Non-Expected Utility Preferences in the Hylland & Zeckhauser Model

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There are many situations in which individuals must be allocated to positions with limited capacities and in which money is not an acceptable means of eliciting individuals' preferences. Aanund Hylland and Richard Zeckhauser (1979) created a mechanism that can be used in such situations that results in an efficient allocation. However, in the nearly 50 years since they made that discovery, much empirical research has called into question one of the central assumptions of the mechanism: that all individuals follow expected utility theory. In this paper, I attempt to resolve this problem by replacing expected utility theory with three different non-expected utility theories: subjective expected value theory (SEVT), simple decision weighted utility theory (SDWUT), and rank dependent utility theory (RDU). For SEVT and SDWUT, a counterexample can be constructed to show a case in which the mechanism will fail to achieve an efficient outcome. For RDU, a counterexample can be constructed to show that the existence of an efficient outcome cannot be proven using the same method Hylland and Zeckhauser used. Thus, the mechanism seems to be reliant on all individuals being expected utility maximizers.

I. Introduction

There are many situations in which individuals must be assigned to positions with limited capacities and in which money is not an acceptable means of eliciting individuals' preferences. Examples include allocating legislators or faculty members to committees, college students to oversubscribed classes, and even allocating energy production among different power plants.[1] Creating a mechanism that elicits honest preferences, results in an efficient allocation, and is adaptable to various distributional objectives without using money was thought to be impossible until Hylland and Zeckhauser (1979) developed a mechanism that could do just that. In the nearly 50 years since their paper was published, new research and discoveries in the field of decision theory have called into question one of the central assumptions of their paper: that individuals follow expected utility theory.[2] Many empirical studies that will be mentioned throughout this paper show that a lot of people treat risk and probabilities differently than expected utility theory would predict. For these individuals, maximizing expected utility may not maximize their actual utility; instead, they would be better off maximizing some non-expected utility theory that better represents how they value prospects. Thus, it is important to know whether Hylland and Zeckhauser's mechanism can accommodate various non-expected utility theories.

In this paper, I will test three specific non-expected utility theories in the Hylland and Zeckhauser mechanism: subjective expected value theory (SEVT), simple decision weighted utility theory (SDWUT), and rank dependent utility theory (RDU). SEVT induces a probability weighting function that can represent

[1] Examples identified by Hylland and Zeckhauser (1979).

[2] Expected utility theory states that individuals attempt to maximize the sum of the probability of each outcome multiplied by the utility of each outcome, or $EU = \sum pu$ where p is the probability of some outcome and u is the utility of some outcome.

pessimism towards low probabilities and aversion to randomization towards higher probabilities.[3] SDWUT includes a utility weighting function in addition to a probability weighting function that can represent the idea that two changes in utility of the same magnitude, one from a starting point very close to the reference point and another from one farther away, will not be valued the same.[4] RDU also includes a probability weighting function, but unlike SEVT and SDWUT, it is monotonic since the weight attached to each probability is dependent on both the true probability of that consequence and the rank of that consequence relative to others.[5] While expected utility theory can represent risk aversion through a concave Bernoulli utility function, it cannot account for any of these other attitudes.

Having identified these theories as representative of how some individuals actually think, I decided to test each one in Hylland and Zeckhauser's mechanism. I will go into detail about how the mechanism works in Section II, but for now, it is sufficient to know that the mechanism operates by assigning hypothetical endowments of pseudo-currency to each individual being allocated, creating a pseudomarket that exchanges probability shares in positions, then allocating those shares to the individuals based on the market clearing prices determined by the pseudomarket. For SEVT, SDWUT, and RDU, assuming market clearing prices have been determined, the outcome will be Pareto optimal when any one of them replaces expected utility theory in the mechanism. However, it could not be proven for any of them that market clearing prices will always exist. For SEVT and SDWUT, a counterexample can be easily constructed to show an exact case in which no market clearing prices exist. For RDU, such a counterexample is not as easy to identify. However, a counterexample can be constructed to show that the

[3] See Tversky and Kahneman (1992) and Quiggan (1982).

[4] See Tversky and Kahneman (1992).

[5] See Quiggan (1982).

existence of market clearing prices cannot be proven using the same method Hylland and Zeckhauser used.

In the rest of this paper, I will begin by going further in depth on how the mechanism works. Then, in Section III, I will show the central role that expected utility theory plays in the mechanism by explaining how demand works in the mechanism and how market clearing prices are proven to exist. I will also go through an example that shows how the market clearing prices are determined in practice. In Section IV, I prove that non-expected utility theories with curved or non-parallel indifference curves will still result in a Pareto optimal outcome as long as market clearing prices have been determined. In Section V, I explain what the shortcomings of expected utility theory are. In Section VI, I explain SEVT and SDWUT in greater depth and go through a counterexample in which market clearing prices do not exist. In Section VII, I explain RDU in greater detail and show a counterexample in which market clearing prices cannot be proven to exist using the same method that Hylland and Zeckhauser used. Finally, in Section VIII, I discuss the results and conclude.

II. Overview of Hylland & Zeckhauser's Mechanism

Hylland and Zeckhauser have three main goals they are attempting to achieve with this mechanism (296). The first goal is to elicit preferences that are honest from the participants of the game. The second goal is to efficiently allocate the individuals to positions based on those preferences. Finally, the last goal is to meet prescribed distributional objectives, which may vary based on the unique situation the mechanism is being used for. In some scenarios, we may want to treat everyone equally, but in others, we may want to favor certain individuals.

Hylland and Zeckhauser begin with some basic assumptions (296). They assume that giving lotteries over jobs is permissible. They state that each person's preferences are assumed to concern solely his or her own assignment, meaning he or she is indifferent to where others get assigned. They also assume that the jobs have no preference over who is assigned to them and that interests external to individuals' preferences, such as institutional interests, play no role.

The algorithm is formulated as follows (296-297). Let I be the total number of individuals, and let J be the total number of jobs (or positions). Let there be M_j individuals assigned to job j, and assume that $\sum_j M_j = I$, meaning there are exactly enough jobs for the number of individuals being allocated. Let u_{ij} be the utility of individual i getting job j. Let $w_i = (w_{i1}, \dots, w_{ij}, \dots, w_{iJ})$ be i's reported vector of von Neumann-Morgenstern utilities and let $W = (w_1, \dots, w_i, \dots, w_I)$ be the total submission of preferences. It is important to note that u_{ij}, which represents the actual utility individual i gets from job j, and w_{ij}, individual i's reported utility for job j, which may or may not be truthful, are distinct. They will only be equal when individual i reports his or her true utilities. Let $p_i = (p_{i1}, \dots, p_{ij}, \dots, p_{iJ})$ be the assignment of probabilities over positions given to individual i and let $P = (p_1, \dots, p_I)$ be the total assignment. Let $q = (q_1, \dots, q_j, \dots, q_J)$ be the price vector where q_j is the price of a probability share in position j. Finally, let B_i be the initial budget given to individual i.

Hylland and Zeckhauser then make three more important assumptions (298-301). The first is:

(1) $$\sum_j p_{ij} = 1 \text{ for all i}$$

This states that the sum of all probabilities in individual i's probability vector is 1, guaranteeing that individual i gets one and only one position.

The second assumption is:

(2) $$\sum_i p_{ij} = M_j \text{ for all j}$$

This tells us that the sum of the probabilities demanded by all individuals for position j must sum up to the total number of openings for position j. This ensures that each job is exactly filled to the number of openings.

Finally, the last assumption is:

(3) $$\text{Maximize } EU = \sum_j p_{ij}u_{ij}$$
$$\text{Subject to } 0 \leq p_{ij} \leq 1 \text{ and } \sum_j p_{ij}q_j \leq B_i$$

This tells us that agents act according to Expected Utility Theory, meaning they seek to maximize the equation above. Players are also subject to the constraints that each probability p_{ij} must be in the range $[0,1]$ and that the total assignment given to each individual must be priced less than or equal to that person's budget constraint.

Given all the assumptions made above, Hylland and Zeckhauser's procedure can be summarized in four steps (300). First, hypothetical endowments of a pseudo-currency are distributed to individuals. Next, von-Neumann Morgenstern utilities are elicited from individuals. This is simply a vector with J entries that gives the individual's utility for each position. Third, a pseudomarket is employed to assign lotteries to each individual in a way that produces an efficient outcome. The "commodity" being exchanged in this pseudomarket is probability shares. For instance, if the price of position j is q_j, then a 0.5 chance at position j would cost $0.5q_j$. The pseudomarket uses the reported von-Neumann Morgenstern utilites to determine the market clearing prices resulting in an efficient allocation. In many cases, a trial-and-error procedure will do the job of determining the market clearing prices, but there may be cases in which adjusting prices up or down based on excess demand or supply will not converge. In such cases, Hylland and Zeckhauser turn to their adaptation of Scarf's mechanism that he developed in sections 4.5-4.7 of his book *The Computation of Economic Equilibria*. The application of this mechanism clears the market and is arbitrarily close to the Pareto frontier. Everyone's assignment will be arbitrarily close to being optimal at the given prices, which, according to Hylland and Zeckhauser, is fine for all practical purposes. It is also important to note that, when two or more lotteries have equal expected utilities for an individual, the least expensive one is always chosen.

Finally, the last step is to employ a specialized mechanism to conduct the lottery. This mechanism takes the probability vectors as determined by the market clearing prices and offers each individual the precise lottery on positions that the individual has purchased using his or her hypothetical endowment.[6]

For this procedure to be successful and achieve Hylland and Zeckhauser's goals, it must ensure that individuals provide honest preferences (307). Hylland and Zeckhauser claim that we can be confident that players will provide honest preferences when the pseudomarket assigns probability shares for each individual as he or she would choose in a real market (which the mechanism does) and when no individual has a noticeable ability to manipulate market prices through manipulation of his or her own reported preferences. Assuming agents take prices as given, we do not have to worry about players manipulating their preferences.

The final important aspect of this procedure to note is that it is both ex-ante and ex-post Pareto optimal (298). Ex-ante Pareto optimality implies that there is no other assignment of lotteries that is weakly preferred by all players and strictly preferred by at least one. This is measured before the lottery is conducted, which in this case, is done by using expected utility theory. Ex-post Pareto optimality implies that after the lottery is conducted, the resulting allocation is Pareto optimal. Hylland

[6] A formal treatment of this procedure is given in Appendix C of Hylland and Zeckhauser (1979) Discussion Paper 51.

and Zeckhauser claim that, because the procedure is ex-ante Pareto optimal, it is also ex-post Pareto optimal.

III. Expected Utility Theory in the Mechanism

A. Demand in the Mechanism

To understand how market clearing prices are determined by the pseudomarket, it is necessary to understand how demand is determined in the setting presented by the mechanism. The assumption of expected utility theory is essential to this determination. As a reminder, the pseudomarket chooses an assignment P such that each p_i is chosen to maximize expected utility

(3) $$\text{Maximize } EU = \sum_j p_{ij} u_{ij}$$

subject to the following constraints where q_j is the price of job j:

(A) $$\sum_j p_{ij} = 1 \text{ for all i}$$

(B) $$0 \leq p_{ij} \leq 1 \text{ for all j}$$

(C) $$\sum_j p_{ij} q_j \leq B_i$$

When these conditions hold, there is a solution to the optimization problem in which each individual i purchases shares in exactly two positions (301). Thus, it is reasonable to represent an individual i's demand assuming that there are two positions: position A and position B. First, we can represent constraint (A), or the probability constraint, graphically with the solid, black line below:

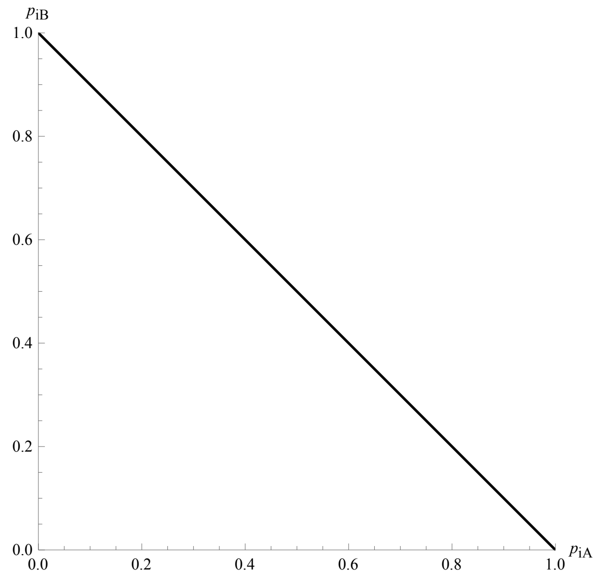

FIGURE 1

Figure 1 represents the probability constraint for each individual when there are only two positions. The x-axis is the probability share of position A, and the y-axis is the probability share of position B. The probabilities in each individual's von Neumann Morgenstern utility vector must sum to 1, meaning it must fall on the line above.

Individual i must select probability shares such that his or her probability vector falls on this line. Otherwise, the sum of the total probabilities will be less than or greater than 1, and the probability constraint will not be met.

Individual i's budget constraint line will be determined by the prices q_A and q_B of positions A and B, respectively. Each player must select a distribution that lies either on or below the budget constraint. Depending on the prices, the budget constraint line will look different. First, consider the case where $q_A > B_i$ and $q_B < B_i$:

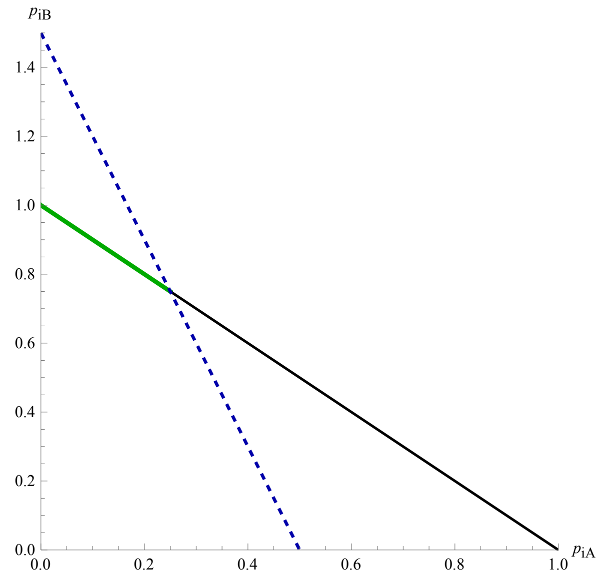

FIGURE 2

Figure 2 shows how demand looks for individual i when $q_A > B_i$ and $q_B < B_i$. The x-axis is the probability share of position A, and the y-axis is the probability share of position B. The solid, black line represents the probability constraint, and the dashed, blue line represents the budget constraint. The thicker, green segment of the probability constraint represents the bundles that individual i can demand.

The dashed, blue line is the budget constraint of individual i. As can be seen in the graph above, if a player was to only demand shares of position A, then he or she would receive ${B_i}/{q_A}$, which is less than 1, and if a player was to only demand shares of position B, he or she would receive ${B_i}/{q_B}$, which is greater than 1. Because individual i can only demand below the budget constraint and on the probability constraint, individual i can only demand distributions on the thicker, green portion of the probability constraint.

We know that if the utility of position B is greater than that of position A, meaning $u_B > u_A$, then individual i will demand $p_{iB} = 1$ and $p_{iA} = 0$, since this will maximize utility. If $u_B < u_A$, individual i will demand as much of p_{iA} as possible, which is the point where the probability constraint and budget constraint intersect. This can be calculated by solving the following system of equations:

$$p_{iA} + p_{iB} = 1, \qquad q_A p_{iA} + q_B p_{iB} = B_i$$

Solving for p_{iA} and p_{iB} results in:

$$p_{iA} = {(B_i - q_B)}\Big/{(q_A - q_B)}, \qquad p_{iB} = {(q_A - B_i)}\Big/{(q_A - q_B)}$$

These will be the probability shares that individual i will demand when $u_B < u_A$.

Now, suppose $q_A < B_i$ and $q_B > B_i$. The graph will look like this:

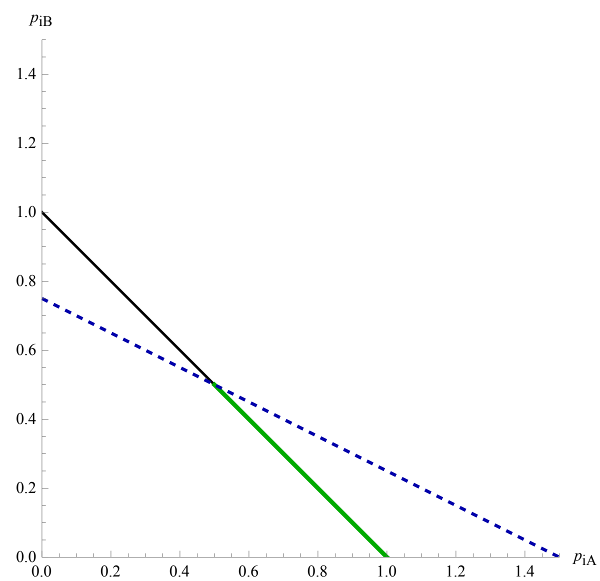

FIGURE 3

Figure 3 shows how demand looks for individual i when $q_A < B_i$ and $q_B > B_i$. The x-axis is the probability share of position A, and the y-axis is the probability share of position B. As in Figure 2, the solid, black line represents the probability constraint, and the dashed, blue line represents the budget constraint. The thicker, green segment of the probability constraint represents the bundles that individual i can demand.

Once again, individual i can only demand probability vectors on the thicker, green line segment. Using the same logic as above, when $u_A > u_B$, individual i will demand $p_{iA} = 1$ and $p_{iB} = 0$. When $u_A < u_B$, individual i will demand the point at the intersection of the two constraints, or where $p_{iA} = {(B_i - q_B)}/{(q_A - q_B)}$ and $p_{iB} = {(q_A - B_i)}/{(q_A - q_B)}$. Finally, when $u_A = u_B$, individual i will be indifferent over any distribution on the thicker, green line segment.

Now, suppose $q_A > B_i$ and $q_B > B_i$. The graph will look like this:

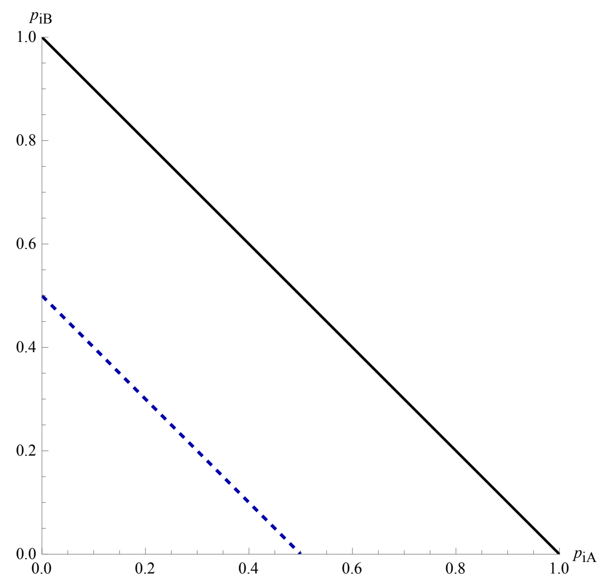

FIGURE 4

Figure 4 shows how demand looks for individual i when $q_A > B_i$ and $q_B > B_i$. The x-axis is the probability share of position A, and the y-axis is the probability share of position B. The solid, black line represents the probability constraint, and the dashed, blue line represents the budget constraint. Because the entire budget constraint falls below the probability constraint, it will be impossible to meet the probability constraint.

The individual cannot satisfy constraint (A) because the budget constraint falls completely below the probability constraint. Thus, there is no probability vector that can be demanded. If this was the case in an example with two total jobs, then new prices would have to be determined to meet the probability constraint. If this game has more than two jobs, then it may be necessary to mix between two different jobs than the ones being represented above.

Now, suppose $q_A < B_i$ and $q_B < B_i$. The graph will look like this:

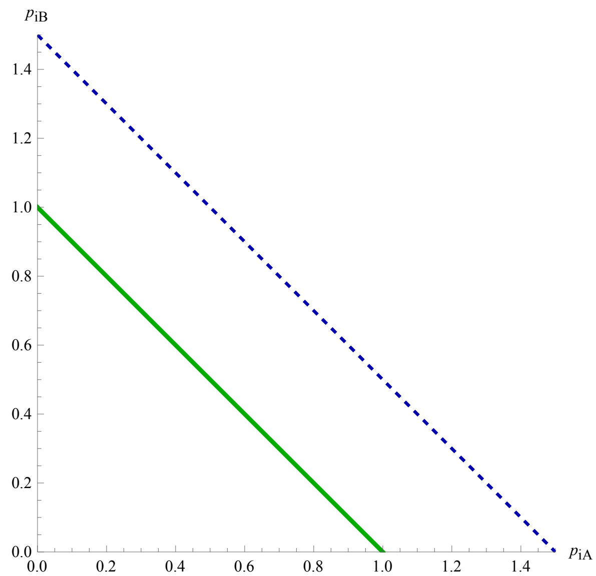

FIGURE 5

Figure 5 shows how demand looks for individual i when $q_A < B_i$ and $q_B < B_i$. The x-axis is the probability share of position A, and the y-axis is the probability share of position B. The solid, green line represents the probability constraint, and the dashed, blue line represents the budget constraint. Because the entire probability constraint falls below the budget constraint, any bundle on the probability constraint can be demanded.

Here, all points on the probability constraint are part of the green line, meaning all points on the probability constraint can be demanded. Thus, if $u_A > u_B$, individual i will demand $p_{iA} = 1$ and $p_{iB} = 0$. If $u_A < u_B$individual i will demand $p_{iA} = 0$ and $p_{iB} = 1$. Finally, if $u_A = u_B$, individual i will be indifferent over the entire probability constraint.

The same results will hold if either $q_A = B_i$ or $q_B = B_i$ and the other q_j is less than or equal to B_i, like in the graph below:

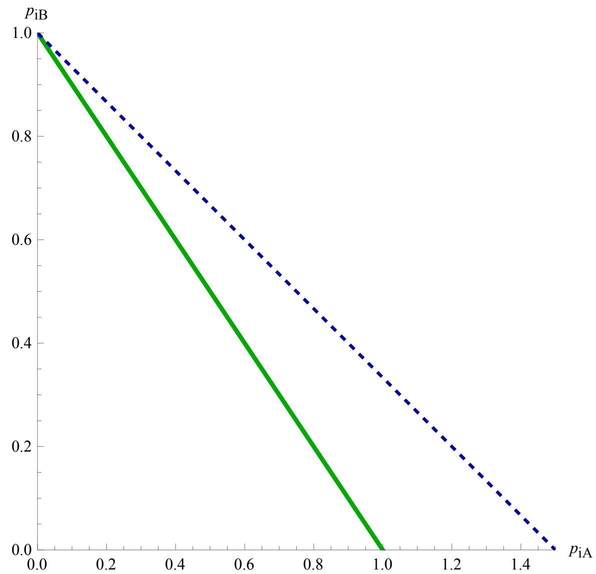

FIGURE 6

Figure 6 shows how demand looks for individual i when $q_B = B_i$ and $q_A < B_i$. The x-axis is the probability share of position A, and the y-axis is the probability share of position B. The solid, green line represents the probability constraint, and the dashed, blue line represents the budget constraint. Because the entire probability constraint falls below the budget constraint, any bundle on the probability constraint can be demanded.

The probability constraint and the budget constraint will intersect at one of the axes, and the rest of the probability constraint will be below the budget constraint, meaning individual i will demand $p_{ij} = 1$ of whichever utility is higher and will be indifferent over the entire line when the utilities are equal.

When the other q_j is greater than B_i, on the other hand, then the constraint lines will intersect at one of the axes, and the rest of the budget constraint line will lie below the probability constraint, such as in the graph below:

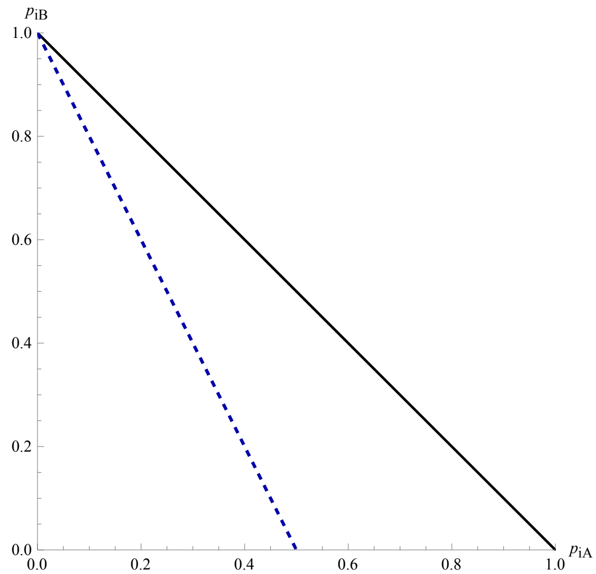

FIGURE 7

Figure 7 shows how demand looks for individual i when $q_B = B_i$ and $q_A > B_i$. The x-axis is the probability share of position A, and the y-axis is the probability share of position B. The solid, black line represents the probability constraint, and the dashed, blue line represents the budget constraint. Because the entire budget constraint except for $(0, 1)$ falls below the probability constraint, it will be impossible to meet the probability constraint unless $(0, 1)$ is demanded.

This means that no matter what the utilities are, individual i will be forced to play $p_{iA} = 1$ and $p_{iB} = 0$ if $q_A = B_i$ or $p_{iA} = 0$ and $p_{iB} = 1$ if $q_B = B_i$.

The final case to consider is when $q_A = q_B = B_i$. In this case, the probability constraint and the budget constraint will be the same line. Thus, whichever position has the higher utility will be demanded fully by individual i, and if the utilities are the same, individual i will be indifferent over the entire line.

B. Proof of Existence of Market Clearing Prices

Hylland and Zeckhauser prove the existence of market clearing prices using Kakutani's fixed-point theorem. Getting to the point at which Kakutani's theorem can be used requires several steps that I will walk through in this section.[7]

The proof only considers price vectors in which at least one price is 0, so let

$$Q = \{\bar{q} = (q_1, q_2, \dots, q_N) | q_j \geq 0 \text{ for all j}, q_j = 0 \text{ for some j}\}$$

Letting i be fixed, for each $\bar{q} \in Q$, define $F_i(\bar{q})$ as the set of lotteries individual i can buy without exceeding his or her budget. $F_i(\bar{q})$ is closed and bounded. Also, because at least one price is always 0, it is non-empty.

Let u_i be i's utility function. Since all agents are expected utility maximizers, u_i is linear. Next, define $G_i(\bar{q})$ as the set of lotteries i can choose that, first, maximizes utility, and from among those, has the lowest price. Because $G_i(\bar{q})$ is determined from $F_i(\bar{q})$ by maximizing and minimizing continuous functions, $G_i(\bar{q})$, like $F_i(\bar{q})$, will be nonempty. Also, since $u(\bar{x})$ and $\bar{x} \cdot \bar{q}$ are both linear functions of $\bar{x}$, it is clear that $G_i(\bar{q})$ is a convex set.

The next step is to prove that $G_i(\bar{q})$ is an upper semicontinuous correspondence. This is done through proof by contradiction. Let $\bar{q}^{(k)}, k = 1, 2, \dots$ be a convergent series of points in Q. Let $\bar{x}^{(k)}, k = 1, 2, \dots$ be a convergent series with $\bar{x}^{(k)} \in G_i(\bar{q}^{(k)})$ for every k. Let $\bar{q}$ and $\bar{x}$ be the limits of the two sequences. The next step will be to prove that $\bar{x} \in G_i(\bar{q})$.

Because $\bar{x} \cdot \bar{q} \leq B_i$, $\bar{x}$ is within individual i's budget and is thus an element of $F_i(\bar{q})$. Suppose, for the sake of contradiction, that $\bar{x}$ is not an element of $G_i(\bar{q})$, and instead some $\bar{y} \in G_i(\bar{q})$ is chosen. It must then be the case that either the utility of

[7] See Hylland and Zeckhauser (1979) Discussion Paper 51 Appendix A for this proof in their words.

$\bar{y}$ is greater than the utility of $\bar{x}$ or the utilities are equal and the price of $\bar{y}$ is less than the price of $\bar{x}$.

If the first case is true, meaning $u_i(\bar{y}) > u_i(\bar{x})$, a sequence can be constructed such that $\bar{y}^{(k)} \in F_i(\bar{q}^{(k)})$ for all k and $\lim(\bar{y}^{(k)}) = \bar{y}$. If $\bar{y} \cdot \bar{q}^{(k)} \leq B_i$, then each $\bar{y}^{(k)}$ will be an element of $F_i(\bar{q}^{(k)})$, so $\bar{y}^{(k)} = \bar{y}$ can be chosen. Otherwise, define $a_k = {B_i}/{\bar{y} \cdot \bar{q}^{(k)}}$ and let j_k be such that ${q_{j_k}}^{(k)} = 0$. Then, define $\bar{y}^{(k)}$ as:

$$\bar{y}_j{}^{(k)} = a_k \cdot y_j, \text{for } j \neq j_k$$

$$\bar{y}_{j_k}{}^{(k)} = a_k \cdot y_{j_k} + 1 - a_k$$

Clearly, $\bar{y}^{(k)} \in F_i(\bar{q}^{(k)})$. Assuming $B_i > 0$ and knowing that $\lim_{k\to\infty} \bar{y} \cdot \bar{q}^{(k)} = \bar{y} \cdot \bar{q} \leq B_i$, it is evident that a_k will approach 1 and $\lim_{k\to\infty} \bar{y}^{(k)} = \bar{y}$. Since, by assumption, $u_i(\bar{y}) > u_i(\bar{x})$, the continuity of u_i means $u_i(\bar{y}^{(k)}) > u_i(\bar{x}^{(k)})$ for some k. This contradicts the previous assumption that $\bar{x}^{(k)} \in G_i(\bar{q}^{(k)})$.

If the second case is true, meaning $u_i(\bar{y}) = u_i(\bar{x})$ and $\bar{y} \cdot \bar{q} < \bar{x} \cdot \bar{q}$, there exists a k for which $\bar{y} \cdot \bar{q}^{(k)} < \bar{x}^{(k)} \cdot \bar{q}^{(k)}$. Then, $\bar{y} \in F_i(\bar{q}^{(k)})$ and $\bar{x}^{(k)} \in G_i(\bar{q}^{(k)})$ imply that $u_i(\bar{x}^{(k)}) > u_i(\bar{y})$. Since both $\bar{x}$ and $\bar{y}$ cost at most B_i, there is some λ such that $((1-\lambda) \cdot \bar{y} + \lambda \cdot \bar{x}^{(k)}) \cdot \bar{q} \leq B_i$ for $0 < \lambda \leq 1$. Define $\bar{x}^* = (1-\lambda) \cdot \bar{y} + \lambda \cdot \bar{x}^{(k)}$. It is clear that $\bar{x}^* \in F_i(\bar{q})$. By linearity of u_i, $u_i(\bar{x}^*) > u_i(\bar{y})$, which contradicts $\bar{y} \in G_i(\bar{q})$. Therefore, G_i is upper semicontinuous.

The next step is to define $G(\bar{q})$ as the set comprised of all possible total demand vectors with prices given by $\bar{q}$ in which each individual i selects a vector contained in $G_i(\bar{q})$. Given that $G(\bar{q})$ is an infinite sequence that is closed, bounded, and has a convergent subsequence and that $G_i(\bar{q})$ is a closed and bounded subset of all lotteries, it is straightforward to prove that G is an upper semicontinuous correspondence.

The market clearing price vector will be a $\bar{q}$ for which the vector $(M_1, M_2, \dots, M_N) \in G(\bar{q})$. Because G is an upper semicontinuous correspondence, Kakutani's fixed-point theorem can be applied to prove that a market clearing price vector will always exist. However, a few problems must be dealt with first; to use Kakutani's theorem, Q must be defined on a compact and convex set.

To deal with the problem of non-convexity, Hylland and Zeckhauser propose creating a new set that is homeomorphic to Q. This set S transforms the vectors in Q into a set that is convex and can be used to solve the convexity problem. To solve the non-compactness problem, Hylland and Zeckhauser find a subset S_0 of S that is bounded and closed. This subset is determined by choosing a ${q_j}^*$ so large that excess demand is not possible given what the players can afford with their budgets. The correspondence S_0 is convex and compact, so Kakutani's theorem can be used on it. The result is then transformed back to Q to give us the market clearing price vector.

C. *Example Showing Determination of Market Clearing Prices*

To show more clearly how market clearing prices are determined, consider the following example. Suppose there are 2 positions $j = A, B$ with 2 openings each and 4 individuals $i = 1, 2, 3, 4$. The 4 individuals report their von Neumann-Morgenstern utilities as follows:

	Position A	Position B
Individual 1	100	0
Individual 2	100	20
Individual 3	100	60
Individual 4	40	100

Suppose each player is given a budget of $B_i = 1$. To find market clearing prices, each individual's demand must be investigated. Immediately, it is clear that position A would be oversubscribed if $q_A \leq 1$ because individuals $1, 2,$ and 3 prefer A to B and would thus demand $p_A = 1$ if they could afford it. Thus, assuming $q_A > 1$, the graph for each player will look like Figure 2 from Section III, shown again below:

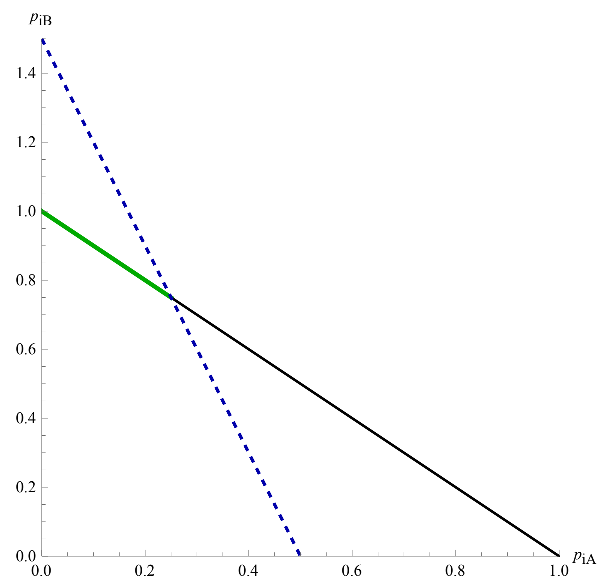

FIGURE 2

Figure 2 shows how demand looks for individual i when $q_A > B_i$ and $q_B < B_i$. The x-axis is the probability share of position A, and the y-axis is the probability share of position B. The solid, black line represents the probability constraint, and the dashed, blue line represents the budget constraint. The thicker, green segment of the probability constraint represents the bundles that individual i can demand.

Because individuals $1, 2,$ and 3 have utilities such that $u_A > u_B$, they will demand as much of position A as possible, which is the furthest right point on the thicker, green segment of the probability constraint. On the other hand, individual 4 has utilities such that $u_A < u_B$. This means individual 4 will want to consume as far left on the green line segment as possible.

It is important to remember assumption (2) before proceeding from here. As a reminder, assumption (2) states that the total number of probability shares demanded for each job j must sum to the total number of openings available for that job. In this case, $M_j = 2$ for $j = A, B$.

Returning to the example at hand, individuals $1, 2,$ and 3 will demand the intersection point of the budget constraint and the probability constraint since it is the furthest right point on the green line segment. To find the intersection point of the two lines, the following system of equations must be solved:

$$p_{iA} + p_{iB} = 1, \qquad q_A p_{iA} + q_B p_{iB} = 1$$

Solving for p_{iA} and p_{iB} gives:

$$p_{iA} = {}^{(1 - q_B)}\!\big/_{(q_A - q_B)}, \qquad p_{iB} = {}^{(q_A - 1)}\!\big/_{(q_A - q_B)}$$

Prices also must be selected such that assumption (2) is met. This gives the following two equations:

$$p_{1A} + p_{2A} + p_{3A} + p_{4A} = 2, \qquad p_{1B} + p_{2B} + p_{3B} + p_{4B} = 2$$

Since $p_{4B} = 1$ and $p_{4A} = 0$, these two equations become:

$$p_{1A} + p_{2A} + p_{3A} = 2, \qquad p_{1B} + p_{2B} + p_{3B} = 1$$

Because individuals $1, 2,$ and 3 will all demand the same probability distribution, which is the point of intersection of the budget constraint and probability constraint, the equations for p_{iA} and p_{iB} can be plugged into the equations above to get:

$$3(1 - q_B) \big/ (q_A - q_B) = 2, \qquad 3(q_A - 1) \big/ (q_A - q_B) = 1$$

Since one price will always be 0, $q_B = 0$ because if $q_A = 0$, individuals $1, 2,$ and 3 will all demand $(1,0)$, which would violate assumption (2) since $M_A = 3 \neq 2$. Thus, the equations simplify to:

$$q_A = {}^{3}\!/_{2}, \qquad q_B = 0$$

These prices will result in individuals $1, 2,$ and 3 demanding the vector $({}^{2}\!/_{3}, {}^{1}\!/_{3})$, which is the intersection point of the budget constraint and the probability constraint, and individual 4 demanding the vector $(0,1)$. These prices satisfy all assumptions and constraints and are the market clearing prices.

IV. Proof of Pareto Optimality

Assuming market clearing prices have been determined, Hylland and Zeckhauser prove that the cleared market will result in a Pareto optimal allocation. As I will show in this section, this claim does not rely on the assumption that all agents are expected utility maximizers. Thus, if market clearing prices can be identified, then non-expected utility theories with non-parallel or non-linear indifference curves can be used in place of expected utility theory.

Hylland and Zeckhauser begin their proof by defining an assignment P that is the result of the market-clearing price vector q and an assignment P' that is assumed to be better for somebody and worse for nobody. Suppose i_1 is the person made better off. Then, $p'_{i1} \cdot q > B_i$ and $p'_{i1} \cdot q > p_{i1} \cdot q$. If $p'_{i1} \cdot q > B_i$ were not true, then individual i_1 would have initially demanded that allocation since we assume that agents demand the best bundle they can afford. Thus, since we know that $p_{i1} \cdot q \leq B_i$ and $p'_{i1} \cdot q > B_i$, we can conclude that $p'_{i1} \cdot q > p_{i1} \cdot q$. This claim does require the assumption that agents are expected utility maximizers; it only requires that agents follow the weak axiom of revealed preferences.[8]

The next claim the proof makes is that, given what was just proved above, there is some other player (denoted i_2) for which $p'_{i2} \cdot q < p_{i2} \cdot q$. This claim also does not require the assumption that agents are expected utility maximizers because of assumption (2), which states $\sum_i p_{ij} = M_j = \sum_i p'_{ij}$. Thus, since $p'_{i1} \cdot q > p_{i1} \cdot q$, there must be some p'_{i2} that is less than the p_{i2} from the original assignment for the constraint to hold. Because p'_{i2} and p_{i2} are being multiplied by the same price vector q, it is clear that $p'_{i2} \cdot q < p_{i2} \cdot q$.

The final claim states that "i_2 must be worse off with the lottery p'_{i2} than with p_{i2}; otherwise i_2 would not have chosen p_{i2} in the first place" (302). To prove this claim, it is important to remember the assumption that "it is essential that the least-expensive lottery be chosen if utility is equal" (302). For sake of contradiction, assume that this is not the case and that between two or more lotteries with equal utility, the more expensive one is chosen. The result would be an outcome that is not Pareto optimal; there is a scenario in which at least one individual can be made better off with no individuals being made worse off. That scenario would occur

[8] The weak axiom of revealed preferences states that when some bundle a is purchased over some other bundle b when both are affordable, the agent reveals that he or she prefers a to b. Then, if b was to be chosen over a in some other budget set, it must be that a is no longer feasible in the new budget set. See Samuelson (1938).

when the player with the two lotteries that have equal utilities (we will call this player i_X) chooses to instead take the lottery that is less expensive. This would result in additional shares of more highly valued positions being made available for other players to demand. The more expensive lottery is more expensive because it includes more shares in positions that are highly valued by the market (as determined by the market clearing prices) than the other lottery in question. The market clearing prices are determined by individuals' utilities for each position, so the higher-priced shares are more highly desired. Thus, some other player could conceivably be made better off when player i_X chooses the less expensive lottery because that would allow for there to be more shares of highly valued positions available for everyone else.

Once again, the final claim of the proof states that "i_2 must be worse off with the lottery p'_{i2} than with p_{i2}; otherwise i_2 would not have chosen p_{i2} in the first place" (302). Below is a graph showing indifference curves (red, dotted lines) induced by expected utility theory in the Hylland & Zeckhauser model graphed on the budget constraint and probability constraint:

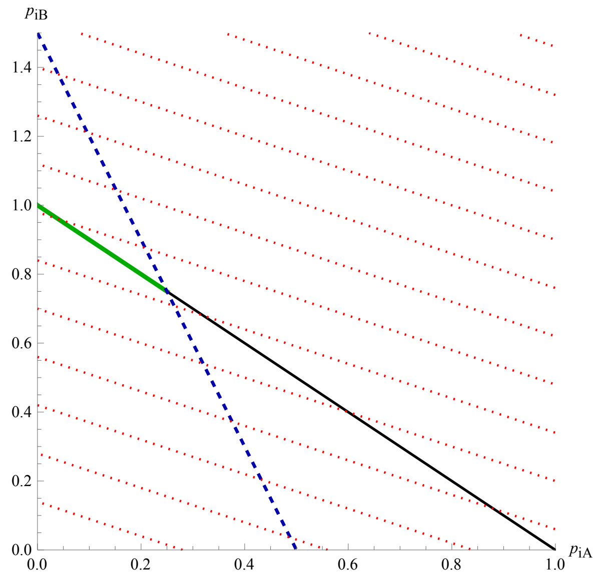

FIGURE 8

Figure 8 shows individual i's indifference curves (red, dotted lines) subject to expected utility theory. The x-axis is the probability share of position A, and the y-axis is the probability share of position B. Once again, the solid, black line represents the probability constraint, the dashed, blue line represents the budget constraint, and the thicker, green segment of the probability constraint represents the bundles that individual i can demand. Individual i will seek to maximize utility by demanding the point on the green line segment that intersects the highest indifference curve.

These indifference curves are all linear and parallel. Because only probability vectors on the probability constraint can be demanded, no other point on the indifference curve that the chosen bundle lies on can be demanded. This explains why Hylland and Zeckhauser's proof works; the lottery that i_2 is forced to take when i_1 takes p'_{i1} will result in i_2 demanding a probability vector p'_{i2} that is on a less-preferred indifference curve than the indifference curve p_{i2} is on.

Now, suppose we no longer assume that indifference curves must be parallel. Refer once again to the following graph for individual i_2:

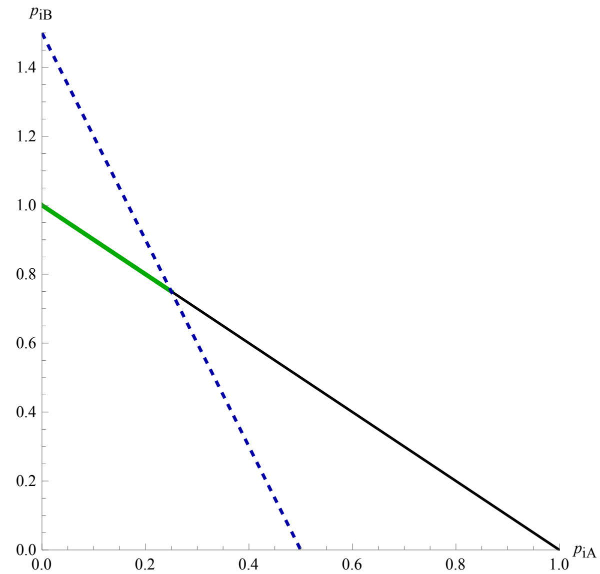

FIGURE 2

Figure 2 shows how demand looks for individual i when $q_A > B_i$ and $q_B < B_i$. The x-axis is the probability share of position A, and the y-axis is the probability share of position B. The solid, black line represents the probability constraint, and the dashed, blue line represents the budget constraint. The thicker, green segment of the probability constraint represents the bundles that individual i can demand.

Because indifference curves cannot intersect in the first quadrant under the premise of rationality, the non-parallel indifference curves must intersect in some other quadrant. This then results in a situation very similar to that of expected utility theory. Any linear indifference curve will only intersect the probability constraint at one point (unless the indifference curve is the same line as the probability constraint). Thus, it can be assumed that each individual is already demanding the bundle on the highest possible indifference curve. The lottery that i_2 is forced to take when i_1 takes p'_{i1} will result in i_2 demanding a probability vector p'_{i2} that is on a less-preferred indifference curve than the indifference curve p_{i2} is on. Therefore, the proof still holds.

Now, suppose we assume that indifference curves can be curved. Suppose we have a curved indifference curve that intersects the probability constraint at p_{i2} and at least one other point within individual i_2's budget constraint on the probability constraint. Individual i_2 would be indifferent between these points because they are on the same indifference curve, meaning they have the same utility. Now, suppose one of the other points where the indifference curve and probability constraint intersect, p'_{i2}, is such that $p'_{i2} \cdot q < p_{i2} \cdot q$. This would prove Hylland and Zeckhauser wrong if not for their assumption that when utility is equal, the lottery that is the least expensive is always chosen. Because this lottery p'_{i2} is within individual i_2's budget constraint, then we assume that i_2 would have demanded that bundle in the first place because it has equal utility to p_{i2} but is less expensive. Thus, we find that Hylland and Zeckhauser's proof still holds with curved indifference curves.

Therefore, assuming market clearing prices can be determined, Hylland and Zeckhauser's mechanism will still result in a Pareto optimal outcome even when we allow agents to have utility functions that are nonparallel or nonlinear.

V. Shortcomings of Expected Utility Theory

Before attempting to implement specific non-expected utility theories into the Hylland and Zeckhauser mechanism, it is important to discuss the shortcomings of expected utility theory. I will discuss not only what these shortcomings are, but also how they apply to the types of allocation problems the Hylland and Zeckhauser mechanism solves.

Von Neumann and Morgenstern (1944) were among the first to determine axioms for expected utility theory. They derived it from three axioms: ordering, continuity, and independence. Ordering means that expected utility theory satisfies both completeness and transitivity. Completeness means that for any two prospects q and r, either $q \succcurlyeq r$ or $r \succcurlyeq q$ or both. Transitivity implies that for some prospects

q, r, s, if $q \succcurlyeq r$ and $r \succcurlyeq s$, then $q \succcurlyeq s$. Continuity means that for all q, r, s where $q \succcurlyeq r$ and $r \succcurlyeq s$, there is some p such that $(q, p; s, 1-p) \sim r$.

The third axiom, independence, is the most controversial and is relaxed by nearly all non-expected utility theories. Independence states that for prospects q, r, s, $if\ q \succcurlyeq r$ then $(q, p; s, 1-p) \succcurlyeq (r, p; s, 1-p)$ for all p.

Allais (1953) was the first to explain why expected utility theory may not accurately depict how individuals value prospects due to violations of the independence axiom. There are two main effects Allais identified that result in these violations: the common consequence effect and the common ratio effect. The common consequence effect can be shown through the following hypothetical choice problem. Allais creates two situations, each with two different prospects. The first situation has the prospects $s_1 = (\$1\text{M}, 1)$ and $r_1 = (\$5\text{M}, 0.1; \$1\text{M}, 0.89; 0, 0.01)$. The second situation has prospects $s_2 = (\$1\text{M}, 0.11; 0, 0.89)$ and $r_2 = (\$5\text{M}, 0.1; 0, 0.9)$. Allais believes that in the first situation, people will choose the guaranteed \$1M that prospect s_1 offers instead of r_1, which offers the chance of getting even more money, but also the small chance of coming away with nothing. However, in the second situation, Allais believes people will choose r_2 since both prospects result in a high chance of coming away with nothing, but r_2 gives the small chance of coming away with \$5M, which Allais believes is preferred to the slightly larger, but still small, chance of coming away with \$1M.

If Allais is right, then this would be a violation of expected utility theory, which states that one must either choose only s or only r in both situations. The violation comes from Savage's (1972) "sure-thing principle," which is a principle of expected utility theory. In the first situation, both prospects give a 0.89 chance of coming away with \$1M. In the second situation, both prospects give a 0.89 chance at coming away with \$0. The sure-thing principle states that it should not matter whether there is 0.89 chance at $1M, $0, or any other dollar amount since otherwise, the two sets of prospects are exactly equal. Thus, individuals who choose s_1 and r_2 violate expected utility theory. According to Starmer (2000), many empirical studies since Allais' initial hypothesis have supported this type of violation of expected utility theory.

Now, let us explore the other effect Allais hypothesized, which is the common ratio effect. Suppose we have another pair of hypothetical situations. In the first, we get a choice between a guaranteed $750 or a 0.8 chance at $1000. In the second situation, we get a choice between a 0.25 chance at $750 or a 0.2 chance at $1000. Allais hypothesized, and much empirical evidence since then has shown, that in the first situation, agents will frequently choose the guaranteed $750, but in the second, choose the 0.2 chance at $1000. Expected utility theory would tell us that an agent must choose the $750 in both situations or the $1000 in both situations since the ratio of the probabilities is the same. Hence the name "common ratio effect."

The goal of non-expected utility theories is to better represent these violations of independence that Allais discovered. There are two broad categories of non-expected utility theories: conventional and non-conventional. In this paper, I only explore conventional theories, which means agents act as if they are optimizing some underlying preference function. Within this broad category, I am only exploring theories that have decision weights, meaning they assume that individuals assign subjective attitudes to objective probabilities. This is done by putting the probability p of some outcome through a function $\pi(p)$. Examples of individuals subjectively weighting objective probabilities can be seen in the papers Ali (1977), Thaler and Ziemba (1988), and Jullien and Salanié (1997). All of these papers explore racetrack betting and show that many of the bettors they studied are oversensitive to the chance of winning on longshots and, at the same time, oversensitive to the chance of losing on a favorite.

The functional form of $\pi(p)$ is a very important part of all conventional non-expected utility theories. Quiggan (1982) proposes an inverted s-shaped function, which has been supported by research from Preston and Baratta (1948) and Prelec (1998). Figure 9 below shows what this function looks like:

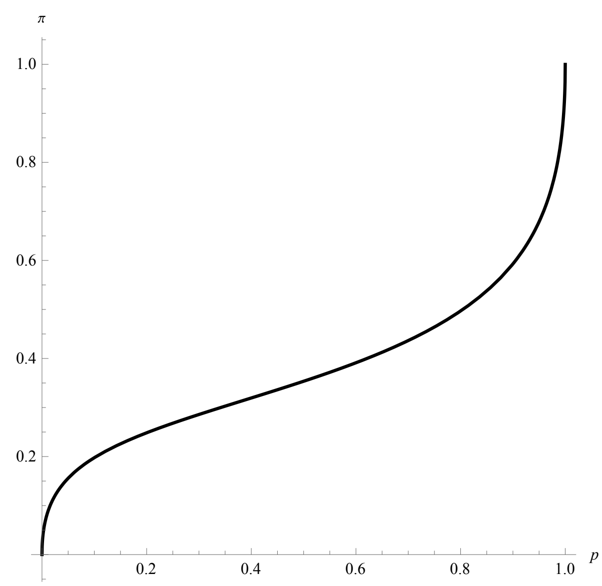

FIGURE 9

Figure 9 shows the inverted s-shaped function $\pi(p) = p^{\delta} \Big/ \left[(p^{\delta} + (1-p)^{\delta})^{\frac{1}{\delta}}\right]$ for $\delta = 0.5$ often used to weight probabilities in conventional non-expected utility theories. The values on the x-axis are the objective probabilities, and the values on the y-axis are the subjective, weighted probabilities.

There are several functional forms that achieve inverted s-shaped graphs. In this paper, I opt to use the single parameter weighting function developed by Tversky and Kahneman (1992):

$$\pi(p) = p^{\delta} \Big/ \left[(p^{\delta} + (1-p)^{\delta})^{\frac{1}{\delta}}\right] \text{ for } 0 < \delta < 1$$

The function is concave below the inflection point p^* and is convex above it, meaning lower probabilities are overweighted, reflecting real-life behavior of many agents. It can be viewed as a form of pessimism, which is closely related to risk aversion. The convex part of the curve implies greater aversion to randomization, so above the inflection point, agents are more averse to randomization. Additionally, in the middle range of probabilities, agents act more closely to what expected utility theory would predict than at the extremes (Quiggan 1982).

The three non-expected utility theories I will be applying to the Hylland and Zeckhauser mechanism are subjective expected value theory (SEVT), simple decision weighted utility theory (SDWUT), and rank dependent utility theory (RDU). Each one builds off the other. SEVT uses a probability weighting function but keeps utilities as they are. SDWUT uses a utility weighting function in addition to a probability weighting function. Finally, RDU takes into account the ranked order of preferences and, unlike the other two, is monotonic.

VI. Subjective Expected Value Theory and Simple Decision Weighted Utility Theory

A. Overview of the Theories

Both SEVT and SDWUT take on the following general form:

$$V = \sum_i \pi(p_i) \cdot u(x_i)$$

With SEVT, $u(x_i) = x_i$, meaning utilities are kept as they are and are not transformed. Edwards (1955, 1962) was the first discuss this theory, and Handa (1977) was the first to recommend the use of $\pi(p_i)$ as the decision weight. It is the most basic form of decision weighted utility theories since the only transformation it makes from expected utility theory is the weighting of probabilities through $\pi(p_i)$.

SDWUT extends beyond SEVT by no longer requiring $u(x_i) = x_i$, meaning in addition to probabilities being weighted, utilities can also be weighted. The idea of having a utility weighting function was first introduced by Kahneman and Tversky (1979). They proposed a utility function that is concave for gains and convex for losses from some reference point. Their reasoning is that two changes in utility of the same magnitude, one from a starting point very close to the reference point and another from one farther away, will not be valued the same. A concave utility function for gains and a convex function for losses takes this into account.

In addition, the utility function ought to be steeper for losses than it is for gains. This is because a loss is thought of by many agents as more impactful than an equivalent gain.

A common utility weighting function that applies well to individuals in many of the situations that this mechanism will encounter is $u(x_i) = {x_i}^{\gamma}$ for $0 < \gamma < 1$. This function sets the reference point at 0, which is a reasonable reference point for most applications of the mechanism since the mechanism uses a pseudomarket with fake money. Everyone is effectively starting from nothing in the context of the relevant allocation problem. Whichever position is allocated to a given individual will be seen as a gain. Figure 10 below shows what the utility weighting function looks like for $\gamma = 0.95$:

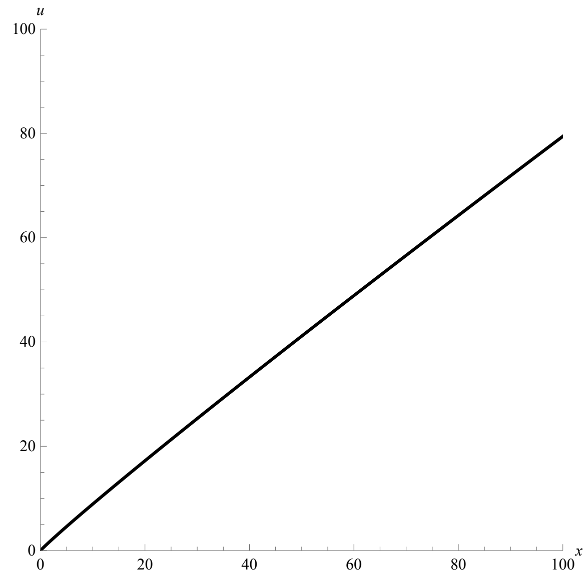

FIGURE 10

Figure 10 shows the utility weighting function $(x_i) = x_i{}^{\gamma}$ for $\gamma = 0.95$. The values on the x-axis are the initial utilities, and the values on the y-axis are the weighted utilities. Utility weighting functions such as this are used in many conventional non-expected utility theories.

B. Implementation Into Hylland & Zeckhauser Mechanism

Substituting SEVT in for expected utility theory in the Hylland and Zeckhauser mechanism causes it to no longer be able to determine market clearing prices in some cases, which is essential to the success of the mechanism. In this section, I will lay out a specific counterexample that shows a case in which market clearing prices cannot be determined. Also, it is important to note that because SEVT is a special case of SDWUT, this counterexample rules out both SEVT and SDWUT as potential substitutes for expected utility theory in the mechanism.

Suppose there are four individuals $I = 1, 2, 3, 4$ and two positions $J = A, B$ each with two openings. Thus, there are 4 total job openings and 4 individuals. The individuals have the following von Neumann-Morgenstern utilities:

	Position A	Position B
Individual 1	100	0
Individual 2	100	0
Individual 3	100	70
Individual 4	0	100

Suppose each individual is subject to SEVT. For individuals 1 and 2, regardless of whether they are subject to EUT or SEVT, they will choose to demand as high of a probability share in position A as they can afford. They both have 0 utility for position B, so the only way to maximize utility will be to spend their entire budget on position A. The same can be said for individual 4, the only difference being that she prefers position B to A, so she will demand as much of position B as she can afford.

Individual 3, however, will be a more unique case. Individual 3's utility will be subject to the following function:

$$Utility_3 = 100\pi(p_1) + 70\pi(p_2)$$

The function $\pi(p_i)$ is the inverted s-shaped function $\pi(p) = p^\delta \Big/ \left[(p^\delta + (1-p)^\delta)^{\frac{1}{\delta}}\right]$ with $\delta = 0.5$. Below is a graph that shows what individual 3's indifference curves look like:

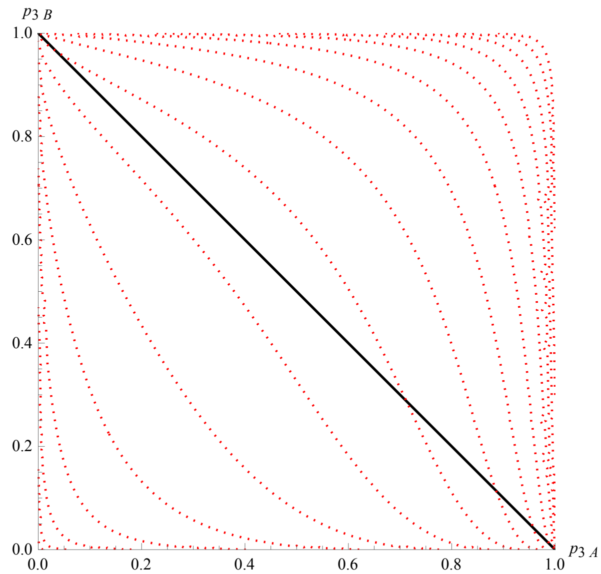

FIGURE 11

Figure 11 shows individual 3's indifference curves (red, dotted lines) as determined by SEVT graphed on top of the probability constraint. The x-axis is individual 3's probability share of position A, and the y-axis is individual 3's probability share of position B.

A particular indifference curve of interest is the one beginning at $(0, 1)$ that intersects the probability constraint a second time at $(0.8566 \ldots, 0.1444 \ldots)$. The figure below isolates that specific indifference curve:

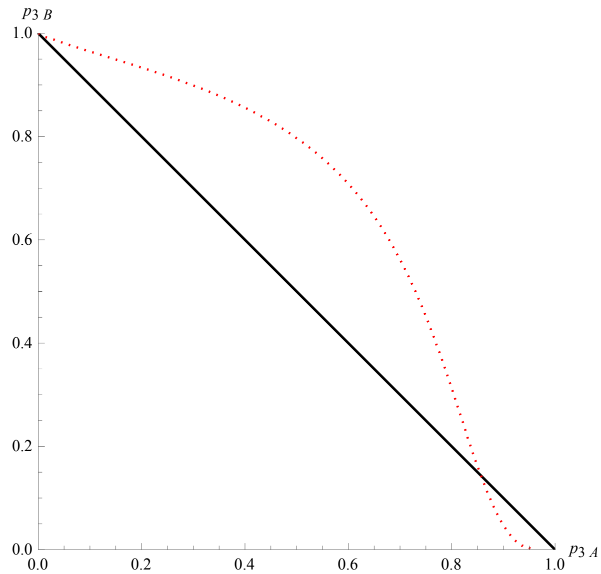

FIGURE 12

Figure 12 isolates a specific indifference from Figure 11 that intersects the probability constraint at both $(0, 1)$ and $(0.8566 \ldots, 0.1444 \ldots)$. The x-axis is individual 3's probability share of position A, and the y-axis is individual 3's probability share of position B. This indifference curve will be used to show why it is impossible to attain market clearing prices for this counterexample.

Now, consider the possible combinations of market clearing prices (q_A, q_B). One of the assumptions of the Hylland and Zeckhauser algorithm and the proof of market clearing prices is that at least one price must be 0. It is clear to see that making $q_A = 0$ would not work because individuals 1, 2, and 3 all prefer position A to position B and would want to demand (1, 0). However, this clearly violates constraint (2) since the sum of the total probability shares demanded for position 2 is greater than the number of openings of position 2. Thus, it must be that $q_B = 0$, which makes sense intuitively since only individual 4 prefers position B to position A.

Given $q_B = 0$, I will now go through individual 3's demand with every possible price of position A. When q_A is in the range $[0, 1]$, individual 3 will demand (1, 0) and will receive utility of 100. When q_A is in the range $(1, 1.17)$, which corresponds to the part of the graph where p_A is in the range $(0.86, 1)$, individual 3 will demand as much of position A as possible, or $(\frac{1}{q_A}, (1 - \frac{1}{q_A}))$.[9] When $q_A = 1.17$, individual 3 will be indifferent between $(0, 1)$ and $(0.86, 0.14)$ because both have equal utility of 70. Finally, when q_A is greater than 1.17, individual 3 will demand $(0, 1)$.

Now, I will show that there is no possible value of q_A whereby market clearing prices can be determined. I will start by considering $0 \leq q_A \leq 1$. This set of prices will not work because individuals 1, 2, and 3 will all demand (1, 0), which will result in constraint (2) being violated because $\sum_i p_{iA} = 3 \neq M_A$.

Now, let us assume that $1 < q_A < 1.17$. In this case, individual 3 will demand $(\frac{1}{q_A}, (1 - \frac{1}{q_A}))$. Individual 4, as always, will demand (0, 1), and individuals 1 and 2 will want to demand as much of p_A as they can afford, so they will also demand $(\frac{1}{q_A}, (1 - \frac{1}{q_A}))$. With q_A in this range, there is no way for constraint (2) to be met. Individuals 1, 2, and 3 all demand $p_A = \frac{1}{q_A}$, and there is no value in the range $1 < q_A < 1.17$ where constraint (2) is met.

Next, let us assume that $q_A = 1.17$. It is now the case that individual 3 is indifferent between $(0.86, 0.14)$ and $(0, 1)$. However, demanding either of these bundles will result in constraint (2) not being met. First, with the bundle $(0, 1)$, $\sum_i p_{iA} < M_A$ and $\sum_i p_{iB} > M_B$. With the bundle $(0.86, 0.14)$, $\sum_i p_{iA} > M_A$ and $\sum_i p_{iB} < M_B$. Thus, $q_A = 1.17$ is not a viable solution.

Finally, let us assume that $q_A > 1.17$. In this case, individual 3 will always demand the vector $(0, 1)$. As were the cases before, individual 4 will still demand $(0, 1)$, and individuals 1 and 2 will demand as much of position A as they can. Once again, there is no possible q_A in which constraint (2) is met. Individuals 1 and 2

[9] All decimals are rounded to the hundredth place in this section.

will demand $p_A = \frac{1}{q_A}$, while individuals 3 and 4 will demand $p_A = 0$, which will always result in $\sum_i p_{iA} < M_A$. Q.E.D.

Therefore, this counterexample shows that when SEVT is used instead of expected utility theory, Hylland and Zeckhauser's mechanism no longer works because it cannot always attain market clearing prices. It is worth noting that it was only necessary in this counterexample for individual 3 to have preferences subject to SEVT. The other three individuals could have been expected utility maximizers, and the counterexample would still have caused the mechanism to not produce market clearing prices.

VII. Rank Dependent Utility Theory

A. Overview of the Theory

One drawback of both SEVT and SDWUT is that both are non-monotonic. This makes it possible, as was shown in both counterexamples above, that the sure option could be chosen over potential outcomes that mix between the worse option and better option that stochastically dominate the sure option. Some people may be subject to this type of thinking if they are very averse to randomization, but there are likely many people who are not.

RDU was developed to solve this monotonicity problem. First proposed by Quiggan (1982), RDU solves the problem by making the weight attached to each probability dependent on both the true probability of that consequence and on the rank of that consequence relative to others. Like any decision weighted theory, RDU takes on the general form:

$$V = \sum_i w_i \cdot u(x_i)$$

In SEVT and SDWUT, $w_i = \pi(p_i)$. However, in RDU, it is a bit more complicated:

$$w_i = \pi(p_i + \cdots + p_n) - \pi(p_{i+1} + \cdots + p_n)$$

Consequences are indexed from worst to best such that x_1 is the worst and x_n is the best. It is important to note here that SEVT and SDWUT are not special cases of RDU, so we cannot use the previous counterexample to disprove RDU.

The same π used before is used once again for RDU. In the words of Gonzalez and Wu (1999, 135), it describes the "psychophysics of risk in agents' heads." It distorts objective probabilities into subjective interpretations of them. The weighting function w, on the other hand, decides how the probability weights enter the actual value function, V. It subtracts the individual's weighted probability of getting an outcome better than i from the weighted probability of getting outcome i or better.

RDU satisfies a weakened form of independence called "co-monotonic independence" (Wakker, Weber 1994). The standard independence axiom states that for any two prospects with common outcome x occurring with probability p, substituting x for some other outcome y will not affect the preference order of the two prospects. Co-monotonic independence states that this axiom is true if and only if the substitution of y for x does not affect the ranked order of the outcomes in either prospect. The idea of ranking prospects also has the benefit that really "good" or "bad" outcomes relative to the others will receive very high or low weights, respectively.

B. Implementation Into Hylland & Zeckhauser Mechanism

Because RDU is monotonic, it does not make Hylland and Zeckhauser's mechanism fail to attain market clearing prices as obviously as SEVT and SDWUT did. There is no indifference curve crossing the probability constraint at two unique points, as can be seen in the graph below of the indifference curves (red, dotted lines) subject to RDU for the same individual 3 in the previous counterexample:

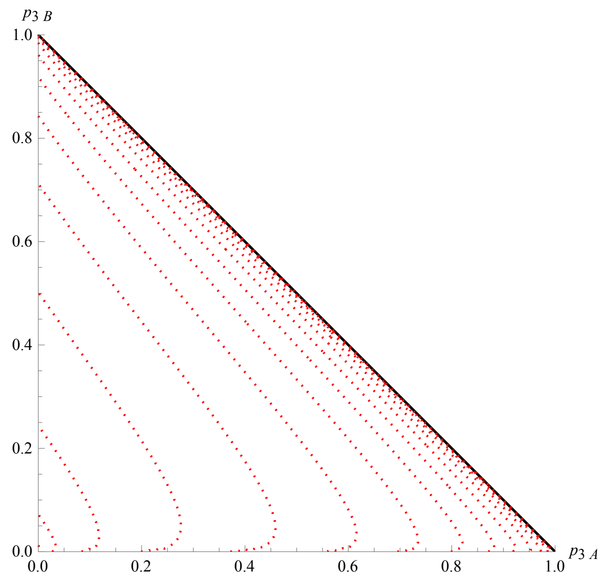

FIGURE 13

Figure 13 shows individual 3's indifference curves (red, dotted lines) as determined by RDU graphed on top of the probability constraint. The x-axis is individual 3's probability share of position A, and the y-axis is individual 3's probability share of position B.

Instead, using the proof of the existence of market clearing prices from Section III, a counterexample can be created to show that the mechanism with RDU cannot be proven to attain market clearing prices using the same method that Hylland and Zeckhauser used.

To reiterate from Section III, it is necessary for $G_i(\overline{q})$ to be a convex set for market clearing prices to exist. In other words, the set of bundles that meet the requirements of the mechanism, meaning the bundles that maximize utility and are least cost when utilities are equal, must be a convex set for each individual and for any given price vector q that satisfies the constraints of the problem. It is here where a potential counterexample arises.

Suppose we have three individuals $I = 1, 2, 3$ and three positions $J = A, B, C$ with the following utilities:

	Position A	Position B	Position C
Individual 1	100	0	0
Individual 2	0	0	100
Individual 3	100	70	0

It is clear that individuals 1 and 2 will demand as much of positions A and C, respectively, as they can. For individual 3, it is less clear what he or she will demand. There are some price vectors q for which individual 3 will not want to demand as much of position A as possible, even though it is the most preferred position for that individual. This can be seen clearly in Figure 13 above that shows individual 3's indifference curves.

To construct the counterexample, let us set $q = (2.69, 1.40, 0)$.[10] This price bundle was calculated so that the budget constraint intersects one of individual 3's indifference curves at two distinct points, as shown in the graph below:

[10] All decimals are rounded to the hundredth place in this section.

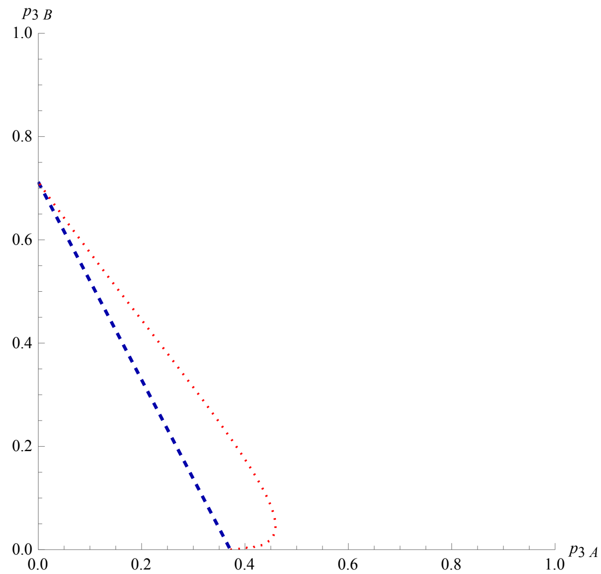

FIGURE 14

Figure 14 isolates a specific indifference from Figure 13 that intersects the budget constraint for the price vector $q = (2.69, 1.40, 0)$ at two points. The x-axis is individual 3's probability share of position A, and the y-axis is individual 3's probability share of position B. This indifference curve will be used to show a violation of one of the central assumptions of Hylland and Zeckhauser's proof of the existence of market clearing prices.

The budget constraint intersects the indifference curve at its x and y intercepts, which are $(0.37, 0)$ and $(0, 0.71)$, respectively. Both points are on the budget constraint, meaning they cost the same, and to ensure they are on the probability constraint, the excess demand must be placed placed in position 3 since that position costs \$0 per probability share. That gives us the points $(0.37, 0, 0.63)$ and $(0, 0.71, 0.29)$. It is easy to see from the graph above that both bundles will maximize utility because they are on the same indifference curve, and it is the highest up indifference curve on or below the budget constraint. Thus, because both bundles satisfy all necessary constraints, maximize utility, and are equal in cost, both are part of the set $G_i(\bar{q})$ for $i = 3$ and $q = (2.69, 1.40, 0)$.

The violation arises here: the set $G_3(\bar{q})$ for $q = (2.69, 1.40, 0)$ is not convex. For the set to be convex, every point on the line segment connecting the two points in the set must be weakly preferred to those points. It is easy to see in the graph that this will not be the case in this example. The indifference curve intersecting $(0.37, 0, 0.63)$ and $(0, 0.71, 0.29)$ is entirely above the line segment connecting those two points (which is the same line as the budget constraint). Thus, every point on that line segment is on a lower indifference curve than the points in the set $G_3(\bar{q})$ for $q = (2.69, 1.40, 0)$. Thus, the set is not convex. Q.E.D.

Therefore, the assumption that $G_i(\bar{q})$ is always a convex set is no longer true, meaning market clearing prices cannot be proven to exist using the method Hylland and Zeckhauser used.

VIII. Discussion and Conclusion

The counterexample for SEVT and SDWUT makes it clear that market clearing prices will not always exist when either of those non-expected utility theories are implemented into the mechanism. While the result for RDU is not as conclusive, since the counterexample only shows that the existence of market clearing prices cannot be proven through the same method that Hylland and Zeckhauser used, it certainly does raise a lot of doubt upon the potential of finding a way to prove that market clearing prices do always exist with RDU. Thus, it seems that Hylland and Zeckhauser's mechanism is heavily reliant on individuals being expected utility maximizers. This can be a cause for concern, since as I explained throughout this paper, many individuals violate the tenants of expected utility theory and act in ways only non-expected utility theories predict.

The results of this paper do not mean that Hylland and Zeckhauser's model is not compatible with any non-expected utility theories. There could be a method of proof different from the one Hylland and Zeckhauser used that proves the existence of market clearing prices when RDU is used. In addition, this paper only explores three non-expected utility theories. There are many more that exist that could be tested as well. However, many of these are built from the foundation of SDWUT and RDU, making it unlikely for a lot of them to work. Still, it may be worthwhile to test some of these other non-expected utility theories in future research.

The conclusion of this research is not that Hylland and Zeckhauser's model should be considered completely useless. However, it does call into question whether the resulting allocation of the mechanism truly maximizes total utility. If even one individual thinks in a way that is better represented by some non-expected utility theory than expected utility theory, then the mechanism will likely not assign a lottery to that individual that maximizes his or her utility. A mechanism that is adaptable to both expected utility theory and various non-expected utility theories would be a very valuable tool in many real-life allocation problems and thus constitutes further research.

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