A p-adic Jacquet-Langlands Correspondence

by

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To my maternal grandparents

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ABSTRACT

The Jacquet-Langlands correspondence asserts that an automorphic representation π of $\mathrm{GL}_2(\mathbf{A})$ transfers to an automorphic representation of a quaternion algebra over $\mathbf{Q}$ if and only if the local component π_v is square-integrable at places v that are ramified in the quaternion algebra. It is known that the local representation π_∞ of $\mathrm{GL}_2(\mathbf{R})$ associated to a cusp form of weight one is not square-integrable. Thus, weight 1 forms do not transfer to a quaternion algebra ramified at infinity. Nevertheless, fixing a prime p split in the quaternion algebra that is ramified at ∞, we will discuss a p-adic formulation of the Jacquet-Langlands correspondence that includes cuspidal newforms of weight 1 that are supercuspidal at p.

CHAPTER I
Introduction

This thesis takes place at the intersection of two themes in Langlands Programme—Langlands' functoriality and its compatibility with p-adic variation of automorphic forms.

Let G be a connected reductive group over $\mathbf{Q}$. Recall that the (Weil form) L-group ${}^LG$ of a reductive group G over $\mathbf{Q}$ is ${}^\vee G(\mathbf{C}) \rtimes W_{\mathbf{Q}}$ where ${}^\vee G(\mathbf{C})$ is the dual reductive group of G and $W_{\mathbf{Q}}$ is the Weil group of $\mathbf{Q}$; and an L-homomorphism is a group homomorphism between the L-groups of two connected reductive groups G and H which is identity on the Weil group factor:

$$\begin{array}{ccc} {}^LH & \longrightarrow & {}^LG \\ & \searrow \quad \swarrow & \\ & W_{\mathbf{Q}} & \end{array} .$$

The general principle of functoriality is then that, an L-homomorphism between L-groups ${}^LH \to {}^LG$ of two connected reductive groups H and G over $\mathbf{Q}$ gives rise to a map

$$\left\{ \begin{matrix} L\text{-packets of automorphic} \\ \text{representations of } H \end{matrix} \right\} \to \left\{ \begin{matrix} L\text{-packets of automorphic} \\ \text{representations of } G \end{matrix} \right\}.$$

The simplest example of functoriality arises when G is quasi-split over $\mathbf{Q}$ and H is an inner form of G, viz., H is a connected reductive group such that $H_{/\overline{\mathbf{Q}}}$ is isomorphic to $G_{/\overline{\mathbf{Q}}}$ via an isomorphism

$$G_{/\overline{\mathbf{Q}}} \xrightarrow{\phi} H_{/\overline{\mathbf{Q}}}$$

defined over $\overline{\mathbf{Q}}$ that satisfies the following property: for every $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}|\mathbf{Q})$, the isomorphism $\phi^{-1} \circ \phi^{\sigma} = \phi^{-1} \circ \sigma \circ \phi \circ \sigma^{-1}$ is an inner automorphism of G. The L-groups of inner forms are *equal* and so the identity map is a morphism of L-groups.

In this thesis, G will be the group $\mathrm{GL}_{2/\mathbf{Q}}$, and H one of its inner forms. A standard fact is that the group H is then the group of units in a quaternion algebra over $\mathbf{Q}$. It is also well known that a quaternion algebra B over $\mathbf{Q}$ is characterized upto isomorphism by the set Σ of places v of $\mathbf{Q}$ such that the $\mathbf{Q}_v$-algebra $\mathbf{Q}_v \otimes_{\mathbf{Q}} B$ is not isomorphic to the matrix algebra $M_{2\times 2}(\mathbf{Q}_v)$; the set Σ is necessarily finite and of even cardinality and conversely given any finite subset Σ of places of $\mathbf{Q}$ of even cardinality, there is a quaternion algebra over $\mathbf{Q}$ (unique upto isomorphism) that is ramified at precisely places in Σ.

Fix a finite subset Σ of places of $\mathbf{Q}$ of even cardinality. Let B be the quaternion algebra over $\mathbf{Q}$ that is ramified precisely at places in Σ. The classical Jacquet-Langlands correspondence is an injective map from the infinite-dimensional admissible automorphic representations of the group $B^\times$ to that of GL_2, together with a description of the image.

Given an automorphic representation π of $\mathrm{GL}_2(\mathbf{A})$, there is an automorphic representation π' of $B^\times(\mathbf{A})$ such that, for all $v \notin \Sigma$, the representations π'_v and π_v are equivalent under an isomorphism $(B \otimes_{\mathbf{Q}} \mathbf{Q}_v)^\times \simeq \mathrm{GL}_2(\mathbf{Q}_v)$ if and only if π_v is square-integrable for all $v \in \Sigma$. It follows from the proof of Jacquet-Langlands correspondence that multiplicity one holds for the group $B^\times$; therefore, when such a π' exists, it is unique. We denote π' by π^{JL}. The correspondence $\pi^{\mathrm{JL}} \mapsto \pi$ then induces an injective map from the set of classical Hecke eigensystems on $B^\times$ to that on GL_2 (cf. Proposition II.21); see Definition II.20 for the notion of "classicality".

Let us now fix an odd[1] prime p once and for all, and turn to the theme of p-adic variation. It is a theorem of Hida [Hid88] that Hecke eigensystems on GL_2 and $B^\times$ vary in p-adic families. We will exclusively deal with the case where B is ramified at ∞ (the so-called *definite quaternion algebras*) and split at p—for an exposition of Hida theory in this (arguably elementary) case, see [LV12, §3], [Buz04, §4 ff], [Hsi21, §4.4], [BD07, §2], and [BDI10, §1-§4]; for Hida theory with tame levels deeper than the maximal order at the ramified places, see [Dal23b].

The spectral theory of the U_p-operator underlying Hida theory is reasonably well-understood, reaching a crescendo in the work of Coleman-Mazur [CM98], Buzzard [Buz07] and Chenevier [Che05] for the purpose of providing context to our thesis. Fix a natural number N that is relatively prime to p (the so-called *tame level*). Let $\mathcal{W}_N$ denote the rigid analytic space $\mathrm{Spf}\, \mathbf{Z}_p[\![(\mathbf{Z}/N\mathbf{Z})^\times \times \mathbf{Z}_p^\times]\!]^{\mathrm{rig}}$; the $\mathbf{C}_p$-points of $\mathcal{W}_N$ are continuous characters $(\mathbf{Z}/N\mathbf{Z})^\times \times \mathbf{Z}_p^\times \to \mathbf{C}_p^\times$; in particular, this space contains the classical weight characters $\kappa_{(k,\psi)}$ given by

$$\kappa_{(k,\psi)} : (a, z) \mapsto \langle z \rangle^k \psi(a, \log\langle z \rangle \bmod p^m);$$

here $\langle z \rangle = z\omega^{-1}(z)$ where $\omega : \mathbf{Z}_p^\times \to \mathbf{Z}_p^\times$ denotes the Teichmüller character, log denotes the branch of the p-adic logarithm that satisfies $\log p = 0$, and $\psi : (\mathbf{Z}/Np^{m+1}\mathbf{Z})^\times \to \mathbf{C}_p^\times$ is a

[1] This an assumption of convenience because it lightens the notation and does not take away from the other ideas.

Dirichlet character (for some $m \geqslant 0$). Coleman-Mazur (for $N = 1$) and Buzzard (for general N) have constructed a rigid analytic variety

$$\mathrm{wt} : \mathcal{E}^{\mathrm{GL}_2} \to \mathcal{W}_N$$

over the space $\mathcal{W}_N$ whose fiber over a weight κ is the set of Hecke eigensystems λ on GL_2 of tame level N and weight κ such that the $\lambda(U_p) \neq 0$ ("finite slope" eigensystems).

We restrict now to the case of definite quaternion algebras. Let D be a quaternion algebra over $\mathbf{Q}$ that is split at p. Under the hypothesis that the primes that divide the discriminant d divide the tame level N exactly (thus forcing that the eigensystems that are in the image of the classical Jacquet-Langlands correspondence with tame level N are twists of Steinberg at places dividing d), Chenevier [Che05] has constructed an eigenvariety $\mathrm{wt} : \mathcal{E}^{D^\times} \to \mathcal{W}_N$ for the group $D^\times$ and an immersion of eigenvarieties

$$\begin{array}{ccc} \mathcal{E}^{D^\times} & \longrightarrow & \mathcal{E}^{\mathrm{GL}_2} \\ & {\scriptstyle \mathrm{wt}}\searrow \quad \swarrow{\scriptstyle \mathrm{wt}} & \\ & \mathcal{W}_N & \end{array}$$

interpolating the Jacquet-Langlands correspondence. Amongst classically studied cuspidal newforms for the group GL_2, those of weight 1 are interesting from this perspective; indeed if f is a cuspidal newform of weight 1 for the group GL_2, then:

- The automorphic representation π_f associated to f does not transfer to $D^\times$ since $\pi_{f,\infty}$ is not square-integrable.
- The local representation $\pi_{f,q}$ is never a twist of Steinberg at primes q dividing the level of the form f.

Thus, weight 1 forms are excluded from the classical Jacquet-Langlands correspondence and from Chenevier's p-adic Jacquet-Langlands correspondence. However, a cuspidal p-stabilized newform f of weight 1 with $a_p(f) \neq 0$ (equivalently, that f is p-ordinary) is a p-adic limit of cuspidal newforms of weight $\geqslant 2$ by Hida theory. For the sake of an informal discussion, let us suppose that $\mathcal{F} = \{\mathcal{F}_\kappa\}_{\kappa \in \mathcal{W}_N}$ is a Hida family of cusp forms of weight $\kappa = (k, \chi)$ with $k \geqslant 2$ specializing to f in weight 1. Since the automorphic types of arithmetic specializations and specializations to weight 1 are rigid in Hida families [Dim14, §6], if we start with f such that f is supercuspidal at primes dividing d, then the Hecke eigensystems corresponding to $\mathcal{F}_\kappa$ transfer to $D^\times$ for weights $\kappa = (k, \chi)$ such $k \geqslant 2$. This leads us naturally to inquire about the existence of p-adic interpolation of the Jacquet-Langlands correspondence that includes this situation.

Constructing a $D^\times$-eigenvariety and a p-adic analytic interpolation of the Jacquet-Langlands correspondence that both include weight 1 points however is subtle owing to the following observations:

1. If q is a prime dividing the quaternion algebra and $\mathscr{O}_q$ is an order in $D_q := \mathbf{Q}_q \otimes_{\mathbf{Q}} D$ deeper than the maximal order in the division algebra, then the dimension of $(\pi_q^{\mathrm{JL}})^{\mathscr{O}_q^\times}$, whenever non-zero, may be more than one; globally, for an order $\mathscr{O}$ of D of "level" N, the relationship between $\pi^{U_1(Nd)}$ and $(\pi^{\mathrm{JL}})^{\widehat{\mathscr{O}}}$ is more complicated. The situation is however understood in the literature; see [Piz80, HPS89, PRV05, Mar20].

2. there may be more than one Galois-conjugacy class of Hida families through f; in view of the automorphic rigidity we mentioned above, the local structure at f^{JL} of a putative $D^\times$-eigenvariety that includes weight 1 forms is at least as complicated as that of the GL_2-eigenvariety at f.

In any event, even if we succeeded in constructing the $D^\times$-eigenvariety that includes weight 1 forms and produced a p-adic Jacquet-Langlands correspondence at the level of eigenvarieties, we'd merely have a spectral correspondence without an understanding of how the interpolation happens in any model for the space of p-adic modular forms on $D^\times$. Such an understanding is crucial to us for the arithmetic applications we have in mind.

Instead we ask if there is a "p-adic automorphic representation" on $D^\times$ that witnesses this phenomenon. In his influential paper [Eme06], Emerton debuted:

- completed cohomology as a proxy for the space of p-adic modular forms; and
- a notion of classical p-adic automorphic representation; see Definition 3.1.5 in loc. cit.

To describe these ideas, let us fix some notations. Let L be a finite extension of $\mathbf{Q}_p$, let $\mathscr{O}$ be the ring of integers in L and let k be the residue field of $\mathscr{O}$. We fix a uniformizer ϖ in $\mathscr{O}$; we also fix an algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$ and an isomorphism $\iota : \mathbf{C} \to \overline{\mathbf{Q}}_p$. Let $K^{(p)}$ be a compact open subgroup in $D^\times(\mathbf{A}^{(p\infty)})$; let K_∞ denote the group $D^\times(\mathbf{R})^1$ of norm 1 elements in $D^\times(\mathbf{R})$. We look at the family of finite sets

$$Y(K_p K^{(p)}) = D^\times(\mathbf{Q}) \setminus D^\times(\mathbf{A}) \,/\, \mathbf{R}_+^\times K_\infty K_p K^{(p)}$$

obtained by varying the level subgroup $K_p \subset D^\times(\mathbf{Q}_p) = \mathrm{GL}_2(\mathbf{Q}_p)$ at p; if $K_p' \subset K_p$, then there is a canonical surjection $Y(K_p' K^{(p)}) \twoheadrightarrow Y(K_p K^{(p)})$. The completed cohomology $\widetilde{\mathrm{H}}^0(K^{(p)})$ of tame level $K^{(p)}$ for the group $D^\times$ is defined to be

$$\widetilde{\mathrm{H}}^0(K^{(p)})_{\mathscr{O}} = \varprojlim_{s} \varinjlim_{K_p} \mathrm{H}^0(Y(K_p K^{(p)}), \mathscr{O}/\varpi^s \mathscr{O})$$

as s ranges over non-negative integers and K_p over compact open subgroups of $D^\times(\mathbf{Q}_p)$. The L-vector space $\widetilde{\mathrm{H}}^0(K^{(p)})_L = L \otimes \widetilde{\mathrm{H}}^0(K^{(p)})_{\mathscr{O}}$ is then an L-Banach space for the sup norm, and in which the $\mathscr{O}$-module $\widetilde{\mathrm{H}}^0(K^{(p)})_{\mathscr{O}}$ is the unit ball. One checks (v. Lemma II.15) that $\widetilde{\mathrm{H}}^0(K^{(p)})_{\mathscr{O}}$ (resp. $\widetilde{\mathrm{H}}^0(K^{(p)})_L$) is the space of $\mathscr{O}$-valued (resp. L-valued) continuous functions on the profinite set $Y_{K^{(p)}} = \varprojlim_{K_p} Y(K_pK^{(p)})$.

Classical p-adic automorphic representations of the group $D^\times$ are constructed out of certain infinite dimensional smooth admissible automorphic representations π of $D^\times(\mathbf{A})$ over $\mathbf{C}$. Factor π into

$$\pi = \pi_\infty \otimes \pi_{\mathrm{fin}} = \pi_\infty \otimes \pi_p \otimes \pi_{\mathrm{fin}}^{(p)}$$

where π_∞ is an admissible representation of $D^\times(\mathbf{R})$, and π_p and $\pi_{\mathrm{fin}}^{(p)}$ are respectively infinite dimensional smooth admissible representations of $D^\times(\mathbf{Q}_p)$ and $D^\times(\mathbf{A}^{(p\infty)})$. If the infinitesimal character of π_∞ coincides with that of an irreducible finite dimensional algebraic representation W of the group $D^\times(\mathbf{R})$ over $\mathbf{C}$, then we say that π is *W-allowable* and *the classical p-adic automorphic representation* $\widetilde{\pi}$ associated to π is the representation of $D^\times(\mathbf{A}^{(\infty)}) = D^\times(\mathbf{Q}_p) \times D^\times(\mathbf{A}^{(p\infty)})$

$$(\pi_p \otimes_{\mathbf{C}} W) \otimes_{\mathbf{C}} \pi_{\mathrm{fin}}^{(p)} \otimes_{\mathbf{C},\iota} \overline{\mathbf{Q}}_p$$

defined[2] over $\overline{\mathbf{Q}}_p$.

Emerton observed that the classical p-adic automorphic representations are summands of the space $\widetilde{\mathrm{H}}^0(K^{(p)})_{L,\mathrm{lalg}}$ of locally algebraic vectors in $\widetilde{\mathrm{H}}^0(K^{(p)})_L$ (cf. Proposition 3.2.4 of loc. cit.). For convenience, let us base change to $\overline{\mathbf{Q}}_p$; then, his theorem states that there is an isomorphism

$$\widetilde{\mathrm{H}}^0(K^{(p)})_{L,\mathrm{lalg}} \otimes \overline{\mathbf{Q}}_p = \bigoplus_W \bigoplus_\pi \left((\pi_p \otimes_{\mathbf{C}} W) \otimes_{\mathbf{C}} (\pi_{\mathrm{fin}}^{(p)})^{K^{(p)}}\right)^{\oplus m_{D^\times}(\pi)} \otimes_{\mathbf{C},\iota} \overline{\mathbf{Q}}_p$$

of $\overline{\mathbf{Q}}_p$-vector spaces where the first direct sum runs over irreducible algebraic representations of $D^\times(\mathbf{R})$ over $\mathbf{C}$ and the second direct sum runs over automorphic representations of $D^\times(\mathbf{A})$ that are W-allowable. In particular, passing to the inductive limit over $K^{(p)}$, we get an isomorphism of $\overline{\mathbf{Q}}_p$-representations of $D^\times(\mathbf{A}^{(\infty)})$

$$\varinjlim_{K^{(p)}} \widetilde{\mathrm{H}}^0(K^{(p)})_{L,\mathrm{lalg}} \otimes \overline{\mathbf{Q}}_p = \bigoplus_W \bigoplus_\pi \left((\pi_p \otimes_{\mathbf{C}} W) \otimes_{\mathbf{C}} \pi_{\mathrm{fin}}^{(p)}\right)^{\oplus m_{D^\times}(\pi)} \otimes_{\mathbf{C},\iota} \overline{\mathbf{Q}}_p.$$

A cuspidal newform f of weight 1 which is supercuspidal at primes dividing d is only a

[2] As Emerton shows in ¶3.1 of loc. cit., the representation descends uniquely to a finite extension of $\mathbf{Q}_p$ but we suppress that here.

p-adic limit of algebraic vectors in $\widetilde{\mathrm{H}}^0(K^{(p)})_L$. Thus to locate weight 1 forms in completed cohomology, we should work with the larger space $\widetilde{\mathrm{H}}^0(K^{(p)})_{L,\mathrm{la}}$ of locally analytic vectors in $\widetilde{\mathrm{H}}^0(K^{(p)})_L$.

Note that, if π is an automorphic representation of $D^\times(\mathbf{A})$ to which a p-adic Galois representation ρ_π may be associated (e.g., C-algebraic in the sense of [BG14]), the local components π_v of π may be recovered from the Weil-Deligne representation $\mathrm{WD}_v(\rho_\pi|_{\mathscr{G}_{\mathbf{Q}_v}})$ via the (classical) local Langlands correspondence; the web of theorems of this flavor go under the name "local-global compatibility" and was first established in Carayol [Car86] for $v \neq p$, and in Saito [Sai97] for $v = p$; for automorphic representation π_f of $\mathrm{GL}_2(\mathbf{A})$ associated to a cuspidal newform f of weight 1, the local-global compatibility can be deduced from Hida theory (cf. [Wil88]). This suggests that we ought to think of the local component $\widetilde{\pi}_p$ as arising from a p-adic local Langlands correspondence while the smooth local representations at primes different from p still come from the classical local Langlands correspondence.

Motivated by these considerations, and the intervening developments in the p-adic Langlands programme, we are led to the following conjectural p-adic Jacquet-Langlands correspondence:

Conjecture I.1. *Let D be the quaternion algebra of discriminant $d\infty$ over $\mathbf{Q}$; we suppose that $p \nmid d$. Fix a continuous absolutely irreducible representation $\overline{\rho} : \mathscr{G}_{\mathbf{Q}} \to \mathrm{GL}_2(k)$. Assume that $\overline{\rho}$ satisfies the following conditions*[3]*:*

($\mathbf{Mod}_{\overline{\rho}}$) *$\overline{\rho}$ is GL_2-modular;*

($\mathbf{Gen}_p$) *$\overline{\rho}|_{\mathscr{G}_{\mathbf{Q}_p}}$ is not equivalent to*

$$\begin{pmatrix} 1 & * \\ 0 & \overline{\chi} \end{pmatrix} \otimes \psi$$

for some character $\psi : \mathscr{G}_{\mathbf{Q}_p} \to k^\times$ and $\overline{\chi}$ is the mod p cyclotomic character;[4]

($\mathbf{SI}_d$) *for $v \mid d$, the local representation $\overline{\rho}|_{\mathscr{G}_{\mathbf{Q}_v}}$ is irreducible, or is of the form*

$$\begin{pmatrix} 1 & * \\ 0 & \overline{\chi}^{-1} \end{pmatrix} \otimes \psi_v$$

where ψ_v is some character $\mathscr{G}_{\mathbf{Q}_v} \to k^\times$ and $\overline{\chi}$ is the mod p cyclotomic character.

Suppose that $\lambda : \mathbf{T}(U_1(N)) \to L$ is a classical eigensystem on GL_2 and to which there is an

[3] See the discussion in §IV.1.

[4] For more on this point, see Remark 6.1.23 of [Eme11].

associated p-adic Galois representation ρ_λ defined over L; suppose[5] *also that $\overline{\rho}_\lambda$ is equivalent to $\overline{\rho}$. There exists a tame level $K^{(pd\infty)}$ of $D^\times(\mathbf{A}^{(pd\infty)})$ (cf. [Mar20]) such that there is a continuous $\mathbf{T}(K^{(pd\infty)})[D^\times(\mathbf{Q}_p)\times\prod_{q|d}D^\times(\mathbf{Q}_q)]$-equivariant embedding*

$$\mathrm{LL}_p(\rho_\lambda|_{\mathscr{G}_{\mathbf{Q}_p}})\,\widehat{\bigotimes_L}\,\otimes_{q|d}\mathrm{LL}_q(\mathrm{WD}_q(\rho_\lambda|_{\mathscr{G}_{\mathbf{Q}_q}}))\hookrightarrow\varinjlim_{K_d}\widehat{\mathrm{H}}^0(K_dK^{(pd\infty)})_L.$$

Here, on the left-hand side, the notation $\widehat{\bigotimes_L}$ refers to the completed tensor product, LL_v denotes the classical local Langlands correspondence for $v\mid d$, and LL_p denotes the p-adic local Langlands correspondence; on the right-hand side, the injective limit is taken over open compact subgroups K_d of $\prod_{q|d}D^\times(\mathbf{Q}_q)$.

In this thesis, under stronger assumptions than in our conjecture, we prove the p-adic Jacquet-Langlands correspondence before passing to the inductive limit over all levels at primes dividing d. Our main theorem is the following:

Theorem I.2. *Let $\mathbb{G}=D^\times$ where D is a definite quaternion algebra of discriminant $d\infty$; we assume that $p\nmid d$ so $D\otimes\mathbf{Q}_p$ is the matrix algebra. Fix a finite extension L of $\mathbf{Q}_p$ with ring of integers $\mathscr{O}$, and residue field k_L.*

Fix a continuous absolutely irreducible representation $\overline{\rho}:\mathscr{G}_{\mathbf{Q}}\to\mathrm{GL}_2(k_L)$ that is modular and satisfies the hypotheses

(Irr_p) *$\overline{\rho}|_{\mathscr{G}_{\mathbf{Q}_p}}$ is irreducible*

and (SI_d) above. Let Σ_0 be a finite set places not containing p so that $\overline{\rho}$ is unramified outside $\Sigma=\Sigma_0\cup\{p\}$.

Let $\pi=\mathbf{V}(\rho^m|_{\mathscr{G}_{\mathbf{Q}_p}})$ be the p-adic local Langlands correspondent of the universal modular deformation ρ^m.

Let $\lambda:\mathbf{T}_{\overline{\rho},\Sigma}\to\mathscr{O}$ be any system of Hecke eigenvalues; let $\mathfrak{p}=\ker\lambda$. Then:

1. *There is a non-zero $\mathbb{G}(\mathbf{Q}_p)$-equivariant map*

$$\pi/\mathfrak{p}\pi\to\widehat{\mathrm{H}}^0_{\overline{\rho},\mathscr{O}}[\mathfrak{p}]$$

of $\mathbf{T}_{\overline{\rho},\Sigma}$-modules.

[5] This assumption forces that N is divisible by the Artin conductor of $\overline{\rho}$; see, generally, the discussion in §IV.1, and [Liv89] in particular.

2. *If λ is associated to a Galois representation $\rho_\lambda : G_{\mathbf{Q}} \to \mathrm{GL}_2(L)$, then, $(\pi/\mathfrak{p}\pi)$ is the locally analytic representation $\Pi(\rho_\lambda|_{\mathscr{G}_{\mathbf{Q}_p}})$ associated to $\rho_\lambda|_{\mathscr{G}_{\mathbf{Q}_p}}$ by the p-adic local Langlands correspondence and so every non-zero map of (1) above extends to a non-zero map*

$$\Pi(\rho_\lambda|_{\mathscr{G}_{\mathbf{Q}_p}}) \hookrightarrow \widehat{\mathrm{H}}^0_{\overline{\rho},L}\,.$$

We close with the following conjecture, strongly inspired by the theory of eigenvarieties:

Conjecture I.3. *Suppose that f is a cuspidal newform of weight 1. Suppose that $\overline{\rho_f}$ satisfies the conditions (Gen$_p$) and (SI$_d$) outlined in the Introduction. Let Σ_0 be a finite set places not containing p so that $\overline{\rho}$ is unramified outside $\Sigma = \Sigma_0 \cup \{p\}$. Then there is a prime ideal $\mathfrak{p}_f$ of $\mathbf{T}_{\overline{\rho}_f,\Sigma}$ such that the Hecke eigenvalues encoded by $\mathfrak{p}_f$ coincide with those of f at places $\ell \notin \Sigma$.*

Together with part 2 of our main theorem, this provides a p-adic Jacquet-Langlands correspondence for cuspidal newforms of weight 1.

CHAPTER II
Generalities on Completed Cohomology

In this chapter, we closely follow Emerton's ICM survey [Eme14]; cf. also the survey by Calegari and Emerton [CE12]. The fundamental properties of completed cohomology are established in Emerton's seminal paper [Eme06].

We spend the first six sections of this chapter on generalities concerning completed cohomology. Before we move on to the Galois side in the next chapter, we mention that the completed cohomology may be identified with the space of p-adic automorphic forms à la Hida (resp. Buzzard) for ordinary (resp. finite slope) families in the context of definite quaternion algebras (§II.7). In the final section (§II.8), we shall reformulate the classical Jacquet-Langlands correspondence in terms of the p-adic Hecke algebras.

Notations for this chapter

- Let $\mathbb{G}$ be a reductive linear algebraic group over $\mathbf{Q}$; we also fix a prime p.
- We write G_∞ for the real points $\mathbb{G}(\mathbf{R})$ of $\mathbb{G}$, and write G for the $\mathbf{Q}_p$-points $\mathbb{G}(\mathbf{Q}_p)$ of $\mathbb{G}$. Thus G_∞ is a reductive Lie group while G is a p-adic Lie group.
- Fix a maximal $\mathbf{Q}$-split torus A_∞ in the center of $\mathbb{G}$, and a choice of maximal compact subgroup K_∞ of G_∞.

II.1: The definition

For a open compact subgroup $K_{\text{fin}} \subset \mathbb{G}(\mathbf{A}^{(\infty)})$, we form the double quotient

$$Y(K_{\text{fin}}) = \mathbb{G}(\mathbf{Q}) \setminus \mathbb{G}(\mathbf{A}) \;/\; A_\infty^\circ K_\infty^\circ K_{\text{fin}}.$$

Note that if $K'_{\text{fin}} \subset K_{\text{fin}}$ are two open compact subgroups, then there is a natural surjection $Y(K'_{\text{fin}}) \twoheadrightarrow Y(K_{\text{fin}})$.

We fix an open compact subgroup $K^{(p)} \subset \mathbb{G}(\mathbf{A}^{(p\infty)})$, the so-called *tame level*. Taking $K_{\mathrm{fin}} = K_p K^{(p)}$ as K_p ranges over open compact subgroups of G, we get a projective system

$$\{Y(K_p K^{(p)})\}_{K_p \subset G, K_p \text{open compact}}.$$

Note that this projective system has an action of G and the component group $\pi_0 = A_\infty K_\infty / A_\infty^\circ K_\infty^\circ$.

Definition II.1.

1. The completed cohomology of tame level $K^{(p)}$ is defined as

$$\widetilde{\mathrm{H}}^i(K^{(p)})_{\mathscr{O}} := \varprojlim_s \varinjlim_{K_p} \mathrm{H}^i\left(Y(K_p K^{(p)}), \mathscr{O}/\varpi^s\right)$$

   where the projective limit is taken over open compact subgroups K_p of G, and the injective limit is taken over integers $s \geqslant 0$. The $\mathscr{O}$-module $\widetilde{\mathrm{H}}^i(K^{(p)})_{\mathscr{O}}$ is equipped with its ϖ-adic topology; this coincides with the projective limit topology suggested by the presentation here–viz., if we endow the $\mathscr{O}/\varpi^s$-modules

$$\varinjlim_{K_p} \mathrm{H}^i\left(Y(K_p K^{(p)}), \mathscr{O}/\varpi^s\right)$$

   with the discrete topology, the projective limit topology on $\widetilde{\mathrm{H}}^i(K^{(p)})_{\mathscr{O}}$ coincides with the ϖ-adic topology of the $\mathscr{O}$-module $\widetilde{\mathrm{H}}^i(K^{(p)})_{\mathscr{O}}$.

2. The naïve completed cohomology $\widehat{\mathrm{H}}^i(K^{(p)})_{\mathscr{O}}$ is the p-adic completion

$$\widehat{\mathrm{H}}^i(K^{(p)})_{\mathscr{O}} = \varprojlim_s \mathrm{H}^i/\varpi^s \mathrm{H}^i$$

   of the injective limit H^i given by

$$\mathrm{H}^i = \varinjlim_{K_p} \varprojlim_s \mathrm{H}^i\left(Y(K_p K^{(p)}), \mathscr{O}/\varpi^s\right) \simeq \varinjlim_{K_p} \mathrm{H}^i\left(Y(K_p K^{(p)}), \mathscr{O}\right).$$

Note that these $\mathscr{O}$-modules fit in a short exact sequence

$$0 \to \widehat{\mathrm{H}}^i(K^{(p)})_{\mathscr{O}} \to \widetilde{\mathrm{H}}^i(K^{(p)})_{\mathscr{O}} \to T_p \mathrm{H}^{i+1} \to 0 \qquad \text{(II.1.1)}$$

where $T_p \mathrm{H}^{i+1}$ is the projective limit $\varprojlim_s \mathrm{H}^{i+1}[\varpi^s]$; the notation $\mathrm{H}^{i+1}[\varpi^s]$ refers to the $\mathscr{O}/\varpi^s$-module of ϖ^s-torsion elements in H^{i+1}.

We are interested in the cases where $\mathbb{G}$ is the group GL_2, or the units $D^\times$ in a quaternion algebra D over $\mathbf{Q}$ split at p and ramified at ∞.

Lemma II.2. *Let i equal 1 or 0 accordingly as $\mathbb{G} = \mathrm{GL}_2$, or $D^\times$ where D is a quaternion algebra over $\mathbf{Q}$ as above. Then, the natural map $\widehat{\mathrm{H}}^i(K^{(p)})_{\mathscr{O}} \to \widetilde{\mathrm{H}}^i(K^{(p)})_{\mathscr{O}}$ is an isomorphism.*

Proof. In the cases above, note that H^{i+1} is the zero module; hence $T_p\mathrm{H}^{i+1} = 0$. The claim now follows from the exact sequence (II.1.1). $\square$

Definition II.3. For an $\mathscr{O}$-algebra A, we will write

$$\widetilde{\mathrm{H}}^i(K^{(p)})_A = A \otimes_{\mathscr{O}} \widetilde{\mathrm{H}}^i(K^{(p)})_{\mathscr{O}}$$
$$\widehat{\mathrm{H}}^i(K^{(p)})_A = A \otimes_{\mathscr{O}} \widehat{\mathrm{H}}^i(K^{(p)})_{\mathscr{O}}$$

for the A-module obtained by change of scalars. (Often, for us, A is one of the residue field k of $\mathscr{O}$, or torsion $\mathscr{O}$-modules $\mathscr{O}/\varpi^s\mathscr{O}$, or the field of fractions L of $\mathscr{O}$.)

The L-vector spaces $\widetilde{\mathrm{H}}^i(K^{(p)})_L$ and $\widehat{\mathrm{H}}^i(K^{(p)})_L$ are Banach spaces, containing $\widetilde{\mathrm{H}}^i(K^{(p)})_{\mathscr{O}}$ and $\widehat{\mathrm{H}}^i(K^{(p)})_{\mathscr{O}}$ respectively as unit balls.

II.2: Completed cohomology as a G-representation

The action of G on the projective system $\{Y(K_pK^{(p)})\}_{K_p\subset G}$ gives rise to an action of G on the $\mathscr{O}$-module $\widetilde{\mathrm{H}}^i = \widetilde{\mathrm{H}}^i(K^{(p)})_{\mathscr{O}}$. The following fundamental result makes completed cohomology amenable to study using largely algebraic and soft analytic techniques:

Theorem II.4. *The G-action of $\widetilde{\mathrm{H}}^i$ affords a continuous ϖ-adically admissible representation of G; that is*

- *$\widetilde{\mathrm{H}}^i$ is ϖ-adically complete as an $\mathscr{O}$-module;*
- *the action map $G \times \widetilde{\mathrm{H}}^i \to \widetilde{\mathrm{H}}^i$ is continuous when $\widetilde{\mathrm{H}}^i$ is given its ϖ-adic topology; note that this amounts to the fact that $\widetilde{\mathrm{H}}^i/\varpi^s$ is a smooth G-representation for all s; and*
- *$\widetilde{\mathrm{H}}^i/\varpi^s$ is admissible (i.e., for each open compact subgroup K_p of G, the $\mathscr{O}$-submodule of K_p-invariants is finitely generated over $\mathscr{O}$).*

In particular, $\widetilde{\mathrm{H}}^i$ being admissible is equivalent to the fact that its Schikof dual $(\widetilde{\mathrm{H}}^i)^* := \mathrm{Hom}_{\mathscr{O}}(\widetilde{\mathrm{H}}^i, \mathscr{O})$ is finitely generated as $\mathscr{O}[\![K_p]\!]$-module for some (and hence every) open compact subgroup K_p of G.

Proof. This is the main theorem in [Eme06]. For the equivalence, see [ST02a, Theorem 3.5]. $\square$

II.3: Emerton's spectral sequence: recovering cohomology at finite levels from completed cohomology

The manner in which the completed cohomology encodes the cohomology of the spaces at finite levels is a bit subtle. To illustrate this phenomenon, we consider a topological situation analogous to the arithmetic case of interest to us – viz., the tower of coverings of the circle $Y_0 = Y = \mathbf{R}/\mathbf{Z}$ of p-power degree. For $n \geqslant 0$, let $Y_n = \mathbf{R}/p^n\mathbf{Z}$ so there are natural covering maps $Y_m \to Y_n$ with Galois group $p^n\mathbf{Z}/p^m\mathbf{Z}$; the projective system $\{Y_n\}_n$ has an action of the group $G = \mathbf{Z}_p$. In this situation,

$$\mathrm{H}^i(Y_n, \mathbf{Z}/p^s\mathbf{Z}) = \begin{cases} \mathbf{Z}/p^s\mathbf{Z}, & \text{for } i = 0, 1 \\ 0, & \text{otherwise} \end{cases}.$$

Since the induced map $\mathrm{H}^0(Y_n, \mathbf{Z}/p^s\mathbf{Z}) \to \mathrm{H}^0(Y_{n+1}, \mathbf{Z}/p^s\mathbf{Z})$ is the identity, we get

$$\widetilde{\mathrm{H}}^0 = \varprojlim_s \varinjlim_n \mathrm{H}^0(Y_n, \mathbf{Z}/p^s\mathbf{Z}) = \mathbf{Z}_p.$$

Note that the action of G on $\widetilde{\mathrm{H}}^0$ is trivial. On the other hand, the induced map $\mathrm{H}^1(Y_n, \mathbf{Z}/p^s\mathbf{Z}) \to \mathrm{H}^1(Y_{n+1}, \mathbf{Z}/p^s\mathbf{Z})$ is multiplication by p; therefore,

$$\widetilde{\mathrm{H}}^1 = \varprojlim_s \varinjlim_n \mathrm{H}^1(Y_n, \mathbf{Z}/p^s\mathbf{Z}) = 0.$$

Neverthless, we can recover the cohomology $\mathrm{H}^\bullet(Y_n, \mathbf{Z}_p)$ of Y_n from $\widetilde{\mathrm{H}}^\bullet$:

Theorem II.5. *There is a first quadrant E_2-spectral sequence*

$$\mathrm{H}^i_{\mathrm{cts}}(K_p, \widetilde{\mathrm{H}}^j(K^{(p)})_{\mathscr{O}}) \implies \mathrm{H}^{i+j}(Y(K_pK^{(p)}), \mathscr{O})$$

abutting to the $\mathscr{O}$-cohomology of the congruence quotients at finite p-power levels.

In our topological example, the spectral sequence

$$\mathrm{H}^i_{\mathrm{cts}}(p^n\mathbf{Z}_p, \widetilde{\mathrm{H}}^j_{\mathbf{Z}_p}) \implies \mathrm{H}^{i+j}(Y_n, \mathbf{Z}_p)$$

collapses on the second page since the only non-zero terms are in the row $j = 0$ and the differentials have bidegree $(2, -1)$, so we do recover the expected infinity page.

Proof of Theorem II.5. In this setting, taking $K_p' \subset K_p$ to be an open compact subgroup, we have the Hochschild Serre spectral sequence for continuous cohomology

$$\mathrm{H}^i_{\mathrm{cts}}(K_p/K'_p, \mathrm{H}^j(Y(K'_pK^{(p)}), \mathscr{O}/\varpi^s)) \implies \mathrm{H}^{i+j}(Y(K_pK^{(p)}), \mathscr{O}/\varpi^s).$$

Since inductive limit commutes with cohomology, we get a spectral sequence:

$$\mathrm{H}^i_{\mathrm{cts}}(K_p, \varinjlim_{K'_p} \mathrm{H}^j(Y(K'_pK^{(p)}), \mathscr{O}/\varpi^s)) \implies \mathrm{H}^{i+j}(Y(K_pK^{(p)}), \mathscr{O}/\varpi^s).$$

Now, we consider the projective limit over s of these spectral sequences and argue that the prosystem $\left\{\varinjlim_{K'_p} \mathrm{H}^j(Y(K'_pK^{(p)}), \mathscr{O}/\varpi^s)\right\}_s$ can be replaced by the prosystem $\left\{\widetilde{\mathrm{H}}^j(K^{(p)})_{\mathscr{O}} \otimes \mathscr{O}/\varpi^s\right\}_s$. This step is subtle, and relies on finiteness properties of the spaces $Y(K_{\mathrm{fin}})$ and its cohomology groups. We refer the interested reader to the details in Emerton's paper [Eme06, Proposition 2.1.11 and Theorem 2.2.11 (v)]. □

II.4: Completed cohomology of local systems

Let W be a finitely generated torsion-free $\mathscr{O}$-module equipped with a continuous representation of an open compact subgroup K_p of G; suppose also that K_p is sufficiently small so that the left translation by $\mathbb{G}(\mathbf{Q})$ has trivial stabilizers on $\mathbb{G}(\mathbb{A})/A^\circ_\infty K^\circ_\infty K_p K^{(p)}$ (i.e., $\mathbb{G}(\mathbf{Q}) \cap x^{-1}A^\circ_\infty K^\circ_\infty K_p K^{(p)} x = \{1\}$ for all $x \in \mathbb{G}(\mathbf{A})$).

Then, the $\mathscr{O}$-module W defines a local system $\mathcal{V}_W \to Y(K'_pK^{(p)})$ for all open compact subgroups $K'_p \subset K_p$ by

$$\mathcal{V}_W = \mathbb{G}(\mathbf{Q}) \setminus \left((\mathbb{G}(\mathbb{A})/A^\circ_\infty K^\circ_\infty K^{(p)}) \times W\right)/K'_p$$

where $\gamma \in K'_p$ acts on pairs (g, w) with $g \in (\mathbb{G}(\mathbb{A})/A^\circ_\infty K^\circ_\infty K^{(p)})$ and $w \in W$ via

$$\gamma \cdot (g, w) = (g\gamma, \gamma^{-1}w),$$

and the action of $\mathbb{G}(\mathbf{Q})$ is by left translation on the first factor.

One can then define completed cohomology with local system coefficients:

Definition II.6. The completed cohomology $\widetilde{\mathrm{H}}^i(K_pK^{(p)}, \mathcal{V}_W)_{\mathscr{O}}$ with coefficients in the local system $\mathcal{V}_W$ is given by

$$\widetilde{\mathrm{H}}^i(K_pK^{(p)}, \mathcal{V}_W) = \varprojlim_s \varinjlim_{K'_p} \mathrm{H}^i\left(Y(K'_pK^{(p)}), W/\varpi^s W\right)$$

where, the inductive limit is taken over compact open subgroups K'_p of K_p and the projective limit over non-negative integers s.

Let us now observe that the $W/\varpi^s W$ is finite and hence every small enough open compact subgroup of K_p fixes $W/\varpi^s W$ pointwise. This leads to an isomorphism

$$\varinjlim_{K'_p} \mathrm{H}^i(Y(K'_p K^{(p)}), W/\varpi^s W) \xrightarrow{\simeq} \varinjlim_{K'_p} \mathrm{H}^i(Y(K'_p K^{(p)}), \mathscr{O}/\varpi^s \mathscr{O}) \otimes W/\varpi^s W$$

(where the injective limits are taken over all open compact subgroups of K_p); one then has the following corollary:

Corollary II.7 (Theorem 2.2.17 of [Eme06]). *Let $W^\vee$ denote the $\mathscr{O}$-dual of W endowed with the contragredient K_p-action. There is a first quadrant E_2-spectral sequence*

$$E_2^{i,j} = \mathrm{Ext}^i_{\mathscr{O}[\![K_p]\!]}(W^\vee, \widetilde{\mathrm{H}}^j(K^{(p)})_{\mathscr{O}}) \implies H^{i+j}(Y(K_p K^{(p)}), \mathcal{V}_W)$$

that computes the cohomology of local systems at finite levels.

This is to be compared with Hida theory, say over GL_2, where the object that interpolates over p-power levels (and fixed weight) has specializations to automorphic forms of all possible weights. But, unlike Hida theory, we have a substantial upgrading of structure on our model for the space of p-adic modular forms—viz., the completed cohomology; the completed cohomology modules now admit the action of the entire group $\mathbb{G}(\mathbf{Q}_p)$ (and the tame level Hecke algebra) as opposed to just the U_p-operator (and the tame Hecke algebra). This makes it possible to view irreducible subrepresentations of completed cohomology as "p-adic automorphic representations" of the group G.

We close this section with a simpler description of the space of global sections of the local system $\mathcal{V}_W$. This helps connect completed cohomology to p-adic automorphic forms when $Y(K_{\mathrm{fin}})$ are zero-dimensional.

Lemma II.8. *The $\mathscr{O}$-module $\mathrm{H}^0(K_p K^{(p)}, \mathcal{V}_W)$ is $\mathbb{G}(\mathbf{A})$-equivariantly identified with the space*

$$\left\{ f : \mathbb{G}(\mathbf{Q}) \backslash \mathbb{G}(\mathbf{A})/A_\infty^\circ K_\infty^\circ K^{(p)} \to W \,\middle|\, \begin{array}{c} f \text{ is continuous and} \\ f(g\gamma) = \gamma^{-1} f(g) \text{ for all } g \in \mathbb{G}(\mathbf{A}), \gamma \in K_p \end{array} \right\}.$$

Proof. This is standard. If s is a section to the natural map $\pi : \mathcal{V}_W \to Y(K_p K^{(p)})$, then for all $\gamma \in Y(K_p K^{(p)})$, there exists $g \in \mathbb{G}(\mathbf{A})$ and $w_\gamma \in W$ such that $\pi([(g, w_\gamma)]) = \gamma$ and $s(\gamma) = [(g, w_\gamma)]$; that $[(g, w_\gamma)]$ lies in the fiber over γ means that there exists $k \in K_p$ such that $gk = \gamma$. Define

$$\widetilde{s} : \mathbb{G}(\mathbf{Q}) \backslash \mathbb{G}(\mathbf{A})/A_\infty^\circ K_\infty^\circ K^{(p)} \to W$$

by $\widetilde{s}(\gamma) = k^{-1}w_\gamma$. One now checks that $\widetilde{s}$ satisfies the properties asserted of f in the lemma, and that the map $s \mapsto \widetilde{s}$ is a $\mathbb{G}(\mathbf{A})$-equivariant isomorphism of the $\mathscr{O}$-module of sections of $\mathcal{V}_W$ onto the $\mathscr{O}$-module given in the lemma. $\square$

II.5: Hecke action on completed cohomology

Recall that a connected reductive group $\mathbb{G}$ over $\mathbf{Q}$ extends to a reductive group over $\mathbf{Z}[\frac{1}{\Sigma_0}]$ (also abusively denoted $\mathbb{G}$) where Σ_0 is a finite set of primes. In particular, for $v \notin \Sigma_0$, the group $\mathbb{G}_{/\mathbf{Q}_v}$ admits a distinguished maximal compact subgroup (up to conjugacy), viz., $\mathbb{G}(\mathbf{Z}_v)$; this is called a hyperspecial maximal compact subgroup of $\mathbb{G}(\mathbf{Q}_v)$.

Recall that we have fixed the tame level $K^{(p)} \subset \mathbb{G}(\mathbf{A}^{(p\infty)})$; by further enlarging Σ_0 if necessary, we may (and do) assume that $K_v^{(p)}$ is the hyperspecial maximal compact subgroup for all $v \notin \Sigma_0$. The algebra $\mathcal{H}_v := \mathcal{H}(\mathbb{G}(\mathbf{Q}_v)//K_v^{(p)})$ of $\mathscr{O}$-valued functions of the double coset space $K_v^{(p)} \setminus \mathbb{G}(\mathbf{Q}_v)/K_v^{(p)}$ is called the *spherical Hecke algebra*. It is commutative and acts naturally by continuous endomorphisms on the cohomology groups $\mathrm{H}^i(Y(K_pK^{(p)}), \mathcal{V}_W)$ for any finitely generated $\mathscr{O}[\![K_p]\!]$-module W.

Letting i range over all cohomological degrees, K_p over all compact open subgroups of G, and W over all continuous representations of K_p on finitely generated torsion $\mathscr{O}$-modules, the product

$$\prod_i \prod_{K_p} \prod_W \mathrm{End}_{\mathrm{cts}}\, \mathrm{H}^i(Y(K_pK^{(p)}), \mathcal{V}_W) \tag{II.5.1}$$

is a profinite ring. We now define the p-adic Hecke algebra for the group $\mathbb{G}$ as an $\mathscr{O}$-subalgebra of this profinite ring:

Definition II.9. The p-adic Hecke algebra $\mathbf{T}(K^{(p)})$ of tame level $K^{(p)}$ is the closure of the $\mathscr{O}$-subalgebra of the profinite ring (II.5.1) generated by $\mathcal{H}_\ell$ for $\ell \notin \Sigma_0$. If we wish to emphasize the underlying group $\mathbb{G}$, we will then write $\mathbf{T}^{\mathbb{G}}(K^{(p)})$ for this Hecke algebra.

We have the following facts about the p-adic Hecke algebra:

Lemma II.10. *Let $\mathbb{G}$ be the group GL_2 or one of its inner forms. Fix a tame level $K^{(p)}$ in $\mathbb{G}(\mathbf{A}^{(p\infty)})$.*

1. *The p-adic Hecke algebra $\mathbf{T}(K^{(p)})$ is reduced, p-torsionfree and p-adically complete.*
2. *The $\mathscr{O}$-algebra $\mathbf{T}(K^{(p)})$ is commutative and acts faithfully on $\widehat{\mathrm{H}}^i(K^{(p)})_{\mathscr{O}}$ and $\widehat{\mathrm{H}}^i(K^{(p)})_L$.*
3. *The $\mathscr{O}$-algebra $\mathbf{T}(K^{(p)})$ is semi-local and its maximal ideals are in bijection with the systems of Hecke eigenvalues $\lambda : \mathbf{T}(K^p) \to \overline{k}$.*

4. *The maximal ideals of the $\mathscr{O}$-algebra $\mathbf{T}(K^p)[1/p]$ are in bijection with the systems of Hecke eigenvalues $\lambda : \mathbf{T}(K^p) \to \overline{\mathscr{O}}$.*

Proof. The first two claims follow from classical theory once we recognize that

$$\mathbf{T}(K^{(p)}) = \varprojlim_{K_p} \mathbf{T}(K_p K^{(p)});$$

here K_p ranges over open compact subgroups of G, and $\mathbf{T}(K_p K^{(p)})$ is the $\mathscr{O}$-algebra generated by the image of the Hecke operators T_ℓ and S_ℓ (for $\ell \neq p$ and $\ell \notin \Sigma_0$) in $\operatorname{End} \mathrm{H}^{i_0}(Y(K_p K^{(p)}), \mathscr{O})$ (where i_0 is the only relevant cohomological degree for $\mathbb{G}$).

The finiteness of the set of mod p eigensystems of a given tame level (equivalently, the set of maximal ideals in $\mathbf{T}(K^{(p)})$) is a theorem of Jochnowitz [Joc82, Theorem 2.2]. □

II.6: Analytic vectors and locally algebraic vectors

Let Π be an L-Banach representation of the p-adic Lie Group G. Let $C^{\mathrm{la}}(G, \Pi)$ denote the space of locally L-analytic Π-valued functions on G. It is a Hausdorff locally convex barrelled topological L-vector space. The natural right translation action of G on $C^{\mathrm{la}}(G, \Pi)$ is continuous (cf. Lemma 2.2 of [ST02b]). Schneider and Teitelbaum [ST02b] considered the space of locally analytic vectors in a Banach representation as a topological G-module by embedding it into $C^{\mathrm{la}}(G, \Pi)$ via the orbit map $v \mapsto (g \mapsto g^{-1}v)$.

Definition II.11.

1. A vector $v \in \Pi$ is said to be *locally analytic* if the orbit map $g \mapsto g \cdot v$ is a locally analytic Π-valued function on G.

   The L-subspace
   $$\Pi^{\mathrm{la}} := \{v \in \Pi : v \text{ is locally analytic}\}$$
   of analytic vectors in Π is G-invariant and is equipped with the closed subspace topology inherited via the embedding $\Pi^{\mathrm{la}} \hookrightarrow C^{\mathrm{la}}(G, \Pi)$.

2. We say that the representation Π is *locally analytic* if every vector in Π is locally analytic. It is shown in [ST02b, Lemma 3.7] that the topology defined on Π^{la} coincides with that of Π if the representation Π is locally analytic.

An important theorem of Schneider-Teitelbaum [ST03, Theorem 7.1] is that, for an admissible L-Banach representation Π, the space Π^{la} of locally analytic vectors in Π is dense in Π and the functor $\Pi \mapsto \Pi^{\mathrm{la}}$ is exact on the category of admissible L-Banach representations.

Let $\mathfrak{g}$ be the Lie algebra of G; we write exp for the exponential map to G that is defined in a small neighbourhood of 0 in $\mathfrak{g}$. Then $\mathfrak{g}$ acts continuously on $C^{\text{la}}(G,\Pi)$ (and hence on Π^{la}):

$$(\mathfrak{x}f)(g) = \frac{d}{dt} f(g \exp(t\mathfrak{x}))\Big|_{t=0}.$$

By the universal property of the enveloping algebra $U(\mathfrak{g})$, there is then an action on $U(\mathfrak{g})$ on Π^{la} by continuous linear endomorphisms.

Remark II.12. Emerton [Eme11, Definition 3.5.3] has considered a different topology on the space of locally analytic vectors; it is finer in general but in the cases of interest to us—viz., when the Banach L-representation Π is an admissible continuous representation—Emerton's topology coincides with that considered by Schneider-Teitelbaum (v. Remarks after Theorem 3.5.7 in loc. cit.).

We now turn to locally algebraic vectors in Π.

Definition II.13.

1. Let W be an algebraic representation of G over L. We say that a vector $v \in \Pi$ is locally W-algebraic if there exists an open subgroup H of G, a natural number n, and an H-equivariant homomorphism $W^n \to \Pi$ whose image contains the vector v. The space $\Pi_{W\text{-lalg}}$ is a G-invariant subspace of Π.

2. We say that a vector $v \in \Pi$ is *locally algebraic* if there is an algebraic representation W of G over L such that v is locally W-algebraic. Denote by Π_{lalg} the space of locally algebraic vectors in Π. We say that Π is locally algebraic if $\Pi = \Pi_{\text{lalg}}$; i.e., if every vector in Π is locally algebraic.

This definition is due to Emerton [Eme11, Proposition-Definition 4.2.6]; it is equivalent to that considered by D. Prasad in the appendix to [STP01] (and used by Colmez in [Col10]). This equivalence follows from the characterisation of locally algebraic representations offered by Theorem 1 in the Appendix to [STP01]) and by Emerton [Eme17, Proposition 4.2.8].

We close with a general result due to Emerton [Eme17] that clarifies the structure of the space of locally algebraic vectors in a locally analytic representation of G:

Proposition II.14. *Let Π be a locally analytic representation of G over L; and let W be an algebraic representation of G over L. Let $\widehat{G}$ denote the set of isomorphism classes of irreducible algebraic representations of G over L. Then:*

1. *the evaluation morphism $\mathrm{Hom}_{\mathfrak{g}}(W,\Pi)\otimes_L W \to \Pi_{W\text{-lalg}}$ is an isomorphism of topological L-vector sapces and $\Pi_{W\text{-lalg}}$ is a closed subspace of Π.*

2. The natural map

$$\bigoplus_{W\in\widehat{G}} \Pi_{W\text{-lalg}} \to \Pi_{\mathrm{lalg}}$$

is an isomorphism.

II.7: Relation to automorphic forms: the case of definite quaternion algebras

In this subsection, we let $\mathbb{G}$ be the group $D^\times$ where D is a definite quaternion algebra of discriminant $d\infty$. We retain the other running notations in the previous sections.

Lemma II.15. *Let $A = \mathscr{O}$, or L. The space $\widetilde{\mathrm{H}}^0(K^{(p)})_{\mathscr{O}}$ is identified with the space of continuous A-valued functions on the profinite set*

$$Y_{K^{(p)}} := \varprojlim_{K_p} Y(K_pK^{(p)})$$

Proof. Owing to the compactness of $Y_{K^{(p)}}$, the natural map $L\otimes\mathscr{C}^0(Y_{K^{(p)}},\mathscr{O}) \to \mathscr{C}^0(Y_{K^{(p)}},L)$ is an isomorphism. Thus, it suffices to prove the claim for $A=\mathscr{O}$.

A function $f : Y_{K^{(p)}} \to \mathscr{O}$ is continuous if and only if the composition of f with natural quotients $\pi_s : \mathscr{O} \to \mathscr{O}/\varpi^s\mathscr{O}$ to the discrete space $\mathscr{O}/\varpi^s\mathscr{O}$ are continuous; this means that $\pi_s\circ f$ is locally constant. The profinite topology on $Y_{K^{(p)}}$ means that there is a compact open subgroup $K_p(s)$ of G such that $\pi_s\circ f$ is pulled back from an *arbitrary* function $Y(K_p(s)K^{(p)}) \to \mathscr{O}/\varpi^s\mathscr{O}$ between two finite sets. This discussion gives us the identification claimed in the lemma. ☐

Theorem II.16. *Let W be an algebraic representation of G over L. Factor an automorphic representation π of $\mathbb{G}$, definable[1] over L, into $\pi_p\otimes\pi_f^{(p)}\otimes\pi_\infty$ where π_p is a smooth L-representation of $\mathrm{GL}_2(\mathbf{Q}_p)$, $\pi_f^{(p)}$ a smooth representation of $\mathbb{G}(\mathbf{A}^{(p\infty)})$, and π_∞ a representation of G_∞. Then there are positive integers $m_{\mathbb{G}}(\pi)$ and an isomorphism*

$$\mathrm{H}^0(K^{(p)},\mathcal{V}_{W^\vee})_L \cong \bigoplus_{\pi:\pi_\infty=W^\vee}\left(\pi_p\otimes(\pi_f)^{K^{(p)}}\right)^{\oplus m_{\mathbb{G}}(\pi)}$$

Proof. This is called Matsushima's formula in the theory of automorphic forms and is standard. The proof is therefore omitted. ☐

[1]See Lemma 3.1.4 and the discussion thereafter in [Eme06].

Theorem II.17. *Let W be an irreducible algebraic representation of G over L. There is an isomorphism*

$$\mathrm{H}^0(K^{(p)}, \mathcal{V}_{W^\vee})_L \cong \mathrm{Hom}_{\mathfrak{g}}(W, \widehat{\mathrm{H}}^0(K^{(p)})_L^{\mathrm{la}})$$

of L-vector spaces.

Proof. This is Corollary 2.2.25 of Emerton [Eme06]. □

From Proposition II.14, we get:

Corollary II.18. *Let W be an irreducible algebraic representation of G over L. There is an isomorphism*

$$W \otimes_L \mathrm{H}^0(K^{(p)}, \mathcal{V}_{W^\vee}) \cong \widehat{\mathrm{H}}^0(K^{(p)})_{L,W-\mathrm{lalg}}$$

of L-vector spaces.

This brings us finally to the characterization of the space of locally algebraic vectors in completed cohomology as the space generated by classical p-adic automorphic representations in the sense of Emerton (and recalled in the introduction to this thesis). Concretely, we have:

Corollary II.19 (Proposition 3.2.4 of [Eme06])**.** *Let W be an irreducible algebraic representation of G over L. There is an isomorphism*

$$\widehat{\mathrm{H}}^0(K^{(p)})_{L,W-\mathrm{lalg}} \cong \bigoplus_{\pi:\pi_\infty = W^\vee} \left(\pi_p \otimes W \otimes (\pi_f)^{K^{(p)}}\right)^{\oplus m_{\mathbb{G}}(\pi)}$$

where π ranges over all automorphic representations definable over L and such that $\pi_\infty = W^\vee$.

The comparison between the p-adic modular forms for GL_2 and the completed cohomology $\widehat{\mathrm{H}}^1$ is not as straightforward; since we do not use such considerations, we content ourselves with a reference to the Emerton's answer on Mathoverflow [44].

II.8: Reformulation of the Jacquet-Langlands correspondence

In this subsection, we shall decorate our Hecke algebras with the underlying group $\mathbb{G}$ as the Jacquet-Langlands correspondence concerns the Hecke algebras acting on completed cohomology of multiple groups. Fix an isomorphism $\iota : \mathbf{C} \to \overline{\mathbf{Q}}_p$ as in the last subsection.

Definition II.20. Let $\mathbb{G}$ be an inner form of $\mathrm{GL}_{2/\mathbf{Q}}$; let Σ be the set of places v such that $\mathbb{G}_{\mathbf{Q}_v}$ is ramified. We say that an eigensystem $\lambda : \mathbf{T}^{\mathbb{G}} \to \overline{\mathbf{Q}}_p$ is *classical modular* if there exists an automorphic representation π_λ of the group $\mathbb{G}(\mathbf{A})$ with the following property:

for all $v \notin \Sigma \cup \{\infty\}$, if π_v is unramified, then the Satake parameters $\{\alpha_v(\pi), \beta_v(\pi)\}$ of the local representation π_v of $\mathrm{GL}_2(\mathbf{Q}_v)$ are related to the eigensystem λ via

$$\lambda(T_v) = \iota\alpha_v(\pi) + \iota\beta_v(\pi);$$

here T_v is the Hecke operator at place v.

The classical Jacquet-Langlands correspondence can then be formulated in terms of the p-adic Hecke algebras as follows:

Proposition II.21. *Let B be a quaternion algebra over $\mathbf{Q}$ of discriminant Δ. Let $\mathfrak{O}$ be an order of level*[2] *M in B. Write $\widehat{\mathfrak{O}}^\times$ for the open compact subgroup $(\mathfrak{O} \otimes \widehat{\mathbf{Z}})^\times$ of $B^\times(\mathbf{A})$. For an integer r, let $U_1(r) \subset \mathrm{GL}_2(\mathbf{A})$ denote the open compact subgroup*

$$U_1(r) = \left\{ \gamma \in \mathrm{GL}_2(\mathbf{A}) : \gamma_p \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod p^{v_p(r)} \text{ for all } p \right\}.$$

There is a morphism of $\mathscr{O}$-algebras

$$\mathbf{T}^{\mathrm{GL}_2}(U_1(\Delta M))_{\mathscr{O}} \to \mathbf{T}^{B^\times}(\widehat{\mathfrak{O}}^\times)_{\mathscr{O}}$$

preserving the Hecke operators away from the discriminant Δ of the quaternion algebra B such that the induced map

$$\left\{ \begin{array}{c} \lambda : \mathbf{T}^{B^\times}(\widehat{\mathfrak{O}}^\times)_{\mathscr{O}} \to \overline{\mathbf{Q}}_p \\ \lambda \textit{ is classical modular} \end{array} \right\} \to \left\{ \begin{array}{c} \lambda : \mathbf{T}^{\mathrm{GL}_2}(U_1(\Delta M))_{\mathscr{O}} \to \overline{\mathbf{Q}}_p \\ \lambda \textit{ is classical modular} \end{array} \right\}$$

is a bijection onto the set of those eigensystems λ that satisfy the property that the local representation $\pi_{\lambda,v}$ of $\mathrm{GL}_2(\mathbf{Q}_v)$ is square-integrable for all $v \mid \Delta$.

[2] We say that an order $\mathfrak{O}$ in B has level M if $\mathfrak{O}$ has index M in a maximal order in B. This notion is best behaved for Eichler orders, or more generally if $v_p(M)$ is odd for $p \mid M$. But orders with $v_p(M) = 2$ are particularly relevant to the context in which this thesis is situated; see [Dal21] (equivalently, the papers [Dal23b] and [Dal23a]) and the references to the works of Hijikata, Pizer and Shemanske therein.

CHAPTER III

The p-adic Local Langlands Correspondence

Our aim in this chapter is modest—we wish to state Colmez's p-adic local Langlands correspondence, its compatibility with an earlier construction of Berger-Breuil in the trianguline case, and the compatibility of this correspondence with the classical local Langlands correspondence (including in weight 1).

The p-adic local Langlands correspondence is a functor Π from the category of two-dimensional representations of L-representations of $\mathscr{G}_{\mathbf{Q}_p}$ to the category of locally analytic L-Banach representations of $\mathrm{GL}_2(\mathbf{Q}_p)$. The construction of this correspondence is due to Colmez [Col10]; it uses the notion of (φ, Γ)-modules introduced by Fontaine to classify p-adic local Galois representations; this background is discussed in §III.1, III.2 and III.3. The construction and properties of the functor $\Pi(-)$, and the properties of its inverse $\mathbb{V}(-)$ are stated in §III.4; the construction of $\mathbb{V}(-)$ is discussed in §III.5. The reformulation of this correspondence using deformation theory, due to Kisin [Kis10], is discussed in §III.6.

In addition to the in-text references, we are indebted to Berger's IHP course notes from 2010 [Ber] for a crisp introduction to the theory (φ, Γ)-modules, to Colmez's big paper [Col10] for a systematic exposition of the p-adic local Langlands correspondence, and to Breuil's Séminaire Bourbaki exposé [Bre12] for a quick and accessible introduction to the work of Colmez, Emerton and Kisin on this subject.

Notations for this chapter

- For finite extensions F of $\mathbf{Q}_p$, we will normalize the valuation on F so that $v_F(F^*) = \mathbf{Z}$ (by setting the valuation of uniformizers in F to be 1).
- Let L be a finite extension of $\mathbf{Q}_p$; let $\mathscr{O}_L$ be the ring of integers in L, ϖ a fixed uniformizer in $\mathscr{O}$. Write k_L for the residue field of L. We will sometimes supress the subscript L in $\mathscr{O}_L$ and k_L if no confusion is imminent. These, and more generally the ϖ-torsion $\mathscr{O}$-modules $\mathscr{O}/\varpi^n$, are to serve as the coefficients for representations of $\mathscr{G}_{\mathbf{Q}_p}$ and $\mathrm{GL}_2(\mathbf{Q}_p)$.

- Here are the various categories of representations of $\mathscr{G}_{\mathbf{Q}_p}$ that we shall work with:
  - $\operatorname{Rep}_{\mathrm{tors}} \mathscr{G}_{\mathbf{Q}_p}$, the category of continuous representations of $\mathscr{G}_{\mathbf{Q}_p}$ on $\mathscr{O}_L$-modules of finite length;
  - $\operatorname{Rep}_{\mathscr{O}_L} \mathscr{G}_{\mathbf{Q}_p}$, the category of continuous representations V of $\mathscr{G}_{\mathbf{Q}_p}$ on free $\mathscr{O}_L$-modules of finite rank such that $V/p^nV \in \operatorname{Rep}_{\mathrm{tors}} \mathscr{G}_{\mathbf{Q}_p}$ for all n; and
  - $\operatorname{Rep}_L \mathscr{G}_{\mathbf{Q}_p}$, the category of continuous representations V of $\mathscr{G}_{\mathbf{Q}_p}$ on finite dimensional L-vector spaces such that V admits a $\mathscr{G}_{\mathbf{Q}_p}$-invariant lattice $V_0 \in \operatorname{Rep}_{\mathscr{O}_L} \mathscr{G}_{\mathbf{Q}_p}$.
- Let $\chi : \mathscr{G}_{\mathbf{Q}_p} \to \mathbf{Z}_p^*$ denote the cyclotomic character of $\mathscr{G}_{\mathbf{Q}_p}$. For $\sigma \in \mathscr{G}_{\mathbf{Q}_p}$, the p-adic integer $\chi(\sigma)$ is determined by requiring

$$\sigma(\zeta_n) = \zeta_n^{\chi(\sigma) \bmod p^n}$$

  for all $n \geqslant 0$; here ζ_n is a primitive p^nth root of unity.
- In this chapter, we will write *p-adic representation of $\mathscr{G}_{\mathbf{Q}_p}$* to mean an object of one of the categories above; we will write *$\mathscr{O}_L$-representation of $\mathscr{G}_{\mathbf{Q}_p}$* to mean an object of $\operatorname{Rep}_{\mathrm{tors}} \mathscr{G}_{\mathbf{Q}_p}$ or $\operatorname{Rep}_{\mathscr{O}_L} \mathscr{G}_{\mathbf{Q}_p}$.
- For a p-adic representation V of $\mathscr{G}_{\mathbf{Q}_p}$, we write $\check{V}$ to denote the *Tate dual* of V given by

$$\check{V} = \begin{cases} \operatorname{Hom}_{\mathscr{O}_L}\left(V, (L/\mathscr{O}_L) \otimes \chi\right), & V \in \operatorname{Rep}_{\mathrm{tors}} \mathscr{G}_{\mathbf{Q}_p} \\ \operatorname{Hom}_{\mathscr{O}_L}\left(V, \mathscr{O}_L \otimes \chi\right), & V \in \operatorname{Rep}_{\mathscr{O}_L} \mathscr{G}_{\mathbf{Q}_p} \\ \operatorname{Hom}_L\left(V, L \otimes \chi\right), & V \in \operatorname{Rep}_L \mathscr{G}_{\mathbf{Q}_p} \end{cases}$$

- Here are the various categories of representations of $G = \mathrm{GL}_2(\mathbf{Q}_p)$ that we shall work with:
  - $\operatorname{Rep}_{\mathrm{tors}} G$, the category of smooth[1] admissible[2] representations Π of G on $\mathscr{O}_L$-modules of finite length, and such that Π admits a central character;

[1]this means that the action of G on Π is locally constant; i.e., every vector $v \in \Pi$ is fixed by an open compact subgroup in G.

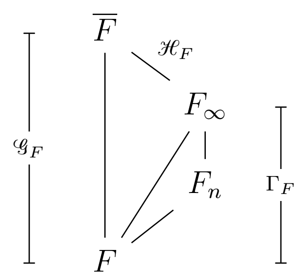
[2]this means that Π^K has finite length as an $\mathscr{O}_L$-module for every open compact subgroup K of G; this condition turns out to be superfluous in the presence of the other conditions defining the category $\operatorname{Rep}_{\mathrm{tors}} G$; cf. [BL94, Bre03a]

– $\mathrm{Rep}_{\mathcal{O}_L} G$, the category of representations of G on $\mathcal{O}_L$-modules Π that are separated, p-adically complete and p-torsionfree, and such that $\Pi/p^n\Pi \in \mathrm{Rep}_{\mathrm{tors}} G$ for all n; and

– $\mathrm{Rep}_L G$, the category of representations of G on L-vector spaces Π that admit a G-invariant $\mathcal{O}_L$-lattice $\Pi_0 \in \mathrm{Rep}_{\mathcal{O}_L} G$.

III.1: The field of norms construction

Let F be a finite extension of $\mathbf{Q}_p$. Let $F_n = F(\mu_{p^n})$ and $F_\infty = F(\mu_{p^\infty})$. Fontaine's classification of $\mathcal{G}_F$-representations over $\mathcal{O}$ (resp. L) relies on the *dévissage* suggested by the field diagram (with the corresponding Galois groups marked) below—



or, equivalently, the exact sequence

$$1 \to \mathcal{H}_F \to \mathcal{G}_F \to \Gamma_F \to 1.$$

The group Γ_F is a subgroup of finite index in $\mathbf{Z}_p^*$; so it is essentially pro-cyclic. A remarkable discovery of Fontaine-Wintenberger, via their field of norms construction, is that the group $\mathcal{H}_F$ is (abstractly) isomorphic to the Galois group of $\mathbf{F}_q(\!(T)\!)$ where $\mathbf{F}_q$ is the residue field of F_∞. Moreover, owing to the Frobenius map on $\mathbf{F}_q(\!(T)\!)$, Fontaine obtains a classification of $\mathcal{O}[\![\mathcal{G}_{\mathbf{F}_q(\!(T)\!)}]\!]$-modules in terms of modules over a Cohen ring for $\mathbf{F}_q(\!(T)\!)$ which comes equipped with a lift φ_q of the Frobenius[3], and is then able to classify $\mathcal{O}[\![\mathcal{G}_F]\!]$-modules in terms of (φ_q, Γ_F)-modules.

Let us now discuss the field of norms construction for infinite strictly arithmetically profinite (sAPF) extensions $K|F$ (v. [Win83])—while we make explicit use of this construction only for the infinite p-cyclotomic extension of F, we need the functoriality of this construction to explain the mysterious isomorphism between $\mathcal{H}_F$ and the Galois group of $\mathbf{F}_q(\!(T)\!)$. While the definition of an sAPF extension can be succintly stated using the upper ramification filtration for the extension $K|F$, and the associated Herbrand function $\psi_{K|F}$ as in loc.

[3]The Witt vector construction would provide Cohen rings for perfect residue fields, equipped with lift of Frobenius; in our case however, since $\mathbf{F}_q(\!(T)\!)$ is not perfect, Fontaine constructs the Cohen ring as a subring of $W(\mathbf{F}_q(\!(T)\!)^{\mathrm{perf},\wedge})$ where $\mathbf{F}_q(\!(T)\!)^{\mathrm{perf},\wedge}$ is the completion of the perfect closure of $\mathbf{F}_q(\!(T)\!)$.

cit., it is more intuitive to work with an equivalent[4] definition that explicates the tower of elementary subextensions in an sAPF extension:

Definition III.1. An infinite extension $K|F$ is said to be strictly arithmetically profinite (sAPF) if

- the field K can be be written as an increasing union $K = \bigcup_{n \geqslant -1} F_n$ of finite extensions F_n of F such that
  - $F_{-1} = F$,
  - $F_0|F$ is unramified, and
  - $F_1|F_0$ is totally ramified of degree prime to p (i.e., $F_1|F_0$ is tamely ramified)
  - $F_{n+1}|F_n$ is an extension of degree p^{r_n} for $n \geqslant 1$;
- letting ϖ_n be an uniformizer in F_n, the quantity $v_{F_n}(\sigma\varpi_n - \varpi_n) - 1$ is independent of the embedding $\sigma : F_n \to F^{\text{sep}}$ into a fixed separable closure F^{sep} of F, distinct from the inclusion;
- setting $i_0 = 0$ and $i_n = v_{F_n}(\sigma\varpi_n - \varpi_n) - 1$ for $n \geqslant 1$, the rational numbers i_n satisfy the following properties:
  - The numbers i_n are strictly increasing and tend to $+\infty$ as $n \to \infty$,
  - The numbers b_n given by
  $$b_n = \sum_{k=1}^{n} \frac{i_k - i_{k-1}}{p^{\sum_{j=1}^{k-1} r_j}}$$
  satisfy $\lim_{n\to\infty} b_n = +\infty$.

Example III.2.

1. The prototypical example to which we will apply this theory is the p-cyclotomic extension $F(\mu_{p^\infty})$ of the field F [Ben22, ¶6.1.2].

   In the special case that F is an unramified extension of $\mathbf{Q}_p$, one has that $F_{-1} = F_0 = F$ and $F_n = F(\mu_{p^n})$ for $n \geqslant 1$. It can be checked easily that $i_0 = 0$, $i_k = (p-1)p^{k-1}$ for $k \geqslant 1$, and $b_n = \frac{(p-1)^2(n-1)}{p} + (p-1)$.

2. Let $K|F$ be an sAPF extension. Then, a finite extension M of K is also an sAPF extension of F (cf. [Win83, ¶1.2.3]).

[4] We do not rely on this equivalence here but see [Win83, §1] (especially ¶1.4.2 in loc. cit.).

Let $\mathscr{E}(K|F)$ denote the directed set of finite extensions of F contained in K. We set

$$\mathscr{X}(K|F) = \varprojlim_{E \in \mathscr{E}(K|F)} E \tag{III.1.1}$$

with the transition maps $E'' \to E'$ given by the norm map $N_{E''|E'}$. In words, $\mathscr{X}(K|F)$ is the set of all norm-compatible sequences in K. This construction is called the *field of norms* of the extension $K|F$ after the following theorem of Fontaine-Wintenberger (cf. [Fon72, Win83]).

Theorem III.3. *Let $K|F$ be an sAPF extension. Let $\alpha = (\alpha_E)_E$ and $\beta = (\beta_E)_E$ be two elements in $\mathscr{X}(K|F)$.*

1. *For each finite extension $E \in \mathscr{E}(K|F)$, the limit*

$$\lim_{E' \in \mathscr{X}(K|E)} N_{E'|E}(\alpha_E + \beta_E)$$

exists in E. Denoting this limit by s_E, the sequence $(s_E)_E$ is a norm-compatible sequence in K. That is, for finite extensions $F \subset E' \subset E''$ of F contained in K, one has

$$N_{E''|E'}(s_{E''}) = s_{E'}.$$

2. *With the sum $\alpha + \beta$ and the product $\alpha\beta$ defined by*

$$\alpha + \beta = (s_E)_E$$
$$\alpha \cdot \beta = (\alpha_E \beta_E)_E$$

the set $\mathscr{X}(K|F)$ is a field of characteristic p.

3. *The field K is an sAPF extension of F_1 and the fields $\mathscr{X}(K|F)$ and $\mathscr{X}(K|F_1)$ are canonically isomorphic. The map*

$$k_F \to \mathscr{X}(K|F_1)$$
$$x \mapsto ([x^{[E:F_1]^{-1}}])_E$$

is an isomorphism of k_F onto the residue field of $\mathscr{X}(K|F_1)$. Here, $[x] \in F$ is the Teichmüller lift of $x \in k_F$; recall that Teichmüller representatives have distinguished p-power roots so our element $[x]^{[E:F_1]^{-1}}$ is unambiguously defined.

4. *If M/K is a finite extension, then M is an sAPF extension of F and the field $\mathscr{X}(M|F)$ is a finite separable extension of $\mathscr{X}(K|F)$. For an arbitary algebraic separable extension*

M of K, set

$$\mathscr{X}(M|F) = \varinjlim_{M' \in \mathscr{E}(M/K)} \mathscr{X}(M'|F)$$

The functor

$$\left\{ \begin{array}{c} \textit{algebraic separable} \\ \textit{extensions of } K \end{array} \right\} \to \left\{ \begin{array}{c} \textit{algebraic separable} \\ \textit{extensions of } \mathscr{X}(K|F) \end{array} \right\}$$
$$M \mapsto \mathscr{X}(M|F)$$

induces an equivalence of categories which preserves the Galois correspondence.

As a consequence of this theorem, we see that the absolute Galois group of the fields K (a field in mixed characteristic $(0, p)$) and $\mathscr{X}(K|F)$ (a field in characteristic p) are isomorphic! Applying this to $K = F_\infty$, we see that the group $\mathscr{H}_F$ is isomorphic to the Galois group of $\mathscr{X}(F_\infty|F)$!

For our purposes, it is useful to have a construction of a "canonical, functorial" complete perfect field containing $\mathscr{X}(K|F)$ for an sAPF extension $K|F$. To this effect, let's first discuss a general construction, referred to as Fontaine's functor $\mathrm{R}(-)$ in the pre-perfectoid literature and as the "tilting" construction in the modern literature.

Theorem III.4. *Let K be an extension of $\mathbf{Q}_p$ complete for the valuation v (normalized so that $v(p) = 1$); suppose that the residue field k_K is perfect. Let $[-] : k_K \to K$ denote the Teichmüller lift. Let $K^\flat$ denote the set of p-power compatible sequences in K, i.e.,*

$$K^\flat = \varprojlim_{x \mapsto x^p} K.$$

Then:

1. *$K^\flat$ is a perfect field of characteristic p for addition and multiplication laws given by*

$$(x_n) + (y_n)_n = ((x+y)_n)_n \qquad \text{where} \qquad (x+y)_n = \lim_{m \to \infty} (x_{n+m} + y_{n+m})^{p^m};$$
$$(x_n) \cdot (y_n)_n = ((xy)_n)_n \qquad \text{where} \qquad (xy)_n = x_n y_n.$$

2. *The field $K^\flat$ is complete for the valuation given by $v^\flat((x_n)_n) = v(x_0)$.*

3. *The map*

$$k_K \to K^\flat$$
$$x \mapsto ([x^{p^{-n}}])_n$$

is an isomorphism of k_K onto the residue field of the ring $\mathcal{O}_{K^\flat}$ of integers in $K^\flat$.

Proof. See Théorème 4.1.2 and ¶4.1.4 in [Win83]. □

We wish to use the tilting correspondence for perfectoid fields [Sch12, §3] and its compatiblity with the field of norms construction [Win83, §4.3]. Let us now recall the definition of perfectoid fields:

Definition III.5. A field K equipped with an absolute value $|\cdot|_K : K \to \mathbf{R}_+$ such that $|p|_K < 1$ is said to be *perfectoid* if

1. $|K|_K$ is non-discrete;
2. K is complete for $|\cdot|_K$; and
3. the Frobenius map $x \mapsto x^p$ on $\mathcal{O}_K/p\mathcal{O}_K$ is surjective.

For example, the field $\widehat{F}_\infty$ is perfectoid. The tilting construction for perfectoid fields preserves the Galois correspondence and hence induces an isomorphism of Galois group of a perfectoid field with that of its tilt:

Theorem III.6. *Let K be a perfectoid field of characteristic 0. Fix an algebraic closure K^{sep} of K and denote by $\mathbf{C}_K$ the completion of K^{sep}. Then the tilting functor $E \mapsto E^\flat$ preserves the Galois correspondence and induces an isomorphism $\mathrm{Gal}(K^{\mathrm{sep}}|K) \to \mathrm{Gal}(K^{\flat,\mathrm{sep}}|K^\flat)$; moreover, writing $\mathbf{C}_{K^\flat}$ for the completion of an algebraic closure of $K^\flat$, one has that $\mathbf{C}_{K^\flat} = \mathbf{C}_K^\flat$.*

Proof. A proof is offered at the end of Section 5 in [Sch12]. A somewhat more elementary proof is offered in the book by Fargues and Fontaine on their eponymous curve [FF18, Théorème 3.2.1]. We also refer the reader to the exposition in [Ben22, Chapter 5]. □

We now have the theorem due to Fontaine-Wintenberger [Win83] that constructs an embedding of the field of norms $\mathscr{X}(K|F)$ of an sAPF extension $K|F$ into the tilt of the completion of K identifying the completion of the perfect closure of the field of norms with $\widehat{K}^\flat$:

Theorem III.7. *Let $K|F$ be an sAPF extension; recall that $F_1|F$ is the maximal tamely ramified extension of F contained in K. Let $\mathscr{E}_n$ denote the set of all finite extensions E of F contained in K such that $p^n \mid [E : F_1]$. Let $(\alpha_E)_E \in \mathscr{X}(K|F)$. For every n, the family*

$(\alpha_E^{p^{-n}[E:F_1]})_{E\in\mathscr{E}_n}$ converges to an element $x_n \in \widehat{K}$ and the sequence $(x_n)_n$ defines an element in $\widehat{K}^\flat$. Moreover, the map

$$\begin{aligned}\Lambda_{K|F} : \mathscr{X}(K|F) &\to \widehat{K}^\flat \\ (\alpha_E)_E &\mapsto (x_n)_n\end{aligned}$$

is a continuous embedding, identifying $\widehat{K}^\flat$ as the completion of the inseparable closure of $\mathscr{X}(K|F)$.

Proof. This is Corollaire 4.3.4 in [Win83]. □

We are now ready to introduce the various *anneaux gnomiques*[5] that underly the theory of (φ, Γ)-modules.

Definition III.8. Recall that F is a finite extension of $\mathbf{Q}_p$; let $F_\infty = F(\mu_{p^\infty})$ denote the infinite p-cyclotomic extension of F. Fix an algebraic closure F^{sep} of F and a completion $\mathbf{C}_F$ of the algebraic closure of F.

1. Let $\widetilde{\mathbf{E}}_F$ denote the field $\widehat{F}_\infty^\flat$. Note that $\widetilde{\mathbf{E}}_F$ admits an action of Γ_F, and a Frobenius automorphism (like any perfect field of characteristic p).

2. Let $\widetilde{\mathbf{E}}$ denote the field $\mathbf{C}_F^\flat$. Then, by[6] Theorem III.6, $\widetilde{\mathbf{E}}$ is the completion of the algebraic closure of $\widetilde{\mathbf{E}}_F$, the Galois group $\mathscr{G}_F$ of the extension $F^{\text{sep}}|F$ acts on the $\widetilde{\mathbf{E}}$ and moreover $\widetilde{\mathbf{E}}^{\mathscr{H}_F} = \widetilde{\mathbf{E}}_F$.

3. Let $\mathbf{E}_F$ denote the image of field $\mathscr{X}(F_\infty|F)$ in $\widetilde{\mathbf{E}}_F$.

4. Let $\mathbf{E}$ denote the separable closure of $\mathbf{E}_{\mathbf{Q}_p}$ in $\widetilde{\mathbf{E}}$. By Theorem III.3, the field $\mathbf{E}$ is the union

$$\mathbf{E} = \bigcup_M \mathbf{E}_M$$

as M ranges over finite extensions of $\mathbf{Q}_p$; and the Galois group of $\mathbf{E}$ over $\mathbf{E}_F$ is the group $\mathscr{H}_F$.

5. Let $\widetilde{\mathbf{A}} = W(\widetilde{\mathbf{E}})$. Then $\widetilde{\mathbf{A}}$ is a complete discrete valuation ring with residue field $\widetilde{\mathbf{E}}$ and comes equipped with a lift of the Frobenius on $\widetilde{E}_{\mathbf{Q}_p}$. To wit, every element in $\widetilde{\mathbf{A}}$ may

[5] We owe the bilingual pun to Colmez' survey article [Col19b, §§2.3.3].

[6] We note that F, being a finite extension of $\mathbf{Q}_p$, is not perfectoid; but the field $\widehat{F}_\infty$ is, and we have $\mathbf{C}_{\widehat{F_\infty}} = \mathbf{C}_F$ (with the obvious meaning for $\mathbf{C}_{\widehat{F_\infty}}$).

be written uniquely as $\sum_{j=0}^{\infty} p^j[x_j]$ with $x_j \in \widetilde{\mathbf{E}}$ where $[-]$ is the Teichmüller lift, and the Frobenius endomorphism on $\widetilde{\mathbf{A}}$ in this presentation is given as

$$\varphi\left(\sum_{j=0}^{\infty} p^j[x_j]\right) = \sum_{j=0}^{\infty} p^j[x_j^p].$$

6. Let $\varepsilon = (\zeta_n)_n$ be a sequence of p-power roots of unity with $\zeta_1 \neq 1$, and $\zeta_{n+1}^p = \zeta_n$ for all $n \geqslant 1$. Then ε defines a well-defined element in $\widetilde{\mathbf{E}}$. Set $T = [\varepsilon] - 1 \in \widetilde{\mathbf{A}}$. The action of φ on T is given by $(1+T)^p - 1$.

7. We recall from the construction of Witt vectors[7] that there is a bijection of sets

$$\widetilde{\mathbf{A}} = W(\widetilde{\mathbf{E}}) \to \widetilde{\mathbf{E}}^{\mathbf{N}} \tag{III.1.2}$$

and that $\widetilde{\mathbf{E}}$ is complete for the valuation $v^\flat$ (v. 2 of Theorem III.4). There are, therefore, two topologies on $\widetilde{\mathbf{A}}$:

   - The *strong topology* on $\widetilde{\mathbf{A}}$ is that obtained by transport of structure via the bijection (III.1.2) when $\widetilde{\mathbf{E}}$ is endowed with with the discrete topology, and the product $\widetilde{\mathbf{E}}^{\mathbf{N}}$ with the product topology. A neighborhood basis of 0 for this topology is given by $\{p^j\widetilde{\mathbf{A}}\}_{j\in\mathbf{Z}_{\geqslant 0}}$.
   - The *weak topology* on $\widetilde{\mathbf{A}}$ is that obtained by transport of structure via the bijection (III.1.2) when $\widetilde{\mathbf{E}}$ is endowed with with the $v^\flat$-adic topology, and the product $\widetilde{\mathbf{E}}^{\mathbf{N}}$ with the product topology. A neighborhood basis of 0 for this topology is given by $\{T^kW(\widetilde{\mathbf{E}}^+) + p^{n+1}\widetilde{\mathbf{A}}\}_{k,n\in\mathbf{Z}_{\geqslant 0}}$ where $\widetilde{\mathbf{E}}^+$ is the $v^\flat$-adic valuation ring in $\widetilde{\mathbf{E}}$.

8. The action of $\mathscr{G}_F$ on $\widetilde{\mathbf{E}}$ extends to a continuous action of $\mathscr{G}_F$ on $\widetilde{\mathbf{A}}$ (for its *weak* topology). The action of φ and $\mathscr{G}_F$ on T is given by

$$\varphi(T) = (1+T)^p - 1 \text{ and } \sigma(T) = (1+T)^{\chi_F(\sigma)} - 1$$

where χ_F is the cyclotomic character of $\mathscr{G}_F$.

9. Let $\mathbf{A}_F$ denote the closure of $\mathscr{O}_F[T, T^{-1}]$ in $\widetilde{\mathbf{A}}$ for the p-adic topology (the strong

[7] For the construction of Witt vectors, and the topological considerations in the next point, please see §6 and §16 of [Ber]; see also §1.5 of [Sch17].

topology); this may be described as the ring

$$\left\{ \sum_{k \in \mathbf{Z}} a_k T^k \,\middle|\, \begin{array}{c} a_k \in \mathscr{O}_F, \\ v_p(a_k) \to \infty \text{ as } k \to -\infty \end{array} \right\}.$$

Here v_p is the valuation on F normalized so that $v_p(p) = 1$. Note that $\mathbf{A}_F$ is stable under the action of φ and $\mathscr{G}_F$.

10. Let $\widetilde{\mathbf{B}} = \widetilde{\mathbf{A}}[\frac{1}{p}]$; this is the field of fractions of the valuation ring $\widetilde{\mathbf{A}}$.

11. Let $\mathbf{B}$ denote the completion of the maximal unramified extension of $\mathbf{B}_F$ contained in $\widetilde{\mathbf{B}}$. For a finite extension K of F, let $\mathbf{B}_K$ be the unramified extension of $\mathbf{B}_F$ with residue field $\mathbf{E}_K$; one then obtain $\mathbf{B}$ as the completion of the union of the fields $\mathbf{B}_K$ in $\widetilde{\mathbf{B}}$. The field $\mathbf{B}$ is stable under φ and the action of $\mathscr{G}_F$ on $\widetilde{\mathbf{B}}$. Conversely, the fields $\mathbf{B}_K$ can be recovered from $\mathbf{B}$ using the Galois action: namely, $\mathbf{B}_K = \mathbf{B}^{\mathscr{H}_K}$.

12. Set $\mathbf{A} = \mathbf{B} \cap \widetilde{\mathbf{A}}$; then $\mathbf{A}$ is a discrete valuation ring with field of fractions of $\mathbf{B}$ and residue field $\mathbf{E}$. The field $\mathbf{A}$ is stable under φ and the action of $\mathscr{G}_F$ on $\widetilde{\mathbf{B}}$. Note that $\mathbf{A}^{\varphi=1} = \mathscr{O}_F$ and $\mathbf{B}^{\varphi=1} = F$.

While we have introduced what appears to be a menagerie of rings, the symbology for these rings has a system to it—

- the rings in characteristic p go by flavors of $\mathbf{E}$; the ones with $\widetilde{\cdot}$ over them are completion of the perfect closure of those without it; the ones with a subscript are unramified extensions ordered by the natural inclusion among the subscript.

- the "integral" rings in characteristic 0 go by flavors of $\mathbf{A}$; ones with $\widetilde{\cdot}$ over them have perfect residue field, and

- the fields in characteristic 0 go by flavors of $\mathbf{B}$; they are obtained by inverting p in the corresponding $\mathbf{A}$-version.

We are almost exclusively interested in the case $F = \mathbf{Q}_p$ (though the development of this case with some details would have amounted to similar amount of labor). We specialize to this case for the rest of this thesis.

III.2: (φ, Γ)-modules

The anneau-nomics of the last section, with $F = \mathbf{Q}_p$, gives us a ring $\mathbf{A}$ that satisfies the following properties:

- $\mathbf{A}$ is a discrete valuation ring with residue field $\mathbf{E}$ and is equipped with an endomorphism φ and a continuous action of $\mathscr{G}_{\mathbf{Q}_p}$ that commutes with φ;
- $(\mathscr{O}_L \cdot \mathbf{A})^{\varphi=1} = \mathscr{O}_L$; and
- the ring $(\mathscr{O}_L \cdot \mathbf{A})^{\mathscr{H}_{\mathbf{Q}_p}}$ and the field $(\mathscr{O}_L \cdot \mathbf{A})^{\mathscr{H}_{\mathbf{Q}_p}}$ have explicit description as subrings of the ring of bi-infinite formal power series with coefficients in L (cf. 9 of Definition III.8); moreover they support an endomorphism φ and a residual action of $\Gamma_{\mathbf{Q}_p} = \mathbf{Z}_p^*$.

We begin by lightening the notation and making some of these objects explicit.

Let $\mathscr{E}$ denote[8] the field

$$\mathscr{E} := \left\{ \sum_{k\in\mathbf{Z}} a_k T^k \;\middle|\; \begin{array}{c} a_k \in L, \\ (v_p(a_k))_{k\in\mathbf{Z}} \text{ bounded below}, \\ v_p(a_k) \to \infty \text{ as } k \to -\infty \end{array} \right\}.$$

The field $\mathscr{E}$ is complete for the discrete valuation $v^{\{0\}}$ defined by

$$v^{\{0\}}\left(\sum_{k\in\mathbf{Z}} a_k T^k\right) = \inf_{k\in\mathbf{Z}} v_p(a_k).$$

Let $\mathscr{O}_{\mathscr{E}}$ denote[9] the ring of integers in $\mathscr{E}$; it consists of those elements of $\mathscr{E}$ with coefficients in $\mathscr{O}_L$

$$\mathscr{O}_{\mathscr{E}} = \left\{ \sum_{k\in\mathbf{Z}} a_k T^k \;\middle|\; \begin{array}{c} a_k \in \mathscr{O}_L, \\ v_p(a_k) \to \infty \text{ as } k \to -\infty \end{array} \right\}.$$

The residue field $k_{\mathscr{E}}$ of $\mathscr{E}$ is $k_L(\!(T)\!)$.

We will study $\mathscr{O}_{\mathscr{E}}$ not with the topology defined by the valuation $v^{\{0\}}$ (the *strong topology*), but instead the *weak topology*—the coarsest topology that renders the natural reduction map $\mathscr{O}_{\mathscr{E}} \to k_{\mathscr{E}}$ continuous, where $k_{\mathscr{E}}$ is given the T-adic topology. To describe this weak topology, let us set

$$\mathscr{O}_{\mathscr{E}}^+ = \mathscr{O}_L[\![T]\!], \qquad \mathscr{E}^+ = \mathscr{O}_{\mathscr{E}}^+\left[\frac{1}{p}\right], \qquad \text{and} \qquad k_{\mathscr{E}}^+ = k_L[\![T]\!].$$

The set of open ideals $\{p^k\mathscr{O}_{\mathscr{E}} + T^n\mathscr{O}_{\mathscr{E}}^+ : k, n \in \mathbf{N}\}$ is a neighbourhood basis of 0 for the weak topology on $\mathscr{O}_{\mathscr{E}}$. (See 7 of Definition III.8.) The field $\mathscr{E}$, being the increasing union $\bigcup_{m\in\mathbf{N}} p^{-m}\mathscr{O}_{\mathscr{E}}$, is then given the inductive limit topology.

[8] We will sometimes write $\mathscr{E}_L$ if we wish to emphasize that the coefficients of the formal Laurent series in $\mathscr{E}$ come from L. This is the field $L \cdot \mathbf{B}_{\mathbf{Q}_p}$ of the last section.

[9] This is the ring $\mathscr{O}_L \cdot \mathbf{A}_{\mathbf{Q}_p}$ of the last section.

Definition III.9. The topological field $\mathscr{E}$ is called the *Fontaine field*; it comes equipped with a lift φ of Frobenius on $k_{\mathscr{E}_{\mathbf{Q}_p}}$ given by $\varphi(T) = (1+T)^p - 1$ and extended to $\mathscr{E}$ so as to be L-linear and continuous. There is an L-linear continuous action of $\Gamma = \mathbf{Z}_p^\times$ on $\mathscr{E}$ given by

$$\sigma_a(T) = (1+T)^a - 1 = \sum_{n=1}^{\infty} \binom{a}{n} T^n.$$

Note that the actions of φ and Γ commute with each other, and preserve $\mathscr{O}_{\mathscr{E}}$.

The field $\mathbf{B}$ of the last section is just the completion of the maximal unramified extension of $\mathscr{E}_{\mathbf{Q}_p}$ and the ring $\mathbf{A}$ is just the ring of integers in $\mathbf{B}$. The rings $\mathscr{O}_{\mathscr{E}}$ and $\mathscr{E}$ serve as base rings for (φ, Γ)-modules while the rings $\mathbf{A}$ and $\mathbf{B}$ will feature in Fontaine's equivalence between the category of (φ, Γ)-modules and p-adic Galois representations.

Definition III.10. Let A be any ring equipped with an endomorphism $\varphi : D \to D$ and an action of $\Gamma = \mathbf{Z}_p^\times$ that commutes with φ. (For us, the ring A is either $\mathscr{O}_{\mathscr{E}}$, or $\mathscr{E}$.)

1. A (φ, Γ)-module D over A is an A-module of finite type equipped with an A-semilinear endomorphism φ and an A-semilinear action of Γ that commute with each other.

2. A (φ, Γ)-module D over $\mathscr{O}_{\mathscr{E}}$ is said to be *étale* if $\varphi(D)$ generates D as an $\mathscr{O}_{\mathscr{E}}$-module. A (φ, Γ)-module D over $\mathscr{E}$ is said to be étale if there exists an $\mathscr{O}_{\mathscr{E}}$-lattice Δ which is stable under φ and Γ such that Δ is étale as an $\mathscr{O}_{\mathscr{E}}$-module.

The following categories of étale (φ, Γ)-modules will be central in our considerations, in view of their equivalence to categories of p-adic Galois representations (v. Theorem III.13):

- the category $\Phi\Gamma^{\text{ét}}_{\text{tors}}$ of étale (φ, Γ)-modules of finite length over $\mathscr{O}_{\mathscr{E}}$;
- the category $\Phi\Gamma^{\text{ét}}(\mathscr{O}_{\mathscr{E}})$ of étale (φ, Γ)-modules that are free of finite rank over $\mathscr{O}_{\mathscr{E}}$; and
- the category $\Phi\Gamma^{\text{ét}}(\mathscr{E})$ of étale (φ, Γ)-modules that are free of finite dimension over $\mathscr{E}$.

III.2.1: The left inverse ψ of φ for an étale (φ, Γ)-module

Let D be an étale (φ, Γ)-module over $\mathscr{O}_{\mathscr{E}}$. Then, every element $x \in D$ can be written uniquely as

$$x = \sum_{i=0}^{p-1} (1+T)^i \varphi(x_i)$$

for some $x_i \in D$; the operator $\psi : D \to D$ defined by setting

$$\psi\left(\sum_{i=0}^{p-1} (1+T)^i \varphi(x_i)\right) = x_0$$

satisfies the following properties:

(i) ψ is $\mathscr{O}$-linear and is a left inverse to φ;

(ii) $\psi(\varphi(a)x) = a\psi(x)$ for $a \in \mathscr{O}_{\mathscr{E}}$ and $x \in D$;

(iii) $\psi(a\varphi(x)) = \psi(a)x$ for $a \in \mathscr{O}_{\mathscr{E}}$ and $x \in D$; and

(iv) ψ is Γ-equivariant.

III.2.2: The Tate dual of a (φ, Γ)-module

Let $\Omega^1_{\mathscr{O}_{\mathscr{E}}}$ denote the $\mathscr{O}_{\mathscr{E}}$-module of continuous $\mathscr{O}_L$-differentials of $\mathscr{O}_{\mathscr{E}}$. Then, $\Omega^1_{\mathscr{O}_{\mathscr{E}}}$ is a free module of rank 1 generated by dT or $\frac{dT}{1+T}$; it has the structure of an *étale* (φ, Γ)-module over $\mathscr{O}_{\mathscr{E}}$ via

$$\sigma_a\left(\frac{dT}{1+T}\right) = a\frac{dT}{1+T}, \qquad a \in \mathbf{Z}_p^* \qquad \text{and} \qquad \varphi\left(\frac{dT}{1+T}\right) = \frac{dT}{1+T}.$$

We can now define the *Tate dual* of a (φ, Γ)-module:

Definition III.11. The *Tate dual* of an étale (φ, Γ)-module D is defined as

$$\check{D} = \begin{cases} \mathrm{Hom}_{\mathscr{O}_{\mathscr{E}}}\left(D, (\mathscr{E}/\mathscr{O}_{\mathscr{E}})\frac{dT}{1+T}\right), & D \in \Phi\Gamma^{\text{ét}}_{\text{tors}} \\ \mathrm{Hom}_{\mathscr{O}_{\mathscr{E}}}\left(D, \mathscr{O}_{\mathscr{E}}\frac{dT}{1+T}\right), & D \in \Phi\Gamma^{\text{ét}}(\mathscr{O}_{\mathscr{E}}) \\ \mathrm{Hom}_{\mathscr{E}}\left(D, \mathscr{E}\frac{dT}{1+T}\right), & D \in \Phi\Gamma^{\text{ét}}(\mathscr{E}) \end{cases}.$$

III.2.3: Examples

Many naturally arising p-adic analytic objects turn out to be examples of (φ, Γ)-modules. Conversely the coefficient rings for (φ, Γ)-modules introduced above have nice p-adic analytic interpretations. This idea, together with the insights of Berger-Breuil [BB10], contains the germ of the construction of the functor $\Pi(-)$ that realizes the p-adic local Langlands correspondence (see, for example, Remarque II.1.1 in [Col10] and Remark 6 in [Dos12]). So we include this discussion here.

To minimize repetition, let M be a topological $\mathscr{O}_L$-module; we will need to take $M = \mathscr{O}_L$ and $M = L$ to interpret the rings $\mathscr{O}^+_{\mathscr{E}}, \mathscr{O}_{\mathscr{E}}, \mathscr{E}^+$, and $\mathscr{E}$.

1. Let $\mathscr{C}^0(\mathbf{Z}_p, M)$ denote the space of M-valued continuous functions on $\mathbf{Z}_p$. The space $\mathscr{C}^0(\mathbf{Z}_p, \mathscr{O}_L)$ is an orthonormalizable $\mathscr{O}_L$-module (v. Definition III.18 (2)); Mahler's theorem states that $\mathscr{C}^0(\mathbf{Z}_p, \mathscr{O}_L)$ is the completion of the free $\mathscr{O}_L$-module generated by the

$\mathscr{O}_L$-linearly independent subset $\{\binom{x}{n} : n \geqslant 0\}$ (v. Théorème I.1.3 of [Col10]). The L-vector space $\mathscr{C}^0(\mathbf{Z}_p, L)$ is an L-Banach space, being complete for the norm

$$\|f\| = \sup_{x \in \mathbf{Z}_p} |f(x)|;$$

in paticular, the $\mathscr{O}_L$-module $\mathscr{C}^0(\mathbf{Z}_p, \mathscr{O}_L)$ is the unit ball in $\mathscr{C}^0(\mathbf{Z}_p, L)$.

2. Let $\mathscr{D}_0(\mathbf{Z}_p, M)$ denote the $\mathscr{O}_L$-module consisting of continuous $\mathscr{O}_L$-linear homomorphisms $\mathscr{C}^0(\mathbf{Z}_p, \mathscr{O}_L) \to M$. For $M = \mathscr{O}_L$ or L, we endow the space $\mathscr{D}_0(\mathbf{Z}_p, M)$ with the topology given by the norm

$$\|\mu\| = \sup_{f \in \mathscr{C}^0(\mathbf{Z}_p, M) \setminus \{0\}} \frac{|\mu(f)|}{\|f\|}.$$

Lemma III.12 (Théorème I.1.4 in [Col10])**.**
For $f \in \mathscr{E}$, define the L-valued function $\phi_f : \mathbf{Z}_p \to L$ by

$$\phi_f(x) = \mathrm{res}_0\left((1+T)^x f(T) \frac{dT}{1+T}\right).$$

For $\mu \in \mathscr{D}_0(\mathbf{Z}_p, M)$, define its Amice transform *by*

$$A_\mu(T) = \int_{\mathbf{Z}_p} (1+T)^x \mu = \sum_{n=0}^{\infty} \left(\int_{\mathbf{Z}_p} \binom{x}{n} \mu\right) T^n.$$

Then, the map A induces isomorphisms

$$\mathscr{D}_0(\mathbf{Z}_p, \mathscr{O}_L) \cong \mathscr{O}_{\mathscr{E}}^+ \qquad \textit{and} \qquad \mathscr{D}_0(\mathbf{Z}_p, L) \cong \mathscr{E}^+$$

and together with ϕ assembles into the following exact sequences:

$$0 \to \mathscr{D}_0(\mathbf{Z}_p, \mathscr{O}_L) \xrightarrow{A} \mathscr{O}_{\mathscr{E}} \xrightarrow{\phi} \mathscr{C}^0(\mathbf{Z}_p, \mathscr{O}_L) \to 0$$
$$0 \to \mathscr{D}_0(\mathbf{Z}_p, L) \xrightarrow{A} \mathscr{E} \xrightarrow{\phi} \mathscr{C}^0(\mathbf{Z}_p, L) \to 0$$

III.3: Fontaine's equivalences

We are now ready to state the theorem due to Fontaine that classifies p-adic representations of $\mathscr{G}_{\mathbf{Q}_p}$ in terms of semi-linear algebraic data.

Theorem III.13.

1. *If D is an étale (φ,Γ)-module, then $\mathbb{V}(D) = ((\mathscr{O}_L \cdot \mathbf{A}) \otimes_{\mathscr{O}_{\mathscr{E}}} D)^{\varphi=1}$ is a p-adic representation of $\mathscr{G}_{\mathbf{Q}_p}$.*

2. *If V is a p-adic representation of $\mathscr{G}_{\mathbf{Q}_p}$, then $\mathbb{D}(V) = \left(\mathbf{A} \otimes_{\mathbf{Z}_p} V\right)^{\mathscr{H}}$ is an étale (φ,Γ)-module.*

3. *The functors $\mathbb{V}$ and $\mathbb{D}$ are exact, inverses of each other and induce the following equivalences of categories:*

$$\mathrm{Rep}_{\mathrm{tors}}\,\mathscr{G}_{\mathbf{Q}_p} \leftrightarrow \Phi\Gamma^{\acute{e}t}_{\mathrm{tors}}, \qquad \mathrm{Rep}_{\mathscr{O}_L}\,\mathscr{G}_{\mathbf{Q}_p} \leftrightarrow \Phi\Gamma^{\acute{e}t}(\mathscr{O}_{\mathscr{E}}), \qquad \mathrm{Rep}_L\,\mathscr{G}_{\mathbf{Q}_p} \leftrightarrow \Phi\Gamma^{\acute{e}t}(\mathscr{E}).$$

4. *The functors $\mathbb{V}$ and $\mathbb{D}$ commute with taking Tate duals.*

III.3.1: (φ,Γ)-modules of dimension 1

Let $\delta : \mathbf{Q}_p^\times \to \mathscr{O}^\times$. Via local class field theory (normalized so that the geometric frobenius maps to p), we get a $\mathscr{G}_{\mathbf{Q}_p}$-representation $T(\delta)$ of rank 1. Explicitly, $T(\delta)$ is the Galois character $\sigma \mapsto \mathrm{unr}^{-1}_{\delta(p)}\,\delta(\chi(\sigma))$ where unr_λ is the unramified character that sends the geometric frobenius to λ.

Consider the $\mathscr{O}_{\mathscr{E}}$-module $\mathscr{O}_{\mathscr{E}}(\delta) = \mathscr{O}_{\mathscr{E}}\, e_\delta$ of rank 1 on which φ and $\gamma \in \Gamma$ act by the semilinear extensions of $\varphi(e_\delta) = \delta(p)e_\delta$ and $\gamma(e_\delta) = \delta(\chi(\gamma))e_\delta$. It can be checked that

$$\mathbb{D}(T(\delta)) = \mathscr{O}_{\mathscr{E}}(\delta).$$

Fontaine's equivalence implies, moreover, that every (φ,Γ)-module over $\mathscr{O}_{\mathscr{E}}$ of rank 1 is of the form $\mathscr{O}_{\mathscr{E}}(\delta)$ for a continuous $\mathscr{O}^\times$-valued valued character of $\mathbf{Q}_p^\times$.

III.4: The p-adic local Langlands correspondence

Let C be an algebraically closed field of characteristic 0 endowed with the discrete topology. Recall that the classical local Langlands correspondence is a bijection between the set of isomorphism classes of semisimple Weil-Deligne representations of $\mathbf{Q}_p$ of dimension 2 over C and the set of isomorphism classes of smooth (admissible) irreducible representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ over C such that the L-functions and the ε-factors of correspondents match.

On the other hand, the category of p-adic representations (i.e., L-vector spaces endowed with continuous $\mathscr{G}_{\mathbf{Q}_p}$-action where L is given its p-adic topology *as opposed to the discrete topology*) is much richer.

However, Fontaine [Fon94], and Colmez and Fontaine [CF00] have shown that there is an equivalence between the category of potentially semistable representations of $\mathscr{G}_{\mathbf{Q}_p}$ with coefficients in L and that of weakly admissible filtered $(\varphi, N, \mathscr{G}_{\mathbf{Q}_p}, L)$-modules; this equivalence is mediated by a certain period ring $\mathrm{B}_{\mathrm{pst}} = \bigcup_L \mathrm{B}_{\mathrm{st},K}$ of Fontaine. While we won't need more about the period ring $\mathrm{B}_{\mathrm{pst}}$, let us describe this semi-linear algebraic data briefly. A filtered $(\varphi, N, \mathscr{G}_{\mathbf{Q}_p}, L)$-module over L is a $\mathbf{Q}_p^{\mathrm{unr}} \otimes L$-module D equipped with the following structures:

- a *Frobenius endomorphism* $\varphi : D \to D$ that is L-linear and $\mathbf{Q}_p^{\mathrm{unr}}$-semilinear;
- a nilpotent $\mathbf{Q}_p^{\mathrm{unr}} \otimes L$-linear *monodromy operator* $N : D \to D$ that satisfies $N\varphi = p\varphi N$;
- an $\mathbf{Q}_p^{\mathrm{unr}}$-semi-linear, and L-linear action of $\mathscr{G}_{\mathbf{Q}_p}$ that is continuous for the discrete topology; and
- a decreasing, separated, exhaustive filtration of $\overline{D} = \overline{\mathbf{Q}}_p \otimes_{\mathbf{Q}_p^{\mathrm{unr}}} D$ by $\overline{\mathbf{Q}}_p$-vector spaces that are stable under the action of $\mathscr{G}_{\mathbf{Q}_p}$.

We omit the definition of weak admissibility (or, what amounts to the same after Colmez-Fontaine (loc. cit.), admissibility) and just refer the reader to [Fon94, §4.4] and [BM02, Definition 3.1.1.1].

For example, there are two distinguished "families" of two-dimensional weakly admissible filtered (φ, N)-modules cut out using p-adic Hodge theoretic conditions. It is known that potentially semi-stable representations of $\mathscr{G}_{\mathbf{Q}_p}$ are Hodge-Tate; after a suitable Tate twist, we may suppose that the Hodge-Tate weights are $(0, k-1)$ for an integer $k \geqslant 1$. The potentially semistable representations of $\mathscr{G}_{\mathbf{Q}_p}$ that are irreducible and crystalline correspond to filtered $(\varphi, N, \mathscr{G}_{\mathbf{Q}_p}, L)$-modules over L in which the filtration is determined by k—

$$D = \overline{\mathbf{Q}}_p e_1 \oplus \overline{\mathbf{Q}}_p e_2$$

$$\varphi = \begin{pmatrix} 0 & -1 \\ p^{k-1}\mu & \nu \end{pmatrix} \qquad (\mu \in \overline{\mathbf{Z}}_p^*, \nu \in \mathfrak{m}_{\overline{\mathbf{Z}}_p}) \qquad \text{and} \qquad N = 0$$

$$\operatorname{Fil}^i D = \begin{cases} D, & \text{if } i \geqslant 0 \\ \overline{\mathbf{Q}}_p e_1, & \text{if } i \in [1, k-1] \\ 0, & \text{if } i \geqslant k \end{cases}$$

and non-crystalline ones correspond to those in which the filtration is determined by two parameters, the integer k and an $\mathscr{L} \in \overline{\mathbf{Q}}_p$ (we fix a square root of p for the description below)—

$$\begin{aligned} D &= \overline{\mathbf{Q}}_p e_1 \oplus \overline{\mathbf{Q}}_p e_2 \\ \varphi &= \begin{pmatrix} \sqrt{p}^k \mu & 0 \\ 0 & \sqrt{p}^{k-2}\mu \end{pmatrix} \qquad (\mu \in \overline{\mathbf{Z}}_p^*) \qquad \text{and} \qquad N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \operatorname{Fil}^i D &= \begin{cases} D, & \text{if } i \geqslant 0 \\ \overline{\mathbf{Q}}_p(e_1 + \mathscr{L} e_2), & \text{if } i \in [1, k-1] \\ 0, & \text{if } i \geqslant k \end{cases} \end{aligned}$$

(For more details, see [BM02, Example 3.1.2.2] which in turn refer to Theorem A and its proof in Appendix A in the celebrated paper of Fontaine-Mazur [FM95] announcing the conjectural characterization of geometric Galois representations.)

Moreover, Fontaine [Fon94, §4.2.1] has associated a $\mathbf{Q}_p^{\mathrm{unr}}$-linear Weil-Deligne representation to a $(\varphi, N, \mathscr{G}_{\mathbf{Q}_p}, L)$-module over L which is functorial—given a (φ, N)-module $(D, \varphi, N, \rho_0, \operatorname{Fil}^\bullet)$ over L, then the associated Weil-Deligne representation $\mathrm{WD}(D, \varphi, N, \rho_0, \operatorname{Fil}^\bullet)$ has the underlying vector space D, the monodromy operator N, and the representation ρ of the Weil group on D given by

$$\rho(w) = \rho_0(w)\varphi^{-v(w)}.$$

Thus there is a forgetful functor $\rho \mapsto \mathrm{WD}((B_{\mathrm{pst}} \otimes \rho)^{\mathscr{G}_{\mathbf{Q}_p}})$ from the category of potentially semistable p-adic representations (of dimension 2) to the category of semisimple Weil-Deligne representations of $\mathbf{Q}_p$ (of dimension 2) which forgets the Hodge filtration on the Dieudonne module $D_{\mathrm{pst}}(\rho) := (B_{\mathrm{pst}} \otimes \rho)^{\mathscr{G}_{\mathbf{Q}_p}}$.

This shows that if one desired to lift the classical local Langlands correspondence to potentially semistable p-adic representations, one needs to be able to recover the Hodge filtration of $D_{\mathrm{pst}}(\rho)$ (equivalently, the parameters k and $\mathscr{L}$) from the putative p-adic Langlands correspondent $\Pi(\rho)$ for a potentially semistable p-adic Galois representation ρ. In a stroke of ingenuity, Breuil considered the locally algebraic representation

$$\mathrm{LAlg}(\rho) = \mathrm{Sym}^{k-2}\, \mathbf{Q}_p^{\oplus 2} \otimes \mathrm{LLC}(\rho)$$

(that encodes the positive jump in the Hodge filtration) and conjectured that the set of commensurability classes of $\mathrm{GL}_2(\mathbf{Q}_p)$-invariant $\mathscr{O}$-lattices in $\mathrm{LAlg}(\rho)$ is in bijection with the set of isomorphism classes of potentially semi-stable p-adic Galois representations ρ whose local Langlands correspondent is the smooth representation $\mathrm{LLC}(\rho)$ (so that the topology of the completion encodes the $\mathscr{L}$-invariant).

It is now tantalizing to ask if there is a functorial correspondence between the category *all* p-adic Galois representations and a suitable category of p-adic Banach representations of the group $\mathrm{GL}_2(\mathbf{Q}_p)$ that extends the construction of Schneider-Teitelbaum [ST06], Breuil-Schneider [BS07] and Breuil-Berger [BB10].

Thanks to the work of Colmez [Col10] (see references therein to his earlier work in the case of p-adic principal series) and Colmez-Dospinescu-Paškūnas [CDP14], there's now a well-defined p-adic local Langlands functoriality for the group $\mathrm{GL}_2(\mathbf{Q}_p)$.

Let us introduce the constructions that underpin this theory in the following proposition:

Proposition III.14. *Let D be an irreducible étale (φ, Γ)-module over $\mathscr{O}_{\mathscr{E}}$ (resp. $\mathscr{E}$) of dimension 2. Let $\alpha_D : \mathbf{Q}_p^\times \to \mathscr{O}^\times$ be such that $\wedge^2 D = \mathscr{O}_{\mathscr{E}}(\alpha_D)$ (resp. $\wedge^2 D = \mathscr{E}(\alpha_D)$). Set $\delta_D(x) = (x|x|)^{-1}\alpha_D(x)$. For an étale (φ, Γ)-module D over $\mathscr{E}$, let Δ be an $\mathscr{O}_{\mathscr{E}}$-lattice stable by φ and Γ.*

1. *If D is an étale (φ, Γ)-module over $\mathscr{O}_{\mathscr{E}}$, then, there is a unique smallest ψ-stable compact $\mathscr{O}_{\mathscr{E}}^+$-submodule $D^\natural$ that generates D. If D is an étale (φ, Γ)-module over $\mathscr{E}$, then we set $D^\natural = L \cdot \Delta^\natural$.*

2. *The monoid*
$$P^+ = \begin{pmatrix} \mathbf{Z}_p \setminus \{0\} & \mathbf{Z}_p \\ 0 & 1 \end{pmatrix}$$
acts on D by
$$\begin{pmatrix} p^k a & b \\ 0 & 1 \end{pmatrix} z = (1+T)^b \varphi^k(\sigma_a(z))$$
for $z \in D, a \in \mathbf{Z}_p^*, b \in \mathbf{Z}_p$, and $k \in \mathbf{Z}_{\geqslant 0}$. The endomorphisms φ, ψ and the P^+-action on D can be packaged into a sheaf $D \boxtimes - = (U \mapsto D \boxtimes U, \mathrm{Res}_\bullet)$ on $\mathbf{Z}_p$ satisfying the following properties:

   (a) *$D \boxtimes \mathbf{Z}_p = D$;*

   (b) *the sheaf $D \boxtimes -$ is P^+-equivariant[10]—that is $D \boxtimes \gamma \mathbf{Z}_p = \gamma D \boxtimes \mathbf{Z}_p$ for all $\gamma \in P^+$;*

   (c) *after identifying $D \boxtimes \left(\begin{smallmatrix} p^k & a \\ 0 & 1 \end{smallmatrix}\right)\mathbf{Z}_p$ with the $\mathscr{O}$-submodule $(1+T)^a\varphi^k(D)$ of D, the restriction map*
   $$\mathrm{Res}_{a+p^k\mathbf{Z}_p} : D \boxtimes \mathbf{Z}_p \to D \boxtimes (a + p^k\mathbf{Z}_p)$$
   is given by $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \circ \varphi^k \circ \psi^k \circ \left(\begin{smallmatrix} 1 & -a \\ 0 & 1 \end{smallmatrix}\right)$.

   (d) *the sum of the restriction maps $\mathrm{Res}_U \oplus \mathrm{Res}_V : D \boxtimes W \to (D \boxtimes U) \oplus (D \boxtimes V)$ is an isomorphism whenever U, V, and W are opens in $\mathbf{Z}_p$ such that $W = U \sqcup V$.*

[10]Note that the monoid P^+ also acts continuously on $\mathbf{Z}_p$ by $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)x = ax + b$ for $a \in \mathbf{Z}_p \setminus \{0\}$ and $b \in \mathbf{Z}_p$.

Note that these properties identify $D \boxtimes \mathbf{Z}_p^*$ with the $\mathscr{O}$-submodule $D^{\psi=0}$.

3. *The sheaf $D \boxtimes -$ extends further to a sheaf on $\mathbf{Q}_p$ that is equivariant for the action of the mirabolic group $P(\mathbf{Q}_p) = \left(\begin{smallmatrix} \mathbf{Q}_p^* & \mathbf{Q}_p \\ 0 & 1 \end{smallmatrix}\right)$ and whose space of global sections $D \boxtimes \mathbf{Q}_p$ is the $\mathscr{O}$-module $\varprojlim_\psi D$—*

   (a) *The action of the group $P(\mathbf{Q}_p)$ acts on $z = (z^{(n)})_{n \in \mathbf{Z}} \in D \boxtimes \mathbf{Q}_p$ is defined as follows:*

   - $((\begin{smallmatrix} p^k & 0 \\ 0 & 1 \end{smallmatrix}) \cdot z)^{(n)} = z^{(n+k)}$ *for $k \in \mathbf{Z}$;*
   - $((\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}) \cdot z)^{(n)} = \sigma_a(z^{(n)})$ *for $a \in \mathbf{Z}_p^*$;*
   - $((\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}) \cdot z)^{(n)} = (1+T)^{bp^n} z^{(n)}$ *for $b \in \mathbf{Q}_p$ and $n \geqslant -v_p(b)$;*

   (b) *There is a P^+-equivariant injection $\iota : D \to D \boxtimes \mathbf{Q}_p$ given by $\iota(x) = (\varphi^n(x))_n$; identify the image of ι with $D = D \boxtimes \mathbf{Z}_p$. We therefore get a projection map $\mathrm{Res}_{\mathbf{Z}_p} : D \boxtimes \mathbf{Q}_p \to D \boxtimes \mathbf{Z}_p$ by $(x^{(n)})_n \mapsto \iota(x^{(0)})$.*

   (c) *For open subsets U of $\mathbf{Z}_p$, let $D \boxtimes U$ denote the image of $D \boxtimes U$ under ι and the restriction map $\mathrm{Res}_U : D \boxtimes \mathbf{Q}_p \to D \boxtimes U$ is the projector $\iota \circ \mathrm{Res}_U \circ \iota^{-1} \circ \mathrm{Res}_{\mathbf{Z}_p}$.*

   (d) *For open subsets U of $\mathbf{Q}_p$, choose an integer k such that $p^k U \subset \mathbf{Z}_p$ and define $D \boxtimes U$ to be*
   $$D \boxtimes U = \begin{pmatrix} p^{-k} & 0 \\ 0 & 1 \end{pmatrix} (D \boxtimes p^k U)$$
   and the restriction map $\mathrm{Res}_U : D \boxtimes \mathbf{Q}_p \to D \boxtimes U$ to be $(\begin{smallmatrix} p^{-k} & 0 \\ 0 & 1 \end{smallmatrix}) \circ \mathrm{Res}_{p^k U} \circ (\begin{smallmatrix} p^k & 0 \\ 0 & 1 \end{smallmatrix})$.

4. *The construction in 3 works also for $D^\natural$ in place of D.*

5. *D extends to a $\mathrm{GL}_2(\mathbf{Q}_p)$-equivariant sheaf $(U \mapsto D \boxtimes_{\delta_D} U, \mathrm{Res}_\bullet)$ of $\mathscr{O}$-modules on $\mathbf{P}^1(\mathbf{Q}_p)$; the $\mathscr{O}$-module of its global sections is therefore a representation of $\mathrm{GL}_2(\mathbf{Q}_p)$; viewing $\mathbf{P}^1(\mathbf{Q}_p)$ as the space obtained by gluing two copies of $\mathbf{Z}_p$ along $\mathbf{Z}_p^*$ via the involution $x \mapsto x^{-1}$ (corresponding to the action of the element $w = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$), the sheaf $D \boxtimes -$ has the following properties:*

   - *The restriction of the sheaf to either copy of $\mathbf{Z}_p$ is identified with P^+-equivariantly with the sheaf $D \boxtimes \mathbf{Z}_p$ of 2;*
   - *The action of w on $D \boxtimes \mathbf{Z}_p^*$ is given by the formula*
   $$w(z) = \lim_{n \to \infty} \sum_{i \in (\mathbf{Z}/p^n\mathbf{Z})^*} \delta_D(i)(1+T)^{i^{-1}} \sigma_{-i^{-2}} \varphi^n \psi^n \left((1+T)^{-i} z\right);$$

- *The space* $D \boxtimes_{\delta_D} \mathbf{P}^1$ *of global sections is identified with the* $\mathscr{O}$*-submodule*

$$\left\{z = (z_1, z_2) \in D \times D : \operatorname{Res}_{\mathbf{Z}_p^*} z_2 = w(\operatorname{Res}_{\mathbf{Z}_p^*} z_1)\right\}$$

  of $D \times D$*; in this representation,* z_1*, and* z_2 *can be recovered from* z *via restriction maps by the formulae* $z_1 = \operatorname{Res}_{\mathbf{Z}_p} z$ *and* $z_2 = \operatorname{Res}_{\mathbf{Z}_p} w \cdot z$*;*

- *If* U *and* V *are compact opens in* $\mathbf{Q}_p$ *such that* $U \sqcup wV = \mathbf{P}^1$ *(with* $0 \in U$*), then the map*

$$z \mapsto (\operatorname{Res}_U(z), \operatorname{Res}_V(wz)) : D \boxtimes_{\delta_D} \mathbf{P}^1 \to D \boxtimes_{\delta_D} U \oplus D \boxtimes_{\delta_D} V$$

  is an isomorphism; therefore, an element $z \in D \boxtimes_{\delta_D} \mathbf{P}^1$ *is determined uniquely by the pair* $\operatorname{Res}_{\mathbf{Z}_p} z$ *and* $\operatorname{Res}_{p\mathbf{Z}_p} wz$*, or by the pair* $\operatorname{Res}_{p\mathbf{Z}_p} z$ *and* $\operatorname{Res}_{\mathbf{Z}_p} wz$*; and*

- *The action of* $\mathrm{GL}_2(\mathbf{Q}_p)$ *on* $z = (z_1, z_2) \in D \boxtimes \mathbf{P}^1$ *can be described as follows—*
  - *the center* $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ *acts by the character* δ_D*; i.e., one has* $a \cdot z = \delta_D(a) z$*;*
  - *for* $a \in \mathbf{Z}_p^*$*, the element* $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$ *acts by* $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot z = (\sigma_a(z_1), \delta(a)\sigma_{a^{-1}}(z_2))$*;*
  - *the element* $\left(\begin{smallmatrix} p^k & 0 \\ 0 & 1 \end{smallmatrix}\right)$ *(*$k \in \mathbf{N}$*) acts by* $\left(\begin{smallmatrix} p^k & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot z = \big(\varphi^k(z_1), \delta(p^k)\psi^k(z_2)\big)$*;*
  - *the element* $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ *acts by* $w \cdot z = (z_2, z_1)$*.*

6. *The map*

$$\begin{aligned} \operatorname{Res}_{\mathbf{Q}_p} : D \boxtimes \mathbf{P}^1 &\to D \boxtimes \mathbf{Q}_p \\ z &\mapsto \left(\operatorname{Res}_{\mathbf{Z}_p}\left(\begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix} z\right)\right)_{n \in \mathbf{N}} \end{aligned}$$

   is $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$*-equivariant. If* $\operatorname{Res}_{\mathbf{Z}_p} z = z_1$*, then the map* $\operatorname{Res}_{\mathbf{Q}_p}$ *is given by*

$$\operatorname{Res}_{\mathbf{Q}_p} z = \iota(z_1) = (\varphi^n(z_1))_{n \in \mathbf{N}}.$$

7. *the* $\mathscr{O}$*-submodule*[11] $D^\natural \boxtimes \mathbf{P}^1 = \{z \in D \boxtimes_{\delta_D} \mathbf{P}^1 : \operatorname{Res}_{\mathbf{Q}_p} z \in D^\natural \boxtimes \mathbf{Q}_p\}$ *is stable under the* $\mathrm{GL}_2(\mathbf{Q}_p)$*-action on* $D \boxtimes \mathbf{P}^1$*;*

8. *the quotient* $\Pi(D) := D \boxtimes_{\delta_D} \mathbf{P}^1 / D^\natural \boxtimes \mathbf{P}^1$ *is an irreducible representation of* $\mathrm{GL}_2(\mathbf{Q}_p)$*.*

[11] Despite the notation, note that $D^\natural \boxtimes \mathbf{P}^1$ is not the $\mathscr{O}$-module of global sections of a sheaf on $\mathbf{P}^1(\mathbf{Q}_p)$.

We now state the properties of the functor $\Pi(-)$ constructed in the proposition above:

Theorem III.15. *Let D be an étale (φ, Γ)-module over $\mathscr{E}$ of dimension 2; and let $\delta_D : \mathbf{Q}_p^* \to L^*$ be the character $(x|x|)^{-1} \det D$ (where $\det D$ is viewed as a character of $\mathbf{Q}_p^*$ by local class field theory).*

1. *The functor $D \mapsto \Pi(D)$ induces an exact sequence of L-Banach spaces*

$$0 \to \Pi(D)^* \otimes \delta_D \to D \boxtimes \mathbf{P}^1 \to \Pi(D) \to 0.$$

2. *(Compatiblity with the classical local Langlands correspondence)*
The set $\Pi(D)^{\mathrm{alg}}$ of locally algebraic vectors in $\Pi(D)$ is non-zero if and only if $\Pi(D)$ is de Rham with distinct Hodge-Tate weights. Moreover, letting[12] *D_{pst} be the filtered (φ, N)-module over L associated to D, the space $\Pi(D)^{\mathrm{alg}}$ of algebraic vectors in $\Pi(D)$ is isomorphic to*

$$\mathrm{LLC}(\mathrm{WD}(D_{\mathrm{pst}})) \otimes (\mathrm{Sym}^{b-a-1} \otimes \det{}^a)$$

where a and b are the Hodge-Tate weights of D. Here Sym^k is the k-th symmetric power of the standard representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ on $\mathbf{Q}_p^{\oplus 2}$.

Going in the opposite direction, we have the functor $\mathbf{V}$:

Theorem III.16. *There is an exact covariant functor*

$$\mathbf{V} : \mathrm{Rep}_? \, \mathrm{GL}_2(\mathbf{Q}_p) \to \mathrm{Rep}_? \, \mathscr{G}_{\mathbf{Q}_p}$$

for $? \in \{\mathrm{tors}, \mathscr{O}, L\}$ with the following properties:

1. *The functor $\mathbf{V}$ induces a bijection between the isomorphism classes of absolutely irreducible non-ordinary*[13] *objects in $\mathrm{Rep}_L G$ and two dimensional absolutely irreducible objects in $\mathrm{Rep}_L \mathscr{G}_{\mathbf{Q}_p}$.*

2. *The functor $\mathbf{V}$ realizes the $\mathrm{mod}\ p$ local Langlands correspondence and its image contains all representations $\overline{\rho} : \mathscr{G}_{\mathbf{Q}_p} \to \mathrm{GL}_2(k_L)$ such that*

$$\overline{\rho} \not\sim \psi \otimes \begin{pmatrix} 1 & * \\ 0 & \overline{\chi} \end{pmatrix}$$

for any k_L-valued character ψ of $\mathscr{G}_{\mathbf{Q}_p}$ and any $*$ (zero or otherwise).

[12] It is a theorem of Berger that every de Rham L-representation of $\mathscr{G}_{\mathbf{Q}_p}$ is potentially semistable [Ber02].
[13] We say that an absolutely irreducible admissible representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ on an L-Banach space is ordinary if it is a subquotient of a unitary parabolic induction of a unitary character of $\mathbf{Q}_p^{\times}$.

3. *If* Π *has central character* δ, *then* $\mathbf{V}(\Pi)$ *has determinant* $\delta\chi$.

4. $\mathbb{D}(\mathbf{V}(\Pi(D)))$ *is isomorphic to* D.

5. *Let* π *be a smooth irreducible infinite dimensional admissible representation of* $\mathrm{GL}_2(\mathbf{Q}_p)$ *defined over* L. *If* Π *is an admissible absolutely irreducible non-ordinary unitary completion of* $\pi \otimes (\mathrm{Sym}^{k-1} L^{\oplus 2} \otimes \det{}^a)$, *then* $\mathbf{V}(\Pi)$ *is potentially semistable with Hodge-Tate weights* a *and* $a+k$ *and* $\mathrm{WD}(D_{\mathrm{pst}}(\mathbf{V}(\Pi)))$ *is the Weil-Deligne representation associated to* π *by the classical Local Langlands correspondence.*

6. *The functor* $\Pi \mapsto D_{\mathrm{pst}}(\mathbf{V}(\Pi)))$ *induces a bijection between the admissible absolutely irreducible non-ordinary unitary completions of* $\pi \otimes (\mathrm{Sym}^{k-1} L^{\oplus 2} \otimes \det{}^a)$ *(with* π *as in 5), and the set of isomorphism classes of weakly admissible absolutely irreducible filtered* (φ, N)*-modules over* L *whose underlying Weil-Deligne representation is* $\mathrm{LLC}(\pi)$ *and whose underlying filtration has jumps at the integers* a *and* $a+k$.

The Hodge-Tate weights of the p-adic Galois representation ρ_f associated to a cuspidal newform f of weight 1 are $(0,0)$ so it is not even clear how one should formulate the compatibility of the p-adic local Langlands correspondence with the classical one. Yet, Colmez [Col19a] discovered a weight-shifting technique and is able to formulate and prove a version of local-global compatibility for weight 1 forms. To discuss this result, we should work with (φ, Γ)-modules over the Robba ring $\mathscr{R}$ of $\mathbf{Q}_p$ with coefficients in L.

Theorem III.17. *Suppose that* D *is an irreducible étale* (φ, Γ)*-module over* $\mathscr{R}$ *whose Hodge-Tate weights are* $(0,0)$*.Let* δ_D *denote the character* $(x|x|)^{-1} \det D$. *Then:*

1. *For every* $k \in \mathbf{Z}$, *there exists an extension of* D *to a* $\mathrm{GL}_2(\mathbf{Q}_p)$*-equivariant sheaf on* $\mathbf{P}^1$ *whose space of global sections is a locally analytic representation of* $\mathrm{GL}_2(\mathbf{Q}_p)$ *with the following properties:*
   - *The center of* $\mathrm{GL}_2(\mathbf{Q}_p)$ *acts by the character* $x^k \delta_D$;
   - *The Casimir operator acts by multiplication by* $\frac{1}{2}(k^2 - 1)$.

2. *For every* $k \in \mathbf{Z}$, *there exists a locally analytic representations* $\Pi(D,k)$ *of* $\mathrm{GL}_2(\mathbf{Q}_p)$ *and a smooth representation* π *uniquely determined by the following properties:*
   - *for* $k \in \mathbf{Z}$, *the representation* $\Pi(D,k)$ *fits in an exact sequence*

$$0 \to \Pi(D,k)^* \otimes x^k \delta_D \to D \boxtimes_{x^k \delta_D} \mathbf{P}^1 \to \Pi(D,k) \to 0;$$

- *for $k \geqslant 1$, there is an isomorphism $\Pi(D,k)^{\mathrm{alg}} \cong \pi \otimes \mathrm{Sym}^{k-1} L^{\oplus 2}$ of $\mathrm{GL}_2(\mathbf{Q}_p)$-representations.*

3. *When D is de Rham (and hence potentially semistable), the smooth representation π in 2 is the classical local Langlands correspondent of the Weil-Deligne representation associated to the filtered (φ, N)-module associated to D.*

III.5: The construction of $\mathbb{V}(-)$

The construction of the functor $\mathbb{V}(-)$ proceeds in four steps. Briefly:

- we re-interpret $\mathscr{O}_L$-modules equipped with a continuous action of the monoid $P^+ = \left(\begin{smallmatrix} \mathbf{Z}_p\setminus\{0\} & \mathbf{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$ as a (φ,Γ)-module over $\mathscr{O}_{\mathscr{E}}^+ = \mathscr{O}_L[\![T]\!]$;

- Let $\Pi \in \mathrm{Rep}_{\mathrm{tors}}\,\mathrm{GL}_2(\mathbf{Q}_p)$; we shall construct a module $M(\Pi)$ equipped with a continuous action of P^+; this gives rise to a (φ,Γ)-module over $\mathscr{O}_{\mathscr{E}}^+$. Thus, the $\mathscr{O}_{\mathscr{E}}$-module $\mathbf{D}(\Pi) := \mathscr{O}_{\mathscr{E}} \otimes_{\mathscr{O}_{\mathscr{E}}^+} M(\Pi)$ is a (φ,Γ)-module over $\mathscr{O}_{\mathscr{E}}$.

- We now extend the construction to objects Π in $\mathrm{Rep}_{\mathscr{O}_L}\,\mathrm{GL}_2(\mathbf{Q}_p)$ by taking inverse limit of $\mathbf{D}(\Pi/p^r\Pi)$ over r, and to objects Π in $\mathrm{Rep}_L\,\mathrm{GL}_2(\mathbf{Q}_p)$ by taking $\mathbf{D}(\Pi_0)[1/p]$ for an $\mathscr{O}$-lattice Π_0 in Π invariant under the action of $\mathrm{GL}_2(\mathbf{Q}_p)$.

- The $\mathscr{G}_{\mathbf{Q}_p}$-representation $\mathbf{V}(\Pi)$ is then set to be the *Tate dual* of $\mathbb{V}(\mathbf{D}(\Pi))$; here $\mathbb{V}(-)$ is the Fontaine's functor from the appropriate category of (φ,Γ)-modules to the corresponding category of representations of $\mathscr{G}_{\mathbf{Q}_p}$ and $\mathbf{D}(-)$ is the functor constructed above.

We shall say a few more words about the first two steps.

III.5.1: P^+-modules and (φ,Γ)-modules over $\mathscr{O}_{\mathscr{E}}^+$

Suppose that M is an $\mathscr{O}_L$-module equipped with a continuous action of the monoid P^+. Then M can be viewed as a module over the continuous group algebra $\mathscr{O}_L[\![\left(\begin{smallmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{smallmatrix}\right)]\!]$. Recall that Amice transform identifies this continuous group algebra with $\mathscr{O}_{\mathscr{E}}^+ = \mathscr{O}_L[\![T]\!]$ via the map

$$\lambda \mapsto \int_{\mathbf{Z}_p} (1+T)^x \lambda(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right))$$

which identifies the element $[(\begin{smallmatrix}1&1\\0&1\end{smallmatrix})]-1$ with T. Moreover, there is a continuous endomorphism $\varphi : M \to M$ and a continuous action of $\Gamma = \mathbf{Z}_p^*$ on M that commutes with φ given as follows—

$$\varphi(v) = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} v \qquad \text{and} \qquad \sigma_a(v) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} v.$$

Recall that $\mathscr{O}_L[\![(\begin{smallmatrix}1&\mathbf{Z}_p\\0&1\end{smallmatrix})]\!]$ is endowed with a continuous endomorphism φ and a continuous action of Γ that commutes with φ given by

$$\varphi\left(\left[\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\right]\right) = \left[\begin{pmatrix} 1 & pa \\ 0 & 1 \end{pmatrix}\right] \text{ and } \sigma_\beta\left(\left[\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\right]\right) = \left[\begin{pmatrix} 1 & \beta a \\ 0 & 1 \end{pmatrix}\right]$$

(where $\beta \in \mathbf{Z}_p^*$). The matrix identity

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & ab \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

implies that the endomorphism φ and the action of Γ on M are semilinear over that on $\mathscr{O}_{\mathscr{E}}^+$. In other words, M may be viewed as a (φ, Γ)-module over $\mathscr{O}_{\mathscr{E}}^+$.

III.5.2: The (φ, Γ)-module $\mathbf{D}(\Pi)$ for torsion Π

Let Π be an admissible smooth $\mathrm{GL}_2(\mathbf{Q}_p)$-representation of finite length over $\mathscr{O}$. Thus, Π is an $\mathscr{O}/\varpi^n\mathscr{O}$-module for some n. Let $W \subset \Pi$ be any finitely generated $\mathscr{O}$-module that is stable under $\mathrm{GL}_2(\mathbf{Z}_p)$ and such that $\Pi = \mathrm{GL}_2(\mathbf{Q}_p)W$; for the existence of such a W, see Lemma III.1.6 of [Col10]. Set

$$D_W^\natural(\Pi) = \sum_{m \geqslant 0} \begin{pmatrix} p^m & \mathbf{Z}_p \\ 0 & 1 \end{pmatrix} W.$$

Then, the matrix identities

$$\begin{pmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p^m & \mathbf{Z}_p \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p^m & \mathbf{Z}_p \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p^m & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p^m & ab \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \text{ for } a \in \mathbf{Z}_p^*, b \in \mathbf{Z}_p$$

and the $\mathrm{GL}_2(\mathbf{Z}_p)$-stability of W imply that $D^\natural_W(\Pi)$ is stable under the action of $\left(\begin{smallmatrix}1 & \mathbf{Z}_p\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}\mathbf{Z}_p^* & 0\\ 0 & 1\end{smallmatrix}\right)$. Consider the $\mathscr{O}$-linear dual of $D^\natural_W(\Pi)$:

$$\mathbf{D}^+_W(\Pi) = \mathrm{Hom}_{\mathscr{O}}(D^\natural_W(\Pi), \mathscr{O}/\varpi^n\mathscr{O}).$$

Then, $\mathbf{D}^+_W(\Pi)$ is naturally a P^+-module whence a (φ,Γ)-module over $\mathscr{O}^+_{\mathscr{E}}$. Colmez then proves that $\mathscr{O}_{\mathscr{E}} \otimes_{\mathscr{O}^+_{\mathscr{E}}} \mathbf{D}^+_W(\Pi)$ is independent of the choice of W and is the (φ,Γ)-module $\mathbf{D}(\Pi)$ required in the case of torsion representation Π of $\mathrm{GL}_2(\mathbf{Q}_p)$.

III.6: Relation to deformation theory

The main results in this section are due to Kisin [Kis10]; we follow Emerton [Eme11]. We will use the language of deformation groupoids developed in the Appendix to [Kis09b].

Let $\mathrm{Comp}(\mathscr{O}_L)$ denote the category of complete Noetherian local $\mathscr{O}_L$-algebras whose residue fields are a finite extension of k_L.

Fix a residual representation $\overline{\rho} : \mathscr{G}_{\mathbf{Q}_p} \to \mathrm{GL}_2(k_L)$ such that

(III.6.1) $$\overline{\rho} \not\sim \psi \otimes \begin{pmatrix} 1 & * \\ 0 & \overline{\chi} \end{pmatrix}$$

for any k_L-valued character ψ of $\mathscr{G}_{\mathbf{Q}_p}$ and any $*$ (zero or otherwise).

Let $\overline{\pi} = \Pi(\overline{\rho})$ be the k_L-representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ attached to $\overline{\rho}$ by the mod p local Langlands correspondence (v. Theorem III.16). We wish to relate the groupoid over $\mathrm{Comp}(\mathscr{O}_L)$ consisting of the category of deformations of $\overline{\rho}$ and isomorphisms between them to that of $\overline{\pi}$ using the p-adic local Langlands correspondence.

III.6.1: The deformation groupoids

Definition III.18. Fix $A \in \mathrm{Comp}(\mathscr{O}_L)$; let $\mathfrak{m} := \mathfrak{m}_A$ denote its maximal ideal.

1. A deformation of $\overline{\rho}$ to A is a pair (V, ι) where V is a free A-module of rank 2 equipped with a continuous action of $\mathscr{G}_{\mathbf{Q}_p}$ together with an $A/\mathfrak{m}$-linear and $\mathscr{G}_{\mathbf{Q}_p}$-equivariant isomorphism $\iota : V/\mathfrak{m}V \xrightarrow{\sim} A/\mathfrak{m} \otimes_{k_L} \overline{\rho}$.

2. An A-module M is *orthonormalizable* if M is $\mathfrak{m}$-adically complete, separated and is such that $M/\mathfrak{m}^s M$ is free over $A/\mathfrak{m}^s$ for all $s \geqslant 0$. (Note that M is then the completion of a free A-module of rank $\dim_{A/\mathfrak{m}} M/\mathfrak{m}M$.)

3. An $A[\mathrm{GL}_2(\mathbf{Q}_p)]$-module M is said to be *orthonormalizable admissible* if M is orthonormalizable as an A-module and $M/\mathfrak{m}^s A$ is a smooth admissible representation of G over $A/\mathfrak{m}^s$ for every $s \geqslant 0$.

4. A deformation of $\overline{\pi}$ to A is a pair (π, ι) where π is an orthonormalizable admissible $A[\mathrm{GL}_2(\mathbf{Q}_p)]$-module together with an $A/\mathfrak{m}$-linear and $\mathrm{GL}_2(\mathbf{Q}_p)$-equivariant isomorphism $\iota : \pi/\mathfrak{m}\pi \xrightarrow{\sim} A/\mathfrak{m} \otimes_{k_L} \overline{\pi}$.

5. The deformation functor $\mathrm{Def}_{\overline{\rho}}$ for $\overline{\rho}$ is a category fibered in groupoids over $\mathrm{Comp}(\mathcal{O})$: for any $A \in \mathrm{Comp}(\mathcal{O})$ with maximal ideal $\mathfrak{m}$, the groupoid $\mathrm{Def}_{\overline{\rho}}(A)$ has as objects deformations of $\overline{\rho}$ to A, and as morphisms the A-linear and $\mathcal{G}_{\mathbf{Q}_p}$-equivariant isomorphisms that are compatible with ι.

6. The deformation functor $\mathrm{Def}_{\overline{\pi}}$ for $\overline{\pi}$ is a category fibered in groupoids over $\mathrm{Comp}(\mathcal{O})$: for any $A \in \mathrm{Comp}(\mathcal{O})$ with maximal ideal $\mathfrak{m}$, the groupoid $\mathrm{Def}_{\overline{\pi}}(A)$ has as objects deformations of $\overline{\pi}$ to A, and as morphisms the A-linear and $\mathrm{GL}_2(\mathbf{Q}_p)$-equivariant isomorphisms that are compatible with ι.

7. Let $\mathrm{Def}^*_{\overline{\pi}}$ be the subfunctor of $\mathrm{Def}_{\overline{\pi}}$ whose fiber $\mathrm{Def}^*_{\overline{\pi}}(A)$ over $A \in \mathrm{Comp}(\mathcal{O})$ is the subgroupoid consisting of deformations π of $\overline{\pi}$ to A on which the center of $\mathrm{GL}_2(\mathbf{Q}_p)$ acts by the character $\det \mathbf{V}(\pi) \cdot \chi$ (where the Galois character $\det \mathbf{V}(\pi) \cdot \chi$ is regarded as a character of $\mathbf{Q}_p^\times$ by local class field theory).

8. The full subgroupoid of $\mathrm{Def}_{\overline{\rho}}$ consisting of the Zariski closure of the set of crystalline points in the generic fiber of $\mathrm{Def}_{\overline{\rho}}$ is denoted by $\mathrm{Def}^{\mathrm{cris}}_{\overline{\rho}}$.

The functor $\mathbf{V}$ induces a natural transformation

$$\mathbf{V} : \mathrm{Def}_{\overline{\pi}} \to \mathrm{Def}_{\overline{\rho}}$$

between the deformation functors $\mathrm{Def}_{\overline{\pi}}$ and $\mathrm{Def}_{\overline{\rho}}$; since the determinant of $\mathbf{V}(\pi)$ is determined by the central character of π, the natural transformation above induces a fully faitful morphism of groupoids (v. [Kis10, Lemma 2.5])

$$\mathrm{Def}^*_{\overline{\pi}} \to \mathrm{Def}_{\overline{\rho}}\,.$$

Together with the subgroupoid $\mathrm{Def}^{\mathrm{cris}}_{\overline{\rho}}$, we have the following diagram of groupoids:

(III.6.2)
$$\begin{array}{ccc} & & \mathrm{Def}^{\mathrm{cris}}_{\overline{\rho}} \\ & & \big\downarrow \\ \mathrm{Def}^{*}_{\overline{\pi}} & \xrightarrow{\mathbf{V}} & \mathrm{Def}_{\overline{\rho}}\,. \end{array}$$

Let $\mathrm{Def}^{\mathrm{cris}}_{\overline{\pi}}$ be the fiber product of $\mathrm{Def}^{*}_{\overline{\pi}}$ and $\mathrm{Def}^{\mathrm{cris}}_{\overline{\rho}}$ over $\mathrm{Def}_{\overline{\rho}}$; the groupoid $\mathrm{Def}^{\mathrm{cris}}_{\overline{\pi}}(A)$ (for $A \in \mathrm{Comp}(\mathscr{O})$) consists of pairs (Π, ρ) such that

- Π is a deformation of $\overline{\pi}$ to A whose central character is $\det \mathbf{V}(\Pi) \cdot \chi$;
- ρ is a deformation of $\overline{\rho}$ to A;
- for every maximal ideal x in the "generic fiber"[14] $A[1/p]$, the specialization $\rho(x)$ of ρ at x is crystalline; and
- $\mathbf{V}(\Pi) = \rho$.

By definition, it comes equipped with a natural transformation

(III.6.3)
$$\mathrm{Def}^{\mathrm{cris}}_{\overline{\pi}} \to \mathrm{Def}^{\mathrm{cris}}_{\overline{\rho}}\,.$$

III.6.2: Reformulation of the p-adic local Langlands correspondence

The following is Theorem 3.3.13 of [Eme11]:

Theorem III.19. *If $\overline{\rho}$ satisfies (III.6.1), then the Colmez' functor $\mathbf{V}$ induces a fully faithful embedding*

$$\mathrm{Def}^{*}_{\overline{\pi}} \xrightarrow{\mathbf{V}} \mathrm{Def}_{\overline{\rho}}$$

and the restriction (III.6.3) of $\mathbf{V}$ to $\mathrm{Def}^{\mathrm{cris}}(\overline{\pi})$ induces an equivalence onto to the groupoid $\mathrm{Def}^{\mathrm{cris}}(\overline{\rho})$.

III.6.3: The case of absolutely irreducible $\overline{\rho}$

In the case that $\overline{\rho} : \mathscr{G}_{\mathbf{Q}_p} \to \mathrm{GL}_2(k_L)$ is absolutely irreducible, our deformation functors above are representable by complete local $\mathscr{O}$-algebras and the main theorems permit an

[14] It is easily checked that the maximal ideals of $A[1/p]$ are kernels of L-algebra homomorphisms $A[1/p] \to L'$ for a finite extension L' of L; the set of maximal ideals of $A[1/p]$ is a way to algebraically phrase the rigid analytic space associated to A.

easier exposition. Since we plan to restrict ourselves to this case in our main theorem, we include this discussion here.

Our assumption that the representation $\overline{\rho}$ is absolutely irreducible implies, by Schur's lemma, that $\mathrm{End}_{\mathscr{G}_{\mathbf{Q}_p}}(\overline{\rho}) = k_L$. By Schlessinger's criterion [Maz89, Proposition 1], the functor $\mathrm{Def}_{\overline{\rho}}$ is representable by a complete local $\mathscr{O}$-algebra $R(\overline{\rho})$; in particular, there are bijections

$$\mathrm{Hom}_{(\mathrm{Comp}(\mathscr{O}))}(R(\overline{\rho}), A) \xrightarrow{\simeq} |\,\mathrm{Def}_{\overline{\rho}}(A)|$$

functorial in $A \in \mathrm{Comp}(\mathscr{O})$. In particular, plugging in $A = R(\overline{\rho})$, the image of the identity homomorphism is the universal deformation $\rho^{\mathrm{univ}} : \mathscr{G}_{\mathbf{Q}_p} \to \mathrm{GL}_2(R(\overline{\rho}))$ of $\overline{\rho}$.

It follows from [Kis10, Lemma 2.2] that $\mathrm{End}_{\mathscr{G}_{\mathbf{Q}_p}}(\overline{\pi}) = k_L$. Thus, the functor $\mathrm{Def}_{\overline{\pi}}$ is representable by a complete local $\mathscr{O}$-algebra $R(\overline{\pi})$.

Since $\mathbf{V}$ is exact, it induces a local morphism $\mathbf{V} : R(\overline{\rho}) \to R(\overline{\pi})$ of complete local $\mathscr{O}$-algebras. The subgroupoid $\mathrm{Def}^{*}_{\overline{\pi}}$ consisting of deformations of $\overline{\pi}$ whose central character is $\det(\mathbf{V}(\pi)) \cdot \chi$ is representable by a quotient $R(\overline{\pi})^{\det}$ of $R(\overline{\pi})$; similarly, the subgroupoid consisting of the Zariski closure of the set of crystalline deformations of $\overline{\rho}$ is represented by the quotient

$$R(\overline{\rho})^{\mathrm{cris}} = R(\overline{\rho})/\cap\,\mathfrak{p}$$

as $\mathfrak{p}$ ranges over kernel of homomorphisms $x : R(\overline{\rho}) \to \overline{\mathbf{Z}}_p$ such that the specialization of $\rho^{\mathrm{univ}}(x[1/p]) : \mathscr{G}_{\mathbf{Q}_p} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ is crystalline. (It is known that the intersection $\cap\mathfrak{p}$ is zero—that the crystalline points are dense in the space of deformations of $\overline{\rho}$—but we do not need this here.)

There is then a diagram of complete local $\mathscr{O}$-algebras in which the vertical arrow is surjective:

$$\begin{array}{ccc} R(\overline{\rho}) & \longrightarrow & R(\overline{\pi})^{\det} \\ \downarrow & & \\ R(\overline{\rho})^{\mathrm{cris}} & & \end{array}$$

We let $R(\overline{\pi})^{\mathrm{cris}}$ denote the fiber product (viz., tensor product) of $R(\overline{\rho})^{\mathrm{cris}}$ and $R(\overline{\pi})^{\det}$ over $R(\overline{\rho})$; which comes equipped with the map

$$R(\overline{\rho})^{\mathrm{cris}} \to R(\overline{\pi})^{\mathrm{cris}} \tag{III.6.4}$$

The following is then an immediate corollary of III.19:

Theorem III.20. *The map* (III.6.4) *is an isomorphism.*

CHAPTER IV

A p-adic Jacquet-Langlands Correspondence

IV.1: Setup

Let D be a definite quaternion algebra over $\mathbf{Q}$ of discriminant $d\infty$; thus d is a square-free integer that is a product of an odd number of primes. Our fixed prime p is such that $(p, d) = 1$. Let $\mathbb{G}$ denote the $\mathbf{Z}$-algebraic group $D^\times$, the group of units in D.

We also fix a finite extension L of $\mathbf{Q}_p$, and write $\mathscr{O}$ and k_L for the ring of integers in L and the residue field of $\mathscr{O}$ respectively. Fix a uniformizer ϖ in $\mathscr{O}$.

Fix a continuous absolutely irreducible representation $\overline{\rho} : \mathscr{G}_{\mathbf{Q}} \to \mathrm{GL}_2(k_L)$. Let Σ_0 be a finite set places not containing p so that $\overline{\rho}$ is unramified outside $\Sigma = \Sigma_0 \cup \{p\}$. Assume that $\overline{\rho}$ is modular. (This follows from that *a priori* weaker assumption that $\overline{\rho}$ is odd after the proof of Serre's conjecture by Khare-Wintenberger [Kha06, KW09a, KW09b] and Kisin [Kis09a]. It follows from loc. cit. that there exists a modular form f with level equal to the Artin conductor $N(\overline{\rho})$ of $\overline{\rho}$ and such that $\overline{\rho_f}$ is equivalent to $\overline{\rho}$; the equivalence of this stronger statement with the weaker Serre's conjecture was known earlier; see [Kha06, p. 558] and the references in [RS01] for more details. In particular, the modularity assumption implies that there exists a newform f whose level is divisible only by places in Σ and such that $\overline{\rho_f}$ is equivalent to $\overline{\rho}$.)

We assume that $\overline{\rho}$ satisfies the following conditions:

(Irr_p) $\overline{\rho}|_{\mathscr{G}_{\mathbf{Q}_p}}$ is irreducible;

(SI_d) for $v \mid d$, the local representation $\overline{\rho}|_{\mathscr{G}_{\mathbf{Q}_v}}$ is irreducible, or is of the form

$$\begin{pmatrix} \psi_v & * \\ 0 & \psi_v \overline{\chi}^{-1} \end{pmatrix}$$

where ψ_v is some character $\mathscr{G}_{\mathbf{Q}_v} \to \overline{\mathbf{F}}_p^\times$ and $\overline{\chi}$ is the mod p cyclotomic character.

The local assumption at p is one of convenience for purposes of deformation theory; it must be possible to establish our results under the assumption (b) of Theorem 1.2.1 in [Eme11]. Presently however, our assumption excludes forms that are ordinary at p. On the other hand, the assumption (SI_d) is crucial as we need to ensure square integrability at the places ramified in the quaternion algebra D (cf. with the discussions in §4 of [BDJ10]).

IV.2: $\overline{\rho}$-part of $\widehat{\mathrm{H}}^0$ and $\mathbf{T}$

Let Σ_0 be as above; and let $\Sigma = \Sigma_0 \cup \{p\}$. Let G_{Σ_0} denote the group $\prod_{v\in\Sigma_0}\mathbb{G}(\mathbf{Q}_v)$. Let $K_0^{\Sigma} := \prod_{\ell\notin\Sigma}\mathrm{GL}_2(\mathbf{Z}_\ell)$. The Hecke algebras $\mathbf{T}$ at finite pro-p levels, and at the infinite p-power level are all for the group $D^\times$ (so we mostly suppress it).

Definition IV.1.

1. We say that an open compact subgroup $K_{\Sigma_0} \subset G_{\Sigma_0}$ is an *allowable level* for $\overline{\rho}$ if there is a maximal ideal $\mathfrak{m}$ of $\mathbf{T}(K_{\Sigma_0}) := \mathbf{T}(K_{\Sigma_0}K_0^{\Sigma})$ having residue field k and which is associated to $\overline{\rho}$; that is, the ideal $\mathfrak{m}$ satisfies

$$T_\ell \bmod \mathfrak{m} = \mathrm{Tr}\,\overline{\rho}(\mathrm{Frob}_\ell)$$
$$\ell S_\ell \bmod \mathfrak{m} = \det\overline{\rho}(\mathrm{Frob}_\ell)$$

   for all $\ell \notin \Sigma$ (more concretely, there exists a cuspidal automorphic form on $D^\times$ of tame level K_{Σ_0} whose associated Galois representation lifts $\overline{\rho}$). Since the elements T_ℓ and S_ℓ ($\ell \notin \Sigma$) topologically generate the Hecke algebra $\mathbf{T}(K_{\Sigma_0})$, the maximal ideal $\mathfrak{m}$ is uniquely determined by the requirements above and is denoted by $\mathfrak{m}_{\overline{\rho}}$.

   (In view of the parenthetical remark before the definition, note that any sufficiently small open compact subgroup K_{Σ_0} is allowable for $\overline{\rho}$.)

2. For an allowable level K_{Σ_0}, we write $\mathbf{T}(K_{\Sigma_0})_{\overline{\rho}}$ for the completion of $\mathbf{T}(K_{\Sigma_0})$ at $\mathfrak{m}$.

For an compact open subgroup K_p of G, let

$$\mathbf{T}(K_pK_{\Sigma_0})_{\overline{\rho}} := \mathbf{T}(K_{\Sigma_0})_{\overline{\rho}} \otimes_{\mathbf{T}(K_{\Sigma_0})} \mathbf{T}(K_pK_{\Sigma_0}K_0^{\Sigma}).$$

This is complete local $\mathscr{O}$-algebra. Following Carayol [Car94, Théorème 3], there exists a deformation $\rho(K_pK_{\Sigma_0})$ of $\overline{\rho}: \mathscr{G}_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{T}(K_pK_{\Sigma_0})_{\overline{\rho}})$ to the ring $\mathbf{T}(K_pK_{\Sigma_0})_{\overline{\rho}}$ such that the characteristic polynomial of $\rho(K_pK_{\Sigma_0})(\mathrm{Frob}_\ell)$ is

$$X^2 - T_\ell X + \ell S_\ell$$

for all $\ell \notin \Sigma$. Since

$$\mathbf{T}(K_{\Sigma_0})_{\overline{\rho}} = \varprojlim_{K_p} \mathbf{T}(K_p K_{\Sigma_0})_{\overline{\rho}},$$

by taking inverse limit of the representations $\rho(K_p K_{\Sigma_0})$, we get a deformation of $\overline{\rho}$ to the complete local $\mathscr{O}$-algebra $\mathbf{T}(K_{\Sigma_0})_{\overline{\rho}}$.

Recall now that there is the universal deformation ring $R_{\overline{\rho},\Sigma}$ that parametrizes deformations, to complete $\mathscr{O}$-algebras, of $\overline{\rho}$ that are unramified outside Σ (cf. §1.2 of [Maz89]). The ring $R_{\overline{\rho},\Sigma}$ is then a complete local $\mathscr{O}$-algebra with residue field k, and comes with the universal deformation ρ^{univ}; viz., a continuous representation $\rho^{\mathrm{univ}} : \mathscr{G}_{\mathbf{Q}} \to \mathrm{GL}_2(R_{\overline{\rho},\Sigma})$ such that, if $\rho : \mathscr{G}_{\mathbf{Q}} \to \mathrm{GL}_2(A)$ is a deformation of $\overline{\rho}$ to a complete local $\mathscr{O}$-algebra A, then there is a local homomorphism $\mathrm{spc}_A : R_{\overline{\rho},\Sigma} \to A$ such that $\rho = \mathrm{GL}_2(\mathrm{spc}_A) \circ \rho^{\mathrm{univ}}$. It follows from Carayol's Lemma [Car94, Théorème 2] for representations on finite free modules over complete local $\mathscr{O}$-algebras and the Chebotarev density theorem that $R_{\overline{\rho},\Sigma}$ is generated by $\mathrm{Tr}\,\rho^{\mathrm{univ}}(\gamma)$ as γ varies over a dense subset of $\mathscr{G}_{\mathbf{Q}}$. In particular, the elements $t_\ell \in R_{\overline{\rho},\Sigma}$ ($\ell \notin \Sigma$) given by

$$t_\ell = \mathrm{Tr}\,\rho^{\mathrm{univ}}(\mathrm{Frob}_\ell)$$

topologically generate the ring $R_{\overline{\rho},\Sigma}$. In particular, the element $s_\ell \in R_{\overline{\rho},\Sigma}$ ($\ell \notin \Sigma$) given by

$$s_\ell = \ell^{-1} \det \rho^{\mathrm{univ}}(\mathrm{Frob}_\ell)$$

belongs to the closure of the subring generated by the t_ℓ's.

For each allowable level K_{Σ_0}, there is then a local homomorphism

$$\phi(K_{\Sigma_0}) : R_{\overline{\rho},\Sigma} \to \mathbf{T}(K_{\Sigma_0})_{\overline{\rho}}$$

uniquely determined by the universal property; moreover, it follows from this that $\phi(K_{\Sigma_0})(t_\ell) = T_\ell$ and $\phi(K_{\Sigma_0})(s_\ell) = S_\ell$. In particular, the map $\phi(K_{\Sigma_0})$ is surjective and that $\mathbf{T}(K_{\Sigma_0})_{\overline{\rho}}$ is generated by the T_ℓ's alone.

For every pair of allowable levels $K'_{\Sigma_0} \subset K_{\Sigma_0}$, there is then a commutative diagram

(IV.2.1)
$$\begin{array}{ccc} R_{\overline{\rho},\Sigma} & & \\ {\scriptstyle \phi(K'_{\Sigma_0})}\downarrow & \searrow^{\phi(K_{\Sigma_0})} & \\ \mathbf{T}(K'_{\Sigma_0})_{\overline{\rho}} & \longrightarrow & \mathbf{T}(K_{\Sigma_0})_{\overline{\rho}} \end{array}$$

in which the vertical arrows are (and hence the horizontal arrow is) surjective.

We christen the the Hecke algebra obtained by passing to the inverse limit over allowable levels K_{Σ_0}, the "$\overline{\rho}$-part of $\mathbf{T}$":

Definition IV.2. Write

$$\mathbf{T}_{\overline{\rho},\Sigma} := \varprojlim \mathbf{T}(K_{\Sigma_0})_{\overline{\rho}}$$

as the inverse limit ranges over allowable levels K_{Σ_0} for the residual representation $\overline{\rho}$. Note that the algebra $\mathbf{T}_{\overline{\rho},\Sigma}$ acts on $\widehat{\mathrm{H}}^0(K_{\Sigma_0})$ for any allowable level K_{Σ_0} via the quotient $\mathbf{T}(K_{\Sigma_0})_{\overline{\rho}}$.

Moreover, the transition maps preserve the Hecke operators T_ℓ ($\ell \notin \Sigma$), and hence there is a well-defined element $T_\ell \in \mathbf{T}_{\overline{\rho},\Sigma}$ which acts by T_ℓ on $\widehat{\mathrm{H}}^0(K_{\Sigma_0})$ for any $\ell \notin \Sigma$ and any allowable level K_{Σ_0}.

The following lemma shows that the $\mathscr{O}$-algebra $\mathbf{T}_{\overline{\rho},\Sigma}$ is isomorphic to the Hecke algebra $\mathbf{T}(K_{\Sigma_0})_{\overline{\rho}}$ for a sufficiently small allowable level K_{Σ_0}; in particular, $\mathbf{T}_{\overline{\rho},\Sigma}$ is complete and local.

Lemma IV.3.

1. *For allowable levels $K'_{\Sigma_0} \subset K_{\Sigma_0}$, the induced map*

$$\mathbf{T}(K'_{\Sigma_0})_{\overline{\rho}} \to \mathbf{T}(K_{\Sigma_0})_{\overline{\rho}}$$

is a surjection of complete local Noetherian $\mathscr{O}$-algebras.

2. *If furthermore K_{Σ_0} is a sufficiently small allowable level, then the induced map $\mathbf{T}(K'_{\Sigma_0})_{\overline{\rho}} \to \mathbf{T}(K_{\Sigma_0})_{\overline{\rho}}$ of part 1 is an isomorphism.*

Proof.

1. This follows from the surjectivity of the vertical arrows in the commutative diagram (IV.2.1).

2. This relies on the observations of Ron Livne [Liv89] and Henri Carayol [Car89] about the behaviour of prime-to-p Artin conductors under reduction modulo p; from the discussion in §1 of [Car89] for example, it follows that there is an integer N_{Σ_0} such that the Artin conductor of every modular deformation ρ of $\overline{\rho}$ divides N_{Σ_0}; thus, as quotients of $R_{\overline{\rho},\Sigma}$, the $\mathscr{O}$-algebras $\mathbf{T}(K_{\Sigma_0})_{\overline{\rho}}$ coincide whenever K_{Σ_0} contains an open subgroup of G_{Σ_0} of level N_{Σ_0}. $\square$

The commutative diagram (IV.2.1) packages to a local homomorphism

$$\phi_\Sigma : R_{\overline{\rho},\Sigma} \to \mathbf{T}_{\overline{\rho},\Sigma}$$

and gives rise to a continuous representation $\rho^{\mathrm{m}} : \mathscr{G}_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{T}_{\overline{\rho},\Sigma})$; we view this representation as the *universal modular deformation* of $\overline{\rho}$; cf. for example, §2 of [Car94] for the aptness of this name.

Finally, using the Hecke action on $\widehat{\mathrm{H}}^0$, we can isolate its $\overline{\rho}$-part:

Definition IV.4. For allowable levels K_{Σ_0} and coefficients $A = L$ or $\mathscr{O}$, let

$$\widehat{\mathrm{H}}^0(K_{\Sigma_0})_A = \widehat{\mathrm{H}}^0(K_{\Sigma_0}K_0^{\Sigma})_A$$
$$\widehat{\mathrm{H}}^0(K_{\Sigma_0})_{A,\overline{\rho}} = \mathbf{T}(K_{\Sigma_0})_{\overline{\rho}} \otimes_{\mathbf{T}(K_{\Sigma_0})} \widehat{\mathrm{H}}^0(K_{\Sigma_0})_A$$

Finally, the $\overline{\rho}$-part of $\widehat{\mathrm{H}}^0$ is the inductive limit over levels K_{Σ_0} allowable for $\overline{\rho}$:

$$\widehat{\mathrm{H}}^0_{A,\overline{\rho}} = \varinjlim_{K_{\Sigma_0}} \widehat{\mathrm{H}}^0(K_{\Sigma_0})_{A,\overline{\rho}}$$

IV.3: Crystalline points of $\operatorname{Spec} \mathbf{T}_{\overline{\rho},\Sigma}[1/p]$

Definition IV.5. Fix an isomorphism $\iota : \mathbf{C} \to \overline{\mathbf{Q}}_p$.

1. We say that an eigensystem $\lambda : \mathbf{T}_{\overline{\rho},\Sigma} \to \overline{\mathbf{Q}}_p$ is *classical modular* if there exists an automorphic representation π_λ of the group $D^\times(\mathbf{A})$ with the following property:

   for all places v not dividing $d\infty$, if $\pi_{\lambda,v}$ is unramified, then the Satake parameters $\{\alpha_v(\pi), \beta_v(\pi)\}$ of the local representation $\pi_{\lambda,v}$ of $\mathrm{GL}_2(\mathbf{Q}_v)$ are related to the eigensystem λ via $\lambda(T_v) = \iota\alpha_v(\pi) + \iota\beta_v(\pi)$.

2. We say that a classical modular eigensystem λ of $D^\times$ is crystalline if the local representation $\pi_{\lambda,p}$ of $\mathrm{GL}_2(\mathbf{Q}_p)$ of the associated automorphic representation π_λ is unramified.

À la Emerton's Proposition 5.4.1 in [Eme11], we now prove:

Proposition IV.6. *If $K_{\Sigma_0} \subset G_{\Sigma_0}$ is an allowable level for $\overline{\rho}$, then the space of $\mathrm{GL}_2(\mathbf{Z}_p)$-algebraic vectors $\left(\widehat{\mathrm{H}}^0(K_{\Sigma_0})_{L,\overline{\rho}}\right)_{\mathrm{GL}_2(\mathbf{Z}_p)\text{-}alg}$ is dense in $\widehat{\mathrm{H}}^0(K_{\Sigma_0})_{L,\overline{\rho}}$.*

(Recall that a vector v in a (continous) G-module is $\mathrm{GL}_2(\mathbf{Z}_p)$-algebraic if the representation $\langle \mathrm{GL}_2(\mathbf{Z}_p)v\rangle$ is an algebraic representation of $\mathrm{GL}_2(\mathbf{Z}_p)$. Cf. with our discussions in §II.6.)

Proof. Tracing through the proof of Proposition 5.4.1 in [Eme11], it suffices to show that there is an isomorphism of ϖ-adically admissible K_p-representations

$$\widehat{\mathrm{H}}^0(K_{\Sigma_0})_{L,\overline{\rho}} \cong \mathcal{C}(K_p, L)^r \tag{IV.3.1}$$

for some $r > 0$ and a suitable compact open subgroup K_p of G. Admitting this fact, let us complete the proof of Proposition IV.6 by bootstrapping (IV.3.1) to a $\mathrm{GL}_2(\mathbf{Z}_p)$-equivariant embedding of $\widehat{\mathrm{H}}^0(K_{\Sigma_0})_{L,\overline{\rho}}$ into $\mathcal{C}(\mathrm{GL}_2(\mathbf{Z}_p), L)^s$ for some $s > 0$. Such an embedding would immediately prove our claim—via the theory of Mahler expansions, it can be checked that the space $\mathcal{C}(X(\mathbf{Z}_p), L)_{\mathrm{alg}}$ is a dense subspace of $\mathcal{C}(X(\mathbf{Z}_p), L)$ for an affine scheme X of finite type over $\mathbf{Z}_p$ (v. Lemma A.1 of [Paš14]).

Taking the topological L-linear dual $\mathrm{Hom}_L^{\mathrm{cts}}(-, L)$ in (IV.3.1), we get an isomorphism

$$\widehat{\mathrm{H}}^0(K_{\Sigma_0})'_{L,\overline{\rho}} \xrightarrow{\simeq} \mathscr{D}_0(K_p, L)^r.$$

It follows then that $\widehat{\mathrm{H}}^0(K_{\Sigma_0})'_{L,\overline{\rho}}$ is projective over $\mathscr{D}_0(\mathrm{GL}_2(\mathbf{Z}_p), L)$ since

$$\mathrm{Hom}_{\mathscr{D}_0(\mathrm{GL}_2(\mathbf{Z}_p),L)}(\widehat{\mathrm{H}}^0(K_{\Sigma_0})'_{L,\overline{\rho}}, -) = \mathrm{Hom}_{\mathscr{D}_0(K_p,L)}(\widehat{\mathrm{H}}^0(K_{\Sigma_0})'_{L,\overline{\rho}}, -)^{\mathrm{GL}_2(\mathbf{Z}_p)/K_p}$$

is the composition of two exact functors; indeed, $\widehat{\mathrm{H}}^0(K_{\Sigma_0})'_{L,\overline{\rho}}$ is free over $\mathscr{D}_0(K_p, L)$ and taking invariants for a finite group action is exact in characteristic 0. As a projective $\mathscr{D}_0(\mathrm{GL}_2(\mathbf{Z}_p), L)$-module, $\mathscr{D}_0(K_p, L)$ is a direct summand of a free module $\mathscr{D}_0(\mathrm{GL}_2(\mathbf{Z}_p), L)^s$ for some $s > 0$. Now, undoing the duality and recalling that the restriction to the Dirac distributions supported on elements of $\mathrm{GL}_2(\mathbf{Z}_p)$ induces an isomorphism [ST02a, Corollary 2.2]

$$\mathrm{Hom}_L^{\mathrm{cts}}(\mathscr{D}_0(\mathrm{GL}_2(\mathbf{Z}_p), L), L) \cong \mathcal{C}(\mathrm{GL}_2(\mathbf{Z}_p), L),$$

we get an embedding of the sort surmised in the last paragraph.

For the isomorphism (IV.3.1), we follow the ideas in §5.3 in [Eme11]. In our case, the proof is a pleasant tour through non-commutative generalizations of standard fare commutative algebra.

Recall that the *Jacobson radical* $J(\Lambda)$ of a ring Λ is the two-sided ideal of Λ of elements $r \in \Lambda$ that annihilate every simple left (equivalently, right) Λ-module. Moreover, the Jacobson radical is also proper since we assume $1 \neq 0$ in our rings. We say that Λ is *local* if its Jacobson radical $J(\Lambda)$ is a maximal ideal in Λ. One has a version of *Nakayama's lemma* in this generality: if M is a finitely generated (left) Λ-module such that $J(\Lambda)M = M$, then $M = 0$. Recall finally that a left Λ-module M is *projective* if the functor $\mathrm{Hom}_\Lambda(M, -)$ is exact. We need the following case of a general theorem of Kaplansky [Kap58]:

Theorem IV.7. *A finitely generated projective module M over a local ring Λ is free.*

Proof. The homological algebraic characterizations of (finitely generated) projective modules are available to us irrespective of whether Λ is commutative or not; in particular, we have the lifting property across surjections for maps out of projective modules and the fact that surjections onto projective modules split.

Since Λ is local, the quotient $\Lambda/J\Lambda$ of Λ by its Jacobson radical $J = J(\Lambda)$ is a division ring, therefore, every $\Lambda/J\Lambda$-module is free. Thus, there is an isomorphism

$$(\Lambda/J\Lambda)^{\oplus n} \to P/JP.$$

Composing this isomorphism with the natural projections from $\Lambda^{\oplus n}$ and P respectively onto $(\Lambda/J\Lambda)^{\oplus n}$ and P/JP, we have

$$\begin{array}{ccc} \Lambda^{\oplus n} & \twoheadrightarrow & (\Lambda/J\Lambda)^{\oplus n} \\ {\scriptstyle \exists}\,\downarrow & \searrow & \downarrow {\scriptstyle \simeq} \\ P & \twoheadrightarrow & P/JP \end{array}.$$

There's then a lift of the isomorphism to a Λ-linear map $\phi : \Lambda^{\oplus n} \to P$ making the diagram commute; the commutativity of the diagram implies, in particular, that $\mathrm{Im}(\phi) + JP = P$. Nakayama's lemma then tells us that $\mathrm{Im}(\phi) = P$; that is, ϕ is surjective. Since P is projective, ϕ splits; setting $K = \ker \phi$, there is an isomorphism $\widetilde{\phi} : \Lambda^{\oplus n} \to P \oplus K$ such that the composition with the projection to P equals ϕ. Reducing mod J, we learn that $JK = K$ since ϕ modulo J is an isomorphism. By Nakayama again, we have that $K = 0$, or that ϕ is an isomorphism. This gives us our theorem. □

It is a theorem of Lazard that, for a finitely generated pro-p group G, the completed group algebra $A[\![G]\!]$ over a commutative local ring A is complete, both left- and right-Noetherian, and local (v. Théorème 2.2.2 and ¶2.2.4 of Chapitre II, and Proposition 2.2.4 of Chapitre V in [Laz65]). (In particular, we may take $A = \mathscr{O}/\varpi^s\mathscr{O}$, or $\mathscr{O}$.)

The argument that there is an isomorphism (IV.3.1) proceeds as follows:

- Suppose that K_p is a compact open subgroup of G, and K_{Σ_0} is an allowable level for $\overline{\rho}$ such that $K_{\text{fin}} = K_p K_{\Sigma_0} K_0^{\Sigma}$ is neat. If W is a smooth K_p-module over $\mathscr{O}/\varpi^s\mathscr{O}$ then the space $\mathrm{H}^0(K_{\text{fin}}, \mathcal{V}_W)$ is isomorphic to $W^{\oplus r}$ for some r independent of W. (The r is simply the cardinality of the finite set $Y(K_{\text{fin}})$.)

- Since we have

$$\mathrm{Hom}_{\mathscr{O}[K_p]}(W, \mathrm{H}^0(K_{\Sigma_0}K_0^{\Sigma}, \mathscr{O}/\varpi^s\mathscr{O})) \simeq \mathrm{H}^0(K_p K_{\Sigma_0} K_0^{\Sigma}, \mathcal{V}_W)$$

it follows that $\mathrm{H}^0(K_{\Sigma_0}K_0^{\Sigma}, \mathscr{O}/\varpi^s\mathscr{O})$ is injective as an $(\mathscr{O}/\varpi^s\mathscr{O})[K_p]$-module; therefore its Pontryagin dual $\mathrm{H}^0(K_{\Sigma_0}K_0^{\Sigma}, \mathscr{O}/\varpi^s\mathscr{O})^{\vee}$ is a projective $(\mathscr{O}/\varpi^s\mathscr{O})[\![K_p]\!]$-module. By our Theorem IV.7 and the discussion after it, we get that

$$\mathrm{H}^0(K_{\Sigma_0}K_0^{\Sigma}, \mathcal{O}/\varpi^s\mathcal{O})^\vee \simeq (\mathcal{O}/\varpi^s\mathcal{O})[\![K_p]\!]^r$$

for an r independent of s. Undoing the Pontryagin duality by noting that the Pontryagin dual of $(\mathcal{O}/\varpi^s\mathcal{O})[\![K_p]\!]$ is $\mathcal{C}(K_p, \mathcal{O}/\varpi^s\mathcal{O})$, we get that

$$\widehat{\mathrm{H}}^0(K_{\Sigma_0})_{\mathcal{O}/\varpi^s\mathcal{O}} \simeq \mathcal{C}(K_p, \mathcal{O}/\varpi^s\mathcal{O})^r$$

for an r independent of s.

- In particular, since the local ring $\mathbf{T}(K_{\Sigma_0})_{\overline{\rho}}$ is a direct summand of $\mathbf{T}(K_{\Sigma_0})$, we get that $\widehat{\mathrm{H}}^0(K_{\Sigma_0})_{\mathcal{O}/\varpi^s\mathcal{O},\overline{\rho}}$ is also injective and the arguments above apply equally to $\widehat{\mathrm{H}}^0(K_{\Sigma_0})_{\mathcal{O}/\varpi^s\mathcal{O},\overline{\rho}}$ to give us an isomorphism

$$\widehat{\mathrm{H}}^0(K_{\Sigma_0})_{\mathcal{O}/\varpi^s\mathcal{O},\overline{\rho}} \simeq \mathcal{C}(K_p, \mathcal{O}/\varpi^s\mathcal{O})^r$$

for some r independent of s.

- Since $\widehat{\mathrm{H}}^0(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}$ is the projective limit

$$\varprojlim \widehat{\mathrm{H}}^0(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}/\varpi^s\, \widehat{\mathrm{H}}^0(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}$$

and the quotient $\widehat{\mathrm{H}}^0(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}/\varpi^s\, \widehat{\mathrm{H}}^0(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}}$ is isomorphic to $\mathcal{C}(K_p, \mathcal{O}/\varpi^s\mathcal{O})^r$ for some $r > 0$ independently of s as an admissible smooth K_p-representation, we get an isomorphism

$$\widehat{\mathrm{H}}^0(K_{\Sigma_0})_{\mathcal{O},\overline{\rho}} \cong \mathcal{C}(K_p, \mathcal{O})^r.$$

- by extending scalars from $\mathcal{O}$ to L, the required isomorphism follows. □

We shall show that the space of locally algebraic vectors generated by Hecke eigenvectors corresponding to classical modular eigensystems that are crystalline is dense in $\widehat{\mathrm{H}}^0_{L,\overline{\rho},\Sigma}$ (analogous to Corollary 5.4.5 of [Eme11]).

Corollary IV.8. *Let $\mathcal{C}$ denote the set of closed points $\mathfrak{p} \in \operatorname{Spec} \mathbf{T}_{\overline{\rho},\Sigma}[1/p]$ that are classical and whose associated Galois representations are crystalline locally at p. The submodule $\bigoplus_{\mathfrak{p}\in\mathcal{C}} \widehat{\mathrm{H}}^0_{L,\overline{\rho},\Sigma}[\mathfrak{p}]^{\mathrm{alg}}$ is dense in $\widehat{\mathrm{H}}^0_{L,\overline{\rho},\Sigma}$.*

Proof. This follows from the characterization of locally algebraic vectors in $\widehat{\mathrm{H}}^0(K_{\Sigma_0})_L$ (v. Corollary II.19) and passing to the inductive limit over K_{Σ_0} in our previous proposition. □

We deduce that the set of classical modular eigensystems that are regular crystalline is Zariski dense in the whole space.

Corollary IV.9. *Let $\mathcal{C}$ denote the set of closed points $\mathfrak{p} \in \operatorname{Spec} \mathbf{T}_{\overline{\rho},\Sigma}[1/p]$ that are classical and whose associated Galois representations are crystalline locally at p. Then $\mathcal{C}$ is dense in* $\operatorname{Spec} \mathbf{T}_{\overline{\rho},\Sigma}$.

Proof. Note that a $t \in \bigcap_{\mathfrak{p}\in\mathcal{C}} \mathfrak{p}$ annihilates $\bigoplus_{\mathfrak{p}\in\mathcal{C}} \widehat{\mathrm{H}}^0_{L,\overline{\rho},\Sigma}[\mathfrak{p}]^{\mathrm{alg}}$; since this space is dense in $\widehat{\mathrm{H}}^0_{L,\overline{\rho},\Sigma}$, it follows that t also annihilates $\widehat{\mathrm{H}}^0_{L,\overline{\rho},\Sigma}$. But $\widehat{\mathrm{H}}^0_{L,\overline{\rho},\Sigma}$ is a faithful $\mathbf{T}_{\overline{\rho},\Sigma}$-module, so $t = 0$. This proves that $\mathcal{C}$ is Zariski dense as desired. □

We close with the following proposition:

Proposition IV.10. *Suppose that $\rho_f : \mathscr{G}_{\mathbf{Q}} \to \mathrm{GL}_2(L)$ is the Galois representation associated to a cuspidal newform f of weight $k \geqslant 2$ that is unramified at p (equivalently, crystalline locally at p) and $\overline{\rho}_f \simeq \overline{\rho}$. Then, there is an injective map*

$$\Pi\left(\rho_f|_{\mathscr{G}_{\mathbf{Q}_p}}\right) \hookrightarrow \widehat{\mathrm{H}}^0(K_{\Sigma_0})_{L,\overline{\rho}}$$

for a level K_{Σ_0} allowable for $\overline{\rho}$.

Proof. Under the assumptions on $\overline{\rho}$ at places dividing d, the Jacquet-Langlands correspondence implies that f transfers to $D^\times$. Therefore, by Corollary II.19 and by definition of allowable levels, there is an allowable level K_{Σ_0} and a continuous $\mathrm{GL}_2(\mathbf{Q}_p)$- and Hecke-equivariant injection

$$\operatorname{Sym}^{k-2} L^{\oplus 2} \otimes \mathrm{LLC}(\rho|_{\mathscr{G}_{\mathbf{Q}_p}}) \hookrightarrow \widehat{\mathrm{H}}^0(K_{\Sigma_0})_{L,\overline{\rho}}. \quad \text{(IV.3.2)}$$

By Theorem III.16 (6), the representation $\Pi\left(\rho_f|_{\mathscr{G}_{\mathbf{Q}_p}}\right)$ is then the unitary completion of

$$\mathrm{LAlg}(\rho_f|_{\mathscr{G}_{\mathbf{Q}_p}}) = \operatorname{Sym}^{k-2} L^{\oplus 2} \otimes \mathrm{LLC}(\rho|_{\mathscr{G}_{\mathbf{Q}_p}})$$

with respect to a finitely generated $\mathrm{GL}_2(\mathbf{Q}_p)$-stable lattice (the key point is that, when $k \geqslant 2$, the representation $\rho_f|_{\mathscr{G}_{\mathbf{Q}_p}}$ is trianguline and the set of commensurability classes of lattices in $\mathrm{LAlg}(\rho_f|_{\mathscr{G}_{\mathbf{Q}_p}})$ is singleton; cf. the main theorem in [Bre03b]). Since the map (IV.3.2) and its target is complete, the map extends to a non-zero map

$$\Pi\left(\rho_f|_{\mathscr{G}_{\mathbf{Q}_p}}\right) \to \widehat{\mathrm{H}}^0(K_{\Sigma_0})_{L,\overline{\rho}}.$$

But since the source is irreducible, this non-zero map is also injective. □

IV.4: The multiplicity module

Let $\overline{\pi} = \Pi(\overline{\rho})$ be the smooth k-representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ attached to $\overline{\rho}$ by Colmez's functor. Let π be a deformation of $\overline{\pi}$ to the local $\mathscr{O}$-algebra $\mathbf{T}_{\overline{\rho},\Sigma}$ such that $\mathbf{V}(\pi) = \rho^{\mathrm{m}}|_{\mathscr{G}_{\mathbf{Q}_p}}$, the universal modular deformation of $\overline{\rho}$ introduced in Section IV.2

Since $\mathbf{V}$ is an exact functor, for any prime ideal $\mathfrak{p}$ of $\mathbf{T}_{\overline{\rho},\Sigma}$, we have that

$$\mathbf{V}(\pi/\mathfrak{p}\pi) = \rho^{\mathrm{m}}|_{\mathscr{G}_{\mathbf{Q}_p}}(\mathfrak{p})$$

where $\rho^{\mathrm{m}}|_{\mathscr{G}_{\mathbf{Q}_p}}(\mathfrak{p})$ is the composition of ρ^{m} with the map $\mathrm{GL}_2(\mathbf{T}_{\overline{\rho},\Sigma}) \to \mathrm{GL}_2(\mathbf{T}_{\overline{\rho},\Sigma}/\mathfrak{p})$. Since Π and $\mathbf{V}$ are exact, if $\mathfrak{p}$ is the prime ideal associated to a classical modular eigensystem f on $D^\times$ (that is crystalline locally at p), then $\pi/\mathfrak{p}\pi$ is $\Pi(\rho_f|_{\mathscr{G}_{\mathbf{Q}_p}})$.

In Proposition IV.10, we have shown that, for prime ideals $\mathfrak{p}$ of $\mathbf{T}_{\overline{\rho},\Sigma}$ which are classical modular and crystalline locally at p, there is an injective map $\pi/\mathfrak{p}\pi \hookrightarrow \widehat{\mathrm{H}}^0_{L,\overline{\rho},\Sigma}$; since the source is annihilated by $\mathfrak{p}$, the image is a $\mathbf{T}_{\overline{\rho},\Sigma}$-submodule of $\widehat{\mathrm{H}}^0_{L,\overline{\rho},\Sigma}[\mathfrak{p}] \simeq \widehat{\mathrm{H}}^0_{\mathscr{O},\overline{\rho},\Sigma}[\mathfrak{p}]$; and in Corollary IV.9, we have seen that the set of such prime ideals is dense in $\mathbf{T}_{\overline{\rho},\Sigma}$.

We now wish to show that for all prime ideals $\mathfrak{p}$ of $\mathbf{T}_{\overline{\rho},\Sigma}$, there is an injection

$$\Pi(\rho^{\mathrm{m}}|_{\mathscr{G}_{\mathbf{Q}_p}}(\mathfrak{p})) \hookrightarrow \widehat{\mathrm{H}}^0_{\mathscr{O},\overline{\rho},\Sigma}[\mathfrak{p}].$$

In analogy with Emerton's work [Eme11], we therefore study the "multiplicity module"

$$\mathfrak{X} = \mathrm{Hom}_{\mathbf{T}_{\overline{\rho},\Sigma}[\mathrm{GL}_2(\mathbf{Q}_p)]-\mathrm{cont}}(\pi, \widehat{\mathrm{H}}^0_{\mathscr{O},\overline{\rho},\Sigma})$$

Proposition IV.11.

1. *The $\mathbf{T}_{\overline{\rho},\Sigma}$-module $\mathfrak{X}$ is cofinitely generated.*

2. *The $\mathscr{O}$-dual $\mathrm{Hom}_{\mathscr{O}}(\mathfrak{X},\mathscr{O})$ of $\mathfrak{X}$ is finitely generated as a $\mathbf{T}_{\overline{\rho},\Sigma}$-module.*

Proof.

1. Recall from [Eme11, Definition C.1] that we need to check the following properties for $\mathfrak{X}$:

   - $\mathfrak{X}$ is ϖ-adically complete, separated and $\mathscr{O}$-torsion free;
   - When $\mathbf{T}_{\overline{\rho},\Sigma}$ is given its $\mathfrak{m}_{\overline{\rho}}$-adic topology and $\widehat{\mathrm{H}}^0_{\mathscr{O},\overline{\rho},\Sigma}$ its ϖ-adic topology, the morphism affording the $\mathbf{T}_{\overline{\rho},\Sigma}$-module structure

$$\mathbf{T}_{\overline{\rho},\Sigma} \times \widehat{\mathrm{H}}^0_{\mathscr{O},\overline{\rho},\Sigma} \to \widehat{\mathrm{H}}^0_{\mathscr{O},\overline{\rho},\Sigma}$$

is continuous; and

- the quotient $(\mathfrak{X}/\varpi\mathfrak{X})\,[\mathfrak{m}_{\overline{\rho}}]$ is finite dimensional over k.

Since $\widehat{\mathrm{H}}^0$ is ϖ-adically complete, separated and $\mathscr{O}$-torsion free, so is $\mathfrak{X}$. The continuity of the action map is by definition of the Hecke algebra. To check that quotient $(\mathfrak{X}/\varpi\mathfrak{X})\,[\mathfrak{m}_{\overline{\rho}}]$ is finite dimensional over k, we argue as in Emerton [Eme11, Theorem 6.3.12]. Note that the reduction mod ϖ of the map

$$\mathfrak{X} \to \mathrm{Hom}_{\mathbf{T}_{\overline{\rho},\Sigma}[\mathrm{GL}_2(\mathbf{Q}_p)]-\mathrm{cont}}(\pi, \widehat{\mathrm{H}}^0_{\mathscr{O},\overline{\rho},\Sigma})$$

induces an injective map

$$\mathfrak{X}/\varpi\mathfrak{X} \to \mathrm{Hom}_{k[\mathrm{GL}_2(\mathbf{Q}_p)]}(\pi_{\overline{\rho}}/\varpi\pi_{\overline{\rho}}, \widehat{\mathrm{H}}^0_{k,\overline{\rho},\Sigma}).$$

Since $\mathscr{O}/\varpi = \mathbf{T}_{\overline{\rho},\Sigma}/\mathfrak{m}_{\overline{\rho}} = k$, and $\pi_{\overline{\rho}}$ is a deformation of $\Pi(\overline{\rho})$ to the local $\mathscr{O}$-algebra $\mathbf{T}_{\overline{\rho},\Sigma}$, it follows that $\pi_{\overline{\rho}}/\varpi\pi_{\overline{\rho}}$ is isomorphic to $\Pi(\overline{\rho})$ as a $k[\mathrm{GL}_2(\mathbf{Q}_p)]$-module.

Passing to $\mathfrak{m}_{\overline{\rho}}$-torsion parts, we have an injective map

$$\begin{aligned}(\mathfrak{X}/\varpi\mathfrak{X})\,[\mathfrak{m}_{\overline{\rho}}] &\hookrightarrow \mathrm{Hom}_{k[\mathrm{GL}_2(\mathbf{Q}_p)]}(\Pi(\overline{\rho}), \widehat{\mathrm{H}}^0_{k,\overline{\rho},\Sigma}[\mathfrak{m}_{\overline{\rho}}]) \\ &\xrightarrow{\sim} \mathrm{Hom}_{k[\mathrm{GL}_2(\mathbf{Q}_p)]}(\Pi(\overline{\rho}), \widehat{\mathrm{H}}^0_{k,\overline{\rho},\Sigma}).\end{aligned}$$

Let $\overline{W}$ be any finite dimensional k-subspace of $\Pi(\overline{\rho})$; since $\Pi(\overline{\rho})$ is irreducible, the k-subspace $\overline{W}$ generates $\Pi(\overline{\rho})$. Since $\Pi(\overline{\rho})$ is smooth, there is an open compact subgroup K_p of G which fixes $\overline{W}$ pointwise. Restricting to K_p-action, we have an inclusion

$$\mathrm{Hom}_{k[\mathrm{GL}_2(\mathbf{Q}_p)]}(\Pi(\overline{\rho}), \widehat{\mathrm{H}}^0_{k,\overline{\rho},\Sigma}) \hookrightarrow \overline{W}^\vee \otimes_k (\widehat{\mathrm{H}}^0_{k,\overline{\rho},\Sigma})^{K_p}.$$

The space $(\widehat{\mathrm{H}}^0_{k,\overline{\rho},\Sigma})^{K_p}$ is finite dimensional by arguments in the proof of Proposition 5.3.13 and Corollary 5.3.14 of [Eme11] and therefore so is $(\mathfrak{X}/\varpi\mathfrak{X})\,[\mathfrak{m}_{\overline{\rho}}]$.

2. This follows from [Eme11, Proposition C.5] in view of 1. □

IV.5: The theorem

Theorem IV.12. *Let $\mathbb{G} = D^\times$ where D is a definite quaternion algebra of discriminant $d\infty$; we assume that $p \nmid d$ so $D \otimes \mathbf{Q}_p$ is the matrix algebra. Fix a finite extension L of $\mathbf{Q}_p$ with ring of integers $\mathscr{O}$, and residue field k_L.*

Fix a continuous absolutely irreducible representation $\overline{\rho} : \mathscr{G}_{\mathbf{Q}} \to \mathrm{GL}_2(k)$ that is modular and satisfies the hypotheses (Irr$_p$) and (SI$_d$) in Section IV.1. Let Σ_0 be a finite set places not containing p so that $\overline{\rho}$ is unramified outside $\Sigma = \Sigma_0 \cup \{p\}$.

Let $\pi = \mathbf{V}(\rho^m|_{\mathscr{G}_{\mathbf{Q}_p}})$ be the p-adic local Langlands correspondent of the universal modular deformation ρ^m of $\overline{\rho}$.

Let $\lambda : \mathbf{T}_{\overline{\rho},\Sigma} \to \mathscr{O}$ be any system of Hecke eigenvalues; let $\mathfrak{p} = \ker\lambda$. Then:

1. *There is a non-zero $\mathbb{G}(\mathbf{Q}_p)$-equivariant map*

$$\pi/\mathfrak{p}\pi \to \widehat{\mathrm{H}}^0_{\overline{\rho},\mathscr{O}}[\mathfrak{p}]$$

   of $\mathbf{T}_{\overline{\rho},\Sigma}$-modules.

2. *If λ is associated to a Galois representation $\rho_\lambda : G_{\mathbf{Q}} \to \mathrm{GL}_2(L)$, then, $(\pi/\mathfrak{p}\pi)$ is the locally analytic representation $\Pi(\rho_\lambda|_{\mathscr{G}_{\mathbf{Q}_p}})$ associated to $\rho_\lambda|_{\mathscr{G}_{\mathbf{Q}_p}}$ by the p-adic local Langlands correspondence and so every non-zero map of (1) above extends to a non-zero map*

$$\Pi(\rho_\lambda|_{\mathscr{G}_{\mathbf{Q}_p}}) \hookrightarrow \widehat{\mathrm{H}}^0_{\overline{\rho},L}\,.$$

Proof. For avoidance of notational clutter, let $\mathbf{T} = \mathbf{T}_{\overline{\rho},\Sigma}$. The proof proceeds in two steps:

Step 1 Consider the module

$$\mathfrak{X} = \mathrm{Hom}_{\mathbf{T}[\mathrm{GL}_2(\mathbf{Q}_p)]-\mathrm{cts}}(\pi, \widehat{\mathrm{H}}^0_{\overline{\rho},\mathscr{O}})$$

Then $\mathfrak{X}[\mathfrak{p}]$ is identified with the $\mathbf{T}$-module of maps

$$\mathrm{Hom}_{\mathbf{T}[\mathrm{GL}_2(\mathbf{Q}_p)]-\mathrm{cts}}(\pi/\mathfrak{p}\pi, \widehat{\mathrm{H}}^0_{\overline{\rho},\mathscr{O}}[\mathfrak{p}]).$$

The module $\mathfrak{X}$ allows to argue that $\mathfrak{X}[\mathfrak{p}] \neq 0$ *for all primes* $\mathfrak{p}$ can be deduced from $\mathfrak{X}[\mathfrak{p}] \neq 0$ *for a Zariski dense subset of primes.* To make this reduction, we argue as follows:

- There is a natural isomorphism (Proposition C.14 of [Eme11])

$$\mathrm{Hom}_L(L \otimes \mathfrak{X}[\mathfrak{p}], L) \to L \otimes (\mathrm{Hom}_{\mathscr{O}}(\mathfrak{X}, \mathscr{O}) \otimes \mathbf{T}/\mathfrak{p})$$

  thus, setting $M = \mathrm{Hom}_{\mathscr{O}}(\mathfrak{X}, \mathscr{O})$, if $M/\mathfrak{p}M \neq 0$, then $\mathfrak{X}[\mathfrak{p}] \neq 0$.

- If there is a Zariski dense set of primes $\mathfrak{p}$ such that $M/\mathfrak{p}M \neq 0$, then M is a faithful $\mathbf{T}$-module: if $t \in \mathbf{T}$ belongs to the annihilator of M, then $t \in \mathfrak{p}$ for a

Zariski dense set of primes so t belongs to every prime. Since $\mathbf{T}$ is reduced, we deduce $t = 0$ as required.

- Recall that M is finitely generated (v. Proposition IV.11). Thus, if $M = \mathfrak{p}M$ for a prime $\mathfrak{p}$, then, there would exist a $t \in \mathfrak{p}$ such that $(1+t)M = 0$ by Nakayama's lemma. But M is a faithful $\mathbf{T}$-module (by Step 2 in light of the discussion above), so we've arrived at a contradiction.

Step 2 We recall from Proposition IV.10 that there are injective (in particular, non-zero!) maps

$$\pi/\mathfrak{p}_\lambda \pi = \Pi(\rho_\lambda|_{\mathscr{G}_{\mathbf{Q}_p}}) \hookrightarrow \widehat{\mathrm{H}}^0_{\mathscr{O},\overline{\rho},\Sigma}[\mathfrak{p}_\lambda]$$

for Hecke eigensystems λ of weight at least 2 and crystalline at p; thus, $\mathfrak{X}[\mathfrak{p}_\lambda] \neq 0$. Since the set $\mathcal{C}$ of such eigensystems is dense in $\operatorname{Spec} \mathbf{T}_{\overline{\rho},\Sigma}$, it follows that $\mathfrak{X}[\mathfrak{p}] \neq 0$ for all primes $\mathfrak{p}$. $\square$

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