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A TOPOLOGICAL STUDY

OF

THE LEVEL CURVES OF HARMONIC FUNCTIONS

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INTRODUCTION

It is known that the level curves of any function $f(x,y)$ which is harmonic in a simply connected domain form a curve family which is regular (locally homeomorphic to parallel lines) in the neighborhood of every point, with the exception at most of an isolated set of points at each of which the curve family has a singularity of the multiple saddle point type. The proof that these local properties are sufficient to characterize topologically the level curve families of such harmonic functions is the main task of this paper. This generalizes some of the results of several papers by W. Kaplan* in which curve families which were regular (without singularities) in the entire plane were considered. It was proved in these papers that (1) every such curve family is the level curve family of a harmonic function; (2) every such family is the solution family of a system of differential equations $\frac{dx}{dt} = f(x,y)$, $\frac{dy}{dt} = g(x,y)$; and (3) the family can be decomposed into the sum of a denumerable collection of non-overlapping subfamilies each homeomorphic to the parallel lines of a half-plane. These results are all extended in this paper to the

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* A detailed bibliography of papers referred to in the introduction and the body of the text is appended. Roman numerals in brackets refer to the bibliography.

more general type of curve family with isolated branch points (i.e., multiple saddle points).

Section 1.0 is devoted to enumerating, without proof, some of the more important properties of curve families F which are regular in some region R of the oriented plane π. These theorems essentially stem from the work of Poincare and Bendixson on curve families defined by a system of differential equations. That the theorems listed in this section are topological in character was shown by Kaplan, from which source they are quoted. In particular, neighborhoods of points and arcs, the existence of cross-sections, and the limit points in π of an open curve are discussed. Several important classical theorems are given; for example, Theorem 1.5-3 which states that interior to every closed curve is a singular point, and Theorem 1.4-4 which says that an open directed curve which is bounded but has no singular point as a limit point in one direction is asymptotic to a closed curve.

Finally, the important Theorem 1.6-1 due to Whitney is given. This theorem states that if the curve family is orientable it is always possible to find a function f(p,t) (p a point of R, t a real parameter) which may be interpreted as defining a continuous flow of particles along the curves of the family.

In Section 2.0 we restrict ourselves to curve families which are regular everywhere in π except for isolated singularities. This set of singularities is then shown to be both closed and denumerable. The notion of index of a regular curve family is defined and two theorems are quoted from Kerekjarto. The first of these (Theorem 1.2-1) gives the arithmetic relation between the sum of the indices at all singular points and the topological invariants of the surface (with boundary) on which the curve family is assumed to be defined; the second (Theorem 1.2-2) gives the index at a branch point in terms of its multiplicity. Figure 1 gives a few pictures of isolated singularities, and Figure 2 an example of a curve family with only isolated singularities.

In Section 3.0 we make our final restrictions on F, namely that it be regular in a region R consisting of all of π except for an isolated (denumerable) set of singular points B, at each of which the curve family has a singularity of the branch point type (Figure 1a). A series of theorems is then proved which makes it possible to divide the curves of F into two classes, the regular curves, which extend to infinity in each direction, and the branched curves, which have a branch point as limit point in at least one direction. It is shown that the collection of branched curves decomposes into subfamilies which are actually connected, one-dimensional complexes with branch points as vertices and branched curves as 1-cells; these subfamilies are called trees (Figures 3 and 4). The use of this term is justified by showing that not only does F contain no closed curves, but that there are also no polygons formed by branched curves, i.e., the trees contain no 1-cycles (Theorem 3.2-1).

At this point two important potential difficulties present themselves, both concerned with the distribution of the branched curves as subsets of π. First, it must be shown that the individual trees are not pathologically imbedded in π; and second, the question of how the collection of trees is distributed on π must be examined more closely, as it may be seen, for example, that the collection of trees crossing a single cross-section pq may be so large that the points of intersection with pq are dense on pq (see Figure 3). The first question is settled by establishing a numbering system for the curves of a tree (Theorem 3.4-1) which in a certain sense chafacterizes the tree

(Theorem 3.4-2); and then noting that the tree may be mapped homeomorphically onto a model tree in the xy-plane made up of straight line segments and that, by virtue of a theorem of Adkisson and Maclane, the homeomorphism may be extended to all of π. The second question is taken up in the next section.

Finally, we note that the complementary domains of a tree and their boundary curves (which will be called maximal chains) are discussed and a notation established for them. They play an important role in later sections, particularly 5.0.

Section 4.0 has as its purpose the generalization of the theorem of Kaplan which states that any family F, regular in a simply connected domain, may be given as the family of level curves of a continuous function without relative extrema. The desired generalization (Theorem 4.1-4) is achieved by making certain <u>cuts</u> in π extending from branch points to infinity along trees, i.e., by removing certain chains of branched curves and their endpoints from each tree; this being done so as to leave a simply connected open subset R* of π in which F*, the curves of F filling R*, is regular without singularities (Figure 4, cuts indicated by heavy lines). Then, applying the theorem of Kaplan mentioned above, there must exist a function f* continuous on R* and with the curves of F* as level curves. It is then shown that this function may be extended to a function f defined on all of π, which has the curves of F as level curves and takes the same value on every curve of a given tree.

The cutting operation described above is made possible by Theorem 4.1-1, which in effect settles the second possible difficulty mentioned above. This theorem states in essence that if we choose any point p of π and consider the closed concentric circular discs K_n of center p and radius n, then for any n, there are at most a finite number of trees which intersect K_n on more than one of their curves. This enables us to choose cuts so that they recede to infinity but still include every branch point.

In Section 5.0, it is shown that there exists a decomposition of F into a denumerable collection of subfamilies, $S(\alpha)$, each consisting of all curves crossing a cross-section $\Gamma(\alpha)$ which extends from a point on a curve C*, called the initial curve of the subfamily, to infinity. Each set $S(\alpha)$ is homeomorphic as a curve family to the lines y = constant of the upper half-plane, with C* mapping onto the line y = 0 and $\Gamma(\alpha)$ onto the x-axis. Two families $S(\alpha)$ and $S(\beta)$ of the decomposition can overlap only on their initial curves. In the event that our curve family F is exactly the level curves of the real part of an analytic function, then this is actually a decomposition of the Riemann surface of the inverse function into fundamental domains. Figure 4 shows such a decomposition by indicating with dotted lines the cross-sections $\Gamma(\alpha)$.

Finally, in Section 6.0, we show that every curve family F has a complementary family G (Theorem 6.1-2) where we mean by a complementary family of F a curve family of the same type, with the same singularities and to the same multiplicity, and such that each of its curves is a cross-section of F. The method of proof is to first note the existence of a complementary family G* of F* in the region R* (= π - the 'cuts'), which was demonstrated by Kaplan, and then to modify G* near the cuts in such a way that when we replace the cuts we get a family G of the desired type. The existence of continuous functions f and g with the curves of these families as level curves enables us to define a map T of π into the complex w-plane as follows: T(p) = (u,v), where u = f(p) and v = g(p). This map carries the curves of F onto the lines u = constant and is light and interior; hence, by Stoilow, it is topologically equivalent to an analytic function. Thus, there is a homeomorphism h from either (1) the domain D_∞ consisting of the z-plane, or (2) the domain D_1 consisting of $|z| < 1$, onto π such that $\varphi(z) = T[h(z)]$ is analytic. Then the level curves of $\varphi(z)$ are homeomorphic under h to the family F, i.e., F is homeomorphic to the level curve family of a harmonic function. It also follows at once that F is homeomorphic to a family defined by a system of differential equations.

1.0 GENERAL PROPERTIES OF REGULAR CURVE FAMILIES IN THE PLANE

This section contains the statement of basic definitions and theorems from W. Kaplan [IV] and [VI] which will be used in this paper. Proofs will be omitted.

1.1 Curve Families Filling a Region

An open curve will mean a homeomorphic image of an open interval, a closed curve a homeomorphic image of a circle, and a half-open curve will mean a homeomorphic image of a half-open interval. A curve will mean any one of these three. A family F of curves will be said to fill a subset R of the Euclidean plane π if every curve of F is in R and every point of R lies on one and only one curve of F. If U is a subset of R such that each curve C of F intersects U in a set $U \cap C$ each of whose components is a curve, then we denote by $F[U]$ the curve family filling U whose curves are the components of $C \cap U$ for all C in F. If the curve family F fills R and the curve family G fills S, then F and G will be called homeomorphic if there is a homeomorphism of R onto S such that the image of each curve in F is a curve in G. If p is a point of R, R filled by a curve family F, then C_p will denote the curve of F through p. i(R) will denote the interior of R.

1.2 Regularity

Henceforth, F will denote a curve family filling a subset R of π, the oriented Euclidean plane. If R_o denotes the rectangle $|x| \leq 1$, $|y| \leq 1$, of the xy-plane, and F_o' the family of lines y = constant filling R_o, then F will be said to be regular at a point p of R if there is a set U(p) to which p is interior (relative to R) and such that $F[\overline{U(p)}]$ is homeomorphic to F_o'. F is then regular in R if F is regular at every point of R. A cross-section of F (through the point r) is an arc pq (to which r is interior), such that pq lies in a subset R' of R which is open relative to R, and such that each curve of F[R'] meets pq at most once. An r-neighborhood of a point p of R will mean a set U(p) which (1) contains p, (2) is open relative to R, (3) whose closure $\overline{U(p)}$ lies in R, and is moreover such that (4) $F[\overline{U(p)}]$ is homeomorphic to the family F_o' filling the rectangle R_o (above) of the xy-plane in such a way that the inverse images of the lines $|x| = 1$ are cross-sections.

Theorem 1.2-1: If a family F fills an open region R and is regular in R, then each curve of F is either open or closed in π. [IV,1]

Theorem 1.2-2: If a family F fills any region R and is regular in R, then every point p of R has an arbitrarily small r-neighborhood U(p), and there is a cross-section pq with p as endpoint. If p is in the interior of R, then there is a cross-section through p. Moreover if st is any arc lying on a curve C of F, then there is, within any ϵ-neighborhood U_ϵ(st), an r-neighborhood containing st. [VI,1 and IV,8]

Theorem 1.2-3: Let $R = R_1 \cup R_2$ where $F[R_1]$ and $F[R_2]$ are both defined. If p is an interior point of R and $F[R_1]$ and $F[R_2]$ are both regular at p, then F is regular at p. [VI,2]

1.3 The S-Families

By a homeomorphism $y' = f(y)$ defined for $0 \leq y \leq 2$, and with $f(0) < f(1) = 1 < f(2)$, points $(0,y)$ on the line segment $x = o$, $0 \leq y \leq 2$, can be identified with points $(1,y')$ on the line $x = 1$. With this identification made, the rectangle $0 \leq x \leq 1$, $0 \leq y \leq 2$ plus the points $(1,y')$ for $0 \leq y' < f(0)$ and $f(2) < y' \leq 2$ becomes homeomorphic to a region G of π and the lines, $y =$ constant, filling the rectangle become a curve family F_1 filling G. Any curve family homeomorphic to $F_1[i(G)]$, where $i(G)$ denotes the interior of G, is called an <u>open s-family</u>. Any curve family homeomorphic to $F_1[i(G) \cap G^*]$ where G^* is the set of images of points (x,y) of the rectangle with $y \geq 1$, will be called a <u>half-open s-family</u>.

<u>Theorem 1.3-1</u>: <u>Let F be a regular curve family filling the set R of π. Let C be a closed curve of F such that F is regular at every point of C. If C is in the interior of R, then there is a set R_o such that $F[R_o]$ has C as an element and is an open s-family. If C is in $R-i(R)$, then there is a set R_o such that $F[R_o]$ has C as an element and is a half-open s-family.</u> [VI, iii]

1.4 The Sets L(C+) and L(C-)

If C is any open curve in F and it has been given a direction, then by a <u>positive (negative) limit point</u> of C will be meant any point q which is the limit of a sequence $p_n = f(t_n)$, where C is the image of $0 < t < 1$ under f and $t_n \rightarrow 1$ ($t_n \rightarrow 0$). The set of all positive (negative) limit points of the directed curve C will be denoted by $L(C+)$ (by $L(C-)$). $L(C)$ is defined by $L(C) = L(C+) \cup L(C-)$. Clearly, $L(C) \cap C$ is empty since C is homeomorphic to $0 < t < 1$.

<u>Theorem 1.4-1</u>: <u>If C is an element of a regular curve family F, and $L(C+)$ contains a closed curve D of F, then $L(C+) = D$.</u>

Theorem 1.4-2: If C is an element of a regular curve family F and if L(C+) = D, a closed curve of F, then to every point p of C corresponds an ϵ-neighborhood, $U_\epsilon(p)$, such that every curve of F crossing $U_\epsilon(p)$ has D as its limiting curve in (at least) one direction. [IV,10]

Theorem 1.4-3: If F is a regular curve family filling R and p is in R and, moreover, in L(C+) for some curve C of F, then every point of the curve D_p of F through p is in L(C+). [IV,7]

Theorem 1.4-4: If C is a directed open curve of F which is bounded in the positive direction, but has no boundary point of R as positive limit point, then L(C+) contains (and hence is equal to) a closed curve of F. [IV,11]

The above theorems still hold if we replace L(C+) by L(C-).

1.5 Bays

Let the curve C of F meet the cross-section pq at points t and u interior to pq. Denote by $(tu)_1$ and $(tu)_2$ respectively the arcs on pq and C determined by t and u and moreover assume t and u taken so that these arcs intersect only at t and u, hence forming a simple closed curve K. If K contains neither p nor q in its interior, it is called a bay.

Theorem 1.5-1: If C is a closed curve of a regular curve family F, and D is a curve of F such that L(D+) = C = L(D-), then an arc of D forms part of a bay in F. The bay is interior to C if, and only if, D is. [IV,9c]

Theorem 1.5-2: Interior to a bay of a regular curve family F filling R there is a boundary point of R. [IV,12]

Theorem 1.5-3: Let C be a closed curve of a regular curve family F filling R. Then interior to C there is a boundary point of R. [IV,13]

1.6 Orientable Regular Families

A regular curve family F filling the open region R is said to be orientable if it is possible to assign a direction to each curve of F in such a fashion that for each point p of R there is an r-neighborhood in which the arcs are all similarly directed. Whitney [XV] has proved the following theorem:

Theorem 1.6-1: If F is orientable and fills R, there is a function f(p,t) defined for each p in R and t in $-\infty < t < +\infty$, and simultaneously continuous in both variables, which assigns to (p,t) the unique point q = f(p,t) in R lying on the curve C through p. f(p,0) = p and f(p,t) moves continuously in the positive (negative) direction on C as t increases (decreases). If C is an open curve, then for p fixed and on C, f(p,t) is a homeomorphism of $-\infty < t < \infty$ onto C.

2.0 CURVE FAMILIES WITH ISOLATED SINGULARITIES

2.1 Isolated Singularities

By an isolated singularity of a regular curve family filling a region R will be meant any isolated boundary point b of R, i.e., there is a neighborhood of b which contains only the point b of π - R. From this point on we will only deal with families F which are regular in the entire plane except for isolated singularities. In [VI], W. Kaplan has completely classified the structure of such a family in any neighborhood of an isolated singular point (containing no other singular point).

Theorem 2.1-1: If the curve family F fills the region R consisting of the entire plane π except for isolated singular points, and F is regular in

R, then the set of singularities is closed in π.

Proof: Let $S(R) = \pi - R$ denote the set of singular points. Suppose that p is a limit point of S(R), then p is not in S(R) since the points of this set are isolated from each other. We shall also prove p cannot be a regular point whence it follows that S(R) has no limit points and is therefore closed. Now, corresponding to any regular point p, there is a set U(p), to which p is interior relative to R, and a homeomorphism f of $\bar{U}$ onto R_o carrying $F[\bar{U}]$ onto F'_o (Section 1.2). It follows that either f(p) is interior to R_o, or that if f(p) is on an edge of R_o, then the inverse image of the entire edge is on the boundary of R, since p is an interior point of U relative to R. The first is impossible if p is to be a limit point of S(R), and the second is impossible if S(R) is to be an isolated set of points.

Theorem 2.1-2: The singularities of the family F above are denumerable.

Proof: Any non-denumerable subset of π must have a point of accumulation which can certainly not be an isolated point.

2.2 Index

Following the definition given in Kerekjarto, [IX], p. 251 ff., we define the index of an isolated singularity on a surface as follows: Let K be any simple closed curve containing the isolated singularity b but no other singularities in its interior and let $U_1, \ldots, U_n$ be a covering of K by r-neighborhoods. Then, it is clear that we may replace K by a simple closed curve K' in $\bigcup_{i=1}^{n} U_i$ and such that K' is a polygon composed of sides which are alternately (1) arcs of curves of F and (2) cross-sections of F. Every vertex of the polygon K' is the intersection of a cross-section and a curve of F; we call it an internal vertex if the curve of F which forms the side at that vertex enters the interior of K' at the vertex and in the other case we call it an external vertex. If we denote the number of internal vertices by e and external vertices by a then the index is $\rho(b) = 1 - \frac{a - e}{4}$ (see Figures 1a and 1b). The following theorem due to Hamburger is proved in Kerekjarto, loc. cit.

Theorem 2.2-1: If $\mathcal{F}$ is a closed two-dimensional manifold (with boundary) of genus p and r boundary curves, and F is a curve family which is regular on $\mathcal{F}$ except for isolated singularities b_i, i = 1, . . ., n, then

$$\sum_{i=1}^{n} \rho(b_i) = 2 - (2p + r) \qquad \text{if } \mathcal{F} \text{ is orientable.}$$

$$\sum_{i=1}^{n} \rho(b_i) = 2 - (p + r) \qquad \text{if } \mathcal{F} \text{ is non-orientable.}$$

We also quote the following from the same source:

Theorem 2.2-2: If b is an isolated singularity of a regular curve family F, and the number of sets L(C+) and L(C-) which equal b is k, then the index, $\rho(b) = 1 - k/2$. (See Figure 1a.)

3.0 CURVE FAMILIES WHOSE SINGULARITIES ARE BRANCH POINTS

3.1 Branch Points

If b is any boundary point of R, $R \in \pi$, and F is a regular curve family filling R, and if b is such that there is a neighborhood U(b) for which $F[U(b) - b]$ is homeomorphic to the level curve family of the real part of $f(z) = z^n$, $n > 1$, under the homeomorphism g carrying U(b) onto $|z| < 1$ with b going onto z = 0; then we say that b is a branch point of F, that n is the multiplicity of b, and that the neighborhood U(b) together with the homeomorphism g is an admissible neighborhood (U,g) of b (see Figure 1a for a branch point of multiplicity 4). In the case of a branch point b of multiplicity n, then there are precisely 2n curves in F[U] which may be directed so that L(C+) = b. It follows that the multiplicity is independent of the choice of the neighborhood U. A branch point is clearly an isolated singularity of F; hence, if F fills the entire plane except for branch points, Theorems 2.1-1 and 2.1-2 will apply to F and R. Henceforth, this will be the only type of curve family considered; thus F will always mean a curve family regular in π - B where B is a set of branch points; and hence B is closed, discrete and denumerable, and R open. Such a family will be called a branched regular curve family filling $\pi = R \cup B$.

Theorem 3.1-1: The level curves of a function f(x,y) harmonic in a simply-connected domain are a branched regular curve family filling the plane.

This theorem is well known and the proof will not be given. A detailed proof may be found in Morse [XI], pp. 6-7. Throughout most of this paper we will use the Euclidean plane π as a homeomorphic model for an open simply-connected domain. It should be noted, however, that the converse of the above theorem, proved in Section 6.0, states that given a branched regular curve family F filling an open simply-connected domain, then there exists a function f(x,y) harmonic on the finite plane, or such a function harmonic on the unit circle, whose level curves are homeomorphic to F.

3.2 Chains and Polygons of Branched Curves

As remarked above, from this point on, only branched regular curve familes F filling the oriented plane, π, will be considered. The collection of branch points will be denoted by B and the region π - B, in which F is regular, by R. We may assume the orientation in π given by a definite fixed homeomorphism h of the xy-plane (or z-plane) onto π, and we will use only admissible and r-neighborhoods whose associated homeomorphism to the xy-plane is such that if we return to the neighborhood in π via h then the resulting homeomorphism of the neighborhood onto itself is orientation preserving.

We shall also assume that all the curves of F are directed, so that there shall be no ambiguity in the use of the symbols L(C+) and L(C-), although we shall at times find it convenient to redirect curves of F. If L(C) = 0, we call C a regular curve, and if L(C+) = $b \varepsilon B$, i.e., a branch point, we shall say that C is a branched curve, branched at the positive end at b; we also call b the positive endpoint of C in this case. Similarly if L(C-) = $b' \varepsilon B$. We call C doubly-branched if both L(C+) and L(C-) have endpoints, and half-branched if only one has. It will subsequently be shown that these are the only possibilities, i.e., L(C+) = 0 or = b, a single branch point, (and similarly L(C-)), so we shall not give any name to the as yet possible type of curve which might have more than one point in L(C+), (or L(C-)).

If $b \varepsilon B$ the curves C which have b as endpoint together with their endpoints are called the star of b, St(b); and without their endpoints, except b, the open star of b, open St(b). If b is of multiplicity n, then there are at most 2n curves in St(b); it will be shown later that there are exactly 2n, i.e., that the two endpoints of a curve of F cannot coincide. It is useful to note that by virtue of this remark and the fact that B is denumerable there are at most a denumerable number of branched curves in F.

If $C_1, \ldots C_n$ are $n \geq 2$ distinct branched curves of F with their endpoints, which may be so directed that $L(C_i+) = b_i = L(C_{i+1}-)$, $b_i \varepsilon B$ and b_i

distinct for $i = 1, \ldots, n-1$, and if in addition neither $L(C_1-)$ nor $L(C_n+)$ is any of the b_i's, then we call $C_1, \ldots C_n$ a simple polygon of branched curves or a chain of branched curves according to whether or not there is a $b_o \varepsilon B$ such that $L(C_1-) = b_o = L(C_n+)$. A single curve will be called a chain if its endpoints do not coincide and will be shown below to always be a chain, i.e., as already remarked it will be shown that a curve cannot have two coincident endpoints. In brief, the curves $C_1, \ldots C_n$, together with their endpoints, for $n \geq 1$ will form a chain, if the set $\bigcup_{i=1}^{n} C_i$ is homeomorphic to a closed line segment and a simple polygon if $\bigcup_{i=1}^{n} C_i$ is homeomorphic to a simple closed curve.

We shall call curves C, C' clockwise adjacent if they may be directed so that $L(C+) = b = L(C'-)$, $b \varepsilon B$ of multiplicity n, and in the map of some admissible neighborhood on $|z| < 1$ they map onto the radii $\theta = (1/n)\pi$ and $\theta' = 0$, respectively, of the level curves of the real part of z^n. Because of our restrictions and conventions on orientation above this definition clearly is independent of the neighborhood chosen, depending only on the orientation of π. C', C are a counterclockwise adjacent pair if C, C' are a clockwise adjacent pair, and in either case we shall call them adjacent. A chain $C_1, \ldots C_n$ is called an adjacent chain if C_i, C_{i+1} are clockwise adjacent for each i or if, for each i, they are counterclockwise adjacent. We shall also consider infinite chains of branched curves: $\{\ldots, C_{-i}, \ldots C_{-1}, C_o, C_1, \ldots C_j, \ldots\}$. If this collection is such that for every $k < m$ the curves $C_k, \ldots C_m$ form a chain, we shall call the collection an infinite chain, and every set $C_k, \ldots C_m$ a subchain. If the collection has no first or last element we shall often call it doubly infinite and in the opposite case half infinite. An infinite chain will be called adjacent if every subchain is adjacent. (Figures 3 and 4 illustrate many of the terms defined above.)

Theorem 3.2-1: A branched regular curve family F filling π can contain neither a closed curve nor a simple polygon of branched curves. (See Figure 5.)

Proof: We suppose that F does contain a closed curve or simple polygon K. We define on the xy-plane a family F' filling the surface $\mathcal{J} = \{(x,y) \mid x^2 + y^2 \leq 4\}$ as follows: (1) we fill the annular domain $A_1 = \{(x,y) \mid 1 \leq x^2 + y^2 \leq 4\}$ with concentric circles and (2) we map K with its interior onto $A_2 = \{(x,y) \mid x^2 + y^2 \leq 1\}$, so that the image K' of the curve K is the circle $x^2 + y^2 = 1$. $F'[A_1]$ is regular and $F'[A_2]$ is regular except for a possible finite number of isolated singular points $b_1', \ldots, b_n'$ lying interior to or on K'. Any such singular point must be the image of a branch point of F lying inside K, or on K (if K is a polygon). If b' lies on K' and is a singular point of F', then in some neighborhood of b' there must be at least three curves C_1, C_2, C_3 which can be directed so that $L(C_i+) = b'$, i.e., at least one from the interior of K'. Hence the index $\rho(b') \leq -1/2$ by Theorem 2.2-2. If b' is interior to K' and it is the image of a branch point of multiplicity n, then the index $\rho(b') = 1 - n \leq -1$, again by Theorem 2.2-2. Now by Theorem 1.5-3 the family F' must contain at least one singularity, hence the sum of the indices of F' filling $\mathcal{J}$ is $\leq -1/2$, i.e., is negative. This, however, contradicts Theorem 2.2-1, which says that the sum of the indices must be 1 = 2 - (2p+r) = 2 - (2.0 + 1) since the genus of $\mathcal{J}$ is 0 and it has 1 boundary curve. Thus it is impossible for F to contain a closed curve or a simple polygon.

Theorem 3.2-2: A branched regular curve family F filling π can contain no bays. (See Figure 6.)

<u>Proof</u>: Suppose F contains a bay formed by the arc $(tu)_1$ on the cross-section pq and the arc $(tu)_2$ on the curve C of F. We will let K denote the simple closed curve $(tu)_1 \cup (tu)_2$ and $\mathcal{D}^*(K)$ its interior. Then p and q lie in $\mathcal{D}^\#(K)$, the complementary domain to $\mathcal{D}^*(K)$, and we assert that $F_1 = F\left[K \cup \mathcal{D}^*(K)\right]$ is a regular curve family except for possible isolated singularities in $\mathcal{D}^*(K)$. It is clear that F_1 is regular in $\mathcal{D}^*(K)$ except at branch points, since $\mathcal{D}^*(K)$ is an open set of π. And at every point of $K - (tu)_1$, F_1 is regular, since, if s is any such point, then a regular neighborhood U(s) in $\pi - (tu)_1$ will furnish a regular neighborhood, $U \cap \left[K \cup \mathcal{D}^*(K)\right]$, of s in F_1. This is true since the image of this intersection under the homeomorphism of U(s) onto R_o in the xy-plane will be the image y = 0 of C together with all of the rectangle to one side of this line. Similarly, if s is any point of $K - (tu)_2$, we may choose an r-neighborhood of s in $\pi - (tu)_2$ such that the cross-section $(tu)_1$ in the neighborhood maps on a line x = 0 of R_o ([IV], Lemma p. 158). The image of the intersection of this neighborhood with $K \cup \mathcal{D}^*(K)$ is the line x = 0 plus all of the rectangle to one side of this line, which will clearly be an r-neighborhood. Finally, if we take an r-neighborhood of t or u such that $(tu)_1$ maps on x = 0 and t on the origin, so that C is the line y = 0, then the image of that part of this neighborhood in $K \cup \mathcal{D}^*(K)$ will be the part of R_o in one quadrant plus the part of the lines x = 0, y = 0 bounding it; again this is an r-neighborhood.

Now we map $K \cup \mathcal{D}^*(K)$ homeomorphically onto the right half R_1 of the circular disk $R = \left\{(x,y) \mid x^2 + y^2 \leq 1\right\}$ in such a way that $(tu)_1$ maps onto the diameter x = 0. This maps F_1 on a family F_1' regular in R_1 except for possible singularities in the interior. Reflecting R_1 in the y-axis onto R_2, the right semi-circle, will give us a family F_2', image of F_1', regular in R_2 (and on its boundary) except for possible branch points in its interior. Hence we have defined a family F, by $F\left[R_1\right] = F_1'$ and $F\left[R_2\right] = F_2'$, regular in R except for possible branch points, by Theorem 1.2-3. The family F contains at least two branch points, since by Theorem 1.5-2, the bay must contain a singularity. The index at every such singular point is at most -1, but the sum of the indices must, as in the previous theorem, be +1. Hence our assumption that F contained a bay is contradictory.

Theorem 3.2-3: If L(C+) is bounded it consists of a single branch point. (See Figure 7a.)

Proof: First, we note that by virtue of Theorem 1.4-4, together with Theorem 3.2-1, every curve of F which is bounded in the positive direction must contain at least one branch point in L(C+). Second, note that L(C+) is bounded and closed, hence it can contain at most a finite number of branch points. Finally, if L(C+) contains more than a single branch point, i.e., if $L(C+) \neq b\epsilon B$, then for each branch point p it contains, it must contain also at least two adjacent curves of St(p). This is clear if we examine the image of the admissible neighborhood of p, i.e., the level curve family of $\mathcal{R}(z^n)$. The image of C cannot coincide with any of the 2n radial curves $\Theta = (k/n)\pi$, k = 0, 1, . . ., 2n, since p is not its endpoint; hence it must clearly intersect the neighborhood an infinite number of times in at least one of the sectors between these radii, and therefore have positive limit points on the two radii bounding that sector; whence L(C+) contains the curves on which lie the inverse images of the two radii, i.e., two adjacent curves of St(p), by Theorem 1.4-3.

Thus, if we assume that L(C+) contains more than a single branch point, then it must contain a certain finite collection of curves branched at these points, and hence a collection of chains. In this collection we will consider chains $C_1^i, \dots C_{n_i}^i$ which are maximal in the sense that a chain is maximal if there is no longer chain in the collection containing it as a sub-chain. For such a chain, the initial and final curves must each be half-branched, for if either, say for example C_1^i, were branched at each end, then (since there are no polygons in F), at the end not already linked to the chain, e.g., at the negative end, there would be a branch point, b = $L(C_i'-)$, and in St(b) would be a curve C' in L(C+) adjacent to the end curve C_1' of the chain and yet which was not already in the chain. C' could thus be added to the chain to form a new and longer chain, C', $C_1^i, \dots C_{n_i}^i$. This is contrary to the definition of maximal chain. For the i-th maximal chain we shall let L_i denote the limit set of the unbranched end of an arbitrarily chosen but fixed end curve of the chain. Now number a subcollection of the maximal chains as follows: Choose any one of the chains as the first, then take any maximal chain of L_1 as the second and in general choose as the k-th chain any maximal chain of L_{k-1}. Clearly, L_i must contain a maximal chain since it must just as L(C+) contain a branch point together with other points of L(C+) and hence two adjacent curves, which can be extended to a maximal chain. Moreover, at any stage L_i may not contain the i-th chain itself for then one of the end curves of the chain would be contained in its own limit set. Moreover, it cannot contain any preceding chain for we have the sequence $L(C+) \supset L_1 \supset L_2 \supset \dots$, and again we get a curve contained in its own closure. By this process we soon exhaust all of the n branch points of L(C+), although on the assumption that L(C+) could contain more than a single branch point which we made initially, the process set up above cannot terminate. Hence it is seen that our assumption cannot be true and that the theorem is correct.

Theorem 3.2-4: An arc pq of $R = \pi - B$ is a cross-section if and only if it meets each (finite) chain (including one-element chains, i.e., curves) of F at most once. (See Figure 7b.)

Proof: If pq meets each curve only once, it is by definition a cross-section.

Let K be either a single curve or a chain of curves of F and let pq be a cross-section which is assumed to meet K more than once. We may find points t, u on pq such that the arcs $(tu)_1$ and $(tu)_2$ on K intersect only at t and u, since the two curves can intersect only a finite number of times on any closed arc on K, as a consequence of the definition of a cross-section plus the fact that for any curve C, $L(C) \cap C = 0$. We denote by K' the simple closed curve $(tu)_1 \cup (tu)_2$. By Theorem 3.2-3 together with the fact that the number of branched curves with endpoints in K' is finite, we can find a curve C passing through a point r interior to K' and which leaves K' in both directions. Let m,n be the points on $(tu)_1$ and on C on opposite sides of r at which C first leaves K'. Then $(mn)_1$ on $(tu)_1$ and $(mn)_2$ on C form a simple closed curve interior to K' and intersecting the boundary of K' along $(mn)_1$ but at no other points. It follows that t and u are exterior to this simple closed curve which is therefore a bay, formed by the cross-section $(tu)_1$ and the curve C. This contradicts Theorem 3.2-2. Thus, it is necessary that a cross-section have only one point on each curve of F or chain of F.

Theorem 3.2-5: L(C+) is either empty or contains a single branch point.

Proof: We have already proved this theorem in the event that L(C+) is bounded, and we have also shown that if L(C+) contains more than a single branch point, then it must contain at least two curves adjacent at that branch point. From this we can conclude that if the theorem is untrue, then $L(C+)$ must contain a regular point p. Consider the image in R_o of an r-neighborhood of p. Let (x_n, y_n) be a sequence of points approaching (0,0), the image of p. These clearly lie on an infinite number of different lines $y = y_n$, each of which is an image of an arc of C. Hence C crosses any cross-section through p an infinite number of times, which contradicts Theorem 3.2-4.

3.3 Trees

In this section we define an equivalence relation which decomposes the oriented plane, π, into a collection of disjoint closed sets, each of which is a sum of curves of F and points of B, and each of which is a topological tree of a certain type which we define below:

Definition: Let the closed set T of the oriented plane be decomposable into the sum of an at most denumerable collection of subsets C_i, each closed in π, and satisfying the four following conditions:

(1) Each set C_i is the homeomorphic image of either a closed, half-open, or open line segment (whence we will refer to it as a curve).

(2) Each set C_i has at most an endpoint in common with any C_j, $i \neq j$; and if we denote by St(b) the collection of all curves with b as endpoint, then St(b) consists of a finite even number of curves ≥ 4.

(3) There is a unique finite chain $c(C_i, C_j) = (C_i, C_{i_1}, \ldots C_{i_k}, C_j)$ from C_i to C_j for every i,j; i.e., each curve of the chain having an endpoint in common with the preceding curve as in the definitions of 3.2.

(4) The sets open St(b), consisting of the curves of St(b) without their endpoints opposite b, and open C_i, consisting of C_i without its endpoints,

are both open sets in T (as a subspace of π).

Then we say that T is a tree. (See Figure 8.) Our use of this term is much less general than is usual, but since we consider only this specialized type of tree throughout, there should be no confusion in the use of the term.

The decomposition of any tree T into sets C_i is unique quite clearly, except for the numbering, and therefore we may speak without ambiguity of the curves of T and the endpoints of curves (or, i.e., branch points) of T. Note that a tree is connected and, in fact, arcwise connected by (3) and that by the uniqueness of the chains of (3) there can be no closed curve in T. Condition (4) plus the fact that T is closed in π is equivalent to the following statement: If (p_n) is any sequence of points of T and $p_n \to p \epsilon \pi$, then $p \epsilon T$ and all the points of p_n after some N will lie either on a single curve C_i of T or on St(p) depending on whether p is not or is an endpoint of some curve of T. In the language of combinatorial topology each tree, as described above, is a locally-finite, connected, one-dimensional complex containing no one-cycles. In order to exhibit this, it would be necessary to introduce arbitrarily an infinite number of vertices tending to infinity on each curve of the tree homeomorphic to a half-open line segment. Once this is done, the statement is clearly true.

It is clear that any regular curve C of a curve family F is a tree with the decomposition being $C_1 = C$. Now, among the elements of our family F we define the relation joins as follows: C is said to join C' if and only if there is a finite chain c(C,C') of curves of F from C to C'. If we add to this definition that every curve joins itself, then this is easily shown to be an equivalence relation on the curves of F. We denote by T_C the equivalence class of C, including with each curve its endpoint, i.e., T_C is the set of all curves of F which join C together with their endpoints. These equivalence classes are disjoint sets and will be shown below to be trees in the sense of our definition.

Theorem 3.3-1: An arc pq on π is a cross-section of F if and only if it lies entirely in $R = \pi - B$ and has at most one point of intersection with each set T_C.

Proof: If pq has only one point in common with each set T_C, since T_C is itself a sum of curves of F with their endpoints, then it will have at most one point in common with each curve of F and hence be a cross-section by definition.

On the other hand, by Theorem 3.2-4, it is necessary that pq meet any set T_C at most once if it is a cross-section, since if pq met T_C at points r,s, then either C_r, C_s are the same curve or else there is a chain $c(C_r, C_s)$ either of which is impossible by that theorem.

Theorem 3.3-2: Each set T_C of a branched regular curve family F is a tree in the sense of our definition.

Proof: In the event that C is a regular curve the theorem is trivial since $T_C = C$, as already noted. Now let T_C contain a singular curve, then it follows that it contains only such and at most a countable number, since there are at most a countable number of singular curves in F. Each curve of F, together with its endpoints, will constitute a curve C_i of the decomposition of T_C. Each such set is closed in π, since we include endpoints, and is homeomorphic to either a closed or half-open segment, the latter if the curve extends to infinity in one direction. Thus (1) is satisfied. Condition (2) is, however, also satisfied since each set C_i has at most an endpoint in common with any set C_j, $i \neq j$, and, if b is any endpoint, then St(b) contains at least four curves and always an even number, 2n = twice the multiplicity of b as a branch point. Likewise (3) is satisfied; i.e., the existence of a chain $c(C,C')$ from $C \in T_C$ to $C' \in T_C$ is part of the definition of T_C, and the uniqueness is due to the fact that there can be no polygons of branched curves of F by Theorem 3.2-1. Finally, we prove simultaneously that condition (4) is satisfied and that T_C is closed as a subset of π. Let p_n be any sequence of points of T_C with a point p of π as limit point. Now if p is a regular point of F, then we take an r-neighborhood U(p) and note that unless every p_n lies on the same curve C_{p_k}, which is necessarily C_p itself, we have a cross-section through p which must cross T_C more than once, contrary to Theorem 3.3-1; and, if p is a branch point, then taking an admissible neighborhood of p, we observe that unless we assume all the points p_n, $n>N$, to lie on St(p) we arrive at the same contradictory conclusion by considering a cross-section from p into one of the sectors of the admissible neighborhood. Thus we conclude that the theorem must be true.

We return to a discussion of a tree T which conforms to our definition, but is not necessarily a tree consisting of curves of a branched regular curve family. As previously noted, the decomposition of T into curves is unique, and hence we may refer without ambiguity to <u>the</u> curves and <u>the</u> branch points (or endpoints) of T. Since T is assumed to be imbedded in an oriented plane, a cyclic order is induced on the curves of St(b); hence our definitions of adjacent curves and adjacent chains and so on apply at once to the curves of T. These concepts will be used below.

It is convenient at this point to give some attention to a theorem due to Adkisson and Maclane [I] which states that if $\bar{T},\bar{T}'$ are two homeomorphic Peano continua lying on spheres S,S' respectively, then a homeomorphism from

$\bar{T}$ to $\bar{T}'$ can be extended to a homeomorphism of S to S' if and only if it preserves the relative sense of every pair of triods of $\bar{T}$. By a triod, $t = [\alpha, \beta, \gamma]$, of $\bar{T}$ is meant any set of three arcs α, β, γ in $\bar{T}$ which have only a single point, called the vertex, in common. A homeomorphism is said to preserve the relative sense of triods of $\bar{T}$ if every two triods t_1, t_2 which have the same sense (i.e., both clockwise or both counterclockwise) on S are carried into two triods t_1', t_2' of $\bar{T}'$ which have the same sense on S'. Let us denote by $\bar{\pi}$ the plane π plus the point ∞ and by $\bar{T}$ the tree T plus the point ∞. Assuming for the moment that the set $\bar{T}$ is a Peano continua, the theorem above is applicable to our situation, and it is a direct consequence of this theorem that if T,T' are two homeomorphic trees on π and the xy-plane respectively, then any homeomorphism between them may be extended to a homeomorphism of the planes if and only if the relative sense of the curves of $St(b_1), St(b_2)$ is preserved for every pair of branch points b_1, b_2 of T. In order to show that this is a consequence of the theorem, it must be shown that the relative sense of every pair of triods of $\bar{T}$ is preserved if this is true for every triod of T. This follows from Theorem 6 of the same paper which states that two non-intersecting triods $t_1 = [\alpha_1, \beta_1, \gamma_1]$, $t_2 = [\alpha_2, \beta_2, \gamma_2]$ have opposite sense on a sphere S if and only if there exists on S a Θ-graph whose vertices are the vertices of t_1 and t_2 and whose three (non-intersecting) arcs contain respectively the legs α_1 and α_2, β_1 and β_2, γ_1 and γ_2. Now it is clear from condition (3) in the definition of a tree (the arcwise connectedness) that given any triod with vertex at ∞, it is possible to find at least one triod with vertex at a branch point of T which does not intersect it but is, with it, part of a Θ-graph. Finally, note that in a tree $\bar{T}$ the branch points and ∞ are the only possible vertices of triods. The conclusion is immediate that we may restate the theorem of

Adkisson and Maclane, as we have above, for our own purpose here. It remains to prove that $\overline{T}$ is a Peano space. This will be done in 3.4 and also in that section a numbering system for the curves of T will be established by the use of which it becomes apparent that it is possible to map the plane π onto the xy-plane by a homeomorphism which carries T onto a tree T' consisting entirely of closed and half-open straight line segments (each curve with two endpoints becoming a single line segment, each curve with one endpoint a line segment plus a ray extending to ∞). This makes it clear that a tree as defined above actually coincides with our intuitive notion, and that no matter how badly 'twisted' it may be it can actually be straightened out, by a homeomorphism of the entire plane, into a rectilinear tree.* Although this result is not completely proved until Section 3.4, it will be established there independently of the remainder of this section, and it will be convenient to assume it at this point to be used in the theorems of this section. (See Figure 8.)

We now consider relations between a tree T and its complementary domains. In this connection it is convenient to consider a special class of adjacent chains (of curves of T) which we shall call maximal chains. An adjacent chain of curves of a tree is said to be <u>maximal</u> if it is not a subchain of any adjacent chain. It is an immediate consequence of our definitions that a chain of adjacent curves is maximal if and only if (1) it is doubly infinite, or (2) it is half infinite and its initial (or terminal) curve has only one endpoint, or (3) it is a finite chain and both its initial and terminal curves have each only one endpoint (i.e., a curve of a tree with only one endpoint extends to infinity in the direction opposite to that with the endpoint).

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* In fact, this may be done so that any particular given chain goes onto the x-axis.

Moreover, since a tree is a closed subset of π, so also is every maximal chain a closed subset and is in fact an open curve extending to infinity in each direction, thus dividing the plane into two Jordan domains.

Theorem 3.3-3: If T is itself a single curve, then it is its only maximal chain. When T contains more than one curve, then (1) each curve of T is contained in exactly two maximal chains which intersect only on this curve and (2) every branch point is contained in exactly 2n maximal chains whose only common point is the branch point itself. Conversely, the intersection of any two maximal chains can be empty, be a single branch point, or, at most a curve of the tree.

Proof: Let $C_1, \ldots C_k$ be any clockwise adjacent chain of two or more curves. Now if C_1 has only one endpoint, then there is no curve C' adjacent to C_1 such that $C', C_1, \ldots C_k$ is a clockwise adjacent chain; but, if C_1 has two endpoints, then there is exactly one curve C' such that $C', C_1, \ldots C_k$ is a clockwise adjacent chain. Similar remarks apply to C_k. If neither C_1 nor C_k has more than a single endpoint, then the chain is maximal; in any other case we may extend the chain, one curve at a time added to the initial or final curve, until we arrive at endcurves which have only one endpoint, or, if we do not come to a curve with one endpoint, indefinitely. In any of these cases, the resulting chain is maximal since it is an open curve extending to infinity in both directions. Thus every such finite adjacent chain which is not already maximal can be extended to a unique maximal chain.

If we begin with a single curve, C with at least one endpoint b, then there is one curve clockwise adjacent to C in St(b) and one counterclockwise adjacent. Thus in St(b) we have C,C' and C,C", unique adjacent chains containing

C, one clockwise and one counterclockwise. Hence, C is contained in just exactly two maximal chains, one of which contains C,C', the other C,C". Similarly, if b is a branch point, there are just 2n pairs of adjacent curves in St(b), whence b is contained in 2n maximal chains.

Now consider the converse. If two chains intersect, they surely must have a branch point b in common. If this is their only point of intersection in open St(b), then they can intersect at no other point, since the tree is arcwise connected and can contain no closed polygon. If they intersect along two curves of St(b), they must be adjacent curves since the chains are adjacent chains; hence by the preceding remarks on uniqueness they must coincide. This leaves only the possibility that they intersect along a single curve of St(b), and is this case again, since there are no closed curves in the tree, they either have no other intersection or they coincide.

Theorem 3.3-4: Every maximal chain of a tree T divides the plane into two domains, whose complete boundary it is; and one of these domains contains no points of T.

Proof: The first part of this theorem is just the Jordan curve theorem. The second part is clear intuitively, but not too easily stated. Using the Theorem of Adkisson and Maclane, we first map π onto the xy-plane so that the maximal chain becomes the x-axis and every curve of T a chain of line segments and moreover, so that the orientation is preserved, i.e., every clockwise adjacent pair of π will still be clockwise adjacent on the xy-plane, and conversely. Now the contention is that all of the image of T, except what is on the x-axis, will lie in one half-plane, say the upper half-plane. If this is not the case, then there will be a point (u,v) of the upper half-plane and a point (x,y) of the lower half-plane, each in the image T' of T. Then, from C'(u,v), the image-curve containing (u,v), there is a chain to any image-curve on the x-axis, i.e., on the given maximal chain. Let C be the last line segment on the last curve (of some such chain) to lie in the upper half-plane (except for one endpoint); i.e., the endpoint p of C lies on the x-axis, but the rest of the curve lies in the upper half-plane. Similarly, we may choose a line segment C' of T' which lies in the lower half-plane except for one endpoint q. Clearly, p and q are branch points. Now let C_1, C_2, C_3, C_4 be curves of the maximal chain, i.e., line segments on the x-axis, numbered from left to right such that p is the common endpoint of the first pair, q of the second. Then necessarily, C_1,C_2 and C_3,C_4 are each adjacent in the same sense, say clockwise. Then it is clear that if $[C_1,C,C_2]$, a triod with vertex p, are in counterclockwise order, then $[C_3,C',C_4]$, a triod with vertex q, will be in clockwise order and conversely, since we may easily form a Θ-graph whose arcs contain the legs of these triods, and apply Theorem 6 [I] (referred to above), which would be impossible if C_1,C_2 and C_3,C_4 are each counterclockwise adjacent and equally impossible if they were both clockwise adjacent. Thus all of T' must lie in the closed upper half-plane, or conversely; whence, the theorem is immediate.

Now let C be a <u>directed</u> curve of T, a tree consisting of more than one curve. Then we have seen that C determines exactly two maximal curves which we shall denote by C* and C# with the following convention. As we move along C* in the direction corresponding to the positive direction on C, then the complementary domain of C* "to the right" (this can clearly be defined in a topologically invariant manner, by a method similar to that above) will contain no points of T, and as we move along C# in the direction corresponding to the positive direction on C, the complementary domain "to the left" will contain no points of T. These domains will be denoted $\mathcal{D}^*(C^*)$ and $\mathcal{D}^\#(C^\#)$ respectively and also by $\mathcal{D}^*(C)$ and $\mathcal{D}^\#(C)$ respectively. Now as proved above, C^* is the common boundary of two Jordan domains, and the notation for one of them was given above as $\mathcal{D}^*(C^*)$, the other will be denoted by $\mathcal{D}^\#(C^*)$. Similarly, $C^\#$ divides the plane into the domains $\mathcal{D}^\#(C^\#)$ and $\mathcal{D}^*(C^\#)$. When T is just a single curve then C, C^* and $C^\#$ are all the same curve, and $\mathcal{D}^*(C^\#) = 0 = \mathcal{D}^\#(C^*)$. If we reverse the direction on C, we must replace # by * throughout. (See Figure 9.)

If we remove open C from $C^* \cup C^\#$ we get either two or four half-open arcs extending to infinity from the endpoint(s) of C; two if C has one endpoint, four if it has two. We let $\sigma^*(C+)$ denote the arc from the positive endpoint of C lying on C^*, and $\sigma^\#(C+)$ the arc from the positive endpoint of C lying on $C^\#$. Similarly, we use the notation $\sigma^*(C-)$ and $\sigma^\#(C-)$ for the arcs at the other endpoint. We also let $\sigma(C+)$ stand for $\sigma^*(C+)$ plus $\sigma^\#(C+)$, and $\sigma(C-)$ for $\sigma^\#(C-)$ plus $\sigma^*(C-)$, and finally, $\sigma(C)$ for $\sigma(C+) \cup \sigma(C-)$.

The collection of all curves C^* and $C^\#$ are then just the maximal chains of T. As already noted above each of these maximal chains bounds two domains, one of which contains no points of T. A converse to this also holds, i.e., denoting by $\bar{\pi}$ the extended plane and $\bar{T}$ the points of T plus the point at infinity, we have:

Theorem 3.3-5: If T is a tree of π, then $\bar{\pi} - \bar{T}$ consists of an at most countable collection of Jordan domains, each bounded by a simple closed curve in $\bar{T}$ containing the point at infinity. The necessary and sufficient condition that a curve of $\bar{T}$ bound one of these domains is that it be a maximal chain of curves of T.

Corollary 1: If T is a tree of a regular curve family F, then each complementary domain is a sum of sets T_C of F.

Corollary 2: The complementary domains, if infinite in number, tend uniformly to infinity with any sequence p_n of their boundary points.

Proof: By the theorem quoted from [I] at the beginning of this section, π may be mapped on the xy-plane so that the image of T is rectilinear and even so that a given arc (or chain) of T goes onto the x-axis. It is clear then that the complementary domains are Jordan domains. The number of open sets on the plane is countable, hence the number of complementary domains must be countable also. A boundary curve of a complementary domain must contain the point at infinity, since T contains no closed curves. Finally, it is clear that π may be mapped onto the xy-plane so that a given complementary domain maps onto the upper half-plane and its boundary onto the x-axis. Thus each such boundary must be a maximal chain.

Corollary 1 follows from the fact that each set T_C is connected and disjoint from every other such set. If Corollary 2 were not true we would obtain an immediate contradiction to either property (4) of a tree or the fact that T is a closed subset of π.

3.4 A Numbering System for the Curves of a Tree

To facilitate further proofs it will be convenient to establish a system for numbering the curves of a tree of π. The numbering proceeds from an arbitrarily chosen, directed curve C of T, which we shall call the base curve of the tree. Using the orientation of the plane together with the existence of a unique chain from the base curve C to each curve of T, we set up a 1-1 correspondence between curves of T and a collection of signed finite sequences, the particular collection depending on both T and C, the sign of the sequences depending only on the direction of C. (See Figure 8b.)

To the curve C itself we assign, ambiguously, the sequences ± 0, and we write $C = C(\pm 0)$. If C has a positive endpoint, we denote it by b(+0) and, numbering in clockwise order, the curves of St b(+0) by $[C(+0)]$, C(+01), C(+02), . . ., $C[+0u(+0)]$, where u(+0) is defined as the number of curves in St(b(+0)) less one, i.e., as twice the multiplicity of that branch point less one. We then denote, if it exists, the endpoint of C(+0k) opposite b(+0) by b(+0k). We follow exactly the same procedure at the other endpoint, if there is one, of $C(\pm 0)$. This endpoint is denoted by b(-0) and the curves of St(b(-0)) are numbered, again in the clockwise direction, $[C(-0)]$, C(-01), C(-02), . . ., $C[-0u(-0)]$. If the chain c(C,C') from the base curve C to another curve C' of T contains n curves, we shall say that C' is of order n with respect to C. The process above then has numbered every curve of T of order 1 or 2 by exactly one finite sequence of one or two elements respectively (except for the ambiguity in the numbering of C itself). Moreover, it assigns a unique sequence to the endpoints of the curve, with the endpoint being numbered with the same number as the curve of lowest order having it as endpoint. Two curves C,C' of the same order will be clockwise adjacent (in that order) if the final integer of the sequence of C is one less than that of the sequence for C'; and two curves C,C' with C of lower order than C' will be clockwise adjacent if the sequence of C' is that of C with a final integer 1 added to it. Finally, the chain from the base curve to a curve C' consists of the curves whose numbering sequences are successive "lower segments" of the sequence numbering C', i.e., if $\alpha = 0p_2 \ldots p_n$ numbers C', then $\alpha_1 = 0$, $\alpha_2 = 0p_2$, . . . $\alpha_{n-1} = 0p_2 \ldots p_{n-1}$, $\alpha = \alpha_n = 0p_2 \ldots p_{n-1}p_n$ number the curves of the chain from the base curve to C'.

Now if we assume that everything said above is true for every curve of order n, it is very simple to show that it may be extended in toto to the curves of order n + 1; i.e., let C' be any curve of T of order n + 1. Then C' is the terminal curve of a chain c(C,C') of n + 1 curves, all of which except C itself have already received their unique numbering, the next-to-last of them by a sequence α of n terms, which sequence also numbers the common endpoint $b(\alpha)$ of this curve and C'. As before, we number the curves in $St(b(\alpha))$ in clockwise order as $\left[C(\alpha)\right]$, $C(\alpha,1)$, $C(\alpha,2)$, . . . $C\left[\alpha,u(\alpha)\right]$. In this process C' will receive a unique numbering, and the statements above will follow through.

With the help of a little new terminology, we will express these facts in a theorem. As above, α, β, etc., will denote finite signed sequences of positive integers and α,k will be the sequence whose first n elements correspond to those of α, but whose final element is k; i.e., we adjoin one more element, k, to α. Given two collections of sequence, A,A*, we denote by $A \cup A^*$ the collection of all signed sequences obtained by giving those in A positive sign and those in A* negative. Using this notation we shall call a collection A of finite sequences <u>admissible</u> if:

(1) Every sequence has 0 as first element, positive integers for the other elements, and 0 is a sequence of A.

(2) $\alpha,k \in A$ implies $\alpha,k-1 \in A$ if $k \neq 1$ and implies $\alpha \in A$ if $k = 1$.

(3) For each $\alpha \in A$ there is defined an odd integer $u(\alpha) \geq 0$ and $\neq 1$ such that if $u(\alpha) > 0$ then $\alpha,1$; $\alpha,2$; . . . $\alpha,u(\alpha)$ are in A but not $\alpha,u(\alpha)+1$; and, if $u(\alpha) = 0$, then there is no sequence of A with α as lower segment, i.e., of the form $\alpha,p_{n+1}p_{n+2}\cdot\cdot\cdot p_{n+k}$. (Note: If $u(\alpha) = 0$ we call α a <u>terminal</u> sequence.)

Theorem 3.4-1: Given a tree T, a curve C of T, and a direction on C, then there exist two unique admissible collections of finite sequences, A,A* such that there is a 1-1 correspondence between the curves of T and the signed sequences $A \cup A^*$ (except for ± 0 being assigned to C), and such that there is further a 1-1 correspondence between the endpoints of the curves of T and the signed sequences of the collection: $A \cup A^*$ - [all terminal sequences], these correspondences being as described above and having in particular the properties:

(1) If $C(\alpha)$ is any curve of T, then $C(\pm 0)$, $C(\alpha_2)$, . . $C(\alpha_{n-1})$, $C(\alpha)$ is the chain from the base curve to $C(\alpha)$.

(2) $C(\alpha,k)$ has the endpoint $b(\alpha)$ in common with the lower order curve $C(\alpha)$ and, if α,k is not a terminal sequence, the endpoint $b(\alpha,k)$ at the opposite end.

(3) $C(\alpha)$, $C(\beta)$ of the same order n are clockwise [counterclockwise] adjacent if and only if $\alpha_{n-1} = \beta_{n-1}$ and $\alpha = \alpha_{n-1},k$; $\beta = \beta_{n-1},k+1$ $[\beta = \beta_{n-1},k-1]$. $C(\alpha),C(\beta)$ of different order are clockwise [counterclockwise] adjacent if and only if $\beta = \alpha,1$ $[\beta = \alpha,u(\alpha)]$.

It is obvious but tedious to prove that maximal chains, the sets $\mathscr{S}^*(C+)$, $\mathscr{S}^{\#}(C+)$ and so on are numbered by sequences with certain characteristic properties. We shall not develop this aspect, but will state one or two important properties below:

Theorem 3.4-2: Two trees T,T' of π, or a tree T of π and a tree T' of the xy-plane, are homeomorphic under a homeomorphism which may be extended to all of π if and only if we may choose and direct a base curve from each so that the numberings of the two trees are then identical.

Proof: If the two trees are homeomorphic under such a homeomorphism and on the same plane π so that orientation will be the same for each, or on the xy-plane, from which π takes its orientation, then it is trivial that for any directed curve C of T we may choose the homeomorph C' of C in T', and giving it the direction induced by C, and using C,C' as base curves, we will get precisely the same numbering for each tree.

On the other hand let T,T' be two trees with identical numberings. We first show that they are homeomorphic. We let $f(\alpha):C(\alpha)\rightarrow C'(\alpha)$ be any homeomorphism of $C(\alpha)$ onto $C'(\alpha)$ such that $b(\alpha_{n-1})$ maps onto $b'(\alpha_{n-1})$, then $f(\alpha)$ coincides with $f(\alpha_{n-1})$ at $b(\alpha_{n-1})$, the only point where their domains overlap. The map $f:T\rightarrow T'$ defined by $f(x) = f(\alpha)x$, for α such that $x\varepsilon C(\alpha)$, is 1-1 and is continuous on each of a family of closed sets covering T. Now let x_n be any sequence of points on T such that $x_n \rightarrow x\varepsilon T$, then by property (4) of trees, for $n \geqq N, x_n$ will lie on C_x or St(x), the latter if x is a branch point. From the continuity of f on C_x and St(x) for every $x\varepsilon T$, it follows that $f(x_n)\rightarrow f(x)$. Hence, since x_n was any sequence and x any point, f is continuous on T. It follows in the same manner that f^{-1} is continuous on T'. Thus f is a homeomorphism from T to T'.

Now in view of the fact that for every branch point p of T, the sense in St(p) must be preserved by f as defined above, and in view of our earlier discussion of the theorem of Adkisson and Maclane, it remains only to show that $\overline{T}$ is a Peano continuum to complete the proof of this theorem, where $\overline{T} = T\cup\infty\epsilon\overline{\pi} = \pi\cup\infty$. First, it is clear that T is a Peano continuum: it is connected, and also locally connected and locally compact due to the fact that open C and open St(p) are open sets in T. Moreover, T is closed in π, and on $\overline{\pi}$, ∞ is a limit point of T but is also the only limit point of T, thus $\overline{T}$ is a closed, connected and hence a compact subset of $\bar{\pi}$. Finally, $\bar{T}$ is locally connected, for a compact continuum cannot fail to be locally connected at a single point (Whyburn XVI, 12.3, p. 19).

We now remark that this theorem makes it possible to construct a rectilinear model of any tree T on the xy-plane, and assures us that there will be a homeomorphism of π onto the xy-plane carrying T onto this model. The model is constructed by considering any numbering $A \cup A^*$ of T, and, using line segments of length ≥ 1 as our elements, building up the model piece by piece: We begin with a base segment corresponding to the sequence ± 0, add segments corresponding to the 2nd order sequences, 3rd order, etc., each time moving further out from our base segment so that its distance from any n-th order segment approaches infinity with n. In this process it is clearly possible to construct the model so that the image of any one particular chain is a straight line, e.g., the x-axis.

3.5 Semi-r-neighborhoods and Cross-sections

For an arc pq, lying on an adjacent chain of curves $C_1, \ldots C_n$, it is possible to get a serviceable analog of the r-neighborhood of an arc on a regular curve (cf, Theorem 1.2-2). By suitably directing C_1, we have both pq and $C_1, \ldots C_n$ as arcs on C_1^*, the latter containing the former. We will define an open semi-r-neighborhood of pq as any open set $U \subset \mathcal{D}^*(C_1^*)$ together with a homeomorphism g of $\bar{U}$ onto the rectangle $\bar{R}_1$ of the xy-plane, where $R_1 = \left\{(x,y) \mid -1 \leq x \leq 1,\ 0 \leq y \leq 1\right\}$, with g having the properties:

(1) g carries $F\left[\bar{U}\right]$ onto the lines y = constant in $\bar{R}_1$

(2) $g^{-1}(\lambda_i)$ are cross-sections, where λ_i, i = 1,2 are, respectively, that part of the lines x = -1 and x = +1 in $\bar{R}_1$.

(3) pq is mapped into the set $\{(x,y) \mid y = 0, -1 < x < 1\}$

We shall finally call the set $U(pq) = U \cup g^{-1}(\{(x,0), -1 \leq x \leq 1\})$ a semi-r-neighborhood of pq. U(pq) contains no branch points except those on C_1^* itself and lies entirely in $\mathcal{D}^*(C_1^*) \cup C_1^*$, which contains no points of T_{C_1} except those on C_1^*. We shall find it convenient to refer to a semi-r-neighborhood of a single point p, by which we shall mean one side of a regular neighborhood of p if p is a regular point and one sector of an admissible neighborhood if p is a branch point.

Theorem 3.5-1: Any arc pq on an adjacent chain of curves $C_1, \ldots C_n$, $n > 1$, has an arbitrarily small semi-r-neighborhood (there exists a neighborhood contained in $U_\epsilon(pq)$ for any $\epsilon > 0$) in the complementary domain of T of which the maximal chain containing the given adjacent chain is the boundary.

Proof: We change the curve family F as follows: Let C_1 be directed so that C_1^* contains the chain $C_1, \ldots C_n$. We leave F unchanged in $\mathcal{D}^*(C_1^*)$, the complementary domain of T_{C_1} in which U is to lie; but we map the lines y = constant of the lower half-plane, including the x-axis onto $C_1^* \cup \mathcal{D}^\#(C_1^*)$ so that the x-axis is mapped onto C_1^*. Then by Theorem 1.2-3 this new family is regular in π and agrees with F in $C_1^* \cup \mathcal{D}^*(C_1^*)$. C_1^* is a regular curve of this new family; hence by Theorem 1.2-2, there is an arbitrarily small r-neighborhood of pq, call it V. Then $U(pq) = V \cap \left[C_1^* \cup \mathcal{D}^*(C_1^*)\right]$ will be our desired semi-r-neighborhood.

Theorem 3.5-2: If a sequence of points q_n on distinct curves C_n approach the point p, where p is a regular point or a branch point, then there is a curve C which may be so directed that p lies on C* and an infinite subsequence q_m of $\{q_n\}$ lies in $\mathcal{D}^*(C^*)$. If p' is any other point on C*, then there is a sequence of points r_m on the same curves C_m containing the q_m such that $r_m \to p'$.

Proof: Let T be the tree of F which contains p, then there are at most a finite number of complementary domains of T on whose boundary p lies, and since the q_n lie on distinct curves, there must be a subsequence q_m of these points lying in one of these complementary domains. The maximal chain which bounds this domain, can, for some suitable directed C be given as $\mathcal{D}^*(C^*)$. Now let p' be any other point on C*. We may take a semi-r-neighborhood U(pp') of pp' in $\mathcal{D}^*(C^*)$ and, if $f:U(pp') \rightarrow R_1$, then the curves C_m containing q_m map onto lines $y = k_m$ for $m \geq M$. If $f(p') = (x',0)$, then the points $r_m = f^{-1}(x',k_m)$ will be the desired sequence.

Theorem 3.5-3: An arc pqr is a cross-section of F if and only if (1) it contains no branch points, (2) one of the domains $\mathcal{D}^*(C_q)$, $\mathcal{D}\#(C_q)$ contains p, the other q, and (3) pq and qr are each cross-sections.

Proof: We first assume that the arc pqr is a cross-section through q. Then (1) and (3) follow by definition of cross-section. By the Lemma stated in [IV], p. 158, there is an r-neighborhood of q, V(q), such that the image of pqr in R_0 is the y-axis. Every curve crossing V(q) crosses pqr; hence, no curve has more than one line y = constant as image in R_0. The point q itself maps on (0,0) and C_q on the x-axis; hence, V - C_q splits into two domains, one containing p and the other r. Moreover, one of these domains lies in $\mathcal{D}^*(C_q)$ and the other in $\mathcal{D}\#(C_q)$ for q is a point on the common boundary of these two domains and hence every neighborhood of q contains points of each domain.

Now, if we assume that pqr is an arc with the properties (1), (2) and (3), we may show that it is a cross-section by showing that it intersects any set T_C at most once. This is clear at once if we remember that a set T_C cannot have points in each of the domains $\mathcal{D}^*(C_q)$ and $\mathcal{D}\#(C_q)$ so that if pqr had more than one point in common with T_C, each such common point would have to lie in the same domain, i.e., both on pq, or both on qr. This is impossible, however, since both of these arcs are cross-sections. It follows that pqr is a cross-section.

The following corollary is immediate:

Corollary 1: If C,C' both intersect a cross-section pq, and each is directed to cross pq in the same direction, then either $\mathcal{D}^*(C) \supset \mathcal{D}^*(C')$ or $\mathcal{D}^\#(C) \subset \mathcal{D}^\#(C')$.

Corollary 2: If an arc is such that any point on it is interior to a subarc which is a cross-section, then it is a cross-section.

Proof: Let pq be such an arc; then we may cover pq with a finite number of r-neighborhoods which overlap. Then, applying the theorem repeatedly a finite number of times, gives the result desired.

4.0 THE FAMILY F AS THE LEVEL CURVES OF A CONTINUOUS FUNCTION

In this section it will be shown that there is a continuous function f(x,y) whose level curves are exactly the family F. The proof of this statement will depend on our ability to remove certain branched curves of F together with their branch points so as to leave a subset R* of the plane π, which is open connected, and simply connected and is such that $F^* = F\left[R^*\right]$ is a regular curve family filling R*. It will then follow from [IV] that there is a continuous function f*(x,y) defined on R* and having the family F* as level curves. Finally, it is shown that f*(x,y) may be extended to a continuous function on all of the plane with the curves of F as level curves. In this and the next section we will restrict the use of the term tree to those sets T_C containing singular curves.

4.1 The Numbering of the Trees of F

Theorem 4.1-1: If K is any compact subset of π, then there are at most a finite number of distinct trees of F which intersect K on more than one curve of the tree. Moreover, no more than a finite number of curves from any one tree can intersect K.

Proof: The second part of the theorem is an immediate consequence of the fact that any point p which is a limit of a sequence of points p_n of the tree must be a point of the tree; and, in addition to this, for $n \geqslant N$, p_n must lie in St(p) if p is a branch point, or on C_p if p is a regular point. If an infinite sequence of curves of a single tree intersected K, we could, by compactness of K, choose a sequence of points on distinct curves of this sequence which has a limit point, and hence could not conform to the requirements above for a convergent sequence of points on a tree.

We prove the first part of the theorem by assuming it false and arriving at a contradiction. Let T_i $(i = 1, 2, \ldots)$ be an infinite collection of trees, each intersecting the compact set K on two curves C_i, C_i'. By compactness of K we may choose a sequence of the T_i's and a point $p_i \varepsilon C_i \cap K$ together with a point $q_i \varepsilon C_i' \cap K$ in each T_i of the sequence such that there exists $p = \lim_{i \to \infty} p_i$ and $q = \lim_{i \to \infty} q_i$. By Theorem 3.5-2 we may assume p and q are each regular points and that all of the points p_n lie in the same complementary domain of the tree containing p and similarly with q. Moreover, it may be assumed that p and q are distinct, for otherwise, in an r-neighborhood of the point p = q we could easily find a cross-section intersecting both C_i and C_i' for some i. This is impossible by Theorem 3.3-1. It follows that p will have a semi-r-neighborhood U(p) containing an infinite subsequence of the p_i's and not intersecting a similar semi-r-neighborhood U(q). This infinite subsequence will determine an infinite subsequence of the q_i's, which will itself have an infinite subsequence approaching q and lying entirely in the semi-r-neighborhood U(q) of q. This is really a sub-subsequence of the original q_i's and we will renumber the original sequence so as to denote the sub-subsequence by q_i, i = 1, 2, ... The subsequence of the p_i's determined by this sub-sequence $\{q_i\}$ will be denoted p_i (i = 1, 2, . . ., i.e., we renumber the terms of the original sequence). Then we have for all i: $p_i \epsilon U(p)$, $q_i \epsilon U(q)$ and $p_i, q_i \epsilon T_i \wedge K$. (See Figure 11.)

Now there will exist in U(p) a cross-section γ which contains three points p_{i_1}, p_{i_2}, p_{i_3} from the sequence $\{p_i\}$, and a corresponding cross-section γ' in U(q) containing points q_{i_1}, q_{i_2}, q_{i_3} from the sequence $\{q_i\}$, all points so chosen that p_{i_2} lies between p_{i_1} and p_{i_3} on γ and similarly q_{i_2} lies between q_{i_1} and q_{i_3} on γ'. Then, denoting by c_j the chain $c(C_{i_j}, C'_{i_j})$ in T_{i_j}, j = 1, 2, 3, we may define the following three arcs, λ_{i_1}, λ_{i_2}, λ_{i_3} having only their endpoints p_{i_2} and q_{i_2} in common: λ_{i_1} is the arc $(p_{i_2}p_{i_1})$ on γ plus the arc $p_{i_1}q_{i_1}$ on c_1 plus the arc $q_{i_1}q_{i_2}$ on γ'. λ_{i_3} is similarly defined with 1 replaced by 3 in the subscripts above, and finally λ_{i_2} is the arc $p_{i_2}q_{i_2}$ on c_2. Two of these arcs, say $\lambda_{j_1}, \lambda_{j_2}$, must form a simple closed curve Γ containing the third λ_{j_3} in its interior. But this is impossible since each arc contains a branch point, in particular the arc λ_{j_3}, thus enclosed in the interior (in our example) would contain a branch point; and from this branch point issues a chain of curves of T_{j_3}, all distinct from λ_{j_3}, which must leave Γ at some point r. This point r cannot be on T_{j_1} or T_{j_2} since two trees cannot intersect, nor can it be on γ or γ' since then this cross-section would have two points on the same tree, which is ruled out by Theorem 3.3-1. Hence, we conclude that our initial assumption is impossible and that the theorem must be true.

This theorem will be used to give a method of numbering all non-trivial trees, i.e., trees containing singular curves. We choose any regular point p on π and let K_n designate the circle (with its interior) of center p and radius n. Now the number of trees cutting K_1 is, of course, denumerable and we number them in any order as T_{11}, T_{12}, T_{13}, . . . and choose from each a curve C_{11}, C_{12}, C_{13}, . . . respectively, which itself intersects K_1. By the above theorem these choices of curves will be unique for all except a finite number of the trees, and for these C_{1j} is chosen at random from any one of the finite number of curves of the tree cutting K_1. Next we number the trees which intersect K_2 but <u>not</u> K_1 as T_{21}, T_{22}, T_{23}, . . . etc., and let C_{21}, C_{22}, C_{23}. . . . respectively be curves of these trees which themselves intersect K_2. Proceeding with this process we number the trees cutting K_n but not K_{n-1} as T_{n1}, T_{n2}, T_{n3}, . . . etc., and choose from each curves C_{n1}, C_{n2}, . . . cutting K_n. This process will clearly number all the trees of F, and we choose the curves C_{ij} as base curves of the trees, hence determining within each tree T_{ij} a numbering of its curves by sets of finite sequences $A_{ij} \cup A^*_{ij}$ as described in 3.4. Our method of numbering the trees guarantees that for $m > n$ no tree T_{mj} intersects K_n and, moreover, for all n, there are at most a finite number of curves of the set $\bigcup_{m,j \; m \leq n} (T_{mj} - C_{mj})$ which intersect K_n. For future reference we shall call the above method of numbering trees a <u>standard</u> numbering of the trees of F.

With these preliminaries we are able to define the curves which we are going to remove from each tree in order to make the region $R = \pi - B$ simply connected. Let T_{ij} be any tree (with base curve C_{ij}) from the standard numbering given above; and thus with numbering sequences $A_{ij} \cup A^*_{ij}$. Let $b(\alpha)$ be any branch point of this tree with sequence $\alpha = e0k_2 \; . \; . \; . \; k_{n-1}k_n$ with $k_n \neq 1$ (where e denotes the sign + or -). Any such $b(\alpha)$ will be called the initial point of a cut, and the cut, $\lambda(b)$, will consist of all curves $C(\alpha,1)$, $C(\alpha,1,1)$, $C(\alpha,1,1,1)$, . . . and so on ad infinitum, or until a terminal sequence $\alpha,1,1$ . . ., 1 is reached, i.e., each cut is a chain of adjacent curves extending from $b(\alpha)$ to infinity. We assume endpoints of the curves included, of course, as part of the cut; thus each cut is of the form $\sigma^*[C(\alpha)+]$ or $\sigma\#[C(\alpha)-]$, the latter if $b(\alpha)$ is the positive endpoint of $C(\alpha)$ and the former if it is the negative (see heavy lines in Figure 4). Each $\lambda(b)$ is, again, an arc from $b(\alpha)$ to infinity and includes all branch points numbered by sequences of the form $\alpha,1,1$ . . ., 1. It is clear that every branch point of the tree is on one and only one cut $\lambda(b)$ and that no two cuts intersect at any point. We denote the collection of all half-open arcs $\lambda(b)$ on T by $\tilde{T}$, and by $\tilde{\mathcal{J}}$ the sum of the sets $\tilde{T}$ over all the trees of F. The set $R^* = R - \tilde{\mathcal{J}}$ contains no branch points and is a union of curves of F. Let F* denote the family $F[R^*]$ filling R*.

Theorem 4.1-2: R* is an open, arcwise connected, and simply connected domain, and F* is regular in R*.

Proof: Let q be any point of R* and let K_n be the first circle with center at p (in the standard numbering scheme) which contains q in its interior. Now consider how much of K_n is removed when $\tilde{\mathcal{J}}$ is subtracted from π. None of the base curves C_{ij} are in $\tilde{\mathcal{J}}$ since none of them are in a set $\lambda(b)$ for these curves are assigned the sequence ± 0 in the numbering, which sequence is not of the form $\alpha,1$, . . ., 1. And by Theorem 4.1-1 there can then be at most a finite number of other curves (than base curves) of any T_{ij} in K_n. Hence there is surely an r-neighborhood of q in $K_n - \tilde{\mathcal{J}}$ and R* is therefore open and F* regular.

We wish to show that R* is arcwise connected. Since every point has an r-neighborhood in R*, it is clear that R* is locally-connected. Hence, if it is connected, it is arcwise connected. Now since each cut $\lambda(b)$ is an arc, extending from a point b to infinity, the set $\tilde{\mathfrak{J}} \cup \infty$ on the extended plane $\bar{\pi}$ can clearly be deformed continuously along itself to a single point, the point at infinity. It follows from Eilenberg [III], Theorem 6, p. 77, that R* is connected.

Finally, if K is any closed curve in R* containing a point q of $\tilde{\mathfrak{J}}$, then q lies on a cut $\lambda(b)$ which extends to infinity from b and hence must intersect K, contrary to the assumption that K is in R*. Thus R* is simply connected.

<u>Theorem 4.1-3: Let $B' \subset B$ be the set of all initial points of cuts $\lambda(b)$, then we may define a collection of disjoint, open sets</u> $\{V_b \mid b \in B'\}$ <u>such that</u> $V_b \supset \lambda(b)$.

Proof: Referring to the closed circular discs K_n of our standard numbering of the trees of F, we have noted already that only a finite number of the cuts $\lambda(b)$ intersect any K_n. We denote by $B'_n = \{b^n_j\}$ the finite subset of B' whose elements b^n_j are for j = 1, . . ., j_n those initial points of cuts which intersect K_n but not K_{n-1}. Now, using the normality of π we are able to find disjoint open sets covering the disjoint closed sets $\lambda_1(b^1_j) = \lambda(b^1_j) \cap K_1$. We define $V_1(b^1_j)$ as the intersection of the so chosen open sets covering $\lambda_1(b^1_j)$ with the <u>interior</u> of K_1. Then we find disjoint open sets covering each of the closed sets $\lambda_2(b^i_j) = \lambda(b^i_j) \cap [K_2 - i(K_1)]$, i = 1,2 and such, moreover, that the open sets covering $\lambda_2(b^2_j)$ do not intersect K_1. Finally, we define $V_2(b^i_j)$, i = 1,2 as the intersections of these open sets with the interior of K_2. Then the sets $V_2(b^2_j)$ and $V_1(b^1_k) \cup V_2(b^1_k)$ are non-intersecting open sets lying in the interior of K_2 and covering $\lambda(b^i_j) \cap i(K_2)$, i = 1,2, for all b^i_j's in B'_1 or B'_2.

This process is continued indefinitely, covering every intersection of a $\lambda(b)$ with a K_n. Then, given any $b \varepsilon B'$ it will be in B'_n for some n, hence will be of the form b^n_i, and the cut $\lambda(b^n_i)$ with it as initial point is covered by $V(b^n_i) = \bigcup_{j=n}^{\infty} V_j(b^n_i)$.

Theorem 4.1-4: Let F be a branched regular curve family filling the plane π. Then there exists a function f(p) such that:

(1) f(p) is defined and continuous for all p in π.

(2) for every real number k the locus f(p) = k consists of an at most countable infinite collection of trees (including regular curves) of F.

(3) in every neighborhood of any point p in π there are points q for which $f(q) > f(p)$ and points r for which $f(r) < f(p)$.

Proof: We assume a standard numbering of the non-trivial trees of F and that thus the cuts $\lambda(b)$ and the sets $\widetilde{\mathcal{J}}$ and R*, etc., are determined. This theorem was proved in [IV] by W. Kaplan for curve families regular throughout an open, simply connected domain; thus we may assume that there is a function f*(p) defined and continuous in R* and with the properties above. We must show that this function can be extended to a function f(p) with properties 1-3 above. The proof has three sections, A, B and C.

(A) First it is necessary to prove that, given any tree T of F, the value of f* is the same on each curve of T[R*], i.e., on all curves of T which lie in R*. Let $C(\pm 0)$ be the base curve of T in the numbering; we shall proceed by induction on the order of the curves of T. If $C(\pm 0)$ has no endpoint, then it is a regular curve, lies entirely in R* and the result is trivial. Assume it has a positive endpoint b(+0). Then C(+01) is in $\lambda(b(+0))$ and hence not in R* or T[R*], but the other curves of St(b(+0)) are all in T[R*]. To prove that

$f^*(p)$ has the same value on each of these it is only necessary to prove that it has the same value on each pair of adjacent curves among them, for then the value of f^* on $C(+02)$ is the same as that on $C(+03)$ and so on until finally we have the value on $C\left[+0u(+0)\right]$ the same as that on $C(+0)$. It is quite obvious that this must be so, however, for if C, C' are adjacent curves of $T\left[R^*\right]$ and $p \varepsilon C$, $q \varepsilon C'$, then there is a semi-r-neighborhood $U(pq)$ in R^*; and, if $p_n \varepsilon U$ is a sequence of points approaching p, then there is a sequence $q_n \varepsilon C_{p_n}$ with $q_n \varepsilon U$ and q_n approaching q. But, since $q_n \varepsilon C_{p_n}$ we have $f^*(q_n) = f^*(p_n)$ and hence $f^*(q) = \lim_{n \to \infty} f^*(q_n) = \lim_{n \to \infty} f^*(p_n) = f^*(p)$. This same procedure actually tells us even more, i.e., that if $C_1, \ldots C_n$ is any chain of adjacent curves with both $C_1, C_n \subset R^*$, then f^* must have the same value on C_1, C_n.

Now let $C(\alpha,k)$ be a curve of $T\left[R^*\right]$ whose sequence is positive and of order $n + 1$, and assume that f^* has the same value on each curve of $T\left[R^*\right]$ numbered by a positive sequence of order n or less. The sequence α is of the form $\alpha = 0k_2 \ldots k_{n-1}k_n$; and we consider two cases: (1) $k_n \neq 1$ and (2) $k_n = 1$; in either event $k \neq 1$ since $C(\alpha,k)$ is in R^*. In case (1) $b(\alpha)$ is the initial point of a cut, hence $C(\alpha,1)$ is the only curve of $St(b(\alpha))$ in the cut, and moreover, the $St(b(\alpha))$ contains the curve $C(\alpha)$ of order n. It follows by precisely the same argument as above that f^* has the same value on each of the curves of $St(b(\alpha))$ in R^* and in particular on $C(\alpha,k)$ as it has on the n-th order curve $C(\alpha)$ and hence that it has on $C(\pm 0)$. In case (2) both the curves $C(\alpha)$ and $C(\alpha,1)$ of $St(b(\alpha))$ are in a cut. But the curves $C(\alpha_{n-1},2)$, $C(\alpha_{n-1},1) = C(\alpha)$, and $C(\alpha,u(\alpha))$ form an adjacent chain with the first and last curves in R^*. On the first curve f^* has the same value as on $C(\pm 0)$ since it is of order n, hence it has this value also on the last, $C(\alpha,u(\alpha))$. Now, by going from adjacent curve to adjacent curve, we see that this must be the value of f^* on each curve of $St(b(\alpha))$ in R^* and in particular on $C(\alpha,k)$. This completes the first step in the proof.

(B) Next we define $f(p)$ at every point of π as follows: $f(p) = f^*(p)$ for $p \varepsilon R^*$, and $f(p)$ = value of f^* on $T[R^*]$ for $p \varepsilon T$. $f(p)$ will then be continuous at each point of R^* since R^* is an open subset of π and thus the extension cannot affect the continuity of f in that domain. Now every point of $\tilde{\mathcal{J}} = \pi - R^*$ lies on a cut $\lambda(b)$, which in turn lies in a neighborhood $V(\lambda(b))$ not containing points of any other cut. What must be shown is that $f(p)$ is continuous at an arbitrary point q of an arbitrary cut $\lambda(b)$. Now let q_n be any sequence of points approaching the point q of $\lambda(b)$. We shall denote by T the tree containing q; then since $f(p)$ is constant on T we shall assume that each q_n lies on a distinct curve and none of them is in T. This involves no loss of generality since the result is trivial otherwise. Moreover, we may restrict ourselves to sequences lying in a single complementary domain of T, the reason being that any sequence q_n, with $q_n \notin T$ can be decomposed into a finite number of such subsequences, containing all the terms of q_n, but no two having a term in common, since the number of complementary domains of T containing q on their boundary is finite. Now, if for each of these subsequences we have $f(q_{n_i}) \to f(q)$, then $f(q_n) \to f(q)$. Thus we need now to consider only a sequence $q_n \to q$ such that for some $C^* \ni q$, $q_n \varepsilon \mathcal{D}^*(C^*)$ for all n. C^* is then in T, and since no cut separates π, there is a curve C' on C^* which is in R^*. Let p be any point of C' and $U(qp)$ a semi-r-neighborhood of qp in $\mathcal{D}^*(C^*)$. Then, by Theorem 3.5-2 there is in U a sequence $p_n \to p$ with $C_{p_n} \equiv C_{q_n}$ and hence $f(p_n) = f(q_n)$. But p is in R^* and $f(p)$ is continuous in R^*, therefore $\lim f(q_n) \equiv \lim f(p_n) = f(p)$. But this is exactly what is needed for p,q are both on T and hence $f(p) = f(q)$, so f is continuous at q.

Property (2) of the theorem is trivial for f(p) since it is satisfied by f* in R* and we have added only a denumerable number of curves to the domain of f* to get the domain of f.

(C) Finally we must prove property (3), i.e., that f(p) has no weak relative extrema. This is clearly equivalent to the following, at least for regular points: if p is a regular point, then f takes a different value on every curve of every r-neighborhood of p, or again equivalently, is monotone on every cross-section. Since any arc pq on a curve C has an r-neighborhood, this implies that a function satisfies property (3) at every point of a curve or no point of a curve. As to branch points, we can show at once that the condition is satisfied there, for there is always a curve of St(b) in R*, hence in any neighborhood of b we may find a point q of this curve and a neighborhood of this point q inside that of b. Now f(q) = f(b) and in this neighborhood of q there will be points q_1, q_2 at which f is respectively <, > f(q), since we are in R*, where we know f to have property (3). Since q_1, q_2 are in the given neighborhood of b, we have proved our contention.

Now we wish to show that if f has property (3) on every curve of St(b) except one, C, where b is any branch point, then f has property (3) on C also. Let the curves of St(b) be numbered counterclockwise $C = C_1, C_2, \ldots C_{2m}$, m being the order of the branch point b. In U(b), an admissible neighborhood, we shall let s_i denote any arc into the sector bounded by C_i, C_{i+1}, such that s_i without b, its endpoint is a cross-section, e.g., in the image of U on $|z| < 1$ we could take for s_i radii into the respective sectors. Then we indicate by s_i^+ that f increases as we move from b on s_i, by s_i^- that f decreases. Clearly C_i has property (3) if and only if s_{i-1}^+ implies s_i^- and s_{i-1}^- implies s_i^+. Hence if we have s_1^+, then we have by induction s_j^+ for even j, and in particular s_{2m}^+, whence C_1 has property (3).

Now let the curves of any cut $\lambda(b)$ be numbered $C_1, C_2, \ldots$ beginning with the initial curve and proceeding out from b. C_1 is the only curve of St(b) not in R* and hence it must have property (3). If the n-th curve C_n has property (3) then C_{n+1} is the only curve of $St(C_n \cap C_{n+1})$ not having this property since the other curves (than C_n) are in R*, thus C_{n+1} also must satisfy the desired property. This proves by induction that every curve of every cut has property (3) and hence f(p) has the property for all points of π.

Corollary: The branched regular curve family F is orientable as a regular curve family in $R = \pi - B$.

Proof: Exactly as in W. Kaplan [IV], Remark 2, p. 184-5.

5.0 DECOMPOSITION OF F INTO HALF-PARALLEL SUBFAMILIES

It is the purpose of this section to describe a decomposition of the curve family F into a sum of subsets, which overlap at most along their boundaries, and such that each of them is homeomorphic as a curve family to the lines y = k filling the upper half of the xy-plane.

5.1 Extended Cross-sections

Theorem 5.1-1: Let p be any regular point of π, C_p the curve of F through p, and let C be a curve containing a point q such that there is a cross-section pq. Then there will be a cross-section from p to an arbitrary point q' of T_C if and only if $q' \varepsilon C^*$, where C is directed so that $p \varepsilon \mathcal{D}^*(C)$. Moreover, if $q' \varepsilon C^*$ and U(qq') is any semi-r-neighborhood of qq', we may choose the cross-section qq' as follows: $qq' \equiv qrq'$ where qr lies on pq and rq' is in U(qq').

Proof: Suppose q' to lie on C^* and let U(qq') be any semi-r-neighborhood of qq'. Now moving along pq from p the cross-section pq lies entirely inside U(qq') from some point on, so we may choose some r on pq, with rq interior to U, letting prq now denote pq. We direct C_r so that $pr \subset \mathcal{D}^*(C_r)$ and $rq \subset \mathcal{D}\#(C_r)$, which we can do by Theorem 3.4-5 since prq is a cross-section. We replace rq by a cross-section rq' in U which is found as follows: U is homeomorphic to a rectangle R_1 in the xy-plane by definition, and we join in R_1 the image of r to that of q' by a straight line, whose inverse image we take for rq'. Since the straight line is a cross-section of the lines $y = k$ (image of F) rq' will be also a cross-section, and will lie in the same domain $\mathcal{D}\#(C_r)$ as rq, since each cross the same curves in U. Hence, by Theorem 3.4-5, we know that prq' is a cross-section.

It remains only to prove that if C' is any curve of T_C not on C^*, then there is no cross-section to C' from p. Now p lies in $\mathcal{D}^*(C^*)$ and C' in $\mathcal{D}\#(C^*)$, hence any such cross-section, if it existed, would have to cross C^* and thus would have two points on T_C, contrary to the assumption that it is a cross-section.

Theorem 5.1-2: Let the trees of F be numbered as in Section 4, i.e., in a standard numbering, using the concentric circles K_n of center p and radius n; further, let the cuts $\tilde{\mathcal{T}}$ be removed from F, leaving $F^* = F[R^*]$. Then, outside every circle K_n lies at least one curve of F^* which can be reached from p by a cross-section lying in $R^* \cap \mathcal{D}^*(C_p)$. (See Figure 12.)

Proof: Denote by $\{C\}$ the collection of all curves in $\mathcal{D}^*(C_p)$ which can be reached by a cross-section from p lying in $R^* \cap \mathcal{D}^*(C_p)$. We direct each curve of $\{C\}$ so that $\mathcal{D}^*(C) \subset \mathcal{D}^*(C_p)$. The existence of a cross-section from p to $q \varepsilon C$ makes this possible, i.e., direct C so that $\mathcal{D}^*(C) \geqslant pq$. $\{C\}$ will certainly not be empty since we assume p to be a regular point.

Now define on the curves of $\{C\}$ the positive real-valued function $d(C) = \underset{p \varepsilon C}{\text{GLB}} \{\text{distance from x to p}\}$. We have at once that C is outside K_n if and only if $d(C) > n$. Also it is clear that $\mathcal{D}^*(C) \geqslant \mathcal{D}^*(C')$ implies that $d(C) < d(C')$. To prove the theorem we must show that the numbers d(C) are unbounded. We assume that this is not so; then there is a least upper bound d' of d(C) for C in $\{C\}$. To show that this is impossible we choose $N > d'$ and consider intersections of curves of $\{C\}$ with K_N. Every curve of $\{C\}$ will then intersect K_N if d(C) is bounded by d', although by Theorem 4.1-1 only a finite number of these curves lie completely inside K_N. All but a finite number of curves of $\{C\}$ in fact, not only have both endpoints outside K_N, but contain within themselves the only intersection of T_C with K_N. Hence, we may choose an infinite sequence of curves C_m of $\{C\}$ such that $d(C_m) \rightarrow d'$ and $T_{C_m} \cap K_N \equiv C_m \cap K_N$, and $C_m \cap K_N$ contains neither endpoint of C_m. Having chosen such a sequence we find a subsequence q'_m of points from C_m which approach a regular point q as a limit and all lie on one side of the image of C_q in an r-neighborhood U(q) (i.e., in the upper or lower half of R_o, the image of U(q)). This may be done as follows: First, by compactness of K_N we may find $q_m \varepsilon C_m \cap K_N$ (a subsequence of the m's) which converges to some point q'. Second, if q' is a regular point, we let q = q' and choose a subsequence q'_m of the q_m's all of whose points lie in one side only of U(q). Third, if q' is a branch point, V(q') an admissible neighborhood of q', then an infinite subsequence of the q_m's will lie in one sector of V. If q is any regular point on either of the adjacent curves bounding this sector there will be a sequence of points q'_m on the same curves C_m as the sequence approaching q' and such that $q'_m \rightarrow q$. The q'_m will lie on the same side of C_q in any r-neighborhood of q and is thus the desired sequence. Finally, we may choose a subsequence of q'_m which we will denote by r_n such that if qs is a cross-section from q to s in U(q), where s lies on the same side of U(q) as the q'_m, then the intersections $C_n \cap qs$ tend monotonely to q on qs, (C_n denoting the curve on which r_n lies). Thus we have $d(C_n) \to d'$ monotonely since $\mathcal{D}^*(C_n) \supset \mathcal{D}^*(C_{n+1}) \supset C_q$ for all n. We direct C_q so that $\mathcal{D}^*(C_n) \supset \mathcal{D}^*(C_q)$.

Now choose in $\mathcal{D}^\#(C_q)$ a semi-r-neighborhood W of qq" where q" is any point of $C_q^\#$ which is in R*. W is chosen so that its interior lies in R*, which is possible by Theorem 4.1-4. Now for $n \geq n_o$, r_n will lie in W and since we have $\mathcal{D}^*(C_{n_o}) \supset C_q^\#$ and $\mathcal{D}^\#(C_{n_o}) \supset C_p$, we may extend the cross-section $pr_{n_o} \subset R^* \cap \mathcal{D}^*(C_p)$ to a cross-section $pr_{n_o}q'' \subset R^* \cap \mathcal{D}^*(C_p)$ by merely adding to it the cross-section $r_{n_o}q''$ in $W \cap \mathcal{D}^*(C_{n_o})$ which is the inverse image of the straight line joining the images of r_{n_o} and q" in R_1, the image of W. This will be a cross-section by Theorem 3.4-5. Now since q" is a regular point of a curve $C_{q''}$, if we take its direction such that $C_{q''}^\# \equiv C_q^\#$, we have $\mathcal{D}^\#(C_{q''}) \supset C_p$ and $\supset C_n$; and $\mathcal{D}^*(C_{q''}) \subset \mathcal{D}^*(C_n)$ for all n, whence $d(C_{q''}) \geq d'$. Now it is easy, however, by taking an r-neighborhood of q" (which will lie in R*) to extend $pr_{n_o}q''$ to a slightly larger cross-section $pr_{n_o}q''s$, and since $C_s \subset \mathcal{D}^*(C_{q''})$, we have at once that $\mathcal{D}^*(C_{q''}) \supset \mathcal{D}^*(C_s)$, where C_s is directed as a curve of $\{C\}$. Hence $d(C_s) > d(C_{q''}) \geq d'$. This is contrary to the assumption that d' is a bound of d(C). Hence d(C) is unbounded, which is what was to be proved.

By an <u>extended cross-section</u>, we shall mean any open or half-open arc in $R = \pi - B$ which meets each curve of F at most once. An extended cross-section is said to <u>tend properly to infinity</u> in R in a given direction on it, if it tends to infinity in that direction in such a way that the curves meeting it tend uniformly to infinity with their intersection points with the cross-section. We shall also speak of an extended cross-section in R^* which will be an extended cross-section as above, and lie entirely in $R^* = \pi - \tilde{\mathcal{J}}$, i.e., it meets only curves of F^*.

Theorem 5.1-3: If p is any regular point on a curve C of F^*, then there is an extended cross-section in R^* from p, which lies in $\lambda^*(C_p)$ and tends properly to infinity.

Proof: We consider a curve C in F^* and p any point on it. As before K_n will denote a circle with center at p and radius n; and for any point s we shall let $Q_n(s)$ denote a circle with center at s and radius so chosen that $Q_n(s)$ contains K_n. Now we choose a regular curve C_1 in $\lambda^*(C_p) \cap R^*$ for which there is a cross-section pq_1 in $\lambda^*(C_p) \cap R^*$ from p to q_1 on C_1. Direct C_1 so that $\lambda^*(C_p) \supset \lambda^*(C_1)$ and choose in $\lambda^*(C_1) \cap R^*$ a curve C_2 outside of $Q_1(q_1)$ and such that a cross-section q_1q_2 in $\lambda^*(C_1) \cap R^*$ exists with q_2 on C_2. Having chosen C_n and $q_n \varepsilon C_n$ in this manner, we choose for C_{n+1} any regular curve outside of $Q_n(q_n)$ for which there is a cross-section q_nq_{n+1} in $\lambda^*(C_n) \cap R^*$ to q_{n+1} on C_{n+1}. We direct C_{n+1} so that $\lambda^*(C_n) \supset \lambda^*(C_{n+1})$. We can continue this process indefinitely by Theorem 5.1-2. Then the curves pq_1, pq_1q_2, $pq_1q_2q_3 \ldots$ will all be cross-sections by Theorem 3.4-5. They approach a curve Γ extending from p to infinity in $\lambda^*(C_p) \cap R^*$ which is an extended cross-section extending from p to infinity in R^*. The curves intersecting Γ tend uniformly to infinity with any sequence of their points of intersection tending to infinity on Γ; since if r on Γ is beyond q_n, then C_r lies outside K_n. Thus Γ is an extended cross-section tending properly to infinity in R^*.

5.2 Half-parallel Subfamilies of F

Theorem 5.2-1: The set S of curves of F crossing an extended, half-open cross-section Γ tending to infinity from a point p on C_p is homeomorphic to the family of parallel lines y = k, $k \geq 0$, filling the upper half of the xy-plane. If C_p^* is directed so that $\mathcal{D}^*(C_p)$ contains Γ, and if C' is any open arc on C_p^*, $p \in C'$, then this homeomorphism may be chosen to map C' onto the x-axis, and Γ onto the y-axis, $y \geq 0$.

Proof: The set S fills a region of the plane in which it is clearly a regular curve family for, if q is any point on the boundary curve $C' \subseteq C_p^*$, we have in S a semi-r-neighborhood U(pq) within which we can find an arbitrarily small r-neighborhood of q. And, if q is a point on some other curve C of S, then we denote by p' the intersection of C with Γ, and there will exist an r-neighborhood U(p'q) by Theorem 1.2-2, which will lie in S (since every curve in it crosses Γ). Within this neighborhood again, we may find an arbitrarily small r-neighborhood of q.

The family S is not only regular, but orientable, for each curve of S crosses Γ exactly once and thus Γ divides S into two regions A and B and we shall say a curve has positive direction if this direction on it carries us from A into B. Then, by Theorem 1.6-1, there is a function f(p,t) defined on S with the properties described in that theorem. We let τ, $0 \leq \tau < \infty$, be a parameter on Γ and restrict p to Γ giving us $f[p(\tau),t]$ a homeomorphism from the upper half-plane to S, cf [IV].

We shall mean by a half-parallel subfamily of F the collection of all curves of F which intersect an extended cross-section Γ tending from a point p on a curve C_p properly to infinity. And we shall mean by a complete half-parallel subfamily of F the curve C_p^* together with all curves of F crossing Γ (C_p being so directed that $\partial^*(C_p) \supset \Gamma$). Each of these sets is homeomorphic to the lines y = k, k $\geq$ 0 of the half-plane; the first will be denoted by S and the second by S*. Clearly $S^* \supset S$ and when C_p is a regular curve they are identical. C_p is called the initial curve of S, C_p^* of S*.

If $\Gamma(q)$ is any half-open cross-section of F tending from a regular point q properly to infinity, then the boundary of $S(\Gamma)$, the collection of curves intersecting Γ is best described in terms of maximal chains C*, C# and the sets $\delta(C+)$, $\delta(C-)$ defined in Section 3. We shall refer to these latter two sets as mixed maximal chains, since they consist of two subchains of maximal chains, one clockwise adjacent, the other counterclockwise adjacent, e.g., $\delta(C+) = \delta^*(C+) \cup \delta\#(C+)$ (which may be empty). $\delta(C)$ will denote $\delta(C+) \cup \delta(C-)$. It is empty if and only if C is a regular curve.

Theorem 5.2-2: The boundary of $S(\Gamma)$ is a collection of maximal chains C*, C# and mixed maximal chains $\delta(C)$ where $\delta(C)$ is on the boundary if and only if C is in $S(\Gamma)$. From each set T_C of F there is either (1) no point, (2) exactly one maximal chain, or (3) a set $\delta(C)$ of T_C on the boundary of $S(\Gamma)$. (1), (2) and (3) are mutually exclusive. (See Figure 13, T_1 for case (2) and T_2 for case (3).)

Proof: Suppose $C \varepsilon S(\Gamma)$ is a singular curve, then $\delta(C)$ is in the boundary of $S(\Gamma)$, for (1) if we consider any point q on $\delta(C)$ there exists a semi-r-neighborhood U(pq) containing q and $p = C \cap \Gamma$ (since C lies on an adjacent chain with q); choosing a sequence of points $p_n \to p$, $p_n \varepsilon U \cap \Gamma$, we can find a sequence $q_n \varepsilon U$ such that $q_n \varepsilon C_{p_n}$ for all n and $q_n \to q$. Whence q is a limit point of points of $S(\Gamma)$. But (2), if q is in $\delta(C)$ it is on a curve of T_C other than

C and C_q cannot intersect Γ hence C_q is not in $S(\Gamma)$, and thus q is on the boundary of $S(\Gamma)$. Moreover, no other curves of T_C can in this case be on the boundary of $S(\Gamma)$, for $S(\Gamma)$ is clearly contained in $\mathcal{D}^*(C) \cup C \cup \mathcal{D}^\#(C)$, a complementary domain of $\sigma(C)$, whereas every other curve of T_C lies in one or two other complementary domains of $\sigma(C)$. (Note: $\sigma(C)$ divides π into at most three Jordan domains.)

On the other hand, suppose that C is a curve of F on the boundary of $S(\Gamma)$. Then, directing C so that $\mathcal{D}^*(C)$ contains the initial point of Γ, we note that if p is a point on C, limit point of a sequence p_n of $S(\Gamma)$, then there is a semi-r-neighborhood U(pq) of any arc pq on C* and a sequence $q_n \rightarrow q$ with $q_n \epsilon C_{p_n}$, and hence a sequence in $S(\Gamma)$, from which we conclude that q is either in $S(\Gamma)$ or on its boundary. If C* does not cross Γ, then q will be on the boundary and C* is a boundary curve of $S(\Gamma)$. When this is the case C* divides into two domains $\mathcal{D}^*(C^*) = S(\Gamma)$ and $\mathcal{D}^\#(C^*) = T_C - C^*$, whence no other point of T_C than those of C* is on the boundary of $S(\Gamma)$. But, if C* crosses Γ at a point p on a curve C', then we are back in the previous case and $\sigma(C') = [C^* \cup C^\#] - C'$ is the boundary in T_C of $S(\Gamma)$.

Theorem 5.2-3: Let $\Gamma(q)$ be a cross-section from q on the curve C_q of F*, and tending properly to infinity in R* in each direction. Further, let h be any homeomorphism of R* onto the xy-plane, then $h[\Gamma(q)]$ is a cross-section of the family $h[F^*]$ (filling the xy-plane) which tends properly to infinity in both directions on the xy-plane.

Proof: On the xy-plane we let K_n denote a circle of radius n, center h(q) and we must show that for every n there are points q_n', r_n' on $\Gamma' = h[\Gamma(q)]$ such that every curve of h(F*) intersecting Γ' at points beyond q_n', r_n' will lie outside K_n. If this is not the case we will be able to find a sequence of points

t_n' on Γ' such that each $C_{t_n'}$ intersects a fixed one K_N of the circles K_n. Now the inverse image of K_N is a simple closed curve in R* containing q in its interior. We will denote by C_n the inverse image of $C_{t_n'}$ and by t_n the inverse image of t_n'. Every C_n must then intersect K and hence intersect some circle with center at q which contains K. But this contradicts the assumption that $\Gamma(q)$ tended properly to infinity in R*, since we have a sequence t_n approaching infinity on $\Gamma(q)$, but the curves C_{t_n} do not approach infinity. Hence the theorem must be true.

W. Kaplan introduced the notion of admissible collections of finite sequences in order to number the half-parallel subsets of a regular curve family filling an open simply connected domain. The concept is so similar to that already considered in the numbering of curves of a tree that we shall be able to use the same notation as in that section. We shall, however, reserve the term admissible for collections of the type of Section 3.4 and, after Kaplan [IV], we shall call a collection A of finite sequences <u>allowable</u> if

(1) A contains the one element sequence 1 and no other one element sequences, and

(2) $\alpha, k \epsilon A$ implies $\alpha, k-1 \epsilon A$ if $k > 1$ and implies $\alpha \epsilon A$ if $k = 1$.

Now, if we have a regular curve family F' filling the xy-plane, and if we have assigned to each point (x,y) an extended cross-section $\Gamma(x,y)$ tending properly to infinity in both directions, then for any fixed curve C_1 it was shown in [IV] that we can decompose $F'[C_1 \cup \mathcal{D}^*(C_1)]$ into a collection of non-overlapping, half-parallel subfamilies $S(\alpha)$ which will be numbered by the finite sequences $\{\alpha\}$ of an allowable collection A. Each half-parallel family $S(\alpha)$ will be the set of all curves intersecting a cross-section $\Gamma(\alpha)$ tending from an initial curve C_α to infinity and lying on some $\Gamma(x,y)$ as chosen above; C_α will be the only curve of $S(\alpha)$ mapped onto the x-axis in the homeomorphism of $S(\alpha)$ onto the lines $y = k \geq 0$ and the complete boundary of $S(\alpha)$ will be, in addition to C_α, just exactly the curves $C_{\alpha,k}$. Note that when we write C_α we mean to indicate that C_α is an initial curve of some $S(\alpha)$ in the decomposition of F', whereas $C(\alpha)$ will as above indicate that C is the curve of a numbered tree which has been assigned the signed sequence α in the numbering of the tree.

As a corollary to the preceding Theorem 5.2-3 plus the proof of the facts mentioned in the preceding paragraph from [IV] we can immediately state the following theorem:

<u>Theorem 5.2-4</u>: <u>Given the family $F^* = F[R^*]$ and an arbitrary regular curve C_1 of F^*, we can decompose $F^*[C_1 \cup \lambda^*(C_1)]$ which is the same as $F[C_1 \cup \lambda^*(C_1) \cap R^*]$ into a collection of non-overlapping half-parallel subsets $S(\alpha)$, each $S(\alpha)$ being all curves intersecting a cross-section $\Gamma(\alpha)$ tending from a curve C_α in F^* properly to infinity in R^*</u>. (See Figures 4 and 13.)

In order to study the relation between an arbitrary tree T of F and a given decomposition of F^* into sets $S(\alpha)$ ($\alpha \varepsilon A$, as described above), it is convenient to adopt some new notation. A(T) will denote the subset of A containing all sequences α such that $S(\alpha) \cap T \neq 0$; and $A_n(T)$ the subset of all sequences of A(T) of order n. We denote by N(T) the smallest integer N such that $A_n(T)$ is not empty. It is clear that $\Gamma(\alpha)$ can have at most one point on T, and $S(\alpha) \cap T$ is a curve of F^* or is empty. If $\Gamma(\alpha) \cap T$ is the initial point of $\Gamma(\alpha)$ we say that $\Gamma(\alpha)$, or $S(\alpha)$, <u>begins at T</u>; in this case $C_\alpha = S(\alpha) \cap T$. When $\Gamma(\alpha) \cap T$ is a point of $\Gamma(\alpha)$ other than the initial point, then $\Gamma(\alpha)$, or $S(\alpha)$ is said to <u>straddle T</u>. In the former case $S(\alpha)$ lies in one domain of T, in the latter in two. Using these notations, we may state the following properties:

(1) If α,β are distinct elements of A with $\beta \varepsilon A(T)$, and α either an element of $A(T)$ or such that points of T lie on the boundary of $S(\alpha)$; then $S(\alpha)$, $S(\beta)$ cannot each have a point in the same complementary domain of T.

(2) If $A_N(T)$, $N = N(T)$, has one element α, then either $S(\alpha)$ straddles T, or if $S(\alpha)$ begins at T, then $C_\alpha{}^* \cap R^* = S(\alpha) \cap T$, i.e., $C_\alpha{}^*$ has just one curve in R*. (See T_2 in Figure 13 for $S(\alpha)$ straddling T_2.)

If $A_N(T)$ has more than one element, then every element of $A_N(T)$ is of the form β, k for fixed β of order N-1 and $C_{\beta,k}$ for $\beta,k \varepsilon A_N(T)$ are just those curves in R* of a maximal chain C*. (See Figure 13, the tree T_1.)

(3) Let γ be an element of $A_{N+k}(T)$, then every lower segment of γ of order $\geq N(T)$ is in A(T), i.e., for $0 \leq j \leq k$ we have $\gamma_{N+j} \varepsilon A_{N+j}(T)$.

(4) A necessary condition that $S(\alpha)$ straddle T is that $\alpha \varepsilon A_N(T)$ and is the only element of $A_N(T)$.

First we prove (1). Let $\mathcal{D}^*(C)$ be a complementary domain of T, bounded by C* on T. Suppose that $S(\alpha)$ and $S(\beta)$ both have points in $\mathcal{D}^*(C)$. Then there is a point p_1 on $\Gamma(\alpha)$, p_2 on $\Gamma(\beta)$, each in $\mathcal{D}^*(C)$. Now since $\beta \varepsilon A(T)$, $\Gamma(\beta)$ has a point q_2 on C* and p_2q_2, an arc on $\Gamma(\beta)$, lies in $\mathcal{D}^*(C) \cup C^*$. In either of the possibilities for α mentioned above, there would be a point q_1 on C* which was a limit point of points q_n in $S(\alpha)$. If $\alpha \varepsilon A(T)$ then q_1 may be taken on $\Gamma(\alpha)$, otherwise q_n will be in $\mathcal{D}^*(C)$, since $S(\alpha) \subset \mathcal{D}^*(C)$. It follows by arguments used many times above that there is a cross-section from q_1 on C* into $\mathcal{D}^*(C)$, which always may be shown to cross a curve also crossed by p_2q_2. This curve would have to be in both $S(\alpha)$ and $S(\beta)$ which is impossible since α,β were assumed distinct.

Lemma: If $\alpha \varepsilon A(T)$ and $\alpha, k \notin A(T)$, then no sequence γ of $A(T)$ can have α, k as a lower segment.

Proof: $C_{\alpha,k}$ lies on the boundary of $S(\alpha)$ but is not in T, nor is any curve of $S(\alpha,k)$ in T by hypothesis. Assuming $C_{\alpha,k}$ directed so that $S(\alpha,k) = \mathcal{D}^*(C_{\alpha,k})$, we have two possibilities: (1) the entire curve $C^{\#}_{\alpha,k}$ is on the boundary of $S(\alpha)$ and is all of this boundary on $T' = T_{C_{\alpha,k}}$, or (2) there exists $C = T' \cap S(\alpha)$ such that $C_{\alpha,k} \in \delta(C)$ where $\delta(C)$ is on the boundary of $S(\alpha)$ and is all of this boundary on T'. In case (1) every curve of $C^{\#}_{\alpha,k} \cap R^*$, being on the boundary of $S(\alpha)$, is a curve $C_{\alpha,k'}$ for some k'. We have $\mathcal{D}\#(C^{\#}_{\alpha,k}) \supset T$, since it contains $S(\alpha)$ which intersects T. In case (2) $\delta(C) = \delta(C+) \cup \delta(C-)$ divides π into three domains (or two if one of the sets $\delta(C\pm)$ is empty); one of these which we denote D_1 contains C and hence $S(\alpha)$ and T. The others contain all other curves of T'. $\delta(C)$ is the complete boundary in T' of $S(\alpha)$, hence every curve of $\delta(C) \cap R^*$ is a curve $C_{\alpha,k'}$ for some k'.

The remainder of the proof depends on the fact that $S(\beta) \cup S(\beta,k)$ is always a connected set. If there exists any sequence $\gamma = \alpha, k, n_1, \ldots n_r$ such that γ is in A(T) then, $S(\gamma)$ must clearly have points in $\mathcal{D}\#(C^{\#}_{\alpha,k})$ above in case (1) or in D_1 in case (2), these being the domains of T' in which T lies. Moreover, the set $\bigcup_{j=0}^{r} S(\alpha,k,n_1 \ldots k_j)$ is connected, and $S(\alpha,k)$ which is in this set lies in $\mathcal{D}^*(C_{\alpha,k})$ in case (1), and in D_2 or D_3 in case (2). Thus this set has points on either $C^{\#}_{\alpha,k}$ or $\delta(C)$, i.e., for $j \neq 0$ there is a curve of $C^{\#}_{\alpha,k}$ or $\delta(C)$ as the case may be in $S(\alpha,k,n_1, \ldots n_j)$. But each such curve as already pointed out is a curve $C_{\alpha,k'}$ which is a contradiction.

The lemma implies in particular, that if α and $\alpha, n_1, \ldots n_r \varepsilon A(T)$ then $\alpha, n_1, \ldots n_j \varepsilon A(T)$, $j \leq r$. Hence (3) will follow if we prove that every sequence of A(T) contains a lower segment in $A_N(T)$.

Now we turn to an examination of the possibilities for $A_N(T)$ and completion of the proof of (3). Suppose that α is an element of $A_N(T)$. Then either (i) $S(\alpha)$ straddles T, or (ii) begins at T. In the former case let $C = S(\alpha) \cap T$, then $\delta(C)$ is the complete boundary of $S(\alpha)$ in T and we know that every curve in R* of $\delta(C)$ is in the collection $\{C_{\alpha,k}\}$. Moreover, $\delta(C)$ divides π into three (or two) domains $D_1, D_2, (D_3)$ of which the first contains C_1, and of T, only the curve C. Now let γ be any sequence of A(T). $S(\gamma)$ must, by (1), lie in D_2 or D_3. But $\bigcup_{i=1}^{n} S(\gamma_i)$ is a connected set containing C_1 (i.e., $C_{\alpha=1}$), hence points of D_1 and also points of D_2 or D_3. It must then contain a curve $C_{\alpha,k}$ of $\delta(C)$, and therefore $S(\alpha)$, i.e., α is a lower segment of γ. Since this is only possible if γ is of order $\geq N$ we conclude α is the only element of $A_N(T)$.

In the case (ii) where $S(\alpha)$ begins at T, we have C_α on the boundary of $S(\beta)$, where β is of order N-1 and $\alpha = \beta, k$. In fact, $C_\alpha^\#$ is the complete boundary on T of $S(\beta)$ [the curves of $C_\alpha^\# \cap R^*$ are all of the set $C_{\beta,k}$, which therefore are in $A_N(T)$]; and we have $\mathcal{D}^\#(C_\alpha^\#) \supset S(\beta)$, $\mathcal{D}^*(C_\alpha^*) \supset S(\alpha)$. Now let us suppose that $\gamma \in A(T)$, then $S(\gamma)$ by (1) cannot lie in $\mathcal{D}^\#(C_\alpha^\#)$, hence must lie in $\mathcal{D}^*(C_\alpha^\#)$. But $\bigcup_{i=1}^{n} S(\gamma_i)$ is connected and has a point in $\mathcal{D}^\#(C_\alpha^\#)$, namely, any point of C_1. Thus this set has a point on $C_\alpha^\# \cap R^*$ and hence a curve $C_{\alpha,k'}$. It follows that every sequence of A(T) has a lower segment in $A_N(T)$. This proves (1) and completes the proof of (3).

To prove (4) we need show only that if $S(\alpha)$ straddles T then no lower segment of α is in A(T). If $\alpha = \beta, k$, so that β is the lower segment α_{n-1}, then if any lower segment of α is in A(T), β is also by our lemma. Then C_α, being on the boundary of $S(\beta)$, we necessarily have $S(\beta)$, $S(\alpha)$ in different domains of T_{C_2}. This is impossible unless $T = T_{C_2}$ for we would otherwise have points of T in two different domains of T_{C_2}.

As above we consider the branched regular curve family F with a regular curve C_1 of F and the decomposition of the corresponding $F^*[C_1 \cup \mathcal{D}^*(C_1)]$ into sets $S(\alpha)$ with initial curves $C(\alpha)$. Then we have the following:

<u>Theorem 5.2-5</u>: <u>The complete half-parallel subfamilies</u> $S^*(\alpha) = S(\alpha) \cup C^*_\alpha$ <u>decompose</u> $F[\mathcal{D}^*(C_1) \cup C_1]$ <u>into a family of half-parallel subsets which intersect only at points of their initial curves, i.e.,</u> $S^*(\alpha) \cap S^*(\beta) = C$ <u>where</u> $C^* = C^*_\alpha$ <u>and</u> $C\# = C^{\#}_\beta$.

<u>Proof</u>: First to prove that every curve of $F[C_1 \cup \mathcal{D}^*(C_1)]$ is included in this decomposition we note that every curve of $F^*[C_1 \cup \mathcal{D}^*(C_1)]$ is automatically included, being already in a set $S(\alpha)$ of the decomposition of that part of the simply connected region R^* included in $\mathcal{D}^*(C_1)$. We have only to consider curves of $\tilde{\mathcal{J}}$; let C be a curve of $F[C_1 \cup \mathcal{D}^*(C_1)]$ which is not in R^* and let T denote the tree which contains it. Then no cross-section $\Gamma(\gamma)$ has a point on C. C will be on the boundary of two distinct sets $S(\alpha)$ and $S(\beta)$ in $\mathcal{D}^*(C)$ and $\mathcal{D}\#(C)$ respectively. They cannot coincide since if they did then it would mean that $S(\alpha) = S(\beta)$ would straddle T, for otherwise the set $S(\alpha)$ lies in a single domain of T. Moreover, in this case, since $\Gamma(\alpha)$ would have to lie in two domains both having C as common boundary (and only C), it would have to contain a point of C, which is clearly impossible if C is not in R^*.

Now, if either α or β, say α, is of order $>N(T)$ then, since α, k for some k is in A(T), by (3), α must also be in A(T). Then by (4), C_α must lie on T, whence we have at once that $C^*_\alpha = C^* \supset C$ and hence C is in $S^*(\alpha)$. Thus it remains to show that either α or β must be of order $>N$. Assume α is of order $<N$, then by (1) all of C^* is on the boundary of $S(\alpha)$ and every curve of $C^* \cap R^*$ is in the set $\{C_{\alpha,k}\}$. Now, since β, k' for some k' is in A(T), β is of

order $\geq$N-1: If β is of order N-1, it must then be equal to α by (1); or if it is order N, then it is of the form α,k for some k. This latter would mean that the common boundary of the domains containing $S(\alpha)$ and $S(\beta)$ would be the curve $C_{\alpha,k}$ which must then coincide with the curve C, contrary to assumption that C is not in R*. Hence β in this case must be of order >N. On the other hand, if α is of order N, then either β is of order >N or C_α and C_β lie on the same maximal curve $C_\alpha^{\#} = C_\beta^{\#}$, and in this case quite clearly, C_α^* and C_β^* could not have a boundary curve C in common. Hence either α or β is of order >N and we have shown that in this event C is in either $S^*(\alpha)$ or $S^*(\beta)$.

Next it must be shown that if C_α is the initial curve of a set $S(\alpha)$, then for any $S(\beta)$ which intersects C_α^*, the intersection must be along C_β. Let C be the curve of intersection, i.e., $C = C_\alpha^* \cap S(\beta)$. Thus, α,β A(T) where T is the tree containing C_α. Now $S(\alpha)$ and $S(\beta)$ cannot have points in the same complementary domain of T, which means in particular that $S(\beta)$ cannot straddle T, since one complementary domain of C is $\mathcal{D}^*(C_\alpha^*)$. Hence $C_\beta = C$ which was to be proved.

<u>Corollary</u>: <u>The family F can be decomposed into complete half-parallel subfamilies which overlap only along their initial curves.</u>

<u>Proof</u>: We merely begin with any regular curve C_1 and decompose both $C_1 \cup \mathcal{D}^*(C_1)$ and $C_1 \cup \mathcal{D}^{\#}(C_1)$ as above.

6.0 THE FAMILY F AS THE LEVEL CURVES OF A HARMONIC FUNCTION

It is the purpose of this section to prove that corresponding to any branched, regular curve family F, there exists a harmonic function whose level curves form a family homeomorphic to F. This is a generalization of a similar theorem proved by W. Kaplan [V] for regular curve families filling π. The method here closely parallels that of [V]. A mapping T_1 from π to the w-plane is defined which carries the curves of F onto the lines u = constant. It is noted that T_1 is light and interior and hence topologically equivalent to an analytic function. This gives the desired theorem at once.

6.1 Complementary Curve Families

Given a branched regular curve family F filling π, we shall call another such family, G, filling π complementary to F if (1) the singularities of G are exactly those of F and each is of the same type, i.e., a point b is an n-th order branch point of G if and only if it is an n-th order branch point of F; and (2) every curve of G is a cross-section* of F. It follows at once from this definition and Theorem 3.2-4 that if G is complementary to F, then F is complementary to G. Hence we may speak of two complementary families, F and G, filling π. They will have a common set of singular points, B.

The major result of this section is to establish that every branched regular curve family F has a complementary family G. In [IV] it is shown that this, in effect, is true when B = 0, i.e., for any regular family filling π. This result immediately gives us a family G* complementary to F* in $R^* = \pi - \tilde{\mathcal{J}}$, for we may by [IV] map F* onto a family F' filling the xy-plane and defined by differential equations, $\frac{dx}{dt} = f(x,y)$; $\frac{dy}{dt} = g(x,y)$. The orthogonal trajectories define a family G' complementary to F' and the inverse image G* of G' is then the desired complementary family to F*. The method we shall use to establish

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* We must extend the definition of a cross-section slightly as follows: an open, or half-open arc is a cross-section if every closed sub-arc on it is a cross-section.

the existence of a family G, complementary to F will be to first consider F^* and its complementary family G^*, both defined in R^* and then to modify G^* slightly near the boundary of R^*, i.e., near the cuts $\lambda(b)$, so that it will become a family G of the desired type when $\tilde{\mathcal{J}}$, the boundary of R^*, is added to R^*. Theorem 4.1-3 tells us that we may cover $\tilde{\mathcal{J}}$ with a collection $\{V[\lambda(b)]\}$ of disjoint open sets; we shall assume such a covering, and moreover, assume that each $V \subset U_\epsilon[\lambda(b)]$ an ϵ-neighborhood of $\lambda(b)$ where $\epsilon > 0$ is fixed. Any modification in G^* will actually take place deep inside V, i.e., in an open set whose closure lies in V. We shall actually discuss the modification for one such V and, assuming similar modifications have taken place in each V, we will denote by $\tilde{G}^*$ the modified G^*. $\tilde{G}^*$ will be shown to be such that when is added to R^* $\tilde{G}^*$ becomes a set $\tilde{G}$ complementary to F. Several preliminary steps must be taken before the transition from G^* to $\tilde{G}^*$ can be adequately described.

First, we must define a <u>semi-r-neighborhood of a cut $\lambda(b)$</u>. We let C be that curve of $St(b) \cap R^*$ which is clockwise adjacent to the initial curve of $\lambda(b)$, i.e., $C \cup \lambda(b)$ is an adjacent chain; and we assume C directed so that $C \cup \lambda(b) \subset C^*$. Next, we let $\tilde{R}_1$ denote the rectangle R_1 without the corner point (1,0), i.e., $\tilde{R}_1 = \{(x,y) \mid 0 \leq y \leq 1, \ -1 \leq x \leq 1\} - \{(1,0)\}$, and F_1 denote the family of lines, y = a, filling R_1. Now let U be a set contained in $\mathcal{D}^*(C)$ $[\equiv \mathcal{D}^*(C^*)]$ together with a homeomorphism $k: \bar{U} \to \tilde{R}_1$ with the properties (1) $F[\bar{U}]$ is mapped homeomorphically by k onto F_1; (2) the inverse image of x = -1 is a cross-section, and the inverse image of the half-open segment consisting of that part of x = 1 in $\tilde{R}_1$ is a cross-section tending to infinity (but not properly) in one direction; and (3) k takes $\lambda(b)$ onto the right half of the x-axis in R_1 with k(b) = (0,0); an arc on C then maps onto the left half of the x-axis. Then we shall refer to that part of U which is mapped onto $\tilde{R}_1$, except for the edges $x = \pm 1$, $y = 1$, as a semi-r-neighborhood of $\lambda(b)$.

Theorem 6.1-1: If $V[\lambda(b)]$ is any open set containing $\lambda(b)$ and γ is any cross-section through $p \in C \cap V$, $C^* \supset C \cup \lambda(b)$ as above, then there is in $\mathcal{D}^*(C)$ a semi-r-neighborhood $U[\lambda(b)]$ with $\bar{U} \subset V$ and bounded on one side by γ (i.e., the image in $\tilde{R}_1$ of γ under k is $x = -1$).

Proof: The proof will consist of two parts, the first, part (A), being the choice of a set U, to be a candidate for the desired semi-r-neighborhood; and second, part (B), being the description of the homeomorphism from $\bar{U}$ to R_1. (See Figure 14.)

(A) We begin by choosing on C a regular point p_o, so chosen that it is inside V and is separated from b on C by p, $p = C \cap \gamma$. Next we choose a sequence $p_1, p_2, \ldots p_n \ldots$ of regular points on $\lambda(b)$ which approach infinity monotonely along $\lambda(b)$. Then $p_n p_{n+1}$ will denote the arc on C* joining these two points; and for each such pair $n \geq 0$, we choose a semi-r-neighborhood U_n, $\bar{U}_n \subset \mathcal{D}^*(C) \cap V$ and having the further property that $\bar{U}_n \subset U_{\epsilon_n}(p_n p_{n+1})$, an ϵ_n-neighborhood of the arc $p_n p_{n+1}$, where $\epsilon_n \to 0$. Moreover, we let U_n be chosen so that $U_{n-1} \cap U_{n+1} = 0$, and we shall assume that when we refer to the image of U_n in R_1 the homeomorphism will always be chosen so that the positive direction on the x-axis corresponds to the direction from b to infinity on $\lambda(b)$. Now let pq be any arc on γ which lies entirely in $\bar{U}_0(p_0 p_1)$; there must be such an arc since p lies between p_0 and p_1 and hence in $\bar{U}_0$, and γ is a cross-section through p. Consider for a moment the image in R_1 of $\bar{U}_0$, let $(\gamma(y), y)$ be the image of pq, defined for $0 \leq y \leq a < 1$ with $p \to (\gamma(0), 0)$ and $q \to (\gamma(a), a)$ and $C_q \to (y = a)$. The image of $\bar{U}_0 \cap \bar{U}_1$ will lie in the lower right hand corner of R_1, and we may find two points (x',a) and (x'',b), the second in the image of $\overline{U}_0 \cap \overline{U}_1$ and with $\gamma(a) < x'$, $a > b$ and so chosen that the points may be connected by a straight line (hence a cross-section of F_1) not intersecting the image of pq. We let q_0, q_1 denote the inverse images respectively of these two points and q_0q_1 the cross-section consisting of the inverse image of the line. Note that, by choice of q_0 as inverse image of (x',a), both q, q_0 lie on the same curve of F. If we now direct all curves C' crossing pq so that $\lambda^*(C) = \lambda^*(C')$, that is, so that $\lambda\#(C') \ni \lambda(b)$, then clearly $\lambda\#(C_q) \ni q_0q_1$ (except for $q_0 \varepsilon C_q$). Now in $\overline{U}_1 \cap \overline{U}_2$ we choose a point q_2 of $\lambda\#(C_{q_1})$ and connect q_1, q_2 by a cross-section lying in $\overline{U}_1$, which may be done again, by taking the inverse image of a straight line connecting their image points in the map of U_1 onto R_1. We repeat this process for all n, each time, however, choosing q_n as indicated but with the additional restriction that $t_n = \gamma \cap C_{q_n}$ is such that t_n approaches p. We thus obtain a sequence of arcs, q_0q_1, $q_0q_1q_2$, . . ., each of which is a cross-section by Theorem 3.5-2, hence they approach a half-open cross-section Γ tending from q_0 to infinity. Every curve crossing pq except C* will cross Γ since $t_n = \gamma \cap C_{q_n} \rightarrow p$ by our choice of q_n. Now the arc from p to infinity on C*, the cross-section pq on γ, the arc qq_0 on C_q and finally the arc Γ from q_0 to infinity form an arc extending to infinity in each direction and thus dividing π into two domains, one interior to $V[\lambda(b)]$. It is this latter domain that we denote by U; it will be our semi-r-neighborhood. It remains to find the map k from U to $\widetilde{R}_1$.

(B) We shall denote by S the collection of all curves of F crossing pq on γ, and by $\widetilde{S}$ the domain of $S - \gamma$ containing $\lambda(b)$ taken together with pq, its boundary in S. Now in Theorem 5.2-1, by use of the function defined by Whitney (Theorem 1.6-1) we were able to map all curves crossing an extended cross-section onto the lines $y = a$ of a half-plane. Hence it is obvious that in a similar manner we can map S by some homeomorphism k_1 onto the lines $y = a$ of the strip $0 \leq y \leq 1$ so that C* maps onto $y = 0$ and pq onto $x = -1$. k_1 then takes $\tilde{S}$ onto R_1'', that part of this strip to the left of $x = -1$, i.e., $R_1'' = \{(x,y) \mid 0 \leq y \leq +1, \ -1 \leq x < \infty\}$. The image of $\bar{U}$ under k_1 will be that portion of R'' bounded by (1) a segment on $y = 1$ joining $(-1,1)$, the image of q, to $(x',1)$, the image of q_0, plus (2) a curve φ_1 given by $(\varphi_1(y),y)$ which is the image of Γ, and hence a cross-section, together with (3) all of the x-axis in R_1'' and (4) the line $x = -1$.

Now let R_1' denote the rectangle R_1 without the line $x = 1$. Then k_2 defined by $k_2:(x,y) \rightarrow (\bar{x},\bar{y})$ where $\bar{x} \equiv x$ for $-1 \leq x \leq 0$, $\bar{x} = \frac{x}{x+1}$ for $0 \leq x \leq 1$ and $\bar{y} \equiv y$, is a homeomorphism from R_1'' onto R_1' holding all of R_1'' to the left of the x-axis fixed, and shrinking each curve $y = a$, along itself to the right of the y-axis. φ_1 goes into a curve φ_2 given by $x = \varphi_2(y)$, where $\lim_{y \to 0} \varphi_2(y) = 1$. The closure of this half-open arc connects a point $(\varphi(1),1)$ on the top edge of R_1' to $(1,0)$, the lower right hand corner, and thus splits R_1' into two domains, the one of which lying to the left of this arc is the image of U under the combined homeomorphisms k_2k_1. This portion of R_1' is then mapped onto $\tilde{R}_1$ by a third homeomorphism k_3 defined as follows $k_3:(\bar{x},\bar{y}) \rightarrow (\bar{\bar{x}},\bar{\bar{y}})$ where $\bar{\bar{x}} = \frac{2+2\bar{x}}{1+\varphi_2(\bar{y})} - 1$, $\bar{\bar{y}} \equiv \bar{y}$. k_3 holds the lines $x = -1$ and $y = 0$ fixed, takes each line $y = a$ along itself and maps φ_2 onto the line φ_3 whose equation is $x = +1$. Hence, $k = k_3k_2k_1$ is a homeomorphism of $\bar{U}$ onto R_1 with the desired properties, and U is a semi-r-neighborhood of $\lambda(b)$ in the sense of our definition.

Since nothing in the above proof depended on the fact that b was the initial point of a cut, we can state the following corollary to the proof above:

Corollary: Let $\lambda(b)$ be a cut with a finite number of curves, the last of which begins at the branch point b' and extends to infinity, and let this last curve be denoted by C_0 and the curve counterclockwise adjacent to it by C_1 (i.e., so that with C_1 properly directed C_0, C_1 form an adjacent chain lying on $C_1^{\#}$). If γ is a cross-section through $p \varepsilon C_1$, then there is, inside any open set $V(\lambda(b))$ containing p, a semi-r-neighborhood U of C_0 with the cross-section γ as one boundary curve.

We shall call a semi-r-neighborhood of type II any which extends thus to infinity along a cut; the earlier defined semi-r-neighborhood (of a finite arc) will then be of type I.

We now proceed to define for each $\lambda(b)$ a certain possibly infinite collection of closed sets $W_0, W_1, \ldots$ all contained inside $V(\lambda(b))$. These are the sets in which G* will be modified. W_0 is the closure of a semi-r-neighborhood of type II, and if the number of curves in $\lambda(b)$ is finite, then there will be a last set W_N of this collection which is also the closure of a neighborhood of type II. All the other sets W_i will be closures of neighborhoods of type I. These sets will be chosen as follows: First, let $b_0 \equiv b$, $b_1, b_2, \ldots$ be the branch points on $\lambda(b)$ and let the curves in R* of each $St(b_i)$ be numbered with two indices, the first being that of b_i, the second being given by a counterclockwise numbering of the $St(b_i)$ proceeding from the first curve to follow counterclockwise after a curve of $St(b_i) \cap \lambda(b)$ to the last to procede a curve of $St(b_i) \cap \lambda(b)$ in the counterclockwise ordering: $C = C_{00}$, $C_{01}, \ldots C_{0n_1}$; $C_{11}, C_{12}, \ldots, C_{1n_2}$; ... etc. (See Figure 15.) Second, choose regular points s_{ij} on each C_{ij} and short cross-sections γ_{ij} through s_{ij}, the γ_{ij} being in each case an arc on a curve of G* and both s_{ij} and γ_{ij} being chosen so as to lie in $V(\lambda)$. Now we choose our sets W_n as follows: $W_0 \subset V(\lambda)$ is the closure of a semi-r-neighborhood of type II, bounded on one side by an arc $r_{00}s_{00}$ on γ_{00} and on one side, of course, by $(s_{00}b_0) \cup \lambda(b)$. Next, in the domain bounded by (the maximal chain of) the adjacent curves C_{00}, C_{01} we choose a semi-r-neighborhood of type I of the arc $s_{00}s_{01}$ on these curves, which is bounded by the arcs $s_{00}t_{00}$ on γ_{00} and $r_{01}s_{01}$ on γ_{01} and whose closure lies in $V(\lambda)$ and will be our W_1. Similarly, we choose $W_2, \ldots W_{n_1-1}$, each a closure of a type I neighborhood in $V(\lambda)$ and bounded by arcs on some γ_{0i}. It may be that b_0 is the only branch point of $\lambda(b)$, in which case the next set W_{n_1} is the last and must be of type II, bounded on one side by an arc $s_{0n_1}t_{0n_1}$ on γ_{0n_1}. Otherwise, we choose for W_n a semi-r-neighborhood of type I of $s_{0n_1}b_0b_1s_{11}$, an arc on the adjacent chain C_{1n_1}, C', C_{11} (C' being the curve of $\lambda(b)$ with endpoints b_0, b_1), the neighborhood being so chosen that its ends are arcs $s_{0n_1}t_{0n_1}$ and $r_{11}s_{11}$ on γ_{0n_1} and γ_{11} respectively, and that it lies in $V(\lambda)$. This process is continued until we have chosen semi-r-neighborhoods on both sides of every curve of $St(b_i)$ in R* for all b_i and on both sides of each curve of $\lambda(b)$. Then $\lambda(b)$ will be contained in the interior of the set $W = \bigcup_i W_i$. W is bounded by an open arc Γ extending to infinity in each direction; and Γ consists either of <u>one</u> infinite cross-section of F*, not in general a curve of G*, plus an infinite number of arcs alternately on curves of F* and on curves of G* (the latter of the form $r_{ij}s_{ij}t_{ij} \subset \gamma_{ij}$); or else Γ consists of a finite number of such alternate arcs on F* and G* plus <u>two</u> half-open cross-sections of F* extending to infinity. The first case occurs when the number of neighborhoods of type II is one, the second when it is two. Γ lies entirely inside $V(\lambda)$ and W, which consists of Γ plus that one of its complementary domains inside $V(\lambda)$, is a closed set. The W_i's clearly intersect on curves of F, namely on $\lambda(b)$ plus arcs b_is_{ij} on each curve in R* of every $St(b_i)$ for b_i in $\lambda(b)$. We denote by $\bar{\lambda}$ the set of all points which lie on the common boundary of two or more W_i's. A point of $\overline{\lambda}$ which is a regular point clearly lies on the intersection of just two such sets, whereas each branch point b_i lies on the intersection of 2m, where m is the multiplicity of b_i. We denote by W_i^* the set $W_i - \overline{\lambda}$ and by W^* the set $W - \overline{\lambda}$, and finally by V^* the set $V(\lambda) - \overline{\lambda}$. Then let $\overline{G}^* = G^*\left[V^*\right]$ and $\overline{F}^* = F^*\left[V^*\right]$.

Now each W_n has associated with it a homeomorphism k_n, of W_n onto R_1 if it is of type I, and onto $\widetilde{R}_1$ if it is of type II. In order that the modification of G^* to $\widetilde{G}^*$ which we are going to make will not destroy the relationship between G^* and F^* we will actually achieve it by a homeomorphism h of $\overline{R}^*$ ($\overline{R}^* = R^* - \overline{\lambda}$) onto itself which is the identity outside of each set W, but which inside such a set carries each curve of $\overline{F}^*$ onto itself, i.e., it may be visualized as "sliding" the points of a curve of $\overline{G}^*$, along the curves of $\overline{F}^*$ to which they belong, to their new position. Actually, we shall describe this operation piecewise, for each W_n^* and, in fact, as a homeomorphism on the image curves in R_1 (or $\widetilde{R}_1$ as the case may be).

We begin by defining a typical homeomorphism f_I on the image of $\overline{F}^*\left[W_i^*\right]$, $\overline{G}^*\left[W_i^*\right]$ under k_i for W_i of type I (see Figure 16a). The image will be $R_1^* = R_1 -$ (x-axis), and we denote the images of the curve families as F_1^*, G_1^*, respectively. The former will, of course, be just the lines $y = a$, $0 \leq a \leq 1$, the latter being a regular curve family filling R_1^*, complementary to F_1^*, and having among its curves the two lines $x = \pm 1$, images of arcs on two of the curves γ_{ij} of G^*. It will be seen that G_1^* consists exactly of the curves whose inverse images cross C', the inverse image of $y = 1$ in R_1^*, for, if we consider any curve of $\overline{G}^*$ with a point inside W_i, it is clear that it must leave W_i in each direction, there being no branch points interior to W_i; and hence, it must either cross C' or have two endpoints on $\overline{\lambda}(b)$. It could scarcely have both endpoints on $\bar{\lambda}(b)$, however, without crossing some curve of F^* twice inside W_i, which is impossible. Moreover, no curve of G^* will cross C' more than once, since C' is a cross-section of G^*. Thus we may define a function F_I mapping R_1^* onto itself as follows: Let $\bar{x} = f(x,y)$ be defined by $f(x,1) \equiv x$ and $f(x,y) =$ constant on each curve of G_1^*, and let $\bar{y} = g(x,y)$ be defined by $g(x,y) \equiv y$. Then it follows from the above remarks and the work of Kaplan [IV] and [VIII] that this is a homeomorphism of R_1^* onto itself which takes each curve of F_1^* onto itself and each curve of G_1^* onto a line $x = b$, $-1 \leq b \leq 1$, the lines $x = \pm 1$ being held pointwise fixed, as is the line $y = 1$, i.e., all of the boundary of R_1^* on which f_I is defined is held pointwise fixed. $h|W_i^*$ is then defined by $k_i^{-1} f_I k_i$, and if thus defined h maps $\bar{F}^*[W_i^*]$ onto itself, takes $\bar{G}^*[W^*]$ homeomorphically onto a new family $\tilde{G}^*[W_i^*]$ which is still complementary to F^* and which is identical to $\bar{G}^*$ on the boundary of W_i^*. Since k_i is actually a homeomorphism of all of W_i onto R_1, it will now map $F[W_i]$ and $\tilde{G}^*[W_i]$ so that the curves $F^*[W_i]$, $\tilde{G}^*[W_i]$ will map onto the lines $y = a$ and $x = b$, respectively. We re-denote k_i by $\tilde{k}_i$ to emphasize that it acts on $\tilde{G}^*$. Thus it is clear that every curve of $\tilde{G}^*[W_i^*]$ has exactly one endpoint, unique to it, on $\bar{\lambda}$ and exactly one endpoint unique to it on the curve of F^* forming the opposite side of W_i. The regularity of G^* which we have achieved at $\bar{\lambda}$ is precisely what is needed. We assume a similar homeomorphism defined for every index i such that W_i is of type I; then h will be defined on every set of W except the one or two neighborhoods of type II.

Now let us suppose that we are dealing with a neighborhood of type II, say W_0, with its associated homeomorphism k_0 onto $\tilde{R}_1$. Again let F_1^*, G_1^* denote the images of the respective families of $\bar{W}_0$ in $\tilde{R}_1^* = R_1 -$ (x-axis), the former being the lines $y = a$, and the line $x = -1$ being a curve of the latter, but not in general the line $x = +1$. f_{II} will be given as the composition of four homeomorphisms of $\tilde{R}_1^*$ onto itself (see Figure 16b). Before we can describe f_1, the first of these, we must note that there is in W_0 at least one curve Ψ of G*, distinct from the arc $r_{00}s_{00}$ on γ_{00} (inverse image of $x = -1$), whose image Ψ_1 in $\tilde{R}_1$ joins a point $(x'',0)$ to a point $(x',1)$, where $-1 < x''$, $x' < 0$, i.e., a curve of G* joining one side of W_0 to the other, and intersecting each at a regular point of R*, i.e., not on $\lambda(b)$. That such a curve exists follows from the fact that in the family G*, regular in R*, the arc $r_{00}s_{00}$ on a curve of G* has an r-neighborhood U (by Theorem 1.2-2) with $\bar{U} \subset R^*$. The curves $C_{s_{00}}$ and $C_{t_{00}}$ have small arcs entirely in this neighborhood, since they are cross-sections of G*, and each of these will be crossed by an infinite number of curves of G* on each side of $s_{00}t_{00}$, one of which will serve our purpose; namely, one crossing that part of each of these arcs which is the inverse image of the segments $(-1,1)$ to $(0,1)$ and $(-1,0)$ to $(-\epsilon,0)$, $1 > \epsilon > 0$. Ψ_1 will be given by a continuous function $x = \Psi_1(y)$, $0 \leq y \leq 1$, and we shall use it to define $f_1 : R_1^* \to R_1^*$ given by $f_1 : (x,y) \to (\bar{x},\bar{y})$ where:

$$\bar{x} = \frac{[1 + \Psi_2(y)]\,x - [\Psi_1(y) - \Psi_2(y)]}{1 + \Psi_1(y)} \quad \text{for } -1 \leq x \leq \Psi_1(y);$$

$$\bar{x} = \frac{[1 - \Psi_2(y)]\,x - [\Psi_1(y) - \Psi_2(y)]}{1 - \Psi_1(y)} \quad \text{for } \Psi_1(y) \leq x \leq +1$$

$$\bar{y} \equiv y$$

(where we have $\Psi_2(y) = (x'-x'')y + x''$, this being the equation of the line joining $(x',1)$ to $(x'',0)$, the curve into which Ψ_1 is mapped by f_1).

The next homeomorphism, $f_2 : R_1^* \to R_1^*$ will carry Ψ_2 into Ψ_3, the line $x = x'$. f_1 is given by $f_2 : (x,y) \to (\bar{x},\bar{y})$ where:

$$\bar{x} = \frac{(1 + x')x + [x' - \varphi_2(y)]}{1 + \varphi_2(y)} \quad \text{for } -1 \leq x \leq \varphi_2(y);$$

$$\bar{x} = \frac{(1 - x')x + [x' - \varphi_2(y)]}{1 - \varphi_2(y)} \quad \text{for } \varphi_2(y) \leq x \leq 1$$

$$\bar{y} \equiv y$$

Each of these homeomorphisms holds the boundary curves $x = \pm 1$, $y = 1$ pointwise fixed. To describe f_3 we first denote by M that portion of $\tilde{R}_1^*$ which lies on or to the left of φ_3, i.e., $M = \{(x,y) \mid -1 \leq x \leq x', 0 \leq y \leq 1\}$. M is bounded on each side by a line x = constant which is the image of a curve of G* under the composition of the above maps, and bounded on top and bottom by an image of a curve of F*. The image of F* in M is the family of lines y = a. Hence by precisely the same argument as in the definition of f_I for the neighborhood of type I above, we may find a homeomorphism $f_3: M \rightarrow M$ which holds the boundary of M pointwise fixed, takes each curve y = a onto itself, and takes the image family of G* onto the lines $x = b$, $-1 \leq b \leq x'$. We extend f_3 to all of $\tilde{R}_1^*$ by defining it as the identity on the rest of this set. Again, f_3 will be a homeomorphism leaving the boundary curves $x = \pm 1$, $y = 1$ pointwise fixed, as well as the curve φ_3 and all of $\tilde{R}_1^*$ to the right of φ_3.

Finally, we define a homeomorphism $f_4: R_1^* \rightarrow R_1^*$, again by giving $f: (x,y) \rightarrow (\bar{x}, \bar{y})$ as follows:

$$\bar{x} = (\varphi_4(y) + 1)\left(\frac{x + 1}{x' + 1}\right) - 1 \quad \text{for } -1 \leq x \leq x'$$

$$\bar{x} = (1 - \varphi_4(y))\left(\frac{x - x'}{1 - x'}\right) + \varphi_4 \quad \text{for } x' \leq x \leq +1$$

$$\bar{y} \equiv y$$

where ψ_4 denotes the line $x = \psi_4(y) = (x' - 1)y + 1$ joining $(x',1)$ to $(1,0)$, this being the image of ψ_3 under f_4. The image of M under f_4 will be denoted by M_1 and will be the trapezoid bounded by ψ_4, the x-axis, the line $x = -1$, and the segment from $(-1,1)$ to $(x',1)$ on the line $y = 1$. f_4 takes the lines $y = a$ onto themselves and the lines $x = b$, $-1 \leq b \leq x'$ of M onto a family of non-intersecting straight lines joining the points of the top edge of M_1 to the bottom (as listed above). f_4 leaves the lines $x = \pm 1$ and $y = 1$ pointwise fixed.

Now we define $f_{II}: R_1^* \rightarrow R_1^*$ as the homeomorphism $f_4 f_3 f_2 f_1$, and we define $h|W_0^*$ as $k_0^{-1} f_{II} k_0$. Then $h|W_0^*$ is a homeomorphism of $W_0^* = W_0 - \bar{\lambda}$ onto itself which is pointwise fixed on the boundary of W_0^* in $\bar{R}^*$, i.e., on $t_{00}s_{00}$, on $C_{t_{00}}$, and on the extended cross-section which bounds one side of W_0. h also takes the curves of $G^*[W_0^*]$ homeomorphically onto a family $\tilde{G}^*$, at the same time mapping each curve of F^* onto itself. Now, if as above for k_1, we re-denote k_0 by $\tilde{k}_0$, then we have a homeomorphism of all of W_0 onto $\tilde{R}_1$ which takes $\bar{\lambda}$ onto the x-axis between $(-1,0)$ and $(1,0)$, with b_0 mapping onto $(0,0)$, and s_{00} onto $(-1,0)$, and which moreover, takes the curves of F onto the lines $y = a$ and takes part of $\tilde{G}^*$ onto the straight lines joining the top and bottom of M_1 as described above, the remainder of $\tilde{G}^*$ mapping onto a regular family filling the rest of R_1. The curve $\tilde{\tau}$ of $\tilde{G}^*$, image of τ under $h|W_0^*$ divides W_0 into two domains, one of which maps onto M_1, the other onto $R_1 - M_1$. We shall denote the one which maps onto M_1, together with its boundary, by $\tilde{W}_0$, the boundary consisting of two curves of $\tilde{G}^*$, namely, $r_{00}s_{00}$ and $\tilde{\tau}$, together with $C_{r_{00}}$ and $s_{00}b_0 \cup \lambda(b)$ in F. It is obvious that M_1 in $\tilde{R}_1$ can be mapped onto $\tilde{R}_1$ by a homeomorphism g which holds $x = -1$ and $y = 0$ pointwise fixed, takes each line $y = a$ into itself, and finally moves the image curves of $\tilde{G}^*$ in M_1 onto the lines $x = b$, $-1 \leq b \leq 1$, keeping, of course, their lower endpoints fixed, thus taking the line $\tilde{\tau}$ onto

$x = 1$. Then $g\tilde{k}_0 : \tilde{W}_0 \to \tilde{R}_1$ with F going onto the lines y = constant and $\tilde{G}^*$ onto the lines x = constant. $\tilde{W}_0$ is then again, like W_0, a semi-r-neighborhood of $\lambda(b)$, hence of type II, but of a kind which is bounded by curves of two complementary families and has associated a homeomorphism $g\tilde{k}_0$ which maps the curves of the respective families onto the lines parallel to the axes on $\tilde{R}_1$. Hereafter, we shall denote $g\tilde{k}_0$ merely by $\tilde{k}_0$. Note that this is similar to the case when we had a semi-r-neighborhood of type I.

Now if W_N is a second neighborhood of type II in W, then it must be the last W_i defined for $\lambda(b)$ and on it we define, in a manner entirely parallel to the above discussion, f_{II}, $h|W_N^*$, $\tilde{W}_N$, $\tilde{k}_N$, etc. Thus we have defined $h|W_i^*$ for all i, and since the W_i are overlapping closed sets of V* (with only a finite number containing any given point) such that h is actually the identity along their overlapping boundaries as well as on Γ, the boundary of W, we have defined a homeomorphism h of W* onto itself ($W^* = W - \bar{\lambda}$). Assume that h is similarly defined for a set $W_\lambda^* \subset V[\lambda(b)]$ for <u>every</u> cut $\lambda(b)$ contained in $\tilde{\mathcal{J}}$, and we define h as the identity outside the W_λ's. We remark that the collection of all the sets W_λ for $\lambda(b)$ in $\tilde{\mathcal{J}}$, together with the set $\pi - \bigcup_{\lambda \in \tilde{\mathcal{J}}} W_\lambda$, is a collection of overlapping closed sets which has a locally finite character, i.e., every neighborhood of any point meets only a finite number of the closed sets. This is clear because the cuts, λ, recede to infinity, and each W_λ lies in an ϵ-neighborhood of the cut λ, $\epsilon > 0$ being fixed. Then it follows that h is a homeomorphism of $\bar{R}^*$ onto itself, where by $\bar{R}^*$ we mean $R^* - \left[\bigcup_{\lambda \in \tilde{\mathcal{J}}} \lambda(b)\right]$. h carries every curve of F* onto itself homeomorphically, and every curve of $G^*[R^*]$ homeomorphically onto a family $\tilde{G}^*$ which is complementary to F* in $\bar{R}^*$ and which coincides with G* except in the interior of the W_λ's.

It remains to prove that by adding the boundary points of $\bar{R}^*$, i.e., $\pi - \bar{R}^*$, the curves of $\tilde{G}^*$ become curves of a family $\tilde{G}$ complementary to F in π. To prove this we must first prove that $\tilde{G}$ is regular in $R = \pi - B$. Now if p is a point of $\bar{R}^*$, this is clear, since $\tilde{G} = \tilde{G}^*$ (which is homeomorphic to $\bar{G}^*$) in some neighborhood of p. In fact, it is clear that there is an arbitrarily small r-neighborhood of p whose closure maps onto $R_0 = \left\{(x,y) \mid |x| \leq 1, \ |y| \leq 1\right\}$ so that the lines x = constant are the images of the curves of $\tilde{G}$, those lines y = constant the image curves of F.

Now, however, suppose that p is a regular point on $\overline{\lambda(b)}$. Then p will be on the common boundary of just two of the neighborhoods W_i, since p is not a branch point. Let W_n, W_m be the two neighborhoods. Then p is interior to $W_n \cup W_m$, and it follows from Theorem 1.2-3 that $\tilde{G}\left[W_n \cup W_m\right]$ is regular at p, since $\tilde{G}$ is regular in W_n and in W_m separately, as may be seen from the existence of the maps $\tilde{k}_n, \tilde{k}_m$ onto R_1 (or $\tilde{R}_1$ as the case may be) with $\tilde{G}$ mapping onto the lines x = constant. It follows that $\tilde{G}$ is regular at every point of R, so that the singularities of $\tilde{G}$ are contained in the set B of singularities of F, and are thus isolated. Now each branch point is in a cut, and hence will be $b_i \varepsilon \lambda(b)$ for some i and some $\lambda(b)$. b_i is on the common boundary of just 2m sets W_n, where m is the multiplicity of b_i. Then it is clear that there are just exactly 2m curves of $\tilde{G}[W]$, one in each of these sets which have b_i as a limit point in one direction. Thus, if W_n has b_i on its boundary, then in the homeomorphism $\tilde{k}_n : W_n \rightarrow R_1$ the point b_i will map onto a point (a,0) and the inverse image of the line x = a is the single curve of $\tilde{G}\left[W_n\right]$ which has b_i as a limit point. It follows at once that b_i is a branch point of multiplicity 2m of $\tilde{G}$. Hence we have established that $\tilde{G}$ is a branched regular curve family with the same branch points as F. Again, just as above, it is clear that it is possible to find an arbitrarily small neighborhood U of each b_i which is homeomorphic to $|z|<1$, and moreover, with a homeomorphism k carrying $F[\overline{U}]$ onto the level curves of $\mathcal{R}(z^m)$ and $G[\overline{U}]$ onto the level curves of $\mathcal{I}(z^m)$.

Finally, to complete the proof that G is complementary to F, we note that by Corollary 2 to Theorem 3.5-3 we have at once that every curve of G is a cross-section of F. This completes the proof of the following:

Theorem 6.1-2: Every branched regular curve family F has at least one complementary family G as described above.

6.2 The Fundamental Theorem

Given any branched regular curve family F on π, we have shown the existence of a complementary family G; and also, we have shown that each of these families is the level curve family of a continuous function f(p) and g(p) respectively. This enables us to define a single-valued mapping T_1 from the plane π to the complex w-plane as follows: $T_1(p) = u + iv$ where $u = f(p)$ and $v = g(p)$. $T_1(p)$ is clearly continuous, because f and g are continuous. Moreover, T_1 is locally a homeomorphism on R and is at most m-to-1 in the neighborhood of an m-th-order branch point. To show this, it is sufficient to consider the special neighborhoods mentioned in the proof of the previous theorem, i.e., for every regular point we consider only a neighborhood U such that there is a homeomorphism of U onto the rectangle R_1 of the xy-plane such that $F[U]$ goes onto the lines y = constant and $G[U]$ onto the lines x = constant. Then T_1 becomes a map of R_1 onto a rectangle in the uv-plane carrying the lines y = constant onto u = constant and x = constant onto v = constant. It is clearly a homeomorphism since it is monotone on each line x = constant and each line y = constant. This is exactly as in [VIII]. It is equally easy to show that in a neighborhood V of a branch point, where $F[V]$ and $G[V]$ map onto $\mathcal{R}(z^m)$ and

$\mathscr{L}(z^m)$ respectively under a homeomorphism of V onto $|z|<1$, T_1 carries V onto an open set and is at most m-to-1, where m is the multiplicity of the branch point. It follows that T_1 is not only interior but light (since for every point there is a neighborhood in which f and g take on the same value only a finite number of times in the neighborhood). It follows from Stoilow [XIII] and Whyburn [XVI] that T_1 is topologically equivalent to an analytic function $W = \varphi(z)$, i.e., there exists a homeomorphism $p = h(z)$ of the plane π onto either the domain $D_1 = \{z \mid |z|<1\}$ or $D_\infty = \{z \mid |z|<\infty\}$ of the z-plane such that $\varphi(z) = T_1[h(z)]$ is analytic. The family F' of level curves of the real part of $\varphi(z)$ are just those curves mapping onto the lines u = constant of the w-plane and hence are homeomorphic to F under h. It is thus proved that:

Theorem 6.2-1: Given any branched regular curve family F there exists a function harmonic in either the finite plane or the unit circle whose level curves are homeomorphic to F.

Since if the function $u(x,y)$ is harmonic in a domain D, its level curves satisfy the differential equations $\frac{dx}{dt} = u_y$, $\frac{dy}{dt} = -u_x$ we have at once:

Theorem 6.2-2: Given any branched regular curve family F, then there is a solution family of a system of differential equations to which it is homeomorphic.

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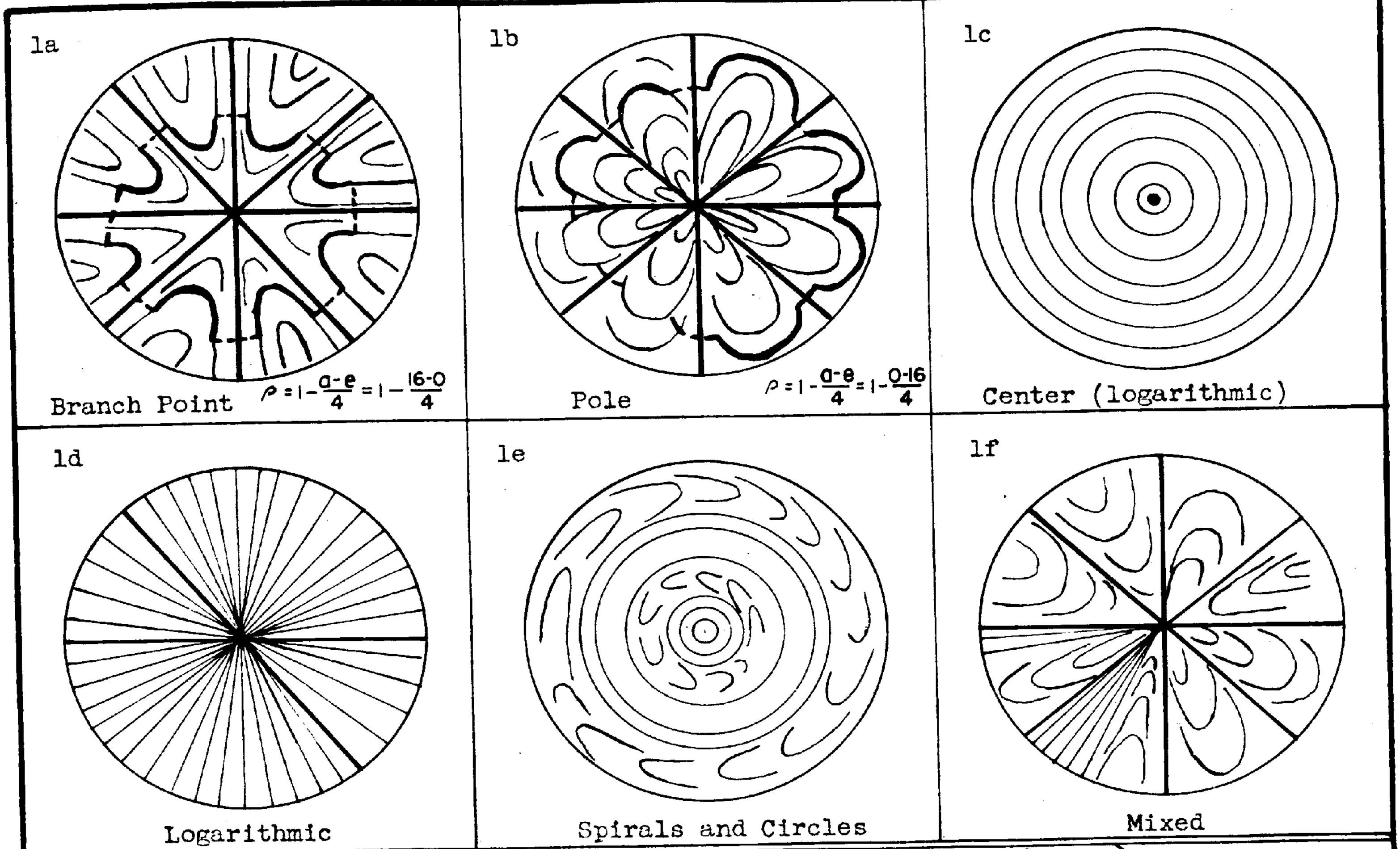
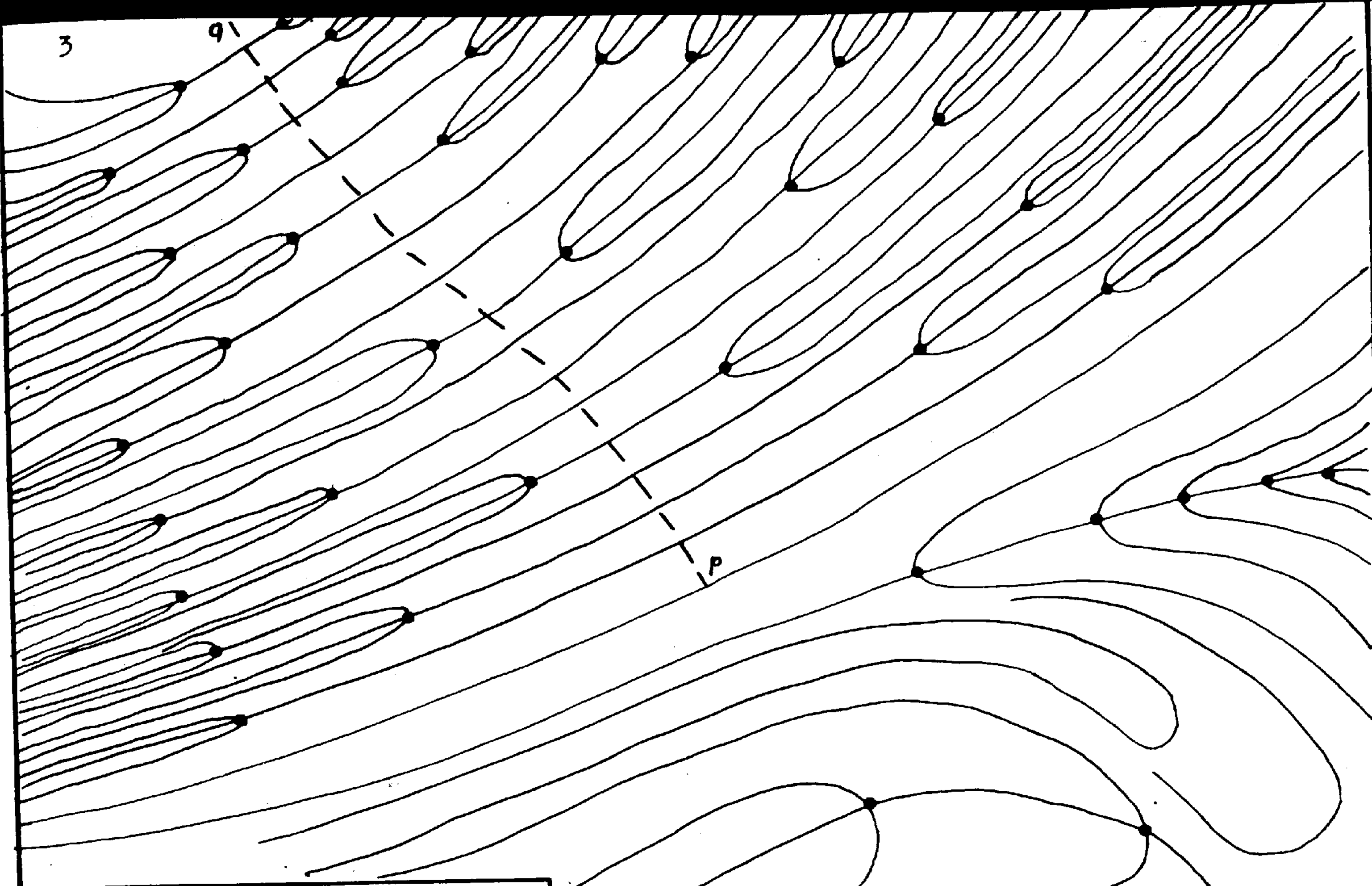

1a
Branch Point
$\rho = 1 - \frac{a-e}{4} = 1 - \frac{16-0}{4}$
1b
Pole
$\rho = 1 - \frac{a-e}{4} = 1 - \frac{0-16}{4}$
1c
Center (logarithmic)
1d
Logarithmic
1e
Spirals and Circles
1f
Mixed

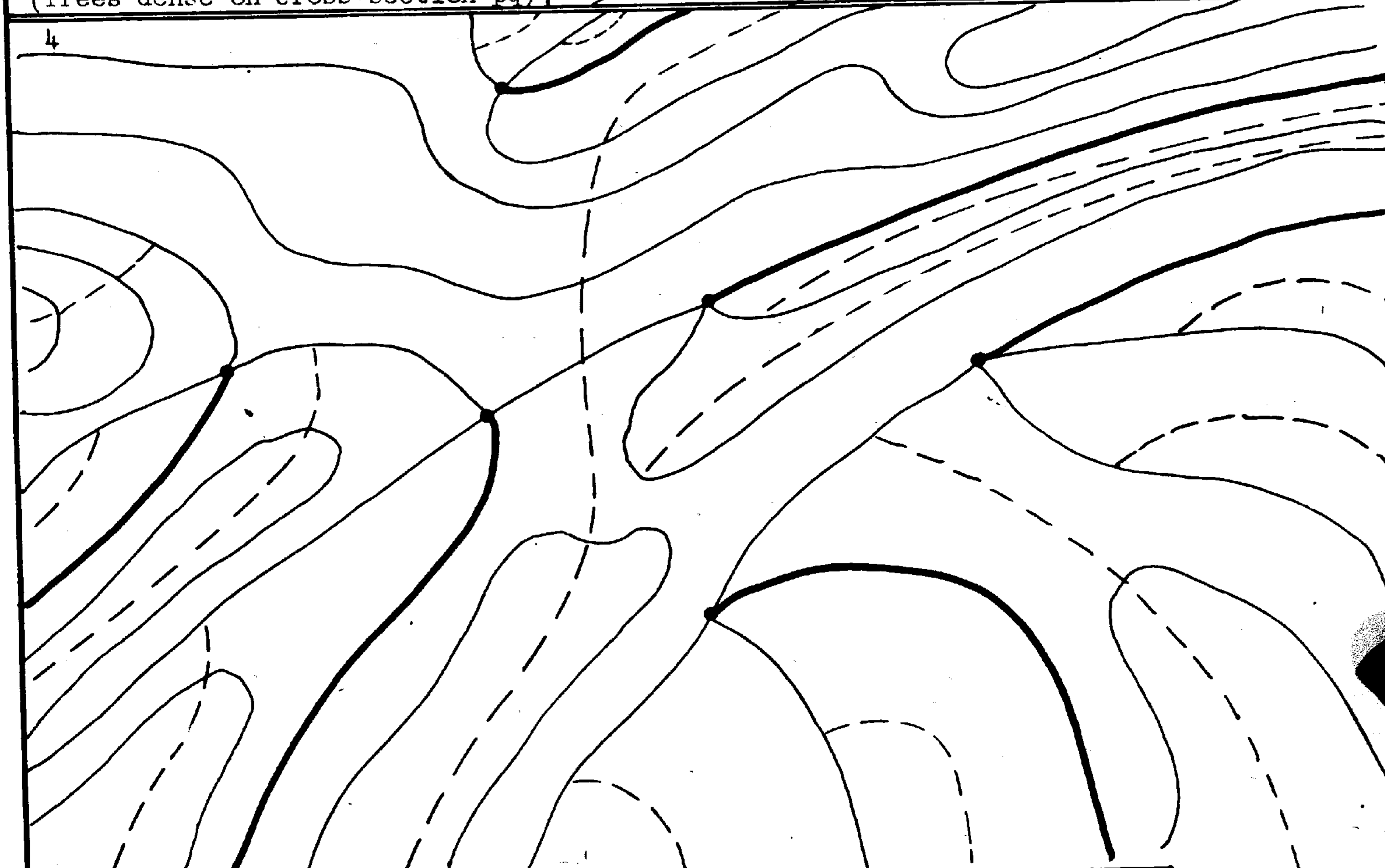

2
Regular curve family
with isolated singularities

3



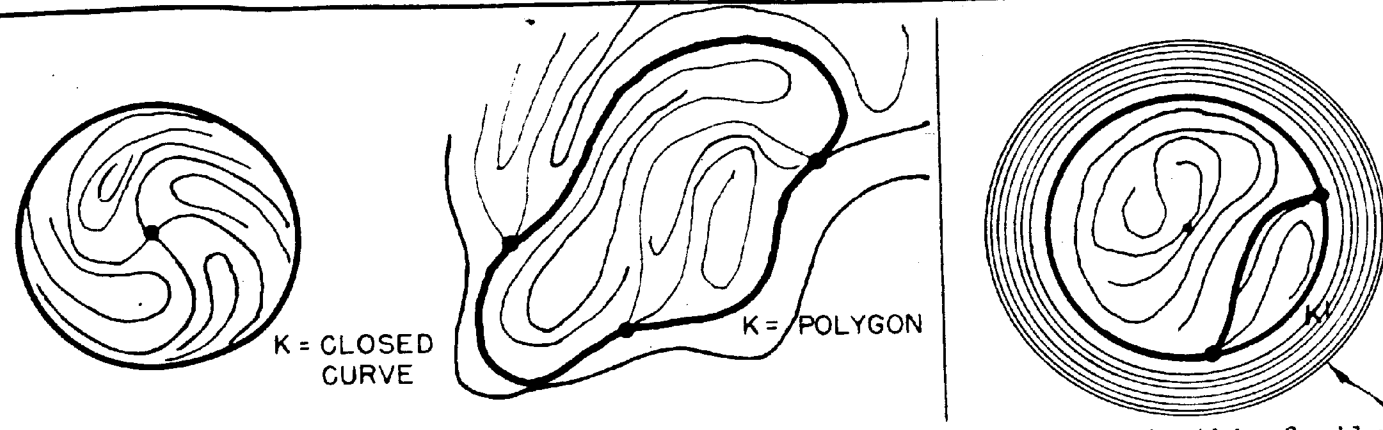
Branched Regular Curve Family
(Trees dense on cross-section pq)

4



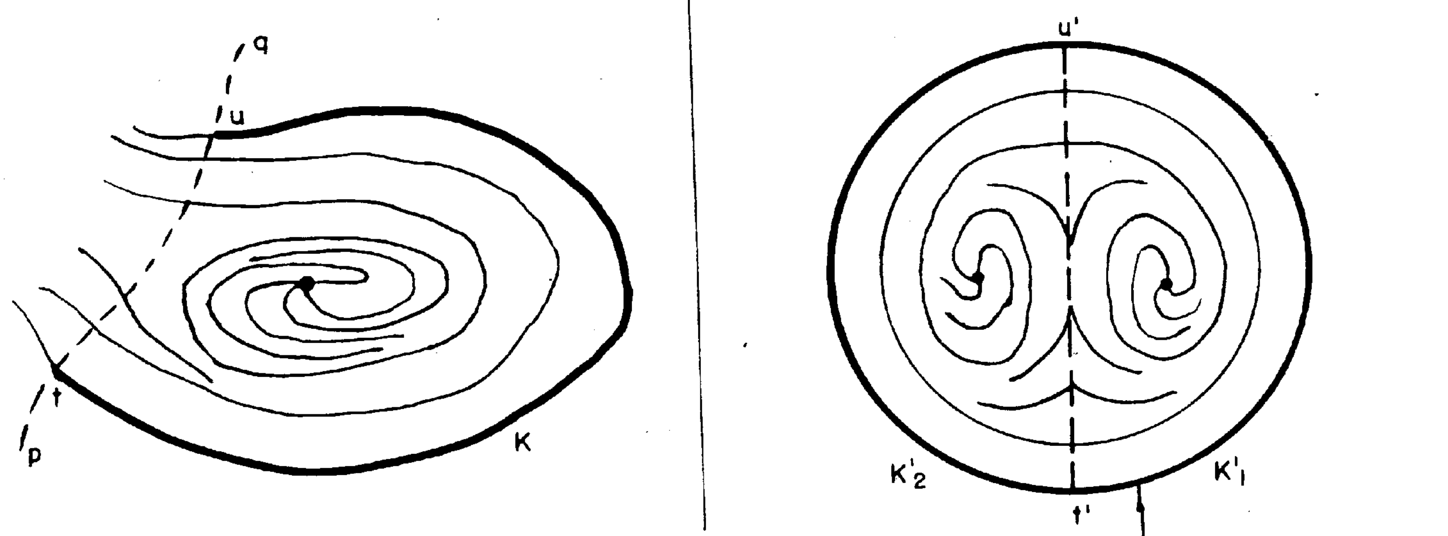
Decomposition of a branched family by cross-sections (dotted lines)
Cuts shown with heavy lines.

5



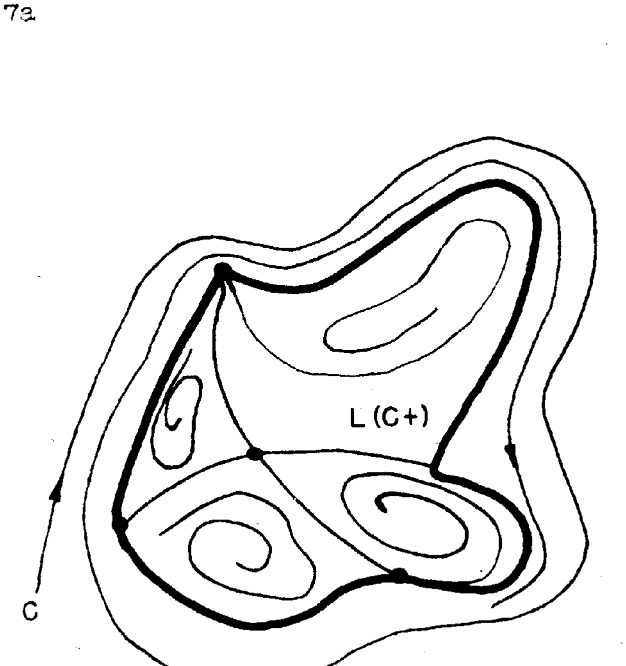
Theorem 3.21. rules out configurations like these two by imbedding in this family

6



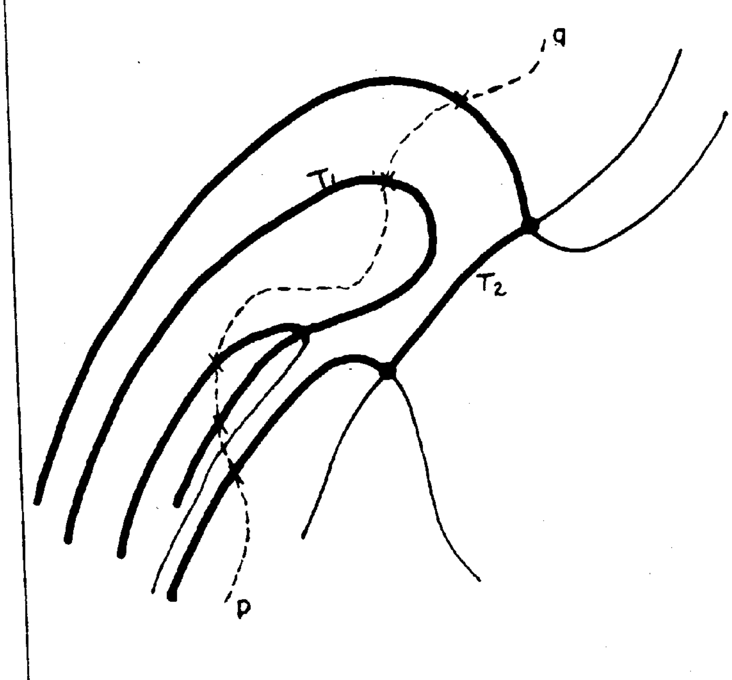
Theorem 3.2-2 rules out bays by imbedding in a family of this type

7a

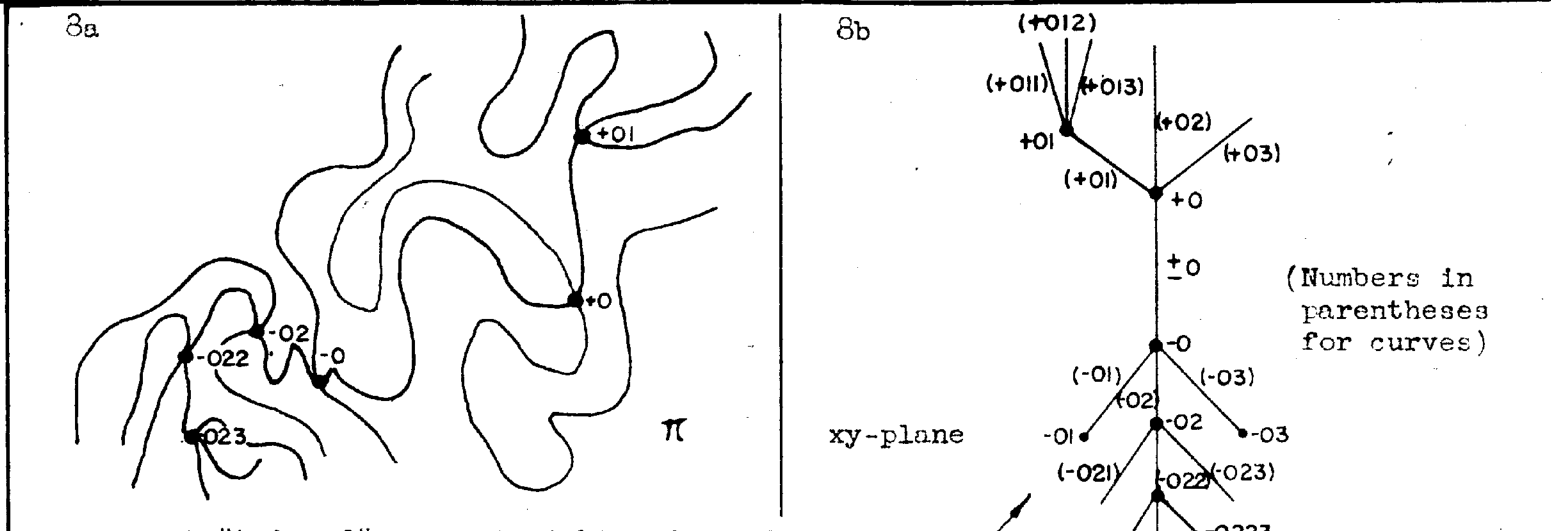


This is impossible by Theorem 3.2-3
C spiraling to a polygon

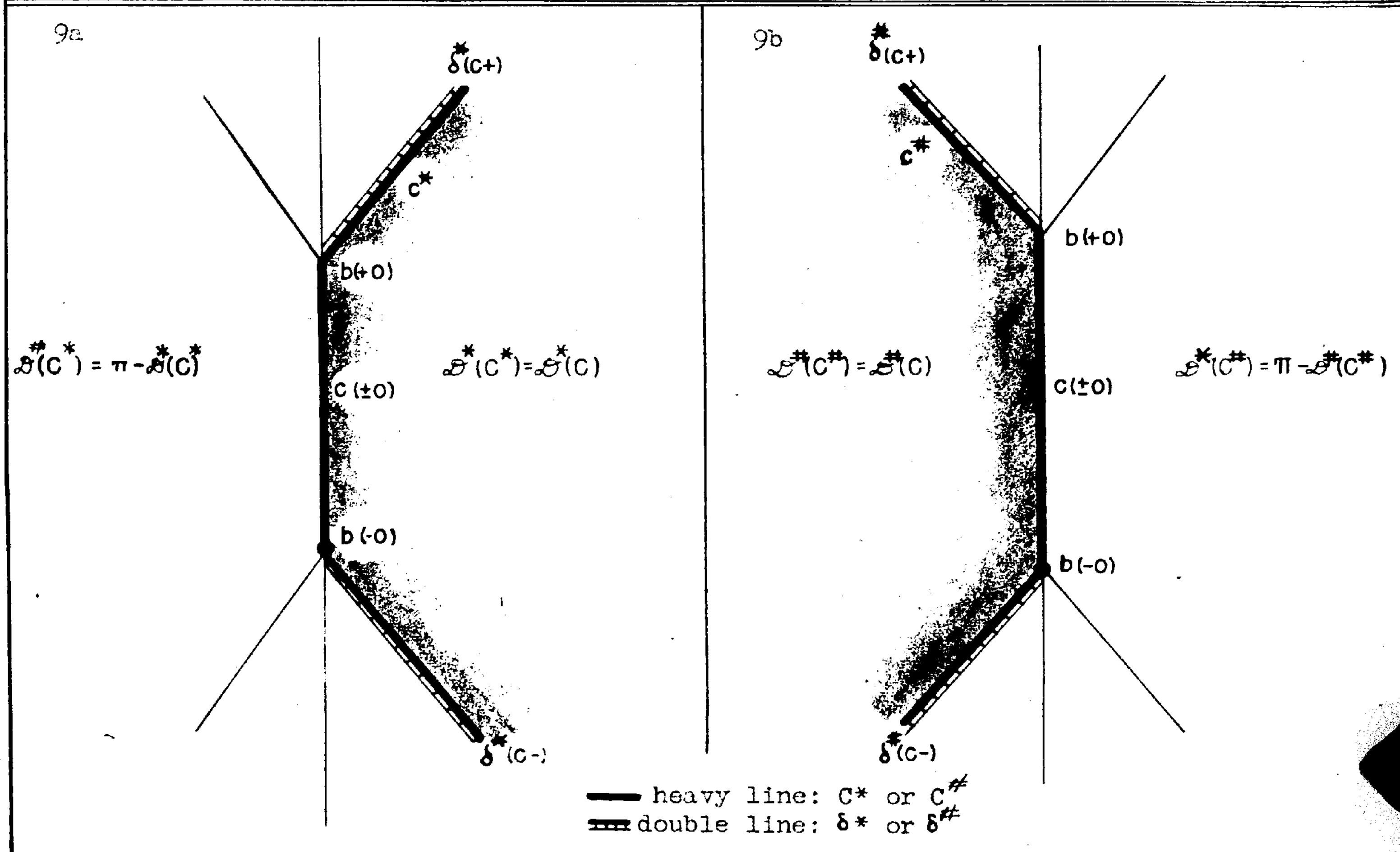
7b

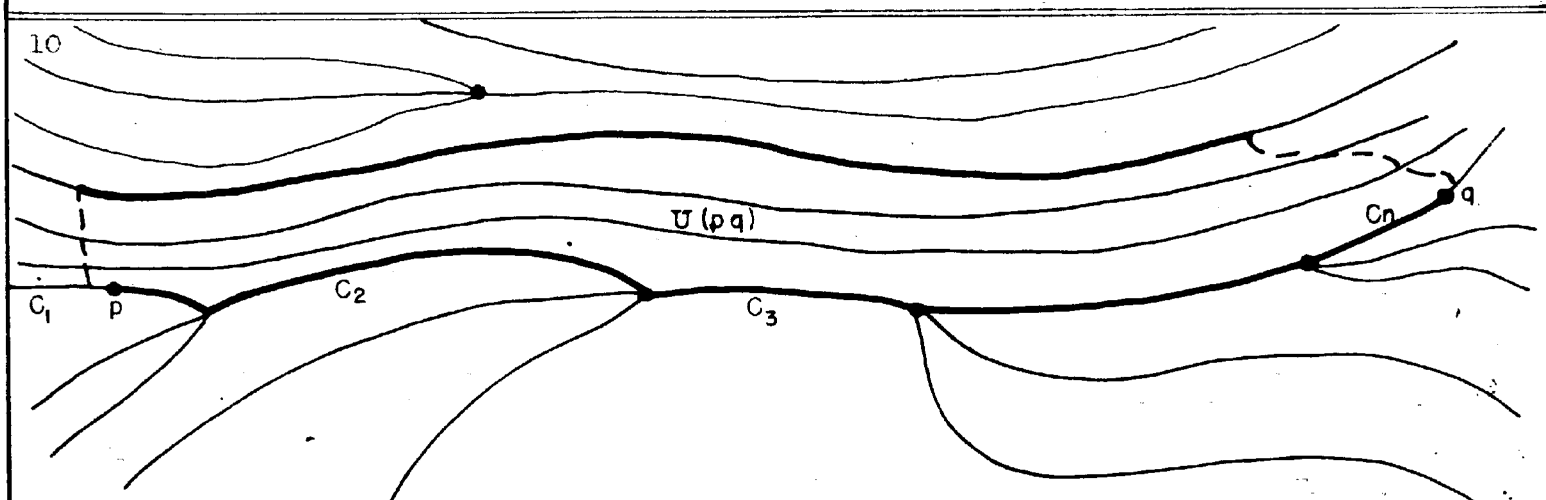


This is impossible by Theorem 3.2-4
Cross-section pq can intersect a tree only once.



A "twisted" tree straightened out by imbedding.





A semi-r-neighborhood (bounded by cross-section ---- and heavy line ———)

11a

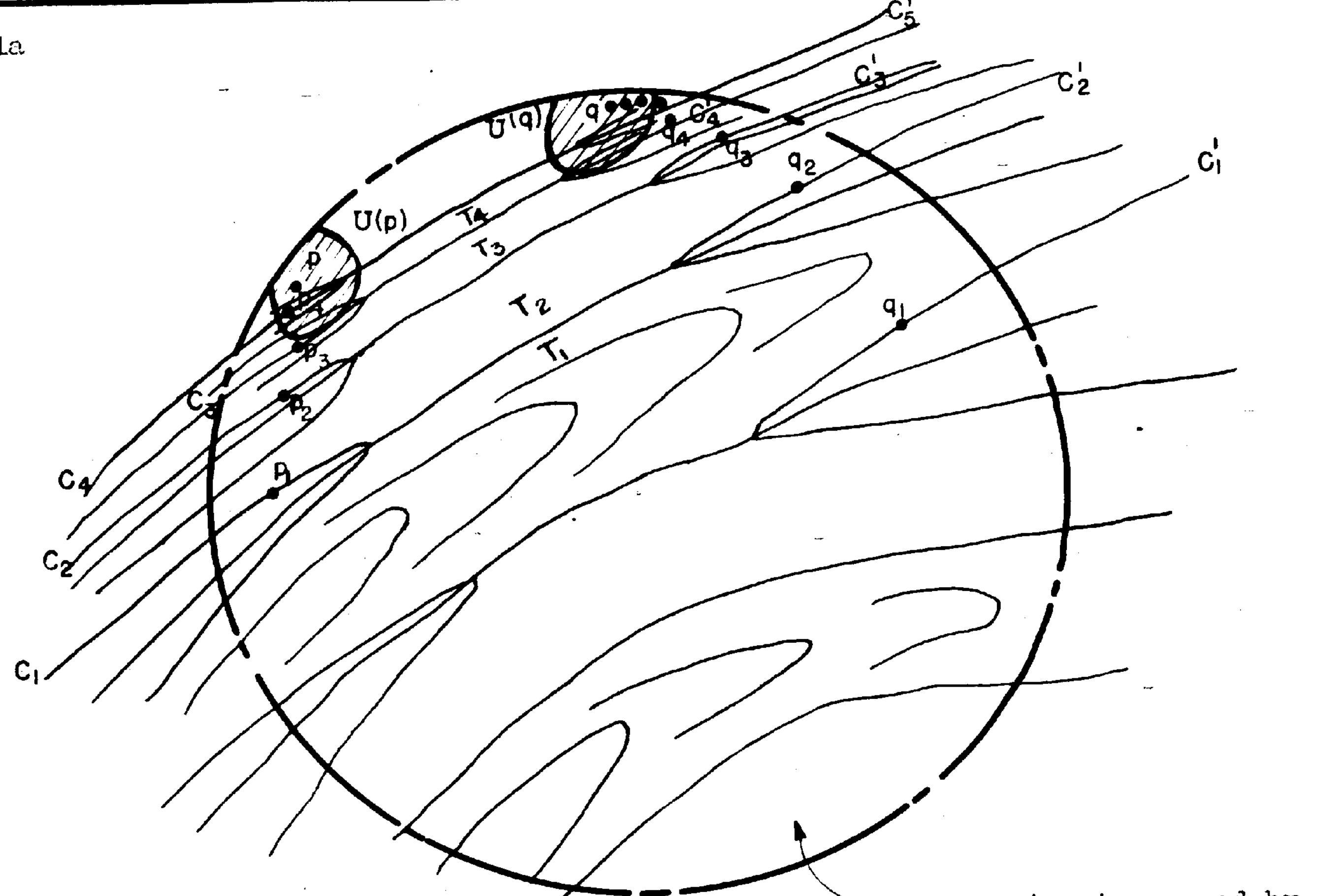
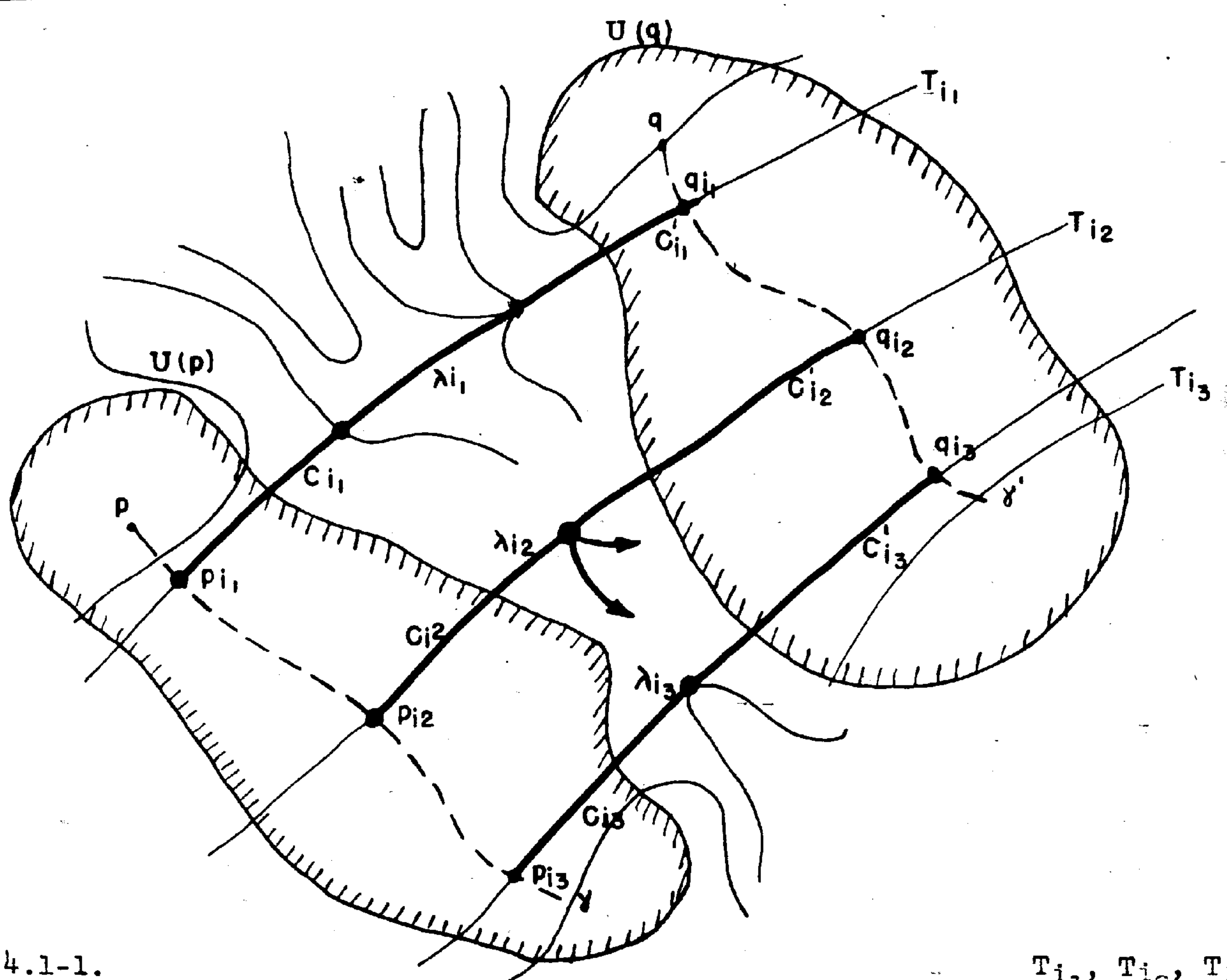


Illustration of proof of Theorem 4.1-1.

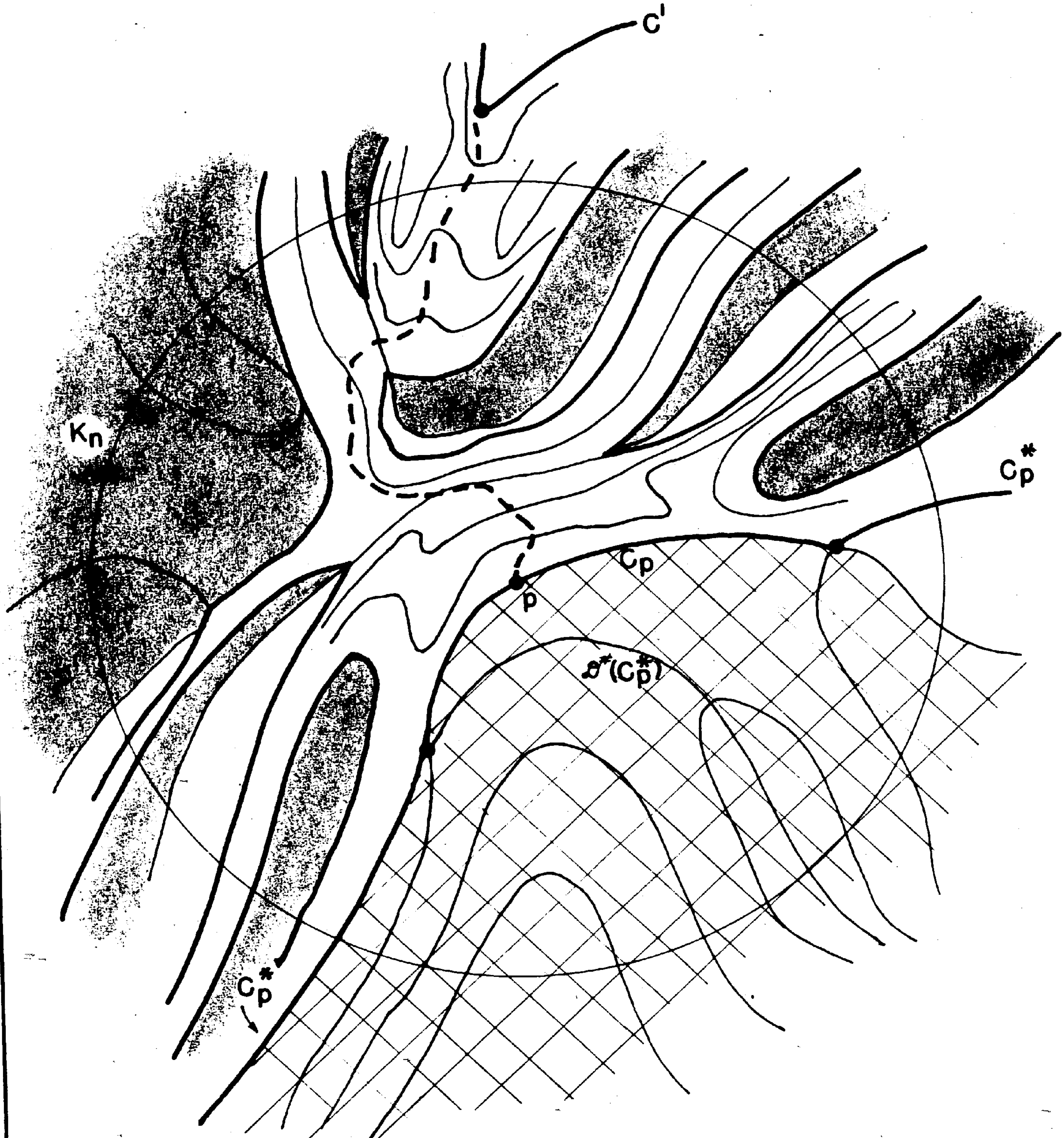
K a compact set crossed by trees T_i at curves C_i, C_i'

11b



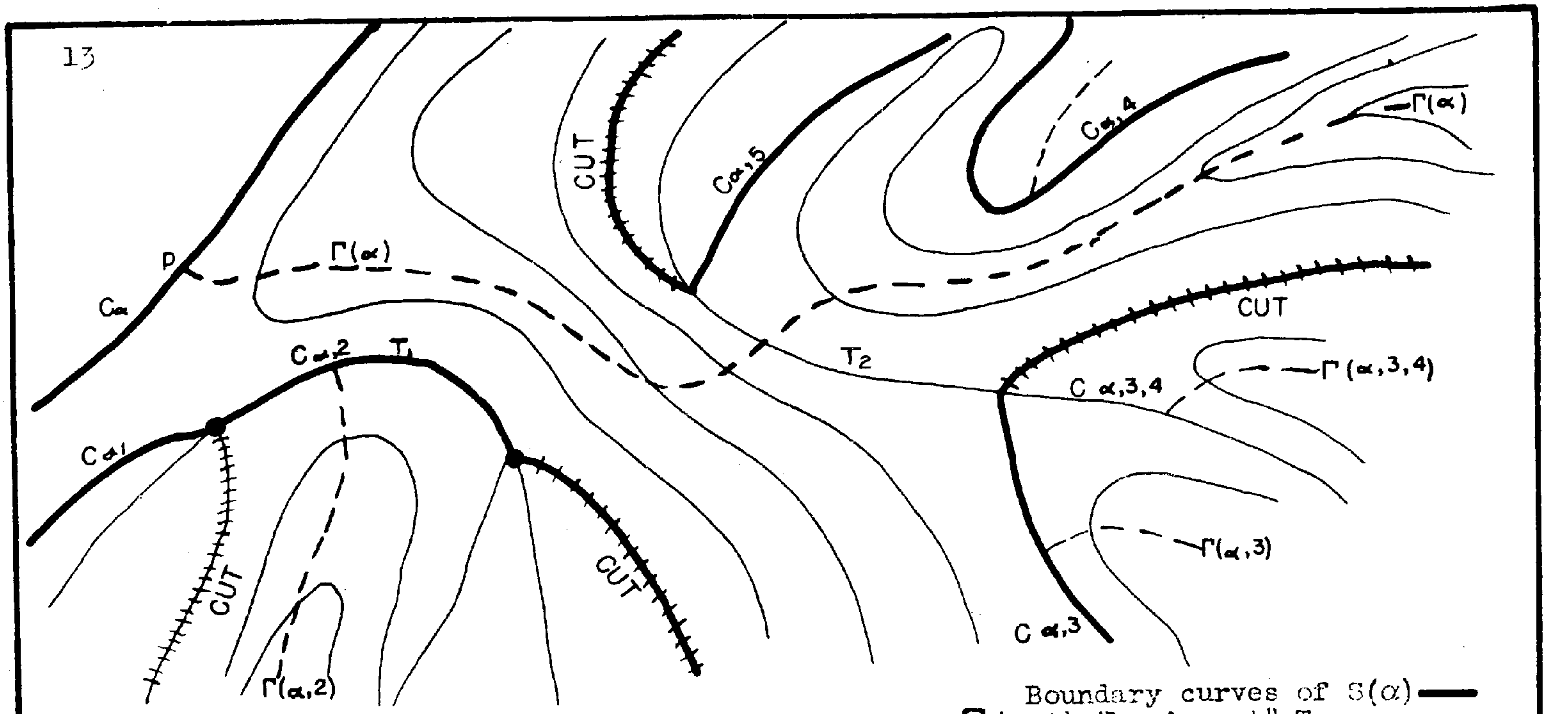
Theorem 4.1-1.

For p and q above, choose neighborhoods U(p) and U(q) and pick three trees T_{i_1}, T_{i_2}, T_{i_3}



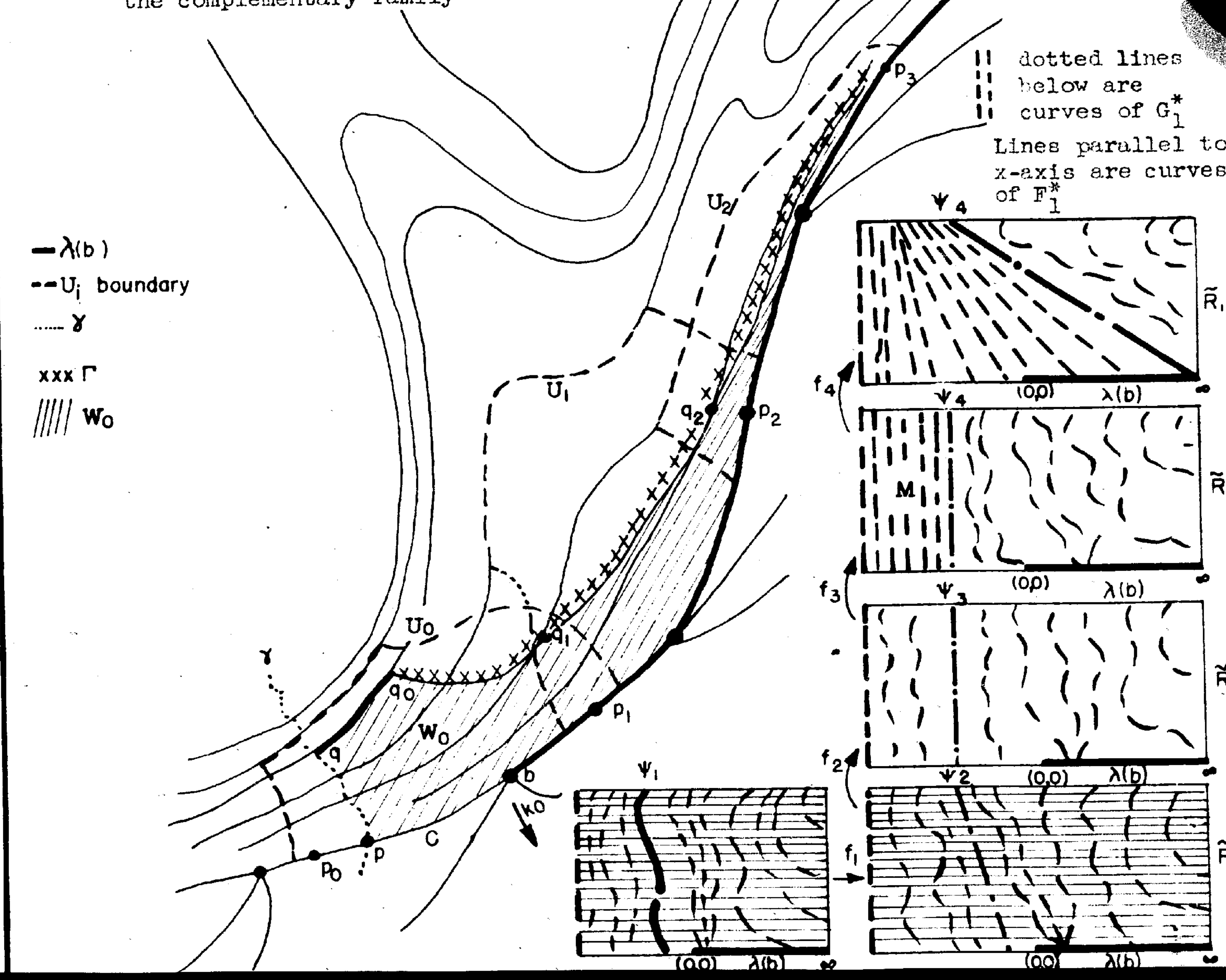
The shaded areas cannot be reached by cross-section from p, the cross-hatched is $\mathcal{D}^*(C_p^*)$. C' is reached by cross-section from p, although outside K_n.

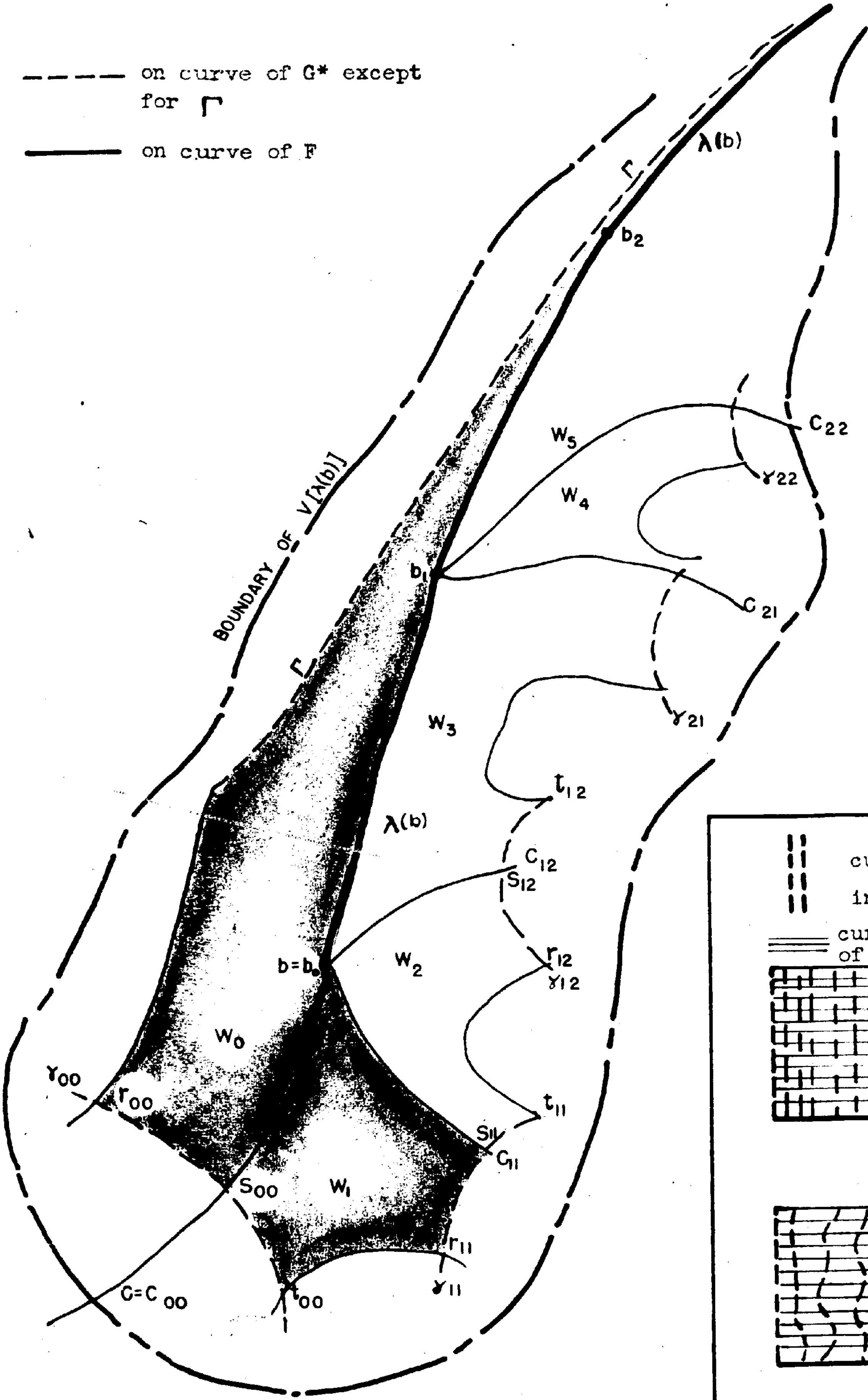
13



The cross-section $\Gamma(\alpha)$ from p on C_α "straddles" T_2. $\Gamma(\alpha,2)$ "begins at" T_1.

14 The semi-r-neighborhood of type II, W_o and its transformation by f_{II} to get the complementary family





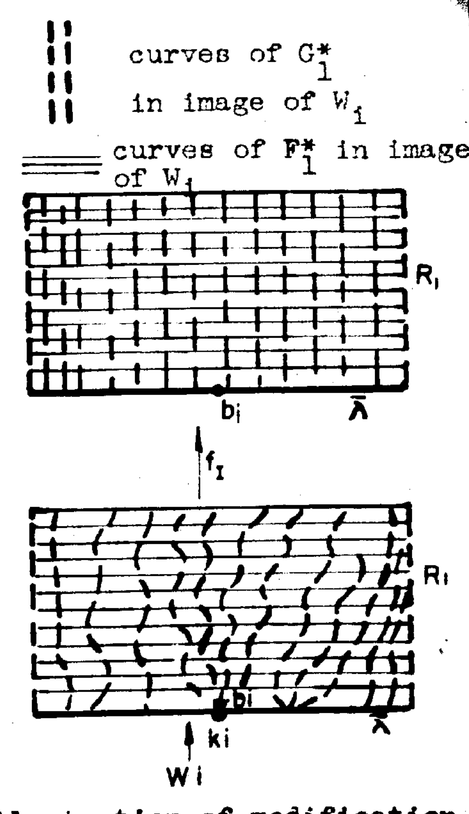


Illustration of modification of G_1^* in a neighborhood W_i of type I.