Probes of Supersymmetric Black Holes in AdS

by

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AdS Anti-de Sitter space

BPS Bogomol'nyi–Prasad–Sommerfield

CFT Conformal Field Theory

HHZ Hosseini–Hristov–Zaffaroni

KN Kerr-Newman

RN Reissner–Nordström

SYM Supersymmetric Yang-Mills

ABSTRACT

Black holes have emerged as crucial conceptual laboratories for testing ideas towards the development of a quantum theory of gravity, as key black hole properties have been formulated in dual quantum terms using the proposed AdS/CFT correspondence. Supersymmetric black holes in AdS in particular have a rich description on both the gravitational and field-theoretic sides of this duality. In this thesis, we will be studying the stability as well as the spacetime structure of these black holes, and will be probing close to and away from the critical parameters at which these black holes become supersymmetric. There is a lack of a general understanding of generically non-supersymmetric black holes, and the goal of this thesis will be to expand with further quantitative explorations and systematize our understanding in that area.

The first part of this thesis will be a survey of Kerr-Newman-AdS$_5$ black hole thermodynamics, away from and approaching the supersymmetric (BPS) regime. We use that limiting process to define thermodynamics directly on the BPS surface, and are able to parametrize these BPS thermodynamics in terms of a new fugacity. We then find that the free energy in this formulation can be directly mapped to the HHZ free energy obtained from the dual field theory.

The second part of this thesis consists of an in-depth analysis of the black hole solution to $\mathcal{N} = 2$ gauged supergravity, in particular the radial profile of its fields from the near-horizon region all the way to asymptotic infinity. We develop the necessary radial flow equations with the help of the $\mathcal{N} = 2$ supersymmetric variations, as well as charge conservation laws based on a nontrivial adaptation of the Noether-Wald procedure in gauged supergravity. We are then able to characterize the radial flow of the black hole solution in great detail. Lastly, we connect the near-horizon limit of this flow to the entropy extremization formalism, and give a full description of the supersymmetric extrema of the theory.

CHAPTER 1

Introduction

1.1 Motivation

Our current understanding of modern physics relies crucially on the two frameworks of general relativity and quantum mechanics. The former is relevant for understanding physics at distant scales and large masses. In contrast, the latter's explanatory success lies at the level of microscopic scales, at the level of the smallest and fundamental constituents of matter. Reconciling these two frameworks with distinct domains of applicability into one consistent physical theory, namely quantum gravity, becomes a necessary follow-up, especially with the knowledge that objects exist involving physical characteristics drawing from both of these domains.

Black holes satisfy this description, as they represent regions of spacetime with both large mass and geometrical curvature and small length scales when compared to the scales of similarly massive but more commonplace gravitational systems. Various developments have been made in the field of general relativity since the first formulation of what we now call black holes, to classify black hole solutions conforming to various boundary conditions, spacetime structure, and matter content. In order for a quantum theory of gravity to account for these black holes with such a range in macroscopic properties, it is crucial to develop a quantum understanding that reproduces these properties in terms of underlying microscopic degrees of freedom. Across this range of black holes, charged and rotating black holes in particular present qualitatively interesting gravitational properties, which we would like to better understand through this quantum gravity formulation.

Furthermore, this formulation has been more reliably analyzed for supersymmetric black holes, that is the subset of black holes whose microscopic properties are determined by the theory of supersymmetry. The task remains to fully incorporate black holes with properties that are either generically non-supersymmetric or close to supersymmetric. The goal of this thesis is to shed light on this range of black holes, in particular by exploring the role of rotation in determining special gravitational features for both supersymmetric and near-supersymmetric black holes, as well as the microscopic features that underpin them.

1.2 Background

In this section, we will review the background of ideas needed for our modern understanding of black holes, the interplay between their macroscopic and microscopic features, and the formulation of this interplay through the modern ideas of supersymmetry and holography.

1.2.1 Properties of black holes:

Black holes are solutions to Einstein's field equations that are characterized by an event horizon, a causal boundary between spacetime events on either side of the surface. A simple example of a black hole would be the Schwarzschild solution, discovered by Karl Schwarzschild in 1916 [1] as the spherically symmetric, static, and asymptotically-free vacuum solution to Einstein's equations. In four dimensions, a Schwarzschild black hole of mass M is expressed by the following line element[1]:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2 \ . \tag{1.1}$$

G here refers to the four-dimensional Newton's constant of gravitation, while t and r refer to the time and radial coordinates that match their respective Minkowski definitions at infinity. $d\Omega_2^2$ is the two-dimensional case of the general $d\Omega_{d-2}^2$, which refers to the $(d-2)$-dimensional spherical line element for spacetimes with $d > 2$.

Dispensing with the vacuum assumption leads to the Reissner–Nordström (RN) family of charged black holes, now accompanied by an electromagnetic field A:

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2GM}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2GM}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_2^2 \ , \\ A &= \frac{Q}{r} dt \ , \end{aligned} \tag{1.2}$$

for electrically-charged black holes with charge Q and mass M. This can be generalized to incorporate angular momentum as well, after relaxing the symmetry properties by swapping

[1] In this thesis, we will be operating in units where $\hbar = c = k_B = 1$

spherical symmetry for more general axisymmetry, yielding a Kerr-Newman (KN) solution parametrized by M, Q and now angular momentum $J = Ma$:

$$\begin{aligned} ds^2 &= -\frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left((r^2+a^2)d\varphi - a dt\right)^2 \ , \\ A &= \frac{Qr}{\rho^2}dt - \frac{aQr\sin^2\theta}{\rho^2}d\varphi \ , \\ \rho^2 &= r^2 + a^2\cos^2\theta \ , \ \Delta = r^2 - 2GMr + a^2 + Q^2 \ . \end{aligned} \tag{1.3}$$

For given Q and J, the existence of an event horizon such that $g_{tt} = 0$ requires the mass M to be bounded above by a minimum value:

$$M^2 \geq Q^2 + \frac{J^2}{M^2} \ . \tag{1.4}$$

A black hole with the minimum possible mass given its macroscopic parameters Q, J, etc., is defined as an extremal black hole. These will be the black holes we focus on in this thesis as they present various properties of interest. One immediate geometrical property is the new symmetry that extremal black holes develop near their horizon: scale invariance. In general, black holes like the ones introduced above (1.3) present isometries based on time-translation and rotational invariance, understood correspondingly in terms of the conserved mass M and angular momentum J. The added symmetry near the horizon takes the form of a long "throat" in the t and r directions that is invariant under rescalings of both of these coordinates, specifically described by a two-dimensional Anti-de Sitter space (AdS) – negative cosmological constant – space [2]. In general, the $(d+1)$-dimensional (globally) AdS_{d+1} space with length scale L is defined with the line element:

$$ds^2 = -\left(1 + \frac{r^2}{L^2}\right)dt^2 + \left(1 + \frac{r^2}{L^2}\right)^{-1}dr^2 + d\Omega_{d-2}^2 \ . \tag{1.5}$$

We will return to these general AdS_{d+1} spacetimes later in this introduction, as they will be important for establishing further ideas explored in this thesis.

1.2.2 Black hole thermodynamics:

In addition to their near-horizon geometry, extremal black holes are noteworthy for another reason, namely for the nontrivial way in which they feature in the key framework of black hole thermodynamics. To motivate this framework, we note that along with macroscopic parameters (M, Q, J), black holes are also characterized by accompanying potentials such as an electrical potential Φ and an angular velocity Ω. If we then study the surface gravity

κ at the horizon of a black hole, we note that it is related to the parameters (M, J, Q) and conjugate potentials (Ω, Φ) in the following way:

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ \ , \tag{1.6}$$

where A is the area of the horizon. Understanding M to be the black hole mass-energy, this relation becomes suggestive of a first law of thermodynamics formulated in terms of black hole quantities, if κ and A can be linked to a black hole-related definition of temperature and entropy respectively. In fact, the proposal of Jacob Bekenstein in 1972, followed by Stephen Hawking's further study in 1974, aim at precisely this idea, formulated in terms of laws of black hole thermodynamics:

- The zeroth law: The surface gravity κ is constant over the horizon of a stationary black hole.
- The first law: The change in the energy dM of a stationary black hole is related to the change in the area dA, angular momentum dJ, and electric charge dQ following the relation (1.6).
- The second law: The change in the horizon area is non-decreasing over time.

The black hole then is associated with a Hawking temperature T_H and Bekenstein-Hawking entropy S_{BH}:

$$T_H = \frac{\kappa}{2\pi} \ , \tag{1.7}$$

$$S_{\text{BH}} = \frac{A}{4G} \ . \tag{1.8}$$

Now returning to extremal black holes, it turns out that extremality is equivalent precisely to:

$$T_H = 0 \ . \tag{1.9}$$

Extending the analogy of the laws of thermodynamics to black holes fails when dealing with the third law of thermodynamics: the behavior of the entropy S when the temperature $T \to 0$ (or Planck-Nernst law). Extremal black holes can have vanishing temperatures (1.9) while also having a finite associated Bekenstein-Hawking entropy S_{BH}. In fact, there also already exist known ordinary and non-gravitational quantum systems that violate the Planck-Nernst statement of the third law [3, 4]. Furthermore, the formal similarities between the classical and black hole laws of thermodynamics, as well as the crucial distinctions that arise out of a specifically quantum treatment (such as the entropy and temperature definitions (1.7)-(1.8)), point out the nontrivial task of a full understanding of black holes via the formalism of quantum gravity.

1.2.3 Black hole microstates and AdS/CFT:

In fact, one of the first challenges in making sense of the thermodynamical characterization of black holes becomes how to interpret the Bekenstein-Hawking entropy S_{BH} (1.8) as reflective of a count of W underlying microstates:

$$S = k_B \log W \ , \tag{1.10}$$

where k_B was temporarily re-introduced for clarity. Interpreting the Bekenstein-Hawking entropy in analogy to an entropy associated with classical microstates W, ties to the more general task of interest of interpreting a gravitational system (say a black hole) in terms of an auxiliary, or dual, microscopic system. This proposed duality is termed *gauge/gravity duality*, as it maps a gravity theory to a gauge theory (meaning a non-gravitational theory of fields), or alternatively, a strongly-coupled theory to a weakly-coupled one [5]. The research program associated with this key idea has been notable, with a range of developments in quantum field theory, gravitational physics and condensed matter [6, 7, 8, 9, 10, 11].

The more specific formulation of this duality relates a theory of quantum gravity defined on an AdS spacetime in $(d+1)$-dimensions, and a Conformal Field Theory (CFT) in d-dimensions. A conformal field theory is a field theory that is invariant under conformal coordinate transformations, i.e. transformations that leave local angles invariant. This AdS/CFT duality has been proposed in its canonical iteration by Maldacena in 1997 [12], between type IIB string theory in $\text{AdS}_5 \times S^5$ (gravity) and $\mathcal{N} = 4$ Supersymmetric Yang-Mills (SYM) theory in four dimensions (field theory). This mapping between a theory and a lower-dimensional one is also termed *holography*, in analogy to the optical holograms that represent three-dimensional images on a two-dimensional surface.

The quantitative statement of the AdS/CFT duality in Maldacena's example is a correspondence between the parameters on both sides of the duality, namely the string coupling g_s and the AdS length scale $L/\sqrt{\alpha'}$ (rescaled by the string length scale) on the gravitational side, and N the rank of the gauge group of the field theory and the Yang-Mills coupling g_{YM} on the field theory side:

$$\begin{aligned} 2\pi g_s &= g_{\text{YM}}^2 \ , \\ \left(\frac{L}{\sqrt{\alpha'}}\right)^4 &= 2 g_{\text{YM}}^2 N \ . \end{aligned} \tag{1.11}$$

This statement is an example of the broader concept of an *AdS-CFT dictionary*: a mapping between parameters and variables between a gravitational theory and the conjectured dual field theory. Such an approach has yielded interesting developments and quantitative matches beyond the scope of quantum field theories and black hole physics, towards condensed-matter areas such as hydrodynamics, quantum chromodynamics, and nonequilibrium statistical mechanics [13].

The aforementioned duality and accompanying dictionary are also manifest at a more fundamental level, in terms of the fundamental features of the two theories on either side. First, we note that the two theories share common symmetries and isometries: in the Maldacena example, the boundary of the AdS_5 spacetime has by definition $SO(4,2)$ as its symmetry group, whereas the $\mathcal{N}=4$ SYM field theory features $SO(2,4) \sim SU(2,2)$ via its conformal (scale- and Poincaré-invariant) symmetry group [6, 7, 14]. Then, we note that the two theories can be mapped into each other via an operator-state correspondence between operators in the field theory, and bulk fields evaluated close to the AdS boundary. In fact, considering for instance a scalar field ϕ in a $(d+1)$-dimensional gravitational theory, the correspondence prescribes an associated operator in the dual theory with a scaling dimension directly related to the field's mass. Close to the AdS boundary, the scalar wave equation for the field ϕ has two independent solutions that drop off as $z^{d-\Delta}$ and z^{Δ} where the boundary is at $z \to 0$. This scaling dimension Δ is related to the scalar mass m^2 via:

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2 L^2}\ , \tag{1.12}$$

where L is the radius associated with the $AdS_{\mathrm{d+1}}$ boundary. Relations similar to (1.12) exist for the remaining bosonic and fermionic fields of the supersymmetric field theory of interest.

1.2.4 Black holes and supersymmetry:

We can see from this setup that supersymmetry features heavily when conveying the central ideas behind the AdS/CFT duality and constructing its emblematic representatives. Supersymmetry is generally understood as the nontrivial extension of the Poincaré spacetime with its distinct incorporation of both bosonic (commuting) and fermionic (anticommuting) generators [15], thus implying symmetry transformations that map bosons to fermionic partners, and vice-versa. The generators in question include the familiar Poincaré translations and Lorentz boosts, as well as additional bosonic generators relating to the conformal group (scale and special conformal transformations) and fermionic supersymmetric generators (supercharges and R-symmetry generators). If these generators are allowed to vary as functions of spacetime, diffeomorphisms are involved and thus gravity has to be properly incorporated along with supersymmetry; this yields a theory of supergravity, with gravitons becoming part of the theory, along with gravitini, their spin-$\frac{3}{2}$ fermionic partners.

The basic ideas behind supersymmetry and supergravity are important for setting up the theories on both sides of the AdS/CFT duality: ten-dimensional supergravity as a low-energy limit to type IIB string theory on the gravitational side, and $\mathcal{N} = 4$ SYM on the field theory side. Furthermore, gravitational solutions to the low-energy theory (small α') are related by the dictionary (1.11) to large N in the dual field theory side. Returning to one of the earliest applications for the duality, reconstructing the Bekenstein-Hawking entropy (1.8) from underlying microstates on the microscopic/field theoretic side, we are led to study the specific subset of supersymmetric black holes, which can preserve a fraction of the symmetries of the broader supergravity theory: they will prove to be the central objects of inquiry in this thesis. These are BPS black holes, named after the Bogomol'nyi–Prasad–Sommerfield (BPS) bound that the energy of dual states in the $\mathcal{N} = 4$ SYM saturate.

In the gravitational picture, BPS black holes in $d + 1$ dimensions are rotating and electrically-charged black holes, similar to the KN black holes introduced in (1.3) except for the crucial difference of an AdS_{d+1} background. BPS black holes in 5D in particular are characterized by three electric charges $Q_{1,2,3}$ and two angular momenta $J_{i=a,b}$. These conserved quantities, in addition to the black hole mass-energy M, reflect conserved quantum numbers on the other side of the duality, as encountered earlier in the prototypical Maldacena example. The four-dimensional $\mathcal{N} = 4$ SYM theory is defined through a thermal partition function on the space $S^1 \times S^3$, with M mapping to the energy eigenvalue of the CFT Hamiltonian, the three Q_I to the 3 Cartans of the $SU(4)$ R-symmetry characteristic of $\mathcal{N} = 4$ supersymmetry, and the two J_i to the two commuting isometries of S^3 [16]. As macroscopic parameters describing the black hole solution, Q_I and J_i are conjugate to the electric potentials Φ_I and angular velocities ω_i. These can be related to the microscopic parameters Δ_I and ω_i respectively, corresponding to the chemical potentials in the dual field theory. Supersymmetry requires that these parameters lead to *zero temperature*, as well as satisfy the following nonlinear charge relation:

$$Q_1Q_2Q_3+\frac{\pi}{4G_5}J_aJ_b = \left(\frac{\pi}{4G_5g^2}+Q_1+Q_2+Q_3\right)\left(Q_1Q_2+Q_2Q_3+Q_1Q_3-\frac{\pi}{4G_5g}(J_a+J_b)\right) \tag{1.13}$$

where $g = L^{-1}$ corresponds to the inverse AdS radius, and G_5 to the five-dimensional Newton constant.

From the gravitational calculation, we find the Bekenstein-Hawking entropy for these black holes:

$$S = 2\pi\sqrt{g^2\left(Q_1Q_2 + Q_2Q_3 + Q_1Q_3\right) - \frac{\pi}{4G_5g^3}(J_1 + J_2)}\ . \tag{1.14}$$

This can be re-expressed to emphasize the scaling of the entropy (1.14) as a function of N using the dictionary $\frac{\pi}{4G_5g^3} = \frac{1}{2}N^2$:

$$S = 2\pi\sqrt{g^2\left(Q_1Q_2 + Q_2Q_3 + Q_1Q_3\right) - \frac{1}{2}N^2(J_1 + J_2)}\ . \tag{1.15}$$

Knowing that each of the Q_I and J_i scales as N^2 for large N, this implies an entropy of the order of N^2 [17]. The AdS/CFT expectation would be to retrieve that entropy on the field theoretic side. BPS black holes in particular have the property that their dual description can be described and counted irrespective of the coupling strength, which is crucial when the CFT is easiest to understand in the weak coupling regime as opposed to the gravitational description that is defined as a strong coupling regime (with its radius of curvature L much greater than the string length scale $\sqrt{\alpha'}$). There have been many nontrivial developments in this supersymmetric calculation, specifically via the technology of the superconformal index [18, 19, 20, 21, 22, 23, 24] in $\mathcal{N} = 4$ SYM theory, with the output being the CFT free energy in terms of chemical potentials (Δ_I, ω_i) and expectedly the entropy in terms of the charges and momenta Q_I and J_i (1.15). The main test for this perspective has been the ability to recover the $O(N^2)$ scaling of the resulting entropy S, which has proven unfeasible unless the aforementioned chemical potential were taken to be complex despite their direct mapping to *real*-valued gravitational parameters Φ_I and ω_i [25, 26, 27].

In both their gravitational description and their dual field theoretic description, supersymmetric black holes have proven to be key objects of study towards a quantum understanding of gravity. One additional property to point out is that they are stable objects, as they are protected under supersymmetry by virtue of being dual to protected BPS states. This is not the case for non-BPS black holes, in particular either non-extremal or whose charges and angular momenta do not satisfy a charge relation like (1.13). An interesting task becomes to quantify the distance from this state of stability, whether nonBPS black holes can be made stable or conversely, how to destabilize a BPS black hole. One approach comes from black hole thermodynamics: the perturbation from stability is defined in the grand canonical ensemble, where the temperature T and the potentials Φ_I and Ω_i are allowed to vary, from their BPS values $T = 0$, $\Phi_I^* = 1$, and $\Omega_i^* = g$, and defines the motivation for the work in **Chapter 2**.

In addition to the macroscopic parameters and the black hole thermodynamics, the spacetime geometry is another interesting gravitational feature of black holes. We noted earlier that extremal black holes in general develop an AdS_2 profile with an enhanced scale invariance. For some supersymmetric black holes, in fact, the entire radial geometry, from the near-horizon region to asymptotic infinity, can offer an enhanced flow structure, with a noteworthy holographic interpretation in terms of the renormalization group flow in the CFT. As part of this flow, the dynamical scalar fields associated with these black hole solutions *attract* to the same values at the horizon regardless of their starting values at radial infinity. With these special properties, the Einstein-Maxwell-AdS action, which accepts these -generically- charged black holes as gravitational solutions, greatly simplifies in the near-horizon region, resulting in a direct way to compute the Bekenstein-Hawking entropy. The catch is that this perspective and associated simplifications have only been developed for supersymmetric black holes arising from ungauged supergravity, which among other aspects stipulates asymptotically-*free* black holes. Defining such a radial flow for BPS black holes from *gauged* supergravity towards a potential generalization of the attraction mechanism is the motivation for the work in **Chapter 3**.

1.3 Overview

This thesis is based on my work with my doctoral advisor Prof. Finn Larsen during my PhD at the University of Michigan, along with collaborators Marina David, Zhihan Liu, and Yangwenxiao Zeng. It spans across two chapters:

- **Chapter 2:** This chapter is based on my work with Prof. Larsen and collaborators Zhihan Liu and Dr. Yangwenxiao Zeng [28]. In this work, we start by reviewing the thermodynamics of generically nonBPS Kerr-Newman-AdS_5 black holes, and examine the role of each of rotation and electric charge in determining the phase diagram and thus thermodynamic stability, in particular as the black hole parameters approach their BPS values. We then develop a formalism of BPS thermodynamics, where these parameters are now at their critical BPS value. The result is a theory of variables along the BPS surface with an underlying continuous parametrization that we interpret as a novel fugacity. The rescaled BPS free energy, complemented with the analytical continuation of this new fugacity is then related to the Hosseini–Hristov–Zaffaroni (HHZ) complexified free energy function [29].
- **Chapter 3:** This chapter is based on my work with Prof. Larsen and collaborator Dr. Marina David [30]. In this work, we proceed with a dimensional reduction of the $\mathcal{N} = 2$ gauged supergravity action from 5D to 2D, and formulate the associated radial conservation laws with particular attention to the subtleties of gauge invariance

in the presence of a Chern-Simons term in the action. In addition, we similarly dimensionally reduce the $\mathcal{N} = 2$ supersymmetric variations, and combine them with the conservation laws into a system of radial BPS flow equations. We then integrate these equations, both from the horizon outwards, and from radial infinity inwards, and discuss the matching of these two approaches. We also connect the near-horizon flow equations with the near-horizon entropy function, and consider a complexification of these equations that maps to the HHZ potential.

1.4 Summary and Outlook

In summary, the work of this thesis corroborates the centrality of BPS black holes in probing various questions pertaining to AdS black hole thermodynamics, gauged supergravity, and the entropy function formalism. We have provided systematic reviews of previously only partially or separately addressed formalisms underpinning these topics, such as full Kerr-Newman-AdS thermodynamics for nonBPS black holes, and very special geometry as an upgrade to the usual STU treatment of gauged supergravity. We have also formulated novel concepts and quantities expanding on the relevant literature, such as defining thermodynamical variables and phase diagrams approaching and on the BPS surface, our generalized attractor flow in gauged supergravity, as well as our complexifications of thermodynamic potentials and of the near-horizon geometry and matter fields in BPS black hole spacetimes.

These developments also offer many follow-up questions of interest, in particular to shed light on the physical meaning of the rescaled thermodynamic potentials and additional fugacities on the BPS surface, in particular as identifications with dual quantities on the $\mathcal{N} = 4$ SYM side become apparent. Additionally, concerning the attractor flow in gauged supergravity, the full integration of the radial flow equations between the near-horizon and infinity recalls the holographic account of the QFT flow between the IR and UV as a radial flow, especially motivated for radial boundaries of the AdS variety. Deepening the understanding of this radial flow, as well as further parsing the suggestive complex structure behind the near-horizon variables, would strengthen the separate ideas underpinning such an attractor formalism in gauged supergravity.

Lastly, we would like to motivate a different but related approach to probing BPS black holes, that also ties in to ideas from black hole thermodynamics, holography and (gauged) supergravity. Namely, we draw from the field of stability studies for BPS, or more generally extremal, black hole solutions by coupling them to a charged scalar field and studying the resulting combined configuration for any field instabilities. This setup can be formulated as a subset of matter fields of the underlying supergravity theory, chosen specifically to satisfy self-contained and consistent field equations, also termed a *consistent truncation*. The prototypical example for this has been termed the holographic superconductor, in reference to the second-order phase transition that the combined black hole+charged scalar system undergoes as a function of temperature, and its corresponding formulation in terms of correlators of operators in the dual CFT. This example has been subsequently enriched as part of the study of the stability of extremal AdS black holes by incorporating rotation as well, thus connecting to well-established studies of rotational instability and superradiance. As part of ongoing (unpublished) research work, we would be exploring this connection between superconductive (alternatively, superfluid) and superradiant instabilities of extremal AdS spacetimes by probing fluctuations of an extremal Kerr-Newman-AdS spacetime that span the spectrum of particles from five-dimensional gauged supergravity, enriching a literature on the subject that has focused on a particularly technically-tractable consistent truncation to only three gauge fields and five scalars to parametrize the black hole background and perturbations thereof.

CHAPTER 2

The Phase Diagram of BPS Black Holes in AdS_5

2.1 General Thermodynamics of AdS_5 Black Holes

In this section we study the thermodynamics of AdS_5 black holes for the case with equal charges ($Q_I = Q$) but general non-equal angular momenta $J_{a,b}$.

Taking Schwarzschild-AdS as a benchmark, the phase diagram deforms smoothly as electric potential and angular velocities are added. Angular momentum proves more destabilizing than charge.

2.1.1 Black Hole Thermodynamics

The Kerr-Newman AdS_5 family of black holes is characterized by four conserved charges (M, Q, J_a, J_b). These physical charges are parameterized by 4 variables (m, q, a, b) through [31]:

$$
\begin{aligned}
M &= \frac{\pi}{4G_5}\frac{m(3-a^2g^2-b^2g^2-a^2b^2g^4)+2qabg^2(2-a^2g^2-b^2g^2)}{(1-a^2g^2)^2(1-b^2g^2)^2}\ , && (2.1)\\
Q &= \frac{\pi}{4G_5}\frac{q}{(1-a^2g^2)(1-b^2g^2)}\ , && (2.2)\\
J_a &= \frac{\pi}{4G_5}\frac{2ma+qb(1+a^2g^2)}{(1-a^2g^2)^2(1-b^2g^2)}\ , && (2.3)\\
J_b &= \frac{\pi}{4G_5}\frac{2mb+qa(1+b^2g^2)}{(1-a^2g^2)(1-b^2g^2)^2}\ . && (2.4)
\end{aligned}
$$

We will assume that all the conserved charges are nonnegative. Their scale is set by the dimensionless parameter

$$
\frac{\pi\ell_5^3}{4G_5} = \frac{1}{2}N^2\ ,
$$

where N refers to the dual $SU(N)$ gauge group of $\mathcal{N} = 4$ SYM. The only other dimensionful parameter entering the thermodynamic formulae is the AdS_5 scale ℓ_5 that is equivalent to the coupling of gauged supergravity $g = \ell_5^{-1}$. Henceforth we set $\ell_5 = 1$ (and so $g = 1$), to avoid clutter, but the AdS_5 scale is easily restored. For example, M, Q are inverse lengths, a, b are lengths, and q, m are lengths squared.

The event horizon is at the coordinate location r_+, a combination of parameters that is ubiquitous in thermodynamic formulae for AdS black holes. It is determined as the largest root of the horizon equation

$$\Delta_r(r) = \frac{(r^2+a^2)(r^2+b^2)(1+r^2)+q^2+2abq}{r^2} - 2m = 0 \ . \tag{2.5}$$

With r_+ given implicitly through this equation, the black hole temperature is

$$T = \frac{r_+^4[1+(2r_+^2+a^2+b^2)]-(ab+q)^2}{2\pi r_+[(r_+^2+a^2)(r_+^2+b^2)+abq]} \ , \tag{2.6}$$

and the electric potential and angular velocities are

$$\Phi = \frac{3qr_+^2}{(r_+^2+a^2)(r_+^2+b^2)+abq} \ , \tag{2.7}$$

$$\Omega_a = \frac{a(r_+^2+b^2)(1+r_+^2)+bq}{(r_+^2+a^2)(r_+^2+b^2)+abq} \ , \tag{2.8}$$

$$\Omega_b = \frac{b(r_+^2+a^2)(1+r_+^2)+aq}{(r_+^2+a^2)(r_+^2+b^2)+abq} \ . \tag{2.9}$$

The linchpin for thermodynamics is, of course, the black hole entropy computed from the area law:

$$S = 2\pi \cdot \frac{1}{2}N^2 \cdot \frac{(r_+^2+a^2)(r_+^2+b^2)+abq}{(1-a^2)(1-b^2)r_+} \ , \tag{2.10}$$

The conserved charges (2.1-2.4) and the thermodynamic potentials (2.6-2.10), with the subsidiary condition (2.5) that determines r_+ implicitly, completely determine the thermodynamics of Kerr-Newman AdS_5 black holes. The nonlinearities evident in these parametric equations contain a great deal of physics.

The black hole mass M is bounded from below by the BPS mass

$$M_{\mathrm{BPS}} = \Phi^* Q + \Omega_a^* J_a + \Omega_b^* J_b \ ,$$

where $\Phi_* = 3$ and $\Omega_a^* = \Omega_b^* = 1$. In this section we consider generic variables, a detailed study of BPS thermodynamics follows later. However, we will find that the *critical* values of the potentials, denoted by stars, play a central role also away from the BPS limit. In particular, we consider only $\Omega_{a,b} \leq \Omega^*_{a,b} = 1$, because the "overspinning" black holes with larger $\Omega_{a,b}$ values are classically unstable due to superradiance [32].

2.1.2 The Grand Canonical Ensemble

In this subsection we study the phase diagram in the grand canonical ensemble, i.e. as a function of temperature T, electric potential Φ, and angular velocities $\Omega_{a,b}$. These potentials correspond to periodicities of the (Euclidean) time, the gauge function and two azimuthal angles, respectively. Therefore, the grand canonical ensemble is the natural description in the path integral formalism where asymptotic boundary conditions are specified as geometrical data.

The Gibbs free energy is given by

$$G = M - TS - \Phi Q - \Omega_a J_a - \Omega_b J_b \ . \tag{2.11}$$

The extensive variables M, Q, $J_{a,b}$, S are presented in (2.1-2.4, 2.10). After trading m for r_+ through the horizon equation (2.5) they become functions of the parameters (r_+, q, a, b). We can explicitly eliminate q in favor of the potential Φ

$$q = \frac{\Phi(r_+^2 + a^2)(r_+^2 + b^2)}{3r_+^2 - ab\Phi} \ , \tag{2.12}$$

by inverting (2.7). However, the parameters (r_+, a, b) must remain as implicit functions of the potentials T, Φ and $\Omega_{a,b}$ because it is impractical to solve (2.6, 2.8-2.9). Roughly, but not precisely, the horizon location r_+ is a proxy for the temperature T, and the rotation parameters a, b represent rotation velocities $\Omega_{a,b}$.

After the elimination of the parameter q, the angular velocities become

$$1 - \Omega_a = \frac{1-a}{r_+^2 + a^2}\left[r_+^2 - r_*^2 + b(1+a)(1 - \frac{1}{3}\Phi)\right] \ , \tag{2.13}$$

$$1 - \Omega_b = \frac{1-b}{r_+^2 + b^2}\left[r_+^2 - r_*^2 + a(1+b)(1 - \frac{1}{3}\Phi)\right] \ . \tag{2.14}$$

These formulae are completely general, albeit presented in a manner that anticipate a special role for the BPS horizon position $r_*^2 = \sqrt{a + b + ab}$ and the critical values of the potentials

$\Omega_a^* = \Omega_b^* = 1$, $\Phi^* = 3$. In a similar spirit, we can present the general temperature as

$$\begin{aligned} T =& \frac{r_+(r_+^2 + a + b - ab)}{2\pi\,(r_+^2 - ab)} \left(\frac{1-\Omega_a}{1-a} + \frac{1-\Omega_b}{1-b} \right) \\ & - \frac{r_+}{6\pi} \frac{(3+\Phi-3a-3b)\,r_+^2 + (a^2+b^2+a^2b+ab^2)\Phi - ab(\Phi-3)}{(r_+^2-ab)(3r_+^2-ab\Phi)}\,(\Phi-3)\ . \end{aligned} \tag{2.15}$$

In this form it is manifest that the temperature vanishes when the potentials take their critical values. We are not able to explicitly invert the three equations (2.13-2.15) any further but, taken together, they relate the physical potentials (T, Ω_a, Ω_b) to the parameters (r_+, a, b) for a given Φ.

After elimination of the parameter q in favor of the electric potential Φ, Gibbs' free energy G becomes:

$$\begin{aligned} G =& -\frac{N^2}{4} \frac{(r_+^2+a^2)(r_+^2+b^2)(r_+^2+a+b-ab)(r_+^2-r_*^2)}{(1-a^2)(1-b^2)(r_+^2-ab)^2} \\ & - \frac{N^2\left(r_+^2+a^2\right)\left(r_+^2+b^2\right)(\Phi-3)}{12(1-a^2)(1-b^2)\left(r_+^2-ab\right)^2\left(3r_+^2-ab\Phi\right)^2}\big[(r_+^2-ab)^2(9r_+^2+3r_+^2\Phi-2ab\Phi^2) \\ & \qquad -3(a+b)^2 r_+^2 (3r_+^2 + r_+^2\Phi - 2ab\Phi)\big]\ . \end{aligned} \tag{2.16}$$

Again, we have pulled out factors that make it manifest that the expression vanishes when the potentials take their critical values.

In the following subsections we study the phase diagram for various ranges or special values of Φ, with any Ω, and for two exceptional values of Ω, for any Φ. When combined, these special cases map out the entire phase diagram.

2.1.2.1 Angular Velocity Ω with Electric Potential $\Phi = 0$

If, in addition to taking the electric potential $\Phi = 0$, we also consider vanishing angular velocities $\Omega_{a,b} = 0$, we are left with the Schwarzschild-AdS black hole. In this case the phase diagram, reproduced in Figure 2.1, is extremely well-known [33]. Some of its features are:

- There is a strictly positive lower bound on the black hole temperature $T \geq T_{\min}$.
- For each temperature $T \geq T_{\min}$ there are two branches of solutions. Black holes on the upper one are "small" in units of the AdS_5 radius. They are qualitatively similar to asymptotically flat black holes. The "large" black holes on the lower branch are influenced significantly by the AdS_5 background.
- A thermal AdS phase has the same boundary conditions at infinity as the black holes

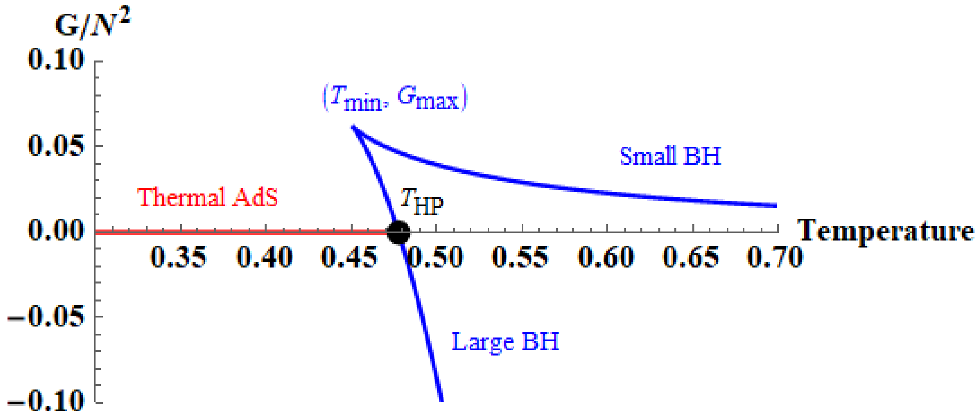

Figure 2.1: AdS-Schwarzschild phase diagram ($\Phi = 0,\ \Omega_{a,b} = 0$). The cusp is at the minimal temperature $T_{\rm min} = \frac{\sqrt{2}}{\pi}$, and the maximal free energy $G_{\rm max} = \frac{N^2}{16}$. The large black hole branch meets the thermal AdS phase at the Hawking-Page temperature $T_{\rm HP} = \frac{3}{2\pi}$.

but vanishing free energy. It dominates when the temperature is below the Hawking-Page temperature $T_{\rm HP}$ [34], where the large black hole also has zero free energy. The large black holes are thermodynamically preferred only when they have temperature $T \geq T_{\rm HP}$.

The Hawking-Page transition is particularly interesting because it can also be interpreted as the confinement/deconfinement transition in QCD-like theories [35, 36, 37, 16] living on the boundary via the AdS/CFT correspondence[38, 33, 39].

As the angular velocities $\Omega_{a,b}$ are turned on and increased, with the electric potential kept at $\Phi = 0$, the qualitative features of the AdS-Schwarzschild phase diagram are preserved. Naturally, there are quantitative changes:

- For any given T, on either branch, the angular velocities lower the free energy. This follows from the first law that gives $\partial_{\Omega_{a,b}} G = -J_{a,b} < 0$.
- The transition temperature $T_{\rm HP}$ decreases as either angular velocity increases, as it must because the large black hole branch is lowered.
- The minimal black hole temperature $T_{\rm min}$ also decreases as either angular velocity increases. The maximal free energy, attained at the cusp, increases. Thus the cusp travels "towards North-West" when angular velocity increases.
- In the limit where the angular velocities $\Omega_{a,b} \to 1$ the height of the cusp diverges $G_{\rm max} \to \infty$ but its temperature approaches a finite (nonzero) limit $T_{\rm min} \to \frac{1}{2\pi}$. In

the strict limit $\Omega_a = \Omega_b = 1$ the "large" branch disappears altogether. This limit is detailed in subsection 2.1.2.6.

The existence of an absolute minimum for the temperature for any angular velocities and the increase in maximal free energy both suggest that, everything else being equal, rotation tends to *destabilize* the black hole.

When $\Phi = 0$, the two angular velocities are "decoupled" in that

$$\Omega_a = \frac{a(1+r_+^2)}{r_+^2 + a^2} \ , \tag{2.17}$$

is a function only of r_+ and a, but not b. The formula for Ω_b is analogous.

The free energy (2.16) also simplifies greatly when $\Phi = 0$:

$$G = -\frac{N^2}{4} \frac{\left(r_+^2 - 1\right)\left(r_+^2 + a^2\right)\left(r_+^2 + b^2\right)}{r_+^2\left(1-a^2\right)\left(1-b^2\right)} \ . \tag{2.18}$$

The Hawking-Page transition, where $G = 0$, corresponds to the horizon parameter $r_+ = 1$. With this value we can invert (2.17) (and its analogue for Ω_b) and find a (b) as a function of Ω_a (Ω_b). These expressions, along with $r_+ = 1$, give a simple formula for the Hawking-Page transition temperature

$$T \underset{\Phi=0}{=} \frac{r_+^4\left(2r_+^2 + a^2 + b^2 + 1\right) - a^2b^2}{2\pi r_+\left(r_+^2 + a^2\right)\left(r_+^2 + b^2\right)} \underset{\text{HP}}{=} \frac{1 + \sqrt{1-\Omega_a^2} + \sqrt{1-\Omega_b^2}}{2\pi} \ . \tag{2.19}$$

It is a decreasing function of the two angular velocities independently, as expected. It interpolates between the AdS-Schwarzschild value $T_{\text{HP}} = \frac{3}{2\pi}$ when there is no rotation, and approaches the finite value $T_{\text{HP}} = \frac{1}{2\pi}$ towards $\Omega_a = \Omega_b = 1$. where G diverges. We discuss the subtle limiting case in subsection 2.1.2.6.

The phase diagram for vanishing potential $\Phi = 0$ and a sample of angular velocities is presented in Figure 2.2.

There have been suggestions that the "thermal gas in AdS" that competes with the black hole phase may not exist, except for AdS-Schwarzschild. The free energy is evaluated by a gravitational path integral with asymptotic boundary conditions that are satisfied by a black hole, but also compatible with a bulk spacetime that has no black hole. The latter has $G = 0$ (up to a Casimir term) and, whatever its precise physical nature, it must be taken into account. It is on this basis that we will consider black holes with $G > 0$ unstable, also for spacetimes with angular velocity and/or electric potential.

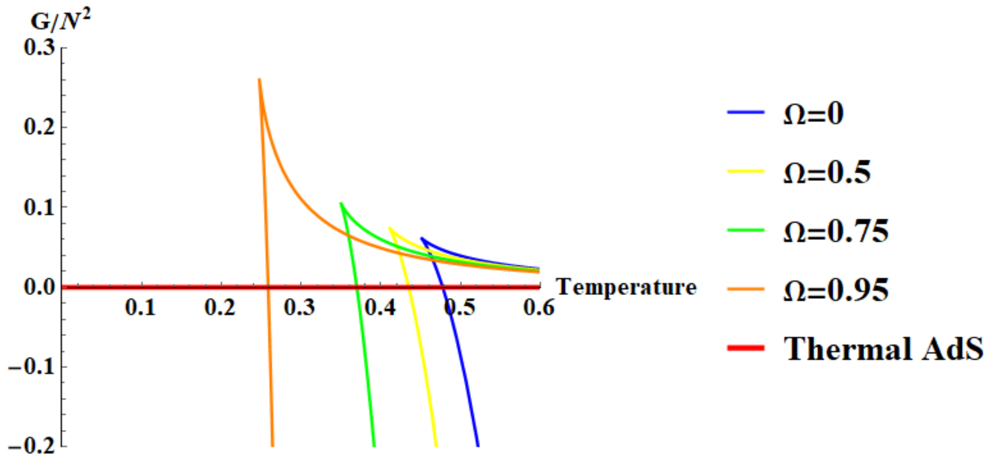

Figure 2.2: Gibbs' free energy G as function of the temperature T. The electric potential vanishes $\Phi = 0$ and the angular velocities increase from right to left as $\Omega_a = \Omega_b = 0,\ 0.5,\ 0.75,\ 0.95$. The cusp moves "North-West" as angular velocities increase, staying above the minimum temperature $T_{\min} = \frac{1}{2\pi} \approx 0.16$ even in the limit $\Omega_a = \Omega_b \to 1$.

2.1.2.2 Subcritical Electric Potential: $0 < \Phi < 3$

Starting from vanishing electric potential Φ, but arbitrary angular velocities (below their critical values $\Omega_a^* = \Omega_b^* = 1$), we now consider increasing the electric potential toward its critical value $\Phi^* = 3$. The phase diagram remains qualitatively similar to the AdS-Schwarzschild case as the electric potential increases, but quantitative features are modified. Some of these changes are similar to an increase in angular velocity at fixed electric potential:

- The free energy at a given temperature decreases, on both the small and the large black hole branches. This follows from the thermodynamic relation $\partial_\Phi G = -Q$.
- The transition temperature $T_{\rm HP}$ decreases, as it must because the entire large black hole branch is lowered. The minimal temperature, attained at the cusp where the two branches meet, also decreases.

However, despite these similarities, increasing electric potential differs significantly from increasing angular velocity:

- The free energy at the cusp $G_{\max}$ *decreases* with increasing electric potential.
- The transition temperature $T_{\rm HP}$ (and so the minimal temperature $T_{\min} < T_{\rm HP}$) decreases with *no minimum nonzero value* as the electric potential approaches the critical value $\Phi = \Phi^*$.

Both of these features indicate that increased potential Φ *stabilizes* the black hole. The phase diagram with electric potential in the range $0 < \Phi < 3$ is presented in Figure 2.3.

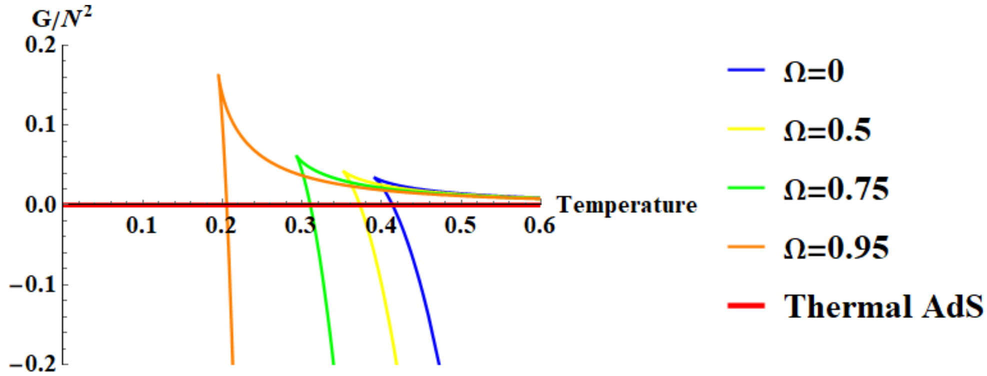

Figure 2.3: Gibbs' free energy G as function of the temperature T. The electric potential $\Phi = 1.5$ is in the range $0 < \Phi < 3$ and the angular velocities increase from right to left as $\Omega_a = \Omega_b = 0,\ 0.5,\ 0.75,\ 0.95$. For a given $\Omega_{a,b}$, the free energy and the temperature are both smaller than for $\Phi = 0$ (compare with Figure 2.2.)

2.1.2.3 Critical Electric Potential $\Phi = \Phi^* = 3$

When Φ increases to the critical value $\Phi^* = 3$, the phase diagram changes *qualitatively*.

The expressions for potentials

$$T|_{\Phi=3} = \frac{r_+(2r_+^2 + a^2 + b^2)\left(r_+^2 + a + b - ab\right)}{2\pi(r_+^2 + a^2)(r_+^2 + b^2)(r_+^2 - ab)}(r_+^2 - r_*^2) \ , \tag{2.20}$$

$$1 - \Omega_a|_{\Phi=3} = \frac{1 - a}{r_+^2 + a^2}(r_+^2 - r_*^2) \ , \tag{2.21}$$

$$1 - \Omega_b|_{\Phi=3} = \frac{1 - b}{r_+^2 + b^2}(r_+^2 - r_*^2) \ , \tag{2.22}$$

and for Gibbs free energy

$$G|_{\Phi=3} = -\frac{N^2(r_+^2 + a^2)(r_+^2 + b^2)\left(r_+^2 + a + b - ab\right)}{4(1 - a^2)(1 - b^2)(r_+^2 - ab)^2}(r_+^2 - r_*^2) \ , \tag{2.23}$$

simplify somewhat. As in earlier formulae, $r_* = \sqrt{a + b + ab}$ is the horizon location for BPS black holes with parameters (a, b).

The equations show that, when $\Phi = \Phi^* = 3$, the requirement of positive temperature $T \geq 0$ is equivalent to $r_+ \geq r_*$. Therefore, taking $\Phi = \Phi^* = 3$ automatically prevents overspinning (it keeps $\Omega_{a,b} \leq 1$) and it also ensures non-positive free energy $G \leq 0$. Because of these inequalities, the phase diagram simplifies greatly when $\Phi = \Phi^*$. There is *no small black hole branch*, no matter what the temperature and angular velocities are. In particular, there is no "cusp" where two branches meet. Additionally, since black hole states have non-positive free energy, the thermal AdS gas with $G = 0$ is never preferred.

When any of the bounds noted in the previous paragraph are saturated, we have $r_+^2 = r_*^2$ and so they all become equalities. Moreover, in this case the black hole is BPS. In other words, the following are equivalent:

- $\Phi = \Phi^*$ and *any* of $T = 0$, $\Omega_a = 1$, $\Omega_b = 1$, $G = 0$.
- $\Phi = \Phi^*$ and *all* of $T = 0$, $\Omega_a = 1$, $\Omega_b = 1$, $G = 0$.
- The black hole is *supersymmetric*, i.e. the mass is exactly the BPS mass: $M = M_{\text{BPS}} = \Phi^* Q + \Omega_a^* J_a + \Omega_b^* J_b$.

The formulae (2.20-2.22) suggest a specific *approach* to the BPS limit: simply fix the parameters a, b and tune black hole parameters so that $r_+^2 - r_*^2 = \epsilon \to 0$. This corresponds to physical potentials approaching the BPS limit *linearly* as

$$T|_{\Phi=3} = \frac{(2+a+b)\sqrt{a+b+ab}}{\pi(1+a)(1+b)(a+b)}\epsilon \ , \tag{2.24}$$

$$1 - \Omega_a|_{\Phi=3} = \frac{1-a}{(1+a)(a+b)}\epsilon \ , \tag{2.25}$$

$$1 - \Omega_b|_{\Phi=3} = \frac{1-b}{(1+b)(a+b)}\epsilon \ , \tag{2.26}$$

with Gibbs free energy (2.16) also approaching zero linearly

$$G|_{\Phi=3} = -\frac{1}{2}N^2 \frac{a+b}{(1-a)(1-b)}\epsilon \ . \tag{2.27}$$

When implementing the BPS limit in this way it is manifest that $G, T, 1-\Omega_{a,b}$ all reach 0 simultaneously, with specified relative rate. In section 2.2 we will identify this approach with the BPS limit itself.

The somewhat formal statements on the BPS limit in the previous paragraphs fail to reflect all physical aspects. Therefore, we now consider the physical realization of thermodynamics already discussed for $\Phi < 3$: start with high temperature and then cool the system all the way to $T = 0$, while keeping the physical angular velocities $\Omega_{a,b}$ fixed. Figure 2.4 shows the resulting free energy G as function of temperature T, for various values of $\Omega_a = \Omega_b$.

To implement the physical thermodynamics underlying Figure 2.4 we would lower the parameter r_+, a proxy for temperature, and simultaneously lower the parameters a, b, in order that the left hand sides of (2.21-2.22) remain constant. The approach to the BPS limit, described in (2.24-2.26), is different because it keeps the parameters a, b fixed. The distinction is dramatic because Figure 2.4 depicts a smooth approach to $T = G = 0$ for

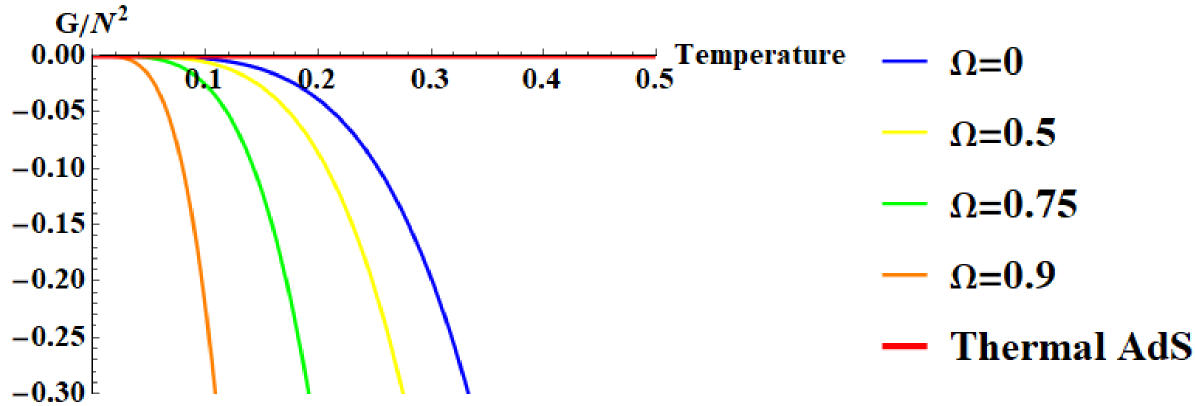

Figure 2.4: The free energy G as function of temperature T for $\Phi = \Phi^* = 3$. The angular velocity $\Omega_a = \Omega_b = 0,\ 0.5,\ 0.75,\ 0.95$ from right to left. The curves approach the origin $G = T = 0$ *without reaching it*. The special case $\Omega_a = \Omega_b = 1$ is not a curve, it is the single point $G = T = 0$.

various $\Omega_{a,b}$, in apparent contradiction with the statement that vanishing temperature $T = 0$ is possible only when $\Omega_a = \Omega_b = 1$ identically.

To resolve the tension, it is sufficient to consider the regime $r_+^2 \gg a^2, b^2, ab$ where formulae simplify so that:

$$T = \frac{r_+^4 - (a+b)^2}{\pi r_+^3} \ ,$$
$$\frac{1-\Omega_a}{1-a} = \frac{1-\Omega_b}{1-b} = 1 - \frac{a+b}{r_+^2} \ ,$$
$$G = -\frac{1}{2} N^2 \frac{r_+^4 - (a+b)^2}{(1-a^2)(1-b^2)} \ . \tag{2.28}$$

We further take $a, b \ll 1$ so $r_+^2 \sim a + b$ (which easily satisfies $r_+^2 \gg a^2, b^2, ab$) in a manner where the rotational velocities can take any value $\Omega_{a,b} = \frac{a+b}{r_+^2}$, as long as $\Omega_a = \Omega_b$. Then

$$T = \frac{1}{\pi} r_+ (1 - \Omega^2) \ , \tag{2.29}$$

and the free energy becomes:

$$G = -\frac{1}{2} N^2 \frac{\pi^4}{(1-\Omega^2)^3} T^4 \ . \tag{2.30}$$

These formulae are not the most general, but they exhibit important features clearly.

The BPS limit (2.24-2.26) lowers r_+ so $r_+^2 - r_*^2 = \epsilon \to 0$. In this case (2.29-2.30) are realized by G, T, $1-\Omega^2$ all approaching zero at the same rate, while r_+ is near the constant r_*.

The low temperature limit with the angular velocity Ω fixed is also described by the simplified free energy (2.30) but, due to (2.29), the strict limit $T \to 0$ requires $r_+^2 \to 0$ (with $a, b \to 0$ such that $\Omega_{a,b} = \frac{a+b}{r_+^2}$ is fixed). The limiting geometry with $r_+ = 0$ is singular, so it is not actually a solution. It is a "small" black hole, not just in the AdS/CFT vernacular, where the term refers to black hole size that is much smaller than the AdS-scale $r_+ \ll \ell_5$, but in the sense that unknown higher derivative curvature corrections dominate the classical "solution". Therefore, the BPS limit is well-controlled only if it is taken as specified in (2.24-2.26).

The free energy (2.30) is reminiscent of the *high* temperature regime $r_+ \gg 1$. In this limit the rotational velocities $\Omega_a = a$, $\Omega_b = b$, the temperature (2.20) simplifies to $T = \frac{r_+}{\pi}$ and the free energy (2.28) becomes

$$G = -\frac{1}{2} N^2 \frac{\pi^4}{(1-\Omega_a^2)(1-\Omega_b^2)} T^4 \ . \tag{2.31}$$

This formula shares with the low temperature expression (2.30) the power T^4 that is characteristic of CFT's in four dimensions. However, factors of $1-\Omega^2$ differ. Taking derivative with respect to temperature , we obtain the expression of the (minus) entropy

$$S = \frac{1}{2} N^2 \frac{4\pi^4}{(1-\Omega_a^2)(1-\Omega_b^2)} T^3. \tag{2.32}$$

A similar formula was presented in [40]. The high temperature regime is dual to the conformal fluid [41, 42].

2.1.2.4 Super-critical Electric Potential $\Phi > 3$

When Φ increases above the critical value $\Phi_* = 3$ all black holes acquire strictly negative free energy. Therefore, they are thermodynamically preferred to thermal AdS_5. The free energy decreases monotonically as Φ increases further. Moreover, it decreases further as the angular velocities increase with fixed Φ and T. All these trends follow straightforwardly from the thermodynamic relations $\partial_\Phi G = -Q$ and $\partial_{\Omega_{a,b}} G = -J_{a,b}$. They are illustrated in Figure 2.5.

To be more specific, we consider the case $\Omega_a = \Omega_b \equiv \Omega$, corresponding to $a = b$. Then we can invert the angular velocity (2.13) and solve for r_+^2

$$r_+^2 = \frac{a(3(1-a\Omega) + \Phi(1-a^2))}{3(\Omega - a)} \ , \tag{2.33}$$

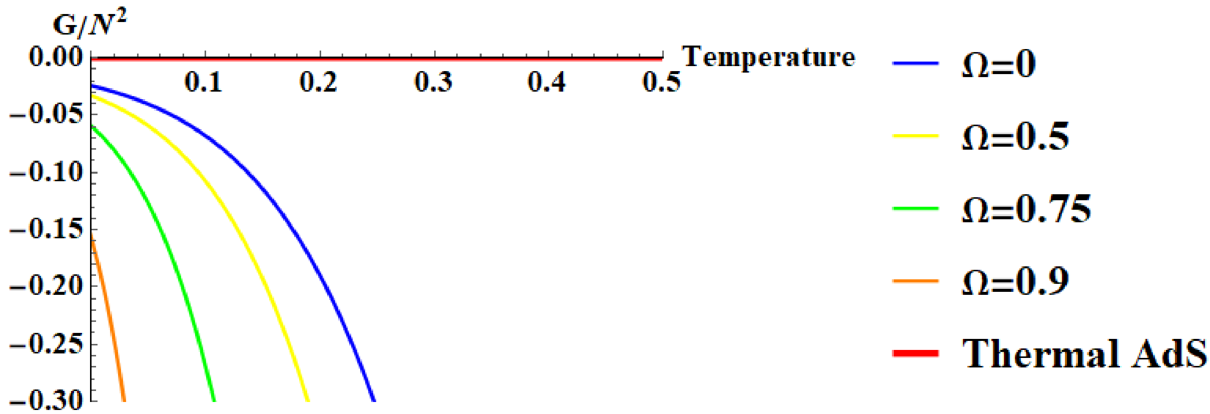

Figure 2.5: The free energy G when $\Phi = 3.5$. The angular velocities $\Omega_a = \Omega_b = 0,\ 0.5,\ 0.75,\ 0.9$ increase from right to left.

and the temperature (2.15) becomes

$$T = \frac{a(3+\Phi-6\Omega^2)-(\Phi-3)\Omega}{6\pi a(1-a\Omega)} r_+ \ . \tag{2.34}$$

The nontrivial solution for extremality ($T=0$)

$$a = \frac{(\Phi-3)\Omega}{3+\Phi-6\Omega^2} \ , \tag{2.35}$$

yields the free energy (2.16) in terms of Φ and $\Omega_a = \Omega_b = \Omega$ at extremality:

$$G_{\rm ext} = -\frac{N^2\left(\frac{1}{3}\Phi-1\right)^2}{16(1-\Omega^2)^2}\left[3\left(\frac{1}{3}\Phi+1\right)^2 - 4\Omega^2\left(1+\frac{2}{3}\Phi\right)\right] . \tag{2.36}$$

This analytical formula gives the strictly negative free energy at $T=0$ for any $\Phi > 3$. In the nonrotating limit $\Omega \to 0$ (further discussed in the following subsection) the formula is regular, so non-rotating black holes in AdS_5 are stable when the electric potential is large. This result applies also in the flat space limit where it is significant for studies of the weak gravity conjecture (WGC) [43, 44].

We are particularly interested in the BPS limit which requires $\Omega = 1$. When $\Phi > 3$ the free energy is singular for this value: the physical range of the angular velocity Ω is $[0,1)$, so Ω can be arbitrarily close to the critical value $\Omega^* = 1$, but it cannot reach it. However, the square bracket in (2.36) vanishes when both $\Omega = 1$ and $\Phi = 3$. Therefore, in the limit $\Phi \to 3$, the singularity at $\Omega = 1$ is suppressed by three powers of $\frac{1}{3}\Phi - 1$. This shows that the free energy vanishes $G \to 0$ when $\Omega \to 1$ and $\Phi \to 3$ at identical "speed".

The reason that the angular velocities $\Omega_{a,b}$ can be arbitrarily close to their critical values $\Omega^*_{a,b} = 1$, but cannot reach it, is that the conserved charges $M, Q, J_{a,b}$ must be finite: they can be arbitrarily large, but they cannot diverge. This in turn requires $a, b \in [0, 1)$, neither of these parameters can be exactly equal to one.

To see this, first consider the temperature (2.15) near $a = 1$ and/or $b = 1$:

$$T \underset{a=1-\epsilon_a, b=1-\epsilon_b}{=} \frac{r_+\left(r_+^2 - (\frac{2}{3}\Phi + 1) - \frac{1}{18}(\Phi^2 - 9)\right)}{\pi\left(r_+^2 - \frac{\Phi}{3}\right)} + \mathcal{O}(\epsilon_a, \epsilon_b) \ . \tag{2.37}$$

Non-negativity of the temperature in this regime shows $r_+^2 \geq (\frac{2}{3}\Phi + 1) > 0$. Therefore, since $\epsilon_{a,b}$ can be small but not strictly zero, the angular velocities (2.13) and (2.14)

$$1 - \Omega_a \underset{a=1-\epsilon_a, b=1-\epsilon_b}{=} \frac{\left(r_+^2 - (\frac{2}{3}\Phi + 1)\right)}{(1 + r_+^2)}\epsilon_a + \mathcal{O}(\epsilon^2) \ , \tag{2.38}$$

$$1 - \Omega_b \underset{a=1-\epsilon_a, b=1-\epsilon_b}{=} \frac{\left(r_+^2 - (\frac{2}{3}\Phi + 1)\right)}{(1 + r_+^2)}\epsilon_b + \mathcal{O}(\epsilon^2) \ . \tag{2.39}$$

can be arbitrarily close, but not equal to, the critical values $\Omega^*_{a,b} = 1$. In other words, when $\Phi > 3$ either of the limits $\Omega_a \to 1$ or $\Omega_b \to 1$ takes *all* the quantum numbers $M, Q, J_{a,b} \to \infty$

2.1.2.5 Nonrotating Black Holes: $\Omega = 0$

In this subsection we turn off rotation by setting $a = b = 0$ and focus on the effect of the electric potential Φ. This case was well developed early on [45, 46] so it serves as an important benchmark for the effects of rotation. Here we will review and compare this case with our previous discussions.

For zero angular velocity $\Omega_a = \Omega_b = 0$, the temperature (2.15) and the free energy (2.16) reduce to [45]

$$T = \frac{2g^2r_+^2 + 1 - \left(\frac{1}{3}\Phi\right)^2}{2\pi r_+} \ , \tag{2.40}$$

$$G = -\frac{N^2}{4}r_+^2\left(g^2r_+^2 + \left(\frac{1}{3}\Phi\right)^2 - 1\right) \ . \tag{2.41}$$

For given (T, Φ), the first equation yields 2, 1, or 0 solutions for r_+; and then the second equation gives the applicable values of the free energy $G(T, \Phi)$. It is presented in Figure 2.6.

The AdS-Schwarzschild black hole, reviewed in the beginning of subsection 2.1.2.1 and plotted in Figure 2.1, is the curve to the right. Increasing electric potential Φ lowers both the minimal temperature $T_{\min}$ and the maximal free energy $G_{\max}$. These effects make the

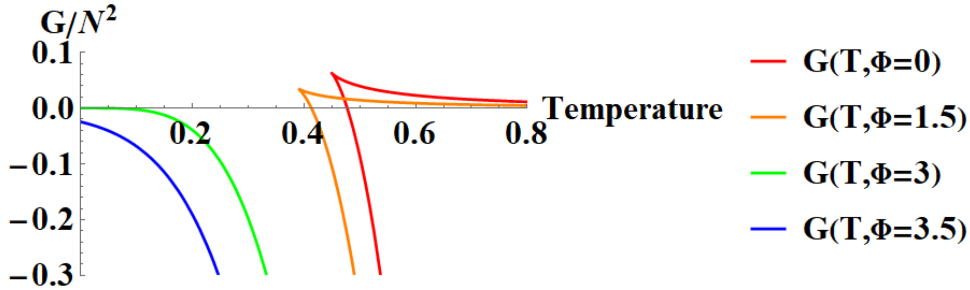

Figure 2.6: The free energy for non-rotating black holes $\Omega = 0$. The electric potential Φ increases from right to left. There is a qualitative transition from AdS-Schwarzschild type at $\Phi < \Phi_* = 3$ to the high potential regime $\Phi \geq \Phi_* = 3$.

black hole more stable.

There is a qualitative change as the potential increases from the regime $\Phi < 3$ to $\Phi \geq 3$. The lower range of potential $\Phi < 3$ is “AdS-Schwarzschild-type”, it has two black hole branches that are joined at a cusp where the temperature reaches its minimum $T_{\min}$ and the free energy its maximum $G_{\max}$. The higher range of potential $\Phi \geq 3$ has just a ”large” black hole branch. Therefore, this range corresponds to particularly stable black holes, as identified by the weak gravity conjecture.

The earlier parts of this subsection considered the effect of rotation on various ranges of Φ. In particular, each curve in Figure 2.6 is the same as first curve shown in Figures 2.2-2.5. The results of those earlier subsections showed that angular velocities strictly below maximal $0 \leq \Omega_{a,b} < 1$ do not change the phase diagrams qualitatively. For $\Phi < 3$, angular velocity lowers the temperature $T_{\min}$ but the free energy $G_{\max}$ increases. For $\Phi > 3$ rotational velocity lowers the free energy of the large black hole branch, the only one there is. The case where the electric potential is exactly critical $\Phi = \Phi^* = 3$ is subtle, as discussed in subsection 2.1.2.3.

Black hole thermodynamics is much simpler in the absence of rotation so more formulae can be made explicit. In this case, we can invert (2.40) and solve for the coordinate location of the horizon r_+:

$$r_+ = \frac{\sqrt{2g^2(\Phi^2 - 9) + (3\pi T)^2} + 3\pi T}{6g^2} \,. \tag{2.42}$$

We picked the *positive* root. Then (2.41) gives the free energy:

$$G = -\frac{N^2}{864 g^6}\left(\sqrt{2g^2\left(\Phi^2-9\right)+(3\pi T)^2}+3\pi T\right)^2\left(\pi T\left(\sqrt{2g^2\left(\Phi^2-9\right)+(3\pi T)^2}+3\pi T\right)\right.$$
$$\left. +g^2\left(\Phi^2-9\right)\right) \,. \tag{2.43}$$

We restored the AdS_5 length scale $g = \ell_5^{-1}$ in order to facilitate comparison to the flat space limit $g \to 0$. When $\Phi < \Phi_* = 3$, the argument of the square roots may become negative so, as we have elaborated on extensively, there is a minimal temperature. For $\Phi > \Phi_* = 3$, the case discussed in the previous subsection, there is a lower bound $r_+ > \frac{1}{6}\ell_5\sqrt{2(\Phi^2-9)}$ and the free energy is strictly negative in the limit.

It is instructive to take $\Phi = \Phi_* = 3$ from the outset and then take $T \to 0$ with the AdS scale $\ell_5 \to \infty$ such that the product $T\ell_5 = \Delta$ is fixed. In this limit the coordinate horizon

$$r_+ = \pi T \ell_5^2 = \pi \Delta \ell_5 \ , \tag{2.44}$$

diverges. In this limit $\ell_5 \to \infty$ the underlying black hole geometry is not ”small”, the position of the coordinate horizon r_+ approaches a finite limit, and here this corresponds to finite spatial extent. Indeed, the underlying ”large” black hole geometry reduces to the asymptotically flat Reissner-Nordström black hole.

The low temperature regime with no rotation also makes contact with a large body of recent literature that studies universal features shared with the SYK model (some entry points to the literature [47, 48]). In this context general thermodynamic arguments show that the approach to extremality is quadratic in temperature [49, 50] and it is interesting that the coefficient of the quadratic temperature dependence can be computed directly in the extremal ($T = 0$) geometry [51, 52]. Here the low-temperature limit of the free energy (2.43) is *linear* in the temperature

$$G = -\frac{N^2\left(\Phi^2-9\right)^2}{432 g^2} - \frac{N^2(\Phi^2-9)^{\frac{3}{2}}}{54\sqrt{2}g^3}\pi T - \frac{N^2\left(\Phi^2-9\right)}{24 g^4}\left(\pi T\right)^2 + \ldots \tag{2.45}$$

for a given $\Phi > \Phi_* = 3$, There is an apparent tension, but it is resolved because our study is in the grand canonical ensemble (fixed potentials), while the quadratic behavior pertains to the canonical ensemble (fixed charges).

2.1.2.6 Maximal Rotational Velocity: $\Omega = 1$

It is instructive to consider black holes with maximal horizon velocities $\Omega_a = \Omega_b = 1$ for any temperature T and electric potential Φ. This example shows that generally such spins are destabilizing, but the BPS case is an exception.

The general expressions for $\Omega_{a,b}$ (2.13-2.14) give the horizon position

$$r_+^2 = r_*^2 - b(1+a)(1-\frac{1}{3}\Phi) \ . \tag{2.46}$$

It also gives the analogous expression with $a \leftrightarrow b$ so, for consistency, the rotational parameters must be identical $a = b$. The general formula for temperature (2.6) becomes

$$T = \tfrac{1}{2\pi}(1 - \tfrac{1}{3}\Phi)\sqrt{(1 + \tfrac{1}{3}\Phi)a^{-1} + \tfrac{1}{3}\Phi} \ . \tag{2.47}$$

Since the temperature is nonnegative $T \geq 0$, we must demand $\Phi \leq 3$. This agrees with the analysis in subsection 2.1.2.4: the rotational velocities can not reach their maximal values $\Omega_{a,b} = 1$ when $\Phi > 3$.

In most of the examples we consider, the horizon location r_+ is *larger* than the corresponding BPS radius r_*, at the same values of a, b. One might wonder if this is a physical condition. For example, it could be that black holes that are excited above the BPS limit must be bigger because they have more entropy. On the other hand, the parameter r_+ is not a physical variable, nor are a, b, so this comparison cannot be taken too seriously. Indeed, according to (2.46), black holes with $\Omega = 1$ give a counterexample because they have $r_+ < r_*$.

The temperature T given in (2.47) is a monotonically decreasing function of the parameter a. Since black holes with finite charges must have $a < 1$, the temperature is bounded from below by the value at $a = 1$:

$$T_{\rm min} = \frac{1}{2\pi}(1 - \frac{1}{3}\Phi)\sqrt{1 + \frac{2}{3}\Phi} \ . \tag{2.48}$$

The inequality $a < 1$ is strict, $a = 1$ is not possible, so the lower bound $T \geq T_{\rm min}$ can be saturated only when $\Phi = 3$. In fact, when $\Phi = 3$, the minimal temperature $T_{\rm min} = 0$ is not just a bound, (2.47) shows it is the only possible temperature. That the temperature must vanish in this case is precisely as expected, because when $\Phi = 3$, $\Omega_{a,b} = 1$ all potentials are critical and so the black hole is BPS $M = M_{\rm BPS}$. Here we focus on the interesting discontinuity between $\Phi \to 3^-$ and $\Phi = 3$.

For maximal rotational velocities $\Omega_{a,b} = 1$ and any generic potential $0 \leq \Phi < 3$, all temperatures $T > T_{\rm min}$ are possible. Gibbs' free energy (2.11) becomes:

$$\begin{aligned} G &= \frac{1}{2}N^2\frac{a(3-\Phi)(\Phi+3)^2}{54(1-a)} \\ &= \frac{1}{4}N^2\frac{\left(1-\frac{1}{9}\Phi^2\right)^3}{(2\pi T)^2 - (1+\frac{2}{3}\Phi)(1-\frac{1}{3}\Phi)^2} \ . \end{aligned} \tag{2.49}$$

In the second expression the rotational parameter a was eliminated by taking advantage of (2.47). This free energy, plotted in Figure 2.7, is positive definite, so the corresponding black holes are always "small". The minimal temperature $T_{\rm min}$ given in (2.48) is such that the

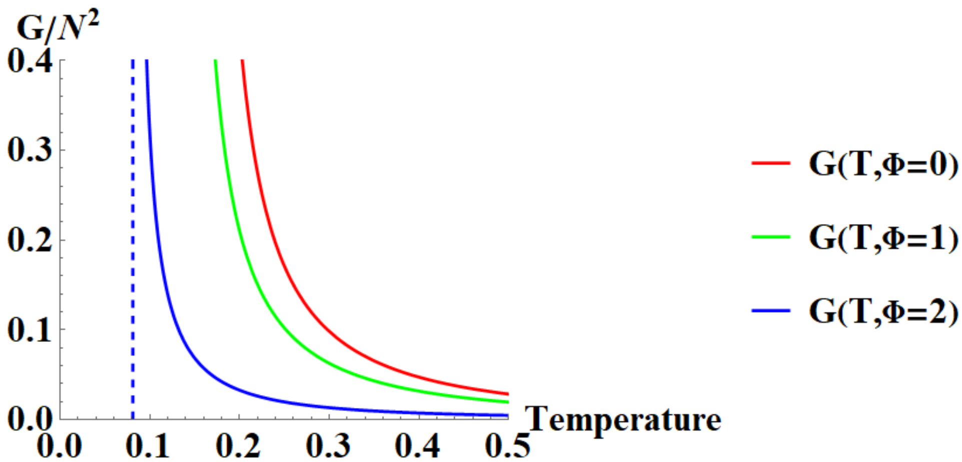

Figure 2.7: Free energy as function of temperature when $\Omega_{a,b} = 1$ and $\Phi \neq 3$. There are *only "small" black holes.* They all have $G > 0$ and negative specific heat (G decreases with temperature). The curves represent $\Phi = 0, 1, 2$ (from right to left). The vertical asymptote indicating the minimal temperature is shown only for $\Phi = 2$, to reduce clutter. As Φ increases to $\Phi \to 3^-$ the minimal temperature decreases to $T_{\rm min} = 0$. The BPS limit where $\Phi = 3$ exactly is not a curve, it is just the origin where $T = 0$ and $G = 0$.

denominator is positive for all temperatures $T > T_{\rm min}$ and vanishes in the limit $T \to T_{\rm min}$. Therefore, the free energy diverges in this limit.

The challenges faced by the approach to the BPS limit $T = 0$, $\Phi = 3$ are stark when $\Omega_{a,b} = 1$ is set from the outset:

- There is a minimal temperature $T_{\rm min}$. However, it decreases monotonically as a function of the potential Φ, from $T_{\rm min} = \frac{1}{2\pi}$ when $\Phi = 0$ to $T_{\rm min} \to 0$ as $\Phi \to 3^-$.
- The free energy diverges if, for a given potential, the temperature T is lowered to $T_{\rm min}$. However, the "height" of the divergence is set by $(1 - \frac{1}{3}\Phi)^3$.
- By increasing $\Phi \to 3^-$ and lowering T simultaneously, the free energy can vanish in the limit, after all. However, the free energy necessarily approaches zero from *above* $G \to 0^+$.

The "safest" approach to the BPS limit are from the large black hole branch $G \to 0^-$. It requires taking electric potential $\Phi = \Phi_* = 3$ first, or at least approaching it from above $\Phi \to 3^+$, and only then tuning other parameters.

2.2 Thermodynamics of Supersymmetric Black Holes

In this section we study the thermodynamics of black holes that are strictly supersymmetric. We consider general aspects, as well as their realization by black holes in AdS_5. As an instructive example, we develop new details on a "benchmark" case that was discussed previously. The following section 2.3 studies more general cases.

2.2.1 General Structure of BPS Thermodynamics

Suppose we have a general free energy[1]

$$G = G(T, \Phi, \Omega) = M - TS - \Phi Q - \Omega J \ , \tag{2.50}$$

and want to study the states that satisfy the BPS mass relation

$$M = \Phi^* Q + \Omega^* J \ , \tag{2.51}$$

where Φ^* and Ω^* are given numbers that depend on the system we consider. The general free energy (2.50) satisfies the first law

$$dG = -SdT - Qd\Phi - Jd\Omega \ ,$$

so it yields the mass

$$M = G + TS + \Phi Q + \Omega J = (1 - T\partial_T - \Phi\partial_\Phi - \Omega\partial_\Omega)G \ .$$

The BPS mass relation (2.51) is satisfied if and only if

$$\Big[1 - T\partial_T G - (\Phi - \Phi^*)\partial_\Phi - (\Omega - \Omega^*)\partial_\Omega\Big]G = 0 \ .$$

This is the condition that the free energy is *homogeneous of degree* 1 in the variables $T, \Phi - \Phi^*, \Omega - \Omega^*$.

Considering any G that is homogeneous of degree 1, we define the BPS free energy

$$W(\Phi', \Omega') = \frac{G}{T} = -S - \Phi' Q - \Omega' J \ . \tag{2.52}$$

[1] In this subsection we refer to a single electric potential Φ and a single angular velocity Ω, to keep terminology simple. However, one or both kinds of potentials can have multiple components, corresponding to several independent charges and/or angular momenta.

The second form used the BPS mass relation (2.51) and introduced primed potentials

$$\Phi' = \frac{\Phi - \Phi^*}{T} \ ,$$
$$\Omega' = \frac{\Omega - \Omega^*}{T} \ . \tag{2.53}$$

The function W inherits a homogeneity property from G and, because we have divided by T, it is in fact precisely homogeneous in the variables $T, \Phi - \Phi^*, \Omega - \Omega^*$, it has "weight" zero. Therefore, it is possible to interpret it as a function of the ratios Φ', Ω', as we have indicated in our notation for W. In the BPS limit we necessarily have temperature $T \to 0$ so these "primed" variables can be identified as derivatives with respect to the temperature, evaluated at $T = 0$. An alternative but equivalent interpretation is that the BPS surface is a projective manifold defined by ratios of thermodynamic variables. Furthermore, viewing the BPS surface as a continuous manifold allows for a generalization beyond previous studies [53] which treated it as a point instead characterized by $\Phi - \Phi^* = 0$ and $\Omega - \Omega^* = 0$, followed by taking $T \to 0$.

The BPS free energy W satisfies the first law

$$dW = -Qd\Phi' - Jd\Omega' \ , \tag{2.54}$$

so

$$Q = -\frac{\partial W}{\partial \Phi'} \ , \qquad J = -\frac{\partial W}{\partial \Omega'} \ . \tag{2.55}$$

It yields the entropy

$$S = (-1 + \Phi' \partial_{\Phi'} + \Omega' \partial_{\Omega'}) W \ . \tag{2.56}$$

However, the BPS mass has been "integrated out" of the BPS thermodynamics, it is not encoded in W and only follows from Q and J via the BPS mass relation (2.51).

In statistical mechanics, the BPS free energy W arises when introducing a BPS partition function as a trace over all BPS states

$$Z_{\rm BPS} = {\rm Tr}_{\rm BPS} \left[e^{\Phi' Q + \Omega' J} \right] \ , \tag{2.57}$$

with no explicit reference to mass. Then the BPS free energy $W = -\ln Z_{\rm BPS}$ [53].

2.2.2 Parameters for BPS Black Holes in AdS_5

In this paper we primarily study AdS_5 black holes with a single charge and two independent angular momenta. We want to make the general considerations in the preceding subsection explicit in this specific case.

The BPS condition, first given in [31], in our convention reads

$$M = 3Q + J_a + J_b \ . \tag{2.58}$$

Comparing with the generic formula (2.51) we have

$$\Phi^* = 3 \ , \tag{2.59}$$

and, for $a = b$, $\Omega^* = 2$. In the general case where $a \neq b$ there are two independent rotational velocities $\Omega_{a,b}$ with BPS values

$$\Omega_a^* = \Omega_b^* = 1 \ . \tag{2.60}$$

The general parametric expressions for the black hole quantum numbers $(M, Q, J_{a,b})$ in (2.1 - 2.4) give the mass excess above the BPS bound

$$M - 3Q - (J_a + J_b) = \frac{1}{2}N^2 \frac{3 + (a+b) - ab}{(1-a)(1+a)^2(1-b)(1+b)^2}\Big(m - q(1+a+b)\Big) \ . \tag{2.61}$$

The prefactor on the right hand side is manifestly positive for all $a, b \in [0, 1[$, so the BPS formula (2.58) yields the condition on the parameters (m, q, a, b):

$$m = q(1 + a + b) \ . \tag{2.62}$$

We can therefore consider m a dependent variable and parameterize the BPS black holes by (q, a, b).

Even when the parameter m satisfies (2.62), the equation (2.5) for the coordinate location of the event horizon is a cubic equation in r^2. Instead of proceeding by brute force, it is useful to recall that BPS black holes are extremal and so we expect a double root. With this clue, and the BPS condition (2.62), it is manageable to find the solution

$$r_*^2 = a + b + ab \ , \tag{2.63}$$

for a specific value of the parameter q:

$$q^* = (1 + a)(1 + b)(a + b) \ . \tag{2.64}$$

Whichever way this solution for particular values of m, q was arrived at, we can, for any m, q, recast the cubic equation (2.5) satisfied by the horizon location r^2 as

$$(r^2 - r_*^2)^2 r^2 + \Big((q - q^*) - (r^2 - r_*^2)(1 + a + b)\Big)^2 + 2r^2(m - q(1 + a + b)) = 0 \ . \qquad (2.65)$$

When the BPS condition (2.62) is satisfied this expression is the sum of two squares so, for BPS black holes, it is manifest that a real solution exists *only* for $q = q^*$ and for this value of q there is indeed a double root when $r^2 = r_*^2$. We will refer to the starred variables as the BPS values.

When m satisfies the BPS condition (2.62) and q takes its BPS value (2.64), the parametric formulae (2.1 - 2.4) for the physical variables $(M, Q, J_{a,b})$ simplify to:

$$M^* = \frac{1}{2}N^2 \frac{(a + b)(3 - a^2 + ab - b^2 - ab(a + b))}{(1 - a)^2(1 - b)^2} \ , \qquad (2.66)$$

$$Q^* = \frac{1}{2}N^2 \frac{a + b}{(1 - a)(1 - b)} \ , \qquad (2.67)$$

$$J_a^* = \frac{1}{2}N^2 \frac{(a + b)(2a + b + ab)}{(1 - a)^2(1 - b)} \ , \qquad (2.68)$$

$$J_b^* = \frac{1}{2}N^2 \frac{(a + b)(a + 2b + ab)}{(1 - a)(1 - b)^2} \ . \qquad (2.69)$$

The potential Φ and the angular velocities $\Omega_{a,b}$ given in (2.7 - 2.9) reduce to the BPS values (2.59-2.60), as they should. The entropy S (2.10) simplifies to:

$$S^* = 2\pi \frac{1}{2}N^2 \frac{(a + b)\sqrt{a + b + ab}}{(1 - a)(1 - b)} \ . \qquad (2.70)$$

As we have stressed, once we impose the BPS mass formula (2.58), black holes exist only when the parameters q, a, b satisfy the relation (2.64). Thus the BPS mass cannot be reached for all black hole variables (Q, J_a, J_b). Rather, according to (2.67), (2.68) and (2.69), the conserved charges must satisfy the constraint

$$\left(Q^{*3} + \frac{1}{2}N^2 J_a^* J_b^*\right) - \left(3Q^* + \frac{1}{2}N^2\right)\left(3Q^{*2} - \frac{1}{2}N^2(J_a^* + J_b^*)\right) = 0 \ . \qquad (2.71)$$

It is surprising that there is this condition on the charges.

2.2.3 The BPS Free Energy

Usually, a good starting point for the study of thermodynamics is the free energy G, an extensive function of the intensive thermodynamic potentials $T, \Phi, \Omega_{a,b}$. However, for BPS black holes the free energy vanishes identically $G = 0$ and the "variables" it would ordinarily be a "function" of are fixed at $T = 0, \Phi^* = 3$, and $\Omega^*_{a,b} = 1$.

The general structure of BPS thermodynamics discussed in subsection 2.2.1 addresses this obstacle. A free energy G that is homogeneous of degree one in T, $\Phi - \Phi^*$, $\Omega_{a,b} - \Omega^*_{a,b}$ depends on variables that nominally depart from their BPS values, but the homogeneity property shows that the mass is *exactly* the BPS mass. Therefore, a free energy of this form describes thermodynamics intrinsic to the BPS surface.

2.2.3.1 Derivation of the BPS Free Energy

In the AdS_5 example we focus on, the explicit free energy G (2.16) of a general AdS_5 black hole depends on the electric potential Φ, as it should, but it is also a function of the parameters r_+, a, b, rather than T, $\Omega_{a,b}$. Conveniently, the expression for G given in (2.16) vanishes when, in addition to $\Phi = 3$, the auxiliary horizon position r_+^2 take the BPS value r_*^2 given in (2.63). At linear order in both $r_+^2 - r_*^2$ and $\Phi - 3$ we have

$$G = -\frac{N^2}{2}\frac{a+b}{(1-a)(1-b)}(r_+^2 - r_*^2) - \frac{N^2}{6}\frac{(a+b)(1-a-b-ab)}{(1-a)(1-b)}(\Phi - 3) \ .$$

The combination $r_+^2 - r_*^2$ has a geometrical interpretation in terms of horizon position, but it is not a traditional thermodynamic potential. It is related to the angular velocities through the two equations (2.13-2.14) which are straightforward to expand at linear order in $\Phi - \Phi^*$, $\Omega_{a,b} - \Omega^*_{a,b}$. The result

$$r_+^2 - r_*^2 = -\frac{(a+b)(1+a)}{2(1-a)}(\Omega_a - 1) - \frac{(a+b)(1+b)}{2(1-b)}(\Omega_b - 1) + \frac{a+b+2ab}{6}(\Phi - 3) \ , \quad (2.72)$$

leads to the free energy

$$G = -\frac{N^2}{12}\frac{(a+b)(2-a-b)}{(1-a)(1-b)}(\Phi - 3) + \frac{N^2}{4}\frac{(a+b)^2(1+a)}{(1-a)^2(1-b)}(\Omega_a - 1) + \frac{N^2}{4}\frac{(a+b)^2(1+b)}{(1-a)(1-b)^2}(\Omega_b - 1) \ . \quad (2.73)$$

Moreover, the condition that the two equations (2.13-2.14) give identical values for $r_+^2 - r_*^2$ yields the *constraint* on the potentials

$$\frac{(a+b)(1+a)}{1-a}(\Omega_a - 1) - \frac{(a+b)(1+b)}{1-b}(\Omega_b - 1) + \frac{1}{3}(a-b)(\Phi - 3) = 0 \ . \quad (2.74)$$

The formula for temperature (2.15), also expanded at linear order in $\Phi - \Phi^*$, $\Omega_{a,b} - \Omega^*_{a,b}$, imposes a second constraint on the potentials

$$T = -\frac{\sqrt{a+b+ab}}{\pi(1-a)}(\Omega_a - 1) - \frac{\sqrt{a+b+ab}}{\pi(1-b)}(\Omega_b - 1) - \frac{\sqrt{a+b+ab}}{3\pi(a+b)}(\Phi - 3) \ . \tag{2.75}$$

We can interpret the two constraints (2.74-2.75) together, as definitions of the auxiliary parameters a, b. Since both constraints are invariant under simultaneous linear scaling of $T, \Phi - \Phi^*$, $\Omega_{a,b} - \Omega^*_{a,b}$, the a, b defined implicitly this way are homogeneous functions of these potentials.

The free energy (2.73) transforms linearly under simultaneous linear scaling of $T, \Phi - \Phi^*$, $\Omega_{a,b} - \Omega^*_{a,b}$ so, as a function of these variables, it is homogeneous of degree one. According to the discussion in subsection 2.2.1, it follows that it describes BPS black holes. To reach this conclusion it is important that the parameters a, b are invariant under such rescalings.

To avoid possible confusion, we reiterate the reasoning. The free energy (2.73) and the constraints (2.74-2.75) were all derived by expanding to linear order in $\Phi - \Phi^*$, $\Omega_{a,b} - \Omega^*_{a,b}$. That could leave the impression that they are approximations that are valid near the BPS limit but in fact they describe the BPS black holes themselves, as these equations contain no information beyond BPS. Once a free energy is presented as a homogeneous function, the additional step of taking the limit $T \to 0$ is possible, but not required, doing so corresponds to a choice of coordinates in projective geometry.

To understand how this is possible, consider the mass excess over the BPS mass (2.61), rewritten using (2.65, 2.72):

$$M - M_* = \frac{N^2}{2}\frac{(3+a+b-ab)(a+b)^2}{4(1-a)(1-b)(1+3a+3b+a^2+3ab+b^2)}\Big((2\pi T)^2 + \varphi^2\Big) \ , \tag{2.76}$$

where

$$\varphi = (\Phi - \Phi^*) - (\Omega_a - \Omega^*_a) - (\Omega_b - \Omega^*_b) = \Phi - \Omega_a - \Omega_b - 1 \ . \tag{2.77}$$

This mass excess $M - M_*$ is *quadratic* in the small variables and so the black hole is BPS because the equality $M = M_*$ holds at linear order.[2]

The BPS energy W (2.52) that is intrinsic to the BPS surface is computed from (2.73)

[2] The quadratic correction in (2.76) is not important in this chapter but it is interesting in its own right [49, 50, 54, 55, 17, 56, 57, 58]. It is described in effective quantum field theory by the Schwarzian theory [47, 59, 60, 61, 62].

by dividing with T:

$$W = \frac{G}{T} = \frac{1}{2}N^2\left(-\frac{(a+b)(2-a-b)}{6(1-a)(1-b)}\Phi' + \frac{(a+b)^2(1+a)}{2(1-a)^2(1-b)}\Omega'_a + \frac{(a+b)^2(1+b)}{2(1-a)(1-b)^2}\Omega'_b\right) \ . \tag{2.78}$$

The primed potentials

$$\Phi' = \frac{\Phi - 3}{T} \qquad\qquad \Omega'_{a,b} = \frac{\Omega_{a,b} - 1}{T} \ , \tag{2.79}$$

were previously introduced in the general BPS formalism through (2.53). Depending on the point of view, they are either coordinates in the projective geometry defining the BPS surface or, via the limiting procedure, thermal derivatives evaluated at $T = 0$.

The parameters a and b are complicated functions of Φ', Ω'_a and Ω'_b defined by the constraints (2.74-2.75) recast in the form:

$$-\frac{\sqrt{a+b+ab}}{1-a}\Omega'_a - \frac{\sqrt{a+b+ab}}{1-b}\Omega'_b - \frac{\sqrt{a+b+ab}}{3(a+b)}\Phi' = \pi \ , \tag{2.80}$$

$$\frac{(a+b)(1+a)}{1-a}\Omega'_a - \frac{(a+b)(1+b)}{1-b}\Omega'_b + \frac{a-b}{3}\Phi' = 0 \ . \tag{2.81}$$

Differentiation of the BPS free energy W (2.78) with respect to the primed potentials must give the conserved charges through the thermodynamic relations (2.55). When computing the derivatives, note that, in addition to the explicit dependence of W on the primed potentials, there is implicit dependence through a, b that is given by (2.80-2.81). It is a consistency check on the various formulae that the black hole charges computed this way do in fact agree with (2.67-2.69).

2.2.3.2 Simplifications of the Constraints

In general, it is impractical to solve the constraints (2.80-2.81) and give a, b as explicit functions of the potentials $\Omega'_{a,b}$ and Φ'. However, we can do so in the regime where $1 - a \sim 1 - b \sim \epsilon$ is small and positive. This is a version of the Cardy-like limit [63, 64].

In this limit, the constraint (2.81) shows that the Φ' term is dominated by the large $\Omega'_{a,b}$ terms. We can partially address this by tuning $\Omega'_a \sim \Omega'_b \sim \epsilon$ but, because a, b are both near unity we have $(a - b) \sim \epsilon$, and so it is insufficient that the $\Omega'_{a,b}$ terms are order unity, they must nearly cancel. Thus:

$$\frac{\Omega'_a}{1-a} = \frac{\Omega'_b}{1-b} \ , \tag{2.82}$$

up to terms of order ϵ. With this equality, and using $a \sim b \sim 1$ extensively, the other constraint (2.80) then yields

$$\frac{\Omega'_a}{1-a} = \frac{\Omega'_b}{1-b} = \frac{\Phi' + 2\pi\sqrt{3}}{12} \ . \tag{2.83}$$

This equation is equivalent to presenting a, b as explicit functions of the BPS potentials $\Phi', \Omega'_{a,b}$, as we wanted to do. With this result it is straightforward to also express the BPS free energy W in (2.78) as function of these variables:

$$W_{\rm Cardy} = -\frac{N^2}{432}\frac{(\Phi' + 2\pi\sqrt{3})^3}{\Omega'_a\Omega'_b} \ . \tag{2.84}$$

This is expression is completely explicit, but it applies only in the Cardy-like limit.

A complementary approach to the awkward constraints (2.80-2.81) that define a, b as functions of the primed potentials is to implement them using Lagrange multipliers. Introducing two multipliers $\Lambda_{1,2}$, we have:

$$\begin{aligned} W = {} & \frac{1}{2}N^2\left[\frac{-(a+b)(2-(a+b))}{6(1-a)(1-b)}\Phi' + \frac{(a+b)^2(1+a)}{2(1-a)^2(1-b)}\Omega'_a + \frac{(a+b)^2(1+b)}{2(1-a)(1-b)^2}\Omega'_b\right] \\ & + \Lambda_1\left(-\frac{\sqrt{a+b+ab}}{1-a}\Omega'_a - \frac{\sqrt{a+b+ab}}{1-b}\Omega'_b - \frac{\sqrt{a+b+ab}}{3(a+b)}\Phi' - \pi\right) \\ & + \Lambda_2\left(\frac{(a+b)(1+a)}{1-a}\Omega'_a - \frac{(a+b)(1+b)}{1-b}\Omega'_b + \frac{a-b}{3}\Phi'\right) \ . \end{aligned} \tag{2.85}$$

Extremization with respect to a, b give conditions that are solved when:

$$\Lambda_1 = \frac{1}{2}N^2\frac{2(a+b)\sqrt{a+b+ab}}{(1-a)(1-b)} \ , \tag{2.86}$$

$$\Lambda_2 = \frac{1}{2}N^2\frac{-(a-b)}{2(1-a)(1-b)} \ , \tag{2.87}$$

without imposing any additional relation on Φ' and $\Omega'_{a,b}$. Substituting these equations into (2.85), we find:

$$\begin{aligned} W_{\rm ext} = -\frac{1}{2}N^2\Big[& \frac{2\pi(a+b)\sqrt{a+b+ab}}{(1-a)(1-b)} - \frac{(a+b)(2a+b+ab)}{(1-a)^2(1-b)}\Omega'_a \\ & - \frac{(a+b)(a+2b+ab)}{(1-a)(1-b)^2}\Omega'_b - \frac{a+b}{(1-a)(1-b)}\Phi'\Big] \ . \end{aligned} \tag{2.88}$$

Importantly, in this equation a, b are auxiliary variables: $W_{\rm ext}$ is defined only after extremizing a, b with primed potentials kept fixed.

At this point differentiation with respect to the primed potentials is trivial, there is no need to take implicit dependence via a, b into account, because these auxiliary variables are anyway evaluated at their extremum. Thus, in (2.88) the coefficient of each BPS potential Φ', $\Omega'_{a,b}$ must reproduce its conjugate conserved charge (2.67, 2.68,2.69), as indeed it does. Additionally, the BPS black hole entropy can be computed from the BPS potential through the formula (2.56), which amounts to extracting the independent-of-primed potentials constant in the BPS free energy W_{ext} given in (2.88). It agrees with the BPS entropy (2.10), as it should.

The free energy (2.88) is thus completely general, unlike the expression (2.84) in the Cardy-limit no assumptions were made on the parameters. The drawback is that, in this formalism, a, b must be extremized over, and the resulting conditions are precisely the constraints (2.80-2.81).

2.2.3.3 Thermodynamics of BPS Black Holes

In our study of BPS thermodynamics, we will facilitate comparisons with the literature by trading the BPS electric potential Φ' for

$$\varphi' = \Phi' - \Omega'_a - \Omega'_b \ , \tag{2.89}$$

which is the obvious BPS analogue of (2.77). The constraints (2.74-2.75) for the angular BPS potentials then become:

$$\Omega'_a = -\frac{1-a}{1+3a+3b+a^2+3ab+b^2}\left[\frac{\pi(a+2b+2ab+b^2)}{\sqrt{a+b+ab}} + \frac{1+a}{2}\varphi'\right] \ , \tag{2.90}$$

$$\Omega'_b = -\frac{1-b}{1+3a+3b+a^2+3ab+b^2}\left[\frac{\pi(2a+b+2ab+a^2)}{\sqrt{a+b+ab}} + \frac{1+b}{2}\varphi'\right] \ . \tag{2.91}$$

For reference, we also record the inverse formula that converts from φ' back to the electric BPS potential Φ':

$$\Phi' = -\frac{3(a+b)}{1+3a+3b+a^2+3ab+b^2}\left[\frac{\pi(1-ab)}{\sqrt{a+b+ab}} - \frac{(2+a+b)}{2}\varphi'\right] \ . \tag{2.92}$$

In the new variables that eliminate Φ', the free BPS energy (2.88) becomes:

$$\begin{aligned}\frac{W}{N^2} =& \frac{\pi(a+b)^2\left[1-2a-2b-a^2-5ab-b^2-a^2b-ab^2\right]}{2(1-a)(1-b)\left(1+3a+3b+a^2+3ab+b^2\right)\sqrt{a+b+ab}} \\ &- \frac{(a+b)^2(3+a+b-ab)}{4(1-a)(1-b)\left(1+3a+3b+a^2+3ab+b^2\right)}\varphi' \ .\end{aligned} \tag{2.93}$$

Even though BPS thermodynamics applies only at strictly vanishing temperature $T = 0$, it is meaningful to construct phase diagrams that depend on the BPS potentials, in a formalism similar to that explored in [65] with its BPS quantum statistical relation, now generalized for purposes of studying the thermodynamic stability of the phases described by W. In order to interpret them, it is useful to introduce the "BPS temperature"

$$\tau \equiv -\frac{2}{\Omega'_a + \Omega'_b} = \frac{4\,(1 + 3a + 3b + a^2 + 3ab + b^2)}{\frac{6\pi(a+b)(1-ab)}{\sqrt{a+b+ab}} + (2 - a^2 - b^2)\,\varphi'} \ , \tag{2.94}$$

that shares some properties with the usual physical temperature. For example, it is positive because physical black holes have $1 - \Omega_i \geq 0$ and so the primed potentials Ω'_i introduced in (2.79) are negative. Also, for small and fixed $1 - \Omega_i$, the effective temperature τ is proportional to the physical temperature T, again by (2.79). Finally, it generalizes the analogous definition in [66, 63].

In the preceding, and in our entire study of the phase diagram for BPS black holes, we have opted to use (a, b, φ') as the variables for all thermodynamic quantities. The a, b arise as conventional parameters for presenting the underlying BPS black hole geometry, they do not have a meaning directly in thermodynamics. When we write them, we refer to them as the functions of Ω'_i and φ' that are defined implicitly by the constraints (2.80-2.81) which cannot be tractably inverted, except in specific cases like the Cardy-like limit where we have the explicit relation (2.83).

2.2.4 The BPS Phase Diagram: a Benchmark Case

In this subsection we review the special case where angular momenta are equal ($a = b$) and $\varphi' = 0$ [53, 66, 63]. It will serve as an important benchmark for our study of the general black holes in section 2.3.

The BPS temperature (2.94) simplifies when $a = b$ and $\varphi' = 0$:

$$\tau = \frac{1 + 5a}{3\pi(1 - a)}\sqrt{1 + \frac{2}{a}} \ . \tag{2.95}$$

This expression diverges at $a = 0$ and $a = 1$. It has a local minimum when the parameter $a = a_{\text{cusp}} = \frac{3\sqrt{3}-4}{11} \approx 0.109$ and $\tau = \tau_{\text{cusp}} \approx 0.809$. It is plotted in Figure 2.8.

The free energy (2.93) when $a = b$ and $\varphi' = 0$ is:

$$\frac{W}{N^2} = \frac{2\pi a^{\frac{3}{2}}(1 - 5a - 2a^2)}{(1 - a)^2\,(1 + 5a)\,\sqrt{2 + a}} \ . \tag{2.96}$$

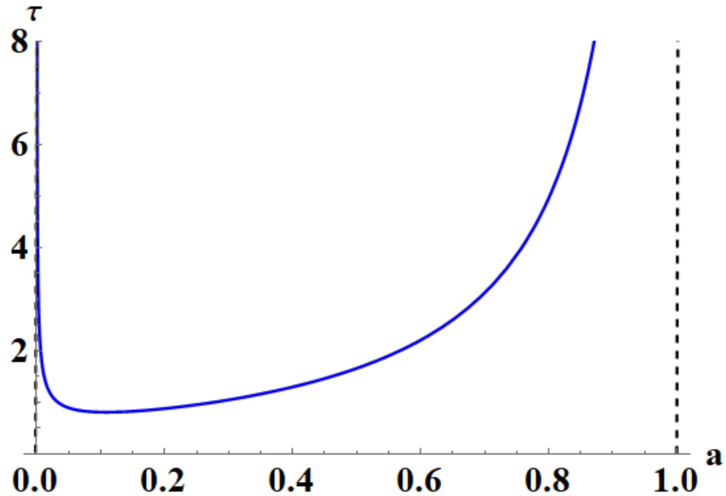

Figure 2.8: The BPS temperature τ as function of the rotation parameter a when $a = b$ and $\varphi' = 0$. The local minimum determines the position of the cusp in the phase diagram. The divergences at $a = 0$ and $a = 1$ correspond to the large-τ extremes of the "small" and "large" black hole branches, respectively.

The expression yields $W = 0$ for $a = 0$ and increases for small and positive a. The BPS temperature $\tau \to \infty$ as $a \to 0^+$ so in the limit of small parameter a the BPS free energy $W \to 0^+$ for large τ. As a increases to the value a_{cusp} where the temperature is at its minimum τ_{cusp}, the free energy increases monotonically to $W_{\text{cusp}}/N^2 = \frac{11\pi}{27\sqrt{273+158\sqrt{3}}} \approx 0.055$. The range $0 < a < a_{\text{cusp}}$ maps out the "small" black hole branch of the BPS phase diagram.

As a increases above a_{cusp}, the free energy decreases from W_{cusp}. It crosses $W = 0$ at $a_{\text{HP}} = \frac{\sqrt{33}-5}{4} \approx 0.186$ which corresponds to the BPS temperature $\tau_{\text{HP}} = 0.863$. Finally, in the limit $a \to 1^-$, the free energy diverges $W \to -\infty$ along with the BPS temperature $\tau \to \infty$. The range where $a_{\text{cusp}} < a < 1$ is the "large" BPS black hole branch. We plot the BPS phase diagram in Figure 2.9

The phase diagram of the BPS black hole is remarkably similar to that of AdS-Schwarzschild, reviewed in subsection 2.1.2.1 [66]. In other words, Figure 2.9 is reminiscent of Figure 2.1. In line with this analogy, we denoted the point where the BPS free energy crosses $W \equiv 0$ by the subscript "HP" that refers to the Hawking-Page transition. However, the notion of thermal AdS is not trivial in the BPS context. It presupposes the existence of a non-interacting gas of BPS particles for any τ, corresponding to a general value of the BPS potential for angular momentum Ω'. This hypothetical phase would always have $W = 0$ and be preferred for the τ's where the BPS black hole has $W > 0$. The equilibrium conditions between such states of matter have not yet been established so we consider the "BPS gas" conjectural.

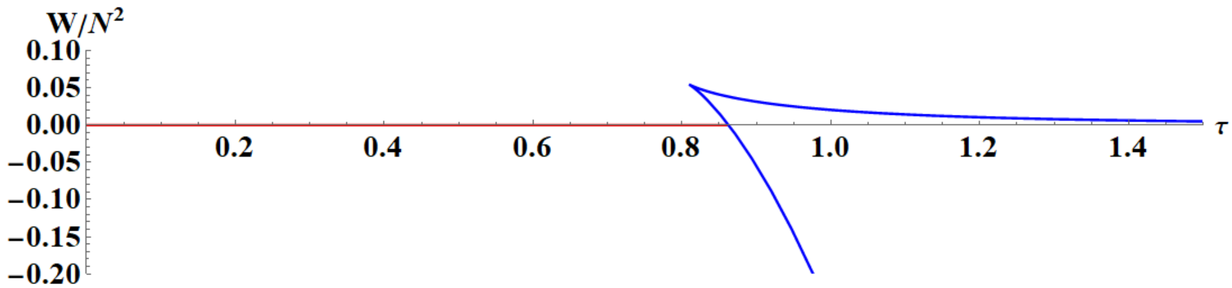

Figure 2.9: BPS free energy W/N^2 vs. BPS temperature τ. This phase diagram displays an upper (small black hole) branch and a lower (large black hole) branch. The two branches coincide at the cusp $(\tau_{\rm cusp}, W_{\rm cusp}/N^2) \approx (0.809, 0.055)$. The large black hole branch dominates over the BPS gas with $W = 0$ (red line) for $\tau > \tau_{\rm HP} \approx 0.863$.

The striking qualitative agreement between the AdS-BPS and AdS-Schwarzschild phase diagrams hides a more detailed distinction between their asymptotic behaviours.

Starting with the Schwarzschild-AdS spacetime, the Gibbs free energy G and the temperature T computed from the parameters (2.1 - 2.6) with $a = b = 0$

$$G = \frac{N^2 r_+^2(1 - r_+^2)}{4} \ , \tag{2.97}$$

$$T = \frac{1 + 2r_+^2}{2\pi r_+} \ , \tag{2.98}$$

give the rescaled free energy $W = G/T$:

$$\frac{W}{N^2} = \frac{\pi}{2}\frac{r_+^3(1 - r_+^2)}{1 + 2r_+^2} \ . \tag{2.99}$$

This allows us to extract the asymptotic behaviors of the two branches by focussing on the extreme regimes $r_+ \ll 1$ and $r_+ \gg 1$.

For very small black holes $r_+ \ll 1$, T diverges as $\sim r_+^{-1}$ and W vanishes as $\sim r_+^3$ so:

$$W \sim T^{-3} \text{ as } T \to \infty \ . \tag{2.100}$$

This is equivalent to $M \sim T^{-2}$ for asymptotically flat 5D Schwarzschild black holes. In contrast, for very large black holes $r_+ \to \infty$, T diverges as r_+ and W diverges as $-r_+^3$ so then:

$$W \sim -T^3 \quad \text{as} \quad T \to \infty \ . \tag{2.101}$$

as expected for a conformal fluid in the dual CFT_4.

Turning now to the BPS black holes, we use equations (2.95-2.96). For very small BPS black holes $a \to 0^+$, τ diverges $\sim a^{-1/2}$ and W vanishes as $\sim a^{3/2}$ so:

$$W \sim \tau^{-3} \quad \text{as} \quad \tau \to \infty \ . \tag{2.102}$$

When $a \to 1^-$ instead, τ diverges as $\sim (1-a)^{-1}$ and W diverges as $\sim -(1-a)^{-2}$ so:

$$W \sim -\tau^{2} \quad \text{as} \quad \tau \to \infty \ . \tag{2.103}$$

Our results show that the asymptotic behaviors (2.100, 2.102) for very small black holes agree but the analogues (2.101, 2.103) for large black holes do not. Thus the qualitative identification between τ and T is not precise.

2.2.5 The HHZ Free Energy

The HHZ free energy has been central to recent progress on the microscopic understanding of BPS black holes in AdS [67, 68, 69, 29, 65, 63, 27, 26, 70]. It was originally motivated by analysis of the near-horizon geometry of supersymmetric black holes rather than conventional black hole thermodynamics. In this subsection we address this gap in the literature.

2.2.5.1 Extremization of the HHZ Free Energy

The HHZ free energy is a function of the HHZ potentials Δ and $\omega_{a,b}$ that are conjugate to the conserved charges Q and $J_{a,b}$: [3]

$$H(\Delta, \omega_{a,b}) = -\frac{N^2}{2}\frac{\Delta^3}{\omega_a \omega_b} \ . \tag{2.104}$$

Unlike the free energy that is familiar from undergraduate studies, the HHZ free energy is complex and the potentials it depends on are complex as well. The latter satisfy the constraint

$$3\Delta - \omega_a - \omega_b = 2\pi i \ , \tag{2.105}$$

so they are genuinely complex numbers [29, 70].

In order to extremize the free energy (2.104) over its arguments Δ, $\omega_{a,b}$, while taking into account the constraint (2.105), it is convenient to Legendre transform to an ensemble with

[3] Our reference to the HHZ free energy generalizes the original research [29]. In particular, we allow general, non-equal angular momenta $J_{a,b}$.

fixed charges $Q, J_{a,b}$. To do so, we introduce the entropy function:

$$S(\Delta, \omega_{a,b}, Q, J_{a,b}) = -\frac{N^2}{2}\frac{\Delta^3}{\omega_a\omega_b} - 3\Delta Q - \omega_a J_a - \omega_b J_b - \Lambda(3\Delta - \omega_a - \omega_b - 2\pi i) \ , \quad (2.106)$$

where Λ is a Lagrange multiplier. Extremization over the potentials then gives:

$$\frac{\partial S}{\partial \Delta} = 0 \quad \Longrightarrow \quad -\frac{N^2}{2}\frac{3\Delta^2}{\omega_a\omega_b} - 3(Q+\Lambda) = 0 \ , \quad (2.107)$$

$$\frac{\partial S}{\partial \omega_i} = 0 \quad \Longrightarrow \quad \frac{N^2}{2\omega_i}\frac{\Delta^3}{\omega_a\omega_b} - (J_i - \Lambda) = 0 \ . \quad (2.108)$$

When these conditions are satisfied, the entropy at the extremum S^* is related to the Lagrange multiplier Λ through

$$S^* = 2\pi i \Lambda \ , \quad (2.109)$$

and Λ is determined by the black hole charges Q, $J_{a,b}$ via the cubic equation:

$$\Lambda^3 + \underbrace{(3Q + \frac{N^2}{2})}_{A}\Lambda^2 + \underbrace{\left(3Q^2 - \frac{N^2}{2}(J_a + J_b)\right)}_{B}\Lambda + \underbrace{(Q^3 + \frac{N^2}{2}J_a J_b)}_{C} = 0 \ . \quad (2.110)$$

The HHZ prescription invokes reality and positivity of the entropy, so that Λ must be a purely imaginary number with negative imaginary part. This is possible only if the cubic polynomial factorizes as $(\Lambda^2 + B)(\Lambda + A)$. It follows that the coefficients of the cubic polynomial satisfy the relation:

$$AB - C = (3Q + \frac{N^2}{2})\left(3Q^2 - \frac{N^2}{2}(J_a + J_b)\right) - (Q^3 + \frac{N^2}{2}J_a J_b) = 0 \ , \quad (2.111)$$

and also that the root with negative purely imaginary part is:

$$\Lambda = -i\sqrt{B} = -i\sqrt{3Q^2 - \frac{N^2}{2}(J_a + J_b)} \ . \quad (2.112)$$

Inserting this value for Λ in (2.109), and expressing Q and $J_{a,b}$ in terms of a and b using their values from (2.67-2.69), we match the BPS black hole entropy S^* (2.70). Moreover, the condition (2.111) is precisely the constraint on black hole charges (2.71).

The HHZ procedure thus gives the correct functions of charges, but we stress that we introduced the HHZ free energy H (2.115) as a function of the HHZ potentials $(\Delta, \omega_{a,b})$ without committing to an associated spacetime interpretation: we derived the black hole entropy via a formal extremization procedure. Some authors simply identify the HHZ formalism with "gravitational thermodynamics", and consider it the target for microscopic CFT considerations. However, the HHZ variables are genuinely complex so their relation to spacetime physics is not clear *a priori*. We therefore distinguish "boundary CFT" (the HHZ free energy) and "bulk interpretation" (black hole thermodynamics) carefully.

The values of the potentials Δ and ω_i (with $i = a, b$) that extremize the HHZ free energy follow from (2.107-2.108) as:

$$\Delta = 2\pi i (Q+\Lambda)^{-1}\left(\frac{1}{J_a-\Lambda}+\frac{1}{J_b-\Lambda}+\frac{3}{Q+\Lambda}\right)^{-1} , \tag{2.113}$$

$$\omega_i = -2\pi i (J_i-\Lambda)^{-1}\left(\frac{1}{J_a-\Lambda}+\frac{1}{J_b-\Lambda}+\frac{3}{Q+\Lambda}\right)^{-1} . \tag{2.114}$$

with the understanding that the Lagrange multiplier $\Lambda = \Lambda(Q, J_{a,b})$ is given by (2.112). Because of the constraint (2.105), they yield the on-shell value of the HHZ free energy (2.104)

$$H = -\frac{N^2}{2}\frac{1}{\omega_a\omega_b}\left(\frac{2\pi i+\omega_a+\omega_b}{3}\right)^3 . \tag{2.115}$$

The values of the potentials Δ, ω_i (2.113-2.114) and the entropy S (2.109) at the extremum are all given in terms of the charges Q, J_i. This is a natural form to present the result of the extremization, and the one that is most appropriate for comparison with microscopic considerations. However, it is awkward that all these expressions are defined only modulo the constraint on the charges (2.111).

The gravitational thermodynamics of BPS black holes expresses the corresponding physical quantities in terms of the parameters (a, b), in a form that automatically solves the constraint on the charges. In particular, the BPS values of the charge Q^* (2.67) and angular momenta J_i^* (2.68-2.69) satisfy the constraint and give equations for ω_i (analogous to similar complexified treatments of these chemical potentials [65]) [4]):

$$\omega_a = \frac{-\pi(1-a)}{a^2+b^2+3(a+b+ab)+1}\left(\frac{a+2b+b^2+2ab}{\sqrt{a+b+ab}}+i(1+a)\right) , \tag{2.116}$$

$$\omega_b = \frac{-\pi(1-b)}{a^2+b^2+3(a+b+ab)+1}\left(\frac{b+2a+a^2+2ab}{\sqrt{a+b+ab}}+i(1+b)\right) . \tag{2.117}$$

We do not explicitly write the corresponding expression for the electric potential $\Delta = \Delta(a, b)$ because it follows from the simple linear relation (2.105). It is worth mentioning that the expressions for $\omega_{a,b}$ and Δ are arrived at by complexifying the chemical potentials

[4]These expressions become identical to ω_1^* and ω_2^* in (3.27) [65] when the branch of the complexified q with $+ir_+$ in (3.3) there is picked instead.

through the complex Lagrange multiplier Λ (2.112), without the need to complexify the gravitational solution itself as in previous such studies [65].

The on-shell value of the HHZ potential H (2.115) can be similarly expressed in terms of a and b:

$$
\begin{aligned}
H(a,b) = &\frac{N^2}{2}\frac{-\pi(a+b)^2(1-2a-2b-a^2-5ab-b^2-a^2b-ab^2)}{(1-a)(1-b)\sqrt{a+b+ab}(a^2+b^2+3(a+b+ab)+1)} \\
&+\frac{N^2}{2}\frac{\pi(a+b)^2(3-ab+a+b)}{2(1-a)(1-b)(a^2+b^2+3(a+b+ab)+1)}i \ .
\end{aligned}
\tag{2.118}
$$

Even though the potentials $\omega_{a,b}$ in (2.116-2.117) and the HHZ potential (2.118) are functions of the parameters a, b that were introduced in the context of gravitational solutions, we stress that these formulae are entirely equivalent to (2.113-2.115) which we consider function of charges. The only difference is that the constraint on charges was solved. As a bonus, the potentials that extremize the HHZ free energy are now written in a form that is convenient for comparison with gravitational thermodynamics.

2.2.5.2 Comparing the HHZ Results with Black Hole Thermodynamics

We now reconsider the conventional thermodynamics of BPS black holes that depends on manifestly real spacetime potentials $\Phi', \Omega'_{a,b}$ and the BPS free energy W (2.93) that is also real. Our gravitational considerations presented results in terms of auxiliary variables (a, b) that are related to black hole charges and an additional physical potential $\varphi' = \Phi' - \Omega'_a - \Omega'_b$ introduced in (2.89). In these variables, the BPS free energy W defined in (2.78) becomes (2.93):

$$
\begin{aligned}
\frac{W}{N^2} = &\frac{\pi(a+b)^2\,(1-2a-2b-a^2-5ab-b^2-a^2b-ab^2)}{2(1-a)(1-b)\,(1+3a+3b+a^2+3ab+b^2)\,\sqrt{a+b+ab}} \\
&-\frac{(a+b)^2(3+a+b-ab)}{4(1-a)(1-b)\,(1+3a+3b+a^2+3ab+b^2)}\varphi' \quad ,
\end{aligned}
\tag{2.119}
$$

and the BPS angular velocities defined in (2.79) are:

$$
\Omega'_a = -\frac{1-a}{2(1+3a+3b+a^2+3ab+b^2)}\left[\frac{2\pi(a+2b+2ab+b^2)}{\sqrt{a+b+ab}} + (1+a)\varphi'\right] \ , \tag{2.120}
$$

$$
\Omega'_b = -\frac{1-b}{2(1+3a+3b+a^2+3ab+b^2)}\left[\frac{2\pi(2a+b+2ab+a^2)}{\sqrt{a+b+ab}} + (1+b)\varphi'\right] \ . \tag{2.121}
$$

The BPS electric potential Φ' defined in (2.79) is not an independent variable because it is given through $\Phi' = \Omega'_a + \Omega'_b + \varphi'$.

We expect that the spacetime free energy W (2.93) must be related to the HHZ free energy H (2.118). Similarly, the spacetime angular velocities $\Omega'_{a,b}$ (2.90-2.91) should be related to the HHZ variables $\omega_{a,b}$ (2.116-2.117), since they are both conjugate to angular momenta $J_{a,b}$. There are indeed striking similarities between the expressions but there are also key differences. As we have stressed already, the HHZ variables are complex while black hole thermodynamics is manifestly real. Moreover, the thermodynamic formulae depend on one more variable φ'[5].

There are several options that address these differences, at least formally. The most conservative is to assign φ' a specific value that is imaginary. Indeed, for $\varphi' = 2\pi i$ we have:

$$H(a,b) = -W(a,b,\varphi' = 2\pi i) \ , \tag{2.122}$$

for the free energy and

$$\omega_i(a,b) = \Omega'_i(a,b,\varphi' = 2\pi i) \quad , \ 3\Delta(a,b) = \Phi'(a,b,\varphi' = 2\pi i) \ , \tag{2.123}$$

for the potentials. Moreover, the definition of φ' reduces precisely to the HHZ constraint (2.105) when $\varphi' = 2\pi i$.

This procedure is well motivated. The general BPS partition function reduces to the superconformal index precisely for $\varphi' = 2\pi i$ because φ' is integral (half-integral) for bosons (fermions), and so this value of the potential is equivalent to inserting $(-)^F$. Therefore, for this value of φ', comparisons between weak and strong coupling in the CFT is justified, with the latter dual to the semiclassical gravity description. According to this line of reasoning, black hole thermodynamics for any real value of φ' corresponds to strongly coupled field theory that is unlikely to be accounted for by computations in weakly coupled CFT.

A distinct procedure, also common in the literature [71, 72, 66, 63, 17, 73], is to identify only the real parts of the HHZ free energy and potentials as physical, and compare them with black hole thermodynamics only for the special value $\varphi' = 0$. Indeed, for the free energy, we find the identity:

$$\text{Re}\, H(a,b) = -W(a,b,\varphi' = 0) = \frac{N^2}{2}\frac{-\pi(a+b)^2(1-2a-2b-a^2-5ab-b^2-a^2b-ab^2)}{(1-a)(1-b)\sqrt{a+b+ab}(a^2+b^2+3(a+b+ab)+1)} \ . \tag{2.124}$$

Sometimes authors even refer to the quantity Re $H(a,b)$ (expressed in terms of the HHZ

[5] We also note that this additional variable is related to the u in [65] through $\varphi' = -2\pi u$.

variables Δ and $\omega_{a,b}$) as the free energy F, rather than the full complex expression [66, 63, 17, 73]. The analogous comparison between the real part of the complex chemical potentials is also successful:

$$\begin{aligned} \text{Re } \omega_a &= \frac{-\pi(1-a)(a+2b+b^2+2ab)}{(a^2+b^2+3(a+b+ab)+1)\sqrt{a+b+ab}} = \Omega'_a(a,b,\varphi'=0) \ , \\ \text{Re } \omega_b &= \frac{-\pi(1-b)(2a+b+a^2+2ab)}{(a^2+b^2+3(a+b+ab)+1)\sqrt{a+b+ab}} = \Omega'_b(a,b,\varphi'=0) \ . \end{aligned} \tag{2.125}$$

The focus on the real part of the potential is heuristic, but it is striking that it "works", in that it yields identities between somewhat elaborate functions. Within the rigorous framework of the superconformal index, these agreements are coincidental.

The feature that allows *both* procedures to work is that the free energy is *linear* in φ'. The analogous feature in the AdS_3/CFT_2 correspondence has an appealing physical interpretation: the BPS black hole cannot be identified with a specific BPS state, it is an average over all chiral primaries [74]. It would be interesting to develop an analogous interpretation in the AdS_5/CFT_4 correspondence.

2.2.5.3 HHZ free Energy and Black Hole Thermodynamics: the Phase Diagram

Choi *et al.* [63] showed that a phase diagram determined from the HHZ free energy (2.104) shares important aspects of the AdS-Schwarzschild black thermodynamics, including the Hawking-Page phase transition and its characteristic "cusp" [45, 33, 75, 76]. This is interesting, because it suggests that the confinement/deconfinement transition in QCD-like theories [35, 36, 38, 37, 16] can be analyzed while preserving supersymmetry, possibly with a great deal of precision. However, it is not obvious that results derived from the HHZ free energy agree with traditional thermodynamics, derived from black hole geometry. In this subsection we verify that they do in fact agree.

The extremization conditions on the entropy function (2.107,2.108) give:

$$J + Q = -\frac{2N^2}{27}\frac{(\omega+\pi i)^2(\omega-2\pi i)}{\omega^3} \ , \tag{2.126}$$

after taking $\omega_a = \omega_b$ and eliminating Λ. Choi *et al.* [63] follow the "heuristic" procedure yielding (2.124-2.125) so they demand that J, Q are positive and real. This requirement relates the real and imaginary parts of $\omega = \omega_R + i\omega_I$ as:

$$\omega_R^2 = \frac{3\omega_I^2(\pi+\omega_I)}{\pi - 3\omega_I} \ . \tag{2.127}$$

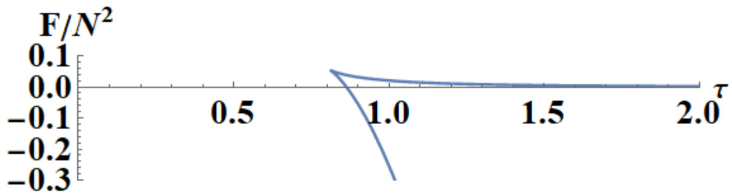

Figure 2.10: The free energy F vs. the BPS temperature τ. This phase diagram was derived from the HHZ potential using the "heuristic" method.

The heuristic procedure interprets $\omega_R = \text{Re}\,\omega$ as the physical potential Ω' through (2.125) and the condition that $\Omega' < 0$ determines phases so:

$$\tau = -\omega_R^{-1} = -\omega_I^{-1}\sqrt{\frac{\pi - 3\omega_I}{3(\pi + \omega_I)}} \ , \tag{2.128}$$

and constrains the imaginary part so $-\pi < \omega_I < 0$. Further, the heuristic procedure posits that it is the real part of the HHZ energy (2.104) that is physical. Since Δ is related to $\omega = \omega_R + i\omega_I$ through the constraint (2.105) we then find

$$F = -\frac{N^2}{2}\text{Re}\,\frac{\Delta^3}{\omega^2} = -\frac{N^2}{2}\text{Re}\,\left(\frac{(2\pi i + 2\omega)^3}{27\omega^2}\right) \tag{2.129}$$

$$= -\frac{N^2}{18}\frac{\pi^3 - 9\pi\omega_I^2 - 8\omega_I^3}{\omega_I^2}\sqrt{\frac{\pi + \omega_I}{3(\pi - 3\omega_I)}} \ . \tag{2.130}$$

The variable ω_I is related to the BPS temperature τ through (2.128) so this equation gives the free energy F as function of the BPS temperature τ which we plot in Figure 2.10.

From visual inspection it seems clear that Figure 2.10 is precisely the same as its analogue in Figure 2.9 that applies to black holes with parameters $a = b$ and $\varphi' = 0$. However, they were arrived at very differently: the former by extremizing the HHZ potential (2.104) (and imposing various reality conditions), the latter from the standard thermodynamic interpretation of BPS black hole geometry.

The analytical comparison is also quite nontrivial. The BPS free energy W derived from gravity is a function of the BPS temperature τ, with both W and τ given in terms of the auxiliary parameter a in (2.95,2.96), and reproduced here for convenience:

$$W = \frac{N^2}{2}\frac{4\pi a^2(1 - 5a - 2a^2)}{(1-a)^2(1+5a)\sqrt{a(a+2)}} \ , \tag{2.131}$$

$$\tau = -\omega_R^{-1} = -\frac{3\pi a(1-a)}{(1+5a)\sqrt{a(a+2)}} \ . \tag{2.132}$$

It must be compared with the real part of the HHZ free energy F (2.129) that is a function of ω_I, the imaginary part of the complexified rotational velocity $\omega = \omega_R + i\omega_I$, which serves as an auxiliary parameter that is related to $\tau = -\omega_R^{-1}$ through (2.128).

As suspected, there is in fact an identification of auxiliary parameters that transforms all the analytical formulae into one another:

$$\omega_I = -\pi \frac{1-a}{1+5a} \ . \tag{2.133}$$

We stress that, given the nonlinear nature of the formulae involved, this agreement was far from preordained. Its success is a sensitive test of the HHZ potential (2.104).

2.3 Thermodynamics of BPS Black Holes: Detailed Study

In this section we study the phase diagram of general BPS black holes. The benchmark BPS black hole introduced in subsection 2.2.4 has equal angular velocities $\Omega'_a = \Omega'_b$ and electric potential Φ' tuned so $\varphi' = 0$. We now explore generic values of the three independent potentials and explain their significance.

2.3.1 Primed Potentials and Their Conjugate Charges

The general BPS black holes we consider in this section are parametrized by the three primed potentials Ω'_a, Ω'_b, and Φ'. The primes remind us that their definitions (2.79) relate them to the corresponding non-BPS potentials $\Omega_{a,b}$ and Φ via a thermal derivative. The primed potentials are the variables that the BPS free energy W

$$W = -S - \Omega'_a J_a - \Omega'_b J_b - \Phi' Q \ , \tag{2.134}$$

defined in (2.52) depends on. In particular, its derivatives read off from the first law of BPS black hole thermodynamics

$$dW = -J_a d\Omega'_a - J_b d\Omega'_b - Q d\Phi' \ , \tag{2.135}$$

yield the black hole charges. Alternative ensembles that are functions of some or all of the charges, rather than their conjugate potentials, can be obtained as usual, by appropriate Legendre transforms.

As we discussed in subsection 2.2.3.3, the interpretation of the phase diagram is simplified by introducing a different basis for the potentials. The BPS temperature τ (2.94) is an analogue of the physical temperature. The modified potential φ' (2.89) is a proxy for the electric field that is defined so the benchmark case, studied in the literature and reviewed in subsection 2.2.4, corresponds to $\varphi' = 0$. To complete a well-defined transformation from the three potentials $\Omega'_{a,b}, \Phi'$ we introduce a third potential:

$$\mu = \Omega'_a - \Omega'_b \ , \tag{2.136}$$

that is sensitive to departures from two equal angular momenta. The combinations of charges that are conjugate to the potentials (τ, φ', μ) follow from the first law of BPS black hole thermodynamics in the form

$$dW = -\frac{1}{\tau^2}(J_a + J_b + 2Q)d\tau - Qd\varphi' - \frac{1}{2}(J_a - J_b)d\mu \ . \tag{2.137}$$

For completeness, we also record the inverse transform from the variables we employ to discuss the phase diagram to the original physical BPS potentials:

$$\Phi' = -\frac{2}{\tau} + \varphi' \ , \tag{2.138}$$

$$\Omega'_a = -\frac{1}{\tau} + \frac{\mu}{2} \ , \tag{2.139}$$

$$\Omega'_b = -\frac{1}{\tau} - \frac{\mu}{2} \ , \tag{2.140}$$

that are conjugate the physical charges Q, J_a, J_b.

The changes of variables above are routine, at the face of it: BPS black holes are characterized by three charges $\{Q, J_a, J_b\}$ which, in the grand canonical ensemble, correspond to three BPS potentials $\{\Phi', \Omega'_a, \Omega'_b\}$. When presenting explicit phase diagrams we further change the basis among the potentials to $\{\tau, \varphi', \mu\}$, in order to clarify physical interpretations and connections to the literature. However, straightforward as these transformations appear, they do not incorporate the fact that all BPS black holes satisfy the constraint between charges (2.71). Mathematically, this means the Legendre transform, from the microcanonical to the canonical ensemble, is singular.

Our physical interpretation of this peculiar feature, discussed at length in the introduction, is that all classical BPS black holes correspond to thermal equilibrium along the direction in parameter space that corresponds to the BPS potential φ'. The formula for the mass excess $M - M_*$ (2.76) offers a perspective. BPS black holes $M = M_*$ must have zero temperature $T = 0$ as well as the potential $\varphi = 0$ but the BPS condition does not specify the "slope"

$\varphi' = \varphi/T$ when the approach is realized as a physical limit where $\varphi \to 0$ and $T \to 0$ simultaneously. This ratio is physical: any supersymmetry implies $M - M^* = 0$ but, once a given supercharge has been committed to, others that differ by a phase φ' are inconsistent with the "preferred" supersymmetry. [6]

The equilibrium condition along the φ' direction was understood in AdS_3, where it corresponds to black holes necessarily taking on the R-charge that is the average of all chiral primaries, in contrast to the index that is independent of the charges of these states [78, 74]. It is important to elucidate the analogous mechanism in AdS_5 so, in the following subsections, we treat φ' as an independent thermodynamic variable. In particular, we do not take $\varphi' = 0$ *a priori* because, even if some argument were to establish $\varphi' = 0$ as the equilibrium value, we would still need to show that the constraint among charges emerges as a derivative with respect to this variable.

2.3.2 The Physical Range of φ' and μ

The BPS free energy W was defined in subsection 2.2.1 as a function of the BPS potentials $\Phi', \Omega'_{a,b}$ but, as have discussed above (and in subsection 2.2.3.3), the variables (τ, μ, φ') are preferable. Unfortunately, in practice the free energy and the various potentials are known only in parametric forms, as functions of (a, b, φ'). For example, the BPS temperature τ was presented in (2.94) and (2.120-2.121) similarly give

$$\mu = \frac{a-b}{2(1+3a+3b+a^2+3ab+b^2)}\left[\frac{2\pi(1+2a+2b+ab)}{\sqrt{a+b+ab}} + (a+b)\varphi'\right] . \qquad (2.141)$$

We need to determine the physical regime of the various parameters. The rotational velocities are bounded by the speed of light $\Omega_i \leq 1$ which, via the definition of primed potentials (2.79), corresponds to $\Omega'_i < 0$. This in turn implies that τ introduced in (2.94) is strictly positive $\tau > 0$ so

$$-\varphi' \leq \frac{6\pi(a+b)(1-ab)}{(2-a^2-b^2)\sqrt{a+b+ab}} . \qquad (2.142)$$

Additionally, we require $0 \leq a, b < 1$, in order that the underlying black hole solutions exist as regular geometries.

The positive temperature condition in the form (2.142) is satisfied for all $\varphi' \geq 0$ but it is

[6] Slowly varying fluctuations of φ' can be identified with the scalar that carries R-symmetry charge in the $\mathcal{N} = 2$ Schwarzian theory describing excitations in the near geometry of the BPS black hole[77, 56, 57, 62] (and references therein).

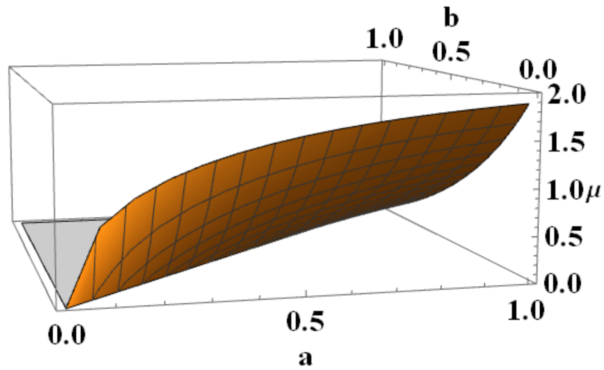

Figure 2.11: The chemical potential μ as a function of the black hole parameters a and b, for $\varphi' = 0$. The maximum is achieved at $(a, b) = (1, 0)$. We plot only $a \geq b$, the mirror for $a \leq b$ (the grey area) follows from antisymmetry of $\mu(a, b)$ under $a \leftrightarrow b$.

nontrivial for $\varphi' < 0$. For $a = b$, it reduces to

$$-\varphi' \underset{a=b}{\leq} 6\pi\sqrt{\frac{a}{2+a}} < 2\sqrt{3}\pi \ .$$

The middle formula is a monotonically increasing function of a and the second inequality would be saturated for $a = 1$. Therefore, for equal angular momenta $a = b$, the BPS potential φ' is bounded from below by $\varphi'_- = -2\sqrt{3}\pi < \varphi'$

For given $a + b$, the function on the right hand side of (2.142) increases monotonically as function of $(a - b)^2$. Therefore, the lower bound on φ' is relaxed when the two angular momenta are unequal. Its global minimum $\varphi'_{\text{min}} = -6\pi$ corresponds to $(a, b) = (1, 0)$ or $(a, b) = (0, 1)$.

We want to similarly analyze the physical range of μ (2.141), the BPS potential that measures the departure from angular momenta of a and b type being equal. Because of the antisymmetry between a and b we have $\mu \equiv 0$ along the line $a = b$ and it is sufficient to study $a \geq b$. When $\varphi' = 0$ we find that μ increases monotonically with $a - b$ when $a + b$ is fixed. The maximal value $\mu_{\text{max}} = \frac{3\pi}{5} \approx 1.88$ is reached when $(a, b) = (1, 0)$. We plot the BPS potential μ in Figure 2.11.

2.3.3 BPS Black Holes with General φ' and Equal Angular Momenta

In this subsection we study the significance of the potential φ'. We take equal angular momenta, i.e. $a = b$ (and so $\Omega'_a = \Omega'_b = \Omega'$) which corresponds to $\mu \equiv 0$.

The BPS free energy (2.93) reduces to

$$\frac{W}{N^2} = \frac{2a}{(1-a)^2(1+5a)} \left(\frac{\pi(1-5a-2a^2)}{\sqrt{1+\frac{2}{a}}} - \frac{a(3-a)\varphi'}{2} \right) , \tag{2.143}$$

and the BPS "temperature" τ (2.94) becomes

$$\tau \equiv -\Omega'^{-1} = \frac{1+5a}{1-a} \frac{1}{\frac{3\pi}{\sqrt{1+\frac{2}{a}}} + \frac{\varphi'}{2}} . \tag{2.144}$$

We will discuss the dependence of W on τ for each sign of φ' in turn.

2.3.3.1 $\varphi' \leq 0$

We first consider the BPS temperature (2.144). For large black holes τ diverges at $a = 1$ and, as a get smaller, it decreases to a minimal temperature τ_{cusp} that is attained at a_{cusp}. This is qualitatively similar to the benchmark case $\varphi' = 0$ that was discussed in subsection 2.2.4 (with $\tau(a)$ plotted in figure 2.8).

The temperature increases again when a decreases below a_{cusp}, as expected for small black holes. However, when $\varphi' < 0$, the denominator in the second factor of τ given in (2.144) reaches zero at some positive value

$$a_{\text{min}} = \frac{2\varphi'^2}{36\pi^2 - \varphi'^2} , \tag{2.145}$$

and then the temperature diverges. Thus, when $\varphi' \leq 0$, the parameter a is limited to the range $1 > a > a_{\text{min}}$.

The parameter value a_{min} increases monotonically when the absolute value $|\varphi'|$ increases. The physical range $1 > a > a_{\text{min}}$ shrinks to zero if it reaches $a_{\text{min}} = 1$ and then no underlying black hole geometry would exist. Therefore, the corresponding value of the potential φ'

$$\varphi'_- = -2\sqrt{3}\pi , \tag{2.146}$$

constitutes a lower limit $\varphi' \geq \varphi'_-$.

The finite range of a changes the BPS free energy W (2.143) qualitatively: at the high temperature end of the small black hole branch (as $a \to a^+_{\text{min}}$), it does not approach 0 but a finite positive value

$$W_{\text{asym}}/N^2 = \frac{4\varphi'^3}{9(12\pi^2 - \varphi'^2)} . \tag{2.147}$$

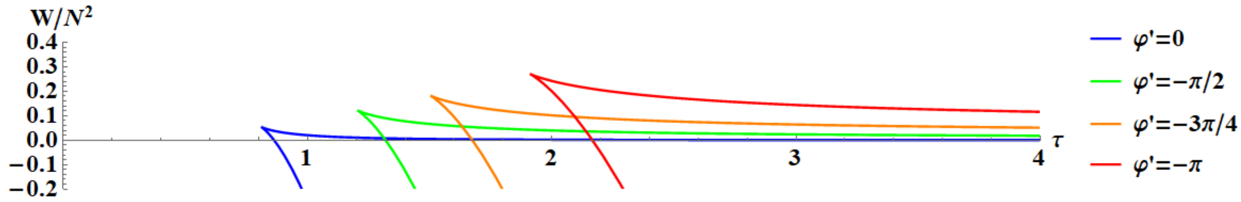

Figure 2.12: The BPS free energy W as function of the BPS temperature τ. The two angular momenta are equal ($\mu = 0$) and $\varphi' = 0, -\pi, -\frac{3\pi}{2}, -2\pi$, from left to right. There are two branches in the phase diagram that meet in a cusp. Small black holes (the upper branch) asymptote to a positive BPS free energy W_{asym} (2.147) at large temperature when $\varphi' < 0$.

Also, for the "big" black hole branch, we have an asymptotic relation

$$W/N^2 \sim -\frac{(\varphi' + 2\sqrt{3}\pi)^3}{432}\tau^2. \tag{2.148}$$

When $\varphi' = 0$, it is in agreement with our benchmark case in subsection 2.2.4.

The phase diagram for various values of $\varphi' \leq 0$ is presented in Figure 2.12.

The minimal temperature for a given φ' is attained at a_{cusp} determined by $\partial_a \tau = 0$ which leads to:

$$\varphi' = \frac{\pi(1 - 8a_{\text{cusp}} - 11a_{\text{cusp}}^2)}{(2 + a_{\text{cusp}})\sqrt{a_{\text{cusp}}(2 + a_{\text{cusp}})}} \ . \tag{2.149}$$

This equation is equivalent to a sextic in a that cannot be solved analytically in general. For $\varphi' = 0$, the condition (2.149) reduces to a quadratic equation with the solution $a_{\text{cusp}} = \frac{3\sqrt{3}-4}{11}$, in agreement with the result in subsection 2.2.4. We also note that $a_{\text{cusp}} = 1$ is a solution at the lower bound $\varphi' = \varphi'_-$ given in (2.146). The BPS temperature τ and BPS free energy W both diverge as $a \to 1^-$ so this shows that, as the lower bound is approached $\varphi' \to \varphi'_-$, the cusp moves to the far upper right corner in Figure 2.12 and no physical black hole remain in the strict limit.

Generally, the right hand side of (2.149) is a monotonic function so, as $|\varphi'|$ increases from zero to its maximum $|\varphi'_-|$, a_{cusp} increases monotonically through $[\frac{3\sqrt{3}-4}{11}, 1[$. The BPS free energy (2.143) at the cusp moves as

$$\frac{dW_{\text{cusp}}}{da} = (\partial_a W)_{\varphi'} + (\partial_{\varphi'} W)_a \, \partial_a \varphi' = \left.\frac{\pi a\,(3 + 14a - 5a^2)}{\sqrt{a(2+a)}(2+a)^2(1-a)^2}\right|_{a_{\text{cusp}}} > 0 \ , \tag{2.150}$$

with $\partial_a \varphi'$ evaluated from (2.149), so it increases monotonically as well. The BPS temperature (2.144) at the cusp increases monotonically entirely through its dependence on φ' because it is a minimum $\partial_a \tau = 0$. Taken together, these arguments establish analytically that the cusp moves monotonically up and to the right when $|\varphi'|$ increases. These trends are also visible in Figure 2.12.

We can analyze the Hawking-Page temperature similarly. The condition that the free energy $W = 0$ yields the algebraic relation for $a_{\rm HP}$:

$$\varphi' = \frac{2\pi(1 - 5a_{\rm HP} - 2a_{\rm HP}^2)}{(3 - a_{\rm HP})\sqrt{a_{\rm HP}(2 + a_{\rm HP})}} \ . \tag{2.151}$$

The derivative of this expression

$$\partial_{a_{\rm HP}}\varphi' = -\frac{6\pi(1 + a_{\rm HP})^2(1 + 3a_{\rm HP})}{(3 - a_{\rm HP})^2(a_{\rm HP}(2 + a_{\rm HP}))^{3/2}} \ , \tag{2.152}$$

is negative in the entire range $0 \leq a \leq 1$. Therefore, the parameter $a_{\rm HP}$ at the Hawking-Page transition depends monotonically on φ'. The corresponding temperature (2.144) moves as

$$\frac{d\tau_{\rm HP}}{da} = (\partial_a\tau)_{\varphi'} + (\partial_{\varphi'}\tau)_a\,\partial_a\varphi' = \left.\frac{3(1 + 3a)}{\pi(1 - a)^3\sqrt{a(2 + a)}}\right|_{a_{\rm HP}} > 0 \ , \tag{2.153}$$

so the Hawking-Page temperature $\tau_{\rm HP}$ is a monotonically increasing function of $|\varphi'|$. This dependence is also apparent in Figure 2.12.

It is instructive to compare the phase diagram for φ' in the range $(-2\pi\sqrt{3}, 0)$ with the benchmark case $\varphi' = 0$ discussed in subsection 2.2.4:

- At a given BPS temperature τ the potential $|\varphi'|$ increases the BPS free energy W on both branches.

- The characteristic temperatures $\tau_{\rm HP}$ and $\tau_{\rm cusp}$ both increase with $|\varphi'|$.

Both effects suggest an instability. Conversely, increased φ' (decreased $|\varphi'|$) *stabilises* the BPS black hole. This is reminiscent of how an electric potential modifies the AdS-Reissner-Nordström black hole, discussed in subsection 2.1.2.2.

2.3.3.2 $\varphi' \geq 0$

When $\varphi' > 0$, the denominator in the second factor of τ (2.144) is strictly positive for all $a \in [0, 1)$. Therefore, unlike when the potential $\varphi' < 0$, the entire range $a \in (0, 1)$ of the parameter a is physical. At the lower end of the range $a = 0$ the free energy $W = 0$ and the temperature is

$$\tau_{\rm max} = \frac{2}{\varphi'} \ . \tag{2.154}$$

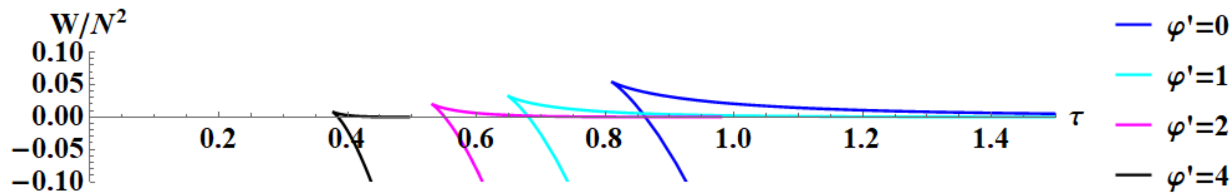

Figure 2.13: The free energy $\frac{W}{N^2}$ as function of BPS temperature τ for various $\varphi' \geq 0$. The values of $\varphi' = 0, 1, 2, 4$ increase from right to left. The small black holes (the upper branch) have $W = 0$ at their maximal temperature $\tau_{\max}$ (2.154) which is $\tau_{\max} = \frac{1}{2}, 1$ for $\varphi' = 4, 2$. The large black hole branch is qualitatively similar to its analogue for $\varphi' < 0$ in Figure 2.12.

This is the maximal temperature on the small black hole branch, a novel feature of the $\varphi' > 0$ regime. In the strict limit $a = 0$ the geometry underlying the thermodynamic formulae reverts to pure AdS_5, it is not a black hole, so $\tau_{\max}$ is a bound, it cannot be reached. The bound on the temperature (2.154) is lowered when the potential φ' increases. The range of allowed temperatures on the small black hole branch shrinks and, in the limit $\varphi' \to \infty$, it disappears altogether. In this limit the large black hole branch starts at $(W, \tau) = (0, 0)$ and there is no small black hole branch. This limit is similar to the special case $\Phi = \Phi^* = 3$ of ordinary (non-BPS) AdS-Reissner-Nordström black holes that was discussed in subsection 2.1.2.3.

The large black hole branch is not modified qualitatively from the benchmark case $\varphi' = 0$ discussed in subsection 2.2.4, or more generally the case $\varphi' \leq 0$ developed earlier in this subsection. The phase diagram for various values of $\varphi' \geq 0$ is plotted in Figure 2.13.

The figure indicates that, as we increase φ' starting from zero, the coordinates of the cusp τ_{cusp}, W_{cusp} both decrease, as does the Hawking-Page temperature τ_{HP} where the large black hole branch crosses $W = 0$. To verify these features analytically, we note that the parameter a_{cusp}, corresponding to the minimal temperature, is given by (2.149) for either sign of φ'. It takes the value $a_{\text{cusp}} = \frac{3\sqrt{3}-4}{11}$ when $\varphi' = 0$ and decreases monotonically when φ' increases to positive values. The result follows because, for and φ', the temperature τ (2.144) has positive derivative $\partial_{\varphi'}\tau > 0$ and the motion (2.150) of the free energy W. The estimate (2.153) similarly establishes that τ_{HP} decreases as φ' become larger.

In summary, the evolution of the cusp, the Hawking-Page temperature, and the asymptotic behavior of the large black hole branch at high temperature, are all smooth for φ' in the entire range (φ'_-, ∞). For these features the value $\varphi' = 0$ plays no special role. In contrast, the asymptotic behavior on the small black hole branch depends sensitively on the sign of φ': for $\varphi' \leq 0$, the temperature $\tau \to \infty$ is reached with an asymptotic value W_{asym} (2.147) of the free energy that is strictly positive except that it vanishes when $\varphi' = 0$. For $\varphi' \geq 0$, the temperature is bounded by $\tau_{\max}$ (2.154) that is finite, except that it diverges when $\varphi' = 0$.

2.3.4 BPS Black Holes with Unequal Angular Momenta

In this paper we have presented nearly all formulae for general parameters a, b but most examples focus on the "equal angular momenta" case $a = b$. This is also the case that has been most studied in the literature, by far. In this subsection we elucidate the significance of "unequal angular momenta" $a \neq b$ by taking the chemical potential $\mu \neq 0$.

We will keep $\varphi' = 0$ so the Gibbs energy W (2.93) simplifies to

$$\frac{W}{N^2} = \frac{\pi(a+b)^2\left[1 - 2a - 2b - a^2 - 5ab - b^2 - a^2b - ab^2\right]}{2(1-a)(1-b)\left(1 + 3a + 3b + a^2 + 3ab + b^2\right)\sqrt{a+b+ab}}\,, \tag{2.155}$$

and the BPS potentials (2.94) and (2.141) read

$$\tau = \frac{2\left(1 + 3a + 3b + a^2 + 3ab + b^2\right)}{3\pi(a+b)(1-ab)}\sqrt{a+b+ab}\,, \tag{2.156}$$

$$\mu = \pi\frac{a-b}{1 + 3a + 3b + a^2 + 3ab + b^2}\frac{1 + 2a + 2b + ab}{\sqrt{a+b+ab}}\,. \tag{2.157}$$

Because of the antisymmetry of μ under the exchange $a \leftrightarrow b$, it is sufficient to analyze $\mu \geq 0$. The phase diagram is shown in Figure 2.14. From the plot, for μ that is positive and small, the phase diagram evolves perturbatively from the benchmark case $\varphi' = \mu = 0$ discussed in subsection 2.2.4 with some changes that appear smoothly:

- A maximal temperature of the small black hole branch develops that decreases when μ increases. The upper branch shrinks.
- For any given τ, on either branch, the BPS potential $\mu > 0$ lowers the free energy. This is expected because, when $a > b$ (2.68-2.69) give $J_a > J_b$, so the first law of BPS thermodynamics (2.137) yields: $\partial_\mu W = -\frac{1}{2}(J_a - J_b) < 0$.
- The Hawking-Page transition temperature $\tau_{\rm HP}$ also decreases as μ get larger, because both black hole branches are lowered.
- The maximal free energy $W_{\rm cusp}$ increases with μ, but the minimal BPS temperature $\tau_{\rm cusp}$ decreases. Thus the cusp travels "towards North-West" as μ increases.

Our interpretation of these features is that increasing μ is *destabilising*: it takes a lower temperature to achieve $W > 0$ and so render the black holes unstable, and positive μ allows a higher maximal free energy $W_{\rm cusp}$, indicating stronger instability. This is a BPS analogue of

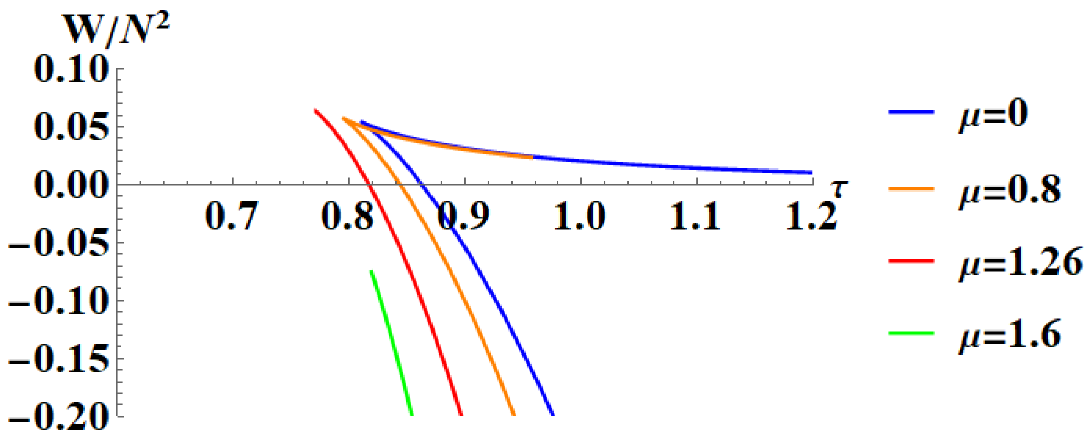

Figure 2.14: The BPS free energy W vs. the BPS temperature τ for select values of the potential $\mu \geq 0$ that parametrizes the asymmetry between angular momenta. $\mu = 0, 0.8, 1.26, 1.6$ from right to left. For small μ, the phase diagram exhibits the familiar lower (large) and upper (small) black hole branches that meet in a cusp. For the small black hole branch, there is a maximal BPS temperature that decreases as function of μ and reaches the cusp at some $\mu = \mu_{\text{crit}}$. The maximal value of W also increases with $\mu \leq \mu_{\text{crit}}$ but then decreases for $\mu > \mu_{\text{crit}}$.

our finding in subsection 2.1.2.1 that, for non-BPS black holes, angular velocities destabilise the AdS Reissner-Nordström black holes.

The preceding comments only apply when the chemical potential μ is sufficiently small. From Figure 2.14 we see that when μ exceeds a certain critical value $\mu_{\text{crit}} \approx 1.26$ the small black hole branch disappears altogether, for such potentials only the "large" black hole solution exists. The evolution when μ is larger than this critical value has the following features:

- For any given τ, increased μ lowers the BPS free energy.
- As μ increases, the minimal temperature τ_{cusp} increases as well but the maximal free energy W_{cusp} decreases.
- The motion of the "cusp" shows that the values τ_{min} and W_{max} evaluated at the cusp with $\mu = \mu_{\text{crit}}$ are the global minimum for the temperature and maximum for the BPS free energy, respectively.
- The Hawking-Page transition temperature τ_{HP} (where $W = 0$) generally decreases as the entire large black hole branch is lowered. However, when the chemical potential is sufficiently large, above $\mu \approx 1.5$, the entire large black hole branch is below $W = 0$ and the transition disappears altogether.

- As discussed in subsection 2.3.2, the parameter μ satisfies an absolute upper bound $\mu_{\max} = \frac{3\pi}{5} \approx 1.885$ when $\varphi' = 0$. As $\mu \to \mu_{\max}$, (2.141) shows that $a \to 1$ and $b = 0$ so the BPS free energy (2.93) $W \to -\infty$. In the strict limit these is no underlying black hole solution.

Here, we observe another effect of perturbing μ when μ is sufficiently large. Our interpretation is that for large μ, it is *stabilising*: the entire black hole would lie below $W = 0$ and thus render the black holes stable, and increasing μ lowers the maximal free energy W_{cusp} indicating the stability. Since our discussion for non-BPS black holes is concentrated in the equal angular momenta case, we do not have an analogous case of it.

2.3.5 Extreme Rotational Asymmetry: $\Omega_b' = 0$

As an extreme example of asymmetry between the two angular momenta, we consider the special case $\Omega_b' \equiv 0$. In this case (2.91) gives

$$\varphi' = -\frac{2\pi\,(2a + b + a^2 + 2ab)}{(1+b)\sqrt{a+b+ab}} \ . \tag{2.158}$$

It follows that φ' must be negative. Formally, we can also achieve $\Omega_b' = 0$ by taking $b = 1$, but then the free energy W (2.93) diverges, so we dismiss this possibility as an unphysical limit.

When $\Omega_b' = 0$ the BPS temperature (2.94) and the BPS free energy (2.93) simplify

$$\begin{aligned} \tau &= -\frac{2}{\Omega_a'} = -\frac{2}{\mu} = \frac{2(1+b)\sqrt{a+b+ab}}{\pi(1-a)(b-a)} \ , \\ W &= \frac{\pi(1+a)(a+b)^2}{2(1-a)(1+b)\sqrt{a+b+ab}} \ . \end{aligned} \tag{2.159}$$

We see that W is positive definite. The phase diagram of τ and W for various fixed φ' is presented in Figure 2.15. There are several features:

- For any given $-2\sqrt{3}\pi < \varphi' < 0$, the BPS free energy W decreases monotonically with τ.

- The lower bound $\tau_{\min}$ decreases in the range $-2\sqrt{3}\pi < \varphi' < -\pi$ and increases when $-\pi < \varphi' < 0$. It reaches its global minimum $\tau_{\min} = \frac{4}{\pi}$ at $\varphi' = -\pi$.

- W displays vertical asymptotes at $\tau_{\min}$ as long as b can approach each 1 (allowed for $-2\pi\sqrt{3} < \varphi' < -\pi$), whereas for $\varphi' > -\pi$, W reaches an upper bound but with no asymptote (such as the $\varphi' = -\frac{3}{4}\pi$ curve in Figure 2.15).

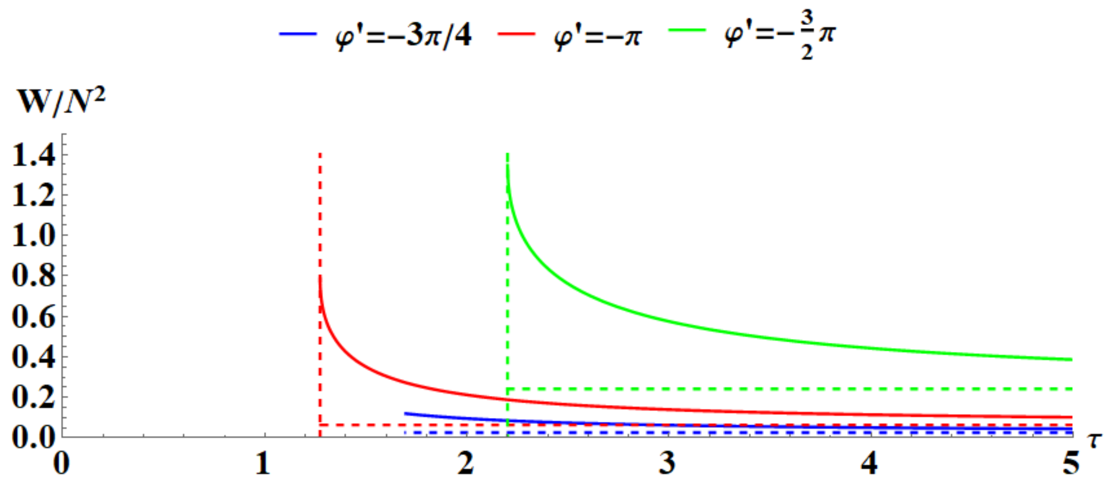

Figure 2.15: W vs. τ when $\Omega_b' = 0$ with $\varphi' = -\frac{3}{2}\pi$, $-\pi$ and $-\frac{3}{4}\pi$ (upper to lower respectively). There are only "small" black holes: they all have $W > 0$ and negative specific heat. For any given φ', there is a minimal BPS temperature which corresponds to the maximal BPS free energy.

- τ diverges when a approaches b from below, leading to an asymptotic value for W at large τ: $\frac{2\varphi'^2}{9(\varphi'^2-12\pi^2)}$.

In summary, the large black hole branch disappears entirely when $\Omega_b' = 0$, there are only small black holes. This is reminiscent of the general (not BPS) black holes with $\Omega = 1$, discussed in subsection 2.1.2.6. One aspect of the special case $\Omega_b' = 0$ is that it corresponds to "maximal asymmetry", there is a BPS potential for rotation along the "a" direction, but none along "b". However, a more illuminating perspective is that the definition of the "primed" potentials (2.79) shows that when the BPS potential $\Omega_b' = 0$, the physical rotational velocity $\Omega_b = 1$ reaches the speed of light. From this point of view the close analogy with the $\Omega = 1$ non-BPS black holes is expected.

2.4 Discussion

In this chapter, we studied the thermodynamics of AdS black holes via analysing their phase diagrams. Compared with previous studies [45, 46], this work has a particular emphasis on the role of rotation. We pointed out that rotation tends to *destabilise* a black hole in the sense that rotation increases the maximal free energy.

We also developed BPS thermodynamics systematically and, in many explicit examples, we pointed out that the phase diagram of the BPS black hole exhibits the similar "cusp" structure as the well-known AdS Schwarzschild black hole phase diagram. We emphasised the role of an important fugacity, φ', that preserves BPS saturation. This fugacity φ' has been set to zero in the BPS limit in previous discussion [63, 18], and paid little attention. We illustrated how φ' brings qualitative change of the phase diagram. Besides, we studied the case where the BPS black holes carry two unequal angular momenta, and discovered a qualitative change of the phase diagram.

However, there are many open questions. For example, we introduced a notion of temperature for BPS black holes in (2.94) as a generalisation to the work by *Choi et al.* in [26]. It is for sure that one should be able to obtain a deeper understanding on this BPS temperature. Another important question is the physical meaning of $W = 0$. Is this phase transition true or, instead, an illusion. We look forward to pursue these and related questions in future research work.

CHAPTER 3

The Attractor Flow for AdS_5 Black Holes in $\mathcal{N} = 2$ Gauged Supergravity

3.1 The Effective 2D Lagrangian

In this section we introduce the action of $\mathcal{N} = 2$ 5D supergravity with coupling to n_V vector multiplets and gauging by Fayet–Iliopoulos couplings, as well as its dimensional reduction to a 2D theory. This also serves to define conventions and notation. For additional details on real special geometry and supersymmetry we refer to Appendices A.2 and A.3, respectively.

3.1.1 The 5D theory

We study five dimensional $\mathcal{N} = 2$ gauged supergravity with bosonic action

$$S = \frac{1}{16\pi G_5} \int_{\mathcal{M}} \mathcal{L}_5 + \frac{1}{8\pi G_5} \int_{\partial\mathcal{M}} d^4x \sqrt{|h|}\, \text{Tr}\, K \ , \tag{3.1}$$

where the 5D Lagrangian density is given by

$$\mathcal{L}_5 = (-\mathcal{R}_5 - 2V) \star_5 1 - G_{IJ} F_5^I \wedge \star_5 F_5^J + G_{IJ} dX^I \wedge \star_5 dX^J - \frac{1}{6} c_{IJK} F_5^I \wedge F_5^J \wedge A_5^K \ . \tag{3.2}$$

We have included the subscript 5 to emphasize that we are in five dimensions and the five dimensional Hodge dual is given by $\star_5$. The Gibbons-Hawking-York boundary term must be included to have a well-defined variation of the action (3.1) and is given by the trace of the second fundamental form K which is integrated over the induced metric h on the boundary. Other conventions and notations regarding differential forms and the Hodge dual are in Appendix A.1.

The field content includes the field strengths $F_5^I = dA_5^I$ where $I = 1, \ldots, n$ and the scalars

X^I, correspond to $n-1$ physical scalars, constrained via the following relation

$$\frac{1}{6}c_{IJK}X^I X^J X^K = 1 \ . \tag{3.3}$$

The scalar potential is given by

$$V = -c^{IJK}\xi_I\xi_J X_K = -\xi_I\xi_J\left(X^I X^J - \frac{1}{2}G^{IJ}\right) \ , \tag{3.4}$$

where ξ_I are the real Fayet–Iliopoulos parameters. The scalars with lowered index

$$X_I = 2G_{IJ}X^J \ , \tag{3.5}$$

obey the analogous constraint

$$\frac{1}{6}c^{IJK}X_I X_J X_K = 1 \ , \tag{3.6}$$

when closure relation (A.11) is satisfied. For further details on definitions, conventions and identities, we refer the reader to Appendix A.2.

Alternatively, the scalar potential can be expressed as

$$V = -\left(\frac{2}{3}W^2 - \frac{1}{2}G^{IJ}D_I W D_J W\right) \ , \tag{3.7}$$

where the superpotential W is

$$W = \xi_I X^I \ , \tag{3.8}$$

and the Kähler covariant derivative D_I takes the constraint (3.6) into acount. Using this form of the potential V, the condition for a supersymmetric minimum becomes

$$D_I W = \xi_I - \frac{1}{3}X_I(\xi\cdot X) \underset{\text{min}}{=} 0 \ . \tag{3.9}$$

According to this equation, the asymptotic values of the scalars $X_{I,\infty}$ must be parallel to ξ_I, in the sense of real special geometry vectors, and the constraint (3.6) determines the proportionality constant between the two:

$$X_{I,\infty} = \left(\frac{1}{6}c^{JKL}\xi_J\xi_K\xi_L\right)^{-1/3}\xi_I \ . \tag{3.10}$$

The value of the potential V at the minimum must be related to the AdS$_5$ length scale ℓ and the cosmological constant in the usual manner

$$V_\infty = -c^{IJK}\xi_I\xi_J X_{K,\infty} \equiv -6\ell^{-2} \ . \tag{3.11}$$

This gives the constraint

$$\frac{1}{6}c^{IJK}\xi_I\xi_J\xi_K = \ell^{-3} \ , \tag{3.12}$$

on the FI-parameters ξ_I and the simple relation for the asymptotic values of the scalars

$$X_{I,\infty} = \ell\xi_I \ . \tag{3.13}$$

Thus the $n_V + 1$ independent FI-parameters ξ_I determine the asymptotic values X^I_∞ of the n_V scalars, as well as the AdS$_5$ scale ℓ.

For contrast, recall that in ungauged supergravity, the scalar fields are moduli as they experience no potential. Then their asymptotic values $X_{I,\infty}$ far from the black hole are set arbitrarily by boundary conditions, which is related to the fact that the spacetime is asymptotically flat. The fact that the value of the scalars X_I at the *horizon* is independent of the asymptotic values $X_{I,\infty}$ is the attractor mechanism for BPS black holes in ungauged supergravity.

As we have seen, the present context is very different in that the asymptotic values of the scalars are set by the theory through the FI-parameters ξ_I, rather than by boundary conditions. This is a generic feature of gauged supergravity, theories with asymptotically AdS vacuum. It precludes an attractor mechanism that is analogous to the one in asymptotically flat space. We will discuss this point in more depth in section 3.3 when we study the linear flow equations derived from supersymmetry.

The equations of motion $\mathcal{E}_\Phi$, where Φ is any field in the theory corresponding to the Lagrangian density (3.2), are the Einstein equation

$$\begin{aligned}\mathcal{E}_g = R_{AB} - \tfrac{1}{2}g_{AB}R + G_{IJ}\left(F^I_{5,AC}F^{J,C}_{5,B} - \tfrac{1}{4}g_{AB}F^I_{5,CD}F^{J,CD}_5\right)\\ - G_{IJ}\left(\nabla_A X^I\nabla_B X^J - \tfrac{1}{2}g_{AB}\nabla_C X^I\nabla^C X^J\right) - g_{AB}V = 0\,,\end{aligned} \tag{3.14}$$

and the matter equations for the Maxwell field A_5^I and the constrained scalars X^I

$$\mathcal{E}_A = d\left(G_{IJ} \star F_5^J\right) + \tfrac{1}{4}c_{IJK}F_5^J \wedge F_5^K = 0, \tag{3.15}$$

$$\begin{aligned}\mathcal{E}_{X^I} = -d\star dX_I + \tfrac{1}{3}X_I X^J d\star dX_J + 2c^{JKL}\xi_K\xi_L\left(\tfrac{2}{3}X_I X_J - c_{IJM}X^M\right)\star 1 + \big(X_J X^L c_{IKL} \\ -\tfrac{1}{2}c_{IJK} - \tfrac{2}{3}X_I X_J X_K + \tfrac{1}{6}X_I c_{JKN}X^N\big)\left(F_5^J \wedge \star F_5^K - dX^J \wedge \star dX^K\right) = 0\ .\end{aligned} \tag{3.16}$$

3.1.2 The effective 2D theory

We do not study all solutions to the 5D theory (3.1), just stationary black holes. Then it is sufficient to consider a reduction to 2D — and eventually to 1D. We impose the metric ansatz[1]

$$ds_5^2 = ds_2^2 - e^{-U_1}d\Omega_2^2 - e^{-U_2}(\sigma_3 + a^0)^2\ , \tag{3.17}$$

with ds_2^2 a general 2D metric and the 1-form ansatz for the gauge potential

$$A_5^I = a^I + b^I(\sigma_3 + a^0)\ . \tag{3.18}$$

In our conventions, the left invariant 1-forms

$$\begin{aligned}\sigma_1 &= \sin\phi\, d\theta - \cos\phi\sin\theta\, d\psi\ ,\\ \sigma_2 &= \cos\phi\, d\theta + \sin\phi\sin\theta\, d\psi\ ,\\ \sigma_3 &= d\phi + \cos\theta\, d\psi\ ,\end{aligned} \tag{3.19}$$

parametrize $SU(2)$ with

$$0 \le \theta \le \pi\ , \qquad 0 \le \phi \le 4\pi\ , \qquad 0 \le \psi \le 2\pi\,. \tag{3.20}$$

The ansatz (3.17) suggests the vielbein

$$\begin{aligned}&e^0 = e^0_\mu dx^\mu\ , \qquad e^1 = e^1_\mu dx^\mu\ , \qquad e^2 = e^{-\frac{1}{2}U_1}\sigma_1\ ,\\ &e^3 = e^{-\frac{1}{2}U_1}\sigma_2\ , \qquad e^4 = e^{-\frac{1}{2}U_2}(\sigma_3 + a^0)\ .\end{aligned} \tag{3.21}$$

We use Greek indices to denote the curved coordinates t and R in 2D. For extremal near-horizon geometries, the 2D coordinates describe the AdS_2 throat of the solution. The dimensional reduction via (3.17) and (3.18) of the 5D Lagrangian (3.2) introduces the scalar

[1]In our conventions the metric has a mostly negative signature.

fields U_1, U_2 and b^I, along with the 1-forms a^0, a^I. All these fields depend only on the 2D coordinates.

The effective 2D Lagrangian density that follows from (3.2) is given by

$$\begin{aligned}\mathcal{L}_2 = \frac{\pi}{G_5} e^{-U_1-\frac{1}{2}U_2}\Big\{&(-\mathcal{R}_2 + 2e^{U_1} - \tfrac{1}{2}e^{2U_1-U_2}) \star 1 - \tfrac{1}{2}dU_1 \wedge \star d\,(U_1+2U_2)\\ &-\tfrac{1}{2}e^{-U_2}da^0\wedge\star da^0 - 2V - G_{IJ}\Big((da^I + b^I da^0)\wedge\star(da^J + b^J da^0) + e^{2U_1}b^I b^J \star 1\\ &+ e^{U_2}db^I\wedge\star db^J - dX^I\wedge\star dX^J\Big) + \tfrac{1}{3}e^{U_1+\frac{1}{2}U_2}c_{IJK}\left(\tfrac{3}{2}b^I b^J da^K + b^I b^J b^K da^0\right)\Big\}\\ &+\frac{\pi}{G_5}d\left((e^{-U_1-\frac{1}{2}U_2}\star d(2U_1+U_2)) - \tfrac{1}{6}b^I b^J a^K\right)\,.\end{aligned} \tag{3.22}$$

We denote the Ricci scalar of the reduced 2D metric $\mathcal{R}_2$ and the Hodge dual is now in 2D. The overall exponential factor $e^{-U_1-\frac{1}{2}U_2}$ comes from the 5D metric on a deformed S^3. The first line in (3.22) is due to the reduction of the 5D Ricci scalar, which introduces additional kinetic and potential terms associated to the scalars U_1 and U_2, as well as for the 1-form a^0 in the beginning of the second line. The terms preceded by G_{IJ} are the reduction of the Maxwell field which yield kinetic terms for the 1-forms a^0, a^I and the scalars b^I, and the reduction of the kinetic term of X^I. The remainder of the third line of (3.22) is the Chern-Simons term. Finally, in the last line, there is a total derivative that is inconsequential for the equations of motion but is required in order that $\mathcal{L}_2$ (3.22) is the dimensional reduction of the 5D Lagrangian (3.2). The latter does not include the Gibbons-Hawking-York boundary term, the extrinsic curvature that appears separately in (3.1).

Boundary terms present an important subtlety that we will return to repeatedly in our study. The Chern-Simons term in the 5D Lagrangian (3.2) is not manifestly gauge invariant, but it transforms to a total derivative under a gauge variation. Gauge invariance could be restored by introducing a total derivative in the action. Such a term does not change the equations of motion but the resulting theory is not covariant in 5D, so there is a tension between important principles. The bulk part of the 2D Lagrangian (3.22) is not only covariant, it is also manifestly gauge invariant: a^I appears only as the field strength da^I. Manifest gauge invariance also applies to a^0 which encodes 5D rotational invariance. These are benefits of reducing to 2D.

From the dimensionally reduced Lagrangian density (3.22), we can derive the equations of motion for the fields U_1, U_2, a^0, a^I and b^I. The solutions to these 2D equations of motion are solutions of the 5D theory. The field equations for the 2D scalar fields are given by

$$\begin{aligned}
\mathcal{E}_{U_1} &= d(e^{-U_1-\frac{1}{2}U_2}\star(dU_1+dU_2)) + e^{-U_1-\frac{1}{2}U_2}\{\big(\mathcal{R}_2+2V-\tfrac{1}{2}e^{2U_1-U_2}\big)\star 1 \\
&\quad + \tfrac{1}{2}dU_1\wedge\star(dU_1+2dU_2) + \tfrac{1}{2}e^{-U_2}da^0\wedge\star da^0 + G_{IJ}\big((b^I da^0+da^I)\wedge\star(b^J da^0+da^J) \\
&\quad -e^{2U_1}b^Ib^J\star 1 + e^{U_2}db^I\wedge\star db^J - dX^I\wedge\star dX^J\big)\} = 0\,,
\end{aligned} \tag{3.23}$$

$$\begin{aligned}
\mathcal{E}_{U_2} &= d(e^{-U_1-\frac{1}{2}U_2}\star dU_1) + \tfrac{1}{2}e^{-U_1-\frac{1}{2}U_2}\{\big(\mathcal{R}_2+2V-2e^{U_1}+\tfrac{3}{2}e^{2U_1-U_2}\big)\star 1 \\
&\quad + \tfrac{1}{2}dU_1\wedge\star(dU_1+2dU_2) + \tfrac{3}{2}e^{-U_2}da^0\wedge\star da^0 + G_{IJ}\big((b^I da^0+da^I)\wedge\star(b^J da^0+da^J) \\
&\quad +e^{2U_1}b^Ib^J\star 1 - e^{U_2}db^I\wedge\star db^J - dX^I\wedge\star dX^J\big)\} = 0\,,
\end{aligned} \tag{3.24}$$

$$\begin{aligned}
\mathcal{E}_{b^I} &= 2d(e^{-U_1+\frac{1}{2}U_2}G_{IJ}\star db^J) - 2e^{-U_1-\frac{1}{2}U_2}G_{IJ}da^0\wedge\star(b^J da^0+da^J) - 2G_{IJ}e^{U_1-\frac{1}{2}U_2}b^J\star 1 \\
&\quad + c_{IJK}b^J da^K + c_{IJK}b^Jb^K da^0 = 0\,,
\end{aligned} \tag{3.25}$$

and the 1-forms satisfy

$$\begin{aligned}
\mathcal{E}_{a^0} &= -d(e^{-U_1-\frac{3}{2}U_2}\star da^0) - 2d(G_{IJ}b^I e^{-U_1-\frac{1}{2}U_2}\star(b^J da^0+da^J)) + \tfrac{1}{3}c_{IJK}d(b^Ib^Jb^K) = 0\ , \\
\mathcal{E}_{a^I} &= -2d\left(e^{-U_1-\frac{1}{2}U_2}G_{IJ}\star\left(b^J da^0+da^J\right)\right) + \tfrac{1}{2}c_{IJK}d\left(b^Jb^K\right) = 0\ .
\end{aligned} \tag{3.26}$$

3.1.3 An effective 1D theory

We conclude the section by reducing the 2D reduced Lagrangian (3.22) to a one-dimensional radial effective theory where all of the functions that appear in the effective Lagrangian (3.22) are set to be exclusively radial functions, with respect to the radial coordinate R. In this additional reduction we pick a diagonal gauge for the 2d line element of (3.17):

$$ds_2^2 = e^{2\rho}dt^2 - e^{2\sigma}dR^2\ . \tag{3.27}$$

The operators d and $\star$ acting on the fields in the Lagrangian (3.22) simplify with this ansatz. For example, the 2D Ricci scalar becomes

$$\mathcal{R}_2 = 2e^{-\rho-\sigma}\partial_R(e^{-\sigma}\partial_R e^{\rho})\,. \tag{3.28}$$

Second derivatives are awkward so it is advantageous to rewrite this term as

$$e^{\rho+\sigma-U_1-\frac{1}{2}U_2}\mathcal{R}_2 = 2\partial_R\left(e^{\rho-\sigma-U_1-\frac{1}{2}U_2}\partial_R\rho\right) + e^{\rho+\sigma-U_1-\frac{1}{2}U_2}(e^{-2\sigma}\partial_R\rho)\partial_R\left(2U_1+U_2\right) \quad . \tag{3.29}$$

The first term is a total derivative, an additional boundary term. To examine the total boundary contribution, we consider a constant radial slice at infinity. As we are now reducing to 1D, the boundary terms must be evaluated at the bounds for the time coordinate. This is trivial since there is no explicit time dependence. After dimensional reduction, the Gibbons-Hawking-York term in (3.1) corresponds to the total derivative

$$\mathcal{L}_{\text{GHY}} = \frac{2\pi}{G_5}\partial_R\left(e^{\rho-\sigma-U_1-\frac{1}{2}U_2}\partial_R(\rho-U_1-\tfrac{1}{2}U_2)\right) \; . \tag{3.30}$$

The total derivative term in (3.29), the Gibbons-Hawking-York term (3.30), and the boundary terms in the last line of (3.22) after dimensional reduction to 1D, precisely cancel. This leaves only the contribution arising from the Chern-Simons term

$$\mathcal{L}_{\text{bdry}} = -\frac{1}{6}\frac{\pi}{G_5}d\left(c_{IJK}b^Ib^Ja_t^K\right) \; . \tag{3.31}$$

This remaining boundary term in (3.31) is crucial as it will affect the conserved charges we seek to compute. We will comment on this in depth in the subsequent section 3.2. In summary, the 1D Lagrangian density takes the form

$$\begin{aligned}\mathcal{L}_1 = \frac{\pi}{G_5}e^{\rho+\sigma-U_1-\frac{1}{2}U_2}\Big[&-e^{-2\sigma}(\partial_R\rho)\partial_R\left(2U_1+U_2\right) - \tfrac{1}{2}e^{-2\sigma}(\partial_RU_1)(\partial_RU_1+2\partial_RU_2)\\ &-G_{IJ}e^{-2\sigma}\left(\partial_RX^I\partial_RX^J - e^{U_2}\partial_Rb^I\partial_Rb^J\right) + \tfrac{1}{2}e^{-U_2-2\rho-2\sigma}(\partial_Ra_t^0)^2\\ &+G_{IJ}e^{-2\rho-2\sigma}(\partial_Ra_t^I+b^I\partial_Ra_t^0)(\partial_Ra_t^J+b^J\partial_Ra_t^0) - 2V + 2e^{U_1} - \tfrac{1}{2}e^{2U_1-U_2}\\ &-G_{IJ}e^{2U_1}b^Ib^J\Big] + \frac{\pi}{G_5}\frac{1}{3}c_{IJK}\left[-\frac{3}{2}b^Ib^J\partial_Ra_t^K - b^Ib^Jb^K\partial_Ra_t^0\right] \; .\end{aligned} \tag{3.32}$$

Having established the effective Lagrangian in 2D (3.22) and 1D (3.32), we proceed in the next subsection with construction of the Noether-Wald surface charges in our theory.

3.2 Noether-Wald surface charges

In this section, we review the Noether-Wald procedure for computing the conserved charge due to a general symmetry [79, 80]. We specifically consider an isometry generated by a Killing vector and a gauge symmetry in the presence of Chern-Simon terms. In each case, we express the conserved charge as a flux integral that is the same for any surface surrounding the black hole.

3.2.1 The Noether-Wald surface charge: general formulae

We consider a theory in D dimensions described by a Lagrangian $\mathcal{L}$ that is presented as a D-form. The Lagrangian depends on fields Φ_i that include both the metric $g_{\mu\nu}$ and matter fields, as well as the derivatives of these fields.

A symmetry ζ is such that the variation of $\mathcal{L}$ with respect to ζ is a closed form (locally), i.e. d acting on a $D-1$ form $\mathcal{J}_\zeta$:

$$\mathcal{L} \underset{\zeta}{\rightarrow} \mathcal{L} + \delta\mathcal{L} = \mathcal{L} + d\mathcal{J}_\zeta \ . \tag{3.33}$$

The variation of the Lagrangian due to *any* change in the fields is given by[2]

$$\begin{aligned} \delta\mathcal{L} &= \delta\Phi_i \frac{\partial\mathcal{L}}{\partial\Phi_i} + (\partial_\mu\delta\Phi_i)\frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi_i} \\ &= \delta\Phi_i\left[\frac{\partial\mathcal{L}}{\partial\Phi_i} - \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi_i}\right)\right] + \partial_\mu\left(\delta\Phi_i\frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi_i}\right) \ . \end{aligned} \tag{3.34}$$

The usual variational principle determines the equations of motion $\mathcal{E}_\Phi$ as the vanishing of the expression in the square bracket. The remaining term, by definition, is the total derivative of the presymplectic potential

$$\Theta^\mu \equiv \delta\Phi_i\frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi_i} \ . \tag{3.35}$$

In our informal notation, the left hand side of this equation is indistinguishable from a vector. However, the Lagrangian is a D-form and the δ-type "derivative" removes an entire 1-form. Therefore, the presymplectic potential Θ becomes a $D-1$ form, with indices obtained by contracting the volume form with the vector that is normal to the boundary. A more precise version of (3.34) reads

$$\delta\mathcal{L} = \delta\Phi_i\left[\frac{\partial\mathcal{L}}{\partial\Phi_i} - \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi_i}\right)\right] + d\Theta[\Phi_i, \delta\Phi_i] \ . \tag{3.36}$$

Comparing this formula for a general variation with its analogue (3.33) for a symmetry

[2] In practice, when we solve for the Einstein equations, we will consider a variation of the metric and will not directly use (3.34).

establishes $d\mathcal{J}_\zeta = d\Theta$ and so the $D-1$ form

$$J_\zeta = \mathcal{J}_\zeta - \Theta[\Phi_i, \delta\Phi_i] \tag{3.37}$$

is closed when the equations of motion $\mathcal{E}_\Phi$ are imposed. This identifies the familiar conserved Noether current associated to the symmetry ζ. The corresponding Noether charge is

$$Q_{\zeta,\text{Noether}} = \int_\Sigma J_\zeta \ , \tag{3.38}$$

where Σ is a Cauchy surface on the background manifold. Conservation amounts to this charge being the same on all Cauchy surfaces. Conceptually, the total charge is the same at all times. That is the point of conservation in a truly dynamical setting, but it is not terribly interesting in a stationary black hole spacetime which is, by definition, independent of time.

For black holes it is important that, given the closed $(D-1)$ form J_ζ, there exists a $(D-2)$-form Q_ζ such that

$$J_\zeta \cong dQ_\zeta \ . \tag{3.39}$$

The Q_ζ is the Noether-Wald *surface* charge. It amounts to a conserved *flux* in the sense of Gauss' law: integration of the flux over any surface enclosing the source gives the same result.

The surface charge Q_ζ is more subtle than the conserved charge integrated over an entire Cauchy surface. The semi-equality $\cong$ reminds us that generally the closed form J_ζ is only d of something locally so, in general, the charge Q_ζ, is only defined up to d of some $D-3$ form. Therefore, it does not necessarily satisfy Gauss' law.

One way around this is to evaluate the surface charge at infinity. For example, the presence of a Chern-Simons term can be interpreted physically as a charge density that obstructs flux conservation but this contribution is subleading at infinity and will not contribute to $Q_{\zeta,\text{Noether}}$.

Alternatively, following [81, 82, 83, 84, 85], we can modify our definition of the surface charge by adding a $D-2$ form to Q_ζ. This new surface charge satisfies a Gauss law and can be integrated at any given surface Σ.

A third approach [86], is the one taken in this paper. It is to compute the surface charges in a dimensionally reduced 2D theory.

Moreover, we integrate by parts such that in the process of dimensional reduction to 2D, we ensure gauge invariance. We will carry this procedure out in section 3.2.4. Therefore, in this case, Q_ζ will satisfy a Gauss law.

The procedure for computing the conserved charges is extremely general. In the following, we make the abstract procedure explicit for two particular symmetries: isometries generated by a spacetime Killing vector ξ and gauge symmetries λ in the presence of Chern-Simons terms.

3.2.2 Killing vector fields

A Killing vector ξ generates a spacetime isometry. It transforms the Lagrangian as

$$\delta_\xi \mathcal{L} = L_\xi \mathcal{L} \ . \tag{3.40}$$

Here L_ξ is the Lie derivative along the Killing vector ξ.
The Lie derivative acting on a general form ω is given by Cartan's magic formula

$$L_\xi \omega = d(i_\xi \omega) + i_\xi d\omega \ . \tag{3.41}$$

Since $\mathcal{L}$ is a D-form it must be closed $d\mathcal{L} = 0$ and then the Lie derivative becomes

$$\delta_\xi \mathcal{L} = L_\xi \mathcal{L} = d(i_\xi \mathcal{L}) + i_\xi (d\mathcal{L}) = d(\xi \cdot \mathcal{L}) \ , \tag{3.42}$$

where $\cdot$ denotes the contraction of ξ with the first index of $\mathcal{L}$. Comparing (3.42) with (3.33), we identify

$$\mathcal{J}_\xi = \xi \cdot \mathcal{L} \ , \tag{3.43}$$

up to a closed form that is unimportant in our application. Thus, for a Killing vector ξ, the Noether current (3.37) becomes

$$J_\xi = \xi \cdot \mathcal{L} - \Theta[\Phi, \mathcal{L}_\xi \Phi] \ . \tag{3.44}$$

The computations show that this current $(D-1)$ form is closed on-shell. In other words, it is conserved when the equations of motion are satisfied.

3.2.3 Incorporating gauge invariance

We now consider a *gauge invariant* Lagrangian and compute the conserved current as defined in (3.37) for the conserved charges of the theory, whether derived from spacetime isometries or gauge invariance.

The relevant gauge invariant Lagrangian is the one defined in (3.1) *without* the Chern-Simons term. In other words, we consider the Lagrangian density

$$\begin{aligned}\mathcal{L}_{5,\mathrm{pot}} &= -\frac{1}{16\pi G_5}\sqrt{g_5}\left(\mathcal{R}_5 + 2V\right) , \\ \mathcal{L}_{5,\mathrm{kin}} &= \frac{1}{16\pi G_5}\sqrt{g_5}\left(-\frac{1}{2}G_{IJ}F^I_{5,AB}F_5^{J,AB} + G_{IJ}\nabla^A X^I \nabla_A X^J\right) .\end{aligned} \tag{3.45}$$

We use early capital Latin indices $A, B, \ldots$ to denote 5D coordinates. The Lagrangian $\mathcal{L}_{5,\mathrm{kin}} + \mathcal{L}_{5,\mathrm{pot}}$ is manifestly gauge invariant

$$\delta_\alpha(\mathcal{L}_{5,\mathrm{pot}} + \mathcal{L}_{5,\mathrm{kin}}) = 0 , \tag{3.46}$$

As detailed in the previous subsection, there is a conserved charge for any Killing vector that generates a spacetime isometry. According to (3.42), the Lagrangian $\mathcal{L}_{5,\mathrm{kin}} + \mathcal{L}_{5,\mathrm{pot}}$ transforms as

$$\delta_\xi(\mathcal{L}_{5,\mathrm{pot}} + \mathcal{L}_{5,\mathrm{kin}}) = \nabla_A\left(\xi^A(\mathcal{L}_{5,\mathrm{pot}} + \mathcal{L}_{5,\mathrm{kin}})\right) . \tag{3.47}$$

The presymplectic potential (3.35) for $\mathcal{L}_{5,\mathrm{pot}}$ is

$$\begin{aligned}\Theta^{A,5}_{\xi,\mathrm{pot}} &= \frac{1}{16\pi G_5}\sqrt{g_5}\left(\nabla_B\nabla^A\xi^B + \nabla_B\nabla^B\xi^A - 2\nabla^A\nabla_B\xi^B\right) \\ &= \frac{1}{16\pi G_5}\sqrt{g_5}\left(\nabla_B\left(\nabla^B\xi^A - \nabla^A\xi^B\right) + 2R^{AB}\xi_B\right) ,\end{aligned} \tag{3.48}$$

where in the second line, we have used the commutator relation for two covariant derivatives. In addition, the presymplectic potential for the kinetic terms $\mathcal{L}_{5,\mathrm{kin}}$ is

$$\Theta^{A,5}_{\alpha,\xi,\mathrm{kin}} = \frac{1}{16\pi G_5}\sqrt{g_5}G_{IJ}\left(2F_5^{I,AB}\left(\xi^C F^J_{5,CB} + \nabla_B\left(\xi^C A^J_{5,C} + \alpha^J_5\right)\right)\right) , \tag{3.49}$$

where we have used the variation

$$\delta A^I_{A,5} = \delta_\xi A^I_{5,A} + \delta_\alpha A^I_{5,A} = \xi^B F^I_{5,BA} + \nabla_A\left(\xi^B A^I_{5,B}\right) + \nabla_A\alpha^I . \tag{3.50}$$

Inserting the variations (3.46) and (3.47) and the presympletic potentials given in (3.48) and

(3.49) into the current density (3.37), we find

$$J^A_{\alpha,\xi} = \frac{1}{16\pi G_5}\sqrt{g_5}\Big[-\nabla_B\left(\nabla^B\xi^A - \nabla^A\xi^B\right) - 2\nabla_B\left(G_{IJ}F^{I,AB}(\xi^C A^J_C + \alpha^J)\right) \\ -2\xi_B\mathcal{E}^B_g - 2\mathcal{E}^A_{J,A_5}(\xi^C A^J_C + \alpha^J)\Big] \ , \tag{3.51}$$

where the second line is proportional to the equations of motion $\mathcal{E}^B_g$ and $\mathcal{E}^A_{J,A_5}$ and vanish on-shell giving

$$J^A_{\alpha,\xi} = -\frac{1}{16\pi G_5}\sqrt{g_5}\nabla_B\Big[\left(\nabla^B\xi^A - \nabla^A\xi^B\right) + 2\left(G_{IJ}F^{I,AB}(\xi^C A^J_C + \alpha^J)\right)\Big]. \tag{3.52}$$

The Noether-Wald surface charges of the theory can now be read off from the current (3.52). To find the conserved charges, we integrate over a surface Σ enclosing the source and we find

$$\begin{aligned} Q_\alpha &= -\frac{1}{8\pi G_5}\int_\Sigma d\Sigma_{AB}\sqrt{g_5}G_{IJ}F^{I,AB}\alpha^J \ , \\ Q_\xi &= -\frac{1}{16\pi G_5}\int_\Sigma d\Sigma_{AB}\sqrt{g_5}\left[\left(\nabla^B\xi^A - \nabla^A\xi^B\right) + 2G_{IJ}F^{I,AB}\xi^C A^J_C\right] \ . \end{aligned} \tag{3.53}$$

3.2.4 Chern-Simons Terms

The charge Q_ξ that corresponds to angular momentum depends explicitly on the gauge field A^J whereas the electric charges Q_α depend on the field strength. When Chern-Simons terms are taken into account, Q_α also depends on the gauge field A^J. This gauge dependence renders the value of the charges ambiguous.

To address the situation, we dimensionally reduce the theory (3.1) to 2D, as was done in subsection 3.1.2 and express the resulting action as a covariant theory in 2D [86]. As part of the process, we must ensure that the field strength does not have a nonzero flux through the squashed sphere. This can be achieved by adding total derivatives before the dimensional reduction to remove the derivatives acting on the gauge potentials and gauge fields that act nontrivally through the squashed sphere.

We now show how this can be done. Let us consider the Lagrangian (3.45) along with the five-dimensional Chern-Simons term of the form

$$\mathcal{L}_{5,\text{CS}} = -\frac{1}{16\pi G_5}\frac{1}{6}c_{IJK}F^I_5 \wedge F^J_5 \wedge A^K_5 \ . \tag{3.54}$$

We are interested in transforming (3.54) by the inclusion of total derivatives such that the potential term associated to the electric charge is manifestly gauge invariant. Note this procedure is not covariant in 5D and therefore we explicitly break covariance along the way. However, because of the dimensional reduction, the 2D Lagrangian still remains covariant.

We consider the ansatz in (3.17) such that the potential and gauge fields are of the form

$$\begin{aligned} A_5^I &= A_{5,A}^I dx^A = A_{5,\mu}^I dx^\mu + A_{5,a}^I dx^a \ , \\ F_5^I &= \frac{1}{2} F_{5,AB}^I \, dx^A \wedge dx^B = \frac{1}{2} F_{5,\mu\nu}^I dx^\mu \wedge dx^\nu + F_{5,\mu a}^I dx^\mu \wedge dx^a + \frac{1}{2} F_{5,ab}^I dx^a \wedge dx^b \ , \end{aligned} \tag{3.55}$$

where lowercase Latin indices denote the indices on the compact space and as before, the Greek indices correspond to the 2D space. Expanding out the Chern-Simons term in component form using (3.55), there are two types of terms, having the following structure of indices: $F_{\mu\nu}^I F_{bc}^J A_a^K$ and $F_{\mu a}^I F_{bc}^J A_\nu^K$. Only for the second expression we must transfer the derivative such that in the process of dimensional reduction, we find it to be gauge invariant in the 2D theory. This means the integration by parts of this term takes the form

$$c_{IJK} \epsilon^{\mu abc\nu} F_{\mu a}^I F_{bc}^J A_\nu^K = 2 c_{IJK} \epsilon^{\mu abc\nu} \left(\partial_\mu (A_\nu^K A_a^I F_{bc}^J) - (\partial_\mu A_\nu^K)(A_a^I F_{bc}^J) \right) \ , \tag{3.56}$$

and the presymplectic potential is found to be

$$\begin{aligned} \Theta_{\alpha,\xi,\text{CS}}^{A,5} = & \frac{1}{16\pi G_5} \sqrt{g_5} \left[\frac{1}{6} c_{IJK} \left(\epsilon^{ABCDE} F_{BC}^I A_D^J \left(\xi^F F_{FE}^K + \nabla_E (\xi^F A_F^K) + \nabla_E \alpha^K \right) \right) \right] \\ & - \frac{1}{8\pi G_5} \sqrt{g_5} \left[c_{IJK} \epsilon^{Aabc\nu} A_a^I F_{bc}^J \nabla_\nu \alpha^K \right] \ , \end{aligned} \tag{3.57}$$

where the last term is the contribution of (3.56) coming from adding a total derivative. To investigate the current and the Noether-Wald surface charges, we proceed to dimensionally reduce over the squashed S^3 where covariance over the 2D spacetime is still maintained. The 5D rotational isometries in φ and ψ take on a different role in the 2D perspective. Moreover, we find that they become 2D gauge transformations of a^0 and a^I coming from the dimensionally reduced potential A^I (3.18).

3.2.5 The 2D conserved charges

The 2D Lagrangian (3.22) inherits some symmetries from the 5D theory (3.2), including gauge symmetry associated with the 5D gauge potential A^I and rotational isometries associated to the Killing vectors ∂_ϕ and ∂_ψ. In the 2D theory, all symmetries become gauge symmetries and have associated charges. We denote the 2D charge originally coming from the 5D rotational isometries J and the 2D charges originally coming from the 5D gauge transformations Q_I. These 2D gauge transformations are associated to a^0 and a^I as they come from the dimensionally reduced potential A^I (3.18). Therefore, we consider the following symmetries

$$\delta_\lambda a^0 = d\lambda, \qquad \delta_\chi a^I = d\chi^I \ , \tag{3.58}$$

with total corresponding conserved current

$$J_{\lambda,\chi} = J_\lambda + J_\chi = \sum_{i=\lambda,\chi} \left(\mathcal{J}_i - \Theta_i\right), \tag{3.59}$$

where J_λ and J_χ are the currents corresponding to λ and χ, respectively, and the second equality is given by (3.37). The effective 2D Lagrangian (3.22) is manifestly gauge invariant and the variations with respect to each symmetry (3.58) yield

$$\delta_\lambda \mathcal{L}_2 = d\mathcal{J}_\lambda = 0 \ , \quad \delta_\chi \mathcal{L}_2 = d\mathcal{J}_\chi = -\frac{\pi}{G_5}\frac{1}{6}c_{IJK} d\left(b^I b^J d\chi^K\right) \ . \tag{3.60}$$

The presymplectic potentials given in (3.36) become

$$\Theta_\lambda = -\frac{\pi}{G_5} e^{-U_1 - \frac{1}{2}U_2}\left[e^{-U_2} \star da^0 + 2G_{IJ} b^I \star \left(b^J da^0 + da^J\right)\right] d\lambda + \frac{\pi}{G_5}\frac{1}{3} c_{IJK} b^I b^J b^K d\lambda \ , \tag{3.61}$$

$$\Theta_\chi = -\frac{\pi}{G_5} e^{-U_1 - \frac{1}{2}U_2}\left[2G_{IJ} d\chi^I \wedge \star\left(b^J da^0 + da^J\right)\right] + \frac{\pi}{G_5}\frac{1}{3} c_{IJK} b^I b^J d\chi^K \ . \tag{3.62}$$

We used the symmetries (3.58) and included the additional total derivative term (3.56). Using the equations of motion (3.26), the on-shell current (3.59) can be recast in the form of (3.39):

$$\begin{aligned} J_{\lambda,\chi} \cong \frac{\pi}{G_5} d\Bigg[\lambda\left(e^{-U_1-\frac{1}{2}U_2}\left[e^{-U_2}\star da^0 + 2G_{IJ}b^I\star\left(b^J da^0 + da^J\right)\right] - \frac{1}{3}c_{IJK}b^I b^J b^K\right) \\ + \chi^I\left(2e^{-U_1-\frac{1}{2}U_2}G_{IJ}\star\left(b^J da^0 + da^J\right) - \frac{1}{2}c_{IJK}b^J b^K\right)\Bigg] . \end{aligned} \tag{3.63}$$

The conserved charges J and Q_I can be directly read off from (3.63) and we find

$$\begin{aligned} J &= \frac{\pi}{G_5}\left[e^{-U_1-\frac{1}{2}U_2}\left[e^{-U_2}\star da^0 + 2G_{IJ}b^I\star\left(b^J da^0 + da^J\right)\right] - \frac{1}{3}c_{IJK}b^I b^J b^K\right] \ , \\ Q_I &= \frac{\pi}{G_5}\left[2e^{-U_1-\frac{1}{2}U_2}G_{IJ}\star\left(b^J da^0 + da^J\right) - \frac{1}{2}c_{IJK}b^J b^K\right] \ . \end{aligned} \tag{3.64}$$

From now on, we use the rescaled charges

$$\widetilde{J} \equiv \frac{4G_5}{\pi} J \ , \qquad \widetilde{Q} \equiv \frac{4G_5}{\pi} Q_I \ . \tag{3.65}$$

The charges and the current are indeed conserved including the charge associated to a^I since we demanded gauge invariance at the level of the Lagrangian in (3.22). This added a total derivative that shifted the charge but did not affect the equations of motion. Moreover, the charges computed in 2D are proportional to those computed in 5D. In the 1D reduction (3.32), the charges take the following form

$$\begin{aligned}
\widetilde{J} &= 4\left(e^{-U_1-\frac{3}{2}U_2-\rho-\sigma}\left(\partial_R a_t^0 + 2G_{IJ}e^{U_2}b^I\left(\partial_R a_t^J + b^J\partial_R a_t^0\right)\right) - \frac{1}{3}c_{IJK}b^Ib^Jb^K\right) \ , \\
\widetilde{Q}_I &= 4\left(2e^{-U_1-\frac{1}{2}U_2-\rho-\sigma}G_{IJ}\left(\partial_R a_t^J + b^J\partial_R a_t^0\right) - \frac{1}{2}c_{IJK}b^Jb^K\right) \ .
\end{aligned} \tag{3.66}$$

These formulae are essential for the radial flow in the black hole background. A *very* rough reading is that each of the conserved charges $\widetilde{J}$ and $\widetilde{Q}_I$ are radial derivatives of their conjugate potentials a_t^0, a_t^I, as in elementary electrodynamics. With this naïve starting point, the overall factors depending on U_1, U_2, ρ and σ serve to take on the non-flat spacetime into account and G_{IJ} incorporates special geometry as required by symmetry. All remaining terms depend on b^I and take rotation into account in a manner that combines kinematics (rotation "looks" like a force) and electrodynamics (electric and magnetic fields mix in a moving frame). These effects defy simple physical interpretations.

From our point of view, the formulae (3.66) for the charges $\widetilde{J}$ and $\widetilde{Q}_I$ are complicated functions of various fields, each of which are themselves nontrivial functions of the radial coordinate. Our construction shows that symmetry guarantees that these *combinations* must be independent of the radial position, within the framework of our *ansatz*.

In the following section we study the conditions that *supersymmetric* AdS_5 black holes must satisfy. The vanishing of the supersymmetry variations of the theory for a subset of the supersymmetries always imposes first-order radial differential equations on the joint geometry/matter configuration. We refer to these first order equations as *flow equations*. They are very constraining but, as usual for first order equations imposed by supersymmetry, they are not sufficient to determine the solution. The raison d'être of this entire section is that the additional data needed, sometimes referred to as the integrability conditions, is furnished by the conserved charges.

3.3 The flow equations

In this section we derive the first order flow equations for AdS_5 black holes. They follow from preservation of supersymmetry, complemented by conservation of the charges. We study the flow equations using two perturbative expansions: one starting at the near-horizon and one starting at the asymptotic boundary. Enforcing the conservation of charges at both the horizon and at the asymptotic boundary allows us to make contact between the two expansions.

3.3.1 Supersymmetry conditions

We study bosonic backgrounds that preserve some supersymmetry [87, 88, 89]. Thus there exists a supersymmetric spinor ϵ^α for which the gravitino and the dilatino variations vanish. This condition amounts to

$$0 = \left[G_{IJ}\left(\frac{1}{2}\gamma^{AB}F^J_{AB} - \gamma^A\nabla_A X^J\right)\epsilon^\alpha - \xi_I\epsilon^{\alpha\beta}\epsilon^\beta\right]\partial_i X^I \ , \tag{3.67}$$

$$0 = \left[(\partial_A - \frac{1}{4}\omega_A^{BC}\gamma_{BC}) + \frac{1}{24}(\gamma_A{}^{BC} - 4\delta_A{}^B\gamma^C)X_I F^I_{BC}\right]\epsilon^\alpha + \frac{1}{6}\xi_I(3A^I_A - X^I\gamma_A)\epsilon^{\alpha\beta}\epsilon^\beta \ , \tag{3.68}$$

where ϵ^α $(\alpha = 1, 2)$ are symplectic Majorana spinors. In the following, we recast these variations as radial flow equations.

For the analysis of supersymmetry, it is convenient to split the 5D spacetime geometry into $(1 + 4)$ dimensions as

$$ds_5^2 = f^2(dt + w\sigma_3)^2 - f^{-1}ds_4^2 \ , \tag{3.69}$$

$$ds_4^2 = g_m^{-1}dR^2 + \frac{1}{4}R^2(\sigma_1^2 + \sigma_2^2 + g_m\sigma_3^2) \, . \tag{3.70}$$

This form highlights the 4D base space ds_4^2 which is automatically Kähler. This can be shown by picking the flat vielbein

$$e^1 = g_m^{-1/2}dR \ , \quad e^2 = \frac{1}{2}R\sigma_1 \ , \quad e^3 = \frac{1}{2}R\sigma_2 \ , \quad e^4 = \frac{1}{2}Rg_m^{1/2}\sigma_3 \ , \tag{3.71}$$

which gives the manifestly closed Kähler 2-form

$$J^{(1)} = \epsilon(e^1 \wedge e^4 - e^2 \wedge e^3) \ . \tag{3.72}$$

The symbol $\epsilon = \pm 1$ denotes the orientation of the base manifold. It should not be confused with the supersymmetry parameter ϵ^α.

The $(1+4)$ split of the 5D gauge potential A^I defined in (3.18) can be expressed as

$$A^I = fY^I(dt + w\sigma_3) + u^I\sigma_3 \ . \tag{3.73}$$

In the rest of the paper, as well as in Appendix A.3, we use lowercase Latin letters for the four spatial indices.

3.3.2 Dictionary between the $(1+4)$ and the $(2+3)$ splits

We can relate the $(1+4)$ split introduced in the previous subsection to simplify the supersymmetric variations (3.67) and (3.68), to the $(2+3)$ split (3.17) and (3.18) that was used earlier to perform the reduction from 5D to 2D. The 5D geometry (3.17) with the diagonal gauge (3.27) for the 2D line element is

$$ds_5^2 = e^{2\rho}dt^2 - e^{2\sigma}dR^2 - e^{-U_1}(\sigma_1^2 + \sigma_2^2) - e^{-U_2}(\sigma_3 + a^0)^2 \ . \tag{3.74}$$

By identifying the metric components of (3.69) and (3.74), we find the dictionary of variables in the $(2+3)$ split of the 5D line element ds_5^2, expressed in terms of the variables in the $(1+4)$ split

$$\begin{aligned} e^{-U_1} &= \frac{1}{4}R^2 f^{-1} \ , \qquad e^{-U_2} = \frac{1}{4}R^2 g_m f^{-1} - f^2 w^2 \ , \qquad b^I = fY^I w + u^I \ , \\ e^{2\sigma} &= f^{-1} g_m^{-1} \ , \qquad a_t^0 = \frac{-f^2 w}{\frac{1}{4}R^2 g_m f^{-1} - f^2 w^2} \ , \qquad e^{2\rho} = f^2 - \frac{f^4 w^2}{(\frac{1}{4}R^2 g_m f^{-1} - f^2 w^2)^2} \ . \end{aligned} \tag{3.75}$$

In this section we primarily use the $(1+4)$ variables fX^I, u^I, w and g_m, along with the conserved charges Q_I and J.

As noted in the previous subsection, the 4D base of the $(1+4)$ split (3.69) is automatically Kähler. In the variables of the $(2+3)$ split in (3.75), the Kähler condition amounts to the relation

$$e^{\sigma+\rho-U_2/2} = \frac{1}{2}R \ , \tag{3.76}$$

between ρ, σ, and U_2. This is explained further in Appendix A.3.1.

3.3.3 The attractor flow equations

The preserved supersymmetries are defined by the projections on the spinors ϵ^α

$$\gamma^0 \epsilon^\alpha = \epsilon^\alpha \ , \tag{3.77}$$

$$\frac{1}{4} J^{(1)}_{mn} \gamma^{mn} \epsilon^\alpha = -\epsilon^{\alpha\beta}\epsilon^\beta \ . \tag{3.78}$$

The $J^{(1)}_{mn}$ are components of the Kähler form $J^{(1)}$ (3.72), and the spatial gamma matrices γ^m satisfy the usual Clifford algebra. The details on the simplification of the equations (3.67) and (3.68) are presented in Appendix A.3. The result is the following differential conditions on the variables fX^I, u^I, w and g_m

$$0 = G_{IJ}\left(\partial_R(fY^I) - \partial_R(fX^I)\right)\partial_i X^J \ , \tag{3.79}$$

$$0 = \left(\partial_{R^2} + \frac{1}{R^2}\right) u^I - \frac{1}{2}\epsilon f^{-1} c^{IJK} X_J \xi_K \ , \tag{3.80}$$

$$0 = \left(\partial_{R^2} - \frac{1}{R^2}\right) w + \frac{1}{2} f^{-1} X_I \left(\partial_{R^2} - \frac{1}{R^2}\right) u^I \ , \tag{3.81}$$

$$0 = -\epsilon R^2(\partial_{R^2} g_m) + 2\epsilon(1-g_m) + 2\xi_I u^I \ . \tag{3.82}$$

The variation (3.79) allows for the electric potential fY^I in (3.73) to be identified with the scalar field fX^I, and thus

$$A^I = fX^I(dt + w\sigma_3) + u^I\sigma_3 \ . \tag{3.83}$$

This sets the variables a^I, a^0 and b^I in the decomposition (3.18) to

$$a^I + b^I a^0 = fX^I dt \ , \ b^I = fX^I w + u^I \ , \tag{3.84}$$

with $a^0 = a^0_t dt$ given in (3.75).

In the limit of ungauged supergravity, fX^I is a harmonic function. Then, the identification of $X^I = Y^I$ means that the corresponding electric potential is also a harmonic function. We may expect the same functional dependence in the case of gauged supergravity.[3]

In the context of a black hole, solutions to the supersymmetry conditions (3.79-3.82) are specified in part by the conserved charges of the theory. The charges in $\widetilde{J}$ and $\widetilde{Q}_I$ in (3.66) are expressed in terms of the $(2+3)$ variables $U_1, U_2, b^I, a^0_t, a^I_t$ but we can recast them in terms of $(1+4)$ variables f, u^I, w, g_m using the dictionary for the geometry (3.75) and the

[3]There are $n_V + 1$ potentials Y^I and n_V scalars X^I so there is freedom to adjust a single integration constant that we do not exploit. It is unclear to us if this freedom is physically significant.

potential (3.84). We can also remove most of the derivatives in the equations (3.66) for the charges $\widetilde{J}$ and $\widetilde{Q}_I$ using the radial equations for the variables u^I, w and g_m (3.80-3.82). Our final expressions of the charges, which we use for the remainder of the section are given by

$$\begin{aligned}\widetilde{J} &= \widetilde{Q}_I u^I + \frac{2}{3}c_{IJK}u^I u^J u^K - R^2 g_m \left(f^{-1}X \cdot u + 2w - \frac{1}{2}\epsilon R^2 f^{-3}\xi \cdot fX \right) \\ &\quad + 2R^2 w(1+\epsilon\xi\cdot u) \ , \\ \widetilde{Q}_I &= -2c_{IJK}u^J u^K - 2\epsilon w R^2 \xi_I - g_m R^4 \partial_{R^2}(f^{-1}X_I) \ .\end{aligned} \tag{3.85}$$

The second of these equations is a first order differential equation for $f^{-1}X_I$. Together with the three radial differential equations (3.80-3.82) for the variables u^I, g_m and w, we find the four equations

$$\left(\partial_{R^2} + \frac{1}{R^2}\right) u^I = \frac{1}{2}\epsilon c^{IJK}(f^{-1}X_J)\xi_K \ , \tag{3.86a}$$

$$\left(\partial_{R^2} + \frac{2}{R^2}\right) g_m = \frac{2}{R^2} + \frac{2}{R^2}\epsilon\xi_I u^I \ , \tag{3.86b}$$

$$\left(\partial_{R^2} - \frac{1}{R^2}\right) w = -\frac{1}{2}f^{-1}X_I \left(\partial_{R^2} - \frac{1}{R^2}\right) u^I \ , \tag{3.86c}$$

$$R^4\partial_{R^2}(f^{-1}X_I) = -\frac{1}{g_m}\left(\widetilde{Q}_I + 2c_{IJK}u^J u^K + 2\epsilon w R^2 \xi_I\right) \ . \tag{3.86d}$$

We refer to the set of first order differential equations (3.86a-3.86d) as the attractor flow equations for black hole solutions to the theory (3.1).

3.3.4 Solution of the attractor flow equations

The attractor flow equations (3.86a-3.86d) are first order differential equations. In this subsection we discuss the boundary conditions needed to specify their solutions completely. This turns out to be surprisingly subtle. We then solve the equations using perturbative expansions.

3.3.4.1 Boundary conditions

The attractor flow equations (3.86a-3.86d) are first order differential equations with ξ_I and $\tilde{Q}_I$ as given parameters. As such the superficial expectation is that the specification of all the unknown functions $u^I, g_m, w, f^{-1}X_I$ at any coordinate R^2 yields the corresponding derivatives at that position. Further iterations should then be sufficient to reconstruct the entire radial dependence, at least in principle. We seek to implement this strategy starting from either asymptotically AdS_5, or from the horizon. We consider each in turn.

For a solution to be asymptotically AdS_5, the metric ansatz (3.69) requires the leading order behavior $f \to R^0$, $g_m \to R^2$, and $w \to R^2$ as $R \to \infty$. With these boundary conditions for f, g_m, and w, (3.86a) and (3.86d) yield $u^I \to c^{IJK}\xi_J\xi_K R^2$ and $X_I \to \xi_I \cdot R^0$ for the matter fields as $R \to \infty$.

Alternatively, we can impose boundary conditions at the horizon of the black hole. There, the near-horizon geometry has a manifest AdS_2 factor of the form

$$ds_2^2 = R^4 dt^2 - \frac{dR^2}{R^2} \ , \tag{3.87}$$

and so $g_m \to R^0$, $f \to R^2$, and $w \to R^{-2}$. With these leading asymptotics, the attractor flow equations (3.86a) and (3.86d) determine $u^I \to R^0$ and $X_I \to R^0$ near the horizon.

Staying with our superficial expectation, we would start from either asymptotically AdS_5 or from the AdS_2 horizon. Mathematically, it could be a concern that the differential equations are coupled and non-linear, because then the expansions might fail to converge. This situation is most likely incompatible with a black hole solution that is regular throughout the entire flow from asymptotically AdS_5 to the AdS_2 horizon, or *vice versa*. Nonlinearity does not appear to pose a conceptual challenge.

In order to study the unknown functions u^I, g_m, w and $f^{-1}X_I$ around a regular point, we multiply by an appropriate factor of R^2. Near the horizon we consider R^2u^I, R^2g_m, R^2w, and $R^2f^{-1}X_I$. At infinity, we expand the functions $R^{-2}u^I, R^{-2}g_m, R^{-2}w$ and $R^{-2}f^{-1}X_I$. After such rescalings the left hand sides of each of the attractor flow equations (3.86a-3.86d) will take the form

$$\left(\partial_{R^2} + \frac{\alpha+\beta}{R^2}\right) P = R^{-2\beta}\left(\partial_{R^2} + \frac{\alpha}{R^2}\right)(R^{2\beta}P), \tag{3.88}$$

for some field P and some integers α and β that can either be positive or negative. The challenge we will encounter repeatedly is that, when $P \sim R^{-2(\alpha+\beta)}$, this expression vanishes. We refer to this situation as a *zero-mode* of the perturbative expansion. What it means is that an attractor flow equation does not reveal a derivative, contrary to expectation. Instead, it yields a constraint between the unknown functions on the right hand side of the equation in question. This constraint will be nonlinear and, in general, difficult to implement. In other words, the initial value problem, at both the horizon and at asymptotic infinity, turns out to be unexpectedly complicated.

In the following subsections we develop this general theme explicitly, first starting from the horizon, and then from asymptotic infinity. We subsequently merge the two perturbative

	$R \to \infty$	$R \to 0$
g_m	R^2	R^0
f	R^0	R^2
w	R^2	R^{-2}
u^I	R^2	R^0
X_I	R^0	R^0

Table 3.1: Asymptotics of the various functions

expansions to seek a global understanding.

3.3.4.2 Perturbative solution starting from the horizon

To satisfy the regularity conditions at the horizon, we take $\beta = 1$ in (3.88) and rewrite the attractor flow equations (3.86a-3.86d) as

$$\partial_{R^2}(R^2 u^I) = \frac{1}{2}\epsilon c^{IJK}(R^2 f^{-1} X_J)\xi_K \ , \tag{3.89a}$$

$$\left(\partial_{R^2} + \frac{1}{R^2}\right)(R^2 g_m) = 2 + \frac{2}{R^2}\epsilon\xi_I(R^2 u^I) \ , \tag{3.89b}$$

$$\left(\partial_{R^2} - \frac{2}{R^2}\right)(R^2 w) = -\frac{1}{2}R^{-2}(R^2 f^{-1} X_I)\left(\partial_{R^2} - \frac{2}{R^2}\right)(R^2 u^I) \ , \tag{3.89c}$$

$$\left(\partial_{R^2} - \frac{1}{R^2}\right)(R^2 f^{-1} X_I) = -R^{-2} g_m^{-1}\left(\widetilde{Q}_I + 2R^{-4} c_{IJK}(R^2 u^J)(R^2 u^K) + 2\epsilon(R^2 w)\xi_I\right) \ . \tag{3.89d}$$

We then expand the unknown functions near the horizon. Since the radial dependence is of the form R^{2n} where n is some integer, the expansion can be written as

$$R^2 u^I = \sum_{n=1}^{\infty} u^I_{(n)} R^{2n} \ , \tag{3.90a}$$

$$R^2 g_m = \sum_{n=1}^{\infty} g_{m,(n)} R^{2n} \ , \tag{3.90b}$$

$$R^2 w = \sum_{n=0}^{\infty} w_{(n)} R^{2n} \ , \tag{3.90c}$$

$$R^2 f^{-1} X_I = \sum_{n=0}^{\infty} x_{I,(n)} R^{2n} \ . \tag{3.90d}$$

With the asymptotic structure of Table 3.1, the expansions for $R^2 u^I$ and $R^2 g_m$ do not start with a constant term. Moreover, with the horizon expansions above, the differential operators on the left hand sides of (3.89c) and (3.89d) are such that the coefficients $w_{(2)}$ and $x_{(1)}$ drop out. These coefficients are the zero-modes that make the initial value problem more complicated. There are no analogous zero-modes for u^I and g_m.

To study the structure of the attractor equations (3.89a-3.89d), we temporarily treat the unknown scalar field $f^{-1}X_I$ as a given function of the radial coordinate R. Then the linear flow equation (3.89a), which is sourced by $f^{-1}X_I$, yields all the coefficients $u^I_{(n)}$ in terms of $x_{I,(n)}$. At this point we know both of the functions $f^{-1}X_I$ and u^I and then the attractor flow equation (3.89b) similarly yields the series coefficients $g_{m,(n)}$ in terms of $x_{I,(n)}$. Given all of $f^{-1}X_I$, u^I, and g_m, it would seem straightforward to exploit (3.89c) and find all the coefficients $w_{(n)}$ in terms of $x_{I,(n)}$. This mostly works, but the zero-mode $w_{(2)}$ can *not* be determined this way. That is the obstacle where, as advertized, the derivatives are such that an expansion coefficient simply drops out.

The final flow equation (3.89d), due to the conserved charge Q_I (3.85), is crucial for the complete story. Assuming for a moment that the zero mode $w_{(2)}$ is given as an initial condition, along with the entire function $f^{-1}X_I$, this equation determines the expansion parameters $x_{I,(n)}$ in terms of the $x_{I,(n)}$ themselves, and so the entire system would appear to be solved. However, this last equation also has a zero mode $x_{I,(1)}$ which cannot be determined by the iterative procedure. In short, a careful analysis must be considered and accordingly, this is what we proceed to do now: we solve the expansion coefficients order by order, following the procedure we have outlined.

First, inserting the expansions (3.90a) and (3.90d) into the flow equations (3.89a), we find

$$\sum_{n=1}^{\infty} n u^I_{(n)} R^{2n-2} = \frac{\epsilon}{2} \sum_{n=0}^{\infty} c^{IJK} \xi_K x_{J,(n)} R^{2n} \ . \tag{3.91}$$

Comparing each order in R^2 leads to a relation between the coefficients $u^I_{(n)}$ and $x_{I,(n)}$:

$$u^I_{(n)} = \frac{\epsilon}{2n} c^{IJK} \xi_K x_{J,(n-1)} \ , \quad n \geq 1 \ . \tag{3.92}$$

We insert this result in the flow equation (3.89b) for the expansion of g_m (3.90b) and find

$$\sum_{n=1}^{\infty} (n+1) g_{m,(n)} R^{2n-2} = 2 + \sum_{n=1}^{\infty} \frac{1}{n} c^{IJK} \xi_I \xi_J x_{K,(n-1)} R^{2n-2} \quad . \tag{3.93}$$

Thus all expansion coeficients of g_m can be expressed in terms of the $x_{I,(n)}$:

$$g_{m,(n)} = \frac{1}{n+1} \left(2\delta_{n,1} + \frac{1}{n} c^{IJK} \xi_I \xi_J x_{K,(n-1)} \right) , \quad n \geq 1 \, . \tag{3.94}$$

The steps we have taken so far leads us to express the functions u^I and g_m solely in terms of $f^{-1}X_I$. This was expected from the general discussion.

We then start with (3.89c) and use the expansions (3.90a-3.90d) to obtain

$$\sum_{n=0}^{\infty}(n-2)w_{(n)}R^{2n-2} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(n-1-k)x_{I,(k)}u^I_{(n+1-k)}\right)R^{2n-2} \ . \tag{3.95}$$

Comparing powers of R^2 we find

$$(n-2)w_{(n)} = -\frac{\epsilon}{4}\sum_{k=0}^{n}\frac{n-1-k}{n+1-k}c^{IJK}\xi_I x_{J,(k)}x_{K,(n-k)} \ . \tag{3.96}$$

We see explicitly that the differential equation (3.89c) fails to express the zero mode $w_{(2)}$ in the expansion (3.90c) in terms of other data. However, the right hand side of (3.96) still reveals important information at $n=2$ since it imposes a constraint on the $x_{I,(n)}$ expansion coefficients

$$0 = \frac{\epsilon}{3}c^{IJK}x_{I,(0)}x_{J,(2)}\xi_K \ . \tag{3.97}$$

Thus determination of $w_{(2)}$ is replaced by a constraint on the functions $f^{-1}X_I$ which we have considered given so far.

To make further progress it remains to study the constants of motion due to the conservation (3.85) of electric charge and angular momentum. The electric charge (3.89d) yields

$$\begin{aligned} &-\sum_{n=1}^{\infty}\sum_{k=1}^{n}(n-1-k)g_{m,(k)}x_{I,(n-k)}R^{2n-2} \\ &\qquad = \widetilde{Q}_I + 2c_{IJK}\sum_{n=1}^{\infty}\sum_{k=1}^{n}u^J_{(k)}u^K_{(n+1-k)}R^{2n-2} + 2\epsilon\xi_I\sum_{n=0}^{\infty}w_{(n)}R^{2n} \ . \end{aligned} \tag{3.98}$$

Comparing each power of R^2 we find the sequence of relations

$$-\sum_{k=1}^{n+1}(n-k)g_{m,(k)}x_{I,(n-k+1)} = \widetilde{Q}_I\delta_{n,0} + 2c_{IJK}\sum_{k=1}^{n+1}u^J_{(k)}u^K_{(n+2-k)} + 2\epsilon\xi_I w_{(n)} \ , \tag{3.99}$$

where it is understood in this relation that the coefficients $g_{m,(k)}$, $u^I_{(k)}$, and $w_{(k\neq 2)}$ depend on the $x_{I,(k)}$ through (3.92), (3.94), and (3.96). Thus the relations (3.99) constrain the given functions $f^{-1}X_I$ significantly. Unfortunately, these constraints are nonlinear and difficult to solve.

For the constant order in the R^2 expansion, we take $n = 0$ in (3.99) and find the electric charge

$$\widetilde{Q}_I = x_{I,(0)}\left(1 + \tfrac{1}{2}c^{JKL}x_{J,(0)}\xi_K\xi_L\right) - \tfrac{1}{2}c_{IJK}c^{JML}c^{KNP}\xi_M\xi_N x_{L,(0)}x_{P,(0)} + \tfrac{1}{4}\xi_I c^{LMN}x_{L,(0)}x_{M,(0)}\xi_N \ . \tag{3.100}$$

The charges $\widetilde{Q}_I$ depend only on the scalar fields at the horizon $x_{I,(0)}$ and the FI-parameters ξ_I. In fact, if positivity conditions are imposed on the $x_{I,(0)}$, the $x_{I,(0)}$ in (3.100) can be inverted in terms of the $\widetilde{Q}_I$, allowing to replace the pair of inputs $(\xi_I, x_{I,(0)})$ by the pair $(\xi_I, \widetilde{Q}_I)$. This fits nicely with the understanding of the physical inputs and charges that go in defining the radial flow at every radial hypersurface.

To rewrite (3.100) in a more canonical form, we simplify the second term involving a triple product of c_{IJK} by contracting (A.11) with $\xi_M\xi_Q x_{L,(0)}x_{P,(0)}$. This gives

$$\widetilde{Q}_I = x_{I,(0)} - \frac{1}{2}\xi_I\left(\frac{1}{2}c^{JKL}x_{J,(0)}x_{K,(0)}\xi_L\right) + \frac{1}{2}c_{IJK}\left(\frac{1}{2}c^{JNO}\xi_N\xi_O\right)c^{KLM}x_{L,(0)}x_{M,(0)} \ . \tag{3.101}$$

This expression makes contact with the form of the charge given in [89].[4] The improvement in our work is that we introduce the charge independently of the radial coordinate so it can be computed at any hypersurface we choose, which — in this case — is the black hole horizon.

Before analyzing the consequences of electric charge conservation (3.99) for $n \geq 1$, we consider the analogous equations due to conservation of the black hole angular momentum J (3.85). As the first line of (3.85) involves the scalar field X^I, we must recast it in terms of the scalar field with a lowered index, as our expansion (3.90d) dictates. Utilizing (A.14), we have

$$f^{-3}fX^I = \frac{1}{2}c^{IJK}(f^{-1}X_J)(f^{-1}X_K) \ . \tag{3.103}$$

[4]The equation agrees with Q_I given in (3.53) of [89] with the following map between notations:

$$q_I = \frac{1}{3}x_{I,(0)}, \ \bar{X}_I = \frac{1}{3}\ell\xi_I \ , \ \bar{X}^I = \frac{1}{2}\ell^2 c^{IJK}\xi_J\xi_K \ , \ Q_{\text{there}} = \frac{\pi}{4G}\widetilde{Q}_{\text{here}} \ . \tag{3.102}$$

Introducing the near-horizon expansions (3.90a-3.90d) we find

$$\begin{aligned}\widetilde{J}\delta_{n,0} = \widetilde{Q}_I u^I_{(n+1)} &+ \frac{2}{3}c_{IJK}\sum_{k=1}^{n+1}\sum_{\ell=1}^{k} u^I_{(\ell)}u^J_{(k+1-\ell)}u^K_{(n+2-k)} - \sum_{k=0}^{n}\sum_{\ell=0}^{n-k} g_{m,(k+1)}x_{I,(\ell)}u^I_{(n+1-\ell-k)} \\ &- 2\sum_{k=0}^{n} g_{m,(k+1)}w_{(n-k)} + \frac{\epsilon}{4}c^{IJK}\xi_I\sum_{k=0}^{n}\sum_{\ell=0}^{n-k} g_{m,(k+1)}x_{J,(\ell)}x_{K,(n-k-\ell)} \\ &+ 2w_{(n)} + 2\epsilon\xi_I\sum_{k=0}^{n} w_{(k)}u^I_{(n-k+1)} \ . \end{aligned} \tag{3.104}$$

As before, it is understood that $g_{m,(k)}$, $u^I_{(k)}$, and $w_{(k)}$ depends on the $x_{I,(k)}$ according to (3.92), (3.94) and (3.96), and here we also need the explicit form of $\widetilde{Q}_I$ (3.101). Thus angular momentum conservation gives another infinite set of relations between the $x_{I,(k)}$. Unfortunately, they are even more nonlinear than their analogues for conservation of electric charge.

For $n = 0$ (3.104) gives the angular momentum expressed in terms of $x_{I(0)}$ and ξ_I

$$\begin{aligned}\widetilde{J} = &\frac{\epsilon}{4}c^{IJK}x_{I,(0)}x_{J,(0)}\xi_K - \frac{\epsilon}{4}c^{IJK}c^{LMN}\xi_I\xi_N\xi_K x_{J,(0)}x_{L,(0)}x_{M,(0)} \\ &+ \epsilon c_{IJK}c^{ILM}c^{JNO}c^{KPQ}\left(\tfrac{1}{8}x_{L,(0)}x_{P,(0)}x_{Q,(0)}\xi_N\xi_O\xi_M + \tfrac{1}{12}x_{N,(0)}x_{P,(0)}\xi_M\xi_O\xi_Q\right) \ ,\end{aligned} \tag{3.105}$$

where we have used the value of $\widetilde{Q}_I$ given in (3.101). To make contact with the form of the angular momentum in [89], we rewrite the formula as[5]

$$\widetilde{J} = \frac{\epsilon}{4}c^{IJK}x_{I,(0)}x_{J,(0)}\xi_K + \frac{1}{36}\left(c^{IJK}\xi_I\xi_J\xi_K\right)\left(c^{LMN}x_{L,(0)}x_{M,(0)}x_{N,(0)}\right) \ . \tag{3.107}$$

Again, we are able to express the final result for the conserved charge entirely in terms of near horizon data. Moreover, since $\widetilde{Q}_I$ and $\widetilde{J}$ depend on the same integration constants $x_{I,(0)}$, the charges are indeed not independent of each other.

We now turn to the $n = 1$ component of (3.99), i.e., electric charge conservation at order R^2 away from the horizon. It amounts to

$$g_{m,(2)}x_{I,(0)} = 4c_{IJK}u^J_{(1)}u^K_{(2)} + 2\epsilon\xi_I w_{(1)} \ . \tag{3.108}$$

[5] This agrees with the angular momentum reported in (3.50) of [89] with the following map between conventions:

$$q_I = \frac{1}{3}x_{I,(0)} \ , \ \bar{X}_I = \frac{1}{3}\ell\xi_I \ , \ J_{\text{there}} = \frac{\pi}{4G}\widetilde{J}_{\text{here}} \ . \tag{3.106}$$

The absence of $x_{I,(1)}$ in this equation is due to the zero-mode in (3.90d). However, a constraint on the $x_{I,(1)}$ will follow, in analogy with the zero-mode $w_{(2)}$ giving the condition (3.97). The values of $g_{m,(2)}$, $u^I_{(1)}$, $u^I_{(2)}$, and $w_{(1)}$ from (3.92), (3.94) and (3.96) give the vector relation

$$\left[\frac{1}{6}c^{JKL}\xi_K\xi_L x_{I,(0)} - \frac{1}{2}c_{IKM}c^{JLM}c^{KNP}\xi_L\xi_N x_{P,(0)} + \frac{1}{2}c^{JKL}\xi_K x_{L,(0)}\xi_I\right] x_{J,(1)} = 0 \ . \qquad (3.109)$$

We see that $x_{I,(1)}$ is constrained even though it is a zero-mode of the differential operator. Using the cubic condition on the c_{IJK} (A.12), we can show that the matrix in square brackets has the null vector

$$x_{I,(1)} = \ell\xi_I \ . \qquad (3.110)$$

It is unique, at least for generic structure constants c_{IJK} and generic charges, which are parametrized by $x_{I,(0)}$. The constraint (3.109) does not determine the overall normalization. However, the scale of the radial coordinate R^2 is arbitrary from the near horizon point of view, so the choice (3.110) involves no loss of generality.

The $n = 2$ component of (3.99) gives another vector-valued relation

$$T_I^J x_{J,(2)} = 2\epsilon\left(w_{(2)} + \frac{\epsilon}{2\ell}\right)\xi_I \ , \qquad (3.111)$$

where we have simplified using (3.110) and

$$T_I^J = -\left(1 + \frac{1}{2}c^{KLM}\xi_K\xi_L x_{M,(0)}\right)\delta_I^J + \frac{1}{12}c^{JKL}\xi_K\xi_L x_{I,(0)} - \frac{1}{3}c_{IKL}c^{KMJ}c^{LNP}\xi_M\xi_N x_{P,(0)} \ . \qquad (3.112)$$

The matrix T_I^J is invertible so, given the inputs $\xi_I, x_{I,(0)}, w_{(2)}$, the coefficient $x_{J,(2)}$ is completely determined by (3.111). However, the value of $x_{J,(2)}$ computed this way fails to satisfy the previously established constraint (3.97). This apparent contradiction can be avoided only if

$$x_{I,(2)} = 0 \ . \qquad (3.113)$$

Because the left hand side of (3.111) vanishes, the right side requires

$$w_{(2)} = -\frac{\epsilon}{2\ell} \ . \qquad (3.114)$$

At this point, we can finally consider generic components of the electric charge conservation (3.99), i.e. the infinite set of equations $n \geq 3$. The coefficients $w_{(n\geq 3)}$ can be eliminated using (3.96) for $w_{(n)}$. The $u^I_{(n)}$ and $g_{m,(n)}$ are similarly traded for $x_{I,(n)}$, this time using (3.92) and (3.94). For all $n \geq 3$ this gives

$$
\begin{aligned}
&-\sum_{k=0}^{n} \frac{n-1-k}{k+2}\left(2\delta_{0,k} + \frac{1}{k+1} c^{JKL} \xi_J \xi_K x_{L,(k)}\right) x_{I,(n-k)} \\
&= \frac{1}{2} c_{IJK} c^{JLM} c^{KNP} \xi_L \xi_N \sum_{k=0}^{n} \frac{1}{(k+1)(n-k+1)} x_{M,(k)} x_{P,(n-k)} \\
&\quad - \frac{1}{2(n-2)} \xi_I \sum_{k=0}^{n} \frac{n-1-k}{n+1-k} c^{JKL} \xi_J x_{K,(k)} x_{L,(n-k)} \ .
\end{aligned}
\tag{3.115}
$$

This messy expression can be reorganized as a recurrence relation giving $x_{I,(n)}$ in terms of the preceding $x_{I,(0\leq k\leq n-1)}$:

$$
\begin{aligned}
&\Big[-\frac{n-1}{2}\left(2 + c^{KLM} \xi_K \xi_L x_{M,(0)}\right) \delta_I^J + \frac{1}{(n+1)(n+2)} c^{JKL} \xi_K \xi_L x_{I,(0)} \\
&- \frac{1}{n+1} c_{IKL} c^{KMJ} c^{LPQ} \xi_M \xi_P x_{Q,(0)} - \frac{1}{(n-2)(n+1)} c^{JKL} \xi_K x_{L,(0)} \xi_I \Big] x_{J,(n)} \\
&= \sum_{k=1}^{n-1} \Big[\frac{(n-1-k)}{(k+1)(k+2)} c^{JKL} \xi_J \xi_K x_{L,(k)} x_{I,(n-k)} + \frac{1}{2} c_{IJK} c^{JLM} c^{KNP} \xi_L \xi_N x_{M,(k)} x_{P,(n-k)} \\
&\quad - \frac{1}{2(n-2)} \xi_I \frac{n-1-k}{n+1-k} c^{JKL} \xi_J x_{K,(k)} x_{L,(n-k)} \Big] \ .
\end{aligned}
\tag{3.116}
$$

The left hand side can be inverted, at least for some specific models of c_{IJK}, such as the STU model ($c_{IJK} = |\epsilon_{IJK}|$ for I, J, K running from 1 to 3). In such cases the recurrence relation (3.116) determines all higher-order $x_{I,(n\geq 3)}$ in terms of the coefficients $x_{I,(0)}$, $x_{I,(1)}$ and $x_{I,(2)}$ as well as the FI-parameters ξ_I. In fact, the constraints (3.110) and (3.113) from low n will be sufficient to show that the series *truncates* at $n = 2$. This is discussed in subsection 3.3.4.4.

At this point we have exhausted the information that comes from the conservation of electric charge $\widetilde{Q}_I$ charge in (3.99). We did not yet study the $\widetilde{J}$ conservation relations (3.104). As noted already, the constant order $n = 0$ determines the angular momentum from a near horizon perspective. We have worked out the first few orders $n \geq 1$ and found either redundant relations, involving already known coefficients such as $x_{I,(0)}$, ξ_I and $x_{I,(1)}$, or relations that tie together higher order $x_{I,(k\geq 2)}$ with lower-order ones. We do not foresee any further constraints due to the $\widetilde{J}$ relations.

In summary, starting from the near horizon region, we have exploited supersymmetry and found the entire black hole solution. The fields u^I, g_m, w and $f^{-1}X_I$ are reported in (3.92), (3.94), (3.96) and (3.111). Additionally, we computed the electric charges $\widetilde{Q}_I$ and the angular momenta in terms of the horizon values of the scalars $x_{I,(0)}$, and the subleading coefficients $x_{I,(1)}$ which, according to (3.110), coincide with the FI-parameters ξ_I.

3.3.4.3 Perturbative solution starting from asymptotic AdS

We now adapt the approach from the previous subsection and expand the unknown functions u^I, g_m, w and $f^{-1}X_I$ at large R, near the asymptotic AdS$_5$ boundary.

Given the asymptotic behaviors listed in Table 3.1, regularity requires taking $\beta = -1$ in (3.88). We then recast the flow equations (3.86a–3.86d) as

$$\left(\partial_{R^2} + \frac{2}{R^2}\right)(R^{-2}u^I) = \frac{1}{2}\epsilon c^{IJK}(R^{-2}f^{-1}X_J)\xi_K \ , \tag{3.117a}$$

$$\left(\partial_{R^2} + \frac{3}{R^2}\right)(R^{-2}g_m) = \frac{2}{R^4} + \frac{2}{R^2}\epsilon\xi_I(R^{-2}u^I) \ . \tag{3.117b}$$

$$\partial_{R^2}(R^{-2}w) = -\frac{1}{2}R^2(R^{-2}f^{-1}X_I)\partial_{R^2}(R^{-2}u^I) \ , \tag{3.117c}$$

$$\begin{aligned}\left(\partial_{R^2} + \frac{1}{R^2}\right)(R^{-2}f^{-1}X_I) = -\frac{R^{-8}}{(R^{-2}g_m)}\Big(&\widetilde{Q}_I + 2c_{IJK}R^4(R^{-2}u^J)(R^{-2}u^K) \\ &+2\epsilon R^4(R^{-2}w)\xi_I\Big) \ .\end{aligned} \tag{3.117d}$$

We define the perturbative expansions at infinity as

$$R^{-2}u^I = \sum_{n=0}^{\infty} \bar{u}^I_{(n)}R^{-2n} \ , \tag{3.118a}$$

$$R^{-2}g_m = \sum_{n=0}^{\infty} \bar{g}_{m,(n)}R^{-2n} \ , \tag{3.118b}$$

$$R^{-2}w = \sum_{n=0}^{\infty} \bar{w}_{(n)}R^{-2n} \ , \tag{3.118c}$$

$$R^{-2}f^{-1}X_I = \sum_{n=1}^{\infty} \bar{x}_{I,(n)}R^{-2n} \ . \tag{3.118d}$$

The bar distinguishes the expansion coefficients at the asymptotically AdS$_5$ boundary from their analogues at the horizon.

As before, we initially specify the entire series $\bar{x}_{I,(n)}$. Additionally, examination of (3.117a–3.117d) shows that $\bar{u}^I_{(2)}$, $\bar{g}_{m,(3)}$, $\bar{w}_{(0)}$, and $\bar{x}_{I,(1)}$ do not appear on the left hand sides of the equations. These are the zero modes that we also regard as inputs, at least provisionally. Among the zero-modes, we can determine $\bar{x}_{I,(1)}$ from the outset because they give the asymptotic values of the scalars

$$\bar{x}_{I,(1)} = \ell \xi_I \,, \tag{3.119}$$

as we found in (3.13), by extremizing the potential of gauged supergravity.

We now proceed to solve for the expansion coefficients of each variable, order by order. Starting with the u^I flow equation (3.86a), and using the expansions (3.118a) and (3.118d), we find

$$\sum_{n=0}^{\infty}(2-n)\bar{u}^I_{(n)} R^{-2n-2} = \frac{1}{2}\epsilon c^{IJK}\xi_J \sum_{n=1}^{\infty} \bar{x}_{K,(n+1)} R^{-2n-2} \quad . \tag{3.120}$$

Comparing inverse powers of R^2, we find $\bar{u}^I_{(n)}$ for $n \neq 2$:

$$(2-n)\bar{u}^I_{(n)} = \frac{1}{2}\epsilon c^{IJK}\xi_J \bar{x}_{K,(n+1)} \ , \quad n \geq 0\,. \tag{3.121}$$

The zero mode $\bar{u}^I_{(2)}$ drops out of the equation. Instead, we find a vectorial constraint on $x_{I,(3)}$

$$c^{IJK}\xi_J \bar{x}_{K,(3)} = 0 \ \ . \tag{3.122}$$

It has the obvious solution

$$\bar{x}_{I,(3)} = 0 \ , \tag{3.123}$$

for all values of I. This solution is unique if the matrix $c^{IJK}\xi_J$ is non-singular. In (A.27), we show that it is indeed invertible

$$(c^{IJK}\xi_J)^{-1} = \frac{1}{2}\ell^3 \left(c_{IJK} c^{JLM} \xi_L \xi_M - \xi_I \xi_K \right) \quad . \tag{3.124}$$

Next, we consider the g_m flow equation (3.86b). The expansions (3.118a) and (3.118b) give

$$\sum_{n=0}^{\infty}(3-n)\bar{g}_{m,(n)} R^{-2n-2} = 2R^{-4} + 2\epsilon \xi_I \sum_{n=0}^{\infty} \bar{u}^I_{(n)} R^{-2n-2}\,. \tag{3.125}$$

The expansion coefficients $\bar{g}_{m,(n)}$ — with the exception of the zero mode $g_{m,(3)}$ — can be expressed in terms of $x_{I,(n)}$ and the zero mode $u^I_{(2)}$ as

$$(3-n)\bar{g}_{m,(n)} = \begin{cases} 2\delta_{1,n} + \frac{1}{2-n} c^{IJK}\xi_I \xi_J \bar{x}_{K,(n+1)} & n \neq 2, \\ 2\epsilon \xi_I \bar{u}^I_{(2)} & n = 2\,. \end{cases} \tag{3.126}$$

In compensation for not determining $\bar{g}_{m,(3)}$, we find the constraint $\xi_I \bar{u}^I_{(3)} = 0$. Rewriting this constraint using (3.121) gives a projection on $x_{I,(4)}$

$$c^{IJK}\xi_I\xi_J\bar{x}_{K,(4)} = 0 \ . \tag{3.127}$$

This constraint is a real special geometry scalar, unlike the vector-valued condition (3.122). It will nevertheless prove useful when simplifying results at large R.

We now turn to the nonlinear flow equation for w (3.86c). After using the expansions (3.118a), (3.118c) and (3.118d), we find

$$\sum_{n=1}^{\infty}(n-1)\bar{w}_{(n-1)}R^{-2n} = -\frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=1}^{n}(n-k)\bar{x}_{I,(k)}\bar{u}^I_{(n-k)}R^{-2n} \ , \tag{3.128}$$

and so

$$(n-1)\bar{w}_{(n-1)} = -\frac{1}{2}\sum_{k=1}^{n}(n-k)\bar{x}_{I,(k)}\bar{u}^I_{(n-k)} \ , \qquad n \geq 1\,. \tag{3.129}$$

For $n = 1$, the left hand side vanishes, so the zero-mode $\bar{w}_{(0)}$ is undetermined. The right hand side also vanishes for $n = 1$ so in this case the equation with a zero-mode offers no additional information. We omit the $n = 1$ case and rewrite (3.129) to

$$\bar{w}_{(n)} = -\frac{1}{2n}\sum_{k=1}^{n}(n+1-k)\bar{x}_{I,(k)}\bar{u}^I_{(n+1-k)} \ , \qquad n \geq 1 \ . \tag{3.130}$$

We have refrained from eliminating $\bar{u}^I_{(k)}$ in favor of $\bar{x}_{I,(n)}$ via (3.121) because, generally, the equation involves the zero mode $\bar{u}^I_{(2)}$ which cannot be removed this way.

The final flow equation (3.117d) was derived by combining supersymmetry with conservation of electric charge. Using the expansions (3.118a–3.118d), we find

$$\begin{aligned} &R^{-4}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\bar{g}_{m,(k)}(n-k)\bar{x}_{I,(n+1-k)}R^{-2n} \\ &= R^{-4}\widetilde{Q}_I\delta_{n,2} + 2c_{IJK}R^{-4}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\bar{u}^J_{(k)}\bar{u}^K_{(n-k)}R^{-2n} + 2\epsilon\xi_I R^{-4}\sum_{n=0}^{\infty}\bar{w}_{(n)}R^{-2n} \ . \end{aligned} \tag{3.131}$$

For all $n \geq 0$, this gives

$$\sum_{k=0}^{n}\bar{g}_{m,(k)}(n-k)\bar{x}_{I,(n+1-k)} = \widetilde{Q}_I\delta_{n,2} + 2c_{IJK}\sum_{k=0}^{n}\bar{u}^J_{(k)}\bar{u}^K_{(n-k)} + 2\epsilon\xi_I\bar{w}_{(n)} \ . \tag{3.132}$$

Again, we have chosen to maintain (3.132) as implicit functions of $\bar{x}_{I,(n)}$ due to the presence of the zero modes.

It is worth examining the first few orders of (3.132) in detail. The $n = 0$ component of (3.132) gives

$$\xi_I \bar{w}_{(0)} = -\epsilon c_{IJK} \bar{u}^J_{(0)} \bar{u}^K_{(0)} = -\frac{\epsilon}{16} c_{IJK} c^{JLM} c^{KNP} \bar{x}_{L,(1)} \xi_M \bar{x}_{N,(1)} \xi_P = -\frac{\epsilon}{2\ell} \xi_I \ , \tag{3.133}$$

where we have used (3.119). Thus, it provides the value of the zero mode $w_{(0)}$

$$\bar{w}_{(0)} = -\frac{\epsilon}{2\ell} \, . \tag{3.134}$$

At the next order, the $n = 1$ component of (3.132) is redundant as it just confirms the value for $\bar{w}_{(1)}$ already obtained from (3.130).

The $n = 2$ component of (3.132) is particularly important, because it relates the electric charge to the expansion parameters at infinity

$$\begin{aligned} \widetilde{Q}_I = \bar{x}_{I,(2)} - \frac{1}{2} \xi_I \left(\frac{1}{2} c^{JKL} \bar{x}_{J,(2)} \bar{x}_{K,(2)} \xi_L \right) + \frac{1}{2} c_{IJK} \left(\frac{1}{2} c^{JNO} \xi_N \xi_O \right) c^{KLM} \bar{x}_{L,(2)} \bar{x}_{M,(2)} \\ + \epsilon \ell \left(\xi_I \xi_J - c_{IJK} c^{KLM} \xi_L \xi_M \right) \bar{u}^J_{(2)} \, , \end{aligned} \tag{3.135}$$

where we imposed (3.119) and (3.123) and recast the charge in a form similar to (3.101). Since the electric charge is conserved, the expression (3.135) for $\widetilde{Q}_I$, written in terms of the expansion parameters at infinity, must be equal to its analogue (3.101) obtained from expansion near the horizon.

We have established that $\widetilde{Q}_I$ at infinity has been determined with the only inputs necessary being the coefficients $\bar{x}_{I,(2)}$, ξ_I and $\bar{u}^I_{(2)}$. We now move on to the components of (3.132) for $n \geq 3$, to establish the recursion relation for the coefficients at infinity.

For $n = 3$, (3.132) simplifies after eliminating the g_m, u^I and w coefficients using (3.121), (3.126) and (3.130) to

$$4\ell^{-2} \bar{x}_{I,(4)} + 2\epsilon \left(\bar{x}_{I,(2)} \xi_J + \frac{1}{3} \xi_I \bar{x}_{J,(2)} - c_{IJK} c^{KLM} \xi_L \bar{x}_{m,(2)} \right) \bar{u}^J_{(2)} = 0 \ . \tag{3.136}$$

This relation indicates that $\bar{x}_{I,(4)}$ is described only with the help of $\bar{x}_{I,(2)}$, ξ_I and $\bar{u}^I_{(2)}$.

Furthermore, we note that for $n \geq 4$, the simplification of (3.132) yields a generalization of (3.136)

$$
\begin{aligned}
&\frac{(n-1)^2}{n-2}\ell^{-2}\bar{x}_{I,(n+1)} \\
&= -(n-1)\left(1+\tfrac{1}{2}(c\cdot\xi\xi\bar{x}_{(2)})\right)\bar{x}_{I,(n)} - 2\epsilon(n-2)\xi_J\bar{u}^J_{(2)}\bar{x}_{I,(n-1)} - (n-3)\bar{g}_{m,(3)}\bar{x}_{I,(n-2)} \\
&\quad - \sum_{k=4}^{n}\frac{n-k}{(k-2)(k-3)}(c\cdot\xi\xi\bar{x}_{(k+1)})\bar{x}_{I,(n+1-k)} + 4c_{IJK}\bar{u}^J_{(2)}\bar{u}^K_{(n-2)} \\
&\quad + 2c_{IJK}c^{JLM}c^{KNP}\xi_L\xi_N\sum_{k=1}^{n-1}{}'\,\frac{\bar{x}_{M,(k+1)}\bar{x}_{P,(n-k+1)}}{4(k-2)(n-k-2)} + \xi_I\sum_{k=2}^{n}{}'\,\frac{n+1-k}{n-1-k}\frac{(c\cdot\xi\bar{x}_{(k)}\bar{x}_{(n-k+2)})}{2n}\,,
\end{aligned}
\tag{3.137}
$$

where the apostrophes on the summation symbols indicate that we exclude the terms in the sum with vanishing denominators. We have imposed the value of $\bar{x}_{I,(1)}$ as given in (3.119) and products of the form $c\cdot xyz$ indicate special geometry contractions under c^{JKL} of the form $c^{JKL}x_Jy_Kz_L$. The expression (3.137) has been expanded to distinguish contributions coming from $\bar{x}_{I,(n+1)}$, given by the left hand side of (3.137), and $\bar{x}_{I,(2\leq k\leq n)}$, given by the right hand side of the equality in (3.137). It becomes clear that a given $\bar{x}_{I,(n+1)}$ depends only on the expansion parameters $\bar{x}_{I,(1)},\bar{x}_{I,(2)},\ldots,\bar{x}_{I,(n)}$. By recursion, i.e. applying (3.137) repeatedly, we can now determine all $\bar{x}_{I,(4\leq k\leq n)}$ in terms of of $\bar{x}_{I,(2)}$, ξ_I, $\bar{u}^I_{(2)}$ and $\bar{g}_{m,(3)}$.

Lastly, we analyse the conservation of angular momentum $\widetilde{J}$ by expanding the first equation in (3.85) at infinity, making sure to rescale the functions u^I, g_m, w and $f^{-1}X_I$ appropriately

$$
\begin{aligned}
\widetilde{J} &= R^2Q_I(R^{-2}u^I) + \tfrac{2}{3}R^6c_{IJK}(R^{-2}u^I)(R^{-2}u^J)(R^{-2}u^K) - R^8(R^{-2}g_m)\big(2R^{-2}(R^{-2}w) \\
&\quad +(R^{-2}f^{-1}X)\cdot(R^{-2}u) - \tfrac{1}{2}\epsilon R^{-2}f^{-3}\xi\cdot fX\big) + 2R^4(R^{-2}w)(1+\epsilon R^2\xi\cdot(R^{-2}u))\,.
\end{aligned}
\tag{3.138}
$$

This can be expanded like (3.104) in terms of the expansions at infinity (3.118a-3.118b), leading to a relation for the component of R^{-2n}. At constant order ($n=0$), we obtain

$$
\begin{aligned}
\widetilde{J} &= \frac{\epsilon}{4}c^{IJK}\bar{x}_{I,(2)}\bar{x}_{J,(2)}\xi_K + \frac{1}{36}\left(c^{IJK}\xi_I\xi_J\xi_K\right)\left(c^{LMN}\bar{x}_{L,(2)}\bar{x}_{M,(2)}\bar{x}_{N,(2)}\right) + \frac{\epsilon}{\ell}\bar{g}_{m,(3)} \\
&\quad + \ell\left(\frac{1}{2}\xi_Ic^{JKL}\xi_J\xi_K\bar{x}_{L,(2)} + \xi_I + \frac{5}{3}\ell^{-3}\bar{x}_{I,(2)}\right)\bar{u}^I_{(2)}\,,
\end{aligned}
\tag{3.139}
$$

where we have imposed (3.119) and (3.123). This expression at most depends on the inputs $\bar{x}_{I,(2)}$, ξ_I, $\bar{g}_{m,(3)}$ and $\bar{u}^I_{(2)}$. All higher order powers of the $\widetilde{J}$ relation at infinity in (3.138) are redundant because they yield relations between the coefficients that have already been established.

In summary, we have studied the first order equations (3.117a-3.117d) due to supersymmetry and conservation of electric charge, by expanding perturbatively near infinity. Given the asymptotic values of the scalars fields (3.119), as well as the conserved charges $\widetilde{Q}_I$, $\widetilde{J}$ defined by fall-off conditions at infinity, the simplest outcome would have been for supersymmetry to determine the entire black hole geometry. Our finding is much more complicated: all physical fields can be expressed as a perturbative series with expansion parameters that depend not only on ξ_I and $\bar{x}_{I,(2)}$ but also the zero-modes $\bar{u}^I_{(2)}$ and $\bar{g}_{m,(3)}$.

3.3.4.4 Summary and discussion of perturbative solutions

The study in the subsection so far focused on technical details. This was needed because the interplay between supersymmetry, boundary conditions, and conserved charges proved to be rather intricate. We now conclude the subsection with a summary of the final results and discussion of their interpretation.

The black hole solution is parametrized primarily by the matter fields: scalar fields $f^{-1}X_I$, with the prefactor f such that the combination $f^{-1}X_I$ is unconstrained by real special geometry, and the magnetic potentials u^I. Because of supersymmetry, the electric potentials fY^I can be identified with the scalar fields fX^I, see (A.74). Given the matter fields, $f^{-1}X_I$ and u^I, as well as supersymmetry, the geometry is specified by a Kähler base that depends on the function g_m, and a fibre encoding rotation through the potential w. All unknown functions $f^{-1}X_I$, u_I, g_m and w can depend only on a single radial coordinate R^2 and they must satisfy specified first order differential equations (3.86a-3.86d).

Supersymmetry is never sufficient to specify an entire solution, because it is first order, and there is always an integrability condition that is of second order. Taking into account the Noether-Wald procedure, we find a second order constraint that satisfies a Gauss' law that was subject to detailed discussion in section 3.2. With this augmentation, the first order differential equations form a complete system. Angular momentum, with its conservation law also discussed in section 3.2, yields nothing new, except for a formula giving the angular momentum in terms of the same parameters that define the electric charge in the near-horizon expansion. In the case of the asymptotic infinity expansion, the electric charge depends on one of the zero modes $\bar{u}^I_{(2)}$ in addition to $\bar{x}_{I,(2)}$, ξ_I whereas the angular momentum depends on both the zero modes $\bar{g}_{m,(3)}$ and $\bar{u}^I_{(2)}$ as well as $\bar{x}_{I,(2)}$ and ξ_I.

Consistent boundary conditions for the differential equations can be specified at any radius, in principle. They must depend, at the very least, on the FI-coupling constants ξ_I and the electric charges Q_I. We find that, when *starting from the black hole horizon*, this data is sufficient. Because of supersymmetry, these parameters specify the entire near horizon geometry, including the squashing of the horizon due to angular momentum. This explains why the electric charge and the angular momentum are only described with the use of the leading $x_{I,(0)}$ term in $f^{-1}X_I$, but any further subleading information about any of the fields u^I, g_m or w requires subleading contributions away from the horizon, with *derivative* information of the $f^{-1}X_I$ expansion (3.90d) supplied by the $x_{I,(1)}$ coefficients.

The linchpin for establishing this claim about the near horizon expansion is the scalar field. In the series expansion for $R^2 f^{-1}X_I$, we have the constant at the horizon $x_{I,(0)}$ and then at $\mathcal{O}(R^2)$, we have $x_{I,(1)}$. For the third expansion coefficent, we find $x_{I,(2)} = 0$ (3.113). With this starting point, the recursion relation (3.116) shows that all $x_{I,(k\geq 2)}$ actually *vanish*. The fact that the scalar field $f^{-1}X_I$ truncates after the first two terms is the near horizon version of the fact that $f^{-1}X_I$ is a harmonic function, as is familiar from ungauged supergratity.

When analyzing the supersymmetry conditions we provisionally considered $f^{-1}X_I$ an input that all other variables were expressed in terms of. The truncation $x_{I,(k\geq 2)} = 0$ has the immediate effect of truncating $u^I_{(n\geq 3)} = g_{m,(n\geq 3)} = w_{(n\geq 3)} = 0$. The expansions at the horizon (3.90a-3.90d) simplify and we find

$$R^2 u^I = \frac{\epsilon}{2} c^{IJK}\xi_J x_{K,(0)} R^2 + \frac{\epsilon\ell}{4} c^{IJK}\xi_J\xi_K R^4\,, \tag{3.140a}$$

$$R^2 g_m = \left(1 + \frac{1}{2} c^{IJK}\xi_I\xi_J x_{K,(0)}\right) R^2 + \frac{1}{\ell^2} R^4\,, \tag{3.140b}$$

$$R^2 w = -\frac{\epsilon}{8} c^{IJK}\xi_I x_{J,(0)} x_{K,(0)} - \frac{\epsilon\ell}{4} c^{IJK}\xi_I\xi_J x_{K,(0)} R^2 - \frac{\epsilon}{2\ell} R^4\,, \tag{3.140c}$$

$$R^2 f^{-1} X_I = x_{I,(0)} + \ell\xi_I R^2. \tag{3.140d}$$

These expressions exactly match the well-known Gutowski-Reall solution [89], with the appropriate identifications of notation[6]. The electric charge $\widetilde{Q}_I$ (3.101) and the angular momentum $\widetilde{J}$ (3.105) computed in the near horizon expansion similarly agree with the familiar results.

The analogous analysis starting from asymptotic AdS_5 turned out to be less straightforward. Recalling that coefficients starting from infinity are denoted by barred expansion coefficients, we find that, when shooting in (going from infinity towards the horizon), we must not only specify ξ_I and $\bar{x}_{I,(2)}$, but also $\bar{u}_{I,(2)}$ and $\bar{g}_{m,(3)}$. The harmonic function we established at the horizon reproduces the Gutowski-Reall solution and has features in common

[6] u^I, g_m, w and $f^{-1}X_I$ match U^I, g, w and $f^{-1}X_I$ respectively in [89] via:

$$q_I = \frac{1}{3}\bar{x}_{I,(2)}\ ,\ \bar{X}_I = \frac{1}{3}\ell\xi_I\ . \tag{3.141}$$

with their very familiar analogues in ungauged supergravity. At infinity, it corresponds to

$$x_{I,(0)} = \bar{x}_{I,(2)} \ , \quad \bar{u}^I_{(2)} = \bar{g}_{m,(3)} = 0 \ . \tag{3.142}$$

With these special values, the recursion relation (3.137) simplifies greatly

$$\begin{aligned}
&\frac{(n-1)^2}{n-2}\ell^{-2}\bar{x}_{I,(n+1)} \\
&= -(n-1)\left(1+\tfrac{1}{2}(c\cdot\xi\xi\bar{x}_{(2)})\right)\bar{x}_{I,(n)} - \sum_{k=4}^{n}\frac{n-k}{(k-2)(k-3)}(c\cdot\xi\xi\bar{x}_{(k+1)})\bar{x}_{I,(n+1-k)} \\
&\quad + 2c_{IJK}c^{JLM}c^{KNP}\xi_L\xi_N\sum_{k=1}^{n-1}{}' \frac{\bar{x}_{M,(k+1)}\bar{x}_{P,(n-k+1)}}{4(k-2)(n-k-2)} + \xi_I\sum_{k=2}^{n}{}' \frac{n+1-k}{n-1-k}\frac{(c\cdot\xi\bar{x}_{(k)}\bar{x}_{(n-k+2)})}{2n} \ .
\end{aligned} \tag{3.143}$$

Since we already know $\bar{x}_{I,(3)} = 0$ from (3.123), and vanishing $\bar{u}^I_{(2)}$ leads to vanishing $\bar{x}_{I,(4)}$ as well via (3.127), it is not difficult to show that the expansion coefficients $\bar{x}_{I,(k\geq 3)}$ all vanish. Thus the perturbative series for $\bar{x}_{I,(n)}$ truncates after two terms, as expected for a harmonic function. The identification (3.142) identifies the subleading coefficient in the harmonic function at infinity with the leading one at the horizon, and *vice versa*.

While (3.142) are the default, it is interesting that asymptotic boundary conditions with nonzero $\bar{u}_{I,(2)}$, $\bar{g}_{m,(3)}$ are consistent with supersymmetry. It has been argued that there may be missing solutions in certain supergravity theories that may not satisfy the canonical nonlinear charge constraint, see for example [90, 91, 92, 93]. Since the value of the conserved charges do not take the canonical form, one way wonder if those parameters are somehow related to these missing solutions.

From this point of view, the possibility of $\bar{u}_{I,(2)}$, $\bar{g}_{m,(3)}$ perturbing asymptotic AdS$_5$ might be desirable. In the following, we discuss this possibility.

First, recall that the electric charge $\widetilde{Q}_I$ and the angular momentum $\widetilde{J}$ are conserved charges, which means that they are the same whether evaluated at infinity or the horizon. Identifying (3.101) with (3.135) we find

$$\begin{aligned}
&x_{I,(0)} - \frac{1}{2}\xi_I\left(\frac{1}{2}c^{JKL}x_{J,(0)}x_{K,(0)}\xi_L\right) + \frac{1}{2}c_{IJK}\left(\frac{1}{2}c^{JNO}\xi_N\xi_O\right)c^{KLM}x_{L,(0)}x_{M,(0)} \\
&= \bar{x}_{I,(2)} - \frac{1}{2}\xi_I\left(\frac{1}{2}c^{JKL}\bar{x}_{J,(2)}\bar{x}_{K,(2)}\xi_L\right) + \frac{1}{2}c_{IJK}\left(\frac{1}{2}c^{JNO}\xi_N\xi_O\right)c^{KLM}\bar{x}_{L,(2)}\bar{x}_{M,(2)} \\
&\quad + \epsilon\ell\left(\xi_I\xi_J - c_{IJK}c^{KLM}\xi_L\xi_M\right)\bar{u}^J_{(2)} \ ,
\end{aligned} \tag{3.144}$$

from matching $\widetilde{Q}_I$, and similarly (3.107) with (3.139) give

$$\begin{aligned} &\frac{\epsilon}{4} c^{IJK} x_{I,(0)} x_{J,(0)} \xi_K + \frac{1}{36} \left(c^{IJK} \xi_I \xi_J \xi_K \right) \left(c^{LMN} x_{L,(0)} x_{M,(0)} x_{N,(0)} \right) \\ &= \frac{\epsilon}{4} c^{IJK} \bar{x}_{I,(2)} \bar{x}_{J,(2)} \xi_K + \frac{1}{36} \left(c^{IJK} \xi_I \xi_J \xi_K \right) \left(c^{LMN} \bar{x}_{L,(2)} \bar{x}_{M,(2)} \bar{x}_{N,(2)} \right) \\ &\quad + \frac{\epsilon}{\ell} \bar{g}_{m,(3)} + \ell \left(\frac{1}{2} \xi_I c^{JKL} \xi_J \xi_K \bar{x}_{L,(2)} + \xi_I + \frac{5}{3} \ell^{-3} \bar{x}_{I,(2)} \right) \bar{u}^I_{(2)} \ , \end{aligned} \tag{3.145}$$

from matching $\widetilde{J}$. These conservation laws are consistent with a UV solution specified by $x_{I,(0)}$ (and the FI-couplings ξ_I) that flows to an IR configuration with $\bar{x}_{I,(2)}$ that may not even remotely agree with (3.142). This consideration suggests that supersymmetry and charge conservation do little to constrain the IR limit of the flow.

However, there is a different source of intuition. If the perturbative series of $f^{-1} X_I$ from infinity did *not* truncate after exactly two terms, the third term would diverge at the horizon $R^2 \to 0$, rather than approaching a constant. Other fields excited at the same order would similarly suggest a singularity. It could happen that, taking into account successive powers R^{-2k} to all orders, there would be a finite limit $R^2 \to 0$, after all, but determining by explicit computation whether this possibility is realized for any $\bar{u}_{I,(2)}$, $\bar{g}_{m,(3)}$ is technically challenging.

From a different perspective, since the conserved charges from the near-horizon expansion do satisfy the typical charge constraint, the possibly new black hole solutions that do not seem to satisfy the typical charge constraint at infinity, would not flow to the expected near-horizon extremal AdS_2 geometry, which implies that these solutions may not be black holes after all.

Moreover, a change in the electric potential $fY^I \to fY^I + \beta^I$ with β^I constant is trivial as it does not change the electromagnetic field strength. However, with the vielbein we have picked, such a shift must be accompanied by $u^I \to u^I - w\beta^I$. Because w includes a term $w \sim R^{-2}$ at large R, such a gauge transformation has the ability to remove $\bar{u}^I_{(2)}$. This mechanism shows the $\bar{u}^I_{(2)}$ are allowed, in principle, but also that they are not physical deformations. Indeed, these coefficients diverge at the horizon, so they correspond to a singular gauge which is ill-advised.

3.4 Entropy Extremization

In this section, we consider the near-horizon limit of the Legendre transform of the radial Lagrangian (3.32), leading to a near-horizon entropy function. Extremizing this entropy function with respect to the near-horizon variables leads to an expression for the entropy in terms of the aforementioned charges.

3.4.1 Near-horizon setup

First, we consider the near-horizon of the line element ds_2^2 (3.27), where we recall that $e^{2\rho}$ and $e^{2\sigma}$ can be expressed in terms of the variables f, g_m, and w as in (3.75). At the horizon $R \to 0$, these variables have known near-horizon behaviors according to Table 3.1. Thus, the near-horizon limit of (3.27) becomes

$$ds^2_{2,\mathrm{nh}} = v\left(\frac{R^4}{\ell_2^2}dt^2 - \frac{dR^2}{R^2}\right) , \tag{3.146}$$

with v and ℓ_2 defined based on the $R \to 0$ behavior of $e^{2\rho}$ and $e^{2\sigma}$:

$$e^{2\sigma}\Big|_{\mathrm{nh}} \equiv \frac{v}{R^2} , \qquad e^{2\rho}\Big|_{\mathrm{nh}} \equiv \frac{v}{\ell_2^2}R^4 . \tag{3.147}$$

Furthermore, $v^{\frac{1}{2}}$ and $\ell_2^{\frac{1}{3}}$ are near-horizon length scales defining the 2D (t, R) part of the line element (3.74)

$$ds^2_{5,\mathrm{nh}} = v\left(\frac{R^4}{\ell_2^2}dt^2 - \frac{dR^2}{R^2}\right) - e^{-U_1}(\sigma_1^2 + \sigma_2^2) - e^{-U_2}(\sigma_3 + a^0)^2 . \tag{3.148}$$

The role of the variable ℓ_2 is elucidated by noting the near-horizon limit of the Kähler condition (3.76):

$$\ell_2 = 2ve^{-\frac{1}{2}U_2} . \tag{3.149}$$

This relation will be used to eliminate ℓ_2 in the rest of the near-horizon analysis.

Having reviewed the near-horizon 2D line element, and in anticipation of applying the entropy function formalism [94, 95, 96, 86, 97, 98, 99, 100, 101] to the Lagrangian density in (3.32), we use the following coordinate transformation

$$\begin{aligned} dt &\to \frac{1}{2}\ell_2 dt , \\ dR &\to \frac{1}{2R}dR , \end{aligned} \tag{3.150}$$

to bring the coordinates (t, R) in (3.146) to the canonical AdS_2 form

$$ds^2_{2,\mathrm{nh}} = \frac{v}{4}\left(R^2 dt^2 - \frac{dR^2}{R^2}\right) , \tag{3.151}$$

where now it is clear that $v^{\frac{1}{2}}$ is related to the AdS_2 length scale.

The coordinate transformation (3.150) will have the effect of rescaling the Lagrangian

2-form $\mathcal{L}_2 = \mathcal{L}_1 dt \wedge dR$ (3.22) by a factor

$$\mathcal{L}_2 \to \frac{\ell_2}{4R}\mathcal{L}_2 \,. \tag{3.152}$$

With the prescription of defining the entropy function through omitting the $dt \wedge dR$ volume form from the dimensionally-reduced action, we anticipate dividing the density $\mathcal{L}_1$ by a factor of $\frac{4R}{\ell_2}$.

Combining the near-horizon behaviors of $e^{2\rho}$ and $e^{2\rho}$ studied above with the dictionary definitions (3.75), e^{-U_1}, e^{-U_2} and b^I can be shown to be constants to leading order in the near-horizon limit based on the leading-order behaviors of f, g_m and w consistent with the small R asymptotics in Table 3.1.

Concerning the matter fields, the electric fields a^I and a^0 in (3.18) become in the near-horizon limit

$$a^I\Big|_{\rm nh} \equiv \frac{e^I R^2}{2v} e^{\frac{1}{2}U_2} dt, \quad a^0\Big|_{\rm nh} \equiv -\frac{e^0 R^2}{2v} e^{\frac{1}{2}U_2} dt \,. \tag{3.153}$$

The total 1D Lagrangian density (3.32) then becomes

$$\begin{aligned}\mathcal{L}_{1,\rm nh} = \frac{\pi}{2G_5} e^{-U_1 - \frac{1}{2}U_2} \frac{4R}{\ell_2} v \Bigg[\frac{1}{v^2} e^{-U_2} (e^0)^2 - \frac{4}{v} + e^{U_1} - \frac{1}{4} e^{2U_1 - U_2} - \frac{1}{2} G_{IJ} e^{2U_1} b^I b^J \\ + \frac{2}{v^2} G_{IJ} (e^I - b^I e^0)(e^J - b^J e^0) - V \Bigg] - \frac{\pi}{2G_5} \frac{4R}{\ell_2} \frac{1}{2} c_{IJK} b^I b^J (e^K - \frac{2}{3} e^0 b^K) \,.\end{aligned} \tag{3.154}$$

Apart from an overall prefactor in the integration measure, every other appearance of ℓ_2 has been re-expressed in terms of v and U_2 by using (3.149). The $\frac{4R}{\ell_2}$ factor has been factored out of the volume element, and we follow the prescription made earlier to exactly cancel it out with the $\frac{\ell_2}{4R}$ factor from (3.152) in order to obtain the Lagrangian density suitable for the entropy function. We also note the presence of the Chern-Simons boundary terms in (3.154) that are crucial for calculating the near-horizon charges.

We now obtain a near-horizon Lagrangian (3.154) that is a function of the variables v, U_1, U_2, e^I, e^0, and b^I. We will next derive the charges Q_I and J from $\mathcal{L}_{1,\rm nh}$, with the goal of Legendre transforming the Lagrangian into an entropy function $\mathcal{S}$ that can be ultimately extremized towards a function purely of the charges $\mathcal{S} = \mathcal{S}(Q_I, J)$.

We note from the earlier Noether procedure (3.64) that Q_I and J were obtained in terms of radially dependent variables. In terms of the near-horizon limit of the electric fields (3.153), this becomes

$$Q_I = \frac{\pi}{G_5}\left[e^{-U_1-\frac{1}{2}U_2}\frac{4}{v}G_{IJ}(e^J - b^J e^0) - \frac{1}{2}c_{IJK}b^J b^K\right] \ , \tag{3.155}$$

$$J = \frac{\pi}{G_5}\left[-\frac{2}{v}e^{-U_1-\frac{3}{2}U_2}e^0 + \frac{4}{v}e^{-U_1-\frac{1}{2}U_2}G_{IJ}b^I(e^J - b^J e^0) - \frac{1}{3}c_{IJK}b^I b^J b^K\right] \ . \tag{3.156}$$

This introduces the charges Q_I and J as conjugates to the electric fields e^I and e^0, allowing for the inversion

$$e^I - b^I e^0 = \frac{v}{16}e^{U_1+\frac{1}{2}U_2}G^{IJ}\left(\widetilde{Q}_J + 2c_{JKL}b^K b^L\right) \ , \tag{3.157}$$

$$e^0 = -\frac{v}{8}e^{U_1+\frac{3}{2}U_2}\left(\widetilde{J} - \widetilde{Q}_I b^I - \frac{2}{3}c_{IJK}b^I b^J b^K\right) \ , \tag{3.158}$$

where now the rescaled $\widetilde{Q}_I$ and $\widetilde{J}$ (3.65) have been used. The near-horizon entropy function can now be defined as a Legendre transform of the Lagrangian density (3.154) with fixed charges

$$\mathcal{S} = 2\pi\left(e^I\frac{\partial\mathcal{L}_{1,\text{nh}}}{\partial e^I} + e^0\frac{\partial\mathcal{L}_{1,\text{nh}}}{\partial e^0} - \mathcal{L}_{1,\text{nh}}\right) \ , \tag{3.159}$$

which, after eliminating the electric fields through (3.157) and (3.158), yields

$$\begin{aligned}\mathcal{S} = \frac{4\pi^2}{G_5}e^{-U_1-\frac{1}{2}U_2}\Bigg[1 + \frac{v}{4}\Bigg(&\frac{1}{4}e^{2U_1-U_2} - e^{U_1} + \frac{1}{64}e^{2U_1+2U_2}\left(\widetilde{J} - \widetilde{Q}_I b^I - \frac{2}{3}c_{IJK}b^I b^J b^K\right)^2 \\ &+V + \frac{1}{128}e^{2U_1+U_2}G^{IJ}\left(\widetilde{Q}_I + 2c_{IKL}b^K b^L\right)\left(\widetilde{Q}_J + 2c_{JMN}b^M b^N\right) + \frac{1}{2}e^{2U_1}G_{IJ}b^I b^J\Bigg)\Bigg] \ .\end{aligned} \tag{3.160}$$

This entropy function depends on the physical variables v, U_1, U_2, b^I, X^I describing the near horizon geometry and matter fields, with the conserved charges $\widetilde{Q}_I$, $\widetilde{J}$ appearing as fixed parameters. At its extremum, it yields the physical variables and the black hole entropy as a function of the charges.

3.4.2 Extremization of the Entropy Function

It is exceedingly simple to extremize with respect to v which appears only as a Lagrange multiplier in front of the large round bracket that comprises nearly all of (3.160). This leaves the extremized value of $\mathcal{S}$:

$$\mathcal{S} = \frac{4\pi^2}{G_5}e^{-U_1-\frac{1}{2}U_2} \ . \tag{3.161}$$

This is exactly the black hole entropy computed via the area law for a horizon defined by the volume 3-form $e^{-U_1-\frac{1}{2}U_2}\sigma_1\wedge\sigma_2\wedge\sigma_3$ with the angular ranges specified in (3.20). However, the explicit dependence of U_1 and U_2 on the charges remains to be determined. For this we must extremize with respect to the remaining variables

$$\partial_{U_1}\mathcal{S}=\partial_{U_2}\mathcal{S}=\partial_{b^I}\mathcal{S}=D_I\mathcal{S}=0\ . \tag{3.162}$$

Here D_I is the Kähler-covariantized derivative with respect to the scalars X^I. It is defined such that $X^I D_I=0$, which is the correct way to vary the scalars while also implementing the constraint (A.9). The conditions (3.162) give

$$0=V-e^{U_1}+\frac{1}{4}e^{2U_1-U_2}+\frac{1}{64}e^{2U_1+2U_2}\mathcal{M}^2+G^{IJ}\frac{1}{128}e^{2U_1+U_2}\mathcal{K}_I\mathcal{K}_J+\frac{1}{2}G_{IJ}e^{2U_1}b^Ib^J\ , \tag{3.163}$$

$$0=-4-vV+\frac{v}{4}e^{2U_1-U_2}+\frac{v}{64}e^{2U_1+2U_2}\mathcal{M}^2+G^{IJ}\frac{v}{128}e^{2U_1+U_2}\mathcal{K}_I\mathcal{K}_J+\frac{v}{2}G_{IJ}e^{2U_1}b^Ib^J\ , \tag{3.164}$$

$$\begin{aligned}0=&-2-\frac{v}{2}V+\frac{v}{2}e^{U_1}-\frac{3v}{8}e^{2U_1-U_2}+\frac{3v}{128}e^{2U_1+2U_2}\mathcal{M}^2+G^{IJ}\frac{v}{256}e^{2U_1+U_2}\mathcal{K}_I\mathcal{K}_J\\&-\frac{v}{4}G_{IJ}e^{2U_1}b^Ib^J\ ,\end{aligned} \tag{3.165}$$

$$0=vD_IV+\frac{v}{128}e^{2U_1+U_2}(D_IG^{JK})\mathcal{K}_J\mathcal{K}_K+\frac{v}{2}e^{2U_1}(D_IG_{JK})b^Jb^K\ , \tag{3.166}$$

$$0=\frac{1}{32}e^{U_2}\left(-e^{U_2}\mathcal{K}_I\mathcal{M}+2G^{JK}c_{IJN}b^N\mathcal{K}_K\right)+G_{IJ}b^J\ , \tag{3.167}$$

where $\mathcal{M}$ and $\mathcal{K}_I$ are shorthand for

$$\mathcal{M}\equiv\widetilde{J}-\widetilde{Q}_Ib^I-\frac{2}{3}c_{IJK}b^Ib^Jb^K\ ,\ \mathcal{K}_I\equiv\widetilde{Q}_I+2c_{IJK}b^Jb^K\ . \tag{3.168}$$

Additionally, D_I acts on the scalars X^J following[7]

$$D_IX^J=\delta_I^J-\frac{1}{3}X_IX^J\ . \tag{3.170}$$

Ideally we seek the most general extremum that solves (3.163-3.167) that is consistent with our ansatz (3.148). However, the extremization equations are highly nonlinear in the variables of interest (v,U_1,U_2,X^I,b^I). Therefore, in the following we specialize and find all supersymmetric solutions.

[7] This can be generalized to other quantities via the product rule on D_I, for instance,

$$D_IX_J=\frac{1}{2}c_{JKL}D_I(X^KX^L)=c_{IJK}X^K-\frac{2}{3}X_IX_J\ . \tag{3.169}$$

Near-horizon supersymmetric conditions

It is straightforward to take the near-horizon limit of the supersymmetry conditions (3.80-3.82), along with the identification $X^I = Y^I$ (A.74). After inverting e^I and e^0 in terms of $\widetilde{Q}_I$ and $\widetilde{J}$, following (3.157) and (3.158)), we obtain the following near-horizon supersymmetric relations

$$0 = \widetilde{Q}_I + 2c_{IJK}b^J b^K - 4e^{-U_2}X_I \ , \tag{3.171}$$

$$0 = \widetilde{J} - \widetilde{Q}_I b^I - \frac{2}{3}c_{IJK}b^I b^J b^K - 4e^{-U_1-U_2}(\xi\cdot X) \ , \tag{3.172}$$

$$0 = b^I - e^{-U_1}\left(X^I(\xi\cdot X) - G^{IJ}\xi_J\right) \ , \tag{3.173}$$

$$0 = e^{U_1} - \frac{4}{v} - 2V \ . \tag{3.174}$$

We also need the near-horizon version of the Kähler condition (3.76). We can trade the variables ρ and σ describing the 2D geometry for f and g_m following (3.75) and eliminate a_t^0 that results in favor of the charges through (3.158). These steps lead to

$$\frac{4}{v} - (\xi\cdot X)^2 = e^{2U_1-U_2} \ . \tag{3.175}$$

We have verified that when the five supersymmetric relations (3.171-3.175) are satisfied, then the five $\mathcal{S}$ extremization equations (3.163-3.167) are satisfied as well. The details of this computation are not instructive so we omit them. The reverse logic would be that *all* extremal solutions within the scope of our ansatz are supersymmetric. This we have not shown, and it is indeed not true, i.e. there are no known nonextremal supersymmetric Lorentzian black holes. Thus the specialization to supersymmetric solutions addresses a genuine subset of the extremal black holes.

In the remainder of this subsection we solve the supersymmetry relations (3.171-3.175) explicitly and find all variables as functions of the charges Q_I and J.

3.4.2.1 Solving for the entropy and the charge constraint

The supersymmetry conditions (3.171-3.175) are all algebraic, but they are far from trivial. Straightforward contractions, followed by taking simple linear combinations, give scalar identities

$$\begin{aligned} X \cdot b &= e^{-U_1} \xi \cdot X \ , \\ \widetilde{Q} \cdot b &= \frac{3}{2}\widetilde{J} - 8e^{-U_1-U_2}\xi \cdot X \ , \\ \xi \cdot b &= e^{U_1-U_2} - 1 \ , \end{aligned} \tag{3.176}$$

which will prove useful later. Our strategy will be to exploit identities like these to find simple combinations of variables that can be expressed entirely in terms of the charges Q_I, J and the couplings ξ_I. Combinations of those will in turn give explicit formulae for physical variables.

In this spirit, we expand $\widetilde{Q}_J\widetilde{Q}_K$ using the square of (3.171), and then simplify the terms that are products of b's using (3.172). We obtain

$$\frac{1}{2}c^{IJK}\widetilde{Q}_J\widetilde{Q}_K = 32e^{-2U_1-U_2}c^{IJK}\xi_J\xi_K + 16e^{-U_1-U_2}X^I - 2\widetilde{J}b^I \ . \tag{3.177}$$

Contracting (3.177) with ξ_I, the first term on the right becomes proportional to $e^{-2U_1-U_2}$, which is related to the black hole entropy through $\mathcal{S}$ (3.161). There will also be a term $(\xi \cdot X)$ that we can eliminate with the help of

$$\widetilde{J} = 8e^{-2U_1}(\xi \cdot X) + \frac{32}{3}e^{-3U_1}c^{IJK}\xi_I\xi_J\xi_K \ , \tag{3.178}$$

which is a simplification of (3.172) with $\widetilde{Q}_I$ and b^I eliminated using (3.171) and (3.173), respectively. These steps give

$$\frac{1}{6}c^{IJK}\xi_I\xi_J\xi_K\left(\frac{\mathcal{S}}{2\pi}\right)^2 = \frac{1}{2}c^{IJK}\xi_I Q_J Q_K - \frac{\pi}{4G_5}2J \ , \tag{3.179}$$

which amounts to an explicit formula for the black hole entropy as function of the conserved charges

$$\mathcal{S} = 2\pi\sqrt{\frac{1}{2}c^{IJK}\ell^3\xi_I Q_J Q_K - N^2 J} \ . \tag{3.180}$$

This is in full agreement with the entropy of supersymmetric extremal AdS_5 black holes [65, 102, 63]. We have expressed the entropy using the untilded charges Q_I and J (3.65) and traded G_5 for N^2 using $\frac{\pi\ell^3}{4G_5} = \frac{1}{2}N^2$ and applied $\frac{1}{6}c^{IJK}\xi_I\xi_J\xi_K = \ell^{-3}$ from (3.12) to explicitly show all the dimensionful quantities.

Continuing with the strategy of evaluating natural combinations of the conserved charges, we evaluate the cubic invariant of the charges $c^{IJK}\widetilde{Q}_I\widetilde{Q}_J\widetilde{Q}_K$ by taking the cube of $\widetilde{Q}_I$ from

(3.171), with the resulting contractions of X_I and b^I such as $c_{IJK}X^Ib^Jb^K$ and X_Ib^I simplified using the b^I relation (3.173) as well as (3.174) and (3.175) for the terms quadratic in ξ. This yields

$$c^{IJK}\widetilde{Q}_I\widetilde{Q}_J\widetilde{Q}_K = 6e^{-U_1-2U_2} + \frac{1}{8}\widetilde{J}c_{IJK}b^Ib^Jb^K \ . \tag{3.181}$$

Alternatively, we can arrive at the cubic product of electric charges by contracting (3.177) with Q_I from (3.171), giving

$$\begin{aligned}&\tfrac{1}{64}c^{IJK}\widetilde{Q}_I\widetilde{Q}_J\widetilde{Q}_K\\ &= 4e^{-U_1-2U_2} + 2e^{-2U_1-U_2} + e^{-2U_1-U_2}c^{IJK}\widetilde{Q}_I\xi_J\xi_K - \tfrac{1}{4}e^{-U_1-U_2}\widetilde{J}(\xi\cdot X) + \tfrac{1}{8}\widetilde{J}c_{IJK}b^Ib^Jb^K \ .\end{aligned} \tag{3.182}$$

Comparing (3.181) and (3.182), we find the identity

$$\frac{1}{2}c^{IJK}\widetilde{Q}_I\xi_J\xi_K + 1 = e^{U_1}\left(e^{-U_2} + \frac{1}{8}\widetilde{J}\ \xi\cdot X\right) \ . \tag{3.183}$$

It is useful because it gives access to a useful combination of U_1, U_2, and $\xi \cdot X$. Indeed, we can simplify the cube of the electric charge in the form (3.181) using the b identity (3.173), and then (3.178) to eliminate $(\xi \cdot X)$, to give

$$\frac{1}{6}c^{IJK}\widetilde{Q}_I\widetilde{Q}_J\widetilde{Q}_K + \widetilde{J}^2 = 64e^{-U_1-U_2}\left(e^{-U_2} + \frac{1}{8}\widetilde{J}\ \xi\cdot X\ \right) \ . \tag{3.184}$$

The right-hand side of this equation differs from that of (3.183) only by a factor proportional $e^{-2U_1-U_2}$ which is the square of the geometric measure on the black hole horizon. As such, it is related to the black hole entropy $\mathcal{S}$ both through the area law (3.161) and as a function of charges (3.180). Collecting these relations, and reintroducing Q_I and J (3.65) in order to align with the conventional units for this result, we find

$$\left(\frac{1}{6}c^{IJK}Q_IQ_JQ_K + \frac{\pi}{4G_5}J^2\right) = \ell^3\left(\frac{1}{2}c^{IJK}\xi_I\xi_JQ_K + \frac{\pi}{4G_5}\right)\left(\frac{1}{2}c^{IJK}\xi_IQ_JQ_K - \frac{\pi}{2G_5}J\right) \ . \tag{3.185}$$

This is the prototypical 5D nonlinear charge constraint [65, 102, 63]. However, the charge constraint (3.185) does not make progress towards solving the supersymmetry equations nor determining the near horizon solution in terms of conserved charges. Rather, it is a relation between the conserved charges that, if taken at face value, all supersymmetric black holes dual to $\mathcal{N} = 4$ SYM must satisfy. This is extremely important and the continuing questions regarding this constraint is one of the motivations for the detailed study reported in this chapter.

Even at this point of our discussion where we are deep into solving certain nonlinear equations, it is worth noting that that the black hole entropy (3.179) and the charge constraint (3.185) can be combined into one complex-valued equation

$$\frac{1}{6}c^{IJK}\left(Q_I+i\frac{\mathcal{S}}{2\pi}\xi_I\right)\left(Q_J+i\frac{\mathcal{S}}{2\pi}\xi_J\right)\left(Q_K+i\frac{\mathcal{S}}{2\pi}\xi_K\right)+\frac{\pi}{4G_5}\left(-J+i\frac{\mathcal{S}}{2\pi}\right)^2=0 \quad . \tag{3.186}$$

The real part gives the constraint (3.185) and the imaginary part gives the formula for the entropy (3.179). Complexified equations are natural in problems involving supersymmetry. Also, (3.186) appears as the condition for a complex saddle point that provides an accounting in $\mathcal{N}=4$ SYM for the entropy of black hole preserving 1/16 of the maximal supersymmetry [65, 103, 63, 102].

3.4.2.2 Near-horizon variables as function of conserved charges

Having discussed the black hole entropy and the constraint on charges, we move on to expressing all other aspects of the near-horizon geometry and the matter content in terms of the fixed charges Q_I and J.

For the following computations, we make use of the expressions (3.171) and (3.172) for J and Q_I simplified using the relation for b^I in (3.173). We find

$$Q_I=\frac{\pi}{G_5}e^{-2U_1}\left[X_Ie^{2U_1-U_2}+X_I(\xi\cdot X)^2-2\xi_I(\xi\cdot X)-4G_{IJ}c^{JKL}\xi_K\xi_L\right] \; , \tag{3.187}$$

$$J=\frac{16\pi}{G_5}e^{-3U_1}\left[\frac{1}{8}e^{U_1}\xi\cdot X+\ell^{-3}\right] \quad . \tag{3.188}$$

Multiplying both sides of (3.183) by J, using the relation (3.188), we can solve for $e^{U_1}\ \xi\cdot X$ in terms of the charges

$$\frac{1}{8}\xi\cdot Xe^{U_1}=\frac{J\left(\frac{1}{2}c^{IJK}Q_I\xi_J\xi_K+\frac{\pi}{4G_5}\right)-\left(\frac{\mathcal{S}}{2\pi}\right)^2\ell^{-3}}{J^2+\left(\frac{\mathcal{S}}{2\pi}\right)^2} \quad . \tag{3.189}$$

This result ties a geometrical quantity — as it appears in the near-horizon Kähler relation (3.175) — to the charges. It also has an additional immediate value as (3.188) relates it to e^{-3U_1}, giving

$$e^{-3U_1}=\frac{G_5}{16\pi}\frac{J^2+\left(\frac{\mathcal{S}}{2\pi}\right)^2}{\frac{1}{2}c^{IJK}Q_I\xi_J\xi_K+\frac{\pi}{4G_5}+J\ell^{-3}} \quad . \tag{3.190}$$

This expression sets the scale of the non-deformed S^3 which has line element $e^{-U_1}(\sigma_1^2+\sigma_2^2+$

σ_3^2).

Due to the rotation of the black hole, the horizon geometry (3.74) is deformed away from S^3. We can quantify the deformation by computing $e^{U_1-U_2}$ via e^{3U_1} from (3.190) and $e^{2U_1+U_2}$ from (3.161):

$$e^{U_1-U_2} = \frac{\frac{4G_5}{\pi}\left(\frac{1}{2}c^{IJK}Q_I\xi_J\xi_K + \frac{\pi}{4G_5} + J\ell^{-3}\right)\left(\frac{\mathcal{S}}{2\pi}\right)^2}{J^2 + \left(\frac{\mathcal{S}}{2\pi}\right)^2} \ . \tag{3.191}$$

The only scalar near-horizon parameter that was not yet computed is the AdS$_2$-volume v. Due to the alternate near-horizon Kähler condition (3.175), the combination ve^{U_1} can be expressed in terms of $\mathcal{S}$, $(\xi \cdot X)e^{U_1}$ and e^{-3U_1}. These three quantities were given as functions of the conserved charges in (3.161), (3.189) and (3.190). After simplifications, we find

$$\frac{v}{4}e^{U_1} = \frac{\pi\ell^3}{4G_5}\frac{\ell^3\left(\frac{1}{2}c^{IJK}Q_I\xi_J\xi_K + \frac{\pi}{4G_5}\right) + J}{\ell^6\left(\frac{1}{2}c^{IJK}Q_I\xi_J\xi_K + \frac{\pi}{4G_5}\right)^2 + \left(\frac{\mathcal{S}}{2\pi}\right)^2}. \tag{3.192}$$

This completes the explicit extremization of the entropy function for the scalar variables which at this point have all been expressed in terms of conserved charges Q_I, J and FI-couplings ξ_I.

We must similarly determine the vectors b^I and X^I at the extremum which may be determined, in principle, by the input vectors ξ_I and $\widetilde{Q}_I$. However, the position of the vector indices I do not match so the full real special geometry enters. We exploit only the vectorial symmetry, and then, X^I and b^I must be linear combinations of *three* vectors: $c^{IJK}\widetilde{Q}_I Q_J$, $c^{IJK}\widetilde{Q}_I\xi_J$, and $c^{IJK}\xi_I\xi_J$. One linear relation of this kind was given in (3.177). To find another, we contract (3.177) with $\widetilde{Q}_I$ and simplify using (3.182). This gives

$$\widetilde{Q} \cdot X = 8e^{-U_2} + 4e^{-U_1} \ . \tag{3.193}$$

Combining this with (3.176) and (3.189), we have all four inner products of b^I, X^I, Q_I and ξ_I. We already determined the scalar combinations $c^{IJK}\xi_I\xi_J\widetilde{Q}_K$ and $c^{IJK}\xi_I\widetilde{Q}_J\widetilde{Q}_K$ from (3.179) and (3.183), so we can establish the vectorial equation

$$\frac{1}{2}c^{IJK}\widetilde{Q}_J\xi_K = b^I + \frac{1}{8}\widetilde{J}e^{U_1}X^I \ . \tag{3.194}$$

Inversion of (3.177) and (3.194) give

$$b^I = \frac{4G_5}{\pi}\cdot\frac{1}{2}\frac{c^{IJK}\xi_J Q_K\left(\frac{\mathcal{S}}{2\pi}\right)^2 - \left(\frac{1}{2}c^{IJK}Q_J Q_K - \frac{1}{2}c^{IJK}\xi_J\xi_K\left(\frac{\mathcal{S}}{2\pi}\right)^2\right)J}{J^2+\left(\frac{\mathcal{S}}{2\pi}\right)^2}\ , \tag{3.195}$$

and

$$X^I = 4e^{-U_1}\frac{Jc^{IJK}\xi_J Q_K + \left(\frac{1}{2}c^{IJK}Q_J Q_K - \frac{1}{2}c^{IJK}\xi_J\xi_K\left(\frac{\mathcal{S}}{2\pi}\right)^2\right)}{J^2+\left(\frac{\mathcal{S}}{2\pi}\right)^2}\ , \tag{3.196}$$

where, once we impose the value of e^{-U_1} given in (3.190), X^I is a function of the entropy (3.180) and the charges of the black hole.

In summary, we have found that the near-horizon limit of the supersymmetric equations implies that the near-horizon fields and variables of the geometry/matter ansatz are given by the charges Q_I and J, through the relations (3.190-3.196), which themselves parametrize a special extremum of the near-horizon entropy function (3.160), for general FI coupling ξ_I and c_{IJK}.

3.4.3 Complexification of the near-horizon variables

Each of the main results derived in the previous subsection are complicated formulae. However, they resemble one another and, in particular, it stands out that several expressions, such as (3.195) and (3.196), share a common denominator. Indeed, there is an elegant way to pair them into complexified near-horizon variables

$$Z^I = b^I - ie^{-\frac{1}{2}U_2}X^I = \frac{G_5}{\pi}\frac{c^{IJK}(Q_J+i\frac{\mathcal{S}}{2\pi}\xi_J)(Q_K+i\frac{\mathcal{S}}{2\pi}\xi_K)}{-J+i\frac{\mathcal{S}}{2\pi}}\ . \tag{3.197}$$

To the extent X^I can be interpreted is an electric field it is indeed natural that its partner is a magnetic field b^I. In addition to the real part b^I being given by (3.195), we recognize in the imaginary part the combination of the factor $e^{-U_1-\frac{1}{2}U_2}$ in the entropy $\mathcal{S}$ and $X^I e^{U_1}$ given respectively by (3.180) and (3.196).

Some discussions of the AdS_5 black hole geometry invoke from the outset principles that are inherently complex, such as the Euclidean path integral or special geometry in four dimensions. This can give conceptual challenges so, in our discussion of entropy extremization, complex variables such as (3.197) are introduced [29, 104] only for their apparent convenience. To make precise connections with the literature, we now to reintroduce the electric fields e^I and e^0 conjugate to the conserved charges $\tilde{Q}_I$ and $\tilde{J}$. For e^0 defined in (3.158), simplification using (3.172), gives an expression for e^0 that depends on ve^{U_1} in (3.192), e^{-3U_1} in (3.190), $\mathcal{S}$ in (3.161), and $(\xi \cdot X)e^{U_1}$ in (3.189). Collecting formulae, we then find

$$e^0 = -\frac{4\pi}{\mathcal{S}}\frac{\pi\ell^3}{4G_5}\frac{J\ell^3\left(\frac{1}{2}c^{IJK}Q_I\xi_J\xi_K + \frac{\pi}{4G_5}\right) - \left(\frac{\mathcal{S}}{2\pi}\right)^2}{\ell^6\left(\frac{1}{2}c^{IJK}Q_I\xi_J\xi_K + \frac{\pi}{4G_5}\right)^2 + \left(\frac{\mathcal{S}}{2\pi}\right)^2} \ . \tag{3.198}$$

This expression for e^0 combines nicely with (3.192) and gives the complex potential

$$\frac{1}{2}e^0 + i\frac{v}{4}e^{U_1} = \frac{\pi\ell^3}{4G_5}\left(\frac{2\pi}{\mathcal{S}}\right)\frac{-J + i\frac{\mathcal{S}}{2\pi}}{\ell^3\left(\frac{1}{2}c^{IJK}Q_I\xi_J\xi_K + \frac{\pi}{4G_5}\right) + i\left(\frac{\mathcal{S}}{2\pi}\right)} \ . \tag{3.199}$$

Given e^0 in (3.198) as well as (3.195) and (3.196), the electrical potentials dual to the vectorial charges become (3.157):

$$e^I = \frac{2\pi}{\mathcal{S}}\frac{\ell^6\left(\frac{1}{2}c^{IJK}Q_JQ_K - \frac{1}{2}c^{IJK}\xi_J\xi_K\left(\frac{\mathcal{S}}{2\pi}\right)^2\right)\left(\frac{1}{2}c^{IJK}Q_I\xi_J\xi_K + \frac{\pi}{4G_5}\right) + \ell^3c^{IJK}Q_J\xi_K\left(\frac{\mathcal{S}}{2\pi}\right)^2}{\ell^6\left(\frac{1}{2}c^{IJK}Q_I\xi_J\xi_K + \frac{\pi}{4G_5}\right)^2 + \left(\frac{\mathcal{S}}{2\pi}\right)^2} . \tag{3.200}$$

As preparation for the complexified version, we combine (3.195) and (3.196) as

$$\begin{aligned}&\frac{v}{2}\left(b^Ie^{U_1} + X^I\xi\cdot X\right)\\ &= \frac{\ell^3c^{IJK}Q_J\xi_K\left(\frac{1}{2}c^{LMN}Q_L\xi_M\xi_N + \frac{\pi}{4G_5}\right) - \ell^3\left(\frac{1}{2}c^{IJK}Q_JQ_K - \frac{1}{2}c^{IJK}\xi_J\xi_K\left(\frac{\mathcal{S}}{2\pi}\right)^2\right)}{\ell^6\left(\frac{1}{2}c^{IJK}Q_I\xi_J\xi_K + \frac{\pi}{4G_5}\right)^2 + \left(\frac{\mathcal{S}}{2\pi}\right)^2} \ ,\end{aligned} \tag{3.201}$$

where we have imposed $\frac{v}{4}e^{U_1}$ in (3.192), e^{-U_1} in (3.190), and $(\xi \cdot X)e^{U_1}$ in (3.189). We then find the complex special geometry vector

$$e^I + i\frac{v}{2}\left(b^Ie^{U_1} + X^I(\xi\cdot X)\right) = \frac{2\pi}{\mathcal{S}}\frac{\frac{1}{2}c^{IJK}(Q_J + i\frac{\mathcal{S}}{2\pi}\xi_J)(Q_K + i\frac{\mathcal{S}}{2\pi}\xi_K)}{\ell^3(\frac{1}{2}c^{LMN}Q_L\xi_M\xi_N + \frac{\pi}{4G_5}) + i\frac{\mathcal{S}}{2\pi}} \ . \tag{3.202}$$

The complex potentials (3.199-3.202) appear commonly in the literature, albeit with the normalization

$$\frac{\omega}{\pi} = \frac{\pi\ell^3}{4G_5}\left(\frac{2\pi}{\mathcal{S}}\right)\frac{-J + i\frac{\mathcal{S}}{2\pi}}{\ell^3\left(\frac{1}{2}c^{IJK}Q_I\xi_J\xi_K + \frac{\pi}{4G_5}\right) + i\left(\frac{\mathcal{S}}{2\pi}\right)} = \frac{1}{2}e^0 + i\frac{v}{4}e^{U_1} \ , \tag{3.203}$$

and

$$\frac{\Delta^I}{\pi} = \frac{2\pi}{\mathcal{S}} \frac{\frac{1}{2}c^{IJK}\ell^3(Q_J + i\frac{\mathcal{S}}{2\pi}\xi_J)(Q_K + i\frac{\mathcal{S}}{2\pi}\xi_K)}{\ell^3\left(\frac{1}{2}c^{IJK}Q_I\xi_J\xi_K + \frac{\pi}{4G_5}\right) + i\left(\frac{\mathcal{S}}{2\pi}\right)} = e^I + i\frac{v}{2}(b^I e^{U_1} + X^I \xi \cdot X) \ . \tag{3.204}$$

The real and imaginary parts of the complexified potentials ω and Δ^I are related to one another through

$$\xi_I e^I + e^0 = 0 \ . \tag{3.205}$$

and from the identities (3.173) and (3.175), we find

$$2\omega + \xi_I \Delta^I = 2\pi i \ . \tag{3.206}$$

This is our version of the well-known complex constraint that is imposed on the chemical potentials conjugate to J and Q_I in analyses involving complex saddle points from the outset. An important example is the Hosseini-Hristov-Zaffaroni (HHZ) extremization principle for 5D rotating BPS black holes [29, 65, 102, 63]. The complexified potentials ω and Δ^I can be exploited to simplify the Lagrangian density (3.154). The linchpin is the identity

$$\frac{\frac{1}{6}c_{IJK}\Delta^I\Delta^J\Delta^K}{\omega^2} = \left(\frac{2\pi^2}{\mathcal{S}}\right) \frac{(-J + \frac{i\mathcal{S}}{2\pi})^2}{\ell^3\left(\frac{1}{2}c^{IJK}Q_I\xi_J\xi_K + \frac{\pi}{4G_5}\right) + i\left(\frac{\mathcal{S}}{2\pi}\right)} \ . \tag{3.207}$$

It is established using the cube of (3.204), along with (A.12) to simplify the products of c_{IJK}, as well as the square of (3.203) and the complexified charge relation (3.186). The same combination of terms appears when evaluating instead

$$\Delta^I Q_I - 2\omega J = -\left(\frac{N^2}{2}\right)\left(\frac{2\pi^2}{\mathcal{S}}\right) \frac{(-J + \frac{i\mathcal{S}}{2\pi})^2}{\ell^3\left(\frac{1}{2}c^{IJK}Q_I\xi_J\xi_K + \frac{\pi}{4G_5}\right) + i\left(\frac{\mathcal{S}}{2\pi}\right)} + \mathcal{S} \ , \tag{3.208}$$

with the use of the definitions of ω and Δ^I in (3.203) and (3.204) respectively, as well as the complex relations (3.186) and (3.206), and exchanging G_5 for N via $\frac{\pi\ell^3}{4G_5} = \frac{N^2}{2}$. This allows us to rewrite the black hole entropy $\mathcal{S}$ as

$$\mathcal{S} = \Delta^I Q_I - 2\omega J + \frac{N^2}{2}\frac{\frac{1}{6}c_{IJK}\Delta^I\Delta^J\Delta^K}{\omega^2} \ . \tag{3.209}$$

Referring back to the real-valued entropy functional $\mathcal{S}$ as the Legendre transform of the on-shell Lagrangian $\mathcal{L}_1$ (3.159), and noting that (e^I, e^0) constitute the real parts of (Δ^I, ω), we obtain the greatly simplified expression

$$2\pi\mathcal{L}_{1,\mathrm{nh}} = -\frac{N^2}{2}\mathrm{Re}\left(\frac{\frac{1}{6}c_{IJK}\Delta^I\Delta^J\Delta^K}{\omega^2}\right) \ . \tag{3.210}$$

We have thus been able to reproduce the standard HHZ entropy function result [29], although while remaining entirely in 5D (no reduction to 4D), with the help of the entropy function formalism. The derivation of (3.209) also makes the Legendre transformation between the entropy $\mathcal{S}$ and the complexified entropy function manifest.

3.5 Discussion

We have analysed the first order attractor flow equations derived from the vanishing of the supersymmetric variations in $D = 5$ $\mathcal{N} = 2$ gauged supergravity with FI-couplings to $\mathcal{N} = 2$ vector multiplets. We focus on solutions with electric charges Q_I and one independent angular momentum J. In order to analyze the flow equations and find the conserved charges, we first assume a perturbative expansion at either the near-horizon geometry or the asymptotic boundary. As usual, the supersymmetry conditions are not sufficient to guarantee a solution to the equation of motion, but we find that the conserved Noether-Wald surface charges fill this gap. This leads to a self-contained set of first order differential equations.

To integrate these differential equations we need boundary conditions, or more generally integration constants. In the present setting, this turns out to be somewhat complicated. Generically, first order differential equations, even coupled ones, just need values at one point to compute the derivative and then, by iteration, the complete solution follows.[8] We find that, whether starting from the black hole horizon or the asymptotic AdS_5, solving the first order equations is subtle. Supersymmetry conditions exhibit zero-modes which fail to provide a derivative, as a first order differential equation is expected to do. On the positive side, in these situations supersymmetry give relations between the first few coefficients near a boundary.

After exploiting conserved charges extensively, the initial value problem simplifies. Indeed, at the horizon, all fields must satisfy the entropy extremization principle, discussed in detail in section 3.4. The relative simplicity of *shooting out* from the horizon can be construed as black hole attractor behavior. The situation starting from asymptotic AdS is much more involved, as detailed in subsection 3.3.4.

[8] Locality is among the major caveats. In principle, first order differential equation give derivatives, and then the derivatives of the derivatives, and so on for the whole series. Generally, it is not easy to prove convergence for a series obtained this way, but this obstacle, and other mathematical fine points, do not appear significant at our level of analysis.

We are far from the first to investigate the attractor flow for rotating AdS_5 black holes. Some notable works are [29, 104]. In our procedure, we have remained in five dimensions, without dimensionally reducing to four dimensions, where the metric no longer contains a fibration. Our approach is complementary, in that the role of rotation is highlighted. Additionally, we have allowed for backgrounds that go beyond the omnipresent STU model. Finally, we have also considered a complexification of the near-horizon variables that elucidates some features of the theory from the near-horizon perspective. This includes the well-known complex constraint on the chemical potentials.

Many open problems persist after our analysis of AdS_5 rotating black holes. For example, we derived the first order attractor flow equations from the supersymmetric variations of the $\mathcal{N} = 2$ gauged theory, but it would be instructive to also derive them from the Lagrangian. After a suitable Legendre transform, the dimensionally reduced Lagrangian can be written as a sum of squares, up to a total derivative. In minimizing the Lagrangian, each square gives a condition that is equivalent to the vanishing of the supersymmetric variations. It would be interesting to extract the flow equations from this method as it can also be more directly related to the entropy extremization once the near-horizon limit is taken. We also expect this now radial entropy function to greatly simplify once the fields and variables in it are suitably complexified, such as was done at the near-horizon level. This would allow for an understanding of the underlying complex structure of the rotating AdS_5 black hole spacetime without the customary reduction to 4D.

Higher derivative corrections in the context of AdS_5 black holes have been studied by [85, 105, 106, 107] and references therein, and it would be interesting to understand the role of higher derivative corrections in the attractor flow. This is also interesting from the entropy extremization point of view and allows us to probe higher derivative corrections to the entropy from the near-horizon, which can be checked via holography. Finally, a similar analysis can then be completed in other dimensions, including the rotating AdS black holes in six and seven dimensions [108, 109]. The product of the scalar fields with one of the parameters of the metric yields a harmonic function and we would expect that one can solve the flow equations using a similar approach via a perturbative expansion. We hope to comment on these ideas in the near future.

APPENDIX A

Conventions, Special Geometry, and Deriving the Radial Flow

A.1 Conventions and notations

In this Appendix, we summarize the conventions and notations used in the various expressions involving differential geometry as well as real special geometry.
We introduce components as

$$\xi = \xi^\mu \frac{\partial}{\partial x^\mu} \ , \quad \omega = \frac{1}{r!}\omega_{\mu_1\ldots\mu_r} \ \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_r} \ . \tag{A.1}$$

In this notation the interior product i_ξ of ω with respect to ξ is

$$\begin{aligned} i_\xi \omega &= \frac{1}{(r-1)!}\xi^\nu \omega_{\nu\mu_2\ldots\mu_r} \ \mathrm{d}x^{\mu_2} \wedge \ldots \wedge \mathrm{d}x^{\mu_r} \\ &= \frac{1}{r!}\sum_{s=1}^{r} \xi^{\mu_s}\omega_{\mu_1\ldots\mu_s\ldots\mu_r}(-1)^{s-1} \ \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \widehat{\mathrm{d}x^{\mu_s}} \wedge \ldots \wedge \mathrm{d}x^{\mu_r} \ . \end{aligned} \tag{A.2}$$

The wide hat indicates that dx^{μ_s} is removed.

The Hodge dual is defined by

$$\star_r \left(\ \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \ldots \wedge \mathrm{d}x^{\mu_r} \right) = \frac{\sqrt{|g|}}{(m-r)!}\varepsilon^{\mu_1\mu_2\ldots\mu_r}{}_{v_{r+1}\ldots v_m} \ \mathrm{d}x^{v_{r+1}} \wedge \ldots \wedge \mathrm{d}x^{v_m} \ , \tag{A.3}$$

where the subscript r denotes the dimension of the spacetime and the totally antisymmetric tensor is

$$\varepsilon_{\mu_1\mu_2\ldots\mu_m} = \begin{cases} +1 & \text{if } (\mu_1\mu_2\ldots\mu_m) \text{ is an even permutation of } (12\ldots m) \\ -1 & \text{if } (\mu_1\mu_2\ldots\mu_m) \text{ is an odd permutation of } (12\ldots m) \\ 0 & \text{otherwise.} \end{cases} \quad . \tag{A.4}$$

The indices on the totally antisymmetric symbol $\varepsilon_{\mu_1\mu_2\ldots\mu_m}$ can be raised by the metric through

$$\varepsilon^{\mu_1\mu_2\ldots\mu_m} = g^{\mu_1 v_1} g^{\mu_2 v_2} \ldots g^{\mu_m v_m} \varepsilon_{v_1 v_2\ldots v_m} = g^{-1}\varepsilon_{\mu_1\mu_2\ldots\mu_m}. \tag{A.5}$$

The Hodge dual of the identity 1 gives the invariant volume element

$$\star_r 1 = \frac{\sqrt{|g|}}{m!}\varepsilon_{\mu_1\mu_2\ldots\mu_m}\, \mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_m} = \sqrt{|g|}\mathrm{d}x^1\wedge\ldots\wedge\mathrm{d}x^m\ . \tag{A.6}$$

We define the r-forms U and V as

$$U = \frac{1}{r!}U_{\mu_1\ldots\mu_r}\,\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_r}, \qquad V = \frac{1}{r!}V_{\mu_1\ldots\mu_r}\,\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_r}\ , \tag{A.7}$$

such that

$$U\wedge\star_r V = V\wedge\star_r U = \frac{1}{r!}U_{\mu_1\ldots\mu_r}V^{\mu_1\ldots\mu_r}\sqrt{|g|}\mathrm{d}x^1\wedge\ldots\wedge\mathrm{d}x^m. \tag{A.8}$$

A.2 Real special geometry

In this appendix we summarize the conventions and formulae needed for manipulations in real special geometry. We study $\mathcal{N}=2$ theories with n_V vector multiplets and $n_H = 0$ hypermultiplets. The starting point is a collection of real 5D scalar fields X^I with $I = 0, 1, \ldots, n_V$. They are subject to the constraint

$$\frac{1}{6}c_{IJK}X^I X^J X^K = 1\ , \tag{A.9}$$

where the structure constants c_{IJK} are real numbers, completely symmetric in I, J, and K, that satisfy the closure relation

$$c_{IJK}c_{J'(LM}c_{PQ)K'}\delta^{JJ'}\delta^{KK'} = \frac{4}{3}\delta_{I(L}c_{MPQ)}\ . \tag{A.10}$$

The index I takes n_V+1 distinct values but, because of the constraint (A.9), there are n_V independent scalar fields, one for each $\mathcal{N}=2$ vector multiplet in 5D. Round brackets $(\cdots)$

indicate symmetrization of indices with weight one so, for example, $c_{IJK} = c_{(IJK)}$.

Using the Euclidean metric to define c^{IJK} with upper indices, meaning $c^{IJK} = \delta^{II'}\delta^{JJ'}\delta^{KK'}c_{I'J'K'}$, the closure relation (A.10) can be rewritten as

$$c_{IJK}c^{J(LM}c^{PQ)K} = \frac{4}{3}\delta_I^{(L}c^{MPQ)} \ . \tag{A.11}$$

We also note the following identities involving symmetrizations

$$c_{IJK}c^{J(LM}c^{PQ)K} = \frac{1}{3}c_{IJK}\left(c^{JLM}c^{PQK} + c^{JLP}c^{MQK} + c^{JPM}c^{LQK}\right) \ , \tag{A.12}$$

$$\delta_I^{(L}c^{MPQ)} = \frac{1}{4}\left(\delta_I^L c^{MPQ} + \delta_I^M c^{LPQ} + \delta_I^P c^{LMQ} + \delta_I^Q c^{LMP}\right) \ . \tag{A.13}$$

Given the scalars X^I and c_{IJK} as inputs, we define the scalar X_I (with lower index) and the metric on field space G_{IJ} as

$$\begin{aligned} X_I &= \frac{1}{2}c_{IJK}X^J X^K \ , \\ G_{IJ} &= \frac{1}{2}\left(X_I X_J - c_{IJK}X^K\right) \ . \end{aligned} \tag{A.14}$$

In manipulations we often use the formulae

$$\begin{aligned} G_{IJ}X^J &= \frac{1}{2}X_I \ , \\ X_I X^I &= 3 \ . \end{aligned} \tag{A.15}$$

The closure relation (A.11) then requires that the inverse matrix G^{IJ} satisfies

$$c^{IJK}X_K = X^I X^J - \frac{1}{2}G^{IJ} \ . \tag{A.16}$$

It follows that, just as G_{IJ} lowers indices on X^J indices (up to a factor of $\frac{1}{2}$), the inverse G^{IJ} raises indices on X_J

$$G^{IJ}X_J = 2(X^I X^J - c^{IJK}X_K)X_J = 2X^I \ . \tag{A.17}$$

We also note the identity

$$(c^{IJK}X_K)(c_{ILM}X^M) = \left(X^I X^J - \tfrac{1}{2}G^{IJ}\right)\left(X_I X_L - 2G_{IL}\right) = \delta_L^J + X^J X_L \, . \tag{A.18}$$

In the literature, it is common to summarize real special geometry through the cubic polynomial

$$\mathcal{V} = \frac{1}{6} c_{IJK} X^I X^J X^K \ . \tag{A.19}$$

The constraint (A.9) is simply $\mathcal{V} = 1$. Differentiating *first* and *then* imposing the constraint $\mathcal{V} = 1$, we find

$$\mathcal{V}_I \equiv \frac{\partial \mathcal{V}}{\partial X^I} = \frac{1}{2} c_{IJK} X^J X^K = X_I \ , \tag{A.20}$$

$$\mathcal{V}_{IJ} \equiv \frac{\partial^2 \mathcal{V}}{\partial X^I \partial X^J} = c_{IJK} X^K \ , \tag{A.21}$$

$$G_{IJ} = -\frac{1}{2} \frac{\partial^2 \ln \mathcal{V}}{\partial X^I \partial X^J} = \frac{1}{2} \left(\mathcal{V}_I \mathcal{V}_J - \mathcal{V}_{IJ} \right) = \frac{1}{2} \left(X_I X_J - c_{IJK} X^K \right) \ . \tag{A.22}$$

The inverse $\mathcal{V}^{IJ}$ of $\mathcal{V}_{IJ}$ (meaning it satisfies $\mathcal{V}^{IJ} \mathcal{V}_{JK} = \delta^I_K$) is given by

$$\mathcal{V}^{IJ} = \frac{1}{2} (X^I X^J - G^{IJ}) \ . \tag{A.23}$$

The STU-model is an important example. In this special case $n_V = 2$ and we shift the labels so $I = 1, 2, 3$ (rather than $I = 0, 1, 2$). The only nonvanishing c_{IJK} are $c_{123} = 1$ and all its permutations. In our normalizations, the STU model has

$$X^1 X^2 X^3 = 1 \ , \quad X_I^{-1} = X_I \ , \quad G_{IJ} = \frac{1}{2} X_I^2 \delta_{IJ} \ .$$

In these formulae there is no sum over $I = 1, 2, 3$. We add a special note about adapting the formalism of real special geometry, this time adapted to ξ_I given by the constraint

$$\frac{1}{6} c^{IJK} \xi_I \xi_J \xi_K = \ell^{-3} \ . \tag{A.24}$$

Following similar steps in terms of defining a raised version of the ξ_I, imposing consistency with the raised c^{IJK} through the condition (A.12), we can define the following

$$\begin{aligned} \xi^I &= \frac{1}{2} c^{IJK} \xi_J \xi_K \ , \\ \xi_I &= \frac{1}{2} \ell^3 c_{IJK} \xi^J \xi^K \ . \end{aligned} \tag{A.25}$$

We then go on defining a version of the G_{IJ} and G^{IJ} for the ξ_I

$$\begin{aligned} \tilde{G}^{IJ} &= 2 \left(\ell^3 \xi^I \xi^J - c^{IJK} \xi_K \right) \ , \\ \tilde{G}_{IJ} &= \frac{1}{2} \ell^3 \left(\xi_I \xi_J - c_{IJK} \xi^K \right) \ , \end{aligned} \tag{A.26}$$

which leads to the crucial inversion identity on the ξ_I:

$$\frac{1}{2}\ell^3(c_{IKM}c^{MNP}\xi_N\xi_P - \xi_I\xi_K)\left(c^{IJL}\xi_L\right) = \delta_K^J \ . \tag{A.27}$$

A final comment: in this article, we take 5D supergravity as the starting point. For an introduction to the geometric interpretation of the 5D fields and the formulae they satisfy in terms of Calabi-Yau compactification of 11D supergravity, we refer to [110].

A.3 Supersymmetry conditions

In this appendix we establish the conditions that our *ansatz* (3.69) must satisfy in order to preserve supersymmetry.

A.3.1 The Kähler condition on the base geometry

We want to establish the conditions on the variables in the 4D base geometry in (3.69) that ensure that it is Kähler. For a given vielbein basis e^a on the base $ds_4^2 = \eta_{ab}e^ae^b$, such as (3.71), the Kähler condition is

$$d(e^1 \wedge e^4 - e^2 \wedge e^3) = 0 \ . \tag{A.28}$$

In the $(1+4)$ split (3.69), the base space (3.70) is automatically Kähler as

$$\left(g_m^{-1/2}\right)\left(\tfrac{1}{2}Rg_m^{1/2}dR \wedge \sigma_3\right) - \tfrac{1}{4}R^2\sigma_1 \wedge \sigma_2 = d\left(\tfrac{1}{4}R^2\sigma_3\right) \ , \tag{A.29}$$

which is automatically closed. We look instead to the $(2+3)$ split in (3.74) to obtain a nontrivial Kähler condition. For that, we rewrite (3.74) in the form $ds_5^2 = f^2(dt+\omega)^2 - f^{-1}ds_4^2$, and find the warp factor

$$f = (e^{2\rho} - e^{-U_2}(a_t^0)^2)^{1/2} \ , \tag{A.30}$$

the 1-form

$$\omega = -f^{-2}e^{-U_2}a_t^0 \ \sigma_3 \ , \tag{A.31}$$

and the 4D base geometry

$$ds_4^2 = fe^{2\sigma}dR^2 + \frac{1}{4}R^2(\sigma_1^2 + \sigma_2^2) + f^{-1}e^{2\rho - U_2}\sigma_3^2 \ . \tag{A.32}$$

To find the condition for which (A.32) is Kähler, we introduce the basis 1-forms

$$e^1 = f^{1/2} e^{\sigma} dR \ , \tag{A.33}$$

$$e^2 = \frac{1}{2} R\sigma_1 \ , \tag{A.34}$$

$$e^3 = \frac{1}{2} R\sigma_2 \ , \tag{A.35}$$

$$e^4 = f^{-1/2} e^{\rho - U_2/2} \sigma_3 \ . \tag{A.36}$$

The Kähler 2-form $J = e^1 \wedge e^4 - e^2 \wedge e^3$ becomes

$$J = e^{\sigma+\rho-U_2/2} dR \wedge \sigma_3 - \frac{1}{4} R^2 \sigma_1 \wedge \sigma_2 \ . \tag{A.37}$$

The Kähler condition demands that J is closed

$$dJ = e^{\sigma+\rho-U_2/2} dR \wedge \sigma_1 \wedge \sigma_2 - \frac{1}{2} R dR \wedge \sigma_1 \wedge \sigma_2 = 0 \ . \tag{A.38}$$

We therefore find

$$e^{\sigma+\rho-U_2/2} = \frac{1}{2} R \ . \tag{A.39}$$

This condition must be satisfied so that the general ansatz (3.74) can support supersymmetry. The Kähler condition allow us to rewrite the base geometry (A.32) as

$$ds_4^2 = f e^{2\sigma} dR^2 + \frac{1}{4} R^2 (\sigma_1^2 + \sigma_2^2 + f^{-1} e^{-2\sigma} \sigma_3^2) \ . \tag{A.40}$$

This form of the base geometry depends on a single function $fe^{2\sigma}$.

A.3.2 Kähler potential

The Kähler condition (A.39) relates the 1-forms e^1 and e^4 in (3.71). If we define a radial coordinate r such that

$$\partial_r R = f^{-\frac{1}{2}} e^{-\sigma} \ , \tag{A.41}$$

the tetrad simplifies so

$$e^1 = dr \ , \tag{A.42}$$

$$e^4 = \partial_r \left(\tfrac{1}{4} R^2 \right) \sigma_3 \ . \tag{A.43}$$

with e^2 and e^3 unchanged. In these coordinates, the unique spin connections solving Cartan's equations $de^a + \omega^a{}_b e^b = 0$ are

$$
\begin{aligned}
{}^4\omega^2{}_1 = {}^4\omega^4{}_3 &= \frac{\partial_r R}{R} e^2 \ , && \text{(A.44)}\\
{}^4\omega^3{}_1 = {}^4\omega^2{}_4 &= \frac{\partial_r R}{R} e^3 \ , && \text{(A.45)}\\
{}^4\omega^4{}_1 &= \left(\frac{\partial_r R}{R} + \frac{\partial_r^2 R}{\partial_r R} \right) e^4 \ , && \text{(A.46)}\\
{}^4\omega^2{}_3 &= \left(\frac{\partial_r R}{R} - \frac{2}{R \partial_r R} \right) e^4 \ , && \text{(A.47)}
\end{aligned}
$$

where the ^{4} superscript distinguishes these 4D spin connections from the 5D spin connections that will appear in later computations. The resulting curvature 2-forms $R^a{}_b = d\omega^a{}_b + \omega^a{}_c \omega^c{}_b$ on the 4D base become

$$
\begin{aligned}
R^2{}_1 = R^4{}_3 &= \frac{\partial_r^2 R}{R} (e^1 e^2 + e^3 e^4) \ , && \text{(A.48)}\\
R^3{}_1 = R^2{}_4 &= \frac{\partial_r^2 R}{R} (e^1 e^3 - e^2 e^4) \ , && \text{(A.49)}\\
R^4{}_1 &= \left(\frac{\partial_r^3 R}{\partial_r R} + 3 \frac{\partial_r^2 R}{R} \right) e^1 e^4 + 2 \frac{\partial_r^2 R}{R} e^3 e^2 \ , && \text{(A.50)}\\
R^2{}_3 &= 2 \frac{\partial_r^2 R}{R} e^1 e^4 + \frac{4}{R^2} ((\partial_r R)^2 - 1) e^3 e^2 \ . && \text{(A.51)}
\end{aligned}
$$

The components of the Riemann curvature are read off from $R^a{}_b = \frac{1}{2} \mathrm{Riem}^a{}_{bcd} e^c e^d$. For a complex manifold they are collected succinctly in the Kähler curvature 2-form with components $\mathcal{R}_{ab} = \frac{1}{2} \mathrm{Riem}_{abcd} J^{cd}$. In the context of our *ansatz* (3.70), we have

$$
\begin{aligned}
\mathcal{R}_{14} &= \epsilon (\mathrm{Riem}_{1423} - \mathrm{Riem}_{1414}) = \epsilon \left(\frac{\partial_r^3 R}{\partial_r R} + 5 \frac{\partial_r^2 R}{R} \right) \ ,\\
\mathcal{R}_{23} &= \epsilon (\mathrm{Riem}_{2323} - \mathrm{Riem}_{1423}) = -\epsilon \left(2 \frac{\partial_r^2 R}{R} + \frac{4}{R^2} ((\partial_r R)^2 - 1) \right) \ , && \text{(A.52)}
\end{aligned}
$$

and so the Kähler curvature 2-form becomes

$$
\begin{aligned}
\mathcal{R} &= \frac{1}{2} \epsilon \left(R \partial_r^3 R + 5 \partial_r R \partial_r^2 R \right) dr \sigma_3 - \frac{1}{2} \epsilon \left(R \partial_r^2 R + 2((\partial_r R)^2 - 1 \right) \sigma_1 \sigma_2 \\
&= \epsilon d \left((\frac{1}{2} R \partial_r^2 R + (\partial_r R)^2 - 1) \sigma_3 \right) \ . && \text{(A.53)}
\end{aligned}
$$

It is manifestly of the form $\mathcal{R} = dP$ where $P = p\sigma_3$ with

$$p = \epsilon\left(\frac{1}{2}R\partial_r^2 R + (\partial_r R)^2 - 1\right) = \epsilon\left(\frac{1}{4}R\partial_R(\frac{1}{f}e^{-2\sigma}) + \frac{1}{f}e^{-2\sigma} - 1\right) \ . \tag{A.54}$$

The second equation follows by repeated use of (A.41). Since $\mathcal{R}$ is the exterior derivative of something, it is clearly closed. Thus the base manifold is Kähler.

The final expression (A.54) depends on the single scalar function $fe^{2\sigma}$ that determines the base geometry (A.40). It encapsulates everything about the curvature of the 4D base.

A.3.3 Supersymmetry conditions

The $\mathcal{N} = 2$ supergravity theory we consider is, in particular, invariant under the fermionic transformations of the gaugino and the gravitino

$$\delta\lambda = \left[G_{IJ}\left(\frac{1}{2}\gamma^{\mu\nu}F^J_{\ \mu\nu} - \gamma^\mu\nabla_\mu X^J\right)\epsilon^\alpha - \xi_I\epsilon^{\alpha\beta}\epsilon^\beta\right]\partial_i X^I \ , \tag{A.55}$$

$$\delta\psi^\alpha_\mu = \left[(\partial_\mu - \frac{1}{4}\omega_\mu^{\nu\rho}\gamma_{\nu\rho}) + \frac{1}{24}(\gamma_\mu^{\ \nu\rho} - 4\delta_\mu^{\ \nu}\gamma^\rho)X_I F^I_{\nu\rho}\right]\epsilon^\alpha + \frac{1}{6}\xi_I(3A^I_\mu - X^I\gamma_\mu)\epsilon^{\alpha\beta}\epsilon^\beta \ , \tag{A.56}$$

where ϵ^α ($\alpha = 1, 2$) are symplectic Majorana spinors. For bosonic solutions to the theory that respect at least some supersymmetry these variations vanish for the spinors ϵ^α that generate the preserved supersymmetry. Supersymmetric black holes in AdS_5 with finite horizon area preserve the supersymmetry generated by the spinors ϵ^α that satisfy the projections

$$\gamma^0\epsilon^\alpha = \epsilon^\alpha \ , \tag{A.57}$$

$$\frac{1}{4}J^{(1)}_{mn}\gamma^{mn}\epsilon^\alpha = -\epsilon^{\alpha\beta}\epsilon^\beta \ . \tag{A.58}$$

Each of these equations impose two projections on the spinor ϵ^α. All these projections commute, so the resulting black holes preserve $2^{-4} = 1/16$ of the maximal supersymmetry.

We seek to work out the conditions that set the supersymmetric variations (A.55) and (A.56) to zero, satisfying the projections (A.57) and (A.58) imposed on purely bosonic solutions. We use the matter ansatz and geometry in (3.18) and (3.69), respectively.

The gamma matrices are defined with respect to a flat 5D space and satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \ , \tag{A.59}$$

with the flat 5D space defined in (3.69) via the following veilbein

$$E^0 = f(dt + w\sigma_3) \ , \tag{A.60}$$

$$E^i = f^{-\frac{1}{2}} e^i \ , \tag{A.61}$$

where e^i with spatial indices refers to the 4D veilbein introduced in (3.71). Furthermore, the gamma matrices γ^μ following the projection (3.78) satisfy

$$-\frac{\epsilon}{2}(\gamma^{23} - \gamma^{14}) = \epsilon\epsilon^{\alpha\beta}\epsilon^\beta \ , \tag{A.62}$$

where $\gamma^{\mu\nu}$ is the antisymmetrized product for $a \neq b$, which means that after squaring (A.62) we obtain

$$\gamma^{1234}\epsilon^\alpha = \epsilon^\alpha \ , \tag{A.63}$$

and thus

$$\gamma^{14}\epsilon^\alpha = -\gamma^{23}\epsilon^\alpha \ . \tag{A.64}$$

This becomes relevant for evaluating inner products of components of 2-forms and γ^{ab} as well as their decomposition into self-dual and anti-self-dual terms.

A.3.3.1 The gaugino equation

Recall that the gaugino equation is given by (A.55), where the 5D 2-form $F^I = dA^I$ can be computed from (3.73)

$$F^I = \partial_R(fY^I)e^{-\sigma}f^{-1}E^1 \wedge E^0 + 4f\left(fY^I\partial_{R^2}w + \partial_{R^2}u^I\right)E^1 \wedge E^4 - \frac{4f}{R^2}\left(fY^I w + u^I\right)E^2 \wedge E^3 \ . \tag{A.65}$$

The spatial F^I_{mn} components can be rearranged into self-dual and anti-self-dual terms

$$\begin{aligned} F^I = {} & \partial_R(fY^I)e^{-\sigma}f^{-1}E^1 \wedge E^0 \\ & + 2f\left(fY^I\left(\partial_{R^2} - \frac{1}{R^2}\right)w + \left(\partial_{R^2} - \frac{1}{R^2}\right)u^I\right)(E^1 \wedge E^4 + E^2 \wedge E^3) \\ & + 2f\left(fY^I\left(\partial_{R^2} + \frac{1}{R^2}\right)w + \left(\partial_{R^2} + \frac{1}{R^2}\right)u^I\right)(E^1 \wedge E^4 - E^2 \wedge E^3) \ . \end{aligned} \tag{A.66}$$

Since $(\gamma^{14}+\gamma^{23})\epsilon^\alpha = 0$ per (A.64), only the anti-self-dual components of $F^{\mu\nu}$ via $F^J_{\mu\nu}\gamma^{\mu\nu}$ contributes to the gaugino variation. We thus simplify $G_{IJ}\frac{1}{2}\gamma^{\mu\nu}F^J_{\mu\nu}$ to find

$$\begin{aligned}
&G_{IJ}\frac{1}{2}\gamma^{\mu\nu}F^J_{\mu\nu}\epsilon^\alpha \\
&= G_{IJ}\left[\partial_R(fY^I)e^{-\sigma}f^{-1}\gamma^{10} + 2f\left(fY^I\left(\partial_{R^2}+\tfrac{1}{R^2}\right)w + \left(\partial_{R^2}+\tfrac{1}{R^2}\right)u^I\right)(\gamma^{14}-\gamma^{23})\right]\epsilon^\alpha \ .
\end{aligned} \tag{A.67}$$

We then move on to the second term of (A.55), noting that X^I is only a function of R

$$G_{IJ}(-\gamma^\mu\nabla_\mu X^J)\epsilon^\alpha = G_{IJ}\left(-\gamma^1 e^{-\sigma}\partial_R X^J\right)\epsilon^\alpha \ . \tag{A.68}$$

Lastly, the third term of (A.55) becomes

$$-\xi_I\epsilon^{\alpha\beta}\epsilon^\beta = +\epsilon\xi_I\gamma^{23}\epsilon^\alpha \ . \tag{A.69}$$

Combining all three contributions, we obtain the following equations

$$G_{IJ}\left[\partial_R(fY^I) - f\partial_R X^I\right]\partial_i X^I = 0 \ , \tag{A.70}$$

$$\left[4G_{IJ}f\left(\left(\partial_{R^2}+\frac{1}{R^2}\right)u^J + fY^J\left(\partial_{R^2}+\frac{1}{R^2}\right)w\right) + \epsilon\xi_I\right]\partial_i X^I = 0 \ . \tag{A.71}$$

Since $X_I\partial_i X^I = \frac{1}{2}\partial_i(X_I X^I) = 0$, the $f\partial_R X^J$ term can be rewritten as $\partial_R(fX^J)$ and thus we obtain

$$\boxed{G_{IJ}\left[\partial_R(fY^J) - \partial_R(fX^J)\right]\partial_i X^I = 0 \ ,} \tag{A.72}$$

This can be reexpressed by defining a vector $\delta^I = fY^I - fX^I$, to imply that $\partial_R\delta^I$ is orthogonal to $\partial_i X^I$, and thus proportional to X^I:

$$\partial_R\delta^I = kX^I \ , \tag{A.73}$$

for some constant k. We will focus on the special solution where δ^I vanishes, meaning

$$X^I = Y^I \ . \tag{A.74}$$

Using this relation, we now move on to (A.71), the second gaugino variation result. It is a projection of the vector quantity in the square brackets, along the direction of $\partial_i X^I$. The immediate consequence of it is that this quantity is proportional to X_I. Rearranging terms, we obtain the ambiguous result

$$\left(\partial_{R^2} + \frac{1}{R^2}\right) u^I = \frac{1}{2}\epsilon f^{-1} c^{IJK} X_J \xi_K + \frac{1}{2} f^{-1} \lambda X^I \ , \tag{A.75}$$

with λ a scalar coefficient that arises from the ambiguity in defining the quantity in square brackets in (A.71) as orthogonal to $\partial_i X^I$. Determining this quantity requires resorting to further supersymmetry relations, which leads us to the vanishing of the gravitino variation (A.56).

A.3.3.2 The gravitino equation

In order to simplify the vanishing of the gravitino equation (A.56), we need to establish the components of the 5D spin connection that appears in the term $-\frac{1}{4}\ \omega_\mu^{\nu\rho}\gamma_{\nu\rho}\epsilon^\alpha$. Based on the vielbein (A.60) and (A.61), we have

$$\begin{aligned}
&{\omega^0}_1 = f^{-1}e^{-\sigma}\partial_R f E^0 + 2f^2\partial_{R^2} w E^4 \ , && {\omega^0}_2 = -\frac{2f^2 w}{R^2} E^3 \ , \\
&{\omega^0}_3 = \frac{2f^2 w}{R^2} E^2 \ , && {\omega^0}_4 = -2f^2 \partial_{R^2} w E^1 \ , \\
&{\omega^2}_1 = {}^4\omega^2_1 - \frac{1}{2} f^{-1} e^{-\sigma}\partial_R f E^2 \ , && {\omega^3}_1 = {}^4{\omega^3}_1 - \frac{1}{2} f^{-1} e^{-\sigma}\partial_R f E^3 \ , \\
&{\omega^4}_1 = {}^4{\omega^4}_1 - \frac{1}{2} f^{-1} e^{-\sigma}\partial_R f E^4 - 2f^2\partial_{R^2} w E^0 \ , && {\omega^2}_3 = {}^4{\omega^2}_3 - \frac{2f^2 w}{R^2} E^0 \ , \\
&{\omega^3}_4 = {}^4{\omega^3}_4 \ , && {\omega^4}_2 = {}^4{\omega^4}_2 \ ,
\end{aligned} \tag{A.76}$$

where ${}^4\omega^m_n$ represents the 4D spin connections (A.44-A.47), and e^m are the 4D tetrad 1-forms, which are related to the E^μ (A.60) and (A.61) via $e^m = f^{1/2} E^m$. We now proceed to evaluate the components of the gravitino variation (A.56), starting with $\mu = 0$:

$$\left(\partial_0 - \frac{1}{4}\omega_0^{\nu\rho}\gamma_{\nu\rho}\right)\epsilon^\alpha = \left(\partial_0 + \gamma^{23} f^2 \left(\partial_{R^2} + \frac{1}{R^2}\right) w - \frac{1}{2} f^{-1} e^{-\sigma}\partial_R f \gamma^1\right)\epsilon^\alpha \ , \tag{A.77}$$

$$\begin{aligned}
\frac{1}{24}(\gamma_0^{\nu\rho} - 4\delta_0^\nu\gamma^\rho) X_I F^I_{\nu\rho}\epsilon^\alpha = \Bigg(&-\gamma^{23} f^2 \left[\left(\partial_{R^2} + \frac{1}{R^2}\right) w + \frac{1}{3} f^{-1} X_I \left(\partial_{R^2} + \frac{1}{R^2}\right) u^I\right] \\
&+ \frac{1}{2} f^{-1} e^{-\sigma}\partial_R f \gamma^1\Bigg)\epsilon^\alpha \ ,
\end{aligned} \tag{A.78}$$

$$\frac{1}{6}\xi_I (3A_0^I - X^I\gamma_0)\epsilon^{\alpha\beta}\epsilon^\beta = \epsilon\frac{1}{3}\xi_I X^I \gamma^{23}\epsilon^\alpha \ , \tag{A.79}$$

where A_0^I stands for the component of the A^I 1-form along the E^0 flat veilbein, which amounts to $A_0^I = f^{-1} A_t^I = f^{-1}(f X^I) = X^I$. Thus, adding up the three contributions in (A.77), (A.78) and (A.79), we note that the terms proportional to γ^1 cancel out identically.

What is left is terms proportional to the identity and to γ^{23}, which when made to vanish separately, lead to two results

$$\partial_0 \epsilon = 0 \ , \tag{A.80}$$

$$f X_I \left(\partial_{R^2} + \frac{1}{R^2} \right) u^I - \epsilon \xi_I X^I = 0 \ . \tag{A.81}$$

This equation is another expression involving a projection of the quantity $\left(\partial_{R^2} + \frac{1}{R^2}\right) u^I$. Rather than a redundant relation, it can in fact be used to further constrain the ambiguity in u^I that arose from the projection in the gaugino variation (A.71). In fact, (A.75) and (A.81) imply that

$$f X_I \left(\frac{1}{2} \epsilon f^{-1} c^{IJK} X_J \xi_K + \frac{1}{2} f^{-1} \lambda X^I \right) - \epsilon \xi_I X^I = 0 \ , \tag{A.82}$$

which immediately means that $\lambda = 0$. The final result is given by

$$\boxed{\left(\partial_{R^2} + \frac{1}{R^2} \right) u^I = \frac{1}{2} \epsilon f^{-1} c^{IJK} X_J \xi_K \ .} \tag{A.83}$$

We now move on to the spatial components of (A.56). For $\mu = 1$:

$$\left(\partial_1 - \frac{1}{4} \omega_1^{\nu\rho} \gamma_{\nu\rho} \right) \epsilon^\alpha = \left(\partial_1 + \gamma^4 f^2 \partial_{R^2} w \right) \epsilon^\alpha \ , \tag{A.84}$$

$$\begin{aligned} \frac{1}{24} (\gamma_1^{\nu\rho} - 4 \delta_1^\nu \gamma^\rho) X_I F^I_{\nu\rho} \epsilon^\alpha = \Bigg(& -\gamma^4 f^2 \left[\left(2\partial_{R^2} - \frac{1}{R^2} \right) w + \frac{1}{3} f^{-1} X_I \left(2\partial_{R^2} - \frac{1}{R^2} \right) u^I \right] \\ & - \frac{1}{2} f^{-1} e^{-\sigma} \partial_R f \gamma^1 \Bigg) \epsilon^\alpha \end{aligned} \ , \tag{A.85}$$

$$\frac{1}{6} \xi_I (3 A_1^I - X^I \gamma_1) \epsilon^{\alpha\beta} \epsilon^\beta = \frac{1}{6} \xi_I X^I \gamma^4 \epsilon^\alpha = \frac{1}{6} f X_I \left(\partial_{R^2} + \frac{1}{R^2} \right) u^I \gamma^4 \epsilon^\alpha \ . \tag{A.86}$$

Again, adding the contributions (A.84), (A.85) and (A.86), and separating out the terms proportional to the identity, γ^1 and γ^4, we obtain

$$\boxed{\left(\partial_{R^2} - \frac{1}{R^2} \right) w + \frac{1}{2} f^{-1} X_I \left(\partial_{R^2} - \frac{1}{R^2} \right) u^I = 0 \ ,} \tag{A.87}$$

as well as the spatial dependence of the spinor ϵ: $\partial_R \epsilon = \frac{1}{2} f^{-1} (\partial_R f) \epsilon$, which leads to $\epsilon = \epsilon_0 f^{1/2}$ for some constant ϵ_0. The $\mu = 2$ and $\mu = 3$ components of (A.56) yield the same condition (A.87), which leaves us with $\mu = 4$ that introduces an additional term due to the appearance of the 4D spin connection terms:

$$\left(\partial_4 - \frac{1}{4}\omega_4^{\nu\rho}\gamma_{\nu\rho}\right)\epsilon^\alpha = \left(\partial_4 - \gamma^1 f^2\partial_{R^2}w - \frac{1}{4}f^{-1}e^{-\sigma}\partial_R f\gamma^{14} + \frac{1}{4}\ {}^4\omega_{mn}\gamma^{mn}\right)\epsilon^\alpha\ , \quad \text{(A.88)}$$

$$\frac{1}{24}(\gamma_4^{\nu\rho} - 4\delta_4^\nu\gamma^\rho)X_I F_{\nu\rho}^I\epsilon^\alpha = \left(\gamma^1 f^2\left[(2\partial_{R^2} - \frac{1}{R^2})w + \frac{1}{3}f^{-1}X_I(2\partial_{R^2} - \frac{1}{R^2})u^I\right]\right. \\ \left. + \frac{1}{4}f^{-1}e^{-\sigma}\partial_R f\gamma^{14}\right)\epsilon^\alpha\ , \quad \text{(A.89)}$$

$$\frac{1}{6}\xi_I(3A_4^I - X^I\gamma_4)\epsilon^{\alpha\beta}\epsilon^\beta = \left(\frac{1}{2}\epsilon\xi_I u^I\gamma^{23} - \frac{1}{6}fX_I(\partial_{R^2} + \frac{1}{R^2})u^I\gamma^1\right)\epsilon^\alpha\ . \quad \text{(A.90)}$$

Combining these terms leads to the condition (A.87) as well as the 4D relation

$$\left(\frac{1}{4}\ {}^4\omega_{mn}\gamma^{mn} + \frac{1}{2}\xi_I u^I\gamma^{23}\right)\epsilon^\alpha = 0\ . \quad \text{(A.91)}$$

Using the 4D spin connections (A.44-A.47), we find that ${}^4\omega_{mn}\gamma^{mn}\epsilon^\alpha = -2p\gamma^{23}\epsilon^\alpha$ with p from (A.54). We relate $fe^{2\sigma}$ to g_m based on (3.75), and find that

$$\boxed{p = \epsilon\left(\frac{1}{2}R^2(\partial_{R^2}g_m) + g_m - 1)\right) = \xi_I u^I\ .} \quad \text{(A.92)}$$

We can now gather the four main supersymmetry relations that were derived:

$$0 = G_{IJ}\left(\partial_R(fY^I) - \partial_R(fX^I)\right)\partial_i X^J\ , \quad \text{(A.93)}$$

$$0 = \left(\partial_{R^2} + \frac{1}{R^2}\right)u^I - \frac{1}{2}\epsilon f^{-1}c^{IJK}X_J\xi_K\ , \quad \text{(A.94)}$$

$$0 = \left(\partial_{R^2} - \frac{1}{R^2}\right)w + \frac{1}{2}f^{-1}X_I\left(\partial_{R^2} - \frac{1}{R^2}\right)u^I\ , \quad \text{(A.95)}$$

$$0 = -\epsilon R^2(\partial_{R^2}g_m) + 2\epsilon(1 - g_m) + 2\xi_I u^I\ . \quad \text{(A.96)}$$

BIBLIOGRAPHY

[1] K. Schwarzschild, *On the gravitational field of a mass point according to Einstein's theory*, *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)* **1916** (1916) 189 [physics/9905030].

[2] G. J. Turiaci, *New insights on near-extremal black holes*, .

[3] R. M. Wald, *The "Nernst theorem" and black hole thermodynamics*, *Physical Review D* **56** (1997) 6467–6474.

[4] R. M. Wald, *The thermodynamics of black holes*, *Living Reviews in Relativity* **4** (2001) 6.

[5] J. Erdmenger, *Introduction to gauge/gravity duality (tasi lectures 2017)*, .

[6] J. Polchinski, *Introduction to gauge/gravity duality*, .

[7] E. D'Hoker and D. Z. Freedman, *Supersymmetric gauge theories and the AdS/CFT correspondence*, .

[8] D. Marolf and J. Polchinski, *Gauge/Gravity Duality and the Black Hole Interior*, *Phys. Rev. Lett.* **111** (2013) 171301 [1307.4706].

[9] S. A. Frolov, R. Roiban and A. A. Tseytlin, *Gauge-string duality for superconformal deformations of N=4 Super Yang-Mills theory*, *JHEP* **07** (2005) 045 [hep-th/0503192].

[10] S. Sachdev, *What can gauge-gravity duality teach us about condensed matter physics?*, *Ann. Rev. Condensed Matter Phys.* **3** (2012) 9 [1108.1197].

[11] M. Ammon, J. Erdmenger, M. Kaminski and P. Kerner, *Superconductivity from gauge/gravity duality with flavor*, *Phys. Lett. B* **680** (2009) 516 [0810.2316].

[12] J. M. Maldacena, *The large n limit of superconformal field theories and supergravity*, *International Journal of Theoretical Physics* **38** (1999) 1113–1133.

[13] M. Natsuume, *AdS/CFT duality user guide*, .

[14] E. Witten, *Anti de sitter space and holography*, .

[15] R. Haag, J. T. Łopuszański and M. Sohnius, *All possible generators of supersymmetries of the s-matrix*, Nuclear Physics B **88** (1975) 257–274.

[16] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas and M. Van Raamsdonk, *The Hagedorn - deconfinement phase transition in weakly coupled large N gauge theories*, Adv. Theor. Math. Phys. **8** (2004) 603 [hep-th/0310285].

[17] F. Larsen, J. Nian and Y. Zeng, *AdS_5 black hole entropy near the BPS limit*, JHEP **06** (2020) 001 [1907.02505].

[18] J. Kinney, J. M. Maldacena, S. Minwalla and S. Raju, *An Index for 4 dimensional super conformal theories*, Commun. Math. Phys. **275** (2007) 209 [hep-th/0510251].

[19] S. Choi, J. Kim, S. Kim and J. Nahmgoong, *Large AdS black holes from QFT*, *arXiv preprint arXiv:1810.12067* (2018) .

[20] A. Cabo-Bizet, D. Cassani, D. Martelli and S. Murthy, *Microscopic origin of the bekenstein-hawking entropy of supersymmetric ads_5 black holes*, Journal of High Energy Physics **2019** (2019) 62.

[21] C.-M. Chang and X. Yin, *1/16 BPS states in $\mathcal{N} = 4$ super-Yang-Mills theory*, Phys. Rev. D **88** (2013) 106005 [1305.6314].

[22] L. Grant, P. A. Grassi, S. Kim and S. Minwalla, *Comments on 1/16 bps quantum states and classical configurations*, Journal of High Energy Physics **2008** (2008) 049–049.

[23] C. Romelsberger, *Counting chiral primaries in n=1 d=4 superconformal field theories*, Nuclear Physics B **747** (2006) 329–353.

[24] R. A. Janik and M. Trzetrzelewski, *Supergravitons from one loop perturbative n=4 sym*, Physical Review D **77** (2008) 085024.

[25] S. Choi, J. Kim, S. Kim and J. Nahmgoong, *Comments on deconfinement in AdS/CFT*, *arXiv preprint arXiv:1811.08646* (2018) .

[26] S. Choi and S. Kim, *Large AdS_6 black holes from CFT_5*, 1904.01164.

[27] F. Benini and P. Milan, *Black Holes in 4D $\mathcal{N}$=4 Super-Yang-Mills Field Theory*, Phys. Rev. X **10** (2020) 021037 [1812.09613].

[28] N. Ezroura, F. Larsen, Z. Liu and Y. Zeng, *The phase diagram of bps black holes in ads_5*, Journal of High Energy Physics **2022** (2022) 33.

[29] S. M. Hosseini, K. Hristov and A. Zaffaroni, *An extremization principle for the entropy of rotating BPS black holes in AdS_5*, JHEP **07** (2017) 106 [1705.05383].

[30] M. David, N. Ezroura and F. Larsen, *The attractor flow for ads_5 black holes in $\mathcal{N} = 2$ gauged supergravity*, .

[31] Z. W. Chong, M. Cvetic, H. Lu and C. N. Pope, *General non-extremal rotating black holes in minimal five-dimensional gauged supergravity*, *Phys. Rev. Lett.* **95** (2005) 161301 [hep-th/0506029].

[32] S. W. Hawking and H. S. Reall, *Charged and rotating AdS black holes and their CFT duals*, *Phys. Rev. D* **61** (2000) 024014 [hep-th/9908109].

[33] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[34] S. W. Hawking and D. N. Page, *Thermodynamics of Black Holes in anti-De Sitter Space*, *Commun. Math. Phys.* **87** (1983) 577.

[35] G. 't Hooft, *On the Phase Transition Towards Permanent Quark Confinement*, *Nucl. Phys. B* **138** (1978) 1.

[36] A. M. Polyakov, *Thermal Properties of Gauge Fields and Quark Liberation*, *Phys. Lett. B* **72** (1978) 477.

[37] L. Susskind, *Lattice Models of Quark Confinement at High Temperature*, *Phys. Rev. D* **20** (1979) 2610.

[38] E. Witten, *Anti-de Sitter space, thermal phase transition, and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505 [hep-th/9803131].

[39] J. Maldacena, *The large-n limit of superconformal field theories and supergravity*, *International journal of theoretical physics* **38** (1999) 1113.

[40] J. Choi, S. Lee and S. Lee, *Near Horizon Analysis of Extremal AdS(5) Black Holes*, *JHEP* **05** (2008) 002 [0802.3330].

[41] S. Bhattacharyya, V. E. Hubeny, S. Minwalla and M. Rangamani, *Nonlinear Fluid Dynamics from Gravity*, *JHEP* **02** (2008) 045 [0712.2456].

[42] M. Rangamani, *Gravity and Hydrodynamics: Lectures on the fluid-gravity correspondence*, *Class. Quant. Grav.* **26** (2009) 224003 [0905.4352].

[43] D. Harlow and H. Ooguri, *A universal formula for the density of states in theories with finite-group symmetry*, 2109.03838.

[44] D. Harlow, B. Heidenreich, M. Reece and T. Rudelius, *The Weak Gravity Conjecture: A Review*, 2201.08380.

[45] A. Chamblin, R. Emparan, C. V. Johnson and R. C. Myers, *Charged AdS black holes and catastrophic holography*, *Phys. Rev. D* **60** (1999) 064018 [hep-th/9902170].

[46] M. M. Caldarelli, G. Cognola and D. Klemm, *Thermodynamics of Kerr-Newman-AdS black holes and conformal field theories*, *Class. Quant. Grav.* **17** (2000) 399 [hep-th/9908022].

[47] J. Maldacena, D. Stanford and Z. Yang, *Conformal symmetry and its breaking in two dimensional Nearly Anti-de-Sitter space*, PTEP **2016** (2016) 12C104 [1606.01857].

[48] S. Sachdev, *Universal low temperature theory of charged black holes with AdS_2 horizons*, J. Math. Phys. **60** (2019) 052303 [1902.04078].

[49] J. Preskill, P. Schwarz, A. D. Shapere, S. Trivedi and F. Wilczek, *Limitations on the statistical description of black holes*, Mod. Phys. Lett. A **6** (1991) 2353.

[50] A. Almheiri and B. Kang, *Conformal Symmetry Breaking and Thermodynamics of Near-Extremal Black Holes*, JHEP **10** (2016) 052 [1606.04108].

[51] F. Larsen, *A nAttractor mechanism for $nAdS_2/nCFT_1$ holography*, JHEP **04** (2019) 055 [1806.06330].

[52] J. Hong, F. Larsen and J. T. Liu, *The scales of black holes with $nAdS_2$ geometry*, JHEP **10** (2019) 260 [1907.08862].

[53] P. J. Silva, *Thermodynamics at the BPS bound for Black Holes in AdS*, JHEP **10** (2006) 022 [hep-th/0607056].

[54] U. Moitra, S. P. Trivedi and V. Vishal, *Extremal and near-extremal black holes and near-CFT_1*, JHEP **07** (2019) 055 [1808.08239].

[55] P. Nayak, A. Shukla, R. M. Soni, S. P. Trivedi and V. Vishal, *On the Dynamics of Near-Extremal Black Holes*, JHEP **09** (2018) 048 [1802.09547].

[56] L. V. Iliesiu and G. J. Turiaci, *The statistical mechanics of near-extremal black holes*, JHEP **05** (2021) 145 [2003.02860].

[57] M. Heydeman, L. V. Iliesiu, G. J. Turiaci and W. Zhao, *The statistical mechanics of near-BPS black holes*, 2011.01953.

[58] F. Larsen and S. Paranjape, *Thermodynamics of Near BPS Black Holes in AdS_4 and AdS_7*, 2010.04359.

[59] A. Kitaev, *A simple model of quantum holography*, in *KITP strings seminar and Entanglement*, vol. 12, p. 26, 2015.

[60] J. Maldacena and D. Stanford, *Remarks on the Sachdev-Ye-Kitaev model*, Phys. Rev. D **94** (2016) 106002 [1604.07818].

[61] A. M. Charles and F. Larsen, *A one-loop test of the near-AdS_2/near-CFT_1 correspondence*, JHEP **07** (2020) 186 [1908.03575].

[62] S. Choi and F. Larsen, *Effective Field Theory of Quantum Black Holes*, 2108.04028.

[63] S. Choi, J. Kim, S. Kim and J. Nahmgoong, *Large AdS black holes from QFT*, 1810.12067.

[64] M. David, J. Nian and L. A. Pando Zayas, *Gravitational Cardy Limit and AdS Black Hole Entropy*, JHEP **11** (2020) 041 [2005.10251].

[65] A. Cabo-Bizet, D. Cassani, D. Martelli and S. Murthy, *Microscopic origin of the Bekenstein-Hawking entropy of supersymmetric AdS_5 black holes*, JHEP **10** (2019) 062 [1810.11442].

[66] S. Choi, J. Kim, S. Kim and J. Nahmgoong, *Comments on deconfinement in AdS/CFT*, 1811.08646.

[67] F. Benini and A. Zaffaroni, *A topologically twisted index for three-dimensional supersymmetric theories*, JHEP **07** (2015) 127 [1504.03698].

[68] F. Benini, K. Hristov and A. Zaffaroni, *Black hole microstates in AdS_4 from supersymmetric localization*, JHEP **05** (2016) 054 [1511.04085].

[69] F. Benini, K. Hristov and A. Zaffaroni, *Exact microstate counting for dyonic black holes in AdS4*, Phys. Lett. B **771** (2017) 462 [1608.07294].

[70] A. Zaffaroni, *AdS black holes, holography and localization*, Living Rev. Rel. **23** (2020) 2 [1902.07176].

[71] S. M. Hosseini and A. Zaffaroni, *Large N matrix models for 3d $\mathcal{N} = 2$ theories: twisted index, free energy and black holes*, JHEP **08** (2016) 064 [1604.03122].

[72] J. T. Liu, L. A. Pando Zayas, V. Rathee and W. Zhao, *One-Loop Test of Quantum Black Holes in anti–de Sitter Space*, Phys. Rev. Lett. **120** (2018) 221602 [1711.01076].

[73] C. Copetti, A. Grassi, Z. Komargodski and L. Tizzano, *Delayed Deconfinement and the Hawking-Page Transition*, 2008.04950.

[74] F. Larsen and S. Lee, *Microscopic entropy of AdS_3 black holes revisited*, JHEP **07** (2021) 038 [2101.08497].

[75] M. Natsuume, *AdS/CFT Duality User Guide*, vol. 903. 2015, 10.1007/978-4-431-55441-7, [1409.3575].

[76] D. Kubiznak, R. B. Mann and M. Teo, *Black hole chemistry: thermodynamics with Lambda*, Class. Quant. Grav. **34** (2017) 063001 [1608.06147].

[77] W. Fu, D. Gaiotto, J. Maldacena and S. Sachdev, *Supersymmetric Sachdev-Ye-Kitaev models*, Phys. Rev. D **95** (2017) 026009 [1610.08917].

[78] R. Dijkgraaf, J. M. Maldacena, G. W. Moore and E. P. Verlinde, *A Black hole Farey tail*, hep-th/0005003.

[79] R. M. Wald, *Black hole entropy is the Noether charge*, Phys. Rev. D **48** (1993) R3427 [gr-qc/9307038].

[80] V. Iyer and R. M. Wald, *Some properties of Noether charge and a proposal for dynamical black hole entropy*, *Phys. Rev. D* **50** (1994) 846 [gr-qc/9403028].

[81] D. Kastor, *Komar Integrals in Higher (and Lower) Derivative Gravity*, *Class. Quant. Grav.* **25** (2008) 175007 [0804.1832].

[82] D. Kastor, S. Ray and J. Traschen, *Enthalpy and the Mechanics of AdS Black Holes*, *Class. Quant. Grav.* **26** (2009) 195011 [0904.2765].

[83] T. Ortín, *Komar integrals for theories of higher order in the Riemann curvature and black-hole chemistry*, *JHEP* **08** (2021) 023 [2104.10717].

[84] P. A. Cano and M. David, *The extremal Kerr entropy in higher-derivative gravities*, *JHEP* **05** (2023) 219 [2303.13286].

[85] D. Cassani, A. Ruipérez and E. Turetta, *Boundary terms and conserved charges in higher-derivative gauged supergravity*, 2304.06101.

[86] A. Sen, *Black Hole Entropy Function, Attractors and Precision Counting of Microstates*, *Gen. Rel. Grav.* **40** (2008) 2249 [0708.1270].

[87] M. Günaydin, G. Sierra and P. Townsend, *The geometry of n= 2 maxwell-einstein supergravity and jordan algebras*, *Nuclear Physics B* **242** (1984) 244.

[88] J. P. Gauntlett, J. B. Gutowski, C. M. Hull, S. Pakis and H. S. Reall, *All supersymmetric solutions of minimal supergravity in five- dimensions*, *Class. Quant. Grav.* **20** (2003) 4587 [hep-th/0209114].

[89] J. B. Gutowski and H. S. Reall, *General supersymmetric AdS(5) black holes*, *JHEP* **04** (2004) 048 [hep-th/0401129].

[90] S. Bhattacharyya, S. Minwalla and K. Papadodimas, *Small Hairy Black Holes in AdS_5xS^5*, *JHEP* **11** (2011) 035 [1005.1287].

[91] J. Markeviciute and J. E. Santos, *Hairy black holes in $AdS_5 \times S^5$*, *JHEP* **06** (2016) 096 [1602.03893].

[92] J. Markeviciute and J. E. Santos, *Evidence for the existence of a novel class of supersymmetric black holes with $AdS_5{\times}S^5$ asymptotics*, *Class. Quant. Grav.* **36** (2019) 02LT01 [1806.01849].

[93] O. J. C. Dias, P. Mitra and J. E. Santos, *New phases of $\mathcal{N} = 4$ SYM at finite chemical potential*, *JHEP* **05** (2023) 053 [2207.07134].

[94] A. Sen, *Black hole entropy function and the attractor mechanism in higher derivative gravity*, *JHEP* **09** (2005) 038 [hep-th/0506177].

[95] A. Sen, *Entropy function for heterotic black holes*, *JHEP* **03** (2006) 008 [hep-th/0508042].

[96] J. R. David and A. Sen, *CHL Dyons and Statistical Entropy Function from D1-D5 System*, JHEP **11** (2006) 072 [hep-th/0605210].

[97] A. Sen, *Entropy Function and AdS(2) / CFT(1) Correspondence*, JHEP **11** (2008) 075 [0805.0095].

[98] A. Sen, *Quantum Entropy Function from AdS(2)/CFT(1) Correspondence*, Int. J. Mod. Phys. A **24** (2009) 4225 [0809.3304].

[99] A. Sen, *Arithmetic of Quantum Entropy Function*, JHEP **08** (2009) 068 [0903.1477].

[100] D. Astefanesei, K. Goldstein, R. P. Jena, A. Sen and S. P. Trivedi, *Rotating attractors*, JHEP **10** (2006) 058 [hep-th/0606244].

[101] A. Castro, J. L. Davis, P. Kraus and F. Larsen, *5D attractors with higher derivatives*, JHEP **04** (2007) 091 [hep-th/0702072].

[102] F. Benini and E. Milan, *A Bethe Ansatz type formula for the superconformal index*, Commun. Math. Phys. **376** (2020) 1413 [1811.04107].

[103] A. Cabo-Bizet and S. Murthy, *Supersymmetric phases of 4d $\mathcal{N} = 4$ SYM at large N*, JHEP **09** (2020) 184 [1909.09597].

[104] P. Ntokos and I. Papadimitriou, *Black hole superpotential as a unifying entropy function and BPS thermodynamics*, 2112.05954.

[105] N. Bobev, K. Hristov and V. Reys, *AdS_5 holography and higher-derivative supergravity*, JHEP **04** (2022) 088 [2112.06961].

[106] D. Cassani, A. Ruipérez and E. Turetta, *Corrections to AdS_5 black hole thermodynamics from higher-derivative supergravity*, JHEP **11** (2022) 059 [2208.01007].

[107] J. T. Liu and R. J. Saskowski, *Four-derivative Corrections to Minimal Gauged Supergravity in Five Dimensions*, 2201.04690.

[108] G. Kántor, C. Papageorgakis and P. Richmond, *AdS_7 black-hole entropy and 5D $\mathcal{N} = 2$ Yang-Mills*, JHEP **01** (2020) 017 [1907.02923].

[109] D. D. K. Chow, *Charged rotating black holes in six-dimensional gauged supergravity*, Class. Quant. Grav. **27** (2010) 065004 [0808.2728].

[110] F. Larsen, *The Attractor Mechanism in Five Dimensions*, Lect. Notes Phys. **755** (2008) 249 [hep-th/0608191].