Extremal Black Holes in Diffeomorphism Covariant Gravity

by

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A thesis submitted in partial fulfillment of the requirements for the degree of Bachelor of Science with Honors

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March 2023

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Abstract

This thesis explores some classical aspects of extremal black holes in a diffeomorphism covariant formulation of gravity (of which general relativity is one). In particular, we first study the existence of Noether currents in the presence of Killing or gauge symmetries, and then generalize it to a theory exhibiting diffeomorphism-invariance in the context of the Noether-Wald formalism. We also explore the implication of the latter on black hole entropy for an arbitrary theory of gravity which may or may not include higher derivatives.

We then study the dynamics of charged static extremal black holes in Eddington-Finkelstein coordinates and encounter the problem of stability of the horizon. From a thermodynamics standpoint, we also realize that a non-extremal black hole cannot be made extremal in a manner consistent with the third law of black hole thermodynamics. To that end, we explore the procedure of studying the near-horizon geometry in the extremal limit of a non-extremal black hole and arrive at a well-known universal result (with a caveat on some exceptions) that all extremal black holes have an AdS_2 factor in their near-horizon geometry.

Motivated by the fact that the near-horizon geometry of an extremal black hole is consistent with the definitions of Wald, we study the entropy function formalism for a theory of gravity coupled to abelian gauge fields, and neutral scalars. A derivation of Wald entropy for the extremal Reissner-Nordström black hole in $D = 4$ dimensions is also given. We also discuss how the entropy function formalism leads to a generalization of the attractor phenomena of black holes without explicit reference to supersymmetry or string theory. Towards the end, we provide some additional evidence for the existence of the horizon instability of an extremal black hole by looking at the redshift effect, pair production at the horizon, and mass inflation.

This thesis has been written keeping in mind an advanced beginner and also includes a useful appendix that could serve as a short "primer" to delve into the field of black hole physics. There, we explore some important geometrical and (classical) thermodynamic features of rotating black holes via brute-force computation.

Acknowledgments

This project would have been impossible without the guidance and support from a number of people. I would first like to thank my advisor, Professor Finn Larsen, without whose guidance I would have ended up studying completely "retro" topics in this field of research. His elementary and intuitive explanations regarding super-technical things were always a charm[†]. His capacity to explain technical things at a broader level trumps the time I spent patiently camping outside his office for several hours to discuss about research every week.

I am especially thankful to Dr. Marina David, who graduated with a PhD from UM last summer and now pursues a postdoc at KU Leuven in Belgium. She was the one who got me organized with not just Mathematica, but also with papers and references. Despite being in a different time zone, and having multiple students under her guidance, she agreed to meet on Zoom every week to resolve my doubts or to discuss about research. For this, I will forever be grateful to her.

I would also like to thank Professor. James Liu who was amiably willing to answer all my questions whenever I had a difficulty in understanding something, especially during the time I was studying general relativity under his instruction.

I am also thankful to my friend Neville Rajappa, with whom I was able to engage myself into all sorts of discussions both within, and outside research, be it about black holes, supersymmetry, or orange juice.

Finally, I would like to thank my parents for always extending their support all the way from 8000 miles away. I will always be grateful for the love they have bestowed upon me and the unwavering encouragment that they have given me to achieve what I have before you today.

[†]Even at times when he went on a straight tangent :)

1 Introduction

Black holes are formed by the collapse of massive stars, resulting in a region of space where the gravitational pull is so strong that nothing, not even light, can escape. The study of these objects is important not only for understanding the fundamental principles of gravity, but also for exploring the frontiers of physics and uncovering new insights into the nature of the space-time, dark energy, and the origins of the Universe. In recent years, the detection of gravitational waves by the Laser Interferometer Gravitational-Wave Observatory (LIGO) and the imaging of the shadow of a supermassive black hole by the Event Horizon Telescope (EHT) have provided crucial experimental evidence for the existence of black holes and gravitational waves, consistent with the predictions of general relativity. Consequently, this has received increased trust in our theoretical models from the scientific community and the public. However, one of the core problems today in theoretical physics is the difficulty in the reconciliation of gravity with quantum mechanics—quantum gravity. Black holes naturally provide an excellent laboratory to test these theories and thus probe into the very structure of space and time.

From a thermodynamic standpoint, the existence of Hawking radiation (found by Hawking almost five decades ago) proved that black holes are not eternal as they radiate with a temperature

$$T_H = \frac{\hbar\kappa}{2\pi k_B} \tag{1.1}$$

where κ is the surface gravity. From this it followed that the entropy of a black hole is just one-quarter of the area of the horizon

$$S_{BH} = \frac{A}{4G\hbar} \tag{1.2}$$

which is the famous Bekenstein-Hawking result. Hawking's result Eq.(1.1) actually emerged out of the semiclassical treatment of black hole where the spacetime geometry is classical but the various fields, such as the electromagnetic field, behave quantum mechanically. This led to the idea that black holes are thermal objects that obey laws analogous to classical thermodynamics [1, 2].

From a general relativistic standpoint, black holes are fundamentally "bald" as they are completely defined by their mass M, charge Q, and angular momentum J—this is famously known as the "no-hair" theorem[†]. Moreover, the physics of black holes is *generally covariant*, since general relativity (GR) itself is a covariant formulation of gravity i.e. the theory is invariant under arbitrary coordinate transformations. In principle, this implies that there are always some symmetries associated with a Lagrangian that one writes for a covariant formulation of gravity. Any consistent formulation of quantum gravity will have this characteristic.

[†]It should be noted that this is rather a surprising result specific to $D = 4$ dimensions. Higher-dimensional black hole solutions, like black rings, are not uniquely described by these parameters, as they have non-spherical topology. See [3] for an introduction.

Generally speaking, symmetries can be more or less in number, depending upon the degrees of freedom a black hole solution allows. However, regardless of this number, every symmetry corresponds to a conserved charge due to *Noether's theorem*. This was used by Wald to show that a generally covariant Lagrangian in an *arbitrary* theory of gravity (that admits black hole solutions) leads to—and as we shall see in Sec.(2.18)—a conserved charge whose integral over a Cauchy-surface gives the Bekenstein-Hawking entropy Eq.(1.2). This remarkable insight solidified the idea that the entropy of a black hole is related to some symmetry of spacetime—ultimately strengthening the connection between thermodynamics and gravity. This, in turn, motivates us to study the surface charges in gravity.

Things get a little awkward when one tries to work with extremal black holes (black holes with minimal mass M that is equal to charge Q and angular momentum J). These objects emit no Hawking radiation $T_H = 0$, and have a special-type of a near-horizon geometry that is of particular interest in holographic theories of quantum gravity. The idea that a higher-dimensional gravity theory is related to a lower-dimensional quantum field theory was first formulated in terms of the AdS/CFT correspondence [4]. Motivated by this, less than two decades ago, it was found in the context of extremal rotating black holes, that its near-horizon region is related to a conformal field theory on its boundary, namely the Kerr$/CFT$ correspondence [5].

Yet another important aspect of extremal black holes, that was only recently investigated over the last decade, is the horizon instability problem. It was proven in [6, 7] that the horizon of an extremal black hole allows a conservation law under a scalar field perturbation which does not decay at late-times asymptotically. In other words, the horizon on an extremal black hole is unstable under even slight perturbations, causing it to support much "hair" exponentially in a finite amount of proper time. Moreover, much recently, during the writing of this thesis, it was suggested in [8] that almost all AdS black holes are singular at the horizon while all curvature scalars still remain finite. Since extremal black holes have an AdS_2 factor in their near-horizon region, this suggestion increases our motivation to study the near-horizon geometry of extremal black holes, at least at a classical level. Indeed, the research in the past decade was focused on AdS_2 models due to its implications for holography [9–11].

The most celebrated part in this field is the exact matching of the microscopic black hole entropy with the Bekenstein-Hawking entropy [12].

$$S_{BH}(Q) = S_{stat}(Q) \tag{1.3}$$

where $S_{stat}(Q) = \ln d(Q)^{\dagger}$ is the microscopic-entropy. Motivated by this discovery and Wald's formalism for calculating entropy in a generally covariant theory, a new method [13] of computing entropy was developed for a wider class of theories with higher derivative terms. The consequence of this was that the equations of motion one obtains while performing such calculations using this method correspond to a well-known phenomena called the *attractor mechanism*,

$^{\dagger}d(Q)$ is the degeneracy of extremal BPS states for a theory carrying same set of charges Q

which describes how the physical properties of a black hole are "attracted" to fixed values (in terms of charges or angular momentum) on its horizon as matter falls into it. In fact the existence of such a mechanism implies that the dynamics of a black hole are governed by a few conserved quantities on the horizon, and that the black hole's internal state is characterized by these quantities, rather than by the specific details of how it was formed. Indeed it suggests that the physics of a black hole is inherently holographic in nature, as evidently observed in nearly-AdS_2 [14].

This thesis studies some theoretical aspects of black holes in diffeomorphism covariant gravity, and aims to communicate and make the advanced beginner aware of some important discoveries in the field of black hole physics. Therefore, it is by no means a comprehensive account/review of all modern literature in black hole physics. Research enthusiasts at all levels are encouraged to refer to the list of resources provided in [15] if they wish to review the current state of the art. This thesis is organized as follows:

- In Sec.2 we first work in Einstein gravity and realize the existence of Noether currents via gauge and/or Killing symmetries. Then, we look at the same aspect more canonically for a diffeomorphism covariant theory of gravity via the Noether-Wald formalism, and also study its implications for black hole entropy.
- In Sec.3 we first study the geometrical features of static charged extremal black holes and encounter the horizon instability problem which is later discussed in Sec.5. Then we look at the method of approaching the extremal limit from non-extremal black holes in a way consistent with the second law of black hole thermodynamics.
- In Sec.4 we study how the entropy function is constructed for a diffeomorphism covariant gravity (that allows extremal black hole solutions) in a way consistent with the definitions of Wald, without using string theory or supersymmetry. We end this section with an observation that the extremization of the entropy function relates to the attractor mechanism.
- Sec.5 is an addendum to the foregoing discussion from various sections.
- Appendix A serves as a short "black hole physics primer" in the Kerr-Newman family, tailored for advanced beginners in the subject. While Appendix B is a useful addendum to Sec.(2).

2 Surface Charges in Gravity

In this section we will use the Lagrangian formulation of general relativity (discussed in Appendix B) to find conserved quantities, for spacetimes that exhibit some kind of a symmetry, via Noether's theorem. These quantities would emerge out of the general covariance of the Einstein-Hilbert action. In particular, we shall see that a spacetime exhibiting a Killing symmetry, gives rise to a non-trivial Noether current (or the Komar current) in its manifestly covariant form. Then, we will turn towards a slightly different and more canonical approach, namely the *Noether-Wald* formalism, for defining conserved charges in general relativity on a covariant phase space. We will also look at how this formalism allows us to reproduce the correct form of black hole entropy for any diffeomorphism covariant theory of gravity (even with higher derivatives).

As we shall later remark, the Komar approach is practical but limited to theories that only exhibit a Killing symmetry. On the other hand, the Noether-Wald approach, although quite technical, is an extremely rigorous way of defining conserved quantities in any diffeomorphism covariant field theory of gravity—of which general relativity is one.

2.1 Covariant Action and Noether Currents

First, consider a generally covariant action of the form

$$S = \int d^4x \sqrt{-g}\, \phi \tag{2.1}$$

where ϕ is an arbitrary scalar.

Now, consider the variation of S under an infinitesimal diffeomorphism generated by a vector field ξ^μ giving rise to a gauge symmetry in spacetime. Then,

$$\begin{aligned}
\delta_\xi S &= \int d^4x\, \delta_\xi(\sqrt{-g}\, \phi) \\
&= \int d^4x\, \left(\delta_\xi\sqrt{-g}\right) \phi + \sqrt{-g}\, (\delta_\xi \phi)) \\
&= \int d^4x \sqrt{-g} \left(\frac{1}{2}\, g_{\mu\nu}(\delta_\xi g^{\mu\nu})\, \phi + \xi^\mu \partial_\mu \phi\right) \\
&= \int d^4x \sqrt{-g} \left(\frac{1}{2}\, g_{\mu\nu}(\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu)\, \phi + \xi^\mu \partial_\mu \phi\right) \\
&= \int d^4x \sqrt{-g}\, (\nabla_\nu\, \xi^\nu \phi + \xi^\mu \partial_\mu \phi) \\
&= \int d^4x \sqrt{-g}\, \nabla_\nu (\xi^\nu \phi)
\end{aligned} \tag{2.2}$$

now, extremizing the action, $\delta_\xi S = 0$, gives

$$0 = \int d^4x \sqrt{-g}\, \nabla_\nu(\xi^\nu \phi) \tag{2.3}$$

Notice that this implies the existence of a conserved quantity in the usual sense, i.e. its divergence equals zero. There are no non-trivial Noether currents associated with this conserved quantity due to the absence of an additional symmetry (the Killing symmetry). This is because there are no isometries of $g_{\mu\nu}$ for a *general* spacetime. Nonetheless, this still leaves us with an important fact: a diffeomorphism generated by a generic vector field ξ^μ generates a redundant symmetry (the gauge symmetry), and leads to a conserved quantity*.

But, how is all this related to the Einstein-Hilbert action? First, note that the action in Eq.(2.1) was simply constructed from an arbitrary scalar ϕ that made no reference or connection to the geometry of spacetime whatsoever. So, the 'gauge symmetry' of the theory that we encountered, was simply a manifestation of the fact that ϕ was independent of the choice of coordinates used to describe the underlying spacetime. But in order to provide an adequate description of gravity, the simplest choice in place of ϕ is that of a coordinate-independent scalar that can be constructed from the metric tensor $g_{\mu\nu}$ and its second derivatives: the Ricci scalar R. The Einstein-Hilbert action is then expressed as:

$$S_{EH} = \int d^4x \sqrt{-g}\, R \tag{2.4}$$

Now, instead of varying S_{EH} with respect to the inverse metric, i.e. $\delta g^{\mu\nu}$, we will vary $g^{\mu\nu}$ with respect to an infinitesimal diffeomorphism $x^\alpha \to x^\alpha + \xi^\mu$ generated by a Killing vector ξ^μ. This is useful because Killing vectors preserve the metric at every point in space-time, and give rise to a special symmetry, namely the *Killing symmetry*. This symmetry can in turn be used to derive *non-trivial* conserved quantities via Noether's theorem. In fact, in black hole mechanics, Killing symmetries give rise to conserved surface charges in the bulk of space-time (aka. Noether-Wald surface charges) [16, 17]. In case of a Kerr black hole, the surface charge corresponding to the two Killing vectors ξ^t and ξ^ϕ is the mass and the angular momentum of the black hole respectively.

Without much ado, let us begin by re-expressing Eq.(B.2) in terms of the variation of S_{EH} with respect to a Killing vector ξ^μ, and keeping the boundary term where ξ^μ (unlike Eq.(2.3)) is non-trivial.

$$\delta_\xi S_{EH} = \int d^4x \sqrt{-g}\, \left(G_{\mu\nu}\, \delta_\xi g^{\mu\nu} + boundary\ term\right)$$

*This inevitably verifies Noether's theorem for diffeomorphisms.

Now, since the metric is symmetric in indices, we have:

$$\delta_\xi S_{EH} = 2 \int d^4x \sqrt{-g} \left(G_{\mu\nu} \left(\nabla^\mu \xi^\nu \right) + boundary\ term \right)$$

and integration by parts leads to

$$\delta_\xi S_{EH} = \int d^4x \sqrt{-g} \left[\underbrace{\nabla^\mu (2\, G_{\mu\nu}\, \xi^\nu)}_{\text{total derivative}} + \underbrace{boundary\ term}_{\text{total derivative}} \right] \tag{2.5}$$

Both terms are total derivatives, which is a good sign because when we extremize the action, a conserved quantity shall emerge. Moreover, the boundary term is $g^{\mu\nu}\delta_\xi R_{\mu\nu}$, which can be simplified to

$$\begin{aligned} g^{\mu\nu}\delta_\xi R_{\mu\nu} &= \nabla_\lambda \left(g^{\mu\nu}\delta_\xi \Gamma^\lambda{}_{\mu\nu} - g^{\mu\lambda}\delta_\xi \Gamma^\nu{}_{\nu\mu} \right) \\ &= \nabla_\mu \left(g^{\mu\alpha} g^{\nu\beta} - g^{\mu\nu} g^{\alpha\beta} \right) \nabla_\nu \left(\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha \right) \end{aligned} \tag{2.6}$$

using the explicit form of $\delta_\xi \Gamma^\mu{}_{\nu\lambda} = \frac{1}{2} g^{\mu\gamma} \left(\nabla_\nu \delta_\xi g_{\gamma\lambda} + \nabla_\lambda \delta_\xi g_{\gamma\nu} - \nabla_\gamma \delta_\xi g_{\nu\lambda} \right)$ and $\delta_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$.

The first total derivative term in Eq.(2.5) can also be simplified by lowering the index and re-expressing it as

$$\nabla^\mu (2\, G_{\mu\nu}\, \xi^\nu) = \nabla^\mu \left(2\, R_{\mu\nu}\, \xi^\nu \right) - \underbrace{\nabla_\mu (\xi^\mu R)}_{=0} \tag{2.7}$$

where the second term vanishes[†] by replacing $\phi \to R$ in Eq.(2.3).

Substituting Eq.(2.5) and (2.6) back into Eq.(2.4), gives

$$\begin{aligned} \delta_\xi S_{EH} &= \int d^4x \sqrt{-g}\, \nabla^\mu \left[2\, R_{\mu\nu}\, \xi^\nu + \left(g^{\mu\alpha} g^{\nu\beta} - g^{\mu\nu} g^{\alpha\beta} \right) \nabla_\nu \left(\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha \right) \right] \\ 0 &= \int d^4x \sqrt{-g} \left(\left[\nabla^\nu, \nabla^\mu \right] \xi_\nu + \nabla^\mu \left(\nabla_\nu \xi^\nu \right) - \frac{1}{2} \nabla_\nu \left(\nabla^\nu \xi^\mu + \nabla^\mu \xi^\nu \right) \right) \\ 0 &= \int d^4x \sqrt{-g} \left(\nabla_\nu \nabla^\mu \xi^\nu - \frac{1}{2} \nabla_\nu \left(\nabla^\nu \xi^\mu + \nabla^\mu \xi^\nu \right) \right) \\ 0 &= \int d^4x \sqrt{-g} \left(\frac{1}{2} \nabla_\nu \left(\nabla^\mu \xi^\nu - \nabla^\nu \xi^\mu \right) \right) \\ 0 &= \int d^4x \sqrt{-g}\, \nabla_\nu (\nabla^{[\mu} \xi^{\nu]}) \end{aligned} \tag{2.8}$$

[†]This simply means that the conserved quantity emerging out of this term is trivial

where in the second line, we have made some simplification using the definition of Ricci tensor in terms of a commutator of covariant derivatives. The resulting quantity in Eq.(2.8) is a total derivative term, and is identified as the Noether current $\mathbf{J}(\xi)$ associated to the Killing symmetry in spacetime[‡] .

$$\mathbf{J}(\xi) = \nabla_\nu(\nabla^{[\mu}\xi^{\nu]}) \tag{2.9}$$

It is also important to keep track of the normalization factor $1/(16\pi G)$, which is typically multiplied in the action of the form Eq.(2.4), in order to produce the correct equations of motion. With this, an appropriate definition of charge for a stationary and axially symmetric spacetime can be written as an integral of the current density $j^\mu = \frac{1}{4\pi G}\nabla_\nu(\nabla^{[\mu}\xi^{\nu]})$ over a space-like three-dimensional hypersurface Σ that extends to a spatial infinity:

$$Q = -\int_\Sigma j^\mu d\Sigma_\mu = -\frac{1}{4\pi G}\int_\Sigma \nabla_\nu(\nabla^{[\mu}\xi^{\nu]})\, d\Sigma_\mu$$

which reduces to a surface integral of a two-form S, via Stokes theorem

$$\begin{aligned} Q &= +\frac{1}{8\pi G}\oint_S dS_{\mu\nu}\sqrt{|\gamma|}\, t_\mu r_\nu\, (\nabla^\mu\xi^\nu - \nabla^\nu\xi^\mu) \\ &= \frac{1}{8\pi G}\oint_S dS_{\mu\nu}\sqrt{|\gamma|}\, t_\mu r_\nu \nabla^\mu\xi^\nu \end{aligned}$$

where t_μ and r_ν corresponds to unit normal vectors in the time-like and space-like (radial) direction respectively, and in the last line we have used the fact that $(\nabla^\mu\xi^\nu - \nabla^\nu\xi^\mu)$ is an antisymmetric tensor. Finally, we write the conserved charge (or the *Komar charge*) as

$$Q = \frac{1}{8\pi G}\oint_S dS\, t_\mu r_\nu \nabla^\mu\xi^\nu \tag{2.10}$$

where $dS = d^{D-2}x\sqrt{\gamma^{D-2}}$ is the area element of the surface in D-dimensions.

Eq.(2.10) is a surface integral over a two-form, and is also known as the *Komar integral* for computing the charge for a stationary and axially symmetric spacetime in the asymptotic limit.

2.2 The Noether Wald Formalism

We just saw that a diffeomorphism ξ^μ generated by a Killing vector gives rise to a non-trivial Noether current, and hence a conserved surface charge for a specific spacetime, namely the one that has a Killing symmetry. This motivates us to seek a general definition for Noether currents and their corresponding surface charges for any given spacetime with a *general* diffeomorphism

[‡]Recall that the Noether current arising out of a Killing symmetry generated by ξ^μ can be written as the divergence of an anti-symmetric tensor in the Einstein-Maxwell theory: $J(\xi) = \nabla_\mu F^{\mu\nu}$, where $F^{\mu\nu} = \nabla^\mu\xi^\nu - \nabla^\nu\xi^\mu$

symmetry.

First, consider a Lagrangian density $\mathcal{L}$ defined on a spacetime manifold $\mathcal{M}$ with fields Φ_i, that include the matter fields Φ_M, the metric $g_{\mu\nu}$, and the derivatives of these fields. An arbitrary variation of the Lagrangian density is

$$\begin{aligned}\delta\mathcal{L} &= \delta\Phi_i \frac{\partial\mathcal{L}}{\partial\Phi_i} + (\partial_\mu\delta\Phi_i)\frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi_i} \\ &= \left[\frac{\delta\mathcal{L}}{\delta\Phi_i} - \partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu\Phi_i)}\right]\delta\Phi_i + \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\Phi_i)}\delta\Phi_i\right)\end{aligned} \tag{2.11}$$

The expression inside the square bracket vanishes when the Euler-Lagrange equations of motion are satisfied, and we identify the term inside the total derivative as the presymplectic potential

$$\Theta^\mu = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\Phi_i)}\delta\Phi_i$$

which allows us to write a more formal expression for an arbitrary variation of Lagrangian density on a D-dimensional manifold $\mathcal{M}$ using Eq.(2.11)

$$\delta\mathcal{L} = \left[\frac{\delta\mathcal{L}}{\delta\Phi_i} - \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\Phi_i)}\right)\right]\delta\Phi_i + d\Theta[\Phi_i,\delta\Phi_i] \tag{2.12}$$

Now, consider a diffeomorphism generated by a vector field ξ^μ on $\mathcal{M}$. Then, plugging in Cartan's magic formula[§], $\delta_\xi\mathcal{L} = L_\xi\mathcal{L} = d\,(\xi\cdot\mathcal{L})$ in Eq.(2.12), one sees that when the equations of motions are satisfied (i.e. on-shell), we have

$$d\mathbf{J} = d\,(\xi\cdot\mathcal{L}) - d\Theta[\Phi_i,\delta\Phi_i]$$

where $d\mathbf{J}$ is a $D-1$ form. Upon integration, we identify $\mathbf{J}_\xi$ as a non-trivially conserved Noether current

$$\mathbf{J}_\xi = \xi\cdot\mathcal{L} - \Theta[\Phi_i, L_\xi\Phi_i] \tag{2.13}$$

The conserved charge associated to $\mathbf{J}_\xi$ is known as the *Noether-Wald* surface charge

$$\mathbf{Q}[\xi] = -\int_S d\Sigma^{\mu_1\cdots\mu_D}\,\Theta[\Phi_i, L_\xi\Phi_i] \tag{2.14}$$

where Σ is a space-like D-dimensional hypersurface and the charge $\mathbf{Q}$ along ξ is integrated over a spatial $D-2$ surface S.

[§]In generally covariant theories, any variation along a diffeomorphism is a Lie derivative along its flow that can be subdivided into several operations of the form: $\mathcal{L}_\xi\,(\omega) = d\,i_\xi(\omega) + i_\xi d(\omega)$ where i_ξ is an involution along ξ^μ, d the exterior product, and ω a k-form [17]

2.2.1 Noether-Wald Charges and Black hole Entropy

A major goal of the research program in quantum gravity is to provide an explanation and also derive the formula for the entropy of a black hole. Such calculations are usually of interest for special type of black holes—extremal black holes—which will not become a subject of our discussion until Section(3.1). For now, it is essential to remark some general ideas and formulae (sans proof) that have been developed by Wald and Iyer [16, 18] and extended by others [19], for calculating black hole entropy within diffeomorphism invariant theories of gravity. We should do so, for two reasons: (a) The intimate relationship between the entropy of a black hole and the Noether-Wald surface charge Eq.(2.14), and (b) Later convenience in deriving the Wald entropy for $SO(2,1) \times SO(D-1)$ invariant extremal black holes using the entropy function formalism (introduced in Sec.(4)).

Until now, we have only been considering two derivative actions in diffeomorphism-invariant theories of gravity. However, it is known that general relativity should be seen as an effective field theory, and therefore it has to be corrected by higher derivative terms. For example, in string theory, we expect α' corrections (which are essentially quantum corrections that take into account the string tension) to the effective Einstein-Hilbert action with higher derivative terms involving the Riemann tensor and other fields

$$I = \int d^4x\sqrt{-g}\left(R + c_1 R^2 + c_2 R_{ab}R^{ab} + c_3 R_{abcd}R^{abcd} + \cdots\right) \tag{2.15}$$

where $c_i = (1, 2, 3, \cdots)$ are dimensionful constants. Note that we are now using I to denote action, as the letter 'S' is usually attributed to entropy in the black hole literature.

The presence of these higher derivative terms would mean that the second law of black hole thermodynamics is only valid in the low energy limit where only two-derivatives are involved in the action. However, Wald [16] constructed a famous derivation of the first law of black hole mechanics, and found that the entropy can still be rigorously defined for theories with higher derivatives provided the condition that the theory is diffeomorphism-invariant. Under this construction, he found that the black hole entropy S is simply 2π times the integral over a space-like D-dimensional hypersurface Σ of the Noether charge $\boldsymbol{Q}$ (a $D-2$ form) associated with the Killing horizon

$$S = 2\pi \int_\Sigma \boldsymbol{Q} \tag{2.16}$$

where the 2π is to normalize the unit surface gravity κ. The equation implies that **black hole entropy is a Noether charge**. Qualitatively speaking, on observing the right-hand-side of the first law

$$\frac{\kappa}{2\pi}\delta S = \delta M - \Omega\,\delta J \tag{2.17}$$

we see that terms like mass M and angular momentum J are involved, which are just the usual

Noether charges in Einstein gravity. So, loosely speaking (without taking away the profundity of the previous sentence in bold), one indeed expects that the addition or subtraction of two or more Noether charges is still a Noether charge. Therefore, the entropy of a black hole is indeed a Noether charge as also implied from the first law directly.

For a general Lagrangian (ignoring higher derivatives) of the form,

$$L = L\left(\Phi_M, \nabla_a \Phi_M, g_{ab}, R_{abcd}, \nabla_e R_{abcd}\right)$$

it was found in [19] that the black hole entropy is given by

$$S = -2\pi \oint \left(\frac{\partial L}{\partial R_{abcd}} - \frac{\partial L}{\partial \nabla_e R_{abcd}} \right) \hat{\epsilon}_{ab} \hat{\epsilon}_{cd} \bar{\epsilon} \tag{2.18}$$

where the integral now is over an *arbitrary* cross-section[†] of the horizon, and $\hat{\epsilon}_{ab}$ is the binormal to the cross-section, and $\bar{\epsilon} = \epsilon_{ab}/\hat{\epsilon}_{ab}$ is the induced volume form on the cross-section. Eq.(2.18) is also known as the **Wald entropy** for a Lagrangian without higher derivatives. For a generalized expression involving higher derivatives (applicable to any diffeomorphism invariant theory of gravity), the reader may refer to [18, 20].

A quick check: One can verify whether Eq.(2.18) produces the right answer in Einstein gravity. For a stationary black hole, a Lagrangian of the form $L(g_{ab}, R_{abcd})$ has no derivatives in the curvature tensor, so Eq.(2.18) reduces to

$$S = -2\pi \int_{\mathcal{H}} E^{abcd} \hat{\epsilon}_{ab} \hat{\epsilon}_{cd} \tag{2.19}$$

where $E^{abcd} = \frac{\partial L}{\partial R_{abcd}} = \frac{1}{32\pi G}\left(g^{ac}g^{bd} - g^{ad}g^{bc}\right)$ on the bifurcate surface $\mathcal{H}$, and the binormal, $\hat{\epsilon}_{ab}$, on the bifurcate Killing horizon becomes $\hat{\epsilon}_{ab} \to \nabla_a \xi_b$ for unit surface gravity $\kappa = 1$. Using this and the fact that the square of the binormal $\hat{\epsilon}_{ab}\hat{\epsilon}^{ab} = -2$, the integrand becomes

$$E^{abcd} \hat{\epsilon}_{ab} \hat{\epsilon}_{cd} = \frac{1}{8\pi G} \sqrt{|\gamma|}\, d\Omega^2 \tag{2.20}$$

where $\sqrt{|\gamma|}$ is the determinant of the metric induced on the two-surface. But, Eq.(2.20) is just the area A of the Killing horizon when integrated over the bifurcation surface $\mathcal{H}$. Hence, we can conclude that Eq.(2.18) reproduces the usual Bekenstein-Hawking-Wald entropy, $S = A/4$, in Einstein gravity.

In Sec.(4), we shall see how Eq.(2.18) was used by Sen [21] to construct the entropy function

[†]Unlike Wald's original construction in [16] that lead to (2.16) with the assumption that there is a *bifurcate* Killing horizon (which is the case for stationary black holes). It happens that even with an arbitrary cross section of the Killing horizon, one indeed obtains (2.16) because cross-sections of Killing horizon are isometric when stationarity is assumed. For non-stationary black holes, we have more candidates of entropy that depend on the Killing field and derivatives as established in [19].

formalism for extremal black hole solutions in higher derivative theories of gravity. For now, we would like to conclude this section with some general remarks that distinguish the Noether-Wald formalism from the Komar approach to define conserved charges in gravity.

2.3 Remarks

The conserved charge we arrived at in Eq.(2.14) is an extremely general definition established through the Noether-Wald formalism in a diffeomorphism-invariant (i.e. invariance under coordinate transformations) theory of gravity. This approach is generally considered to be more powerful and versatile than the *Komar* approach due to the following reasons:

- The Noether-Wald charges are defined in terms of the symmetries of spacetime generated by some *arbitrary* vector field ξ^μ and are therefore applicable to *any* diffeomorphism-invariant theory of gravity, including theories beyond general relativity. The Komar charges, on the other hand, are defined in terms of the geometry of the spacetime and are specific to general relativity.
- The Noether-Wald charges can include contributions from matter fields Φ_M, in which case, we get a term like $\left[\frac{\delta\mathcal{L}}{\delta\Phi_M} - \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\Phi_M)}\right)\right]$ from an arbitrary variation in the Lagrangian density. Note, that this would correspond to the on-shell conservation of the energy-momentum tensor, $\nabla_\mu T^{\mu\nu} = 0$, in general relativity.
- One more important point to note is that the Komar approach is only relevant for spacetimes with Killing symmetries. The Noether-Wald procedure, on the other hand, is more general in the sense that it also allows gauge symmetries, as long as Lagrangian density $\mathcal{L}$ is diffeomorphism invariant.
- In fact, in general relativity, if we only take Φ_i to include the metric tenor $g_{\mu\nu}$, then its variation along ξ^μ is the Lie-derivative, i.e. $\delta_\xi g_{\mu\nu} = L_\xi g_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu$, and consequently the presymplectic potential will be equal to the Noether current $\mathbf{J}(\xi)$ found in Eq.(2.9), i.e.

  $$\Theta[g_{\mu\nu}, L_\xi g_{\mu\nu}] = \mathbf{J}(\xi) = \nabla_\nu(\nabla^{[\mu}\xi^{\nu]}).$$

  So, when a spacetime is stationary and axissymmetric, one can practically use this relation to derive the conserved charge in the form of a Komar surface-integral in D-dimensions, by following the same narrative from Eq.(2.9)–(2.10).
- Eq.(2.10) is deficient in the sense that it is an asymptotic expression i.e. it gives us the conserved charge at an asymptotic infinity. However, a conserved quantity in general should be independent of the location where it is computed as long as it arises from a symmetry of the spacetime (See [22]). The Noether-Wald charge in Eq.(2.14) can be defined anywhere on the Cauchy hypersurface Σ. For $D = 4$, the answers from both expression match due to the Killing symmetry. However, this may not be the case in higher dimensions.

3 Dynamics of Extremal Black Holes

Extremal black holes are objects with minimum mass that is equal to charge and angular momentum. These black holes are often supersymmetric (i.e. they are invariant under supercharges[§]). They emit no Hawking radiation (i.e. $T_H = 0$) and thus are of particular interest in quantum gravity. For example, the entropy for a class of five-dimensional extremal black-holes has been famously calculated in string theory [12]. Moreover, their near-horizon geometries have also been investigated [5] to allow for a description of quantum gravity via the holographic duality. So, investigating the dynamics and geometry of near-extremal black holes is of great importance in current research.

The extremal Reissner-Nordström geometry exhibits a higher degree of symmetry than non-extremal black holes, and its theoretical investigation is more tractable — which is also why it frequently appears in the present literature. So, in following section we will first review the geometry of the extremal Reissner-Nordström black hole. Then, in Sec.(3.2) we look at some conceptual problems we face while going from non-extremality to extremality. Finally, in Sec.(3.3) we show how these problems can be avoided if one instead takes the extremal limit of non-extreme black holes.

3.1 The Extremal Reissner-Nordström

The Reissner-Nordström (RN) family of black holes are spherically-symmetric, asymptotically-flat solutions to the Einstein-Maxwell equations in $D \geq 4$ dimensions. The electromagnetic potential for the RN metric in $D = 4$ dimensions was calculated in Sec.Kerr Newman black hole. The extreme case corresponds to taking $Q = M$ for the standard RN metric in (ref. equation KN section), and the line-element becomes:

$$ds^2 = -\Delta(r)dt^2 + \frac{1}{\Delta(r)}dr^2 + r^2 d\Omega^2 \tag{3.1}$$

where $\Delta(r) = (1 - 2M/r + M^2/r^2) = (1 - M/r)^2$, and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$.

Notice that there is a coordinate singularity at $r = M$, and a curvature singularity at $r = 0$. The latter is easily seen after computing the Kretschmann scalar $R_{\mu\nu\delta\gamma}R^{\mu\nu\delta\gamma}$. The singularity at $r = M$, which also happens to be the location of the horizon of this black hole, can be eliminated by transforming the metric to *Eddington-Finkelstein* coordinates. To see how, we first rewrite Eq.(3.1) as $ds^2 = \Delta(r)\left(-dt^2 + \frac{dr^2}{\Delta(r)^2}\right) + r^2 d\Omega^2$. Then, we introduce a *tortoise* coordinate r^* and define

$$\frac{dr}{\Delta(r)} = dr^*$$

[§]A supercharge Q is a spinor (half-integer spin) that transforms fermions (half-integer spin) into bosons (spin zero or one).

so that $r^* \longrightarrow -\infty$ as $r \longrightarrow M$ (i.e. we put the horizon at $-\infty$ so that the metric is well-behaved at $r = M$). Hence, the metric in tortoise coordinate becomes

$$ds^2 = \Delta(r)\left(-dt^2 + (dr^*)^2\right) + r^2 d\Omega^2 \tag{3.2}$$

where r is a function of r^*. Moreover, the null rays travel along the radial direction (i.e. constant θ, ϕ) satisfying

$$ds^2 = 0 \implies \frac{dr^*}{dt} = \pm 1$$

So, the null radial geodesic are given by

$$t \pm r^* = \text{constant}$$

One can also express r^* as a function of r by integrating dr^* from definition to obtain

$$r^*(r) = r + 2M \ln|r - M| - \frac{M^2}{r - M} - 2M \ln M - M \tag{3.3}$$

However, notice that Eq.(3.3) still breaks down at $r = M$ even after placing the horizon at $-\infty$. To remove this artifact, we need to extend the metric beyond $r = M$ by introducing the ingoing Eddington-Finkelstein coordinates (v, r), where

$$v = t + r^*$$

The improved line-element of the extremal RN metric in these coordinates is

$$ds^2 = -\Delta(r)\, dv^2 + 2\, dv\, dr + r^2 d\Omega \tag{3.4}$$

which is regular at $r = M$.

A few important remarks are in order:

1. The function $\Delta(r)$ for standard RN metric (i.e. $Q < M$) has two real roots satisfying $r_+ - r_- > 0$. So, there are two coordinate singularities corresponding to the inner and outer horizons. Of course, there radii are just special cases of the Kerr-Newman black hole with $a \longrightarrow 0$, i.e.

$$r_+ = M + \sqrt{M^2 - Q^2}, \quad r_- = M - \sqrt{M^2 - Q^2} \tag{3.5}$$

2. In the non-extremal case, as usual, one can utilize a suitable coordinate system, such as Eddington-Finkelstein, to eliminate coordinate singularities. This approach is analogous to the one adopted in the extremal case. Once again, one can obtain an expression similar to Eq.(3.3) for the non-extremal case after rewriting $\Delta(r) = Q^2 - 2Mr + r^2 =$

$(r - r_+)(r - r_-)$ and solving for r^*

$$r^*(r) = r + \frac{1}{2\kappa_+} \ln \frac{r - r_+}{r} + \frac{1}{2\kappa_-} \ln \frac{r - r_-}{r} \tag{3.6}$$

where $\kappa_+ = \frac{r_+ - r_-}{2r_+^2} > 0$ and $\kappa_- = \frac{r_- - r_+}{2r_-^2} < 0$ are constants.

3. There is an important difference between Eq.(3.3) and Eq.(3.6). Notice that r^* is "inverse linear" in r in the extreme case and "logarithmic" in r in non-extreme case. So, there is a clear discontinuity between extremality and non-extremality. This is due the *instability of the horizon* of an extremal black hole [7].

One can also investigate the causal structure to see why there exists a *discontinuity* or a *tear* in the extremal RN spacetime. In Fig.1, notice the sudden "coalescence" of region II when going from non-extremality ($Q < M$) to extremality ($Q = M$).

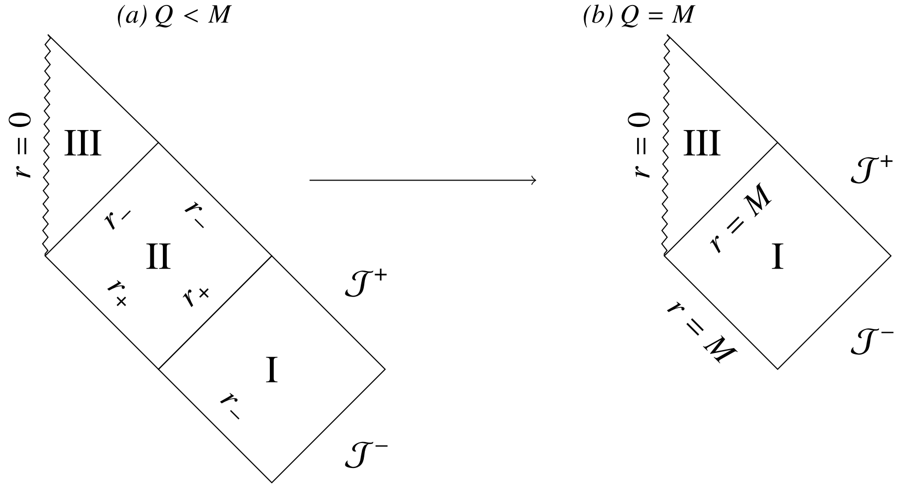

Figure 1: The Penrose diagram of (a) non-extremal, and (b) extremal Reissner-Nordström black hole, in ingoing Eddington-Finkelstein coordinates. The mirror counterparts are not shown here.

The two initially disconnected regions I and III, having different topologies in Fig.1-(a), 'suddenly' become connected in Fig.1-(b). Clearly, there is a discontinuity in how two different regions get connected. As pointed out earlier, this is due to the fundamental instability of the horizon of an extreme black hole. We will discuss about this instability problem in Sec.(5). But for now, we should remark certain subtleties associated with the assertion: "going" from non-extremality to extremality. This is not quite accurate because extreme black holes cannot be consistently obtained from the non-extreme black holes, since they are topologically different objects to begin with [23, 24]. The remarks in the following section will establish this more clearly.

3.2 Non-Extremality to Extremality?

In the previous section we saw that there is an instability between the two horizons of the extremal RN. So, is it really possible to go from a state of non-extremality to extremality? To answer this, let's first discuss two important features of an extreme black hole $Q = M$:

1. An extremal black hole has a vanishing surface gravity κ according to the third law of black hole thermodynamics. It states that it is impossible to achieve $\kappa = 0$ in finite number of operations.

   To see this, consider making a classical argument that one could achieve extremality in a finite number of operations by dropping a charge q with mass m, into a non-extremal RN black hole of mass M and charge $Q < M$. To achieve extremality, we would need $Q + q = M + m$. However, in order for the gravitational attraction to be greater than the coulomb repulsion, we must ensure that $Mm > Qq \ \ (*)$. Then rewriting this as

$$
\begin{aligned}
(Q + q - m)\, m &> Qq \\
Q\, m + q\, m &> Qq + m^2 \\
Q + q &> \frac{Qq}{m} + m
\end{aligned}
$$

   The last equality implies that for extremality one needs to identify $\frac{Qq}{m} == M$, However, condition $(*)$ disallows it, and we would need an infinite number of such operations to violate it. So, regardless of what we can practically do, we will always be left with a non-extremal black hole that has a well-defined surface gravity $\kappa \neq 0$. Therefore it is impossible to consistently reach extremality from non-extremality (a straight consequence of the third law)—*An extreme black hole has to be born extreme*.

2. Another important property is that extremal black holes do not exhibit Hawking radiation because $\kappa = 0$:

$$
\begin{aligned}
T_H &= \frac{\hbar}{2\pi k_B}\kappa \\
&= 0
\end{aligned} \tag{3.7}
$$

   On the other hand, non-extreme black hole has a well-defined Hawking temperature and entropy. So, these two types of black holes correspond to *different* thermodynamic systems, and one cannot be obtained from the other (as seen from the argument made in the previous point).

   A note on the entropy of an extremal black hole:

   Despite the fact that the Hawking temperature vanishes for an extreme black hole, it can still have a "non-zero entropy" since there is a "non-zero horizon area". This observation is rather non-trivial because on one hand, it had been suggested using semiclassical methods, that the entropy in the extremal limit is "zero" and that the traditional Bekenstein-Hawking formula, $S = A/4$, does not hold for extreme black holes [23]. But on the other hand, one indeed obtains a "non-zero entropy" of an extreme black hole by counting the near-horizon microstates[†] [25]. In fact both answers are indeed correct and non-contradictory because these methods calculate the entropy in topologically distinct regions of the extremal limit [24].

Finally, since an extreme black hole has vanishing surface gravity and no bifurcation surface, the Noether-Wald formalism cannot be used. However, if one regards an extremal black hole as an object arising in the extremal limit of a non-extremal black hole. That is, one considers a non-extremal black hole and then takes the extremal limit. Then, in this sense, the definitions of Wald are still applicable. The following section discusses how this limit is consistently obtained from non-extremality.

3.3 Universality of extremal black holes in the extreme limit

In generally covariant theories of gravity, all known asymptotically flat extremal black holes have a physical structure that is universal: The near-horizon geometry is locally AdS_2.

Exception: In string theory, there are extremal black holes with near horizon geometries that are described by the product of AdS_2 and an internal space which is a circle. In this case, the near horizon geometry is locally $AdS_3 \times S^1$, where S^1 is the circle [13].

Let's establish this result in $D = 4$ dimensions for a spherically symmetric extremal black hole (the RN metric). The non-extremal RN metric in Eq.(A.20) is asymptotically flat, i.e. in the limit $r \to \infty$ the metric approaches Minkowski. But, what about the extremal RN metric Eq.(3.1)? It is clearly not asymptotically flat as can be checked by taking $r \to \infty$. So we need to define a new type of limiting procedure to explore the asymptotic structure of the extremal RN metric as $r \to M$. One way to do this is by defining a scaling limit λ as $r \to M(1+\lambda)$ in Eq.(3.1) and expanding around $\lambda = 0$. This gives:

$$ds^2 = \underbrace{-\lambda^2 dt^2 + M^2 \lambda^{-2} d\lambda^2}_{\text{Poincaré patch of } AdS_2} + \underbrace{r^2 d\Omega^2}_{S^2} \tag{3.8}$$

where we have only kept the first non-trivial terms in $O(\lambda)$. So, the near-horizon of the extremal RN black hole is locally the Poincaré patch of AdS_2 (without the two-sphere S^2, because it does not affect the conformal boundary of AdS_2). Eq.(3.8) is also known as the *Robinson-Bertotti* metric. Moreover, **all four-dimensional, asymptotically-flat, spherically-symmetric**

[†]This approach relies on a consistent formulation of quantum gravity without reliance on string theory or supersymmetry, however there is a third approach that is particularly elegant and deserves a mention here — It is precisely in the extreme cases that the microscopic origins of a "non-zero" Bekenstein-Hawking entropy was *first* calculated via string theory [12].

extremal black holes have an $AdS_2 \otimes S^2$ factorization. To see why, we can transform Eq.(3.8) to the global AdS_2 coordinates with

$$t = \frac{\rho \sin\tau}{\cos\tau - \sin\theta}, \qquad \lambda = \frac{\cos\tau - \sin\theta}{\cos\theta}$$

Eq.(3.8) is now brought into the familiar form

$$ds^2 = \frac{\rho^2}{\cos^2\theta}\left(-d\tau^2 + d\theta^2\right) + \rho^2 d\Omega^2 \tag{3.9}$$

which confirms that it is indeed a product of a maximally extended AdS_2 times a two sphere S^2 of the same radius ρ. Note that $AdS_2 \otimes S^2$ is a type of a *compactification* solution in the sense that the Poincaré disk of AdS_2 is "compactified" on the S^2 sphere in the coordinate range $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $-\infty \le \tau \le \infty$.

In fact, a straightforward generalization of the previous statement in bold is that *all* asymptotically-flat, spherically-symmetric black holes in D dimensions have an $AdS_2 \otimes S^{D-2}$ product type compactification in their respective near-horizon geometries owing to the $SO(2,1)\times SO(D-1)$ isometry in the Einstein-Maxwell theory (even with higher derivatives). We can easily verify this by obtaining a general formula for the near-horizon geometry of a spherically-symmetric extremal black hole in $D = 4$[†]. To do so, first we rewrite the non-extremal RN solution (Eq.(A.20)) in the form

$$ds^2 = -f(r, r_\pm)dt^2 + \frac{1}{f(r, r_\pm)}dr^2 + r^2 d\Omega^2 \tag{3.10}$$

where $f(r, r_\pm) = \left(1 - \frac{r_-}{r}\right)\left(1 - \frac{r_+}{r}\right)$, with r_+ and r_- defined from Eq.(3.5). This is consistent with the fact that when we impose the condition of extremality $r = r_- = r_+ = M$, we recover Eq.(3.1).

Now, let's use a slightly different scaling limit that allows us to "zoom" in on the near-horizon region as well as reach extremality. We define the dimensionless coordinates τ, ρ

$$\tau = \frac{\lambda t}{r_+^2}, \qquad \rho = \frac{2r - r_+ - r_-}{2\lambda} \tag{3.11}$$

where λ is now a dimensionful parameter that measures the distance (or the "close-ness") between the inner and the outer horizon,

$$\lambda = \frac{r_+ - r_-}{2} \tag{3.12}$$

Note that ρ blows up when the horizons coincide. This is required because we want to put the horizon at infinity in the limit of extremality, so that observers sitting at asymptotic infinity of

[†]Both $D = 4$ and its generalization to an arbitrary number of dimensions is given in ref.(Sen 0708.1270)

the extremal RN black hole do not observe those redshifts or blueshifts that occur at the inner horizon of the non-extremal RN geometry.

In these new coordinates, Eq.(3.10) becomes

$$ds^2 = -r_+^4\left[\frac{(\lambda(\rho-1)+2\lambda)(\rho-1)}{\lambda(r_+ + \lambda(\rho-1))^2}\right]d\tau^2 + \left[\frac{\lambda(r_+ + \lambda(\rho-1))^2}{(\lambda(\rho-1)+2\lambda)(\rho-1)}\right]d\rho^2 + (r_+ + \lambda(\rho-1))^2\,d\Omega^2 \quad (3.13)$$

and taking the extremal limit $\lambda \to 0$ gives

$$ds^2 = r_+^2\left[-\left(\rho^2-1\right)d\tau^2 + \frac{d\rho^2}{\left(\rho^2-1\right)}\right] + r_+^2\,d\Omega^2 \quad (3.14)$$

Notice that just like Eq.(3.13), Eq.(3.14) is also a solution to the equations of motion for some scaling parameter λ.

The limiting procedure has some interesting consequences:

- The two horizons remain at a finite distance away from each other even at the limit of extremality. This is evident from Eq.(3.14) where $\rho = \pm 1$ is the location of the inner and the outer horizon at $\lambda \to 0$. Note that the r_+^2 factor is just a constant in our new coordinate system that was constructed using the variables $(\rho, \tau, \theta, \phi)$.
- The Penrose diagram in Fig.(1)(b) does not take into account the non-vanishing distance between r_+ and r_-. In fact, we only gave a simplistic representation of the space-time in terms of the ingoing Eddington-Finkelstein coordinates. A more accurate pictorial representation that takes into account the $AdS_2 \otimes S^2$ factor together with the limiting procedure is given in [24].
- A straightforward generalization of Eq.(3.14), consistent with the $SO(2,1)\times SO(D-1)$ isometry of $AdS_2 \otimes S^{D-1}$:

$$ds^2 = v_1\left[-\left(\rho^2-1\right)d\tau^2 + \frac{d\rho^2}{\left(\rho^2-1\right)}\right] + v_2\,d\Omega_{D-2}^2 \quad (3.15)$$

  where $v_i\ (i=1,2)$ are constants to be determined by solving the equations of motion. This generalization allows us to define the classical entropy function formalism by exploiting the $SO(2,1)\times SO(D-1)$ isometry in the near horizon geometry of extremal black holes, to which we will now turn.

4 The Entropy Function Formalism

In an original paper by Sen [21] in 2005, it was first demonstrated that the entropy function formalism could be used to derive the entropy of spherically symmetric extremal black hole solutions in various dimensions. This analysis relied on deriving the equations of motion in diffeomorphism covariant theories, and studying the effect of higher derivative terms on Wald entropy (Eq.(2.18) or its generalization given in [18]). In this development, no direct use of supersymmetry was made; rather these black holes were defined as objects with an $AdS_2 \otimes S^{D-1}$ near-horizon geometry. We shall review this formalism here in $D = 4$ dimensions.

4.1 Constructing the Entropy Function

First, consider an *arbitrary* theory of gravity with metric g_{ab} coupled to some Maxwell gauge fields A^i_μ and neutral scalar fields ϕ_s. We assume that $\sqrt{-g}\,\mathcal{L}$ is the Lagrangian density expressed as a function of the metric g_{ab}, the scalars ϕ_s, the field strengths F^i_{ab}, and the covariant derivatives these fields. Then consistent with the $SO(2,1) \times SO(3)$ symmetry in $D = 4$ dimensions, we already have an expression for the near horizon metric and other fields from Eq.(3.15)

$$\begin{gathered} ds^2 = v_1 \left[-\left(\rho^2 - 1\right) d\tau^2 + \frac{d\rho^2}{\left(\rho^2 - 1\right)} \right] + v_2\, d\Omega_2^2, \\ F^i_{\rho\tau} = e_i, \qquad F^i_{\theta\phi} = \frac{p_i}{4\pi} \sin\theta, \qquad \phi_s = u_s \end{gathered} \tag{4.1}$$

where v_1, v_2, e_i, p_i, u_s are constant and the subscript i denotes the type of gauge field A^i_μ corresponding to the electric or magnetic charge.

A note on dimensional reduction/compactification:

As mentioned earlier in Sec.(2.2.1), the action in this theory may contain higher derivative terms that may, for instance, come from the compactifications of string theory. These compactifications can dimensionally reduce the theory from $D = 4$ to $D = 2$ by compactifiying the higher derivative terms of a four-dimensional manifold $\mathcal{M}_4$ onto a two-dimensional manifold $\mathcal{M}_2$. The theory is then formulated on this compact manifold $\mathcal{M}_2$, where the "effective" physics from the higher derivative terms is described in terms of fields (for instance ϕ_s) that are functions on the manifold $\mathcal{M}_2$.

Moreover, in the presence of higher derivative terms, the near-horizon geometry of extremal black holes may still have an $AdS_2 \otimes S^2$ factorization, but the coefficients of the metric functions may be modified. This higher derivative terms can introduce new degrees of freedom and modify the behavior of the theory, which can in turn affect the geometry of the near-horizon region. The precise form of the near-horizon geometry here would depend on the specific theory and the form of the higher derivative terms. However, the entropy function formalism is still valid for all such modifications, as it does not depend on the specific structure of the higher derivative terms [21].

From the compactification point of view, the action I in $D = 4$ with a Lagrangian $\mathcal{L} = \mathcal{L}\left(g_{ab}, \phi_s, F^i_{ab}\right)$ for this theory reads

$$\begin{aligned} I &= \int d^4x \sqrt{-g^{(4)}}\, \mathcal{L}^{(4)}|_{AdS_2\times S^2} \\ &= \int_{AdS_2} d\tau d\rho \sqrt{-\gamma^{(2)}} \underbrace{\int_{S^2} d\theta d\phi \sqrt{-h^{(2)}}\, \mathcal{L}^{(4)}|_{AdS_2\times S^2}}_{\mathcal{L}^{(2)}|_{AdS_2}} \end{aligned}$$

where we have where we split the four-dimensional determinant $\sqrt{-g^{(4)}}$ into two two-dimensional determinants $\sqrt{-\gamma} = v_1$ and $\sqrt{-h} = v_2$ (or induced metrics in AdS_2 and S^2 respectively), and defined the whole integral over S^2 to be $\mathcal{L}^{(2)}|_{AdS_2}$, which is just a two-dimensional Lagrangian density on the remaining geometry

$$\begin{aligned} \mathcal{L}^{(2)}|_{AdS_2} &:= \int_{S^2} d\theta d\phi \sqrt{-h^{(2)}}\, \mathcal{L}^{(4)}|_{AdS_2\times S^2} \\ &= 4\pi v_2\, \mathcal{L}^{(4)}|_{AdS_2\times S^2} \end{aligned} \tag{4.2}$$

One can think of $\mathcal{L}^{(2)}|_{AdS_2}$ as a two-dimensional theory obtained after integrating out the S^2 sphere from the Lagrangian of the full four-dimensional theory. Moreover, due to the compact nature of v_2 (which effectively arises from the compact object S^2), it becomes a quotient of the AdS_2 by a translation of 4π along AdS_2.

Plugging Eq.(4.2) back into the action I gives

$$\begin{aligned} I &= \int d\tau d\rho \sqrt{-\gamma^{(2)}}\, \mathcal{L}^{(2)}|_{AdS_2} \\ &= \int d\tau d\rho\ 4\pi v_1 v_2\, \mathcal{L}^{(4)}|_{AdS_2\times S^2} \\ &= \int d\tau d\rho\ f(u, v, e\,;\, p) \end{aligned} \tag{4.3}$$

where in the last line we have defined a function

$$f(u, v, e\,;\, p) = 4\pi v_1 v_2\, \mathcal{L}^{(4)}|_{AdS_2\times S^2} \tag{4.4}$$

where the variable and fixed parameters are separated by a "$;$". In our case, the variable parameters are u_s, v_1, v_2, e_i, while the magnetic field parameter p_i is a fixed quantity (due to the Maxwell equation in Eq.(4.8)).

Now, consider the entropy function constructed by Sen in [21] for an $SO(2,1)\times SO(3)$ invariant theory in the near-horizon geometry of an extremal black hole

$$\mathcal{E}\,(u, v, e\,;\, p, q) = 2\pi \left(e_i \frac{\partial f}{\partial e_i} - f(u, v, e\,;\, p) \right) \tag{4.5}$$

where the fixed parameter q entering the function is a direct consequence of Eq.(2.16).

4.2 Deriving Wald Entropy

To derive the Wald entropy from Eq.(4.5) we need to obtain the three equations of motion corresponding to three free parameters u, v, e. First, we extremize the function $\mathcal{E}(u, v, e\,;\, p, q)$ with variables u_s and v_i, which essentially amounts to extremizing the corresponding scalar and metric field equations for the near-horizon geometry from the function f:

$$\frac{\partial \mathcal{E}}{\partial u_s} = \frac{\partial \mathcal{E}}{\partial v_i} = 0, \tag{4.6}$$

$$\frac{\partial f}{\partial u_s} = \frac{\partial f}{\partial v_i} = 0 \tag{4.7}$$

Eq.(4.7) are the equations of motion for parameters u_s and v_i. Now, we still need to find the equation of motion for variable e. This will come from the non-trivial Maxwell equations that are satisfied on the near horizon geometry

$$\partial^\rho F^i_{\theta\phi} = \partial^\rho p_i = 0, \qquad \partial_\rho \left(\frac{\delta I}{\delta F^i_{\rho\tau}} \right) = \partial_\rho \left(\frac{\partial f(u, v, e\,;\, p)}{\partial e_i} \right) = 0 \tag{4.8}$$

where the first equation tells us that the magnetic charge p_i is a constant (i.e. the integral of the magnetic flux at an asymptotic infinity is identified with a constant magnetic charge p_i on the horizon), and the second equation follows from Eq.(4.3), and tells us that $\partial f/e_i$ is a constant and equal to the charge q_i (i.e. the integral of the electric flux at an asymptotic infinity is identified with a constant charge q_i). Thus, the following equation governs the flow of the charge q_i in the near-horizon geometry of the extremal black hole

$$\frac{\partial f}{\partial e_i} = q_i \tag{4.9}$$

Now, extremizing the entropy function with respect to e_i gives us the third required equation

$$\frac{\partial \mathcal{E}}{\partial e_i} = 0 \tag{4.10}$$

The extremization equations (4.6) and (4.10) typically determine the values of u_s, v_i, e_i in terms of the electric and magnetic charges q_i, p_i. Under Sen's construction, the Wald entropy is just the extremum value of the near-horizon solution

$$S_{\text{Wald}}(p_i, q_i) := \mathcal{E}_*(p_i, q_i) \tag{4.11}$$

where the extremum $\mathcal{E}_*(p_i, q_i) = \mathcal{E}(u_*(p_i, q_i), v_*(p_i, q_i), e_*(p_i, q_i))$ is a function of electric and magnetic charges.

The entropy function formalism ultimately boils down to solving some algebraic equations to efficiently compute the entropy of a variety of extremal black holes even in the presence of higher derivative corrections.

Computing the Wald Entropy of Extremal Reissner Nordström in $D = 4$:
To illustrate the formalism in use, let's compute the entropy of an extremal RN black hole. The Lagrangian of an RN black hole is

$$\mathcal{L} = \left[\frac{R}{16\pi G} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right] \tag{4.12}$$

which acts on the $AdS_2 \otimes S^2$ background via Eq.(3.15) in $D = 4$. In this case, we should find

$$\sqrt{-g} = v_1 v_2 \sin\theta, \qquad R = \left[\frac{2}{v_2} - \frac{2}{v_1}\right], \qquad F_{\mu\nu} F^{\mu\nu} = \frac{2}{v_1^2}\, e^2 - \frac{2}{v_2^2} \left(\frac{p}{4\pi}\right)^2 \tag{4.13}$$

To compute the Wald entropy via Eq.(**??**), we need to find the extremum of the entropy function $\mathcal{E}$. Consequently, one requires the knowledge of $f(u, v, e\,;\, p)$. Using Eq.(4.4), we can easily compute f for the near-horizon background as

$$\begin{aligned} f(v_1, v_2, e\,;\, p) &= 4\pi v_1 v_2\, \mathcal{L}^{(4)}|_{AdS_2 \times S^2} \\ &= 4\pi v_1 v_2 \left(\frac{R}{16\pi G} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right) \\ &= 4\pi v_1 v_2 \left(\frac{1}{16\pi G}\left(\frac{2}{v_2} - \frac{2}{v_1}\right) + \frac{1}{2v_1^2}\, e^2 - \frac{1}{2v_2^2}\left(\frac{p}{4\pi}\right)^2\right) \end{aligned} \tag{4.14}$$

Plugging this into the entropy function given in Eq.(4.5) and simplifying gives

$$\mathcal{E}\,(u, v, e\,;\, p, q) = 2\pi \left(e\, q - \frac{1}{4G}\,(2v_1 - 2v_2) - \frac{2\pi v_2}{v_1} e^2 + \frac{2\pi v_1}{v_2}\left(\frac{p}{4\pi}\right)^2\right) \tag{4.15}$$

Now, we one easily extremize $\mathcal{E}$ with variables v_1, v_2, e using equations

$$\frac{\partial \mathcal{E}}{\partial v_1} = 0, \qquad \frac{\partial \mathcal{E}}{\partial v_2} = 0, \qquad \frac{\partial \mathcal{E}}{\partial e} = 0 \tag{4.16}$$

One finds that the extremum value $\mathcal{E}_*(p,q)$ occurs at

$$v_1 = v_2 = \frac{G\left(q^2+p^2\right)}{4\pi}, \qquad e = \frac{q}{4\pi} \tag{4.17}$$

Substituting these extremized values back into Eq.(4.15) we get an expression for the Wald entropy:

$$\begin{aligned} S_{\text{Wald}} = \mathcal{E}_*(p,q) &= 2\pi\left(\frac{q^2}{4\pi} - 2\pi\left(\frac{q}{4\pi}\right)^2 + 2\pi\left(\frac{p}{4\pi}\right)^2\right) \\ &= 2\pi\left(\frac{q^2+p^2}{8\pi}\right) \\ &\boxed{S_{\text{Wald}} = \frac{q^2+p^2}{4}} \end{aligned} \tag{4.18}$$

This is the Wald entropy carried by the near-horizon $AdS_2 \otimes S^2$ patch of the extremal RN black hole, which is clearly in agreement with the Bekenstein-H awking entropy for an extremal RN black hole.

4.3 The Attractor Mechanism

The equations of motion that one obtains from Eq.(4.7) and Eq.(4.9) completely determine the $AdS_2 \otimes S^2$ background in terms of only electric and magnetic charges q_i and p_i. This fact is consistent with the attractor mechanism†, which says that the near-horizon configuration of a black hole (for instance Eq.(3.15)) depends only on the electric and magnetic charges carried by the black hole and not on the asymptotic values of the scalar fields ϕ_s in the background (also known as *moduli*). To see this clearly, first note that the entropy function in Eq.(4.5) is independent of the scalar fields ϕ_s. Thus, if the extremization equations

$$\frac{\partial\mathcal{E}}{\partial v_1} = 0, \qquad \frac{\partial\mathcal{E}}{\partial v_2} = 0, \qquad \frac{\partial\mathcal{E}}{\partial e_i} = 0, \qquad \frac{\partial\mathcal{E}}{\partial u_s} = 0 \tag{4.19}$$

completely determine each set of parameters u, v, e uniquely then the value of $\mathcal{E}_*$, and hence the Wald entropy S_{Wald}, is independent of the asymptotic values of the scalars ϕ_s. This is the generalization of the usual attractor mechanism which involves defining a black hole potential. A discussion about what this implies for the entropy function formalism is given in Sec.(5.3).

†First observed in N=2 extremal magnetic black holes in supergravity [26] and later extended to non-supersymmetric backgrounds in four-dimensions [27].

5 Open Discussion/Comments

This section serves as an addendum to some observations made in the first two sections.

5.1 Why go symplectic?

We made a big jump from working in Einstein gravity to a canonically covariant notion of gravity in Sec.(2.2), perhaps a justification for the usage of the symplectic language of differential forms is required. Noether-Wald formalism is built upon the symplectic structure of abstract spaces of fields through the covariant phase space formalism. Linking this structure to lower-degree forms (like Eq.(2.14)) is usually how one specifies conserved charges in a diffeomorphism covariant gravity (see [17] for a thorough introduction). There is a very instructive analogy between Riemannian and Symplectic geometry (after [28, 29]):

Comparison	Hamiltonian symplectic structure	Minkowski spacetime metric structure
1. Canonical Coordinates	q^1, q^2, p_1, p_2	t, x, y, z
2. Canonical structure	$\Theta = dp_1 \wedge dq^1 + dp_2 \wedge dq^2$	$ds^2 = -dt \otimes dt + dx \otimes dx$ $+dy \otimes dy + dz \otimes dz$
3. Nature of "metric"	antisymmetric	symmetric
4. Field equations	$d\Theta$=0: closed $(D-2)$ form	$R_{\alpha\beta\mu\nu} = 0$: flat spacetime
5. Manifold in D=4	phase space	spacetime
6. Cauchy problem	$d\Theta = 0$	$R_{\mu\nu} = 0$

Table 1: The difference between the symplectic structure and the Minkowski spacetime structure.

The covariant phase space formalism is built upon these canonical coordinates that allow us to define all variables in terms of generalized coordinates q and momenta p. The advantage of this approach is that it removes the explicit dependence of the metric structure on time t, making it easier to quantize gravitational systems using standard techniques from quantum field theory. In particular, the covariant phase space formalism provides a framework for defining a quantum theory of gravity that is consistent with general covariance.

5.2 On the horizon instability

In general relativity, the notion of "stability" can be understood only after the Cauchy initial value problem has been formulated: $R_{\mu\nu} = 0$. The stability of any black hole solution is then proven by perturbing it with a scalar field ϕ and observing if it dies off at asymptotically late times. The more curious case is that of extremal black holes because they exhibit a discontinuity as mentioned in Sec.(3.1).

It is known that for non-extreme black holes, the *redshift effect* is enough to prove their linear stability [30]. Consider the local redshift effect as follows: Observer A and B cross the event horizon $\mathcal{H}^+$ at time t and $t + t'$ respectively. The photons sent by A to B undergo a redshift proportional to $e^{-\kappa v}$ where κ is the surface gravity and v is the ingoing Eddington-Finkelstein time coordinate (see Fig.(2) [30]). This implies that for a non-extreme black hole, the surface gravity $\kappa \neq 0$, so the radiations reaching B will decay. But for an extreme black hole, the surface gravity $\kappa = 0$, and the radiations will not decay! It is as if extreme black holes have "hair" on the horizon.

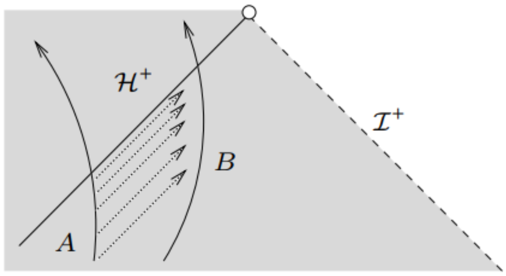

Figure 2: Redshift effect between observers A and B who cross the horizon $\mathcal{H}^+$ at time t and $t + t'$ respectively. The radiations do not decay for $\kappa = 0$ i.e. an extreme black hole appears to have "hair."

Some other possible sources of instability of the horizon of a black hole are:

- *Mass inflation instability*: When an electrically charged black hole is perturbed, its inner horizon becomes a singularity, often referred to as the *Poisson-Israel mass inflation instability*[§]. For example, consider the extreme RN black hole where we had an ingoing Eddington-Finkelstein coordinate Fig.(1). We can also draw its mirror counter part, which will be an outgoing Eddington-Finkelstein coordinate. Together, these would make a maximally extended RN spacetime (see Fig.1 in [32]). The mass inflation instability occurs when the ingoing and the outgoing streams of particles create an outward radial pressure that is stronger than the gravitational force produced by the background. This causes the interior mass to "inflate" exponentially. All gauge-invariant scalars also exponentiate.
- *Pair Production*: In the extremal case, the electric field near the horizon is extremely strong, leading to a large number of charged particles being produced by the *Schwinger effect*. The existence of this effect shows the violation of the Breitenlohner-Freedman bound in the near-extremal RN black hole [33]

[§]The exact mass-inflation solution of the Einstein-Maxwell equations shows that the horizon singularity is weak enough that its tidal forces remain finite but the curvature diverges [31]

It was also recently suggested in [8] that almost all extremal AdS blackholes have a curvature singularity at the horizon where all gauge invariant scalars remain finite and the tidal forces diverge. Although, the numerical techniques used in this paper to showcase the existence of divergences are based on the Euler approximation. Euler's method can be unstable for large step sizes and the error increases exponentially at late times. They also perform mode analysis in the paper to setup their differential equations, however one needs to be careful with rotating cases there is the existence of superradiance due to scalar perturbations. Aretakis's approach [7] is a better one to prove the existence of instability and divergences in such cases.

5.3 On the entropy function and attractor mechanism

In Eq.(3.15) of Sec.(4), we gave a general form of the $AdS_2 \otimes S^2$ and the fields living on it, in an arbitrary theory of gravity with $U(1)$ gauge fields and neutral scalars ϕ_s in $D = 4$ dimensions. Now, because it is a general theory of gravity, any form of the Lagrangian density $\sqrt{-g}\,\mathcal{L}$ should work as long as it is diffeomorphism-invariant. Although, it may be unclear why ϕ_s was included in the first place. For example, we could have a Lagrangian of the form

$$\mathcal{L} = \mathcal{L}\left(R - 2(\partial\phi_s)^2 - f_{ab}(\phi_s)F^a_{\mu\nu}F^{b\ \mu\nu}\right)$$

which includes a kinetic term of ϕ_s and a dilaton-like coupling[†] to the gauge fields. But, as shown in [34], such a Lagrangian still has an extremum $\phi_s = \phi_{s*}$ on the horizon for which $\frac{\partial f}{\phi_{s*}}$ vanishes. Thus, the attractor equations are always manifest for such theories of gravity.

However, the entropy function formalism does not tell us whether the full black-hole solution, interpolating between $AdS_2 \otimes S^{D-1}$ near-horizon geometry and the asymptotically flat Minkowski space, really exists. This is where supersymmetry comes into play to determine the existence of such a solution.
Although note that for the Lagrangian given above, it was shown in [34] that such a solution does in fact exist as long as the matrix of second derivatives of f with respect to the scalar field values at the horizon takes positive eigenvalues at the extremum, without having to invoke supersymmetry. For a more detailed review of the entropy function formalism applied to different types of black holes, the reader is strongly encouraged to refer [13] and the references therein.

[†]For example in Kaluza–Klein theories, after dimensional reduction, the effective theory varies as some power of the volume of compactified space. This volume can appear as a dilaton-field in the lower-dimensional effective theory [34]

A Black Holes in Kerr-Newman Family

This appendix serves as an introduction to some geometrical and thermodynamic features of the Kerr-Newman family at a classical level. Since one of the motivations behind the writing of this thesis is to be accessible to advanced undergraduates, we have chosen to discuss these topics here. Black holes beyond Schwarzschild are usually not covered in a one-semester undergraduate course in general relativity, so it would be nice to introduce these more exciting black holes here. However, it should be noted that we have performed explicit computations at all places, so this appendix should serve as a quick conceptual toolkit rather than a complete reference. The reader is encouraged to refer standard textbooks like [35] and instructive lecture notes by [36]. At a graduate/research level, [37, 38] are better resources.

We begin our analysis with the Kerr black hole in $D = 4$ dimensions. The line element is:

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 + g_{rr}dr^2 + g\theta\theta d\theta^2 \tag{A.1}$$

where the coefficients are

$$\begin{aligned} g_{tt} &= 1 - \frac{2GMr}{\rho^2}, \quad g_{t\phi} = -\frac{2GMar\sin^2\theta}{\rho^2}, \\ g_{\phi\phi} &= \frac{\sin^2\theta}{\rho^2}((r^2+a^2)^2 - a^2\Delta\sin^2\theta), \\ g_{rr} &= \frac{\rho^2}{\Delta}, \quad g_{\theta\theta} = \rho^2 \end{aligned}$$

with

$$\begin{aligned} \Delta(r) &= r^2 - 2GMr + a^2, \\ \rho(r,\theta) &= r^2 + a^2\cos^2\theta \\ a &= J/M \end{aligned}$$

All the metric coefficients are functions of the coordinates r, θ but independent of the t, ϕ. So, one can simply make the identification that the Killing vectors are ∂_t and ∂_ϕ. This fact will be used at several places to uncover interesting dynamics of the geometry. But first, let's understand the geometry.

A.1 Geometry and Thermodynamics of Kerr

In this section, we will discuss some geometrical aspects of Kerr and ultimately use them to derive the first law of black hole thermodynamics.

A.1.1 Horizons

The metric in Eq.(A.1) has been expressed in Boyer-Lindquist coordinates and has a coordinate singularity at $g^{rr} = \Delta/\rho^2 = 0$. This implies that $\Delta(r) = 0$, giving us two values of r that correspond to the outer and inner horizons of the black hole†:

$$r_+ = M + \sqrt{M^2 - a^2}$$
$$r_- = M - \sqrt{M^2 - a^2}$$

A.1.2 Stationary limit and the Ergoregion

Consider static observers in Kerr spacetime. Their four-velocity will be proportional to the Killing vector ∂_t. However, ∂_t will not be timelike everywhere in Kerr, and in fact becomes null when:

$$g_{tt} = 0$$

Solving this gives us two values of r

$$[r_+]_{\text{s.lim}} = M + \sqrt{M^2 - a^2 \cos^2\theta}$$
$$[r_-]_{\text{s.lim}} = M - \sqrt{M^2 - a^2 \cos^2\theta}$$

which correspond to the stationary limits (s.lim) of the outer '+' and inner '–' horizons.

Now, consider the stationary limit of the outer horizon, i.e. $[r_+]_{\text{s.lim}}$. At $\theta = 0$, the stationary limit becomes equal to outer horizon r_+, and at $\theta = \pi/2$, it becomes equal to the Schwarzschild radius $r_s = 2M$. However, at $\theta = \pi$, it becomes:

$$[r_+]_{\text{s.lim}} = M + \sqrt{M^2 - a^2 \cos^2\pi} = M + \sqrt{M^2 + a^2}$$

So, the region between r_+ and $[r_+]_{\text{s.lim}}$ at $\theta = \pi$ is the *ergoregion* or *ergosphere* . Here's a typical illustration of an ergosphere: Observers in the ergoregion shown in Fig.(3) are called *stationary observers* because they have to co-rotate with the black hole with the help of an external agent (like a rocket engine) to ensure that they do not fall-in towards the horizon. This is an example of a non-geodesic motion.

†I will use the Gaussian units hereof: $G = 1$, $c = 1$, $\hbar = 1$

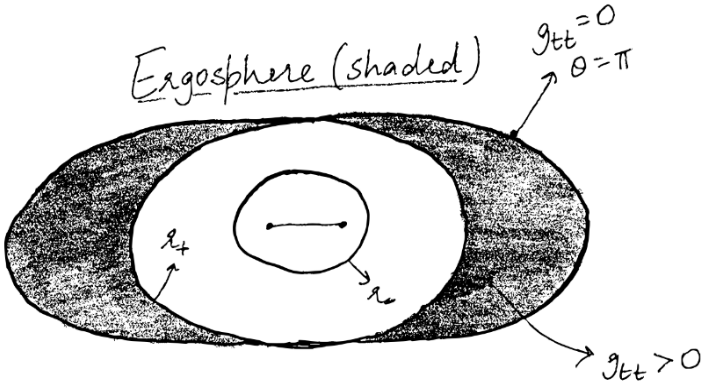

Figure 3: Ergosphere (shaded) of a Kerr black hole. The boundary of the ergosphere is the stationary limit. Observers can remain stationary in this region with the help of a rocket engine.

A.1.3 Rotational velocity, Area, and Entropy

Stationary observers in the ergosphere have to move in the ϕ direction with a uniform velocity Ω. These observers do not perceive any time variation in the black hole's gravitational velocity and move with a four-velocity proportional to the linear combination $\partial_t + \Omega\partial_\phi$. Note that the linear combination of Killing vectors is also a Killing vector, so

$$K = K^u_{(t)} - \Omega K^u_{(\phi)} \tag{A.2}$$

where $K^u_{(t)} = \partial_t = (1, 0, 0, 0)$ is a time-like Killing vector, and $K^u_{(\phi)} = \partial_\phi = (0, 0, 0, 1)$ is a space-like Killing vector that traces out the axial angle.

This linear combination is only future-pointing and time-like everywhere exterior to the horizon; it is null at the horizon itself. So, a stationary observer on the horizon will essentially rotate with the rotational velocity of the black hole Ω, and the linear combination in Eq.(A.2) becomes zero. This is because the linear combination of rotations and time-translation of Killing vectors on a Killing horizon is null for spinning black holes. This fact will later be exploited to calculate the surface gravity of the black hole (Sec.(A.4)). For now, we would like to find Ω such that Eq.(A.2) is zero. First, we compute K from Eq.(A.2)

$$K = (1, 0, 0, 0) - \Omega_{BH}\,(0, 0, 0, 1) = (1, 0, 0, -\Omega_{BH}) \tag{A.3}$$

At the horizon $K = 0$, so in components this will become $0 = K^T g_{\mu\nu} K$, where $g_{\mu\nu}$ is the Kerr metric defined in Eq.(A.1)— but with constant r and θ. The rotational velocity Ω is then computed as follows:

$$
\begin{aligned}
0 &= \begin{pmatrix} 1 & 0 & 0 & -\Omega_{BH} \end{pmatrix} \begin{pmatrix} g_{tt} & 0 & 0 & g_{t\phi} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ g\phi t & 0 & 0 & g_{\phi\phi} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -\Omega \end{pmatrix} \\
0 &= \Omega^2 g_{\phi\phi} + 2g_{t\phi}\Omega_{BH} - g_{tt} \\
\Omega &= \frac{-2g_{t\phi} \pm \sqrt{4g_{t\phi}^2 + 4g_{\phi\phi}g_{tt}}}{2g_{\phi\phi}} \\
\Omega &= \frac{-g_{t\phi}}{g_{\phi\phi}} \pm \frac{\sqrt{\left(r_+^2 - 2Mr_+ + a^2\right)\sin^2\theta}}{g_{\phi\phi}} \\
\Omega &= \frac{-g_{t\phi}}{g_{\phi\phi}} \pm \frac{\sqrt{\Delta|_{r_+}\sin^2\theta}}{g_{\phi\phi}} \\
\Omega &= \frac{-g_{t\phi}}{g_{\phi\phi}} \qquad \ldots \text{(because } \Delta(r_+) = 0\text{)}
\end{aligned}
$$

So, we finally have:

$$\boxed{\Omega = \frac{a}{r_+^2 + a^2}} \tag{A.4}$$

The area of a Kerr black hole can also be calculated. We first need to find the area element δA on the boundary of Kerr, which is a two-sphere. Note that this will only depend on θ and ϕ since at fixed $r = r_+$ and $t = k$, the dr and dt terms vanish. With these parameters, the metric in Eq.(A.1) can be reduced to a two-sphere defined by the line-element

$$ds^2 = g_{\phi\phi}(r_+)d\phi^2 + g_{\theta\theta}(r_+)d\theta^2 \tag{A.5}$$

Now, the determinant of this two-sphere will just be $g_{\phi\phi}(r_+)$ times $g_{\theta\theta}(r_+)$, and has the area element $\delta A = \sqrt{g_{\phi\phi}(r_+) \cdot g_{\theta\theta}(r_+)}$. So, the total area will be:

$$
\begin{aligned}
A &= \iint \delta A \, d\phi \, d\theta \\
&= \iint \sqrt{g_{\phi\phi}(r_+) \cdot g_{\theta\theta}(r_+)} \, d\phi \, d\theta \\
&= \iint \left(r_+^2 + a^2\right) sin\theta \, d\phi \, d\theta \\
&= 4\pi \left(r_+^2 + a^2\right) \\
&\boxed{A = 8\pi M r_+}
\end{aligned} \tag{A.6}
$$

where I have used the usual periodicity condition on θ and ϕ to evaluate the integral. The final equality comes from substituting the value of r_+. Comparing Eq.(A.4) and Eq.(A.6), we find

$$A = \frac{4\pi a}{\Omega} \tag{A.7}$$

Eq.(A.7) implies that faster spinning black holes are smaller in size (and vice versa).

The macroscopic Bekenstein-Hawking entropy is just one-quarter of area:

$$S = A/4 = 2\pi M r_+ \tag{A.8}$$

$$= 2\pi \left(M^2 + \sqrt{M^4 - J^2}\right) \tag{A.9}$$

A.1.4 Verifying the First Law

We are now in a good position to verify the first law of thermodynamics. On taking the total differential of Eq.(A.9) we have

$$dS = 4\pi M dM + \frac{\pi}{\sqrt{M^4 - J^2}} \left(4M^3 dM - 2J dJ\right) \tag{A.10}$$

We can express the angular momentum J as $J = \Omega \left(M^2 + \sqrt{M^4 - J^2}\right)$ from Eq.(A.4), and rewrite Eq.(A.10) as follows

$$\begin{aligned} dS &= 4\pi M dM + \frac{4\pi M^3 dM}{\sqrt{M^4 - J^2}} - \frac{2\pi J dJ}{\sqrt{M^4 - J^2}} \\ &= dM \left(4\pi M + \frac{4\pi M^3}{\sqrt{M^4 - J^2}}\right) - \frac{4\pi M \Omega \left(M^2 + \sqrt{M^4 - J^2}\right) dJ}{\sqrt{M^4 - J^2}} \\ &= dM \left(4\pi M + \frac{4\pi M^3}{\sqrt{M^4 - J^2}}\right) - \Omega \left(\frac{4\pi M^3}{\sqrt{M^4 - J^2}} + 4\pi M\right) dJ \end{aligned}$$

Therefore,

$$dS = \frac{dM}{T} - \frac{\Omega}{T} dJ$$

where we assumed $1/T = \left(4\pi M + \frac{4\pi M^3}{\sqrt{M^4 - J^2}}\right)$. This verifies the first law:

$$\boxed{T dS = dM - \Omega_{BH} dJ} \tag{A.11}$$

A.2 The Kerr-Newman Case

Generalizing our results from the previous section to the Kerr-Newman case ($Q \neq 0$) is straightforward. We can simply make the replacement $2GMr \rightarrow 2GMr - Q^2$ in Eq.(A.1) to arrive at the Kerr-Newman solution. The Kerr-Newman metric becomes:

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}\, dt d\phi + g_{\phi\phi}d\phi^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 \tag{A.12}$$

where the metric coefficients are

$$\begin{aligned}
g_{tt} &= 1 - \left(\frac{2GMr - Q^2}{\rho^2}\right), \quad g_{t\phi} = -a\frac{\left(2GMr - Q^2\right)\sin^2\theta}{\rho^2}, \\
g_{\phi\phi} &= \frac{\sin^2\theta}{\rho^2}((r^2 + a^2)^2 - a^2\Delta\sin^2\theta), \\
g_{rr} &= \frac{\rho^2}{\Delta}, \quad g_{\theta\theta} = \rho^2
\end{aligned}$$

with

$$\begin{aligned}
\Delta(r) &= r^2 - 2GMr + Q^2 + a^2, \\
\rho(r,\theta) &= r^2 + a^2\cos^2\theta \\
a &= J/M
\end{aligned}$$

The geometry and thermodynamics of Kerr-Newman is qualitatively no different from Kerr, except that now there is a charge Q. For Kerr-Newman family of stationary black hole solutions we have:

- The location of the horizons:

$$\begin{aligned}
r_+ &= M + \sqrt{M^2 - a^2 - Q^2} \\
r_- &= M - \sqrt{M^2 - a^2 - Q^2}
\end{aligned} \tag{A.13}$$

- The location of the stationary limit surfaces:

$$\begin{aligned}
[r_+]_{\text{s.lim}} &= M + \sqrt{M^2 + a^2 - Q^2} \\
[r_-]_{\text{s.lim}} &= M - \sqrt{M^2 + a^2 - Q^2}
\end{aligned} \tag{A.14}$$

- The rotational velocity in the outer ergoregion:

$$\begin{aligned}
\Omega &= \frac{a}{r_+^2 + a^2} \\
&= \frac{a}{\left(M + \sqrt{M^2 + a^2 - Q^2}\right)^2 + a^2}
\end{aligned} \tag{A.15}$$

- The area of the largest horizon:

$$\begin{aligned}
A &= 8\pi r_+ M \\
&= 8\pi M\left(M + \sqrt{M^2 - a^2 - Q^2}\right)
\end{aligned} \tag{A.16}$$

- The Bekenstein-Hawking entropy:

$$\begin{aligned} S &= 2\pi r_+ M \\ &= 2\pi M \left(M + \sqrt{M^2 - a^2 - Q^2}\right) \end{aligned} \tag{A.17}$$

These equations above give us the geometry and classical thermodynamics of Kerr (with $Q \to 0$), Reissner-Nordström (with $a \to 0, Q \neq 0$), and Schwarzschild (with $a = Q = 0$).

A.3 The Reissner-Nordström Black hole

The Reissner-Nordström (RN) metric, is a static, asymptotically-flat, spherically symmetric solution of the Einstein-Maxwell equations. As noted previously, it can be obtained from Eq.(A.12) by setting $a \to 0$. The Lagrangian is of the form

$$\mathcal{L} = \sqrt{-g}\left[\frac{1}{16\pi G}R - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right] \tag{A.18}$$

The equations of motion are given by the Maxwell and the Einstein-Maxwell equations

$$\nabla^\mu F_{\mu\nu} = 0, \qquad G_{\mu\nu} = 8\pi G\left(F_{\mu\lambda}F_\nu{}^\lambda - \frac{1}{4}g_{\mu\nu}F_{\mu\nu}F^{\mu\nu}\right) \tag{A.19}$$

The metric in spherically-symmetric form is given by

$$ds^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \frac{1}{\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)}dr^2 + r^2 d\Omega^2 \tag{A.20}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$.

Now, consider the electromagnetic field tensor $F_{\mu\nu}$. We can re-express this in terms of the electromagnetic 4-vector potential:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{A.21}$$

Since the Reissner-Nordström solution corresponds to a static, spherically-symmetric spacetime, we can assume that the potential A is a function of r exclusively. This is because any physical observable U on a static and a spherically-symmetric spacetime only depends on the radial direction[†]. However, we can still make an ansatz for the general form of A as a function of t, r, θ, ϕ. Solving the Einstein field equations will then ultimately demonstrate that A indeed depends on r.

[†]This is because for a spherical symmetry the spatial directions are not unique, and so their contributions to some physical observable U will ultimately cancel out. As for time, since the spacetime is static, no time derivatives survive.

Given the symmetries discussed above, we observe that the partial derivatives of $A(t, r, \theta, \phi)$ with respect to t, θ, and ϕ vanish. So, the most general form of the field tensor $F_{\mu\nu}$ according to equation Eq.(A.21) will become:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -A(t, r, \theta, \phi)' & 0 & 0 \\ A(t, r, \theta, \phi)' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{A.22}$$

where the prime $'$ denotes a derivative with respect to r.

To solve the Einstein field equations, we would first like to know the form of the stress-energy tensor $T_{\mu\nu}$. For curved space time, the stress-energy tensor in terms of the electromagnetic field tensor $F^{\mu\nu}$ takes the form:

$$T_{\mu\nu} = F^{\mu\alpha} F_{\alpha}{}^{\nu} - \frac{1}{4} g_{\mu\nu} F^{\mu\nu} F_{\mu\nu} \tag{A.23}$$

Substituting Eq.(A.22)in Eq.(A.23), we get

$$T_{\mu\nu} = \begin{pmatrix} \frac{r^2 (\boldsymbol{A}')^2}{2\Delta} & 0 & 0 & 0 \\ 0 & \frac{\Delta (\boldsymbol{A}')^2}{2r^2} & 0 & 0 \\ 0 & 0 & -\frac{(\boldsymbol{A}')^2}{2r^2} & 0 \\ 0 & 0 & 0 & -\frac{\csc\theta (\boldsymbol{A}')^2}{2r^2} \end{pmatrix} \tag{A.24}$$

where $\boldsymbol{A}' = A(t, r, \theta, \phi)'$, and $\Delta = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)$.

Using the Einstein field equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$ for the metric (A.20), we arrive at a differential equation of the form

$$(\boldsymbol{A}') + \frac{Q}{r^2} = 0 \tag{A.25}$$

which can also be easily solved for A:

$$A(r) = \frac{Q}{r} \tag{A.26}$$

The form of A now reveals that it only depends on r as was expected.

Now, the electromagnetic field tensor Eq.(A.22) can be written as

$$F_{\mu\nu} = \begin{pmatrix} 0 & \frac{Q}{r^2} & 0 & 0 \\ -\frac{Q}{r^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which gives us the associated component of the electric field trivially,

$$E_r = \frac{Q}{r^2} \tag{A.27}$$

Note that the surface integral $\oint \vec{E} \cdot ds$ of Eq.(A.27) is exactly equal to the charge Q. So essentially, the RN black hole acts like a point charge in the asymptotic limit. This ultimately verifies Gauss's law for a Reisnner-Nordström blackhole.

For the KN black hole, deriving A from the metric itself is quite cumbersome as the metric itself is non-diagonal. Although, it is not impossible to find it out using the coordinate component null-tetrad method, or the differential-form method. For brevity, the derivation of A using these methods have been skipped from our current discussion.

A.4 Surface Gravity

It was pointed out in Sec.(A.1.3) that the vector $K = K^u_{(t)} - \Omega_{BH} K^u_{(\phi)}$ is null at the horizon. To every Killing horizon Σ we can define as in [37]:

$$\left(-K^\mu K_\mu\right)_{;\nu} = 2\kappa K_\nu$$

and by a rearrangement

$$K^\mu \nabla_\mu K^\nu = \kappa K^\nu \tag{A.28}$$

where κ is the surface gravity.

This implies that the Killing vector obeys a geodesic equation and can be equivalently expressed as,

$$\kappa^2 = -\frac{1}{2}(\nabla^\mu K^\nu)(\nabla_\mu K_\nu) \tag{A.29}$$

Solving the above equation reveals that the surface gravity κ of KN black hole is

$$\kappa = \frac{r_+ - M}{r_+^2 + a^2} \tag{A.30}$$

Note that this is the most general result for a rotating black hole. One can easily find the surface gravity for the limiting cases: Kerr, Reissner-Nordström, and Schwarzschild by setting $Q = 0$, $a = 0$, and both $Q = a = 0$ respectively. Moreover, in black hole thermodynamics, the temperature of a black hole is $T = \frac{1}{2\pi}\kappa$. One can check that Eq.(A.30) is indeed consistent with the form of T given above Eq.(A.11).

B Action Variational Principle in GR

A brief review of the Einstein-Hilbert action is given here. We will use the action principle to arrive at important formulae that are extremely useful in Sec.(2).

Here's the Einstein-Hilbert action

$$S_{EH} = \int d^4x \sqrt{-g}\, R \tag{B.1}$$

Let us rewrite $R = g^{\mu\nu}R_{\mu\nu}$ and perform the variation of S_{EH} with the inverse metric $\delta g^{\mu\nu}$, term by term:

$$\begin{aligned} \delta S &= \int d^4x \left(\left(\delta\sqrt{-g}\right) g^{\mu\nu} R_{\mu\nu} + \sqrt{-g} \left(\delta g^{\mu\nu}\right) R_{\mu\nu} + \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} \right) \\ &= \int d^4x \left(\left(-\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \right) g^{\mu\nu} R_{\mu\nu} + \sqrt{-g} \left(\delta g^{\mu\nu}\right) R_{\mu\nu} + \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} \right) \\ &= \int d^4x \sqrt{-g} \left[\left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \delta g^{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu} \right] \\ &= \int d^4x \sqrt{-g} \left(G_{\mu\nu}\, \delta g^{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu} \right) \end{aligned} \tag{B.2}$$

where in the second line we have used the identity $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu}$. The $g^{\mu\nu}\delta R_{\mu\nu}$ term in the last line is known as a boundary term which can be rewritten as a total derivative using the Palatini identity:

$$\delta R_{\mu\nu} = \nabla_\gamma \delta\Gamma^\gamma{}_{\mu\nu} - \nabla_\nu \delta\Gamma^\gamma{}_{\mu\gamma} \tag{B.3}$$

The overall term is usually thrown away to extremize the action and produce the bulk equations of motion. To see why, let's simplify the boundary term as

$$\begin{aligned} g^{\mu\nu} R_{\mu\nu} &= \nabla_\rho \left[g^{\mu\nu} \delta\Gamma^\rho{}_{\nu\mu} - g^{\mu\rho} \delta\Gamma^\beta{}_{\beta\mu} \right] \\ &= \nabla_\mu \left[\nabla_\nu \delta g_{\alpha\beta} \left(g^{\mu\alpha} g^{\nu\beta} - g^{\mu\nu} g^{\alpha\beta} \right) \right] \end{aligned} \tag{B.4}$$

where we have used the definition $\delta\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha} \left(\partial_\mu \delta g_{\nu\alpha} + \partial_\nu \delta g_{\mu\alpha} - \partial_\alpha \delta g_{\mu\nu} \right)$ to go from the first line to the second line. Notice that we can identify the $\left(g^{\mu\alpha} g^{\nu\beta} - g^{\mu\nu} g^{\alpha\beta} \right)$ term from the equation above as general tensor $T^{\mu\nu\alpha\beta}$. This is an important observation because now one can bring the $\sqrt{-g}$ term from Eq.(B.2) inside the brackets and have it multiplied with the boundary term in Eq.(B.4) to become a tensorial density $\nabla_\mu \left[\nabla_\nu \left(\sqrt{-g} \delta g_{\alpha\beta} T^{\mu\nu\alpha\beta} \right) \right]$, whose integral vanishes via the Stokes theorem. This is why the boundary term is typically thrown away. However (see Sec.(2)), in presence of symmetries (like Killing or gauge symmetries), the boundary term arising from variations of the action along a given symmetry becomes the most important quantity to keep, as it corresponds to a conserved on-shell Noether current.

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