Transcendental Thurston Theory and Dynamical Approximations

by

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To my Amma and Appa

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ABSTRACT

A holomorphic map is said to be postsingularly finite (PSF) if it has finitely many singular values and every singular orbit is finite. If every singular value is a critical value, a postsingularly finite map is also called postcritically finite (PCF). As shown by Douady and Hubbard in their study of the Mandelbrot set, these maps are crucial in understanding the structure of parameter spaces, and can often be determined by a finite amount of combinatorial data. Thurston's theory considers a certain class of continuous maps with a finite postsingular set and asks when such a map is equivalent to a holomorphic PSF map. While this theory was initially introduced by William Thurston for topological analogs of rational maps on the sphere, the last few decades have seen a push to generalize this theory to topological analogs of entire maps on the plane, starting with the work of Hubbard, Schleicher and Shishikura on topological versions of exponential functions. In this thesis, we explore the relationship between finite and infinite degree Thurston theory, and use this relationship to establish dynamically meaningful approximations for PSF entire functions by PCF polynomials.

CHAPTER I
Introduction

For an entire function $g : \mathbb{C} \to \mathbb{C}$, a *critical* value of g is a point $y \in \mathbb{C}$ for which exists a point $x \in g^{-1}(y)$ such that $g'(x) = 0$; an *asymptotic* value is a point $a \in \mathbb{C}$ for which there exists an arc $\gamma : [0, \infty) \to \mathbb{C}$ with the property that $\gamma(t) \to \infty$ and $g(\gamma(t)) \to a$ as $t \to \infty$. The critical values and asymptotic values of g are collectively called its *singular* values; they can be equivalently characterized as the points in $\mathbb{C}$ where some branch of g^{-1} does not exist locally.

The structures of the Fatou, Julia and escaping sets of an entire function g are known to be highly dependent on the iterative behavior of the set of singular values S_g (see [Mil06], [Fat26], [EL89], [BKL91] and [Er9]). Generally, the dynamical behavior of entire functions is markedly different from that of polynomials. However, certain classes of entire maps show many similarities with polynomials in their dynamical behavior. One such class is the family of entire maps g with $|S_g| < \infty$, for which Sullivan's theorem on non-wandering domains holds. For general entire functions g, the relationship between the dynamical system $g|S_g : S_g \longrightarrow S_g$ and the existence of Fatou components such as Baker and wandering domains, which are unique to the transcendental setting, has been studied in [BHK⁺93], [Bak63], [Bak76] and [Bak84].

Singular values also come into play in the study of parameter spaces. The complicated structure of the Mandelbrot set, for instance, can be explained using special maps that satisfy the property of *postsingular finiteness* (see [DH84], [Sch04]); an entire map g is said to be postsingularly finite (or PSF) if its postsingular set, defined as $P_g = \overline{\bigcup_{n \geqslant 0} g^{\circ n}(S_g)}$, is finite. If, as in the case of polynomials, there are no asymptotic values, a postsingularly finite map is also said to be *postcritically finite* (or PCF in short).

While it is hard to overstate the importance of PSF functions, a huge challenge in utilizing these is that they are hard to find: it is difficult to answer questions such as "what is a transcendental entire map g with a hundred singular values, with each singular value fixed under g?"

I.1: Classical Thurston Theory

One of the major breakthroughs in complex dynamics is William Thurston's program to construct postcritically finite *rational* maps on $\widehat{\mathbb{C}}$ with prescribed iterative behavior on their set of critical values, by starting with orientation-preserving branched covers of the sphere $\mathbb{S}^2$. For such a map f, we let S_f denote the set of critical values (which may now include the point at infinity), and define P_f as the closure of the union of singular orbits, as above. The function f is called a *Thurston map* if the set P_f is finite. Two Thurston maps f and g on $\mathbb{S}^2$ are said to be *combinatorially equivalent* if there exist homeomorphisms $\varphi_0, \varphi_1 \in \mathrm{Homeo}^+(\mathbb{S}^2)$ which are isotopic rel. P_f such that the following diagram commutes:

$$\begin{array}{ccc} (\mathbb{S}^2, P_f) & \xrightarrow{\varphi_1} & (\mathbb{S}^2, P_g) \\ \Big\downarrow f & & \Big\downarrow g \\ (\mathbb{S}^2, P_f) & \xrightarrow{\varphi_0} & (\mathbb{S}^2, P_g) \end{array}$$

If a Thurston map is combinatorially equivalent to a PCF rational map, then it is said to be *realized*. Otherwise, it is said to be *obstructed*. It is useful to think of realizability of a Thurston map as the existence of a holomorphic model for that map. The core of Thurston theory is a pathway to determine whether a given Thurston map is realized. However, it is known that the case $|P_f| = 1$ does not occur, and if $|P_f| = 2$, then f is Thurston equivalent to $z \mapsto z^{\deg f}$ (see [Hub16, Corollary 10.6.6]). Associated with every Thurston map f with $|P_f| \geqslant 3$ is an operator called its *Thurston pullback map*, denoted σ_f, which acts on the Teichmüller space $\mathrm{T}(\mathbb{S}^2, P_f)$. The realizability of f is equivalent to a condition on the dynamics of σ_f:

Theorem I.1 (Thurston's theorem; [Hub16, Theorem 10.6.4])**.** *A Thurston map f on $\mathbb{S}^2$ with $|P_f| \geqslant 3$ is combinatorially equivalent to a postcritically finite rational function if and only if the Thurston pullback map σ_f has a fixed point in* $\mathrm{T}(\mathbb{S}^2, P_f)$.

For a rational Thurston map f on the sphere with hyperbolic orbifold (see [Hub16, Definition 10.1.9]), Douady and Hubbard also proved Thurston's criterion, which gives an equivalent condition for f being obstructed. Somewhat surprisingly, this condition is purely topological: f is obstructed if and only if it does not have a certain invariant multicurve with special properties under iteration. In [HS94], Hubbard and Schleicher further studied the special case of unicritical polynomials, and provided an answer for the realizability problem in terms of an operator called the *spider* operator, which emulates the behavior of Thurston pullback but is distinct from the latter.

I.2: Thurston Theory on the plane

Thurston's program was first extended by Hubbard, Schleicher and Shishikura ([HSS09]) to a class of maps on the plane which were topological analogs of PSF exponentials. Following their work, there has been a drive to generalize Thurston's work even further, to topological analogs of general PSF entire maps.

Formally, a Thurston map on $\mathbb{R}^2$ is a continuous function $f : \mathbb{R}^2 \to \mathbb{R}^2$ which satisfies three properties: (1) at each point x in the domain, f locally "looks" like $z \mapsto z^d$ for some $d \in \mathbb{N}$ that depends only on x; (2) f is postsingularly finite; (3) f has *stable parabolic type*. Condition (3) is equivalent to the statement that for any complex structure (i.e., maximal holomorphic atlas) $\mathcal{A}$ on $\mathbb{R}^2$ such that $(\mathbb{R}^2, \mathcal{A})$ has the conformal type of $\mathbb{C}$, the Riemann surface $(\mathbb{R}^2, f^*\mathcal{A})$ obtained by pulling back $\mathcal{A}$ also has the conformal type of $\mathbb{C}$. Under this formulation, for every orientation-preserving homeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{C}$, there exists an orientation-preserving homeomorphism $\psi : \mathbb{R}^2 \to \mathbb{C}$ unique up to post-composition with an affine map, such that the function $\varphi \circ f \circ \psi^{-1} : \mathbb{C} \to \mathbb{C}$ is entire. Thurston maps on $\mathbb{R}^2$ can have finite or infinite degree: those of the former type are said to be *polynomial*, and the latter are referred to as *transcendental*. Note that polynomial Thurston maps can be thought of as Thurston maps on $\mathbb{S}^2$ in the classical sense, which additionally satisfy the condition $f^{-1}(\infty) = \{\infty\}$.

As in classical Thurston theory, two Thurston maps f and g on $\mathbb{R}^2$ are said to be combinatorially equivalent if there exist homeomorphisms $\varphi_0, \varphi_1 \in \mathrm{Homeo}^+(\mathbb{R}^2)$ which are isotopic rel. P_f, such that $\varphi_0(P_f) = \varphi_1(P_f) = P_g$, and $g = \varphi_0 \circ f \circ \varphi_1^{-1}$. We say f is realized if it is combinatorially equivalent to a postsingularly finite entire function, and obstructed otherwise. For a Thurston map f on the plane, the Thurston pullbak operator σ_f is similarly defined, but in this setting it acts on the Teichmüller space $\mathrm{T}(\mathbb{S}^2, P_f \cup \{\infty\})$. This operator σ_f has a fixed point if and only if f is Thurston equivalent to a PSF entire map.

By a theorem due to Berstein, Lei, Levy and Rees ([Hub16, Theorem 10.3.8]), a polynomial Thurston map is known to be realized if and only if it doesn't have a topological multicurve called a *Levy cycle*. Conditions for being obstructed are not fully understood for transcendental Thurston maps, outside of the case $|S_f| = 1$ handled by Hubbard, Schleicher and Shishikura, where it was shown that f is realized if and only if it does not have a Levy cycle. Since then, it is conjectured that the Levy cycle criterion holds for larger spaces of transcendental Thurston maps. Presently, understanding equivalent conditions for being obstructed remains one of the most important open questions in complex dynamics. However, the striking similarities between the topological exponentials studied in [HSS09] and unicritical polynomial Thurston maps motivates the question of whether there is a more direct link between Thurston theory in the polynomial and transcendental regimes.

This thesis explores the relationship between Thurston theory in the polynomial and transcendental settings. Our driving force is the philosophy of *approximation*: we are interested in approximating transcendental Thurston maps and their pullback operators by corresponding polynomial objects. We will also present results that utilize this link between polynomial and transcendental Thurston theory to construct dynamically meaningful approximations for transcendental entire functions. Furthermore, in **Chapter IX**, we provide detail on possible future applications of our results in tackling the question of realizability for transcendental Thurston maps.

We summarize our results in the remainder of this chapter.

I.3: Main Results

I.3.1: Approximating Thurston pullback maps

This thesis studies ways to approximate a transcendental Thurston map by polynomial Thurston maps. Through Chapters III, IV and V, we present joint work with Nikolai Prochorov and Bernhard Reinke.

In **Chapter III**, we introduce two notions of convergence for a sequence of Thurston maps (f_n).

Given a sequence of Thurston maps $f_n\colon \mathbb{R}^2 \to \mathbb{R}^2, n \in \mathbb{N}$ and a Thurston map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $P_{f_n} = P_f = A$ for all $n \in \mathbb{N}$, the maps f_n are said to converge to f *topologically* if for every compact subset K of $\mathbb{R}^2$, $f_n|K$ coincides with $f|K$ for all n sufficiently large. While this is seemingly a strict notion of convergence, it is equivalent in a certain sense to *combinatorial convergence*, which, loosely, only requires that loop-lifting under f_n eventually resemble loop-lifting under f.

More formally, for f and f_n as above, we say that $f_n \to f$ *combinatorially* if there exists a point $t \in \mathbb{R}^2 \backslash A$ and points b, b_n with $f(b) = f_n(b_n) = t$ for all $n \in \mathbb{N}$, such that the following condition is satisfied: for every loop $\gamma \subset \mathbb{R}^2 \backslash A$ based at t, there exists $N(\gamma) \in \mathbb{N}$ such that for all $n \geqslant N(\gamma)$,

- the lift of γ under f based at b, denoted $\gamma\uparrow(f, b)$ (also see Definition A.3) is a loop if and only if $\gamma\uparrow(f_n, b_n)$ is a loop;
- if $\gamma\uparrow(f, b)$ is a loop, then it is path-homotopic in $\mathbb{R}^2 \backslash A$ to $\gamma\uparrow(f_n, b_n)$.

It is straightforward that topological convergence implies combinatorial convergence; we also show that if $f_n \to f$ combinatorially, then there exists a sequence $(\widetilde{f}_n)$ of Thurston maps converging topologically to f such that $\widetilde{f}_n$ and f_n are *isotopic* rel. A (i.e., $\widetilde{f}_n = f_n \circ \varphi_n$ for some $\varphi_n \in \mathrm{Homeo}_0^+(\mathbb{R}^2, A)$).

Combinatorial convergence of the sequence (f_n) implies controlled behavior of the sequence of pullback operators (σ_{f_n}).

Theorem I.2 (Mukundan, Prochorov and Reinke; [MPR24, Main Theorem B])**.** *Let $f_n \colon \mathbb{R}^2 \to \mathbb{R}^2, n \in \mathbb{N}$ and $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ be Thurston maps with $P_f = P_{f_n} = A$ for all $n \in \mathbb{N}$. If $f_n \to f$ combinatorially, then $\sigma_{f_n} \to \sigma_f$ locally uniformly on $\mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$.*

In **Chapter IV**, we describe a technique of constructing topological holomorphic maps using covering maps between regular planar graphs, and use it in Proposition IV.15 to construct, for a given Thurston map $f : (\mathbb{R}^2, A) \circlearrowleft$, a sequence of polynomial Thurston maps converging combinatorially to f.

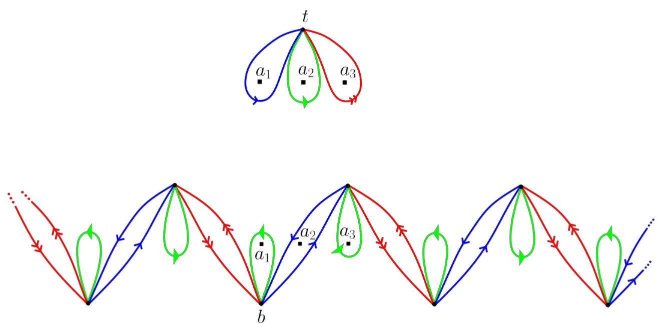


Figure 1.1: Rose graph $\mathcal{R}$ (top) and its pre-image Γ (bottom) for the PSF function $f(z) = \pi\cos(z)/2$, where $P_f = \{a_1, a_2, a_3\} = \{-\pi/2, 0, \pi/2\}$.

We give a rough sketch of this construction here: we draw a rose graph $\mathcal{R}$ based at a vertex $t \notin \mathbb{R}^2 \backslash A$ (see Figure 1.1 above), whose edges form a generating set for $\pi_1(\mathbb{R}^2 \backslash A, t)$. The graphs $\mathcal{R}$ and $\Gamma := f^{-1}(\mathcal{R})$ are both $2|A|-$regular, and the restriction $f|\Gamma : \Gamma \to \mathcal{R}$ is a covering map. For any exhaustion (K_n) of Γ by finite graphs, we construct a sequence of $f_n : \Gamma_n \to \mathcal{R}$, where Γ_n is a finite planar $2|A|-$regular graph (see Figure 1.2) containing K_n . The covering f_n also coincides with f on K_n. To end, we show how to extend each f_n to a polynomial Thurston map with postsingular set A, so that the sequence (f_n) converges to f combinatorially. This discussion leads to the following result:

Theorem I.3 (Mukundan, Prochorov and Reinke; Proposition IV.15, together with Theorem I.4)**.** *For any Thurston map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with $P_f = A$, there exists a sequence of polynomial Thurston maps $f_n : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with $P_{f_n} = A$ such that $\sigma_{f_n} \to \sigma_f$ locally uniformly on $\mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$.*

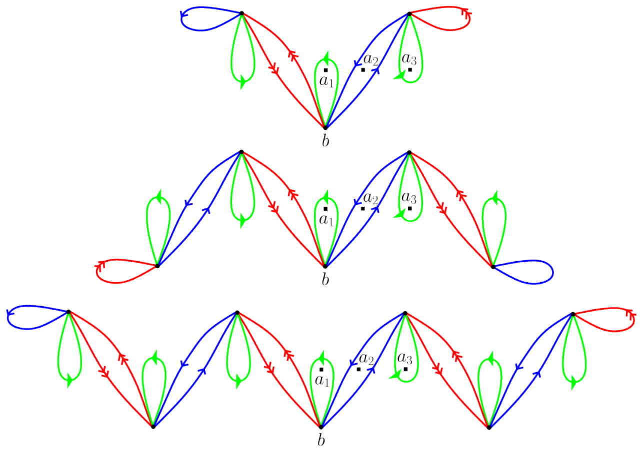


Figure 1.2: Example of a sequence graphs Γ_n approximating Γ in Figure 1.1 for the function $f(z) = \pi\cos(z)/2$. Here the subgraph $K_n \subset \Gamma$ is the union of all directed paths in Γ with $\leqslant 2n+1$ edges starting and ending at the point b.

The contraction properties of pullback maps and the local uniform convergence of $\sigma_{f_n} \to \sigma_f$ imply that if f is realized, then f_n is realized for all n sufficiently large, and the fixed points of the σ_{f_n} converge to the fixed points of σ_f in Teichmüller space.

I.3.2: Dynamical approximations of postsingularly finite entire functions

Say f is an entire map that is realized as the local uniform limit on $\mathbb{C}$ of a sequence of entire maps (f_n). In a broad sense, the approximation $f_n \to f$ is considered to be dynamically meaningful if there is some dynamical property of f that every map f_n also satisfies.

Several major results in transcendental dynamics have been derived using techniques generalized from polynomial dynamics, as well as by developing dynamically meaningful approximations by polynomials that preserve a prescribed property. Devaney, Goldberg and Hubbard illustrated a dynamical approximation for *escaping exponentials* (see [MSRIBDGH86]), and Kisaka ([Kis95]) provided sufficient conditions for the convergence of Julia sets of sequences of entire maps.

The question of whether any dynamical approximation by polynomials exists for a given entire map also has a meaningful formulation at the parameter level. Mihaljević-Brandt ([MB12]) studied the convergence of *non-escaping hyperbolic components* within spaces of entire functions, and the authors of [BDH$^+$00] showed that hyperbolic components (and certain parameter rays) in the space of unicritical polynomials of the form $z \mapsto \lambda(1+\frac{z}{n})^n$ converge to hyperbolic components (resp. parameter *hairs*) of exponential functions of the form $z \mapsto \lambda\exp(z)$.

In this thesis, **Chapter V** is focussed on dynamically approximating postsingularly finite

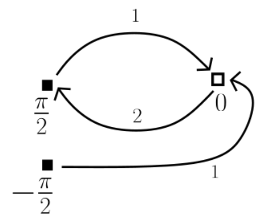


Figure 1.3: $\mathcal{P}_{f,P_f}$ for $f(z) = \frac{\pi}{2}\cos z$. Here, $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ are the singular values.

entire maps by polynomials. Generally, for a Thurston map f, we define its postsingular portrait $\mathcal{P}_{f,P_f}$ as a weighted graph with vertex set P_f, with an oriented edge from v to $f(v)$ with weight $\deg_f(v)$ for each $v \in P_f$. This graph is a visual representation of the dynamical system $f|P_f : P_f \to P_f$ (see Figure 1.3). We say that two postsingular portraits $\mathcal{P}_{f,P_f}$ and $\mathcal{P}_{g,P_g}$ are *dynamically isomorphic* if there exists an isomorphism φ of weighted graphs from $\mathcal{P}_{f,P_f}$ to $\mathcal{P}_{g,P_g}$ such that φ maps S_f bijectively to S_g. Clearly, the existence of such an isomorphism implies that $f|P_f$ is conjugate to $g|P_g$. It is also worthwhile noting that two Thurston maps that are combinatorial equivalent have dynamically isomorphic postsingular portraits.

We state our main result below.

Theorem I.4 (Mukundan, Prochorov, Reinke; [MPR24, Main Theorem A])**.** *Let g be a postsingularly finite entire map. Then there exists a sequence of postcritically finite polynomials (g_n) converging locally uniformly to g such that g and g_n have dynamically isomorphic postsingular portraits for all $n \in \mathbb{N}$.*

For entire maps with finitely many singular values, the iterative behavior of the singular values controls the global dynamics ([Sch10]). In a broad sense, in the setting of Theorem I.4, the "dynamical cores" of the polynomials g_n look more and more like the "dynamical core" of the limiting map g as n tends to infinity. This supplies a contrast with Taylor approximations (h_n) for g, where it is possible that the number of critical values of h_n tends to infinity, resulting in the sets P_{h_n} having a structure very different from that of P_g.

The combinatorics of the PCF polynomials g_n from Main Theorem I.4 also approach the combinatorics of g in the following sense: let $\gamma \subset \mathbb{C}\backslash P_g$ be a simple closed curve and $\widetilde{\gamma}$ be a connected component of $g^{-1}(\gamma)$. If $\widetilde{\gamma}$ is also a simple closed curve, then, as shown in Corollary V.6, for any sufficiently small $\varepsilon > 0$ and sufficiently large $n \in \mathbb{N}$, there exists a unique simple closed curve $\widetilde{\gamma}_n \subset g_n^{-1}(\gamma)$ that lies in the $\varepsilon-$neighbourhood of $\widetilde{\gamma}$. In particular, for all sufficiently large n, the curves $\widetilde{\gamma}_n$ and $\widetilde{\gamma}$ are free-homotopic relative to $P_{g_n} \cup P_g$ and, moreover, $\deg(g|\widetilde{\gamma}) = \deg(g_n|\widetilde{\gamma}_n)$. If $\widetilde{\gamma}$ is not a simple closed curve, or in other words, $\deg(g|\widetilde{\gamma})$ is infinite, then for all sufficiently large n, there exists a unique simple closed curve

$\widetilde{\gamma}_n \subset g_n^{-1}(\gamma)$ such that $d(z, \widetilde{\gamma}_n) \to 0$ for every $z \in \widetilde{\gamma}$ and $\deg(g_n|\widetilde{\gamma}_n) \to \infty$. For a more detailed version see Corollary V.6.

Our strategy for proving Theorem I.4 is to first solve an approximation problem for σ_g on the Teichmüller space $\mathrm{T}(\mathbb{S}^2, P_g \cup \{\infty\})$. Setting $A = P_g$, from Theorem I.3, we get a sequence of polynomial Thurston maps $g_n : (\mathbb{R}^2, A) \circlearrowleft$ such that $\sigma_{g_n} \to \sigma_g$. Since g is realized, the maps g_n are eventually realized, and the fixed points τ_n of σ_{g_n} converge to the fixed point τ of σ_g. We then show that there exist representatives $\psi_n, \varphi_n \in \tau_n$ such that the polynomials $\varphi_n \circ g_n \circ \psi_n^{-1}$ are postcritically finite and converge to g locally uniformly on $\mathbb{C}$.

I.3.3: Case study: unicritical polynomials to exponential functions

In Chapters VI, VII and VIII, we explore Thurston theory specifically for exponential maps and unicritical polynomials. We are mainly motivated by the classical approximation $\lambda(1 + \frac{z}{n})^n \to \lambda \exp(z)$ for every $\lambda \in \mathbb{C}^*$ that has, in the past, helped us decode the dynamics of exponentials. We exhibit approximations for exponential PSF maps and their Thurston pullback operators by unicritical PCF polynomials and their corresponding Thurston pullback operators; these approximations, moreover, are meaningful at a parameter level.

We first describe in brief the parameter spaces of unicritical polynomials and exponential maps. This background material is also recalled in more detail in Chapter II. Fix a degree $n \in \mathbb{N}_{\geqslant 2}$. Every unicritical polynomial is conjugate by affine maps to a polynomial of the form $f_{n,c}(z) = z^n + c$. For $\lambda \in \widehat{\mathbb{C}}$, let $p_{n,\lambda}(z) = \lambda(1 + \frac{z}{n})^n$. For a unicritical polynomial $f_{n,c}$ which is not critically fixed, there exists a unique $\lambda \in \mathbb{C}^*$ so that $f_{n,c}$ is affine conjugate to $p_{n,\lambda}$. On the other hand, for every $\lambda \in \mathbb{C}^*$, there are exactly $(n-1)$ complex numbers c such that $p_{n,\lambda}$ is affine conjugate to $f_{n,c}(z) = z^n + c$; we call these values c the monic representatives for λ.

We note that $S_{p_{n,\lambda}} = \{0\}$, and that $p_{n,\lambda}(0) = \lambda$. If $p_{n,\lambda}$ is postcritically finite, there are exactly two possibilities:

1. 0 is periodic under $p_{n,\lambda}$ with period k (i.e., the postsingular portrait is a cyclic graph of length k)

2. 0 is pre-periodic under $p_{n,\lambda}$, with pre-period $\ell \geqslant 1$ and eventual period $k \geqslant 1$ (in this case, the cyclic graph is chain of length ℓ attached to a cycle of length k).

Let $\mathcal{P}_n$ denote the set of $\lambda \in \mathbb{C}^*$ such that the unicritical polynomial $p_{n,\lambda}$ is postcritically finite. Every $\lambda \in \mathcal{P}_n$ is associated with a finite set of angles $\Theta_n(\lambda) \subset \mathbb{R}/\mathbb{Z}$, such that the parameter ray at each angle in this set, which lives in the space $\{f_{n,c} : c \in \mathbb{C}\}$, lands at some monic representative c for λ if λ is critically pre-periodic, or on a hyperbolic component containing a monic representative c if λ is periodic. We call this set $\Theta_n(\lambda)$ the set of angular coordinates for λ.

Similarly, for $\lambda \in \mathbb{C}^*$, we let $p_\lambda(z) = \lambda \exp(z)$ and denote by $\Lambda \subset \mathbb{C}^*$ the set of λ values for which the orbit under p_λ of the set $S_{p_\lambda} = \{0\}$ is bounded. Let $\mathcal{P} \subset \Lambda$ be the subset of postsingularly finite parameters. We observe that for any $\lambda \in \mathcal{P}$, the singular value 0 is strictly pre-periodic under p_λ, and thus the postsingular portrait of p_λ looks like that in case (2) above.

As for the unicriticals, every $\lambda \in \mathcal{P}$ is associated with a finite set of sequences $\Theta_\infty(\lambda) \subset \mathbb{Z}^{\mathbb{N}}$ called *external addresses*. In broader generality, an external address $\underline{s}$ is an element of $\mathbb{Z}^{\mathbb{N}}$, and for a subset of $\mathbb{Z}^{\mathbb{N}}$ consisting of *exponentially bounded* addresses, each address corresponds to a simple arc, called a *parameter hair*, contained in the complement of Λ. For any $\lambda \in \mathcal{P}$, the set $\Theta_\infty(\lambda)$ consists of all external addresses whose corresponding hairs have λ as a limiting value; in other words, these are exactly the hairs that land at λ.

To summarize, angles and external addresses are combinatorial representations of the location of postsingularly finite parameters in the spaces $\mathcal{P}_n$, $n \in \mathbb{N}_{\geqslant 2}$ and the space $\mathcal{P}$. There is a further analogy between them: for fixed degree n, we can think of angles in $\mathbb{R}/\mathbb{Z}$ in terms of their $n-$adic expansions, which are sequences in $\{0, 1, \cdots, n-1\}^{\mathbb{N}}$

Our main result is the following theorem, proved in **Chapter VIII**:

Theorem I.5 ([Muk23, Theorem A]). *Given $\lambda \in \mathcal{P}$, there exists an $N = N(\lambda) \in \mathbb{N}_{\geqslant 2}$ and a sequence of complex numbers $\lambda_n \in \mathcal{P}_n$, $n \geqslant N$ such that*

1. *the sequence (p_{n,λ_n}) converges to p_λ locally uniformly on $\mathbb{C}$, and for all $n \geqslant N$, the postsingular portrait of p_{n,λ_n} is dynamically isomorphic to the postsingular portrait of p_λ;*

2. *there exists a polynomial $Q \in \mathbb{Z}[X]$ and integers $\ell, k \geqslant 1$ with $\deg Q \leqslant \ell + k - 2$ depending only on λ, and a sequence of angles $\theta_n \in \mathbb{Q}/\mathbb{Z}, n \geqslant N$, such that $\theta_n \in \Theta_n(\lambda_n)$ and $(n-1)\theta_n \equiv \frac{(n-1)Q(n)}{n^\ell(n^k-1)} \pmod 1$, for all $n \geqslant N$.*

The second condition above essentially shows that the $n-$adic expansions of λ exhibit a stability condition as $n \to \infty$. We show in Proposition VIII.4 that the $n-$adic expansions of the θ_n above converge in a combinatorial sense to an external address of λ.

Theorem I.5 is a refinement of Theorem I.4 for the special case of exponentials.

Our approach will rely on building a combinatorial relationship between the sets $\mathcal{P}_n$ and the set $\mathcal{P}$. The polynomials in $\mathcal{P}_n$ can be completely classified by the combinatorial data contained in their *spiders* (see for example [DH84], [HS94], [BFH92]). We recall that a spider for a unicritical polynomial is the finite forward orbit of some dynamical ray landing at its critical value. Hubbard and Schleicher introduced the approach of using spiders to build topological models for unicritical polynomials (see [HS94]). They constructed, for each angle

$\theta \in \mathbb{Q}/\mathbb{Z}$, an abstract graph $S_2(\theta) \subset \mathbb{S}^2$ called the *degree n spider* of θ, and a polynomial Thurston map $\mathcal{F}_{2,\theta} : \mathbb{S}^2 \to \mathbb{S}^2$ which leaves $S_2(\theta)$ invariant. The map $\mathcal{F}_{2,\theta}$ is shown to be combinatorially equivalent to $p_{n,\lambda}$, where λ is the unique parameter in $\mathcal{P}_n$ such that $\theta \in \Theta_2(\lambda)$. Their theory generalises to all degrees $n \geqslant 2$.

The authors of [LSV08] developed a similar approach for exponentials: for every pre-periodic external address $\underline{s}$, there exists a finite graph $S_\infty(\underline{s})$ and a Thurston map $\mathcal{G}_{\underline{s}} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $S_\infty(\underline{s})$ is invariant under $\mathcal{G}_{\underline{s}}$, and $\mathcal{G}_{\underline{s}}$ is Thurston equivalent to p_λ where λ is the landing point of the parameter ray at address $\underline{s}$.

We show in **Chapter** VI that spiders in one degree are realized in the next:

Theorem I.6 ([Muk23, Lemma 1.1])**.** *For every $n \in \mathbb{N}_{\geqslant 2}$, there exist distinct maps $\mathrm{Jump}_{n,j} : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$, $j = 0, 1, 2, \cdots, n-1$ such that for each j,*

1. *considering $\mathbb{Q}/\mathbb{Z}$ as a subset of $[0, 1)$, $\mathrm{Jump}_{n,j}$ is strictly increasing;*
2. *for every $\theta \in \mathrm{Jump}_{n,j}(\theta)$, the spiders $S_n(\theta)$ and $S_{n+1}(\mathrm{Jump}_{n,j}(\theta))$ are congruent.*

The congruency condition above means that the spiders have the same circular order of edges, or 'legs', at infinity. The maps $\mathrm{Jump}_{n,j}$ additionally preserve *landing relations* between angles in $\mathbb{Q}/\mathbb{Z}$: if two angles θ_1, θ_2 correspond to parameter rays to $\mathcal{M}_n$ that land at the same point, then $\mathrm{Jump}_{n,j}(\theta_1)$ and $\mathrm{Jump}_{n,j}(\theta_2)$ land at the same point in $\mathcal{M}_{n+1}$.

In **Chapter** VII, we promote this realization of spiders from one degree to the next, into an embedding of $\mathcal{P}_n$ into $\mathcal{P}_{n+1}$. First we define a poset structure on $\mathcal{P}_n$, defining $\lambda \lhd \widehat{\lambda}$ if there exist $\theta_1, \theta_2 \in \Theta_n(\lambda)$ and $\theta \in \Theta_n(\widehat{\lambda})$ such that the parameter rays $R_n(\theta_1)$ and $R_n(\theta_2)$ land together, and $\theta_1 < \theta < \theta_2$. We call a map from $X \subset \mathcal{P}_n$ to $Y \subset \mathcal{P}_{n'}$ a *combinatorial embedding* if it preserves the poset order defined above, and preserves postsingular portraits.

Theorem I.7 ([Muk23, Lemma 1.2])**.** *For every $n \in \mathbb{N}_{\geqslant 2}$, there exist distinct combinatorial embeddings $\mathcal{E}_{n,j} : \mathcal{P}_n \longrightarrow \mathcal{P}_{n+1}$, for $j = 0, 1, \cdots, n-1$.*

For a fixed $\lambda \in \mathcal{P}$, using Theorems I.6 and I.7, we show that for any external address $\underline{s}$ of λ, there exist a sequence of angles (θ_n) whose degree n spiders are congruent to the spider corresponding to $\underline{s}$, and which satisfy the growth condition (3) of Theorem I.5. We then construct a topological approximation of $\mathcal{G}_{\underline{s}}$ by polynomial Thurston maps $\mathcal{G}_{n,\underline{s}}$ such that for n large enough, $\mathcal{G}_{\underline{s}}$ is combinatorially equivalent to $\mathcal{F}_{n,\theta_n}$. We realize the λ_n required in Theorem I.5 as landing points in degree n of the angle θ_n.

I.4: Notations and Conventions

- The cardinality of a set X is denoted by $|X|$ and the identity map on X by id_X. We denote by (x_n) a sequence of elements $x_n \in X$. If $f\colon X \to Y$ is a map and $U \subset X$, then $f|U$ stands for the restriction of f to U.

- We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$, respectively the sets of positive integers, integers, real and complex numbers. When $X = \mathbb{N}, \mathbb{Z}$ or $\mathbb{R}$ and $k \in X$, the notation $X_{\geqslant k}$ will be used for the set of elements $\{x : x \in X \text{ and } x \geqslant k\}$. We can similarly define $X_{\leqslant k}$. We also let $\mathbb{I} = [0, 1]$. By (x_n), we denote a sequence $x_1, x_2, \cdots, x_n, \cdots$ indexed by the set $\mathbb{N}$.

- If X is a topological space and $U \subset X$, then $\overline{U}$ denotes the closure, $\mathrm{int}(U)$ the interior, and ∂U the boundary of U in X.

- We denote the $2-$dimensional plane by $\mathbb{R}^2$. We identify the $2-$sphere $\mathbb{S}^2$ with the one-point compactification $\mathbb{R}^2 \cup \{\infty\}$ of the plane. The complex plane $\mathbb{C}$ is then the sphere $\mathbb{S}^2$ endowed with the standard complex structure. When we are working with $\mathbb{C}$ purely as a topological surface, we will often conflate it with the topological plane $\mathbb{R}^2$. In the complex plane, we let $\mathbb{D} := \{z \in \mathbb{C}\colon |z| < 1\}$, $\mathbb{D}^* := \mathbb{D}\backslash\{0\}$, $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$. The open disk of radius $r > 0$ centered at $z_0 \in \mathbb{C}$ is denoted by $\mathbb{D}(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}$, and, for simplicity, $\mathbb{D}_r$ is the disk of radius r centered at 0. We denote by $\widehat{\mathbb{C}}$ the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

- When we refer to a map between $\mathbb{R}^2$ and $\mathbb{C}$, we think of $\mathbb{C}$ as a Riemann surface and $\mathbb{R}^2$ as a topological surface. When working with holomorphic maps, we will rarely refer to them as maps on $\mathbb{R}^2$; these will be notated as maps on $\mathbb{C}$. For a sequence of entire maps f_n that converge to the identity, we say $f_n \to \mathrm{id}_{\mathbb{C}}$. If these maps are not holomorphic, we write $f_n \to \mathrm{id}_{\mathbb{R}^2}$ instead.

CHAPTER II
Background

II.1: Thurston maps

Thurston maps are topological analogs of postsingularly finite entire functions. In this section we introduce the background required to define these maps and explore their basic properties. For the notation used for various topological objects in this thesis, refer to Appendix A.

II.1.1: Topological holomorphicity

Let X, Y be oriented topological surfaces and $f : X \to Y$ be continuous.

Definition II.1. The map f is said to be *topologically holomorphic* if it is open, has discrete fibers, and satisfies the property that for every point $x \in X$ where f is locally injective, it is locally an orientation-preserving homeomorphism. We denote by $\mathcal{C}_{hol}(X, Y)$ the set of holomorphic maps from X to Y.

An equivalent set of conditions for topological holomorphicity of f is that for every $x \in X$, there exists a neighborhood U of x, orientation-preserving homeomorphisms $\varphi\colon U \to \mathbb{D}$ and $\psi\colon f(U) \to \mathbb{D}$, and $d \in \mathbb{N}$ such that $\varphi(x) = \psi(f(x)) = 0$ and $\psi \circ (f|U) \circ \varphi^{-1}(z) = z^d$ for all $z \in \mathbb{D}$.; for a proof, see [Sto28] or [Sto56].

If $X = Y = \mathbb{R}^2$, the map f is called a *topological polynomial* if it has finite degree, and *transcendental* otherwise.

II.1.2: The type problem

Let X and Y be oriented topological surfaces. We refer to a maximal holomorphic atlas on X as a *complex structure*, and denote by $\mathcal{A}(X)$ the set of all complex structures on X.

If $X = \mathbb{R}^2$ and $\mathcal{A} \in \mathcal{A}(X)$, then the Riemann surface $(X, \mathcal{A})$ is conformally equivalent to either $\mathbb{C}$ or $\mathbb{D}$. In the former case, we say $\mathcal{A}$ is a *flat* structure; in the latter, we call it a *hyperbolic* structure. The partition $\mathcal{A}(X) = \mathcal{A}_{flat}(X) \cup \mathcal{A}_{hyp}(X)$ decomposes $\mathcal{A}(X)$ into flat and hyperbolic complex structures.

It is known that we can pull back complex structures under topologically holomorphic maps (see [LP20, Stoïlow's factorisation theorem] and [BM17, Lemma A.12]) as the following proposition states.

Proposition II.2. *Let $f \in \mathcal{C}_{hol}(X,Y)$. For every $\mathcal{A} \in \mathcal{A}(Y)$, there exists a unique structure $f^*\mathcal{A} \in \mathcal{A}(X)$, such that the map $f\colon (X, f^*\mathcal{A}) \to (Y, \mathcal{A})$ is holomorphic.*

This implies that for any orientation-preserving homeomorphism $\varphi : Y \longrightarrow S_Y$, where S_Y is a Riemann surface, there exists a Riemann surface S_X and an orientation-preserving homeomorphism $\psi : X \longrightarrow S_X$ such that the function $\varphi \circ f \circ \psi^{-1}\colon S_X \to S_Y$ is holomorphic.

Now assume $X = Y = \mathbb{R}^2$, and let $f \in \mathcal{C}_{hol}(X,Y)$. It is clear that if $\mathcal{A} \in \mathcal{A}_{hyp}(Y)$ and f is non-constant, then $f^*\mathcal{A} \in \mathcal{A}_{hyp}(X)$. However, if $\mathcal{A} \in \mathcal{A}_{flat}(Y)$, it is possible for $f^*\mathcal{A}$ to be either flat or hyperbolic. This ambiguity in the comformal type of $f^*\mathcal{A}$ is called the *type problem*, and was first posed by Speiser in [Spe29]. This problem is hard to solve and a general solution is still unknown; however, special cases have been resolved in [Ahl35], [PeiwsS26] and [Rob39].

The map f is said to have *stable conformal type* if the conformal equivalence class of $(f^*\mathcal{A}, X)$ is constant over all $\mathcal{A} \in \mathcal{A}_{flat}(Y)$. In this case, the conformal type of f is said to be *parabolic* if $(f^*\mathcal{A}, X) \cong \mathbb{C}$ for every $\mathcal{A} \in \mathcal{A}_{flat}(Y)$, and *hyperbolic* otherwise. Later on in this section we will explore a class of maps for which the conformal type is stable.

Quasiconformality

Let $U \subset \widehat{\mathbb{C}}$ be a simply connected domain. An orientation-preserving homeomorphism $\varphi\colon U \to f(U) \subset \widehat{\mathbb{C}}$ is *K-quasiconformal* for $K \geqslant 1$ if for every annulus $V \subset U$ of finite modulus, we have

$$\mathrm{mod}(V)/K \leqslant \mathrm{mod}(\varphi(V)) \leqslant K\mathrm{mod}(V),$$

where $\mathrm{mod}(V)$ and $\mathrm{mod}(\varphi(V))$ are the moduli of the annuli V and $\varphi(V)$ respectively. The infimum of all values K such that φ is K-quasiconformal is called the *dilatation* of φ, and is denoted by $K(\varphi)$ (in particular, φ is $K(\varphi)$-quasiconformal). For further properties and equivalent definitions of quasiconformal maps, see [Ahl06], [Hub06], or [BF14].
Here we list some properties of quasiconformal maps used in this thesis.

Theorem II.3 (Weyl's Lemma; [BF14, Theorem 1.14])**.** *If $\varphi : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ is $1-$quasiconformal, then φ is conformal.*

Theorem II.4 ([BF14, Theorem 1.26])**.** *The set of $K-$quasiconformal homeomorphisms $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ fixing three points $\{a,b,c\}$ is compact in the topology of uniform convergence on compact subsets of $\widehat{\mathbb{C}}\backslash\{a,b,c\}$.*

Proposition II.5. *Let $\varphi_n\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}, n \in \mathbb{N}$ be a sequence of K_n-quasiconformal maps fixing three distinct points $a, b, c \in \widehat{\mathbb{C}}$. Suppose that $K_n \to 1$ as $n \to \infty$, then the sequence (φ_n) converges to $\mathrm{id}_{\widehat{\mathbb{C}}}$ uniformly on compact subsets of $\widehat{\mathbb{C}} \backslash \{a, b, c\}$.*

Proof. Let us consider an arbitrary subsequence (φ_{n_k}) of (φ_n). Theorem II.4 implies that from (φ_{n_k}), we can extract a further subsequence converging to a quasiconformal limit $\varphi\colon \mathbb{C} \to \mathbb{C}$ uniformly on compact subsets of $\widehat{\mathbb{C}} \backslash \{a, b, c\}$. Moreover, from the same theorem, it is evident that $K(\varphi)$ equals 1. Theorem II.3 implies that φ is holomorphic. Since it also fixes three distinct points of $\widehat{\mathbb{C}}$, it coincides with $\mathrm{id}_{\widehat{\mathbb{C}}}$.

As the above argument applies to any arbitrary subsequence of (φ_n), we conclude that (φ_n) converges uniformly on compact subsets of $\widehat{\mathbb{C}} \backslash \{a, b, c\}$ to $\mathrm{id}_{\widehat{\mathbb{C}}}$. □

II.1.3: The Speiser class $\mathcal{S}$

Fix oriented surfaces X and Y, and let $f \in \mathcal{C}_{hol}(X, Y)$. Given a point $y \in Y$,

- y is a *regular* value for f if there exists a neighborhood V of y such that for every connected component U of $f^{-1}(V)$, $f|U$ is a homeomorphism onto V;

- y is a *singular value* for f otherwise.

we denote by S_f the set of singular values of f. It is important to observe that every $y \in S_f$, the map f is branched at y. By definition of regular values, it is clear that the map $f|X \backslash f^{-1}(S_f)\colon X \backslash f^{-1}(S_f) \to Y \backslash S_f$ is a covering.

Let $X = Y = \mathbb{R}^2$ and $f \in \mathcal{C}_{hol}(X, Y)$. If S_f is finite, the map f is said to be of *finite type*. The set of topologically holomorphic maps on $\mathbb{R}^2$ that are of finite type is called the *Speiser class*, and denoted $\mathcal{S}$. It is known that maps in class $\mathcal{S}$ have stable conformal type (see [Tei20a], [Tei20b], [Ere04, pp3-4]). In other words, for all $f \in \mathcal{S}$, the set $f^*(\mathcal{A}_{flat}(\mathbb{R}^2)) \subset \mathcal{A}(\mathbb{R}^2)$ is fully contained in either $\mathcal{A}_{flat}(\mathbb{R}^2)$ or $\mathcal{A}_{hyp}(\mathbb{R}^2)$. Also note that if f is of finite degree, then f belongs to class $\mathcal{S}$ and also has stable parabolic type.

For the rest of Section II.1, we assume $X = Y = \mathbb{R}^2$ and that $f \in \mathcal{S}$. In this case we have a classification of singular values; for every $y \in S_f$, at least one the following is true:

- y is a *critical* value for f; in other words, there exists a point $x \in f^{-1}(y)$ where f is not locally injective;

- y is an *asymptotic* value; or equivalently, there exists an arc $\gamma\colon [0, \infty) \to X$ that leaves every compact set of X as $t \to \infty$, and satisfies $\lim_{t \to \infty} f(\gamma(t)) = y$.

Analogous to [ERG15, Propostion 2.3] and proven in a similar way, we have the following isotopy lifting property applicable to maps in $\mathcal{S}$.

Proposition II.6. *Let $\widehat{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a topologically holomorphic map in class $\mathcal{S}$, with $\varphi_0 \circ f = \widehat{f} \circ \psi_0$ for some $\varphi_0, \psi_0 \in \mathrm{Homeo}^+(\mathbb{R}^2)$.*

Let $A \subset \mathbb{R}^2$ be a finite set containing S_f, and $\varphi_1 \in \mathrm{Homeo}^+(\mathbb{R}^2)$ be isotopic rel. A to φ_0. Then $\varphi_1 \circ f = \widehat{f} \circ \psi_1$ for some $\psi_1 \in \mathrm{Homeo}^+(\mathbb{R}^2)$ isotopic to ψ_0 rel. $f^{-1}(A)$.

II.1.4: Properties of topologically holomorphic maps on $\mathbb{R}^2$

For the next two propositions, assume that f has stable parabolic type.

Proposition II.7. *Let $V \subset \mathbb{R}^2$ be a bounded simply connected domain, and U be a connected component of $f^{-1}(V)$.*

1. *If $V \cap S_f = \{y\}$, then U is simply connected, and exactly one of the following statements is true:*

   (a) there exist orientation-preserving homeomorphisms $\varphi\colon U \to \mathbb{D}$ and $\psi\colon V \to \mathbb{D}$ and an integer $d \in \mathbb{N}$ such that $\psi(y) = 0$ and $\psi \circ (f|U) \circ \varphi^{-1}(z) = z^d$ for all $z \in \mathbb{D}$. In particular, $U \backslash f^{-1}(y)$ is an annulus and $f|(U\backslash f^{-1}(y))\colon U\backslash f^{-1}(y) \to V\backslash\{y\}$ is a covering map of degree d. Additionally, if $\partial V \cap S_f = \varnothing$, then U is bounded;

   (b) there exist orientation-preserving homeomorphisms $\varphi\colon U \to \mathbb{D}$ and $\psi\colon V \to \mathbb{H}$ such that $\psi(y) = 0$ and $\psi \circ (f|U) \circ \varphi^{-1}(z) = \exp(z)$ for all $z \in \mathbb{H}$. In particular, U is unbounded, and the map $f|U\colon U \to V\backslash\{y\}$ is a universal covering.

2. *Else if $V \cap S_f = \varnothing$, then U is simply connected and $f|U\colon U \to V$ is an orientation-preserving homeomorphism. Additionally, if $\partial V \cap S_f = \varnothing$, then U is bounded.*

Proof. When f is an entire function, the same statement can be found, for example, in [For91, Theorem 5.10, 5.11].

For the general case, let $h : \mathbb{R}^2 \to \mathbb{C}$ be an orientation-preserving homeomorphism, and $\mathcal{A}_0$ be the standard complex structure on $\mathbb{C}$. Then $f^*h^*\mathcal{A}_0$ is a flat structure. Let $\widehat{h} : (\mathbb{R}^2, \mathcal{A}_0) \to \mathbb{C}$ be a biholomorphism. Then the map $g = h \circ f \circ \widehat{h}^{-1}$ is entire, and the proposition follows easily.

□

Proposition II.8. *Let V be an unbounded simply connected domain, and U be a connected component of $f^{-1}(V)$. If $V \cap S_f = \varnothing$, then U is an unbounded simply connected domain, and $f|U\colon U \to V$ is an orientation-preserving homeomorphism. If $V = \mathbb{R}^2\backslash W$, where W is a compact simply connected set containing S_f, then U is unbounded, and exactly one of the following is true:*

1. *if f is a topological polynomial of degree d, then there exist orientation-preserving homeomorphisms $\varphi\colon U \to \mathbb{D}^*$ and $\psi\colon V \to \mathbb{D}^*$ such that $\psi \circ (f|U) \circ \varphi^{-1}(z) = z^d$ for all $z \in \mathbb{D}^*$. In particular, U is an annulus, $\mathbb{R}^2 \backslash U$ is compact, and $f|U\colon U \to V$ is a covering map of degree d;*

2. *if f is transcendental, then there exist orientation-preserving homeomorphisms $\varphi\colon U \to \mathbb{D}^*$ and $\psi\colon V \to \mathbb{H}$ such that $\psi \circ (f|V) \circ \varphi^{-1})(z) = \exp(z)$ for all $z \in \mathbb{H}$. In particular, $f|U\colon U \to V$ is a universal covering map.*

Proof. The proof is similar to that of Proposition II.7. □

II.1.5: Postsingular finiteness

Fix a map $f \in \mathcal{C}_{hol}(\mathbb{R}^2)$.

Definition II.9. The map f is said to be *postsingularly finite* (PSF) if its postsingular set, defined as

$$P_f = \overline{\bigcup_{n \geqslant 0} f^{\circ n}(S_f)}\ ,$$

is finite. If all singular values are critical values, then a PSF map is also said to be postcritically finite (PCF).

It is immediate from this definition that postsingularly finite maps are contained in class $\mathcal{S}$.

Definition II.10. The map f is said to be a *Thurston map* if it is postsingularly finite and has stable parabolic type.

We use the notation $f\colon (\mathbb{R}^2, A) \circlearrowleft$ for a Thurston map f with a marked finite set A, with $f(A) \subset A$ and $P_f \subseteq A$. If no information about A is given, or no marked set is specified, we assume that $A = P_f$. Some natural examples of Thurston maps are PCF polynomials and PSF entire functions.

II.1.6: Combinatorial equivalence of Thurston maps

Definition II.11. We say that two Thurston maps $f\colon (\mathbb{R}^2, A) \circlearrowleft$ and $\widehat{f}\colon (\mathbb{R}^2, B) \circlearrowleft$ are *combinatorially equivalent*, and write $f \simeq_{\text{comb}} \widehat{f}$, if there exist homeomorphisms $\varphi, \psi \in \text{Homeo}^+(\mathbb{R}^2)$ such that $\varphi(A) = \psi(A) = B$, φ and ψ are isotopic rel. A, and $\varphi \circ f = \widehat{f} \circ \psi$.

Definition II.12. A Thurston map $f\colon (\mathbb{R}^2, A) \circlearrowleft$ is said to be *realized* if it is combinatorially equivalent to a PSF entire map g. If f is not realized, we say that it is *obstructed*.

Definition II.13. Two Thurston maps $f_1\colon (\mathbb{R}^2, A) \circlearrowleft$ and $f_2\colon (\mathbb{R}^2, A) \circlearrowleft$ are called *isotopic* (rel. A) if there exists $\varphi \in \mathrm{Homeo}_0^+(\mathbb{R}^2, A)$ such that $f_1 = f_2 \circ \varphi$.

We recall from Appendix A that for an oriented topological surface X, we take $\mathrm{Homeo}^+(X, A)$ to be the set of orientation-preserving homeomorphisms that fix the set A pointwise (this is different from the usual notation in the literature), and $\mathrm{Homeo}_0^+(X, A)$ is the subset of maps in $\mathrm{Homeo}^+(X, A)$ that are isotopic to id_X rel. A. The following proposition classifies topologically holomorphic maps with a unique singular value, as well as Thurston maps with a unique postsingular value.

Proposition II.14. *Suppose that $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a topologically holomorphic map such that $|S_f| = 1$. Then $f = \varphi^{-1} \circ g \circ \psi$, for some orientation-preserving homeomorphisms $\varphi, \psi\colon \mathbb{R}^2 \to \mathbb{C}$ and a unique $g \in \{z \mapsto z^d | d \in \mathbb{N}_{\geqslant 2}\} \cup \{z \mapsto \exp(z)\}$.*

If $f\colon (\mathbb{R}^2, A) \circlearrowleft$ is a Thurston map with $|A| = 1$, then f is combinatorially equivalent to $z \mapsto z^d$ for some degree $d \geqslant 2$. In particular, A consists of a unique fixed critical point of f.

Proof. The first part essentially follows from Proposition II.8.

Suppose now that $f\colon (\mathbb{R}^2, A) \circlearrowleft$ is a Thurston map with $|A| = 1$. Thus, f has a fixed singular value, so it cannot be of the form $\varphi^{-1} \circ \exp \circ\, \psi$ for any orientation-preserving homeomorphisms $\varphi\colon \mathbb{R}^2 \to \mathbb{C}$ and $\psi\colon \mathbb{R}^2 \to \mathbb{C}$. Therefore, $\varphi \circ f \circ \psi^{-1}(z) = z^d$ for all $z \in \mathbb{C}$ and $d = \deg(f)$, where $A = \{\varphi(0)\} = \{\psi(0)\}$. In particular, φ and ψ are isotopic rel. A, and f is combinatorially equivalent to $z \mapsto z^d$. ◻

The dynamics of a Thurston map on its marked set can also be represented visually.

Definition II.15. Let $f\colon (\mathbb{R}^2, A) \circlearrowleft$ be a Thurston map with some marked set A. The *marked portrait (rel. A) of f* is a weighted directed abstract graph $\mathcal{P}_{f,A}$ such that the vertex set of $\mathcal{P}_{f,A}$ equals A, and for each vertex $v \in A$ there exists a unique directed edge from v to $f(v)$ with weight $\deg(f, v)$. Additionally, among all vertices of $\mathcal{P}_{f,A}$, we label the ones that are singular values of f.

If $A = P_f$, then, for simplicity, we denote by $\mathcal{P}_f$ the marked portrait rel. P_f of f and call it the *postsingular portrait* of f.

Let $f\colon (\mathbb{R}^2, A) \circlearrowleft$ and $\widehat{f}\colon (\mathbb{R}^2, B) \circlearrowleft$ be Thurston maps. We say that the postsingular portraits $\mathcal{P}_f$ and $\mathcal{P}_{\widehat{f}}$ are *dynamically isomorphic* if there exists a bijective map $\varphi\colon A \to B$ such that

- $\varphi(S_f) = S_{\widehat{f}}$,

(a) $G_1(z) = \pi\cos(z)/2$.   (b) $G_2(z) = \sqrt{\ln 2}(1 - \exp(z^2))$.

Figure 2.1: Examples of postsingular portraits

- φ is an *isomorphism* between the weighted directed abstract graphs $\mathcal{P}_f$ and $\mathcal{P}_{\widehat{f}}$. In other words, there exists an edge $e_{u,v}$ of $\mathcal{P}_f$ joining u with v if and only if there exists an edge $\widehat{e}_{\varphi(u),\varphi(v)}$ of $\mathcal{P}_{\widehat{f}}$ joining $\varphi(u)$ with $\varphi(v)$. The weights of $e_{u,v}$ and $\widehat{e}_{\varphi(u),\varphi(v)}$ coincide.

If two Thurston maps $f\colon (\mathbb{R}^2, A) \circlearrowleft$ and $\widehat{f}\colon (\mathbb{R}^2, B) \circlearrowleft$ have dynamically isomorphic postsingular portraits, then it is clear that $f|A$ and $\widehat{f}|B$ are conjugate dynamical systems. Combinatorial equivalence of f and $\widehat{f}$ is a sufficient condition for them to have dynamically isomorphic portraits.

Example II.16. Given below are two postsingularly finite entire maps that we will use as prototypical examples throughout this thesis.

1. The map $G_1(z) = \pi\cos(z)/2$ has no asymptotic values and two critical values $\pm\pi/2$, with $P_{G_1} = \{0, -\pi/2, \pi/2\}$;
2. The map $G_2(z) = \sqrt{\ln 2}(1 - \exp(z^2))$ has a unique critical value 0 and a unique asymptotic value $\sqrt{\ln 2}$, with $P_{G_2} = \{0, -\sqrt{\ln 2}, \sqrt{\ln 2}\}$.

Figure 2.1 illustrates $\mathcal{P}_{G_1}$ and $\mathcal{P}_{G_2}$. Singular and non-singular vertices of the corresponding graphs are labeled by solid and hollow squares, respectively.

II.2: Conditions for holomorphic realizability

Proposition II.14 tells us that every Thurston map with $|P_f| = 1$ is realized. The core tenet of Thurston's theory is that a Thurston map f with $|P_f| \geqslant 2$ is realized if and only if its Thurston pullback operator σ_f which acts on the Teichmüller space $\mathrm{T}(\mathbb{S}^2, P_f \cup \{\infty\})$ has a fixed point. We describe this operator and its properties in this section.

II.2.1: Teichmüller spaces modelled on a punctured sphere

Let $B \subset \mathbb{S}^2$ be a finite set with $|B| \geqslant 3$.

Definition II.17. The *Teichmüller space of* $\mathbb{S}^2$ *with the marked set* B (or Teichmüller space *modelled on* $\mathbb{S}^2 \backslash B$) is defined as

$$\mathrm{T}(\mathbb{S}^2, B) := \{\varphi | \varphi : \mathbb{S}^2 \to \widehat{\mathbb{C}} \text{ is an orientation-preserving homeomorphism}\}/\sim$$

where $\varphi \sim \psi$ if there exists a Möbius transformation M such that φ is isotopic rel. B to $M \circ \psi$. Given $\tau \in \mathrm{T}(\mathbb{S}^2, B)$, for any $\varphi \in \tau$, we will write $\tau = [\varphi]$.

If $B = A \cup \{\infty\}$ for some finite set $A \subset \mathbb{R}^2$, we note that for every $\tau \in \mathrm{T}(\mathbb{S}^2, B)$, there exists $\varphi \in \tau$ so that $\varphi(\infty) = \infty$, and if $\psi \in \tau$ also satisfies $\psi(\infty) = \infty$, then there exists an affine transformation M such that $M \circ \varphi$ and ψ are isotopic rel. A.

The space $\mathrm{T}(\mathbb{S}^2, B)$ is known to have a natural structure of a complex $(|B|-3)$-dimensional manifold ([Hub06, Theorem 6.5.1]) that is contractible. The *Teichmüller metric* on $\mathrm{T}(\mathbb{S}^2, B)$ is defined as

$$d(\tau_1, \tau_2) := \inf_{\psi} \log K(\psi)$$

where the infimum is taken over all quasiconformal homeomorphisms $\psi \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that for some $\psi_1 \in \tau_1$ and $\psi_2 \in \tau_2$, φ_1 is isotopic to $\psi \circ \varphi_2$ rel. B.

It is known that the Teichmüller metric is complete, and $\mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$ with this metric is a *path metric space* in the sense of [Gro07, Definition 1.7], (for a proof, see [FM12a, Section 11.8] and [Gro07, Sections 1.8bis and 1.8bis+]). By the metric version of Hopf-Rinow theorem ([Gro07, page 8]), every bounded closed set of $\mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$ is compact. For more details about $\mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$, see [Hub06] and [Ahl06].

We also describe a group action on $\mathrm{T}(\mathbb{S}^2, B)$.

Definition II.18. The *mapping class group* $\mathrm{MCG}(\mathbb{S}^2, B)$ is the group of orientation preserving homeomorphisms $\varphi : (\mathbb{S}^2, B) \longrightarrow (\mathbb{S}^2, B)$ where $\varphi(B) = B$, with φ and ψ equivalent if φ is isotopic to ψ rel. B. The group law is given by function composition. We let $\langle h \rangle$ denote the equivalence class of h in $\mathrm{MCG}(\mathbb{S}^2, B)$. The subgroup of homeomorphisms that fix B pointwise is called the *pure* mapping class group, denoted $\mathrm{PMCG}(\mathbb{S}^2, B)$.

The topology of uniform convergence on compact subsets sets of $\mathbb{S}^2 \backslash B$ on the space of homeomorphisms in $\mathrm{Homeo}^+(\mathbb{S}^2)$ that fix the set B also induces a topology on $\mathrm{MCG}(\mathbb{S}^2, B)$. With this topology, $\mathrm{MCG}(\mathbb{S}^2, B)$ is known to be a discrete group.

We can also look at the group $\mathrm{MCG}^{\pm}(\mathbb{S}^2, B)$ of all homeomorphisms $\varphi \in \mathrm{Homeo}^+(\mathbb{S}^2) \cup \mathrm{Homeo}^-(\mathbb{S}^2)$ such that $\varphi(B) = B$, modulo isotopy rel. B. Here, $\mathrm{Homeo}^-(\mathbb{S}^2)$ represents the group of orientation-resversing homeomorphisms of $\mathbb{S}^2$. Note that if $\langle\psi\rangle = \langle\varphi\rangle$ in $\mathrm{MCG}^{\pm}(\mathbb{S}^2, B)$, then ψ and φ are either both orientation-preserving, or both orientation-reversing. From this we get the following sequence of maps which is short exact:

$$1 \longrightarrow \mathrm{MCG}(\mathbb{S}^2, B) \longrightarrow \mathrm{MCG}^{\pm}(\mathbb{S}^2, B) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

Here the map $\mathrm{MCG}^{\pm}(\mathbb{S}^2, B) \longrightarrow \mathbb{Z}/2\mathbb{Z}$ sends each class of homeomorphisms to 0 if they are orientation-preserving, and to 1 if they are orientation-reversing.

Given a group G and an element $g \in G$, the group of inner automorphisms $\mathrm{Inn}(G)$ of G is the collection $\{T_g \in \mathrm{Aut}(G) : g \in G\}$, where T_g is the automorphism $h \mapsto ghg^{-1}$. It is easy to see that for any T_g as above and $H \in \mathrm{Aut}(G)$, $H \circ T_g \circ H^{-1} = H_{H(g)}$. In other words, $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

Definition II.19. Given a group G, the outer automorphism group of G, denoted $\mathrm{Out}(G)$, is defined as the quotient $\mathrm{Aut}(G)/\mathrm{Inn}(G)$.

Given $\varphi \in \mathrm{Homeo}^+(\mathbb{S}^2) \cup \mathrm{Homeo}^-(\mathbb{S}^2)$ with $\varphi(B) = B$, let p be a path from t to $\varphi(t)$. Consider the map $\varphi_{*,p} : \pi_1(\mathbb{S}^2\backslash B, t) \to \pi_1(\mathbb{S}^2\backslash B, t)$ given by $[\gamma] \mapsto [p \cdot \varphi(\gamma) \cdot \overline{p}]$. For a different choice q of path from t to $\varphi(t)$, we see that $\varphi_{*,q} = T_{q\cdot\overline{p}} \circ \varphi_{*,p}$. Furthermore, if we choose a different representative ψ isotopic to φ rel. A, then for some choice of path q from t to $\psi(t)$, we have $\psi_{*,q} = \varphi_{*,p}$. So we can define φ_* as the equivalence class of $\varphi_{*,p}$ in $\mathrm{Out}(\pi_1(\mathbb{S}^2\backslash B, t))$, and we get a homomorphism $\Phi : \mathrm{MCG}^{\pm}(\mathbb{S}^2, B) \longrightarrow \mathrm{Out}(\pi_1(\mathbb{S}^2\backslash B, t))$ defined as $\langle\varphi\rangle \mapsto \varphi_*$. The following result, known as the Dehn-Neilson-Baer theorem, is proved in [FM12b]:

Theorem II.20 ([FM12b, Theorem 8.8])**.** Φ *is a homeomorphism.*

The group $\mathrm{PMCG}(\mathbb{S}^2, B)$ acts on $\mathrm{T}(\mathbb{S}^2, B)$ as $\langle h\rangle \cdot [\varphi] = [\varphi \circ h^{-1}]$. Fricke's Theorem states that this action is properly discontinouous (see [FM12b, Chapter 12] for a proof). The quotient $\mathrm{T}(\mathbb{S}^2, B)/\mathrm{PMCG}(\mathbb{S}^2, B)$ is a complex manifold of dimension $|B| - 3$, and is called the *moduli space* of $\mathbb{S}^2$ with the marked set B. It can be alternatively described as follows:

Definition II.21. The moduli space

$$\mathrm{M}(\mathbb{S}^2, B) = \{\varphi | \varphi : B \hookrightarrow \widehat{\mathbb{C}}\}/\sim$$

where $\varphi \sim \psi$ if $\varphi = M \circ \psi$ for some Möbius map $M \in \mathrm{Aut}(\widehat{\mathbb{C}})$.

We denote by $[[\varphi]]$ the equivalence class of φ in $\mathrm{M}(\mathbb{S}^2, B)$.

The map $\mathrm{T}(\mathbb{S}^2, B) \longrightarrow \mathrm{M}(\mathbb{S}^2, B)$ defined as $[\varphi] \mapsto [[\varphi]]$ is a universal covering, and the fiber over $[[\varphi]]$ is $\mathrm{PMCG}(\mathbb{S}^2, B) \cdot [\varphi]$. For more details and a proof sketch, see [FM12b, Chapter 12].

II.2.2: Thurston pullback

The notion of the Thurston pullback map is classical for rational Thurston maps (e.g., [Hub06, Definition 10.6.1]), but is less well-known in the transcendental setting (see for example [HSS09] for the case of *exponential Thurston maps*).

Any orientation-preserving homeomorphism $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{C}$ can be extended to a homeomorphism from $\mathbb{S}^2$ to $\widehat{\mathbb{C}}$ by setting $\varphi(\infty) = \infty$. With this in mind, for any finite set $A \subset \mathbb{R}^2$ with $|A| \geqslant 2$, the element $[\varphi] \in \mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$ is well-defined.

Proposition II.22. *Let $f\colon (\mathbb{R}^2, A) \circlearrowleft$ be a Thurston map such that $|A| \geqslant 2$ and $\varphi\colon \mathbb{R}^2 \to \mathbb{C}$ be an orientation-preserving homeomorphism. Then there exists an orientation-preserving homeomorphism $\psi\colon \mathbb{R}^2 \to \mathbb{C}$ such that $g_\varphi := \varphi \circ f \circ \psi^{-1}\colon \mathbb{C} \to \mathbb{C}$ is an entire holomorphic map. In other words, the following diagram commutes*

$$\begin{array}{ccc} (\mathbb{R}^2, A) & \xrightarrow{\psi} & (\mathbb{C}, \psi(A)) \\ \big\downarrow f & & \big\downarrow g_\varphi \\ (\mathbb{R}^2, A) & \xrightarrow{\varphi} & (\mathbb{C}, \varphi(A)) \end{array}$$

The map ψ is unique up to post-composition with an affine map. Different choices of φ that represent the same point in $\mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$ yield maps ψ that represent the same point in $\mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$.

In other words, we have a well-defined map $\sigma_f\colon \mathrm{T}(\mathbb{S}^2, A \cup \{\infty\}) \to \mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$ such that $\sigma_f([\varphi]) = [\psi]$, called the Thurston pullback map associated with f. As φ ranges across all maps representing a single point in $\mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$, the function g_φ is uniquely defined up to pre- and post-composition with affine maps.

Proof. The existence of a homeomorphism ψ follows from Proposition II.2, the Uniformization Theorem and the fact that f has stable parabolic type. It is also clear that ψ is unique up to post-composition with an affine map and that post-composing φ with an affine map does not affect ψ.

Due to Proposition II.6, changing φ by isotopy rel. A does not change $[\psi]$. Thus, changing φ within its equivalence class in $\mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$ does not affect $[\psi]$, showing that σ_f introduced as above is well-defined. These arguments also show that g_φ is uniquely defined up to pre- and post-composition with affine maps. $\square$

The operator σ_f is a holomorphic map with respect to the natural complex structure on $\mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$ (for a proof, see, for instance, [BCT14, Section 1.3]). Moreover, it is well-behaved with respect to the Teichmüller metric, as the next two propositions suggest.

Proposition II.23. *Let $B \subset \mathbb{S}^2$ be a finite set with $|B| \geqslant 3$. Every holomorphic map $H : \mathrm{T}(\mathbb{S}^2, B) \to \mathrm{T}(\mathbb{S}^2, B)$ is 1-Lipschitz in the Teichmüller metric; in other words, for all $\tau, \hat{\tau} \in \mathrm{T}(\mathbb{S}^2, B)$, we have $d_{\mathrm{T}}(H(\tau), H(\hat{\tau})) \leqslant d_{\mathrm{T}}(\tau, \hat{\tau})$.*

Proof. This result follows directly from [Hub06, Corollary 6.10.7]. □

Proposition II.24. *Let $f\colon (\mathbb{R}^2, A) \circlearrowleft$ be a Thurston map. Then,*

1. *σ_f is 1-Lipschitz with respect to the Teichmüller metric;*
2. *if f is transcendental, then σ_f is locally uniformly contracting; in other words, for any compact set $K \subset \mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$ there exists $\varepsilon_K > 0$ such that*

$$d_{\mathrm{T}}(\sigma_f(\tau), \sigma_f(\hat{\tau})) \leqslant (1 - \varepsilon_K) d_{\mathrm{T}}(\tau, \hat{\tau})$$

   for every $\tau, \hat{\tau} \in K$;

3. *if f is polynomial, then $\sigma_f^{\circ 2}$ is locally uniformly contracting.*

Proof. The first item is clear by Proposition II.23. The second item follows from [HSS09, Section 3.2]. The last item is proved in [Hub16, Corollary 10.7.8]. □

Remark II.25. If Thurston maps $f\colon (\mathbb{R}^2, A) \circlearrowleft$ and $\hat{f}\colon (\mathbb{R}^2, B) \circlearrowleft$ are combinatorially equivalent, then $\sigma_f\colon \mathrm{T}(\mathbb{S}^2, A \cup \{\infty\}) \to \mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$ and $\sigma_{\hat{f}}\colon \mathrm{T}(\mathbb{R}^2, B \cup \{\infty\}) \to \mathrm{T}(\mathbb{R}^2, B \cup \{\infty\})$ are conjugate by a biholomorphism. In the special case where $A = B$ and f is isotopic rel. A to $\hat{f}$, we have $\sigma_f = \sigma_{\hat{f}}$.

We now state the main theorem of Thurston theory. This result follows from Definition II.11 and Proposition II.22 (also cf. [Hub16, Theorem 10.6.4] and [HSS09, Theorem 3.1]).

Theorem II.26. *A Thurston map $f\colon (\mathbb{R}^2, A) \circlearrowleft$ is realized if and only if the Thurston pullback map $\sigma_f\colon \mathrm{T}(\mathbb{S}^2, A \cup \{\infty\}) \to \mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$ has a fixed point $\tau \in \mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$.*

Proof. Suppose that f is realized by a postsingularly finite entire map $g\colon (\mathbb{C}, B) \circlearrowleft$. Then by Definition II.11, there exist orientation-preserving homeomorphisms $\varphi, \psi\colon \mathbb{R}^2 \to \mathbb{C}$ such that $\varphi(A) = \psi(A) = B$, φ and ψ are isotopic rel. A, and $\varphi \circ f = g \circ \psi$. Clearly, $\tau = [\varphi] = [\psi] \in \mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$ is a fixed point of σ_f.

Now suppose that $\tau = [\varphi] \in \mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$ is a fixed point of σ_f. Let $\psi\colon \mathbb{R}^2 \to \mathbb{C}$ be an orientation-preserving homeomorphism so that $g_\varphi = \varphi \circ f \circ \psi^{-1}$ is entire. Since $[\varphi] = [\psi]$, by post-composing ψ with an affine map we can assume that $\varphi|A = \psi|A$, which in turn implies that φ and ψ are isotopic rel. A. Thus, $g_\varphi\colon (\mathbb{C}, \varphi(A)) \circlearrowleft$ is a postsingularly finite entire map Thurston equivalent to f. □

Remark II.27. Let $f : (\mathbb{R}^2, A) \circlearrowleft$ be a Thurston map. If $|A| = 1$, $\mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$ is not well-defined; however, recall from Proposition II.14 that f is realized by $z \mapsto z^d$ for some degree $d \geqslant 2$.

If $|A| = 2$, since $\mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$ consists of one point, Theorem II.26 immediately implies that f is realized.

Finally, Proposition II.24 and Theorem II.26 lead to the following result called *Thurston's rigidity* (this result is similar to [Hub06, Corollary 10.7.8] in classical Thurston theory).

Proposition II.28. *Let $f : (\mathbb{R}^2, A) \circlearrowleft$ be a Thurston map. Then*

1. *σ_f can have at most one fixed point;*

2. *if $g_1 : (\mathbb{C}, A_1) \circlearrowleft$ and $g_2 : (\mathbb{C}, A_2) \circlearrowleft$ are postsingularly finite entire maps Thurston equivalent to f, then g_1 and g_2 are conjugate by an affine map.*

Proposition II.29. *If $f : (\mathbb{R}^2, A) \circlearrowleft$ and $g : (\mathbb{R}^2, A) \circlearrowleft$ are Thurston maps that are isotopic rel. A, then the operators σ_f and σ_g coincide.*

Proof. There exists a map $\varphi \in \mathrm{Homeo}_0^+(\mathbb{R}^2, A)$ such that $g = f \circ \varphi$. Then for any $\tau = [\psi] \in \mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$, if $\sigma_g(\tau) = \hat{\tau}$, then there exists a representative $\hat{\psi} \in \hat{\tau}$ such that $\hat{\psi}(\infty) = \infty$, and $g_\psi := \psi \circ g \circ \hat{\psi}^{-1} : \mathbb{C} \to \mathbb{C}$ is entire. Also note that $\psi^{-1} \circ g_\psi = g \circ \hat{\psi}^{-1} = f \circ (\varphi \circ \hat{\psi}^{-1})$. By Proposition II.6, since $\varphi \circ \hat{\psi}^{-1}$ is isotopic to $\hat{\psi}^{-1}$ rel. A, there exists a map $\psi_1 \in \mathrm{Homeo}^+(\mathbb{R}^2, A)$ isotopic to ψ rel. A such that $\psi_1^{-1} \circ g_\psi = f \circ \hat{\psi}^{-1}$. In other words, $\sigma_f(\tau) = [\psi_1] = [\psi] = \hat{\tau}$. □

This finishes the summary of basic properties of Thurston maps visited by this thesis. We will now move on to known results concerning our special case study described in Section I.3.3.

II.3: Postsingularly finite maps in complex dynamics

This section reviews the properties of two prominent families of PSF maps: unicritical polynomials and exponentials. For the foundational theory of holomorphic dynamics, see [Mil06] and [Hub16].

II.3.1: The dynamics of unicritical PCF polynomials

Definition II.30. Given a polynomial $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, the filled Julia set K_f of f is the set of points $z \in \hat{\mathbb{C}}$ such that the orbit $z, f(z), f^{\circ 2}(z), \cdots$ is bounded.

For any monic polynomial f with $\deg f \geqslant 2$, it is known that there exists a neighborhood U_∞ of ∞ contained in $\widehat{\mathbb{C}} \backslash K_f$ and a conformal map $\varphi_f \colon U_\infty \to \widehat{\mathbb{C}} \backslash \overline{\mathbb{D}}$ such that $(\varphi_f(z))^{\deg f} = \varphi_f \circ f(z)$. This conformal map is called the Böttcher chart, and it is unique up to post-multiplication by an $(n-1)$th root of unity. If K_f is connected (or equivalently, all the critical points of f are in K_f), then we can take $U_\infty = \widehat{\mathbb{C}} \backslash K_f$. In this case, we may define *dynamical rays* in the plane of f by setting $R_f(\theta) = \varphi_f^{-1}(\{r \exp(2\pi i \theta) : r > 1\})$, for every $\theta \in \mathbb{R}/\mathbb{Z}$. If $\lim_{r \to 1^+} \varphi_f^{-1}(r \exp(2\pi i \theta)$ exists, we say that $R_f(\theta)$ *lands*; this limits belongs to ∂K_f.

A polynomial is said to be unicritical if it has exactly one critical point on the plane. By [Mil06], it is known that any unicritical polynomial is affine conjugate to a polynomial of the form $f_{n,c}(z) = z^n + c$ for some $c \in \mathbb{C}$. It is known that $K_{f_{n,c}}$ is connected if and only if $0 \in K_{f_{n,c}}$.

Operations on angles

All angles in this thesis are taken to be elements of $\mathbb{R}/\mathbb{Z}$. Given distinct angles α, β, the complement of these angles in $\mathbb{R}/\mathbb{Z}$ consists of two connected components or *arcs*. The length of the shorter arc is denoted $d_{\mathbb{R}/\mathbb{Z}}(\alpha, \beta)$. We take the linear order on $\mathbb{R}/\mathbb{Z}$ induced by that on $[0, 1)$. The map $\mu_n : \mathbb{R}/\mathbb{Z} \circlearrowleft$ is defined as $\mu_n(x) = nx$, and we let $\mathcal{O}_n(\theta) = \{n^{j-1}\theta : j \geqslant 1\}$.

Every rational angle is pre-periodic under μ_n with pre-period $\ell \geqslant 0$ and eventual period $k \geqslant 1$ (we will often drop the word 'eventual').

Itineraries and kneading data

Given $\theta \in \mathbb{R}/\mathbb{Z}$, for $j = 0, 1, ..., n-1$, we define the jth static sector with respect to θ as the interval $\left(\frac{\theta+j}{n}, \frac{\theta+j+1}{n}\right) \subset \mathbb{R}/\mathbb{Z}$, and denote it $T^{stat}_{n,j}(\theta)$. Note that $\bigcup_{j=0}^{n-1} T^{stat}_{n,j}(\theta)$ is the complement of the set $\mu_n^{-1}(\theta)$ in $\mathbb{R}/\mathbb{Z}$.

Now suppose $\theta \in T^{stat}_{n,i}(\theta)$. For $j = 0, 1, ..., n-1$, we define the jth dynamic sector with respect to θ to be $T^{stat}_{n,j+i}(\theta)$, and denote it $T^{dyn}_{n,j}(\theta)$. Dynamic sectors are well-defined if and only if $\theta \notin \mu_n^{-1}(\theta)$.

For any angle $t \in \mathbb{R}/\mathbb{Z}$, the itinerary of t with respect to θ, denoted $\Sigma_{n,\theta}(t)$, is the sequence $\nu_1\nu_2\nu_3.... \in \{0, 1, ..., n, *\}^{\mathbb{N}}$ where

$$\nu_m = \begin{cases} j & n^{m-1}t \in T^{dyn}_{n,j}(\theta) \\ * & n^{m-1}t \in \bigcup_{j=0}^{n-1} \partial T^{dyn}_{n,j}(\theta) \end{cases}$$

An itinerary is called $*-$periodic if it is periodic under the shift map $\nu_1\nu_2\nu_3\nu_4.... \mapsto \nu_2\nu_3\nu_4....$ with period k, and the $*$'s occur exactly at indices $k, 2k, 3k$, etc. For an angle θ, the itinerary

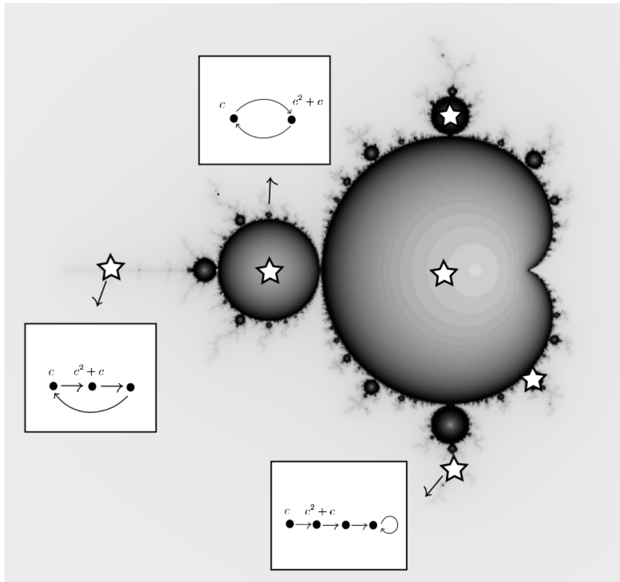


Figure 2.2: Some PCF parameters in $\mathcal{M}_2$ marked by a '$\star$' symbol, along with their portraits

$\Sigma_{n,\theta}(\theta)$ is called the kneading sequence of θ. We note that kneading sequences of rational angles are always periodic or $*-$periodic.

Example II.31. When $n = 2$, the angle $\frac{1}{7}$ has kneading sequence $\overline{00*}$, while the angle $\frac{17}{240} = \frac{17}{2^4(2^4-1)}$ has kneading sequence $0001\overline{0}$.

II.3.2: Parameter spaces of unicritical polynomials

Unicritical non-escaping loci

The set of $c \in \mathbb{C}$ for which the orbit of 0 under $f_{n,c}$ is bounded is called the *non-escaping locus* of the polynomials $\{f_{n,c}, c \in \mathbb{C}\}$, and is commonly known as the Multibrot set of degree n. We denote this set by $\mathcal{M}_n$. We recall some known facts about $\mathcal{M}_n$ here. Proofs can be found in [Hub16, Chapters 9, 10], and [EMS16].

- $\mathcal{M}_n$ is connected and compact, and the map $\Phi_n : \hat{\mathbb{C}} \backslash \mathcal{M}_n \longrightarrow \hat{\mathbb{C}} \backslash \overline{\mathbb{D}}$ given by $\Phi_n(c) = \varphi_{n,c}(c)$, where $\varphi_{n,c}$ is the Böttcher chart of $f_{n,c}$, is a conformal isomorphism. This was proven first for $n = 2$ by Douady and Hubbard in [DH82], and their proof generalizes to higher degrees.

  Given $\theta \in \mathbb{R}/\mathbb{Z}$, the parameter ray at angle θ, denoted $R_n(\theta)$, is the preimage under Φ_n of the set $\{re^{2\pi i\theta} : 1 < r \leqslant \infty\}$. Landing of parameter rays is defined analogously. If $\theta \in \mathbb{Q}/\mathbb{Z}$, then it is known that $R_n(\theta)$ lands.

- Given any hyperbolic component $U \subset \mathcal{M}_n$ (for a definition, see [DH84, Chapter 3] or [Hub16, Chapter 9]), the multiplier map $\rho_U : U \longrightarrow \mathbb{D}$ is a ramified covering of degree $(n-1)$ branched over 0. The unique critical point c_0 of ρ_U is called the center of U, and f_{d,c_0} has a super-attracting cycle of exact period k. For $c \in U$, $f_{n,c}$ has a unique attracting cycle of exact period k. ρ_U extends to a continuous map $\partial U \longrightarrow \partial\mathbb{D}$, and the fiber $\rho_U^{-1}(1)$ consists of $(n-1)$ parabolic parameters $c_1, c_2, ..., c_{n-1}$ on ∂U.

  Of these, there exists a unique parameter, say c_1, which is the landing point of exactly two rays: $R_n(\theta)$ and $R_n(\theta')$, where θ, θ' are periodic under μ_n with exact period k. This point is called the root of U, and the angles θ, θ' are said to form a companion pair. The points $c_2, \cdots, c_{n-1}$ are called the co-roots of U and each c_i is the landing point of exactly one ray $R_n(\theta_i)$, where θ_i has exact period k under μ_n. In the dynamical plane of f_{n,c_0}, the dynamic rays at angles $\theta', \theta, \theta_2, \cdots, \theta_{n-1}$ all land on the Fatou component U_0 containing c_0, and θ, θ' land at a unique point $z_0 \in \partial U_0$, called its root. The dynamic rays at angles θ, θ' separate c_0 from the other points in the postcritical set.

- Let U be a hyperbolic component of $\mathcal{M}_n$, with center c_0. Suppose the parameter rays $R_n(\theta), R_n(\theta')$ with $\theta < \theta'$ land at the root of U. Then $R_n(\theta) \cup R_n(\theta')$ split $\mathcal{M}_n \backslash U$ into two components. The component not containing 0 is called the wake of U, and denoted $\mathcal{W}(U)$ (we will also refer to this as the wake c_0). Furthermore, suppose $\theta_1 < \theta_2 < \cdots < \theta_{n-2}$ are the angles that land at the co-roots of U. Then the subsets of $\mathcal{W}(U)$ bound by a pair of rays of the form $(R_n(\theta)_j, R_n(\theta_{j+1}))$, $(R_n(\theta), R_n(\theta_1))$ or $(R_n(\theta_{n-2}), R_n(\theta'))$ are called sub-wakes of U.

- If $\theta \in \mathbb{Q}/\mathbb{Z}$ is $k-$periodic under μ_n, $R_n(\theta)$ lands on the root or co-root of a hyperbolic component of period k. If θ is pre-periodic under μ_n with pre-period $\ell \geqslant 1$ and eventual period k, then $R_n(\theta)$ lands at a Misiurewicz parameter c whose critical value has pre-period ℓ and period dividing k.

- We call $\theta \in \mathbb{Q}/\mathbb{Z}$ an angular coordinate for $c \in \mathcal{M}_n$ if

  1. c is Misiurewicz, and $R_n(\theta)$ lands on c, or
  2. c is critically periodic, and $R_n(\theta)$ lands on the root or a co-root of the hyperbolic component containing c.

  Given a PCF parameter c, we let $\Omega_n(c)$ denote its set of angular coordinates. Letting

ω denote the complex number $\exp(\frac{2\pi i}{n-1})$, it can be shown that

$$\omega \mathcal{M}_n = \mathcal{M}_n,$$
$$\forall c \in \mathcal{M}_n, \ , \Omega_n(\omega c) = \Omega_n(c) + \frac{1}{n-1}.$$

- If $f_{n,c}$ is postsingularly finite, its filled Julia set is locally connected, so the inverse Böttcher chart $\varphi_{n,c}^{-1} : \widehat{\mathbb{C}} \backslash \overline{\mathbb{D}}$ extends to a continuous surjective map from $\mathbb{S}^1 = \partial \mathbb{D}$ to $\partial K_{f_{n,c}}$. Let γ denote this boundary map; it is called the Carathéodory loop of $f_{n,c}$. Choose any $\theta \in \Omega_n(c)$. For all angles $t, t' \in \mathbb{R}/\mathbb{Z}$, it is known that $\gamma(t) = \gamma(t')$ if and only if $\Sigma_{n,\theta}(t) = \Sigma_{n,\theta}(t')$.

The quotients Λ_n

For all $c \in \mathbb{C}$, $f_{n,c}$ is affine conjugate to $f_{n,\omega c}$, where $\omega = \exp(\frac{2\pi i}{n-1})$. Moreover, for $c \neq 0$, $f_{n,c}$ is affine conjugate to $p_{n,\lambda}(z) = \lambda(1 + \frac{z}{n})^n$, where $\lambda = nc^{n-1}$, and $p_{n,\lambda}$ is affine conjugate to $p_{n,\mu}$ if and only if $\lambda = \mu$.

We define Λ_n as the image of $\mathcal{M}_n \backslash \{0\}$ under the map $c \mapsto nc^{n-1}$. Equivalently, Λ_n is the set of $\lambda \in \mathbb{C}^*$ such that $p_{n,\lambda}$ has connected filled Julia set.

Given $\lambda = nc^{n-1}$, we refer to the parameters $c, \omega c, \omega^2 c, ..., \omega^{n-2} c$ as the monic representatives for λ, and denote this set $M_n(\lambda)$. We call θ an *angular coordinate* for λ if $\theta \in \Omega_n(c)$ for some $c \in M_n(\lambda)$. The set of angular coordinates is denoted $\Theta_n(\lambda)$.

$$\Theta_n(\lambda) = \bigcup_{n=0}^{n-2} \Omega_n(\omega^n c)$$

We shall denote by $\mathcal{P}_n$ the set of PCF parameters λ for which $p_{n,\lambda}$ is postsingularly finite. This consists of all Misiurewicz parameters, and all critically periodic parameters with period $\geqslant 2$.

II.3.3: Dynamics of postsingularly finite exponential maps

The exponential family is simplest among all families of transcendental maps, since for any exponential function f, the set S_f contains only one point. This section covers some basic properties of exponential maps, with a focus on postsingularly finite maps.

For $\lambda \in \mathbb{C}^*$, let $p_\lambda(z) = \lambda \exp(z)$. We note that $S_{p_\lambda} = \{0\}$, and that $p_\lambda(0) = \lambda$. The set of $\lambda \in \mathbb{C}^*$ for which the p_λ−orbit of 0 is bounded is called the *non-escaping locus* of the exponential family $\{p_\lambda | \lambda \in \mathbb{C}^*\}$, denoted Λ. Any $\lambda \notin \Lambda$ is called an *escaping parameter.*

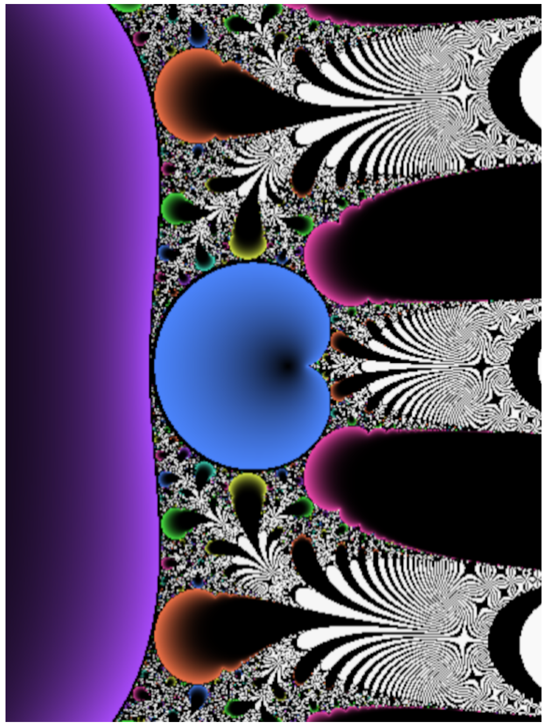


Figure 2.3: The non-escaping locus Λ of the exponential family $\{p_\lambda : \lambda \in \mathbb{C}^*\}$; here the white region represents $\mathbb{C}\backslash\Lambda$

We note that for each λ, the map p_λ is affine conjugate to an exponential map of the form $z \mapsto \exp(z) + c$, with c unique upto translation by $2\pi in$ for some $n \in \mathbb{Z}$. We could also look at the non-escaping locus in the family $\{z \mapsto \exp(z) + c | c \in \mathbb{C}\}$, which forms a natural analog to $\mathcal{M}_n$; however, it is preferable to work in the p_λ normalization since we can then use the fact that $\lim_{n\to\infty} \lambda(1 + \frac{z}{n})^n = \lambda \exp(z)$.

Similar to unicritical polynomials, for $\lambda \in \Lambda$, the structure of the set of points $\{z \in \mathbb{C} : \lim_{n\to\infty} p_\lambda^{\circ n}(z) = \infty\}$, called the *escaping set* of p_λ, is well-understood on a topological level. We list some of its properties in this section. For a more complete description, see [SZ03a], [SZ03b], [Rem06], and [Rem10] .

The escaping set also helps play a vital role in classifying postsingularly finite exponential maps; in fact, the authors of [LSV08] carry out this classification and develop a theory of spiders akin to that of polynomials.

Dynamic and parameter hairs

Just as angles in $\mathbb{R}/\mathbb{Z}$ correspond to dynamical rays, which are subsets of $\widehat{\mathbb{C}}\backslash K_f$ for a polynomial f, in the exponential regime, certain subsets of the escaping set, called *hairs*, can be labelled by sequences of integers called *external addresses*. We formalize this theory below (for more details, see [Rem06]).

Given $\lambda \in \mathbb{C}^*$, choose $c = \ln \lambda$ such that $\operatorname{Im} c \in [-\pi, \pi)$. For $j \in \mathbb{Z}$, define

$$U^{stat}_{\infty,j}(\lambda) = \{z : (2j-1)\pi - \operatorname{Im} c < \operatorname{Im} z < (2j+1)\pi + \operatorname{Im} c\}$$

Note that every connected component of $p_\lambda^{-1}(\mathbb{C}\backslash\mathbb{R}_{\leqslant 0})$ is of the form $U^{stat}_{\infty,j}(\lambda)$ for some j, and that the map $p_\lambda | U^{stat}_{\infty,j}(\lambda) : U^{stat}_{\infty,j}(\lambda) \longrightarrow \mathbb{C}\backslash\mathbb{R}_{\leqslant 0}$ is a conformal isomorphism for each $j \in \mathbb{Z}$. The collection $\{U^{stat}_{\infty,j}(\lambda)\}_{j\in\mathbb{Z}}$ is a partition of the plane, and is called the static partition with respect to λ.

Definition II.32. We call a sequence $\underline{s} \in \mathbb{Z}^{\mathbb{N}}$ an *external address*. Let μ be the left shift map on $\mathbb{Z}^{\mathbb{N}}$. For any $z \in \mathbb{C}$ with $p_\lambda^{\circ n}(z) \notin \mathbb{R}_{\leqslant 0}$ for all n, the external address of z is the sequence $s_1 s_2 \cdots s_m \cdots$ with $p_\lambda^{\circ(m-1)}(z) \in U^{stat}_{\infty,s_m}(\lambda)$ for all $m \in \mathbb{N}$. For any external address $\underline{s}$ and integer r, the external address $r\underline{s}$ is the sequence $rs_1s_2\cdots$.

Let $<$ denote the standard lexicographic order on $\mathbb{Z}^{\mathbb{N}}$: $\underline{s} < \underline{t}$ if at the first index m where $s_m \neq t_m$, we have $s_m < t_m$. Additionally, let $<<$ denote the *cylindrical* order: $\underline{s} << \underline{t}$ if one of the following is true:

- $\underline{s} < \underline{t}$, or
- $\underline{t} < \overline{0} < \underline{s}$

As for angles, for every $j \in \mathbb{Z}$, we let $T^{stat}_{\infty,j}(\underline{s})$ denote the interval $(j\underline{s}, (j+1)\underline{s}) = \{\underline{u} \in \mathbb{Z}^{\mathbb{N}} : j\underline{s} < \underline{u} < (j+1)\underline{s})\}$; we call this a static sector with respect to $\underline{s}$.

An address $\underline{s}$ is said to be pre-periodic if its orbit under μ in $\mathbb{Z}^{\mathbb{N}}$ is finite; it is periodic if this orbit is cyclic (we will often take 'pre-periodic' to mean strictly pre-periodic). If $\underline{s}$ is not periodic with period 1, then there exists a unique $j \in \mathbb{Z}$ such that $\underline{s} \in T^{stat}_{\infty,j}(\underline{s})$. In this case, we define, for $m \in \mathbb{Z}$, the *dynamic* sector with respect to $\underline{s}$ as $T^{dyn}_{\infty,m}(\underline{s}) = T^{stat}_{\infty,j+m}(\underline{s})$.

Remark II.33. There is a direct analogy between external addresses and angles in $\mathbb{R}/\mathbb{Z}$ that correspond to rays for a degree n polynomial : every angle in $\mathbb{R}/\mathbb{Z}$ has an $n-$adic expansion, and can therefore be viewed as an element of $\{0, 1, \cdots, n-1\}^{\mathbb{N}}$. Furthermore, rational angles can be thought of as pre-periodic sequences in $\{0, 1, \cdots, n-1\}^{\mathbb{N}}$.

Definition II.34. Let $\underline{s}$, $\underline{t} \in \mathbb{Z}^{\mathbb{N}}$. The itinerary of $\underline{t}$ with respect to $\underline{s}$, denoted $\Sigma_{\underline{s}}(\underline{t})$, is a sequence $\nu_1\nu_2\nu_3\cdots\nu_m\cdots$ where

$$\nu_m = \begin{cases} j & \mu^{\circ(m-1)}(\underline{t}) \in T^{stat}_{\infty,j}(\underline{s}) \\ * & \mu^{\circ(m-1)}(\underline{t}) = j\underline{s} \text{ for some } j \in \mathbb{Z} \end{cases}$$

Definition II.35. A sequence $\underline{s} \in \mathbb{Z}^{\mathbb{N}}$ is *bounded* if there exists a constant $C > 0$ such that $|s_n| \leqslant C$ for all n.

Let $F(t) = e^t - 1$. An address $\underline{s}$ is said to be *exponentially bounded* if there exist constants $A \geqslant 1$, $x > 0$ such that $|s_n| \leqslant A|F^{\circ(n-1)}(x)|$ for all $n \geqslant 1$.

Fix $\lambda \in \mathbb{C}^*$. The following theorems illustrate the behaviour of escaping points in the dynamic plane of p_λ.

Theorem II.36 ([SZ03b, Theorem 2.3])**.** *If $\lambda \in \Lambda$, then for every bounded $\underline{s}$ there is a unique injective and continuous curve $\gamma_{\underline{s}} : (0, \infty) \longrightarrow \mathbb{C}$ of external address $\underline{s}$ satisfying*

$$\lim_{t\to\infty} \operatorname{Re} \gamma_{\underline{s}}(t) = +\infty$$

which has the following properties: it consists of escaping points such that

$$p_\lambda(\gamma_{\underline{s}}(t)) = \gamma_{\underline{s}}(p_\lambda(t)) \ \forall t > 0$$
$$\gamma_{\underline{s}}(t) = t - c + 2\pi i s_1 + r_{\underline{s}}(t) \ \forall t > 0$$

with $|r_{\underline{s}}(t)| < 2e^{-t}(|K| + C)$, where $C \in \mathbb{R}$ depends only on a bound for $\underline{s}$.

If the singular orbit does escape, then the statement is still true for every bounded address $\underline{s}$ for which there is no $n \geqslant 1$ and $t_0 > 0$ such that $0 = \gamma_{\mu^{\circ n}(\underline{s})}(t_0)$.

For those exceptional $\underline{s}$, there is an injective curve $\gamma_{\underline{s}} : (t^*_{\underline{s}}, \infty) \longrightarrow \mathbb{C}$ with the same properties as before, where $t^*_{\underline{s}} > 0$ is the largest number which has an $n \neq 1$ such that $F^{\circ n}(t^*_{\underline{s}}) = t_0$ and $0 = \gamma_{\mu^{\circ n}(\underline{s})}(t_0)$.

The curve $\gamma_{\underline{s}}$ is called the dynamic ray (or *hair*) at external address $\underline{s}$.

For $\lambda \in \mathbb{C}^*$, let $\arg(\lambda)$ be the argument of λ that in the interval $[-\pi, \pi)$. On the dynamical plane of any p_λ, a point z in the escaping set has external address $\underline{s}$ if and only if $p_\lambda^{\circ(m-1)}(z) \in U^{stat}_{\infty,m}(\lambda)$ for every $m \in \mathbb{N}$.

Theorem II.37 ([LSV08, Theorem 2.6])**.** *For every pre-periodic external address $\underline{s}$ starting with the entry 0, there exists a postsingularly finite exponential map $z \mapsto \lambda \exp(z)$ such that the dynamic ray at external address $\underline{s}$ lands at the singular value 0. Every postsingularly finite exponential map is associated in this way to a positive finite number of pre-periodic external addresses starting with 0.*

As in the polynomial case, we have a notion of parameter rays to the exponential family.

Theorem II.38 ([FRS08, Theorem 1.1])**.** *The set of parameters $\lambda \in \mathbb{C}^*$ for which $p_\lambda^{\circ n}(0) \to \infty$ consists of uncountably many disjoint curves in $\mathbb{C}$. More precisely, every path-connected* *component of this set is an injective curve $\gamma : (0, \infty) \longrightarrow \mathbb{C}$ or $\gamma : [0, \infty) \longrightarrow \mathbb{C}$ with $\lim_{t\to\infty} \gamma(t) = \infty$.*

Every path connected component as in the above theorem above is referred to as a parameter ray. We say that a parameter ray γ as above lands if $\lim_{t\to 0^+} \gamma(t)$ exists. Parameter rays are also defined in [FS09] for the exponential family $\{z \mapsto \exp(z + \kappa), \kappa \in \mathbb{C}\}$.

Let $\mathcal{P}$ denote the collection of $\lambda \in \mathbb{C}^*$ such that p_λ is postsingularly finite.

Theorem II.39 ([LSV08, Theorem 3.4])**.** *For every postsingularly finite exponential map p_λ and every pre-periodic external address $\underline{s}$ with $s_1 = 0$, the dynamic ray at address $\underline{s}$ lands at the singular value if and only if the parameter ray at address $\underline{s}$ lands at λ.*

Theorem II.40 ([LSV08, Corollary 3.5])**.** *Every parameter ray at a strictly pre-periodic external address $\underline{s}$ lands at a postsingularly finite exponential map, and every pre-periodic exponential map is the landing point of a finite positive number of parameter rays at strictly pre-periodic external addresses.*

We let $\Theta_\infty(\lambda)$ denote the set of external addresses $\underline{s}$ such that the parameter ray with address $\underline{s}$ lands at λ (note that for any such $\underline{s}$, we have $s_1 = 0$).

When exponential maps are normalized as $\exp(z) + c$, dynamic rays for these maps and parameter rays in the $c-$plane are analogously defined. We note that if $c = \ln \lambda$ and the dynamic ray at address $\underline{s} = s_1 s_2.....$ lands at c in the dynamic plane of $z \mapsto \exp(z) + c$, then the dynamic ray at address $0(s_2 - s_1)(s_3 - s_1)(s_4 - s_1)....$ lands at 0 in the dynamic plane of p_λ.

II.4: Combinatorial theory

II.4.1: Combinatorics of unicritical PCF polynomials

Postsingularly finite polynomials admit several combinatorial descriptions: in terms of *spiders*, *hubbard trees* and *kneading sequences*. In this section we give a brief overview of these tools for unicritical polynomials. These tools have been used to completely classify postsingularly finite polynomials.

For the rest of Section II.4.1, fix a degree $n \geqslant 2$.

Orbit portraits

Given any repelling cycle $\{z_1, ..., z_r\}$ of a polynomial p, the orbit portrait associated with this cycle is the collection $\{\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_r\}$, where $\mathcal{A}_i$ is the set of angles $\theta \in \mathbb{R}/\mathbb{Z}$ such that the dynamic ray at angle θ in the plane of p lands at z_i.

Definition II.41. A collection $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_r\}$ is called a formal degree n orbit portrait if

1. Each $\mathcal{A}_i$ is a non-empty finite subset of $\mathbb{Q}/\mathbb{Z}$.
2. μ_n maps each $\mathcal{A}_i$ bijectively onto $\mathcal{A}_{i+1}$ and preserves the cyclic order of the angles.
3. Each $\mathcal{A}_i$ is contained in some arc of length less than $1/n$ in $\mathbb{R}/\mathbb{Z}$.
4. Each $\alpha \in \cup_{i=1}^{r} \mathcal{A}_i$ is periodic under μ_n and all such α's have a common period rp for some $p \geqslant 1$.
5. For every m, let $\mathcal{A}_{m,i} = \mathcal{A}_m + \frac{i}{n+1}$. For all pairs $(m, j) \neq (m', j')$, $\mathcal{A}_{m,j}$ and $\mathcal{A}_{m',j'}$ are unlinked.

Given any formal portrait $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_r\}$, there exists $j \in \{1, 2, ..., r\}$ and distinct angles $\alpha, \beta \in \mathcal{A}_j$ such that $d_{\mathbb{S}^1}(\alpha, \beta)$ is uniquely minimal among all arc lengths $d_{\mathbb{S}^1}(\alpha', \beta')$ where α' and β' are distinct angles in some $\mathcal{A}_{j'}$. The pair (α, β) is called the characteristic angle pair of $\mathcal{A}$. Moreover, the portrait $\mathcal{A}$ can be reconstructed from (α, β), and we call $\mathcal{A}$ the degree n orbit portrait generated by (α, β).

[EMS16, Theorem 2.12] states that given a formal degree$-n$ portrait $\mathcal{A} = \{\mathcal{A}_1, ..., \mathcal{A}_r\}$ and p as in point (4) above, there exists a PCF polynomial of the form $f_{n,c}(z) = z^n + c$ such that

1. $f_{n,c}$ is critically periodic with period rp, and
2. $\mathcal{A}$ is the orbit portrait associated with the cycle $(z_1, z_2, ..., z_r)$ of $f_{n,c}$, where, for each $j \in \{1, 2, ..., r\}$, the point z_j is the root of the Fatou component containing $f_{n,c}^{\circ(j-1)}(c)$.

The authors of [BFH92] carry out a combinatorial classification of critically pre-periodic polynomials of a given degree, in terms of angles landing on the orbit of the critical values. The following lemma sheds light on the relationship between characteristic angle pairs; while the result below was proved by Lavaurs for degree $n = 2$, it generalizes naturally to higher degrees.

Lemma II.42 (Lavaurs' Lemma; [Lav86])**.** *Let $P_1 = (\theta_1, \theta_1')$ and $P_2 = (\theta_2, \theta_2')$ be companion angle pairs with the same period k, and U_1, U_2 be the hyperbolic components defined by P_1 and P_2 . If $\theta_1 < \theta_2 < \theta_2' < \theta_{2'}$, then there exists a companion pair $R_3 = (\theta_3, \theta_3')$ with period $< k$ such that such that*

$$\theta_1 < \theta_3 < \theta_2 < \theta_2' < \theta_3' < \theta_1'$$

In other words, R_3 separates R_1 from R_2.

Internal addresses

Definition II.43. Let c be a critically periodic PCF parameter in $\mathcal{M}_n$ with period k.

If $c \neq 0$, let R_1 be the parameter ray pair of lowest period in $\mathcal{M}_n$ that separates c from 0 (by Lemma II.42, there is exactly one ray pair whose period is lowest is among all ray pairs that separate 0 and c). Let s_1 be the period of R_1. For $n \geqslant 2$, inductively define R_n to be the ray pair of lowest period that separates R_{n-1} from c, and let s_1 be the period of R_n. This sequence ends after finitely many (say r) steps, with the last entry $s_r = k$. The *internal address* of c (and its hyperbolic component) is given by the sequence

$$s_0 = 1 \mapsto s_1 \mapsto s_2 \mapsto \cdots \mapsto s_r = k.$$

Example II.44. For the quadratic rabbit parameter $c \approx -0.122561 + 0.744862i$, we have $R_1 = (\frac{1}{7}, \frac{2}{7})$, and $s_1 = 3$. The full internal address is in fact $1 \mapsto 3$.

It is proved in [BS08] showed that kneading sequences and internal addresses are equivalent. Given any sequence $*-$periodic kneading sequence ν in degree n, consider the function

$$\rho_\nu : \mathbb{N} \cup \{\infty\} \longrightarrow \mathbb{N} \cup \{\infty\}$$
$$\rho_\nu(k) = \begin{cases} \inf\{n > k : \nu_{n-k} \neq \nu_n\} & k \neq \infty \\ \infty & k = \infty \end{cases}$$

For any periodic PCF parameter c in $\mathcal{M}_n$ with kneading sequence ν, the internal address of c can be computed as the sequence $1 \mapsto \rho_\nu(1) \mapsto \rho_\mu^{\circ 2}(1) \mapsto \cdots$, upto the entry before ∞.

Note that the internal address of c can be used to determine if c is primitive or a satellite: c is primitive if and only if s_{r-1} does not divide s_r.

Generalised spiders

Definition II.45. Given $n \in \mathbb{Z}_{\geqslant 2}$, a degree n *generalised spider* is a tuple (S, t) where $t \in \mathbb{R}^2$ and $S \subset \mathbb{S}^2$ is an undirected planar graph satisfying the following properties:

1. The vertex set of S is $\{t, \infty\} \cup \{a_1, a_2, \cdots, a_r\}$, where the a_i are not necessarily distinct from each other, but are all distinct from t and ∞.

2. For each $i \in \{1, 2, \cdots, r\}$, there is a unique edge given by a Jordan arc joining ∞ and a_i.

3. The other edges are given by finitely many Jordan arcs $\eta_1, \eta_2, \cdots, \eta_n$ joining ∞ and t which are pairwise disjoint.

Spiders are so named because of the resemblence with the eponymous insect perched on $\mathbb{S}^2$ with the head placed at ∞. We will call each edge of S a 'leg'. We will always assume that the η_j's are labelled so that $\eta_1, \eta_2, \cdots \eta_n$ are in counterclockwise order at ∞.

Definition II.46. Given $n, n' \in \mathbb{Z}_{\geqslant 2}$, let (S, t) and (S', t') be two generalised spiders, and $W \subseteq S$ and $W' \subseteq S'$ be connected subgraphs. The graphs W and W' are said to be *similar* if they satisfy the following properties:

- They have the same number of edges, $t \in W$ if and only if $t' \in W'$, and if $t \in W$, then the local degree of W at t is equal to the local degree of W' at t'.
- There exists a relabelling of the $\gamma-$type edges in W and W' such that for any three legs, say γ_i, γ_j and η_k in W that are in counterclockwise order at ∞, the corresponding legs γ_i', γ_j' and η_k' in W' are in counterclockwise order at ∞.

Remark II.47. We will often describe S and S' with a fixed labellings of their edges; if subgraphs W and W' are similar with respect to the given labelling on S and S', we say that W and W' are *congruent*.

Definition II.48. Given a generalised spider (S, t) as above, a spider map on (S, t) is a continuous map $f : S \longrightarrow S$ that takes vertices to vertices and edges to edges, and satisfies the properties below:

There exists a relabelling of the edges set as $\{\gamma_1, \gamma_2, \cdots, \gamma_r\}$ (and the corresponding vertices a_i), such that f additionally satisfies the following properties:

1. $f(\infty) = \infty$ and $f(t) = a_1$;
2. f maps η_j homeomorphically to γ_1 for all $j \in \{1, 2, \cdots, n\}$;
3. for $j = 1, 2, \cdots, r-1$, f maps γ_j, homeomorphically to γ_{j+1}, and maps γ_r homeomorphically to $\gamma_{\ell+1}$ for some ℓ, or else maps γ_r homeomorphically to η_i for some i;
4. for each $j \in \{1, 2, \cdots, r\}$, the map f preserves the circular order of all legs contained between η_j to $\eta_{j+1(\bmod r)}$.

The above points imply that f has critical points t and ∞, and that the orbit of each critical point under f is finite. Condition (3) above also implies that $\mathbb{S}^2 \backslash f(S)$ is an open topological disk.

A spider map $f : S \to S$ as in the above definition can be extended in the way outlined in the following remark (Remark II.49) to a polynomial Thurston map. Recall that in Appendix A, we explore graphs embedded in $\mathbb{R}^2$. We can similarly think of graphs embedded on the sphere $\mathbb{S}^2$; this includes graphs that contain the point at infinity.

Remark II.49. Suppose G and G' are graphs embedded on $\mathbb{S}^2$ such that $U = \mathbb{S}^2 \backslash G$ and $U' = \mathbb{S}^2 \backslash G'$ are open topological disks. Let f be a homeomorphism from G to G'; we show a way to extend f to a homeomorphism of $\mathbb{S}^2$.

There exist homeomorphisms $\varphi : \mathbb{D} \to U$ and $\psi : \mathbb{D} \to U'$ such that φ extends to a branched cover of degree 2 from $\partial\mathbb{D}$ to G, and ψ extends to a branched cover of degree 2 from $\partial\mathbb{D}$ to G' (this becomes clear when we think of $\mathbb{S}^2 \backslash G$ as cutting the plane along G; the boundary of the new region contains two copies of G).

Then there exists a homeomorphism h of $\partial\mathbb{D}$ such that the following diagram commutes:

$$\begin{array}{ccc} \partial\mathbb{D} & \xrightarrow{h} & \partial\mathbb{D} \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} \\ G & \xrightarrow{f} & G' \end{array}$$

Using Proposition A.5, extend h to $\overline{\mathbb{D}}$, and define, for all $z \in U$, $f(z) = \psi \circ \varphi^{-1}$. This extends f as a homeomorphism of $\mathbb{S}^2$. For different choices of φ, ψ, we get different extensions of f; however, by Proposition A.5, any two such extensions are isotopic rel G. This process is also known as the Alexander trick.

By following the construction in Remark II.49 on every connected component of $\mathbb{S}^2 \backslash S$, we extend any spider map $f : S \longrightarrow S$ to $\mathbb{S}^2$. Since there are n such connected components, and the closure of each component maps onto $\mathbb{S}^2$, we see that this newly defined map f has degree n. By restricting it to $\mathbb{R}^2$, we get a polynomial Thurston map.

Proposition II.50. *Given two maps $f : S \to S$ and $g : S' \to S'$, where (S, t) and (S', t') have the same degree, if there exists a homeomorphism $h : S \to S'$ such that*

1. *$h(\infty) = \infty$ and $h(t) = t'$,*
2. *h preserves the circular order of legs at ∞,*
3. *$h(P_f) = P_g$, and $(g|S') \circ h = h \circ (f|S)$,*

then the extended Thurston maps f, g on $\mathbb{R}^2$ are Thurston equivalent.

Proof. Since h preserves the circular order of legs, by using the Alexander trick on each component of $\mathbb{S}^2 \backslash S$, the map h can be extended to a homeomorphism from $\mathbb{S}^2$ to $\mathbb{S}^2$; note that each connected component of $S^2\backslash S$ maps under h homeomorphically onto a connected component of $S^2\backslash S'$. Let D be an open connected component of $S^2\backslash S'$. On D, let $\widetilde{g} = h^{-1}\circ f\circ h$. Since $\widetilde{g}$ agrees with g on ∂D, it is isotopic to g relative to ∂D. We can similarly define $\widetilde{g}$ on every connected component of $\mathbb{S}^2\backslash S'$ to get a map $\widetilde{g}$ on $\mathbb{R}^2$ such that $\widetilde{g}\circ h = h\circ f$, with $\widetilde{g}$ isotopic to g rel. P_g. □

Proposition II.51. *Given two spider maps $f : S \to S$ and $g : S' \to S'$, where (S, t) and (S', t') have the same degree, assume that their edge sets are labelled so that f and g satisfy conditions (1)-(4) in Definition II.48.*

If S and S' are congruent, then f and g extended to $\mathbb{R}^2$ are Thurston equivalent.

Proof. Let h be a homeomorphism that maps ∞ to ∞, t to t', γ_i to γ_i' for all $i \in \{1, 2, \cdots, r\}$ and η_j to η_j' for all $j \in \{1, 2, \cdots n\}$. It is clear that h preserves the circular order of legs at ∞. Moreover, h can be defined so that $h \circ f|S = (g|S') \circ h$. Therefore the statement follows from Proposition II.50. □

Standard spiders for unicritical polynomials

Fix $n \in \mathbb{Z}_{\geqslant 2}$ and $\lambda \in \mathcal{P}_n$.

A *spider* for $p_{n,\lambda}$ is an invariant graph in its dynamical plane, from which we can recover several of its dynamical properties. While a comprehensive theory for spiders is described in [Hub16, Chapter 10], in this section, we restrict our approach to generalised spiders on the sphere $\mathbb{S}^2$ that can be used to construct a topological model (i.e., a Thurston map) that is combinatorially equivalent to $p_{n,\lambda}$. The construction here generalizes that presented in [HS94] and [Hub16, Chapter 10] for $n = 2$.

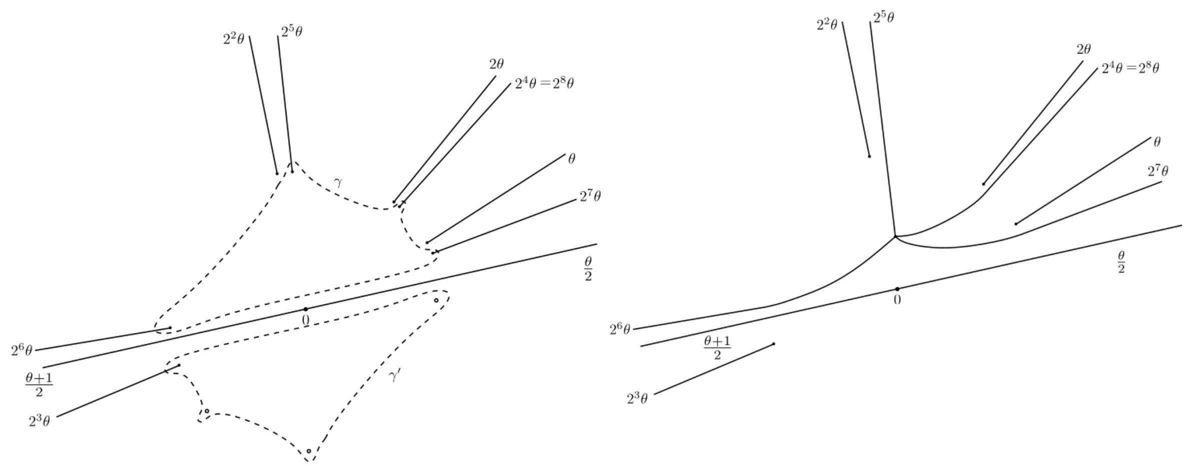
Definition II.52. Let $\theta \in \Theta_n(\lambda)$.

- If θ is pre-periodic under μ_n, the *standard degree n spider* of θ is defined as

$$\widehat{S}_n(\theta) := \bigcup_{j\geqslant 1}\{r \exp(2\pi i n^{j-1}\theta) : r \in [1, \infty]\}$$

- Else if θ is periodic under μ_n with period k, the standard spider $\widehat{S}_n(\theta)$ is the union of the above set and $\{r \exp(2\pi i n^{k-1}\theta) : r \in [0, 1]\}$.
- The *extended* degree n spider $\widehat{S}_n^{ext}(\theta) := S_n(\theta) \cup \bigcup_{j=0}^{n-1}\{r \exp(\frac{2\pi i(\theta+j)}{n}) : r \in [0, \infty]\}$.

In the sense of Definition II.45, we note that $(\widehat{S}_n^{ext}(\theta), 0)$ is a generalized spider.



(a) The standard spider $\widehat{S}_2\left(\frac{17}{240}\right)$. The curve γ forms a Levy cycle

(b) The quotient spider $S_2\left(\frac{17}{240}\right)$. The Levy cycle γ has been contracted to a point

Figure 2.4: Spiders in degree 2

Figure 2.4a shows an example of a standard degree 2 spider. Note that there exists a natural map $\widehat{\mathcal{F}}_{n,\theta} : \widehat{S}_n^{ext}(\theta) \longrightarrow \widehat{S}_n(\theta)$ that simulates the map μ_n on $\mathbb{R}/\mathbb{Z}$:

$$\widehat{\mathcal{F}}_{n,\theta}(r\exp(2\pi it)) = \begin{cases} \infty & r = \infty \\ \exp(2\pi i\theta) & r = 0 \\ r\exp(2\pi int) & nt \neq \theta \\ (r+1)\exp(2\pi i\theta) & nt = \theta \end{cases}$$

For θ periodic under μ_n with period k, the definition of $\widehat{\mathcal{F}}_{n,\theta}$ is the same as above, but we additionally let $\mathcal{F}_{n,\theta}(r\exp(2\pi i n^{k-2}\theta)) = (r-1)\exp(2\pi i n^{k-1}\theta)$ for all $r \geqslant 1$. With this definition, it is clear that 0 is periodic under $\mathcal{F}_{n,\theta}$.

The only critical points of $\mathcal{F}_{n,\theta}$ are $0, \infty$, each with local degree n. By the Alexander trick, we can extend $\mathcal{F}_{n,\theta}$ to a n-sheeted branched self-cover of $\mathbb{S}^2$. As shown in [HS94] and [Hub16, Chapter 10], if θ is an angular coordinate for λ,

- if θ is periodic, $\mathcal{F}_{n,\theta}$ is Thurston equivalent to $p_{n,\lambda}$;
- if θ is pre-periodic, $\mathcal{F}_{n,\theta}$ is realized if and only if the eventual period of θ under μ_n is equal to the eventual period of λ under $p_{n,\lambda}$. If this equality holds, then $\mathcal{F}_{n,\theta}$ is Thurston equivalent to $p_{n,\lambda}$.

Suppose that θ has pre-period $\ell \geqslant 1$ and period $k \geqslant 1$ under μ_n, and k is strictly larger than the eventual period k' of λ under $p_{n,\lambda}$. Then for each $m \geqslant \ell$, the points of the form

$\exp(2\pi i n^m\theta), \exp(2\pi i n^{m+k'}\theta), \exp(2\pi i n^{m+2k'}\theta)$, etc, all have the same itinerary with respect to θ. By drawing loops around all points that share an itinerary, we obtain a Levy cycle C for $\mathcal{F}_{n,\theta}$. Moreover, this is the only Levy cycle upto homotopy relative to the postsingular set of $\mathcal{F}_{n,\theta}$.

In this case, we form a new topological polynomial by contracting each curve in C to a point. More precisely, form a quotient surface of $\mathbb{S}^2$ by shrinking each region in $\mathbb{R}^2$ bounded by some $\gamma \in C$ to a point (we always assume 0 is mapped to 0 in this quotient construction). This gives rise to quotient graphs $S_n\theta$ and $S_n^{ext}(\theta)$ of the standard and extended spiders respectively. The pair $(S_n^{ext}(\theta), 0)$ is a generalized degree n spider. While this quotient construction is dependent on C, we will assume that we have made a choice of C for every $\theta \in \mathbb{Q}/\mathbb{Z}$ (it is also easy to see that the quotient spiders for different choices of C are congruent).

Given the above construction, $\widehat{\mathcal{F}}_{n,\theta}$ descends to a map $\mathcal{F}_{n,\theta} : S_n^{ext}(\theta) \longrightarrow S_n(\theta)$, which in turn can be extended to a topological polynomial $\mathcal{F}_n(\theta) : \mathbb{S}^2 \to \mathbb{S}^2$ using the Alexander trick, unique upto isotopy rel $S_n^{ext}(\theta)$. This extension is Thurston equivalent to $p_{n,\lambda}$.

When θ is known, we let x_1 denote the equivalence class of $\exp(2\pi i\theta)$ in $S_n(\theta)$, and let $x_j = \mathcal{F}_{n,\theta}^{\circ(j-1)}(x_1)$ for all $j \in \mathbb{Z}_{\geqslant 1}$.

II.4.2: Combinatorics of postsingularly finite exponentials

Spiders

Given a strictly pre-periodic external address $\underline{s}$, we recall the construction of the spider graph $S_\infty(\underline{s})$ in [LSV08, Section 5.1] here, with some slight modifications. Let ℓ and k be the pre-period and period of $\underline{s}$ respectively under the left-shift map $\mu : \mathbb{Z}^{\mathbb{N}} \longrightarrow \mathbb{Z}^{\mathbb{N}}$.
First, we extend $\mathbb{C}$ to a bigger space $\mathbb{C}^{ext} = \mathbb{C} \cup \{\widehat{e}_{-\infty}, \widehat{e}_{+\infty}\}$, and declare, for every $r \in \mathbb{R}$, the sets $\{\widehat{e}_{-\infty}\} \cup \{Re(z) < r\}$ and $\{\widehat{e}_{+\infty}\} \cup \{Re(z) > r\}$ to be open. We can think of $\widehat{e}_{+\infty}$ and $\widehat{e}_{-\infty}$ as points at $+\infty$ and $-\infty$ respectively. With this topology, it is clear that the space $\mathbb{C}^{ext}$ is Hausdorff.
Now let $\widehat{e}_1 = 0$. For every $m \in \{2, 3, \cdots, \ell + k\}$, define points $\widehat{e}_m \in \mathbb{C}$ so that $\mathrm{Re}(\widehat{e}_m) = 0$, and

1. $\mathrm{Im}(\widehat{e}_m) \in \big((2s_m - 1)\pi, (2s_m + 1)\pi\big)$;

2. for $m \neq m'$, $\mathrm{Im}\,\widehat{e}_m < \mathrm{Im}\,\widehat{e}_{m'}$ if $\mu^{\circ m}(\underline{s}) < \mu^{\circ m'}(\underline{s})$, and $\widehat{e}_m = \widehat{e}_{m'}$ if $\mu^{\circ m}(\underline{s}) = \mu^{\circ m'}(\underline{s})$.

The second condition implies that if $m > \ell + k$, we can think of the point $\widehat{e}_m$ as $\widehat{e}_{m-k}$. For every $m \in \{1, 2, \cdots, \ell + k\}$, also define γ_m as the closure of the horizontal ray $\{\mathrm{Re}(z) \geqslant 0, \mathrm{Im}(z) = \mathrm{Im}\,\widehat{e}_m\}$ in $\mathbb{C}^{ext}$ (this is in fact equal to $\{\mathrm{Re}(z) \geqslant 0, \mathrm{Im}(z) = \mathrm{Im}\,\widehat{e}_m\} \cup \{\widehat{e}_{+\infty}\}$). Further, for each $r \in \mathbb{Z}$, let p_m be the closure of the horizontal line $\{\mathrm{Im}(z) = (2r-1)\pi\}$ in $\mathbb{C}^{ext}$ (this is simply $\{\mathrm{Im}(z) = (2r-1)\pi\} \cup \{\widehat{e}_{+\infty}, \widehat{e}_{-\infty}\}$).

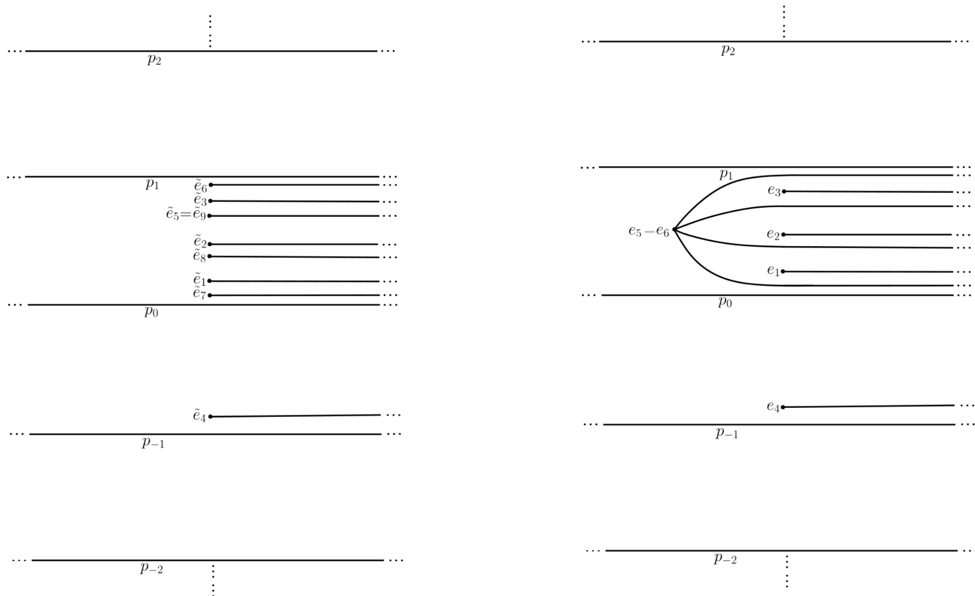


Figure 2.5: The standard spider $\widehat{S}_\infty(\underline{s})$ and the quotient $S_\infty(\underline{s})$ corresponding to the external address $\underline{s} = 000(-1)\overline{0010}$

The standard exponential spider and its corresponding extended spider of $\underline{s}$ are respectively defined as follows:

$$\widehat{S}_\infty(\underline{s}) = \bigcup_{m=1}^{\ell+k} \gamma_m$$
$$\widehat{S}_\infty^{ext}(\underline{s}) = \widehat{S}_\infty(\underline{s}) \cup \bigcup_{r\in\mathbb{Z}} p_r$$

We mark the points in $\{\widehat{e}_{+\infty}, \widehat{e}_{-\infty}\} \cup \{\widehat{e}_1, \widehat{e}_2, \cdots, \widehat{e}_{\ell+k}\}$ as vertices of the spiders above (note that $\widehat{e}_{-\infty}\} \notin \widehat{S}_\infty(\underline{s})$). Also observe that $\mathbb{C}^{ext} \backslash \widehat{S}_\infty(\underline{s})$ is a topological surface that is homeomorphic to an open topological disk. We will often call an edge in either of the graphs above a 'leg'.

As for polynomials, we define a 'spider map' $\widehat{\mathcal{G}}_{\infty,\underline{s}} : \widehat{S}_\infty^{ext}(\underline{s}) \longrightarrow \widehat{S}_\infty(\underline{s})$ as follows:

$$\widehat{\mathcal{G}}_{\infty,\underline{s}}(z) = \begin{cases} 0 & z = \widehat{e}_{-\infty} \\ \widehat{e}_{+\infty} & z = \widehat{e}_{+\infty} \\ \mathrm{Re}(z) + \widehat{e}_{m+1} & \mathrm{Im}(z) = \widehat{e}_m \text{ for some } m \in \{1, 2, \cdots \ell + k\} \\ \exp(\mathrm{Re}(z)) + \widehat{e}_1 & z \in p_r \text{ for some } r \in \mathbb{Z} \end{cases}$$

We note that $\widehat{F}_{\infty,\underline{s}}$ satisfies the following properties:

- $\widehat{\mathcal{G}}_{\infty,\underline{s}}$, and has two branch points, $\widehat{e}_{+\infty}$ and $\widehat{e}_1$.

- for every $m \in \{1, 2, \cdots, \ell+k-1\}$, $\widehat{\mathcal{G}}_{\infty,\underline{s}}(\widehat{e}_m) = \widehat{e}_{m+1}$ and $\widehat{\mathcal{G}}_{\infty,\underline{s}}$ maps γ_m homeomorphically to γ_{m+1}
- for every $r \in \mathbb{Z}$, $\widehat{\mathcal{G}}_{\infty,\underline{s}}$ maps p_r homeomorphically to γ_1.

Next, we construct a transcendental Thurston map from $\widehat{\mathcal{G}}_{\infty,\underline{s}}$.

For each $r \in \mathbb{Z}$, consider the connected component U_r of $\mathbb{C}^{ext} \backslash \widehat{S}^{ext}_{\infty}(\underline{s})$ bounded by p_r and p_{r+1}. It is evident that the closure of U_r in $\mathbb{C}^{ext}$ is homeomorphic to a closed topological disk.

By this discussion and the definition of $\widehat{\mathcal{F}}_{\infty,\underline{s}}$, it is clear that we can use the Alexander trick to extend $\widehat{\mathcal{G}}_{\infty,\underline{s}}$ to a continuous map from $\overline{U_r} \cup \{\widehat{e}_{+\infty}, \widehat{e}_{-\infty}\}$ to $\mathbb{C}^{ext}$ that maps U_r homeomorphically to $\mathbb{C}^{ext} \backslash \widehat{S}_{\infty}(\underline{s})$. Doing this for every $r \in \mathbb{Z}$, $\widehat{\mathcal{G}}_{\infty,\underline{s}}$ can be extended to a continuous map from $\mathbb{C}^{ext}$ to $\mathbb{C}^{ext}$.

From this point, we will restrict this map to $\mathbb{C}$. We see that $\widehat{\mathcal{F}}_{\infty,\underline{s}} : \mathbb{C} \to \mathbb{C}$ is open and has discrete fibers. Furthermore, $\widehat{\mathcal{F}}_{\infty,\underline{s}}$ is locally injective at $z \in \mathbb{C}$ if and only if $z \notin p_r$ for any $r \in \mathbb{Z}$. If $z \notin \cup_r p_r$, then by definition, $\widehat{\mathcal{G}}_{\infty,\underline{s}}$ is locally an orientation-preserving homeomorphism at z. Following from this, we also see that $\widehat{\mathcal{G}}_{\infty,\underline{s}} : \mathbb{C} \longrightarrow \mathbb{C} \backslash \{\widehat{e}_1\}$ is a universal covering, and that $\widehat{e}_1$ is an asymptotic value for $\mathcal{G}_{\underline{s}}$.

To summarize, this discussion shows that $\widehat{\mathcal{G}}_{\infty,\underline{s}}$ is topologically holomorphic. It is also postsingularly finite: the postsingular set is $\{\widehat{e}_1, \widehat{e}_2, \cdots, \widehat{e}_{\ell+k}\}$. We may regard it as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$ by forgetting the standard complex structure. We use the following proposition to show that $\widehat{\mathcal{G}}_{\infty,\underline{s}} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is of stable parabolic type, and is therefore a Thurston map.

Proposition II.53. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2 \backslash \{a\}$ be a universal cover for some $a \in \mathbb{R}^2$. Then f has stable parabolic type.*

Proof. Let $\mathcal{A} \in \mathcal{A}_{flat}(\mathbb{R}^2)$. Then by the Uniformization Theorem, there exists a biholomorphism φ from the Riemann surface $(\mathbb{R}^2, \mathcal{A})$ to the complex plane $\mathbb{C}$. For a unique $X \in \{\mathbb{C}, \mathbb{D}\}$, there exists a biholomorphism $\psi : (\mathbb{R}^2, f^*\mathcal{A}) \to X$. Then the map $\varphi \circ f \circ \psi^{-1} : X \to \mathbb{C}$ would be a universal cover, and this is possible only if $X = \mathbb{C}$. □

We can also identify if the map $\widehat{\mathcal{G}}_{\infty,\underline{s}}$ is realized holomorphically. We recall that the parameter ray at address $\underline{s}$ lands at a unique PSF parameter $\lambda \in \Lambda$. [LSV08] showed that

- if the eventual period of the orbit of λ is k, then $\widehat{\mathcal{G}}_{\infty,\underline{s}}$ is Thurston equivalent to p_λ;
- if the eventual period of the orbit of λ is $k' < k$, then $\widehat{\mathcal{G}}_{\infty,\underline{s}}$ has a Levy cycle C; it can be described as a multicurve $\{\gamma_1, \gamma_2, \cdots, \gamma_{k'}\}$ where, for each $m \in \{1, 2, \cdots, k'\}$, the loop γ_m surrounds $\{\widehat{e}_{\ell+m}, \widehat{e}_{\ell+m+k'}, \cdots, \widehat{e}_{\ell+m+k}\}$.

  However, by collapsing this Levy cycle we may form a map Thurston equivalent equivalent to p_λ. Formally, form a quotient space of $\mathbb{C}^{ext}$ by collapsing each region in $\mathbb{C}$

bounded by a loop in C to a point; this induces quotient graphs $S_\infty^{ext}(\underline{s})$ and $S_\infty(\underline{s})$ of the extended and standard exponential spiders respectively. Then $\widehat{\mathcal{G}}_{\infty,\underline{s}}|\widehat{S}_\infty^{ext}(\underline{s})$ descends to a map $\mathcal{G}_{\underline{s}} : S_\infty^{ext}(\underline{s}) \longrightarrow S_\infty(\underline{s})$. In other words, the following diagram commutes:

$$\begin{array}{ccc} \widehat{S}_\infty^{ext}(\underline{s}) & \xrightarrow{\widehat{\mathcal{G}}_{\infty,\underline{s}}} & \widehat{S}_\infty(\underline{s}) \\ \downarrow \pi & & \downarrow \pi \\ S_\infty^{ext}(\underline{s}) & \xrightarrow{\mathcal{G}_{\underline{s}}} & S_\infty(\underline{s}) \end{array}$$

Here π is the quotient map that describes collapsing C.

By a similar discussion as for $\widehat{\mathcal{G}}_{,\infty,\underline{s}}$, this map $\mathcal{G}_{\underline{s}}$ can be extended to a Thurston map defined on $\mathbb{R}^2$, and moreover, this extension is combinatorially equivalent to p_λ.

Note that the quotient construction in the point above can be done for any $\underline{s}$, and so the graph $S_\infty(\underline{s})$ is well-defined for all pre-periodic addresses $\underline{s}$. We define the legs of $S_\infty(\underline{s})$ to be the images of the legs of $\widehat{S}_\infty(\underline{s})$, and we will call them by the same names (i.e., p_r or γ_m). When $\underline{s}$ is evident, we let e_n denote the class of $\widehat{e}_n$ in $S_\infty(\underline{s})$, and let $A_{\underline{s}} = \{e_1, ..., e_{\ell+k}\}$. We note that $|A_{\underline{s}}| \geqslant 2$ for all adresses $\underline{s}$.

Consider a line $\beta = \{y = b\}$ disjoint from all the legs of $\widehat{S}_\infty^{ext}(\underline{s}) \cap \mathbb{C}$, where b is chosen so that $b < \operatorname{Im} \widehat{e}_m$ if and only if $\overline{0} < \mu^{\circ(m-1)}(\underline{s})$, and β is bound between p_r and p_{r+1} if $r\overline{0} < \beta < (r+1)\overline{0}$. The line β represents the lexicographic position of the external address $\overline{0}$: note that $\mu^{\circ(n-1)}(\underline{s}) << \mu^{\circ(m-1)}(\underline{s})$ if and only if, starting from γ_n and moving counterclockwise in a neighborhood of ∞, we can reach γ_m without intersecting β. This gives a circular order of the legs of $\widehat{S}_\infty^{ext}(\underline{s})$ (and correspondingly, on $S_\infty^{ext}(\underline{s})$), which can be shown to be independent of b.

Poset structure on $\mathcal{P}_n$ and $\mathcal{P}$

We can use parameter rays to define partial orders on the spaces $\mathcal{P}_n$ and $\mathcal{P}$.

Fix $n \in \mathbb{N}_{\geqslant 2}$. In $\mathcal{P}_n$, we say that $\lambda \lhd \widehat{\lambda}$ if there exist angles $\theta_1, \theta_2 \in \Theta_n(\lambda)$ and $\widehat{\theta} \in \Theta_n(\widehat{\lambda})$ such that $\theta_1 < \widehat{\theta} < \theta_2$, and the parameter rays $R_n(\theta_1)$ and $R_n(\theta_2)$ land at the same point of $\mathcal{M}_n$.

Similarly, in $\mathcal{P}$, we say that $\lambda \lhd \widehat{\lambda}$ if there exist external addresses $\underline{s}_1, \underline{s}_2 \in \Theta_\infty(\lambda)$ and $\widehat{\underline{s}} \in \Theta_\infty(\widehat{\lambda})$ such that $\underline{s}_1 < \widehat{\underline{s}} < \underline{s}_2$ and the parameter hairs corresponding to $\underline{s}_1$ and $\underline{s}_2$ land together in Λ.

CHAPTER III
Convergence of Thurston Maps

In this chapter, we introduce various notions of convergence for sequences of Thurston maps. We provide two different points of view (see Definitions III.2 and III.7), and show that they are equivalent.

Afterward, we give a construction that allows us to approximate, in an appropriate sense, an arbitrary transcendental Thurston map by a sequence of polynomial Thurston maps (Proposition IV.15). We use these results to establish Main Theorem I.4. For the notation for topological objects used here, we refer to Appendix A.1.

III.1: Combinatorial convergence

Our first criterion is combinatorial convergence. This is a condition on how a sequence of Thurston maps $f_n : (\mathbb{R}^2, A)$ lift loops in $\mathbb{R}^2 \backslash A$, which we regard as elements of $\pi_1(\mathbb{R}^2 \backslash A)$. We will want to quantify curves that eventually lift to loops under f_n (i.e., loops that belong to $(f_n)_* \pi_1(\mathbb{R}^2 \backslash f_n^{-1}(A))$ for all n sufficiently large). So we first introduce a notion of convergence for subgroups of $\pi_1(\mathbb{R}^2 \backslash A)$.

More generally, we use the following notion of convergence, in the sense of Chabauty ([Cha50]), of sequences of subgroups of a given group.

Definition III.1. Let G be a group endowed with the discrete topology and (H_n) be a sequence of its subgroups. We say that the sequence (H_n) *converges* to a subgroup H of G and write $\lim\limits_{n\to\infty} H_n = H$ if for every $g \in G$, there exists $N = N(g) \in \mathbb{N}$ so that if $g \in H$, then for every $n \geqslant N$, $g \in H_n$; and if $g \notin H$, then $g \notin H_n$ for every $n \geqslant N$.

Definition III.2. Let $f_n \colon (\mathbb{R}^2, A) \circlearrowleft, n \in \mathbb{N}$ be a sequence of Thurston maps. We say that the sequence (f_n) *converges combinatorially* to a Thurston map $f \colon (\mathbb{R}^2, A) \circlearrowleft$ if there exist points $t \in \mathbb{R}^2 \backslash A$ $b \in f^{-1}(t)$, $b_n \in f_n^{-1}(t)$ and paths $p_n \subset \mathbb{R}^2 \backslash A$ $\forall n \in \mathbb{N}$ such that

- $f_n(b_n) = f(b) = t$ $\forall n \in \mathbb{N}$;

- $p_n(0) = b$ and $p_n(1) = b_n \ \forall n \in \mathbb{N}$;
- for every $[\gamma] \in \pi_1(\mathbb{R}^2 \backslash A, t)$, the following hold:
    1. The lifts $\gamma\uparrow(f_n, b_n)$ eventually have the same closing behavior as $\gamma\uparrow(f, b)$ (Definition A.2). Equivalently,

    $$\lim_{n\to\infty} (f_n)_*\pi_1(\mathbb{R}^2 \backslash f_n^{-1}(A), b_n) = f_*\pi_1(\mathbb{R}^2 \backslash f^{-1}(A), b),$$

    in the sense of Definition III.1.
    2. If $\gamma \in f_*\pi_1(\mathbb{R}^2 \backslash f^{-1}(A), b)$, then for all n sufficiently large, the closed curves $\gamma\uparrow(f, b)$ and $p_n \cdot (\gamma\uparrow(f_n, b_n)) \cdot p_n^{-1}$ are homotopic rel A.

III.2: Independence from isotopy

Proposition III.3. *Let $A \subset \mathbb{R}^2$ be a finite set. Two Thurston maps $f, \widehat{f} : (\mathbb{R}^2, A) \circlearrowleft$ are isotopic rel. A if and only if there exist points $t \in \mathbb{R}^2 \backslash A$, $b \in f^{-1}(t)$ and $\widehat{b} \in \widehat{f}^{-1}(t)$ where the following conditions hold:*

1. *$f_*\pi_1(\mathbb{R}^2 \backslash f^{-1}(A), b) = \widehat{f}_*\pi_1(\mathbb{R}^2 \backslash \widehat{f}^{-1}(A), \widehat{b})$;*
2. *there exists a path $p \subset \mathbb{R}^2 \backslash A$ with $p(0) = b$ and $p(1) = \widehat{b}$ such that for all $\gamma \in f_*\pi_1(\mathbb{R}^2 \backslash f^{-1}(A), b)$, the loop $\gamma\uparrow(f, b)$ is homotopic to $p \cdot \gamma\uparrow(\widehat{f}, \widehat{b}) \cdot \overline{p}$.*

Before we prove this proposition, we formulate a simple observation.

Lemma III.4. *Suppose $A \subset \mathbb{R}^2$ is finite with $|A| \geqslant 2$, and φ is an orientation-preserving homeomorphism such that $\varphi(b) = t$ for some $b, t \in \mathbb{R}^2 \backslash A$. Then φ is isotopic rel. A to $\mathrm{id}_{\mathbb{R}^2}$ if and only if there exists a path $p \subset \mathbb{R}^2 \backslash A$ joining b with t such that every loop $\gamma \subset \mathbb{R}^2 \backslash A$ based at b is homotopic rel. A to $p \cdot (\varphi \circ \gamma) \cdot \overline{p}$.*

Proof. ($\Longrightarrow$): Suppose that φ is isotopic rel. A to $\mathrm{id}_{\mathbb{R}^2}$ via an isotopy $(\varphi_s)_{s\in\mathbb{I}}$, where $\varphi_0 = \mathrm{id}_{\mathbb{R}^2}$ and $\varphi_1 = \varphi$. Define $p : \mathbb{I} \to \mathbb{R}^2 \backslash A$ as $p(s) = \varphi_s(b)$ for each $s \in \mathbb{I}$. Then it is evident that for any loop $\gamma \subset \mathbb{R}^2 \backslash A$ based at b and $p \cdot (\varphi \circ \gamma) \cdot \overline{p}$ are homotopic rel. A via a homotopy $H_s := p_s \cdot (\varphi_s \circ \gamma) \cdot \overline{p_s}$, where p_s is a subpath of p joining b with $\varphi_s(b)$ for every $s \in \mathbb{I}$.

($\Longleftarrow$): Now suppose that there exists a path $p \subset \mathbb{R}^2 \backslash A$ joining b with t such that every loop $\gamma \subset \mathbb{R}^2 \backslash A$ based at b is homotopic rel. A to $p \cdot (\varphi \circ \gamma) \cdot \overline{p} \subset \mathbb{R}^2 \backslash A$. Taking γ to be a loop separating a unique point $a \in A$ from the other points of A and using continuity of φ, we can show that $\varphi(a) = a$ for each $a \in A$. The rest follows from Theorem II.20. □

Proof of Proposition III.3. ($\Longrightarrow$) : First suppose that $\widehat{f} = f \circ \varphi$ for some $\varphi \in \mathrm{Homeo}_0^+(\mathbb{R}^2, A)$ and choose a point $b \in \mathbb{R}^2 \backslash f^{-1}(A)$. Let $\widehat{b} = \varphi(b)$ and $t = f(b)$. Given any loop $\gamma \in \pi_1(\mathbb{R}^2 \backslash A, t)$, we note that $\gamma\uparrow(\widehat{f}, \widehat{b}) = \varphi(\gamma\uparrow(f, b))$. This immediately proves item (1). For $r \in \mathbb{I}$, let $\varphi_r \in \mathrm{Homeo}^+(\mathbb{R}^2, A)$ be such that $\varphi_0 = \mathrm{id}_{\mathbb{R}^2}$ and $\varphi_1 = \varphi$. Define continuous maps $p_r : \mathbb{I} \to \mathbb{R}^2 \backslash A$, $r \in \mathbb{I}$ as $s \mapsto \varphi_{sr}(b)$. Then $p_r(0) = b$ and $p_r(1) = \varphi_r(b)$. Letting $p = p_1$, it is evident that for all $\gamma \in \pi_1(\mathbb{R}^2 \backslash f^{-1}(A), b)$, the map $r \mapsto p_r \cdot \varphi_r(\gamma\uparrow(f, b)) \cdot \overline{p_r}$ is a homotopy (rel. A) between $\gamma\uparrow(f, b)$ and $p \cdot \gamma\uparrow(\widehat{f}, \widehat{b}) \cdot \overline{p}$.

($\Longleftarrow$) : Now conversely, suppose the conditions (1) and (2) above are satisfied for some $b, \widehat{b}, t \in \mathbb{R}^2 \backslash A$ and path p as above. Due to condition (1), there exists a homeomorphism $\varphi : \mathbb{R}^2 \backslash f^{-1}(A) \to \mathbb{R}^2 \backslash \widehat{f}^{-1}(A)$ such that the following diagram commutes:

$$\begin{array}{ccc} (\mathbb{R}^2 \backslash f^{-1}(A), b) & \xrightarrow{\varphi} & (\mathbb{R}^2 \backslash \widehat{f}^{-1}(A), \widehat{b}) \\ \big\downarrow f & \swarrow \widehat{f} & \\ (\mathbb{R}^2 \backslash A, t) & & \end{array}$$

Since $f^{-1}(A)$ is discrete, we can extend φ to a homeomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$. Let $\gamma \in f_* \pi_1(\mathbb{R}^2 \backslash A, b)$. By the diagram above, we have $\varphi(\gamma) = f(\gamma) \uparrow (\widehat{f}, \widehat{b})$. Due to condition (2), we know that $\gamma = f(\gamma) \uparrow (f, b) \sim_A p \cdot \varphi(\gamma) \cdot \overline{p}$. In particular, for any $a \in A$, we can take γ above to be a loop separating a from every other point of A. The discussion above shows that $\varphi(a) = a$ for all $a \in A$, and by Theorem II.20.

$\square$

Proposition III.5. *Let $f_n \colon (\mathbb{R}^2, A) \circlearrowleft, n \in \mathbb{N}$ be a sequence of Thurston maps that converges combinatorially to a Thurston map $f \colon (\mathbb{R}^2, A) \circlearrowleft$. If $\widehat{f}_n, \widehat{f} \colon (\mathbb{R}^2, A) \circlearrowleft$ are Thurston maps such that $\widehat{f}_n$ is isotopic to f_n rel. A for all $n \in \mathbb{N}$ and $\widehat{f}$ is isotopic to f rel. A, then $(\widehat{f}_n)$ converges combinatorially to $\widehat{f}$.*

Proof. Let $\varphi_n, \varphi \in \mathrm{Homeo}^+(\mathbb{R}^2, A)$ be such that $f_n = \widehat{f}_n \circ \varphi_n$ for all $n \in \mathbb{N}$ and $f = \widehat{f} \circ \varphi$. By assumption, there exist points $t \in \mathbb{R}^2 \backslash A$, $b \in f^{-1}(t)$, $b_n \in f_n^{-1}(t)$ and paths $p_n \subset \mathbb{R}^2 \backslash A$ for all $n \in \mathbb{N}$ such that all the conditions of Definition III.2 are satisfied. Let $\widehat{b}_n = \varphi_n(b_n)$, and $\widehat{b} = \varphi(b)$. Then, by Proposition III.3,

$$\begin{aligned} \lim_{n\to\infty} (\widehat{f}_n)_* \pi_1(\mathbb{R}^2 \backslash \widehat{f}_n^{-1}(A), \widehat{b}_n) &= \lim_{n\to\infty} (f_n)_* \pi_1(\mathbb{R}^2 \backslash f_n^{-1}(A), b_n) \\ &= f_* \pi_1(\mathbb{R}^2 \backslash f^{-1}(A), b) \\ &= (\widehat{f})_* \pi_1(\mathbb{R}^2 \backslash \widehat{f}^{-1}(A), \widehat{b}). \end{aligned}$$

Also by Proposition III.3, there exist paths $\widehat{p}_n, \widehat{p} : \mathbb{I} \to \mathbb{R}^2 \backslash A$ with $\widehat{p}_n(0) = b_n$ and $\widehat{p}_n(1) = \widehat{b}_n$

for all $n \in \mathbb{N}$, and a path $\widehat{p}$ with $\widehat{p}(0) = b$ and $\widehat{p}(1) = \widehat{b}$ such that for every $n \in \mathbb{N}$,

$$\forall \gamma \in (f_n)_* \pi_1(\mathbb{R}^2 \backslash f_n^{-1}(A), b_n), \text{ the loop } \gamma\!\uparrow\!(f_n, b_n) \sim_A \widehat{p}_n \cdot \gamma\!\uparrow\!(\widehat{f}_n, \widehat{b}_n) \cdot \overline{\widehat{p}_n} \text{ for all } n \in \mathbb{N},$$
$$\forall \gamma \in f_* \pi_1(\mathbb{R}^2 \backslash f^{-1}(A), b), \text{ the loop } \gamma\!\uparrow\!(f, b) \sim_A \widehat{p} \cdot \gamma\!\uparrow\!(\widehat{f}, \widehat{b}) \cdot \overline{\widehat{p}}.$$

Then for every $\gamma \in (\widehat{f})_* \pi_1(\mathbb{R}^2 \backslash \widehat{f}^{-1}(A), b)$, for all sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned} \gamma\!\uparrow\!(\widehat{f}, \widehat{b}) &\sim_A \overline{\widehat{p}} \cdot \gamma\!\uparrow\!(f, b) \cdot \widehat{p} \\ &\sim_A \overline{\widehat{p}} \cdot (p_n \cdot \gamma\!\uparrow\!(f_n, b_n) \cdot \overline{p_n}) \cdot \widehat{p} \\ &\sim_A (\overline{\widehat{p}} \cdot p_n \cdot \widehat{p}_n) \cdot \gamma\!\uparrow\!(\widehat{f}_n, \widehat{b}_n) \cdot (\overline{\widehat{p}_n} \cdot \overline{p_n} \cdot \widehat{p}). \end{aligned}$$

This shows that $(\widehat{f}_n)$ converges combinatorially to $\widehat{f}$. ☐

Convergence to a polynomial Thurston map

If the limiting map f of a combinatorially convergent sequence (f_n) is of finite degree, then we show that the maps f_n form a constant approximation upto isotopy.

Proposition III.6. *Let $f_n \colon (\mathbb{R}^2, A) \circlearrowleft, n \in \mathbb{N}$ be a sequence of Thurston maps converging combinatorially to a Thurston map $f \colon (\mathbb{R}^2, A) \circlearrowleft$. If f is a topological polynomial, then for all sufficiently large n, f_n is isotopic to f rel. A.*

Proof. Choose points $t \in \mathbb{R}^2 \backslash A$, $b \in f^{-1}(t)$, $b_n \in f_n^{-1}(t)$ and paths $p_n \subset \mathbb{R}^2 \backslash A$ that satisfy the conditions of Definition III.2. Since f is a topological polynomial, $f^{-1}(A)$ is a finite set, and the group $f_* \pi_1(\mathbb{R}^2 \backslash f^{-1}(A), b)$ is finitely generated. Let $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ be a generating set for $f_* \pi_1(\mathbb{R}^2 \backslash f^{-1}(A), b)$.

Since the maps f_n converge to f combinatorially, for sufficiently large n, the set Γ is a subset of $(f_n)_* \pi_1(\mathbb{R}^2 \backslash f_n^{-1}(A), b_n)$, and thus, $f_* \pi_1(\mathbb{R}^2 \backslash f^{-1}(A), b)$ is a subgroup of $(f_n)_* \pi_1(\mathbb{R}^2 \backslash f_n^{-1}(A), b_n)$. From the classical theory of covering maps, there exists a covering map $\varphi_n : \mathbb{R}^2 \backslash f^{-1}(A) \to \mathbb{R}^2 \backslash f_n^{-1}(A)$ such that the following diagram commutes:

$$\begin{array}{ccc} (\mathbb{R}^2 \backslash f^{-1}(A), b) & \xrightarrow{\varphi_n} & (\mathbb{R}^2 \backslash f_n^{-1}(A), b_n) \\ & {\scriptstyle f} \searrow & \downarrow {\scriptstyle f_n} \\ & & (\mathbb{R}^2 \backslash A, t) \end{array}$$

In particular, $\deg(\varphi_n)$ and $\deg(f_n)$ are bounded above by $\deg(f)$ for all n sufficiently large. Let $\alpha \subset \mathbb{R}^2 \backslash A$ be a simple loop based at t such that the unique bounded component of $\mathbb{R}^2 \backslash \alpha$ contains the set A. Proposition II.8 implies that for every $\ell \in \mathbb{Z}$, the lift $\alpha^\ell \uparrow (f, b)$ is a loop if and only if $\deg(f)$ divides ℓ. A similar statement holds for the topological polynomials f_n as well. Thus, condition (1) implies that $\deg(f) = \deg(f_n)$ for all sufficiently large n. Hence, the covering map $\varphi_n \colon \mathbb{R}^2 \backslash f^{-1}(A) \to \mathbb{R}^2 \backslash f_n^{-1}(A)$ is an orientation-preserving homeomorphism and, therefore, it can be extended to $\mathbb{R}^2$. Finally, condition (2) and Lemma III.4 imply that φ_n is isotopic rel. A to $\mathrm{id}_{\mathbb{R}^2}$. $\square$

III.3: Topological convergence

Now we move on to our second notion of convergence. Throughout this section, we assume $A \subset \mathbb{R}^2$ is a finite set.

Definition III.7. Let $f_n \colon (\mathbb{R}^2, A) \circlearrowleft, n \in \mathbb{N}$ be a sequence of Thurston maps. We say that (f_n) *converges topologically* to the Thurston map $f \colon (\mathbb{R}^2, A) \circlearrowleft$ if for every compact set $D \subset \mathbb{R}^2$, there exists $N \in \mathbb{N}$ such that for $n \geqslant N$, we have $f_n|D \equiv f|D$.

It is easier to establish $\sigma_{f_n} \to \sigma_f$ if the f_n converge topologically to f. However, to establish that every transcendental Thurston map f, that there exists a sequence of polynomial Thurston maps that converge combinatorial approximations. But by showing the equivalence of combinatorial and topological convergence upto isotopy, we sidestep this difficulty.

Proposition III.8. *Let $f_n \colon (\mathbb{R}^2, A) \circlearrowleft, n \in \mathbb{N}$ be a sequence of Thurston maps. Then (f_n) converges combinatorially to a Thurston map $f \colon (\mathbb{R}^2, A) \circlearrowleft$ if and only if there exists a sequence of Thurston maps $\widehat{f}_n \colon (\mathbb{R}^2, A) \circlearrowleft$ converging topologically to f, where $\widehat{f}_n$ is isotopic rel. A to f_n for every $n \in \mathbb{N}$.*

Before we prove Proposition III.8 we need to obtain the following result of non-dynamical nature.

Proposition III.9. *Suppose that $f_n \colon \mathbb{R}^2 \to \mathbb{R}^2$, $n \in \mathbb{N}$ and $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ are topologically holomorphic maps in class $\mathcal{S}$ such that $S_{f_n} = S_f = A$ for all $n \in \mathbb{N}$. Let $b, b_n \in \mathbb{R}^2$ and $t \in \mathbb{R}^2 \backslash A$ be points such that $f_n(b_n) = f(b) = t$ for every $n \in \mathbb{N}$. Further suppose that*

$$\lim_{n \to \infty} (f_n)_* \pi_1(\mathbb{R}^2 \backslash f_n^{-1}(A), b_n) = f_* \pi_1(\mathbb{R}^2 \backslash f^{-1}(A), b).$$

Then for every bounded domain $D \subset \mathbb{R}^2$ containing b, for sufficiently large n, there exists a continuous injective map $\varphi_n \colon D \hookrightarrow \mathbb{R}^2$ such that the following diagram commutes:

$$\begin{array}{ccc} (D, b_n) & \xrightarrow{\varphi_n} & (\mathbb{R}^2, b) \\ \big\downarrow f & \swarrow f_n & \\ (\mathbb{R}^2, t) & & \end{array}$$

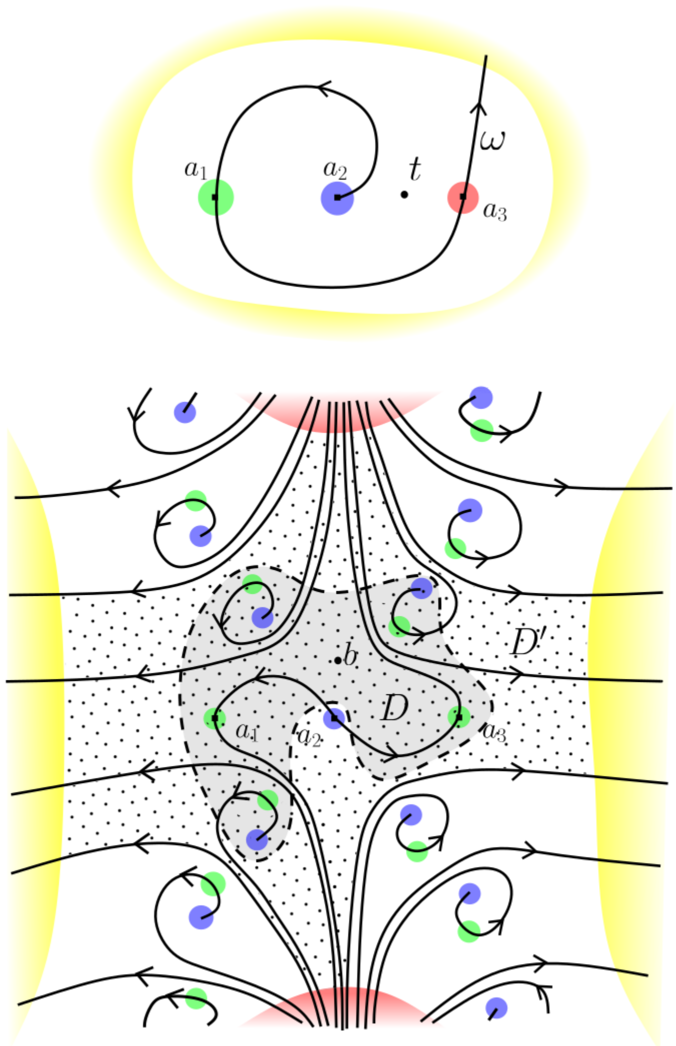


Figure 3.1: Top: the co-domain of $G_2(z) = \sqrt{\ln 2}(\exp(z^2) - 1)$ along with its postsingular set $\{a_1, a_2, a_3\}$; Bottom: an illustration of D', where D is the shaded region bounded by the dotted loop

Proof. Let D be as above. Without loss of generality, we can assume that $\partial D \cap f^{-1}(A) = \varnothing$. Choose a simple continuous curve $\omega\colon [0,\infty) \to \mathbb{R}^2$ such that ω passes through all points of A, $\omega(0) \in A$, and $\lim_{t\to\infty} \omega(t) = \infty$.

Claim 1. Let $K \subset \mathbb{R}^2$ be a bounded set and $W \subset \mathbb{R}^2$ be a closed locally connected set. Then, only finitely many connected components of $\mathbb{R}^2 \backslash f^{-1}(W)$ intersect K.

Proof of Claim 1. Suppose there exist distinct connected components $E_1, E_2, \ldots, E_n, \ldots$ of $\mathbb{R}^2 \backslash f^{-1}(W)$ intersecting K. Pick an arbitrary point $x_n \in E_n \cap K$ for all $n \in \mathbb{N}$. We may assume without loss of generality that the sequence (x_n) converges to $x \in \overline{K} \cap f^{-1}(W)$. Thus, any neighbourhood of x intersects infinitely many connected components of $\mathbb{R}^2 \backslash f^{-1}(W)$, which leads to a contradiction since W is locally connected and f is topologically holomorphic. ∎

Claim 2. There exists a bounded domain $D' \subset \mathbb{R}^2$ containing D such that for every connected component F of $\mathbb{R}^2 \backslash f^{-1}(\omega)$, the set $D' \cap F$ is connected (perhaps, empty).

Proof of Claim 2. For every $a \in A \cup \{\infty\}$ we choose a set U_a such that

1. $U_a = \overline{D(a, r_a)}$ for some $r_a > 0$, and $U_\infty = \mathbb{R}^2 \backslash V_\infty$, where V_∞ is an open ball containing A,
2. $U_a \cap U_{\widehat{a}} = \varnothing$ if $\widehat{a} \in A \cup \{\infty\}$ is distinct from a, and
3. every connected component of $f^{-1}(U_a)$ is either contained in D or disjoint from it.

The last condition can be satisfied if we pick the values r_a to be sufficiently small and U_∞ sufficiently far from $f(D)$. Indeed, due to the previous claim, D intersects with finitely many connected components of $f^{-1}(U_a)$ for every $a \in A \cup \{\infty\}$, and $\partial D \cap f^{-1}(A) = \varnothing$. Thus, the rest easily follows from Propositions II.7 and II.8.

Let $U := \bigcup_{a\in A\cup\{\infty\}} U_a$ and $W_F := F \backslash f^{-1}(U)$ for every connected component F of $\mathbb{R}^2 \backslash f^{-1}(\omega)$. Therefore,

$$
\begin{aligned}
(W_F \cup D) \cap F &= W_F \cup \Big(D \cap (W_F \cup (F \cap f^{-1}(U)))\Big) \\
&= W_F \cup (D \cap F \cap f^{-1}(U)) \\
&= W_F \cup \bigcup_{\substack{a\in A\cup\{\infty\} \\ F\cap f^{-1}(U_a)\subset D}} F \cap f^{-1}(U_a).
\end{aligned}
$$

By Proposition II.8, the map $f|F\colon F \to \mathbb{R}^2 \backslash \omega$ is a homeomorphism. Thus, W_F is connected, and for every $a \in A \cup \{\infty\}$, $F \cap f^{-1}(U_a)$ is a connected set whose boundary intersects W_F. This discussion implies that the set $(W_F \cup D) \cap F$ is connected.

Therefore, $D' := D \cup \bigcup_{F:F\cap D\neq\varnothing} W_F$ is an open set such that $D' \cap F$ is connected for every connected component F of $\mathbb{R}^2\backslash f^{-1}(\omega)$. The set D' is connected since $W_F \cap D = \varnothing$ if and only if $F \cap D = \varnothing$. At the same time, D' is bounded since each $W_F = F \cap f^{-1}(\mathbb{R}^2\backslash(U \cup \omega))$ is bounded by Proposition II.7, only finitely many regions F intersect D.

■

See Figure 3.1 for a depiction of D' for the map G_2 from Example II.16. Due to the claims above, there exists a positive integer m such that for all $x \in D'\backslash f^{-1}(A)$, the point b can be joined to x by a path contained in $D'\backslash f^{-1}(A)$ and intersecting $f^{-1}(\omega)$ at most m times.

Denote by $P_{\omega,k}$ the set of elements of $\pi_1(\mathbb{R}^2\backslash A, t)$ that can be represented by a loop $\alpha : \mathbb{I} \to \mathbb{R}^2\backslash A$ such that $|\alpha^{-1}(\omega)| \leqslant k$. Clearly, $P_{\omega,k}$ is finite for all $k \in \mathbb{N}$.

Now we construct a continuous map $\varphi_n \colon D' \to \mathbb{R}^2$ and prove its injectivity as required. Let D'' be an open Jordan region containing D'. Since the group $f_*\pi_1(D''\backslash f^{-1}(A), b)$ is finitely generated, then by our initial assumptions in this proposition, it is a subgroup of $(f_n)_*\pi_1(\mathbb{R}^2\backslash f_n^{-1}(A), b_n)$ for all n sufficiently large. Therefore, there exists a continuous map $\varphi_n \colon D''\backslash f^{-1}(A) \to \mathbb{R}^2\backslash f_n^{-1}(A)$ such that $f = f_n \circ \varphi_n$ on $D''\backslash f^{-1}(A)$ and $\varphi_n(b) = b_n$. We prove that the map φ_n is injective on $D'\backslash f^{-1}(A)$ for all n sufficiently large.

Suppose φ_n is not injective for some n, choose distinct points $x_1, x_2 \in D'\backslash f^{-1}(A)$ such that $\varphi_n(x_1) = \varphi_n(x_2)$. We join b with x_1 and x_2 by simple paths $\alpha_1 : \mathbb{I} \to D'\backslash f^{-1}(A)$ and $\alpha_2 : \mathbb{I} \to D'\backslash f^{-1}(A)$ in $D'\backslash f^{-1}(A)$, respectively, so that $|\alpha_1^{-1}(f^{-1}(\omega))|, |\alpha_2^{-1}(f^{-1}(\omega))| \leqslant m$. Let $\gamma := f \circ \alpha_1 \cdot \overline{f \circ \alpha_2}$. By definition, $\gamma \uparrow (f, b)$ is not a loop; however, $\gamma \uparrow (f_n, b_n)$ is the loop $\varphi_n \circ \alpha_1 \cdot \overline{\varphi_n \circ \alpha_2}$. Thus,

$$[\gamma] \in G_n := P_{\omega,2m} \cap \Big((f_n)_*\pi_1(\mathbb{R}^2\backslash f_n^{-1}(A), b_n) \ \backslash \ f_*\pi(\mathbb{R}^2\backslash f^{-1}(A), b)\Big).$$

Since $P_{\omega,2m}$ is a finite set, if φ_n fails to be injective for infinitely many n, then there exists $[\widehat{\gamma}] \in \pi_1(\mathbb{R}^2\backslash A, t)$ such that $[\widehat{\gamma}] \in G_n$ for infinitely many n. This, however, is not possible since for all n large enough, $[\widehat{\gamma}] \notin \ (f_n)_*\pi_1(\mathbb{R}^2\backslash f^{-1}(A), t)$.

Finally, since $f^{-1}(A)$ and $f_n^{-1}(A)$ are discrete subsets of $\mathbb{R}^2$, we can extend φ_n to a continuous injective map defined on D' such that $f = f_n \circ \varphi_n$ on D' and $\varphi_n(b) = b_n$ for all sufficiently large n.

□

Proof of Proposition III.8. If the sequence $(\widehat{f}_n)$ converges topologically to f, then it does so combinatorially with respect to $t, b, (b_n), (p_n)$, where $t \in \mathbb{R}^2\backslash A$ and $b \in f^{-1}(t)$ are arbitrary, $b_n := b$ and p_n is a constant loop based at b for every $n \in \mathbb{N}$. Hence, Proposition III.5 implies that the sequence (f_n) converges combinatorially to f.

Now suppose that the sequence (f_n) converges combinatorially to f with respect to some choice of $(b_n), b, t$ and (p_n). Let $(D_m)_m$ be an exhaustion of $\mathbb{R}^2$ by closed topological disks containing A and b in their interiors for every $m \in \mathbb{N}$. Then for sufficiently large n let $m = m(n)$ be the maximal index such that

1. there exists a continuous injective map $\varphi_n \colon D_m \to \mathbb{R}^2$ such that $f|D_m = f_n \circ \varphi_n$ and $\varphi_n(b) = b_n$;
2. for every loop $\gamma \subset \mathbb{R}^2 \backslash A$ based at t such that $[\gamma] \in f_*\pi_1(D_m \backslash f^{-1}(A), b)$, the lifted loops $\gamma\uparrow(f,b)$ and $p_n \cdot \gamma\uparrow(f_n, b_n) \cdot \overline{p_n}$ are homotopic rel. A.

These conditions hold for sufficiently large n because of Definition III.2, Proposition III.9 and the fact that $f_*\pi_1(D_m \backslash f^{-1}(A), b)$ is finitely generated. Moreover, it is easy to show that $m(n)$ converges to infinity as n tends to infinity.

Now we can extend φ_n to an orientation-preserving homeomorphism $\widehat{\varphi}_n$ of $\mathbb{R}^2$. In fact, by the Alexander trick, all such extensions of φ_n are isotopic to each other rel. D_m. Clearly, if $\gamma \subset D_m \backslash f^{-1}(A)$ is a loop based at b, then $\varphi_n \circ \gamma$ is a f_n-lift (based at b_n) of $f \circ \gamma$. In particular, $\gamma \sim_A p_n \cdot (\widehat{\varphi}_n \circ \gamma) \cdot \overline{p_n}$ for every loop $\gamma \subset \mathbb{R}^2 \backslash A$ based at b. Thus, the homeomorphism $\widehat{\varphi}_n$ is isotopic rel. A to $\mathrm{id}_{\mathbb{R}^2}$ by Proposition III.3.

Now consider the sequence $(\widehat{f}_n)$ of Thurston maps, where $\widehat{f}_n = f_n \circ \widehat{\varphi}_n$ for all $n \in \mathbb{N}$. Clearly, $(\widehat{f}_n)$ converges topologically to f.

□

III.4: Convergence of Thurston pullback maps

In this section we use the theory developed in this chapter so far to establish Theorem I.4. We assume throughout that $A \subset \mathbb{R}^2$ is finite.

Proposition III.10. *Let $\tau_n, n \in N$ and τ be points in $\mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$. Suppose there exist representatives $\varphi_n \in \tau_n$, $n \in \mathbb{N}$ and $\varphi \in \tau$ such that $\varphi(\infty) = \infty$ and $\varphi_n(\infty) = \infty$ for all $n \in \mathbb{N}$, and $\varphi_n \to \varphi$ uniformly on compact sets of $\mathbb{R}^2$. Then $d_{\mathrm{T}}(\tau_n, \tau) \to 0$ as $n \to \infty$.*

Proof. Without loss of generality, we may assume that there exists $a \in A$ such that $\varphi_n(a) = \varphi(a)$ for all $n \in \mathbb{N}$.

By the given conditions, in the moduli space $\mathrm{M}(\mathbb{S}^2, A \cup \{\infty\})$, we note that $[[\varphi_n]] \to [[\varphi]]$. Therefore, there exists a sequence of homeomorphisms $q_n \in \mathrm{Homeo}^+(\mathbb{S}^2, A \cup \{\infty\})$ such that $d_{\mathrm{T}}([\varphi_n \circ q_n^{-1}], [\varphi]) \to 0$. Equivalently, for every $n \in \mathbb{N}$, there exists a quasiconformal map $k_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ isotopic to $\varphi_n \circ q_n^{-1} \circ \varphi^{-1}$ rel. $A \cup \{\infty\}$, such that $K(k_n) \to 1$ as $n \to \infty$.

For every $n \in \mathbb{N}$, let M_n be the unique Möbius map satisfying $M_n(\infty) = \infty$, $M_n(b) = b$, and $M_n(\varphi_n \circ \varphi^{-1}(\hat{b})) = \hat{b}$ for some $\hat{b} \in \varphi(A)\backslash\{b\}$. Then $(M_n \circ k_n)$ is a sequence of quasiconformal maps that fix the three points ∞, b and $\widehat{b}$; and $K(M_n \circ k_n) \to 1$. By Proposition II.5, $M_n \circ k_n \to \mathrm{id}_{\widehat{\mathbb{C}}}$ uniformly on compact subsets of $\widehat{\mathbb{C}}\backslash\{\infty, b, \widehat{b}\}$. Note that $M_n \to \mathrm{id}_{\widehat{\mathbb{C}}}$ uniformly on compact subsets of $\mathbb{C}$; this implies that $k_n \to \mathrm{id}_{\widehat{\mathbb{C}}}$ locally uniformly on compact subsets of $\widehat{\mathbb{C}}\backslash\{\infty, b, \widehat{b}\}$. But then we have $\langle q_n \rangle = \langle \varphi^{-1} \circ k_n^{-1} \circ \varphi_n \rangle = \langle \varphi^{-1} \circ k_n^{-1} \circ \varphi \circ \varphi^{-1} \circ \varphi_n \rangle \to \langle \mathrm{id}_{\mathbb{S}^2} \rangle$ in $\mathrm{PMCG}(\mathbb{S}^2, A \cup \{\infty\})$. Since this is a discrete group we must have $\langle q_n \rangle = \langle \mathrm{id}_{\mathbb{S}^2} \rangle$ eventually, thereby proving the proposition. □

Proposition III.11. *Let $\tau, \tau_n \in \mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$. Suppose there exist representatives $\varphi_n \in \tau_n$ and $\varphi \in \tau$ such that $\varphi, \varphi_n : \mathbb{R}^2 \to \mathbb{C}$ are orientation-preserving homeomorphisms, and for each open set $D \subset \mathbb{R}^2$ with compact closure, the map $(\varphi_n \circ \varphi^{-1}) \mid \varphi(D)$ is holomorphic for all $n \geqslant N_D$. Then $d_{\mathrm{T}}(\tau_n, \tau) \to 0$ as $n \to \infty$.*

Proof. Let $\psi_n = \varphi_n \circ \varphi^{-1}$ for each $n \in \mathbb{N}$.

Without loss of generality, we may assume D is an open disk. We also assume that D is large enough so that $\mathbb{D}(a, 1) \subset D$ for every $a \in A$. Suppose without loss of generality that for all $n \in \mathbb{N}$, there exists $b \in \varphi(A)$ such that $\psi_n(b) = b$ and $\psi_n'(b) = 1$ for some $b \in \varphi(A)$. Due to Proposition III.10, it suffices to establish the following:

Claim 1. The sequence (ψ_n) converges locally uniformly on $\varphi(D)$ to $\mathrm{id}_{\varphi(D)}$.

In order to establish the above, we first prove a preliminary statement:

Claim 2. If $z_0 \in \varphi(D)$ satisfies $\psi_n(z_0) \to z_0$ and $\psi_n'(z_0) \to 1$, then given $r > 0$ such that $D(z_0, r) \subset \varphi(D)$, the sequence of maps $\left(\psi_n | D(z_0, \frac{r}{16})\right)$ converges locally uniformly to $\mathrm{id}_{D(z_0, \frac{r}{16})}$.

Proof of Claim 2. Let $h_n(z) = M_n \circ \psi_n$, where $M_n(z) = \frac{1}{\psi_n'(z_0)}(z - \psi_n(z_0)) + \psi_n(z_0)$. Then $h_n(z_0) = z_0$ and $h_n'(z_0) = 1$. Let $r > 0$ be such that $D(z_0, r) \subset \varphi(D)$. Then by the Koebe 1/4-theorem, there exists $N = N_D \in \mathbb{N}$ such that $D(z_0, \frac{r}{4}) \subset h_n(D(z_0, r))$ for all $n \geqslant N$. Consider the sequence $(h_n^{-1} | D(z_0, \frac{r}{16}))$. Again by the Koebe 1/4-theorem, we see that $D(z_0, \frac{r}{16}) \subset h_n^{-1}(D(z_0, \frac{r}{4}))$ for all $n \geqslant N_D$.

Thus we get the sequence $h_n | D(z_0, \frac{r}{16}) : D(z_0, \frac{r}{16}) \hookrightarrow D(z_0, \frac{r}{4})$ satisfying $h_n(z_0) = z_0, h_n'(z_0) = 1$ for all $n \geqslant N$. By Montel's theorem, every subsequence of this sequence has a subsequence that converges to $\mathrm{id}_{D(z_0, \frac{r}{16})}$ locally uniformly. Therefore, $h_n | D(z_0, \frac{r}{16}) \to \mathrm{id}_{D(z_0, \frac{r}{16})}$ locally uniformly. Since $\psi_n = M_n^{-1} \circ h_n$ and M_n converges to $\mathrm{id}_{\mathbb{C}}$ locally uniformly, this proves the claim.

■

Proof of Claim 1. Let U be the set of points $z \in \varphi(D)$ such that (ψ_n) converges locally uniformly to id_{U_z} on some neighborhood U_z of z. By Claim 1, U is an open set, and we see that $b \in U$.

Suppose $U \neq \varphi(D)$, there exists a point $\widehat{z} \in \partial U$ and $r > 0$ such that $D(\widehat{z}, r) \subset \varphi(D)$. We can also choose a point $z_0 \in U \cap D(\widehat{z}, \frac{r}{2})$ such that $16|\widehat{z} - z_0| < \frac{r}{2}$. Then for any δ such that $16|\widehat{z} - z_0| < \delta < \frac{r}{2}$, we observe that $D(z_0, \frac{\delta}{16})$ is contained in D but not in U. By Claim 1, however, $D(z_0, \frac{\delta}{16}) \subset U$. ■

□

Proposition III.12. *Let $f_n : (\mathbb{R}^2, A) \circlearrowleft$, $n \in \mathbb{N}$ be a sequence of Thurston maps that converge combinatorially to a Thurston map $f : (\mathbb{R}^2, A) \circlearrowleft$. For every $\tau \in \mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$, $d_{\mathrm{T}}(\sigma_{f_n}(\tau), \sigma_f(\tau_n)) \to 0$.*

Proof. Due to Proposition II.29 and III.8, we may assume without loss of generality that the sequence (f_n) converges topologically to f. Let $\varphi \in \tau$ such that $\varphi(\infty) = \infty$.

Then there exist maps $\psi \in \sigma_f(\tau)$, $\psi_n \in \sigma_{f_n}(\tau)$ for $n \in \mathbb{N}$ and entire maps g and $g_n, n \in \mathbb{N}$ such that the following diagrams commute:

$$\begin{array}{ccc} (\mathbb{R}^2, A) & \xrightarrow{\psi} & (\mathbb{C}, \psi(A)) \\ \downarrow f & & \downarrow g \\ (\mathbb{R}^2, A) & \xrightarrow{\varphi} & (\mathbb{C}, \varphi(A)) \end{array}$$

$$\begin{array}{ccc} (\mathbb{R}^2, A) & \xrightarrow{\psi_n} & (\mathbb{C}, \psi(A)) \\ \downarrow f_n & & \downarrow g_n \\ (\mathbb{R}^2, A) & \xrightarrow{\varphi} & (\mathbb{C}, \varphi(A)) \end{array}$$

By Proposition III.12, it suffices to show that for any open set $D \subset \mathbb{R}^2$ with compact closure, the map $\psi_n \circ \psi^{-1}|\psi(D)$ is holomorphic for all n sufficiently large.

Let $\widehat{D} = \psi(D)$. Then there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, $f_n|D \equiv f|D$.

Fix an $n \geqslant N$. Let z_0 be a point in $\widehat{D}$ such that $z_0 \notin g^{-1}(\varphi(A))$. Since the point $y_0 := \varphi^{-1}(g(z)) \notin A$, there exists a local inverse $h : V_{y_0} \to \mathbb{R}^2$ for f in a neighborhood $V_{y_0} \ni y$ such that $x_0 := h(y_0) = \psi^{-1}(z_0)$ and $h(V_{y_0}) \subset D$. Moreover, $\varphi(f_n(x_0)) = \varphi(f(x_0)) = g(z_0)$. Since $g(z_0) \notin \varphi(A)$, there exists a local inverse $k_n : \widehat{V}_n \to \mathbb{C}$ for g_n in a neighborhood $\widehat{V}_n$ of $g(z_0)$ such that $k_n(g(z_0)) = z_0$. Let U_n be a neighborhood of z_0 such that $g(U_n) \subset \widehat{V}_n$ and $\varphi^{-1} \circ g(U_n) \subset V_{y_0}$.

Then for every $z \in U_n$,

$$\begin{aligned}\psi_n \circ \psi^{-1}(z) &= (k_n \circ \varphi \circ f_n) \circ (h \circ \varphi^{-1} \circ g)(z)\\ &= k_n \circ \varphi \circ (f_n \circ h) \circ (\varphi^{-1} \circ g)(z)\\ &= k_n \circ g(z).\end{aligned}$$

The last equality is due to the fact that the point $\varphi^{-1} \circ g(z)$ is in V_{y_0}, and since f_n and f coincide on D, we have $f_n \circ h \equiv \mathrm{id}_{V_{y_0}}$.

The above discussion shows that $\psi_n \circ \psi^{-1}$ is holomorphic at every point in $\widehat{D} \backslash g^{-1}(\varphi(A))$. Since $\widehat{D}$ is compact, the intersection $\widehat{D} \cap g^{-1}(\varphi(A))$ is a finite set. Moreover, $\psi_n \circ \psi^{-1}(D) \subset \mathbb{R}^2$ has compact closure. Therefore, $\psi_n \circ \psi^{-1}$ extends to a holomorphic map on $\widehat{D}$. Since this is true for all $n \geqslant N$, the proposition follows. □

Now fix a Thurston map $f : (\mathbb{R}^2, A) \circlearrowleft$ and let $f_n : (\mathbb{R}^2, A) \circlearrowleft$, $n \in \mathbb{N}$ be a sequence of entire Thurston maps that converge topologically to f.

Proof of Theorem I.4. We first show that (σ_{f_n}) converges to σ_f pointwise. Given $\tau = [\varphi] \in \mathrm{T}(\mathbb{S}^2, A \cup \infty)$, let $\sigma_f(\tau) = [\psi]$ and $\sigma_{f_n}(\tau) = \tau_n = [\psi_n]$, where $\psi, \psi_n, \varphi : \mathbb{R}^2 \to \mathbb{C}$ are orientation-preserving homeomorphisms for all $n \in \mathbb{N}$.

Given any compact set $D \subset \mathbb{R}^2$, there exists $N \in \mathbb{N}$ such that $f_n|D = f|D$ for all $n \geqslant N$. In particular, $\psi_n \circ \psi^{-1}$ is holomoprhic on $\varphi(D)$. By Proposition III.12, we have

$$\lim_{n\to\infty} d_{\mathrm{T}}(\sigma_f(\tau), \sigma_{f_n}(\tau)) = 0.$$

Next we show that the convergence $\sigma_{f_n} \to \sigma_f$ is in fact locally uniform on $\mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$. Let $K \subset \mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$ be a compact set. Given $\varepsilon > 0$, cover K by open balls $B_1, B_2, \ldots, B_k$ each of radius $\varepsilon/3$, with B_j centered at $\mu_j \in \mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$ for each $j = 1, 2 \ldots, k$.

Since (σ_{f_n}) converges to σ_f pointwise, for sufficiently large n, the distance $d_{\mathrm{T}}(\sigma_{f_n}(\mu_j), \sigma_f(\mu_j))$ is bounded above by $\varepsilon/3$ for each $j = 1, 2, \ldots, k$. Hence, for sufficiently large n and any $\mu \in K$ we have

$$d_{\mathrm{T}}(\sigma_{f_n}(\mu), \sigma_f(\mu)) \leqslant d_{\mathrm{T}}(\sigma_{f_n}(\mu), \sigma_{f_n}(\mu_j)) + d_{\mathrm{T}}(\sigma_{f_n}(\mu_j), \sigma_f(\mu_j)) + d_{\mathrm{T}}(\sigma_f(\mu_j), \sigma_f(\mu)) \leqslant \varepsilon.$$

Here j is chosen so that $d_{\mathrm{T}}(\mu, \mu_j) \leqslant \varepsilon/3$. We also use the fact that all σ-maps above are 1-Lipschitz (see Proposition II.24).

□

Corollary III.13. *Let $f : (\mathbb{R}^2, A) \circlearrowleft$ be a realized Thurston map and $f_n : (\mathbb{R}^2, A) \circlearrowleft, n \in \mathbb{N}$ be a sequence of Thurston maps converging combinatorially to f. Then f_n is realized for* *sufficiently large n. Moreover, letting $\tau_n \in \mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$ be a unique fixed point of σ_{f_n}, we have $\tau_n \to \tau$, where τ is the fixed point of σ_f.*

Proof. If f is a polynomial Thurston map, by Remark II.25 and Proposition III.6, the equality $\sigma_{f_n} = \sigma_f$ holds for sufficiently large n.

Now suppose that f is transcendental. Let $B \subset \mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$ be a closed ball of radius $r > 0$ centered at τ, the fixed point of σ_f. Since B is closed and bounded, it is compact.

Proposition II.24 implies the existence of $\varepsilon_B > 0$ such that the inequality $d_{\mathrm{T}}(\sigma_f(\mu_1), \sigma_f(\mu_2)) \leqslant (1 - \varepsilon_B) d_{\mathrm{T}}(\mu_1, \mu_2)$ is satisfied for all $\mu_1, \mu_2 \in B$. Thus, for any $\mu \in B$ we have the following:

$$d_{\mathrm{T}}(\sigma_{f_n}(\mu), \tau) \leqslant d_{\mathrm{T}}(\sigma_{f_n}(\mu), \sigma_f(\mu)) + d_{\mathrm{T}}(\sigma_f(\mu), \tau) \leqslant \varepsilon_n + (1 - \varepsilon_B) r, \tag{III.4.1}$$

where we know by Main Theorem I.4 that $\varepsilon_n \to 0$ as $n \to \infty$. In other words, $\sigma_{f_n}(B) \subset B$ for all sufficiently large n. By Proposition II.24, $\sigma_{f_n}^{\circ 2}$ is uniformly contracting on B and, therefore, by the Banach fixed point theorem, when n is large enough, the map σ_{f_n} has a fixed point $\tau_n \in B$.

Lastly, similar to inequality (III.4.1), we have

$$d_{\mathrm{T}}(\tau_n, \tau) \leqslant d_{\mathrm{T}}(\sigma_{f_n}(\tau_n), \sigma_f(\tau_n)) + d_{\mathrm{T}}(\sigma_f(\tau_n), \sigma_f(\tau)) \leqslant \varepsilon_n + (1 - \varepsilon_B) d_{\mathrm{T}}(\tau_n, \tau).$$

This shows that the sequence (τ_n) converges to τ. □

CHAPTER IV
Admissible Quadruples

In this chapter, we show how to construct topologically holomorphic maps in class $\mathcal{S}$ from a covering map between a pair of regular planar embedded graphs. In subsequent sections, we will use this construction to define Thurston maps. The foundational theory of planar graphs we use here is explored in Appendix A.2.

IV.1: Rose graphs and quadruples

Let $\mathcal{R}$ be a directed rose graph based at t that surrounds a finite set $A \subset \mathbb{R}^2$ (see Definition A.7), and suppose $|A| = m$. For $a \in A$, let p_a be the edge of $\mathcal{R}$ that surrounds A, and let P_a be the bounded face of $\mathcal{R}$ containing a. Also denote by P_∞ the unique unbounded face of $\mathcal{R}$.

Let $\Gamma \subset \mathbb{R}^2$ be a $2m$-regular and connected graph, and $\Phi : \Gamma \to \mathcal{R}$ be a covering map such that $\Phi(v) = t$ for all $v \in V(\Gamma)$ and $\Phi(e) \in E(\mathcal{R})$ for all $e \in E(\Gamma)$. The graph Γ can be assumed to be a directed graph with the orientation induced by the map Φ (see Appendix A.2).

Let us assume that $m \geqslant 2$. Consider an arbitrary face $F \in F(\Gamma)$. We label F by P_a if $\Phi(\partial F) = \{p_a\}$. If no such a exists (or equivalently, the image $\Phi(\partial F)$ spans at least two petals of $\mathcal{R}$), we label F by P_∞. We denote by Γ^* the directed planar embedded graph obtained by subdividing each edge of Γ (see Definition A.8).

Definition IV.1. Let $A, \mathcal{R}, \Gamma, \Phi$ be as above. Label the points of A as $a_1, a_2, \cdots, a_m$ such that the edges $p_{a_1}, p_{a_2}, \cdots, p_{a_m}$ are arranged in counterclockwise order around t. We say that the quadruple $\Delta = (A, \mathcal{R}, \Gamma, \Phi)$ is *admissible* if $m = 1$, or

1. every face F of Γ labeled by P_∞ is unbounded, and
2. for each $v \in V(\Gamma)$, the set of edges of Γ^* at v (both incoming and outgoing) can be written in the counterclockwise order as $e_1 = e_{2m+1}, e_2, e_3, \cdots, e_{2m}$ such that the following conditions are satisfied for each $j = 1, 2, \ldots, m$:

- e_{2j-1} is incoming at v and e_{2j} is outgoing at v;
- there exists $F \in F(\Gamma)$ labelled by P_{a_j} such that $e_{2j-1}, e_{2j} \in \partial F$;
- there exists $F' \in F(\Gamma)$ labelled by P_∞ such that $e_{2j}, e_{2j+1} \in \partial F'$.

Remark IV.2. Suppose that $\Delta = (A, \mathcal{R}, \Gamma, \Phi)$ is an admissible quadruple and F is a face of Γ. If F is not labelled by P_∞, then clearly, ∂F is a counterclockwise directed cycle if F is bounded, and is otherwise an infinite directed chain. If F is labelled by P_∞, then ∂F is a unilaterally connected graph (see Definition A.10).

IV.2: Functions on $\mathbb{R}^2$ from quadruples

Natural examples of admissible quadruples are preimages of rose graphs under entire topologically holomorphic maps in class $\mathcal{S}$. More precisely, suppose that $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is topologically holomorphic and rose graph $\mathcal{R}$ surrounds the set A, where $S_f \subset A$. Denote by $\Delta(A, \mathcal{R}, f)$ the quadruple $(A, \mathcal{R}, f^{-1}(\mathcal{R}), \Phi_{\mathcal{R},f})$, where $\Phi_{\mathcal{R},f}(x) = f(x)$ for each $x \in f^{-1}(\mathcal{R})$.

Proposition IV.3. *Let f, A, and $\mathcal{R}$ be as above. Then $\Delta(A, \mathcal{R}, f)$ is an admissible quadruple. Moreover, if $m \geqslant 2$, then for each face F of $f^{-1}(\mathcal{R})$ the following properties hold:*

1. *if F is bounded and labeled by P_a for some $a \in A$, then $f(F) = P_a$ and $|F \cap f^{-1}(A)| = 1$. If $|V(\partial F)| = 1$, then F does not contain any critical points of f and $f|F$ is injective, otherwise, F contains a unique critical point x_F with $\deg(f, x_F) = |V(\partial F)|$;*
2. *if F is unbounded and labeled by P_a for some $a \in A$, then $f(F) = P_a \backslash \{a\}$ and $F \cap f^{-1}(A) = \varnothing$. In particular, $a \in S_f$ and F is an asymptotic tract of f over a;*
3. *if F is labeled by P_∞, then $f(F) = P_\infty$ and $F \cap f^{-1}(A) = \varnothing$. In particular, f restricts to a universal covering map from F to P_∞.*

Proof. First, we show that $f^{-1}(\mathcal{R})$ is connected. Consider any two distinct vertices u and v of $f^{-1}(\mathcal{R})$. There exists a path $\alpha\colon \mathbb{I} \to \mathbb{R}^2 \backslash f^{-1}(A)$ joining u and v. Note that $f \circ \alpha$ is a loop in $\mathbb{R}^2 \backslash A$ based at t. Label A as $a_1, ..., a_m$ such that $p_{a_1}, ..., p_{a_m}$ are counterclockwise around t. Assuming that each p_{a_j} is parameterized by $\alpha_{a_j} : \mathbb{I} \to p_{a_j}$, it follows that $f \circ \alpha$ is homotopic to a loop $\gamma = \gamma_1 \cdot \gamma_2 \cdot \cdots \cdot \gamma_k$ rel. A, where $\gamma_\ell \in \{\alpha_1, \alpha_2, \ldots, \alpha_m, \overline{\alpha_1}, \overline{\alpha_2}, \ldots, \overline{\alpha_m}\}$ for each $\ell \in \{1, 2, \cdots, k\}$. By the homotopy lifting property, the path α is homotopic rel. $f^{-1}(A)$ to $\widetilde{\gamma} := \gamma\!\uparrow\!(f, u)$. In particular, $\widetilde{\gamma}$ joins u with v. Since $\widetilde{\gamma} \subset f^{-1}(\mathcal{R})$, the vertices u and v belong to the same connected component of $f^{-1}(\mathcal{R})$.

Proposition II.8 implies that every face of $f^{-1}(\mathcal{R})$ labeled by P_∞ is unbounded. Finally, the regularity of $f^{-1}(\mathcal{R})$ and admissibility of $\Delta(A, \mathcal{R}, f)$ follow from the fact that f is locally injective and orientation-preserving at every $v \in V(f^{-1}(\mathcal{R}))$. Thus, the quadruple $\Delta(A, \mathcal{R}, f)$ is admissible.

The rest of the statement easily follows from Propositions II.7 and II.8.

□

Remark IV.4. Let $\Delta = (A, \mathcal{R}, \Gamma, \Phi)$ be an admissible quadruple with $m = |E(\mathcal{R})| = 1$. In this case, Γ is a counterclockwise directed cycle or an infinite directed chain. It is easy to see that there exists a map f such that $\Delta(A, \mathcal{R}, f) = \Delta$ satisfying $f = \varphi \circ g \circ \psi^{-1}$ for some orientation-preserving homeomorphisms $\varphi, \psi \colon \mathbb{C} \to \mathbb{R}^2$, where $g(z) = z^d$ if $d = |V(\Gamma)| < \infty$, and $g(z) = \exp(z)$ if Γ is infinite. In particular, using Proposition II.14, we can formulate a statement close in spirit to Proposition IV.3 for the case $m = 1$.

Now assume that $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ is an arbitrary topologically holomorphic map such that $S_f \subset A$ and $\Delta(A, \mathcal{R}, f) = \Delta$. Since $\mathcal{R}$ is a deformation retract of $\mathbb{R}^2 \backslash A$ and Γ is a deformation retract of $\mathbb{R}^2 \backslash f^{-1}(A)$ as Proposition IV.3 suggests, for any $v \in V(\Gamma)$, we see that $f_* \pi_1(\mathbb{R}^2 \backslash f^{-1}(A), v) = \Phi_* \pi_1(\Gamma, v)$, and

$$\Phi_* \pi_1(\Gamma, v) = \{[\Phi(\delta_1) \cdot \Phi(\delta_2) \cdot \dots \cdot \Phi(\delta_\ell)] : \text{each } \delta_i \text{ is a path parameterizing an edge of } \Gamma, \\ \text{and } \delta_1 \cdot \delta_2 \cdot \dots \cdot \delta_\ell \text{ is a loop in } \Gamma \text{ based at } v\}.$$

Now suppose that γ is a loop based at the center of the rose graph $\mathcal{R}$ such that γ is homotopic to $\gamma_1 \cdot \gamma_2 \cdot \dots \cdot \gamma_k$ rel. A, where $\gamma_j \in \{\alpha_1, \alpha_2, \dots, \alpha_m, \overline{\alpha_1}, \overline{\alpha_2}, \dots, \overline{\alpha_m}\}$ and α_j is a parametrization of p_j for each $j = 1, 2, \dots, m$. If we know Φ, we can easily reconstruct $\gamma \uparrow (f, v)$ for any vertex $v \in V(\Gamma)$ up to homotopy rel. $f^{-1}(A)$.

The following result is the converse to Proposition IV.3:

Proposition IV.5. *Let $\Delta = (A, \mathcal{R}, \Gamma, \Phi)$ be an admissible quadruple. Then there exists a topologically holomorphic map $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ of finite type such that $S_f \subset A$ and $\Delta(A, \mathcal{R}, f) = \Delta$.*

Proof. When $|A| = 1$, the desired result follows from Remark IV.4. Thus, we assume $|A| \geqslant 2$ and give an outline of the construction.

First, we define f on Γ simply by setting $f|\Gamma := \Phi$. Choose orientation-preserving homeomorphisms $\varphi_{P_j} \colon \mathbb{D} \to P_j$ and $\varphi_{P_\infty} \colon \mathbb{H} \to P_\infty$. Similarly, since each face F of Γ is simply connected, choose an orientation-preserving homeomorphism $\psi_F : X \to F$ where $X = \mathbb{D}$ if F is bounded, and $X = \mathbb{H}$ if F is unbounded.

Given a face F with label $P \in F(\mathcal{R})$, we define $f|F$ so that

1. if F is bounded, then $f|F := \varphi_P \circ g_d \circ \psi_F^{-1}$, where $d = |V(\partial F)|$ and $g_d(z) = z^d$;

2. if F is unbounded, then $f|F := \varphi_P \circ \exp \circ \psi_F^{-1}$.

Due to admissibility conditions and Remark IV.2, the sets of homeomorphisms $\{\varphi_P\}_{P\in F(\mathcal{R})}$ and $\{\psi_F\}_{F\in F(\Gamma)}$ can be chosen so that the map f we construct above is continuous. Finally, one can show that f acts locally as a power map $z \mapsto z^d$ for some $d \in \mathbb{N}$, and that $S_f \subset A$. Thus, f is topologically holomorphic, has finite type, and satisfies $\Delta(A, \mathcal{R}, f) = \Delta$.

□

Definition IV.6. Two admissible quadruples $\Delta_1 = (A, \mathcal{R}_1, \Gamma_1, \Phi_1)$ and $\Delta_2 = (A, \mathcal{R}_2, \Gamma_2, \Phi_2)$ are said to be *equivalent* if there exist $\psi \in \mathrm{Homeo}_0^+(\mathbb{R}^2, A)$ and $\varphi \in \mathrm{Homeo}^+(\mathbb{R}^2, A)$ such that

1. ψ is an isomorphism between $\mathcal{R}_1$ and $\mathcal{R}_2$;
2. φ is an isomorphism between Γ_1 and Γ_2;
3. $\psi \circ \Phi_1 = \Phi_2 \circ \varphi$.

It is easy to see that Definition IV.6 provides an equivalence relation on the set of all admissible quadruples with a fixed marked set.

Proposition IV.7. *Let $A \subset \mathbb{R}^2$ be finite, and $\mathcal{R}_1$, $\mathcal{R}_2$ be rose graphs surrounding A such that $\mathcal{R}_1$ isotopic to $\mathcal{R}_2$ rel. A. Let $f_1 \colon \mathbb{R}^2 \to \mathbb{R}^2$ and $f_2 \colon \mathbb{R}^2 \to \mathbb{R}^2$ be topologically holomorphic maps such that $S_{f_1} \subset A$ and $S_{f_2} \subset A$. Then $\Delta(A, \mathcal{R}_1, f_1)$ and $\Delta(A, \mathcal{R}_2, f_2)$ are equivalent if and only if there exists a map $\psi \in \mathrm{Homeo}^+(\mathbb{R}^2)$ such that $f_1 = f_2 \circ \psi$.*

Moreover, if f_1 and f_2 are holomorphic, and $\Delta(A, \mathcal{R}_1, f_1)$ and $\Delta(A, \mathcal{R}_2, f_2)$ are equivalent, then the map ψ is an affine transformation.

Proof. Suppose that the admissible quadruples $\Delta(A, \mathcal{R}_1, f_1)$ and $\Delta(A, \mathcal{R}_2, f_2)$ are equivalent. Due to Proposition II.6 we can assume that $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R}$ and that the equivalence between $\Delta(A, \mathcal{R}_1, f_1)$ and $\Delta(A, \mathcal{R}_2, f_2)$ is provided by $\psi = \mathrm{id}_{\mathbb{R}^2}$ and an orientation-preserving homeomorphism φ. By pre-composing f_2 with φ we can further assume $\psi = \varphi = \mathrm{id}_{\mathbb{R}^2}$, or equivalently, $\Delta(A, f_1, \mathcal{R}_1) = \Delta(A, f_2, \mathcal{R}_2) = (A, \mathcal{R}, \Gamma, \Phi)$. Then by the previous discussions, we have that

$$(f_1)_* \pi_1(\mathbb{R}^2 \backslash f_1^{-1}(A), v) = \Phi_* \pi_1(\Gamma, v) = (f_2)_* \pi_1(\mathbb{R}^2 \backslash f_2^{-1}(A), v)$$

for every $v \in V(\Gamma)$. By the classical theory of covering maps, there exists an orientation-preserving homeomorphism $\psi \colon \mathbb{R}^2 \backslash f_1^{-1}(A) \to \mathbb{R}^2 \backslash f_2^{-1}(A)$ such that $f_1 = f_2 \circ \psi$ on $\mathbb{R}^2 \backslash f_1^{-1}(A)$. Since $f_1^{-1}(A)$ is a discrete subset of $\mathbb{R}^2$, we can extend ψ to $\mathbb{R}^2$, still satisfying $f_1 = f_2 \circ \psi$.

Conversely, let us suppose there exists $\psi \in \mathrm{Homeo}^+(\mathbb{R}^2)$ such that $f_1 = f_2 \circ \psi$. By our assumptions on $\mathcal{R}_1$ and $\mathcal{R}_2$, we can find $\psi \in \mathrm{Homeo}_0^+(\mathbb{R}^2, A)$ such that $\psi(\mathcal{R}_1) = \mathcal{R}_2$. Then

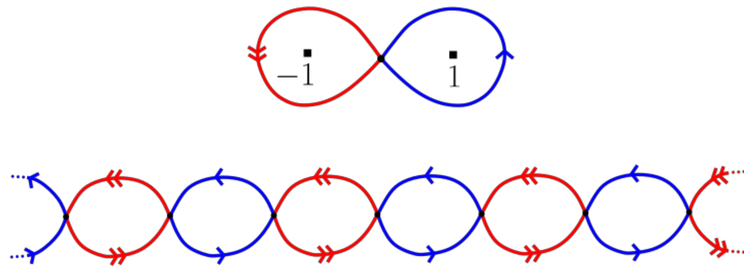


Figure 4.1: Admissible quadruple realized by $g_1(z) = \cos(z)$.

by Proposition II.6, there exists $\varphi \in \mathrm{Homeo}^+(\mathbb{R}^2)$ such that $\psi \circ f_1 = f_2 \circ \varphi$, and the rest easily follows.

If f_1 and f_2 are holomorphic, any homeomorphism ψ satisfying $f_1 = f_2 \circ \psi$ is holomorphic, and therefore affine.

□

We say that a topologically holomorphic map $f: \mathbb{R}^2 \to \mathbb{R}^2$ *realizes* an admissible quadruple $\Delta = (A, \mathcal{R}, \Gamma, \Phi)$ or, equivalently, Δ *defines* f, if $S_f \subset A$ and $\Delta(A, \mathcal{R}, f)$ is equivalent to Δ. In particular, Propositions IV.5 and IV.7 imply that every admissible quadruple Δ defines an entire topologically holomorphic map f of finite type, which is unique up to pre-composition by an orientation-preserving homeomorphism.

Note that an admissible quadruple $\Delta = (A, \mathcal{R}, \Gamma, \Phi)$ is a combinatorial object even though Φ is a continuous map. In fact, to define the map Φ uniquely (up to a certain notion of equivalence introduced below), it is sufficient to know the images $\Phi(e), e \in E(\Gamma)$ and the orientation of the graph Γ induced by Φ. Indeed, suppose that $\Psi: \Gamma \to \mathcal{R}$ is a covering map such that $\Psi(v) = t$ for each vertex $v \in V(\Gamma)$, $\Phi(e) = \Psi(e)$ for each edge $e \in E(\Gamma)$, and the orientations of Γ induced by the maps Φ and Ψ coincide. In this case, it is clear that there exists a homeomorphism $\varphi: \Gamma \to \Gamma$ isotopic rel. $V(\Gamma)$ to id_Γ such that $\Psi = \Phi \circ \varphi$. In particular, the orientation of the graph Γ and the images of its edges under the map Φ uniquely define the equivalence class of the admissible quadruple Δ.

The language of admissible quadruples provides a convenient way of thinking about entire topologically holomorphic (and holomorphic) maps of finite type, which we demonstrate in the following example.

Example IV.8. Let $A = \{-1, 1\}$ be the set represented by solid black squares at the top of Figure 4.1. Denote by $\mathcal{R}_1$ and Γ_1 the planar embedded graphs shown at the top and bottom of Figure 4.1, respectively. The map $\Phi_1: \Gamma_1 \to \mathcal{R}_1$ is a covering that maps each edge of Γ_1 to the unique edge of $\mathcal{R}_1$ of the same color. Arrows on the edges of the graphs $\mathcal{R}_1$ and Γ_1

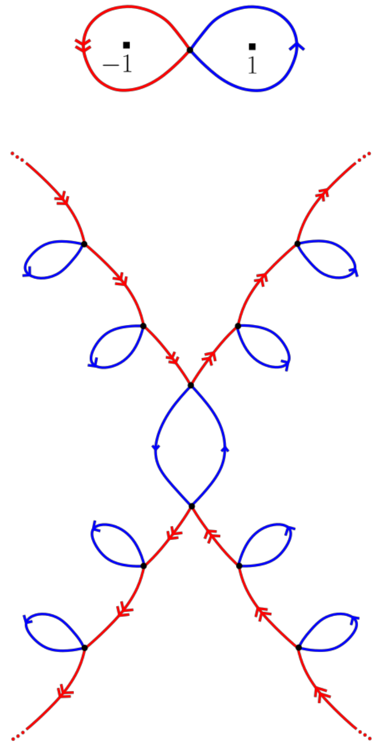


Figure 4.2: Admissible quadruple realized by $g_2(z) = 2\exp(z^2) - 1$.

indicate the orientations of the corresponding graphs. It is straightforward to check that $\Delta_1 = (A, \mathcal{R}_1, \Gamma_1, \Phi_1)$ is an admissible quadruple, which is realized by the map $g_1(z) = \cos(z)$. Thus, any entire holomorphic map realizing Δ_1 has the form $\cos(az + b)$ for some constants $a, b \in \mathbb{C}$ with $a \neq 0$.

Figure 4.2 is analogous to Figure 4.1, and provides another example $\Delta_2 = (A, \mathcal{R}_2, \Gamma_2, \Phi_2)$ of an admissible quadruple. This quadruple can be shown to be realized by the map $g_2(z) = 2\exp(z^2) - 1$. In particular, the planar embedded graph Γ_2 has two unbounded faces that correspond to the asymptotic tracts of g_2 over $w_0 = 1$, two unbounded faces that correspond to the asymptotic tracts of g_2 over ∞, and the only face with the boundary consisting of two edges, corresponding to the unique critical point of g_2 (see Proposition IV.3).

Definition IV.9. Let Δ be an admissible quadruple. If every topologically holomorphic map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ realizing Δ is of parabolic type, then we say that Δ is *parabolic*. If every such map f is hyperbolic, then Δ is called *hyperbolic*.

Proposition IV.7 implies that every admissible quadruple is either parabolic or hyperbolic. Moreover, from Proposition II.2, it follows that every parabolic admissible quadruple Δ with a marked set A defines an entire holomorphic map of finite type with $S_f \subset A$, which is unique up to pre-composition with an affine transformation.

Definition IV.10. An admissible quadruple $\Delta = (A, \mathcal{R}, \Gamma, \Phi)$ is called *dynamically admissible* if Δ is parabolic, $\Gamma \cap A = \varnothing$, and every face F of the graph Γ contains at most one point of A if F is bounded and no points of A, otherwise.

Proposition IV.3 implies that $\Delta(A, \mathcal{R}, f)$ is dynamically admissible if $f\colon (\mathbb{R}^2, A) \circlearrowleft$ is a Thurston map. Also, analogous to Proposition IV.5, we can establish the following statement.

Proposition IV.11. *Let $\Delta = (A, \mathcal{R}, \Gamma, \Phi)$ be a dynamically admissible quadruple. Then there exists a Thurston map $f\colon (\mathbb{R}^2, A) \circlearrowleft$ such that $\Delta(A, \mathcal{R}, f) = \Delta$.*

In a similar way we define a *dynamical equivalence* relation on the set of all dynamically admissible quadruples with a fixed marked set. We say that two dynamically admissible quadruples $\Delta_1 = (A, \mathcal{R}_1, \Gamma_1, \Phi_1)$ and $\Delta_2 = (A, \mathcal{R}_2, \Gamma_2, \Phi_2)$ are dynamically equivalent if there exist $\psi, \varphi \in \mathrm{Homeo}_0^+(\mathbb{R}^2, A)$ such that

1. ψ is an isomorphism between $\mathcal{R}_1$ and $\mathcal{R}_2$;
2. φ is an isomorphism between Γ_1 and Γ_2;
3. $\psi \circ \Phi_1 = \Phi_2 \circ \varphi$.

Finally, the following observation provides a dynamical analog of Proposition IV.7.

Proposition IV.12. *Let $f_1\colon (\mathbb{R}^2, A) \circlearrowleft$ and $f_2\colon (\mathbb{R}^2, A) \circlearrowleft$ be Thurston maps, and $\mathcal{R}_1$, $\mathcal{R}_2$ be rose graphs surrounding A such that $\mathcal{R}_1$ is isotopic rel. A to $\mathcal{R}_2$. Then $\Delta(A, \mathcal{R}_1, f_1)$ and $\Delta(A, \mathcal{R}_2, f_2)$ are dynamically equivalent if and only if f_1 and f_2 are isotopic rel. A.*

Proof. First suppose that $\Delta(A, \mathcal{R}_1, f_1)$ and $\Delta(A, \mathcal{R}_2, f_2)$ are dynamically equivalent via $\psi, \varphi \in \mathrm{Homeo}_0^+(\mathbb{R}^2, A)$. As in the proof of Proposition IV.7, we can assume that $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R}$ and $\psi = \varphi = \mathrm{id}_{\mathbb{R}^2}$. The rest easily follows from Proposition III.3 and the discussion of Section IV.1.

Conversely, suppose that there exists $\psi \in \mathrm{Homeo}_0^+(\mathbb{R}^2)$ such that $f_1 = f_2 \circ \psi$. Choose $\psi \in \mathrm{Homeo}_0^+(\mathbb{R}^2, A)$ with $\psi(\mathcal{R}_1) = \mathcal{R}_2$. By Proposition II.6, there exists $\varphi \in \mathrm{Homeo}_0^+(\mathbb{R}^2, A)$ such that $\psi \circ f_1 = f_2 \circ \varphi$. It directly implies that $\Delta(A, \mathcal{R}_1, f_1)$ and $\Delta(A, \mathcal{R}_2, f_2)$ are dynamically equivalent. □

Given dynamically admissible quadruple $\Delta = (A, \mathcal{R}, \Gamma, \Phi)$, we say that a Thurston map $f\colon (\mathbb{R}^2, A) \circlearrowleft$ *realizes* Δ (or equivalently, Δ *defines* f), if $\Delta(A, \mathcal{R}, f)$ is dynamically equivalent to Δ. Propositions IV.11 and IV.12 imply that every dynamically admissible Δ with a marked set A defines a Thurston map $f\colon (\mathbb{R}^2, A) \circlearrowleft$, unique up to isotopy rel. A.

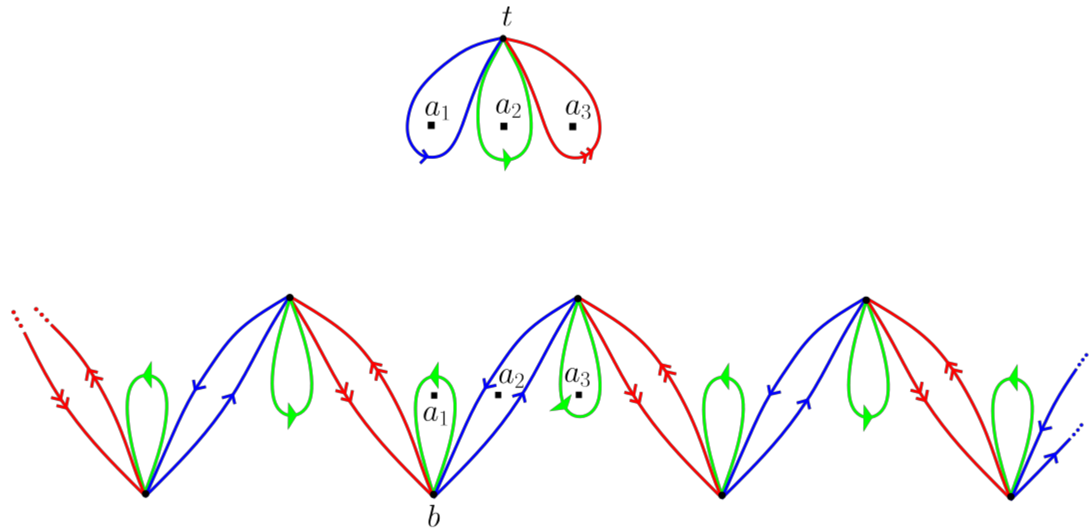


Figure 4.3: Dynamically admissible quadruple realized by the PSF entire map $G_1(z) = \pi\cos(z)/2$, where $P_{G_1} = \{a_1, a_2, a_3\} = \{-\pi/2, 0, \pi/2\}$.

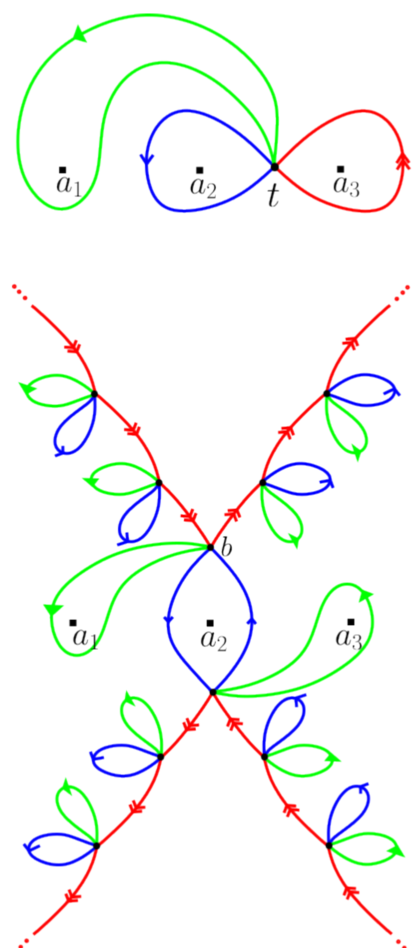


Figure 4.4: Dynamically admissible quadruple realized by the PSF entire map $G_2(z) = \sqrt{\ln 2}\big(1 - \exp(z^2)\big)$, where $P_{G_2} = \{a_1, a_2, a_3\} = \{-\sqrt{\ln 2}, 0, \sqrt{\ln 2}\}$.

Example IV.13. Let $G_1\colon (\mathbb{C}, P_{G_1}) \circlearrowleft$ and $G_2\colon (\mathbb{C}, P_{G_2}) \circlearrowleft$ be the postsingularly finite entire maps defined in Example II.16. The graphs in Figures 4.3 and 4.4 describe dynamically admissible quadruples realized by G_1 and G_2 respectively (compare with Figures 4.1 and 4.2 referenced in Example IV.8). The solid black squares in these figures represent the postsingular values of G_1 and G_2, respectively.

IV.3: Construction of combinatorial approximations of polynomial type

Proposition IV.14. *Let $f_n\colon (\mathbb{R}^2, A) \circlearrowleft, n \in \mathbb{N}$ and $f\colon (\mathbb{R}^2, A) \circlearrowleft$ be Thurston maps, and $\mathcal{R}$ be a rose graph that surrounds A. Then the sequence (f_n) converges combinatorially to f if and only if for every finite subgraph K of $f^{-1}(\mathcal{R})$ and all sufficiently large n, there exists a homeomorphism $\varphi_{K,n} \in \mathrm{Homeo}_0^+(\mathbb{R}^2, A)$ such that $\varphi_{K,n}(K)$ is a subgraph of $f_n^{-1}(\mathcal{R})$ and $\Phi_{f,\mathcal{R}}|K = \Phi_{f_n,\mathcal{R}} \circ \varphi_{K,n}|K$.*

Proof. Sufficiency easily follows from Definition III.2, Proposition III.5, and the discussion of Chapter IV.1. Necessity can be obtained by applying Proposition III.8. □

Proposition IV.15. *Let $f\colon (\mathbb{R}^2, A) \circlearrowleft$ be an arbitrary Thurston map. Then there exists a sequence $f_n\colon (\mathbb{R}^2, A) \circlearrowleft, n \in \mathbb{N}$ of polynomial Thurston maps converging combinatorially to f.*

Proof. Due to Proposition II.14 and Remark IV.4, the case when $|A| = 1$ is trivial and, therefore, we can assume that $|A| \geqslant 2$. Let us choose a rose graph $\mathcal{R}$ surrounding the set A and consider the dynamically admissible quadruple $\Delta(A, \mathcal{R}, f) = (A, \mathcal{R}, \Gamma, \Phi)$. Next, we choose an arbitrary exhaustion of Γ by finite connected subgraphs $K_n = (V_n, E_n)$. We shall construct a sequence of eventually dynamically admissible quadruples $\Delta_n = (A, \mathcal{R}, \Gamma_n, \Phi_n)$, where Γ_n is obtained from K_n by adding several new edges, and the maps Φ_n and Φ coincide on K_n. We describe this more precisely by defining Γ_n and Φ_n algorithmically. For each $n \in \mathbb{N}$, initialize Γ_n as K_n, and Φ_n as $\Phi|K_n$. Let F be an arbitrary face of Γ labelled by a bounded face P of $\mathcal{R}$ with the property that ∂F intersects K_n but it is not a proper subset of K_n.

Claim. $C := \partial F \cap K_n$ is a directed finite chain.

Proof of Claim. First we consider the case when F is bounded: here, ∂F is a counterclockwise directed cycle by Remark IV.2. It suffices to show that C (or equivalently $\partial F \backslash C$) is connected. Supposing the contrary, let e_1 and e_2 be two edges in disjoint components of $\partial F \backslash C$. It is easy to see that each edge $e \in E(\Gamma)$ is a boundary of exactly two faces and one of them is always unbounded. Therefore, for $i = 1, 2$, there exists a continuous curve $L_i\colon [0, +\infty) \to \mathbb{R}^2$ that

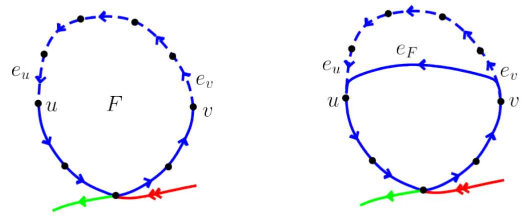


Figure 4.5: Constructing the directed edge e_F. The diagram on the left represents the face F and the graph K_n, and the diagram on the right demonstrates the newly added edge e_F. Black dots and colored (solid and dashed) arcs represent the vertices and directed edges of Γ, respectively. The solid blue arcs are edges of K_n, while the dashed ones are in $\Gamma \backslash K_n$.

joins an interior point $x_i := L_i(0)$ of e_i with ∞ (i.e., $\lim_{t \to +\infty} L_i(t) = \infty$) and intersects Γ only at the point x_i. Let y_1 and y_2 be vertices of C lying in disjoint components of $\partial F \backslash \{\mathrm{int}(e_1), \mathrm{int}(e_2)\}$. Any path in $\mathbb{R}^2$ joining y_1 to y_2 has to intersect $F \cup \{\mathrm{int}(e_1), \mathrm{int}(e_2)\}$ or $L_1 \cup L_2$. In particular, there is no path in K_n joining y_1 to y_2. This forms a contradiction since K_n is connected. The case when F is unbounded is analogous. ■

Let u and v be the endpoints of C (it is possible that $u = v$). Then there exist unique edges $e_u, e_v \in E(\partial F \backslash K_n)$ such that e_u is incident to u, and e_v is incident to v (again, e_u and e_v might coincide). We shall construct an edge e_F with endpoints at v and u (see Figure 4.5) with $e_F \subset e_u \cup e_v \cup F$, so that e_F coincides with the edges e_u and e_v in small neighborhoods of u and v, respectively. In particular, $\mathrm{int}(e_F)$ does not intersect K_n. We add e_F to Γ_n and define $\Phi_n | e_F$ so that Φ_n and Φ coincide on $e_F \cap e_u$ and $e_F \cap e_v$. Then we repeat the above procedure for all faces $F \in F(\Gamma)$ satisfying previously listed properties.

Now consider the sequence of constructed quadruples $\Delta_n = (A, \mathcal{R}, \Gamma_n, \Phi_n)$. Since (K_n) is an exhaustion of Γ, there exists $N \in \mathbb{N}$ such that for each $n \geqslant N$, if $F \in F(\Gamma)$ is bounded and contains a point of A, then $\partial F \subset K_n$. For each $n \geqslant N$, it is straightforward to check that the conditions of Definitions IV.1 and IV.10 are satisfied. In particular, Δ_n is parabolic since Γ_n is a finite graph. By Proposition IV.11, we can construct a polynomial Thurston map $f_n \colon (\mathbb{R}^2, A) \circlearrowleft$ such that $\Delta(A, \mathcal{R}, f_n) = \Delta_n$ for each $n \geqslant N$. Thus, the sequence (f_n) converges combinatorially to f due to Proposition IV.14. □

Remark IV.16. Suppose that we are in the setting of the proof of Proposition IV.15. Let F be a face of Γ_n labelled by a bounded face P of $\mathcal{R}$ (with respect to the quadruple Δ_n) for some $n \geqslant N$. Then there exists a unique face $F' \in F(\Gamma)$ having the same label P (with respect to the quadruple Δ) such that $F \subset F'$. Moreover, exactly one of the following is true:

1. $\deg(f_n | F) = \deg(f_n | F')$. Then Proposition IV.3 implies that F contains a (unique) critical point z_n of f_n if and only if F' contains a (unique) critical point z of f and, moreover, $\deg(f, z) = \deg(f_n, z_n)$.

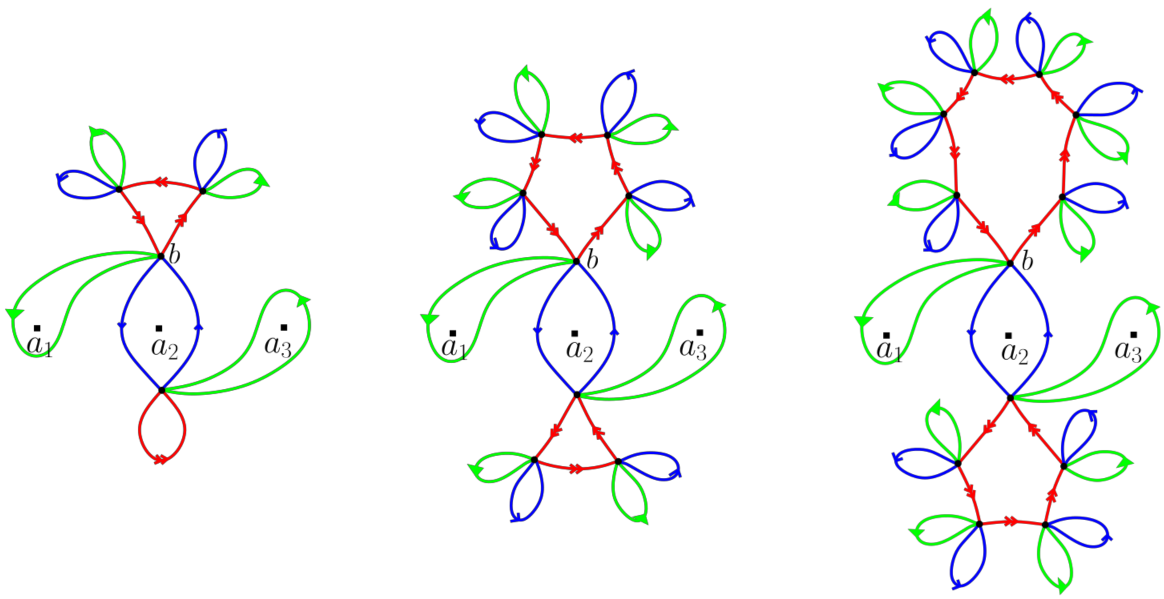


Figure 4.6: Sequence of graphs that define polynomial Thurston maps converging combinatorially to $G_2\colon (\mathbb{C}, P_{G_2}) \circlearrowleft$, where $G_2(z) = \sqrt{\ln 2}(1 - \exp(z^2))$ and $P_{G_2} = \{a_1, a_2, a_3\} = \{-\sqrt{\ln 2}, 0, \sqrt{\ln 2}\}$.

2. $\deg(f_n|F) < \deg(f_n|F')$. If F contains a critical point of f_n, then it is unique in F. In this case F' is either an asymptotic tract of f or it contains a unique critical point z of f and, moreover, $\deg(f, z) > \deg(f_n, z_n)$.

With this discussion and the following example, we show that dynamically admissible quadruples provide a convenient way for constructing "combinatorial" approximations and thinking about combinatorial convergence.

Example IV.17. Consider the postsingularly finite entire map G_2 from Example II.16, realizing the dynamically admissible quadruple $\Delta_{G_2} = \Delta(P_{G_2}, \mathcal{R}, G_2) = (P_{G_2}, \mathcal{R}, \Gamma, \Phi)$ depicted in Figure 4.4 along with a chosen point $b \in V(\Gamma)$. Define the graph $K_n \subset \Gamma$ to be a collection of all vertices and edges of Γ accessible from b via a path in Γ intersecting interiors of at most n edges. It is clear that the sequence (K_n) is an exhaustion of Γ by finite connected graphs. Starting with Δ_{G_2} and (K_n) and applying the construction from the proof of Proposition IV.15, we obtain a sequence of polynomial Thurston maps $f_n\colon (\mathbb{R}^2, P_{G_2}) \circlearrowleft, n \in \mathbb{N}$ converging combinatorially to $G_2\colon (\mathbb{C}, P_{G_2}) \circlearrowleft$, with each f_n defined by a dynamically admissible quadruple $\Delta_n = (P_{G_2}, \mathcal{R}, \Gamma_n, \Phi_n)$. Figure 4.6 illustrates the graphs Γ_n and the maps Φ_n for $n = 1, 2, 3$ (from left to right). As usual, $\Phi_n\colon \Gamma_n \to \mathcal{R}$ maps each edge of Γ_n to the unique edge of $\mathcal{R}$ of the same color, and the set P_{G_2} is represented by solid black squares. We also recall Figure 1.2 from Chapter I as an example of a combinatorial approximation for

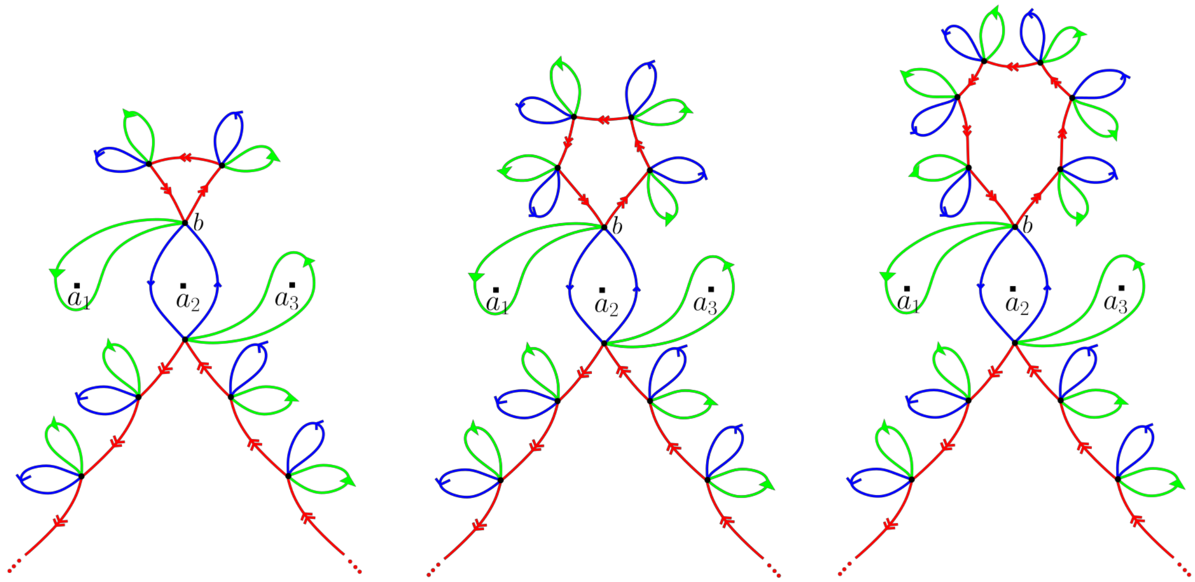


Figure 4.7: Sequence of graphs that define transcendental Thurston maps converging combinatorially to $G_2\colon (\mathbb{C}, P_{G_2}) \circlearrowleft$, where $G_2(z) = \sqrt{\ln 2}(1 - \exp(z^2))$ and $P_{G_2} = \{a_1, a_2, a_3\} = \{-\sqrt{\ln 2}, 0, \sqrt{\ln 2}\}$.

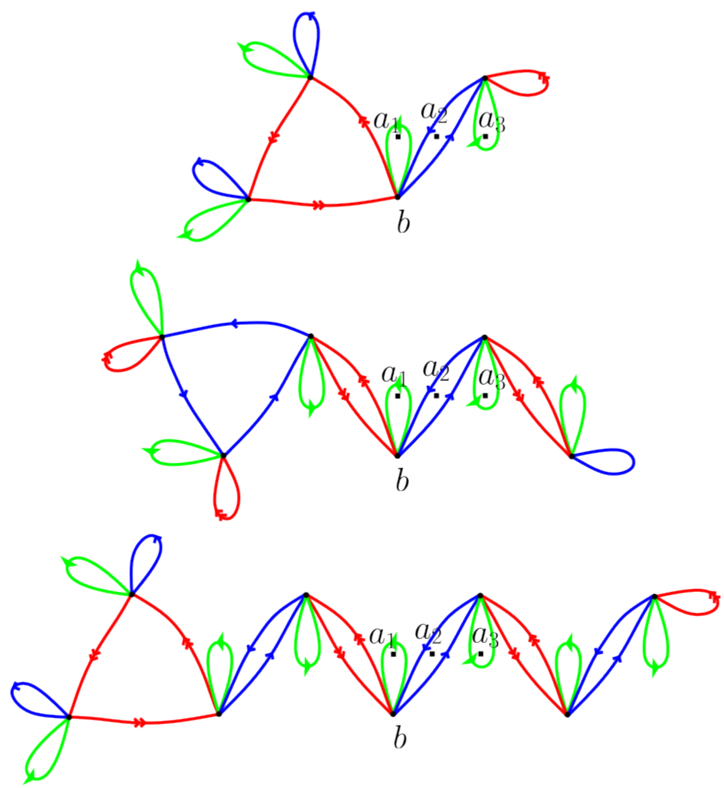


Figure 4.8: Sequence of graphs that define polynomial Thurston maps converging combinatorially to $G_1\colon (\mathbb{C}, P_{G_1}) \circlearrowleft$, where $G_1(z) = \pi\cos(z)/2$ and $P_{G_1} = \{a_1, a_2, a_3\} = \{0, -\pi/2, \pi/2\}$.

the map $G_1(z) = \frac{\pi}{2}\cos z$ from Example II.16, with $\Delta_{G_1} = \Delta(P_{G_1}, \mathcal{R}, G_1) = (P_{G_1}, \mathcal{R}, \Gamma, \Phi)$ given in Figure 1.1 (as well as Figure 4.3).

However, combinatorial approximations need not to be polynomial. Consider the sequence of infinite graphs $(\widehat{\Gamma}_n)$ depicted in Figure 4.7. For each n, we can similarly define a map $\widehat{\Phi}_n : \widehat{\Gamma}_n \to \mathcal{R}$ such that $\widehat{\Delta}_n = (P_{G_2}, \mathcal{R}, \widehat{\Gamma}_n, \widehat{\Phi}_n)$ is a dynamically admissible quadruple. These maps $\widehat{\Phi}_n$ can be constructed so that they have only finitely many critical points and asymptotic tracts (i.e., unbounded faces that map to a bounded face of $\mathcal{R}$). In this special case, a theorem of Nevanlinna ([Nev32]) shows that $\widehat{\Delta}_n$ is parabolic (see also [Cui21, Theorem 4.1]). Then $\widehat{\Delta}_n$ determines a transcendental Thurston map $\widehat{f}_n$ such that the sequence $(\widehat{f}_n)$ converges combinatorially to $G_2 : (\mathbb{C}, P_{G_2}) \circlearrowleft$ as n tends to ∞.

There is also no canonical choice for a sequence of polynomial Thurston maps converging combinatorially to a given transcendental Thurston map. For instance, the sequence of polynomial Thurston maps illustrated in Figure 4.8 (with respect to the rose graph showed at the top of Figure 4.3) converges combinatorially the map $G_1 \colon (\mathbb{C}, P_{G_1}) \circlearrowleft$ from Example II.16, which realizes dynamically admissible quadruple Δ_{G_1} as in Figure 4.3. However, the property of Remark IV.16 cannot be satisfied for these combinatorial approximations.

CHAPTER V
Dynamical Approximations

The main goal of this chapter is to prove Theorem V.9, which is a stronger version of Theorem I.2. To do this, we first construct combinatorial polynomial approximations, and then use upgrade those to analytic approximations.

We will use several combinatorial and topological properties of locally uniform convergence of sequences in $\mathcal{S}$. In particular, we develop techniques of combinatorial nature for finding the limit of a sequence of maps (f_n) in $\mathcal{S}$, where $|S_{f_n}|$ is constant (see Theorem V.7). We refer to Appendix A.1 for some of the notation used in this Chapter. We will be using several properties of holomorphic covering maps, so we will start by exploring these.

V.1: Properties of holomorphic covering maps

We are mainly interested in convergence conditions for sequences of maps whose domains vary, and in the behavior of lifts of loops under every map in a converging sequence.

Definition V.1. Let X be an oriented topological surface. Given a collection $\mathcal{U} = \{U_j\}_{j \in I}$ of open subsets of X, we define the kernel of $\mathcal{U}$, denoted $\ker(\mathcal{U})$ as the set of points $x \in X$ with an open neighborhood V such that $V \in U_j$ for all but finitely many $j \in I$.

Given a topological surface Y and a sequence of continuous maps $g_n : \mathrm{Dom}(g_n) \to \mathrm{Rg}(g_n)$ with $\mathrm{Dom}(g_n) \subset X$ and $\mathrm{Rg}(g_n) \subset Y$ for all $n \in \mathbb{N}$, we say that the sequence (g_n) converges locally uniformly on $U \subset X$ if for every $x \in U$, there exists a neighborhood W of x such that $W \subset \mathrm{Dom}(g_n)$ eventually, and a continuous map $g : W \to Y$ such that (g_n) converges uniformly on W to g.

The kernel of $\{\mathrm{Dom}(g_n)\}_{n \in \mathbb{N}}$ is a natural space where we can hope for the sequence (g_n) to converge. Bargmann proved that with some assumptions of regularity, the desired convergence occurs on some connected component of $\ker(\{\mathrm{Dom}(g_n)\}_{n \in \mathbb{N}})$:

Proposition V.2 ([Bar01, Theorem 1])**.** *Let $g_n : \mathrm{Dom}(g_n) \to \mathrm{Rg}(g_n)$ be a sequence of holomorphic covering maps. Suppose that (g_n) converges locally uniformly in a neighborhood* *of a point $z \in \mathbb{C}$ to a limiting function that is not locally constant at z, and X is the connected component of* $\ker(\{\mathrm{Dom}(g_n)\}_{n\in\mathbb{N}})$ *that contains z.*

Then, letting Y be the connected component of $\ker(\{\mathrm{Rg}(g_n)\}_{n\in\mathbb{N}})$ *containing* $w = \lim_{n\to\infty} g_n(z)$, *there exists a holomorphic covering map $g : X \to Y$ such that (g_n) converges to g locally uniformly on X.*

For a converging sequence of holomorphic covers, sequences consisting of certain local inverse functions also exhibit controlled behavior, as we show below.

Proposition V.3. *Let (g_n) be a sequence of holomorphic coverings that converge locally uniformly on $X \subset \mathbb{C}$ to a holomorphic covering $g : X \to Y$. Let x be a point in X and $V \subset Y$ a bounded Jordan domain containing $g(x) \in V$, such that $\overline{V} \subset Y$.*

1. *Let U be the connected component of $g^{-1}(V)$, and for every $n \in \mathbb{N}$, U_n be the conneced component of $g_n^{-1}(V)$ containing x. If $\varphi_{V,x} : V \longrightarrow U$ and $\varphi_{V,x,n} : V \longrightarrow U_n$ are the inverses of $g|U$ and $g_n|U_n$ respectively, then $\varphi_{V,x,n} \to \varphi_{V,x}$ uniformly.*

2. *There exists a neighborhood W of x such that $W \subset \mathrm{Dom}(g_n)$ and g_n maps W injectively into V for all sufficiently large n.*

For proving the above, we will use the following result of Kisaka:

Proposition V.4 ([Kis95, Theorem 1])**.** *Let $g_n, n \in \mathbb{N}$ and g be entire maps in $\mathcal{S}$ such that (g_n) converges to g locally uniformly on $\mathbb{C}$. If $w \in S_g$, then for some $N \in \mathbb{N}$ and some sequence of points $w_n \in S_{g_n}$, $n \geqslant N$, we have $\lim_{n\to\infty} w_n = w$.*

Proof of Proposition V.3. We first prove item (1). Choose a bounded Jordan domain V' such that $\overline{V} \subset V'$ and $\overline{V'} \subset Y$. As before, we have the local inverse $\varphi_{V',x}\colon V' \to U'$ of g at $g(x)$, where U' is the connected component of $g^{-1}(V')$ containing x. Note that U' is again a Jordan domain, and $U' \backslash \overline{U}$ is an open annulus. Let $\gamma \subset U' \backslash \overline{U}$ be an essential simple closed curve that separates $\mathbb{C} \backslash U'$ from $\overline{U}$. Let U'' be the bounded component of $\mathbb{C} \backslash \gamma$. Note that $\overline{U} \subset U''$ and $\overline{U''} \subset U'$.

We claim that $U_n \subset U''$ for all sufficiently large n. By the injectivity of g on U', the Hausdorff distance $d(g(\gamma), V)$ is strictly greater than zero. Since (g_n) converges uniformly to g on γ, we also have $d(g_n(\gamma), V) > 0$ for sufficiently large n. But this implies that $\gamma \cap U_n = \varnothing$, and since U_n is connected and contains x, we must have $U_n \subset U''$.

Now suppose the sequence $(\varphi_{V,x,n})$ does not converge uniformly to $\varphi_{V,x}$. Then there exists a real number $\varepsilon > 0$ such that for infinitely many n, there exists a point $y_n \in V$ with $|\varphi_{V,x,n}(y_n) - \varphi_{V,x}(y_n)| > \varepsilon$. Without loss of generality, we can assume that (y_n) converges to a point $y \in \overline{V}$. Since $U_n \subset U''$ eventually, the sequence $(\varphi_{V,x,n}(y_n))$ converges to a point $z \in \overline{U''}$, implying that $|z - \varphi_{V',x}(y)| \geqslant \varepsilon$. Since g is injective on $\overline{U''}$, we have $g(z) \neq g(\varphi_{V',x}(y))$, but at the same time,

$$g(z) = \lim_{n\to\infty} g_n(\varphi_{V,x,n}(y_n)) = \lim_{n\to\infty} y_n = y = g(\varphi_{V',x}(y))$$

resulting in a contradiction. This proves item (1).

Let $\lambda = |\varphi'_{V,x}(g(x))|$. For n sufficiently large, we have $\lambda_n = |\varphi'_{V,x,n}(g(x))| > \frac{\lambda}{2}$. By the Koebe 1/4-Theorem, it follows that $D(x, \frac{\lambda}{4}) \subset U$, and that $D(x, \frac{\lambda}{8}) \subset D(x, \frac{\lambda_n}{4}) \subset U_n$. Clearly, $W := D(x, \frac{\lambda}{8})$ satisfies the requirements of item (2).

□

Lastly, we investigate the behavior of lifts of closed loops under a sequence of converging holomorphic covers.

Proposition V.5. *Let (g_n) be a sequence of holomorphic coverings that converge locally uniformly on $X \subset \mathbb{C}$ to a holomorphic covering $g\colon X \to Y$. Further suppose there exist points $x_0 \in X$, $y_0 \in Y$ such that $g_n(x_0) = g(x_0) = y_0$ for all $n \in \mathbb{N}$.*

Then for any loop $\alpha \subset Y$ based at y_0, we have

1. *$\alpha \subset \mathrm{Rg}(g_n)$ eventually, and thus, $\beta_n = \alpha\uparrow(g_n, x_0)$ is eventually well-defined,*
2. *assuming that the β_n and $\beta := \alpha\uparrow(g, x_0)$ are parametrized by $\mathbb{I}$, with $\beta_n(0) = \beta(0) = x_0$ and $g_n(\beta_n(t)) = g(\beta(t))$ for all n and forall $t \in \mathbb{I}$, then (β_n) converges to β uniformly on $\mathbb{I}$, and*
3. *for n sufficiently large, the lifts β_n and β have the same closing behavior (refer to Definition A.2)*

Proof. We can infer from Proposition V.3 that for every $y \in Y$, there exists an open neighborhood V of y in Y such that $V \subset \mathrm{Rg}(g_n)$ eventually. Thus, $Y \subset \ker(\{\mathrm{Rg}(g_n)\}_{n\in\mathbb{N}})$. This in turn implies that some open neighborhood of α in Y is contained in $\mathrm{Rg}(g_n)$ eventually, proving item (1).

To show item (2), we choose bounded Jordan domains $U_1, U_2, \ldots, U_k$ in X covering β and a strictly increasing finite sequence $t_0 := 0, t_1, t_2, \ldots, t_{k-1}, t_k := 1$ of points in $\mathbb{I}$ so that

- $g|\overline{U_j}$ is injective for each $j \in \{1, 2, \cdots, k\}$;
- $\beta(I_j) \subset U_j$ for each $j = 1, 2, \ldots, k$, where $I_j = [t_{j-1}, t_j]$.

Let $u_j := \beta(t_j)$ for each $j \in \{0, 1, \ldots, k-1\}$ and $V_j := g(U_j)$ for each $j \in \{1, 2, \cdots, k\}$. Assume that n is large enough so that $\overline{U_j} \subset \mathrm{Dom}(g_n)$ and $\overline{V_j} \subset \mathrm{Rg}(g_n)$ for each j in the set $\{1, 2, \cdots, k\}$.

We will use induction on $\ell \in \{0, 1, \cdots, k\}$ to see that (β_n) converges to β uniformly on $[0, t_\ell]$. The base case $\ell = 0$ is obvious since $\beta_n(0) = \beta(0) = z_0$. Now we show that if (β_n) converges to β uniformly on $[0, t_\ell]$, then it does so uniformly on $[0, t_{\ell+1}]$. Indeed, since $\beta_n(t_\ell) \to \beta(t_\ell) = u_\ell$, due to Proposition V.3 we have $\varphi_{V_\ell, \beta_n(t_\ell), n} = \varphi_{V_\ell, v_\ell, n}$ for sufficiently large n. Thus,

$$\beta | I_{\ell+1} = \varphi_{U_l, v_l}(\alpha | I_{\ell+1})$$
$$\beta_n | I_{\ell+1} = \varphi_{U_l, \beta_n(t_l), n}(\alpha | I_{\ell+1}) = \varphi_{U_l, v_l, n}(\alpha | I_{\ell+1})$$

Item (3) follows from the two cases below:

- if β is a closed curve, by item (2), $\beta_n(1) \to x_0$ as $n \to \infty$. By Proposition V.3, there exists a neighbourhood of U of x_0 such that for large enough n, $W \subset \mathrm{Dom}(g_n)$ and $g_n | W$ is injective. Therefore, we must have $\beta_n(0) = \beta_n(1)$ for sufficiently large n.
- if β is not a closed curve, by item (2), we have $\beta_n(1) \to \beta(1) \neq x_0$. Thus, for all n sufficiently large, we have $x_0 = \beta_n(0) \neq \beta_n(1)$.

□

V.2: Convergence properties of entire maps in class $\mathcal{S}$

We will now use the discussions in the previous section to establish conditions under which maps in class $\mathcal{S}$ converge. First, however, we are interested in observing the behavior of lifts of loops under maps in class $\mathcal{S}$, all with the same number of singular values, which also converge to a map in class $\mathcal{S}$.

Proposition V.6. *Let $g_n, n \in \mathbb{N}$ and g be entire maps in class $\mathcal{S}$ of finite type such that $|S_{g_n}| = |S_g|$ for all $n \in \mathbb{N}$, and $g_n \to g$ locally uniformly on $\mathbb{C}$.*

Let $\gamma \subset \mathbb{C} \backslash S_g$ be a simple closed curve and $\widetilde{\gamma}$ be a connected component of $g^{-1}(\gamma)$. Then for every $z \in \widetilde{\gamma}$ there exists $\varepsilon = \varepsilon(z, \gamma) > 0$ such that for all n sufficiently large, there is a unique connected component $\widetilde{\gamma}_n$ of $g_n^{-1}(\gamma)$ satisfying $d(z, \widetilde{\gamma}_n) < \varepsilon$. Moreover,

1. *if $\deg(g|\widetilde{\gamma})$ is finite (i.e., $\widetilde{\gamma}$ is a simple closed curve), then for any $\delta > 0$, the following hold true for all n large enough:*
    - *$\widetilde{\gamma}_n$ is a simple closed curve;*

- $\deg(g|\widetilde{\gamma}_n) = \deg(g|\widetilde{\gamma})$;
- $\widetilde{\gamma}_n \subset N_\delta(\widetilde{\gamma})$.

2. *if* $\deg(g|\widetilde{\gamma})$ *is infinite (i.e.,* $\widetilde{\gamma}$ *is an unbounded curve), then* $\deg(g|\widetilde{\gamma}_n) \to \infty$ *as* $n \to \infty$, *and for every bounded Jordan disk* $D \subset \mathbb{C}$ *and every* $\varepsilon > 0$, *we have* $\widetilde{\gamma}_n \cap D \subset N_\varepsilon(\widetilde{\gamma} \cap D)$ *for large enough* n.

Proof. Since $|S_g| = |S_{g_n}|$ for all $n \in \mathbb{N}$, Proposition V.4 implies that $S_{g_n} \to S_g$ in the sense of Hausdorff. Then the holomorphic coverings $g_n|\mathbb{C}\backslash g_n^{-1}(S_{g_n})$ converge to the holomorphic covering $g|\mathbb{C}\backslash g^{-1}(S_g)$ locally uniformly on $\mathbb{C}\backslash g^{-1}(S_g)$. The statement then follows from Proposition V.5 applied to the maps $g_n|\mathbb{C}\backslash g_n^{-1}(S_{g_n})$. □

We are now ready to establish equivalent conditions for convergence in class $\mathcal{S}$, under certain assumptions on the singular value sets and a normality condition.

Theorem V.7. *Let* $g_n, n \in \mathbb{N}$ *and* g *be entire maps in class* $\mathcal{S}$, *and* $B_n \supset S_{g_n}$ *and* $B \supset S_g$ *be finite subsets of* $\mathbb{C}$. *Further assume that* $|B_n| = |B|$ *for all* $n \in \mathbb{N}$ *and* $B_n \to B$ *in the sense of Hausdorff.*

Let $z_0, w_0 \in \mathbb{C}$ *be points such that* $z_0 \notin B \cup \bigcup_{n\in\mathbb{N}} B_n$, *and* $g(z_0) = g_n(z_0) = w_0$ *for all* $n \in \mathbb{N}$. *Then* (g_n) *converges locally uniformly to* g *if and only if the following conditions hold:*

1. $\lim_{n\to\infty} g_n'(z_0) = g'(z_0)$;
2. *for any loop* $\alpha \subset \mathbb{C}\backslash B$ *based at* w_0, *the lifts* $\alpha\uparrow(g_n, z_0)$ *eventually have the same closing behavior as* $\alpha\uparrow(g, x)$.

Proof. ($\Longrightarrow$): If $g_n \to g$ locally uniformly, then condition (1) is obvious and condition (2) easily follows from Proposition V.5 applied to the sequence $g_n|\mathbb{C}\backslash g_n^{-1}(B_n)$ converging locally uniformly on $\mathbb{C}\backslash g^{-1}(B)$ to $g|\mathbb{C}\backslash g^{-1}(B)$.

($\Longleftarrow$): Now suppose that conditions (1) and (2) are satisfied. It suffices to show that any arbitrary subsequence (g_{n_k}) of (g_n) contains a further subsequence converging locally uniformly to g. For the sake of simplicity, we will relabel (g_{n_k}) as (g_n).

Claim 1. There exists an open neighborhood U of z_0 and a subsequence (g_{n_k}) of (g_n) converging uniformly on U to a limiting function that is not locally constant at z_0.

Proof of Claim 1. Let $\mathbb{D}(w_0, r)$ be a disk contained in $\mathbb{C}\backslash B_n$ for all sufficiently large n, and U_n be the connected component of $g_n^{-1}(\mathbb{D}(w_0, r))$ containing z_0. By Proposition II.7, g_n maps U_n biholomorphically to $\mathbb{D}(w_0, r)$ and has an inverse $\varphi_n\colon \mathbb{D}(w_0, r) \to U_n$ satisfying $\varphi_n(w_0) = z_0$.

Note that $\varphi_n'(w_0) = 1/g_n'(z_0)$. Since $g_n'(z_0) \to g'(z_0) \neq 0$, there exists $\lambda > 0$ such that $|\varphi_n'(w_0)| > \lambda$ for every $n \in \mathbb{N}$. By the Koebe 1/4-Theorem, the disk $W = \mathbb{D}(z_0, r\lambda/4)$ is contained U_n for each n. In particular, the map g_n is injective on W for all n sufficiently large.

Since $g_n(W) \subset \mathbb{D}(w_0, r)$, by Montel's Theorem, $\{g_n|W\}_{n\in\mathbb{N}}$ is a normal family. Thus we extract a converging subsequence $(g_{n_k}|W)$ from $(g_n|W)$. Clearly the limiting function for the maps g_{n_k} cannot be locally constant at z_0 since $\lim_{n\to\infty} g_n'(z_0) = g'(z_0) \neq 0$. ■

We again relabel the converging subsequence (g_{n_k}) above as (g_n). Now apply Proposition V.2 to the sequence of maps $h_n = g_n|\mathbb{C}\backslash g_n^{-1}(B_n)$: letting $X \subset \mathbb{C}$ be the connected component of $\ker(\{\mathbb{C}\backslash g_n^{-1}(B_n)\}_{n\in\mathbb{N}})$ that contains z_0, we see that (h_n) converges locally uniformly on X to a holomorphic covering map $h\colon X \to \mathbb{C}\backslash B$. We will now show that $g|X \equiv h$. To begin with, we observe that

$$h_*\pi_1(X, z_0) = g_*\pi_1(\mathbb{C}\backslash g^{-1}(B), z_0) \subset \pi_1(\mathbb{C}\backslash B, w_0),$$

following from condition (2) above and Proposition V.5. By the classical theory of covering maps, there exists a biholomorphism $\varphi\colon \mathbb{C}\backslash g^{-1}(B) \to X$ with $\varphi'(z_0) = 1$ such that the following diagram commutes:

$$\begin{array}{ccc} (\mathbb{C}\backslash g^{-1}(B), z_0) & \xrightarrow{\varphi} & (X, z_0) \\ & \searrow^{g} & \downarrow h \\ & & (\mathbb{C}\backslash B, w_0) \end{array}$$

Claim 2. The map φ extends to a Möbius transformation of $\widehat{\mathbb{C}}$.

Proof of Claim 2. As $g^{-1}(B)$ is a discrete set in $\mathbb{C}$ and φ is injective, every point in $g^{-1}(B)$ is a removable singularity of φ, considered as a map to $\widehat{\mathbb{C}}$. It follows that φ extends to a map from $\mathbb{C}$ to $\widehat{\mathbb{C}}$ which can, moreover, be shown to be injective. Since $\varphi(\mathbb{C})$ is conformally equivalent to $\mathbb{C}$, the set $\varphi(\mathbb{C})$ is obtained by removing a single point from $\widehat{\mathbb{C}}$. This then means that φ extends to an automorphism of $\widehat{\mathbb{C}}$. ■

Claim 2 also implies that X is obtained from $\widehat{\mathbb{C}}$ by removing countably many points which have at most one accumulation point, namely $\varphi(\infty)$.

Claim 3. The map φ equals $\mathrm{id}_{\widehat{\mathbb{C}}}$.

Proof of Claim 3. Let us first prove that $\varphi(\infty) = \infty$. Suppose that $\varphi(\infty) = z \in \mathbb{C}$. There exists a compact set $K \subset \mathbb{C}$ such that $z \in \mathrm{int}(K)$ and $\partial K \subset X$. Note that $g_n \to h$ uniformly on ∂K.

Let $m \in (0, +\infty)$ be the maximum of $|h|$ on ∂K. Then by the maximum modulus principle, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$ and $z \in K$, we have $|g_n(z)| \leqslant m + \varepsilon$. Thus, $|h|$ is bounded by m on $K \cap X$.

On the other hand, one can find a sequence $(z_n) \subset \varphi^{-1}(K) \backslash g^{-1}(B)$ such that $(g(z_n)) \to \infty$. However, as $\varphi(z_n) \in X \cap K$ for all n, $(\varphi(z_n))$ converges to z. This means that $|g(z_n)| = |h(\varphi(z_n))|$ is bounded by m for all $n \in \mathbb{N}$, leading to a contradiction. We have thus shown that $\varphi(\infty) = \infty$.

Thus, $\varphi|\mathbb{C}\colon \mathbb{C} \to \mathbb{C}$ is an affine map satisfying $\varphi(z_0) = z_0$ and $\varphi'(z_0) = 1$, which implies that $\varphi \equiv \mathrm{id}_{\hat{\mathbb{C}}}$.

■

Claim 3 implies that g coincides with h on $X = \mathbb{C} \backslash g^{-1}(B)$. In particular, (g_n) converges locally uniformly on $\mathbb{C} \backslash g^{-1}(B)$ to g. This suffices to conclude that (g_n) converges locally uniformly on $\mathbb{C}$ to g.

□

V.3: Dynamical approximations

In this section, we establish Theorem I.2. We will use the following fact multiple times:

Proposition V.8. *Let $\varphi_n \in \mathrm{Homeo}^+(\mathbb{R}^2), n \in \mathbb{N}$ and $\varphi_n \in \mathrm{Homeo}^+(\mathbb{R}^2)$ be such that $\varphi_n \to \varphi$ uniformly on compact subsets of $\mathbb{R}^2 \backslash X$ for some discrete set X. Then sequence $\varphi_n \to \varphi$ locally uniformly on $\mathbb{R}^2$.*

Proof. Given $x \in X$, it suffices to show $\varphi_n \to \varphi$ locally uniformly at x. Let γ_r be a loop such that the bounded component D_r of $\mathbb{R}^2 \backslash \gamma$ contains x but no other point of X, and $\varphi(\gamma_r) \subset \mathbb{D}(\varphi(x), r)$. Since $\varphi_n(\gamma_r) \to \varphi(\gamma_r)$ uniformly, we must have $\varphi_n(\gamma_r) \subset \mathbb{D}(\varphi(x), r)$ for all n sufficiently large. But this means $\varphi_n(D_r) \subset \mathbb{D}(\varphi(x), r)$ for all n sufficiently large. Therefore, for every $z \in D_r$,

$$|\varphi(z) - \varphi_n(z)| \leqslant |\varphi(z) - \varphi(x)| + |\varphi(x) - \varphi_n(z)| < 2r.$$

This shows $\varphi_n \to \varphi$ locally uniformly at x. □

Theorem V.9. *Let $f_n\colon (\mathbb{R}^2, A) \circlearrowleft$, $n \in \mathbb{N}$ and $f\colon (\mathbb{R}^2, A) \circlearrowleft$ be Thurston maps such that the sequence (f_n) converges combinatorially to f. If f is realized as a postsingularly finite entire function $g\colon (\mathbb{C}, B) \circlearrowleft$, then there exists a sequence of postsingularly finite entire maps $g_n\colon (\mathbb{R}^2, B_n) \circlearrowleft$ such that*

1. $g_n\colon (\mathbb{C}, B_n) \circlearrowleft$ *is Thurston equivalent to* $f_n\colon (\mathbb{R}^2, A) \circlearrowleft$ *for sufficiently large* n*;*

2. the sequence (B_n) *converges to* B *in the sense of Hausdorff topology;*

3. g_n *converges locally uniformly to* g *on* $\mathbb{C}$*.*

Proof. By proposition III.13, we know that f_n's are realized eventually as holomorphic PSF maps g_n. If $|A| = 1$, by Proposition II.14, there exists $d \geqslant 2$ such that $g(z) = z^d$, and $g_n(z) = z^d$ for all n sufficiently large, and we are done.

Now suppose $|A| \geqslant 2$. Without loss of generality, assume that (f_n) converges topologically to f, and that there exist points $b \in \mathbb{R}^2 \backslash f^{-1}(A)$ and $t \in \mathbb{R}^2 \backslash A$ such that $f(b) = f_n(b) = t$ for all $n \in \mathbb{N}$. Since $f \simeq_{\text{comb}} g$, it follows from Proposition II.28 that there exists a unique point $\tau \in \mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$ such that $\tau = [\varphi] = [\psi]$, $g = \varphi \circ f \circ \psi^{-1}$, and $B = \varphi(A)$, where $\varphi\colon \mathbb{R}^2 \to \mathbb{C}$ and $\psi\colon \mathbb{R}^2 \to \mathbb{C}$ are orientation-preserving homeomorphisms isotopic to each other rel. $A \cup \{\infty\}$.

By Corollary III.13, there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$ the map σ_{f_n} has a unique fixed point $\tau_n \in \mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$ and, moreover, the sequence (τ_n) converges to τ. We may assume without loss of generality that $N = 1$.

For each $n \in \mathbb{N}$, we pick homeomorphisms $\varphi_n, \psi_n \in \tau_n$ isotopic rel. $A \cup \{\infty\}$, such that $\varphi_n(\infty) = \psi_n(\infty)$, such that the map $h_n := \varphi_n \circ f_n \circ \psi_n^{-1}$ is entire. Note that the set $B_n := \varphi_n(A)$ contains S_{h_n}. By Proposition II.22, the φ_n and ψ_n can be chosen to satisfy the following conditions for all $n \in \mathbb{N}$:

- $\varphi_n \circ \varphi^{-1}\colon \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ is quasiconformal, with $\lim_{n\to\infty} K(\varphi_n \circ \varphi^{-1}) = 1$;
- $\varphi_n(t) = \varphi(t)$ and $\varphi_n(y) = \varphi(y)$ for some arbitrary point $y \neq t$;
- $\psi_n(b) = \psi(b)$ and $h_n'(\psi_n(b)) = g'(\psi(b))$ for every $n \in \mathbb{N}$.

First note that for each n, the homeomorphism $\varphi_n \circ \varphi^{-1}$ fixes three distinct points $\infty, \varphi(t), \varphi(y) \in \widehat{\mathbb{C}}$. Therefore, by Proposition II.5, $\varphi_n \circ \varphi^{-1} \to \mathrm{id}_{\widehat{\mathbb{C}}}$ locally uniformly on compact subsets of $\widehat{\mathbb{C}} \backslash \{\varphi(t), \varphi(y)\}$ as $n \to \infty$. By Proposition V.8, $\varphi_n \to \varphi$ locally uniformly on $\mathbb{R}^2$.

Claim 1. The sequence (h_n) converges to g locally uniformly on $\mathbb{C}$.

Proof of Claim 1. We prove that the maps g and h_n, $n \in \mathbb{N}$ satisfy all the conditions of Theorem V.7 with respect to the points $z_0 := \psi(b) = \psi_n(b)$, $w_0 := \varphi(t) = \varphi_n(t)$, and the sets B_n and B.

Next we see that $B_n \supset S_{h_n}$ converges to $B \supset S_g$ in the Hausdorff topology of $\mathbb{C}$ since $\varphi_n \to \varphi$ as $n \to \infty$. Due to our choices of φ_n and ψ_n, the equalities $h_n'(z_0) = g'(z_0)$ and

$h_n(z_0) = g(z_0) = w_0$ are satisfied, and we have $w_0 \notin B \cup \bigcup_{n\in\mathbb{N}} B_n$. Hence, condition (1) of Theorem V.7 is satisfied for the maps h_n.

The condition (2) required in Theorem V.7 easily follows from the topological convergence of (f_n) to f and the locally uniform convergence of (φ_n) to φ.

■

With this choice of φ_n and ψ_n, the maps h_n are not necessarily postsingularly finite. We will now construct the required maps g_n from the maps h_n by showing that ψ_n and φ_n have controlled behavior.

Claim 2. The maps $\psi_n \to \psi$ locally uniformly on $\mathbb{R}^2$.

Proof of Claim 2. Let $x \in \mathbb{R}^2$ be a regular point (i.e., not a critical point) of f such that $\psi_n(x) \to \psi(x)$ as $n \to \infty$ (the point b, for instance, satisfies this property). Then there exists a Jordan domain V containing x such that $f|V$ is injective. Assume without loss of generality that $\overline{V}$ is compact. For sufficiently large n, we have $f_n|V = f|V$. Hence, the maps $g|\psi(V)$ and $h_n|\psi_n(V)$ are injective, with inverses $\varphi_{U,\psi(x)} \colon U \to \psi(V)$ and $\varphi_{U_n,\psi_n(x),n} \colon U_n \to \psi_n(V)$, respectively, where $U := h(\psi(V)) = \varphi(f(V))$ and $U_n := h_n(\psi_n(V)) = \varphi_n(f_n(V)) = \varphi_n(f(V))$.

Since $h_n \to g$, using Proposition V.3, we see that the sequence $(\varphi_{U_n,\psi_n(x),n})_n$ converges to $\varphi_{U,\psi(x)}$ uniformly on U. Finally, we have

$$\begin{aligned}\psi|V &= \varphi_{U,\psi(x)} \circ (\varphi|f(V)) \circ (f|V),\\ \psi_n|V &= \varphi_{U_n,\psi_n(x),n} \circ (\varphi_n|f_n(V)) \circ (f_n|V)\\ &= \varphi_{U_n,\psi_n(x),n} \circ (\varphi_n|f(V)) \circ (f|V).\end{aligned}$$

Since $\varphi_n|f(V) \to \varphi|f(V)$ uniformly, we have $\psi_n \to \psi$ uniformly on V.

Let U_f be the set of regular points of f, and note that this is open in $\mathbb{R}^2$. We will first show for all $x \in U_f$, we have $\psi_n(x) \to \psi(x)$. Let $D = \{x \in U_f | \psi_n(x) \to \psi(x)\}$. We know that $D \neq \emptyset$. If $D \neq U_f$, then there exists $y \in U_f \cap \partial D$. Choose a bounded Jordan domain W containing y on which f is injective. By the above discussion, since W contains a point $z \in D$, we have $\psi_n \to \psi$ uniformly on W. This implies that for any such W, the open set $W \cap U_f$ is contained in D, which contradicts the fact that $y \in \partial D$. By this discussion, we also note that $\psi_n \to \psi$ uniformly on compact subsets of U_f. Since the $\mathbb{R}^2 \backslash f^{-1}(A) \subset U_f$ and $f^{-1}(A)$ is discrete, by Proposition V.8, the maps $\psi_n \to \psi$ locally uniformly on $\mathbb{R}^2$. ■

Since $[\varphi_n] = [\psi_n]$, there exists an affine map M_n such that $\varphi_n|A$ and $M_n \circ \psi_n|A$ coincide. Then $g_n := h_n \circ M_n^{-1} \colon (\mathbb{C}, B_n) \circlearrowleft$ is a postsingularly finite entire map. It is sufficient to show that (M_n) converges locally uniformly to $\mathrm{id}_{\mathbb{C}}$ to prove that the sequence (g_n) converges locally uniformly to g.

Note that M_n is an affine map satisfying $M_n(\psi_n(a)) = \varphi_n(a)$ for every $n \in \mathbb{N}$ and $a \in A$. Now the desired statement follows from the fact that for every $a \in A$, $\psi_n(a) \to \psi(a) = \varphi(a)$ and $\varphi_n(a) \to \varphi(a) = \psi(a)$ as $n \to \infty$.

□

The next corollary is an immediate consequence of the proof of Theorem V.9.

Corollary V.10. *Suppose that we are in the setting of Theorem V.9. Assume that the sequence (f_n) converges topologically to the map f and $g = \varphi \circ f \circ \psi^{-1}$, where $\varphi\colon \mathbb{R}^2 \to \mathbb{C}$ and $\psi\colon \mathbb{R}^2 \to \mathbb{C}$ are orientation-preserving homeomorphisms isotopic to each other rel. A. Then for sufficiently large n there exist orientation-preserving homeomorphisms $\varphi_n\colon \mathbb{R}^2 \to \mathbb{C}$ and $\psi_n\colon \mathbb{R}^2 \to \mathbb{C}$ such that $g_n = \varphi_n \circ f_n \circ \psi_n^{-1}$, $\varphi_n \sim \psi_n$ rel. A, $\varphi_n \to \varphi$ and $\psi_n \to \psi$ locally uniformly as $n \to \infty$.*

The following result establishes Theorem I.2 and easily follows from Proposition IV.15 and Theorem V.9.

Corollary V.11. *Let g be a postsingularly finite entire map. Then there exists a sequence of postcritically finite polynomials (g_n) converging locally uniformly to g, such that g_n has the same singular portrait as g for every $n \in \mathbb{N}$.*

Remark V.12. Note that usually there is no canonical choice for a sequence of polynomials (g_n) in Corollary V.11. Constructing different "combinatorial approximations" by Proposition IV.15 and then applying Theorem V.9 lead to different sequences of polynomials.

CHAPTER VI
Persistence of Spiders

We now move on to our special case study involving the exponentials and unicritical polynomials. In this chapter we lay the foundation for the approximation of postsingularly finite exponentials by proving Theorem I.6. We will obtain, for every $n \in \mathbb{N}_{\geqslant 2}$ and $j \in \{0, 1, \cdots, n-1\}$, a monotone increasing map $\mathrm{Jump}_{n,j} : \mathbb{Q}/\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z}$ that preserves spiders and respects landing relations in the set $\mathcal{M}_n$.

Throughout this chapter, fix a degree $n \in \mathbb{N}_{\geqslant 2}$.

VI.1: Construction of $\mathrm{Jump}_{n,j}$

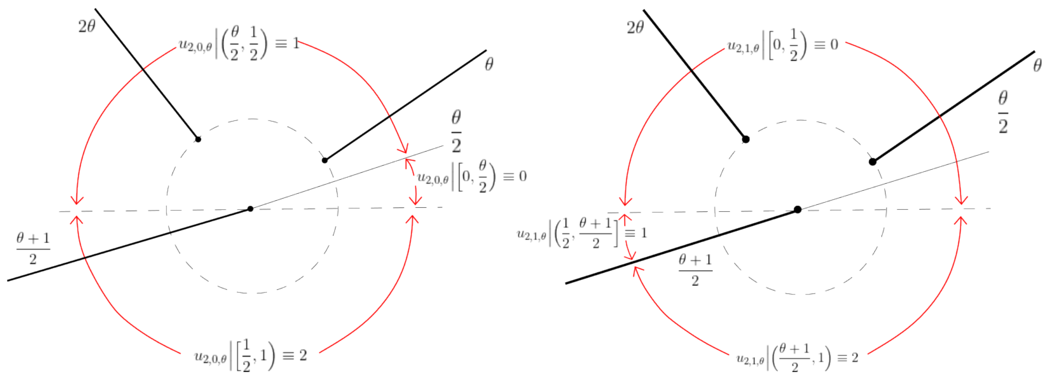
Before we start, we note that for every $\theta \in \mathbb{Q}/\mathbb{Z}$, if θ is strictly pre-periodic under μ_n, then $\mathcal{O}_n(\theta)$ contains no element in $\mu_n^{-1}(\theta)$. However, if θ is $k-$periodic under μ_n, then there exists a unique integer $j =: j_n(\theta) \in \{0, 1, \cdots, n-1\}$ such that $\mu_n^{\circ(k-1)}(\theta) = \frac{\theta + j_n(\theta)}{n}$.
Fix $\theta \in \left[0, \frac{1}{n-1}\right)$. Given an integer $j \in \{0, 1, \cdots, n-1\}$, we define a 'symbol shift' function as follows:

$$u_{n,j,\theta} : \mathbb{R}/\mathbb{Z} \setminus \{(\theta + j)/n\} \longrightarrow \{0, 1, \cdots, n\}$$
$$u_{n,j,\theta}(t) = \begin{cases} m & t \in \left[\frac{m}{n}, \frac{m+1}{n}\right) \text{ for some } m \in \{0, 1, \cdots, j-1\} \\ m+1 & t \in \left[\frac{m}{n}, \frac{m+1}{n}\right) \text{ for some } m \in \{j+1, j+2, \cdots, n-1\} \\ j & t \in \left[\frac{j}{n}, \frac{\theta+j}{n}\right) \\ j+1 & t \in \left(\frac{\theta+j}{n}, \frac{j+1}{n}\right) \end{cases}$$

Additionally, if $\frac{\theta+j}{n} \in \mathcal{O}_n(\theta)$, let

$$u_{n,j,\theta}\left(\frac{\theta + j}{n}\right) = \begin{cases} j & \text{if } \theta \text{ is the smaller angle in a companion pair} \\ j+1 & \text{otherwise} \end{cases}$$

The $u_{n,j,\theta}$ function assigns a symbol to each angle in $\mathcal{O}_n(\theta)$: we first divide $[0, 1)$ into n



(a) The symbol shift function $u_{2,0,\theta}$ (b) The symbol shift function $u_{2,1,\theta}$

Figure 6.1: For $n = 2$ and $\theta \equiv \frac{1}{7} \pmod 1$, the two figures above illustrate the corresponding symbol shift functions. The bold solid lines correspond to angles in $\mathcal{O}_2(\theta)$, and the regular solid lines correspond to angles in $\mu_n^{-1}(\theta)$. The dotted lines represent the angles $0, \frac{1}{2}$. Also note that θ is smaller than its companion angle $\frac{2}{7}$. This comes into play in the definition of $u_{2,1,\theta}$ at $\frac{\theta+1}{2}$.

sub-intervals of the form $\left[\frac{m}{n}, \frac{m+1}{n}\right)$, $m \in \{0, 1, \cdots, n-1\}$. The symbol assigned to an angle t depends on which sub-interval t belongs to. The sub-interval $\left[\frac{j}{n}, \frac{j+1}{n}\right)$ is 'split' at $\frac{\theta+j}{n}$, and the symbols to the left and right of this angle differ by 1. The goal is to push angles in $\left(\frac{\theta+j}{n}, 1\right)$ exactly one sub-interval further. In effect, the map $\mathrm{Jump}_{n,j}$ pushes open a new sector in the spider $S_n^{ext}(\theta)$ (compare Figures 6.2 and 6.4).

Definition VI.1. Given $j \in \{0, 1, \cdots, n-1\}$, the map $\mathrm{Jump}_{n,j}\colon \left[0, \frac{1}{n-1}\right) \cap \mathbb{Q}/\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z}$ is defined as:

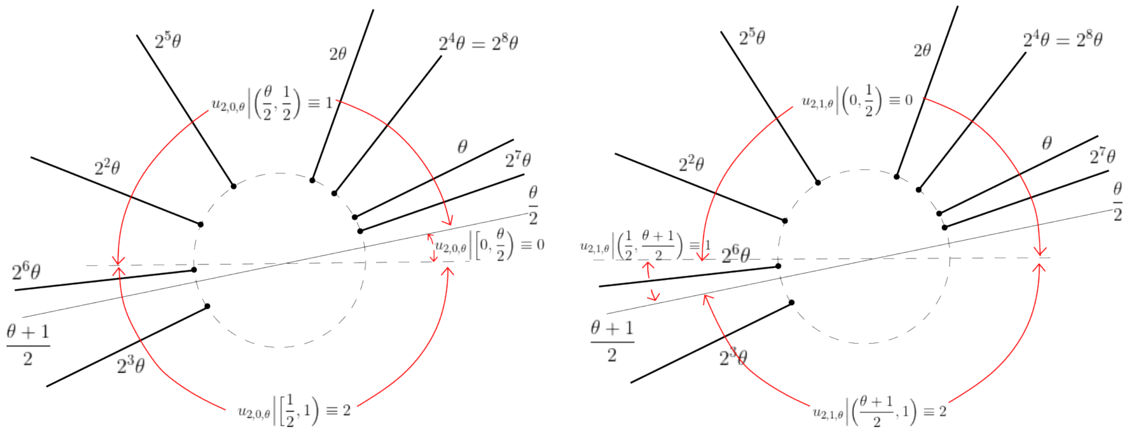
$$\mathrm{Jump}_{n,j}(\theta) \equiv \sum_{m=1}^{\infty} \frac{u_{n,j,\theta} \circ \mu_n^{\circ(m-1)}(\theta)}{(n+1)^m} \pmod 1$$

In other words, the angle $\mathrm{Jump}_{n,j}(\theta)$ has an $(n+1)$-adic expansion $.x_1x_2x_3\cdots$, where $x_m = u_{n,j,\theta} \circ \mu_n^{\circ(m-1)}(\theta)$. For $n = 2$, this defines $\mathrm{Jump}_{n,j}$ on $\mathbb{Q}/\mathbb{Z}$.

For $n > 2$, we extend $\mathrm{Jump}_{n,j}$ to $\mathbb{Q}/\mathbb{Z}$ as follows. Note that for any $\theta' \in \mathbb{Q}/\mathbb{Z}$, there exists a unique $\theta \in \mathbb{Q}/\mathbb{Z} \cap \left[0, \frac{1}{n-1}\right)$ and $m \in \{0, 1, \cdots, n-2\}$ such that $\theta' \equiv \theta + \frac{m}{n-1} \pmod 1$. We set

$$\mathrm{Jump}_{n,j}(\theta') \equiv \mathrm{Jump}_{n,j}(\theta) + \frac{m}{n} \pmod 1$$

Remark VI.2. Let $\theta \in \left[0, \frac{1}{n-1}\right)$ be rational. Suppose there exists $M \in \mathbb{N}$ such that for all



(a) The symbol shift function $u_{2,0,\theta}$ (b) The symbol shift function $u_{2,1,\theta}$

Figure 6.2: Symbol shift functions for $n = 2$ and $\theta \equiv \frac{17}{2^4(2^4-1)} \pmod 1$.

$m \geqslant M$, we have

$$x_m = u_{n,j,\theta} \circ \mu_n^{\circ(m-1)}(\theta) = n$$

Then for all $m \geqslant M$, we have $\mu_n^{\circ(m-1)}(\theta) \in [\frac{n-1}{n}, 1)$. This in turn implies $\mu_n^{\circ(M-1)}(\theta)$ is the angle 0, and thus, $u_{n,j,\theta} \circ \mu_n^{\circ(M-1)}(\theta) = 0$, contradicting our assumption. Thus the $(n+1)-$adic expansion of $\mathrm{Jump}_{n,j}(\theta)$ produced by the symbol shift function does not end in the constant stream $nnnnnn\cdots$.

Example VI.3. Let $n = 2$. For $\theta \equiv \frac{1}{7} \pmod 1 = .\overline{001}$ in base 2, we have, in base 3,

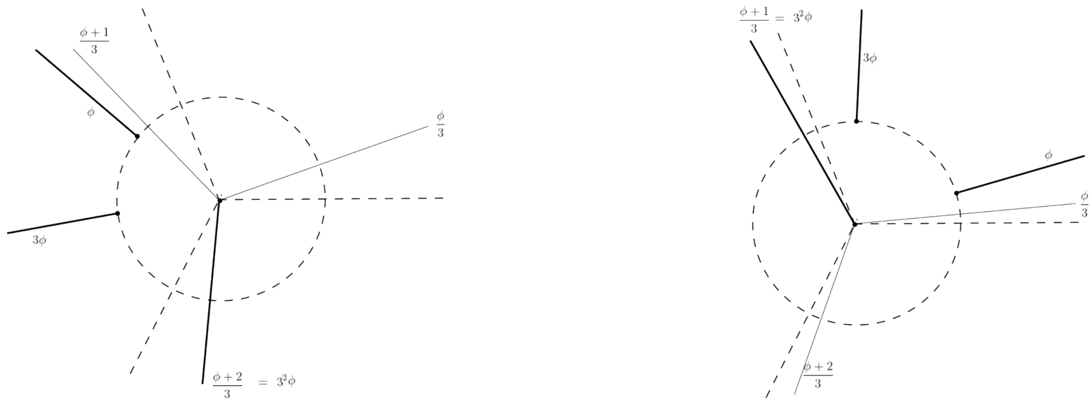
$$\mathrm{Jump}_{2,0}(\theta) = .\overline{112} \equiv \frac{14}{3^3} \pmod 1 \equiv \frac{14}{26} \pmod 1$$
$$\mathrm{Jump}_{2,1}(\theta) = .\overline{001} \equiv \frac{1}{3^3} \pmod 1 \equiv \frac{1}{26} \pmod 1$$

The symbol shift functions $u_{2,0,\theta}$ and $u_{2,1,\theta}$ are illustrated in Figure 6.1.

Example VI.4. Let $n = 2$. The angle $\theta \equiv \frac{17}{2^4(2^4-1)} \pmod 1 \equiv \frac{17}{240} \pmod 1$ is strictly pre-periodic under μ_2, with pre-period 4 and period 4. The sequence $.0001\overline{0010}$ is a $2-$adic expansion for θ.

The angles $\mathrm{Jump}_{2,0}(\theta)$ and $\mathrm{Jump}_{2,1}(\theta)$ are given below in terms of $3-$adic expansions produced by symbol shift:

$$\mathrm{Jump}_{2,0}(\theta) = .1112\overline{1121} \equiv \frac{3323}{3^4(3^4-1)} \pmod 1 \equiv \frac{3323}{6480} \pmod 1$$
$$\mathrm{Jump}_{2,1}(\theta) = .0002\overline{0010} \equiv \frac{163}{3^4(3^4-1)} \pmod 1 \equiv \frac{163}{6480} \pmod 1$$



(a) The standard spider of $\phi = \text{Jump}_{2,0}(\theta) = \frac{14}{3^3-1} = .\overline{112}$ in base 3

(b) The standard spider of $\phi = \text{Jump}_{2,1}(\theta) = \frac{1}{3^3-1} = .\overline{001}$ in base 3

Figure 6.3: The standard degree 3 spiders of $\text{Jump}_{n,0}(\theta)$ and $\text{Jump}_{n,1}(\theta)$, for $n = 2$ and $\theta \equiv \frac{1}{2^3-1} \pmod 1$. The dotted rays indicate the position of the angles $0, \frac{1}{3}$ and $\frac{2}{3}$. Compare with Figure 6.1

See Figure 6.2 for an illustration of the symbol-shift.

VI.2: Monotonicity of $\text{Jump}_{n,j}$

Fix $j \in \{0, 1, \cdots, n-1\}$. In this section we show that $\text{Jump}_{n,j}$ is injective and preserves linear order, when we consider $\mathbb{Q}/\mathbb{Z}$ as a subset of $[0, 1)$.

Proposition VI.5. *Fix $\theta \in \left[0, \frac{1}{n-1}\right)$. Given $s, t \in \mathbb{R}/\mathbb{Z}$ such that $s'_m = u_{n,j,\theta} \circ \mu_n^{\circ(m-1)}(s)$ and $t'_m = u_{n,j,\theta} \circ \mu_n^{\circ(m-1)}(t)$ are defined for all $m \in \mathbb{N}$, let*

$$s' = .s'_1 s'_2 s'_3 \cdots$$
$$t' = .t'_1 t'_2 t'_3 \cdots$$

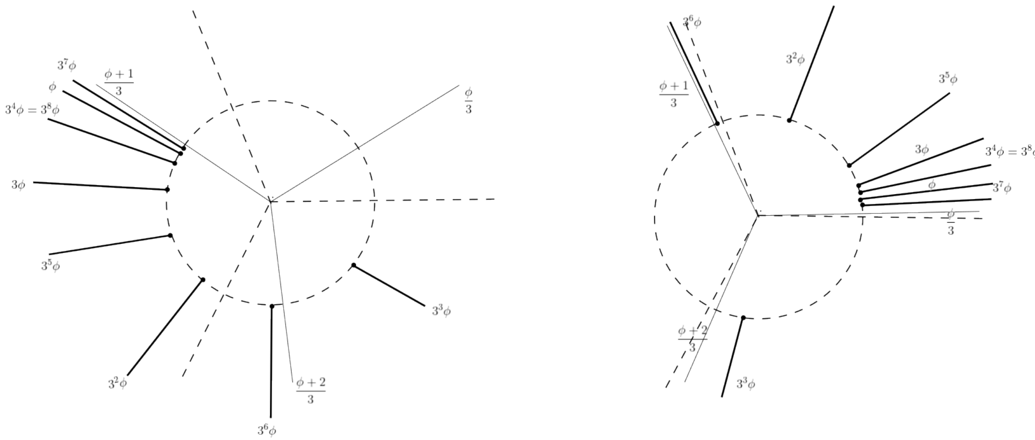
in base $(n+1)$. Then $s < t$ in $[0, 1)$ if and only if $s' < t'$ in $[0, 1)$.

Proof. It is enough to prove one direction, since $s \equiv t \pmod 1$ also implies $s' \equiv t' \pmod 1$.

So assume $s < t$, and choose $n-$adic expansions $s = .s_1 s_2 s_3 \cdots$. and $t = .t_1 t_2 t_3 \cdots$.. so that if either angle is rational, the corresponding expansion does not end in a constant sequence of $(n-1)$'s.

At the first index r where $s_r \neq t_r$, we have

$$s_r < t_r, \text{ implying } s'_r < t'_r$$



(a) The standard spider of $\phi := \text{Jump}_{2,0}(\theta) \equiv \frac{3323}{3^4(3^4-1)} \pmod 1$, equalling $.1112\overline{1121}$ in base 3

(b) The standard spider of $\phi := \text{Jump}_{2,0}(\theta) \equiv \frac{163}{3^4(3^4-1)} \pmod 1$, which equals $.0002\overline{0010}$ in base 3

Figure 6.4: For $n = 2$ and $\theta \equiv \frac{17}{2^4(2^4-1)} \pmod 1$, an illustration of the standard degree 3 spiders of $\text{Jump}_{2,0}(\theta)$ and $\text{Jump}_{1,0}(\theta)$. The dotted rays indicate the position of the angles $0, \frac{1}{3}$ and $\frac{2}{3}$. Compare with Figure 6.2

For all indices $m < r$, by assumption,

$$s_m \leqslant t_m, \text{ and thus, } s'_m \leqslant t'_m$$

This shows that $s' \leqslant t'$. Equality holds if and only if $s'_r + 1 = t'_r$, $s'_m = n$ and $t'_m = 0$ for all $m > r$; however, Remark VI.2 shows that the condition $s'_m = n$ for all $m > r$ can never be true. ☐

Proposition VI.6. *Fix $t \in \mathbb{R}/\mathbb{Z}$ and $j \in \{0, 1, \cdots n-1\}$. If $0 \leqslant \theta < \theta' < \frac{1}{n-1}$ are rational angles such that $t_m = u_{n,j,\theta} \circ \mu_n^{\circ(m-1)}(t)$ and $t'_m = u_{n,j,\theta'} \circ \mu_n^{\circ(m-1)}(t)$ are well-defined for all $m \in \mathbb{N}$, then in base $(n+1)$, we have*

$$.t_1 t_2 t_3 \cdots < .t'_1 t'_2 t'_3 \cdots$$

Proof. The fact that $\theta < \theta'$ implies that $\frac{\theta+j}{n} < \frac{\theta+j'}{n}$. By the definition of $u_{n,j,\theta}$ and $u_{n,j,\theta'}$, we have $t_m \leqslant t'_m$ for all $m \in \mathbb{N}$. ☐

Remark VI.7. Note: for $n > 2$, the statement holds if we assume $0 < \theta < \frac{1}{n-1}$ and $\theta' \equiv \frac{1}{n-1} \pmod 1$.

Proposition VI.8. $\text{Jump}_{n,j}$ *is strictly increasing.*

Proof. It suffices to show the following claim:

Claim. 1. For $m \in \{0, 1, \cdots, n-2\}$, the map $\mathrm{Jump}_{n,j}$ satisfies the formula

$$\mathrm{Jump}_{n,j}\left(\frac{m}{n-1}\right) \equiv \begin{cases} \frac{m+1}{n} \pmod 1 & j = 0 \\ \frac{m}{n} \pmod 1 & j \geqslant 1 \end{cases}$$

2. $\mathrm{Jump}_{n,j}$ is strictly increasing on $\mathbb{Q} \cap \left[0, \frac{1}{n-1}\right)$, and $\mathrm{Jump}_{n,j}\left(\mathbb{Q} \cap \left[0, \frac{1}{n-1}\right)\right) \subset \mathbb{Q}/\mathbb{Z} \cap \left[\mathrm{Jump}_{n,j}(0), \mathrm{Jump}_{n,j}(0) + \frac{1}{n}\right)$.

Together with the definition of $\mathrm{Jump}_{n,j}$, this claim implies that $\mathrm{Jump}_{n,j}$ is strictly increasing on $\mathbb{Q}/\mathbb{Z}$.

Proof of Claim. 1. This is clear from the fact that

$$\mathrm{Jump}_{n,j}(0) \equiv \begin{cases} 0 \pmod 1 & j \geqslant 1 \\ \frac{1}{n} \pmod 1 & j = 0 \end{cases}$$

2. Given rational angles $0 \leqslant \theta < \theta' < \frac{1}{n-1}$, for every $m \in \mathbb{N}$, let $x_m = u_{n,j,\theta'} \circ \mu_m^{\circ(m-1)}(\theta)$. Then by Propositions VI.5 and VI.6, in base $(n+1)$, we have

$$\mathrm{Jump}_{n,j}(\theta) < .x_1x_2x_3\cdots < \mathrm{Jump}_{n,j}(\theta')$$

This shows that $\mathrm{Jump}_{n,j}$ is strictly increasing on $\left[0, \frac{1}{n-1}\right)$.

Note that $\mathrm{Jump}_{n,j}\left(\frac{1}{n-1}\right) \equiv \mathrm{Jump}_{n,j}(0) + \frac{1}{n} \pmod 1$. Thus, to prove point (2), it suffices to show that $\mathrm{Jump}_{n,j}(\theta) < \mathrm{Jump}_{n,j}\left(\frac{1}{n-1}\right)$ for all $\theta \in \left[0, \frac{1}{n-1}\right)$.

Let $.t_1t_2t_3\cdots$ be an $n-$adic expansion for θ that does not end in the constant stream $nnn\cdots$. Then there exists a minimal index $r \geqslant 1$ such that $t_r = 0$, and for all integers $0 < r' < r$, we have $t_{r'} = 1$. Thus, we note that $0 \leqslant \mu_n^{\circ(r-1)}(\theta) < \frac{1}{n}$, and for all $0 < r' < r$, we have $\frac{1}{n} \leqslant \mu_n^{\circ(r'-1)}(\theta) < .1t_1t_2t_3\cdots$ in base n, and $.1t_1t_2t_3\cdots \equiv \frac{\theta+1}{n} \pmod 1$.

We split this into two cases:

- $j = 0$: in this case, for all $0 < r' < r$, we have $u_{n,j,\theta} \circ \mu_n^{\circ(r'-1)}(\theta) = 2$, and $u_{n,j,\theta} \circ \mu_n^{\circ(r-1)}(\theta) \leqslant 1$. Thus, in base $(n+1)$, $\mathrm{Jump}_{n,j}(\theta) < .\overline{2} \equiv \frac{2}{n} \pmod 1 \equiv \mathrm{Jump}_{n,j}\left(\frac{1}{n-1}\right) \pmod 1$.
- $j \geqslant 1$: in this case, for all $0 < r' < r$, we have $u_{n,j,\theta} \circ \mu_n^{\circ(r'-1)}(\theta) = 1$. We also have $u_{n,j,\theta} \circ \mu_n^{\circ(r-1)}(\theta) = 0$. Thus, in base $(n+1)$, $\mathrm{Jump}_{n,j}(\theta) < .\overline{1} \equiv \frac{1}{n} \pmod 1 \equiv \mathrm{Jump}_{n,j}\left(\frac{1}{n-1}\right) \pmod 1$.

■

□

VI.3: $\mathrm{Jump}_{n,j}$ preserves spiders

Next, we show that for any $\theta \in \mathbb{Q}/\mathbb{Z}$ and $j \in \{0, 1, \cdots n-1\}$, the spiders $S_n(\theta)$ and $S_{n+1}\Big(\mathrm{Jump}_{n,j}(\theta)\Big)$ are isomorphic. For the rest of this section, we fix $\theta \in \mathbb{Q}/\mathbb{Z}$ and $j \in \{0, 1, \cdots, n-1\}$. Let ϕ denote the angle $\mathrm{Jump}_{n,j}(\theta)$, and $.x_1x_2x_3\cdots$. be a $n-$adic expansion for θ that does not end in a constant string of $(n-1)$'s.

Proposition VI.9. *1. The angles in $\mathcal{O}_n(\theta)$ and $\mathcal{O}_{n+1}(\phi)$ have the same circular order. In particular, the pre-period and period of ϕ under μ_{n+1} coincide with the pre-period and period respectively of θ under μ_n.*

2. If $\theta \equiv \theta' + \frac{m}{n-1} \pmod 1$ for some angle $\theta' \in \Big[0, \frac{1}{n-1}\Big)$ and some $m \in \{0, 1, \cdots, m-1\}$, then the $\mu_{n+1}-$orbit of ϕ does not intersect $T^{stat}_{n+1,j+m}(\phi)$.

Proof. 1. Without loss of generality, we may assume that $\theta \in \Big[0, \frac{1}{n-1}\Big)$. This is because for all degrees n and all angles $t \in \mathbb{R}/\mathbb{Z}$, $\mu_n^{\circ(m-1)}\Big(t + \frac{1}{n-1}\Big) = \mu_n^{\circ(m-1)}(t) + \frac{1}{n-1}$ for all $m \in \mathbb{N}$, implying that $\mathcal{O}_n(t)$ and $\mathcal{O}_n\Big(t + \frac{1}{n-1}\Big)$ have the same circular order. Item (1) now follows directly from Proposition VI.5.

2. First assume $m = 0$. So $\theta \in \Big[0, \frac{1}{n-1}\Big)$. For every $r \in \mathbb{N}$,

$$\mu_n^{\circ(r-1)}(\theta) \in \left(\frac{m}{n}, \frac{m+1}{n}\right) \implies \mu_{n+1}^{\circ(r-1)}(\phi) \in \begin{cases} \left(\frac{m}{n+1}, \frac{m+1}{n+1}\right) & m \in \{0, 1, \cdots, j-1\} \\ \left(\frac{m+1}{n+1}, \frac{m+2}{n+1}\right) & m \in \{j+1, \cdots, n-1\} \end{cases}$$

Moreover, if $\mu_n^{\circ(r-1)}(\theta) \in \left(\frac{m}{n}, \frac{\theta+m}{n}\right)$ for some $m \in \{0, 1, \cdots, j-1\}$, then $\mu_n^{\circ r}(\theta) < \theta$, in turn implying $\mu_n^{\circ r}(\phi) < \phi$. But this means we must have $\mu_{n+1}^{\circ(r-1)}(\phi) \in \left(\frac{m}{n+1}, \frac{\phi+m}{n+1}\right)$.

Using similar arguments,

- the sector $T^{stat}_{n,m}(\theta)$ corresponds to $T^{stat}_{n+1,m}(\phi)$ for $m < j$, and to $T^{stat}_{n+1,m+1}(\phi)$ for $m > j$, and the orbit points in these sectors are in the same circular order;
- the sector $T^{stat}_{n,j}(\theta)$ corresponds to $T^{stat}_{n+1,j+1}(\phi)$, and the orbit points in these sectors are in the same circular order;

- if there is an orbit point in $\mathcal{O}_n(\theta)$ on the boundary of $T^{stat}_{n,m}(\theta)$ for some $m \in \{0, 1, \cdots n-1\}$, this implies θ is periodic under μ_n, with some period k. In this case, if θ is the smaller angle in a companion pair, then

$$\mu_{n+1}^{\circ(k-1)}(\phi) \equiv \frac{\phi + j}{n+1} \pmod 1 \in \partial T^{stat}_{n+1,j-1}(\phi) \cap \partial T^{stat}_{n+1,j}(\phi)$$

Else,

$$\mu_{n+1}^{\circ(k-1)}(\phi) \equiv \frac{\phi + j + 1}{n+1} \pmod 1 \in \partial T^{stat}_{n+1,j}(\phi) \cap \partial T^{stat}_{n+1,j+1}(\phi)$$

The rest of item (2) follows from the fact that

$$\begin{aligned} \theta &\equiv \theta' + \frac{m}{n-1} \pmod 1 \\ \Longrightarrow \phi &\equiv \mathrm{Jump}_{n,j}(\theta') + \frac{m}{n} \pmod 1 \\ \Longrightarrow T^{stat}_{n+1,j+m}(\phi) &= T^{stat}_{n+1,j}(\mathrm{Jump}_{n,j}(\theta')) + \frac{m}{n} \end{aligned}$$

□

Example VI.10. Let $n = 2$ and consider $\theta \equiv \frac{1}{2^3-1} \pmod 1 \equiv \frac{1}{7} \pmod 1$ as in Example VI.3. In Figure 6.3 we illustrate the spiders $\widehat{S}_3(\mathrm{Jump}_{2,0}(\theta))$ and $\widehat{S}_3(\mathrm{Jump}_{2,1}(\theta))$.

Example VI.11. Let $n = 2$ and consider $\theta \equiv \frac{17}{2^4(2^4-1)} \pmod 1$ as in Example VI.4. Figure 6.4 shows that $\widehat{S}_2(\theta)$, $\widehat{S}_3(\mathrm{Jump}_{2,0}(\theta))$ and $\widehat{S}_3(\mathrm{Jump}_{2,1}(\theta))$ are all isomorphic.

Remark VI.12. Proposition VI.9 implies that the graphs $\widehat{S}_n(\theta) \subset \widehat{S}^{ext}_n(\theta)$ and $\widehat{S}_{n+1}(\phi) \subset \widehat{S}^{ext}_{n+1}(\phi)$ are congruent.

Proposition VI.13. *$\widehat{\mathcal{F}}_{n,\theta}|\widehat{S}_n(\theta)$ and $\widehat{\mathcal{F}}_{n+1,\phi}|\widehat{S}_{n+1}(\phi)$ are conjugate by a homeomorphism $\widehat{h}$ that preserves the circular order of legs and satisfies $\widehat{h}(\infty) = \infty$ and $\widehat{h}(\exp(2\pi i\theta)) = \exp(2\pi i\phi)$.*

Consequently, $\mathcal{F}_{n,\theta}\Big|S_n(\theta)$ and $\mathcal{F}_{n+1,\phi}\Big|S_{n+1}(\phi)$ are conjugate by a homeomorphism $h : S_n(\theta) \to S_{n+1}(\phi)$ that preserves the circular order of legs, and satisfies $h(\infty) = \infty$ and $h(x_1) = \underline{x}_1$, where x_1 is the equivalence class of $\exp(2\pi i\theta)$ in $S_n(\theta)$, and $\underline{x}_1$ is the equivalence class of $\exp(2\pi i\phi)$ in $S_{n+1}(\phi)$.

Proof. For $m \in \mathbb{N}$, let θ_m and ϕ_m denote the angles $\mu_n^{\circ(m-1)}(\theta)$ and $\mu_{n+1}^{\circ(m-1)}(\phi)$ respectively. Define $\widehat{h} : \widehat{S}_n(\theta) \longrightarrow \widehat{S}_{n+1}(\phi)$ as $h(r\exp(2\pi i\phi_n)) = r\exp(2\pi i\phi_n)$. Proposition VI.9 implies that $\widehat{h}$ satisfies the required properties, and by definition of the graphs $S_n(\theta)$ and $S_{n+1}(\phi)$, $\widehat{h}$ descends to a homeomorphism $h : S_n(\theta) \longrightarrow S_{n+1}(\phi)$ as required. In other words, the following diagram commutes:

$$\begin{array}{ccc} \big(\widehat{S}_n(\theta), \exp(2\pi i\theta), \infty\big) & \xrightarrow{\hat{h}} & \big(\widehat{S}_{n+1}(\phi), \exp(2\pi i\phi), \infty\big) \\ \downarrow \pi & & \downarrow \pi \\ \big(S_n(\theta), x_1, \infty\big) & \xrightarrow{h} & \big(\widehat{S}_{n+1}(\phi), \underline{x}_1, \infty\big) \end{array}$$

Here, π is the notation used for the quotient map between a standard or extended spider and its corresponding quotient. □

Remark VI.14. In later sections, we will simply say that $\mathcal{F}_{n,\theta}\Big|S_n(\theta)$ and $\mathcal{F}_{n+1,\phi}\Big|S_{n+1}(\phi)$ are conjugate and assume that the conjugating map satisfy the properties in the above proposition.

Proof of Theorem I.6. The theorem follows from Propositions VI.8 and Remark VI.12. □

Remark VI.15. Let $\theta \in \left[0, \frac{1}{n-1}\right)$, and let ν be the kneading sequence of θ. Suppose $T^{stat}_{n,m}(\theta)$ is the sector containing θ in its interior. Construct a sequence $\nu^j = \nu^j_1\nu^j_2\nu^j_3....$ using the following rule for each $m \in \mathbb{N}$:

- If $\mu_n^{\circ(m-1)}(\theta)$ in one of the sectors $T^{stat}_{n,m}(\theta), T^{stat}_{n,m+1}(\theta), \cdots, T^{stat}_{n,j-1}(\theta)$ (or equivalently, $\nu_m \in \{0, 1, \cdots, j-1-m\}$), then $\nu^j_m := \nu_m$.
- If $\mu_n^{\circ(m-1)}(\theta)$ is in one of the sectors $T^{stat}_{n,j}(\theta), T^{stat}_{n,j+1}(\theta), \cdots, T^{stat}_{n,m-1}(\theta)$ (or equivalently, $\nu_m \in \{j-m, j+1-m, \cdots, n-1\}$), then $\nu^j_m := \nu_m + 1 \pmod n$)

In the resulting sequence ν^j, for all $m, m' \in \mathbb{N}$, we have $\nu^j_m = \nu^j_{m'}$ if and only if $\nu_m = \nu_{m'}$.

Proposition VI.5 shows that ν^j is the kneading sequence of $\mathrm{Jump}_{n,j}(\theta)$.

Remark VI.16 (Distinctness of the $\mathrm{Jump}'_{n,j}\, s$). Given any degree n, let θ be pre-periodic angle under μ_n whose orbit intersects every static sector. For example, θ can be taken to be $0\overline{x_1 x_2 \cdots x_{n-1}}$ in base n, where $x_i = i$ for $i = 1, 2, \cdots n-1$. Then for distinct elements $j, \hat{j}$ in the set $\{0, 1, \cdots, n-1\}$, it is clear that $\mathrm{Jump}_{n,j}(\theta) \neq \mathrm{Jump}_{n,\hat{j}}(\theta)$. Thus $\mathrm{Jump}_{n,j}$ and $\mathrm{Jump}_{n,\hat{j}}$ are different functions.

CHAPTER VII
Combinatorial Embeddings in the Unicritical Family

Definition VII.1. Let $n, n' \in \mathbb{N}_{\geqslant 2}$. Given $X \subseteq P_n$ and $Y \subseteq P_{n+1}$, a function $\mathcal{E} : X \to Y$ is said to be a *combinatorial embedding* if for every $\lambda \in X$,

1. the postsingular portraits of λ and $\widehat{\lambda}$ are isomorphic;
2. if $\widehat{\lambda} \in X$ satisfies $\lambda \lhd \widehat{\lambda}$, then $\mathcal{E}(\lambda) \lhd \mathcal{E}(\widehat{\lambda})$;
3. if λ is hyperbolic and $\widehat{\lambda} \in X$ is also hyperbolic and is a satellite of λ, then $\mathcal{E}(\widehat{\lambda})$ is a satellite of $\mathcal{E}(\lambda)$.

In this chapter, for every $n \in \mathbb{Z}_{\geqslant 2}$, we will construct a set of n distinct combinatorial embeddings $\{\mathcal{E}_{n,j} : \mathcal{P}_n \to \mathcal{P}_{n+1} : j \in \{0, 1, \cdots, n-1\}\}$ using the 'Jump' maps defined in Chapter VI. Fix a degree $n \geqslant 2$. We will construct the embeddings $\mathcal{E}_{n,j}$ using Theorem I.6, and prove Theorem I.7 by exhibiting additional properties of the $\mathrm{Jump}_{n,j}$'s.
First, fix $j \in \{0, 1, \cdots, n-1\}$.

VII.1: Image of $\mathrm{Jump}_{n,j}$

We give here a description of the image of $\mathrm{Jump}_{n,j}$ in $\mathbb{Q}/\mathbb{Z}$. Recall from Proposition VI.9 that every ϕ in the image of $\mathrm{Jump}_{n,j}\Big|\Big[0, \frac{1}{n-1}\Big)$ has the property that at least one static sector does not contain any angles in $\mathcal{O}_{n+1}(\phi)$. We will show here that all such angles are in the image of $\mathrm{Jump}_{n,j}$ for some j.

Fix $j \in \{0, 1, \cdots, n-1\}$.

Proposition VII.2. *Let $\phi \in \mathbb{Q}/\mathbb{Z} \cap \Big[\mathrm{Jump}_{n,j}(0), \mathrm{Jump}_{n,j}(0) + \frac{1}{n}\Big)$. If the μ_{n+1}−orbit of ϕ does not intersect $T^{stat}_{n+1,j}(\phi)$, then $\phi \equiv \mathrm{Jump}_{n,j}(\theta) \pmod 1$ for a unique angle $\theta \in \Big[0, \frac{1}{n-1}\Big)$.*

Proof. Let ℓ, k be the pre-period and period respectively of ϕ under μ_{n+1}. If $\ell = 0$ and $k = 1$ then it is clear that $\phi \equiv \mathrm{Jump}_{n,j}(0) \pmod 1$, since this is the only periodic angle of period 1 in the sector $\Big[\mathrm{Jump}_{n,j}(0), \mathrm{Jump}_{n,j}(0) + \frac{1}{n}\Big)$.

Otherwise, let $.y_1y_2\cdots.y_\ell\overline{y_{\ell+1}y_{\ell+2}\cdots y_{k+\ell}}$ be an $(n+1)-$ adic expansion for ϕ that does not terminate in a constant stream of n's. We define another symbol shift function as follows:

$$w : \mathcal{O}_{n+1}(\phi) \longrightarrow \{0, 1, \cdots, n-1\}$$
$$w(t) = \begin{cases} m & t \in [\frac{m}{n+1}, \frac{m+1}{n+1}) \text{ for some } m \in \{0, 1, .., j-1\} \\ j & t \in [\frac{j}{n+1}, \frac{\phi+j}{n+1}] \\ j & t \in [\frac{\phi+j+1}{n+1}, \frac{j+2}{n+1}) \\ m-1 & t \in [\frac{m}{n+1}, \frac{m+1}{n+1}) \text{ for some } m \in \{j+2, j+3, \cdots, n\} \end{cases}$$

We claim that the angle $\theta \equiv \sum_{m=1}^{\infty} \frac{w\circ\mu_{n+1}^{\circ(m-1)}(\phi)}{n^m} \pmod 1$ is pre-periodic under μ_n with pre-period ℓ and period k, and that $\text{Jump}_{n,j}(\theta) \equiv \phi \pmod 1$.

For every $m \in \mathbb{N}$, for ease of notation, we let $\theta_m \equiv \mu_n^{\circ(m-1)}(\theta) \pmod 1$, $\phi_m \equiv \mu_{n+1}^{\circ(m-1)}(\phi) \pmod 1$ and $x_m = w(\phi_m)$. By this notation, $.x_1x_2\cdots$ is an $n-$adic expansion for θ. The angle θ is clearly rational, and has some pre-period ℓ' and period k' under μ_n. It is clear that $\ell' \leqslant \ell$ and that $k'|k$. We first prove the following claims:

Claim 1. $\theta \in \left[0, \frac{1}{n-1}\right)$.

Claim 2. $\ell' = \ell$ and $k' = k$.

Assuming these claims to be true, by definition of the functions w and $u_{n,j,\theta}$, we can see that $\text{Jump}_{n,j}(\theta) = \phi$. By Proposition VI.8, the angle θ is unique.

Proof of Claim 1. Let $\text{Jump}_{n,j}(0) \equiv \frac{r}{n} \pmod 1$. We note that $r = 0$ if $j \geqslant 1$, and $r = 1$ if $j = 0$.

- if $\phi < \frac{r+1}{n+1}$, we see that $w(\phi) = 0$. Thus $\theta \in \left[0, \frac{1}{n}\right)$.
- if $\phi \in \left[\frac{r+1}{n+1}, \frac{r+1}{n}\right)$, then there exists an index $m > 1$ such that $y_{m'} = r+1$ for all $m' < m$, and $y_m \leqslant r$.
  - if $j = 0$, then $r = 1$. By the above discussion, since $y_{m'} \geqslant j+2$, we have $x_{m'} = w(\phi_{m'}) = y_{m'} - 1 = 1$ for all $m' < m$, and $x_m = w(\phi_m) = 0$.
  - if $j \geqslant 1$, then $r = 0$. Thus $x_{m'} = w(\phi_{m'}) = 1$ for all $m' < m$, and $x_m = w(\phi_m) = 0$.

  In both cases above, we see that $\theta \in \left[\frac{1}{n}, \frac{1}{n-1}\right)$.

■

Proof of Claim 2. Now suppose $k' < k$. Then $\theta_{\ell'+1} \equiv \theta_{\ell'+k'+1} \pmod 1$, which can happen only if $x_m = x_{m+k'}$ for all $m > \ell'$. We note, for all $m \in \mathbb{N}$,

$$\phi_m \in \left[0, \frac{\phi + k'}{n+1}\right] \implies x_m = y_m,$$
$$\phi_m \in \left[\frac{\phi + j' + 1}{n+1}, 1\right) \implies x_m = y_m - 1.$$

So if $x_m = x_{m+k'}$, then $y_m = y_{m+k'}$ or $\{y_m, y_{m+k'}\} = \{k', k'+1\}$.

Let $D = \{m > \ell' | y_m \neq y_{m+k'}\}$. Note that D is non-empty, since $k \in D$. Let $r = \inf D$, and observe that

- $$y_r < y_{r+k'} \implies \phi_r \in \left(\frac{k'}{n+1}, \frac{\phi + k'}{n+1}\right],$$
- since only one angle of the form $\frac{\phi+m}{n+1}$ can belong to $\mathcal{O}_{n+1}(\phi)$, exactly one of the following equalities holds:

$$\phi_r \equiv \frac{\phi + j}{n} \pmod 1, \text{ or } \phi_{r+j} \equiv \frac{\phi + j + 1}{n} \pmod 1.$$

Therefore,

$$\phi_{r+1} < \phi_{r+k'+1}$$
$$\iff .y_{r+1}y_{r+2}\cdots.. < .y_{r+k'+1}y_{r+k'+2}\cdots..$$

So at the first index $m > r$ in D, we must have $y_m < y_{m+k'}$. The above discussion shows that exactly one of the following statements is true:

$$y_m \leqslant y_{m+\ell'} \ \forall m > \ell', \text{ or } y_m \geqslant y_{m+\ell'} \ \forall m > \ell'.$$

Supposing the first condition to be true, we then have, for any $m \in D$,

$$y_m \leqslant y_{m+k'} \leqslant \cdots . \leqslant y_{m+(\frac{k}{k'}-1)j} \leqslant y_{m+k} = y_m,$$

which implies that $y_m = y_{m+k'}$, and thereby contradicts the definition of D. Thus we have shown that $k' = k$.

Suppose $\ell' < \ell$, then there exists an integer $r > 0$ such that $\ell' = \ell - rk$. Then $\theta_{\ell'+1} \equiv \theta_{k+\ell'+1} \pmod 1$, and this holds only if for all $m > \ell'$, we have $x_m = x_{m+k}$. But this means, for all $m > \ell'$ such that $y_m \neq y_{m+k}$, we have $y_m, y_{m+k} \in \{k', k'+1\}$. We note that $y_\ell \neq y_{k+\ell}$. Without loss of generality, suppose, $y_\ell = k'$, $y_{k+\ell} = k'+1$. Then we have $\phi_\ell \in \left[\frac{k'}{n+1}, \frac{\phi+k'}{n+1}\right]$ and $\phi_{k+\ell} \in \left[\frac{\phi+k'}{n+1}, \frac{k'+2}{n+1}\right)$. Therefore,

$$\phi_{\ell+1} < \phi < \phi_{k+\ell+1},$$

which is a contradiction. Thus, we must have $\ell' = \ell$.

■

□

Remark VII.3. This proposition also shows that if $\phi \in \left[\text{Jump}_{n,j}(0) + \frac{m}{n}, \text{Jump}_{n,j}(0) + \frac{m+1}{n}\right)$ for some $m \in \{0, 1, \cdots, n-1\}$, and $\mathcal{O}_{n+1}(\phi)$ does not intersect $T^{stat}_{n+1,j+m}(\phi)$, then there exists a unique angle $\theta \in \left[\frac{m}{n-1}, \frac{m+1}{n-1}\right)$ such that $\phi \equiv \text{Jump}_{n,j}(\theta) \pmod 1$.

For the rest of this section, we fix $\lambda \in \mathcal{P}_n$ and a monic representative $c \in M_n(\lambda)$. We saw that for any $\theta \in \Omega_n(c)$, the map $\mathcal{F}_{n+1,j,\text{Jump}_{n,j}(\theta)}$ defined in the discussion following Definition II.52 is Thurston equivalent to a polynomial $\lambda_{j,\theta}\left(1 + \frac{z}{n+1}\right)^{n+1}$ such that the angle $\phi \equiv \text{Jump}_{n,j}(\theta) \pmod 1 \in \Theta_{n+1}(\lambda_{j,\theta})$. We will show that $\lambda_{j,\theta}$ is independent of the choice of θ and c, allowing us to define $\mathcal{E}_{n,j}(\lambda) = \lambda_{j,\theta}$.

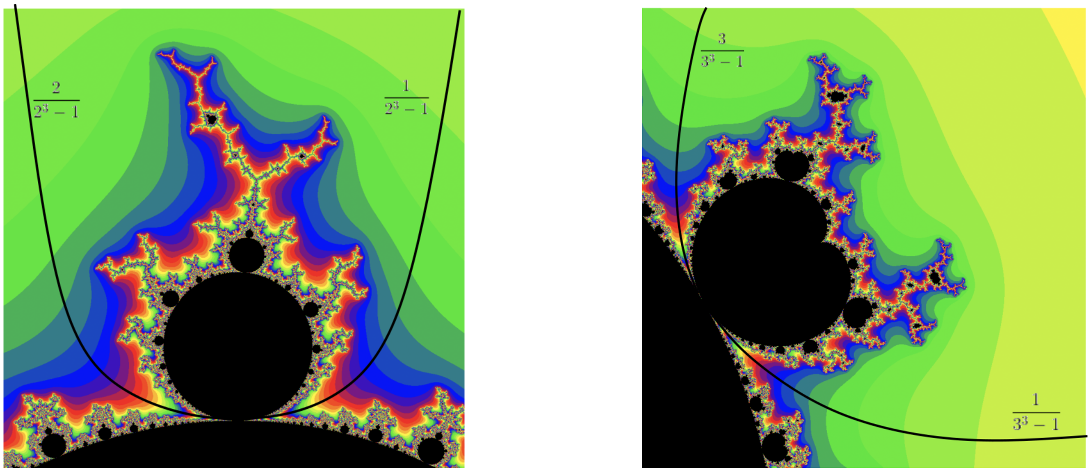
Proposition VII.4. *Let $c' = \exp(\frac{2\pi i m}{n-1})c$ for some $m \in \{0, 1, \cdots, n-1\}$. For $\theta' \equiv \theta + \frac{m}{n-1} \pmod 1$, we have $\lambda_{j,\theta} = \lambda_{j,\theta'}$.*

Proof. $\text{Jump}_{n,j}(\theta)$ is an angular coordinate for a PCf parameter $c_{n,j,\theta} \in M_{n+1}(\lambda_{j,\theta})$. By the definition of the $\text{Jump}_{n,j}$ functions, we know that $\text{Jump}_{n,j}(\theta') \equiv \phi + \frac{m}{n} \pmod 1$, and hence $\text{Jump}_{n,j}(\theta') \in \Theta_{n+1}(\lambda_{j,\theta})$. However, $\text{Jump}_{n,j}(\theta')$ is an angular coordinate for the point $\exp(\frac{2\pi i m}{n})c_{n,j,\theta} \in M_{n+1}(\lambda_{j,\theta'})$, from which the statement follows. □

Due to the above proposition, we can assume without loss of generality that c is in the subwake defined by $\left(0, \frac{1}{n-1}\right)$.

VII.2: Definition for critically periodic parameters

First suppose λ has a $k-$periodic critical point, with $k \geqslant 2$. Throughout this section, we fix $\theta \in \Omega_n(c)$. By our assumption on c, we have $\theta \in \left(0, \frac{1}{n-1}\right)$. Note that the parameter ray $R_n(\theta)$ could land either at the root or a co-root of the hyperbolic component U containing c.



(a) $R_2(\theta)$ and $R_2(\theta')$ (b) $R_3(\text{Jump}_{2,1}(\theta))$ and $R_3(\text{Jump}_{2,1}(\theta'))$

Figure 7.1: $\text{Jump}_{2,1}(\theta)$ and $\text{Jump}_{2,1}(\theta')$ for the companion pair $(\theta, \theta') = (\frac{1}{7}, \frac{2}{7})$ that land at the hyperbolic component containing the "rabbit" parameter $c \approx -0.122561 + 0.744862i$

Landing at a root

We first suppose that θ lands at the root of U, and has companion θ'. Let ϕ and ϕ' denote the angles $\text{Jump}_{n,j}(\theta)$ and $\text{Jump}_{n,j}(\theta')$ respectively.

Proposition VII.5. *ϕ, ϕ' are companion angles under μ_{n+1}.*

Proof. Let $\mathcal{A} = \{\mathcal{A}_1, \cdots, \mathcal{A}_r\}$ be the orbit portrait generated by (θ, θ'), and let $\mathcal{O}(\theta, \theta') = \bigcup_{i=1}^{r} \mathcal{A}_i$. The symbol shift functions $u_{n,j,\theta}$ and $u_{n,j,\theta'}$ coincide on $\mathcal{O}(\theta, \theta')$, since $\mathcal{O}(\theta, \theta') \cap \left(\frac{\theta+j}{n}, \frac{\theta'+j}{n}\right) = \varnothing$. We use the following for ease of notation:

$$\forall m \in \mathbb{N},\ \ \theta_m \equiv \mu_n^{\circ(m-1)}(\theta) \pmod 1\ \ \theta'_m \equiv \mu_n^{\circ(m-1)}(\theta') \pmod 1,$$
$$\phi_m \equiv \mu_{n+1}^{\circ(m-1)}(\phi) \pmod 1,\ \ \phi'_m \equiv \mu_{n+1}^{\circ(m-1)}(\phi') \pmod 1$$

For every $j \in \{1, 2, \cdots r\}$, we define the set $\mathcal{B}_i$ as follows:

$$\mathcal{B}_i = \{\phi_m : \theta_m \in \mathcal{A}_i\} \cup \{\phi'_m : \theta'_m \in \mathcal{A}_i\}$$

The collection $\mathcal{B} = \{\mathcal{B}_1, \cdots, \mathcal{B}_r\}$ is a partition of the union of the μ_{n+1}-orbits of ϕ and ϕ'. In order to show that (ϕ, ϕ') is a companion pair, we first show that the angles in their orbits taken together form a formal orbit portrait.

Let $\mathcal{B}(\theta, \theta') = \bigcup_{i=1}^{r} \mathcal{B}_i$. We will prove that $\mathcal{B}$ satisfies the properties (1) through (5) listed in Definition II.41.

1. This is clear by definition.

2. The fact that μ_{n+1} maps $\mathcal{B}_i$ onto $\mathcal{B}_{i+1}$ is clear since μ_n maps $\mathcal{A}_i$ bijectively onto $\mathcal{A}_{i+1}$. The rest follows from Proposition VI.13.

3. For every $i \in \{0, 1, \cdots r\}$, define

$$\mathcal{B}_i(\theta) = \mathcal{O}_{n+1}(\phi) \cap \mathcal{B}_i$$
$$\mathcal{B}_i(\theta') = \mathcal{O}_{n+1}(\phi') \cap \mathcal{B}_i$$

Since $\mathcal{A}_i \subset T^{stat}_{n,m}(\theta)$ for some $m \in \{0, 1, \cdots, n-1\}$, we have $\mathcal{B}_i(\theta) \subset T^{stat}_{n+1,m'}(\phi)$ and $\mathcal{B}_i(\theta') \subset T^{stat}_{n+1,m'}(\phi')$ for some $m' \in \{0, 1, .., n\}$. It suffices to show that

$$\mathcal{B}_i \subset \left(\frac{\phi' + m'}{n+1}, \frac{\phi + m' + 1}{n+1}\right) = T^{stat}_{n+1,m'}(\phi) \cap T^{stat}_{n+1,m'}(\phi')$$

For any $s \in \{0, 1, \cdots, n\}$, if there exists $\psi \in \mathcal{B}_i \cap \left[\frac{\phi+s}{n+1}, \frac{\phi'+s}{n+1}\right]$, then there exists $r \in \{0, 1, \cdots n-1\}$ such that $\mathcal{A}_i \cap \left[\frac{\theta+r}{n}, \frac{\theta'+r}{n}\right] \neq \varnothing$, which is not possible. Thus, for every $s \in \{0, 1, \cdots, n\}$,

$$\mathcal{B}_i \cap \left[\frac{\phi + s}{n+1}, \frac{\phi' + s}{n+1}\right] = \varnothing$$

The result follows immediately.

4. Since the period of all angles in $\mathcal{B}(\theta, \theta')$ is equal to k, and since $\mathcal{A}$ is a formal orbit portrait, we have $k = rp$ for some $p \geqslant 1$.

5. If $\mathcal{A}_i \subset \left(\frac{\theta'+m}{n}, \frac{\theta+m+1}{n}\right)$ for some $m \in \{0, 1, \cdots n-1\}$, then $\mathcal{B}_i \subset \left(\frac{\phi'+m}{n+1}, \frac{\phi+m+1}{n+1}\right)$.

   Given distinct integers $0 \leqslant i, i' \leqslant n$, by property (2), the sets $\mathcal{B}_{n,i}$ and $\mathcal{B}_{n,i'}$ are unlinked. Given $m' \neq m$, and any i, i', we want to show that $\mathcal{B}_{m,i}, \mathcal{B}_{m',i'}$ are unlinked. If $i = i' = 0$, this follows from the fact that $\mathcal{A}_m, \mathcal{A}_{m'}$ are unlinked. If at least one of i, i' is nonzero and $\mathcal{B}_{m,i}$ and $\mathcal{B}_{m',i'}$ are linked, then without loss of generality, we can find angles $\alpha, \beta \in \mathcal{B}_{m,i}$ and $\eta, \delta \in \mathcal{B}_{m',i'}$ that satisfy

$$\alpha < \eta < \beta < \delta$$

   which implies that

$$(n+1)\alpha < (n+1)\eta < (n+1)\beta < (n+1)\delta$$

The inequalities are strict since $\mathcal{B}_{m+1}$ and $\mathcal{B}_{m'+1}$ are disjoint. But $\mathcal{B}_s$ maps bijectively onto $\mathcal{B}_{s+1}$ for each s, and the above inequality implies that $\mathcal{B}_{m+1}, \mathcal{B}_{m'+1}$ are linked, which is a contradiction.

Thus $\mathcal{B}$ is a formal, non-trivial orbit portrait. As a last step, we show that ϕ, ϕ' are the characteristic angles of this orbit portrait. The interval (ϕ, ϕ') has either the smallest or largest arc length in $\mathcal{B}_1$. We let ℓ_i be the length of the unique complementary arc γ_i of $\mathcal{B}_i$ of length greater than $\frac{1}{n}$. Each γ_i is a critical arc - that is, under multiplication by $n+1$, it covers the circle n times. Furthermore, the γ_i's are the only critical arcs of $\mathcal{B}$.

For $i \neq r$, γ_i is strictly contained in $\mathbb{R}/\mathbb{Z}\backslash\left[\frac{\phi'+m}{n+1}, \frac{\phi+m+1}{n+1}\right]$ for some m, whereas $\gamma_r = \mathbb{R}/\mathbb{Z}\backslash\left[\frac{\phi'+m_0}{n+1}, \frac{\phi+m_0+1}{n+1}\right]$ for some m_0. This proves that $\ell_r = \max_i \ell_i$. Therefore, the critical value arc bounded by $\mu_{n+1}(\partial\gamma_r)$ is the shortest critical value arc among all critical value arcs. But we note that $\mu_{n+1}(\partial\gamma_r) = \{\phi, \phi'\}$, implying that

$$d_{\mathbb{R}/\mathbb{Z}}(\phi, \phi') = \min_{i=1}^{r} \min_{\alpha,\beta\in\mathcal{B}_i} d_{\mathbb{R}/\mathbb{Z}}(\alpha, \beta)$$

This shows that (ϕ, ϕ') is the characteristic angle pair for $\mathcal{B}$, and the result follows. $\square$

Example VII.6. Let $n = 2$. The quadratic rabbit polynomial (so called because its Julia set looks like a rabbit) is given by $z^2 + c$ where $c \approx -0.122561 + 0.744862i$. The root of the hyperbolic component containing the rabbit is the landing point of the companion angles $(\theta, \theta') = (\frac{1}{7}, \frac{2}{7})$. The pair of angles $(\text{Jump}_{2,0}(\theta), \text{Jump}_{2,0}(\theta')) = (\frac{14}{26}, \frac{16}{26})$ is a companion pair forming angular coordinates for a "cubic rabbit" parameter $c_{2,0,\theta} = c_{2,0,\theta'} \approx -0.54056 - 0.52858i$.

Similarly, we also find that $(\text{Jump}_{2,1}(\theta), \text{Jump}_{2,1}(\theta')) = (\frac{1}{26}, \frac{3}{26})$ also constitute angular coordinates for a "cubic rabbit" parameter $c_{2,1,\theta} = c_{2,1,\theta'} \approx 0.54056 + 0.52858i$. See Figure 7.1 for further details.

Landing at a co-root

In this section, we assume that θ lands at a co-root of U, and show that its image under $\text{Jump}_{n,j}$ lands at a co-root as well.

There exists an angle pair (α, α') with period k landing at the root of U. By the previous section, the angles $\psi \equiv \text{Jump}_{n,j}(\alpha) \pmod 1$, and $\psi' \equiv \text{Jump}_{n,j}(\alpha') \pmod 1$ land at the root of a hyperbolic component $V \subset \mathcal{M}_{n+1}$. Since $\text{Jump}_{n,j}$ is order-preserving, the angle $\phi \equiv \text{Jump}_{n,j}(\theta) \pmod 1$ lands in the wake of (ψ, ψ'), and by Proposition VI.5, ϕ, ψ and ψ' all have period k under μ_{n+1}. Additionally, by Remark VI.15, the kneading sequences of ϕ, ψ and

ψ' coincide with ν^j up to and including the index $(k-1)$. The itineraries of ϕ with respect to both ψ and ψ' differ from each other, and from the sequence ν^j (see Remark VI.15) at the kth position. Since $\psi < \phi < \psi'$, we have $\mu_{n+1}^{\circ(k-1)}(\phi) \in \left(\frac{\psi+m}{n+1}, \frac{\psi'+m}{n+1}\right)$ for some $m \in \{0, 1, \cdots, n\}$. We now show that ϕ lands at a co-root of V.

Proposition VII.7. *1.* $\mu_{n+1}^{\circ(k-1)}(\psi) \not\equiv \frac{\psi+m}{n+1} \pmod 1$*, and* $\mu_{n+1}^{\circ(k-1)}(\psi') \not\equiv \frac{\psi'+m}{n+1} \pmod 1$.

2. There exists an angle ϕ' *landing at a co-root of* V *such that* $\mu_{n+1}^{\circ(k-1)}(\psi') \in (\frac{\psi+m}{n+1}, \frac{\psi'+m}{n+1})$.

Proof. 1. First, we note that we have

$$d_{\mathbb{R}/\mathbb{Z}}(\psi, \psi') < d_{\mathbb{R}/\mathbb{Z}}\big(\mu_{n+1}(\psi),\ \mu_{n+1}(\psi')\big) < \cdots .. < d_{\mathbb{R}/\mathbb{Z}}\big(\mu_{n+1}^{\circ(k-1)}(\psi),\ \mu_{n+1}^{\circ(k-1)}(\psi')\big)$$

This in turn implies that

$$d_{\mathbb{R}/\mathbb{Z}}(\psi, \phi) < d_{\mathbb{R}/\mathbb{Z}}\big(\mu_{n+1}(\psi),\ \mu_{n+1}(\phi)\big) < \cdots .. < d_{\mathbb{R}/\mathbb{Z}}\big(\mu_{n+1}^{\circ(k-1)}(\psi),\ \mu_{n+1}^{\circ(k-1)}(\phi)\big)$$

Suppose $\mu_{n+1}^{\circ(k-1)}(\psi) \equiv \frac{\psi+m}{n+1} \pmod 1$, we would then have $d_{\mathbb{R}/\mathbb{Z}}\big(\mu_{n+1}^{\circ(k-1)}(\psi),\ \mu_{n+1}^{\circ(k-1)}(\phi)\big) = \frac{d_{\mathbb{R}/\mathbb{Z}}(\psi,\phi)}{n}$, which is a contradiction. By a similar argument, $\mu_{n+1}^{\circ(k-1)}\psi' \not\equiv \frac{\psi'+m}{n+1} \pmod 1$.

2. For any co-root angle ϕ' of the component V, the angle $\mu_{n+1}^{\circ(k-1)}(\phi')$ cannot be in $[\frac{\psi'+m}{n+1}, \frac{\psi+m+1}{n+1}]$ for any $m \in \{0, 1, \cdots, n\}$, since this would imply that $\phi' \notin (\psi, \psi')$. Therefore each $\mu_{n+1}^{\circ(k-1)}(\phi')$ belongs to $(\frac{\psi+m}{n+1}, \frac{\psi'+m}{n+1})$ for some m that satisfies $\mu_{n+1}^{\circ(k-1)}(\psi) \not\equiv \frac{\psi+m}{n+1} \pmod 1$ and $\mu_{n+1}^{\circ(k-1)}(\psi') \not\equiv \frac{\psi'+m}{n+1} \pmod 1$, by (1). There are $(n-1)$ co-roots for V, and $(n-1)$ values of m that satisfy the latter this property.

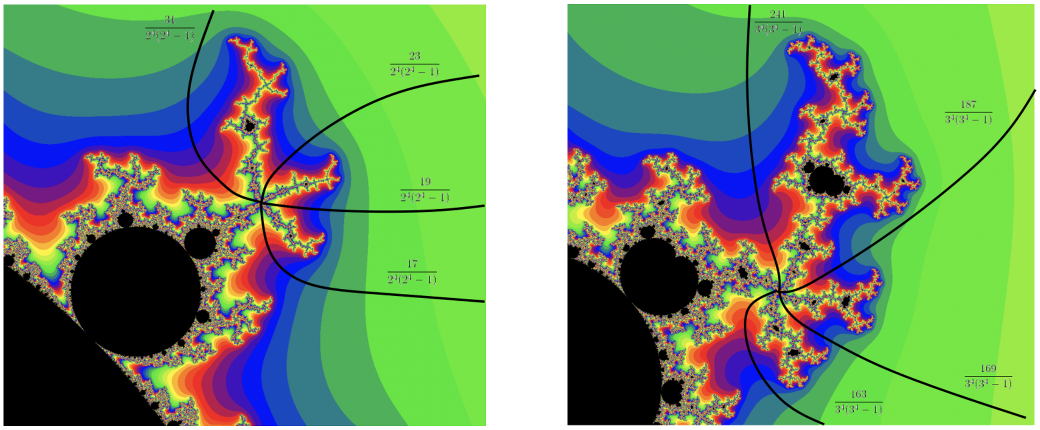
Suppose we have two co-root angles ϕ', ϕ'' with $\mu_{n+1}^{\circ(k-1)}(\phi'), \mu_{n+1}^{\circ(k-1)}(\phi'') \in (\frac{\psi+m}{n+1}, \frac{\psi'+m}{n+1})$, this again contradicts the chain of inequalities given in (1), thus for each $m \in \{0, 1, \cdots, n\}$ with $\mu_{n+1}^{\circ(k-1)}(\psi) \not\equiv \frac{\psi+m}{n+1} \pmod 1$ and $\mu_{n+1}^{\circ(k-1)}(\psi') \not\equiv \frac{\psi'+m}{n+1} \pmod 1$, there exists exactly one co-root angle $\phi' \in (\frac{\psi+m}{n+1}, \frac{\psi'+m}{n+1})$.

□

Proposition VII.8. *ϕ lands at a co-root of V.*

Proof. Let ϕ' be the angle from item (2) in the previous proposition.

The angles ϕ and ϕ' have the same itinerary with respect to ψ, and therefore, the dynamic rays at angles ϕ and ϕ' land at the same point z_0 in the plane of $f_{n+1,\widetilde{c}}$, where $\widetilde{c}$ is the center of V, making z_0 a cut point in the Julia set of $f_{n+1,\widetilde{c}}$ unless $\phi = \phi'$. But the co-root angle ϕ' cannot land at a cut-point, and this forces $\phi \equiv \phi' \pmod 1$. □



(a) Parameter rays corresponding to $\{\theta_i\}_{i=1}^4$ in degree 2

(b) Parameter rays corresponding to $\{\text{Jump}_{2,1}(\theta_i)\}_{i=1}^4$ in degree 3

Figure 7.2: Images under $\text{Jump}_{2,1}$ for the angles $\theta_1 \equiv \frac{17}{2^4(2^4-1)} \pmod 1$, $\theta_2 \equiv \frac{19}{2^4(2^4-1)} \pmod 1$, $\theta_3 \equiv \frac{23}{2^4(2^4-1)} \pmod 1$ and $\theta_4 \equiv \frac{31}{2^4(2^4-1)} \pmod 1$ which land at $c \approx 0.36638 + 0.59152i \in \mathcal{P}_2$.

The discussion in this section shows that for a periodic parameter λ, we may define $\mathcal{E}_i(\lambda) = \lambda_{i,\theta}$ for any angle $\theta \in \Omega_n(c)$, for any choice of $c \in M_n(\lambda)$.

Proposition VII.9. *Given a companion pair (ϕ, ϕ') with $\phi, \phi' \in \Big[\text{Jump}_{n,j}(0), \text{Jump}_{n,j}(0) + \frac{1}{n}\Big)$ landing at a hyperbolic root in $\mathcal{M}_{n+1}$ such that the orbit of ϕ (or ϕ') under μ_{n+1} does not intersect the interior of $T^{stat}_{n+1,j}(\phi)$, there exists a companion pair (α, α') periodic under μ_n, with $\alpha, \alpha' \in \Big[0, \frac{1}{n-1}\Big)$ such that $\phi \equiv \text{Jump}_{n,j}(\alpha) \pmod 1$ and $\phi' \equiv \text{Jump}_{n,j}(\alpha') \pmod 1$.*

Proof. By Proposition VII.2, there exists $\alpha \in \mathbb{Q}/\mathbb{Z} \cap \Big[0, \frac{1}{n-1}\Big)$ such that $\text{Jump}_{n,j}(\alpha) \equiv \phi \pmod 1$. If α lands at a co-root in $\mathcal{M}_n$, this would imply that ϕ lands at a co-root in $\mathcal{M}_{n+1}$, contradicting our assumption. Thus α lands at a root, and has a companion α'. By Proposition VII.5, $\phi' \equiv \text{Jump}_{n,j}(\alpha') \pmod 1$. □

VII.3: Definition for critically pre-periodic parameters

Supposing λ has a strictly pre-periodic critical point, let $\theta \in \Omega_n(c)$. As in the previous section, $\theta \in \Big[0, \frac{1}{n-1}\Big)$. Let us denote by $c_{n,j,\theta}$ the landing point of the ray at angle $\phi \equiv \text{Jump}_{n,j}(\theta) \pmod 1$ in $\mathcal{M}_{n+1}$. Since ϕ is strictly pre-periodic under μ_{n+1}, $c_{n,j,\theta}$ is critically pre-periodic under $z^{n+1} + c_{n,j,\theta}$.

Proposition VII.10. *For $\theta' \in \Omega_n(c)$, both ϕ and $\phi' \equiv \text{Jump}_{n,j}(\theta') \pmod 1$ land at $c_{n,j,\theta}$.*

Proof. Let γ denote the Carathéodory loop of $f_{n,c}$. Then, given any $\alpha \in \Omega_n(c)$ and $t_1, t_2 \in \mathbb{R}/\mathbb{Z}$,

$$\gamma(t_1) = \gamma(t_2) \iff \Sigma_{n,\alpha}(t_1) = \Sigma_{n,\alpha}(t_2)$$

This gives us $\Sigma_{n,\theta}(\theta') = \Sigma_{n,\theta}(\theta) = \Sigma_{n,\theta'}(\theta) = \Sigma_{n,\theta'}(\theta')$. This common sequence is also the kneading sequence of both θ and θ'. Let us call it ν. By Remark VI.15, the angles ϕ and ϕ' have the kneading sequence ν^j, and by Proposition VI.5, it is clear that $\Sigma_{n+1,\phi}(\phi) = \Sigma_{n+1,\phi}(\phi') = \Sigma_{n+1,\phi'}(\phi') = \Sigma_{n+1,\phi'}(\phi) = \nu^j$. So in the dynamical plane of $z^{n+1}+c_{n,j,\theta}$, $c_{n,j,\theta} = \gamma_j(\phi) = \gamma_j(\phi')$, where γ_j is the corresponding Carathéodory loop. Hence the parameter ray at ϕ' to $\mathcal{M}_{n+1}$ also lands at $c_{n,j,\theta}$. □

This proof shows that $c_{n,j,\theta}$ is independent of θ. Due to Proposition VII.4, we may define $\mathcal{E}_{n,j}(\lambda) = \lambda_{j,\theta} = (n+1)c^n_{n,j,\theta}$ for any $c \in M_n(\lambda)$ and any $\theta \in \Omega_n(c)$.

Example VII.11. Let $n = 2$. The parameter rays at the following angles land at $c \approx 0.36638 + 0.59152i \in \mathcal{M}_2$:

$$\theta_1 \equiv \frac{17}{2^4(2^4-1)} \pmod 1, \ \ \theta_2 \equiv \frac{19}{2^4(2^4-1)} \pmod 1$$
$$\theta_3 \equiv \frac{23}{2^4(2^4-1)} \pmod 1, \ \ \theta_4 \equiv \frac{31}{2^4(2^4-1)} \pmod 1$$

Correspondingly, in degree 3, the angles listed below land at $c_{2,1,\theta} = c_{2,1,\theta'} \approx 0.62759 + 0.29869i \in \mathcal{M}_3$:

$$\mathrm{Jump}_{2,1}(\theta_1) \equiv \frac{163}{3^4(3^4-1)} \pmod 1, \ \ \mathrm{Jump}_{2,1}(\theta_2) \equiv \frac{169}{3^3(3^4-1)} \pmod 1$$
$$\mathrm{Jump}_{2,1}(\theta_3) \equiv \frac{187}{3^4(3^4-1)} \pmod 1, \ \ \mathrm{Jump}_{2,1}(\theta_4) \equiv \frac{241}{3^4(3^4-1)} \pmod 1$$

See Figure 7.2 for an illustration.

This finishes the definition of $\mathcal{E}_{n,j}(\lambda)$ for all $\lambda \in \mathcal{P}_n$.

VII.4: Properties of $\mathcal{E}_{n,j}$

In the rest of the chapter, we show that $\mathcal{E}_{n,j}$ is a combinatorial embedding. As in previous sections, fix $\lambda \in \mathcal{P}_n$, $c \in M_n(\lambda)$ within the subwake $\left(0, \frac{1}{n-1}\right)$ in the parameter plane, and $j \in \{0, 1, \cdots, n-1\}$. First suppose λ is a periodic parameter with critical value of period $k \geqslant 2$. Thus c is the center of some hyperbolic component $U \subset \mathcal{M}_n$.

Proposition VII.12. *Suppose that θ lands at the root of U and has companion θ'. Let V be the hyperbolic component in $\mathcal{M}_{n+1}$ associated with the pair $(\text{Jump}_{n,j}(\theta), \text{Jump}_{n,j}(\theta'))$. Then V is primitive if and only if U is primitive.*

Proof. Clearly, $\rho_\nu = \rho_{\nu^j}$ (see Section II.4.1).

A hyperbolic component is a satellite if and only if the penultimate entry of its internal address divides the final entry. From this, the second statement follows. ☐

Proposition VII.13. *If (θ, θ') is a satellite of (ψ, ψ'), then $(\text{Jump}_{n,j}(\theta), \text{Jump}_{n,j}(\theta'))$ is a satellite of $(\text{Jump}_{n,j}(\psi), \text{Jump}_{n,j}(\psi'))$.*

Proof. Let k be the period of ψ, and let ν be the kneading sequence of (θ, θ'). $(\text{Jump}_{n,j}(\theta), \text{Jump}_{n,j}(\theta'))$ is a satellite of a ray pair (α, α') of period k (since its internal address is the same as that of (θ, θ')), and it lies in the wake of $(\text{Jump}_{n,j}(\psi), \text{Jump}_{n,j}(\psi'))$. Now suppose that the internal address of (θ, θ') is given by

$$1 \mapsto s_1 \mapsto \cdots \mapsto s_{r-1} = k \mapsto s_r = k'$$

Then for every $m \in \{1, 2, \cdots, r\}$, there exist ray pairs P_m, Q_m periodic under μ_n, μ_{n+1} respectively that correspond to the entry s_m, and moreover, we have

$$\begin{aligned} P_r &= (\theta, \theta') \\ P_{r-1} &= (\psi, \psi') \\ Q_r &= (\text{Jump}_{n,j}(\theta), \text{Jump}_{n,j}(\theta')) \\ Q_{r-1} &= (\alpha, \alpha') \end{aligned}$$

For any pair $P = (\alpha', \beta')$, let $\text{Jump}_{n,j}(P)$ denote the pair $(\text{Jump}_{n,j}(\alpha'), \text{Jump}_{n,j}(\beta'))$.

For each $m \in \{1, 2, \cdots, r\}$, Q_r is in the wake of $\text{Jump}_{n,j}(P_m)$. Additionally, by definition,

$$Q_1 = \text{Jump}_{n,j}(P_1) \text{ and } Q_r = \text{Jump}_{n,j}(P_r)$$

Let $m' \in \{1, 2, \cdots, r\}$ be the first index where $\text{Jump}_{n,j}(P_{m'}) \neq Q_{m'}$. By Proposition VI.8, Q_r is in the wake of both $Q_{m'}$ and $\text{Jump}_{n,j}(P_{m'})$. Therefore, one of the following is true: either $Q_{m'}$ is in the wake of $\text{Jump}_{n,j}(P_{m'})$ or $\text{Jump}_{n,j}(P_{m'})$ is in the wake of $Q_{m'}$. In either case, by Lemma II.42, there exists a ray pair P of period $p < s_{m'}$ that separates $Q_{m'}$ and $\text{Jump}_{n,j}(P_{m'})$.

But P lies in the wake of $Q_{m'-1}$, and Q_r lies in the wake of P. This suggests that $\rho_\nu(s_{m'-1}) \leqslant p < s_{m'}$, which is a contradiction to the fact that $\rho_\nu(s_{m'-1}) = s_{m'}$. Therefore, for all indices m', we must have $Q_{m'} = \text{Jump}_{n,j}(P_{m'})$. In particular, $(\alpha, \alpha') = Q_{r-1} = (\text{Jump}_{n,j}(\theta), \text{Jump}_{n,j}(\theta'))$. $\square$

More on co-roots

Let $\theta \in \left[0, \frac{1}{n-1}\right)$ be an angle that lands at a co-root of a hyperbolic component U, whose root angles are (α, α'). Let V be the component in $\mathcal{M}_{n+1}$ on which $\text{Jump}_{n,j}(\alpha)$ lands.
Note that in base $(n+1)$,

$$\text{Jump}_{n,j}(\theta) = \overline{.u_{n,j,\theta}(\theta_1)u_{n,j,\theta}(\theta_2)\cdots .u_{n,j,\theta}(\theta_k)(j+1)}$$

where $\theta_m \equiv \mu_n^{\circ(m-1)}(\theta) \pmod 1$. We also define the angles $\psi \equiv \text{Jump}_{n,j}(\theta) \pmod 1$, and $\psi' = \overline{.u_{n,j,\theta}(\theta_1)u_{n,j,\theta}(\theta_2)\cdots .u_{n,j,\theta}(\theta_k)j}$ in base $(n+1)$.

We know that ψ lands at a co-root of V. Note that ψ' has itinerary ν^j with respect to $\text{Jump}_{n,j}(\alpha)$. Also note that $\psi' \equiv \psi - \frac{1}{(n+1)^k-1} \pmod 1$. By Proposition VII.7, we can show that there exists an angle ψ'' landing at a co-root of V whose itinerary with respect to $\text{Jump}_{n,j}(\alpha)$ coincides with that of ψ'. This forces $\psi' \equiv \psi'' \pmod 1$, that is, ψ' lands at a co-root of V.

We will show that the image of $\text{Jump}_{n,j}\Big|\left(\mathbb{Q}/\mathbb{Z} \cap \left[0, \frac{1}{n-1}\right)\right)$ does not intersect (ψ, ψ').

Proposition VII.14. *Given a rational angle $\beta \in (\psi', \psi)$, the $\mu_{n+1}-$orbit of β intersects the interior of $T^{stat}_{n+1,j}(\beta)$.*

Proof. We note that the angles $\psi'_k \equiv (n+1)^{k-1}\psi' \pmod 1$, $\beta_k \equiv (n+1)^{k-1}\beta \pmod 1$ and $\psi_k \equiv (n+1)^{k-1}\psi \pmod 1$ are in counterclockwise order. Let $\varepsilon := d_{\mathbb{R}/\mathbb{Z}}(\beta, \psi')$. Then,

$$\begin{aligned} d_{\mathbb{R}/\mathbb{Z}}(\beta_k, \psi'_k) &= (n+1)^{k-1}\varepsilon \\ d_{\mathbb{R}/\mathbb{Z}}\left(\frac{\beta+j}{n+1}, \frac{\psi'+j}{n+1}\right) &= \frac{\varepsilon}{n+1} \end{aligned}$$

Since ψ'_k is the angle $\frac{\psi'+j}{n+1}$, we note that $\frac{\psi'+j}{n+1}, \frac{\beta+j}{n+1}$ and β_k are also in counterclockwise order, and moreover, $\varepsilon \in \left(0, \frac{1}{(n+1)^k-1}\right)$; from this it follows that

$$\begin{aligned} d_{\mathbb{R}/\mathbb{Z}}\left(\beta_k, \frac{\beta+j}{n+1}\right) &= d_{\mathbb{R}/\mathbb{Z}}(\beta_k, \psi'_k) - d_{\mathbb{R}/\mathbb{Z}}\left(\frac{\beta+j}{n+1}, \frac{\psi'+j}{n+1}\right) \\ &= (n+1)^{k-1}\varepsilon - \frac{\varepsilon}{n+1} \\ &< \frac{1}{n+1} \end{aligned}$$

Arguing similarly,

$$d_{\mathbb{R}/\mathbb{Z}}\Big(\beta_k,\ \frac{\beta+j+1}{n+1}\Big) < \frac{1}{n+1}$$

Since $n \geqslant 2$, this means that

$$\beta_k \in \Big[\frac{\beta+j}{n+1}, \frac{\beta+j+1}{n+1}\Big]$$

The angles ψ', ψ are consecutive among angles periodic of period k under μ_{n+1}, hence β cannot be periodic of period k. Thus,

$$\beta_k \in \Big(\frac{\beta+j}{n+1}, \frac{\beta+j+1}{n+1}\Big)$$

□

For any $\delta \in \Big[0, \frac{1}{n-1}\Big)$, by Proposition VI.9, the μ_{n+1}-orbit of $\mathrm{Jump}_{n,j}(\delta)$ does not intersect $T^{stat}_{n+1,j}(\delta)$. By the above proposition, we see that the image of $\mathrm{Jump}_{n,j}$ does not intersect (ψ', ψ).

Critically pre-periodic parameters

Now assume that λ has pre-period $\ell \geqslant 1$ and period $k \geqslant 1$. Choose an angular coordinate $\theta \in \Omega_n(c)$, and let $c_{n,j,\theta}$ be the landing point of $\phi \equiv \mathrm{Jump}_{n,j}(\theta) \pmod 1$.

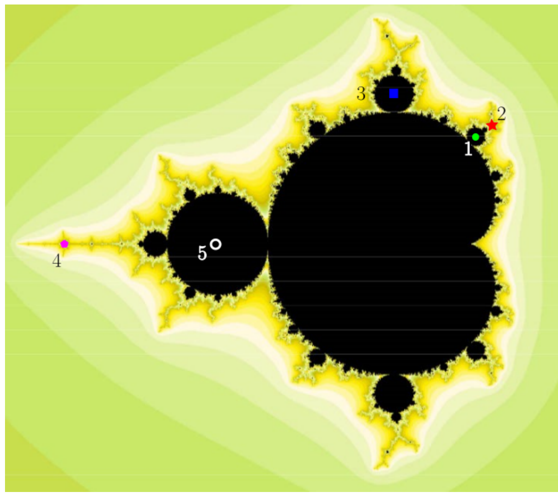
Proposition VII.15. $\Omega_{n+1}(c_{n,j,\theta}) = \mathrm{Jump}_{n,j}(\Omega_n(c))$*; in particular, c and $c_{n,j,\theta}$ have the same number of angular coordinates.*

Proof. By Proposition VII.10, $\mathrm{Jump}_{n,j}(\Omega_n(c)) \subset \Omega_{n+1}(c_{n,j,\theta})$. Given $\phi' \in \Omega_{n+1}(c_{n,j,\theta})$ with $\phi' \neq \phi$, the μ_{n+1}-orbit of ϕ' does not intersect $T^{stat}_{n+1,j}(\phi')$. By Proposition VII.2, there exists an angle $\theta' \in \Big[0, \frac{1}{n-1}\Big)$ with $\mathrm{Jump}_{n,j}(\theta') \equiv \phi' \pmod 1$.

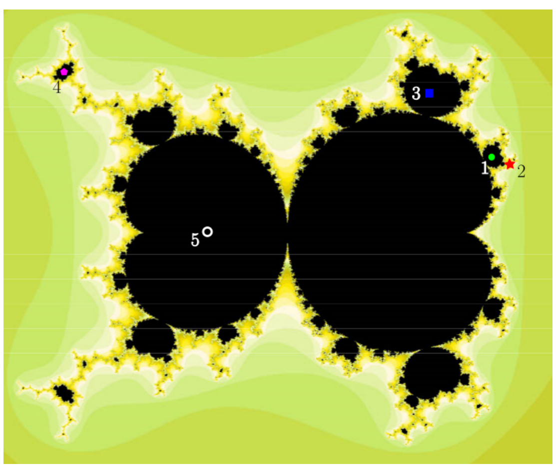
Let ν be the kneading sequence of θ. Note that the itinerary of ϕ' with respect to ϕ is ν^j. By construction, we can show that the itinerary of θ' with respect to θ is ν. This implies that in the dynamical plane of $z^n + c$, the ray at angle θ' lands at c. Thus, the parameter ray $R_n(\theta')$ lands at c in $\mathcal{M}_n$, implying $\phi' \in \mathrm{Jump}_{n,j}(\Omega_n(c))$. □

Proof of Theorem I.7. Given $\lambda \in \mathcal{P}_n$, choose $c \in M_n(\lambda)$ within the subwake $\Big(0, \frac{1}{n-1}\Big)$ in the parameter plane. By Proposition VI.13, λ and $\mathcal{E}_{n,j}(\lambda)$ have the same dynamics on their postsingular sets, which shows (1) in Definition VII.1. The property (3) is clear by Proposition VII.12.

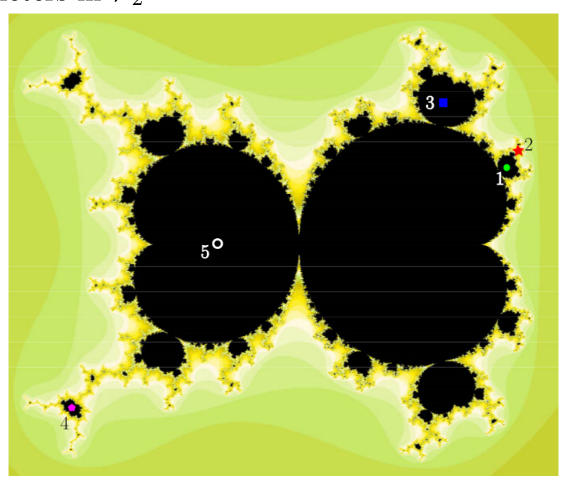
We show that property (2) is true: given $\lambda \lhd \mu$, there exist angles $\theta, \theta' \in \Theta_n(\lambda)$ that land at the same point in $\mathcal{M}_n$, and $\alpha \in \Theta_n(\mu)$ such that $\theta < \alpha < \theta'$. Without loss of generality, we may



(a) A few PCF parameters in $\mathcal{P}_2$



(b) An illustration of $\mathcal{E}_{2,0} : \mathcal{P}_2 \longrightarrow \mathcal{P}_3$



(c) An illustration of $\mathcal{E}_{2,1} : \mathcal{P}_2 \longrightarrow \mathcal{P}_3$

Figure 7.3: Images of a few parameters in $\mathcal{P}_2$ under $\mathcal{E}_{2,0}$ and $\mathcal{E}_{2,1}$. Parameters in the top picture are mapped to those in the bottom row pictures with matching number labels

assume that the landing point in $\mathcal{M}_n$ of the parameter rays at angles θ, θ' is within the subwake $\left(0, \frac{1}{n-1}\right)$. By monotonicity of $\text{Jump}_{n,j}$, we have $\text{Jump}_{n,j}(\theta) < \text{Jump}_{n,j}(\alpha) < \text{Jump}_{n,j}(\theta')$. By definition of $\mathcal{E}_{n,j}$, for all $\widehat{\lambda} \in \mathcal{P}_n$, $\text{Jump}_{n,j}(\Theta_n(\widehat{\lambda})) \subseteq \Theta_{n+1}(\mathcal{E}_{n,j}(\widehat{\lambda}))$. This shows that $\mathcal{E}_{n,j}(\lambda) \lhd \mathcal{E}_{n,j}(\mu)$.

Lastly, we show $\mathcal{E}_{n,j}$ is injective. Suppose $\mathcal{E}_{n,j}(\lambda) = \mathcal{E}_{n,j}(\lambda') = \mu$ for $\lambda \neq \lambda'$. Pick monic representatives c, c' for λ, λ' respectively that are in the sub-wake $\left(0, \frac{1}{n-1}\right)$.

- If c is critically periodic, let θ, θ' be the companion pair in $\Omega_n(c)$ and α, α' be the companion pair in $\Omega_n(c')$. Without loss of generality, assume

$$0 < \theta < \theta' < \alpha < \alpha' < \frac{1}{n-1}$$

  which implies

$$0 < \text{Jump}_{n,j}(\theta) < \text{Jump}_{n,j}(\theta') < \text{Jump}_{n,j}(\alpha) < \text{Jump}_{n,j}(\alpha') < \frac{1}{n}$$

  By Proposition VII.5, $(\text{Jump}_{n,j}(\theta), \text{Jump}_{n,j}(\theta'))$, and $(\text{Jump}_{n,j}(\alpha), \text{Jump}_{n,j}(\alpha'))$ are companion pairs. The above inequality implies that they land on different hyperbolic components, but since both components are in the sub-wake $(0, \frac{1}{n})$, the centers of these components cannot both be monic representatives for μ. This presents a contradiction.

- If c (and therefore c') are critically pre-periodic, choose $\theta, \theta' \in \Omega_n(c)$ and $\alpha, \alpha' \in \Omega_n(c')$. Again without loss of generality, we may assume

$$0 < \theta < \theta' < \alpha < \alpha' < \frac{1}{n-1}$$
$$0 < \text{Jump}_{n,j}(\theta) < \text{Jump}_{n,j}(\theta') < \text{Jump}_{n,j}(\alpha) < \text{Jump}_{n,j}(\alpha') < \frac{1}{n}$$

  Let x be the landing point of $\text{Jump}_{n,j}(\theta)$. Proposition VII.15, $\Omega_{n+1}(x) = \text{Jump}_{n,j}(\Omega_n(c))$, and so $\text{Jump}_{n,j}(\alpha)$ and $\text{Jump}_{n,j}(\alpha')$ land at $y \neq x$. Since x and y are both in the sub-wake $(0, \frac{1}{n})$, they are not both monic representatives of μ, which is a contradiction.

□

Figure 7.3 illustrates $\mathcal{E}_{2,0}, \mathcal{E}_{2,1} : \mathcal{P}_2 \longrightarrow \mathcal{P}_3$ on a few input points.

Remark VII.16 (Distinctness of the $\mathcal{E}'_{n,j}$s). Given any degree n, let θ be pre-periodic angle under μ_n whose orbit intersects every static sector. For example, as in Remark VI.16, we can take θ to be $0\overline{x_1 x_2 \cdots x_{n-1}}$ in base n where $x_i = i$ for $i = 1, 2, \cdots n-1$. Then for distinct elements $j, \widehat{j}$ in $\{0, 1, \cdots, n-1\}$, we saw that $\text{Jump}_{n,j}(\theta) \neq \text{Jump}_{n,\widehat{j}}(\theta)$. Moreover, it is easy to see that the angles $\text{Jump}_{n,j}(\theta)$ and $\text{Jump}_{n,\hat{j}}(\theta)$ do not share a degree $n+1$ kneading sequence . Let $\lambda \in \mathcal{P}_n$ be the unique element with $\theta \in \Theta_n(\lambda)$. By the above discussion, we have $\mathcal{E}_{n,j}(\lambda) \neq \mathcal{E}_{n,\hat{j}}(\lambda)$.

CHAPTER VIII

Dynamical Approximations for Postsingularly Finite Exponentials by Unicriticals

Our focus in this chapter is to prove Theorem I.5. With this end in view, throughout this chapter we fix $\lambda \in \mathcal{P}$ and $\underline{s} \in \Theta_\infty(\lambda)$. Let ℓ and k be the pre-period and period respectively of $\underline{s}$ under left shift. First we define a sequence of polynomial Thurston maps $\mathcal{G}_{n,\underline{s}} : (\mathbb{R}^2, A_{\underline{s}}) \circlearrowleft$ that converge topologically to $\mathcal{G}_{\underline{s}}$.

Next we obtain a sequence of angles (θ_n) such that $\mathcal{G}_{n,\underline{s}}$ and $\mathcal{F}_{n,\theta_n}$ are Thurston equivalent for every n. We prove that these angles θ_n also satisfy the growth condition

$$(n-1)\theta_n \equiv \frac{(n-1)Q(n)}{n^\ell(n^k-1)} \pmod 1$$

for some polynomial Q with integer coefficients that satisfies $\deg Q \leqslant \ell + k - 2$. Letting λ_n be the unique point in $\mathcal{P}_n$ such that $\theta_n \in \Theta_n(\lambda_n)$, we then show that $p_{n,\lambda_n} \to p_\lambda$ as required.

VIII.1: Construction of the maps $\mathcal{G}_{n,\underline{s}}$

Let $N(\underline{s}) = 1 + 2(\max_m |s_m| + 1)$.

Recall the construction of the graphs $S_\infty^{ext}(\underline{s})$ and $S_\infty(\underline{s})$ from Section II.4.2, and let e_1 denote the singular value of the map $\mathcal{G}_{\underline{s}} : S_\infty^{ext}(\underline{s}) \to S_\infty(\underline{s})$.

Choose a real number $R > 0$ such that $\mathbb{D}_R$ contains $A_{\underline{s}}$ and $\mathbb{D}(e_1, 1/R)$ does not contain any other points of $A_{\underline{s}}$. Let $\widehat{\beta}_R = \partial\mathbb{D}_R$ and consider the lift $\beta = \mathcal{G}_{\underline{s}}^{-1}(\widehat{\beta}_R)$. This is a simple, unbounded arc on $\mathbb{R}^2$. Let γ_1 be the leg of $S_\infty^{ext}(\underline{s})$ landing at e_1. Since $\widehat{\beta}_R$ intersects γ_1 exactly once, the path β intersects each leg p_r exactly once, and divides $\mathbb{R}^2$ into two connected components: one of them contains $\mathcal{G}_{\underline{s}}^{-1}(A_{\underline{s}})$, and the other one contains some right half plane.

Similarly, let $\widehat{\alpha}_R = \partial\mathbb{D}(e_1, 1/R)$. Then $\alpha_R = \mathcal{G}_{\underline{s}}^{-1}(\widehat{\alpha}_R)$ is a simple unbounded arc in $\mathbb{C}$, and divides $\mathbb{C}$ into two connected components: one of them contains $\mathcal{G}_{\underline{s}}^{-1}(A_{\underline{s}})$, and the other one contains some left half plane.

Let $N = N(\underline{s})$. We note that $|s_m| \leqslant \frac{N-3}{2}$ for all $m \in \mathbb{N}$. Given $n \geqslant N$, let $r_n = -\lfloor n - 3/2 \rfloor$.

Then $A_{\underline{s}}$ is contained in the infinite closed strip bounded by p_{r_n} and p_{r_n+n-1}, and this strip intersects exactly n lines $\{p_{r_n}, p_{r_n+1}, \cdots, p_{r_n+n-1}\}$. Let $(R_n)_{n \geqslant N}$ be a sequence of real numbers such that $R_n \to \infty$, the disk $\mathbb{D}_{R_n}$ contains $A_{\underline{s}}$, and the disk $\mathbb{D}(e_1, 1/R_n)$ does not contain any points of $A_{\underline{s}}$ other than e_1. For $n \geqslant N$, let D_n be the compact region bounded by $\alpha_{R_n}, \beta_{R_n}, p_{r_n}$ and p_{r_n+n-1}, and D'_n be the closure of the connected component of $\mathbb{C} \backslash (\alpha_{2R_n} \cup p_r \cup p_{n+r-1})$ containing $A_{\underline{s}}$.

Let $\widehat{K}_n = S_\infty(\underline{s}) \cap D'_n$. Define a new graph K_n from $\widehat{K}_n$ by contracting all points on $\alpha_{2R_n} \cap \widehat{K}_n$ to a single new vertex a_n, and attaching a new vertex at ∞. This can be done in such a way that $K_n \cap D_n = S_\infty(\underline{s}) \cap D_n$. By this construction, it is easy to see that (K_n, a_n) is a generalized spider.

Proposition VIII.1. *For every $n \geqslant N(\underline{s})$, the map $\mathcal{G}_{\underline{s}}|D_n$ as above can be extended to a degree n Thurston map $\mathcal{G}_{n,\underline{s}} : (\mathbb{R}^2, A_{\underline{s}}) \circlearrowleft$. The sequence $(\mathcal{G}_{n,\underline{s}})$ converges toopologically to $\mathcal{G}_{\underline{s}}$.*

Proof. Let $\mathcal{G}_{n,\underline{s}} = \mathcal{G}_{\underline{s}}|D_n$.

Let P be the set $\{p_{r_n}, p_{r_n+1}, \cdots, p_{r_n+n-1}\}$. For every $j \in \{r_n, r_n+1, \cdots r_n+n-1\}$, let $\widehat{q}_j$ be the unique edge of K_n that begins at a_n and coincides with p_j on D_n, and $q_j \subset \widehat{q}_j$ be the segment from a_n to the unique point $b_n \in p_j \cap \alpha_{R_n}$. Observe that the intersection $\gamma_1 \cap \partial\mathbb{D}(e_1, 1/R_n)$ contains a single point, namely $\mathcal{G}_{\underline{s}}(b_n)$.

We extend $\mathcal{G}_{n,\underline{s}}$ to q_j (and hence, $\widehat{q}_j$) by mapping q_j homeomorphically to $\gamma_1 \cap \overline{\mathbb{D}(e_1, 1/R_n)}$ in such a way that $\mathcal{G}_{n,\underline{s}}(a_n) = e_1$. Similarly, for each $j \in \{r_n, r_n+1, \cdots r_n+n-2\}$, let Δ_j be the triangular region bounded by q_j, q_{j+1} and α_{R_n}. Using the Alexander trick, we can define $\mathcal{G}_{n,\underline{s}}$ on $\mathrm{int}(\Delta_j)$ so that it maps homeomorphically to $\mathbb{D}(e_1, \frac{1}{R_n}) \backslash \gamma_1$.

Similarly, for every j as above, we extend $\mathcal{G}_{n,\underline{s}}$ to the unbounded component of $D'_n \backslash (p_j \cup p_{j+1} \cup \beta_{R_n})$ so that said component maps homeomorphically onto $\mathbb{C} \backslash (\overline{\mathbb{D}_{R_n}} \cup \gamma_1)$.

Lastly, let Δ be the unique unbounded face of $\mathbb{C} \backslash \widehat{K}_n$. By the Alexander trick, we can extend $\mathcal{G}_{n,\underline{s}}$ to $\overline{\Delta}$ so that $\mathrm{int}(\Delta)$ is mapped homeomorphically to $\mathbb{C} \backslash \gamma_1$.

By this construction, for every $n \geqslant N(\underline{s})$, $\mathcal{G}_{n,\underline{s}}$ maps K_n onto $S_\infty(\underline{s})$, and its restriction to the plane is a degree n Thurston map with postsingular set $\mathcal{A}_{\underline{s}}$. Since (D_n) is an exhaustion of the plane and $\mathcal{G}_{n,\underline{s}}|D_n = \mathcal{G}_{\underline{s}}$, we see that the sequence $(\mathcal{G}_{n,\underline{s}})$ converges topologically to $\mathcal{G}_{\underline{s}}$. □

Remark VIII.2. For every $n \geqslant N(\underline{s})$, we note that the legs of the generalized spider (K_n, a) coincide near ∞ with the legs of $S_\infty^{ext}(\theta)$ bounded between p_{r_n} and p_{r_n+n-1}.

VIII.2: Construction of the angles (θ_n)

Definition VIII.3. Given $\ell \in \mathbb{Z}_{\geqslant 0}$, $k \in \mathbb{Z}_{\geqslant 1}$ and $Q \in \mathbb{Z}[X]$, let $\kappa_Q : \mathbb{Z}_{\geqslant 2} \longrightarrow \mathbb{Q}/\mathbb{Z}$ be the function $n \mapsto \frac{(n-1)Q(n)}{n^\ell(n^k-1)} \pmod 1$.

Proposition VIII.4. *For $n \geqslant \max_m |s_m|$, let $\theta_n(\underline{s})$ be the angle with n-adic expansion $.x_1(n)x_2(n)\cdots x_\ell(n)\overline{x_{\ell+1}(n)x_{\ell+2}(n)\cdots x_{\ell+k}(n)}$ given by*

$$x_m(n) = \begin{cases} s_m & s_m \geqslant 0 \\ n - |s_m| & s_m < 0 \end{cases}$$

for all $m \in \mathbb{N}$.

Then there exists an integer $j \in \{0, 1, \cdots, N(\underline{s}) - 1\}$ and a polynomial $Q \in \mathbb{Z}[X]$ with $\deg Q \leqslant \ell + k - 2$ such that

$$\begin{aligned} \theta_{n+1}(\underline{s}) &\equiv \mathrm{Jump}_{n,j}(\theta_n(\underline{s})) \pmod 1 \\ (n-1)\theta_n(\underline{s}) &\equiv \kappa_q(n) \pmod 1 \end{aligned}$$

for all $n \geqslant N(\underline{s})$.

Remark VIII.5. We note that the expansions $.x_1(n)x_2(n)\cdots x_\ell(n)\overline{x_{\ell+1}(n)x_{\ell+2}(n)\cdots x_{\ell+k}(n)}$ of $\theta_n(\underline{s})$ given above also "converge" to $\underline{s} = s_1 s_2 \cdots$ in a combinatorial sense.

Before we prove the above proposition, we will need one more statement.

Proposition VIII.6. *Given integers $\ell \geqslant 0, k \geqslant 1$ and polynomials $Q, \widehat{Q} \in \mathbb{Z}[X]$, if there exists a polynomial $H \in \mathbb{Z}[X]$ such that $\widehat{Q}(X) - Q(X) = H(X)X^\ell(X^k - 1 + X^{k-2} + . \cdots + 1)$, then the functions κ_Q and $\kappa_{\widehat{Q}}$ coincide.*

Proof. We note that in $\mathbb{Q}[X]$,

$$\begin{aligned} \frac{(X-1)Q(X)}{X^\ell(X^k-1)} - \frac{(X-1)\widehat{Q}(X)}{X^\ell(X^k-1)} &= \frac{(X-1)H(X)X^\ell(X^{k-1} + X^{k-1} + \cdots + 1)}{X^\ell(X^k-1)} \\ &= H(X) \end{aligned}$$

Since $H \in \mathbb{Z}[X]$, we have $\kappa_Q(n) - \kappa_{\widehat{Q}}(n) = H(n) \equiv 0 \pmod 1$ for all $n \in \mathbb{Z}_{\geqslant 2}$. ☐

Proof of Proposition VIII.4. Let $N = N(\underline{s}) = 1 + 2(\max_m |s_m| + 1)$. For all $m \in \mathbb{N}$, and

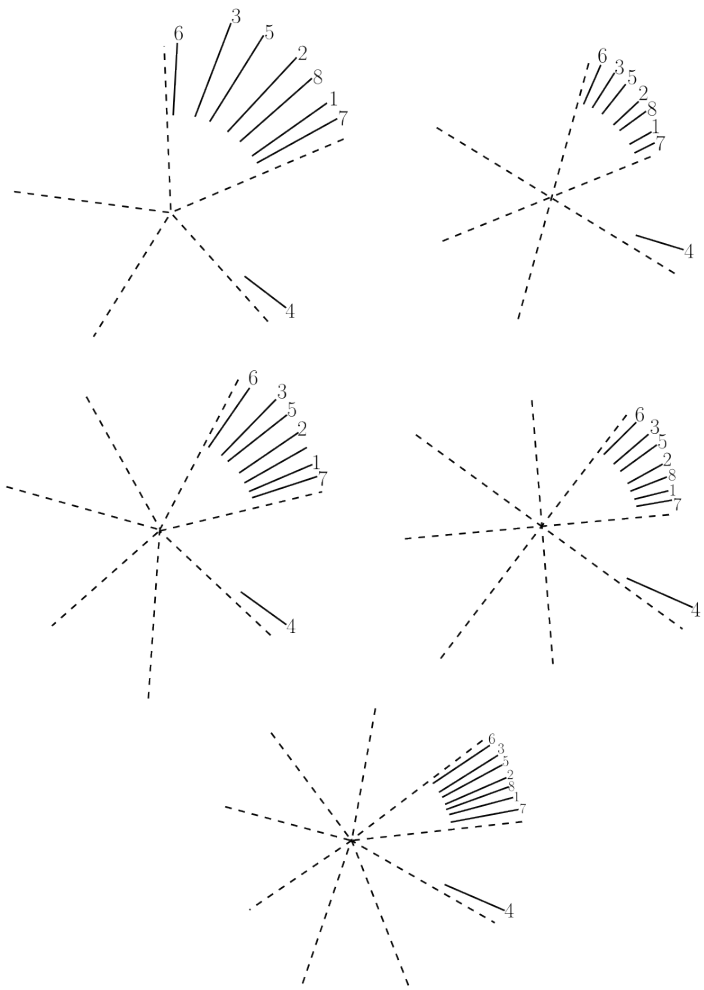


Figure 8.1: The standard spiders $\widehat{S}_n(\theta_n(\underline{s}))$ for $n = 5, 6, 7, 8, 9$, when $\underline{s}$ is set to $000(-1)\overline{0010}$. The dotted lines indicate the additional legs in $S_n^{ext}(\theta_n(\underline{s}))$. The leg labelled m corresponds to $\mu_n^{\circ(m-1)}(\theta_n(\underline{s}))$. Compare with Figure 2.5.

$n \geqslant N$,

$$s_m \geqslant 0 \implies 0 \leqslant x_m(n) \leqslant \frac{N-3}{2}$$
$$s_m < 0 \implies \frac{N+3}{2} \leqslant x_m(n) \leqslant n-1$$

Recall that $s_1 = 0$. The interval $(\frac{N-3}{2}, \frac{N+3}{2})$ contains at least one integer j in $\{1, 2, \cdots, N-1\}$, and for any such j, for all $n \geqslant N$, it is easy to see that

$$\theta_n \in \Big[0, \frac{1}{n-1}\Big)$$
$$\theta_{n+1}(\underline{s}) \equiv \mathrm{Jump}_{n,j}(\theta_n(\underline{s})) \pmod 1$$

We can think of the coefficients $x_m(n)$ as values taken by polynomials $x_m \in \mathbb{Z}[X]$. More particularly, for each $m \in \mathbb{N}$, define

$$x_m(X) = \begin{cases} s_m & s_m \geqslant 0 \\ X - |s_m| & s_m < 0 \end{cases}$$

Let $\widehat{Q} \in \mathbb{Z}[X]$ be the polynomial given by

$$\widehat{Q}(X) = \sum_{m=1}^{\ell} x_m\big(X\big) X^{\ell-m}(X^k - 1) + \sum_{m=1}^{k} x_{\ell+m}\big(X\big) X^{k-m}$$

Then,

$$\begin{aligned} \frac{\widehat{Q}(n)}{n^\ell(n^k-1)} &= \sum_{m=1}^{\ell} \frac{x_m(n)}{n^m} + \sum_{m=1}^{k} \frac{x_{\ell+m}(n)}{n^{m+\ell-k}(n^k-1)} \\ &= \sum_{m=1}^{\ell} \frac{x_m(n)}{n^m} + \sum_{m=\ell+1}^{\ell+k} \frac{x_m(n)}{n^m\Big(1-\frac{1}{n^k}\Big)} \\ &\equiv \theta_n(\underline{s}) \pmod 1 \end{aligned}$$

Since $x_1(X) = s_1 = 0$ and $x_2(X)$ is either constant or linear in X, the degree of $\widehat{Q}$ is less than or equal to $\ell + k - 1$. If $\deg \widehat{Q} \leqslant \ell + k - 2$, set $Q = \widehat{Q}$. Otherwise, note that $\widehat{Q}$ has degree $\ell + k - 1$ if and only if $x_2(X) = X - |s_2|$. In this case, the leading coefficient of $\widehat{Q}$ is 1, and we set $Q(X) = \widehat{Q}(X) - X^\ell(X^{k-1} + X^{k-2} + ... + X + 1)$.

The polynomial $Q \in \mathbb{Z}[X]$ thus defined has degree $\leqslant \ell + k - 2$, and by Proposition VIII.6, it satisfies

$$\kappa_Q(n) \equiv \kappa_{\hat{Q}}(n) \equiv (n-1)\theta_n(\underline{s}) \pmod 1 \ \ \forall n \in \mathbb{Z}_{\geqslant 2}$$

□

Example VIII.7. Given $r \in \mathbb{Z}$, for $\underline{s} = 0\overline{r}$, we have two separate cases:

- If $r \geqslant 0$, then $N(\underline{s}) = 2r + 3$, $\theta_n(\underline{s}) = \frac{r}{n(n-1)} = 0\overline{r}$ in base n, for all $n \geqslant N(\underline{s})$, and $Q(X) = r$.
- If $r < 0$, then $N(\underline{s}) = 2|r| + 3$, $\theta_n(\underline{s}) = \frac{n-|r|}{n(n-1)} = 0\overline{n-|r|}$ in base n, for all $n \geqslant N(\underline{s})$, and $Q(X) = -|r| = r$.

Example VIII.8. Let $\underline{s} = 000(-1)\overline{0010}$. We have $N(\underline{s}) = 5$, and for all $n \geqslant N$,

$$\theta_n(\underline{s}) = .000(n-1)\overline{0010} = \frac{n^5 - n^4 + 1}{n^4(n^4-1)}$$

We note that $Q(X) = X^4 + X - 1$.

In fact, letting $\theta = \frac{17}{2^4(2^4-1)}$, we have $\theta_n(\underline{s}) = \mathrm{Jump}_{n-1,1} \circ \mathrm{Jump}_{n-2,1} \circ \cdots \circ \mathrm{Jump}_{2,1}(\theta)$ for all $n \geqslant N(\underline{s})$. Compare this with Example VI.4.

Next we show that for all $n \geqslant N(s)$, with $\theta_n(\underline{s})$ defined as in Proposition VIII.4, the map $\mathcal{G}_{n,\underline{s}}$ from Section VIII.1 and the polynomial Thurston map $\mathcal{F}_{n,\theta_n}$ from Definition II.4.1 are Thurston equivalent.

Proposition VIII.9. *Let $N := N(\underline{s})$, $\theta := \theta_N(\underline{s})$ be as defined in Proposition VIII.4. Let $r = \frac{3}{2} - \frac{N}{2}$. The circular order of the legs of $S_N^{ext}(\theta)$ at ∞ coincides with the circular order of the set of addresses $\mathcal{O}_\infty(\underline{s}) \cup \{r\underline{s}, (r+1)\underline{s}, \cdots, (r+n-1)\underline{s}\}$.*

Proof. It suffices to prove the following claims:

Claim 1. The angles in $\mathcal{O}_N(\theta)$ and the addresses in $\mathcal{O}_\infty(\underline{s})$ have the same circular order.

Claim 2. For every $j \in \{r, r+1, \cdots, \frac{N-3}{2}\}$,

$$\text{if } j \geqslant 0, \text{ then } \mu^{\circ(m-1)}(\underline{s}) \in T^{stat}_{\infty,j}(\underline{s}) \implies \mu_N^{\circ(m-1)}(\theta) \in T^{stat}_{N,j}(\theta)$$
$$\text{if } j < 0, \text{ then } \mu^{\circ(m-1)}(\underline{s}) \in T^{stat}_{\infty,j}(\underline{s}) \implies \mu_N^{\circ(m-1)}(\theta) \in T^{stat}_{N,N+j}(\theta)$$

Proof of Claim 1. Suppose $\mu_N^{\circ(m-1)}(\theta) < \mu_N^{\circ(m'-1)}(\theta)$ for some integers $m \neq m'$, then

$$x_m(N)x_{m+1}(N)\cdots < x_{m'}(N)x_{m'+1}(N)\cdots$$

where the digits $x_m(N)$ are as defined in Proposition VIII.4. Let $r \geqslant 0$ be the least integer such that $x_{m+r}(N) \neq x_{m'+r}(N)$. Then for all $r' < r$, there are two possibilities:

- If $0 \leqslant x_{m+r'}(N) \leqslant \frac{N-3}{2}$, then $s_{m+r'} = x_{m+r'}(N) = x_{m'+r'}(N) = s_{m+r'}$.
- Else, we have $\frac{N+3}{2} \leqslant x_{m+r'}(N) \leqslant N-1$, and thus, $s_{m+r'} = x_{m+r'}(N) - N = x_{m'+r'}(N) - N = s_{m'+r'}$

This implies, for all $r' < r$, that $s_{m+r'} = s_{m'+r'}$.

At the index $m+r$ we have $x_{m+r}(N) < x_{m'+r}(N)$. There are only three possibilities:

- If $x_{m'+r}(N) \leqslant \frac{N-3}{2}$, then $s_{m+r} = x_{m+r}(N) < x_{m'+r}(N) = s_{m'+r}$.
- If $x_{m+r}(N) \geqslant \frac{N+3}{2}$, then $s_{m+r} = x_{m+r}(N) - N < x_{m'+r}(N) - N = s_{m'+r}$.
- If $x_{m+r}(N) \leqslant \frac{N-3}{2}$ and $x_{m'+r}(N) \geqslant \frac{N+3}{2}$, then $s_{m+r} > 0$ and $s_{m'+r} < 0$.

In the first two cases, we directly get $\mu^{\circ(n-1)}(\underline{s}) < \mu^{\circ(m-1)}(\underline{s})$. In the third case, $\mu^{\circ(n-1)}(\underline{s}) > \overline{0}$ and $\mu^{\circ(m-1)}(\underline{s}) < \overline{0}$. Thus $\mu^{\circ(n-1)}(\underline{s}) << \mu^{\circ(m-1)}(\underline{s})$.

Lastly, if $\mu_N^{\circ(m-1)}(\theta) = \frac{\theta+r}{N}$, then $x_m(N) = r$ and $x_{m'}(N) = 0$ for all $m' > m$. The latter condition implies $s_{m'} = 0$ for all $m' > m$.

- If $0 \leqslant r \leqslant \frac{N-3}{2}$, then $\mu^{\circ(m-1)}(\underline{s}) = r\overline{0}$.
- If $\frac{N-3}{2} \leqslant r \leqslant N-1$, then $\mu^{\circ(m-1)}(\underline{s}) = -c\overline{0}$.

■

Proof of Claim 2. First assume $j \geqslant 0$. If $\mu^{\circ(m-1)}(\underline{s}) \in T^{stat}_{\infty,j}(\underline{s})$, then there are two cases:

- If $j\underline{s} < \mu^{\circ(m-1)}(\underline{s}) < (j+1)\overline{0}$, then we note that $s_m = j$. Thus $x_m(N) = s_m = j$. This assumption also implies that $\underline{s} < \mu^{\circ m}(\underline{s}) < \overline{0}$. By the proof in Claim one, we have $\theta < \mu_N^{\circ m}(\theta)$ in $[0,1)$, and thus, $\frac{\theta+j}{N} < \mu_N^{\circ(m-1)}(\theta) < \frac{j+1}{N}$.
- Else, we must have $(j+1)\overline{0} < \mu^{\circ(m-1)}(\underline{s}) < (j+1)\underline{s}$. In this case, $x_m(N) = s_m = j+1$, and by an argument similar to the one above, we see that $\frac{j+1}{N} < \mu_N^{\circ(m-1)}(\theta) < \frac{\theta+j+1}{N}$ are in counterclockwise order.

This shows that $\mu_N^{\circ(m-1)}(\theta) \in T^{stat}_{N,j}(\theta)$.

The case $j < 0$ follows from a similar discussion, and using the fact that if $s_m < 0$, then $x_m(N) = N + s_m$ by definition. ■

□

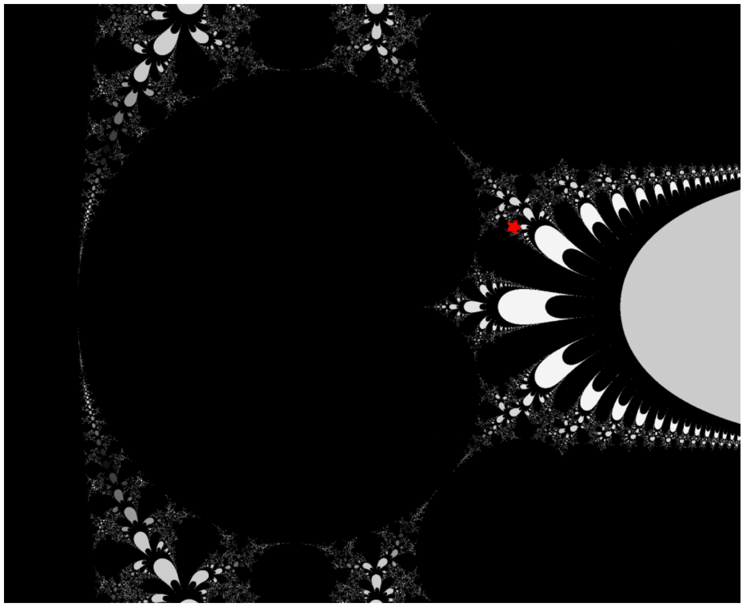


Figure 8.2: The star marks the position of $\lambda \approx 1.16302 + 0.71056i$, the landing point of the parameter ray at address $000(-1)\overline{0010}$ in the exponential parameter plane, indicated in blue

Proposition VIII.10. *Let $N = N(\underline{s})$. For all $n \geqslant N(\underline{s})$, the maps $\mathcal{F}_{n,\theta_n}$ and $\mathcal{G}_{n,\underline{s}}$ are Thurston equivalent.*

Proof. Using the general argument followed in Proposition VIII.9, it is easy to see that for every $n \in N(\underline{s})$, the generalized spider K_n from Section VIII.1 is congruent to $S_n^{ext}(\theta_n(\underline{s}))$ for all $n \geqslant N$. Therefore, by Proposition II.51,the maps $\mathcal{F}_{n,\theta_n(\underline{s})}$ and $\mathcal{G}_{n,\underline{s}}$ are Thurston equivalent for every $n \in \mathbb{N}$.

□

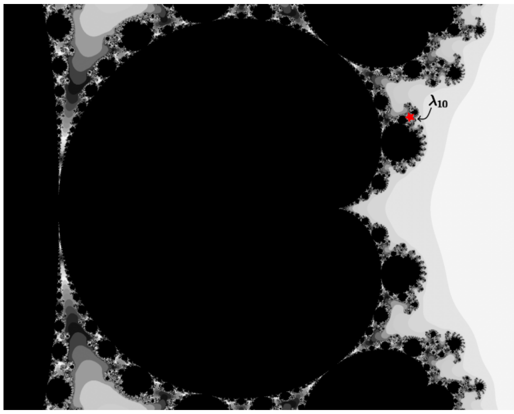
VIII.3: Proof of Theorem I.5

Let $\mathcal{G}_{n,\underline{s}} : (\mathbb{R}^2, A_{\underline{s}}) \circlearrowleft, n \geqslant N := N(\underline{s})$ be the sequence of Thurston maps constructed in Proposition VIII.1. By Main Theorem I.4, the operators $\sigma_n := \sigma_{\mathcal{G}_{n,\underline{s}}}$ defined on $\mathrm{T}(\mathbb{S}^2, A_{\underline{s}} \cup \{\infty\})$ converge locally uniformly to the operator $\sigma := \sigma_{\mathcal{G}_{\underline{s}}}$. Let $\tau \in \mathrm{T}(\mathbb{S}^2, A_{\underline{s}} \cup \{\infty\})$ be the fixed point of σ.

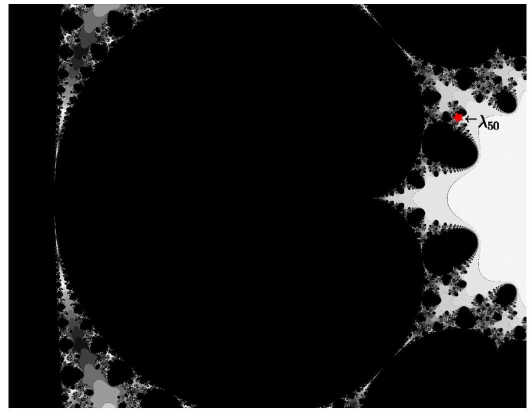
Fix $n \geqslant N$. Let $\lambda_n \in \mathcal{P}_n$ be the unique point such that $\theta_n(\underline{s})) \in \Theta_n(\lambda_n)$. Since $\mathcal{G}_{n,\underline{s}} \simeq_{\text{comb}} \mathcal{F}_{n,\theta_n(\underline{s})} \simeq_{\text{comb}} p_{n,\lambda_n}$, there exists a unique fixed point $\tau_n \in \mathrm{T}(\mathbb{S}^2, A_{\underline{s}} \cup \{\infty\})$ for the operator σ_n.

Proposition VIII.11. *The sequence of polynomials (p_{n,λ_n}) converges to p_λ locally uniformly.*

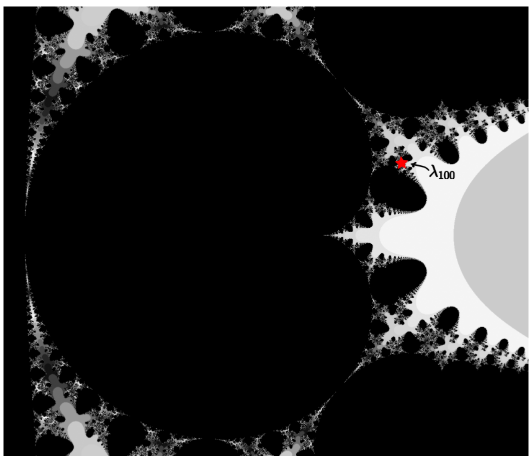
Proof. It is possible to prove this using Theorem V.9; however, we give a different proof here.



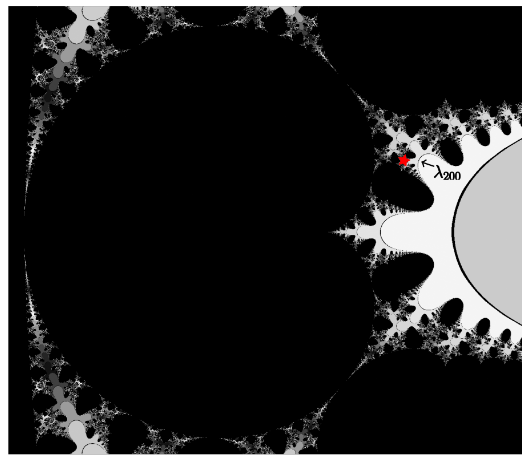
(a) $\lambda_{10} \approx 1.1176 + 0.86608i$



(b) $\lambda_{50} \approx 1.1545 + 0.74281i$



(c) $\lambda_{100} \approx 1.1575 + 0.72671i$



(d) $\lambda_{200} \approx 1.15891 + 0.71869i$

Figure 8.3: Approximating parameters (λ_n) for $\lambda \approx 1.16302 + 0.71056i$, angular coordinates given by $000(n-1)\overline{0010}$ in base n. Compare with Figure 8.2

Recall that $e_2 = \mathcal{G}_{\underline{s}}(e_1) = \mathcal{G}_{n,\underline{s}}(e_1)$ and $e_3 = \mathcal{G}_{\underline{s}}(e_2) = \mathcal{G}_{n,\underline{s}}(e_2)$ for all $n \geqslant N$. It suffices to show that $\lambda_n \to \lambda$.

By Corollary III.13, we know that $d_{\mathrm{T}}(\tau_n, \tau) \to 0$ as $n \to \infty$. Let $\varphi \in \tau$, $\varphi_n \in \tau_n, n \geqslant N$ be representatives that satisfy

$$\begin{aligned} \varphi(\infty) = \varphi_n(\infty) &= \infty \\ \varphi(e_1) = \varphi_n(e_1) &= 0 \\ \varphi(e_2) &= \lambda \\ \varphi_n(e_2) &= \lambda_n \end{aligned}$$

for all $n \geqslant N$.

There exists $\psi \in \tau$ that satisfies $\psi(\infty) = \infty$, $\psi(e_1) = 0$, and $\psi(e_2) = \lambda$. We note that

$\varphi \circ \mathcal{G}_{\underline{s}} \circ \psi^{-1} \equiv p_\lambda$.

Similarly, for every $n \neq N$, there exists $\psi_n \in \tau_n$ that satisfies $\psi_n(\infty) = \infty$, $\psi_n(e_1) = 0$ and $\psi_n(e_2) = \lambda_n$. We note that $\varphi_n \circ \mathcal{G}_{n,\underline{s}} \circ \psi_n^{-1} \equiv p_{n,\lambda_n}$. Let $\widetilde{\varphi}_n = \frac{\lambda}{\lambda_n}\varphi_n$. Note that $\widetilde{\varphi}_n \in \tau_n$, and $\widetilde{\varphi}_n \circ \mathcal{G}_{n,\underline{s}} \circ \psi_n^{-1} \equiv p_{n,\lambda}$.

Since $\tau_n \to \tau$, there exists a quasiconformal map $k : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ isotopic to φ rel. $A_{\underline{s}} \cup \{\infty\}$ and for every $n \geqslant N$, a quasiconformal map $k_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ isotopic to $\widetilde{\varphi}_n$ rel. $A_{\underline{s}} \cup \{\infty\}$ such that $K(k_n \circ k^{-1}) \to 1$. Then the sequence of maps $k_n \circ k^{-1}$ fix the three points ∞, λ and 0. By Propositions II.5 and V.8, $h_n := k_n \circ k^{-1}|\mathbb{C} \to \mathrm{id}_{\mathbb{C}}$ uniformly on compact subsets of $\mathbb{C}$. Let $c = \varphi(e_3) = k(e_3)$ and $c_n = h_n(c)$; observe that

$$\widetilde{\varphi}_n(e_3) = k_n(e_3) = c_n \to c = \varphi(e_3)$$

We also note that $c_n = p_{n,\lambda}(\lambda_n)$, and $c = p_\lambda(\lambda)$. Letting $r_n = |c_n/\lambda|$, $r = |c/\lambda|$, $w_n = \arg(c_n/\lambda)$ and $w = \arg(c/\lambda)$, (where $\arg(c/\lambda), \arg(c_n/\lambda)$ are all chosen to be in $[-\pi, \pi)$), we have

$$\lambda = \ln r + wi + 2\pi i(s_2 - s_1) = \ln r + wi + 2\pi i s_2$$
$$\lambda_n = n\Big(r_n^{1/n} \exp\Big(\frac{iw_n}{n}\Big) \exp\Big(\frac{2\pi i(x_2 - x_1)}{n}\Big) - 1\Big)$$

where $x_1x_2x_3....$ is the $n-$adic expansion of $\theta_n(\underline{s})$ given in Proposition VIII.4. In particular, we have

$$x_1 = s_1 = 0,$$
$$x_2 = \begin{cases} s_2 & s_n \geqslant 0 \\ n + s_2 & \text{otherwise} \end{cases}$$

Hence, we have $\exp(2\pi i(x_2 - x_1)/n) = \exp(2\pi i s_2/n)$. So the equation for λ_n becomes

$$\lambda_n = n\Big((r_n/r)^{1/n} \exp\Big(\frac{iw_n + 2\pi i s_2}{n}\Big) - 1\Big)$$

Let U be a bounded neighborhood of c that contains no other point in $\widetilde{\varphi}(A_{\underline{s}})$, and V be the connected component of $p_\lambda^{-1}(U)$ containing λ. Similarly, let V_n be the connected component of $p_{n,\lambda}^{-1}(U)$ containing λ. Then, using an argument similar to the one in the proof of Proposition V.3, we see that for n sufficiently large, there exist local inverses $q_n : U \longrightarrow V_n$ of $p_{n,\lambda}|V_n$, such that $q_n \to q$ uniformly on U, where $q : U \longrightarrow V$ is the local inverse of $p_\lambda|V$.

We note that q_n and q have the following formulae: for all $z \in U$, we have

$$q(z) = \ln|z/\lambda| + i\arg(z/\lambda) + 2\pi i s_2$$
$$q_n(z) = n\Big(|z/\lambda|^{1/n}\exp\Big(\frac{i\arg(z/\lambda) + 2\pi i s_2}{n}\Big) - 1\Big)$$

where $\arg(z/\lambda)$ is chosen to be in $[-\pi, \pi)$. Comparing with the formula for λ_n, we see that for n sufficiently large so that $c_n \in U$, we have

$$\lambda_n = q_n(c_n) \to q(c) = \lambda$$

□

Proof of Theorem I.5. The theorem follows from Propositions VIII.4, VIII.9, and VIII.11. □

Example VIII.12. Let $\lambda = 2\pi i r$, where $r \in \mathbb{Z}\backslash\{0\}$. The orbit of 0 under p_λ is $0 \longrightarrow 2\pi i r \circlearrowleft$, and λ has a unique external address, $0\overline{r}$. We have

$$\theta_n(\underline{s}) = \begin{cases} \frac{r}{n(n-1)} & r > 0 \\ \frac{n-|r|}{n(n-1)} & r < 0 \end{cases}$$
$$Q(X) = r$$
$$\lambda_n = n\Big(\exp(2\pi i r/n) - 1\Big)$$

Example VIII.13. Let λ be the landing point of the ray at address $\underline{s} = 000(-1)\overline{0010}$ (its approximate value is $1.16302 + 0.71056i$; see Figure 8.2). The orbit of 0 under p_λ has the form $0 \longrightarrow \lambda \longrightarrow p_\lambda(\lambda) \longrightarrow p_\lambda^{\circ 2}(\lambda) \longrightarrow p_\lambda^{\circ 3}(\lambda) \longrightarrow p_\lambda^{\circ 4}(\lambda) \longrightarrow p_\lambda^{\circ 5}(\lambda) \circlearrowleft$.

In Example VIII.8, we computed $\theta_n(\underline{s}) \equiv \frac{n^5 - n^4 + 1}{n^4(n^4-1)} \pmod 1$, with $n-$adic expansion $000(n-1)\overline{0010} =$. In Fig 8.3 we have indicated the position of $\lambda_n(\underline{s})$ for $n = 10, 50, 100$ and 200, along with approximate values.

CHAPTER IX

Future Scope

In this chapter, we pose some questions and start some discussions that are highly relevant to the results presented in this thesis, and connect these to well-known open problems in transcendental Thurston theory.

IX.1: Obstructed Thurston maps

By Theorem I.2 and Proposition IV.15, we know that for every transcendental Thurston map $f : (\mathbb{R}^2, A) \circlearrowleft$, there exists a sequence of polynomial Thurston maps $f_n : (\mathbb{R}^2, A) \circlearrowleft$ such that $\sigma_{f_n} \to \sigma_f$ locally uniformly on $\mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$. We also showed in Corollary III.13 that if f is realized, then f_n is realized for all n sufficiently large. We are interested in the converse:

Question IX.1. *If f_n is realized for all sufficiently large n, is f realized?*

We give a simple sufficient condition for f to be realized: if there exists a compact set $K \subset \mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$ such that a subsequence of fixed points τ_{n_k} of $\sigma_{f_{n_k}}$ are contained in K, then by completeness of the Teichmüller metric, some subsequence of (τ_{n_k}) converges to a point $\tau \in \mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$. It is easy to see that τ is a fixed point of σ_f.

If no such set K exists, or equivalently, the sequence (τ_n) of fixed points of σ_{f_n} contains no bounded subsequence, we then need to consider the *augmented Teichmüller space* $\widehat{\mathrm{T}}(\mathbb{S}^2, A \cup \{\infty\})$ and classify all the limit points of the sequence (τ_n) in this space. Note that $\widehat{\mathrm{T}}(\mathbb{S}^2, A \cup \{\infty\})$ is the metric completion of $\mathrm{T}(\mathbb{S}^2, A \cup \{\infty\}$ endowed with the Weil-Petersson metric (see [Hub16, Chapter 7.7]). A related problem is the boundary behavior of σ_f. Selinger showed in [Sel12] that for a classical Thurston map $g : \mathbb{S}^2 \longrightarrow \mathbb{S}^2$, the operator σ_g extends continuously to $\widehat{\mathrm{T}}(\mathbb{S}^2, P_f)$, and has a fixed point in this space. This raises another question, which is especially curious since all the σ_{f_n}'s admit a continuous extension to $\widehat{\mathrm{T}}(\mathbb{S}^2, A \cup \{\infty\})$:

Question IX.2. *Does σ_f extend to a continuous map on $\widehat{\mathrm{T}}(\mathbb{S}^2, A \cup \{\infty\})$?*

A fixed point in $\widehat{\mathrm{T}}(\mathbb{S}^2, A \cup \{\infty\})$ for σ_f would also provide us with a topological obstruction for f in the form of a topological multicurve.

If f is obstructed by a *Levy cycle*, however, we can show that the f_n's are eventually obstructed as well.

Definition IX.3. Let $f : (\mathbb{R}^2, A) \circlearrowleft$ be a Thurston map. A *Levy cycle* for f is a multicurve $G = \{\gamma_0 = \gamma_n, \gamma_1, \gamma_2, \cdots, \gamma_{n-1}\}$ such that

- for every $i \in \{0, 1, \cdots, n-1\}$, the curve γ_i is an essential simple closed curve contained in $\mathbb{R}^2 \backslash A$;
- if $i \neq j$, then γ_i and γ_j are disjoint;
- for every $i \in \{0, 1, \cdots, n-1\}$, there exists a connected component η of $f^{-1}(\gamma_{i+1})$ such that η is homotopic to γ_{i-1} rel. A, and the map $f|\eta : \eta \to \gamma_{i+1}$ has degree one.

By a theorem credited to Berstein, Lei, Levy and Rees ([Hub16, Theorem 10.3.8]), it is known that a polynomial Thurston map is obstructed if and only if it has a Levy cycle. This creterion was extended in [HSS09] for topological exponential Thurston maps. This proof from [HSS09] generalises to show that if a transcendental Thurston map has a Levy cycle, then it is obstructed. It is not known, however, that the converse is true.

Proposition IX.4. *Let $f_n : (\mathbb{R}^2, A) \circlearrowleft$ be a sequence of Thurston maps that converge topologically to a Thurston map $f : (\mathbb{R}^2, A) \circlearrowleft$. If $G = \{\gamma_0 = \gamma_r, \gamma_1, \gamma_2, ..., \gamma_{r-1}\}$ is a Levy cycle for f, then for sufficiently large n, G is a Levy cycle for f_n.*

Proof. Let $j \in \{1, ..., r\}$, and let η_j be the connected component of $f^{-1}(\gamma_j)$ that is an essential closed curve that is isotopic to γ_{j-1} rel. A, and $f|\eta_j : \eta_j \to \gamma_j$ is a homeomorphism. Let $D \subset \mathbb{R}^2$ be a compact set such that $\eta_j \subset \text{int}(D)$ for all j. Then there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, $f_n|D \equiv f|D$. This implies that η_j is a connected component of $f_n^{-1}(\gamma_j)$, and $f_n|\eta_j$ is a homeomorphism onto γ_j . Thus for all $n \geqslant N$, the multicurve G is a Levy cycle for f_n. □

The above proposition constitutes a partial converse to Corollary III.13.

IX.2: Approximating tree lifting operators

Let $f : \mathbb{C} \to \mathbb{C}$ be a postsingularly finite holomorphic polynomial. By a theorem of Douady and Hubbard, it is known that there exists a finite tree $H_f \subset \mathbb{C}$ such that

1. $P_f \subset V(H_f)$;
2. every vertex of degree less than or equal to 2 is an element of P_f;

3. $f(H_f) = H_f$.

This tree is unique, and can be defined as a union of *regulated arcs* in the filled Julia set of f (we refer to [Hub16, Section 10.4, Definition 10.4.7] for a much more detailed discussion).

Hubbard trees are a powerful combinatorial tool to study PCF polynomials. The question of which Hubbard trees are realized by polynomials, and whether the tree H_f and the dynamical system $f : H_f \circlearrowleft$ are sufficient to determine f were dicussed by Poirier (see [Poi10]).

The preimage $f^{-1}(H_f)$ contains H_f; indeed, H_f is the convex hull of P_f in $f^{-1}(H_f)$. More generally, for any polynomial Thurston map $f : (\mathbb{R}^2, P_f) \circlearrowleft$ with $P_f = A$, for any finite tree $T \subset \mathbb{C}$ that satisfies the first two conditions above, let $\widehat{T} \subset f^{-1}(T)$ be the convex hull of A in $f^{-1}(T)$. Then it can be shown that $\widehat{T}$ also satisfies conditions (1) and (2). The authors of [BLMW22] used this property to construct an operator $\lambda_f : \mathcal{T}_A \to \mathcal{T}_A$, where $\mathcal{T}_A$ is the set of finite trees in $\mathbb{C}$ that satisfy conditions (1) and (2), modulo isotopy rel. A . This space $\mathcal{T}_A$ is countable, and can be realized as a *spine* for $\mathrm{T}(\mathbb{S}^2, A \cup \{\infty\})$. The results of [BLMW22] show that f is realized if and only if for every $[T] \in \mathcal{T}_A$, the λ_f orbit of f lands in a $2-$neighborhood of $[H_f]$ after finitely many steps.

In our context of approximations, if $f_n : (\mathbb{R}^2, A) \circlearrowleft$ is a series of polynomial Thurston maps that converge combinatorially to a transcendental Thurston map $f : (\mathbb{R}^2, A) \circlearrowleft$, we can pose several questions about the sequence of tree-lifting operators $\lambda_{f_n} : \mathcal{T}_A \circlearrowleft$. For example, do they converge? Another question is whether it is possible to define the operator λ_f in this setting. Recent work in ([PRS23]) which defines a *homotopy Hubbbard tree* for PSF exponential maps and further examines lifting properties of finite trees, leaves the scope for this open.

APPENDIX A
Basic Topology

This chapter summarizes the notation used for various topological objects throughout this thesis.

A.1: Loops, paths and homeomorphisms

If X is a topological space, then a *path* α in X is a continuous map $\alpha\colon \mathbb{I} \to X$. Points $x = \alpha(0)$ and $y = \alpha(1)$ are called *endpoints* of the path α and we say that α *joins* x with y. The interior of the path α is the set $\operatorname{int}(\alpha) := \alpha((0,1))$. The path α is called a *loop* if $\alpha(0) = \alpha(1)$, otherwise we say that α is a *non-closed path*. We say that the path α *starts* at $x \in X$ if $\alpha(0) = x$. When α is a loop, we also say that α is *based at* x.

A non-closed path α is called *simple* if it has no self-intersections (i.e., $\alpha\colon \mathbb{I} \to X$ is injective). A loop α is called a *simple* if it has no self-intersections except at endpoints (i.e., $\alpha|(0,1)$ is injective). A loop α is called *constant* if the map $\alpha\colon \mathbb{I} \to X$ is constant. We often conflate paths and their images. For instance, for a path α as above and $Y \subset X$, we write $\alpha \subset Y$ to indicate that $\alpha(\mathbb{I}) \subset Y$. We say that $\gamma \subset X$ is a *simple closed curve* if $\gamma = \alpha(\mathbb{I})$ for some simple loop α in X.

If α and β are two paths in X, then we denote by $\alpha \cdot \beta$ their *concatenation*; in other words, the path that first traverses α and then β. By $\overline{\alpha}$ we denote the path in X such that $\overline{\alpha}(t) = \alpha(1-t)$ for all $t \in \mathbb{I}$. For $n \in \mathbb{Z}$, we define α^n to be a constant loop based at $\alpha(0)$ if $n = 0$, the concatenation of α with itself n times, if $n > 0$, and the concatenation of $\overline{\alpha}$ with itself $|n|$ times if $n < 0$.

Definition A.1. Two paths $\alpha\colon \mathbb{I} \to X$ and $\beta\colon \mathbb{I} \to X$ are called *path-homotopic* (or simply *homotopic*) if there exists a continuous map $H\colon \mathbb{I} \times \mathbb{I} \to X$ called a *homotopy* so that $H(t,0) = \alpha(t)$, $H(t,1) = \beta(t)$ for all $t \in \mathbb{I}$, and $H(0,s) = \alpha(0) = \beta(0)$, $H(1,s) = \alpha(1) = \beta(1)$ for all $s \in \mathbb{I}$.

Let A be a finite subset of X. We say that two paths α and β are *path-homotopic relative* A (abbreviated as "α and β are homotopic rel. A" and denoted $\alpha \sim_A \beta$) if $\alpha \subset X \backslash A$ and

$\beta \subset X \backslash A$ are homotopic in $X \backslash A$, in other words, $H(s,t) \in X \backslash A$ for all $(s,t) \in \mathbb{I} \times \mathbb{I}$, where H is the corresponding homotopy.

Definition A.2. Given a topological space X and paths α and β in X, we say they have the same closing behavior if either both α and β are closed loops, or if neither is a closed loop.

If X is a topological space and $x \in X$, then by $\pi_1(X,x)$ we denote the *fundamental group* of X based at x, or in other words, the set of all homotopy equivalence classes of loops in X based at x endowed with the operation of path concatenation. If $f\colon X \to Y$ is a continuous map between topological spaces X and Y such that $f(x) = y$, then the group homomorphism $f_*\colon \pi_1(X,x) \to \pi_1(Y,y)$ is defined as $f_*([\alpha]) = [f \circ \alpha]$ for any loop $\alpha \subset X$ based at x, where $[\cdot]$ denotes a homotopy equivalence class. Finally, we say that path $\alpha \subset Y$ *lifts under* f to a path $\beta \subset X$, if $\alpha = f \circ \beta$, and β is called the f*-lift* (or *lift under* f) of α.

Definition A.3. If $f\colon X \to Y$ is a covering map and $f(x) = y$, then every path α starting at y has a unique f-lift starting at x, which is denoted by $\alpha{\uparrow}(f,x)$.

By definition, the path $\alpha{\uparrow}(f,x)$ is a loop if and only if $[\alpha] \in f_*\pi_1(X,x)$.

If (X,d) is a metric space, $x \in X$ ad α is a path in X, the distance $d(x,\alpha) = \inf_{t\in\alpha} d(x,t)$. For a path α, and a real number $\varepsilon > 0$, $N_\varepsilon(\alpha) := \{x \in X | d(x,\alpha) < \varepsilon\}$.

For a topological space X, we denote by $\mathrm{Homeo}^+(X)$ of all orientation-preserving self-homeomorphisms of X . Commonly in the literature, the notation $\mathrm{Homeo}^+(X,A)$ is used for the set of maps in $\mathrm{Homeo}^+(X)$ that fix the set A, however, we will take $\mathrm{Homeo}^+(X,A)$ to mean the set of maps in $\mathrm{Homeo}^+(X)$ that fix A pointwise. We use the notation $\mathrm{Homeo}_0^+(X)$ for the subgroup of $\mathrm{Homeo}^+(X)$ consisting of homeomorphisms that are isotopic to id_X. Similarly, $\mathrm{Homeo}_0^+(X,A)$ for the subgroup of $\mathrm{Homeo}^+(X,A)$ consisting of all homeomorphisms isotopic rel. A to id_X.

Definition A.4. Suppose that X and Y are topological spaces. We say that homeomorphisms $\varphi\colon X \to Y$ and $\psi\colon X \to Y$ are *isotopic* if there exists a continuous map $H\colon X \times \mathbb{I} \to Y$ called an *isotopy* such that $H(x,0) = \varphi(x)$ and $H(x,1) = \psi(x)$ for all $x \in X$, and $H(\cdot,t)\colon X \to Y$ is a homeomorphism for every $t \in \mathbb{I}$. We say that φ and ψ are *isotopic rel.* A for some $A \subset X$, if $H(x,t) = x$ for all $(x,t) \in A \times \mathbb{I}$.

We now list a property of homeomorphisms of disks that we use several times in this thesis.

Proposition A.5. *1. Every orientation-preserving homeomorphism $\varphi : \partial\mathbb{D} \to \partial\mathbb{D}$ extends to a homeomorphism from $\overline{\mathbb{D}}$ to $\overline{\mathbb{D}}$.*

2. *Any two such extensions are isotopic rel.* $\partial\mathbb{D}$.

Proof. There is a more general version of this proposition proved in [Hub16, Proposition C2.1], however we will restate it here. With φ as above, extend it to $\overline{\mathbb{D}}$ by setting, for all $t \in \mathbb{I}, z \in \partial\mathbb{D}$, $\varphi(tz) = t\varphi(z)$. This is called the radial extension of φ.

To show item (2), let $\widetilde{\varphi}$ be another extension of φ to $\overline{\mathbb{D}}$. For every $t \in \mathbb{I}$, define a homeomorphism $\varphi_t : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ as follows:

$$\varphi_t(z) = \begin{cases} \varphi(z) & |z| \geqslant t \\ t\widetilde{\varphi}(z/t) & 0 \leqslant |z| \leqslant t \end{cases}$$

Then φ is an isotopy rel. $\partial\mathbb{D}$ between $\varphi_0 = \varphi$ and $\varphi_1 = \widetilde{\varphi}$. □

A.2: Planar embedded graphs

A *planar embedded graph* is a pair $G = (V, E)$, where

1. V is a discrete (in particular, countable) set of points in $\mathbb{R}^2$, and
2. E is a set of simple paths and simple loops (viewed as subsets of $\mathbb{R}^2$) such that their endpoints belong to V, their interiors are pairwise disjoint and lie in $\mathbb{R}^2\backslash V$, and every compact set $K \subset \mathbb{R}^2$ intersects finitely many elements of E.

The sets V and E are called the *vertex set* and *edge set* of G, respectively. Our notion of a planar embedded graph allows *multi-edges* (i.e., distinct edges that connect the same pair of vertices), and *loop-edges* (i.e., edges that connect a vertex to itself).

A planar embedded graph $G = (V, E)$ is said to be *finite* if V and E are finite sets. The *degree* of a vertex v in G, denoted by $\deg_G(v)$, is the number $n_1 + 2n_2$, where n_1 and n_2 are the numbers of simple paths and simple loops in E incident to v, respectively (the second condition above ensures this is always finite). We say that G is *k-regular* if $\deg_G(v) = k$ for every $v \in V$. A *subgraph* of G is a planar embedded graph $G' = (V', E')$ such that $V' \subset V$ and $E' \subset E$.

The subset $\mathcal{G} := V \cup \bigcup_{e\in E} e$ of $\mathbb{R}^2$ is called the *realization* of G. A *face* of the graph G is a connected component of $\mathbb{R}^2\backslash\mathcal{G}$. The set of all faces of G is denoted by $F(G)$. By the definition of a planar embedded graph, the set $\mathcal{G}$ is closed in $\mathbb{R}^2$, and thus, every face F of G is open. If F is a face of G, then we denote by ∂F the subgraph of G forming the (topological) boundary of F in $\mathbb{R}^2$. The graph G is called connected if its realization $\mathcal{G}$ is connected (or equivalently, path-connected). It follows that G is connected if and only if each face of G is simply connected. We will often conflate a planar embedded graph with its realization.

Definition A.6. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be planar embedded graphs. The graph G_1 is said to be *isomorphic* to G_2 if there exists a homeomorphism $\varphi \in \text{Homeo}^+(\mathbb{R}^2)$ that maps vertices and edges of G_1 into vertices and edges of G_2, that is, $\varphi(V_1) = V_2$ and $\varphi(E_1) = E_2$. In this case, we call φ an *isomorphism* between G_1 and G_2.

If φ is isotopic rel. A to $\text{id}_{\mathbb{R}^2}$ for some set $A \subset \mathbb{R}^2$, we say that G_1 is *isotopic* rel. A to G_2.

Suppose that $f: U \to W$ is a covering map, where U and W are open subsets of $\mathbb{R}^2$. If $G = (V, E)$ whose realization is a subset of U, then the preimage $f^{-1}(G)$ has a natural graph structure, given by

$$V(f^{-1}(G)) = f^{-1}(V)$$
$$E(f^{-1}(G)) = \{\alpha | \alpha \subset \mathbb{R}^2 \text{ is a simple path or a simple loop, and } f(\alpha) \in E\}$$

We define some common types of embedded graphs below.

Definition A.7. Let $G = (V, E)$ be a planar embedded graph.

- G is said to be a *cycle* if it is finite, connected and 2-regular.
- If G is infinite, connected and $2-$regular, it is called an *infinite chain*. Any finite, connected subgraph of an infinite chain is called a *finite chain*. Note that if a finite chain has more than one vertex, it has exactly two vertices of degree one, which we call its *endpoints*.
- G is called a *rose graph* if it satisfies the following conditions:
  1. it has a single vertex, called the *center*;
  2. for every edge $e \in E(G)$, the bounded connected component of $\mathbb{R}^2 \backslash G$ does not intersect G.

  We say that G *surrounds* a finite set $A \subset \mathbb{R}^2$ if every bounded face of $\mathcal{R}$ contains a unique point of A, and every point of A is contained in some bounded face of $\mathcal{R}$.

Definition A.8. The graph $G' = (V', E')$ is the result of *subdivision of an edge* $e \in E(G)$ of the planar embedded graph $G = (V, E)$ if G' is obtained from G by adding a new vertex in the interior of e. More precisely, there exists $v \in \text{int}(e)$ such that $V' = V \cup \{v\}$ and $E' = (E \backslash \{e\}) \cup \{e_1, e_2\}$, and e_1, e_2 are the closures of the connected components of $e \backslash \{v\}$. In particular, subdividing an edge does not change the realization of the graph, and the resulting graph is uniquely defined up to an isotopy relative to the set V.

Let $G = (V, E)$ be a planar embedded graph and $e \in E$ be one of its edges. We say that a continuous map $\alpha \colon \mathbb{I} \to e$ is a *parametrization* of e if $\alpha|(0,1)$ is bijective onto $\mathrm{int}(e)$. Two parametrizations α_1 and α_2 of e are considered equivalent if the function $\alpha_1^{-1} \circ \alpha_2$ is increasing on $(0,1)$. We note that every edge $e \in E(G)$ admits two distinct equivalence classes of parametrizations. We call each of these equivalence classes a *direction* of the edge e.

Definition A.9. We say that a graph G is *directed* if each of its edges is endowed with a unique direction (called the *forward* direction). A choice of forward directions for all edges of a graph G is also called an *orientation* of G. Directions that are omitted from the orientation of G are called *backward*.

In a similar way, we introduce notions of forward and backward parametrizations of the edges of the directed graph G. If v is a vertex of G and e is an edge incident to v, then there is a natural way to call e *incoming* or *outgoing* at v unless e is a loop-edge.

Suppose that $\Phi \colon G_1 \to G_2$ is a continuous map, where G_1 is a planar embedded graph and G_2 is a directed planar embedded graph, such that $\Phi^{-1}(V(G_2)) = V(G_1)$ and $\Phi|\,\mathrm{int}(e)$ is injective for each edge $e \in E(G_1)$. The map Φ and the orientation of the graph G_2 naturally induce an orientation of the graph G_1. Indeed, we choose a forward direction for $e \in E(G_1)$ so that if α is a parametrization of e, then α is forward if and only if $\Phi \circ \alpha$ is a forward parametrization of the edge $\Phi(e) \in E(G_2)$. Similarly, there is a natural way to define forward directions for a subgraph and graph obtained by a subdivision of an edge of a directed planar embedded graph.

Definition A.10. We say that a directed planar embedded graph $G = (V, E)$ is *unilaterally connected* if for every pair of vertices $u, v \in V$, there exists a path α with endpoints at u and v that is obtained by concatenation of forward parametrizations of edges of G.

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