j,j} \int_{|t|^{1/2b}<|w|<1} \chi(f \circ \mathrm{Log}_U)\theta_{j,t} \wedge \overline{\theta_{j,t}} = |C_j|^2 \chi(0,0) \int_0^{1/2ab} f(u) du.$$

It follows from Proposition III.5 that $\frac{1}{l_{e_P}+r_{e_P}} = \sum_{j=1}^{g'} |C_j|^2$. This gives us the first term on the right-hand side in the lemma.

For the third case when $j,k > g'$, note that $\overline{A(t)}^{-1}_{j,k} = O(1)$ and $\theta_{j,t} \wedge \overline{\theta_{k,t}} = O(|w|^{-2+\frac{2}{a}}) dw \wedge d\overline{w}$. Therefore, we can apply the dominated convergence theorem. The pointwise limit of the integrand as $t \to 0$ is given by

$$\overline{A(t)}^{-1}_{j,k} \cdot \chi\left(\frac{t}{w}, w\right) \cdot f\left(\frac{\log|w|}{a\log|t|}\right) \theta_{j,t} \wedge \overline{\theta_{k,t}} \;\to\; \left(\lim_{t\to 0} \overline{F'(0)_{j,k}}\right) \cdot \chi(0,w) \cdot f(0) \cdot \frac{\psi_j \wedge \overline{\psi_k}}{a^2}.$$

After interchanging the limit and the integral and using the fact that $\psi = 0$ unless $a = 1$ (see Remark III.1), we get the second term on the right-hand side of the lemma. ☐

Corollary III.13. Let the notation be as in the Lemma III.12. Then, as $t \to 0$

$$\int_{U\cap X_t} \chi(f \circ \mathrm{Log}_U)\mu_t = \chi(P)\frac{1}{l_{e_P}+r_{e_P}} \int_0^{1/ab} f(u)du + f(0)\int_{U\cap E_1} \chi\widetilde{\mu_0} + f(1)\int_{U\cap E_1} \chi\widetilde{\mu_0}.$$

Proof. Note that

$$\int_{U\cap X_t} \chi(f\circ \mathrm{Log}_U)\mu_t = \int_{|t|^{1/2a}<|z|<1} \chi(f\circ \mathrm{Log}_U)\mu_t + \int_{|t|^{1/2b}<|w|<1} \chi(f\circ \mathrm{Log}_U)\mu_t.$$

Applying the previous lemma for both the terms on the right-hand side, we are done. ☐

Corollary III.14. Let $V = \bigcup_i U_i$ be a neighborhood of $\mathscr{X}_0$ where U_i are adapted coordinate charts. Let χ_i be a partition of unity with respect to the cover U_i. Let $\mathrm{Log}_V = \sum_i \chi_i \mathrm{Log}_{U_i}$ be a global log function on V. Let f be a continuous function on Γ. Then, as $t \to 0$,

$$\int_{X_t} (f \circ \mathrm{Log}_V)\mu_t \to \int_\Gamma f\mu_{Zh}.$$

Proof. Note that

$$\int_{X_t} (f \circ \mathrm{Log}_V)\mu_t = \sum_i \int_{U_i \cap X_t} \chi_i (f \circ \mathrm{Log}_V)\mu_t.$$

Since $\mathrm{Log}_V - \mathrm{Log}_U = O\left(\frac{1}{\log|t|^{-1}}\right)$, as $t \to 0$,

$$\int_{U_i \cap X_t} \chi(f \circ \mathrm{Log}_V - f \circ \mathrm{Log}_{U_i})\mu_t \to 0.$$

Therefore, the limit we are interested in is the same as the limit of

$$\sum_i \int_{U_i \cap X_t} \chi_i (f \circ \mathrm{Log}_{U_i})\mu_t.$$

The result just follows from the using the previous two lemmas and using that $\int_E \widetilde{\mu}_0 = g(\widetilde{E})$ for all irreducible components E of $\mathscr{X}_0$. □

The following Corollary is equivalent to Theorem D′.

Corollary III.15. Let h be a continuous function on $\mathscr{X}^{\mathrm{hyb}}$. Then, $\int h\mu_t \to \int h\mu_{Zh}$ as $t \to 0$.

Proof. Let $f = h|_\Gamma$ and let $\widetilde{h} = f \circ \mathrm{Log}_V$. By the previous lemma, the result is true for $\widetilde{h}$ i.e. $\int \widetilde{h}\mu_t \to \int f\mu_t$ as $t \to 0$. Thus, it is enough to show that $\int (h - \widetilde{h})\mu_t \to 0$ as $t \to 0$. Pick $\epsilon > 0$. Since $h - \widetilde{h} = 0$ on Γ and since $h - \widetilde{h}$ is continuous on $\mathscr{X}^{\mathrm{hyb}}$, there exists $0 < r < 1$ such that $|h - \widetilde{h}| < \epsilon$ on all $\pi^{-1}(r\mathbb{D})$. Thus, $|\int (h - \widetilde{h})\mu_t| \leq \epsilon g$ for all $|t| < r$. Letting $\epsilon \to 0$, we get that $\int (h - \widetilde{h})\mu_t \to 0$ as $t \to 0$. □

III.6.3: Extending the convergence to X^{hyb}

The convergence theorem on $\mathscr{X}^{\mathrm{hyb}}$ has the drawback that it depends on the choice of a normal crossing model. To remedy this, we consider the convergence on X^{hyb}. Recall that $X^{\mathrm{hyb}} = \varprojlim \mathscr{X}^{\mathrm{hyb}}$ and does not depend on the choice of an nc model of X. We would like to extend the convergence to X^{hyb} by patching the convergence results for $\mathscr{X}^{\mathrm{hyb}}$ for all nc models $\mathscr{X}$ of X.

To do this, note that we have a canonical measure, $\mu_{0,X}$ on $X^{\mathrm{an}}_{\mathbb{C}((t))} = \varprojlim \Gamma_{\mathscr{X}}$ induced by the Zhang measure on all the $\Gamma_{\mathscr{X}}$'s. This follows from the fact that if $\mathscr{X}' \geq \mathscr{X}$ and we consider the retraction $\Gamma_{\mathscr{X}'} \to \Gamma_{\mathscr{X}}$, then the pushforward of the Zhang measure on $\Gamma_{\mathscr{X}'}$ to $\Gamma_{\mathscr{X}}$ is the same as the Zhang measure on $\Gamma_{\mathscr{X}}$. The compatibility of these measures thus prove Theorem D in the case when X has a semistable reduction.

III.6.4: Ground field extension

Now we need to treat the general case of the Theorem D i.e. the case when X does not necessarily have semistable reduction. To do this, note that after performing a base change by $\mathbb{D}^* \to \mathbb{D}^*$ given by $u \mapsto u^n$, X will have semistable reduction (see Section III.2.2). So, we only need to understand what happens after we perform such a base change.

So consider the map $\mathbb{D}^* \to \mathbb{D}^*$ given by $u \mapsto u^n$. Let Y be the base change of X along this map i.e. we have a Cartesian diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \arrow[d] \\ \mathbb{D}^* \arrow[r, "u \mapsto u^n"] & \mathbb{D}^* \end{tikzcd} .$$

At the level of varieties, this corresponds to doing a base field extension $\mathbb{C}((t)) \to \mathbb{C}((u))$ and $Y_{\mathbb{C}((u))} = X_{\mathbb{C}((t))} \times_{\mathbb{C}((t))} \mathrm{Spec}\ \mathbb{C}((u))$. Thus, we have a surjective map $Y^{\mathrm{an}}_{\mathbb{C}((u))} \to X^{\mathrm{an}}_{\mathbb{C}((t))}$. This map is compatible with $Y \to X$ in the sense that the map $Y^{\mathrm{hyb}} \to X^{\mathrm{hyb}}$ is continuous. We would like to relate the convergence of Bergman measures on X_t to the convergence of Bergman measures on Y_u.

Note that if X has a semistable model, then so does Y. To see this, pick the minimal nc model of $\mathscr{X}$ and base change it to get a model $\widetilde{\mathscr{Y}}$ of Y. The model $\widetilde{\mathscr{Y}}$ is not regular, but can be made regular after blowing up at each singular point $\lfloor \frac{n}{2} \rfloor$ times to get a model $\mathscr{Y}$ of Y. Then $\mathscr{Y}$ is the minimal nc model of Y. It is easy to see that under the base change operation, $\Gamma_{\mathscr{Y}}$ is obtained by scaling the lengths of all edges in $\Gamma_{\mathscr{X}}$ by a factor of n. Thus, we see that the Zhang measure on Γ_Y is compatible with the Zhang measure on Γ_X if assume that X has a semistable reduction. Similarly, the Zhang measures on $Y^{\mathrm{an}}_{\mathbb{C}}((u))$ and $X^{\mathrm{an}}_{\mathbb{C}((t))}$ are compatible if we assume that X has a semistable reduction.

Note that this not necessarily true if X does not have semistable reduction. Starting with X, we can always perform a suitable base change so that Y has semistable reduction. Let $\mathscr{Y}$ and $\mathscr{X}$ be nc models of Y and X respectively. Then, we have a map $Y^{\mathrm{an}}_{\mathbb{C}}((u)) \to X^{\mathrm{an}}_{\mathbb{C}((t))}$, which gives rise to a local isometry $\Gamma_{\mathscr{Y}} \to \Gamma_{\mathscr{X}}$. Let μ_0 be the Zhang measure on $Y^{\mathrm{an}}_{\mathbb{C}}((u))$. Since the map $p : Y^{\mathrm{hyb}} \to X^{\mathrm{hyb}}$ is continuous, we get that the Bergman measure μ_t on X_t converge to the pushforward measure $p_*(\mu_0)$ supported on the image of $\Gamma_{\mathscr{Y}}$ in $X^{\mathrm{an}}_{\mathbb{C}((t))}$, thus completing the proof of Theorem D.

III.7: Metrized curve complex hybrid space

In this section, we prove Theorem E. To do this, we first construct the metrized curve complex hybrid space. Let $X \to \mathbb{D}^*$ be a family of curves with semistable reduction. Let $\mathscr{X}$ be an nc model of X.

III.7.1: Metrized curve complexes

The *metrized curve complex*, $\Delta_{\mathrm{CC}}(\mathscr{X})$, associated to $\mathscr{X}$ is a topological space which is obtained from $\widetilde{\mathscr{X}_{0,\mathrm{red}}}$ by adding line segments joining the points that lie over the same nodal point. More precisely,

$$\Delta_{\mathrm{CC}}(\mathscr{X}) = \left(\widetilde{\mathscr{X}_{0,\mathrm{red}}} \sqcup \bigsqcup_{e \in E(\Gamma_{\mathscr{X}})} [0, l_e] \right) / \sim,$$

where $P' \sim 0$ and $P'' \sim l_{e_P}$ for $P', P'' \in \widetilde{\mathscr{X}_{0,\mathrm{red}}}$ that lie over a node P and $0, l_{e_P} \in [0, l_{e_P}]$. We call the image of an irreducible component of $\mathscr{X}_{0,\mathrm{red}}$ as a *curve* in $\Delta_{\mathrm{CC}}(\mathscr{X})$ and the image of $[0, l_e]$ as an *edge* in $\Delta_{\mathrm{CC}}(\mathscr{X})$.

We have a continuous map $\Delta_{\mathrm{CC}}(\mathscr{X}) \to \mathscr{X}_{0,\mathrm{red}}$ obtained by collapsing all the edges of $\Delta_{\mathrm{CC}}(\mathscr{X})$ to the associated nodes. We also have a continuous map $\Delta_{\mathrm{CC}}(\mathscr{X}) \to \Gamma_{\mathscr{X}}$ obtained by collapsing the curves to the associated vertices.

We define a measure μ_{CC} on $\Delta_{\mathrm{CC}}(\mathscr{X})$ as follows. Let $\widetilde{\mu_0}$ denote the Bergman measure on the positive genus components of $\widetilde{\mathscr{X}_{0,\mathrm{red}}}$.

$$\mu_{\mathrm{CC}} = \widetilde{\mu_0} + \sum_{e \in E(\Gamma_{\mathscr{X}})} \frac{dx|_e}{l_e + r_e},$$

where $\frac{1}{l_e+r_e}$ is the coefficient that shows up in the Zhang measure(see Section III.2.5) and $dx|_e$ is Lebesgue measure on the edge e normalized to have length l_e.

We say that a point $Q \in \Delta_{\mathrm{CC}}(\mathscr{X})$

- is in the *interior of a curve* if it lies on a curve but not on an edge.
- is in the *interior of an edge* if it lies on an edge but not on a curve.
- is an *intersection point* if lies on a curve as well as an edge.

III.7.2: Curve complex hybrid space

We define the *curve complex hybrid space*, $\mathscr{X}^{\mathrm{hyb}}_{\mathrm{CC}}$, which as a set is given by

$$\mathscr{X}^{\mathrm{hyb}}_{\mathrm{CC}} = X \sqcup \Delta_{\mathrm{CC}}(\mathscr{X}).$$

We declare the topology on $\mathscr{X}^{\mathrm{hyb}}$ to be the weakest topology satisfying the following.

- $X \hookrightarrow \mathscr{X}^{\mathrm{hyb}}_{\mathrm{CC}}$ is an open immersion.
- $\mathscr{X}^{\mathrm{hyb}}_{\mathrm{CC}} \to \mathscr{X}$ given by collapsing all edges in $\Delta_{\mathrm{CC}}(\mathscr{X})$ is a continuous map.
- $\mathscr{X}^{\mathrm{hyb}}_{\mathrm{CC}} \to \mathscr{X}^{\mathrm{hyb}}$ given by collapsing all curves in $\Delta_{\mathrm{CC}}(\mathscr{X})$ to points is a continuous map.

We now describe a neighborhood basis of a point $Q \in \Delta_{\mathrm{CC}}(\mathscr{X})$.

- If Q is an interior point of a curve, then an adapted coordinate chart centered at Q gives a neighborhood basis of Q.

- If Q is an interior point of an edge, let P denote the node associated to the edge containing Q. Let U be an adapted neighborhood chart around P. Let $\alpha, \beta \in e_P \simeq [0, l_{e_P}]$ such that $\alpha < Q < \beta$. If we view $(\alpha, \beta) \subset [0, l_{e_P}]$, then,

$$\{x \in U \setminus \mathscr{X}_0 \mid \mathrm{Log}_U(x) \in (\alpha, \beta)\} \cup (\alpha, \beta)$$

  is a neighborhood of Q. As we vary U, α and β, we get a neighborhood basis of Q.

- If Q is an intersection point, let P denote the node associated to the edge containing Q. Let U be an adapted coordinate chart centered at P with coordinates z, w with $|z|, |w| < 1$ such that the projection $\mathscr{X} \to \mathbb{D}$ is given by $(z, w) \mapsto z^a w^b$. Let $E_1 \cap U = \{z = 0\}$ and $E_2 \cap U = \{w = 0\}$, where E_1, E_2 are irreducible components of $\mathscr{X}_{0,\mathrm{red}}$. WLOG, assume that $\widetilde{E_1}$ is the irreducible component of $\widetilde{\mathscr{X}_{0,\mathrm{red}}}$ containing Q. We identify $e_P \simeq [0, \frac{1}{ab}]$ with v_{E_1} identified with 0. Pick $0 < \epsilon < \frac{1}{2ab}$. Then,

$$\left\{(z, w) \in U \setminus \mathscr{X}_0 \;\middle|\; \frac{\log|w|}{a\log|t|} < \epsilon\right\} \cup (E_1 \cap U) \cup [0, \epsilon)$$

  is a neighborhood of Q. Varying U and ϵ, we get a neighborhood basis of Q.

III.7.3: Convergence of Bergman measures

To show that the Bergman measures μ_t on X_t converge to μ_{CC} on $\Delta_{\mathrm{CC}}(\mathscr{X})$, we can use a partition of unity argument to reduce the problem to studying the convergence on a neighborhood of each point in $\Delta_{\mathrm{CC}}(\mathscr{X})$.

Consider a point Q and consider a neighborhood V of Q as described at the end of Section III.7.2. We need to show that the measures μ_t on $X_t \cap V$ converges weakly to μ_{CC}

on $\Delta_{\mathrm{CC}}(\mathscr{X}) \cap V$.

If Q is an interior point of a curve, then this computation has been worked out in Lemma III.11. If Q is an interior point of an edge, then a minor modification of Lemma III.12 yields the result. So, it remains to prove the result in the result in the case when Q is an intersection point.

Lemma III.16. Let Q be an intersection point in $\Delta_{\mathrm{CC}}(\mathscr{X})$ and let V be a neighbourhood of Q in $\mathscr{X}_{\mathrm{CC}}^{\mathrm{hyb}}$ mentioned at the end of Section III.7.2. Let f be a continuous compactly supported function on V. Then, as $t \to 0$,

$$\int_{V \cap X_t} f\mu_t \to \int_{V \cap \Delta_{\mathrm{CC}}(\mathscr{X})} f\mu_{\mathrm{CC}}.$$

Proof. Let f be a compactly supported continuous function on V. Let $f_0 = f|_{V \cap \Delta_{\mathrm{CC}}(\mathscr{X})}$. Note that $V \cap \Delta_{\mathrm{CC}}(\mathscr{X})$ is homeomorphic to a *half-dumbbell*

$$D = \{(w, v) \in \mathbb{D} \times [0, \epsilon) \mid \text{Either } w = 0 \text{ or } v = 0\} \subset \mathbb{D} \times [0, \epsilon).$$

Let $r : \mathbb{D} \times [0, \epsilon) \to D$ be a strong deformation retract.

Consider the compactly supported continuous function $h : V \to \mathbb{R}$ defined by

$$h(z, w) = f_0\left(r\left(w, \frac{\log|w|}{a\log|t|}\right)\right)$$

for $(z, w) \in V \cap X$ and by

$$h(x) = f_0(x)$$

for $x \in \Delta_{\mathrm{CC}}(\mathscr{X})$.

We first prove that

$$\int_{V \cap X_t} h\mu_t \to \int_{V \cap \Delta_{\mathrm{CC}}(\mathscr{X})} f_0\mu_{\mathrm{CC}}.$$

To see this, recall the following facts from Sections III.5 and III.6:

$$\mu_t = \frac{i}{2}\sum_{j,k=1}^{g} \overline{A(t)^{-1}}_{j,k}\theta_{j,t}\wedge\overline{\theta_{k,t}},$$

where

$$\overline{A(t)^{-1}}_{j,k}\theta_{j,t}\wedge\overline{\theta_{k,t}} = O\left(\frac{1}{\log|t|^{-1}}\right)$$

when either $j\le g'$ and $k>g'$ or $j>g'$ and $k\le g'$,

$$\overline{A(t)^{-1}}_{j,k}\theta_{j,t}\wedge\overline{\theta_{k,t}} = \frac{1}{\log|t|^{-1}}\left(\frac{C_j\overline{C_k}\delta_{jk}}{a^2}\frac{dw\wedge d\overline{w}}{|w|^2} + O(|w|^{-1+\frac{1}{a}})dw\wedge d\overline{w}\right)$$

when $j,k\le g'$, and

$$\overline{A(t)^{-1}}_{j,k}\theta_{j,t}\wedge\overline{\theta_{k,t}} \to \delta_{j,k}\frac{\psi_j\wedge\overline{\psi_k}}{a^2}$$

when $j,k>g'$.

Thus, if either $j\le g'$ and $k>g'$ or $j>g'$ and $k\le g'$, then as $t\to 0$,

$$\overline{A(t)^{-1}}_{j,k}\int_{V\cap X_t} h\theta_{j,t}\wedge\overline{\theta_{k,t}}\to 0. \tag{III.7.1}$$

We also get that

$$\sum_{j,k=g'+1}^{g}\overline{A(t)^{-1}}_{j,k}\int_{V\cap X_t} h\theta_{j,t}\wedge\overline{\theta_{k,t}}\to\sum_{j=g'+1}^{g}\int_{U\cap E_1}\left(\lim_{t\to 0} h\left(\frac{t}{w},w\right)\right)\frac{\psi_j\wedge\overline{\psi_j}}{a^2}.$$

Note that $\lim_{t\to 0} h(\frac{t}{w},w) = \lim_{t\to 0} f_0(r(w,\frac{\log|w|}{a\log|t|})) = f_0(r(w,0)) = f_0(w,0)$. Note that $\psi = 0$ unless $a=1$ (see Remark III.1). Also, recall that $\widetilde{\mu_0} = \frac{i}{2}\sum_{j=g'+1}^{g}\psi_j\wedge\overline{\psi_j}$. Thus,

$$\frac{i}{2}\sum_{j,k=g'+1}^{g}\overline{A(t)^{-1}}_{j,k}\int_{V\cap X_t} h\theta_{j,t}\wedge\overline{\theta_{k,t}}\to\int f_0\widetilde{\mu_0}. \tag{III.7.2}$$

Now, it remains to consider the limit of

$$\frac{i}{2}\sum_{j,k=1}^{g'}\overline{A(t)^{-1}}_{j,k}\int_{V\cap X_t} h\theta_{j,t}\wedge\overline{\theta_{k,t}}$$

as $t\to 0$. But this is the same as the limit of

$$\sum_{j=1}^{g'}\frac{|C_j|^2}{2\pi a^2\log|t|^{-1}}\int_{V\cap X_t} h\frac{dw\wedge d\overline{w}}{|w|^2}=\sum_{j=1}^{g'}\frac{a|C_j|^2}{2\pi a^2\log|t|^{-1}}\int_{\{1>|w|>|t|^{a\epsilon}\}} h\frac{dw\wedge d\overline{w}}{|w|^2}$$

as $t\to 0$. The factor a appears on the right-hand side since

$$V\cap X_t\to\{w\in\mathbb{D}^*\mid |t|^{a\epsilon}<|w|<1\}$$

is an a-sheeted cover. Consider a change of variables $u=\frac{\log|w|}{a\log|t|}$ and $\theta=\arg(w)$. Then, the above integral is the same as

$$\sum_{j=1}^{g'}\frac{|C_j|^2}{2\pi}\int_0^{\epsilon}\int_0^{2\pi} h\left(\frac{t}{|t|^{au}e^{i\theta}},|t|^{au}e^{i\theta}\right)d\theta du.$$

Note that

$$\lim_{t\to 0} h\left(\frac{t}{|t|^{au}e^{i\theta}},|t|^{au}e^{i\theta}\right)=\lim_{t\to 0} f_0\left(r\left(|t|^{au}e^{i\theta},u\right)\right)=f_0(r(0,u))=f_0(0,u)$$

almost everywhere for $u\in[0,\epsilon]$. Also recall that $\sum_{j,k=1}^{g'}|C_j|^2=\frac{1}{l_{e_P}+r_{e_P}}$. Thus, we get that

$$\frac{i}{2}\sum_{j,k=1}^{g'}\overline{A(t)^{-1}}_{j,k}\int_{V\cap X_t} h\theta_{j,t}\wedge\overline{\theta_{k,t}}\to\frac{1}{l_{e_P}+r_{e_P}}\int_0^{\epsilon} f_0 de. \tag{III.7.3}$$

Using Equations (III.7.1), (III.7.2) and (III.7.3), we get that

$$\int_{V\cap X_t} h\mu_t\to\int_{V\cap\Delta_{\mathrm{CC}}(\mathscr{X})} f_0\mu_{\mathrm{CC}}.$$

To show that $\int_{V\cap X_t} f\mu_t \to \int_{V\cap\Delta_{\mathrm{CC}}(\mathscr{X})} f_0\mu_{\mathrm{CC}}$, note that $h-f$ is a compactly supported continuous function on V such that $(h-f)|_{\Delta_{\mathrm{CC}}(\mathscr{X}_0)} = 0$. Thus, given $\epsilon' > 0$, there exists an t_0 such that $|h-f| < \epsilon'$ on $V\cap X_t$ for $|t| < |t_0|$. Thus,

$$\left|\int_{V\cap X_t} f\mu_t - \int_{V\cap X_t} h\mu_t\right| < \epsilon' g.$$

Taking $\epsilon' \to 0$, we get that

$$\lim_{t\to 0}\int_{V\cap X_t} f\mu_t = \lim_{t\to 0}\int_{V\cap X_t} h\mu_t = \int_{V\cap\Delta_{\mathrm{CC}}(\mathscr{X})} f_0\mu_{\mathrm{CC}}.$$

$\square$

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