# The Cartier Core Map and $F$-Graded Systems

by

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*To my mom*

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# ABSTRACT

This dissertation studies singularities in positive characteristic rings and the operators that define these singularities. One approach we take is via the obstructions to strong $F$-regularity: given a commutative Noetherian $F$-finite ring $R$ of prime characteristic and a Cartier algebra $\mathcal{D}$, we define a self-map, called the Cartier core, on the Frobenius split locus of the pair $(R, \mathcal{D})$ by sending a point $P$ to the splitting prime of $(R_P, \mathcal{D}_P)$. We prove the Cartier core map is continuous, containment preserving, and fixes the $\mathcal{D}$-compatible ideals. We show the Cartier core map can be extended to arbitrary ideals $J$, where it outputs the largest $\mathcal{D}$-compatible ideal contained in $J$ in the case that the pair $(R, \mathcal{D})$ is Frobenius split. The other approach we take is by studying $F$-graded systems of ideals in $R$, which are sequences of ideals giving rise to Cartier algebras on $R$. We identify how properties of these systems (or modifications of these systems) affect the singularity properties of the corresponding Cartier algebra. In particular, we show that in a regular local ring for a special class of such systems called $p$-families, strong $F$-regularity and $F$-splitting are the same. Further, we make use of this and a new operation we introduce called *$p$-stabilization* to get a criterion that in a regular local ring, a system is strongly $F$-regular exactly when its $p$-stabilization is $F$-split. Finally, we associate a combinatorial object to systems built out of monomial ideals in such a way that encapsulates the behavior of the $p$-stabilization.

# CHAPTER I

# Introduction

One goal of commutative algebra is to better understand different classes of singularities of commutative Noetherian rings. For a ring $R$ of prime characteristic $p$, we have a powerful tool at our disposal, namely the Frobenius map $F : R \to R$ defined by $F(r) = r^p$. The Frobenius map is a ring homomorphism, and in particular it induces an interesting $R$-module structure on $R$ by extension of scalars. This new Frobenius twisted $R$-module structure carries information about the singularities of $R$. For example, a famous theorem of Kunz says that $R$ is regular exactly when the Frobenius map is flat [Kun69].

Two mild classes of singularities which will be of great interest to us are Frobenius splitting and strong $F$-regularity. A ring is *Frobenius split* if, true to its name, the Frobenius map splits as a map of $R$-modules. This notion was introduced by Hochster and Roberts in their 1976 work on invariant rings [HR76], and was used to great effect in work on Schubert varieties by Mehta and Ramanathan, who coined the term Frobenius splitting [MR85]. A ring is *strongly $F$-regular* if it has an abundance of splittings of the Frobenius and of certain related maps. See Section II.1 for a more detailed discussion of these singularity classes.

Both of these notions have been shown to be closely related to interesting singularities for complex varieties: strong $F$-regularity is known to behave similarly to Kawamata log terminality [Smi97, Har98, HW02], and Frobenius splitting is conjectured to behave similarly to log canonicity. It is important and difficult to understand the realm of rings that are Frobenius split but not strongly $F$-regular. Further, as these singularity classes are founded on determining whether certain maps split, it is of interest to understand more about these splittings as well. One of our main objects of study will be *Cartier algebras* (see Definition II.3.4).

One approach we take is looking at the *compatibility* of an ideal with a Cartier algebra. Compatible ideals are those such that every map of the Cartier algebra sends the ideal back into itself, and the structure of these ideals can carry information about the singularities of the ring. For example, the $F$-pure centers of a ring (introduced by Schwede as an analog of log canonical centers) are precisely the *prime* uniformly $F$-compatible ideals [Sch10]. As a more specific example, the splitting prime of a local ring, an ideal introduced by Aberbach and Enescu that contains all the elements "obstructing" strong $F$-regularity, is an $F$-pure center. The notion of a splitting prime can also be considered with respect to a Cartier algebra, see for example [BST12].

From this point on, we fix a commutative Noetherian ring $R$ of prime characteristic $p$. We further assume that $R$ is $F$-finite, that is, that the Frobenius map is a finite map.

In Chapter III, we take a new perspective on the compatible ideals of a ring. More specifically, fix a Frobenius split pair $(R, \mathcal{D})$, where $\mathcal{D}$ is a Cartier algebra. We then introduce the *Cartier core map*

$$\mathrm{C}_{\mathcal{D}} : \operatorname{Spec} R \to \operatorname{Spec} R$$

which assigns to each prime $P \in \operatorname{Spec} R$ the splitting prime $\mathrm{C}_{\mathcal{D}}(P)$ corresponding to the pair $(R_P, \mathcal{D}_P)$. The Cartier core map can be considered more generally as a map taking an ideal $I$ of $R$ to the ideal of elements which are always mapped into $I$ by $\mathcal{D}$. The name was first introduced by Badilla-Cespédes, in the special setting without reference to a Cartier algebra [BC21]. Our main result is to show that this Cartier core map is continuous in the Zariski topology and describe its image:

**Theorem I.0.1** (Theorem III.1.9)**.** *Let $R$ be an $F$-finite Noetherian ring of characteristic $p$, let $\mathcal{D}$ be a Cartier algebra, and let $\mathcal{U}_{\mathcal{D}}$ be the Frobenius split locus of $(R, \mathcal{D})$. Then the*

*Cartier core map*

$$\mathcal{U}_{\mathcal{D}} \to \operatorname{Spec} R \qquad\qquad P \mapsto C_{\mathcal{D}}(P)$$

*is a continuous containment preserving map on $\mathcal{U}_{\mathcal{D}}$ which fixes the $\mathcal{D}$-compatible ideals. The image of $C_{\mathcal{D}}$ is the set of prime $\mathcal{D}$-compatible ideals and is always finite. The image coincides with the set of minimal primes of $R$ precisely when the pair $(R, \mathcal{D})$ is strongly $F$-regular.*

In Chapter IV, we pivot to studing sequences of ideals called *$F$-graded systems* (see Definition IV.1.1), primarily working over regular rings. These arise naturally when considering Cartier algebras, and this connection allows us to define a notion of strong $F$-regularity and Frobenius splitting for $F$-graded systems (see Definition IV.2.1). A special class of $F$-graded systems are *$p$-families* (see Definition IV.1.5), named by Hernandéz and Jeffries [HJ18], which are independently of interest as they appear when defining the Hilbert-Kunz multiplicity [Mon83] and the $F$-signature [Tuc12]. However, not much is known about the classes of Cartier algebras these $p$-families correspond to. As it turns out, for $p$-families, Frobenius splitting and strong $F$-regularity collapse into the same condition:

**Theorem I.0.2** (Theorem IV.2.3). *Let $(R, \mathfrak{m})$ be an $F$-finite regular local ring. Let $\mathfrak{b}_\bullet$ be a $p$-family in $R$. Then $\mathfrak{b}_\bullet$ is Frobenius split if and only if it is strongly $F$-regular.*

We also describe a new operation on $F$-graded systems called *$p$-stabilization*, which turns an $F$-graded system into a closely related $p$-family in a way that preserves strong $F$-regularity. In particular, when combined with the previous theorem, we can show:

**Theorem I.0.3** (Corollary IV.3.4). *Let $(R, \mathfrak{m})$ be a regular local ring, and let $\mathfrak{a}_\bullet$ be an $F$-graded system in $S$ with $\mathfrak{a}_1 \neq 0$. Let $\widetilde{\mathfrak{a}}_\bullet$ be the $p$-stabilization of $\mathfrak{a}_\bullet$. Then $\mathfrak{a}_\bullet$ is strongly $F$-regular if and only if $\widetilde{\mathfrak{a}}_\bullet$ is Frobenius split.*

When the ideals in an $F$-graded system are all monomial, we can also use combinatorics to gain insight into the properties of the system. By taking advantage of the correspondence between monomials in $k[x_1, \ldots, x_d]$ and points in $\mathbb{N}^d$, we define an *associated $p$-body* in

$(\mathbb{N}[1/p])^d$ (see Definition IV.4.2) to a monomial $F$-graded system. Conversely we define an *associated p-family* to subsets of $(\mathbb{N}[1/p])^d$ (see Definition IV.4.5). This construction extends Hernandéz and Jeffries's notion of an associated $p$-body for a $p$-family, and gives a concrete way to encapsulate the asymptotic behaviour of an $F$-graded system. Further, we show that it is intimately connected to the $p$-stabilization:

**Theorem I.0.4** (Theorem IV.4.6)**.** *If $\mathfrak{b}_\bullet$ is $F$-graded, then the associated $p$-family of the associated $p$-body of $\mathfrak{b}_\bullet$ is the $p$-stabilization, i.e., $\mathfrak{a}_\bullet^{\Delta(\mathfrak{b}_\bullet)} = \widetilde{\mathfrak{b}}_\bullet$. If $\Delta \subset (\mathbb{N}[1/p])^d$, then $\Delta(\mathfrak{a}_\bullet^\Delta) = \Delta + (\mathbb{N}[1/p])^d$.*

*In particular, this gives a correspondence between $p$-stable $F$-graded systems and subsets of $(\mathbb{N}[1/p])^d$ which are invariant under adding $(\mathbb{N}[1/p])^d$.*

## I.1: Outline

We now proceed to a more detailed description of the rest of this dissertation.

In Chapter II, we give necessary background. In Section II.1, we describe the $F$-singularities mentioned in this introduction in more detail. In Section II.2, we see the splitting prime and the notion of "compatibility" of an ideal, along with a discussion of "test elements" which gives useful result on using a single element to test for strong $F$-regularity. In Section II.3 we define Cartier algebras and describe how the above $F$-singularities can be generalized to this setting. Finally, in Section II.4 we state results on the structure of $\operatorname{Hom}_R(F^e_* R, R)$ when $R$ is a homomorphic image of a regular local $F$-finite ring.

In Chapter III, we start by introducing the Cartier core of an ideal with respect to a Cartier algebra, and proving the core properties of how it behaves, including the aforementioned Theorem III.1.9.

We then focus on the setting of the full Cartier algebra of a quotient of a regular ring in Section III.2. In particular, we give the following explicit formula for the Cartier core of an arbitrary ideal, making use of a criterion for strong $F$-regularity due to Glassbrenner

[Gla96]:

**Theorem I.1.1** (Theorem III.2.1)**.** *Let $S$ be a regular $F$-finite ring, let $I \subseteq J$ be ideals of $S$, and let $R = S/I$. Then the Cartier core of the ideal $J/I$ in the ring $R$ is*

$$C_R(J/I) = \left( \bigcap_{e>0} J^{[p^e]} :_S (I^{[p^e]} :_S I) \right) /I.$$

This presentation of the Cartier core allows us to prove that the Cartier core map commutes with basic operations such as localizing, adjoining a variable, and in the case of quotients of polynomial rings, with homogenization (see Lemma III.1.7, Proposition III.2.4, Proposition III.2.6).

As an application of these techniques, in Section III.3 we give an exact description of the Cartier core map in the case of Stanley-Reisner rings.

**Theorem I.1.2** (Theorem III.3.1, Corollary III.3.2)**.** *Let $R$ be a Stanley-Reisner ring over a field that has prime characteristic and is $F$-finite. Let $Q$ be any prime ideal of $R$. Then*

$$C_R(Q) = \sum_{\substack{P \in \mathrm{Min}(R) \\ P \subseteq Q}} P.$$

*Further, if $J$ is any ideal, then*

$$\mathrm{C}_R(J) = \sum_{\substack{\mathcal{Q} \subset \mathrm{Min}(R) \\ \left(\bigcap_{P \in \mathcal{Q}} P\right) \subset J}} \left( \bigcap_{P \in \mathcal{Q}} P \right).$$

These formulas for the Cartier core map extend existing work on computing certain uniformly $F$-compatible ideals and $\mathcal{D}$-compatible ideals for Stanley-Reisner rings:

- Aberbach and Enescu computed the splitting prime of a Stanley-Reisner ring [AE05];
- Vassilev computed the test ideal for a Stanley-Reisner ring [Vas98];
- Enescu and Ilioaea computed the test ideal for pairs $(R, \psi)$ where $R = k[x_1, \ldots, x_n]/I$ is a Stanley-Reisner ring and where $\psi = \Phi \circ F^e_* \left((x_1 \cdots x_n)^{p^e - 1}\right)$, for $\Phi$ a generator of $\mathrm{Hom}(F^e_* S, S))$ [EI20]; and

- Badilla-Cespédes computed the Cartier core for monomial prime ideals [BC21].

In Chapter IV, we start in Section IV.1 by giving an introduction to $F$-graded systems and $p$-families and their basic properties, including some examples of each. Then in Section IV.2 we define strong $F$-regularity and Frobenius splitting for $F$-graded systems, and futher see how these singularities can be identified. This culminates in proving the above-mentioned Theorem IV.2.3, showing that for $p$-families, Frobenius splitting and strong $F$-regularity are in fact the same.

This then motivates our introduction of $p$-stabilization in Section IV.3. Here we show the above mentioned Theorem IV.3.3, and further discuss the basic properties of $p$-stabilization. At this point we see some examples of applying $p$-stabilization to our favorite selection of examples in Section IV.3.1. We also suggest in Section IV.3.2 some other operations on $F$-graded systems which could be interesting future objects of study.

Finally, in Section IV.4 we introduce the combinatorial construction of the associated $p$-body, and explain the connection with $p$-stabilization. We then in Section IV.4.1 illustrate how this can be used to actually compute the $p$-stabilization of some of the examples introduced back in Section IV.3.1.

# CHAPTER II

# Background

## II.1: Frobenius Splitting and Strong $F$-Regularity

One advantage of working in positive characteristic is the *Frobenius map.* Given a ring $R$ with characteristic $p > 0$, the Frobenius map

$$F : R \to R \text{ defined by } F(r) = r^p$$

is a ring homomorphism.

**Theorem II.1.1** ([Kun69, Cor. 2.7])**.** *A reduced Noetherian ring $R$ in prime characteristic is regular if and only if the Frobenius is a flat map.*

This is a first clue that studying the Frobenius can give useful information about the singularities of a ring. To clarify our study of the Frobenius map, we introduce some alternate notation. We will write $F_*R$ for the codomain of the map. As a ring, this Frobenius pushforward $F_*R = \{F_*r \mid r \in R\}$ is exactly the same as $R$, just with this formal symbol $F_*$ prepended everywhere. For example, multiplication is $(F_*r)(F_*s) = F_*(rs)$. The benefit of this notation is that it clarifies the $R$-module structure induced by $F$. The Frobenius map is now written as $F : R \to F_*R$ so that $F(r) = F_*(r^p)$, and the $R$-module action is now written $rF_*s = F_*(r^p s)$. We can iterate the Frobenius, writing $F^e : R \to F^e_*R$, where $F^e(r) = F^e_*(r^{p^e})$ and $rF^e_*s = F^e_*(r^{p^e}s)$.

**Definition II.1.2.** A ring $R$ is *$F$-finite* if $F_*R$ is a finitely generated $R$-module.

For the rest of this dissertation, we will work exclusively with $F$-finite rings. This is because $F$-finite rings are "nice" (e.g., every $F$-finite ring is excellent [Kun76, Thm. 2.5], and every $F$-finite ring is the quotient of an $F$-finite regular ring [Gab04]). They are also common: any ring essentially of finite type over a perfect field (or more generally, over an $F$-finite field) is also $F$-finite.

**Definition II.1.3.** A ring $R$ is *Frobenius split*, or simply *$F$-split*, if the Frobenius splits as a map of $R$-modules, i.e., if there exists a map $\varphi \in \mathrm{Hom}_R(F_*R, R)$ with $\varphi(F_*1) = 1$.

Closely related is the following definition:

**Definition II.1.4.** A ring $R$ is *$F$-pure* if the Frobenius is a pure map of $R$-modules, i.e., if for any $R$-module $N$, the induced map $N \otimes_R R \to N \otimes_R F_*R$ is injective.

Clearly $F$-splitting implies $F$-purity, but in fact for Noetherian $F$-finite rings the two conditions are equivalent [HR76, Cor. 5.2].

We will also be interested in a strengthening of $F$-splitting, introduced in [HH89]:

**Definition II.1.5.** A ring $R$ is *strongly $F$-regular* if for all non-zero divisors $c$, there exists $e > 0$ and $\varphi \in \mathrm{Hom}_R(F^e_*R, R)$ with $\varphi(F^e_*c) = 1$.

To make clearer the connection between $F$-splitting and strong $F$-regularity, it is useful to note the following result:

**Lemma II.1.6.** *The following are equivalent:*

1. *$R$ is $F$-split.*
2. *For some $e > 0$, there is a map $\varphi \in \mathrm{Hom}_R(F^e_*R, R)$ with $\varphi(F^e_*1) = 1$.*
3. *For all $e > 0$, there is a map $\varphi \in \mathrm{Hom}_R(F^e_*R, R)$ with $\varphi(F^e_*1) = 1$.*

*Proof.* Suppose as in (2) that $\varphi \in \mathrm{Hom}_R(F^e_*R, R)$ has $\varphi(F^e_*1) = 1$. Thus by composing with the iterated Frobenius, we see $\varphi \circ F^{e-1}(F_*1) = \varphi(F^e_*1) = 1$, and so $R$ is Frobenius split.

Suppose as in (1) that $\pi \in \mathrm{Hom}_R(F_*R, R)$ has $\pi(F_*1) = 1$. Then a degree $e$ map $\varphi$ with $\varphi(F^e_*1) = 1$ can be used to build a degree $e+1$ map via $\pi \circ (F_*\varphi)$, and so by induction we can get such maps for all $e$.

Finally, (3) $\implies$ (2) is clear. $\square$

Taking $c = 1$ as our non-zerodivisor, we thus we immediately see that strong $F$-regularity implies Frobenius splitting. Further, strong $F$-regularity implies several other nice features:

**Proposition II.1.7** ([HH90], [Smi97, Thm. 3.1])**.** *If $R$ is Noetherian, $F$-finite, and strongly $F$-regular, then $R$ is Cohen-Macaulay, normal, and pseudo-rational.*

Being pseudo-rational is a characteristic-free generalization of the notion of rational singularities for complex varieties due to Lipman, see [Lip69, LT81].

**Example II.1.8** ([HH89, Thm. 3.1])**.** Regular rings and direct summands of strongly $F$-regular rings are also strongly $F$-regular.

**Example II.1.9.** Stanley-Reisner rings are $F$-split (see, e.g., [Rei76, Lemma 10] for a proof over perfect fields). However, any Stanley-Reisner ring which is not just a polynomial ring is not strongly $F$-regular, since it is not normal.

It will also be useful for us to have a way to test for strong $F$-regularity, by way of a "test element":

**Theorem II.1.10** ([HH89, Thm. 3.3])**.** *Let $R$ be an $F$-finite Noetherian ring, and suppose that $g \in R^\circ$ has the property that $R[g^{-1}]$ is strongly $F$-regular. Then $R$ is eventually Frobenius split along $g$ if and only if $R$ is strongly $F$-regular.*

*Proof.* Take any non-zerodivisor $c \in R$, and consider the "evaluation at $c$" map

$$\mathrm{Hom}_R(F^f_*R, R) \to R \text{ which has } \phi \mapsto \phi(F^f_*c).$$

Since $R[g^{-1}]$ is strongly $F$-regular, when we tensor with $R[g^{-1}]$ there is some large enough $f$ such that this map is surjective in the localization, i.e., there is some $\psi \in \mathrm{Hom}_R(F^f_*R, R)$ and some $m$ such that $\psi(F^f_* c) = g^m$. Because $g$ is eventually split, this means $R$ is in particular $F$-split, and so there exists $\pi \in \operatorname{Hom}_R(F_* R, R)$ with $\pi(F_* 1) = 1$. We can always replace $m$ by a larger $m$ and compose $\pi$ with its pushforwards, so we reduce to the case where

$$\psi \in \operatorname{Hom}_R(F^f_* R, R),\ \psi(F^f_* c) = g^{p^\ell} \quad \text{and} \quad \pi \in \operatorname{Hom}_R(F^\ell_* R, R),\ \pi(F^\ell_* 1) = 1.$$

Now

$$\pi \circ \psi(F^{f+\ell}_* c) = \pi(F^\ell_* g^{p^\ell}) = g\pi(F^\ell_* 1) = g.$$

Let $\varphi \in \operatorname{Hom}_R(F^e_* R, R)$ be our given splitting of $g$. Now finally $\varphi \circ (F^e_* \pi) \circ (F^{e+\ell}_* \psi)$ is our desired splitting of $c$. □

## II.2: Compatible Ideals

For a local ring, Aberbach and Enescu introduced the *splitting prime* as a way to measure the difference between Frobenius splitting and strong $F$-regularity. The elements in the splitting prime are obstructions to strong $F$-regularity, in the sense that they are precisely those elements that cannot be taken to 1 be any element of $\operatorname{Hom}_R(F^e_* R, R)$. In particular, the splitting prime of a domain is zero precisely for strongly $F$-regular rings [AE05]. Aberbach and Enescu's splitting prime can also be described as the largest uniformly $F$-compatible ideal in the sense of Schwede [Sch10]. They first defined the splitting prime in terms of the injective hull of the residue field, $E_R(k)$, but we will instead state the following equivalent version of their definition.

**Definition II.2.1** ([AE05, Thm. 3.3])**.** The *splitting prime* of a reduced local ring $(R, \mathfrak{m})$ is

$$\mathcal{P}(R) = \left\{ r \in R \mid \varphi(F^e_* r) \in \mathfrak{m}\ \forall e > 0,\ \forall \varphi \in \operatorname{Hom}_R(F^e_* R, R) \right\}.$$

We can also give a description of this ideal in terms of *uniform $F$-compatibility*, which we now work towards explaining.

**Definition II.2.2.** Consider a map $\varphi \in \operatorname{Hom}_R(F^e_* R, R)$. An ideal $J \subset R$ is *$\varphi$-compatible* if $\varphi(F^e_* J) \subset J$. An ideal is *uniformly $F$-compatible* if it is $\varphi$-compatible for all $e \in \mathbb{N}$ and all $\varphi \in \operatorname{Hom}_R(F^e_* R, R)$.

**Example II.2.3.** The splitting prime is uniformly $F$-compatible since for every map $\psi \in \operatorname{Hom}_R(F^f_* R, R)$ and $r \in \mathcal{P}(R)$, we have $\varphi\left(F^e_*(\psi(F^f_* r))\right) = (\varphi \circ F^e_* \psi)(F^{e+f}_* r) \in \mathfrak{m}$ for all $e > 0$ and all $\varphi \in \operatorname{Hom}_R(F^e_* R, R)$.

**Definition II.2.4** ([Sch10, Def. 4.1])**.** Let $R$ be an $F$-finite reduced ring. A point $\mathfrak{p} \in \operatorname{Spec} R$ is a *center of $F$-purity* if for all $r \in \mathfrak{p}R_{\mathfrak{p}}$ and for all $e > 0$, the map

$$R_{\mathfrak{p}} \longrightarrow F^e_* R_{\mathfrak{p}} \xrightarrow{F_*(\cdot r)} F^e_* R_{\mathfrak{p}}$$

does not split.

If $(R, \mathfrak{m})$ is Frobenius split (so that the splitting prime $\mathcal{P}(R)$ is a proper ideal), then in fact the splitting prime is the unique largest center of $F$-purity [Sch10, Rmk. 4.4].

Further, by design, there is a connection between the ideas of uniform $F$-compatibility and centers of $F$-purity:

**Proposition II.2.5** ([Sch10, Prop. 4.6])**.** *Centers of $F$-purity are precisely the uniformly $F$-compatible prime ideals.*

## II.3: Cartier Algebras

We will use the $R$-module structure on $F^e_* R$, but first we need a cohesive way to consider only certain maps in $\operatorname{Hom}_R(F^e_* R, R)$. First, given any map $\psi \in \operatorname{Hom}_R(F^d_* R, R)$, we write $F^e_* \psi : F^{e+d}_* R \to F^e_* R$ for the *Frobenius pushforward* of the map, where

$$(F^e_* \psi)(F^{e+d}_* r) = (F^e_* \psi)(F^e_*(F^d_* r)) = F^e_*(\psi(F^d_* r)).$$

Now we define a (non-commutative) multiplication on the abelian group $\bigoplus_e \operatorname{Hom}_R(F^e_* R, R)$ as follows: given maps $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ and $\psi \in \operatorname{Hom}_R(F^d_* R, R)$, we define their product as

$$\phi \star \psi = \phi \circ F^e_* \psi. \tag{II.3.1}$$

More concretely, for any $r \in R$ we have

$$(\phi \star \psi)(F^{e+d}_* r) = \phi \left( F^e_* (\psi(F^d_* r)) \right).$$

**Definition II.3.1.** The *full Cartier algebra* on $R$ is the graded non-commutative ring

$$\mathcal{C}_R = \bigoplus_{e \geq 0} \operatorname{Hom}_R(F^e_* R, R),$$

where multiplication is as defined in Equation (II.3.1).

Note that $F^0_* R$ here means $R$ as an $R$-module, so that $(\mathcal{C}_R)_0 = \operatorname{Hom}_R(R, R) \cong R$. We will often write the "multiplication by $c$" map as simply $c$, and its pushforward as $F^e_* c$, so that $(F^e_* c)(F^e_* r) = F^e_*(cr)$. However, this copy of $R$ is rarely central in $\mathcal{C}_R$, because for $\phi$ of degree $e$, we have $r \star \phi = \phi \star r^{p^e}$. Therefore, $R$ is central only if $R = \mathbb{F}_p$.

**Definition II.3.2.** A *Cartier (sub)algebra* $\mathcal{D}$ is a graded subring of $\mathcal{C}_R$ such that $\mathcal{D}_0 = R$. In particular, $\mathcal{D}$ has the form $\mathcal{D} = \bigoplus_e \mathcal{D}_e$ where $\mathcal{D}_e \subseteq \operatorname{Hom}_R(F^e_* R, R)$ for all $e \geq 0$.

**Example II.3.3** ([Sch11, Rmk. 3.10])**.** Let $(R, \mathfrak{a}^t)$ be a pair where $\mathfrak{a}$ is an ideal and the formal exponent $t$ is a positive real number. Then the corresponding Cartier algebra $\mathcal{C}^{\mathfrak{a}^t}$ has

$$\mathcal{C}^{\mathfrak{a}^t}_e = \operatorname{Hom}_R(F^e_* R, R) \star \mathfrak{a}^{\lceil t(p^e - 1) \rceil}.$$

We can now restate many of the basic definitions from Section II.1 in Cartier algebra setting.

**Definition II.3.4.** Let $R$ be a ring of prime characteristic, and let $\mathcal{D}$ be a Cartier algebra on $R$.

- The pair $(R, \mathcal{D})$ is *$F$-finite* if $R$ is $F$-finite, i.e., $F_* R$ is a finite $R$-module. We assume every ring $R$ in this dissertation will be $F$-finite.

- The pair $(R, \mathcal{D})$ is *Frobenius split* or *(sharply) $F$-pure* if there exists some $e > 0$ and some $\phi \in \mathcal{D}_e$ with $\phi(F^e_* 1) = 1$.

- If $c$ is an element of $R$, then the pair $(R, \mathcal{D})$ is *eventually Frobenius split along $c$* or *$F$-pure along $c$* if there exists some $e > 0$ and some $\phi \in \mathcal{D}_e$ with $\phi(F^e_* c) = 1$.

- The pair $(R, \mathcal{D})$ is *strongly $F$-regular* if it is eventually Frobenius split along every $c$ which is not in any minimal prime of $R$.

We will follow the example of Blickle, Schwede, and Tucker and omit the adjective "sharp" when discussing $F$-purity of pairs [BST12, Def. 2.7]. Observe that if $\phi \in \mathcal{D}_e$ is a splitting of $F^e$, then there is a splitting in any multiple of the degree, given by $\phi^n \in \mathcal{D}_{en}$.

The notion of "compatibility" from Definition II.2.2 also makes sense in this setting:

**Definition II.3.5.** An ideal $J \subset R$ is $\mathcal{D}$-compatible if $J$ is $\varphi$-compatible for all $e \in \mathbb{N}$ and for all $\varphi \in \mathcal{D}_e$.

**Remark II.3.6** (Localizing a Cartier algebra)**.** Since we will consider only pairs $(R, \mathcal{D})$ where $R$ is Noetherian and $F$-finite, this means that for any ring $S$ such that $R \to S$ is flat, we have by [Mat89, Thm. 7.11],

$$S \otimes_R \operatorname{Hom}_R(F^e_* R, R) \cong \operatorname{Hom}_S(S \otimes_R F^e_* R, S).$$

In the case that $S$ is a localization of $R$ we further know that $S$ commutes with the Frobenius; that is, for any multiplicative set $W$,

$$W^{-1}R \otimes_R \operatorname{Hom}_R(F^e_* R, R) \cong \operatorname{Hom}_{W^{-1}R}(F^e_*(W^{-1}R), W^{-1}R).$$

We will use this isomorphism freely: if $\frac{r}{w} \otimes \phi$ is a pure tensor in $W^{-1}R \otimes \operatorname{Hom}_R(F^e_* R, R)$, we will identify this with the map in $\operatorname{Hom}_{W^{-1}R}(F^e_*(W^{-1}R), W^{-1}R)$ which sends $F^e_*(\frac{s}{u})$ to $\frac{r\phi(F^e_*(su^{p^e-1}))}{wu}$. This identification is easier to understand if we first rewrite $F^e_*(\frac{s}{u})$ as

$$F^e_*\left(\frac{su^{p^e-1}}{u^{p^e}}\right) = \frac{1}{u} \cdot \frac{F^e_*(su^{p^e-1})}{1} = \frac{F^e_*(su^{p^e-1})}{u}.$$

Thus we have a natural containment $W^{-1}R \otimes_R \mathcal{D}_e \subseteq (\mathcal{C}_{W^{-1}R})_e$. We can therefore construct a new Cartier algebra $W^{-1}\mathcal{D}$ on $W^{-1}R$ using this isomorphism, so that

$$(W^{-1}\mathcal{D})_e = W^{-1}R \otimes \mathcal{D}_e.$$

When localizing at a prime ideal $P$, we write this Cartier algebra as $\mathcal{D}_P$.

Now that we have the setup to discuss localizations of Cartier algebras, we can state and prove the following result on the Frobenius split locus in the setting of Cartier algebra pairs.

**Theorem II.3.7.** *Let $R$ be a Noetherian $F$-finite ring, and $\mathcal{D}$ a Cartier algebra. Then the set of primes $P$ of $R$ at which $(R_P, \mathcal{D}_P)$ is $F$-pure is open. Further, the pair $(R, \mathcal{D})$ is $F$-pure if and only if the localized pair $(R_P, \mathcal{D}_P)$ is $F$-pure for all primes $P$.*

*Proof.* For any $e$, we get a module map $\Psi_e : \mathcal{D}_e \to R$ via evaluation at $F^e_*1$. The pair $(R, \mathcal{D})$ is $F$-pure exactly when this map is surjective for some $e > 0$, or equivalently, when there exists an $e > 0$ such that $R/\operatorname{im}\Psi_e = 0$. The localization $(\Psi_e)_P$ corresponds to the evaluation map $(\mathcal{D}_P)_e \to R_P$, so the pair $(R_P, \mathcal{D}_P)$ is *not* $F$-pure if and only if $R_P/\operatorname{im}(\Psi_e)_P \neq 0$ for all $e$. Thus the non-$F$-pure locus is precisely the closed set $\bigcap_{e>0} \mathbb{V}(\operatorname{im}\Psi_e)$.

For the second statement, if $(R, \mathcal{D})$ is $F$-pure, then there exists some $e > 0$ and $\phi \in \mathcal{D}_e$ with $\phi(F^e_*1) = 1$. By definition, the localization $\phi_P : F^e_*(R_P) \to R_P$ is in $(\mathcal{D}_P)_e$, and so $(R_P, \mathcal{D}_P)$ is also $F$-pure.

Conversely, if each $(R_P, \mathcal{D}_P)$ is $F$-pure, then the complements of the sets $\mathbb{V}(\operatorname{im}\Psi_e)$ give an open cover of $\operatorname{Spec} R$. Since $\operatorname{Spec} R$ is compact, only finitely many are needed, say, the complements of $\mathbb{V}(\operatorname{im}\Psi_{e_1}), \ldots, \mathbb{V}(\operatorname{im}\Psi_{e_t})$. Taking $e = e_1 \cdots e_t$ to be the product of these indices, we must have that $(R_P, \mathcal{D}_P)$ has a splitting in $\mathcal{D}_e$ for every prime $P$. Thus the map $\Psi_e$ is surjective, since it is surjective at every prime. $\square$

This proof in fact shows that for any $c$, the set

$$\{P \in \operatorname{Spec} R \mid (R_P, \mathcal{D}_P) \text{ is not eventually Frobenius split along } c\}$$

is closed. Further, it also shows that $(R, \mathcal{D})$ is eventually Frobenius split along $c$ if and only if $(R_P, \mathcal{D}_P)$ is for every prime ideal $P$. In particular, this shows that just like in the non-pair setting (see [HH89, Thm. 3.1], $(R, \mathcal{D})$ is strongly $F$-regular if and only if every $(R_P, \mathcal{D}_P)$ is as well.

## II.4: Testing for Splittings in Quotients of Regular Rings

We have already seen one example (namely Kunz's theorem, Theorem II.1.1) of how a regular ring behaves nicely with regards to the Frobenius. It turns out that the Hom sets we are interested in also have a nice module structure in the regular setting. In fact, the following result is true for Gorenstein rings, though we will only use it in the regular case.

**Lemma II.4.1** ([Fed83, Lemma 1.6])**.** *If $S$ is an $F$-finite regular local ring, then the maps $\mathrm{Hom}_S(F^e_*S, S)$ form a free rank one $F^e_*S$-module.*

We will typically use $\Phi^e$ to refer to the generator of $\mathrm{Hom}_S(F^e_*S, S)$.

**Example II.4.2.** Let $S = \mathbb{F}_p[x_1, \ldots, x_d]$ be a polynomial ring. Then $F^e_*S$ is a free $S$-module with basis $\{F^e_* x^\alpha \mid 0 \le \alpha_i < p \ \forall i\}$. The *standard monomial generator* for $\mathrm{Hom}_S(F^e_*S, S)$ is defined on this basis to be

$$\Phi^e(F^e_* x^\alpha) = \begin{cases} 1 & \alpha_1 = \alpha_2 = \cdots = \alpha_d = p^e - 1 \\ 0 & \text{else.} \end{cases}$$

Now we see an example of how to use the $F_*S$-module structure to get other maps in $\mathrm{Hom}_S(F^e_*S, S)$ from our generator $\Phi$. We can present the *standard monomial splitting* as $(F_* x^{(p-1)\mathbb{1}}) \cdot \Phi = \Phi \circ (F_* x^{(p-1)\mathbb{1}})$, so that

$$\left(\Phi \circ (F_* x^{(p-1)\mathbb{1}})\right)(F_* x^\alpha) = \Phi(F_* x^{(p-1)\mathbb{1}+\alpha}) = \begin{cases} 1 & \alpha_1 = \cdots = \alpha_d = 0 \\ 0 & \text{else.} \end{cases}$$

Further, Fedder goes on to give a nice description the image of an ideal under any map in $\mathrm{Hom}_S(F^e_*S, S)$ (since we now know all such maps are of the form $\Phi^e \star s$):

**Lemma II.4.3** ([Fed83, Lemma 1.6])**.** *Let $(S, \mathfrak{m})$ be an $F$-finite regular local ring. Let $\Phi^e$ be the generator of* $\operatorname{Hom}_S(F^e_* S, S)$*, and let $I, J \subset S$ be ideals, and $s \in S$ an element. Then*

$$(\Phi^e \star s)(F^e_* I) \subset J \;\Leftrightarrow\; s \in J^{[p^e]} : I.$$

Taking $I = J$, this immediately indicates which maps $\varphi = \Phi^e \star s \in \operatorname{Hom}_S(F^e_* S, S)$ descend to maps in $\operatorname{Hom}_{S/I}(F^e_*(S/I), S/I)$ and gives the following isomorphism as a corollary:

**Lemma II.4.4** (Fedder's Lemma, [Gla96, Lemma 2.1])**.** *Let $S$ be an $F$-finite regular local ring and let $R = S/I$ for some ideal $I$. Then*

$$\operatorname{Hom}_R(F^e_* R, R) \cong F^e_* \left( \frac{I^{[p^e]} :_S I}{I^{[p^e]}} \right)$$

*as $R$-modules.*

It will also be useful of us to rephrase this result in the language of Cartier algebras:

**Example II.4.5** (Fedder's Lemma, rephrased)**.** Let $(S, \mathfrak{m})$ be a regular local ring, let $I$ be an ideal, and let $R = S/I$. Then the Cartier algebra $\mathcal{D}$ on $S$ composed of all maps which lift from $\mathcal{C}^R$ to $\mathcal{C}^S$ is

$$\mathcal{D}_e = \operatorname{Hom}_S(F^e_* S, S) \star \left( I^{[p^e]} : I \right).$$

This description of $\operatorname{Hom}_R(F^e_* R, R)$ is the core of Fedder's criterion and of Glassbrenner's criterion:

**Proposition II.4.6** ([Fed83, Prop. 1.7])**.** *Let $(S, \mathfrak{m})$ be an $F$-finite regular local ring of prime characteristic $p$, and let $I$ be an ideal of $S$. Then $R$ is $F$-pure if and only if $(I^{[p]} : I) \not\subseteq \mathfrak{m}^{[p]}$.*

**Proposition II.4.7** ([Gla96, Lemma 2.2])**.** *Let $(S, \mathfrak{m})$ be an $F$-finite regular local ring of prime characteristic $p$. Let $I$ be an ideal of $S$. Then the map $S/I \to F^e_*(S/I)$, where $1 \mapsto F^e_* c$, splits as an $(S/I)$-module map exactly when $c \notin \mathfrak{m}^{[p^e]} : (I^{[p^e]} : I)$.*

# CHAPTER III

# The Cartier Core Map

In this chapter, we study obstructions to strong $F$-regularity by introducing the *Cartier core map* with respect to a Cartier algebra. We'll first analyze the behavior of this map in general, and then focus on the setting of the full Cartier algebra in a quotient of a regular ring.

We assume that all our rings are commutative Noetherian of prime characteristic $p$, and are $F$-finite. The work in this chapter originally appeared in [Bro23].

## III.1: The Cartier Core Map

Fix a pair $(R, \mathcal{D})$, where $R$ is an $F$-finite Frobenius split ring and where $\mathcal{D}$ is a Cartier algebra (see Definition II.3.2). In this section we will define an explicit continuous map

$$\mathrm{C}_{\mathcal{D}} : \operatorname{Spec} R \to \operatorname{Spec} R$$

that has some especially nice properties. The image of this map is the set of $\mathcal{D}$-compatible primes of $\operatorname{Spec} R$, which in the case $\mathcal{D} = \mathcal{C}_R$ is the set of (generic points of) $F$-pure centers. If $R$ is not Frobenius split, we can instead define $\mathrm{C}_{\mathcal{D}}$ on the open locus of Frobenius split points. More generally, the map $\mathrm{C}_{\mathcal{D}}$ can be viewed as an endomorphism defined on the set of *all ideals* of $R$ (not necessarily proper), and is especially interesting on the class of radical ideals in a Frobenius split ring.

**Definition III.1.1.** Let $R$ be an $F$-finite ring of prime characteristic. Let $J$ be an ideal of $R$. Let $\mathcal{D} \subseteq \mathcal{C}_R$ be a Cartier algebra. Then the *Cartier core* of $J$ in $R$ with respect to $\mathcal{D}$ is

$$\mathrm{C}_{\mathcal{D}}(J) = \{r \in R \mid \phi(F^e_* r) \in J \ \ \forall e > 0, \ \forall \phi \in \mathcal{D}_e\}.$$

We will write $\mathrm{C}_R(J)$ to mean the Cartier core with respect to the full Cartier algebra $\mathcal{C}_R$, or just $\mathrm{C}(J)$ when the ring and Cartier subaglebra are clear from context. In the case that $\mathcal{D} = \mathcal{C}_R$, the Cartier core $\mathrm{C}_R(J)$ is also denoted (e.g., in [BC21]) as $\mathcal{P}(J)$.

**Notation III.1.2.** The *e-th Cartier contraction* of $J$ with respect to $\mathcal{D}$ is

$$A_{\mathcal{D}_e}(J) = \{r \in R \mid \phi(F^e_* r) \in J \ \ \forall \phi \in \mathcal{D}_e\}.$$

We can express the Cartier core in terms of the Cartier contractions as

$$\mathrm{C}_{\mathcal{D}}(J) = \bigcap_{e>0} A_{\mathcal{D}_e}(J).$$

We can also express the Frobenius pushforward of the $e$-th Cartier contraction as

$$F^e_*(A_{\mathcal{D}_e}(J)) = \bigcap_{\phi \in \mathcal{D}_e} \phi^{-1}(J).$$

When $\mathcal{D} = \mathcal{C}_R$, the $e$-th Cartier contraction $A_{\mathcal{D}_e}(J)$ is sometimes denoted by $J_e$.

Note that for an $F$-finite pair $(R, \mathcal{D})$, $A_{\mathcal{D}_e}(J)$ and $\mathrm{C}_{\mathcal{D}}(J)$ are ideals. Both are clearly additively closed, so it suffices to check that if $a \in A_{\mathcal{D}_e}(J)$ and $r \in R$, then $ra \in A_{\mathcal{D}_e}(J)$. For any $\phi \in \mathcal{D}_e$, we have $\phi(F^e_*(ra)) = (\phi \star r)(F^e_* a)$, which is in $J$ since $a \in A_{\mathcal{D}_e}(J)$.

The Cartier core was defined for the case $\mathcal{C}_R = \mathcal{D}$ by Badilla-Céspedes [BC21, Def. 4.12] as a generalization of Aberbach and Enescu's splitting prime [AE05] and of Brenner, Jeffries, and Núñez Betancourt's differential core [BJN19]. Here we generalize this definition to the context of pairs, similar to Blickle, Schwede, and Tucker's generalization of the splitting prime to the context of pairs [BST12].

To motivate the definition of the Cartier core, note that the condition $J \subseteq \mathrm{C}_{\mathcal{D}}(J)$ (in other words, $\phi(F^e_*(J)) \subseteq J$ for all $e$ and for all $\phi \in \mathcal{D}_e$) is precisely the condition that $J$ is

$\mathcal{D}$-compatible (see Definition II.3.5). In the case where $\mathcal{D}$ is the full Cartier algebra, this is equivalent to saying $J$ is uniformly $F$-compatible. In fact, it is known that when $R$ is $F$-pure, $\mathrm{C}_R(J)$ is the largest uniformly $F$-compatible ideal contained in $J$ [BC21, Prop. 4.11]. We will see in Corollary III.1.20 that when the pair $(R, \mathcal{D})$ is Frobenius split, the Cartier core $\mathrm{C}_{\mathcal{D}}(J)$ is the largest $\mathcal{D}$-compatible ideal contained in $J$.

Further, as the next two results show, the Cartier core of a prime ideal $P$ carries information about the localization $(R_P, \mathcal{D}_P)$.

**Proposition III.1.3** ([BST12, Prop. 2.12])**.** *Let $(R, \mathcal{D})$ be an $F$-finite pair and let $P$ be a prime ideal of $R$. Then $r \notin \mathrm{C}_{\mathcal{D}}(P)$ if and only if the pair $(R_P, \mathcal{D}_P)$ is $F$-pure along $r/1$. In particular, $(R_P, \mathcal{D}_P)$ is $F$-pure if and only if $\mathrm{C}_{\mathcal{D}}(P)$ is proper.*

*Proof.* Since $\mathcal{D}_P = \mathcal{D} \otimes R_P$, saying $\phi(F^e_*(r)) \in P$ for some $\phi \in \mathcal{D}_e \subseteq \mathrm{Hom}_R(F^e_*R, R)$ is equivalent to saying $\phi(F^e_*(r/1)) \in PR_P$, viewing $\phi \in (D_P)_e \subseteq \mathrm{Hom}_{R_P}(F^e_*(R_P), R_P)$.

The pair $(R_P, \mathcal{D}_P)$ is $F$-pure if and only if there is some $\phi \in \mathcal{D}_P$ such that $\phi(F^e_*(1))$ is a unit, i.e., not in $PR_P$, which by the above is equivalent to having $1 \notin \mathrm{C}_{\mathcal{D}}(P)$. □

**Proposition III.1.4** (Cf. [BST12, Thm. 2.11,Prop. 2.12])**.** *Let $(R, \mathcal{D})$ be an $F$-finite pair and let $P$ be a prime ideal of $R$. Then the pair $(R_P, \mathcal{D}_P)$ is strongly $F$-regular if and only if $\mathrm{C}_{\mathcal{D}}(P)$ is contained in some minimal prime of $R$.*

*Proof.* The pair $(R_P, \mathcal{D}_P)$ is strongly $F$-regular if and only if $(R_P, \mathcal{D}_P)$ is $F$-pure along every non-zero divisor, i.e., $\mathrm{C}_{\mathcal{D}}(P)$ is contained in the union of the minimal primes of $R$. Since $\mathrm{C}_{\mathcal{D}}(P)$ is an ideal, by prime avoidance this is equivalent to the containment of $\mathrm{C}_{\mathcal{D}}(P)$ in some minimal prime of $R$. □

Now that we have provided some motivation for the Cartier core construction, we will discuss some of its nice properties.

**Proposition III.1.5.** *Let $(R, \mathcal{D})$ be an $F$-finite pair. If $J_1 \subseteq J_2$ in $R$, then $\mathrm{C}_{\mathcal{D}}(J_1) \subseteq \mathrm{C}_{\mathcal{D}}(J_2)$.*

*Proof.* For every $e$, $A_{\mathcal{D}_e}(J_1) \subseteq A_{\mathcal{D}_e}(J_2)$, since if $\phi(F^e_* r) \in J_1$ for some $\phi \in \mathcal{D}_e$, we also have $\phi(F^e_* r) \in J_2$. Taking the intersection over all $e$ gives our result. □

**Proposition III.1.6** (Cf. [BC21, Prop 4.6])**.** *Let $\{J_\alpha\}$ be an arbitrary collection of ideals in an $F$-finite ring $R$, and let $\mathcal{D}$ be a Cartier algebra. Then*

$$\mathrm{C}_{\mathcal{D}}\left(\bigcap_\alpha J_\alpha\right) = \bigcap_\alpha \mathrm{C}_{\mathcal{D}}(J_\alpha).$$

*Proof.* We see that

$$\begin{aligned}
\mathrm{C}_{\mathcal{D}}\left(\bigcap_\alpha J_\alpha\right) &= \left\{ r \in R \mid \phi(F^e_* r) \in \bigcap_\alpha J_\alpha \ \forall e, \forall \phi \in \mathcal{D}_e \right\} \\
&= \bigcap_\alpha \{ r \in R \mid \phi(F^e_* r) \in J_\alpha \ \forall e, \forall \phi \in \mathcal{D}_e \} \\
&= \bigcap_\alpha \mathrm{C}_{\mathcal{D}}(J_\alpha)
\end{aligned}$$

□

In particular, the set of Cartier cores with respect to $\mathcal{D}$ is closed under arbitrary intersection. We will see in Proposition III.1.15 that this set is also closed under arbitrary sum for $F$-pure pairs.

Our next goal is to show that the Cartier core construction commutes with localization. To do so, we need the following lemma.

**Lemma III.1.7.** *Let $(R, \mathcal{D})$ be an $F$-finite pair, let $Q$ be a $P$-primary ideal of $R$, and let $W$ be a multiplicative set avoiding $P$, so that $W \cap P = \emptyset$. Then*

$$\mathrm{C}_{W^{-1}\mathcal{D}}(QW^{-1}R) \cap R = \mathrm{C}_{\mathcal{D}}(Q).$$

*Proof.* By Remark II.3.6, $W^{-1}\mathcal{D}_e$ is generated by the maps $\frac{\phi}{w} : F_*(W^{-1}R) \to R$ for $\phi \in \mathcal{D}_e$ and $w \in W$, where $\frac{\phi}{w}(F_*(\frac{s}{u})) = \frac{\phi(F^e_*(su^{p^e-1}))}{wu}$. We will start by showing that $\frac{s}{1} \in A_{W^{-1}\mathcal{D}_e}(QW^{-1}R)$ if and only if $s \in A_{\mathcal{D}_e}(Q)$.

By definition, $\frac{s}{1} \in A_{W^{-1}\mathcal{D}_e}(QW^{-1}R)$ if and only if $\psi(F^e_*(\frac{s}{1})) \in QW^{-1}R$ for all $\psi \in W^{-1}\mathcal{D}_e$. This is equivalent to having

$$\frac{\phi(F^e_*(s))}{w} \in QW^{-1}R$$

for all $\phi \in \mathcal{D}_e$ and all $w \in W$. This means that we can write $\frac{\phi(F^e_*(s))}{w} = \frac{j}{u}$ for some $j \in Q$, $u \in W$, i.e., there exists $v \in W$ such that $vu\phi(F^e_*(s)) = vwj$. The latter is in $Q$, but $vu \notin P$, so by $P$-primaryness of $Q$ we must then have $\phi(F^e_* s) \in Q$. This holds for all $\phi$ exactly when $s \in A_{\mathcal{D}_e}(Q)$.

Now we have shown our first claim, which implies $A_{\mathcal{D}}(Q) = A_{W^{-1}\mathcal{D}_e}(Q) \cap R$. Intersecting both sides over all $e > 0$, we see

$$\mathrm{C}_{W^{-1}\mathcal{D}}(Q) \cap R = \mathrm{C}_{\mathcal{D}}(Q). \qquad \square$$

**Theorem III.1.8.** *Let $(R, \mathcal{D})$ be an $F$-finite pair, let $J$ be an ideal of $R$, and let $W$ be a multiplicative set avoiding every prime in* $\mathrm{Ass}(J)$. *Then*

$$\mathrm{C}_{W^{-1}\mathcal{D}}(JW^{-1}R) \cap R = \mathrm{C}_{\mathcal{D}}(J) \quad \textit{and} \quad \mathrm{C}_{\mathcal{D}}(J)W^{-1}R = \mathrm{C}_{W^{-1}\mathcal{D}}(JW^{-1}R).$$

*Proof.* Write $J = Q_1 \cap \cdots \cap Q_t$ a minimal primary decomposition of $J$ with corresponding primes $P_i = \sqrt{Q_i}$. Then since intersection commutes with applying $\mathrm{C}_{\mathcal{D}}$ and with contraction,

$$\mathrm{C}_{W^{-1}\mathcal{D}}(J) \cap R = \bigcap_{i=1}^{t} (\mathrm{C}_{W^{-1}\mathcal{D}}(Q_i) \cap R).$$

By Lemma III.1.7, since $W \cap P_i = \emptyset$ we have $\mathrm{C}_{W^{-1}\mathcal{D}}(Q_i) \cap R = \mathrm{C}_{\mathcal{D}}(Q_i)$ and so

$$\mathrm{C}_{W^{-1}\mathcal{D}}(J) \cap R = \bigcap_{i=1}^{t} \mathrm{C}_{\mathcal{D}}(Q_i) = \mathrm{C}_{\mathcal{D}}(J).$$

For the second equality, we note

$$\mathrm{C}_{W^{-1}\mathcal{D}}(J) = (\mathrm{C}_{W^{-1}\mathcal{D}}(J) \cap R)W^{-1}R = \mathrm{C}_{\mathcal{D}}(J)W^{-1}R$$

since contracting then extending to a localization preserves ideals. $\square$

Now that we have established the preliminary results for arbitrary ideals, we move to considering prime ideals. Our main results of the rest of this section can be summarized in the following theorem.

**Theorem III.1.9.** *Let $R$ be an $F$-finite Noetherian ring, and let $\mathcal{D}$ be a Cartier algebra. Then the Cartier core construction with respect to $\mathcal{D}$ induces a well-defined, continuous, and containment preserving map on the $F$-pure locus of the pair $(R, \mathcal{D})$ which fixes $\mathcal{D}$-compatible ideals. The image of the map is the set of $\mathcal{D}$-compatible ideals in $\mathcal{U}_{\mathcal{D}}$ and is always finite. The image is the set of minimal primes of $R$ precisely when the pair $(R, \mathcal{D})$ is strongly $F$-regular.*

*Proof.* We have already seen in Proposition III.1.5 that the Cartier core is containment preserving, even without restricting to primes. Corollary III.1.12 will show that the map $\mathrm{C} : \mathcal{U}_{\mathcal{D}} \to \mathcal{U}_{\mathcal{D}}$ is well-defined. Theorem III.1.23 will show that this map is continuous, and Proposition III.1.21 discusses the finiteness of the image. Theorem III.1.19 will show that the image is precisely the set of $F$-pure $\mathcal{D}$-compatible ideals, which combined with Proposition III.1.16 shows that all the $\mathcal{D}$-compatible ideals in $\mathcal{U}_{\mathcal{D}}$ are fixed.

The one statement that doesn't have a stand-alone proof elsewhere is the last one. $(R, \mathcal{D})$ is strongly $F$-regular if and only if each $(R_P, \mathcal{D}_P)$ is strongly $F$-regular. By Proposition III.1.4, this occurs exactly when each $\mathrm{C}_{\mathcal{D}}(P)$ is contained in a minimal prime of $R$. But since $\mathrm{C}_{\mathcal{D}}(P)$ is prime, this is equivalent to having $\mathrm{C}_{\mathcal{D}}(P)$ be a minimal prime. $\square$

It is known that the splitting prime, which in our notation is $\mathrm{C}_R(\mathfrak{m})$ for $(R, \mathfrak{m})$ local, is indeed prime [AE05, Thm. 3.3], even in the case of an arbitrary Cartier algebra [BST12, Prop. 2.12]. After localizing, the same proof works here, which we repeat for the reader's convenience.

**Proposition III.1.10** ([BST12, Prop. 2.12])**.** *Let $R$ be a Noetherian $F$-finite ring. If $P$ is prime ideal of $R$ and $\mathrm{C}_{\mathcal{D}}(P)$ is proper, then $\mathrm{C}_{\mathcal{D}}(P)$ is prime.*

*Proof.* Suppose $c_0, c_1 \notin \mathrm{C}_{\mathcal{D}}(P)$. Then we will show $c_0c_1 \notin \mathrm{C}_{\mathcal{D}}(P)$. Our assumption means that $(R_P, \mathcal{D}_P)$ is $F$-pure along each $c_i$, i.e., there exists an $e_i$ and $\psi_i \in (D_P)_{e_i}$ such that $\psi_i(F^{e_i}_* c_i) = 1$. Then applying the map $\psi_1 \circ F^{e_1}_* \psi_0 \circ F^{e_0+e_1}_*(c_1^{p^{e_0}-1})$ to $F^{e_0+e_1}_*(c_0c_1)$, where we are writing $F^{e_0+e_1}_*(c_1^{p^{e_0}-1})$ to mean multiplication by this ring element, we get

$$\begin{array}{ccccccc} F_*^{e_0+e_1}(R_P) & \xrightarrow{F_*^{e_0+e_1}(c_1^{p^{e_0}-1})} & F_*^{e_0+e_1}(R_P) & \xrightarrow{F_*^{e_1}\psi_0} & F_*^{e_1}(R_P) & \xrightarrow{\psi_1} & R_P \end{array}$$

$$F_*^{e_0+e_1}(c_0c_1) \longmapsto F_*^{e_0+e_1}(c_0c_1^{p^{e_0}}) = F_*^{e_1}(c_1F_*^{e_0}(c_0)) \longmapsto F_*^{e_1}c_1 \longmapsto 1.$$

Rewriting this map as $\psi_1 \circ F_*^{e_1}\psi_0 \circ F_*^{e_0+e_1}(c_1^{p^{e_0}-1}) = \psi_1 \star \psi_0 \star c_1^{p^{e_0}-1}$, we see that it is in $(\mathcal{D}_P)_{e_0+e_1}$, and thus that that $(R_P, \mathcal{D}_P)$ is also $F$-pure along $c_0c_1$, as desired. $\square$

**Proposition III.1.11.** *Let $R$ be a Noetherian $F$-finite ring, and let $\mathcal{D}$ be a Cartier algebra. If $Q$ is a $P$-primary ideal of $R$ and $\mathrm{C}_\mathcal{D}(P)$ is proper, then $\mathrm{C}_\mathcal{D}(Q) \subseteq Q$.*

*Proof.* Since $\mathrm{C}_\mathcal{D}(P)$ is proper, there is some $e > 0$ and $\psi \in \mathcal{D}_e$ with $\psi(F_*^e 1) \notin P$. Consider $r \notin Q$ and the map $\psi \circ (F_*^e(r^{p^e-1})) = \psi \star r^{p^e-1}$ in $\mathcal{D}_e$. Then by $P$-primaryness,

$$r\psi(F_*1) = \psi(F_*^e r^{p^e}) = (\psi \star r^{p^e-1})(F_*^e r) \notin Q,$$

and so $r \notin \mathrm{C}_\mathcal{D}(Q)$ as desired. $\square$

**Corollary III.1.12.** *Let $(R, \mathcal{D})$ be an $F$-finite pair, with $F$-pure locus $\mathcal{U}_\mathcal{D} \subseteq \operatorname{Spec} R$. Then the Cartier core construction induces a well-defined map $\mathrm{C}_\mathcal{D} : \mathcal{U}_\mathcal{D} \to \mathcal{U}_\mathcal{D}$.*

Recall from Theorem II.3.7 that because $R$ is $F$-finite, the $F$-pure locus is an open subset of $\operatorname{Spec} R$.

*Proof.* Let $P$ be a prime ideal in $\mathcal{U}_\mathcal{D}$. Then $(R_P, \mathcal{D}_P)$ is Frobenius split, so Proposition III.1.3 gives that $\mathrm{C}_\mathcal{D}$ is proper, and thus prime by Proposition III.1.10. This gives a map $\mathrm{C}_\mathcal{D} : \mathcal{U}_\mathcal{D} \to \operatorname{Spec} R$.

Then Proposition III.1.11 says $\mathrm{C}_\mathcal{D}(P) \subseteq P$. Since the $F$-pure locus is open, this means $\mathrm{C}_\mathcal{D}(P)$ must also be in the $F$-pure locus. $\square$

**Corollary III.1.13** (Cf. [Sch10, Cor. 4.8])**.** *Suppose the pair $(R, \mathcal{D})$ is $F$-finite and $F$-pure. If $P$ is a minimal prime of $R$, then $\mathrm{C}_\mathcal{D}(P) = P$.*

*Proof.* Since $(R, \mathcal{D})$ is $F$-pure, $\mathrm{C}_\mathcal{D}(P) \subseteq P$. Since $\mathrm{C}_\mathcal{D}(P)$ is prime by Proposition III.1.10 and $P$ is minimal, we must have that $\mathrm{C}_\mathcal{D}(P) = P$. $\square$

**Corollary III.1.14** (Cf. [BC21, Prop. 4.5]). *If the pair $(R, \mathcal{D})$ is $F$-finite and $F$-pure, then for any ideal $J$ we have $\mathrm{C}_{\mathcal{D}}(J) \subseteq J$.*

*Proof.* Write $J = Q_1 \cap \cdots \cap Q_t$, where the $Q_i$ give a primary decomposition of $J$. Then by Proposition III.1.6,

$$\mathrm{C}_{\mathcal{D}}(J) = \mathrm{C}_{\mathcal{D}}(Q_1) \cap \cdots \cap \mathrm{C}_{\mathcal{D}}(Q_t).$$

Since $(R, \mathcal{D})$ is Frobenius split, for every prime $P$ the pair $(R_P, \mathcal{D}_P)$ is also Frobenius split, and thus has $\mathrm{C}_{\mathcal{D}}(P)$ proper by Proposition III.1.3. By Proposition III.1.11, each $\mathrm{C}_{\mathcal{D}}(Q_i) \subseteq Q_i$. Intersecting, we get that $\mathrm{C}_{\mathcal{D}}(J) \subseteq J$ as desired. □

**Proposition III.1.15** (Cf. [Sch10, Lemma 3.5]). *Let $(R, \mathcal{D})$ be an $F$-finite, $F$-pure pair, and let $\{J_\alpha\}_{\alpha \in \mathcal{A}}$ be a collection of ideals with $\mathrm{C}_{\mathcal{D}}(J_\alpha) = J_\alpha$ for all $\alpha \in \mathcal{A}$. Then we have*

$$\mathrm{C}_{\mathcal{D}}\left(\sum_\alpha J_\alpha\right) = \sum_\alpha \mathrm{C}_{\mathcal{D}}(J_\alpha)$$

*Proof.* Since $J_\beta \subseteq \sum J_\alpha$, we have $\mathrm{C}_{\mathcal{D}}(J_\beta) \subseteq \mathrm{C}_{\mathcal{D}}\left(\sum_\alpha J_\alpha\right)$ for all $\beta \in \mathcal{A}$ by Proposition III.1.5, and so

$$\sum_\alpha \mathrm{C}_{\mathcal{D}}(J_\alpha) \subseteq \mathrm{C}_{\mathcal{D}}\left(\sum_\alpha J_\alpha\right).$$

For the reverse containment, we use our assumption that $\mathrm{C}_{\mathcal{D}}(J_\alpha) = J_\alpha$ and Corollary III.1.14 to see that

$$\mathrm{C}_{\mathcal{D}}\left(\sum J_\alpha\right) = \mathrm{C}_{\mathcal{D}}\left(\sum \mathrm{C}_{\mathcal{D}}(J_\alpha)\right) \subseteq \sum \mathrm{C}_{\mathcal{D}}(J_\alpha)$$

which is our desired opposite inclusion. □

**Proposition III.1.16.** *If the pair $(R, \mathcal{D})$ is $F$-finite and $F$-pure, then for any ideal $J$ in $R$,*

$$\mathrm{C}_{\mathcal{D}}(J) = \mathrm{C}_{\mathcal{D}}\left(\mathrm{C}_{\mathcal{D}}(J)\right).$$

*Proof.* By Corollary III.1.14, we know that $\mathrm{C}_{\mathcal{D}}(J) \subseteq J$. Then $\mathrm{C}_{\mathcal{D}}\left(\mathrm{C}_{\mathcal{D}}(J)\right) \subseteq \mathrm{C}_{\mathcal{D}}(J)$ by Proposition III.1.5, so it suffices to show the other direction.

Consider $f \notin \mathrm{C}_{\mathcal{D}}\left(\mathrm{C}_{\mathcal{D}}(J)\right)$. Thus there exists $e > 0$ and $\phi \in \mathcal{D}_e$ with $\phi(F^e_* f) \notin \mathrm{C}_{\mathcal{D}}(J)$. Then there must also exist $e'$ and $\phi' \in \mathcal{D}_{e'}$ with $\phi'(F^{e'}_* \phi(F^e_*(f)) \notin J$. This term can be rewritten as $(\phi' \star \phi)(F^{e'+e}_*(f)) = \phi'\left(F^{e'}_*(\phi(F^e_* f))\right)$, and so $f \notin \mathrm{C}_{\mathcal{D}}(J)$. $\square$

**Remark III.1.17.** If the pair $(R, \mathcal{D})$ is $F$-finite and $F$-pure, then combining the results on the forms of containments (Corollary III.1.14, Proposition III.1.5, and Proposition III.1.16) shows that $\mathrm{C}_{\mathcal{D}}$ is a relative interior operation on ideals of $R$, in the sense of Epstein, R.G., and Vassilev [EGV21, Def. 2.2].

The following result is known when $\mathcal{D} = \mathcal{C}_R$ [BC21], and for triples $(R, \Delta, \mathfrak{a}^t)$ [Sch10]. The proof in the Cartier algebra setting proceeds similarly to Badilla-Céspedes' proof, with a little care needed for the exponents used.

**Proposition III.1.18** (Cf. [BC21, Rmk. 4.14], [Sch10, Cor. 3.3])**.** *If the pair $(R, \mathcal{D})$ is $F$-finite and $F$-pure, then for any ideal $J$, the Cartier core $\mathrm{C}_{\mathcal{D}}(J)$ is radical.*

*Proof.* Suppose $r \in \sqrt{\mathrm{C}_{\mathcal{D}}(J)}$. Then there exists some $n$ so that $r^{p^n} \in \mathrm{C}_{\mathcal{D}}(J)$. Since the pair is $F$-pure, there also exists some $\psi \in \mathcal{D}_d$ so that $\psi(F^d_* 1) = 1$. Take $e = nd$, so that there is $\phi \in \mathcal{D}_e$ with $\phi(F^e_* 1) = 1$, and so that Proposition III.1.16 gives $r^{p^e} \in \mathrm{C}_{\mathcal{D}}(J) = \mathrm{C}_{\mathcal{D}}(\mathrm{C}_{\mathcal{D}}(J))$. Then

$$\phi(F^e_*(r^{p^e})) = r\phi(F^e_* 1) = r \in \mathrm{C}_{\mathcal{D}}(J). \qquad \square$$

The hypothesis that $(R, \mathcal{D})$ be $F$-pure is necessary. Consider $R = k[x]/\langle x^2 \rangle$ where $k$ is an $F$-finite field, and let $\mathcal{D} = \mathcal{C}_R$. This ring $R$ is non-reduced, so it is not $F$-pure. For any ideal $J$ in $k[x]$, use $\overline{J}$ to denote the image of $J$ in $R$. Now using the presentation from Theorem III.2.1, we compute

$$A_e(\overline{\langle x^2 \rangle}) = \overline{\langle x^2 \rangle^{[p^e]} :_{k[x]} \left(\langle x^2 \rangle^{[p^e]} :_{k[x]} \langle x^2 \rangle\right)} = \overline{\langle x^{2p^e} \rangle :_{k[x]} \langle x^{2p^e - 2} \rangle} = \overline{\langle x^2 \rangle}.$$

Intersecting over all $e$, we see that $\mathrm{C}_R(\overline{\langle x^2 \rangle}) = \overline{\langle x^2 \rangle}$, a non-radical ideal.

**Theorem III.1.19** (Cf. [BC21, Prop. 4.9, Thm. 4.10])**.** *If the pair $(R, \mathcal{D})$ is $F$-finite and $F$-pure, then the set of Cartier cores with respect to $\mathcal{D}$, i.e., the set $\{\mathrm{C}_{\mathcal{D}}(J) \mid J \text{ an ideal of } R\}$, is precisely the set of $\mathcal{D}$-compatible ideals.*

*Proof.* An ideal $J$ is $\mathcal{D}$-compatible precisely if $\phi(F^e_*(J)) \subseteq J$ for all $e$ and for all $\phi \in \mathcal{D}_e$, and thus by construction $J$ is $\mathcal{D}$-compatible if and only if $J \subseteq \mathrm{C}_{\mathcal{D}}(J)$. By Corollary III.1.14, if the pair $(R, \mathcal{D})$ is $F$-pure then this is equivalent to having $J = \mathrm{C}_{\mathcal{D}}(J)$. This shows that every $\mathcal{D}$-compatible ideal is a Cartier core.

Conversely, the Cartier core $\mathrm{C}_{\mathcal{D}}(J)$ is $\mathcal{D}$-compatible since by Proposition III.1.16 we have $\mathrm{C}_{\mathcal{D}}(J) = \mathrm{C}_{\mathcal{D}}(\mathrm{C}_{\mathcal{D}}(J))$. □

**Corollary III.1.20** (Cf. [BC21, Prop. 4.11])**.** *If the pair $(R, \mathcal{D})$ is $F$-finite and $F$-pure and $J$ is an ideal of $R$, then $\mathrm{C}_{\mathcal{D}}(J)$ is the largest $\mathcal{D}$-compatible ideal contained in $J$.*

*Proof.* $\mathrm{C}_{\mathcal{D}}(J)$ is $\mathcal{D}$-compatible by the previous result. If another $\mathcal{D}$-compatible ideal $J'$ has $\mathrm{C}_{\mathcal{D}}(J) \subseteq J' \subseteq J$, then by Proposition III.1.5 we have $\mathrm{C}_{\mathcal{D}}(\mathrm{C}_{\mathcal{D}}(J)) \subseteq \mathrm{C}_{\mathcal{D}}(J') \subseteq \mathrm{C}_{\mathcal{D}}(J)$, and by Proposition III.1.16 we in fact have $\mathrm{C}_{\mathcal{D}}(J') = J' = \mathrm{C}_{\mathcal{D}}(J)$. □

The following result, originally due to Schwede [Sch09, Cor. 5.10] and to Kumar and Mehta [KM09, Thm. 1.1], captures another nice property of the Cartier core map. Recent work of Datta and Tucker [DT21, Prop. 3.4.1] provides an alternate proof that uses similar language to the rest of this dissertation.

**Proposition III.1.21** ([DT21, Prop. 3.4.1])**.** *If $(R, \mathcal{D})$ is an $F$-finite, $F$-pure pair, then there are only finitely many Cartier cores with respect to $\mathcal{D}$, i.e., there are only finitely many $\mathcal{D}$-compatible ideals.*

**Remark III.1.22.** If additionally $R$ is local, one can in fact get concrete bounds on the number of $\mathcal{D}$-compatible ideals. Using Theorem 4.2 of [ST10] or the argument from Remark 3.4 of [HW15], the number of prime Cartier cores with respect to $\mathcal{D}$ of coheight $d$ is bounded above by $\binom{n}{d}$, where $n$ is the embedding dimension of $R$.

**Theorem III.1.23.** *Let $(R, \mathcal{D})$ be an $F$-finite pair, and let $\mathcal{U}_{\mathcal{D}}$ denote the $F$-pure locus of $(R, \mathcal{D})$. Then the map $\mathrm{C}_{\mathcal{D}} : \mathcal{U}_{\mathcal{D}} \to \mathcal{U}_{\mathcal{D}}$ is continuous under the Zariski topology.*

*Proof.* We will show that the inverse image of the closed set $V = \mathbb{V}(J) \cap \mathcal{U}_{\mathcal{D}}$ is also closed, where $J$ is an ideal of $R$. Let $K$ be the intersection of all Cartier cores containing $J$ which come from primes, so that

$$K = \bigcap_{\substack{P \in \mathcal{U}_{\mathcal{D}} \\ \mathrm{C}_{\mathcal{D}}(P) \in \mathbb{V}(J)}} \mathrm{C}_{\mathcal{D}}(P).$$

Since the set of Cartier cores with respect to $\mathcal{D}$ is closed under infinite intersection by Proposition III.1.6, $K = \mathrm{C}_{\mathcal{D}}(K)$ is also a Cartier core. We claim that $\mathrm{C}_{\mathcal{D}}^{-1}(V) = \mathbb{V}(K) \cap \mathcal{U}_{\mathcal{D}}$.

Suppose $P \in \mathrm{C}_{\mathcal{D}}^{-1}(V)$. Then since $P \in \mathcal{U}_{\mathcal{D}}$, we have $\mathrm{C}_{\mathcal{D}}(P) \subseteq P$ by Proposition III.1.11. Since $\mathrm{C}_{\mathcal{D}}(P) \in \mathbb{V}(J)$, we have $K \subseteq \mathrm{C}_{\mathcal{D}}(P)$ by construction. Thus $K \subseteq P$ and so $\mathrm{C}_{\mathcal{D}}^{-1}(V) \subseteq \mathbb{V}(K) \cap \mathcal{U}_{\mathcal{D}}$.

Conversely, if $P \in \mathbb{V}(K) \cap \mathcal{U}_{\mathcal{D}}$, then $K \subseteq P$ and by Proposition III.1.5,

$$J \subseteq K = \mathrm{C}_{\mathcal{D}}(K) \subseteq \mathrm{C}_{\mathcal{D}}(P).$$

Thus $\mathbb{V}(K) \cap \mathcal{U}_{\mathcal{D}} \subseteq \mathrm{C}_{\mathcal{D}}^{-1}(V)$. □

## III.2: Quotients of Regular Rings

Now that we have seen some abstract properties of the Cartier core map, $\mathrm{C}_{\mathcal{D}}$, we shift our focus to actually computing it. In this section we give a concrete description of the Cartier core in the case when $R$ is presented as a quotient of a regular local ring $(S, \mathfrak{m})$, and $\mathcal{D}$ is the full Cartier algebra $\mathcal{C}_R$ on $R$. We will then use this concrete description to show that the Cartier core commutes with adjoining a variable and with homogenization (in the case that our regular ring is a polynomial ring).

As we saw in Section II.4 of the background, Fedder's and Glassbrenner's criteria give a clear description of the elements which are obstructions to strong $F$-regularity, in the sense that there are no splittings along these elements (see Proposition II.4.7). For the following, we use $\overline{J}$ to denote the image of an ideal $J$ in a quotient ring, and similarly $\bar{c}$ to denote the image of an element $c$.

**Theorem III.2.1.** *Let $S$ be a regular $F$-finite ring, let $I \subseteq J$ be ideals of $S$, and let $R = S/I$. Fix $e \geq 1$, $c \in S$. Then there exists some $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ with $\phi(\bar{c}) \notin \overline{J}$ if and only if $c \notin J^{[p^e]} : (I^{[p^e]} : I)$. In particular,*

$$A_{e;R}(\overline{J}) = \overline{J^{[p^e]} :_S (I^{[p^e]} :_S I)} \quad \text{and} \quad C_R(\overline{J}) = \overline{\bigcap_e J^{[p^e]} :_S (I^{[p^e]} :_S I)}.$$

*Proof.* The representations of $A_e$ and $\mathrm{C}_R$ follow directly from the first statement, so it suffices to prove that $c \notin J^{[p^e]} : (I^{[p^e]} : I)$ if and only if there is some $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ with $\phi(\bar{c}) \notin \overline{J}$.

For our fixed $e$, let $E : \operatorname{Hom}_R(F^e_* R, R) \to R$ be the "evaluation at $c$" map, so that $E(\phi) = \phi(F^e_* c)$. Our goal is to show $\operatorname{im}(E) \subseteq \overline{J}$ if and only if $c \in J^{[p^e]} : (I^{[p^e]} : I)$. By Remark II.3.6, we can view the localization of $E$ as a map $\operatorname{Hom}_{R_P}(F^e_*(R_P), R_P) \to R_P$ so that $(\operatorname{im} E)_P \cong \operatorname{im}(E_P)$. Since localization also commutes with Frobenius and with ideal colon, we can without loss of generality assume that $(S, \mathfrak{m})$ is local.

Let $\Psi$ be a generator of $\operatorname{Hom}_S(F^e_* S, S)$ as an $F^e_* S$ module. By Lemma II.4.4, the maps $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ are exactly those maps induced by something of the form $\Psi \circ F^e_*(s)$ where $s \in I^{[p^e]} : I$. Thus

$$\phi(F^e_*(\bar{c})) = \overline{(\Psi \circ F^e_* s)(F^e_* c)} = \overline{\Psi(F^e_*(sc))}$$

and so there exists $\phi$ with $\phi(F^e_*(\bar{c})) \notin \overline{J}$ if and only if there exists $s \in I^{[p^e]} : I$ with $\Psi(F^e_*(sc)) \notin J$, i.e., if and only if

$$\Psi\left(F^e_*\left(c(I^{[p^e]} : I)\right)\right) = (F^e_* c \star \Psi)(I^{[p^e]} : I) \not\subseteq J.$$

Using [Fed83, Lemma 1.6], this occurs if and only if

$$F^e_*(c) \notin (J F^e_* S) : (F^e_*(I^{[p^e]} : I)) = F^e_*(J^{[p^e]}) : F^e_*(I^{[p^e]} : I).$$

Since $S$ is regular, the flat Frobenius commutes with colon and is injective, thus this is equivalent to

$$c \notin J^{[p^e]} : (I^{[p^e]} : I). \qquad \square$$

We will frequently move between considering $\mathrm{C}_R(\overline{J})$ in $R$ and its lift $\bigcap_{e>0} J^{[p^e]} : (I^{[p^e]} : I)$ in $S$, which we will denote as either $\widetilde{\mathrm{C}}_R(J)$ or $\widetilde{\mathrm{C}}_R(\overline{J})$. Similarly, we will denote the lift of $A_{e;R}(\overline{J})$ as $\widetilde{A}_{e;R}(J)$ or $\widetilde{A}_{e;R}(\overline{J})$.

**Remark III.2.2.** In Chapter IV we will introduce the terminology of *$F$-graded systems*, which in a regular local ring $(S, \mathfrak{m})$ correspond exactly to the Cartier algebras. We'll then be able to state Proposition IV.2.5, which gives a formula for the Cartier core $\mathrm{C}_{\mathcal{D}}(J)$ of an ideal $J \subseteq S$ with respect to any Cartier algebra $\mathcal{D} \subseteq \mathcal{C}^S$.

We now prove results which let us connect Cartier cores of related ideals computed in different, related rings.

**Lemma III.2.3.** *Let $S_1 \to S_2$ be a flat map of regular $F$-finite rings. Consider ideals $I \subseteq J_1$ in $S_1$, and ideal $J_2$ in $S_2$ contracting to $J_1$. Let $R_1 = S_1/I$ and $R_2 = S_2/IS_2$. Then*

$$C_{R_1}(\overline{J_1})R_2 \subseteq C_{R_2}(\overline{J_2}).$$

*Proof.* Finite intersections always commute with flat base change. Thus for any sequence of ideals $\{K_e\}_{e\in\mathbb{N}}$ and for any $n$,

$$\left(\bigcap_{e=1}^{\infty} K_e\right) S_2 \subseteq \left(\bigcap_{e=1}^{n} K_e\right) S_2 = \bigcap_{e=1}^{n}(K_e S_2)$$

and in particular we must have $\left(\bigcap_{e=1}^{\infty} K_e\right) S_2 \subseteq \bigcap_{e=1}^{\infty}(K_e S_2)$. Colon commutes with flat base change when the ideals are finitely generated [Mat89, Thm. 7.4]. Thus

$$\left(\bigcap_{e\geq 1} J_1^{[p^e]} : (I^{[p^e]} : I)\right) S_2 \subseteq \bigcap_{e\geq 1} \left((J_1S_2)^{[p^e]} : ((IS_2)^{[p^e]} : IS_2)\right) \subseteq \bigcap_{e\geq 1} \left(J_2^{[p^e]} : ((IS_2)^{[p^e]} : IS_2)\right),$$

which by using Theorem III.2.1 to pass to the quotient gives

$$C_{R_1}(\overline{J_1})R_2 \subseteq C_{R_2}(\overline{J_2}). \qquad \square$$

In the case of a general flat map, even a general faithfully flat map, containment is the best we can do. For example, consider $S_1 = k[x^p]$ and $S_2 = k[x]$ where $k$ is a perfect field. The inclusion of $S_1$ into $S_2$ is faithfully flat since it corresponds to the Frobenius on the regular ring $k[x]$. Now consider $I = J_1 = \langle x^p \rangle \subset S_1$ and $J_2 = \langle x \rangle \subset S_2$. Then $R_1 = S_1/I \cong K$ which is Frobenius split, so $C_{R_1}(\overline{J}_1) = \overline{J}_1$. But $R_2 = S_2/IS_2 = k[x]/\langle x^p \rangle$ is not reduced, thus cannot be Frobenius split. Since $\overline{J}_2$ is a prime ideal, this means $C_{R_2}(\overline{J}_2) = R_2$.

However, it turns out that in the case of adjoining a variable, we can get a stronger result.

**Proposition III.2.4.** *Let $R$ be a quotient of a regular $F$-finite ring, let $J$ be an ideal of $R$, and let $J'$ be an ideal of $R[x]$ such that $JR[x] \subseteq J' \subseteq JR[x] + \langle x \rangle$. Then*

$$C_R(J)R[x] = C_{R[x]}(J') \quad \textit{and} \quad C_{R[x]}(J') \cap R = C_R(J).$$

*Proof.* By Proposition III.1.5,

$$C_{R[x]}(JR[x]) \subseteq C_{R[x]}(J') \subseteq C_{R[x]}(JR[x] + \langle x \rangle).$$

Our first step will be to show

$$C_{R[x]}(JR[x]) \supseteq C_{R[x]}(JR[x] + \langle x \rangle),$$

which will then give us $C_{R[x]}(JR[x]) = C_{R[x]}(J') = C_{R[x]}(JR[x] + \langle x \rangle)$.

To do so, note that by assumption we can write $R = S/I$ where $S$ is a regular $F$-finite ring, and so we can also write $R[x] = S[x]/IS[x]$. We use $\widetilde{\ }$ to denote lifting an ideal from $R$ or $R[x]$ to $S$ or $S[x]$, as appropriate. Consider $S[x]$ to be $\mathbb{N}$-graded by $x$. Since $\widetilde{JR[x] + \langle x \rangle}$, the lift of $JR[x] + \langle x \rangle$ to $S[x]$, is homogeneous, as is $IS[x]$, our lift of the Cartier core

$$\widetilde{\mathrm{C}}_{R[x]}(J[x] + \langle x \rangle) = \bigcap_{e>0} \widetilde{JR[x] + \langle x \rangle} : (IS[x]^{[q]} : IS[x])$$

is also homogeneous. Consider some homogeneous $g$ in this lift of the Cartier core. Ideal colon commutes with flat maps, and $S \to S[x]$ and the Frobenius are both flat. Thus for every $q = p^e$ we have

$$IS[x]^{[q]} : IS[x] = (I^{[q]} : I)S[x].$$

Since $g \in \widetilde{A}_e(J)$, we must have $g(I^{[q]} : I) \subseteq (\widetilde{JR[x] + \langle x \rangle})^{[q]}$. However, any element of $(\widetilde{JR[x] + \langle x \rangle})^{[q]}$ of degree less than $q$ must be expressible in terms of elements of $\widetilde{JR[x]}^{[q]}$. In particular, if $q > \deg g$ then $g(I^{[q]} : I) \subseteq \widetilde{JR[x]}^{[q]}$. Thus for $e \gg 0$, we have

$$g \in \widetilde{JR[x]}^{[q]} : (IS[x]^{[q]} : IS[x]) = \widetilde{A}_{e;R[x]}(JR[x]).$$

By [BC21, Prop. 4.15], since $\mathrm{C}_{R[x]}(JR[x]) = \bigcap_{e \gg 0} A_{e;R[x]}(JR[x])$, this tells us that

$$\mathrm{C}_{R[x]}(JR[x] + \langle x \rangle) \subseteq \mathrm{C}_{R[x]}(JR[x])$$

as desired.

Now we have shown $C_{R[x]}(JR[x]) = C_{R[x]}(J')$, and it suffices to show $C_R(J)R[x] = C_{R[x]}(J[x])$. To do so, we will show that adjoining a variable commutes with infinite intersection. Consider an arbitrary ideal $K = \bigcap_\alpha K_\alpha$ in $S$. As a set, each $K_\alpha S[x]$ is polynomials with coefficients in $K_\alpha$, and so the polynomials in $\bigcap_\alpha K_\alpha S[x]$ are those with coefficients in $K_\alpha$ for every $\alpha$, which is precisely $KS[x]$, as desired.

This lets us repeat the argument in Lemma III.2.3 but with equalities, and thus

$$C_R(J)R[x] = C_{R[x]}(J[x]) = C_{R[x]}(J')$$

as desired. The contraction result then follows directly from the fact that adjoining a variable is faithfully flat, so that

$$C_R(J) = C_R(J)R[x] \cap R = C_{R[x]}(J') \cap R. \qquad \square$$

If $R$ is a quotient of a polynomial ring by a homogeneous ideal, we can also look at how the Cartier core behaves under homogenization. More concretely, take $R = S/I$ for $S = k[x_1, \ldots, x_d]$ and $I$ a homogeneous ideal of $S$, so that $R$ is $\mathbb{N}$-graded. If $f \in R$, we let $f^h$ denote the minimal homogenization of $f$ in $R[t]$, so that

$$f^h = t^{\deg f} f\left(\frac{x_1}{t}, \ldots, \frac{x_n}{t}\right).$$

If $J$ is an ideal of $R$, we define its homogenization in $R[t]$ to be $J^h = \langle f^h \mid f \in J \rangle$.

For any degree-preserving lift of $f$ to $S$, there is a corresponding lift of $f^h$ to $S[t]$ so that the lift of the homogenization is the homogenization of the lift. This means we can freely consider a given homogenization to live either in $R[t]$ or in $S[t]$. Further, the ideals $(\widetilde{J^h})$ and $(\widetilde{J})^h$ are the same: $(\widetilde{J^h})$ is generated by the lifts of the homogenizations of elements of $J$, and $(\widetilde{J})^h$ is generated by homogenizations of lifts of elements of $J$.

There is also a corresponding dehomogenization map $\delta : R[t] \to R$ defined by $\delta(t) = 1$, which ensures that $\delta(f^h) = f$.

We recall the following straightforward facts about homogenization.

**Lemma III.2.5.** *Let $R$ be a quotient of a polynomial ring by a homogeneous ideal. Let $I, J$ be ideals of $R$, and $\{I_\alpha\}$ a family of ideals. Let $f$ be an element of $R$. Then the following statements all hold.*

- $f \in I$ *if and only if* $f^h \in I^h$.
- $(I : J)^h = I^h : J^h$ *and* $\left(\bigcap I_\alpha\right)^h = \bigcap(I_\alpha^h)$.
- $(I^h)^{[p^e]} = (I^{[p^e]})^h$.

*Proof.* For the first two bullets, see Problems 3.15 and 3.17 in [EH12]. For the third bullet, use Proposition 3.15 of [EH12] and Theorem 6.2 of [HT92]. □

Using these facts, we will prove the following useful result.

**Proposition III.2.6.** *Let $R = S/I$ where $S$ is a polynomial ring over an $F$-finite field and $I$ is a homogeneous ideal. Let $J$ be an ideal of $R$. Then*

$$(C_R(J))^h = C_{R[t]}(J^h) \quad \text{and} \quad C_R(J) = \delta\left(C_{R[t]}(J^h)\right)$$

*Proof.* If we lift to $S[t]$ using Theorem III.2.1 and the above discussion on lifting and homogenization, then

$$\begin{aligned}
(\widetilde{C_R(J)})^h &= \left(\widetilde{C}_R(J)\right)^h \\
&= \left(\bigcap_{e>0} (\widetilde{J})^{[q]} : (I^{[q]} : I)\right)^h \\
&= \bigcap_{e>0} \left((\widetilde{J}^h)^{[q]} : ((I^h)^{[q]} : I^h)\right) \\
&= \bigcap_{e>0} \left((\widetilde{J^h})^{[q]} : (I^{[q]} : I)\right) \\
&= \widetilde{C}_{R[t]}(J^h)
\end{aligned}$$

and so contracting back to $R[t]$ via Theorem III.2.1,

$$(C_R(J))^h = C_{R[y]}(J^h).$$

The last statement follows directly from dehomogenizing each side of the equation. □

## III.3: Formula for Stanley-Reisner Rings

A ring $R$ is a *Stanley-Reisner ring* if it can be written as $R = S/I$, where $S$ is a polynomial ring and $I$ is a square-free monomial ideal. The following theorem gives a complete description of the Cartier core map for $\operatorname{Spec} R$ where $R$ is a Stanley-Reisner ring.

**Theorem III.3.1.** *Let $R$ be a Stanley-Reisner ring over a field that has prime characteristic and is $F$-finite. Let $Q$ be any prime ideal. Then*

$$C_R(Q) = \sum_{\substack{P \in \mathrm{Min}(R) \\ P \subseteq Q}} P.$$

*In particular, the set of prime Cartier cores of $R$, i.e., the set of generic points of $F$-pure centers of $R$, is the set of sums of minimal primes.*

This theorem extends some earlier results. Aberbach and Enescu showed that the splitting prime of a Stanley-Reisner ring, which is its largest proper uniformly $F$-compatible ideal, is the sum of the minimal primes [AE05, Prop 4.10]. For the reader's convenience, we will reprove this in our proof of Theorem III.3.1. At the other extreme, Vassilev showed that the test ideal of a Stanley-Reisner ring, which is its smallest non-zero uniformly $F$-compatible ideal, is $\sum_{i=1}^{t} \bigcap_{j \neq i} P_j$ where $P_1, \ldots, P_t$ are the minimal primes of $R$ [Vas98, Thm. 3.7]. In a related but different direction, for a specific choice of $\phi : F^e_* R \to R$, Enescu and Ilioaea showed that the $\phi$-compatible primes of $R$ are precisely the prime monomial ideals which *contain* a minimal prime of $R$. They used this to give a combinatorial description the test ideal of the pair $(R, \phi)$ [EI20, Prop. 3.9, Prop. 3.10].

Badilla-Céspedes showed that if $P'$ is a prime monomial ideal, then $\mathrm{C}(P')$ as well as each $A_e(P')$ is also a monomial ideal, and more explicitly that $A_e(P') = (P')^{[p^e]} + \mathrm{C}(P')$ in this setting [BC21, Lemma 4.16,Prop 4.17]. Meanwhile, Àlvarez Montaner, Boix, and Zarzuela gave a concrete description of $I^{[p^e]} :_S I$ in terms of the minimal primes of $I$, which could be used to explicitly compute the Cartier contractions for any ideal $J$ [ÀMBZ12, Prop 3.2].

*Proof of Theorem III.3.1.* Our proof will proceed as follows: First we will reduce to the case where every minimal prime is contained in $Q$. Then we will homogenize and trap $Q^h$ between a sum of minimal primes and the homogeneous maximal ideal, and use Proposition III.1.5 and the convenient form of monomial primes to get our desired equality.

Let $\widetilde{\ }$ denote the lift of any ideal to $S$, let $I' = \bigcap_{P \in \mathrm{Min}(R),\ P \subseteq Q} \widetilde{P}$ be the intersection of the minimal primes contained in $Q$, and let $R' = S/I'$. Then

$$R_Q \cong S_{\widetilde{Q}}/I_{\widetilde{Q}} = S_{\widetilde{Q}}/I'_{\widetilde{Q}} \cong R'_Q$$

and so by Lemma III.1.7,

$$C_R(Q)R_Q = C_{R_Q}(Q) = C_{R'_Q}(Q) = C_{R'}(Q)R'_Q.$$

Stanley-Reisner rings are $F$-pure [HR76, Prop. 5.8], and so $C_R(Q) \subseteq Q$ by Corollary III.1.14, and thus when we lift back to $S$ using Theorem III.2.1, we see

$$\bigcap_{e>0} \widetilde{Q}^{[p^e]} : (I^{[p^e]} : I) = \bigcap_{e>0} \widetilde{Q}^{[p^e]} : (I'^{[p^e]} : I').$$

Thus we can use $I'$ as our new $I$, and so we can assume $P \subseteq Q$ for all minimal primes $P$.

Relabel the variables so that $\sum_{P \in \mathrm{Min}(R)} P = \langle x_1, \ldots, x_c \rangle$ and define $A = k[x_1, \ldots, x_c]/I$, so that $R = A[x_{c+1}, \ldots, x_d]$. Now we homogenize, so $Q^h \subseteq \mathfrak{m}$ where $\mathfrak{m}$ is the homogeneous maximal ideal in $S[t]$. Then Proposition III.1.5 tells us

$$C_{R[t]}\left(\sum_{P \in \mathrm{Min}(R)} P^h\right) \subseteq C_{R[t]}(Q^h) \subseteq C_{R[t]}(\overline{\mathfrak{m}}).$$

Each minimal prime $P$ of $R$ remains a minimal prime of $R[t]$ after homogenizing, so Corollary III.1.13 says $C_{R[t]}(P^h) = P^h$, and Proposition III.1.15 then says that their sum is also preserved by the Cartier core map. Applying Proposition III.2.4 to $\overline{\mathfrak{m}}$, we get

$$\langle x_1, \ldots, x_c \rangle R[t] = C_A(P_1 + \cdots + P_t) A[x_{c+1}, \ldots, x_d, t] = C_{R[t]}(\overline{\mathfrak{m}}).$$

Thus

$$\langle x_1, \ldots, x_c \rangle = \mathrm{C}_{R[t]}\left(\sum_{P \in \mathrm{Min}(R)} P^h\right) \subseteq C_{R[t]}(Q^h) \subseteq C_{R[t]}(\overline{\mathfrak{m}}) = \langle x_1, \ldots, x_c \rangle$$

and by Proposition III.2.6,

$$(C_R(Q))^h = C_{R[t]}(Q^h) = \langle x_1, \ldots, x_c \rangle.$$

Dehomogenizing the homogenization always gives back the original ideal, and so

$$C_R(Q) = \langle x_1, \ldots, x_c \rangle = \sum_{P \in \mathrm{Min}(R)} P.$$

For the last statement of the theorem, note that since each minimal prime of $R$ corresponds to an ideal of $S$ which is generated by variables, any sum of minimal primes is also prime, and thus is fixed by the Cartier core map. $\square$

Since taking the Cartier core commutes with intersection, Theorem III.3.1 immediately gives a formula for the Cartier core of any radical ideal in terms of the Cartier cores of its minimal primes. The following corollary instead gives a formula for the Cartier core of an arbitrary ideal which is more analogous to the previous one.

**Corollary III.3.2.** *Let $R$ be a Stanley-Reisner ring over a field that has prime characteristic and is $F$-finite. Let $J$ be any ideal. Then*

$$\mathrm{C}_R(J) = \sum_{\substack{\mathcal{Q}\subseteq \mathrm{Min}(R) \\ \left(\bigcap_{P\in\mathcal{Q}} P\right)\subseteq J}} \left(\bigcap_{P\in\mathcal{Q}} P\right).$$

*Proof.* First, we will show our desired formula gives an ideal contained in $\mathrm{C}_R(J)$. The restriction on the $P$'s appearing ensures that the resulting ideal is contained in $J$. Further, each $P$ appearing is minimal, so $P = \mathrm{C}_R(P)$ by Corollary III.1.13. Using Propositions III.1.6 and III.1.15 (our intersection and sum results) to apply $\mathrm{C}_R$ to the formula and then using Proposition III.1.5 to preserve the containment gives

$$\sum_{\substack{\mathcal{Q}\subseteq \mathrm{Min}(R) \\ \left(\bigcap_{P\in\mathcal{Q}} P\right)\subseteq J}} \left(\bigcap_{P\in\mathcal{Q}} P\right) = \mathrm{C}_R\left(\sum_{\substack{\mathcal{Q}\subseteq \mathrm{Min}(R) \\ \left(\bigcap_{P\in\mathcal{Q}} P\right)\subseteq J}} \left(\bigcap_{P\in\mathcal{Q}} P\right)\right) \subseteq \mathrm{C}_R(J).$$

To show equality, we will show that summing over a specific smaller subset in fact already yields $\mathrm{C}_R(J)$, and so the larger sum above must yield $\mathrm{C}_R(J)$ as well. Since $R$ is Frobenius split, $\mathrm{C}_R(J)$ is radical by Proposition III.1.18, so we can write $\mathrm{C}_R(J) = \bigcap_{i=1}^n Q_i$ as the intersection of its minimal primes. By the same argument as in the proof of Theorem III.1.23, applying Proposition III.1.10, Corollary III.1.14, and Proposition III.1.16 shows that $\mathrm{C}_R(Q_i) = Q_i$ for each $i$.

Now since the Cartier core commutes with intersection, we use Theorem III.3.1 to see

$$\mathrm{C}_R(J) = \bigcap_i \mathrm{C}_R(Q_i) = \bigcap_i \left(\sum_{\substack{P\in \mathrm{Min}(R) \\ P\subseteq Q_i}} P\right).$$

Writing $R = S/I$ as a quotient of a polynomial ring and lifting back up to $S$, this says that the lift of $\mathrm{C}_R(J)$ is an intersection of sums of monomial ideals. Sum and intersection of monomial ideals commute [EH12, Problem 1.17], and so passing back to the quotient gives

$$\mathrm{C}_R(J) = \sum_{\substack{P_1,\dots,P_n\in \mathrm{Min}(R) \\ P_i\subseteq Q_i}} \left(\bigcap_{i=1}^n P_i\right).$$

For each possibility for $P_1, \ldots, P_n$ in the above sum, we have $\bigcap_{i=1}^n P_i \subseteq \mathrm{C}_R(J) \subseteq J$, and so this sum is a subset of our desired formula, which thus must be equal to $\mathrm{C}_R(J)$ as well. $\square$

# CHAPTER IV

# $F$-Graded Systems and $p$-Families

In this chapter, we transition to thinking about Cartier algebras through a new lens, namely that of *$F$-graded systems*. We'll define notions of $F$-splitting and strong $F$-regularity in this setting, and we'll be especially interested in the special subclass of $F$-graded systems called $p$-families.

As before, we assume that all rings in this chapter are commutative Noetherian of prime characteristic $p$, and are $F$-finite.

## IV.1: Preliminaries on $F$-Graded Systems

**Definition IV.1.1** ([Bli13, Def 3.20])**.** Let $R$ be a commutative Noetherian ring of prime characteristic $p$, and let $\{\mathfrak{b}_n\}_{n\in\mathbb{N}}$ be a sequence of ideals. We say $\mathfrak{b}_\bullet$ is an *$F$-graded system of ideals* if

1. $\mathfrak{b}_0 = R$, and
2. $\mathfrak{b}_e^{[p^f]}\mathfrak{b}_f \subseteq \mathfrak{b}_{e+f}$ for all $e, f \geq 0$.

We will refer to $\mathfrak{b}_e$ as the *degree $e$* piece (or degree $e$ ideal) of the system.

Blickle first introduced a version of this definition in [Bli13, Def 3.20]. We use the definition as it later appears in [BST12, Def 4.7].

When describing an $F$-graded system (as in the following example), we will typically omit stating the degree zero piece.

**Example IV.1.2** (Three main examples)**.** For any ring $R$ of prime characteristic $p$ and any ideal $J$ in $R$, the following three systems of ideals are all $F$-graded:

1. Setting $\mathfrak{a}_e = \prod_{i=0}^{e-1} J^{[p^i]}$ for $e > 0$.

2. Setting $\mathfrak{b}_e = J^{[p^e]} : J$ for $e > 0$.

3. Setting $\mathfrak{c}_e = J^{\lceil t(p^e-1)\rceil}$ for all $e > 0$, and for some fixed $t \in \mathbb{R}_{>0}$.

*Proof.* For the second system, we will show by induction on $n$ that $\mathfrak{a}_n \supset \mathfrak{a}_{n-e}^{[p^e]}\mathfrak{a}_e$ for all $0 \leq e \leq n$. The base cases of $n = 0$ and $n = 1$ are clear. Now suppose we want to prove the statement for $n+1$. If $e = 0$ or $e = n+1$, again the result is clear. Otherwise,

$$\mathfrak{a}_{n+1-e}^{[p^e]}\mathfrak{a}_e = \left(\prod_{i=0}^{n-e} J^{[p^i]}\right)^{[p^e]} \left(\prod_{j=0}^{e-1} J^{[p^j]}\right) = \left(\prod_{i=0}^{n-e} J^{[p^{i+e}]}\right)\left(\prod_{j=0}^{e-1} J^{[p^j]}\right) = \prod_{i=0}^{n} J^{[p^i]} = \mathfrak{a}_{n+1}.$$

Note that $(J^{[p^e]} : J)^{[p^f]} \subseteq J^{[p^{e+f}]} : J^{[p^f]}$, since for any $x \in J^{[p^e]} : J$, we have

$$x^{p^f} J^{[p^f]} \subseteq (xJ)^{[p^f]} \subseteq (J^{[p^e]})^{[p^f]}.$$

This containment makes it clear that

$$(\mathfrak{b}_e^{[p^f]}\mathfrak{b}_f)J \subseteq (J^{[p^{e+f}]} : J^{[p^f]})(J^{[p^f]} : J)J \subseteq J^{[p^{e+f}]},$$

and so

$$\mathfrak{b}_e^{[p^f]}\mathfrak{b}_f \subseteq J^{[p^{e+f}]} : J = \mathfrak{b}_{e+f}.$$

For the last system, since

$$p^f \lceil t(p^e-1)\rceil + \lceil t(p^f-1)\rceil \geq \lceil t(p^{e+f}-p^f)\rceil + \lceil t(p^f-1)\rceil \geq \lceil t(p^{e+f}-p^f) + t(p^f-1)\rceil,$$

we immediately have

$$\mathfrak{c}_e^{[p^f]}\mathfrak{c}_f \subseteq J^{p^f\lceil t(p^e-1)\rceil} J^{\lceil t(p^f-1)\rceil} \subseteq J^{\lceil t(p^{e+f}-1)\rceil} \subseteq \mathfrak{c}_{e+f}.$$ □

**Remark IV.1.3.** A similar argument as for system $\mathfrak{a}_\bullet$ above also tells us that for *any* $F$-graded system $\mathfrak{d}_\bullet$, we have $\mathfrak{d}_n \supset \prod_{i=0}^{n-1}\mathfrak{d}_1^{[p^i]}$, and $\mathfrak{d}_{en} \supset \prod_{i=0}^{n-1}\mathfrak{d}_e^{[p^i]}$. In particular, system $\mathfrak{a}_\bullet$ above is the minimal $F$-graded system that has $\mathfrak{b}_1 = J$.

The following straightforward result of Blickle, Schwede, and Tucker illustrates the original motivation behind $F$-graded systems: they are a useful way to describe Cartier algebras (as defined in Definition II.3.2).

**Lemma IV.1.4** ([BST12, Lemma 4.9])**.** *If $(R, \mathfrak{m})$ is an $F$-finite local ring, then every $F$-graded system of ideals $\mathfrak{a}_\bullet$ of $R$ defines a Cartier subalgebra $\mathcal{C}^{\mathfrak{a}_\bullet}$ on $R$ by setting $\mathcal{C}_e^{\mathfrak{a}_\bullet} := \mathcal{C}_e^R \star \mathfrak{a}_e$ for all $e \geq 0$. Furthermore, if $R$ is Gorenstein, then* every *Cartier subalgebra $\mathcal{D}$ arises uniquely in this manner.*

In light of this lemma, we have in fact seen examples two and three of Example IV.1.2 before: the system $\mathfrak{b}_\bullet$ appeared in Example II.4.5 as the $F$-graded system defining a Cartier algebra on the regular local ring $R$ which is the lift of the full Cartier algebra on $R/J$, and the system $\mathfrak{c}_\bullet$ appeared in Example II.3.3 as the $F$-graded system defining the Cartier algebra for the pair $(R, J^t)$.

An interesting special case of $F$-graded systems are $p$-families of ideals, which were introduced by Hernández and Jeffries for a different purpose:

**Definition IV.1.5** ([HJ18, Def 5.1])**.** A *$p$-family of ideals* is a sequence of ideals $I_\bullet$ such that $I_e^{[p]} \subseteq I_{e+1}$ for all $e$.

Note that $p$-families are indeed $F$-graded; iterating the definition shows $I_e^{[p^f]} \subseteq I_{e+f}$ for any $f \geq 1$, and so

$$I_e^{[p^f]} I_f \subseteq I_{e+f} I_f \subseteq I_{e+f}.$$

**Example IV.1.6** (cf. [HJ18, Ex. 5.4–5.7])**.** The following systems are all examples of $p$-families:

- the classic example of $I^{[p^\bullet]}$.
- (Ex. 5.4) the sequence of Cartier contractions $A_e(J)$ of an ideal under the full Cartier algebra (see Notation III.1.2) also gives a $p$-family. In the setting of a quotient of a regular ring, where $R = S/I$ and $I \subseteq J$ are ideals of $S$, recall from Theorem III.2.1 that these are of the form $A_e(J/I) = \left(J^{[p^e]} :_S (I^{[p^e]} :_S I)\right)/I$.

- (Ex. 5.5) If $I_\bullet$ is a graded family of ideals (in the typical sense), then $\mathfrak{a}_e := I_{p^e}$ is a $p$-family.

  As a variant of this example, if $I_\bullet$ is a graded family of ideals (in the typical sense), then $\mathfrak{a}_e := I_{p^e-1}$ is an $F$-graded system.

- (Ex. 5.6) $p$-families are preserved under arbitrary termwise product, sum, and intersection; by expansion and contraction to or from another ring; and by termwise saturation with respect to a fixed ideal.

- (Ex. 5.7) Fix $t \in \mathbb{R}_{>0}$ and $f \in R$, and let $\mathfrak{a}_\bullet$ be a $p$-family. Then $\mathfrak{b}_e := \mathfrak{a}_e : f^{\lceil tp^e \rceil - 1}$ is also a $p$-family.

### IV.1.1: New Systems From Old

We can also modify existing $F$-graded systems to get new ones. In this subsection, we observe some basic operations on $F$-graded systems as a prelude to our more detailed study of the operation of "$p$-stabilization" in Section IV.3.

First, we see that in a polynomial ring, the termwise operation of taking initial ideals with respect to a fixed monomial order preserves the property of being an $F$-graded system. For a polynomial $r$, we write $\mathrm{in}_<(r)$ for the leading term (or simply $\mathrm{in}(r)$ if the monomial order is clear). Similarly, we write $\mathrm{in}_<(I)$ or simply $\mathrm{in}(I)$ for the initial term ideal of a given ideal $I$. See [EH12] for more background.

**Proposition IV.1.7.** *Let $\mathfrak{a}_\bullet$ be an $F$-graded system in $S = K[x_1, \ldots, x_n]$. Fix a monomial term order. Then the system of initial ideals* $\mathrm{in}\,\mathfrak{a}_\bullet$*, where*

$$\mathrm{in}\,\mathfrak{a}_e = \langle \mathrm{in}(r) \mid r \in \mathfrak{a}_e \rangle,$$

*is also $F$-graded. Similarly, if $\mathfrak{b}_\bullet$ is a $p$-family, then* $\mathrm{in}\,\mathfrak{b}_\bullet$ *is also a $p$-family.*

*Proof.* Taking leading terms commutes with products and powers, i.e., $\mathrm{in}(r^q) = \mathrm{in}(r)^q$ and $\mathrm{in}(rs) = \mathrm{in}(r)\,\mathrm{in}(s)$. Thus our desired condition is preserved. More explicitly, if $r \in \mathfrak{a}_e$ and

$s \in \mathfrak{a}_f$, then

$$(\mathrm{in}(r))^{pf}\,\mathrm{in}(s) = \mathrm{in}(r^{pf} s) \in \mathrm{in}(\mathfrak{a}_e^{[p^f]}\mathfrak{a}_f) \subseteq \mathrm{in}\,\mathfrak{a}_{e+f}.$$

Removing the $s$'s from the argument gives the $p$-family result. $\square$

Likewise, the termwise operation of taking integral closure also preserves the property of being an $F$-graded system.

**Proposition IV.1.8.** *Let $\mathfrak{a}_\bullet$ be an $F$-graded system in some ring $R$. Then the system $\overline{\mathfrak{a}}_\bullet$ of termwise integral closures $\overline{\mathfrak{a}}_e = \overline{\mathfrak{a}_e}$ is also $F$-graded. Similarly, if $\mathfrak{b}_\bullet$ is a $p$-family, then $\overline{\mathfrak{b}}_\bullet$ is also a $p$-family.*

*Proof.* For any ideal $I$, we have $F^e(\overline{I})F^e_*R \subseteq \overline{F^e(I)F^e_*R}$ by persistence of integral closure applied with the Frobenius [HS06, Rmk 1.1.3(7)], so that $F^e_*\overline{I}^{[q]} \subseteq \overline{F^e_* I^{[q]}}$. In other words, $\overline{I}^{[q]} \subseteq \overline{I^{[q]}}$. This gives the $p$-family result, since $\overline{\mathfrak{b}_e}^{[p]} \subseteq \overline{\mathfrak{b}_e^{[p^e]}} \subseteq \overline{\mathfrak{b}_{e+1}}$.

It is also true [HS06, Rmk 1.3.2(4)] that $\overline{I} \cdot \overline{J} \subseteq \overline{IJ}$. Thus we see

$$\overline{\mathfrak{a}_e}^{[p^f]}\overline{\mathfrak{a}_f} \subseteq \overline{\mathfrak{a}_e^{[p^f]}}\,\overline{\mathfrak{a}_f} \subseteq \overline{\mathfrak{a}_e^{[p^f]}\mathfrak{a}_f} \subseteq \overline{\mathfrak{a}_{e+f}}$$

as desired. Removing the $\mathfrak{a}_f$'s from the argument gives the $p$-family result. $\square$

Finally, it is illustrative to observe that the short-term behavior of an $F$-graded system may not be representative of the long-term behavior, in the sense that one can "splice" a smaller sequence together with a larger one:

**Lemma IV.1.9** (Splicing Lemma)**.** *Let $T$ be a commutative ring of prime characteristic $p$ and let $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ be $F$-graded systems. If there exists an index $E$ such that $\mathfrak{a}_e \subseteq \mathfrak{b}_e$ for all $e \leq 2E$, then the system $\mathfrak{c}_\bullet$ with*

$$\mathfrak{c}_e = \begin{cases} \mathfrak{a}_e & e \leq E \\ \mathfrak{b}_e & e > E \end{cases}$$

*is $F$-graded.*

*Proof.* The grading condition clearly holds if $e, f > E$. It is not hard to check that the condition also holds if $e, f \leq E$ because of the containment between $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$:

$$\mathfrak{c}_e^{[p^f]}\mathfrak{c}_f \subseteq \mathfrak{a}_e^{[p^f]}\mathfrak{a}_f \subseteq \mathfrak{a}_{e+f} \subseteq \mathfrak{b}_{e+f}.$$

So suppose $e \leq E$ and $f > E$. Then

$$\mathfrak{c}_e^{[p^f]}\mathfrak{c}_f = \mathfrak{a}_e^{[p^f]}\mathfrak{b}_f \subseteq \mathfrak{b}_e^{[p^f]}\mathfrak{b}_e \subseteq \mathfrak{b}_{e+f} = \mathfrak{c}_{e+f}$$
$$\mathfrak{c}_f^{[p^e]}\mathfrak{c}_e = \mathfrak{b}_f^{[p^e]}\mathfrak{a}_e \subseteq \mathfrak{b}_f^{[p^e]}\mathfrak{b}_e \subseteq \mathfrak{b}_{e+f} = \mathfrak{c}_{e+f}.$$

□

## IV.2: Detecting $F$-Singularities of an $F$-Graded System

In continuation of our theme from the previous chapter of studying $F$-singularities of Cartier algebras, we can now meaningfully define $F$-singularities for an $F$-graded system.

**Definition IV.2.1.** Let $\mathfrak{a}_\bullet$ be an $F$-graded system. Then $\mathfrak{a}_\bullet$ is *Frobenius split* if the corresponding Cartier algebra $\mathscr{C}^{\mathfrak{a}_\bullet}$ is Frobenius split. Likewise, $\mathfrak{a}_\bullet$ is *strongly $F$-regular* if the corresponding Cartier algebra $\mathscr{C}^{\mathfrak{a}_\bullet}$ is strongly $F$-regular.

**Remark IV.2.2.** Recalling Definition II.3.4, we can also restate the above definition even more explicitly as follows: let $\mathfrak{a}_\bullet$ be an $F$-graded system on the ring $R$.

- The system $\mathfrak{a}_\bullet$ is Frobenius split if there exists some $e > 0$ and some $\varphi \in \mathrm{Hom}_R(F^e_* R, R)$ and $a \in \mathfrak{a}_e$ with $\varphi(F^e_* a) = 1$.
- The system $\mathfrak{a}_\bullet$ is strongly $F$-regular if for every $c$ not in any minimal prime of $R$, there exists some $e > 0$ and some $\varphi \in \mathrm{Hom}_R(F^e_* R, R)$ and $a \in \mathfrak{a}_e$ with $\varphi(F^e_*(ac)) = 1$.

It can be difficult to verify that a given Cartier algebra is strongly $F$-regular, especially if one does not have an explicit "strong test element" as in Theorem II.1.10, and the same difficulty holds for $F$-graded systems. However, we shall see that for $p$-families, strong $F$-regularity and $F$-splitting collapse into the same condition. More specifically, our main result of this section is the following:

**Theorem IV.2.3.** *Let $(S, \mathfrak{m})$ be a regular local ring. Let $\mathfrak{b}_\bullet$ be a $p$-family in $S$. Then $\mathfrak{b}_\bullet$ is $F$-split if and only if it is strongly $F$-regular.*

Before proving this theorem we will first prove some useful tools for detecting strong $F$-regularity of an $F$-graded system. The first is a version of Fedder's criterion for $F$-graded systems:

**Lemma IV.2.4** ([BST12, Lemma 4.12]). *Let $(S, \mathfrak{m})$ be an $F$-finite regular local ring, let $\mathfrak{a}_\bullet$ be an $F$-graded system, and let $\mathcal{C}^{\mathfrak{a}_\bullet} := \bigoplus_e \mathcal{C}_e^S \star \mathfrak{a}_e$ be the corresponding Cartier algebra. Fix $c \in S$. Then there is some $\psi \in \mathcal{C}_e^{\mathfrak{a}_\bullet}$ such that $\psi(F_*^e c) = 1$ if and only if $c \notin \mathfrak{m}^{[p^e]} : \mathfrak{a}_e$.*

*Proof.* Recall Lemma II.4.3, Fedder's result which says $(\Phi^e \star s)(F_*^e I) \subseteq J$ if and only if $s \in J^{[p^e]} :_S I$, where $I, J$ are ideals of $S$, $s \in S$, and $\Phi^e$ is the free generator of $\operatorname{Hom}_S(F_*^e S, S)$. Since every map $\psi \in \mathcal{C}_e^{\mathfrak{a}_\bullet}$ is of the form $\psi = \Phi^e \star a$ for $a \in \mathfrak{a}_e$, this means that $\psi(F_*^e c) \in \mathfrak{m}$ for all such $\psi$ if and only if

$$(\Phi^e \star a)(F_*^e c) = \Phi^e(F_*^e ac) = (\Phi^e \star c)(F_*^e a) \in \mathfrak{m}$$

for all $a \in \mathfrak{a}_e$. In other words, this occurs whenever

$$\Phi^e(F_*^e(\mathfrak{a}_e c)) = (\Phi^e \star c)(F_*^e \mathfrak{a}_e) \subseteq \mathfrak{m}$$

which by Fedder's criterion happens if and only if $c \in \mathfrak{m}^{[p^e]} : \mathfrak{a}_e$. $\square$

We can also revisit the Cartier core from Chapter III, as we can now give a formula for the Cartier core of any ideal with respect to any Cartier algebra in a regular ring. In particular, since we are working over a domain, the system $\mathfrak{a}_\bullet$ is strongly $F$-regular exactly when the Cartier core of every ideal is zero.

**Proposition IV.2.5.** *As above, let $(S, \mathfrak{m})$ be an $F$-finite regular local ring, let $\mathfrak{a}_\bullet$ be an $F$-graded system, and let $\mathcal{C}^{\mathfrak{a}_\bullet} := \bigoplus_e \mathcal{C}_e^S \star \mathfrak{a}_e$ be the corresponding Cartier algebra. Then*

$$\mathrm{C}_{\mathcal{C}^{\mathfrak{a}_\bullet}}(J) = \bigcap_e J^{[p^e]} : \mathfrak{a}_e.$$

*Proof.* By Lemma II.4.3 and the same argument as Lemma IV.2.4, we have that $\psi(F^e_* c) \in J$ for all $c \in \mathcal{C}^{\mathfrak{a}_\bullet}_e$ if and only if $c \in J^{[p^e]} : \mathfrak{a}_e$. Intersecting over all $e$ gives the desired result. $\square$

In order to put Lemma IV.2.4 to good use, we'll next observe that the idea of using a "test element" for strong $F$-regularity also works in the setting of Cartier algebras. This is effectively the same argument as in the setting of [HH89, Thm. 3.3] (which we saw in the background as Theorem II.1.10), but we include the proof here for completeness.

**Fact IV.2.6.** *Let $R$ be a Noetherian $F$-finite ring, and $\mathcal{D}$ a Cartier algebra on $R$. Let $g \in R$ be a non-zero divisor. Then $(R, \mathcal{D})$ is strongly $F$-regular if and only if both $(R[g^{-1}], \mathcal{D}[g^{-1}])$ is strongly $F$-regular and $g$ eventually splits with respect to $\mathcal{D}$ (i.e., there exists some $e$ and some $\varphi \in \mathcal{D}_e$ with $\varphi(F^e_* g) = 1$).*

*Proof.* Take any non-zerodivisor $c \in R$, and consider the "evaluation at $c$" map $\mathcal{D}_f \to R$ which has $\phi \mapsto \phi(F^f_* c)$. Since $(R[g^{-1}], \mathcal{D}[g^{-1}])$ is strongly $F$-regular, when we tensor with $R[g^{-1}]$ there is some large enough $f$ such that this map is surjective in the localization, i.e., there is some $\psi \in \mathcal{D}_f$ and some $m$ such that $\psi(F^f_* c) = g^m$. Because $g$ is eventually split with respect to $\mathcal{D}$, this means $(R, \mathcal{D})$ is in particular $F$-split, and so there exists $\pi \in \mathcal{D}_\ell$ with $\pi(F^\ell_* 1) = 1$. We can always replace $m$ by a larger $m$, and replace $\ell$ by a multiple of $\ell$, so we reduce to the case where

$$\psi \in \mathcal{D}_f, \ \psi(F^f_* c) = g^{p^\ell} \quad \text{and} \quad \pi \in \mathcal{D}_\ell, \ \pi(F^\ell_* 1) = 1.$$

Now

$$\pi \star \psi(F^{f+\ell}_* c) = \pi(F^\ell_* g^{p^\ell}) = g\pi(F^\ell_* 1) = g.$$

Let $\varphi \in \mathcal{D}_e$ be our given splitting of $g$. Now finally $\varphi \star \pi \star \psi$ is our desired splitting of $c$. $\square$

In particular, it will be useful for us to have a ready-made collection of elements with which to test strong $F$-regularity, which is what the following corollary provides.

**Corollary IV.2.7.** *Let $\mathcal{C}^{\mathfrak{a}_\bullet} = \mathcal{C}^S \star \mathfrak{a}_\bullet$, where $\mathfrak{a}_\bullet$ is an $F$-graded system in a strongly $F$-regular ring $S$. Let $g$ be a non-zero element of $S$ such that $g \in \sqrt{\mathfrak{a}_e}$ for all $e \gg 0$. Then $\mathcal{C}^{\mathfrak{a}_\bullet}$ is* *strongly $F$-regular if and only if the element $g$ eventually splits with respect to $(S, \mathcal{C}^{\mathfrak{a}_\bullet})$. In particular, for any non-zero $g \in \mathfrak{a}_1$, we have that $\mathcal{C}^{\mathfrak{a}_\bullet}$ is strongly $F$-regular if and only if $g$ is eventually $F$-split.*

*Proof.* By the previous fact, it suffices to show that $(S[g^{-1}], \mathcal{C}^{\mathfrak{a}_\bullet}[g^{-1}])$ is strongly $F$-regular. Consider any element $c \in S$. Since $S$ is strongly $F$-regular, there exists some $e$ and $\varphi \in \mathcal{C}^S_e$ such that $\varphi(F^e_* c) = 1$. Now choose $f$ and $a$ such that $\psi \in \mathcal{C}^S_f$ is an $F$-splitting and $g^a \in \mathfrak{a}_{e+f}$, and let $n = \left\lceil \frac{a}{p^{e+f}} \right\rceil$. Then

$$g^n = g^n(\psi \star \varphi)(F^{e+f}_* c) = \psi \star \varphi \star g^{np^{e+f}}(F^{e+f}_* c).$$

By design, $g^{np^{e+f}} \in \mathfrak{a}_{e+f}$ so $\psi \star \varphi \star g^{np^{e+f}} \in \mathcal{C}^{\mathfrak{a}_\bullet}_{e+f}$. Once we localize, this shows $c/1$ is eventually $F$-split with respect to $(\mathcal{C}^{\mathfrak{a}_\bullet}[g^{-1}])_{e+f}$, and so by a standard result this means that in fact any element $c/g^t \in S[g^{-1}]$ is eventually $F$-split with respect to this Cartier algebra.

For the "in particular," note that if $g \in \mathfrak{a}_1$, then $g^{\sum_{i=0}^{e-1} p^i} \in \mathfrak{a}_e$, and thus $g \in \sqrt{\mathfrak{a}_e}$ for all $e$. □

Using any element from $\mathfrak{a}_1$ as our test element combined with our understanding of splittings from Lemma IV.2.4, we get a simplified criterion for checking $F$-splitting and strong $F$-regularity of an $F$-graded system.

**Corollary IV.2.8.** *Let $(S, \mathfrak{m})$ be a regular local ring. Let $\mathfrak{a}_\bullet$ be an $F$-graded system in $S$ with $\mathfrak{a}_1 \neq 0$.*

- *$\mathfrak{a}_\bullet$ is $F$-split if and only if there exists $e > 0$ such that $\mathfrak{a}_e \not\subseteq \mathfrak{m}^{[p^e]}$.*
- *$\mathfrak{a}_\bullet$ is strongly $F$-regular if and only if there exists $e > 0$ such that $\mathfrak{a}_1\mathfrak{a}_e \not\subseteq \mathfrak{m}^{[p^e]}$.*

Now we are ready to prove the main theorem of this section:

*Proof of Theorem IV.2.3.* If $\mathfrak{b}_\bullet$ is strongly $F$-regular, it is by definition also $F$-split. Conversely, suppose $\mathfrak{b}_\bullet$ is $F$-split. Then by Corollary IV.2.8, there exists some $e > 0$ and some $c \in \mathfrak{b}_e \setminus \mathfrak{m}^{[p^e]}$. Since $\mathfrak{b}_\bullet$ is a $p$-family, $c^{p^f} \in \mathfrak{b}_e^{[p^f]} \subseteq \mathfrak{b}_{e+f}$ for all $f \geq 0$. By Corollary IV.2.7, this $c$ is a test element. Further, the ideal $\mathfrak{m}^{[p^e]} : c$ is proper, and so by Krull's intersection theorem there exists some $f$ such that

$$c \notin (\mathfrak{m}^{[p^e]} : c)^{p^f} \supset (\mathfrak{m}^{[p^e]} : c)^{[p^f]} = \mathfrak{m}^{[p^{e+f}]} : c^{p^f} \supset \mathfrak{m}^{[p^{e+f}]} : \mathfrak{b}_{e+f}.$$

Thus by Lemma IV.2.4, $\mathcal{C}^{\mathfrak{a}_\bullet}$ is eventually $F$-split along $c$, which completes the proof. $\square$

## IV.3: $p$-Stabilization

In light of Theorem IV.2.3, it is natural to consider whether we can use $p$-families to get results for $F$-graded systems more generally. Our goal in this section is thus to present a useful construction which turns $F$-graded systems into $p$-families, in such a way that strong $F$-regularity is preserved.

**Definition IV.3.1.** Let $\mathfrak{a}_\bullet$ be an $F$-graded system. The *$p$-stabilization* of $\mathfrak{a}_\bullet$ is $\widetilde{\mathfrak{a}}_\bullet$, where

$$\widetilde{\mathfrak{a}}_e := \left\{ r \;\middle|\; r^{p^f} \in \mathfrak{a}_{f+e} \text{ for all } f \gg 0 \right\}.$$

From this definition, we immediately get the following result:

**Fact IV.3.2.** *The $p$-stabilization of any $F$-graded system is a $p$-family.*

*Proof.* If $r \in \widetilde{\mathfrak{a}}_e$, then for all $f \gg 0$, we have $(r^p)^{p^f} = r^{p^{f+1}} \in \mathfrak{a}_{(e+1)+f}$. $\square$

Further, strong $F$-regularity is indeed preserved:

**Theorem IV.3.3.** *Let $(S, \mathfrak{m})$ be a regular local ring. Let $\mathfrak{a}_\bullet$ be an $F$-graded system in $S$ with $\mathfrak{a}_1 \neq 0$, and let $\widetilde{\mathfrak{a}}_\bullet$ be the $p$-stabilization of $\mathfrak{a}_\bullet$. Then $\mathfrak{a}_\bullet$ is strongly $F$-regular if and only if $\widetilde{\mathfrak{a}}_\bullet$ is strongly $F$-regular.*

Combining this theorem and Theorem IV.2.3 with the fact that $\widetilde{\mathfrak{a}}_\bullet$ is a $p$-family, the following corollary is immediate.

**Corollary IV.3.4.** *Let $(S, \mathfrak{m})$ be a regular local ring. Let $\mathfrak{a}_\bullet$ be an $F$-graded system in $S$ with $\mathfrak{a}_1 \neq 0$, and let $\widetilde{\mathfrak{a}}_\bullet$ be the $p$-stabilization of $\mathfrak{a}_\bullet$. Then $\mathfrak{a}_\bullet$ is strongly $F$-regular if and only if $\widetilde{\mathfrak{a}}_\bullet$ is $F$-split.*

Now we proceed towards proving Theorem IV.3.3, the main theorem of this section. It will help to first have some constraints on how far apart $\mathfrak{a}_e$ and $\widetilde{\mathfrak{a}}_e$ can get, which the following lemma addresses.

**Lemma IV.3.5.** *Let $\mathfrak{a}_\bullet$ be an $F$-graded system. Then for every $e$, $\mathfrak{a}_1 \cdot \mathfrak{a}_e \subseteq \widetilde{\mathfrak{a}}_e$.*

*Proof.* For any $f \in \mathbb{N}$, we have

$$\mathfrak{a}_{e+f} \supset \mathfrak{a}_e^{[p^f]} \mathfrak{a}_f \supset \mathfrak{a}_e^{[p^f]} \cdot \prod_{i=0}^{f-1} \mathfrak{a}_1^{[p^i]}.$$

In particular, this means that for $r \in \mathfrak{a}_1$, $s \in \mathfrak{a}_e$, we have

$$\mathfrak{a}_{e+f} \ni s^{p^f} r^{\sum_{i=0}^{f-1} p^i} = s^{p^f} r^{\frac{p^f-1}{p-1}}.$$

Thus $(sr)^{p^f} \in \mathfrak{a}_{e+f}$ for all $f \geq 1$, which means that $rs \in \widetilde{\mathfrak{a}}_e$ as desired. $\square$

In the case that $\mathfrak{a}_1 = 0$, the previous lemma is of course not very informative.

**Remark IV.3.6.** Ideally, we would have some kind of containment that showed "something related to $\widetilde{\mathfrak{a}}_e$" is contained in "something related to $\mathfrak{a}_e$". There are unfortunately some limitations here. Taking $\mathfrak{a}_e = \langle x^{p^e + c^e} \rangle$ and $\widetilde{\mathfrak{a}}_e = \langle x^{p^e+1} \rangle$ for integer $0 < c < p$ as in Example IV.3.9 shows that for any monomial ideal $I = \langle x^a \rangle$, we have that eventually, $I \cdot \widetilde{\mathfrak{a}}_e \not\subseteq \mathfrak{a}_e$. So there is no fixed multiplicative factor that will work in this direction.

We are now ready to prove the main result of this section.

*Proof of Theorem IV.3.3.* To show the contrapositive, assume that $\mathcal{D}^{\mathfrak{b}_\bullet}$ is not strongly $F$-regular. By Theorem IV.2.3, this means $\mathcal{D}^{\mathfrak{b}_\bullet}$ is also not $F$-split, and by Corollary IV.2.8 this tells us that for all $e$, $\mathfrak{b}_e \subseteq \mathfrak{m}^{[p^e]}$. But then Lemma IV.3.5 further tells us that

$$\mathfrak{a}_1 \mathfrak{a}_e \subseteq \mathfrak{b}_e \subseteq \mathfrak{m}^{[p^e]}$$

for all $e$, and so $\mathcal{D}^{\mathfrak{a}_\bullet}$ cannot be strongly $F$-regular.

For the other direction, now assume that $\mathcal{D}^{\mathfrak{b}_\bullet}$ is strongly $F$-regular. Thus there exists an $e$ such that $\mathfrak{b}_1 \mathfrak{b}_e \not\subseteq \mathfrak{m}^{[p^e]}$, i.e., there is $f \in \mathfrak{b}_1$, $g \in \mathfrak{b}_e$ with $fg \notin \mathfrak{m}^{[p^e]}$. On the one hand, this also means that $(fg)^{p^{e'}} \notin \mathfrak{m}^{[p^{e+e'}]}$ for all $e'$. On the other hand, by definition of $\mathfrak{b}$, for all $e' \gg 0$, $g^{p^{e'}} \in \mathfrak{a}_{e+e'}$ and $f^{p^{e'}} \in \mathfrak{a}_{1+e'}$. Let $e_0$ be an $e' \gg 0$ which satisfies both conditions, i.e., $g^{p^{e_0}} \in \mathfrak{a}_{e+e_0}$ and $f^{p^{e_0}} \in \mathfrak{a}_{1+e_0}$.

Then $(fg)^{p^{e_0}} \in \mathfrak{a}_{1+e_0}\mathfrak{a}_{e+e_0} \setminus \mathfrak{m}^{[p^{e+e_0}]}$, which in particular means $f^{p^{e_0}} \notin \mathfrak{m}^{[p^{e+e_0}]} : \mathfrak{a}_{e+e_0}$. Since $f \in \mathfrak{b}_1$, this means $f$, and also $f^{p^{e_0}}$, is in $\sqrt{\mathfrak{a}_{e'}}$ for all $e' \gg 0$, and thus $f^{p^{e_0}}$ is a test element in the sense of Corollary IV.2.7. □

Now that we have seen a useful application of $p$-stabilization, we will note some other properties of this construction.

**Proposition IV.3.7** (Basic properties of $p$-stabilization)**.** *Let $R$ be a Noetherian $F$-finite ring, and let $\mathfrak{a}_\bullet$ be an $F$-graded system of $R$.*

1. *$\widetilde{\widetilde{\mathfrak{a}}}_e = \widetilde{\mathfrak{a}}_e$ for all $e$, i.e., any $p$-stabilized system is itself* $p$-stable.

2. *If further $\mathfrak{b}_\bullet$ is an $F$-graded system with $\mathfrak{a}_e \subseteq \mathfrak{b}_e$ for all $e \gg 0$, then $\widetilde{\mathfrak{a}}_e \subseteq \widetilde{\mathfrak{b}}_e$ for all $e$.*

3. *If $\mathfrak{a}_\bullet$ is a $p$-family, then $\mathfrak{a}_e \subseteq \widetilde{\mathfrak{a}}_e$ for all $e$.*

*Thus $p$-stabilization behaves like a "closure" operation on $p$-families.*

*Proof.* We will prove these out of order, starting with the second and third:

2. Let $r \in \widetilde{\mathfrak{a}}_e$, so that $r^{p^f} \in \mathfrak{a}_{e+f}$ for all $f \gg 0$. But by taking sufficiently large $f$, we get $r^{p^f} \in \mathfrak{a}_{e+f} \subseteq \mathfrak{b}_{e+f}$, and thus $r \in \widetilde{\mathfrak{b}}_e$.

3. If $r \in \mathfrak{a}_e$, then since it is a $p$-family, of course $r^{p^f} \in \mathfrak{a}_{e+f}$ for all $f$, and thus $r \in \widetilde{\mathfrak{a}}_e$.

Now we return to the first property on the list:

1. By using the other two properties just proven and the fact that $\widetilde{\mathfrak{a}}_\bullet$ is a $p$-family, we automatically get $\widetilde{\mathfrak{a}}_e \subseteq \widetilde{\widetilde{\mathfrak{a}}}_e$ for all $e$. Conversely, take $r \in \widetilde{\widetilde{\mathfrak{a}}}_e$. Then for all $f \gg 0$ and $g \gg 0$, $r^{p^{f+g}} = (r^{p^f})^{p^g} \in \mathfrak{a}_{(e+f)+g}$ as desired.

□

As a caution to the reader, we note that the restriction to $p$-families in the third property of Proposition IV.3.7 is necessary: in Theorem IV.3.16, we will see an example of an $F$-graded system where in fact $\widetilde{\mathfrak{a}}_e \subseteq \mathfrak{a}_e$ for all $e$, namely the system where $\mathfrak{a}_e = I^{[p^e]} : I$ for some fixed ideal $I$, yields the smaller stabilization $\widetilde{\mathfrak{a}}_e = I^{[p^e]}$. We also note that the containment in the third property can indeed sometimes be strict:

**Example IV.3.8.** Fix a constant $c \in \mathbb{N}$ with $c \geq 1$, and let $\mathfrak{a}_e = \langle x^{p^e+c} \rangle \subset k[x]$. One can easily check that this is a $p$-family. However, $\widetilde{\mathfrak{a}}_e = \langle x^{p^e+1} \rangle$, since monomial $x^a \in \widetilde{\mathfrak{a}}_e$ means $(x^a)^{p^f} \in \langle x^{p^{e+f}+c} \rangle$ for all $f \gg 0$, i.e., $p^e + \frac{c}{p^f} \leq a$ for all $f \gg 0$. In particular, if $c \neq 1$, the $p$-stabilization is larger than the original family.

In fact, even exponential growth is not enough if the growth factor is less than $p$:

**Example IV.3.9.** Fix a constant $c \in \mathbb{N}$ and consider the system $\mathfrak{a}_e = \langle x^{p^e+c^e} \rangle$. This is a $p$-family if and only if $c^{e+1} \leq p \cdot c^e$ for all $e$. One can further check that a monomial $x^a \in \widetilde{\mathfrak{a}}_e$ if and only if $p^e + \frac{c^{e+f}}{p^f} \leq a$ for all $f \gg 0$. In particular, if $0 < c < p$, then $\widetilde{\mathfrak{a}}_e = \langle x^{p^e+1} \rangle$.

In light of these examples and the first property of $p$-stabilization (Proposition IV.3.7), it would be interesting to identify the $F$-graded systems that are $p$-*stable*, i.e., the systems $\mathfrak{a}_\bullet$ such that $\widetilde{\mathfrak{a}}_e = \mathfrak{a}_e$ for all $e$. To be $p$-stable, it is clearly necessary for the system to be a $p$-family to begin with. We now work towards giving a sufficient condition.

**Lemma IV.3.10.** *Let $S$ be a Noetherian $F$-finite ring, and let $\mathfrak{a}_\bullet$ be an $F$-graded system. Then for any fixed surjective degree $d$ map $\varphi \in \mathcal{C}_d^S$, we have*

$$\widetilde{\mathfrak{a}}_e \subseteq \bigcup_{m>0} \bigcap_{n \geq m} \varphi^{\star n}(F_*^{nd} \mathfrak{a}_{e+nd}).$$

*Proof.* Since $\varphi$ is surjective, in particular there is some $s \in S$ with $\varphi(F_*^d s) = 1$, so that further $\varphi^{\star n}(F_*^{nd} s^{1+p^d+\cdots+p^{(n-1)d}}) = 1$ for all $n$. Then consider any $r \in \widetilde{\mathfrak{a}}_e$, so that $r^{p^f} \in \mathfrak{a}_{e+f}$ for all $f \gg 0$. In particular, for $n \gg 0$ we have

$$r = r \cdot \varphi^{\star n}(F_*^{nd} s^{(p^{nd}-1)/(p^d-1)}) = \varphi^{\star n}(F_*^{nd}(r^{p^{nd}} s^{(p^{nd}-1)/(p^d-1)})) \in \varphi^{nd}(F_*^{nd} \mathfrak{a}_{e+nd}).$$

This gives the desired containment. $\square$

**Proposition IV.3.11.** *Let $S$ be a ring, and let $\mathfrak{b}_\bullet$ be a $p$-family in $S$. Suppose there exists some $d$ and some map $\varphi \in \mathcal{C}^S_d$ such that $\varphi$ is surjective and for all $e$, we have $\varphi(F^d_* \mathfrak{b}_{e+d}) \subseteq \mathfrak{b}_e$. Then $\mathfrak{b}_\bullet$ is $p$-*stable*, i.e., $\widetilde{\mathfrak{b}}_e = \mathfrak{b}_e$ for all $e$.*

*Proof.* Note that the condition iterates in the sense that

$$\varphi^{\star n}(F^{nd}_* \mathfrak{b}_{e+nd}) \subseteq \varphi^{\star(n-1)}(F^{(n-1)d}_* \mathfrak{b}_{e+(n-1)d}) \subseteq \cdots \subseteq \mathfrak{b}_e.$$

Thus

$$\widetilde{\mathfrak{b}}_e \subseteq \bigcup_{m>0} \bigcap_{n \geq m} \varphi^{\star n}(F^{nd}_* \mathfrak{a}_{e+nd}) \subseteq \bigcup_{m>0} \bigcap_{n \geq m} \mathfrak{b}_e = \mathfrak{b}_e.$$

Since $\mathfrak{b}_\bullet$ is already a $p$-family, $\mathfrak{b}_e \subseteq \widetilde{\mathfrak{b}}_e$ by the third property of $p$-stabilization (Proposition IV.3.7) and so $\mathfrak{b}_\bullet$ is stable, as desired. □

**Remark IV.3.12.** The requirement that $\mathfrak{b}_\bullet$ be a $p$-family in the previous proposition is indeed necessary. Consider the $F$-graded system $\mathfrak{b}_e = \langle x^{1+p+\cdots+p^{e-1}} \rangle$ in $k[x]$, and let $\varphi$ be the standard monomial splitting in $\operatorname{Hom}_{k[x]}(F_* k[x], k[x])$. Then

$$\varphi\left(F_* \mathfrak{b}_{e+1}\right) = \varphi\left(F_* \langle x^{1+p+\cdots+p^{e-1}} \rangle\right) = x^{1+\cdots+p^{e-2}} \varphi\left(F_* \langle x \rangle\right) = \mathfrak{b}_{e-1} \varphi(F_* \langle x \rangle) = x \mathfrak{b}_{e-1},$$

so we have the desired containment. But the system is not $p$-stable. We shall see in Theorem IV.3.15 that in fact $\widetilde{\mathfrak{b}}_e = x \mathfrak{b}_{e-1}$.

These results prompt the following conjecture:

**Conjecture IV.3.13.** *Let $S$ be a strongly $F$-regular ring, and let $\mathfrak{b}_\bullet$ be a $p$-family in $S$. Then $\mathfrak{b}_\bullet$ is* stable, *i.e., $\widetilde{\mathfrak{b}}_e = \mathfrak{b}_e$ for all $e$, if and only if there exists some $d$ and some map $\varphi \in \mathcal{C}^S_d$ such that $\varphi$ is surjective and for all $e$, we have $\varphi(F^d_* \mathfrak{b}_{e+d}) \subseteq \mathfrak{b}_e$.*

This conjecture is in fact true when $\mathfrak{b}_\bullet$ is a system of monomial ideals, as we shall see in Proposition IV.4.8.

**Remark IV.3.14.** A non-strongly $F$-regular ring, and even a non-$F$-split ring can certainly have stable $p$-families. For example, the $p$-family $\mathfrak{b}_e = \langle 1 \rangle$ for all $e$ is stable in any prime characteristic ring, so our conjecture certainly wouldn't hold in such settings. We can't just generalize the condition to being any map. For example, in $S = k[x]$, the family $\mathfrak{b}_e = \langle x^{p^e+2} \rangle$ is not stable, because

$$x^{p^2+2} \cdot x^{p-2} = (x^{p+1})^p \in \mathfrak{b}_2$$

but $x^{p+1} \notin \mathfrak{b}_1$. However, taking $\varphi = x^2 \star \Phi$, where $\Phi$ generates $\mathrm{Hom}_S(F_*S, S)$, we see that for any $rx^{p^{e+1}+2} \in \mathfrak{b}_{e+1}$,

$$\varphi(F_*(rx^{p^{e+1}+2})) = x^{p^e}\varphi(F_*(rx^2)) = x^{p^e+2}\Phi(r) \in \mathfrak{b}_e.$$

### IV.3.1: Application: Our Three Main Examples

Now that we've seen some benefits of $p$-stabilization, we will show how to compute the $p$-stabilization for (some special cases of) our main examples from Example IV.1.2. A key tool for computing the $p$-stabilization of a monomial idea will be the associated $p$-body, introduced in Section IV.4. In fact, several of the proofs here will be deferred to Section IV.4.1, once we have developed the necessary machinery.

**Theorem IV.3.15.** *Let $I$ be a monomial ideal in $k[x_1, \ldots, x_d]$ with minimal monomial generating set $\{x^\nu \mid \nu \in \mathcal{V}\}$, and let $\mathfrak{a}_e = \prod_{i=0}^{e-1} I^{[p^i]}$. Define*

$$J = \left\langle x^{\lceil \sum_\nu c_\nu \nu \rceil} \;\middle|\; c_\nu \in \mathbb{R}_{\geq 0},\ \sum_{\nu \in \mathcal{V}} c_\nu = \frac{1}{p-1} \right\rangle.$$

*Then the $p$-stabilization is*

$$\widetilde{\mathfrak{a}}_e = J \cdot \mathfrak{a}_e = J \cdot \prod_{i=0}^{e-1} I^{[p^i]}.$$

This theorem will be proven on page 60.

**Theorem IV.3.16.** *Let $S$ be a regular local ring, fix a non-zero ideal $I$ in $S$, and define $\mathfrak{a}_e = I^{[p^e]} : I$. Then $\widetilde{\mathfrak{a}}_e = I^{[p^e]}$.*

*Proof.* We see first that

$$(I^{[p^e]})^{[p^f]} = I^{[p^{e+f}]} \subseteq I^{[p^{e+f}]} : I,$$

and so $I^{[p^e]} \subseteq \widetilde{\mathfrak{a}}_e$. On the other hand, if $r^{p^f} \in \mathfrak{a}_{e+f} = I^{[p^{e+f}]} : I$, then

$$I \subseteq I^{[p^{e+f}]} : r^{p^f} = (I^{[p^e]} : r)^{[p^f]}.$$

But this is supposed to hold for all sufficiently large $f$. If $I^{[p^e]} : r$ is a proper ideal, then Krull's intersection theorem would tell us that $I = 0$. Otherwise, this means that $r \in I^{[p^e]}$ as desired. □

**Theorem IV.3.17.** *Let $S = k[x_1, \ldots, x_d]$, let $\mathfrak{m}$ be the homogeneous maximal ideal, and fix $t \in \mathbb{R}_{\geq 0}$. Let $\mathfrak{a}_e = \mathfrak{m}^{\lceil t(p^e-1) \rceil}$. Then the p-stabilization is*

$$\widetilde{\mathfrak{a}}_e = \mathfrak{m}^{\lceil tp^e \rceil}.$$

This theorem will be proven on page 63.

### IV.3.2: Other Possible Operations on $F$-Graded Systems

There are other operations one could conceivably define on an $F$-graded system. In Section IV.1, we already saw that taking termwise integral closures (Proposition IV.1.8) and, in a polynomial ring, taking termwise initial ideals (Proposition IV.1.7) are both ways one can get a new $F$-graded system from an old one. In this section, we will suggest some possible operations that act more holistically on an $F$-graded system, as well as pointing out the limits of our current understanding of them.

First, we consider a definition that appears very similar to that of $p$-stabilization.

**Definition IV.3.18.** The *sporadic p-stabilization* is

$$\widetilde{\mathfrak{a}}_e^{\infty} := \left\{ r \;\middle|\; r^{p^f} \in \mathfrak{a}_{f+e} \text{ for infinitely many } f \right\}.$$

By effectively the same proof as Fact IV.3.2, we get

**Fact IV.3.19.** *The sporadic p-stabilization of any $F$-graded system is a p-family.*

However, these are legitimately different constructions, as the following examples show:

**Example IV.3.20.** Define an $F$-graded system in $\mathbb{F}_p[x]$ as follows:

$$\mathfrak{a}_e := \begin{cases} \langle x^{p^e+1} \rangle & e \text{ is odd} \\ \langle x^{p^e} \rangle & e \text{ is even} \end{cases}$$

One can straightforwardly check that this is an $F$-graded system, by considering the possible parities of $e$ and $f$ in $\mathfrak{a}_e^{[p^f]}\mathfrak{a}_f$. However, one can also check, e.g., using the idea of Example IV.4.7, that

$$\widetilde{\mathfrak{a}}_e = \langle x^{p^e+1} \rangle \quad \text{and} \quad \widetilde{\mathfrak{a}}_e^{\infty} = \langle x^{p^e} \rangle.$$

**Example IV.3.21.** Take any $F$-graded system $\mathfrak{a}_\bullet$, and define a new $F$-graded system $\mathfrak{b}_\bullet$ via

$$\mathfrak{b}_e = \begin{cases} 0 & e \text{ is odd} \\ \mathfrak{a}_e & e \text{ is even.} \end{cases}$$

Now $\widetilde{\mathfrak{b}}_e = 0$ for all $e$, but $\widetilde{\mathfrak{b}}_e^{\infty} \supset \widetilde{\mathfrak{a}}_e$ which in particular is often not zero.

There are other differences in these constructions as well. For example, if we replace $\widetilde{\mathfrak{a}}_\bullet$ by $\widetilde{\mathfrak{a}}_\bullet^{\infty}$, then Lemma IV.3.5 holds in a more general way—for any $g$ and every $e$, $\mathfrak{a}_g \cdot \mathfrak{a}_e \subseteq \widetilde{\mathfrak{a}}_e^{\infty}$. However, we do not know whether there is a version of Theorem IV.3.3 that holds for the sporadic $p$-stabilization construction.

Our definition of $p$-stabilization evokes the idea of taking the *Frobenius closure* of an ideal. One might then want to consider a notion of stabilization built on *tight closure* instead. This leads to the following definition.

**Definition IV.3.22.** Let $\mathfrak{a}_\bullet$ be an $F$-graded system. The *tight stabilization* is $\widetilde{\mathfrak{a}}_\bullet^*$, where

$$\widetilde{\mathfrak{a}}_e^* := \left\{ r \;\middle|\; \exists c \in R^\circ \text{ s.t. } cr^{p^f} \in \mathfrak{a}_{f+e} \text{ for all } f \gg 0 \right\}.$$

However, this proposed definition has proven more challenging to work with, so we simply pose the following questions:

**Question IV.3.23.** *Is the tight stabilization of an $F$-graded system always $F$-graded? When it is, do variants of any of the basic properties of $p$-stabilization (Proposition IV.3.7) hold?*

Alternatively, in the vein of viewing the $p$-stabilization as the $p$-family that is asymptotically closest to the original system (and in particular, even when starting with a $p$-family, the construction "fills in" missing elements in the earlier ideals), one could try and do this "filling in" in a way more focused on being $F$-graded instead of being a $p$-family. More concretely, this leads to the following definition.

**Definition IV.3.24.** Let $\mathfrak{a}_\bullet$ be an $F$-graded system. The *Cartier stabilization* is $\widetilde{\mathfrak{a}}^C_\bullet$, where

$$\widetilde{\mathfrak{a}}^C_e := \left\{ r \mid r^{(p^{fe}-1)/(p^e-1)} \in \mathfrak{a}_{fe} \text{ for all } f \gg 0 \right\}.$$

This is intended to be an analog of $p$-stabilization that behaves more like a "closure" on $F$-graded systems. Specifically:

**Proposition IV.3.25** (Basic properties of Cartier stabilization)**.** *Let $\mathfrak{a}_\bullet$ be an $F$-graded system such that $\widetilde{\mathfrak{a}}^C_\bullet$ is also $F$-graded.*

1. *If $\mathfrak{b}_\bullet$ is an $F$-graded system with $\mathfrak{a}_e \subseteq \mathfrak{b}_e$ for all $e \gg 0$, then $\widetilde{\mathfrak{a}}^C_e \subseteq \widetilde{\mathfrak{b}}^C_e$ for all $e$.*

2. *$\mathfrak{a}_e \subseteq \widetilde{\mathfrak{a}}^C_e$ for all $e$.*

*Proof.* 1. Let $r \in \widetilde{\mathfrak{a}}_e$, so that $r^{(p^{ef}-1)/(p^e-1)} \in \mathfrak{a}_{ef}$ for all $f \gg 0$. But by taking sufficiently large $f$, we get $r^{(p^{ef}-1)/(p^e-1)} \in \mathfrak{a}_{ef} \subseteq \mathfrak{b}_{ef}$, and thus $r \in \widetilde{\mathfrak{b}}^C_e$.

2. If $r \in \mathfrak{a}_e$, then

$$r^{(p^{fe}-1)/(p^e-1)} = r \cdot r^{p^e} \cdots r^{p^{e(f-1)}} \in \prod_{i=0}^{e} \mathfrak{a}_e^{[p^i]} \subseteq \mathfrak{a}_{ef}.$$

This immediately ensures that $\mathfrak{a}_e \subseteq \widetilde{\mathfrak{a}}^C_e$ for all $e$. □

However, as we would ultimately hope for a more complete analog of $p$-stabilization, we again must pose the following questions:

**Question IV.3.26.** *Is the Cartier stabilization of an $F$-graded system always $F$-graded? When the resulting systems are $F$-graded, is this operation actually a "stabilization," i.e., is it true that $\widetilde{\mathfrak{a}}^C_e = \widetilde{\widetilde{\mathfrak{a}}_e^C}^C$ for all $e$?*

In the case that $\widetilde{\mathfrak{a}}_\bullet^C$ is $F$-graded, we know that the Cartier stabilization does indeed preserve properties of $F$-singularities, and in an even stronger way than $p$-stabilization does:

**Proposition IV.3.27.** *Let $(S, \mathfrak{m})$ be a local ring, and suppose $\mathfrak{a}_\bullet$ is an $F$-graded system such that $\widetilde{\mathfrak{a}}_\bullet^C$ is also an $F$-graded system. Then:*

- *$\mathfrak{a}_\bullet$ is $F$-split if and only if $\widetilde{\mathfrak{a}}_\bullet^C$ is $F$-split.*
- *$\mathfrak{a}_\bullet$ is strongly $F$-regular if and only if $\widetilde{\mathfrak{a}}_\bullet^C$ is strongly $F$-regular.*

*Proof.* Because of the containment between the systems, both of the implications in the forward direction are clear. We now proceed to the other direction.

Let $c \in S$ be a non-zerodivisor, and suppose $\widetilde{\mathfrak{a}}_\bullet^C$ is eventually $F$-split along $c$. Then there is some $\psi \in \mathcal{C}_e^S$ and $r \in \widetilde{\mathfrak{a}}_e^C$ such that $(\psi \star r)(F_*^e c) = 1$. But then for all $g \gg 0$, we have $r^{(p^{eg}-1)/(p^e-1)} \in \mathfrak{a}_{eg}$, and in particular,

$$\psi^{\star g} \star (cr)^{(p^{eg}-1)/(p^e-1)}(F_*^{eg} 1) = \psi^{\star g} \star r^{(p^{eg}-1)/(p^e-1)} \star c^{(p^{eg}-1)/(p^e-1)-1}(F_*^{eg} c) = 1.$$

Taking $c = 1$ gives the first statement. □

## IV.4: The Associated $p$-Body

We now pivot to *monomial $F$-graded systems*, i.e., $F$-graded systems in a polynomial ring for which every ideal is a monomial ideal. In this setting, we will develop a new technique for computing the $p$-stabilization introduced in Section IV.3, which comes by way of a geometric construction (the associated $p$-body) that is interesting in its own right. In this section, we assume the ambient ring is a polynomial ring over an $F$-finite field.

**Notation IV.4.1.** If $I$ is a monomial ideal, then $\log I = \{\alpha \in \mathbb{N}^d | : | x^\alpha \in I\}$ is the set of exponent vectors. For $n \in \mathbb{N}$, we write $[n] := \{1, 2, \ldots, n\}$. Whenever possible, we will use lowercase greek letters to denote vectors (be they in $\mathbb{N}^d$, $(\mathbb{Z}[1/p])^d$, or $\mathbb{R}^d$). It will often be useful to consider the termwise partial order $\leq$ so that given two vectors $\alpha, \beta$, we will write

$\alpha \leq \beta$ if $\alpha_i \leq \beta_i$ for all coordinates $i$. Finally, we use $\mathbb{1}$ to denote the vector which has every coordinate equal to 1.

Given a monomial ideal in $k[x_1, \ldots, x_d]$, it is often the case that looking at corresponding diagrams of exponent vectors in $\mathbb{N}^d$ can shed light on the algebraic picture. A version of this correspondence is also illuminating for $F$-graded systems, which we now describe.

**Definition IV.4.2.** Let $\mathfrak{a}_\bullet$ be a monomial $F$-graded system in $k[x_1, \ldots, x_d]$. Then the *associated $p$-body* in $\left(\mathbb{Z}\left[\frac{1}{p}\right]\right)^d$ is

$$\Delta(\mathfrak{a}_\bullet) := \bigcup_{f>0} \bigcap_{e \geq f} \frac{1}{p^e} \log \mathfrak{a}_e = \{\alpha \mid p^e \alpha \in \log \mathfrak{a}_e \ \forall e \gg 0\}\,.$$

**Proposition IV.4.3.** *Let $\mathfrak{a}_\bullet$ be a monomial $F$-graded system, and let $\Delta = \Delta(\mathfrak{a}_\bullet)$. Then*

$$\Delta = \Delta + \left(\mathbb{N}\left[\frac{1}{p}\right]\right)^d.$$

*Proof.* Since $\underline{0} \in \left(\mathbb{N}\left[\frac{1}{p}\right]\right)^d$, the containment $\subset$ is clear.

Take $\alpha \in \Delta$ and let $\beta \in \left(\mathbb{N}\left[\frac{1}{p}\right]\right)^d$. We will show that $\alpha + \beta \in \Delta$. Choose $G \in \mathbb{N}$ such that $p^G \beta \in \mathbb{N}^d$. By definition of $\Delta$, there exists $g \geq G$ such that for all $e \geq g$, we have $x^{p^e \alpha} \in \mathfrak{a}_e$. Since our choice of $G$ ensures $x^{p^e \beta} \in S$, this means that $x^{p^e \alpha} x^{p^e \beta} \in \mathfrak{a}_e$, so that $\alpha + \beta \in \frac{1}{p^e} \log \mathfrak{a}_e$ for all $e \gg 0$, as desired. $\square$

**Remark IV.4.4.** As we will be primarily interested in using $p$-bodies in the context of the $p$-body/$p$-stabilization correspondence (see Theorem IV.4.6 below), $\Delta(\mathfrak{a}_\bullet)$ is functionally equivalent to the subset of $\mathbb{R}^d$ obtained by taking $\Delta(\mathfrak{a}_\bullet) + \mathbb{R}^d_{\geq 0}$, since the $(\mathbb{Z}[1/p])^d$ points are the same.

This perspective also explains the source of the name. Working in the setting of a local ring accompanied by a sufficiently nice valuation, Hernandez and Jeffries introduce a subset of $\mathbb{R}^d$ called the *associated $p$-body* to a collection of subsets in a semigroup called a *$p$-system* [HJ18, Def. 4.4]. In Section 5 of their paper they then apply this to $p$-families to get the associated $p$-body living in $\mathbb{R}^d$.

**Definition IV.4.5.** Let $\Delta$ be any subset of $\left(\mathbb{N}[\frac{1}{p}]\right)^d$. Then we can define an *associated p-family* $\mathfrak{a}_\bullet^\Delta$, where

$$\mathfrak{a}_e^\Delta := \left\langle x^\beta \;\middle|\; \frac{1}{p^e}\beta \in \Delta \right\rangle.$$

This is a $p$-family because for any $x^\beta \in \mathfrak{a}_e^\Delta$, we have $\frac{p\beta}{p^{e+1}} = \frac{1}{p^e}\beta \in \Delta$ and so $x^{p\beta} \in \mathfrak{a}_{e+1}^\Delta$.

**Theorem IV.4.6** ($p$-body/$p$-stabilization correspondence)**.** *If $\mathfrak{b}_\bullet$ is $F$-graded, then the associated $p$-family of the associated $p$-body of $\mathfrak{b}_\bullet$ is the $p$-stabilization, i.e., $\mathfrak{a}_\bullet^{\Delta(\mathfrak{b}_\bullet)} = \widetilde{\mathfrak{b}}_\bullet$. If $\Delta \subseteq (\mathbb{N}[1/p])^d$, then $\Delta(\mathfrak{a}_\bullet^\Delta) = \Delta + (\mathbb{N}[1/p])^d$.*

*In particular, this gives a correspondence between $p$-stable $F$-graded systems and subsets of $(\mathbb{N}[1/p])^d$ which are invariant under adding $(\mathbb{N}[1/p])^d$.*

*Proof.* A straightforward computation shows

$$\begin{aligned}\mathfrak{a}_e^{\Delta(\mathfrak{b}_\bullet)} &= \left\langle x^\beta \;\middle|\; \frac{1}{p^e}\beta \in \Delta(\mathfrak{b}_\bullet)\right\rangle = \left\langle x^\beta \;\middle|\; \frac{p^f}{p^e}\beta \in \log(\mathfrak{b}_f)\ \forall f \gg 0\right\rangle \\ &= \left\langle x^\beta \;\middle|\; x^{p^{f-e}\beta} \in \mathfrak{b}_f\ \forall f \gg 0\right\rangle = \widetilde{\mathfrak{b}}_e.\end{aligned}$$

For the statement about $\Delta$, since $\mathfrak{a}_\bullet^\Delta$ is a $p$-family, we have $\frac{1}{p^e}\log \mathfrak{a}_e^\Delta \subseteq \frac{1}{p^{e+1}}\log \mathfrak{a}_{e+1}^\Delta$. Thus

$$\Delta(\mathfrak{a}_\bullet^\Delta) = \bigcup_{f>0}\bigcap_{e\geq f}\frac{1}{p^e}\log(\mathfrak{a}_e^\Delta) = \bigcup_{f>0}\frac{1}{p^f}\log(\mathfrak{a}_f^\Delta).$$

The generating monomials of $\mathfrak{a}_f^\Delta$ come from the lattice points in $\Delta \cap \left(\frac{1}{p^f}\mathbb{N}^d\right)$. Since $\mathfrak{a}_f^\Delta$ is an ideal, the monomials correspond to the lattice points in $\left(\Delta \cap \left(\frac{1}{p^f}\mathbb{N}^d\right)\right) + \frac{1}{p^f}\mathbb{N}^d$, i.e., we have

$$\frac{1}{p^f}\log(\mathfrak{a}_f^\Delta) = \left(\Delta \cap \left(\frac{1}{p^f}\mathbb{N}^d\right)\right) + \frac{1}{p^f}\mathbb{N}^d,$$

and so taking the union over all $f > 0$ gives our desired result. □

Now we will see our first example of using an associated $p$-body to find the $p$-stabilization:

**Example IV.4.7.** If $\Delta \subseteq \mathbb{N}[1/p]$ is invariant under adding $\mathbb{N}[1/p]$, then there exists some $a \in \mathbb{R}$ such that either $\Delta = [a, \infty) \subset \mathbb{N}[1/p]$ or $\Delta = (a, \infty) \subset \mathbb{N}[1/p]$. In particular, any $p$-stable system $\mathfrak{b}_\bullet$ looks like either $\mathfrak{b}_e = \langle x^{\lceil ap^e \rceil}\rangle$ or $\langle x^{\lfloor ap^e \rfloor + 1}\rangle$ for $e > 0$.

This example also prompts the return of the question: which (monomial) $F$-graded systems are $p$-stable? The $p$-body/$p$-stabilization correspondence (Theorem IV.4.6) gives a geometric answer to this question. But there is also a description in terms of a distinguished map, in support of Conjecture IV.3.13:

**Proposition IV.4.8.** *Let $\mathfrak{b}_\bullet$ be a monomial $p$-family. Then $\mathfrak{a}_\bullet^{\Delta(\mathfrak{b}_\bullet)} = \mathfrak{b}_\bullet$ if and only if for all $e$, $\varphi(F_*\mathfrak{b}_{e+1}) \subseteq \mathfrak{b}_e$, where $\varphi \in \operatorname{Hom}_S(F_*^S, S)$ is the standard monomial splitting which sends $F_*1 \mapsto 1$ and the other standard monomial generators of $F_*S$ to $0$.*

*Proof.* If $\varphi(F_*\mathfrak{b}_{e+1}) \subseteq \mathfrak{b}_e$ for all $e$, then Proposition IV.3.11 ensures that $\widetilde{\mathfrak{b}_\bullet} = \mathfrak{b}_\bullet$.

Conversely, suppose that $\mathfrak{a}_\bullet^{\Delta(\mathfrak{b}_\bullet)} = \mathfrak{b}_\bullet$. To show $\varphi(F_*\mathfrak{b}_{e+1}) \subseteq \mathfrak{b}_e$, it suffices to consider what happens on monomials. So, consider $x^\alpha \in \mathfrak{b}_{e+1}$. By assumption, $\alpha/p^{e+1} \in \Delta$, which means for all $f \gg 0$ we have $\frac{\alpha}{p^{e+1}} \in \frac{1}{p^{e+1+f}} \log \mathfrak{b}_{e+1+f}$. We also know that for a general monomial, the standard monomial splitting sends

$$\varphi(F_*x^\alpha) = \begin{cases} x^{(n-1)\mathbb{1}} & \alpha = np\mathbb{1} \text{ for some positive integer } n \\ 0 & \text{else.} \end{cases}$$

In the latter case the result is clearly in $\mathfrak{b}_e$, so suppose $\alpha = np\mathbb{1}$. But then $\frac{1}{p^e}\alpha = \frac{n}{p^e}\mathbb{1} \in \Delta$, and so $x^{n\mathbb{1}} \in \mathfrak{a}_e^{\Delta(\mathfrak{b}_\bullet)} = \mathfrak{b}_e$ as desired. $\square$

### IV.4.1: Application: Our Three Main Examples

Again, now that we have seen some benefits of the associated shape, we will show how to compute it for (some special cases of) our main examples. This will also allow us to describe the $p$-stabilizations of the corresponding ideals that were promised in Section IV.3.1.

**Theorem IV.4.9.** *Let $I$ be a monomial ideal in $k[x_1, \ldots, x_d]$. Suppose that $I$ has minimal monomial generating set $\{x^v \mid v \in \mathcal{V}\}$, and let $\mathfrak{a}_e = \prod_{i=0}^{e-1} I^{[p^i]}$. Consider the set of vectors*

$$\mathcal{W} = \left\{ \sum_{i=1}^{\infty} v(i)\frac{1}{p^i} \,\middle|\, v : \mathbb{N}_{>0} \to \mathcal{V} \right\} \subset \mathbb{R}^d.$$

*Then*

$$\Delta(\mathfrak{a}_\bullet) = \left\{ u \in (\mathbb{Z}[1/p])^d \;\middle|\; \exists w \in \mathcal{W} \textit{ s.t. } u \geq w \right\}.$$

*In other words,* $\Delta(\mathfrak{a}_\bullet)$ *is the* $(\mathbb{N}[1/p])^d$*-invariant subset generated by* $\mathcal{W}$*.*

An example of the process of computing this associated $p$-body is shown in Figure IV.1 for starting ideal $I = \langle x^3, y^6 \rangle$ in characteristic 3.

*Proof.* Note that each sum in $\mathcal{W}$ actually converges, since there are only finitely many vectors in $\mathcal{V}$. First, suppose that $\mu \in \mathbb{Z}[1/p]^d$ and that there exists some $\omega \in \mathcal{W}$ with corresponding function $v : \mathbb{N}_{>0} \to \mathcal{V}$ such that $\mu \geq \omega$. Consider any $e \in \mathbb{N}$ such that $p^e\mu \in \mathbb{Z}^d$. By assumption,

$$p^e\mu \geq p^e\omega \geq p^e \left( \sum_{i=1}^{e} v(i)\frac{1}{p^i} \right) = \sum_{i=0}^{e-1} v(e-i)p^i.$$

In particular, this means that $x^{p^e\mu}$ is a multiple of $x^{v(e)}(x^{v(e-1)})^p \cdots (x^{v(1)})^{p^{e-1}} \in \prod_{i=0}^{e-1} I^{[p^i]}$.

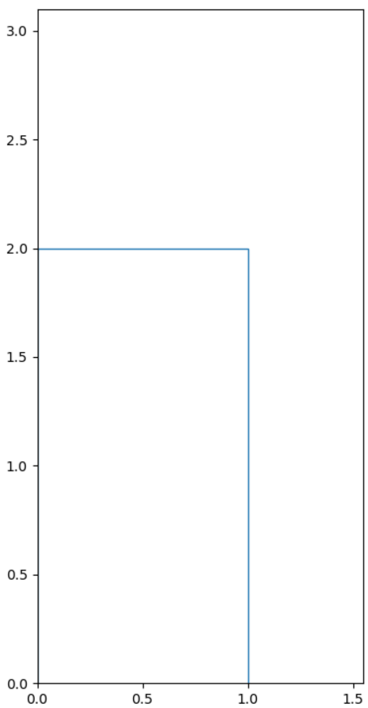
For the other direction, suppose $\mu \in \Delta$. This means for all $e \gg 0$, $\mu \in \frac{1}{p^e}\log \mathfrak{a}_e$. More specifically, for all such $e$, there exists $v^{(e)} : [e] \to \mathcal{V}$ such that $\sum_{i=1}^{e} v^{(e)}(i)\frac{1}{p^i} \leq \mu$. Since $\mathcal{V}$ is a finite set, we can iteratively define a function $\widetilde{v} : \mathbb{N}_{>0} \to \mathcal{V}$ via successively choosing vectors as follows:

$\widetilde{v}(1) =$ a vector appearing infinitely often in the set $\{v^{(e)}(1) \mid e \gg 0\}$.

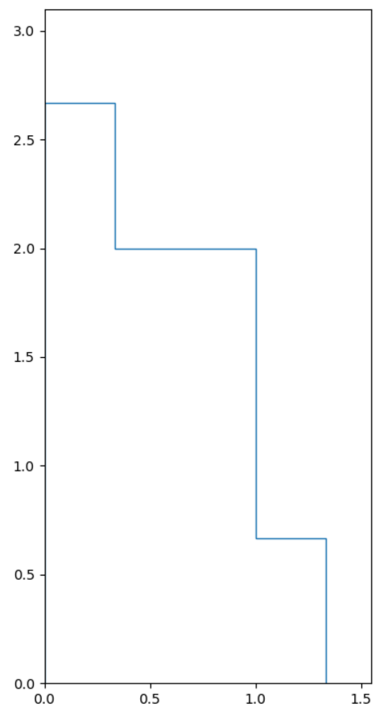
$\widetilde{v}(n) =$ a vector appearing infinitely often in the set $\{v^{(e)}(n) \mid e \gg 0,\ v^{(e)}(i) = \widetilde{v}(i)\ \forall i < n\}$.

By design, for every $e$, the restriction $\widetilde{v}|_{[e]}$ agrees with the restriction $v^{(f)}|_{[e]}$ for some $f > e$ (in fact, for infinitely many such $f$). This ensures that the $e$th partial sum of $\sum_{i=1}^{\infty} \widetilde{v}(i)\frac{1}{p^i}$ matches the $e$th partial sum of the vector corresponding to $v^{(f)}$, which in particular is a lower bound for $\mu$. $\square$

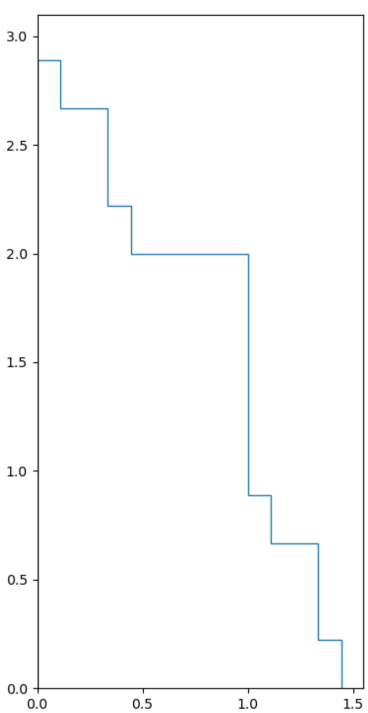
Now that we have a formula for this associated $p$-body, we can use it to give a description of the $p$-stabilization of the minimal generator system.



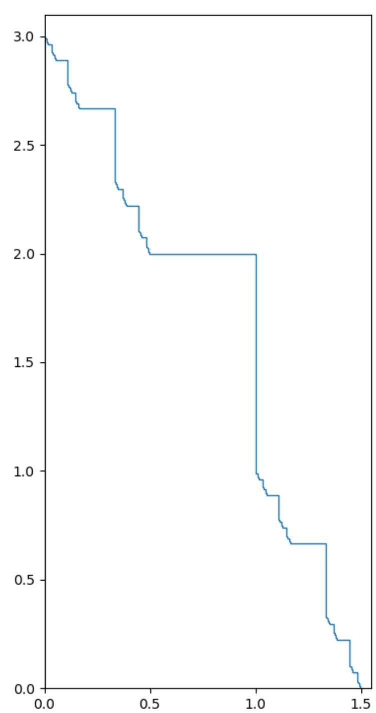

(a) $e = 1$




(b) $e = 2$




(c) $e = 3$




(d) $e = 8$

Figure IV.1: Plots of $\frac{1}{p^e} \log \mathfrak{a}_e$ when $I = \langle x^3, y^6 \rangle$, $\mathfrak{a}_e = \prod_{i=0}^{e-1} I^{[p^i]}$, and $p = 3$. The relevant lattice points of $\frac{1}{p^e}\mathbb{N}$ lie above and to the left of the blue line in each subfigure.

*Proof of Theorem IV.3.15.* As in the setup of Theorem IV.4.9, let

$$\mathcal{W} = \left\{ \sum_{i=1}^{\infty} v(i) \frac{1}{p^i} \,\middle|\, v : \mathbb{N}_{>0} \to \mathcal{V} \right\} \subset \mathbb{R}^d.$$

Then by Theorem IV.4.9,

$$\Delta(\mathfrak{a}_\bullet) = \left\{ u \in (\mathbb{Z}[1/p])^d \mid \exists w \in \mathcal{W} \text{ s.t. } u \geq w \right\}.$$

Since $\mathfrak{a}_e^{\Delta(\mathfrak{a}_\bullet)} = \widetilde{\mathfrak{a}}_e$ by Theorem IV.4.6, it suffices to understand the vectors $\beta \in \mathbb{Z}^d$ with $\frac{1}{p^e}\beta \in \Delta(\mathfrak{a}_\bullet)$, i.e., to understand the vectors $\lceil p^e \alpha \rceil$ for $\alpha \in \Delta$. In particular, it suffices to understand our generators from $\mathcal{W}$, so take

$$\left\lceil \sum_{i=1}^{\infty} v(i) \frac{p^e}{p^i} \right\rceil = \sum_{i=1}^{e} v(i) p^{e-i} + \left\lceil \sum_{i=1}^{\infty} v(e+i) \frac{1}{p^i} \right\rceil.$$

The term outside of the ceiling corresponds to monomials in $\prod_{i=0}^{e-1} I^{[p^i]}$, so we only need to understand the ceiling term, which is of the form $\lceil \omega \rceil$ for $\omega \in \mathcal{W}$. Since there are only finitely many vectors in $\mathcal{V}$, we can factor $\omega = \sum_\nu c_\nu \nu$, where each $\sum_\nu c_\nu = \sum_{i=1}^{\infty} \frac{1}{p^i} = \frac{1}{p-1}$ as desired. $\square$

For the next $F$-graded system, we will take another approach and use our already-computed $p$-stabilization to find the associated $p$-body.

**Theorem IV.4.10.** *For any fixed monomial ideal $I$, if $\mathfrak{a}_e = I^{[p^e]} : I$, then*

$$\Delta(\mathfrak{a}_\bullet) = \log I.$$

*Proof.* We know from Theorem IV.3.16 that $\widetilde{\mathfrak{a}}_e = I^{[p^e]}$, and so by Theorem IV.4.6, it suffices to show that $\Delta(I^{[p^\bullet]}) = \log I$.

But then $\log I = \frac{1}{p^e} \log I^{[p^e]}$, and so

$$\bigcap_{f>0} \bigcup_{e \geq f} \frac{1}{p^e} \log I^{[p^e]} = \log I$$

as desired. $\square$

Finally, we will compute the associated $p$-body of the rounding system specifically when starting with the homogeneous maximal ideal.

**Theorem IV.4.11.** *Let* $\mathfrak{a}_e = \mathfrak{m}^{\lceil t(p^e-1)\rceil}$ *for* $t \in \mathbb{R}_{\geq 0}$. *Then*

$$\Delta(\mathfrak{a}_\bullet) = \left\{ \alpha \in (\mathbb{N}[\frac{1}{p}])^d \;\middle|\; |\alpha| \geq t \right\}.$$

*Proof.* Suppose that $\alpha \in (\mathbb{N}[1/p])^d$ and $|\alpha| \geq t$. Choose $E$ such that $p^E\alpha \in \mathbb{N}^d$. Now for all $e \geq E$, we have $p^e|\alpha| \geq p^e t$ and $p^e\alpha \in \mathbb{N}^d$, so that

$$|p^e\alpha| \geq \lceil p^e t \rceil \geq \lceil (p^e-1)t \rceil,$$

and in particular, $p^e\alpha \in \log \mathfrak{m}^{\lceil (p^e-1)t\rceil}$, so that $\alpha \in \Delta$.

Conversely, suppose that $\alpha \in (\mathbb{N}[1/p])^d$ but $|\alpha| < t$. Now choose $F$ such that $|\alpha| < t - \frac{t}{p^F}$. Then for all $e \geq F$, we have

$$|\alpha| < t - \frac{t}{p^F} \leq t(1-\frac{1}{p^e}) \implies p^e|\alpha| < (p^e-1)t \leq \lceil (p^e-1)t \rceil,$$

and in particular $p^e\alpha \notin \log \mathfrak{m}^{(p^e-1)t\rceil}$ for all such $e$, so that $\alpha \notin \Delta$. $\square$

Again, this allows us to compute the $p$-stabilization of this system as well:

*Proof of Theorem IV.3.17.* We simply note that $\frac{1}{p^e}\alpha \in \Delta$ if and only if $\frac{1}{p^e}|\alpha| \geq t$ if and only if $\alpha \geq tp^e$. Since $|\alpha| \in \mathbb{N}^d$, this gives the desired ceiling statement. $\square$

## IV.5: Numerical Properties

In the last section of this chapter, we suggest some natural questions relating to an invariant of $F$-graded systems.

**Definition IV.5.1.** Let $R$ be a ring of prime characteristic $p$ with $\dim R = d$ which is local or standard graded with (homogeneous) maximal ideal $\mathfrak{m}$, and let $\mathfrak{a}_\bullet$ be an $F$-graded system of $R$ which is *eventually* $\mathfrak{m}$*-primary.* The *volume* of $\mathfrak{a}_\bullet$ is

$$\operatorname{vol}(\mathfrak{a}_\bullet) := \lim_{e\to\infty} \frac{\ell(R/\mathfrak{a}_e)}{p^{ed}}.$$

This limit is analogous to the Hilbert-Kunz multiplicity (first considered by Kunz in [Kun76], and studied in-depth by Monsky [Mon83]) and to the $F$-signature (introduced by this name by Huneke and Leuschke in [HL02], but not shown to exist until Tucker's work a decade later [Tuc12]). In fact this volume is an extension of the notion of the volume of a $p$-family, introduced in [HJ18], which already encompasses these two examples of the Hilbert-Kunz multiplicity and the $F$-signature. When working with $p$-families, Hernandéz and Jeffries have completely characterized the rings for which this volume is always guaranteed to exist:

**Theorem IV.5.2** ([HJ18, Thm. 1.2])**.** *Let $(R, \mathfrak{m})$ be a ring of prime characteristic $p > 0$ with $\dim R = d$. Then $\operatorname{vol}(\mathfrak{a}_\bullet)$ exists for every $p$-family $\mathfrak{a}_\bullet$ of $\mathfrak{m}$-primary ideals of $R$ if and only if the $R$-module dimension of the nilradical of the completion of $R$ is less than $d$.*

Even for the case of $F$-graded systems, leveraging pre-existing work on the Hilbert-Kunz multiplicity and $F$-signature can show cases when this volume exists:

**Proposition IV.5.3.** *Let $(R, \mathfrak{m}, k)$ be an $F$-finite local domain of dimension $d$, and let $\mathfrak{a}_\bullet$ be an $F$-graded sequence of ideals such that $\mathfrak{m}^{[p^e]} \subseteq \mathfrak{a}_e$ for all $e$ and such that $\mathfrak{a}_1 \neq 0$. Then* $\operatorname{vol}(\mathfrak{a}_\bullet)$ *exists.*

*Proof.* The key component of this proof is [PT18, Thm. 4.3], which shows that in fact for *any* sequence of ideals $I_\bullet$ and constant $0 \neq c \in R$ with $\mathfrak{m}^{[p^e]} \subseteq I_e$ and $cI_e^{[p]} \subseteq I_{e+1}$ for all $e \in \mathbb{N}$, the limit $\lim_{e\to\infty} \frac{\ell_R(R/I_e)}{p^e}$ exists. In our case, the additional requirement that $\mathfrak{a}_\bullet$ be $F$-graded and $\mathfrak{a}_1 \neq 0$ means we can simply take $c$ to be any non-zero element of $\mathfrak{a}_1$, since then

$$\mathfrak{a}_e^{[p]} c \subseteq \mathfrak{a}_e^{[p]} \mathfrak{a}_1 \subseteq \mathfrak{a}_{e+1}$$

as desired. $\square$

Beyond existence, of particular interest to us is how this invariant relates to the associated $p$-body, and more generally how it relates the $p$-stabilization.

**Conjecture IV.5.4.** *If $\mathfrak{a}_\bullet$ is an $F$-graded system in the polynomial ring $k[x_1, \ldots, x_d]$, then*

$$\mathrm{vol}(\mathfrak{a}_\bullet) = \mathrm{vol}_{\mathbb{R}^d}\left(\mathbb{R}^d_{\geq 0} \setminus \Delta(\mathfrak{a}_\bullet)\right).$$

Working in the setting of valuations into $\mathbb{Z}^d$ instead of exponent vectors on a monomial, Hernandéz and Jeffries prove a version of this conjecture for $p$-families. The conjecture is also supported by computational evidence in the case of the three main examples of $F$-graded systems from Example IV.1.2.

Finally, given the $p$-body/$p$-stabilization correspondence (Theorem IV.4.6), the above conjecture also prompts our closing question:

**Question IV.5.5.** *For an $F$-graded system $\mathfrak{a}_\bullet$, is $\mathrm{vol}(\mathfrak{a}_\bullet) = \mathrm{vol}(\widetilde{\mathfrak{a}}_\bullet)$?*

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