**Self-Conjugate Cobordism and the Rectified Adams-Novikov Spectral Sequence**

by

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*To My Grandparents*

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# ABSTRACT

This thesis considers the problem of computing the cobordism groups associated to manifolds with self-conjugate and double-real structures. In the first two chapters, we discuss the historical and mathematical background relevant to the problem, and highlight the parallels with our own arguments. In Chapter 3, we introduce a new spectral sequence, called the rectified Adams-Novikov spectral sequence, which we show converges to the relevant cobordism groups. This is a further generalization of both the classical Adams spectral sequence and the generalized Adams-Novikov spectral sequence. In particular, our spectral sequence relies on the resolution of the classical complex cobordism group as a comodule over two specific Hopf algebroids, one for each of self-conjugate and double-real cobordism. We give a complete computation of the algebraic structure of these Hopf algebroids, showing each is polynomial and giving a determination of the respective coproduct structures. Additional useful properties of these Hopf algebroids are also shown. In the case of self-conjugate cobordism, we show that our spectral sequence collapses, and we discuss the potential for collapse of the spectral sequence associated to double-real cobordism.

In Chapter 4, we discuss Sage computations which allow us to compute the self-conjugate and double-real cobordism groups to degree 16, which doubles the height of previous computations. We produce code which symbolically solves for the image of each polynomial generator in our given Hopf algebroids under their coproduct maps. We construct the reduced cobar complex and associated differentials coming from our spectral sequence, and compute the homology to recover the homotopy groups. Additional intermediate computations are also included. We conclude by including a list of tables containing the result of the computations given in Chapter 4.

# CHAPTER 1

# Introduction

The field of algebraic topology has shaped and been shaped by the development of cobordism theory. Implicit in the work in the work of Poincaré, the first definitions of cobordism was made explicit by Pontryagin in [Pon50]. At its simplest, cobordism relates $n$-dimensional manifolds which form the boundary of manifolds in dimension $n+1$. René Thom was the first to observe that cobordism classes of manifolds (originally a purely geometric construction) were in bijection with certain homotopy classes of maps [Tho54]. Thom's work specifically concerned smooth manifolds, both with and without orientation, but generalizations due to the independent work of Pontryagin [Pon50, Pon59], and later Lashof [Las63], solidified the connection between homotopy classes of maps and cobordisms of smooth manifolds with more general stable normal structures.

These results motivated Milnor [Mil60] and Novikov [Nov60] to independently compute the cobordism ring associated to manifolds with stable complex structure on their normal bundle, denoted $\Omega_*^{\mathbb{C}}$. Both answered the problem conclusively, providing a complete calculation of $\Omega_*^{\mathbb{C}}$ along with the image under the Hurewicz homomorphism. Each proof hinged on the application of a recently developed computational tool of Adams. His work in [Ada58, Ada59], on mod $p$ singular homology operations and the Steenrod Algebra $\mathcal{A}_*$ motivated his introduction of the Adams Spectral Sequence:

$$\mathrm{Cotor}_{\mathcal{A}_*}(H_*(\mathbb{S};\mathbb{F}_p), H_*(X;\mathbb{F}_p)) \Rightarrow \pi_*(X) \otimes \mathbb{F}_p.$$

The application of this tool by Milnor and Novikov is particularly notable as it is a complete solution to a geometric problem that uses strictly algebraic techniques. The relevant geometry simply provides context for the existence of the module structure maps. This result showed that there was room for a more unified approach to these geometrically motivated problems, and along with Atiyah's work on $K$-theory and generalized cohomology theories, began to steer homotopy theory towards developing exactly this approach, which would come to be called spectral algebra.

The seeds of this idea were contained in Thom's original work. His result (and its subsequent generalizations) utilized the Pontryagin-Thom construction, a certain quotient of the universal classifying bundle determined by the tangential structure being classified. The resulting space, called the Thom space, carried a large amount of homotopical information. Additionally, this construction was done dimension-wise, meaning there were a sequence of Thom spaces, each related to the next by a series of connecting maps coming from geometric suspension. These connecting maps allowed for the study of so-called "stable" homotopical data, i.e. the information that persists as dimension was increased.

Milnor points out that this structure could be encoded by the recently defined "spectrum", a term introduced by Lima in [Lim59], with further revisions of the definition due to Spanier [Spa59] to solve problems related to stable duality as recounted in [May99]. Additionally, Milnor also mentions that the structure is similar to Adams "stable object", mentioned in [Ada59]. Since these foundational observations, the accepted definition of spectra and stability have been significantly overhauled. In particular, we want to make important note that, despite the similarities highlighted by Milnor, the "spectrum" of Lima and Spanier and the "stable object" of Adams are distinct, and neither align with the modern perspective of spectra as the objects in the stable homotopy category as noted in [May80], where a more complete and thorough account of the historical development can be found. With the modern context of spectra, the computations of Thom can be viewed as computing $\pi_*(MO)$, the homotopy groups associated to the spectra real unoriented cobordism spectrum $MO$, while Milnor-Novikov computed $\pi_*(MU)$, the groups associated to the complex cobordism spectrum $MU$. Atiyah's work on complex and real $K$-theory computes the homotopy groups of the spectra $KU$ and $KO$. Singular homology is computed utilizing the Eilenberg-Maclane spectrum $H\mathbb{F}_p$, and Atiyah's work on generalized homology theories, when combined with Brown's Representability Theorem [Bro82], allow any generalized homology theory to be studied by studying the representing spectra.

The spectral perspective on cobordism can also be extended to other flavors of cobordism, including oriented cobordism ($MSO$), symplectic cobordism ($MSp$), and framed cobordism ($Mfr$). Additionally, this thesis treats the theories of self-conjugate cobordism ($MSC$) and double-real cobordism ($MO[2]$). While the computation of $\pi_*(MU)$ mentioned above is the most celebrated, the initial work of Thom showed $MO$ and $MSO$ were quite tractable, giving a complete computation of $\pi_*(MO)$, and $\pi_*(MSO) \otimes \mathbb{Q}$, with work by Milnor and others further characterizing the torsion of $\pi_*(MSO)$. Given how accessible real and complex cobordism are, one might expect symplectic cobordism to follow similarly. However, the work of Kochman in [Koc80, Koc82, Koc93] shows that $MSp$ is highly complex, with [Koc93] specifically highlighting that in the classical Adams Spectral Sequence, the differentials $d^r$ are non-trivial for all $r \geq 2$. The spectrum $Mfr$ is even more intractable. Pontryagin's initial work in [Pon50] showed that $\pi_*(Mfr) \cong \pi_*(\mathbb{S})$, or equivalently, computing framed cobordism groups is equivalent to computing the stable homotopy groups of spheres, a problem which has been at the center of stable homotopy theory since its creation.

The self-conjugate cobordism ring walks the line between the tractable and intractable. Smith and Stong [SS68b] computed $\pi_*(MSC)\otimes\mathbb{Z}[\frac{1}{2}]$ to be polynomial, showing that the only torsion for $\pi_*(MSC)$ will be of the form $\mathbb{Z}/2^n\mathbb{Z}$. Some multiplicative relations were derived by Gozman [Goz77], motivated by the work of Buchstaber-Novikov[BN71], but ultimately the computation of $\pi_*(MSC)$ resisted a complete solution until now. This thesis presents original computer computations and observations which contributed to the complete algebraic characterization of $\pi_*(MSC)$, which is joint with Hu, Kriz, and Somberg, [HKRS23].

This includes the observation that the spectral sequence presented in Theorem 3.1.1 is distinct from the generalized Adams-Novikov spectral sequence. In particular, we note that the rectified Adams-Novikov spectral sequence we present here utilizes advances in spectral algebra and the collapse leverages results in motivic homotopy theory, both of which were unavailable at the time of the original work on $MSC$.

As mentioned above, since Lima's initial definition of a spectrum, decades of work has gone into developing the theory of spectra, including the construction of the derived category of spectra and the introduction of a symmetric monoidal smash product. This acts as a tensor product for spectra, allowing for more explicit adaptations of algebraic constructions in the category of spectra and would making the transition from topology to algebra more natural. Developing these tools required modifications of the classical definitions as naïve spectra fail to admit such a suitable smash product as shown in [Lew91]. Specifically, the categories of orthogonal spectra [May80, MM02], symmetric spectra [HSS00], and $\mathbb{S}$-modules [EKMM97] were each introduced, along with a corresponding smash product, to alleviate certain defects in the naïve category. Each of these categories admits a model structure which were shown to be Quillen equivalent, and therefore independent of the choice of model when working in the derived setting. We ultimately work in the derived category, but at the strict spectral level, we chose to work in the category of $\mathbb{S}$-modules. This is primarily for the convenience offered when working with the smash product over certain ring spectra, but we do highlight the involvement of the author's advisor in its development [EKMM97].

By taking advantage of the increased algebraic freedom granted by the development of a symmetric monoidal smash product, we produce the spectrum $MU \wedge_{MSC} MU$ , which when paired with $MU$, allow us to consider the Hopf algebroid $(\pi_*(MU), \pi_*(MU \wedge_{MSC} MU))$. We use this to produce the spectral sequence:

$$\mathrm{Cotor}_{\pi_*(MU \wedge_{MSC} MU)}(\pi_*(MU), \pi_*(MU)) \Rightarrow \pi_*(MSC)$$

which we show collapses in Theorem 3.5.1 without extensions, in a spectral analog of the algebraic resolution in [GM74]. We note the parallel with the generalized Adams-Novikov spectral sequence

$$\mathrm{Cotor}_{E_*(E)}(E_*(\mathbb{S}), E_*(X)) \Rightarrow \pi_*(X)$$

which for certain spectrum $E$, produces a spectral sequence over the Hopf algebroid $(\pi_*(E), \pi_*(E \wedge E))$, more commonly denoted $(E_*, E_*E)$. As such, we call the above spectral sequence the rectified Adams-Novikov spectral sequence.

We also want to note that when $E$ is taken to be $MU$ or the Brown-Peterson spectrum $BP$, we have an alternative perspective coming from the study of formal groups laws. The work of Quillen [Qui69] and Landweber[Lan75] show that there exists natural isomorphisms between the Lazard ring admitting the universal formal group law $L$, and $MU_*$, while also showing the associated Hopf algebroid characterizing strict isomorphisms of formal group laws, $LB$, is isomorphic as Hopf algebroids to $MU_*MU$, with equivalent results holding for $p$-typical formal groups and the $BP_*$ analogous statements. Our characterization of the Hopf algebroid $(MU_*, \pi_*(MU \wedge_{MSC} MU))$ takes advantage of this perspective, further motivated by results of [BN71].

The rest of thesis is organized as follows. In Chapter 2, we fully give the definitions which parallel the historical introduction given in this chapter. We give the concrete definitions needed to construct the spectral sequence of Theorem 3.1.1 , and we provide the context via formal group laws for our discussion of the Hopf algebroid $((MU_*), \pi_*(MU \wedge_{MSC} MU))$. In Chapter 3, we give the general construction of the rectified Adams-Novikov spectral sequence. We characterize $((MU_*), \pi_*(MU \wedge_{MSC} MU))$ algebraically. We prove the collapse of this spectral sequence over $((MU_*), \pi_*(MU \wedge_{MSC} MU))$ is obtained by using the "motivic loop", via the motivic homotopy theory over $\mathbb{C}$ introduced by Hu, Kriz, and Ormsby in [HKO11]. The collapse follows using results of [GWX21] and [IWX20].

Additionally, to simplify the above concrete calculations stemming from $MSC$, we find it useful to work with the spectrum $MO[2]$, the double-real cobordism ring, first introduced by Kitchloo and Wilson in [KW15]. In some sense, this spectrum considers the "real" part of $MSp$, but was significantly less studied. We produce a similar Hopf algebroid $(MU_*, \pi_*(MU \wedge_{MO[2]} MU))$, which is easier to characterize, but whose associated rectified Adams-Novikov spectral sequence is harder to compute. This characterization is also obtained via the same formal group methods discussed for $(MU \wedge_{MSC} MU)_*$.

We use the simplified algebraic structure to produce an alternative characterization of the rectified Adams-Novikov spectral sequence associated to $MSC$, whose $E_2$-term reduces to the cohomology of a polynomial algebra (whose generators correspond to permanent cycles in $\mathrm{Ext}^1$), acting on $MU_*$. The action is complicated but can be characterized using the connections to formal groups described above, allowing concrete computer computations, given in Chapter 4. Finally, Chapter 5 contains the results of these computations, in addition to intermediate computational results. These intermediate computations include characterization of the right unit associated to each Hopf algebroid, and the determination of primitive generators for $\pi_*(MU \wedge_{MO[2]} MU)$.

# CHAPTER 2

# Cobordism, Algebra, and Spectra

Here we recall the necessary background material needed to proceed with Chapter 3 and Chapter 4. If the reader is interested in deeper context, we direct the reader to the following references. For Section 2.1, we refer the reader to [Ada69, LMSM86] for details related to the foundations of spectra, and for further details on the symmetric monoidal smash product, we refer to [EKMM97] . For Section 2.2, we refer to [Sto68], although for the generalized result in Theorem 2.2.1 the original reference of [Las63] is particularly readable. For the information of Sections 2.4 and 2.5, we refer to Appendices A and B of [Rav86] respectively. For Section 2.3, the original definition of $MO[2]$ is given in [KW15], while a definition of $MSC$ and related results are given in [SS68b].

## 2.1 Spectra and Spectral Algebra

We present the the present settled definitions of the core constructions as a reference.

**Definition 2.1.1.** *A prespectrum is a collection of based topological spaces $(E_n)_{n=0}^{\infty}$, and based maps $\sigma_n : \Sigma E_n \to E_{n+1}$. If the the corresponding adjoint map $\tilde{\sigma} : E_n \to \Omega E_{n+1}$ is an homeomorphism, then we say $E$ is a spectrum. A map of (pre)spectra $f : E \to F$ is a* *collection of maps $f_n : E_n \to F_n$ such that the diagram:*

$$\begin{array}{ccc} E_n & \xrightarrow{\tilde{\sigma}_{E,n}} & \Omega E_{n+1} \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle \Omega f_{n+1}} \\ F_n & \xrightarrow{\tilde{\sigma}_{F,n}} & \Omega F_{n+1} \end{array}$$

*commutes (strictly).*

Importantly, given any space $X$, we can produce a prespectrum by letting $X_n = \Sigma^n X$, and letting the $\sigma_n$ be the trivial homeomorphism. This is called the suspension prespectrum of $X$, and is denoted $\Sigma^\infty X$. If one attempts this with $S^0$, we obtain the sphere prespectrum. However, the corresponding adjoints $\tilde{\sigma}_n$ are not homeomorphisms, meaning that the naive construction of a "sphere spectrum" fails. As such, we recall that the forgetful functor from spectra to prespectra has a left adjoint, called "spectrification" which suitably produces a spectrum given any prespectrum. Therefore, the sphere spectrum $\mathbb{S}$ is defined as the spectrification of the suspension prespectrum $\Sigma^\infty S^0$.

With these definitions in mind, the concept of the $n$-sphere spectrum $\mathbb{S}^n$, homotopy classes of maps $[E, F]$, and homotopy groups of a spectrum $\pi_n(E) := [\mathbb{S}^n, E]$ can be intuited based on their traditional topological analogs, or their existence can be taken on faith for the purposes of this thesis. The following notion of weak equivalence is space-level construction that has been adapted to the category of spectra.

**Definition 2.1.2.** *A weak equivalence of spectra $f : E \to F$ is a map of spectra such that $f_* : \pi_n(E) \to \pi_n(F)$ is an isomorphism for all $n$.*

For suitably chosen fibrations and cofibrations, and with the definition of weak equivalence provided above, the category of spectra can be given a model category structure. The resulting derived category is the setting for most of modern homotopy theory. Additionally, as referenced in the introduction, there was significant work done to produce a symmetric monoidal smash product in this setting, allowing for more explicit adaptations of algebraic constructions in the category of spectra. It should not surprise the reader that the definition and construction of such a stable smash product is quite technical and dependant on the initial model category. Specifically, we note that the constructions given below require working with $\mathbb{L}$-spectra. Briefly, $\mathbb{L}$-spectra generalize spectra as defined above by indexing over all finite dimensional subspaces of $\mathbb{R}^\infty$ as opposed to the natural numbers. This indexing is required to satisfy additional coherence conditions enabling the construction of the desirable smash product. A more complete description can be found in [EKMM97]. Therefore, for the purposes of this thesis, we simply assert the existence and give the following properties:

**Theorem 2.1.1.** *There exists a smash product $\wedge$ on $(\mathbb{L})$-spectra, such that when restricted to an appropriate full subcategory, the operation $\wedge$ is a symmetric monoidal product in the derived setting.*

At this point, we note that for our purposes, the suitable subcategory we will work in is the category of $\mathbb{S}$-modules, defined as follows:

**Definition 2.1.3.** *A $(\mathbb{L})$-spectrum $E$ is an $\mathbb{S}$-module if there exists a map $\lambda : \mathbb{S} \wedge E \to E$ which is a strict isomorphism of spectra.*

Now, that a smash product is given, we can begin strengthening the analogy with algebra. As such, let us introduce the following definition:

**Definition 2.1.4.** *A spectrum $R$ is a ring spectrum if there exists a map $\mu : R \wedge R \to R$, and unit map $\eta : \mathbb{S} \to R$. The spectrum $R$ is $A_\infty$ if $\mu$ is associative up to arbitrary higher homotopies. $R$ is $E_\infty$ if it is $A_\infty$ and $\mu$ commutes up to arbitrary higher homotopies.*

It should also be noted that our results in later chapters are in the derived stetting. In light of the Quillen equivalences (which preserve smash products) between orthogonal spectra, symmetric spectra and $\mathbb{S}$-modules, these results are therefore are independent of the setting, and $\mathbb{S}$-modules have been chosen strictly for convenience. Now, that we have defined

**Definition 2.1.5.** *Let $R$ be an $\mathbb{S}$-module. $R$ is an $\mathbb{S}$-algebra if it also an $A_\infty$-ring spectrum. If $R$ is $E_\infty$, then we say $R$ is an commutative $\mathbb{S}$-algebra.*

Now that we have the notion of an algebra, left and right modules over an algebra are defined by asking that the spectral analogs simply satisfy the necessary commutative diagrams. This leads to the following:

**Definition 2.1.6.** *Given an $\mathbb{S}$-algebra $R$ and left and right $R$-module spectra $M$ and $N$ respectively, $M \wedge_R N$ is the $R$-module spectrum defined as the coequalizer of the diagram:*

$$M \wedge R \wedge N \rightrightarrows M \wedge N \longrightarrow M \wedge_R N$$

*where the maps are defined analogously to the maps in the traditional tensor product.*

The category of $\mathbb{S}$-modules admits all limits and colimits, making this construction well-defined. It is relevant for later that the functors $(-) \wedge E$ and the generalized $(-) \wedge_R E$ define monads in the category of $\mathbb{S}$-modules. We will leverage this fact in Chapter 3 to construct and define the Rectified Adams-Novikov spectral sequence.

## 2.2 Cobordism and the Pontryagin-Thom Isomorphism

Now that we have discussed the spectral background of this thesis, we shall proceed to background on cobordism.

**Definition 2.2.1.** *Given two $n$-dimensional manifolds $M$ and $N$, we say that $M$ and $N$ are cobordant if there exists an $(n+1)$-dimensional manifold $W$ such that $M$ and $N$ form the boundary of $W$, or more precisely: $\partial W = M \sqcup N$.*

This is a loose definition of cobordism, and does not consider any underlying structure on the manifolds, such as orientation, almost-complex structure, or framing. However, this definition proves to be an equivalence relation amongst manifolds, allowing us to classify $n$-dimensional manifolds up to cobordism. Additionally, if we let $\varnothing$ be the empty manifold in each dimension, then the disjoint union operator defines an addition on cobordant manifolds whose zero is the class $[\varnothing]$, yielding a group structure. The cross product on manifolds produces a well-defined multiplication operation, transforming our collection of manifolds into a ring.

Next, we see that our definition of cobordism can be adapted to more specialized classes of manifolds. Since our definition is largely structure agnostic, it suffices to introduce the right notion of structure, and then check compatibility. As alluded to in Chapter 1, this formalization of structure is due Lashof:

**Definition 2.2.2.** *Let $M$ be a manifold with normal bundle $\nu$, and let $\nu(i)$ denote the map $M \to BO$ classifying $\nu$. Fix a collection of spaces $(B_n)$ and fibrations $f_n : B_n \to BO(n)$ indexed over $n \in \mathbb{N}$. Then, a $(B,f)$-structure on a manifold $M$ is the collection of homotopy classes of lifts*

$$\begin{array}{ccc} & \overset{\widetilde{\nu(i)}}{\nearrow} & B_n \\ & & \downarrow f_n \\ M & \xrightarrow{\nu(i)} & BO(n) \end{array}$$

*for all sufficiently large $n$, along with maps $g_n : B_n \to B_{n+1}$ making the diagram:*

$$\begin{array}{ccc} B_n & \xrightarrow{g_n} & B_{n+1} \\ \downarrow f_n & & \downarrow f_{n+1} \\ BO(n) & \hookrightarrow & BO(n+1) \end{array}$$

*commute, where the lower map is the standard inclusion. Additionally, we ask that $g_n \circ \widetilde{\nu(i)}_n = \widetilde{\nu(i)}_{n+1}$. If such maps exist, we say $M$ is a $(B,f)$-manifold.*

The notation $\nu(i)$ is indicative of this construction arising from an embedding of $M \hookrightarrow \mathbb{R}^n$ for sufficiently large $n$. Considerations about choice of embedding are treated in Lashof's original text and notably, the $(B,f)$-structure of a given manifold is shown to be depend only on homotopy class of the embedding. As such, for a given $(B,f)$-structure, we can consider the collection of all closed $(B,f)$-manifolds. Note that given a $(B,f)$-manifold $M$, we can consider the "opposite" $(B, f)$-manifold $-M$. This space is the underlying manifold $M$, but whose $(B, f)$-structure is the one induced by the outer normal from the inclusion $M \cong M \times 1 \subset M \times [0, 1]$. Examples include orientable manifolds with the opposite orientation or the conjugate complex structure. For non-orientable manifolds the structure on $-M$ coincides with the structure on $M$. This lets us proceed as with the following definition

**Definition 2.2.3.** *Given two $n$-dimensional closed $(B, f)$-manifolds $M$ and $N$, we say that $M$ and $N$ are $(B, f)$-cobordant if there exists a $(n+1)$-dimensional $(B, f)$-manifold $W$ such that $M$ and $-N$ form the boundary of $W$, or more precisely: $\partial W = M \sqcup -N$. Let $\Omega_n^B$ denote the group of cobordism classes of $n$-dimensional $(B, f)$-manifolds by $\Omega_n^B$ (where the operation is given by disjoint union and inverses are given by the equivalence class $[-M]$). If there is an induced $(B, f)$-structure on $M \times N$ for any $(B, f)$-manifolds $M$, $N$, then we denote the graded ring of $(B, f)$-manifolds by $\Omega_*^B$.*

Our original definition applies to the trivial $(B, f)$-structure with $B = BO$. New examples of $(B, f)$-structures include manifolds with stable complex normal bundles ($B = BU$) and orientable manifolds ($B = BSO$), with the usual identifications between $BU$, $BSO$ and $BO$ serving as the structure maps.

At this point, we note that the additional structure we have placed on these manifolds can also be encoded as structures on the tangent bundles of each manifold. These bundles can be classified as pullbacks of the universal bundle $\gamma_{\mathbb{C}}^n$, $\gamma_{\mathbb{R}}^n$ and $\gamma_{SO}^n$. Namely, we always have a pullback diagram:

$$\begin{array}{ccc} \tau_M & \longrightarrow & \gamma_{\mathbb{C}}^n \\ \downarrow & & \downarrow \\ M & \longrightarrow & BU(n) \end{array}$$

which is unique up to the homotopy class of the map $M \to BU(n)$. The situation is analogous for $BO(n)$ and $BSO(n)$ for real unoriented and oriented cobordism respectively.

Additionally, there are embeddings:

$$BU(1) \hookrightarrow \cdots \hookrightarrow BU(n) \hookrightarrow BU(n+1) \hookrightarrow \dots$$

inducing pullbacks:

$$\begin{array}{ccc} \gamma^n_{\mathbb{C}} \oplus \underline{1}_{\mathbb{C}} & \longrightarrow & \gamma^{n+1}_{\mathbb{C}} \\ \downarrow & & \downarrow \\ BU(n) & \hookrightarrow & BU(n+1) \end{array}$$

Applying the Thom space to this construction we get a series of maps:

$$Th(\gamma^n_{\mathbb{C}} \oplus \underline{1}_{\mathbb{C}}) \sim Th(\gamma^n_{\mathbb{C}}) \wedge S^2 \to Th(\gamma^{n+1}_{\mathbb{C}}).$$

However, this is precisely the definition of a prespectrum $D$, with $D_{2n} = Th(\gamma^n_{\mathbb{C}})$. The spectrum associated to this prespectrum we denote by $MU$. Analogous constructions give $MO$ and $MSO$. At this point, it seems that this construction has left our original motivation of cobordism far behind. However, the following theorem allows us to use the tools of spectral algebra to study and in some cases, completely classify manifolds up to cobordism.

**Theorem 2.2.1** (Pontryagin-Thom)**.** *For manifolds with $(B, f)$-structure, where $B_i = BG_i$.*

$$\Omega^G_* \cong \pi_*(MG)$$

Namely, we have $MU_* = \Omega^U_*$, $MO_* = \Omega^O_*$ and $MSO_* = \Omega^{SO}_*$.

## 2.3 Double-Real and Self-Conjugate Cobordism

Let us now introduce the additional structures on manifolds we will study.

**Definition 2.3.1** (Double Real Manifold 1))**.** *A manifold has a double-real structure if its stable normal bundle, $\nu_M$, splits as $2\xi_M$ for some real bundle $\xi_M$. Formally, there is some*

*$N$ large enough such that $\tau_M$ and $\xi_M$ satisfy:*

$$\tau_M \oplus 2\xi_M = \tau_M \oplus \nu_M = \underline{N}$$

**Definition 2.3.2** (Self-Conjugate Manifold 1))**.** *A manifold is self conjugate if its stable normal bundle $\nu$ is isomorphic to its own complex conjugate. Formally:*

$$\nu \cong \overline{\nu}$$

These structures give rise to double-real and self-conjugate cobordism theories $\Omega_*^{O[2]}$ and $\Omega_*^{SC}$, respectively. Additionally, these structures can be characterized as pull-backs, just as real and complex cobordism were in Section 2.2. In the case of $MO[2]$, we have the pullback:

$$\begin{array}{ccc} 2\xi & \longrightarrow & 2\gamma_{\mathbb{R}}^n \\ \downarrow & & \downarrow \\ M & \longrightarrow & BO(n) \end{array}$$

The embeddings $BO(n) \hookrightarrow BO(n+1)$ induce maps $2\gamma_{\mathbb{R}}^n \oplus \underline{2} \to 2\gamma_{\mathbb{R}}^{n+1}$, allow us to form a prespectrum with $D_{2n} = Th(2\gamma_{\mathbb{R}}^n)$. The resulting spectrum is denoted $MO[2]$. This characterization lets us make the following equivalent definition to the one above

**Definition 2.3.3** (Double Real Manifold 2))**.** *Let $G_{2i} = G_{2i+1} = O(i)$. Then, the $(B, f)$-structure corresponding to manifolds with a double real structure is given by $B_n = BG_n$ with maps $f_{2i} : BO(i) \to BO(2i)$ induced by the map $O(i) \to O(2i)$ given by*

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

*and the map $f_{2i+1} : BO(i) \to BO(2i+1)$ is similarly given by appending a final* 1.

When treating self-conjugate cobordism, we note that we have two maps $id_n : U(n) \to$

$U(n)$ and conjugation $c_n : U(n) \to U(n)$, and can form the homotopy equalizer, here denoted $SC(n)$.

$$SC(n) \longrightarrow U(n) \overset{id_n}{\underset{c_n}{\rightrightarrows}} U(n)$$

We note that the homotopy equalizer of a topological group remains a topological group. Therefore, we can take the classifying space to obtain $BSC(n)$, which classifies the virtual bundle $\gamma_{\mathbb{C}} - \overline{\gamma}_{\mathbb{C}}$. Letting $\gamma^n_{SC}$ denote the universal self-conjugate $n$-bundle, we get the diagram

$$\begin{array}{ccc} \nu & \longrightarrow & \gamma^n_{SC} \\ \downarrow & & \downarrow \\ M & \longrightarrow & BSC(n) \end{array}$$

which allows us to perform the construction of the spectrum $MSC$, again by taking iterated Thom spaces of $\gamma^n_{SC}$. Additionally, we can present the $(B, f)$-structure associated to self conjugate manifolds.

**Definition 2.3.4** (Self-Conjugate Manifold 2))**.** *Let $B_{2n} = B_{2n+1} = BSC(n)$, and maps $f_{2n} : BSC(n) \to BU(n) \to BO(2n)$ given by composing the map induced by the equalizer with the standard inclusion of $BU(n)$ into $BO(2n)$, and $f_{2n+1}$ given by the trivial inclusion. The maps $g_{2n}$ are induced by pulling back the inclusions $BU(n) \hookrightarrow BU(n+1)$ and $g_{2n+1}$ is again taken to be the trivial inclusion. This defines the self-conjugate $(B, f)$-structure.*

We see that in the cases of both $MSC$ and $MO[2]$, we satisfy the necessary conditions to apply the Pontryagin-Thom isomorphism, and solidify our approach to classify cobordism classes of self-conjugate and double-real manifolds by computing $\pi_*(MSC)$ and $\pi_*(MO[2])$.

Before moving onto the algebraic background relevent to this thesis, we first want to consider an example of a family of manifolds with both double-real and self-conjugate structure. Namely, we can equip $\mathbb{R}P^{4k+1}$ (and by extension $\mathbb{R}P^{\infty}$ ) with these structures. In the case of $\mathbb{R}P^{4k+1}$ we note that this exists as a subspace of $\mathbb{C}P^{2k+1}$, and therefore we can pullback the canonical complex stable normal bundle over $\mathbb{C}P^{2k+1}$, $\gamma_{\mathbb{C}}$ to the stable normal bundle over$\mathbb{R}P^{4k+1}$. As our projective space is a real manifold, the pullback $i^*\gamma_{\mathbb{C}}$ bundle splits as

$2\gamma_{\mathbb{R}}$. This gives $\mathbb{R}P^{4k+1}$ a natural double-real structure. Additionally, we can note that as conjugation fixes the real subspaces, the normal bundle of $\mathbb{R}P^{4k+1}$ is classified over $BSC$ by the construction described above as well.

## 2.4 Hopf Algebras and Algebroids

Now that we have connected the geometric origins of cobordism to the modern spectral approach, we might be asking why this approach is preferred. We see quite quickly that the extra structure provided by working spectrally allow us to completely compute $MU_*$. We now give the result due to Milnor and Novikov:

**Theorem 2.4.1** (Milnor-Novikov). *$MU_* = \mathbb{Z}[x_1, x_2, \dots]$ where $|x_i| = 2i$. Under the Hurewicz homomorphism,*

$$h : MU_* = \mathbb{Z}[x_1, x_2, \dots] \to H_*(MU, \mathbb{Z}) = \mathbb{Z}[b_1, b_2 \dots]$$

*we have*

$$h(x_i) = \begin{cases} -pb_i & i = p^k - 1 \\ -b_i & \text{else} \end{cases}$$

*modulo decomposable elements.*

As mentioned in Chapter 1, this computation relies on the Adams spectral sequence, a computational tool which computes the homotopy groups of a spectrum $X$ by producing a resolution of $X$ as a fibered sequence of spectra, and iteratively computing the homotopy groups of the fibers associated to the resolution. One obtains a long exact sequence on homotopy groups which we use to define an exact couple which defines a spectral sequence. Specifically, with the Adams spectral sequence, the spectrum $X$ is resolved as $H\mathbb{F}_p$-comodule spectrum for a prime $p$. Then, given the map $X \to H\mathbb{F}_p$, we note that by taking homology, we get that $H_*(X; \mathbb{F}_p)$ is a $H_*(H\mathbb{F}_p)$-comodule. From here, we can identify the spectral sequence as depicted in Chapter 1 with the sequence:

$$\mathrm{Ext}_{H_*(H\mathbb{F}_p)}(H_*(\mathbb{S}), H_*(X)) \Rightarrow \pi_*(X) \otimes \mathbb{F}_p$$

where coefficients have been omitted for clarity. At this point, it is important to point out that $H_*(H\mathbb{F}_p)$ admits the structure of a Hopf algebra, which makes the computations of certain families of Ext groups more tractable.

**Definition 2.4.1.** *A Hopf algebra over a commutative ring $K$ is an algebra $A$, along with additional structure maps:*

- *The coproduct $\Delta : A \to A \otimes_K A$*
- *The conjuguation $c : A \to A$*

*such that the dual coalgebra $A^*$ is an algebra with respect to $\Delta^*$, making $A$ a bialgebra, and $c$ makes the expected diagrams commute.*

The importance $H^*(H\mathbb{F}_p)$ plays in homotopy theory relates specifically to its role describing the stable cohomology operations for $H\mathbb{F}_p$-cohomology. This motivates the following definition:

**Definition 2.4.2.** *The mod $p$ Steenrod Algebra $\mathcal{A}^*$ is the algebra of mod $p$ cohomology operations. Specifically, $\mathcal{A}^* = H^*(H\mathbb{F}_p)$ and its dual $\mathcal{A}_* = H_*(H\mathbb{F}_p)$.*

The role $\mathcal{A}_*$ plays in the classical Adams spectral sequence can be generalized, provided the algebraic structure given by the Hopf algebra is also generalized. We introduce an abridged definition of a Hopf algebroid to provide a suitably general context. The following is adapted from [Rav86].

**Definition 2.4.3** (Abridged)**.** *A Hopf algebroid over a commutative ring $K$ is a cogroupoid object in the category of $K$-algebras. Concretely, this is a pair of $K$-algebras $(A, \Gamma)$ with the following structure maps:*

1. *The left unit $\eta_L : A \to \Gamma$, making $\Gamma$ a left $A$-module.*

2. *The right unit $\eta_R : A \to \Gamma$ making $\Gamma$ a right $A$-module*

3. *The coproduct $\Delta : \Gamma \to \Gamma \otimes_A \Gamma$, (as the tensor product of bimodules), where $\Delta$ is an $A$-bimodule map.*

4. *The counit $\epsilon : \Gamma \to A$, as an $A$-bimodule map.*

5. *and the conjugation $c : \Gamma \to \Gamma$*

*These maps satisfy the compatibility conditions for a cogroupoid object, namely those which turn $Hom(A, B)$ and $Hom(\Gamma, B)$ into the objects and morphisms of a groupoid for any $K$-algebra $B$.*

As an example, we note that by replacing $H\mathbb{F}_p$ with $MU$ or the $p$-local Brown-Peterson spectrum $BP$, we produce $MU_*MU$ (or respectively $BP_*BP$). These are not Hopf algebras, but do satisfy the conditions of a Hopf algebroid.

In general, when a spectrum $E$ satisfies certain technical conditions, we can construct the generalized Adams-Novikov spectral sequence over the Hopf algebroid $E_*E$ given by:

$$\mathrm{Cotor}_{E_*(E)}(E_*(\mathbb{S}), E_*(X)) \Rightarrow \pi_*(X).$$

This generalizes the Adams spectral sequence by resolving $X$ as a series of $E_*E$-comodules. When $E$ is taken as $MU$ or $BP$, the Hopf algebroid structure maps can be derived spectrally, but the following section gives a more concrete description, allowing us to proceed with concrete calculations.

## 2.5 Formal Group Laws and $(MU_*, MU_*MU)$

As we saw above, cobordism admits a classically geometric definition, but intersects conveniently with the tools used in homotopy theory. To fully study the cobordism rings $MSC_*$

and $MO[2]_*$, we present another surprising connection between cobordism and algebra. This perspective makes strong use of the notion of a formal group law, which we now define.

**Definition 2.5.1.** *A formal group law over a ring $R$ is a power series $F(x, y) \in R[[x, y]]$ which satisfies the following properties:*

1. $F(x, 0) = F(0, x) = x$
2. $F(x, y) = F(y, x)$
3. $F(x, F(y, z)) = F(F(x, y), z)$

*It is convenient to write $x +_F y$ for $F(x, y)$.*

The language "formal group law" is suggestive of these desired conditions, having clear identity, commutativity and associativity conditions (and indeed the historical origin makes the connection explicit). The following proposition gives some useful constructions which allow us to simplify the notation of working with formal group laws:

**Proposition 2.5.1.** *Given a formal group law $F$ over $R$, there is a formal power series $i_F(x) \in R[[x]]$ such that $x +_F i_F(x) = 0$. We let $x +_F x := [2]_F(x)$, $i_F(x) := [-1]_F(x)$, and inductively define $[n]_F(x) := x +_F [n-1]_F(x)$, for $n \geq 0$. We can define $[-n]_F(x) := [n]_F([-1]_F(x))$. It is clear from these constructions that for any two integers $r_1$ and $r_2$, $[r_1 r_2]_F(x) = [r_1]_F([r_2]_F(x))$ and $[r_1 + r_2]_F(x) = [r_1]_F(x) +_F [r_2]_F(x)$*

An important note about $[-1]_F(x)$ which we need for Chapter 4 is that this power series can be recursively determined for given $F$. A more important construction due to Lazard will help us characterize all formal group laws over commutative rings with unit.

**Definition 2.5.2.** *Let $F(x, y) = x + y + \sum a_{i,j} x^i y^j$ be a power series with indeterminate coefficients $a_{i,j}$. Let $I$ be the ideal of $\mathbb{Z}[a_{i,j}]$ generated by the relations obtained from requiring $F(x, y)$ satisfy the definition of a formal group law. We define the Lazard ring to be $L := \mathbb{Z}[a_{i,j}]/I$.*

The following lemma characterizes $F(x, y)$ as the universal formal group law.

**Lemma 2.5.1.** *Given a formal group law $(R, G)$, where $R$ is a commutative ring with unit, there is a unique ring homomorphism $\theta_R : L \to R$ such that $G(x, y) = x + y + \sum \theta_R(a_{i,j}) x^i y^j$.*

Lazard went beyond the above universal characterization, and characterized the ring $L$ concretely. The following modern statement of the theorem is adapted from [Rav86], with the original French proof given in [Laz55].

**Theorem 2.5.1** (Lazard)**.** *Let $L$ be the Lazard Ring. Then*

1. *$L = \mathbb{Z}[x_1, x_2, \ldots]$ with $|x_i| = 2i$ for $i \geq 0$.*

2. *The $x_i$ can be chosen such that their image in $L \otimes \mathbb{Q} = \mathbb{Q}[m_1, m_2, \ldots]$ is defined by*

$$\begin{cases} pm_i & i = p^k - 1 \\ m_i & else \end{cases}$$

   *modulo decomposables.*

3. *$L$ is a subring of $\mathbb{Z}[m_1, m_2, \ldots]$*

If we compare this with the statement of Theorem 2.4.1, we see a surprising similarity. Quillen was the first to make the connection concrete with the following result.

**Theorem 2.5.2** ([Qui69])**.** *The natural map $\theta_{MU_*} : L \to MU_*$ is an isomorphism.*

Now, one can consider maps between formal group laws over a fixed ring $R$. As we ultimately want to connect formal group laws to Hopf algebroids, we only define the morphisms between formal group laws which are invertible.

**Definition 2.5.3.** *A strict isomorphism of formal group laws $F, G$ over a ring $R$, is a power series $f(x) \in R[[x]]$ of the form $f(x) = x + \sum_{i=1}^{\infty} r_i x^{i+1}$, such that $F(f(x), f(y)) = f(G(x, y))$.*

In the same way $L$ characterizes formal group laws, we can define another object which characterizes these strict isomorphisms.

**Definition 2.5.4.** *Define the ring* $LB = L \otimes \mathbb{Z}[b_1, b_2, \ldots]$. *For a strict isomorphism of formal group laws* $f = x + \sum f_i x^{i+1} \in R[[x]]$, *there is a map* $\theta : LB \to R$ *such that* $\theta(b_i) = f_i$

In addition to the above, as discussed in [Rav86], one can say that $L$ represents the functor $FGL(-)$, assigning to a ring $R$ the set of formal group laws over $R$, $FGL(R)$. In turn, $LB$ represents the functor $SI(-)$, assigning to $R$ the set, $SI(R)$, whose elements are strict isomorphisms between objects in $FGL(R)$. In this way, we can see $L$ and $LB$ form a pair $(L, LB)$ such that $\mathrm{Hom}(L, R)$ and $\mathrm{Hom}(LB, R)$ are the objects and morphisms of a groupoid for any $K$-algebra $R$. By definition, this endows the pair $(L, LB)$ with the structure of a Hopf algebroid. The connections between $(L, LB)$ and $(MU_*, MU_*MU)$ are made concrete with the following result:

**Theorem 2.5.3** ([Lan67], [Nov67]). *The map* $\theta_{MU_*} : L \to MU$ *extends to a Hopf algebroid isomorphism between* $(L, LB)$ *and* $(MU_*, (MU \wedge MU)_*)$

Now that we have connected cobordism to the language of formal groups, and noted how much structure is gained by working over a Hopf algebroid, we may proceed to our main results.

# CHAPTER 3

# The Rectified Adams-Novikov Spectral Sequence and its Applications

Now that we have covered the prerequisite material, we can finally construct the rectified Adams-Novikov spectral sequence and characterize the Hopf algebroids over which our spectral sequence will converge to $\pi_*(MO[2])$ and $\pi_*(MSC)$. In particular, in the coming sections, we show that both $LS = \pi_*(MU \wedge_{MO[2]} MU)$ and $LSC = \pi_*(MU \wedge_{MSC} MU)$ are polynomial, and in the case of $\pi_*(MU \wedge_{MO[2]} MU)$ primitively generated with respect to its coproduct structure. Additionally, we aim to present our results in a way which highlights the parallels with the Hopf algebroid $(MU_*, \pi_*(MU \wedge MU)) = (L, LB)$.

## 3.1 Constructing the Spectral Sequence

We can now finally begin exploring the tools needed to compute $MSC_*$ and $MO[2]_*$. The principle object of interest is the following spectral sequence.

**Theorem 3.1.1.** *Fix an $E_\infty$-ring spectrum $E$, over which $MU$ is an $E_\infty$-algebra, and let $\Gamma := \pi_*(MU \wedge_E MU)_*$. If $\Gamma$ is flat over $MU_*$, then there is a spectral sequence*

$$Cotor_\Gamma(MU_*, MU_*) \Rightarrow \pi_*(E).$$

*This is the descent spectral sequence associated to the monad $X \mapsto X \wedge_E MU$.*

*Proof.* While the following general construction is not new, the specifics warrant a closer examination. Given a monad $\mathbb{T} : \mathcal{C} \to \mathcal{C}$, and object $E$, there is an associated cosimplicial object, $G(E)$, given by the Godement construction:

$$\mathbb{T}X \rightrightarrows \mathbb{T}^2X \rightrightarrows \mathbb{T}^3X \rightrightarrows \dots$$

whose face maps are given by $\mathbb{T}^{n-k}\eta\mathbb{T}^k$ for $0 \leq k \leq n$, where $\eta$ is the unit of the monad. The unshown degeneracy maps are those given similarly by $\mathbb{T}^{n-k}\mu\mathbb{T}^k$, where $\mu$ is the multiplication of the monad. By taking the total space of the cosimplicial object, we obtain a spectrum $Tot(G(X))$, where we have a canonical map $X \to Tot(G(X))$, which we recall is an equivalence when $X$ is connected and of finite type. (One can see this by decomposing $X$ as a colimit of finite cell spectrum and mimicking argument of the simplicial equivalence $|SSet(X)| \to X$ in spaces.) Therefore, we have $\pi_*(Tot(G(X))) \cong \pi_*(X)$. From here, what remains is to compute $\pi_*(Tot(G(X)))$. For us, the monad is given by $X \mapsto MU \wedge_E X$, and by taking $X = E$, the construction above simplifies to:

$$MU \rightrightarrows MU \wedge_E MU \rightrightarrows MU \wedge_E MU \wedge_E MU \rightrightarrows \dots$$

This reduces to computing the cohomology of the following cochain complex:

$$MU_* \to (MU \wedge_E MU)_* \to (MU \wedge_E MU \wedge_E MU)_* \to \dots$$

By letting $\Gamma = \pi_*(MU \wedge_E MU)$, and noting that $\Gamma$ is flat over $MU_*$, this above complex becomes:

$$MU_* \to \Gamma \otimes_{MU_*} MU_* \to \Gamma^{\otimes_{MU_*}2} \otimes_{MU_*} MU_* \to \Gamma^{\otimes_{MU_*}3} \otimes_{MU_*} \dots\dots$$

Notice that the objects in this complex coincide with the objects in the cobar complex associated to $\text{Cotor}_\Gamma(MU_*, MU_*)$. The differentials of our complex are induced by the structure maps of the monad, giving:

$$d_n(\gamma_1|\dots|\gamma_n|x) = id_1 \otimes \cdots \otimes id_n \otimes \eta_R(x) + \sum_{j=1}^{n}(-1)^{j+1} id_1 \otimes \Delta_j(\gamma_j) \otimes id_n \otimes id_0.$$

where $\eta_R$ and $\Delta$ correspond to the place of $\eta$ and $\mu$ in the monadic construction. This is precisely the definition of the differential associated to the cobar complex of $\text{Cotor}_\Gamma(MU_*, MU_*)$. Therefore, we have

$$\text{Cotor}_\Gamma(MU_*, MU_*) \Rightarrow \pi_*(Tot(G(X)) \sim \pi_*(E)$$

which is our desired result. □

We will see below that the spectra $MO[2]$ and $MSC$ both satisfy the hypothesis of this theorem, and so the spectral sequences specialize to

$$\text{Cotor}_{LSC}(MU_*, MU_*) \Rightarrow \pi_*(MSC) \qquad \text{and} \qquad \text{Cotor}_{LS}(MU_*, MU_*) \Rightarrow \pi_*(MO[2]).$$

We present computations of $\pi_*(MSC)$ and $\pi_*(MO[2])$ for a limited range obtained using the techniques described in Chapter 4 in Table 3.1. A larger table can be found in Chapter 5.

Table 3.1: Limited computations of $\pi_*(MSC)$

| $s\backslash t-s$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $\mathbb{Z}$ | | 0 | | $\mathbb{Z}$ | | 0 | | $\mathbb{Z}^2$ | | 0 | | $\mathbb{Z}^3$ |
| 1 | | $(4)$ | | $\mathbb{Z}$ | | $(2,16)$ | | $\mathbb{Z}^2$ | | $(8,64)$ | | $\mathbb{Z}^4$ | |
| 2 | | | 0 | | $(2)$ | | $(4)$ | | $(2,4,8)$ | | $(\mathbb{Z},2,4)$ | | $(2,4^2,8,32)$ |
| 3 | | | | 0 | | 0 | | 0 | | $(2)$ | | $(2)$ | |
| 4 | | | | | 0 | | 0 | | 0 | | 0 | | 0 |

We want to remark that this spectral sequence is distinct, yet related, to the classical

Adams-Novikov spectral sequence. We examine the Hurewicz homomorphism for $MSC$

$$\pi_*(MSC) \to MU_*MSC \to H_*MSC.$$

We say more in Section 3.3 (and described concretely for $MO[2]$ in Lemma 3.2.4), but assert that for odd primes, there is a Thom isomorphism giving the identification

$$H_*(MSC; \mathbb{F}_p) \cong H_*(MO; \mathbb{F}_p) \otimes \Lambda[e_1, e_2, \dots]$$

We see that the element $a_1 \in \pi_1(MSC)$ is represented by a class $(\overline{s}_1)$ in $(t, s)$-Ext degree $(2, 1)$. Geometrically, this map sends $a_1$ to the first Stiefel-Whitney class for the stable normal bundle of $\mathbb{R}P^1$ in $H_*(MO)$, and therefore survives all maps. As it is 4-torsion, there must be 4-torsion in $MU_*MSC$. Analogous statements hold in the case of $MO[2]$. However, under the induced maps of the classical Adams-Novikov spectral sequence, the image of this element does not survive. Therefore, we can see that the classical cobar complex has torsion, while the new cobar complex is torsion-free with respect to this element. As our construction is distinct from the classical Adams-Novikov spectral sequence, we make the following definition:

**Definition 3.1.1.** *The spectral sequence given in Theorem 3.1.1 is called the rectified Adams-Novikov spectral sequence.*

To determine if the rectified Adams-Novikov spectral sequence collapses and to perform the computational calculations of $\pi_*(MSC)$ and $\pi_*(MO[2])$, we need to determine the algebraic structure of the Hopf algebroids $(MU_*, \pi_*(MU \wedge_{MO[2]} MU))$ and $(MU_*, \pi_*(MU \wedge_{MSC} MU))$.

## 3.2 Structure of $\pi_*(MU \wedge_{MO[2]} MU)$

We now want to introduce the Hopf algebroids $(MU_*, \pi_*(MU \wedge_{MO[2]} MU))$ and $(MU_*, \pi_*(MU \wedge_{MSC} MU))$. Before beginning with the algebraic characterization, we first want to take a moment to note that both of these naturally inherit a Hopf algebroid structure from $(MU_*, \pi_*(MU \wedge MU))$, induced by the natural coequalizer map defining $MU \wedge_{MO[2]} MU$ and $MU \wedge_{MSC} MU$ and then taking homotopy groups.

**Lemma 3.2.1.** *As an $MU_*$-algebra,*

$$\pi_*(MU \wedge_{MO[2]} MU) = MU_*[s_1, s_3, s_5, \dots]$$

*for indeterminants $s_i$, where $|s_{2i+1}| = 4i + 2$.*

*Proof.* We first show the result locally at a prime $p$. For an odd prime, we start by computing $\pi_*(MO[2]) \otimes \mathbb{F}_p$. The Thom isomorphism gives us that

$$H^*(MO[2]; \mathbb{F}_p) \cong H^*(BO; \mathbb{F}_p) \cong \mathbb{F}_p[p_1, p_2, \dots]$$

where $p_i$ are the symplectic Pontryagin classes, with $|p_i|$ is in degree $|4i|$ [Bro82]. Then, we note that as $H^*(MO[2]; \mathbb{F}_p)$ is concentrated in even dimension and is a module over the dual Steenrod Algebra $\mathcal{A}_*$. By [Rav86], it is a module over a certain polynomial subalgebra, $P_* \subset \mathcal{A}_*$. In our case, as $p$ is an odd prime, we have

$$P_* = \mathbb{F}_p[\xi_1, \xi_2, \dots],$$

where $|\xi_i| = 2(p^i - 1)$, where we note that as $p$ is odd, this is divisible by 4. Then, we note that there is a surjection $H_*(MU, \mathbb{F}_p) \twoheadrightarrow P_*$ induced by the map $MU \to H\mathbb{F}_p$. We recall now that the map $t : MO[2] \to MU$ (induced by complexification $BO \to BU$) itself induces a map $t^* : H^*(MU) \to H^*(MO[2])$, such that $t^*(c_{2i}) = (-1)^i p_i$, where $c_i$ denotes the $i^{th}$

Chern class. The dual of this map therefore composes to a surjection: $H_*(MO[2];\mathbb{F}_p) \twoheadrightarrow P_*$.

This surjection is sufficient to apply the result in [Rav86, A1.1.17] to decompose $H_*(MO[2];\mathbb{F}_P)$ as the following tensor product

$$H_*(MO[2];\mathbb{F}_p) = P_* \otimes_{\mathbb{F}_p} C$$

where $C = \mathbb{F}_p[u'_1, u'_2, \dots]$ where $|u'_i| = 4i$ and $i$ is not of the form $(p^k-1)/2$. By [SS68a], this is then sufficient for us to conclude that there is a homotopy equivalence between $MO[2]$ and a wedge of $BP$, and so in the derived setting, we have $MO[2] = \bigvee \Sigma^{2n_i} BP$. This tells us that $\pi_*(MO[2]) \otimes \mathbb{F}_p = \mathbb{F}_p[u''_1, u''_2, \dots]$ where $|u''_i| = 4i$. Additionally, these $BP$-summands map equivalently to the $BP$-summands of $MU$ under the map $MO[2] \to MU$ described above. Therefore, these copies are identified in $MU \wedge_{MO[2]} MU$. Over $p$, $MU$ is a free $MO[2]$-module, so we get that

$$\pi_*(MU \wedge_{MO[2]} MU) \cong \pi_*(MU) \otimes_{\pi_*(MO[2])} \pi_*(MU).$$

Finally, we want to note that since $\pi_*(MO[2]) \otimes \mathbb{F}_p$ is polynomial on generators $u''_i$ in dimension $4i$, which coincide with the $BP$-summands giving the elements $x_{2i}$, we get that

$$\pi_*(MU \wedge_{MO[2]} MU) \cong MU_*[s_1, s_3, s_5, \dots].$$

where we see that the polynomial generators $s_i$ are in degree $4k+2$, and so we have the desired form.

For an even prime, we need to work a little harder. We start similarly, by recalling the $\mathbb{F}_2$-homologies of $MU$ and $MO[2]$. Again, by the Thom isomorphism we have

$$H_*(MO[2];\mathbb{F}_2) \cong H_*(BO;\mathbb{F}_2) \cong \mathbb{F}_2[a_1, a_2, \dots]$$

and for $MU$ we have:

$$H_*(MU;\mathbb{F}_2) \cong \mathbb{F}_2[a_2, a_4, \dots]$$

where the elements $a_i$ are dual to the universal Stiefel-Whitney classes $w_1^i$. Then, we leverage the $\mathbb{F}_2$-Eilenberg-Moore spectral sequence.

$$\mathrm{Tor}^{H_*(MO[2])}(H_*(MU), H_*(MU)) \Rightarrow H_*(MU \wedge_{MO[2]} MU)$$

The $E_2$-page of this becomes $\Lambda_{\mathbb{F}_2}[b_1, b_3, \dots]$, where the $b_{2i+1}$ is in topological degree $2i+1$ and algebraic degree 1, for a total degree of $2i+2$. Then, we can leverage the Dyer-Lashoff operations described in [Pri75], to get that $b_{4i+3}$ is identified with $b_{2i+1}^2$ in total degree $4i+4$, and will therefore vanish after taking homology. Therefore, we can conclude the above $E_2$-page collapses to give $\mathbb{F}_2[b_{4i+1}]$, where $|b_{4i+1}|$ is in total dimension $4i+2$. We are now able to leverage [Rav86, A1.1.17] and [SS68a] again, by noting we are concentrated in even dimension and surject onto $P_* = \mathbb{F}_2[\zeta_1^2, \zeta_2^2, \dots]$, to conclude that at $p = 2$, $MU \wedge_{MO[2]} MU$ is also a wedge of copies of $BP$, with the same number in each degree as at odd primes. Finally, we note that the above constructions respect multiplicative structure, so we can obtain the desired ring structure and conclude $\pi_*(MU \wedge_{MO[2]} MU) = MU_*[s_1, s_3 \dots]$. □

We also have the following corollary, showing that $MO[2]$ satisfies the conditions on Theorem 3.1.1.

**Corollary 3.2.1.** *$\pi_*(MU \wedge_{MO[2]} MU)$ is a free $MU_*$-module.*

The specifics of the coproduct for $(MU_*, \pi_*(MU \wedge_{MO[2]} MU))$ are saved until after we have defined our next object. In fact, this next object will give us the foothold we need to parse the induced Hopf algebroid structure on $(MU_*, \pi_*(MU \wedge_{MO[2]} MU))$. Following the parallels with $(L, LB)$, we introduce the following:

**Definition 3.2.1.** *Let $(L, LB)$ be as in Lazard's Theorem. Let $s(x) = \sum_{i \geq 1} s_i x^i$ be the power series on indeterminants $s_i$. We define $LS$ as a quotient of $L\{s_1, s_2, \dots\}$, (the free algebra* *over $L$ generated by words in the $s_i$) determined by the identifications of $s_i$ with polynomials $f_i(b_1, \ldots, b_i) \in LB \otimes \mathbb{Q}$ as determined by the relations*

$$b(x) = x + [2]_F(x) \cdot s(x) \qquad x \cdot i_F(x) = b(x) \cdot b(i_F(x))$$

The above definition can be greatly simplified with the following lemma.

**Lemma 3.2.2.** $LS = L[s_1, s_3, s_5, \ldots]$

*Proof.* First, we note that the first relation implies that each $b_i \equiv 2s_i$ modulo decomposable elements of strictly lower degree. We see this by replacing with definitions:

$$\begin{aligned} x + \sum_{i\geq 1} b_i x^{i+1} &= x + [2]_F(x) \cdot s(x) \\ \sum_{i\geq 1} b_i x^{i+1} &= [2]_F(x) \cdot s(x) \\ \sum_{i\geq 1} b_i x^{i+1} &= (2x + \sum_{j\geq 2} c_j x^j)(\sum_{i\geq 1} s_i x^i) \\ \sum_{i\geq 1} b_i x^{i+1} &= \sum_{i\geq 1} 2s_i x^{i+1} + (\sum_{j\geq 2} c_j x^j)(\sum_{i\geq 1} s_i x^i) \end{aligned}$$

This implies that each $s_i$ is linearly independent, and indeed form a polynomial basis for $L[s_1, s_2, \ldots]$. We now show that

$$s_{2i} \equiv (-1)^{i-1} s_i^2 + 2\sum_{k=1}^{i-1} (-1)^{k-1} s_{i-k} s_k \mod I,$$

where $I$ is the augmentation ideal of $LS$. This follows again from a manipulation of the underlying power series. If we replace $i_F(x)$ by $[-1]_F(x)$ and compose the first relation with the second, we get:

$$
\begin{aligned}
x \cdot [-1]_F(x) &= (x + [2]_F(x) \cdot s(x)) \cdot ([-1]_F(x) + [2]_F([-1]_F(x)) \cdot s([-1]_F(x))) \\
x \cdot [-1]_F(x) &= x \cdot [-1]_F(x) + [-1]_F(x) \cdot [2]_F(x) \cdot s(x) + x \cdot [-2]_F(x) s([-1]_F(x)) \\
&\quad + [2]_F(x) \cdot [-2]_F(x) \cdot s(x) \cdot s([-1]_F(x)) \\
0 &= [-1]_F(x) \cdot [2]_F(x) \cdot s(x) + x \cdot [-2]_F(x) s([-1]_F(x)) \\
&\quad + [2]_F(x) \cdot [-2]_F(x) \cdot s(x) \cdot s([-1]_F(x)).
\end{aligned}
$$

This reduction gives the following:

$$
[-1]_F(x) \cdot [2]_F(x) \cdot s(x) + x \cdot [-2]_F(x) s([-1]_F(x)) = -[2]_F(x) \cdot [-2]_F(x) \cdot s(x) \cdot s([-1]_F(x)).
$$

We see that the right hand side features the factor $s(x) \cdot s([-1]_F(x))$, which we will soon see allows us to relate $s_{2n}$ on the left to $s_n^2$ on the right. We note that it suffices to reduce the series $[2]_F(x)$, $[-2]_F(x)$, and $[-1]_F(x)$ to their leading terms only, as the higher terms are in the augmentation ideal. These terms are $2x$, $-2x$ and $-x$ respectively. Then our above greatly relation simplifies to

$$
-2x^2 \cdot s(x) - 2x^2 \cdot s(-x) \equiv 4x^2 s(x) \cdot s(-x).
$$

From here, if we substitute the definition of $s(x)$,

$$
\begin{aligned}
-2x^2 \left( \sum_{i \geq 1} s_i x^i \right) - 2x^2 \left( \sum_{i \geq 1} s_i (-x)^i \right) &\equiv 4x^2 \left( \sum_{i \geq 1} s_i x^i \right) \left( \sum_{i \geq 1} s_i (-x)^i \right) \\
-4x^2 \left( \sum_{i \geq 1} s_{2i} x^{2i} \right) &\equiv 4x^2 \sum_{i \geq 2} \left( \sum_{1 \leq k \leq i}^{i} (-1)^k s_{i-k} s_k \right) x^i.
\end{aligned}
$$

By comparing coefficients, we get

$$s_{2i} \equiv \sum_{1 \leq k \leq 2i}^{2i} (-1)^{k-1} s_{i-k} s_k$$

$$s_{2i} \equiv (-1)^{i-1} s_i^2 + 2 \sum_{k=1}^{i-1} (-1)^{k-1} s_{i-k} s_k$$

which is the desired result. □

Now that we have defined our objects $(MU_*, \pi_*(MU \wedge_{MO[2]} MU))$ and $(L, LS)$ analogously to $(MU_*, \pi_*(MU \wedge MU))$ and $(L, LB)$, we can prove the analogous isomorphism.

**Theorem 3.2.1.** *The pair $(L, LS)$ form a Hopf algebroid.*

*Proof.* We give $(L, LS)$ the Hopf algebroid structure induced by $(L, LB)$. Define

$$\Delta(s_i) := \Delta(f(b_1, \dots b_i)) = f(\Delta(b_1), \dots \Delta(b_i))$$

where we then re-express the $b_i$ as polynomials in the $s_i$. We define the conjugation and counit maps similarly. The unit remains the same. These satisfy the necessary axioms as a consequence of satisfying them over $LB \otimes \mathbb{Q}$. □

Next, we want to identify $(L, LS)$ with $(MU_*, \pi_*(MU \wedge_{MO[2]} MU))$. This motivates the theorem:

**Theorem 3.2.2.** *The pairs $(L, LS)$ and $(MU_*, MU \wedge_{MO[2]} MU)$ are isomorphic as Hopf algebroids.*

Before proving this, we need the following lemma:

**Lemma 3.2.3.** *There is an isomorphism $MU_*(\mathbb{R}P^\infty) \cong MU_*[[x]]/\langle [2]_F(x) \rangle$, where $[2]_F(x)$ is $F(x, x)$ where $F$ is the universal formal group law over $MU_*$.*

*Proof of Lemma.* Our first step is to construct a specific cofiber sequence of the form

$$\mathbb{R}P^\infty \to \mathbb{C}P^\infty \to \mathbb{C}P^\infty.$$

We begin by recalling two standard facts. First, we note that $MU_*(\mathbb{C}P^\infty) = MU_*[[x]]$. Next, we note that $\mathbb{C}P^\infty$ admits a canonical normal bundle $\gamma_{\mathbb{C}}$ such that when one takes the Thom Space $Th(\gamma_{\mathbb{C}})$ one obtains an equivalence $Th(\gamma_{\mathbb{C}}) \sim \mathbb{C}P^\infty$. Next, the Thom space for an arbitrary bundle $\zeta$ is given by the cofiber sequence:

$$S(\zeta) \to D(\zeta) \to Th(\zeta).$$

Now, if we take $\zeta = \gamma_{\mathbb{C}}$, we get the cofiber sequence:

$$S(\gamma_{\mathbb{C}}) \to D(\gamma_{\mathbb{C}}) \to Th(\gamma_{\mathbb{C}}).$$

where $S(\gamma_{\mathbb{C}}) \subset D(\gamma_{\mathbb{C}}) \subset \gamma_{\mathbb{C}}$ are the fiberwise sphere and disc bundles over $\mathbb{C}P^\infty$. In particular, we highlight the equivalences $D(\gamma_{\mathbb{C}}) \sim \mathbb{C}P^\infty$ and $Th(\gamma_{\mathbb{C}}) \sim \mathbb{C}P^\infty$. Next, we note that $\mathbb{R}P^\infty$ includes into $\mathbb{C}P^\infty$ as

$$\begin{array}{ccccc} S^1 & \longrightarrow & S^\infty & \longrightarrow & \mathbb{C}P^\infty \\ & & \downarrow & \nearrow & \\ & & \mathbb{R}P^\infty & & \end{array} .$$

Therefore, if we consider the Thom space cofiber sequence associated to $(\gamma_{\mathbb{C}})^2$ over $\mathbb{C}P^\infty$, we get

$$S((\gamma_{\mathbb{C}})^2) \to D((\gamma_{\mathbb{C}})^2) \to Th((\gamma_{\mathbb{C}})^2).$$

By considering the inclusion diagram, we see tha $S((\gamma_{\mathbb{C}})^2) \sim \mathbb{R}P^\infty$ so up to equivalence of spaces, we obtain a cofiber sequence

$$\mathbb{R}P^\infty \to \mathbb{C}P^\infty \to \mathbb{C}P^\infty.$$

Then, by applying $MU^*(-)$ to the cofiber sequence, we get a long exact sequence:

$$\cdots \to MU^*(\mathbb{C}P^\infty) \to MU^*(\mathbb{C}P^\infty) \to MU^*(\mathbb{R}P^\infty) \to MU^{*+1}(\mathbb{C}P^\infty) \to \ldots.$$

We note that we can equivalently right this as

$$\cdots \to MU^*[[x]] \to MU^*[[x]] \to MU^*(\mathbb{R}P^\infty) \to MU^{*+1}[[x]] \to \ldots.$$

This sequence is determined by the image of $x$. As this arises as the $MU_*$-orientation of the bundle $\gamma_{\mathbb{C}}^2$, this acts by $[2]_F(x)$. This is a non-zero divisor in $MU^*[[x]]$, and therefore, the connecting homomorphism is forced to be zero. This then forces our sequence to be of the form:

$$MU^*[[x]] \to MU^*[[x]] \to MU^*(\mathbb{R}P^\infty) \to 0$$

and therefore $MU^*(\mathbb{R}P^\infty) \cong MU^*[[x]]/\langle [2]_F(x) \rangle$. □

*Proof of Theorem.* To show that these Hopf algebroids are now isomorphic, it suffices to show that the map $MU_*MU \to MU \wedge_{MO[2]} MU_*$ respects and imposes the same relations given in the definition of $LS$. We appeal to geometry. If we consider the standard complex orientation $x \in MU_*(\mathbb{C}P^\infty) = MU_*[[x]]$, then we can consider the maps

$$\eta_L^*, \eta_R^* : MU_*[[x]] \cong MU_*(\mathbb{C}P^\infty) \to (MU \wedge MU)_*(\mathbb{C}P^\infty) \cong MU_*[b_1, b_2, \ldots][[x]].$$

Under these maps, we get $\eta_L(x) = x$ in $MU_*[b_1, b_2, \ldots][[x]]$ and $\eta_R(x) = b(x)$ in $MU_*[b_1, b_2, \ldots][[x]]$. Now, if we replace $\mathbb{C}P^\infty$ by $\mathbb{R}P^\infty$, we obtain a similar result, with the difference being

$$\eta_R^*(x) \equiv b(x) \mod [2]_F(x)$$

by Lemma 3.2.3.

Additionally, the orientation $\mathbb{R}P^\infty \to MU$ factors through $MO[2]$, as $\mathbb{R}P^\infty$ has a double-real structure. This means that the orientations must coincide in $MU \wedge_{MO[2]} MU$, and so we get that $b(x) = x + [2]_F(x) \cdot s(x)$, for some some elements $s_i$ in $MU \wedge_{MO[2]} MU_*$.

To recover the second relation, we examine the second Chern class

$$c_2 : BU(2) \to \Sigma^4 MU$$

Now note again the standard fact that $\mathbb{C}P^\infty \sim BSO(2)$, defining a map $\mathbb{C}P^\infty \to BU(2)$ which factors through $BO(2)$. By factoring through $BO(2)$, the resulting Chern class will factor through $MO[2]$. The second Chern class gives $x \cdot i_F(x)$, and so by comparing the left and right units, we recover $x \cdot i_F(x) = b(x) \cdot b(i_F(x))$, giving the final relation. We may stop here as the bundle $2\gamma_{\mathbb{R}}$ over $\mathbb{R}P^\infty$ classifies $MO[2]$-bundles, and we have determined how this bundle factors through $MU$, and therefore any additional relations are generated by the two discussed. The Hopf algebroid structure is inherited, and so the isomorphism $(L, LB) \cong (MU_*, MU_*MU)$ induces the isomorphism $(L, LS) \cong (MU_*, \pi_*(MU \wedge_{MO[2]} MU)$. □

Now that we have shown the isomorphism between $(L, LS)$ and $(MU_*, \pi_*(MU \wedge_{MO[2]} MU))$, we see this mirrors the isomorphism $(L, LB) \cong (MU_*, MU_*MU)$. Due to the connection of $(L, LB)$ with the study of formal groups, it is natural to ask if there is a similar connection for $(L, LS)$.

Our relations defining $LS$ are motivated by the work of Buchstaber and Novikov. In [BN71], the pair works closely with the 2-valued formal group laws. Specifically, the 2-valued formal groups they study are parametrized by the element $x \cdot i_F(x)$. We see that our relation $xi_F(x) = b(x) \cdot b(i_F(x))$ can be interpreted as preserving this parameter. Additionally, our other relation imposes a relation on strict isomorphisms which are congruent to the identity up to series $[2]_F(x)$. Therefore, we can say that $LS$ represents strict isomorphisms of formal groups which preserve the coordinate of the 2-valued formal group, and identity on the 2-torsion component of the formal group. However, this result is more metaphorical than concrete, and should be treated as such.

Now that we have algebraically computed $LS$, and discussed its connections to $(L, LB)$, we further expand on its Hopf algebra structure. In particular, we show that we can find an alternative polynomial basis $\{\overline{s}_{2i+1}\}$ which have a nicely characterized coproduct. To that end, we recall the following definition.

**Definition 3.2.2.** *Let $(A, R)$ be an arbitrary Hopf algebroid. An element $s \in R$, is said to be primitive if*

$$\Delta(s) = s \otimes 1 + 1 \otimes s.$$

It is worth noting that a primitive element $s$ represents a permanent cycle $(s)$ in $\mathrm{Cotor}^1_R(A, A)$, and any permanent cycle will be represented by such an element. With this in mind, we present the next lemma.

**Lemma 3.2.4.** *$(L, LS)$ is primitively generated. Specifically, there are elements $\overline{s}_{2i+1} \equiv s_{2i+1}$ modulo decomposables such that*

$$\Delta(\overline{s}_{2i+1}) = \overline{s}_{2i+1} \otimes 1 + 1 \otimes \overline{s}_{2i+1}.$$

*Proof.* We identify the primitive generators $\overline{s}_{2i+1}$ as the classes $[\mathbb{R}P^{4k+1}]_{MO[2]}$ as follows. We start by examining the classical mod 2 Adams Spectral Sequence, given by

$$E_2 = \mathrm{Ext}_{\mathcal{A}_*}(H^*MO[2], \mathbb{F}_2) \Rightarrow \pi_*(MO[2]) \otimes \mathbb{Z}_2.$$

We note that while this will not collapse, we can still examine the elements in filtration degree 0 to learn about the eventual structure of $\pi_*(MO[2])$. Specifically, we see that as $H^*(MO[2], \mathbb{F}_2) \cong \mathbb{F}_2[w_1, w_2, \ldots]$, the dual polynomial generator $a_i$ such that $\langle a_i, w_1^i \rangle = 1$

lives in bidegree $(i, 0)$ of the $E_2$-page. Namely, in $(4k+1, 0)$, we get the generator dual to $a_{4k+1}$, which corresponds to $w(\tau_{\mathbb{R}P^{4k+1}})$ and so the class represented by $[\mathbb{R}P^{4k+1}]$ exists in $(4k+1, 0)$.

Let $Q_nM$ denote the submodule of $M$ of degree $n$ indecomposables. Next, we recall that for $x \in MU_*$, there is a class, denoted $m(x)$, called the Milnor class, which detects the image of $x \in Q_nMU_*$. This class is constructed as follows. Recall that the universal Chern classes $c_1, c_2, \ldots$ are the generators $H^*(MU, \mathbb{Z}) = \mathbb{Z}[c_1, c_2, \ldots]$. These can be expressed as symmetric polynomials $c_i := \sigma_i(b_1, b_2, \ldots)$ in $H_*(MU; \mathbb{Z}) = \mathbb{Z}[b_1, b_2, \ldots]$. The $n^{th}$ Milnor class is then defined to be the polynomial $p(c_1, c_2 \ldots)$ such that

$$m_n := p(c_1, c_2, \ldots) = b_1^n + b_2^n + \ldots.$$

Then, if we let $x \in MU_*$ be the cobordism class of manifolds with representative $M$, we defined the Milnor number on $x = [M]$ to be $m_n([M]) := m_n(\nu_M)$, where $\nu_M$ is the stable normal bundle of $M$. Analogously, we have that for double-real manifolds, the Milnor class $m_n(x)$ for $x$ in $MO[2]_*$ detects the image of $x$ in $Q_nMO[2]_*$, where we replace Chern classes $c_i$ by the Stiefel-Whitney classes $w_i$, and the stable normal bundle $\nu_M$ is replaced by $\xi_M$, one half of the normal bundle, for $M$ being a manifold-representative of the class $x$. Therefore, if we can show that $m_{4k+1}([\mathbb{R}P^{4k+1}]) \neq 0$, then $[\mathbb{R}P^{4k+1}]_{MO[2]}$ is indecomposable in $\pi_{4k+1}(MO[2])$. First, we note that the tangent bundle is subject to the relation

$$\tau_{[\mathbb{R}P^{4k+1}]} \oplus \underline{1} = (4k+2)\gamma_R^1.$$

Therefore, if we work over virtually, our normal bundle $\nu_{\mathbb{R}P^{4k+1}}$ is represented by $(-4k-2)\gamma_R^1$. Therefore, our half normal bundle is represented by $(-2k-1)\gamma_{\mathbb{R}}^1$. As $\gamma_{\mathbb{R}}^1$ has nontrivial first Stiefel-Whitney number, the Milnor class

$$m([\mathbb{R}P^{4k+1}] = m(\zeta) = m((-2k-1)\gamma_{\mathbb{R}}^1) = m(\gamma_{\mathbb{R}}^1)^{-2k-1} = 1 \neq 0$$

Finally, we note that as these classes are indecomposable, in the context of the rectified Adams-Novikov spectral sequence, they must also be indecomposable. This forces them into bi-degree $(4k+1,0)$ or $(4k+2,1)$. However, as $\pi_*(MU \wedge_{MO[2]} MU)$ is entirely even-dimensional, $[\mathbb{R}P^{4k+1}]_{MO[2]}$ must persist from a class in $(4k+2,1)$. As such this, must be a permanent cycle, and represented by a primitive element. Since it is indecomposable, there must be an indecomposable, primitive in $(4k+2,1)$ for all $k$. This is equivalent to the statement that there exist $\overline{s}_{2k+1}$ such that $\overline{s}_{2k+1} \equiv s_{2k+1}$ modulo decomposables. □

## 3.3 Structure of $\pi_*(MU \wedge_{MSC} MU)$

We now define and prove the analogous statements for $MSC$.

**Lemma 3.3.1.**

$$\pi_*(MU \wedge_{MSC} MU) = MU_*[B_1, B_2, \dots]$$

*where* $|B_i| = 2i$.

*Proof.*

Case 1 ($p$ odd): For $MSC$, we again proceed by considering the odd prime and even prime case separately. We suppress coefficients for the sake of brevity. We note that at odd primes,

$$H^*(MSC) = \mathbb{F}_p[p_1, p_2, \dots] \otimes \Lambda_{\mathbb{F}_p}[e_1, e_2, \dots]$$

where the $p_i$ are as described above and $|e_k| = 4k-1$ [SS68b]. Geometrically, the $e_k$ also transgress from the Chern class $c_{2k}$. Let $R := \mathbb{F}_p[p_1, p_2, \dots]$. Note that this gives $H^*(MSC) = R \otimes \Lambda_{\mathbb{F}_p}[e_1, e_2, \dots]$. Additionally, we note that as the elements $p_i$ are the image of elements $c_{2i}$ in $H^*(MU)$, we can equivalently decompose $H^*(MU)$ as $R \otimes F$, where $F := \mathbb{F}_p[c_1, c_3, \dots]$. Now, we compute $H^*(MU \wedge_{MSC} MU)$ to then apply the classical Adams spectral sequence. We start with the Eilenberg-Moore Spectral Sequence which gives:

$$\mathrm{Tor}^{H^*(MSC)}(H^*(MU), H^*(MU)) \Rightarrow H^*(MU \wedge_{MSC} MU)$$

In light of our above refactoring of $H^*(MSC)$ and $H^*(MU)$, we get

$$\mathrm{Tor}^{R \otimes \Lambda_{\mathbb{F}_p}[e_1, e_2, \ldots]}(R \otimes F, R \otimes F) \Rightarrow H^*(MU \wedge_{MSC} MU).$$

Now, we may apply a change-of-base isomorphism to get:

$$\mathrm{Tor}^{R \otimes \Lambda_{\mathbb{F}_p}[e_1, e_2, \ldots]}(R \otimes F, R \otimes F) \cong R \otimes \mathrm{Tor}^{\Lambda_{\mathbb{F}_p}[e_1, e_2, \ldots]}(F, F)$$

Now, as $e_i$ transgresses from $c_{2i}$, they will act trivially on the elements $c_{2i+1}$, and so this reduces further:

$$R \otimes \mathrm{Tor}^{\Lambda_{\mathbb{F}_p}[e_1, e_2, \ldots]}(F, F) = R \otimes F \otimes F \otimes \mathrm{Tor}^{\Lambda_{\mathbb{F}_p}[e_1, e_2, \ldots]}(\mathbb{F}_p, \mathbb{F}_p)$$

This final term is Tor of an exterior algebra, and is given by $\mathbb{F}_p[e_1', e_2', \ldots]$, where the $e_i'$ transgress from $e_i$ and have topological degree $4i - 1$ and algebraic degree 1 for overall degree $4i$. Combined with the classes $c_{2i+1}$ having total degree $|4i + 2|$, we conclude that

$$H^*(MU \wedge_{MSC} MU) = H^*(MU) \otimes \mathbb{F}_p[u_1', u_2', \ldots]$$

for some polynomial generators $u_i'$ where $|u_i'| = 2i$. Now, as we are concentrated an $\mathcal{A}_*$ comodule and concentrated in even degrees, applying Milnor-Moore lets us conclude

$$\pi_*(MU \wedge_{MSC} MU) = MU_*[B_1, B_2, \ldots]$$

for some polynomial generators $B_i$.

Case 2 ($p$ even): Now, moving onto the $p = 2$ case, we can proceed similarly. Note that in this case $H_*(MSC) = H_*(MU) \otimes \Lambda(a_1, a_2, \dots)$ [SS68b], where $|a_i| = 2i - 1$. Therefore, we get another Eilenberg-Moore Spectral Sequence, giving us:

$$\mathrm{Tor}^{H_*(MSC)}(H_*(MU), H_*(MU)) \Rightarrow H_*(MU \wedge_{MSC} MU).$$

An identical change-of-ring isomorphism gives

$$\mathrm{Tor}^{H_*(MSC)}(H_*(MU), H_*(MU)) = H_*(MU) \otimes \mathrm{Tor}^{\Lambda(a_1, a_2, \dots)}(\mathbb{F}_2, \mathbb{F}_2)$$

Examining the Tor term, we see that this comes from applying the homology of an exterior algebra over characteristic 2 and so we have:

$$\mathrm{Tor}^{\Lambda(a_1, a_3, \dots)}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2[a'_1, a'_2, \dots]$$

where the induced $a'_i$ again has topological degree $2i - 1$ but algebraic degree 1, and therefore $|a'_i| = 2i$. Combining this, the $E_2$-page of our original spectral sequence becomes:

$$H_*(MU) \otimes \mathbb{F}_2[a'_1, a'_2, \dots] \Rightarrow H_*(MU \wedge_{MSC} MU).$$

We can again note that as this is concentrated in entirely even degree, this is a $P_*$-comodule algebra, of the form:

$$H_*(MU \wedge_{MSC} MU) = P_* \otimes \mathbb{F}_2[x_i | i \neq 2^k] \otimes \mathbb{F}_2[a'_1, a'_2, \dots]$$

Applying Milnor-Moore and [SS68a] gives that:

$$\pi_*(MU \wedge_{MSC} MU) = MU_*[a'_1, a'_2, \dots] = MU_*[B_1, B_2, \dots].$$

where the $a_i'$ have been relabeled as $B_i$.

Now, we ensure that the geometric meaning of the $B_i$ is consistent across primes. We do this by examining the image under the Hurewicz homomorphism. Specifically, note that $H_*(MU \wedge_{MSC} MU)$ satisfies a Thom isomorphism, giving:

$$H_*(MU \wedge_{MSC} MU) = H_*(BU \times_{BSC} BU).$$

The space $BU \times_{BSC} BU$ is homeomorphic to the space $BU \times (BU/BSC)$ where the second factor of $BU$ is the antidiagonal in $BU \times BU$. Then, we have a homotopy equivalence

$$BU \times (BU/BSC) \sim BU \times BU$$

which we obtain after recalling that $BSC$ is as the fiber of

$$BSC \to BU \to BU.$$

Finally, we note that this means the Thom isomorphism identifies $H_*(MU \wedge_{MSC} MU)$ with $H_*(MU \wedge BU) = H_*(MU)[b_1', b_2', \ldots]$. This is a global result, and therefore $p$-locally, our generators agree. $\square$

Again, we have the corollary showing that $MSC$ satisfies the conditions on Theorem 3.1.1.

**Corollary 3.3.1.** *$\pi_*(MU \wedge_{MSC} MU)$ is a free $MU_*$-module.*

Now, we continue to proceed analogously to *Section* 3.2.

**Definition 3.3.1.** *Let $(L, LB)$ be as in Lazard's Theorem. Let $B(x) = \sum_{i \geq 1} B_i x^{i+1}$. We define $LSC := L[B_1, B_2, \ldots]$ where $B_i = f(b_1, \ldots, b_i) \in LB \otimes \mathbb{Q}$ as determined by the relation*

$$b(x) = \frac{B(x) \cdot i_F(x)}{B(i_F(x))}$$

To give $(L, LSC)$ a Hopf algebroid structure, we see that the we need to simply demonstrate the inherited relations from $(L, LB)$. However, in this case, there is an added layer of complication due to the lack of clear geometric motivation aligning the generators. Notice that $(L, LB)$ and $(L, LSC)$ are abstractly isomorphic as algebras, so we take special care to the characterization of the coproduct structure for $(L, LSC)$.

**Theorem 3.3.1.** *The pair $(L, LSC)$ form a Hopf algebroid.*

*Proof.* First, we need to show $LSC$ inherits the Hopf algebroid structure from $LB$. We first recall that the computation of $\Delta : LB \to LB \otimes LB$ is determined by the composition:

$$\Delta(b(x)) := b_r \circ b_l(x)$$

where $b_l(x) := (b \otimes 1)(x)$ and $b_r(x) = (1 \otimes b)(x)$. Therefore, we need to verify that when composing with the defining relation in Definition 3.3.1, the coproduct axioms still hold. Now, let $B_l(x)$ denote the analogous series to $b_l$ in $LSC \otimes LSC$, and similarly for $B_r(x)$. To start, let $g(x)$ denote the following:

$$g(x) := \frac{B_l(x) \cdot i_F(x)}{B_l(i_F(x))}$$

Additionally, note that as the coefficients $i_F(x)$ lie in $MU_*$, they transform via the right unit. Therefore $\eta_R(i_F(x)) := b(i_F(b^{-1}(x)))$ in $LB$. Then, we see that if we aim to compute $\Delta : LSC \to LSC \otimes LSC$. To simplify the notation, let $i_g(x)$ denote the composition:

$$i_g(x) := g(i_F(g^{-1}(x))).$$

Note that this gives $\eta_R(i_F(x))$ in $LSC$. Next, we note that as

$$b(x) = \frac{B(x)i_F(x)}{B(i_F(x))},$$

we must also have that

$$\Delta(b(x)) = \Delta\left(\frac{B(x)i_F(x))}{\Delta(B(i_F(x)))}\right).$$

However, as we also have $\Delta b(x) = b_r \circ b_l(x)$ we can obtain the relation

$$\frac{\Delta(B(x))}{\Delta(B(i_F(x)))} \cdot i_F(x) = \frac{B_r(g(x))i_g(g(x))}{B_r(i_g(g(x)))}$$

where again, we note that $g(x)$ is the image of $b_l(x)$ in $LSC$ in the following way. If we expand terms, we see that on the right we obtain

$$\frac{\Delta(B(x))}{\Delta(B(i_F(x)))} \cdot i_F(x) = \frac{B_r(\frac{B_l(x)\cdot i_F(x)}{B_l(i_F(x))}) \cdot \frac{B_l(i_F(x))\cdot x}{B_l(x)}}{B_r(\frac{B_l(i_F(x))\cdot x}{B_l(x)})} = \frac{B_r(\frac{B_l(x)\cdot i_F(x)}{B_l(i_F(x))})}{B_r(\frac{B_l(i_F(x))\cdot x}{B_l(x)})} \cdot \frac{B_l(i_F(x)) \cdot x}{B_l(x)}$$

where we recall that $i_F(i_F(x)) = [-1 \cdot -1](x) = x$. Dividing by $i_F(x)$, we see that

$$\frac{\Delta(B(x))}{\Delta(B(i_F(x)))} = \frac{B_r(\frac{B_l(x)\cdot i_F(x)}{B_l(i_F(x))})}{B_r(\frac{B_l(i_F(x))\cdot x}{B_l(x)})} \cdot \frac{B_l(i_F(x)) \cdot x}{B_l(x) \cdot i_F(x)}$$

leading to the guess that

$$\Delta(B(x)) = B_r\left(\frac{B_l(x) \cdot i_F(x)}{B_l(i_F(x))}\right) \cdot \frac{B_l(i_F(x))}{i_F(x)}.$$

It suffices now to verify that this satisfies the necessary axioms. Unity follows clearly by replacing $B_l$ and $B_r$ with $x$ as necessary. Associativity goes as follows. We introduce the following terms to condense notation. Let $B_1 := B \otimes 1 \otimes 1$, $B_2 := 1 \otimes B \otimes 1$, and $B_3 := 1 \otimes 1 \otimes B$, and let $z$ and $\overline{z}$ be defined as follows:

$$z := \frac{B_1(x) \cdot i_F(x)}{B_1(i_F(x))} \qquad \overline{z} := \frac{B_1(i_F(x)) \cdot x}{B_1(x)}$$

Then, we see that

$$(\Delta \otimes 1)\Delta(B(x)) = B_3\left(\frac{B_2(z)\cdot B_1(i_F(x))\cdot x}{B_2(\overline{z})\cdot B_1(x)}\right)\cdot B_2(\overline{z})\cdot \frac{B_3(x)}{x\cdot i_F(x)}$$

$$(1\otimes \Delta)\Delta(B(x)) = B_3\left(\frac{B_2(z)\overline{z}}{B_2(\overline{z})}\right)\cdot \frac{B_2(\overline{z})}{\overline{z}}\cdot \frac{B_1(i_F(x))}{i_F(x)}$$

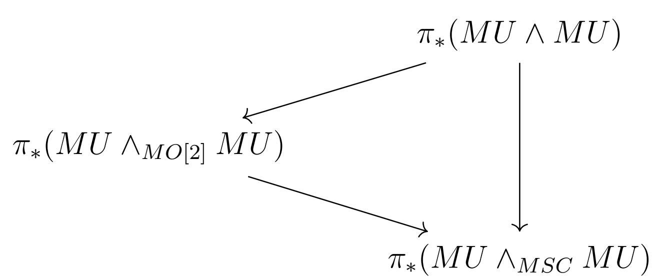
When the definitions are unwrapped, these are quickly verified to coincide.

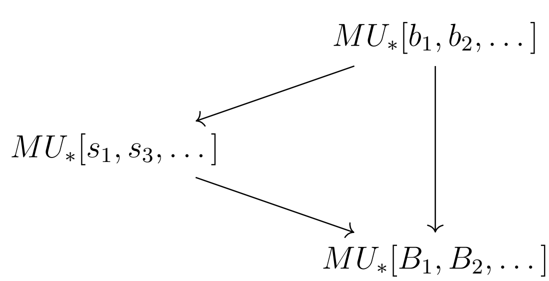
We see now that $(L, LSC)$ has inherited a coproduct via its defining relation. The left unit is given by strict inclusion, and the right unit is given by composing $\eta_R$ with the defining relation. The same holds for the conjugation. The counit is determined by sending $B_i$ to zero, just as the counit in $(L, LB)$ was determined. Therefore, we see that $(L, LSC)$ forms a Hopf algebroid. □

Finally, we may conclude our algebraic description of $LSC$ with the following theorem

**Theorem 3.3.2.** *$(L, LSC)$ and $(MU_*, MU \wedge_{MSC} MU_*)$ are isomorphic as Hopf algebroids.*

*Proof.* First, we need to note that the map structure map $\mathbb{S} \to MSC$ factors through $MO[2]$, meaning that we have a diagram:



where the algebraic computations are given in the below diagram. Now, we compare the underlying geometry. We do note that seeing as $BSC$ is the fiber of the map $B(\mathrm{id}-c) : BU \to BU$, the canonical inclusion $\mathbb{C}P^\infty \to BU(2)$ factors through $BSC$, meaning as discussed in the proof of Theorem 3.2.1, the Chern class $c_2$ induces the relation $xi_F(x) = b(x)b(i_F(x))$ must hold. In particular, we see that if we define the series $\overline{b}(x) := b(x)/x$, we see that $1 = \overline{b}(x)\overline{b}(i_F(x))$. Next, we note that with this notation, we are able to translate the map $(\mathrm{id}-c)$ classifying self-conjugate bundles as a relation on characteristic classes $\overline{B}(x) \cdot \overline{B}(i_F(x))^{-1}$., where we note that conjugation sends $x$ to $i_F(x)$, since conjugation acts via inversion on the universal formal group law over $MU$. Therefore, the vertical map in the diagram sends $\overline{b}(x) \mapsto \overline{B}(x) \cdot \overline{B}(i_F(x))^{-1}$. We see that this trivially respects the relation $1 = \overline{b}(x)\overline{b}(i_F(x))$ by recalling $[-1]_F([-1]_F(x)) = [(-1)^2]_F(x) = x$, and therefore, we have no additional relations on generators just as in the case of $\pi_*(MU \wedge_{MO[2]} MU)$.

Unwrapping the definitions shows that this is identical to the defining relation defined in Definition 3.3.1, which we have already shown gives a Hopf algebroid structure compatible with the one inherited from $LB$. As we already know that $LB$ is isomorphic to $\pi_*(MU \wedge MU)$, and the Hopf algebroid structure of $\pi_*(MU \wedge_{MSC} MU)$, then there must be an onto Hopf algebroid homomorphism from $LSC$ to $\pi_*(MU \wedge_{MSC} MU)$. It remains to show that there are no further defining relations in $\pi_*(MU \wedge_{MSC} MU)$. However, as we have already geometrically identified the generators $B_i$ in the calculation of $\pi_*(MU \wedge_{MSC} MU)$, we see that the generators are linearly independent and have not introduced anymore relations. $\square$

Finally, before proceeding, we would like to make one observation about the 2-local structure of $MU \wedge_{MSC} MU_*$. This is necessary for Theorem 3.4.1, where we also prove a similar statement about a construction related to $LS$.

**Lemma 3.3.2.** *The Hopf algebroid* $(MU_*, MU \wedge_{MSC} MU_*)^\wedge_2$ *is bipolynomial, i.e.* $Hom(MU \wedge_{MSC} MU_*, \mathbb{Z}_2)$ *is also polynomial.*

*Proof.* It suffices to consider the formula given by

$$\Delta(B(x)) = B_r\left(\frac{B_l(x)\cdot i_F(x)}{B_l(i_F(x))}\right)\cdot\frac{B_l(i_F(x))}{i_F(x)}.$$

We note by a generalized version of the Borel-Hopf structure theorem [Cro00], if $\Delta(B_i) = B_{i'}\otimes B_{i'} + \dots$, where $i = 2^n k$ for an odd value $k$, and $i' = 2^{n-1}k$, then our algebra will be bipolynomial. Therefore, we need to examine the coefficient of $x^{i+1}$ and show that $B_{i'}\otimes B_{i'}$ appears. We note that it again suffices to consider the above equation mod $I$, the augmentation ideal. We see this now becomes:

$$\Delta(B(x)) \equiv B_r\left(\frac{B_l(x)\cdot -x}{B_l(-x)}\right)\cdot\frac{B_l(-x)}{-x}.$$

Now, note that we may factor out a copy of $\frac{-x}{B_l(-x)}$ from inside $B_r$, as our series has no constant term, and this factor cancels with $\frac{B_l(-x)}{-x}$. From here, for degree reasons it suffices to examine just the term

$$(1\otimes B_{i'})(B_l(x))^{i'+1}\left(\frac{-x}{B_l(-x)}\right)^{i'}$$

Now, we note that the power series $\frac{-x}{B_l(-x)}$, is of the form

$$\frac{-x}{B_l(-x)} = 1+\sum_{j=1}^{\infty} c_j x^j,$$

where $c_j \equiv (-1)^{j-1}B_j\otimes 1$ modulo terms of the form $B_{j_1}\dots B_{j_m}\otimes 1$. Therefore, $B_{i'}\otimes 1$ appears in degree $i'+1$ of $B_l(x)$ and in the coefficient of the degree $i'$ term of $\frac{-x}{B_l(-x)}$. With this in mind, we have reduced computing the $B_{i'}\otimes B_{i'}$ in degree $i+2$ of the coproduct to a straightforward counting argument. Note that the leading coefficient of $\frac{-x}{B_l(-x)}$ was one and that $B_{i'}\otimes 1x^{i+1}$ cannot distribute to any higher order terms of $\frac{-x}{B_l(-x)}$ if we wish to have the form $B_{i'}\otimes B_{i'}$. Therefore, there will be $(i'+1)$ copies of $B_i'\otimes B_i'$ coming from the expansion of

$$(1 \otimes B_{i'})(B_l(x))^{i'+1} (1 + \dots)^{i'}$$

Similarly, if we note that the leading term of $B_l(x)$ is $x$, we analogously obtain $i'$ terms of the form $(-1)^{i'-1} B_{i'} \otimes B_{i'}$ from the expansion

$$(1 \otimes B_{i'})(x + \dots)^{i'+1} \left( \frac{-x}{B_l(-x)} \right)^{i'}.$$

By adding these two, we see that the coefficient of $B_{i'} \otimes B_{i'}$ will be either $2i' + 1$ or 1. Both of these are odd, and so as we are 2-local, both are units. □

## 3.4 Primitive Elements and The Witt Construction

We showed at the end of Section 3.2, that the algebra $LS$ is primitively generated with respect to its coproduct. The algebras $LB$ and $LSC$ are not primitively generated which can be easily checked by examining $\Delta(b_2)$ and $\Delta(B_2)$, and attempting to solve for a primitive in degree 4. As a consequence, these Hopf algebroids are (from the naïve perspective) much more difficult to work with. We will see, however, that there is a useful connection between $LS$ and $LSC$ that allows us to leverage the existence of primitives in $LS$ to show collapse of the rectified Adams-Novikov spectral sequence associated to $MSC_*$. To do this, we need the following construction, generalized from [Sch70].

**Definition 3.4.1.** *Given a Hopf algebroid $(A, R)$, and a collection of primitives $S \subset R$ such that $R = R_0[S]$, the Witt construction $(A, W_S(R))$ at a prime $p$ is a Hopf algebroid defined by:*

$$W_S(R) := R[s_i | s \in S, i \geq 0].$$

*The coproduct $\Delta(s_i)$ is determined by the requirement that the "ghost component" $w_i$ be* *primitive, where*

$$w_i := p^i s_i + p^{i-1} s_{i-1}^p + \dots p^{i-2} s_{i-2}^{p^2} + \dots p s_1^{p^{i-1}} + s^{p^i}.$$

*where* $|s_i| = p^i |s|$.

Next, as this construction is dependent on the choice of set $S$, we show there is a certain amount of independence

**Lemma 3.4.1.** *The Hopf algebroid* $(A, W_S(R))$*, up to isomorphism, is dependent only on the permanent cycle a primitive element* $s \in S \subset R$ *represents in* $Cotor^1_R(A, A)$*.*

*Proof.* Suppose $s$ and $s'$ are two primitives, such that they both converge to $(s)$ at $E_\infty$ of $\mathrm{Cotor}^1_R(A, A)$. As they are permanent cycles already, we know that $s - s' \in \mathrm{im}\ d_1$. If we examine the differential in the associated cobar complex we see that this is given precisely by $(\eta_L - \eta_R) : A \to R$. As they represent the same permanent cycle, we obtain $s = s' + (\eta_L - \eta_R)(a)$. Now, we show that $\Delta(s) = \Delta(s')$, and therefore determine the same coproduct on the interated $s_i$ and $s'_i$, implying the Hopf algebroid structures will be isomorphic. The calculation is as follows. First, we apply $\Delta$:

$$\Delta(s) = \Delta(s' + (\eta_L - \eta_R)(a))$$
$$s \otimes 1 + 1 \otimes s = \Delta(s') + \Delta((\eta_L - \eta_R)(a))$$
$$s \otimes 1 + 1 \otimes s = s' \otimes 1 + 1 \otimes s' + \Delta((\eta_L - \eta_R)(a))$$

noting that $s$ and $s'$ are primitive by definition. Now that we have applied the coproduct, we substitute $s$ for $s' + (\eta_L - \eta_R)(a)$ as appropriate, and cancel terms which appear on both sides:

$$(s' + (\eta_L - \eta_R)(a)) \otimes 1 + 1 \otimes (s' + (\eta_L - \eta_R)(a)) = s' \otimes 1 + 1 \otimes s' + \Delta((\eta_L - \eta_R)(a))$$

$$s' \otimes 1 + (\eta_L - \eta_R)(a) \otimes 1 + 1 \otimes s' + 1 \otimes (\eta_L - \eta_R)(a) = s' \otimes 1 + 1 \otimes s' + \Delta((\eta_L - \eta_R)(a))$$

$$(\eta_L - \eta_R)(a) \otimes 1 + 1 \otimes (\eta_L - \eta_R)(a) = \Delta((\eta_L - \eta_R)(a))$$

At this point, we note that passing over the tensor transforms the left unit into the right unit. Therefore, we can unpack the above to get:

$$\eta_L(a) \otimes 1 - \eta_R(a) \otimes 1 + 1 \otimes \eta_L(a) - 1 \otimes \eta_R(a) = \Delta((\eta_L - \eta_R)(a))$$

$$\eta_L(a) \otimes 1 - \eta_R(a) \otimes 1 + \eta_R(a) \otimes 1 - \eta_L(a) \otimes 1 = \Delta((\eta_L - \eta_R)(a))$$

$$0 = \Delta((\eta_L - \eta_R)(a))$$

Knowing this, we see that $\Delta(s) = \Delta(s') + \Delta((\eta_L - \eta_R)(a))$ becomes $\Delta(s) = \Delta(s') + 0$. □

Now, we apply this Witt construction to our Hopf Algebra $LS$, where we take the set $S$ to be the basis of primitives $S = \{\bar{s}_1, \bar{s}_3, \dots\}$. The properties of the resulting Hopf algebroid $(L, W_S(LS))$ are desirable as we shall soon see, but first we address a matter of notation. The new elements in $W_S(LS)$ are denoted $\bar{s}_{2i+1,j}$. The degree of these elements is $2 \cdot 2^j(2i+1)$, meaning we can uniquely relabel $s_{2i+1,j}$ as $s'_n$ for $n = 2^j(2i+1)$. This lets us conclude $W_S(LS) = MU_*[s'_1, s'_2, s'_3, \dots]$. Next, we highlight how the addition of the induced elements modifies the structure of $LS$.

**Lemma 3.4.2.** *For $p = 2$, the Hopf algebroid $W_S(LS)^\wedge_2$ is bipolynomial.*

*Proof.* Again, by [Cro00], if $\Delta(\bar{s}_{i,j}) \equiv \bar{s}_{i,j-1} \otimes \bar{s}_{i,j-1}$, then $W_S(LS)$ will be bipolynomial. Therefore, it suffices to examine the coproduct of the additional Witt elements. First, we note that the elements $s'_{2i+1}$ remain primitive. Next, we note that the coproduct for elements $s'_n$ where $n = 2^j \cdot (2i+1)$ depends only on $j$. Therefore, we need to determine just the structure of a generic $s_j$, induced by a primitive $s$. We proceed by induction. To deduce $s_1$, we have the following condition:

$$\Delta(2s_1 + s_0^2) = (2s_1 + s_0^2) \otimes 1 + 1 \otimes (2s_1 + s_0^2)$$

Therefore we have:

$$\begin{aligned}
2\Delta(s_1) + (s_0 \otimes 1 + 1 \otimes s_0)^2 &= (2s_1 + s_0^2) \otimes 1 + 1 \otimes (2s_1 + s_0^2) \\
2\Delta(s_1) + s_0^2 \otimes 1 + 1 \otimes s_0^2 + 2s_0 \otimes s_0 &= (2s_1 + s_0^2) \otimes 1 + 1 \otimes (2s_1 + s_0^2) \\
2\Delta(s_1) + 2s_0 \otimes s_0 &= 2s_1 \otimes 1 + 1 \otimes 2s_1 \\
\Delta(s_1) &= s_1 \otimes 1 + 1 \otimes s_1 - s_0 \otimes s_0.
\end{aligned}$$

Next, we assume that $\Delta(s_{j-1}) = s_{j-1} \otimes 1 + s_{j-1} \otimes 1 \pm s_{j-2} \otimes s_{j-2} \pm \ldots$, where we have omitted terms of higher algebraic order. Then to determine $\Delta(s_j)$, we get:

$$2^j \Delta(s_j) + 2^{j-1}\Delta(s_{j-1})^2 + \cdots = (2^j s_j + 2^{j-1}s_{j-1}^2 + \ldots) \otimes 1 + 1 \otimes (2^j s_j + 2^{j-1}s_{j-1}^2 + \ldots)$$

By induction hypotheses, we get

$$\begin{aligned}
2^j \Delta(s_j) + 2^{j-1}(s_{j-1}^2 \otimes 1 + 2s_{j-1} \otimes s_{j-1} + 1 \otimes s_{j-1}^2) + \ldots = (2^j s_j + 2^{j-1}s_{j-1}^2 + \ldots) \otimes 1 \\
+ 1 \otimes (2^j s_j + 2^{j-1}s_{j-1}^2 + \ldots)
\end{aligned}$$

This gives

$$2^j \Delta(s_j) = 2^j s_j \otimes 1 + 1 \otimes 2^j s_j - 2^j s_{j-1} \otimes s_{j-1} + \ldots$$

and so we see that $\Delta(s_j) = s_j \otimes 1 + 1 \otimes s_j - s_{j-1} \otimes s_{j-1} + \ldots$ satisfies the form of the induction hypothesis. $\square$

**Theorem 3.4.1.** *There is an isomorphism*

$$(L, W_S(LS))^{\wedge}_2 \cong (L, LSC)^{\wedge}_2$$

*Proof.* This is a corollary of Ravenel and Wilson. By Lemma 3.3.2 and Lemma 3.4.2, both Hopf algebroids are bipolynomial. Therefore by [RW74], they are isomorphic. □

## 3.5 Collapse for $MSC_*$ and Notes on $MO[2]_*$

Now, we can now present the main result.

**Theorem 3.5.1.** *The $E_2$-page of the rectified Adams-Novikov spectral sequence for $MSC$:*

$$Cotor_{LSC}(MU_*, MU_*) \Rightarrow \pi_*(MSC)$$

*collapses.*

This proof deviates from the classical techniques, and relies on the work of [GWX21]. We rely on the underlying algebraicity of $BSC$ and $BU$ to generalize to the motivic setting and note that after change-of-base, we specialize to our desired result.

*Proof.* First, we consider the permanent cycles $\alpha_{2k}$ and $\alpha_{2k+1}$, for $k \geq 0$, with $\alpha_0 = 1$. These classes are represented by the manifolds $\alpha_{2k+1} := [\mathbb{R}P^{4k+1}]$, and $\alpha_{2k} = [N_{4k+3}]$, where $N_{4k+3}$ are the Landweber manifolds detailed in [SS68b]. These are known permanent cycles and therefore represent classes in $MSC_*$. Importantly, this means we can form an $MSC$-module spectrum

$$F_{(\alpha_1,\ldots,\alpha_n,\ldots)}MSC = \mathrm{holim}_n \Sigma^{1-n} MSC/(\alpha_1, \ldots, \alpha_n).$$

By smashing with $MU$, we can construct the spectrum

$$F_{(\alpha_1,\ldots,\alpha_n,\ldots)}MU = \mathrm{holim}_n \Sigma^{1-n} MU \wedge_{MSC} MSC/(\alpha_1, \ldots, \alpha_n).$$

Additionally, we have the equivalence

$$MU \wedge_{MSC} F_{(\alpha_1,\dots,\alpha_n,\dots)}(MSC) \sim F_{(\alpha_1,\dots,\alpha_n,\dots)}(MU) \tag{3.5.1}$$

Now, we see that this construction gives us a map from $F_{(\alpha_1,\dots,\alpha_n,\dots)}(MU)$ to the cobar $MSC$-resolution constructed in Theorem 3.1.1. In fact, as the elements $\alpha_i$ were chosen to be those representing permanent cycles of our descent spectral sequence, this induces an isomorphism on the $E_2$-page of the corresponding spectral sequences. This implies an isomorphism of the corresponding homotopy groups, and therefore an equivalence $F_{(\alpha_1,\dots,\alpha_n,\dots)}(MU) \sim MSC$ in the category of $MSC$-modules. Additionally when combined with Eq. (3.5.1), we see that $MU$ and $F_{(\alpha_1,\dots,\alpha_n,\dots)}(MSC)$ are strongly dual in the derived category. More importantly, the dual structure implies they are inverse objects in this setting.

To make use of this, we now transition to the motivic setting. As it is known there is no $p$-torsion in $\pi_*(MSC)$ for odd $p$ by [SS68b], we make work in the 2-complete setting over $\mathbb{C}$. By leveraging the algebraicity of the maps $BSC \to BU \to BU$, and noting that $BU$ is equivalent to $BGL$, we produce a motivic analog of $MSC$, denoted $MSC^{Mot}$. Then, we note the results of Levine and Morel in [LM01], which gives

$$MGL_\star \cong MU_*[\tau].$$

Therefore, if we apply our previous constructions to this 2-complete motivic setting, we get a motivic analog of the rectified Adams-Novikov spectral sequence we constructed above. Namely, we obtain

$$\pi_\star(MGL \wedge_{MSC^{\mathrm{mot}}} MGL) \cong LSC[\tau].$$

This yeilds the spectral sequence:

$$\mathrm{Cotor}_{LSC}(MU_*, MU_*)[\tau] \Rightarrow MSC_*^{mot}$$

Now, we apply the work of [GWX21] to conclude that by change-of-base from $\mathbb{S}^{mot}$ to $\mathbb{S}/\tau$, we get that the spectral sequence

$$\mathrm{Cotor}_{LSC}(MU_*, MU_*) \Rightarrow \left(MSC^{mot}/\tau\right)_*.$$

collapses. Therefore, we now need to show $MSC^{Mot}$ has no $\tau$-torsion or $\mathbb{Z}$-multiplicative $\tau$-extensions. Showing this will imply that $(MSC^{Mot}/\tau)_* \cong MSC_*$ and therefore will be done. To start, we note that our construction giving $F_{(\alpha_i)}MSC \cong MU$ holds in the motivic setting, and therefore $F_{(\alpha_i)}MSC^{Mot} \cong MGL$ are inverse in the motivic derived setting. Then, suppose there is an non-zero element $\beta \in MSC_*^{Mot}$ which is $\tau$-torsion. Then, $\beta$ must act as 0 on $MGL_*$. However, as we have already noted, as these spectra are invertible and inverse to one another, implying that $\beta$ must also act as zero on $MSC_*^{Mot}$. Since $\beta$ was a non-zero element, we see that this implies a contradiction, and so there can be no $\tau$-torsion in $MSC_*^{Mot}$.

To show the absence of multiplicative extensions, we proceed similarly. As we are 2-complete, suppose there is some $x, y \in MSC_*^{Mot}$ such that $2^m x = \tau y$. Again, by strong duality, there must be corresponding some action on $MU_*[\tau]$. However, as we have already noted, the operations here are $MU_*MU[\tau]$, which has no $\tau$-extensions. Therefore, we see that $\mathrm{Ext}_{LSC}(MU_*, MU_*)$ collapses. This was our original spectral sequence of interest, and given the extensions and torsion are independant of $\tau$, we have the necessary result by the canonical comparison map to the non-motivic setting. □

We now note that the above is insufficient to generalize when $MSC$ is replaced with $MO[2]$. Specifically, we do not have the isomorphism onto the $E_2$ page of the necessary spectral sequences implied by Eq. (3.5.1) in the $MSC$ context.

# CHAPTER 4

# Implementing $\mathrm{Cotor}_\Gamma(MU_*, MU_*)$ in Sage

Having spent the previous portions of this thesis detailing the underlying algebraic structure of the rectified Adams-Novikov spectral sequence, and how it allows us to compute $\pi_*(MSC)$ and $\pi_*(MO[2])$, it is time to actually present computations of $\pi_*(MSC)$ and $\pi_*(MO[2])$. While we have showed the RANSS collapses for $MSC$, we do not have a simple characterization of the $E_\infty$-page of this spectral sequence. Therefore, we have written code which allows for the computation of homology of the reduced cobar complex associated to the RANSS for $MSC$ and $MO[2]$. We highlight some of the technical results from Chapter 3 which allow us to (somewhat) simplify the computation and allow for the solving of extensions. Our computations are implemented in `Sage` [S$^+$YY]. Before beginning our computations, the following declarations are made, which initialize our symbolic variables.

```
#steps necessary to compute the right unit in MO2 and subsequently WLSC.
var('y z');
degree=10; #sets n for m_n, coefficient of m_i x^(i+1)
m_var=var(['m_{}'.format(i) for i in (1..degree)]);
b_var=var(['b_{}'.format(i) for i in (1..degree)]);
x_var=var(['x_{}'.format(i) for i in (1..degree)]);
s_var=var(['s_{}'.format(i) for i in (1..degree)]);
cb_var=var(['cb_{}'.format(i) for i in (1..degree)]);
nb_var=var(['nb_{}'.format(i) for i in (1..degree)]);
nm_var=var(['nm_{}'.format(i) for i in (1..degree)]);
nx_var=var(['nx_{}'.format(i) for i in (1..degree)]);
```

```
ns_var=var(['ns_{}'.format(i) for i in (1..degree)]);
all_var=m_var+b_var+x_var+nb_var+nm_var+cb_var+s_var+ns_var;

M=PolynomialRing(QQ, all_var);
MU.<y,z>=PolynomialRing(M); # Creates a ring Q[m_1, b_1, m_2, b_2, ...]
P=LazyPowerSeriesRing(MU);

m_list=[0, 1]+list(m_var);
b_list=[0, 1]+list(b_var);
cb_list=[0, 1]+list(cb_var);
nm_list=[0, 1]+list(nm_var);
s_list=[0]+list(s_var);
m_yz_list=[0,y+z];
```

## 4.1 Structure Maps for $(L, LB)$

To proceed with a direct calculation of $\pi_*(MO_2)$ and $\pi_*(MSC)$, we need formulas for the differentials of the Cobar complex associated to the respective Cotor groups. To compute these differentials, we need formulas for the right unit and coproducts of $LS$ and $LSC$. As described in the Chapter 3, the Hopf algebroid structures of both $LS$ and $LSC$ are determined by the structure of $LB$. As such, we must first compute these maps for $(L, LB)$, and then make the appropriate substitutions for $LS$ and $LSC$ respectively.

We start by computing the right unit. Recall the relevant structure formulas for $(L, LB)$ as given in [Rav86]:

$$\sum_{i\geq 0} \eta_R(m_i) = \sum_{i\geq 0} m_i \left(\sum_{j\geq 0} c(b_j)\right)^{i+1} \qquad \sum_{i\geq 0} c(b_i) \left(\sum_{j\geq 0} b_j\right)^{i+1} = 1$$

where $b_i$ and $m_i$ are the coefficients of the power series:

$$\exp_F(x) := x + \Sigma_{i=1}^{\infty} b_i x^{i+1} \qquad \log_F(x) := x + \Sigma_{i=1}^{\infty} m_i x^{i+1}.$$

These functions denote the power series which define the $\mathbb{Q}$-isomorphism of the universal and additive formal group laws. We should note that the $b_i$ here are distinct from the $b_i$ used in the definition of $LB$.

Notice that the structure formulas determining $\eta_R$ do not involve the elements $x_i$. Therefore, if we wish to proceed, we need to first express the polynomial generators $x_i$ of $MU_*$ as polynomials in $\mathbb{Z}[m_i]$, such that up to decomposables, $x_i$ satisfies the conditions of Lazard's Theorem, Theorem 2.5.1. The choice of $x_i$ is not unique, and our method simply computes one of many compatible choices. Note that by this theorem, while $\exp_F$ and $\log_F(x)$ are power series over base rings with rational coefficients, the images of the associated coproduct and right unit for $L \to LB$ consist entirely of $\mathbb{Z}$-coefficients. Therefore, we are free to perform our operations over $\mathbb{Q}$, and provided we satisfy the conditions of Lazard's Theorem, we will have suitable choices.

Given $\exp_F$ and $\log_F$ are inverses, by composing the series and matching coefficients, we may solve for $b_i$ in terms of $m_i$ over $\mathbb{Q}$. The following snippet of code initializes all variables, and then utilizing the `LazyPowerSeries` package in Sage, creates two functions, `log` and `exp`, composes them, extracts the coefficients, sets them equal to zero and solves for each $b_i$ as a polynomial in the $m_i$.

```
def compute_b_as_m(m_list, b_list, P):
    log=P(m_list+[0]);
    exp=P(b_list+[0]);
    logexp=log(exp);
    coeffs=logexp.coefficients(degree+2);
    relation_eqs=[];
    for i in range(degree):
        relation_eqs.append(SR(coeffs[i+2])==0)

    b_sol=solve(relation_eqs, b_var);
    b_as_m=[];
    for i in range(degree):
```

```
        b_as_m.append(SR(b_sol[0][i].left()==b_sol[0][i].right().expand())
);

    return b_as_m
```

Next, we note that the universal formal group law $F(x,y) = \sum a_{i,j}x^i y^j$ can also be expressed as $\exp_F(\log_F(x) + \log_F(y))$. Again, by comparing coefficients, it is now possible to collect the $a_{i,j}$ entirely in terms of the $m_k$, as we have already expressed the $b_{k'}$ in terms of $m_k$. This is shown in the following snippet, where $\exp_F(x)$ and $\log_F(x)$ are denoted by $b(x)$ and $m(x)$ respectively:

```
def compute_aij_as_m(m_yz_list, b_list, b_as_m, P, MU):
    b_poly=P(b_list+[0]);
    m_yz=P(m_yz_list+[0]);
    an_coeffs_sim=[0];
    F=b_poly(m_yz);
    for i in range(degree):
        an_coeffs_sim.append(F.coefficient(i+2));

    an_coeffs_new=[];
    for i in range(degree):
        an_coeffs_new.append(SR(an_coeffs_sim[i+1]).subs(b_as_m));
    aij_as_m_array=[];
    for i in range(degree):
        aij_as_m_array.append(MU(an_coeffs_new[i]).coefficients());

    return aij_as_m_array
```

Specifically, we now have a collection of expressions

$$a_{i,j} = \binom{i+j}{j} m_k + \text{decomposables}$$

for all $i + j - 1 = k$ Then, as we know the leading coefficients of $m_k$, we can perform an extended Euclidean algorithm with all terms for $i < j < k$ to determine the coefficients $p_{i,j}$ such that:

$$x_k := \sum_{i+j=k+1, i<j} p_{i,j} a_{i,j} \equiv u_k m_k.$$

where $u_k$ is as determined in Theorem 2.5.1. This is detailed as follows:

```
def compute_x_as_m_m_as_x(aij_as_m):
    xi_list=[]
    for i in range(degree):
        xi_eq_list=aij_as_m[i];
        mi_coeffs=[];
        gcd=[];
        euc_coeffs=[];
        for j in range(len(xi_eq_list)):
            eq=xi_eq_list[j];
            monomial=eq.monomials()[len(eq.monomials())-1]
            mi_coeffs.append(eq.monomial_coefficient(monomial));
        for k in range(floor(len(mi_coeffs)+1/2)):
            if k==0 and floor(len(mi_coeffs)+1/2)!=1:
                gcd=xgcd(mi_coeffs[0],mi_coeffs[1]);
                euc_coeffs.append([gcd[1],gcd[2]]);
            elif k==0 and floor(len(mi_coeffs)+1/2)==1:
                euc_coeffs.append(-1);
            else:
                gcd=xgcd(gcd[0],mi_coeffs[k]);
                euc_coeffs.append([gcd[1],gcd[2]]);
        xi=0
        for k in range(floor(len(mi_coeffs)+1/2)):
            if k==0 and floor(len(mi_coeffs)+1/2)!=1:
                xi=euc_coeffs[k][0]*xi_eq_list[k]+euc_coeffs[k][1]*
    xi_eq_list[k+1]
            elif k==0 and floor(len(mi_coeffs)+1/2)==1:
                xi=euc_coeffs[0]
```

```
            else:
                xi=euc_coeffs[k][0]*xi+euc_coeffs[k][1]*xi_eq_list[k];
        xi_list.append(xi);
    xi_list[0]=2*m_1;

    x_as_m=[]
    for i in (1..degree):
        x_as_m.append(SR(x_var[i-1]==xi_list[i-1]))

    m_as_x_sol=solve(x_as_m, m_var);
    m_as_x=[];
    for i in range(degree):
        m_as_x.append(SR(m_as_x_sol[0][i].left()==m_as_x_sol[0][i].right()
   .expand()));

    return x_as_m, m_as_x
```

Next, as we know know $x_i$ in terms of $m_i$, it is possible to compute the $\eta_R(x_i)$ using the formula listed above for $\eta_R(m_i)$. This requires us to first compute the $c(b_j)$ in terms of $b_i$. We present the code:

```
def compute_cb_as_b(b_list, cb_list, P):
    cb_poly=P(cb_list+[0])
    b_poly=P(b_list+[0])
    b_cb_poly=cb_poly(b_poly)
    b_cb_coeffs=b_cb_poly.coefficients(degree+2)
    b_cb_coeffs_list=[]
    for i in range(degree+2):
        b_cb_coeffs_list.append(SR(b_cb_coeffs[i])==0)
    b_cb_eqs=b_cb_coeffs_list[2::]
    b_cb_solved=solve(b_cb_eqs, cb_var)
    cb_as_b=[]
    for i in range(len(b_cb_solved[0])):
```

```
        cb_as_b.append(SR(cb_var[i]==(SR(b_cb_solved[0][i].right()).expand
    ())))
    return cb_as_b
```

We can then take this and immediately use it to compute $\eta_R(m_i)$.

```
def compute_nm_as_m(m_list, nm_list, cb_list, cb_as_b, P):
    m_poly=P(m_list+[0]);
    nm_poly=P(nm_list+[0])
    cb_poly=P(cb_list+[0])
    nm_def_poly=m_poly(cb_poly)
    nm_rel_poly=nm_poly-nm_def_poly
    nm_m_coeffs=nm_rel_poly.coefficients(degree+2)
    nm_m_coeffs_list=[]
    for i in (0..degree+1):
        nm_m_coeffs_list.append(SR(nm_m_coeffs[i])==0)

    nm_m_coeffs_list=nm_m_coeffs_list[2::]
    nm_m_solved=solve(nm_m_coeffs_list, nm_var)
    nm_as_m=[]
    for i in range(len(nm_m_solved[0])):
        nm_as_m.append(SR(nm_var[i]==(SR(nm_m_solved[0][i].right()).expand
    ())).subs(cb_as_b).expand())
    return nm_as_m
```

We are now free to make the final substitutions to compute $\eta_R(x_i)$, by applying $\eta_R$ to our relations of $x_i$ in terms of $m_i$, substituting our solved values for $\eta_R(m_i)$, and then again substituting our relations for $m_i$ in terms of the $x$. Again, we note that the final relations only make sense a priori over $\mathbb{Q}$, but as shown by Lazard, in combination with the rest of the relations, will give us formula with with coefficients in $LB$. The code is given here:

```
def compute_nx_as_b(m_as_x, x_as_m, nm_as_m):
    temp_var=[]
    nx_as_nm=[]
```

```
    nx_as_m=[]
    nx_as_b=[]
    for i in range(len(list(nm_var))):
        temp_var.append(SR(m_var[i]==nm_var[i]))
    for i in range(len(x_as_m)):
        nx_as_nm.append(SR(nx_var[i]==x_as_m[i].right().subs(temp_var)).
   expand())
    for i in range(len(nx_as_nm)):
        nx_as_m.append(SR(nx_var[i]==nx_as_nm[i].right().subs(nm_as_m)).
   expand())
    for i in range(len(nx_as_m)):
        nx_as_b.append(SR(nx_var[i]==nx_as_m[i].right().subs(m_as_x)).
   expand())
    return nx_as_b
```

Next, we want to compute the coproduct $\Delta : LB \to LB \otimes LB$. As described in [Rav86], we have a polynomial which defines the generating relations:

$$\sum_{i\geq 0} \Delta(b_i)x^{i+1} = \sum_{j\geq 0} b''_j \left( \sum_{i\geq 0} b'_i x^{i+1} \right)^{j+1}$$

where $b'_i$ becomes $b_i \otimes 1$ and $b''_j$ becomes $1 \otimes b_j$. For computational notation, we denote $b_i \otimes 1$ as `bl_i` and $1 \otimes b_i$ by `br_i` (for left and right respectively). We again compose the necessary series and collect coefficients. We include the code:

```
def compute_cob_as_blr(cob_var, bl_list, br_list, degree, R):
    cob_list=[0,1]+list(cob_var)
    cob_poly=R(cob_list+[0])
    bl_poly=R(bl_list+[0])
    br_poly=R(br_list+[0])
    cob_as_blr_poly=cob_poly-br_poly(bl_poly)
    cob_as_blr_relations=cob_as_blr_poly.coefficients(degree+2)
    cob_as_blr_relation_eqs=[]
    for i in range(degree+2):
```

```
        cob_as_blr_relation_eqs.append(SR(cob_as_blr_relations[i])==0)
    cob_as_blr_solved=solve(cob_as_blr_relation_eqs, cob_var)[0]
    cob_as_blr=[]
    for cob in cob_as_blr_solved:
        cob_as_blr.append(cob.left()==cob.right().expand())
    return cob_as_blr
```

## 4.2 Structure Maps for $(L, LS)$ and Solving for Primitives

Just as before, we detail the necessary initializions to proceed with the computations.

```
#steps necessary to compute the coproduct in MO2, and subsequently WLSC
cob_var=var(['cob_{}'.format(i) for i in (1..degree)]);
cos_var=var(['cos_{}'.format(i) for i in (1..degree)]);
br_var=var(['br_{}'.format(i) for i in (1..degree)]);
bl_var=var(['bl_{}'.format(i) for i in (1..degree)]);
sl_var=var(['sl_{}'.format(i) for i in (1..degree)]);
sr_var=var(['sr_{}'.format(i) for i in (1..degree)]);
xl_var=var(['xl_{}'.format(i) for i in (1..degree)]);
xr_var=var(['xr_{}'.format(i) for i in (1..degree)]);
more_var=cos_var+cob_var+br_var+bl_var+sr_var+sl_var+xl_var+xr_var

CoPolyRing=PolynomialRing(QQ, more_var);
R=LazyPowerSeriesRing(CoPolyRing)
LS=ZZ[s_var[0::2]+x_var]
LS_LS=ZZ[sl_var[0::2]+sr_var[0::2]+xl_var+xr_var]

bl_list=[0,1]+list(bl_var)
br_list=[0,1]+list(br_var)

s_as_sl=[]
s_as_sr=[]
sl_even_as_sl_odd=[]
```

```
sl_even_as_sl_odd=[]
b_as_br=[]
b_as_bl=[]
b_as_cob=[]
x_as_nx=[]
x_as_xl=[]
x_as_xr=[]
#generate symbolic conversion relations
for i in range(degree):
    s_as_sl.append(SR(s_var[i]==sl_var[i]))
    s_as_sr.append(SR(s_var[i]==sr_var[i]))
    b_as_bl.append(SR(b_var[i]==bl_var[i]))
    b_as_br.append(SR(b_var[i]==br_var[i]))
    b_as_cob.append(SR(b_var[i]==cob_var[i]))
    x_as_nx.append(SR(x_var[i]==nx_var[i]))
```

Before continuing to the structure maps for $(L, LS)$, it is important to highlight that the results in the previous section were sufficient to compute the coefficients of the universal formal group law as polynomials in the $x_i$. Specifically, we can compute $[2]_F(x)$ and $[-1]_F(x)$, which are each used in the defining relations for $(L, LS)$. In both cases, we leverage the isomorphism $F(x, y) = \exp_F(\log_F(x) + \log_F(y))$ to draw our desired conclusions. We note if $F(x, i_F(x)) = \exp_F(\log_F(x) + \log_F(i_F(x))) = 0$ we can set $i_F(x) = \exp_F(-\log_F(x))$. The following code performs the necessary composition and substitutions for $[-1]_F(x)$.

```
def compute_ifx_coeff_list(b_list, m_as_x, b_as_m, P):
    b_poly=P(b_list+[0])
    m_as_x_list=[0,1]
    for m in m_as_x:
        m_as_x_list.append(m.right())
    m_as_x_poly=P(m_as_x_list+[0]);
    ifx=b_poly(-1*m_as_x_poly);
    ifx_coeffs_sim=ifx.coefficients(degree+2);
    ifx_coeffs_inter=[0]
```

```
    ifx_coeffs_final=[]
    for i in range(degree+1):
        ifx_coeffs_inter.append(SR(ifx_coeffs_sim[i+1]).subs(b_as_m));
    for j in range(degree+1):
        ifx_coeffs_final.append(SR(ifx_coeffs_inter[j]).subs(m_as_x).
   expand());
    return ifx_coeffs_final
```

Similarly, we can express $[2]_F(x)$ as $\exp_F(2\log_F(x))$. This computation is included in the next set of results.

As we noted previously, we have computed the Hopf algebroid structure on $(L, LB)$ and so we can now compute the structure for $(L, LS)$ and $(L, LSC)$. Recall that the structure of $LS$ is defined by

$$b(x) = x + [2]_F(x)s(x) \qquad s(x) = \sum_{i\geq 0} s_i x^i$$

along with the relation

$$x i_F(x) = b(x)b(i_F(x)).$$

By performing a similar series of compositions and substitutions, we can compute the coproduct and right unit of $LS$ in terms of the $s_i$. This consists of two stages. The first relation allows us to relate $s_i$ to $b_i$. Then, using the second relation, we can express each $s_{2i}$ as a function of $s_{2j+1}$ for all $j < i$, as noted in the proof of Lemma 3.2.2. This is done in one step (along side the computation for $i_F(x)$), in the following code:

```
def compute_s_as_b_relations(ifx_coeff_list, s_var, b_var, b_as_x, m_as_x,
    degree, P):
    s_list=[0]+list(s_var);
    b_list=[0,1]+list(b_var);
    b_as_x_list=[0,1]
    for b in b_as_x:
        b_as_x_list.append(b.right())
    m_as_x_list=[0,1]
```

```
    for m in m_as_x:
        m_as_x_list.append(m.right())

    s_poly=P(s_list+[0]);
    b_poly=P(b_list+[0]);
    b_F_poly=P(b_as_x_list+[0]);
    m_F_poly=P(m_as_x_list+[0]);
    ix_poly=P(ifx_coeff_list+[0])
    x_poly=P([0,1,0])
    xix_poly=x_poly*ix_poly

    F2x_rational=b_F_poly(2*m_F_poly);
    F2x_coeffs=F2x_rational.coefficients(degree+2)
    F2x_integral_coeffs=[]
    for coeff in F2x_coeffs:
        F2x_integral_coeffs.append(SR(coeff).expand())
    F2x_poly=P(F2x_integral_coeffs+[0])



    b_as_s_poly=x_poly+F2x_poly*s_poly
    b_as_s_relations_poly=b_as_s_poly-b_poly
    b_as_s_relations=b_as_s_relations_poly.coefficients(degree+2)
    b_as_s_relations_list=[]
    for i in range(degree+2):
        b_as_s_relations_list.append(SR(b_as_s_relations[i]).expand()==0)
    s_as_b_solved=solve(b_as_s_relations_list, s_var)[0]
    b_as_s_solved=solve(s_as_b_solved, b_var)[0]
    s_as_b=[]
    b_as_s=[]
    for i in range(len(s_as_b_solved)):
        s_as_b.append(SR(s_as_b_solved[i].left()==s_as_b_solved[i].right()
  .expand()))
```

```
        b_as_s.append(SR(b_as_s_solved[i].left()==b_as_s_solved[i].right().expand()))

    s_even_as_s_odd_poly=b_as_s_poly*b_as_s_poly(ix_poly)-xix_poly

    s_even_as_s_odd_relations=s_even_as_s_odd_poly.coefficients(degree+3)
    s_even_as_s_odd_relations_list=[]
    for i in range(degree+3):
        s_even_as_s_odd_relations_list.append(SR(s_even_as_s_odd_relations[i]).expand()==0)

    even_s_var=s_var[1::2]
    s_even_as_s_odd_relations_solved=solve(s_even_as_s_odd_relations_list[0::2], even_s_var)
    s_even_as_s_odd=[]
    for i in s_even_as_s_odd_relations_solved[0]:
        s_even_as_s_odd.append(SR(i.left()==i.right().expand()))

    return s_as_b, b_as_s , s_even_as_s_odd
```

Now, given that we have $\eta_R(x_i)$ in terms of $b_i$, $b_i$ in terms of $s_i$, and the $s_{2i}$ in terms of $s_{2j+1}$, we can immediately obtain $\eta_R(x_i)$ for $LS$ by performing the following simple substitutions:

```
def compute_b_as_s_odd(b_as_s, s_as_b, s_even_as_s_odd):
    b_as_s_odd=[]
    for b in b_as_s:
        b_as_s_odd.append(b.left()==b.right().subs(s_even_as_s_odd).expand())
    return b_as_s_odd, s_as_b[::2]

def compute_nx_as_s_odd(nx_as_b, b_as_s_odd):
    nx_as_s_odd=[]
    for nx in nx_as_b:
        nx_as_s_odd.append(nx.left()==nx.right().subs(b_as_s_odd).expand()
```

```
    )
    return nx_as_s_odd
```

Next, we can compute the coproduct in $LS$. The initial substitutions are straightforward. As we know $s_i$ in terms of $b_i$, we apply $\Delta$ to obtain $\Delta(s_i)$ in terms of $\Delta(b_i)$, and then use the computation of $\Delta(b_i)$ we obtained in the previous section to obtain $\Delta(s_i)$ in terms of the variables `bl_i` and `br_i`. Next by knowing $b_i$ in terms of the $s_{2k+1}$, we can obtain `bl_i` in terms of `sl_i` and equivalently for the elements on the right of the tensor. Now, there is a bit of nuance. The coproduct of $LB$ did not involve elements of $MU_*$; that is to say, it was free of all $x_i$. However, we have seen that due to the $[2]_F(x)$ and $[-1]_F(x)$ terms in the defining relations for $LS$, the relations between $b_i$ and $s_i$ do involve the $x_i$. Therefore when we substitute $br_i$ for $sr_i$, we need to respect that elements $x_i$ are appearing on the right of the tensor. Our calculations are simplified if we consider $LB \otimes LB$ as a polynomial algebra $MU_*[b_i \otimes b_j]$, so therefore, we need to pass $x_i$ on the right of the tensor over to the left so we can accurately regard them as coefficients. This involves transforming them via the right unit, so the term $1 \otimes x_i$ becomes $\eta_R(x_i) \otimes 1$. More practically, we perform a substitution `xr_i` by $\eta_R(x_i)$ expressed in terms of `sl_i` and `xl_i`. We include the two preliminary substitutions (translating the `b_as_s` relation to the left and right and translating `xr_i` to $\eta_R(x) \otimes 1$) and the final coproduct substitution here:

```
def compute_blr_as_slr(b_as_s_odd, s_as_sl, s_as_sr, x_as_xl, x_as_xr,
    br_var, bl_var, degree):
    bl_as_sl=[]
    br_as_sr=[]
    for i in range(degree):
        bl_as_sl.append(SR(bl_var[i]==SR(b_as_s_odd[i].right()).subs(
    s_as_sl).subs(x_as_xl)))
        br_as_sr.append(SR(br_var[i]==SR(b_as_s_odd[i].right()).subs(
    s_as_sr).subs(x_as_xr)))
    return bl_as_sl, br_as_sr

```

```
def compute_xr_as_nxl(nx_as_s_odd, s_as_sl, x_as_xl, xr_var):
    xr_as_nxl=[]
    for i in range(degree):
        xr_as_nxl.append(SR(xr_var[i]==SR(nx_as_s_odd[i].right()).subs(s_as_sl).subs(x_as_xl)))
    return xr_as_nxl

def compute_cos_as_slr(s_as_b, b_as_cob, cob_as_blr, bl_as_sl, br_as_sr, degree):
    cos_as_cob=[]
    cos_as_blr=[]
    cos_as_slr=[]
    cos_as_slrn=[]
    cos_as_s_final=[]

    for i in range(degree):
        cos_as_cob.append(SR(cos_var[i]==SR(s_as_b[i].right()).subs(b_as_cob).subs(x_as_xl)))
    for i in range(degree):
        cos_as_blr.append(SR(cos_var[i]==SR(cos_as_cob[i].right()).subs(cob_as_blr)))
    for i in range(degree):
        cos_as_slr.append(SR(cos_var[i]==SR(cos_as_blr[i].right()).subs(bl_as_sl+br_as_sr).expand()))
    for i in range(degree):
        cos_as_slrn.append(SR(cos_var[i]==SR(cos_as_slr[i].right()).subs(xr_as_nxl).expand()))
    for i in range(degree):
        cos_as_s_final.append(SR(cos_var[i]==SR(cos_as_slrn[i].right()).subs(s_as_sl).expand()))
    return cos_as_s_final[::2]
```

Now, we can procedurally obtain the image of the generators $s_i$ under the coproduct in

$LS$, and by extension, all polynomials in $LS$. We are now technically able to proceed with the computation of the associated cobar complex and start computing $\pi_*(MO[2])$. However, this is computationally unfeasible, and would require numerous substitutions, passing $\eta_R(x)$ over the tensor to collect all of the $x_i$ terms. Therefore, we would like to find a basis of primitives $\overline{s}_{2i+1}$. This not only simplifies calculations for $LS$, but also make computations of $LSC$ possible, which we will see in Section 4.3.

To start the process of solving for primitives, we recall first that we already asserted such a basis exists in Chapter 3. Therefore, we can safely assume such terms exist, and use this to help simplify our solving process. We know that on $s_{2i+1}$, each coproduct is of the form

$$\Delta(s_{2i+1}) = s_{2i+1} \otimes 1 + 1 \otimes s_{2i+1} + f(s_{k'} \otimes s_k)$$

where $f$ is a polynomial in $LS \otimes LS$.

It suffices to find some homogenous degree $2(2i+1)$ polynomial $g \in LS$, such that

$$\Delta(g) = g \otimes 1 + 1 \otimes g - f(s_{k'} \otimes s_k).$$

To do this, we enumerate all monomials of the desired topological degree, compute the coproducts, subtract the primitive part, and isolate the error term. Then, it is possible to construct a system of linear diophantine equations to solve for coefficients which eliminate the original error term of $\Delta(s_{2i+1})$. Importantly, we are able to do this because we know such a solution must exist, and therefore our code can simply find it.

We now examine the implementation of this process more closely. Let us start by enumerating the necessary monomials in degree $2(2i+1)$. We can classify these into three groups: those which are strictly in $\mathbb{Z}[s_i]$ (denoted `s_only` terms); those which are strictly in $MU_*$ (denoted `x_only` terms); and those which involve both the $s_i$ and $x_i$ terms (denoted `xs` terms). To condense our notation, we denote monomials using multiindex notation. We note that the `x_only` terms can be disregarded entirely, as $\Delta(x_i) = x_i \otimes 1$, as these are elements of the base ring for the Hopf algebroid. We generate these classes of terms again utilizing the `LazyPowerSeries` package. We construct three generating functions such that the coefficient of $x^i$ is a polynomial containing all monomials of degree $2i$. We then extract these monomials while disregarding the associated coefficients. This is implemented here:

```

def compute_initial_terms(x_var, s_var, order, degree, term_gen_ring):
    x_list=[0]+list(x_var)
    s_list=[0]
    for i in range(degree):
        if i%2==0:
            s_list.append(s_var[i])
        else:
            s_list.append(0)
    xs_list=[]
    for i in range(degree+1):
        xs_list.append(x_list[i]+s_list[i])

    weight_list=[0,1,1,1,1,1,1,1,1];
    weight_poly=term_gen_ring(weight_list)
    xs_generator_poly=term_gen_ring(xs_list)
    x_generator_poly=term_gen_ring(x_list)
    s_generator_poly=term_gen_ring(s_list)
    s_term_poly=weight_poly(s_generator_poly);
    x_term_poly=weight_poly(x_generator_poly);
    total_term_poly=weight_poly(xs_generator_poly);
    all_xs_term_list=LS(total_term_poly.coefficient(order)).monomials()
    s_terms=LS(s_term_poly.coefficient(order)).monomials()
    x_terms=LS(x_term_poly.coefficient(order)).monomials()
    xs_term_list=[]
    for term in all_xs_term_list:
        if term in x_terms:
            continue
```

```
        elif term in s_terms:
            continue
        else:
            xs_term_list.append(term)
    return xs_term_list, s_terms
```

Next, we detail why we have chosen to separate the `s_only` and `xs` terms. If we consider a monomial $s_J$ for some index set of powers $J$, the coproduct is of the form

$$\Delta(s_J) = s_J \otimes 1 + 1 \otimes s_J + \text{error}$$

where error denotes the nonprimitive part. Ideally, we would simply compute the error portion for all $s_J$ and $x_I s_J$ and then construct the necessary diophantine system. However, then`xs` terms must be given special attention with regards to their coproduct. Therefore similarly to the `x_only` terms above, the coproduct here becomes

$$\Delta(x_I s_J) = \Delta(x_I)\Delta(s_J) = x_I \otimes 1(s_J \otimes 1 + 1 \otimes s_J + \text{error}) = x_I s_J \otimes 1 + x_I \otimes s_J + \text{error}.$$

We again have an issue with the side of the $x_I$, as we want to isolate the error term by subtracting $x_I s_J \otimes 1 + 1 \otimes x_I s_J$, but cannot simply replace $x_I \otimes s_J$ by $1 \otimes \eta_R(x_I)s_J$, as it is important that our error terms are consistently expressed in basis where all $x_i$ are on the left. We can solve this dilemma by noting that $\eta_R(x_I) \otimes s_J = 1 \otimes x_I s_J$. Therefore, it suffices to add $-\eta_R(x_I) \otimes s_J + 1 \otimes x_I s_J$ to the coproduct, yielding:

$$\Delta(x_I s_J) = x_I s_J \otimes 1 + 1 \otimes x_I s_J - \eta_R(x_I) \otimes s_J + x_I \otimes s_J + \text{error}.$$

where we now include $-\eta_R(x_I) \otimes s_J + x_I \otimes s_J$ as part of our error term.

As it stands, we have only computed the image of the necessary generators, that is to say, we have $\eta_R(x_i)$ and $\Delta(s_{2j+1})$. To generalize this, we make use of the `R.hom()` function associated to a `Ring` object in Sage. This will allow us to simply loop over the monomials and extract the error terms systematically. First, we define the rings:

```
LS=ZZ[s_var[0::2]+x_var]
LS_LS=ZZ[sl_var[0::2]+sr_var[0::2]+xl_var+xr_var]
```

Then, we produce the coproduct and the right unit on `xl_i`. We also include some of the helper functions used to format our existing results to the proper input. Additionally, we define maps which are strict left and right inclusions. These are use to isolate the error term of the coproduct in the next step of our process.

```
def compute_coprod_LS_list(cos_as_slr, xl_var):
    coprod_LS_list=[]
    for cos in cos_as_slr:
        coprod_LS_list.append(cos.right())
    for xi in xl_var:
        coprod_LS_list.append(xi)
    return coprod_LS_list

def compute_left_to_right_unit(sr_var, xr_var, nx_as_s_odd, x_as_xl,
    s_as_sl):
    right_unit=[]
    for i in range(degree):
        right_unit.append(((SR(SR(nx_as_s_odd[i]).right().subs(x_as_xl).
    subs(s_as_sl)))).expand())
    prim_refine=flatten([sl_var[0::2]]+[sr_var[0::2]]+[right_unit]+[xr_var
    ])
    return prim_refine
```

```
coprod_LS_list=compute_coprod_LS_list(cos_as_slr, xl_var)
coprod=LS.hom(coprod_LS_list, LS_LS)

left_to_right_unit=compute_left_to_right_unit(sr_var, xr_var, nx_as_s_odd,
     x_as_xl, s_as_sl)
left_to_right_refine=LS_LS.hom(left_to_right_unit, LS_LS)

```

```
right_incl=LS.hom(sr_var[0::2]+xr_var, LS_LS)
left_incl=LS.hom(sl_var[0::2]+xl_var, LS_LS)
```

Finally, we can now compute the error terms. First, we enumerate and isolate the errors, based on if the monomial is in `xs_terms` or `s_only_terms`. We make the appropriate corrections in the case of the `xs_terms` and for all monomials subtract the primitive part.

```
def compute_primitive_errors(xs_terms, s_only_terms, x_as_xl, s_as_sr,
    LS_LS ):
     primitive_errors=[];
     for term in xs_terms:
          xterm=LS_LS(SR(term).subs(x_as_xl).subs(s_as_sr))
          r_xterm=LS_LS(SR(term).subs(x_as_xr).subs(s_as_sr))
          correction= -1*left_to_right_refine(xterm)+r_xterm
          primitive_errors.append(SR(coprod(term)+correction-right_incl(term
    )-left_incl(term)).subs(xr_as_nxl).expand())
     for term in s_only_terms:
               primitive_errors.append(SR(coprod(term)-right_incl(term)-
    left_incl(term)))
     return primitive_errors
```

This results in a list of all error terms corresponding to monomials in topological degree $2(2i+1)$. Due to the lexicographic ordering of the `monomials()` command used to extract the terms for `xs_terms` and `s_only_terms`, the error of the $\Delta(s_{2i+1})$ will always be the final entry of `primitive_errors`. Therefore, we can proceed to setup a system of equations to solve for terms which cancel the errors. Symbolically, we create a polynomial ring $\mathbb{Z}[p_i]$ over placeholder variables `p_i`. This allows us to solve for the coefficients of the primitive term:

$$\overline{s}_{2i+1} = s_{2i+1} + \sum_{x_I s_J} p_i x_I s_J.$$

We loop over the `primitive_error` list, constructing a single overall error term. We now attempt to solve for the variables $p_i$ which force this term to be zero. We extract the coefficients of this overall error term, which gives a list of equations in the `p_i`, which we denote `error_coeffs`. At this point, we could simply proceed to use the `solve()` function over a symbolic ring to solve $p_i$. However, `solve()` exclusively works over $\mathbb{Q}$, which does not produce specific solutions, but a general solution. In our case, their are infinitely many solutions over $\mathbb{Q}$. As the number of terms grows similarly to the number of integer partitions, we see that manually searching for integer solutions from the generic rational solution is unfeasible. Therefore, we setup a linear system of equations and leverage Smith Normal Form to solve for integer coefficients. We convert the list of equations into a system $A \cdot P = C$. By applying smith normal form to $A$, we obtain $U, V$ such that $UAV = B$, where $B$ is in Smith Form. From here we see that $BV^{-1}P = U \cdot C$. Then, if we let $k$ be the rank of $A$ and $n-1$ be the number of variables we are solving for, we can conclude that $P = V \cdot C'$, where $C'_i = (U \cdot C)_i$ if $1 \leq i \leq k$, $C'_i = 0$ if $k < i < n$ and $C'_n = 1$. This $P$ contains the the solved values for `p_i` so all that is left is to dot $P$ with the list of monomials to obtain the primitive term. This is included here:

```
def lin_dioph_solv(A,C):
    smith=A.smith_form()
    B=smith[0]
    U=smith[1]
    V=smith[2]
    D=U*C
    k=B.rank()
    m,n=[len(A.rows()), len(A.columns())]
    temp=[]
    for i in range(n):
        if i<k:
            temp.append(D[i][0]/B[i,i])
        elif i==n-1:
            temp.append(1)
        else:
            temp.append(0)
```

```
    X=V*matrix(temp).transpose()
    return X
```

```
def compute_primitive_generator(xs_terms, s_only_terms, primitive_errors):
    p=var(['p_{}'.format(i) for i in range(len(primitive_errors))]);
    Dummy_Var_Ring=ZZ[p]
    Dummy_LS=Dummy_Var_Ring[sl_var[0::2]+sr_var[0::2]+xl_var+xr_var]
    s_error_term=primitive_errors[len(primitive_errors)-1] #error of s_2n+1
    for i in range(len(primitive_errors)-1):
        s_error_term=s_error_term+p[i]*primitive_errors[i]
    error_poly=Dummy_LS(s_error_term)
    error_coeffs=error_poly.coefficients()
    prim_mat=[]
    for term in error_coeffs:
        coeff_vec=[]
        for i in range(len(p)+1):
            if i!=len(p):
                coeff_vec.append(int(Dummy_Var_Ring(term).monomial_coefficient(Dummy_Var_Ring(p[i]))))
            else:
                coeff_vec.append(int(Dummy_Var_Ring(term).constant_coefficient()))
        prim_mat.append(coeff_vec)
    prim_mat=matrix(prim_mat)
    A=prim_mat[:,:-1]
    C=-1*prim_mat[:,-1]
    prim_coeffs = lin_dioph_solv(A,C)
    term_list=xs_terms+s_only_terms
    term=(matrix(term_list)*(prim_coeffs))[0][0]
    return term
```

Additionally, we want to point out entries for $k < i < n$ are in fact free entries, and not forced to be zero. Modifying these entries allows us to generate all possible integer coefficients for primitives. This is in fact one area for significant improvement, as the resulting primitives have coefficients which are quite large. Being able to minimize these coefficients could potentially improve performance when computing the homology of the cobar complex later.

Our final step requires us to note that we have $\eta_R(x_i)$ in terms of the $s_i$. A simple substitution (which we will not include) allows us to produce `nx_as_sbar` from `nx_as_s_odd`. We now have completely determined (computationally) the Hopf algebra structure of $(L, LS)$. We are now able to proceed with characterizing $(L, LSC)$.

## 4.3 Structure Maps for $(L, LSC)$ and the Witt Construction

To characterize $(L, LSC)$, we should start by highlighting the most important strucutral difference between $LS$ and $LSC$. The existence of primitives in $LS$, the cobar calculation can be done globally over $\mathbb{Z}$. This is not a luxery shared by $LSC$, which we have already addressed in Chapter 3. Therefore, to proceed with a meaninful computation of the cobar complex associated to $LSC$, we need to think outside the box. As a consequence of [RW74], 2-locally, we have an isomorphism:

$$W_S(LS) \cong LSC$$

where $W_S(LS)$ is the Witt construction applied to the primitives $\overline{s}_{2i+1}$, which we computationally solved for in Section 4.2. Then, since each induced $s_{2i+1,j}$ has a coproduct which can be recursively determined, and is exclusively in terms of $\overline{s}$, with no $x_i$. Therefore, we do not have to worry about passing $x_i$ over the tensor to collect coefficients. This means that we are a modified version of the cobar complex for $(L, LS)$ to $W_S(LS)$. This will only allow us to compute the 2-torsion associated to $\pi_*(MSC)$. However, as we have already noted, at odd primes $\pi_*(MSC)$ is polynomial, meaning the only torsion is 2-torsion. Therefore, we can work over $\mathbb{Z}$ and ignore the $p$-torsion for odd $p$.

Working 2-locally with the Witt construction provides two major benefits to the code. We have already mentioned the first, being that the $s_{i,j}$ do not contain $x_i$. The second is similar, as $\eta_R(x_i)$ will involve any of the new induced elements, meaning we have already effectively computed $\eta_R$ for $(L, LSC)^\wedge_2$. Therefore, the only necessary step to determine the Hopf algebra structure is to compute the coproduct on the induced $s_{i,j}$. The determination of this coproduct is done by requiring the ghost component

$$\overline{s}_{2i+1,0}^{2^j} + 2\overline{s}_{2i+1,1}^{2^{j-1}} + \dots 2^j\overline{s}_{2i+1,j}$$

be primitive. This uses a modified version of the error algorithm detailed above. This is shown here:

```
var('y z');
degree=4; #sets n for m_n, coefficient of m_i x^(i+1)
s_var=var(['s_{}'.format(i) for i in (0..degree)]);
sl_var=var(['sl_{}'.format(i) for i in (0..degree)]);
sr_var=var(['sr_{}'.format(i) for i in (0..degree)]);
cos_var=var(['cos_{}'.format(i) for i in (0..degree)]);
all_var=s_var+sl_var+sr_var+cos_var
WLS=PolynomialRing(ZZ,all_var)

cos_as_s=[SR(cos_0== sl_0+sr_0)]
cos_ghost=0
left_ghost=0
right_ghost=0
for i in range(1,degree+1):
    for j in range(i+1):
        cos_ghost=2^j*cos_var[j]^(2^(i-j))+cos_ghost
        left_ghost=2^j*sl_var[j]^(2^(i-j))+left_ghost
        right_ghost=2^j*sr_var[j]^(2^(i-j))+right_ghost
```

```
    cos_as_s.append(SR(cos_ghost==left_ghost+right_ghost))
    cos_as_s_solved=solve(cos_as_s, cos_var[0:i+1])[0]
    cos_temp=[]
    for term in cos_as_s_solved:
        print(term.expand())
        cos_temp.append(term.expand())
    cos_as_s=cos_temp
```

We note that the recursion relation is independent of which primitive $\overline{s}_{2i+1}$ is used as the initial term, and therefore allows us to compute many relations simultaneously. Note that $\overline{s}_{2i+1,j}$ has degree $|2^{j+1}(2i+1)|$, and for the sake of notational simplicity, as each $n \in \mathbb{N}$ can be written uniquely as $2^j(2i+1)$, we can re-lable $\overline{s}_{2i+1,j}$ by $\overline{s}_n$ for the corresponding $n$. This also lets us see how the Witt construction is "filling-in" the missing $s_{2i}$ from $LS$, and using them as primitive analogs in place of the $B_{2i}$ present in $(L, LSC)$.

## 4.4 The Cobar Complex

We are now able to proceed with the construction of the cobar complex and the associated computation of its homology. Algebraically, the cobar complex is given by:

$$d_s : \overline{LS}^{\otimes s} \otimes MU_* \to \overline{LS}^{\otimes s+1} \otimes MU_*$$

where the differential is defined by the map:

$$d_s(s_1|\dots|s_s|x_0) := id_1 \otimes \cdots \otimes id_s \otimes \eta_R(x) + \sum_{j=1}^{s}(-1)^{j+1} id_1 \otimes \Delta_j(s_j) \otimes id_s \otimes id_0.$$

Additionally, recall that $\overline{LS}$ is the cokernel of the left unit, which given that $\eta_L$ was the inclusion, gives $\overline{LS} := \mathbb{Z}[\overline{s}_{2i+1}]$.

Therefore, we can continue to leverage `Sage` and the packages associated to the `Ring` class to help us construct the cobar complex. The process centralizes around how we can produce differentials which are manageable in terms of computational resources. As taking homology requires extracting elementary divisors, if we can keep the differential matrices small, we will be able to compute to much higher total degree than otherwise possible. First, we note that our differentials respect topological degree, so it is possible to work levelwise in $t$ in our generation of the cobar complex. This means we are able to compute just the degree $t$ part of $d_s$, denoted $d_{t,s}$ as we loop over $t$. Secondly, note that the tensor $\overline{LS}^{\otimes s}$ can be treated as polynomial, with generators $\overline{s}_{2i+1,\ell} := 1 \otimes \ldots \overline{s}_{2i+1} \cdots \otimes 1$ where $\overline{s}_{2i+1}$ is in the $\ell^{th}$ index. We denote these variables by `s_i_j` (this notation is similar to the notation initially used in the Witt construction, but as we have already addressed above, we have chosen to relabel those terms for ease of notation).

Now, we have completed the setup necessary to begin the process of constructing the differentials. First, we will be looping over $t$. Therefore, our code takes a fixed $t$, and will compute all $d_{t,s}$ $0 < s < t$. As such it also suffices to fix $s$ in the range $0 < s < t$. We let `C_i` denote $(\overline{LS}^{\otimes s} \otimes MU_*)_t$ and `C_i1` denote $(\overline{LS}^{\otimes s+1} \otimes MU_*)_t$. Therefore to construct the differential matrix $d_{t,s}$ we need to enumerate the generators of these two modules. We start by producing the exponent vector corresponding to each generator. Using the `WeightedIntegerVectors(t, weights)` function included in `Sage`, we enumerate all vectors whose entries sum to $t$ weighted according to the entries of the vector `weights`. Therefore, we produce weight vectors for the `x_i` and `s_i_j`, concatenate these into a single weight vector, and then using the `WeightedIntegerVectors` function, produce a list of possible exponent vectors. However, this list needs to be refined, as it does not for the $s$ degree being correct. Therefore, we need to refine the list to include only those terms who also live in degree $s$. We use the `enumerate_generator_exponents_MO2()` and `reduce_Weight_MO2()` classes detailed here:

```
def enumerate_generator_exponents_MO2(t,s):
    x_weights=[2*i for i in range(1,top_degree+1)]
    s_weights=[2*i for i in range(1,top_degree+1, 2)]
    weights=[]
```

```
    weights.append(x_weights[0:t])
    for i in range(0,s):
        weights.append(s_weights[0:floor((t+1)/2)])
    flat_weights=flatten(copy(weights))
    temp=copy(flat_weights)
    temp.append(s_weights[0:floor((t+1)/2)])
    flat_weights1=flatten(copy(temp))
    wi=WeightedIntegerVectors(2*t,flat_weights)
    wi1=WeightedIntegerVectors(2*t,flat_weights1)
    exp_vec=reduceWeight_MO2(wi,t,s)
    exp_vec1=reduceWeight_MO2(wi1,t,s+1)
    return exp_vec, exp_vec1
```

```
def reduceWeight_MO2(weights, t,s):
    if t==0:
        return weights
    reduced=[]
    for term in weights:
        s_terms=np.array(term[t::])
        s_count=floor((t+1)/2)
        term_len=len(s_terms)/s_count
        s_reshape=np.reshape(s_terms, (term_len, s_count))
        term_count=0
        for i in range(0,len(s_reshape)):
            if np.count_nonzero(s_reshape[i])!=0:
                term_count=term_count+1
        if term_count==s:
            reduced.append(term)
    return reduced
```

Now that we have enumerated the generators of our source and our target, we now need to construct the differential maps. We will then loop over our source generators, compute the image of the differential for each generator, and extract the coefficients of each generator in the target. This becomes a matrix which we use to compute the homology. Specifically, we use the `Ring.hom()` function to compute each of the component maps to the differential, and then take a sum. There are two types of component maps to the differential. We start by defining a list, denoted `diff_array`, which will contain all component maps of the differential $d_{t,s}$. Then, we specify the image of each polynomial generator of `C_i`. For the $j^{th}$ component map, we have `s_i_k`↦`s_i_k` for $k < j$, `s_i_k`↦`s_i_k+s_i_{k+1}` for $k = j$, and finally `s_i_k`↦`s_i_{k+1}` if $k > j$, and is the identity on `x_i`. Our final map is the identity on all `s_i_k` for $1 \leq k \leq s$, but sends `x_i` to $\eta_R(x_i)$ where each $s_i$ has been replaced by `s_i_s+1`. We construct this array with the `gen_maps_MO2()` function, provided here:

```
def gen_maps_MO2(C_i, C_i1, nx_as_sbar t,s):
    diff_array=[]
    C_i_s=list(C_i.gens()[t::])
    C_i1_s=list(C_i1.gens()[t::])
    C_i_x=list(C_i.gens()[0:t])
    s_count=floor((t+1)/2)
    C_i1_s_last=C_i1_s[-s_count::]
    for j in range(0,s):
        coprod_array=copy(C_i_x)
        for i in range(0,len(C_i.gens()[t::])):
            coprod_array.append(C_i1(SR((i<(j+1)*s_count)*C_i_s[i]+(i>=j*
    s_count)*C_i1_s[i+s_count])))
        diff_array.append(C_i.hom(coprod_array, C_i1))
    sbar_as_s_last=[]
    for i in range(0,len(C_i1_s_last)):
        sbar_as_s_last.append(SR(sbar_var[i]==C_i1_s_last[i]))
    right_unit_array=[]
    for i in range(0, len(C_i_x)):
        if i < len(nx_as_sbar):
            right_unit_array.append(nx_as_sbar[i].subs(sbar_as_s_last))
        else:
            right_unit_array.append(C_i_x[i])
```

```
    for s in C_i_s:
        right_unit_array.append(s)
    diff_array.append(C_i.hom(right_unit_array, C_i1))
    return diff_array
```

Finally, we are now able to produce the $d_{t,s}$ matrix. As we have already specfified, this simply loops over our module generators, computes the differential, and extracts the coefficients of the module generators in the target. The `compute_di_mat_MO2` combines the work we have already done to enumerate the terms and generate the maps. It is given here:

```
def compute_di_mat_MO2(t,s, x_var, s_var, nx_as_sbar):
    exp_vec, exp_vec1=enumerate_generator_exponents_MO2(t,s)
    C_i=ZZ[x_var[0:t]+tuple(s_var[0:s, 0:floor((t+1)/2)].flatten())];
    C_i1=ZZ[x_var[0:t]+tuple(s_var[0:s+1, 0:floor((t+1)/2)].flatten())];
    C_i_gen=[]
    C_i1_gen=[]
    for w in exp_vec:
        if len(w)==0:
            C_i_gen.append(0)
        else:
            C_i_gen.append(C_i({tuple(w):1}))
    for v in exp_vec1:
        if len(v)==0:
            C_i1_gen.append(0)
        else:
            C_i1_gen.append(C_i1({tuple(v):1}))
    diff_summands=gen_maps_MO2(C_i, C_i1, nx_as_sbar, t, s)
    diff_ts=matrix(len(C_i1_gen), len(C_i_gen),sparse=true )
    for col in range(0,len(C_i_gen)):
        term=0
        for i in range(0,len(diff_summands)):
            term=term+(-1)^(i+1)*diff_summands[i](C_i_gen[col])
        for row in range(0,len(C_i1_gen)):
```

```
            if term!=0:
                diff_ts[row,col]=term.coefficient(C_i1_gen[row])
    return diff_ts
```

Once the differential matrix is constructed, there are several ways to compute the homology. This can be done by constructing a `ChainComplex()` object in `Sage`, or simply extracting the elementary divisors and performing the necessary rank computations manually.

We want to note that the above code snippets are for exclusively $LS$. This is primarily due to the indexing considerations needed to skip over even $s_i$ while still preserving the weight vectors. Aside from the indexing considerations, the computation of $W_S(LS)$ differs in the `gen_maps_WLS()` function, which requires modification to how it constructs the coproduct maps. Instead of simply mapping to a primitive, the induced Witt elements need to map according to the induced coproduct. We passes this to the function as `cos_as_s` and make the appropriate index-shifting substitutions just as we did in the case of $LS$. The code for this is included here:

```
def gen_maps_WLS(C_i, C_i1,nx_as_s, cos_as_s t,s):
    diff_array=[]
    C_i_s=list(C_i.gens()[t::])
    C_i1_s=list(C_i1.gens()[t::])
    C_i_x=list(C_i.gens()[0:t])
    C_i1_s_last=C_i1_s[-t::]
    for j in range(0,s):
        coprod_array=copy(C_i_x)
        slr_as_sii1=[]
        for i in range(0,t):
            slr_as_sii1.append(SR(slbar_var[i]==C_i_s[j*(t)+i]))
            slr_as_sii1.append(SR(srbar_var[i]==C_i1_s[(j+1)*t+i]))
        for i in range(0,len(C_i_s)):
            if (i<j*t):
                coprod_array.append(C_i1(SR(C_i_s[i])))
```

```
            elif (i>=(j+1)*t):
                coprod_array.append(C_i1(SR(C_i1_s[i+t])))
            else:
                coprod_array.append(C_i1(SR(cos_as_s[i-j*t].subs(slr_as_sii1))))
                #coprod_array.append(C_i1(SR((i<(j+1)*s_count)*C_i_s[i]+(i>=j*s_count)*C_i1_s[i+s_count])))
        diff_array.append(C_i.hom(coprod_array, C_i1))
    sbar_as_s_last=[]
    for i in range(0,len(C_i1_s_last)):
        sbar_as_s_last.append(SR(sbar_var[i]==C_i1_s_last[i]))

    right_unit_array=[]
    for i in range(0, len(C_i_x)):
        if i < len(nx_as_sbar):
            right_unit_array.append(nx_as_sbar[i].subs(sbar_as_s_last))
        else:
            right_unit_array.append(C_i_x[i])
    for s in C_i_s:
        right_unit_array.append(s)
    diff_array.append(C_i.hom(right_unit_array, C_i1))
```

# CHAPTER 5

# Tables

This chapter contains the tables generated by the computations in Chapter 4. We include the computations for the following results discussed above:

- The map of Hopf algebroids $LB \to LS$, specifying the image of each polynomial generator $b_i \in LB$.
- A primitive generator $\overline{s}_i$ in $LS$ for $i = 1, 3, 5, 7$.
- The image of the right unit $\eta_R : L \to LS$ on the generators $x_i$, using both the naive generators ($s_{2i+1}$)and primitive generators ($\overline{s}_{2i+1}$).
- The coproduct for the Witt elements $s_{i,j}$.
- The $E_\infty$-page of the rectified Adams-Novikov spectral sequence computing $\pi_*(MSC)$. In the absence of extensions via Theorem 3.5.1, this is equivalent to $\pi_{t-s}(MSC)$.
- The $E_\infty$-page of the rectified Adams-Novikov spectral sequence computing $\pi_*(MO[2])$. As we have not resolved the extension problem, we are unable to claim this is $\pi_{t-s}(MO[2])$.

Table 5.1: Image of generators of $LB$ in $LS$

| $b_i$ | Image Under $LB \to LS$ |
|---|---|
| $b_1$ | $2s_1$ |
| $b_2$ | $2s_1^2 + s_1x_1$ |
| $b_3$ | $2s_3 - s_1^2x_1 - s_1x_1^2 - 2s_1x_2$ |
| $b_4$ | $-2s_1^4 - 6s_1^3x_1 - 6s_1^2x_1^2 - 2s_1x_1^3 - 4s_1^2x_2 - 2s_1x_1x_2 + 4s_1s_3 + 3s_3x_1 + s_1x_3$ |
| $b_5$ | $2s_5 + s_1^4x_1 + 3s_1^3x_1^2 + 3s_1^2x_1^3 + s_1x_1^4 - 5s_1^2x_1x_2 - 3s_1x_1^2x_2 - 2s_1s_3x_1 - 2s_3x_1^2$<br>$+ 2s_1x_2^2 - 7s_1^2x_3 - 5s_1x_1x_3 - 2s_3x_2 - 6s_1x_4$ |
| $b_6$ | $4s_1^6 + 22s_1^5x_1 + 48s_1^4x_1^2 + 52s_1^3x_1^3 + 28s_1^2x_1^4 + 6s_1x_1^5 + 8s_1^4x_2 + 8s_1^3x_1x_2 - 8s_1^2x_1^2x_2$<br>$- 5s_1x_1^3x_2 - 8s_1^3s_3 - 26s_1^2s_3x_1 - 28s_1s_3x_1^2 - 10s_3x_1^3 + 6s_1^2x_2^2 + 57s_1x_1x_2^2 - 18s_1^3x_3$<br>$- 36s_1^2x_1x_3 - 15s_1x_1^2x_3 - 8s_1s_3x_2 - 2s_3x_1x_2 + 50s_1x_2x_3 - 12s_1^2x_4 + 6s_1x_1x_4 + 2s_3^2$<br>$+ 4s_1s_5 + 5s_5x_1 + 3s_3x_3 + 2s_1x_5$ |
| $b_7$ | $2s_7 - 2s_1^6x_1 - 11s_1^5x_1^2 - 24s_1^4x_1^3 - 26s_1^3x_1^4 - 14s_1^2x_1^5 - 3s_1x_1^6 + 3s_1^4x_1x_2 + 17s_1^3x_1^2x_2$<br>$- 35s_1^2x_1^3x_2 + 18s_1x_1^4x_2 + 4s_1^3s_3x_1 + 13s_1^2s_3x_1^2 + 14s_1s_3x_1^3 + 5s_3x_1^4 - 1672s_1^2x_1x_2^2$<br>$+ 194s_1x_1^2x_2^2 + 7s_1^4x_3 + 30s_1^3x_1x_3 - 21s_1^2x_1^2x_3 + 23s_1x_1^3x_3 - 10s_1s_3x_1x_2 - 10s_3x_1^2x_2$<br>$- 2s_1x_2^3 - 1607s_1^2x_2x_3 + 203s_1x_1x_2x_3 - 651s_1^2x_1x_4 + 77s_1x_1^2x_4 - s_3^2x_1 - 2s_1s_5x_1$<br>$- 3s_5x_1^2 + 2s_3x_2^2 - 14s_1s_3x_3 - 13s_3x_1x_3 + 12s_1x_3^2 + 20s_1x_2x_4 - 62s_1^2x_5 + 6s_1x_1x_5$<br>$- 2s_5x_2 - 6s_3x_4 - 18s_1x_6$ |
| $b_8$ | $-10s_1^8 - 84s_1^7x_1 - 302s_1^6x_1^2 - 602s_1^5x_1^3 - 718s_1^4x_1^4 - 512s_1^3x_1^5 - 202s_1^2x_1^6 - 34s_1x_1^7$<br>$- 24s_1^6x_2 - 84s_1^5x_1x_2 - 52s_1^4x_1^2x_2 - 2s_1^3x_1^3x_2 - 26s_1^2x_1^4x_2 + 57s_1x_1^5x_2 + 24s_1^5s_3$<br>$+ 142s_1^4s_3x_1 + 336s_1^3s_3x_1^2 + 398s_1^2s_3x_1^3 + 236s_1s_3x_1^4 + 56s_3x_1^5 - 20s_1^4x_2^2 - 3592s_1^3x_1x_2^2$<br>$- 6066s_1^2x_1^2x_2^2 - 720s_1x_1^3x_2^2 + 58s_1^5x_3 + 284s_1^4x_1x_3 + 396s_1^3x_1^2x_3 + 210s_1^2x_1^3x_3$<br>$+ 113s_1x_1^4x_3 + 32s_1^3s_3x_2 + 40s_1^2s_3x_1x_2 - 32s_1s_3x_1^2x_2 - 39s_3x_1^3x_2 - 8s_1^2x_2^3 - 7589s_1x_1x_2^3$<br>$- 3392s_1^3x_2x_3 - 5762s_1^2x_1x_2x_3 - 670s_1x_1^2x_2x_3 + 24s_1^4x_4 - 1296s_1^3x_1x_4 - 2228s_1^2x_1^2x_4$<br>$- 102s_1x_1^3x_4 - 12s_1^2s_3^2 - 8s_1^3s_5 - 34s_1s_3^2x_1 - 34s_1^2s_5x_1 - 24s_3^2x_1^2 - 52s_1s_5x_1^2 - 28s_5x_1^3$<br>$+ 12s_1s_3x_2^2 + 167s_3x_1x_2^2 - 62s_1^2s_3x_3 - 132s_1s_3x_1x_3 - 67s_3x_1^2x_3 - 7306s_1x_2^2x_3 + 2s_1^2x_3^2$<br>$+ 3s_1x_1x_3^2 + 52s_1^2x_2x_4 - 3002s_1x_1x_2x_4 - 132s_1^3x_5 - 226s_1^2x_1x_5 - 19s_1x_1^2x_5 - 4s_3^2x_2$<br>$- 8s_1s_5x_2 - 2s_5x_1x_2 + 154s_3x_2x_3 - 24s_1s_3x_4 + 42s_3x_1x_4 - 107s_1x_3x_4 - 277s_1x_2x_5$<br>$- 36s_1^2x_6 - 102s_1x_1x_6 + 4s_3s_5 + 4s_1s_7 + 7s_7x_1 + 5s_5x_3 + 6s_3x_5 + s_1x_7$ |

Table 5.2: Primitive Generators of $LS$

| $\overline{s_i}$ | Primitive Element |
|---|---|
| $\overline{s}_1$ | $s_1$ |
| $\overline{s}_3$ | $s_3 - 4s_1^3 + s_1^2x_1 + 2s_1x_1^2 - 29s_1x_2$ |
| $\overline{s}_5$ | $s_5 + 168s_1^5 - 223s_1^4x_1 + 101s_1^3x_1^2 - 10s_1^2x_1^3 + 112s_1^3x_2 - 51s_1^2x_1x_2 + 8s_1^2s_3 - 10s_1s_3x_1$<br>$- 7s_1^2x_3 - 43s_1x_1x_3 - 19s_3x_2 + 15s_1x_4$ |
| $\overline{s}_7$ | $s_7 + 5523530266907576x_1^6s_1 - 66276962336827839x_1^5s_1^2 + 441817643504713883x_1^4s_1^3$<br>$- 1767184345002182444x_1^3s_1^4 + 4241104531376304967x_1^2s_1^5$<br>$- 5654714110715752774x_1s_1^6 + 3231265206140931960s_1^7 - 10801266771352x_1^4x_2s_1$<br>$+ 86314318348477x_1^3x_2s_1^2 - 344913488560468x_1^2x_2s_1^3 + 689483218080877x_1x_2s_1^4$<br>$- 551586543370108x_2s_1^5 + 97815318506x_1^2x_2^2s_1 + 525252153x_1^3x_3s_1$<br>$- 523712956582x_1x_2^2s_1^2 - 2123651282x_1^2x_3s_1^2 + 698294198920x_2^2s_1^3$<br>$+ 2833874063x_1x_3s_1^3 + 3824882x_3s_1^4 + 1965210x_1^3s_1s_3 - 6785712x_1^2s_1^2s_3$<br>$- 329464x_1s_1^3s_3 - 15429344s_1^4s_3 + 88404175296x_2^3s_1 + 41178119x_1x_2x_3s_1$<br>$- 234495086x_1^2x_4s_1 + 3789289x_2x_3s_1^2 + 944578098x_1x_4s_1^2 - 1259489640x_4s_1^3$<br>$+ 3838454x_1x_2s_1s_3 - 15575236x_2s_1^2s_3 + 232517x_3^2s_1 - 19638940x_2x_4s_1$<br>$- 75502x_1x_5s_1 - 3280x_5s_1^2 + 491410x_1x_3s_3 - 1964996x_3s_1s_3 - 982496x_1s_3^2$<br>$+ 3929968s_1s_3^2 - 3254x_1s_1s_5 + 6492s_1^2s_5 + 6161x_6s_1 - 81x_4s_3 + 3199x_2s_5$ |

Table 5.3: Right Unit in $LS$ Using Naive Generators

| $x_i$ | Image under $\eta_R$ using $s_i$ |
|---|---|
| $x_1$ | $x_1 - 4s_1$ |
| $x_2$ | $x_2 + 2s_1^2 - s_1x_1$ |
| $x_3$ | $x_3 + 8s_1^3 + 2s_1^2x_1 + 4s_1x_1^2 + 8s_1x_2 - 4s_3$ |
| $x_4$ | $x_4 - 6s_1^4 - 66s_1^3x_1 - 11s_1^2x_1^2 + 7s_1x_1^3 - 38s_1^2x_2 + s_1x_1x_2 + 36s_1s_3 - 9s_3x_1 - 9s_1x_3$ |
| $x_5$ | $x_5 - 236s_1^5 - 3287s_1^4x_1 + 290s_1^3x_1^2 + 489s_1^2x_1^3 - 80s_1x_1^4 - 1856s_1^3x_2 + 284s_1^2x_1x_2$ <br> $- 56s_1x_1^2x_2 + 1788s_1^2s_3 - 882s_1s_3x_1 + 104s_3x_1^2 - 104s_1x_2^2 - 437s_1^2x_3 + 136s_1x_1x_3$ <br> $+ 108s_3x_2 + 48s_1x_4 - 2s_5$ |
| $x_6$ | $x_6 - 76s_1^6 - 562s_1^5x_1 + 1259s_1^4x_1^2 + 730s_1^3x_1^3 - 51s_1^2x_1^4 - 37s_1x_1^5 - 514s_1^4x_2 - 17s_3x_3$ <br> $+ 578s_1^3x_1x_2 - 50s_1^2x_1^2x_2 - 24s_1x_1^3x_2 + 488s_1^3s_3 - 1102s_1^2s_3x_1 - 80s_1s_3x_1^2 + 60s_3x_1^3$ <br> $+ 170s_1^2x_2^2 - 1383s_1x_1x_2^2 - 622s_1^3x_3 - 131s_1^2x_1x_3 + 42s_1x_1^2x_3 - 204s_1s_3x_2 - 50s_1x_5$ <br> $+ 17s_3x_1x_2 - 1299s_1x_2x_3 - 286s_1^2x_4 - 457s_1x_1x_4 + 34s_3^2 + 100s_1s_5 - 25s_5x_1$ |
| $x_7$ | $x_7 + 16136s_1^7 + 115326s_1^6x_1 - 313265s_1^5x_1^2 - 103150s_1^4x_1^3 + 46890s_1^3x_1^4 + 6613s_1^2x_1^5$ <br> $- 1476s_1x_1^6 + 104916s_1^5x_2 - 179339s_1^4x_1x_2 + 9324s_1^3x_1^2x_2 - 2102s_1^2x_1^3x_2$ <br> $- 1216s_1x_1^4x_2 - 98040s_1^4s_3 + 282912s_1^3s_3x_1 - 31264s_1^2s_3x_1^2 - 19520s_1s_3x_1^3$ <br> $+ 2398s_3x_1^4 - 36136s_1^3x_2^2 + 246958s_1^2x_1x_2^2 - 66575s_1x_1^2x_2^2 + 112066s_1^4x_3$ <br> $- 20663s_1^3x_1x_3 - 16778s_1^2x_1^2x_3 + 1986s_1x_1^3x_3 + 56572s_1^2s_3x_2 - 5090s_1s_3x_1x_2$ <br> $744s_3x_1^2x_2 - 652s_1x_2^3 + 228545s_1^2x_2x_3 - 62689s_1x_1x_2x_3 + 49656s_1^3x_4 + 69934s_1^2x_1x_4$ <br> $- 22604s_1x_1^2x_4 - 19960s_1s_3^2 - 17516s_1^2s_5 + 5006s_3^2x_1 + 8790s_1s_5x_1 - 1004s_5x_1^2$ <br> $+ 656s_3x_2^2 + 10028s_1s_3x_3 - 1621s_3x_1x_3 - 937s_1x_3^2 + 1728s_1x_2x_4 + 8894s_1^2x_5$ <br> $- 2429s_1x_1x_5 - 538s_5x_2 - 356s_3x_4 - 120s_1x_6 - 4s_7$ |
| $x_8$ | $x_8 + 6390s_1^8 + 27300s_1^7x_1 - 336810s_1^6x_1^2 + 37262s_1^5x_1^3 + 164628s_1^4x_1^4 + 25966s_1^3x_1^5$ <br> $- 6133s_1^2x_1^6 - 823s_1x_1^7 + 56732s_1^6x_2 - 216506s_1^5x_1x_2 + 60734s_1^4x_1^2x_2 + 24067s_1^3x_1^3x_2$ <br> $- 9987s_1^2x_1^4x_2 - 2698s_1x_1^5x_2 - 54408s_1^5s_3 + 325262s_1^4s_3x_1 - 157222s_1^3s_3x_1^2$ <br> $- 77205s_1^2s_3x_1^3 + 6782s_1s_3x_1^4 + 1367s_3x_1^5 - 65698s_1^4x_2^2 + 1068250s_1^3x_1x_2^2$ <br> $- 354279s_1^2x_1^2x_2^2 - 198553s_1x_1^3x_2^2 + 133706s_1^5x_3 - 102161s_1^4x_1x_3 - 52668s_1^3x_1^2x_3$ <br> $- 2508s_1^2x_1^3x_3 - 1663s_1x_1^4x_3 + 96552s_1^3s_3x_2 - 53602s_1^2s_3x_1x_2 + 3861s_1s_3x_1^2x_2$ <br> $- 1578s_3x_1^3x_2 + 1144s_1^2x_2^3 - 1411503s_1x_1x_2^3 + 972818s_1^3x_2x_3 - 346348s_1^2x_1x_2x_3$ <br> $- 197166s_1x_1^2x_2x_3 + 73278s_1^4x_4 + 323970s_1^3x_1x_4 - 131800s_1^2x_1^2x_4 - 54388s_1x_1^3x_4$ <br> $- 30896s_1^2s_3^2 - 24928s_1^3s_5 + 26080s_1s_3^2x_1 + 27280s_1^2s_5x_1 + 761s_3^2x_1^2 - 813s_1s_5x_1^2$ <br> $- 716s_5x_1^3 - 2736s_1s_3x_2^2 + 16856s_3x_1x_2^2 + 30860s_1^2s_3x_3 + 2669s_1s_3x_1x_3$ <br> $- 1784s_3x_1^2x_3 - 1360251s_1x_2^2x_3 - 6720s_1^2x_3^2 - 4050s_1x_1x_3^2 - 8926s_1^2x_2x_4$ <br> $- 568885s_1x_1x_2x_4 + 38168s_1^3x_5 - 12542s_1^2x_1x_5 - 5340s_1x_1^2x_5 + 1580s_3^2x_2$ <br> $+ 1424s_1s_5x_2 + 278s_5x_1x_2 + 16328s_3x_2x_3 + 5436s_1s_3x_4 + 6249s_3x_1x_4$ <br> $- 18180s_1x_3x_4 - 51553s_1x_2x_5 + 6444s_1^2x_6 - 8892s_1x_1x_6 - 1268s_3s_5 - 756s_1s_7$ <br> $+ 189s_7x_1 + 317s_5x_3 + 634s_3x_5 + 189s_1x_7$ |

Table 5.4: Right Unit in $LS$ Using Primitive Generators

| $x_i$ | Image under $\eta_R$ using using $\overline{s}_i$ |
|---|---|
| $x_1$ | $x_1 - 4\overline{s}_1$ |
| $x_2$ | $x_2 + 2\overline{s}_1^2 - \overline{s}_1 x_1$ |
| $x_3$ | $x_3 - 8\overline{s}_1^3 + 6\overline{s}_1^2 x_1 + 12\overline{s}_1 x_1^2 - 108\overline{s}_1 x_2 - 4\overline{s}_3$ |
| $x_4$ | $x_4 + 138\overline{s}_1^4 - 138\overline{s}_1^3 x_1 - 74\overline{s}_1^2 x_1^2 + 25\overline{s}_1 x_1^3 + 1006\overline{s}_1^2 x_2 - 260\overline{s}_1 x_1 x_2 + 36\overline{s}_1\overline{s}_3$<br>$- 9\overline{s}_3 x_1 - 9\overline{s}_1 x_3$ |
| $x_5$ | $x_5 + 7316\overline{s}_1^5 - 9145\overline{s}_1^4 x_1 - 1798\overline{s}_1^3 x_1^2 + 2169\overline{s}_1^2 x_1^3 - 288\overline{s}_1 x_1^4 + 50964\overline{s}_1^3 x_2$<br>$- 26046\overline{s}_1^2 x_1 x_2 + 2820\overline{s}_1 x_1^2 x_2 + 1804\overline{s}_1^2\overline{s}_3 - 902\overline{s}_1\overline{s}_3 x_1 + 104\overline{s}_3 x_1^2 + 1926\overline{s}_1 x_2^2$<br>$- 451\overline{s}_1^2 x_3 + 50\overline{s}_1 x_1 x_3 + 70\overline{s}_3 x_2 + 78\overline{s}_1 x_4 - 2\overline{s}_5$ |
| $x_6$ | $x_6 - 17580\overline{s}_1^6 + 26370\overline{s}_1^5 x_1 - 15720\overline{s}_1^4 x_1^2 + 4765\overline{s}_1^3 x_1^3 + 435\overline{s}_1^2 x_1^4 - 157\overline{s}_1 x_1^5$<br>$- 6090\overline{s}_1^4 x_2 + 5820\overline{s}_1^3 x_1 x_2 - 17773\overline{s}_1^2 x_1^2 x_2 + 2632\overline{s}_1 x_1^3 x_2 - 40\overline{s}_1^3\overline{s}_3 + 30\overline{s}_1^2\overline{s}_3 x_1$<br>$- 466\overline{s}_1\overline{s}_3 x_1^2 + 60\overline{s}_3 x_1^3 + 77948\overline{s}_1^2 x_2^2 - 14665\overline{s}_1 x_1 x_2^2 + 10\overline{s}_1^3 x_3 + 4011\overline{s}_1^2 x_1 x_3$<br>$- 999\overline{s}_1 x_1^2 x_3 + 3668\overline{s}_1\overline{s}_3 x_2 - 458\overline{s}_3 x_1 x_2 - 1792\overline{s}_1 x_2 x_3 - 1786\overline{s}_1^2 x_4 - 82\overline{s}_1 x_1 x_4$<br>$+ 34\overline{s}_3^2 + 100\overline{s}_1\overline{s}_5 - 25\overline{s}_5 x_1 - 17\overline{s}_3 x_3 - 50\overline{s}_1 x_5$ |
| $x_7$ | $x_7 + 129250608245659905 04\overline{s}_1^7 - 2261885644299048338 2\overline{s}_1^6 x_1$<br>$+ 1696441812531502968 9\overline{s}_1^5 x_1^2 - 706873737982650446 2\overline{s}_1^4 x_1^3$<br>$+ 176727057411265629 4\overline{s}_1^3 x_1^4 - 26510784937867965 3\overline{s}_1^2 x_1^5$<br>$+ 2209412106762403 2\overline{s}_1 x_1^6 - 220634457733722 0\overline{s}_1^5 x_2 + 275793115175630 7\overline{s}_1^4 x_1 x_2$<br>$- 137965622925725 6\overline{s}_1^3 x_1^2 x_2 + 34525792510405 8\overline{s}_1^2 x_1^3 x_2 - 4320506698041 8\overline{s}_1 x_1^4 x_2$<br>$+ 63716264\overline{s}_1^4\overline{s}_3 - 63716264\overline{s}_1^3\overline{s}_3 x_1 - 82158032\overline{s}_1^2\overline{s}_3 x_1^2 + 23531192\overline{s}_1\overline{s}_3 x_1^3$<br>$+ 2398\overline{s}_3 x_1^4 + 2804576673280\overline{s}_1^3 x_2^2 - 2097705754920\overline{s}_1^2 x_1 x_2^2 + 391260208705\overline{s}_1 x_1^2 x_2^2$<br>$- 15929066\overline{s}_1^4 x_3 + 11351515475\overline{s}_1^3 x_1 x_3 - 8481074759\overline{s}_1^2 x_1^2 x_3 + 2097039388\overline{s}_1 x_1^3 x_3$<br>$+ 848413048\overline{s}_1^2\overline{s}_3 x_2 - 212257712\overline{s}_1\overline{s}_3 x_1 x_2 - 18332\overline{s}_3 x_1^2 x_2 + 353623473714\overline{s}_1 x_2^3$<br>$- 212177217\overline{s}_1^2 x_2 x_3 + 222133432\overline{s}_1 x_1 x_2 x_3 - 5038038404\overline{s}_1^3 x_4 + 3778446396\overline{s}_1^2 x_1 x_4$<br>$- 937986528\overline{s}_1 x_1^2 x_4 + 15699912\overline{s}_1\overline{s}_3^2 + 8452\overline{s}_1^2\overline{s}_5 - 3924978\overline{s}_3^2 x_1 - 4226\overline{s}_1\overline{s}_5 x_1$<br>$- 1004\overline{s}_5 x_1^2 + 233558\overline{s}_3 x_2^2 - 7849956\overline{s}_1\overline{s}_3 x_3 + 1964019\overline{s}_3 x_1 x_3 + 929131\overline{s}_1 x_3^2$<br>$- 78757622\overline{s}_1 x_2 x_4 - 4226\overline{s}_1^2 x_5 - 304437\overline{s}_1 x_1 x_5 + 12258\overline{s}_5 x_2 - 680\overline{s}_3 x_4$<br>$+ 24524\overline{s}_1 x_6 - 4\overline{s}_7$ |

Table 5.4: Right Unit in $LS$ Using Primitive Generators (Continued)

| | |
|---|---|
| $x_8$ | $x_8 + 2442836495842446822454\overline{s}_1^8 - 4885672991684893644908\overline{s}_1^7 x_1$<br>$+ 4275015992614865375986\overline{s}_1^6 x_1^2 - 2137560121207818767216\overline{s}_1^5 x_1^3$<br>$+ 668011979703992816963\overline{s}_1^4 x_1^4 - 133608918159402911717\overline{s}_1^3 x_1^5$<br>$+ 16702134764928194523\overline{s}_1^2 x_1^6 - 1043947220445535421\overline{s}_1 x_1^7$<br>$- 4169991246674935532\overline{s}_1^6 x_2 + 6254987684385782252\overline{s}_1^5 x_1 x_2$<br>$- 3910672750532957148\overline{s}_1^4 x_1^2 x_2 + 1304425052390812108\overline{s}_1^3 x_1^3 x_2$<br>$- 2447919472515130088\overline{s}_1^2 x_1^4 x_2 + 20414394198528371\overline{s}_1 x_1^5 x_2 + 12064790808\overline{s}_1^5 \overline{s}_3$<br>$- 15080988510\overline{s}_1^4 \overline{s}_3 x_1 - 12536834722\overline{s}_1^3 \overline{s}_3 x_1^2 + 8344941203\overline{s}_1^2 \overline{s}_3 x_1^3$<br>$- 1114195088\overline{s}_1 \overline{s}_3 x_1^4 + 1367\overline{s}_3 x_1^5 + 530069627032198\overline{s}_1^4 x_2^2$<br>$- 528985473218224\overline{s}_1^3 x_1 x_2^2 + 173065312642696\overline{s}_1^2 x_1^2 x_2^2 - 18487072905523\overline{s}_1 x_1^3 x_2^2$<br>$- 3016197702\overline{s}_1^5 x_3 + 2146327092972\overline{s}_1^4 x_1 x_3 - 2139379040468\overline{s}_1^3 x_1^2 x_3$<br>$+ 797094321151\overline{s}_1^2 x_1^3 x_3 - 99086932820\overline{s}_1 x_1^4 x_3 + 160618631816\overline{s}_1^3 \overline{s}_3 x_2$<br>$- 80355000606\overline{s}_1^2 \overline{s}_3 x_1 x_2 + 10050417730\overline{s}_1 \overline{s}_3 x_1^2 x_2 - 15182\overline{s}_3 x_1^3 x_2$<br>$+ 66834870860252\overline{s}_1^2 x_2^3 - 16708723041106\overline{s}_1 x_1 x_2^3 - 40197340034\overline{s}_1^3 x_2 x_3$<br>$+ 52056357855\overline{s}_1^2 x_1 x_2 x_3 - 10502362624\overline{s}_1 x_1^2 x_2 x_3 - 952247487042\overline{s}_1^4 x_4$<br>$+ 952199927010\overline{s}_1^3 x_1 x_4 - 355812838231\overline{s}_1^2 x_1^2 x_4 + 44319484490\overline{s}_1 x_1^3 x_4$<br>$+ 2971035056\overline{s}_1^2 \overline{s}_3^2 + 4877952\overline{s}_1^3 \overline{s}_5 - 1485517528\overline{s}_1 \overline{s}_3^2 x_1 - 3658464\overline{s}_1^2 \overline{s}_5 x_1$<br>$+ 185692505\overline{s}_3^2 x_1^2 + 616729\overline{s}_1 \overline{s}_5 x_1^2 - 716\overline{s}_5 x_1^3 + 44669060\overline{s}_1 \overline{s}_3 x_2^2 - 11465471\overline{s}_3 x_1 x_2^2$<br>$- 1485517528\overline{s}_1^2 \overline{s}_3 x_3 + 742841519\overline{s}_1 \overline{s}_3 x_1 x_3 - 92878274\overline{s}_3 x_1^2 x_3 - 712072\overline{s}_1 x_2^2 x_3$<br>$+ 175778351\overline{s}_1^2 x_3^2 - 43936132\overline{s}_1 x_1 x_3^2 - 14884412206\overline{s}_1^2 x_2 x_4 + 3720880952\overline{s}_1 x_1 x_2 x_4$<br>$- 2438976\overline{s}_1^3 x_5 - 56472768\overline{s}_1^2 x_1 x_5 + 14263270\overline{s}_1 x_1^2 x_5 - 22512\overline{s}_3^2 x_2 + 2383096\overline{s}_1 \overline{s}_5 x_2$<br>$- 604333\overline{s}_5 x_1 x_2 + 22351\overline{s}_3 x_2 x_3 - 36780\overline{s}_1 \overline{s}_3 x_4 + 21558\overline{s}_3 x_1 x_4 - 22935\overline{s}_1 x_3 x_4$<br>$- 33167\overline{s}_1 x_2 x_5 + 4664160\overline{s}_1^2 x_6 - 1173321\overline{s}_1 x_1 x_6 - 1268\overline{s}_3 \overline{s}_5 - 756\overline{s}_1 \overline{s}_7 + 189\overline{s}_7 x_1$<br>$+ 317\overline{s}_5 x_3 + 634\overline{s}_3 x_5 + 189\overline{s}_1 x_7$ |

Table 5.5: Image of Witt elements $s_i, j$ in $W_S(LS)$

| Witt Element $s_{i,j}$ | Image under Coproduct |
|---|---|
| $s_{1,0} = s'_1$ | $s_{1,0} \otimes 1 + 1 \otimes s_{1,0}$ |
| $s_{1,1} = s'_2$ | $-s_{1,0} \otimes s_{1,0} + s_{1,1} \otimes 1 + 1 \otimes s_{1,1}$ |
| $s_{3,0} = s'_3$ | $s_{3,0} \otimes 1 + 1 \otimes s_{3,0}$ |
| $s_{1,2} = s'_4$ | $- s_{1,0}^3 \otimes s_{1,0} - 2s_{1,0}^2 \otimes s_{1,0}^2 - s_{1,0} \otimes s_{1,0}^3 + s_{1,0}s_{1,1} \otimes s_{1,0} + s_{1,0} \otimes s_{1,0}s_{1,1}$ $- s_{1,1} \otimes s_{1,1} + s_{1,2} \otimes 1 + 1 \otimes s_{1,2}$ |
| $s_{5,0} = s'_5$ | $s_{5,0} \otimes 1 + 1 \otimes s_{5,0}$ |
| $s_{3,1} = s'_6$ | $-s_{3,0} \otimes s_{3,0} + s_{3,1} \otimes 1 + 1 \otimes s_{3,1}$ |
| $s_{7,0} = s'_7$ | $s_{7,0} \otimes 1 + 1 \otimes s_{7,0}$ |

Table 5.6: $\pi_{t-s}(MSC)$

| $s \backslash t-s$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $\mathbb{Z}$ | | 0 | | $\mathbb{Z}$ | | 0 | | $\mathbb{Z}^2$ | | 0 | | $\mathbb{Z}^3$ |
| 1 | | (4) | | $\mathbb{Z}$ | | $(2, 16)$ | | $\mathbb{Z}^2$ | | $(8, 64)$ | | $\mathbb{Z}^4$ | |
| 2 | | | 0 | | (2) | | (4) | | $(2, 4, 8)$ | | $(\mathbb{Z}, 2, 4)$ | | $(2, 4^2, 8, 32)$ |
| 3 | | | | 0 | | 0 | | 0 | | (2) | | (2) | |
| 4 | | | | | 0 | | 0 | | 0 | | 0 | | 0 |

| $s \backslash t-s$ | 13 | 14 | 15 | 16 |
|---|---|---|---|---|
| 0 | | 0 | | $\mathbb{Z}^5$ |
| 1 | $(2, 4, 32, 256)$ | | $\mathbb{Z}^7$ | |
| 2 | | $(\mathbb{Z}^2, 2, 4^3)$ | | $(2^2, 4^3, 8, 16, 32, 128)$ |
| 3 | $(2^3)$ | | $(2^3, 4)$ | |
| 4 | | (0) | | (2) |

Table 5.7: $E_\infty$-page of the RANNS converging to $\pi_*(MO[2])$

| $s \backslash t-s$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $\mathbb{Z}$ | | 0 | | $\mathbb{Z}$ | | 0 | | $\mathbb{Z}^2$ | | 0 | | $\mathbb{Z}^3$ | | 0 | | $\mathbb{Z}^5$ |
| 1 | | (4) | | 0 | | $(2,16)$ | | 0 | | $(8,64)$ | | 0 | | (2,4,32,256) | | 0 | |
| 2 | | | (2) | | 0 | | $(2,4)$ | | (2) | | $(2,4^2)$ | | (4) | | $(2^2,4^2,8)$ | | $(2^2)$ |
| 3 | | | | (2) | | 0 | | $(2^2)$ | | (2) | | $(2^5)$ | | $(2^2)$ | | $(2^7,4)$ | |
| 4 | | | | | (2) | | 0 | | $(2^2)$ | | (2) | | $(2^5)$ | | $(2^4)$ | | $(2^9)$ |
| 5 | | | | | | (2) | | 0 | | $(2^2)$ | | (2) | | $(2^5)$ | | $(2^4)$ | |
| 6 | | | | | | | (2) | | 0 | | $(2^2)$ | | (2) | | $(2^5)$ | | $(2^4)$ |
| 7 | | | | | | | | (2) | | 0 | | $(2^2)$ | | (2) | | $(2^5)$ | |
| 8 | | | | | | | | | (2) | | 0 | | $(2^2)$ | | (2) | | $(2^5)$ |
| 9 | | | | | | | | | | (2) | | 0 | | $(2^2)$ | | (2) | |
| 10 | | | | | | | | | | | (2) | | 0 | | $(2^2)$ | | (2) |
| 11 | | | | | | | | | | | | (2) | | 0 | | $(2^2)$ | |
| 12 | | | | | | | | | | | | | (2) | | 0 | | $(2^2)$ |

Note: Entries shown in red (those with $22 \leq t \leq 26$ with $s \leq \frac{t}{2} - 2$) are not computationally verified. They are projected based on extending the existing $t+s$-diagonal by multiplying the permanent cycle $(\bar{s}_1)$

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