Essays on Auction Theory

by

Qian Li

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Doctoral Committee:

Associate Professor David Miller, Chair
Professor Amanda Friedenberg
Associate Professor Vijay Subramanian
Assistant Professor Mu Zhang

Qian Li

liqia@umich.edu

ORCID ID: 0009-0001-3700-1061

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ABSTRACT

This dissertation focuses on auction theory, especially multi-unit auction, since bidder's behaviour will differ a lot in multi-unit auctions than in single-unit auctions. In Chapter II, "*Multi-Unit Auction In Discrete Type Space*", we study a multi-unit pay-as-bid auction where there are two discrete types of bidders and each type of bidder demands two units. We find closed-form solutions for symmetric Bayesian Nash Equilibria for different proportions of types in the population and one main feature is identical bidding behaviour, where one particular type of bidder will use identical bidding prices for both of the units. This chapter also finds that distributions for mixed strategy equilibrium from different types will have overlapping support in bidding spaces. These two features will lead to inefficient allocations. The identical bidding behaviour is also reported in empirical literature studying treasury bill auctions. We also compare expected revenue between formats of multi-unit auctions and confirm that revenue equivalence does not hold in multi-unit settings with ambiguous ranking between revenue from pay-as-bid and Vickrey auctions, while both dominate uniform-price auction in expected revenue. We also show through examples that the identical bidding behaviour can also be extended to higher-unit settings.

In Chapter III, "*Package Bidding with Distinct Objects*", we study a combinatorial auction where two discrete types of bidders are competing for two distinct objects. In this chapter we assume different bidders will have different favoured objects. In the setting of a combinatorial auction, we allow each bidder to propose an extra price for the bundle, besides prices for each single object. In this chapter, we focus on closed-form solutions for symmetric Bayesian Nash Equilibria with restriction to

equilibria where bidders bid pure strategies on single objects and mixed strategies on the bundle. We continue to study their performance in terms of welfare.

In Chapter IV, "*Smooth Ambiguity Averse Level-k Auction*", we return to the context of single-unit auction and re-examine the elimination process of dominated bids in a first-price sealed bid auction. In particular, we study the first-price auction in a discretized unit interval and construct the upper and lower bounds of feasible bids in the process of elimination of implausible bids, with the help of smooth ambiguity averse model proposed by Klibanoff, Marinacci and Mukerji (2005) which allows bidders to aggregate smoothly over the support of subjective beliefs. We numerically construct upper and lower bounds of feasible bids in each round of elimination as well as the convergent stable bids for each type in discretized type and bidding spaces. We compare our result of stable bounds with results from Bayesian Nash Equilibrium for first price auctions. We also find that my approach is also similar to the level-k theory.

CHAPTER I

Introduction

This dissertation focuses on bidders' behaviour in auctions, especially in multi-unit auctions. Auction is a tool to allocate goods among consumers by presenting goods to bidder who propose higher prices, and is also widely used in practice. The goal of auction is usually to give the goods to buyers who value them the highest, since such an allocation will achieve highest social welfare. Efficient allocation can be observed in single-unit auctions but when multiple-units are being sold simultaneously both empirical evidence and theoretical results would suggest that efficient allocations could not be guaranteed primarily due to complicity of bidders' strategies.

The first two chapters of the dissertation construct symmetric Bayesian Nash Equilibria for multi-unit auctions with two indivisible goods and further assume there are no synergies between the goods nor the goods are substitutes. On consumer side, we assume that the type space is binary and consumers have diminishing marginal valuations toward the units. We study two formats of pay-as-bid multi-unit auctions respectively in the next two chapters. The first auction only allows bidding prices for single units and assume that all units being sold in the auction are identical. The second auction is built upon the first model but with several subtle differences: objects in auctions are distinct and different bidders have different favoured objects. What's more, the second auction permits each bidder to propose a price for the two objects as a bundle, besides the two bidding prices for each single objects. The last chapter, on the other hand, studies a completely different topic and focuses on why bidders' bidding strategy in real experiments of single-unit first-price auctions would differ from predictions of Bayesian Nash Equilibrium. To validate experimental data, we employ the smooth ambiguity averse model proposed by Klibanoff, Marinacci and

Mukerji (2005).

In Chapter II, "*Multi-Unit Auction In Discrete Type Space*", we construct closed-form symmetric Bayesian Equilibrium for a full range of parameters as well as all possible orderings of private values. One main feature of the theoretical results is identical bidding behaviour. By observing bidders' behaviour in this multi-unit auction, this chapter is able to provide a detailed interpretation of identical bidding behaviour: bidders understand that in this multi-unit auction their higher (lower) bids are competing only against opponents' lower (higher) bids. And accordingly, a bidder would have incentive to decrease her higher bid to extract as high payoff as possible, and raise her lower bid for a better chance of winning at the same time until higher and lower bids from the same bidder are identical. Such identical bidding behaviours are also observed in empirical literature studying treasury bill auctions. Additionally, we find that distributions for mixed strategy equilibrium from different types will have overlapping of support in bidding spaces. As long as equilibria feature identical bidding and overlapping of support, the allocations are likely to be efficient. We are also able to propose another equilibrium under a specific range of parameters where both bidders are bidding pure bids equivalent to the private value of one bidder's less favoured objects. Such a result will guarantee efficient allocation.

We also compare expected revenue between pay-as-bid auction, Vickrey auction and uniform-price auction, and confirm that revenue equivalence does not hold in multi-unit auctions, and one main reason is that different formats of multi-unit auctions lead to different allocations. We compute via numerical examples that ranking between revenue from pay-as-bid and Vickrey auctions would be ambiguous, which will depend on value of parameters but uniform-price auction will always generate the lowest revenue under our set-up. We continue to construct two examples by extending our theoretical results with two-unit into environment when there are four units. Both examples are illustrations that identical bidding behaviour established by my theorems can be approximations of empirical evidence.

In Chapter III, "*Package Bidding with Distinct Objects*", we study a combinatorial auction where two discrete types of bidders are competing for two distinct objects. We assume different bidders will have different favoured objects. In the setting of combinatorial auction, we allow each bidder to propose an extra price for the bundle, with the requirement that prices for the bundle should be no smaller than summation of prices for single objects. In this chapter, we focus on closed-form solutions for symmetric Bayesian Nash Equilibria with restriction to equilibria where bidders bid pure strategies on single objects and mixed strategies on the bundle. The paper proposes two different Bayesian Nash Equilibria, with the first result featuring bidding zero for all single units and the second result being similar to one main result found in the previous chapter where pure bids are equivalent to a bidders' private valuation of the less favoured object. We continue to study their performance in terms of welfare.

In Chapter IV, "*Smooth Ambiguity Averse Level-k Auction*", we return to the context of single-unit auction and re-examined the elimination process of dominated bidding prices in a first-price single-unit auction. In particular, we study the first price auction in a discretized unit interval and construct the upper and lower bounds of feasible bids in the process of elimination of implausible bids, with the help of smooth ambiguity averse model proposed in Klibanoff, Marinacci and Mukerji (2005). KMM (2005) allows bidders to aggregate smoothly over the support of subjective beliefs. To be more specific, this chapter proposes a new elimination process of implausible bids by constructing upper and lower bounds of plausible bids. This chapter incorporates KMM (2005) by assuming opponents are best responding to beliefs that each bidder bid pure strategy and the pure bids are distributed uniformly over the range of plausible bids. Model from KMM (2005) also includes an ambiguity averse coefficient, and the paper will assume that bidders of each type share the identical range of ambiguity averse attitudes. This chapter constructs opponent's upper(lower) bounds of plausible bids when opponent's ambiguity aversion reaches highest (lowest).

We construct upper and lower bounds of feasible bids in each round of elimination as well as the convergent stable bids for each type in discretized type and bidding spaces. It turns out that the survival bids will display underbidding relative to Bayesian Nash Equilibria since previous study finds out that the ambiguity averse coefficients for almost every individual are very small. But if we raise the ambiguity aversion to high values, overbidding is observed. We interpret such pattern with the help of the utility function proposed by KMM (2005) and point out that bidders' behaviour will resemble bidders with maxmin preferences who focus on worst scenarios to bid very high prices to avoid losing. This chapter concludes by comparing it with level-k theory. This chapter is similar to how Level-1 responds to Level-0 in level-k theory, but introduction of ambiguity aversion guarantees my model my model will end up with a range of plausible bids in each round of elimination until convergence, while for level-k theory the level 1 bidders will probably best response by single bids.

CHAPTER II

Multi-Unit Auction In Discrete Type Space

2.1 Introduction

In auction theory literature not too much attention has been given to multi-unit pay-as-bid auctions, where the monetary payment for each unit is the winning price for that unit. But in reality multi-unit auction is not rare: the sale of treasury bill auction is an example of multi-unit auction with identical goods. Aalsmeer flower auction is an example of multi-unit auctions for indivisible objects. Unlike single-unit auctions where bidders need to propose a price higher than any other's bid for the win, a bidder does not need to outbid her opponent's highest bid in order to win her first unit in a multi-unit auction. On the contrary, a bidder will get her first unit as long as her highest bid is higher than her opponent's smallest bid when there are two bidders competing for two units. The mechanism underlying unit assignment mentioned above, is the main difference between multi-unit and single-unit auctions for indivisible goods. Complications of multi-unit auctions arise not only because we need to solve multiple optimal bidding functions at the same time but also because bidders will have incentives to decrease higher bids and increase their lower bids since all bidders understand that their higher bids are competing with opponents' lower bids and vice versa. And such behaviours usually lead to inefficient allocations in terms of auction results.

We will be looking at a particular version of pay-as-bid multi-unit auction by making the following assumptions: two identical and indivisible units are being sold and two ax-ante identical bidders with multi-unit demand are participating the auction; bidders' type spaces are binary with diminishing marginal valuations;

bidders have incomplete information about each other's types. Bidders are risk-neutral and only care about monetary payoff. To be more precise, we focus on case where "high" type of bidders has private valuations $\bar{v} = (\bar{v}_1, \bar{v}_2)$ and "low" type of bidders have private valuations $\underline{v} = (\underline{v}_1, \underline{v}_2)$, with value ordering $\bar{v}_1 > \underline{v}_1 > \underline{v}_2 > \bar{v}_2 \geq 0$. We will also report mixed strategy equilibria in cases where ranking for private valuations is $\bar{v}_1 > \underline{v}_1 > \bar{v}_2 > \underline{v}_2 \geq 0$ or $\bar{v}_1 > \bar{v}_2 > \underline{v}_1 > \underline{v}_2 \geq 0$ but results in those cases are far less complicated. We further assume bidders have additive valuation, meaning each bidder's value for the two units as a whole is simply the summation of marginal values of the two units. So we will refer to result where ordering for private valuations is $\bar{v}_1 > \underline{v}_1 > \underline{v}_2 > \bar{v}_2$ our **main results**. Both bidders have a common prior that a low type opponent will appear with probability $p \in [0, 1]$ and a high type opponent will appear with probability $1 - p$.

We will construct symmetric mixed strategy equilibrium for these multi-unit auctions. Our main results are that when high type has marginal valuations $\bar{v} = (\bar{v}_1, \bar{v}_2)$ with value ordering $\bar{v}_1 > \underline{v}_1 > \underline{v}_2 > \bar{v}_2 \geq 0$:

1. as long as p is not too large (i.e. $p < \frac{\underline{v}_2}{\underline{v}_1}$), low type bidders' equilibrium bids are generally perfectly correlated (with a few exceptions);

2. when p is very small (i.e. $p < \frac{\underline{v}_2}{\bar{v}_1 + \underline{v}_2}$), we expect high type to put atoms at the lower bound of distribution of her first bid;

3. when p is large enough (i.e. $p > \frac{\underline{v}_2}{\underline{v}_1}$), we find equilibria where distributions of first bid of both high and low types are degenerate on $\underline{v}_2$ while second bid of low type is mixing strictly below $\underline{v}_2$.

One feature of our equilibrium result is that we always have a functional (conditionally deterministic) relationship between two bids of low type. In most cases, low type will be bidding identical bids so the functional relationship is identical function. But we still have a few exceptions where two bids from low type are distinct but connected by an increasing and differentiable function. Since our results are mixed strategy equilibrium, overlapping of support for bids of low and high types will be inevitable. Identical bids from bidders, together with overlapping of support of high and low type leads to the next feature of our results: equilibrium allocations tend to be inefficient. Intuitively speaking, inefficiency arises from the fact that bidders understand their higher bids are competing with opponents' lower bids and they will accordingly make their first bids lower in exchange for higher net payoff. Given that bidders understand they will face lower higher bids from opponents, they will respond by bidding higher second bids for a better chance of winning. Efficiency is guaranteed in our equilibrium when $p > \frac{\underline{v}_2}{\underline{v}_1}$ where higher bids are equivalent to $\underline{v}_2$, the marginal valuation of second unit of low type. So low type will be able to not get any positive expected payoff from her second bid and second bids of low type will mix by distributions aggressively enough to prevent first bids from deviating.

Maskin and Riley (1985) studied a single-unit first price-auction with private valuation where high type had valuation v_H and low type had valuation $v_L < v_H$. Low type bidders would bid their private valuations and high type bidder would use mixed strategy by randomizing over an interval between v_L and v_H. We treat this single-unit private value model as the single-unit benchmark to our model since we have binary types of bidders as well and we will also report mixed strategy equilibria randomizing above the smallest marginal valuation $\bar{v}_2$. The equilibrium strategy implies that our benchmark scenario will achieve efficiency.

Simultaneous auctions, where multiple single-unit auctions are run simultaneously, are comparable to multi-unit auctions. Szentes (2007) and Szentes and Rosenthal (2003) studied two identical bidders with three and two objects simultaneous auctions respectively. Both auctions were complete information auctions with discrete valuations where bidders had multi-unit demands. They also allowed for complementarities (super-additive) or substitutes (sub-additive) among objects for the bidders, while our model only consider additive valuation. Szentes (2007) established conditions for symmetric mixed strategy equilibrium when goods are substitutes or complements. Szentes and Rosenthal (2003) found symmetric mixed strategy equilibrium, which was a probability measure with support being surfaces of tetrahedron describing combinations of equilibrium bids. Results in those scenarios were not necessarily efficient either since overlapping of support is inevitable when symmetric bidders are bidding the same strategy. Gentry, Komarovaz and Schiraldi (2020) studied empirical evidence of synergies in pay-as-bid simultaneous auctions. They modeled simultaneous auctions of heterogeneous objects with private valuations in Michigan Department of

Transportation highway procurement auctions, and their estimation found evidence of cost reduction for highly complementary projects while increment in cost on the other end of complementarities.

It is easy to find analogies of the 3 most frequently used forms of multi-unit auctions in single-unit settings. Uniform-price auction in the multi-unit setting is analogous to second-price auction in single-unit setting where winners pay the highest rejected price as their prices, and the first-price auction in multi-unit realm is usually called pay-as-bid auction or discriminatory auction. Vickrey auction and 2nd-price auction are identical for single unit auctions but are distinct when there are multi-units. Ausbel et al. (2014) solved equilibrium strategy for uniform-price auction, pay-as-bid auctions and Vickrey auctions with divisible goods when demand is constant or downward sloping. They also compared efficiency (and revenue) of pay-as-bid and uniform-price auctions with private and interdependent valuations under many assumptions. They found conditions for pay-as-bid auctions or uniform-price auction to achieve efficiency with perfectly divisible goods and constant marginal valuations, although they also established in general ranking in terms of efficiency was ambiguous under constant marginal valuations. Ausbel et al. (2014) found that with diminishing linear demand and increasing linear supply, expected revenue from linear equilibrium of pay-as-bid auctions were strictly higher than that of uniform-price auctions, but none was able to achieve efficiency.

Branco (1996) showed that deterministic mechanism (i.e. sellers announced that she would implement a specific allocation for sure) was efficient for multi-unit demand pay-as-bid auction where (asymmetric) bidders with private and interdependent valuations were competing for homogeneous indivisible objects. Branco (1996) also proposed conditions (i.e. required minimum bids for kth unit and bid monotonically w.r.t. signals) for some common single-demand auctions (e.g. pay-as-bid, uniform price and sequential auctions) to be efficient by restricting only to homogeneous bidders. Engelbrecht-Wiggans and Kahn (1998) studied a pay-as-bid auction similar to our set-up. They assumed bidders with diminishing marginal valuations competed for two objects in a pay-as-bid multi-unit auction as well. They proposed a system of differential equations derived from first order conditions from expected payment as equilibrium bids and constructed an example of pure strategy equilibrium by using a specific marginal distribution, where the optimal bid is a function of combinations of valuations. Engelbrecht-Wiggans and Kahn (1998) established the existence of both pooling and separating equilibrium in multi-unit auction, where pooling equilibrium describes the behaviour that one bidder is bidding identically for both bids while separating equilibrium is that one bids differently. Their paper differed from ours by the following aspects: they assumed bidders' valuation come from atomless distributions while we assumed discrete distribution with binary types of bidders. Their results were more of a characterization of equilibrium properties since they only showed that there will be positive probability that the auction ends in a pooling equilibrium without solving the general model. Anwar (2007) extended the affiliated model [1] from Milgrom and Weber (1981) to multi-unit demand environment. Anwar (2007) assumed that a bidder's valuation is a non-decreasing function of her own private information about the object, the highest information from other bidders and an additional common signal about the object. The multi-unit auction studied in Anwar (2007) is competition for $k \geq 2$ objects. Anwar (2007) solved the unique pure strategy equilibrium where bidders would bid identical bids for all objects with a simplification by restricting to case of constant marginal valuations.

One characteristic of our findings, bids from low type are identical (i.e. conditionally deterministic), can be found in literature. We can see pooling equilibrium in multi-unit auctions from both Engelbrecht-Wiggans and Kahn (1998) and Anwar (2007) as mentioned in the previous paragraphs. Empirical evidence where bidders tend to bid identically can also be found. Hortaçsu and McAdams (2010) studied bidding behaviour from Turkish Treasury auction market and modelled the auction as multi-unit auction with indivisible but identical objects. They found that bidders submitted bids as step-functions, indicating that bidders used identical prices for certain ranges of quantities. Cassola, Hortaçsu and Kastl (2013) also found out that bidders would bid by similar step-functions when studying European banks' demand for short-term funds before and after the 2007 subprime market crisis, although their model is to study multi-unit auction with divisible objects.

We may also be able to derive other implications from the pooling equilibrium.

[1] Each bidder has private information that is positively correlated with the bidder's value of the good.

Ausbel et al. (2014) mentioned differential bid-shading where bidders shaded bids differently across units. Given that our model found pooling equilibrium for low type, we can treat our pooling equilibrium as bidders shaded more for higher marginal valuations. Besides the pooling equilibrium which prevails in majority of our results, we are able to find some separating equilibrium for low type for some small range of p. We get separating equilibrium by assuming that it is the interior solution to maximization problem where low type maximizes her expected payoff from second bid given any first bid in the support of joint bids, while we may interpret pooling equilibrium as boundary solutions since first and second bids in such equilibrium are at their extreme. We can conclude for separating equilibrium that first and second bids of low type are related by an increasing function, which is strictly smaller than the identity function.

Establishment of revenue equivalence theorem has always been a topic discussed in auction literature. In fact we can compare revenue generated from our pay-as-bid auction and a hypothetical uniform-price auction, where common monetary payment for each unit is the highest losing bid. The pay-as-bid auction will generate positive revenue by its own rule: the monetary payment for each unit in a pay-as-bid auction is the winning price for that unit and it is highly unlikely for bidders to win a unit by bidding zero. And accordingly we should expect winning prices and expected revenue in pay-as-bid auction to be strictly positive. The uniform-price auction, on the other hand, has an obvious equilibrium where bidders are bidding truthfully for their first units and 0 for their second units. Such a bidding strategy leads to zero revenue since the highest losing price is always 0. So without any computation we can conclude that pay-as-bid auction will dominate uniform-price auction in terms of revenue given our multi-unit setting and accordingly we do not have a version of revenue equivalence. Besides, we can also compare expected revenue from our pay-as-bid auction with Vickrey auctions, where a bidder i who wins k_i units will pay the highest losing k_i bids from her opponent. Truthful bidding is an equilibrium for Vickrey auction and each bidder will win one unit and pay the marginal valuation of her opponent's second unit. Our comparison indicates an ambiguous relationship between expected revenue of pay-as-bid and Vickrey auctions: Vickrey auction generates higher expected revenue when p is relatively low but pay-as-bid auction will have higher revenue when p is high.

Our comparison above fits with consensus from literature: in multi-unit setting, the revenue equivalence theorem prevalent from single-unit environment does not hold in general. Revenue equivalence is possible when assignment from different auctions turn out to be identical, with Engelbrecht-Wiggans (1988) being an good example showing the result. And when different formats of auctions lead to different assignment, revenue equivalence does not apply. Theoretical and empirical literature draw different conclusions regarding revenue ranking among different auction formats. Tenorio (1999) studied a two-agent three-unit multi-unit auction where capacity of demand of identical bidders may be either two or three. Tenorio (1999) proved revenue generated from different formats of auctions were equivalent as long as bidders have the same units of demand, but revenue from auctions where bidders' demand is three-unit is higher than that where bidders' demand is two-unit. What's more, as mentioned in the previous paragraphs, Ausbel et al. (2014) showed that revenue ranking between uniform-price and pay-as-bid auctions are ambiguous: when demand is flat they provided examples where each type of auction dominated. When demand is downward sloping, they found that pay-as-bid auctions would dominate in terms of revenue. Hortaçsu and McAdams (2010) conducted counter-factual simulation to compute a hypothetical revenue if the auction were switched to the format of uniform-price. But they could not reject the hypothesis that the two formats (pays-as-bid and uniform-price) generated same level of revenue.

2.2 Example

We will illustrate one numerical example of our results in this section before showing any theoretical results. The auction we look into is a multi-unit auction with two identical units. Any bidder will be a high or low type with probability $\frac{1}{4}$ or $\frac{3}{4}$ respectively. We suppose high type's marginal valuation for the two units is $(3, 0)$ while low type's marginal valuation is $(2, 1)$. The format of the auction is pay-as-bid, meaning that the monetary payment for each unit is the winning price for that unit. We normalize high type's marginal valuation to be zero so that high type's bid will only be one non-negative price. On the contrary, low type will be submitting two non-negative prices.

We report a mixed strategy equilibrium where a high type will be bidding by CDF $F_H(x) = \frac{x^2}{(1-x)(3-x)}$ on support $[0, \frac{3}{4}]$. In the meanwhile, a low type will be bidding her two bids identically and mixing by a common CDF $G_L(x) = \frac{3x}{3-x}$ over the same support. A good way to understand this equilibrium is to look at low type's expected payoff from her higher bid b_{l1}, which is $\frac{1}{4}\frac{3b_{l1}}{3-b_{l1}}(2-b_{l1}) + \frac{3}{4}(2-b_{l1}) = \frac{9(2-b_{l1})}{4(3-b_{l1})}$. It is not hard to notice that derivative with respect to b_{l1} is negative, implying that the low type should pick the smallest feasible price as her higher bid, and low type's higher bid should be no smaller than her lower bid. So a low type will be bidding identically due to monotonicity of expected payoff from her marginal bid.

This numerical example highlights the main findings of our theorems: low type will be bidding identically for her two bids. Another feature arises from the bidding strategies is overlapping of support. With bidders mixing their bids in identical support, it is likely that our result leads to inefficient allocation of units. A low type may win both the units while efficient allocation is always to make each bidder get one object regardless of type. Additionally, there are some cases where over some region the low type may not choose to bid identically, but there will be a conditionally deterministic relationship between bids of low type.

2.3 Model

There are two identical indivisible objects being auctioned off. Each of two bidders, $i = 1, 2$, demand up to two units of the object. In particular, bidder i's valuations are given by (v_{i1}, v_{i2}), where v_{i1} indicates the bidder's value of the first unit obtained and v_{i2} indicates the bidder's value of the second unit obtained. Note $v_{i1} > v_{i2} \geq 0$.

Bidders can be one of two types: high or low. The high type has valuations $\overline{v} = (\overline{v}_1, \overline{v}_2)$ and the low type has valuations $\underline{v} = (\underline{v}_1, \underline{v}_2)$. Note, $\overline{v}_1 > \underline{v}_1 > \underline{v}_2 > \overline{v}_2 \geq 0$. So the high type has high-variance in their valuations and the low type has low-variance in their valuations. Let $V = \{\overline{v}, \underline{v}\}$ be the set of possible valuations (or types). The bidders' types are drawn independently from a common prior. And we denote $p \in (0,1)$ for the probability that bidder i is the low type.

The objects are auctioned off in a multi-unit pay-as-bid auction: the bidders simultaneously submit bids for both units of the object. In particular, bidder i's bid is given by a vector $b_i = (b_{i1}, b_{i2})$, where $b_{i1} \geq b_{i2} \geq 0$. b_{i1} is bidder i's **first bid** (i.e., bid for the first unit) and b_{i2} is her **second bid** (i.e., bid for the second unit). So, b_{i1} denotes i's payment if she only gets one unit of the object and $b_{i1} + b_{i2}$ represents her payment if she gets both units of the object. Furthermore, we let $\mathcal{B}_i$ to be the set of possible bids of i, i.e., $\mathcal{B}_i = \{(b_{i1}, b_{i2}) : (b_{i1}, b_{i2}) \in R^2_+, b_{i1} \geq b_{i2}\}$.

The winner of the auction is determined by the profile of bids $(b_{11}, b_{12}, b_{21}, b_{22})$. If $b_{i1} > b_{-i1}$, the allocation is determined by comparing bidder i's second bid b_{i2} to bidder $-i$'s first bid b_{-i1}. Each bidder wins exactly one unit if $b_{i1} > b_{-i1}$ and $b_{-i1} > b_{i2}$. Bidder i wins both units if $b_{i1} > b_{-i1}$ and $b_{-i1} < b_{i2}$. Moreover, if $b_{i1} > b_{-i1}$ and $b_{-i1} = b_{i2}$, bidder i wins the first with probability one and the players split the second unit with $.5 : .5$ probability. Finally, if $b_{11} = b_{21}$ then each bidder i wins exactly one unit of the object.

The payoffs depend on the profile of bids and the type of the bidder. In particular, the ex-post payoff function of a bidder of type (v_{i1}, v_{i2}) is given by

$$\Pi_i(b_{i1}, b_{i2}, b_{-i1}, b_{-i2} \mid v_{i1}, v_{i2}) = \begin{cases} v_{i1} + v_{i2} - b_{i1} - b_{i2} & \text{if } b_{i1} > b_{-i1} \text{ and } b_{i2} > b_{-i1} \\ v_{i1} - b_{i1} + \frac{1}{2}(v_{i2} - b_{i2}) & \text{if } b_{i1} > b_{-i1} \text{ and } b_{i2} = b_{-i1} \\ \frac{1}{2}(v_{i1} - b_{i1}) & \text{if } b_{-i1} > b_{i1} \text{ and } b_{-i2} = b_{i1} \\ v_{i1} - b_{i1} & \text{if } b_{i1} = b_{-i1} \text{ or} \\ & b_{i1} > b_{-i2} \text{ and } b_{-i1} > b_{i2} \\ 0 & \text{otherwise} \end{cases}$$

Let $\Delta(\mathcal{B}_i)$ be the set of probability distributions over $\mathcal{B}_i$. A strategy for bidder i is a mapping $\sigma_i : V \to \Delta(\mathcal{B}_i)$. So, $\sigma_i(v_i)$ is bidder i's mixed bid when she is of type $v_i = (v_{i1}, v_{i2})$. It will be convenient to denote distribution of opponent's mixed bid (b_{-i1}, b_{-i2}) as $\mathbb{P}$ and to write $\mathbb{P}_{\sigma_i(v_i)}$ as the distribution induced by mixed bid $\sigma_i(v_i)$.

Write $\mathbb{E}_{\mathbb{P}}[\Pi_i(b_{i1}, b_{i2}, b_{-i1}, b_{-i2} \mid v_i)]$ for bidder i's expected payoff from (b_{i1}, b_{i2}) given that her value is v_i and the distribution induced by (b_{-i1}, b_{-i2}) is $\mathbb{P}$, i.e., $\mathbb{E}_{\mathbb{P}}[\Pi_i(b_{i1}, b_{i2}, b_{-i1}, b_{-i2} \mid v_i)] = \mathbb{P}(b_{-i2} < b_{i1}, b_{-i1} < b_{i2})(v_{i1} + v_{i2} - b_{i1} - b_{i2}) + \mathbb{P}(b_{-i1} < b_{i1}, b_{-i1} = b_{i2})(v_{i1} - b_{i1} + \frac{1}{2}(v_{i2} - b_{i2})) + \mathbb{P}(b_{-i1} > b_{i1}, b_{-i2} = b_{i1})\frac{1}{2}(v_{i1} - b_{i1}) + (1 - \mathbb{P}(b_{-i2} < b_{i1}, b_{-i1} < b_{i2}) - \mathbb{P}(b_{-i1} < b_{i1}, b_{-i1} = b_{i2}) - \mathbb{P}(b_{-i1} > b_{i1}, b_{-i2} = b_{i1}))(v_{i1} - b_{i1})$.

With this, bidder i's interim expected payoffs from bidding (b_{i1}, b_{i2}) given that her value is v_i and her opponent chooses σ_{-i} is given by $\pi_i(b_{i1}, b_{i2}, \sigma_{-i} \mid v_i) = p\mathbb{E}_{\mathbb{P}_{\sigma_{-i}(\underline{v})}}[\Pi_i(b_{i1}, b_{i2}, b_{-i1}, b_{-i2} \mid v_i)] + (1-p)\mathbb{E}_{\mathbb{P}_{\sigma_{-i}(\overline{v})}}[\Pi_i(b_{i1}, b_{i2}, b_{-i1}, b_{-i2} \mid v_i)]$.

The paper restricts to **symmetric** Bayesian Nash equilibria, (σ_1^*, σ_2^*). So, we always have $\sigma_1^* = \sigma_2^*$ and, secondly for each i and each $v_i \in V$, $\sigma_i^*(v_i)$ maximizes $\pi_i(b_{i1}, b_{i2}, \sigma_{-i}^* \mid v_i)$.

2.4 Preliminary Results

2.4.1 Separation of Marginal Bidding Distributions

Consider a bidder i in the auction, who given his type, bids (b_{i1}, b_{i2}) with $b_{i1} \geq b_{i2} \geq 0$. Suppose (b_{-i1}, b_{-i2}) are her opponent's bids with $b_{-i1} \geq b_{-i2} \geq 0$. Recall that we let $\mathbb{P}$ be the distribution induced by (b_{-i1}, b_{-i2}) in definition of $\mathbb{E}_{\mathbb{P}}[\Pi_i(b_{i1}, b_{i2}, b_{-i1}, b_{-i2} \mid v_i)]$. We invent another notations with B_{i1}, B_{i2} to be the marginal CDFs for bids b_{i1}, b_{i2} respectively, i.e. $B_{-i1}(x) = \mathbb{P}(b_{-i1} \leq x)$ and $B_{-i2}(y) = \mathbb{P}(b_{-i2} \leq y)$. We will show in later subsections (without invoking result in this subsection) that tie happens with zero probability when $p < \frac{\underline{v}_2}{\underline{v}_1}$ or upper bounds of support of distributions are below $\underline{v}_2$. And when both type bids $\underline{v}_2$ with $p > \frac{\underline{v}_2}{\underline{v}_1}$, there is no tie since assignment rule will simply let each bidder get one unit. So it is safe for us only to care about events $\{b_{i1} > b_{-i1}, b_{i2} > b_{-i1}\}$ and $\{b_{i1} > b_{-i2}, b_{-i1} > b_{i2}\}$, since all other events from our definition of ex-post payoff are involved in ties and will be of zero probability.

We have simplified $\mathbb{E}_{\mathbb{P}}[\Pi_i(b_{i1}, b_{i2}, b_{-i1}, b_{-i2} \mid v_i)] = \mathbb{P}(b_{-i2} \leq b_{i1}, b_{-i1} \leq b_{i2})(v_{i1} + v_{i2} - b_{i1} - b_{i2}) + \mathbb{P} \leq b_{i1}, b_{-i1} > b_{i2})(v_{i1} - b_{i1})$. Terms $\mathbb{P}(b_{-i2} \leq b_{i1}, b_{-i1} \leq b_{i2})$ and $\mathbb{P}(b_{-i2} \leq b_{i1}, b_{-i1} > b_{i2})$ are probabilities when one bidder wins exactly 2 and 1 units, which associates with the joint distribution of opponent's bids. Notice also that by arguing ties happen at zero probability, we are free to interchange between weak and strict inequalities for expressions in the probability notations.

If we want to look for equilibrium strategies, we could try to use first order approaches. But first order partial derivative on the joint distribution function will further complicate the computational process [2]. But if we denote $B_{-i1}(x) = \mathbb{P}(b_{-i1} \leq x)$ and $B_{-i2}(y) = \mathbb{P}(b_{-i2} \leq y)$, the following lemma essentially shows that instead of focusing on joint distributions we are able to simplify our computation by using only marginal distributions B_{-i1} and B_{-i2} in the computation of expected payoff.

Lemma II.1
$\mathbb{E}_{\mathbb{P}}[\Pi_i(b_{i1}, b_{i2}, b_{-i1}, b_{-i2} \mid v_i)] = B_{-i1}(b_{i2})(v_{i2} - b_{i2}) + B_{-i2}(b_{i1})(v_{i1} - b_{i1})$.

Proof. Expected payoff of bidding $b_{i1} \geq b_{i2}$ is $\mathbb{E}_{\mathbb{P}}[\Pi_i(b_{i1}, b_{i2}, b_{-i1}, b_{-i2} \mid v_i)] =$
$\mathbb{P}(b_{-i2} \leq b_{i1}, b_{-i1} \leq b_{i2})(v_{i1} + v_{i2} - b_{i1} - b_{i2}) + \mathbb{P}(b_{-i2} \leq b_{i1}, b_{-i1} > b_{i2})(v_{i1} - b_{i1})$.
Note that
$\mathbb{P}(b_{-i2} \leq b_{i1}, b_{-i1} \leq b_{i2}) = \mathbb{P}((b_{-i2} \leq b_{i1}) \cap (b_{-i1} \leq b_{i2})) = \mathbb{P}(b_{-i1} \leq b_{i2})) = B_{-i1}(b_{i2})$
by the ordering of b_{i1}, b_{i2} and b_{-i1}, b_{-i2}.

$\mathbb{P}(b_{-i2} \leq b_{i1}, b_{-i1} > b_{i2}) = \mathbb{P}((b_{-i2} \leq b_{i1}) \cap (b_{-i1} > b_{i2}))$
$= \mathbb{P}(b_{-i2} \leq b_{i1}) - \mathbb{P}((b_{-i2} \leq b_{i1}) \cap (b_{-i1} \leq b_{i2}))$ by Carathéodory's criterion. And it can be simplified to $\mathbb{P}(b_{-i2} \leq b_{i1}, b_{-i1} > b_{i2}) = \mathbb{P}(b_{-i2} \leq b_{i1}) - \mathbb{P}(b_{-i1} \leq b_{i2})$ or equivalently $B_{-i2}(b_{i1}) - B_{-i1}(b_{i2})$.

So we can write the expected payoff as
$\pi_i = B_{i1}(b_{-i2})(v_{i1} + v_{i2} - b_{i1} - b_{i2}) + (B_{-i2}(b_{i1}) - B_{-i1}(b_{i2}))(v_{i1} - b_{i1})$
$= B_{-i1}(b_{i2})(v_{i2} - b_{i2}) + B_{-i2}(b_{i1})(v_{i1} - b_{i1})$. ■

Implication of this lemma is that in the multi-unit auction, for any bidder, her second bid is competing with her opponent's first bid and vice versa.

[2] Actually $\frac{\partial}{\partial x}F_{X,Y}(x,y) = \int\limits_{-\infty}^{y} f_{X,Y}(x,t)dt = \int\limits_{-\infty}^{y} f_{Y|X}(t|x)f_X(x)dt$
$= \int\limits_{-\infty}^{y} f_{Y|X}(t|x)dt \times f_X(x) = \mathbb{P}(Y \leqslant y|X = x)f_X(x)$

2.4.2 Second Bid From High Type

In the proof of lemma II.1, we are assuming no ties happen with positive probability. And we will argue that it is safe to make such an assumption by showing several results regarding atoms on distributions. But before we do the proof, we can first simplify our analysis by showing second bid of low type will never win in equilibrium. Before we do the proof, we can first simplify our analysis by showing second bid of low type will never win in equilibrium. With lemma II.1 established, we always suppose F_{H1}, F_{H2} are marginal distributions of high type's first and second bids respectively while G_{L1}, G_{L2} are marginal distributions of low type's first and second bids in the following parts.

Theorem II.1 *For any equilibrium distribution, a high type will win at most 1 object.*

Proof. We will show an equivalent statement in order to prove this theorem: no type will put lower bound of first bids lower than $\bar{v}_2$. So second bid of high type will not outbid any first bid and accordingly high type will at most win one object.

If second bid of high type outbids another bid, it must be that at least one of high and low types is putting positive probability on $\bar{v}_2$ or smaller values on support of F_{H1} or G_{L1}. Without loss of generality, we assume F_{H1} is putting positive probability. If a high type is using $(\bar{v}_2^+, \bar{v}_2)$ as her two bids, her expected payoff will be no smaller than $(1-p)(\bar{v}_1 - \bar{v}_2)$, since $\bar{v}_2^+$ will definitely outbid second bid of high type. And accordingly to support bids lower than $\bar{v}_2$, the expected payoff from bids in that region must be no smaller than $(1-p)(\bar{v}_1 - \bar{v}_2)$. In particular, high type's expected payoff from bidding at exactly her lower bounds, which are no greater than $\bar{v}_2$ in this scenario, should be no smaller than $(1-p)(\bar{v}_1 - \bar{v}_2) > 0$. If a high type gets positive payoff by breaking ties at $\bar{v}_2$, she will have incentive to deviate to bid slightly higher than $\bar{v}_2$ and win higher payoff. For atoms at values strictly lower than $\bar{v}_2$ and atomless distributions, there are two sources of this positive payoff for high type:

1. When lower bounds of F_{H1}, F_{H2} does not coincide and lower bound of F_{H2} is lower than $\bar{v}_2$, lowest first bid of high type can outbid second bid of high type with positive probability. But this indicates that

(a) When lower bound of support from G_{L1} is no smaller than that from F_{H2}, high type will deviate her lower bound of F_{H2} to higher values for strictly higher payoff since the current lower bound for F_{H2} is not able to outbid any first bids.

(b) When lower bound of support from G_{L1} is smaller than that from F_{H2}, low type will deviate to higher lower bounds by a similar reason since bidding at the current lower bound will not outbid any bids.

2. When lower bounds of F_{H1}, F_{H2} coincide and lower bound of F_{H2} is lower than $\bar{v}_2$, high type may get her positive payoff at her lowest bids

   (a) by outbidding bids of low type with positive probability. But this means low type will deviate her lower bounds to values no lower than that of high type.

   (b) if both F_{H1}, F_{H2} put atoms at the lower bound of their supports. We argue this arrangement of distributions is not an equilibrium distribution since high type will have incentive to raise lower bound to break the tie and get strictly higher payoff.

   (c) or if only F_{H2} has an atom at lower bound. But high type will move the atom at bottom of support from F_{H2} to higher values since by bidding at the atom second bid of high type will outbid first bids of high and low type with zero probability.

So we conclude that there will be no equilibrium when first bid of high type is lower than $\bar{v}_2$ or F_{H1} has an atom at $\bar{v}_2$. For low type, bids no greater than $\bar{v}_2$ will be dominated by $(\bar{v}_2^+, \bar{v}_2^+)$ by a similar reason. ■

The intuition is clear: with our set-up, marginal valuation of second good of high type is the lowest. So first bid of both types will have strong incentive to bid at least $\bar{v}_2$, which will guarantee a positive expected payoff as long as there is positive probability of facing high-type opponents. This behaviour will incentivize second bid from low type to put zero probability on values below $\bar{v}_2$ because otherwise she will be "wasting" probability on a unwinnable range. Such a theorem is in consistent with with our single-unit benchmark (Maskin and Riley, 1985) where high type is randomizing between v_L and v_H, which makes it impossible for low type to win.

With theorem III.1 established, we will normalize $\bar{v}_2 = 0$ to simplify our analysis. With our existing tie-breaking rules, we will encounter several interesting scenarios: when one bidder is bidding $(0, 0)$, a high type can get 1.5 objects by bidding one positive bid and one zero bid. But high type's marginal valuation toward second object is 0 so we want some new rules to get rid of the possibility that high type will get more than one object. We can add a few new auction rules besides the existing tie-breaking rules. And we will call the following rules assumption II.1:

Assumption II.1

1. *Bidding* $(0, 0)$ *is not allowed;*
2. *As long as some bidder submits a bid containing* 0*, she can get at most one object;*

The first rule requires a bidder to bid either a single zero or at least one positive bid. The second rule has two implications: high type will not get an extra 0.5 object by bidding one positive price and one zero price when her opponent is bidding zero. Low type will not be bidding zero when she submits two bids since it is a weakly dominated strategy: under the new rule, by bidding zero low type is essentially giving up one bid since the only object she can win is through her first bid as her total win in the auction is capped at one. Similarly, we can conclude that low type will always bid 2 prices. If she only bids one price, she is able to get weakly better payoff by adding another bidding price as long as the new bidding price is smaller than the marginal valuation of her less favoured unit. And hence we conclude that for a low type bidding only one price is a weakly dominated strategy. So high type should just submit one bid while low type should submit two bids In all, our rules will solve the problem mentioned in the previous paragraph: when two high types meet each will get one object regardless of bids and when one low type bids zero, she is guaranteed to get one object when facing a high type.

We have a direct result from introduction of assumption II.1:

Lemma II.2 *High type may put an atom at* 0 *while low type will never put an* *atom at* 0.

The intuition for this result is that high type will automatically wins one object when facing another high type, so bidding zero means high type will get a high net surplus when she faces a high type with a trade-off of losing to low type with certainty. On the other hand, low type can always get more payoff by submitting two bids and will have incentive to do so.

2.4.3 No Ties Happen with Positive Probability

With $\bar{v}_2$, high type's marginal valuation of second object, being normalized to 0, we can do the proof mentioned in subsection 4.1 to show that no tie will happen with positive probability when $p < \frac{\underline{v}_2}{\underline{v}_1}$ or upper bounds of support of distributions are below $\underline{v}_2$. Ties may happen when high type submits $\underline{v}_2$ and low type submits $(\underline{v}_2, \underline{v}_2)$ with $p > \frac{\underline{v}_2}{\underline{v}_1}$. But we will argue that our tie-breaking rule dictates that each unit wins only 1 unit in this scenario, so the "tie" in this scenario can be trivially resolved.

Before checking atoms at positive values, we first take a look at gaps in support of marginal distributions. It turns out we can make the following conclusions regarding gaps on marginal distributions:

Lemma II.3

1. *There can be no gaps of interval on marginal distribution of second bid for low type.*
2. *If first bids of high and low types both have gaps in the support of distributions, the gaps must have intersection with zero measure.*

Proof. For the first point, suppose the gap from support of marginal distribution of second bid of low type is interval (a, b) with $a < b$. Then first bids from high and low types will put zero probability in interval (a, b) as well since bidding those values will be dominated by bidding a while holding second bid constant. First we assume low type does not put any atom at a. Mathematically speaking, we can hold second bid constant and only compare the probability of winning any unit by bidding a or $x \in (a, b)$: $\mathbb{P}(b_{-i2} \leq a, b_{-i1} \leq b_{i2}) = \mathbb{P}(b_{-i2} \leq x, b_{-i1} \leq b_{i2})$ and

$\mathbb{P}(b_{-i2} \leq a, b_{-i1} \geq b_{i2}) = \mathbb{P}(b_{-i2} \leq x, b_{-i1} \geq b_{i2})$. In a word, bidding in interval (a, b) will give the same probability of winning as bidding a when facing second bids of high and low types but one has to pay more. If low type puts an atom at a on her marginal distribution of second bid, bidders of any type will prefer to bid (a^+, x) with $x \leq a$ than bid (a, x) or any bid (b_1, b_2) in intersection of joint support of one bidder's mixed strategy and set $\{(b_1, b_2) : b_2 \leq b_1 \leq a\}$ [3]. So putting an atom at a on marginal distribution of second bid will lead to first bids of any bidder to bid no lower than a. And hence we exclude possibility of interval $[a, b)$.

If we look again at the comparison last paragraph, we know that bidding exactly at or slightly higher than b is dominated by bidding a as well: by bidding in right neighbourhood of b, when $\epsilon > 0$ is sufficiently small bidding $x \in (b, b + \epsilon)$ will give almost the same probability of winning as bidding a, but bidders have to much higher price when they win. We exclude possibility of interval $[a, b)$ (i.e. atom at $\{b\}$) for the following reason: if low type puts an atom at b on the marginal distribution of her second bid, bidders will respond by putting a gap at singleton set $\{b\}$ on marginal distributions for first bids, since bidding slightly higher than b will break the tie and generate strictly higher payoff than bidding at b. Knowing this, a bid (x, b) with $x > b$ for low type will be dominated by bidding (x, a), which means low type will not put any positive probability at b at the first place. In all we conclude that if second bid of low type is putting a positively measured gap in the support, first bids from high and low type will respond by putting a larger gap (a, b') where $b' > b$ in the support.

Knowing this, second bid of low type will not bid at b since it is dominated by bidding at a (while holding first bid constant) when she knows that distribution of first bids will respond to put a larger gap. So we conclude we can not have an equilibrium where low type puts a positively measured gap in the marginal distribution of her second bid. And hence we have our first conclusion regarding gaps in marginal distribution.

For the second point, assume that intersection of gaps in support of distributions of first bids from high and low type is interval (c, d) with $c < d$. Low type will use similar deviation method mentioned in the previous paragraphs, i.e. second bid of

[3] To support a mixed strategy equilibrium, a necessary condition would be bids in intersection of support of bids for low type and set $\{(b_1, b_2) : b_2 \leq b_1 \leq a\}$ generate identical payoff.

low type will not bid in interval (c, d) or prices slightly higher than d since all bids in that area are dominated by bidding c while holding first bid constant. Such a behaviour by low type will lead to a gap in the support of distributions of second bids from low type. ■

As long as we conclude that low type will not put any gap of interval on marginal distribution of her second bid and gaps from distributions of first bids of high and low type have zero-measured intersection, we can show some results regarding ties and atoms in the multi-item auction.

Lemma II.4 *No equilibrium distribution will put atoms at positive values smaller than upper bound of support.*

Proof. Suppose second bid of low type puts an atom at x, which is lower than the upper bound. Lemma II.3 has established that at least one from marginal distributions of first bids of high and low types will have support containing neighbourhood of x. So we can look at deviations case by case. If support of distribution for first bid of low type contains neighbourhood of x, we may consider bids (x, b) where x is her first bid and b is her second bid. She will have incentive to deviate her first bid to x^+ while holding second bid constant. Such a deviation will lead to strictly higher payoff for low type since it breaks the tie at x, where there is an atom with positive measure. Note that to support a mixed strategy equilibrium, a necessary condition would be bids in intersection of support of bids for low type and set $\{(b_1, b_2) : b_2 \leq b_1 \leq x\}$ generates identical payoff [4]. Since bid $(x, b) \in \{(b_1, b_2) : b_2 \leq b_1 \leq x\}$ is dominated by (x^+, b), bids in set $\{(b_1, b_2) : b_2 \leq b_1 \leq x\}$ will also be dominated by (x^+, b). Similarly if distribution of high type contains neighbourhood of x, it is easy to see that bidding x^+ will generate strictly higher payoff than bidding x^- or x. And accordingly by a similar reason bids lower than x will be dominated by bid x^+.

If distribution of a high type or first bid of low type puts an atom at y, which is smaller than the upper bound of bids, we will have a similar argument as the previous paragraph since lemma II.3 shows that low type will not have a positively measured gap. Suppose low type is bidding (b, y) (or (b, y^-)) as her pair of bids $(b > y)$, and we will see that deviating the second bid to y^+ generates higher payoff.

[4] (b_1, b_2) denote first and second bids for low type respectively.

When first bid of low type is already higher than y (i.e. $b > y$), we allow it to stay at b, but when first bid of low type is no greater than y we can also increase it to be y^+. The slight increment in first bid will not change payment from first bid. ■

Lemma II.5 *No equilibrium distribution will put atoms at upper bounds of support when the upper bound is smaller than $\underline{v}_2$.*

Proof. When upper bound of bids is smaller than $\underline{v}_2$ and some type chooses to put an atom at the upper bound, an obvious deviation will be bidding slightly higher than the upper bound. And by a similar reason discussed in the proof of lemma II.4, such a behaviour will generate strictly higher payoff. Note that this argument works for any p. ■

These three lemmata imply that equilibrium distributions will be atomless when upper bound of support is smaller than $\underline{v}_2$ unless high type puts an atom at 0. Since we normalize $\bar{v}_2$ to be 0, proof of lemma II.1 will only be restricted to low type. And since low type is bidding positive bids and atom only happens at 0, ties will not appear with positive probability in equations used in lemma II.1. What's more, distributions will have the same upper bound in this case since ties only happen with zero probability.

Lemma II.6 *When $p < \frac{\underline{v}_2}{\underline{v}_1}$, no equilibrium distribution will put atoms at upper bounds of support when the upper bound is equal to $\underline{v}_2$.*

Proof. With common upper bound being equivalent to $\underline{v}_2$, a high type will get $\bar{v}_1 - \underline{v}_2$ when she bids at $\underline{v}_2$ and a low type will get $\underline{v}_1 - \underline{v}_2$ when both her bids are at upper bound. Since we normalize marginal valuation of second object of high type to zero, low type will get at least $(1-p)\underline{v}_1$ by deviating first (and second) bid to slightly above 0. Given $p < \frac{\underline{v}_2}{\underline{v}_1}$, we have $(1-p)\underline{v}_1 \geq \underline{v}_1 - \underline{v}_2$. So putting upper bound of bids at $\underline{v}_2$ will not be an equilibrium strategy at the first place when $p < \frac{\underline{v}_2}{\underline{v}_1}$. ■

Given the tie-breaking rules introduced previously, these four lemmas imply that when $p < \frac{\underline{v}_2}{\underline{v}_1}$ ties will not appear with positive probability in equations used in lemma II.1: the only atom in this situation happens at 0 but low type is bidding positive bids and high type only bids one bid. Such results imply that we do not need to worry about ties in our proof of lemma II.1 when $p < \frac{\underline{v}_2}{\underline{v}_1}$.

On the other hand, when $p > \frac{\underline{v}_2}{\underline{v}_1}$, a bidder may choose to set upper bounds on $\underline{v}_2$. If one bidder puts upper bound at $\underline{v}_2$, we expect the upper bound of the other bidder's bids to be at $\underline{v}_2$ as well since otherwise this bidder will decrease her upper bound to avoid wasting probability on a range that is too high. With this observation, we consider the case where low type bidding $(\underline{v}_2, \underline{v}_2)$ again: low type's expected payoff from second bid is 0 since $\underline{v}_2 - \underline{v}_2 = 0$. If distribution of high type or first bid of low type puts non-zero probability on values lower than $\underline{v}_2$, low type will be deviating her second bid to lower values in order for strictly higher payoff. So we have to conclude that when $p > \frac{\underline{v}_2}{\underline{v}_1}$, to support bid $(\underline{v}_2, \underline{v}_2)$ for low type, first bids of low type and high type must put zero probability on values lower than $\underline{v}_2$, i.e. the atom at $\underline{v}_2$ must be of size 1 [5].

Results in the last paragraph indicate that for equation in lemma II.1, we have to consider possible ties at $\underline{v}_2$ and 0 since there may be two atoms. Low type is bidding positive bids so atom at 0 will not lead to ties. For atom at $\underline{v}_2$, our tie breaking rules will dictate each bidder to get their first object when high type bids $\underline{v}_2$ and low type bids $(\underline{v}_2, \underline{v}_2)$, which seems to be in contradiction with equations in lemma II.1. But note that low type's marginal valuation of second object is $\underline{v}_2$. Low type will not get any net surplus from her second bid in this scenario regardless of winning the second object or not. So our equation in lemma II.1 works trivially for atoms at $\underline{v}_2$.

With lemmas established in this subsection, we may conclude common upper bound for all cases.

Corollary II.1 *Equilibrium distributions should have the common upper bound.*

Proof. When upper bound is smaller than $\underline{v}_2$ or $p < \frac{\underline{v}_2}{\underline{v}_1}$, we know from lemmata in this subsection that no ties happen with positive probability. And hence any bidder can get the object with certainty by bidding at a common upper bound, while bidding beyond the upper bound only implies paying strictly higher and getting lower net payoff.

When upper bound is $\underline{v}_2$ and $p > \frac{\underline{v}_2}{\underline{v}_1}$, we know that ties happen when high type bids $\underline{v}_2$ and low type bids $(\underline{v}_2, \underline{v}_2)$. Tie breaking rules will assign each bidder one

[5]We will discuss this scenario in more detail in lemma II.8 and theorem II.10.

object in this case. If low type raises her first bid, she still gets one object but she pays more; if low type raises her both bid, she still gets two objects but she gets less net payoff Since marginal valuation of low type's second object is $\underline{v}_2$. In all, low type will not want to increase her bids. High type will not raise her bids as well since it only means she pay more as well. ■

2.4.4 Pure Strategy Equilibrium

Lemma II.7 *Bidding the smaller marginal valuation for both bids is a pure strategy equilibrium when $p = 0$ or 1.*

Proof. If $p = 0$ (or 1), bidding 0 (or $\underline{v}_2$) for both objects will be an equilibrium. Bidding at the marginal valuations of the second object guarantees each bidder exactly one object. Increasing bids will decrease net payoff: firstly it would be a strictly dominated strategy for low type to use a second bid higher than marginal valuation of that object, and secondly increasing first bid will only mean the bidder pays more for the only object she can win. Decreasing just one bid will not change the allocation but decreasing both bids will lead to 0 payoff since the highest two bids will both come from opponent. ■

We have mentioned that when $p \geq \frac{\underline{v}_2}{\underline{v}_1}$, high type may bid $\underline{v}_2$ and low type will bid $(\underline{v}_2, \underline{v}_2)$, and now we will formally show this is actually a pure strategy equilibrium:

Lemma II.8 *For $p \in (0, 1)$, there is a unique pure strategy symmetric equilibrium in our multi-item auction when $p \geq \frac{\underline{v}_2}{\underline{v}_1}$.*

Proof. First we suppose in this proof that high type is bidding non-negative (b_{h1}, b_{h2}) with $b_{h1} \geq b_{h2}$ and low type is bidding non-negative (b_{l1}, b_{l2}) with $b_{l1} \geq b_{l2}$. We propose $b_{h1} = b_{l1} = b_{l2} = \underline{v}_2$ and $b_{h2} = 0$ as the equilibrium strategy. Each type is getting one object by the current pure strategy under our tie-breaking rule. If high type increases her first bid, she still wins 1 object but she has to pay more. If she decreases her first bid she will win nothing when facing a low type. Range of p will indicate that her payoff will be (weakly) lower since $\bar{v}_1 - \underline{v}_2 \geq (1 - p)\bar{v}_1$, where the right hand side is the highest payoff high type can get by bidding lower than $\underline{v}_2$ [6]. For a low type, decreasing only one bid does not change the allocation of

[6] If a high type bids lower than $\underline{v}_2$, she can only win when facing another high type. So a high type would rather bid 0 when she is bidding below $\underline{v}_2$.

auction. Decreasing both bids will strictly lower her payoff because when she faces another low type she can not win and range of p guarantees $\underline{v}_1 - \underline{v}_2 \geq (1-p)\underline{v}_1$. Second bid of low type is exactly at the marginal valuation of her second object and hence reason to eliminate increasing bids for low type is similar to the reason used in lemma II.7.

We now move on to check uniqueness. We still assume the non-negative bids from high type are $b_{h1} \geq b_{h2}$ and bids from low type are $b_{l1} \geq b_{l2}$. If $b_{l2} = \underline{v}_2$, tie-breaking rule will predict b_{L1} will either be $\underline{v}_2$ (or slightly higher than $\underline{v}_2$ [7]). We can analyze all possible cases:

1. If $\bar{v}_1 - \underline{v}_2 \geq (1-p)\bar{v}_1$, or equivalently $p > \frac{\underline{v}_2}{\bar{v}_1}$, high type will bid $\underline{v}_2$ (or $\underline{v}_2 + \epsilon$)
   (a) if $p > \frac{\underline{v}_2}{\underline{v}_1}$, we are in the proposed equilibrium;
   (b) if $p \in (\frac{\underline{v}_2}{\bar{v}_1}, \frac{\underline{v}_2}{\underline{v}_1})$, high type is glad to bid $\underline{v}_2$ (or $\underline{v}_2 + \epsilon$) but low type will want to deviate to bid close to 0 since bidding just above 0 will give $(1-p)\underline{v}_1$, which is greater than $\underline{v}_1 - \underline{v}_2$ under this range of p. And high type will consequently deviate to just outbid low type so that high type could get an payoff close to $\bar{v}_1$.
2. If $p < \frac{\underline{v}_2}{\bar{v}_1}$, first bid of high type will be just above 0 since $(1-p)\bar{v}_1 > \bar{v}_1 - \underline{v}_2$. Low type will have incentive to decrease her second bid to just outbid high type. First bid of low type can either stay at $\underline{v}_2$ or be just above 0. The former choice generates payoff $\underline{v}_1 - \underline{v}_2 + (1-p)\underline{v}_2$ for low type while the second choice generates payoff of at least $(1-p)(\underline{v}_1 + \underline{v}_2)$. Given $p < \frac{\underline{v}_2}{\bar{v}_1} < \frac{\underline{v}_2}{\underline{v}_1}$ we have $\underline{v}_1 - \underline{v}_2 + (1-p)\underline{v}_2 \leq (1-p)(\underline{v}_1 + \underline{v}_2)$. So low type should also deviate her first bid to just above 0. It is easy to check the deviating payoff $(1-p)(\underline{v}_1 + \underline{v}_2) \geq \underline{v}_1 - \underline{v}_2$ when $p \leq \frac{2\underline{v}_2}{\underline{v}_1 + \underline{v}_2}$. But note that $\frac{2\underline{v}_2}{\underline{v}_1 + \underline{v}_2} > (\frac{\underline{v}_2}{\underline{v}_1} >)\frac{\underline{v}_2}{\bar{v}_1}$. So as long as first bid of high type is close to 0, it it optimal for low type to decrease her bids to slightly outbid high type.

Similarly, suppose $b_{l2} < \underline{v}_2$, we can still first conclude that $b_3 = b_{l2}$ or $b_{l2} + \epsilon$ because of the tie-breaking rules and we want a symmetric equilibrium. We can

[7] If a low type knows another low type is bidding $\underline{v}_2 + \epsilon$ and $\underline{v}_2$ ($\epsilon > 0$), she will respond by bidding $\underline{v}_2 + \epsilon$ and $\underline{v}_2$ because she can only win 0.5 objects on average if her first bid is $\underline{v}_2$. If a low type knows another low type is bidding $\underline{v}_2$ and $\underline{v}_2$, she will respond by bidding $\underline{v}_2$ and $\underline{v}_2$.

still do a case-by-case analysis:

1. When $p > \frac{b_{l2}}{\bar{v}_1}$, $b_{h1} = b_{l2}$ (or $b_{l2} + \epsilon$ as above) since $\bar{v}_1 - b_{l2} > (1-p)\bar{v}_1$. Low type has incentive to raise second bid (and hence her first bid) slightly to outbid b_{h1};

2. When $p < \frac{b_{l2}}{\bar{v}_1}$, b_{h1} is just above to 0. Low type will have incentive to decrease her second bid. She has 2 choices for her first bid now: either stay at b_{l2} or just be above 0. But bidding first bid at b_{l2} will generate lower payoff than bidding first bid at 0: $\underline{v}_1 - b_4 + (1-p)\underline{v}_2 < (1-p)(\underline{v}_1 + \underline{v}_2)$ since $p < \frac{b_{l2}}{\bar{v}_1} < \frac{b_{l2}}{\underline{v}_1}$. So low type will deviate her first bid to just above 0 as well. Note that $(1-p)(\underline{v}_1 + \underline{v}_2) \geq \underline{v}_1 - b_4$ when $p < \frac{\underline{v}_2 + b_{l2}}{\underline{v}_1 + \underline{v}_2}$. And it is easy to check $\frac{\underline{v}_2 + b_{l2}}{\underline{v}_1 + \underline{v}_2} > \frac{b_{l2}}{\bar{v}_1}$. So it is optimal for low type to decrease both bids when high type is bidding close to 0.

■ We will see later that this pure strategy equilibrium is just a specific case of the mixed strategy equilibria.

Remark II.2 *Note that pure strategy in this subsection is efficient since high and low type each get one object.*

2.5 Mixed Strategy Equilibrium

In this section, we will formally show the symmetric mixed strategy Bayesian Nash Equilibrium by range of p, the probability of low type. With theorem III.1, we will always assume that first bid of high type follows distribution F_H and first and second bids of low type follow distributions G_{L1} and G_{L2} respectively. High type will bid b_{h1} where $b_{h1} \geq 0$ and low type will bid by (b_{l1}, b_{l2}) with $b_{l1} \geq b_{l2} \geq 0$.

The mixed strategy equilibria will have two main features: support for bids of high is a subset of support of bids for low types (i.e. overlapping support) and low type will bid identically for two objects, where the first feature implies when high type bids low and high types will share common support. As argued in introduction, since bidders understand that their higher bids are competing with other's lower bids, they will have incentive to bid lower higher bids for higher net surplus. On the other hand, knowing first bids will be lower, bidder will be submitting higher second bids for a better chance of winning. The overlapping support and identical bids will make our mixed strategy equilibria inefficient since there will be positive probability that one low type gets both objects. An efficient allocation should let each bidder get exactly one object since we assume high type has valuation $\bar{v}_1, \bar{v}_2$ while low type has valuation $\underline{v}_1, \underline{v}_2$ with $\bar{v}_1 > \underline{v}_1 > \underline{v}_2 > \bar{v}_2$. The top two highest marginal valuations will always come from each bidder's highest valuation.

2.5.1 $\bar{v}_1 \geq \underline{v}_1 + \underline{v}_2$

For $p < \frac{\underline{v}_2}{\underline{v}_1}$, we will introduce the mixed strategy equilibria by different range of marginal valuations and by range of p.

2.5.1.1 when $p \leq \frac{\underline{v}_2}{2\bar{v}_1 - \underline{v}_1}$

We can summarize results in this subsection by a theorem:

Theorem II.3 *Suppose $\bar{v}_1 \geq \underline{v}_1 + \underline{v}_2$ and $p \leq \frac{\underline{v}_2}{2\bar{v}_1 - \underline{v}_1}$. Low type will be bidding the same price for her bids with distribution $G_{L1}(x) = G_{L2}(x) = \frac{(1-p)x}{p(\bar{v}_1 - x)}$ and high type is bidding according to distribution $F_H(x) = \frac{(\bar{v}_1 - \underline{v}_1 - \underline{v}_2 + x)x}{(\underline{v}_2 - x)(\bar{v}_1 - x)} + \frac{\underline{v}_2 - (2\bar{v}_1 - \underline{v}_1)p}{(\underline{v}_2 - x)(1-p)}$ on common support $[0, \bar{v}_1 p]$*

This theorem implies that when p is low or when a low type appears rarely, high type will focus on getting a high net payoff when she wins. And high type will accomplish this goal by putting an atom at 0. Lemma II.9 and II.10 will be dealing with equilibrium distributions of low and high type respectively:

Lemma II.9 *Low type will be bidding the same price for her bids with distribution $G_{L1}(x) = G_{L2}(x) = \frac{(1-p)x}{p(\bar{v}_1 - x)}$ on support $[0, \bar{v}_1 p]$.*

Proof. High type will be facing indifferent condition:
$(1-p)[\bar{v}_1 - b_{h1}] + p[G_{L2}(b_{h1})(\bar{v}_1 - b_{h1})] = \bar{v}_1(1-p)$. Solving the indifferent condition, we can get $G_{L2}(x) = \frac{(1-p)x}{p(\bar{v}_1 - x)}$. And $G_{L2}(x) = 1$ when $x = \bar{v}_1 p$.

With $G_{L2}(\cdot)$ calculated, we now compute low type's expected payoff of her first bid (denoted as b_{l1}): $\frac{(1-p)b_{l1}}{p(\bar{v}_1 - b_{l1})} p(\underline{v}_1 - b_{l1}) + (1-p)(\underline{v}_1 - b_{l1}) = \frac{\bar{v}_1}{\bar{v}_1 - b_{l1}}(1-p)(\underline{v}_1 - b_{l1})$.
Note that first order derivative of low type's expected payoff w.r.t. b_{l1} is negative, which implies that low type's expected payoff of her first bid is a decreasing

function. If first and second bids for low type are (b_{l1}, b_{l2}) respectively, low type should pick the smallest $b_{l1} \geq b_{l2}$. So for any given y, we must let $b_{l1} = b_{l2}$ and hence $G_{L1}(x) = G_{L2}(x) = \frac{(1-p)x}{p(\bar{v}_1 - x)}$.

We now argue that low type will not deviate to bid differently: for a mixed strategy equilibrium, a bidder will be having a fixed expected payoff for all bids that she is randomizing with. We have computed that expected payoff for first bid for low type is a decreasing function. Fixed expected payoff implies expected payoff for second bid of low type must be an increasing function. If a bidder is bidding identically by $(b_{l1}, b_{l2}) = (x, x)$ currently for any positive and real x, she will not want to deviate only one bid since derivative of expected payoff of first bid is negative and derivative of expected payoff of second bid is positive. If the bidder deviates both bids to $(b_{l1}, b_{l2}) = (z_1, z_2)$ where $z_1 > x$ and $z_2 < x$, we can treat this scenario as deviating one bid from $(b_{l1}, b_{l2}) = (z_1, z_1)$ or (z_2, z_2) to $(b'_{l1}, b'_{l2}) = (z_1, z_2)$. And hence our previous argument still works since the monotone condition for expected payoff of first and second bid will give bidder incentive to decrease her first bid and increase her second bid until they are identical. No bidders will bid higher than the upper bound since biding exactly at the upper bound means that a low type will win both objects with certainty. And hence bidding above upper bound only indicates lower payoff. ■

We call results where the low type is bidding identical bids the **perfectly correlated equilibrium**.

We now construct the distribution for high type's first bid. Our tie-breaking rules introduced in section 2 and the new auction rules discussed in subsection 3.2 guarantees high type to get one object when facing another high type. So high type may choose to put an atom at 0 for high net surplus when there are high probability that she faces another high type in the population.

Lemma II.10 *High type is bidding according to distribution* $F_H(x) = \frac{(\bar{v}_1 - \underline{v}_1 - \underline{v}_2 + x)x}{(\underline{v}_2 - x)(\bar{v}_1 - x)} + \frac{\underline{v}_2 - (2\bar{v}_1 - \underline{v}_1)p}{(\underline{v}_2 - x)(1-p)}$ *when* $p \leq \frac{\underline{v}_2}{2\bar{v}_1 - \underline{v}_1}$ *with support* $[0, p\bar{v}_1]$.

Proof. With G_{L1}, G_{L2} computed, we look at low type's indifferent condition to compute high type's distribution when $p = \frac{\underline{v}_2}{2\bar{v}_1 - \underline{v}_1}$:

$$p[G_{L2}(b_{l1})(\underline{v}_1 - b_{l1}) + G_{L1}(b_{l2})(\underline{v}_2 - b_{l2})] + (1-p)[(\underline{v}_1 - b_{l1}) + F_H(b_{l2})(\underline{v}_2 - b_{l2})] = \underline{v}_1(1-p).$$

With $b_{l1} = b_{l2}$, we have
$(1-p)F_H(b_{l1})(\underline{v}_2 - b_{l1}) = (1-p)b_{l1} - p\frac{(1-p)b_{l1}}{p(\bar{v}_1 - b_{l1})}(\underline{v}_1 + \underline{v}_2 - 2b_{l1})$
$= (1-p)b_{l1} - \frac{(1-p)b_{l1}}{(\bar{v}_1 - b_{l1})}(\underline{v}_1 + \underline{v}_2 - 2b_{l1}) = \frac{\bar{v}_1 - \underline{v}_1 - \underline{v}_2 + b_{l1}}{\bar{v}_1}(1-p)b_{l1}$. So
$F_H(x) = \frac{(\bar{v}_1 - \underline{v}_1 - \underline{v}_2 + x)x}{(\underline{v}_2 - x)(\bar{v}_1 - x)}$. When $\bar{v}_1 \geq \underline{v}_1 + \underline{v}_2$, $F_H(x)$ is always positive. $F_H(x) = 1$ when $x = \frac{\bar{v}_1 \underline{v}_2}{2\bar{v}_1 - \underline{v}_1}$. Comparing upper bounds for G_L's and F_H, we conclude that when $p = \frac{\underline{v}_2}{2\bar{v}_1 - \underline{v}_1}$ F_H is an atomless distribution.

When $p < \frac{\underline{v}_2}{2\bar{v}_1 - \underline{v}_1}$ we have to put an atom with size T on F_H. $G_{L1} = G_{L2}$ is still true since G_{L2} is computed from high type's indifferent condition. So indifferent condition for low type is
$p[G_{L2}(b_{l1})(\underline{v}_1 - b_{l1}) + G_{L1}(y)(\underline{v}_2 - b_{l2})] + (1-p)[(\underline{v}_1 - b_{l1}) + F_H(b_{l2})(\underline{v}_2 - b_{l2})] =$
$\underline{v}_1(1-p) + (1-p)T\underline{v}_2$ with $b_{l1} = b_{l2}$, we have $F_H(x) = \frac{(\bar{v}_1 - \underline{v}_1 - \underline{v}_2 + x)x}{(\underline{v}_2 - x)(\bar{v}_1 - x)} + \frac{T\underline{v}_2}{\underline{v}_2 - x}$. We need to solve T. Let $x = \bar{v}_1 p$, $1 = \frac{T\underline{v}_2}{\underline{v}_2 - \bar{v}_1 p} + \frac{(\bar{v}_1 - \underline{v}_1 - \underline{v}_2 + \bar{v}p)\bar{v}_1 p}{(\underline{v}_2 - \bar{v}p)\bar{v}_1(1-p)}$; $1 = \frac{T\underline{v}_2}{\underline{v}_2 - \bar{v}_1 p} + \frac{(\bar{v}_1 - \underline{v}_1 - \underline{v}_2 + \bar{v}_1 p)p}{(\underline{v}_2 - \bar{v}p)(1-p)}$;
$\frac{T\underline{v}_2}{\underline{v}_2 - \bar{v}_1 p} = 1 - \frac{(\bar{v}_1 - \underline{v}_1 - \underline{v}_2 + \bar{v}_1 p)p}{(\underline{v}_2 - \bar{v}_1 p)(1-p)}$; $T\underline{v}_2 = (\underline{v}_2 - \bar{v}_1 p) - \frac{(\bar{v}_1 - \underline{v}_1 - \underline{v}_2 + \bar{v}_1 p)p}{1-p}$; $T = \frac{\underline{v}_2 - (2\bar{v}_1 - \underline{v}_1)p}{\underline{v}_2(1-p)}$. So
$F_H(x) = \frac{(\bar{v}_1 - \underline{v}_1 - \underline{v}_2 + x)x}{(\underline{v}_2 - x)(\bar{v}_1 - x)} + \frac{\underline{v}_2 - (2\bar{v}_1 - \underline{v}_1)p}{(\underline{v}_2 - x)(1-p)}$. It is easy to see that when $x < p\bar{v}_1 < \underline{v}_2$, function F_H is increasing. This F_H function coincide with the distribution computed last paragraph when $p = \frac{\underline{v}_2}{2\bar{v}_1 - \underline{v}_1}$.

The high type will not bid above the upper bound since a high type can secure the object by bidding $\bar{v}_1 p$ and bidding above that value only means paying more to get the object. ■

Remark II.4 *There will be a positive probability that one low type gets both objects due to common support of mixed strategy equilibrium distributions. And the equilibrium strategy is not necessarily efficient.*

Graphical Illustration
We can illustrate the theorems in this subsection by showing plots of density functions with $\bar{v}_1 = 3, \underline{v}_1 = 2, \underline{v}_2 = 1$. Let $p = \frac{1}{5}$, and probability density functions will be with support being $[0, \frac{3}{5}]$. Note that there will be an atom of size $\frac{1}{4}$ for distribution of mixed strategy of high type at 0.

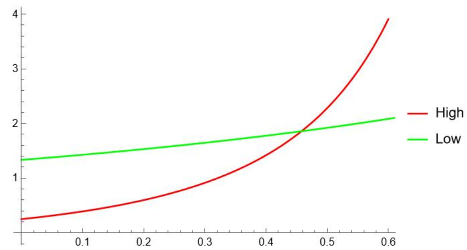

Figure 2.1: Illustration of theorem II.3

2.5.1.2 $\frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1} < p < \frac{\underline{v}_2}{\underline{v}_1}$

In this range of p, we show that high type is randomizing in interval $[a_1, a_2]$ with $0 < a_1 < a_2$ since low type is now appearing with a decent probability and high type will have to bid higher to guarantee some wins. Both bids of low type are randomized in interval $[0, a_1] \cup [a_1, a_2]$. If we denote bids from low type as (b_{l1}, b_{l2}), We will have three different indifferent conditions for low type:
$p[G_{L2}(b_{l1})(\underline{v}_1 - b_{l1}) + G_{L1}(b_{l2})(\underline{v}_2 - b_{l2})] + (1-p)(\underline{v}_1 - b_{l1}) = \underline{v}_1(1-p)$ when $b_{l2} \le b_{l1} \le a_1$, $p[G_{L2}(b_{l1})(\underline{v}_1 - b_{l1}) + G_{L1}(b_{l2})(\underline{v}_2 - b_{l2})] + (1-p)(\underline{v}_1 - b_{l1}) = \underline{v}_1(1-p)$ when $b_{l2} \le a_1 \le b_{l1}$, and
$p[G_{L2}(b_{l1})(\underline{v}_1 - b_{l1}) + G_{L1}(b_{l2})(\underline{v}_2 - b_{l2})] + (1-p)[(\underline{v}_1 - b_{l1}) + F_H(b_{l2})(\underline{v}_2 - b_{l2})] = \underline{v}_1(1-p)$ when $a_1 \le b_{l2} \le b_{l1}$.

We call support of the three indifferent conditions R_1, R_2 and R_3 respectively, i.e. $R_1 = \{(b_{l1}, b_{l2}) : b_{l2} \le b_{l1} \le a_1\}$, $R_2 = \{(b_{l1}, b_{l2}) : b_{l2} \le a_1 \le b_{l1}\}$ and $R_3 = \{(b_{l1}, b_{l2}) : a_1 \le b_{l2} \le b_{l1}\}$.

First, we summarize results in this subsection via a theorem by ranges of bids and probability low type appears in the population:

Theorem II.5 *Suppose* $\bar{v}_1 \ge \underline{v}_1 + \underline{v}_2$ *and* $\frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1} < p < \frac{\underline{v}_2}{\underline{v}_1}$.

1. *High type will bid by distribution* $F_H(x) = \frac{x}{\underline{v}_2 - x} - \frac{\bar{v}_1 p - a_2 + (1-p)x}{(1-p)(\bar{v}_1 - x)(\underline{v}_2 - x)}(\underline{v}_1 + \underline{v}_2 - 2x)$ *with support* $[a_1, a_2]$, *where* $a_2 = \frac{p\underline{v}_1 + \underline{v}_2}{2}$ *and* a_1 *solves* $F_H(a_1) = 0$.

2. *In region* $R_3 = \{(b_{l1}, b_{l2}) : a_1 \le b_{l2} \le b_{l1}\}$, *low type will bid by distributions* $G_{L1}(x) = G_{L2}(x) = \frac{p\bar{v}_1 - a_2 + (1-p)x}{p(\bar{v}_1 - x)}$ *with support* $[a_1, a_2]$.

3. *In region* $R_1 = \{(b_{l1}, b_{l2}) : b_{l2} \le b_{l1} \le a_1\}$, *letting* $p^* = \frac{2\bar{v}_1^2\underline{v}_1 - \bar{v}_1(\underline{v}_1+\underline{v}_2)^2 + \underline{v}_2(\underline{v}_1^2+\underline{v}_2^2)}{-\underline{v}_1^3 + 2\bar{v}_1^2\underline{v}_2 + \underline{v}_1^2\underline{v}_2 + \underline{v}_1\underline{v}_2^2 + \underline{v}_2^3 + 2\bar{v}_1(\underline{v}_1^2 - 2\underline{v}_1\underline{v}_2 - \underline{v}_2^2)}$

$-\frac{\sqrt{(\bar{v}_1-\underline{v}_2)^2(-2\bar{v}_1+\underline{v}_1+\underline{v}_2)^2(-\underline{v}_1^2+\underline{v}_2^2)}}{-\underline{v}_1^3+2\bar{v}_1^2\underline{v}_2+\underline{v}_2^2\underline{v}_2+\underline{v}_1\underline{v}_2^2+\underline{v}_2^3+2\bar{v}_1(\underline{v}_1^2-2\underline{v}_1\underline{v}_2-\underline{v}_2^2)}$ *and*
$C = \frac{(\bar{v}_1-\underline{v}_1)^2+\underline{v}_2^2+\underline{v}_1\underline{v}_2-2\bar{v}_1\underline{v}_2+p(\bar{v}_1\underline{v}_1-\underline{v}_1^2+\bar{v}_1^2-\bar{v}_1\underline{v}_2)}{2p(-2\bar{v}_1+\underline{v}_1+\underline{v}_2)}$
$+\frac{1}{2}\frac{(\bar{v}_1-\underline{v}_1)\sqrt{\bar{v}_1^2(1+p)^2+\underline{v}_2^2(2-2p+p^2)+2\underline{v}_1\underline{v}_2(1-p+p^2)+\underline{v}_1^2(1-2p+2p^2)-2\bar{v}_1[\underline{v}_2(2-p+p^2)+\underline{v}_1(1-p+2p^2)]}}{p(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}$

(a) *when* $\frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_2} < p < p^*$, *low type will bid the same according to* $G_{L1}(x) = G_{L2}(x) = \frac{(1-p)x}{p(\underline{v}_1+\underline{v}_2-2x)}$;

(b) *when* $p^* < p < \frac{\underline{v}_2}{\underline{v}_1}$, *low type will bid her 1st bid according to* $G_{L1}(x) = \frac{C}{\underline{v}_2-x}$ *and 2nd bid by* $G_{L2}(y) = \frac{(1-p)y-pC}{p(\underline{v}_1-y)}$ *on interval* $[a_3, a_1]$ *for* C *defined above. The two marginal distributions are related by* $G_{L1}(x) = G_{L2}(h(x))$ *where* $h(x) = \frac{Cp(\underline{v}_1+\underline{v}_2-x)}{Cp+(1-p)\underline{v}_2-(1-p)x}$. *Low type will bid identically by* $G_{L1}(x) = G_{L2}(x) = \frac{(1-p)x}{p(\underline{v}_1+\underline{v}_2-2x)}$ *in* $[0, a_3]$. $a_3 < a_1$ *solve* $\frac{C}{\underline{v}_2-x} = \frac{(1-p)x-pC}{p(\underline{v}_1-x)}$.

4. *Region* $R_2 = \{(b_{l1}, b_{l2}) : b_{l2} \leq a_1 \leq b_{l1}\}$ *has zero probability under distributions of bids for low type.*

The first two points are shown in lemma II.11. Point 3.a comes from lemma II.12 and point 3.b is dealt in lemma II.13. Point 4 is a direct result of point 2. There will be a positive probability that one low type gets both objects due to common support of mixed strategy equilibrium distributions. And the equilibrium strategy is not necessarily efficient.

Lemma II.11 *High type will bid by distribution* $F_H(x) = \frac{x}{\underline{v}_2-x} - \frac{\bar{v}_1 p-a_2+(1-p)x}{(1-p)(\bar{v}_1-x)(\underline{v}_2-x)}(\underline{v}_1+\underline{v}_2-2x)$ *with support* $[a_1, a_2]$. *In region* $R_3 = \{(b_{l1}, b_{l2}) : a_1 \leq b_{l2} \leq b_{l1}\}$, *low type will bid by distributions* $G_{L1}(x) = G_{L2}(x) = \frac{p\bar{v}_1-a_2+(1-p)x}{p(\bar{v}_1-x)}$ *with support* $[a_1, a_2]$. $a_2 = \frac{p\underline{v}_1+\underline{v}_2}{2}$ *and*
$a_1 = \frac{-\bar{v}_1+\underline{v}_1+2\underline{v}_2-\bar{v}_1 p-\underline{v}_2 p+\sqrt{(\bar{v}_1-\underline{v}_2-2\underline{v}_2+\bar{v}_1 p+\underline{v}_2 p)^2-(2-2p)(\underline{v}_1\underline{v}_2+\underline{v}_2^2-2\bar{v}_1\underline{v}_2 p+\underline{v}_1^2 p-2\bar{v}_1\underline{v}_2 p+\underline{v}_1\underline{v}_2 p)}}{1-p}$.

Proof. For high type, the indifferent condition will be $(1-p)[\bar{v}_1 - b_{h1}] + p[G_{L2}(b_{h1})(\bar{v}_1 - b_{h1})] = (\bar{v}_1 - a_1)(1-p) + p(\bar{v}_1 - a_1)G_{L2}(a_1) \iff G_{L2}(x) = \frac{1}{p(\bar{v}_1-x)}[(1-p)(x-a_1) + p(\bar{v}_1 - a_1)G_{L2}(a_1)]$. By $G_{L2}(a_2) = 1$ we have $G_{L2}(a_1) = 1 + \frac{a_1-a_2}{p(\bar{v}_1-a_1)}$. Plugging $G_{L2}(a_1)$ into high type's indifferent condition and it will become $(1-p)[\bar{v}_1 - b_{h1}] + p[G_{L2}(b_{h1})(\bar{v}_1 - b_{h1})] = \bar{v}_1 - a_2 \iff G_{L2}(x) = \frac{p\bar{v}_1-a_2+(1-p)x}{p(\bar{v}_1-x)}$.

With $G_{L2}(x) = \frac{p\bar{v}_1 - a_2 + (1-p)x}{p(\bar{v}_1 - x)}$, expected payoff for first bid of low type on region R_3 will be

$pG_{L2}(b_{l1})(\underline{v}_1 - b_{l1}) + (1-p)(\underline{v}_1 - b_{l1}) = \frac{p\bar{v}_1 - a_2 + (1-p)b_{l1}}{(\bar{v}_1 - b_{l1})}(\underline{v}_1 - b_{l1}) + (1-p)(\underline{v}_1 - b_{l1})$
$= \frac{(\bar{v}_1 p - a_2) + (1-p)\bar{v}_1}{\bar{v}_1 - b_{l1}}(\underline{v}_1 - b_{l1})$, which will have a negative derivative w.r.t b_{l1}. So for region R_3 we will still have the perfectly correlated equilibrium for low type.
Plugging $G_{L1}(x) = G_{L2}(x)$ into indifferent condition for low type on region R_3, we have $(1-p)F_H(b_{l2})(\underline{v}_2 - b_{l2}) = (1-p)b_{l2} - p\frac{\bar{v}_1 p - a_2 + (1-p)b_{l2}}{p(\bar{v}_1 - b_{l2})}(\underline{v}_1 + \underline{v}_2 - 2b_{l2})$ and hence $F_H(x) = \frac{x}{\underline{v}_2 - x} - \frac{\bar{v}_1 p - a_2 + (1-p)x}{(1-p)(\bar{v}_1 - x)(\underline{v}_2 - x)}(\underline{v}_1 + \underline{v}_2 - 2x)$. We can solve $a_2 = \frac{p\underline{v}_1 + \underline{v}_2}{2}$ by letting $F_H(a_2) = 1$ and
$a_1 = \frac{-\bar{v}_1 + \underline{v}_1 + 2\underline{v}_2 - \bar{v}_1 p - \underline{v}_2 p + \sqrt{(\bar{v}_1 - \underline{v}_1 - 2\underline{v}_2 + \bar{v}_1 p + \underline{v}_2 p)^2 - (2-2p)(\underline{v}_1\underline{v}_2 + \underline{v}_2^2 - 2\bar{v}_1\underline{v}_1 p + \underline{v}_1^2 p - 2\bar{v}_1\underline{v}_2 p + \underline{v}_1\underline{v}_2 p)}}{1-p}$ by solving $F_H(a_1) = 0$.

Computation will show that $F_H(x)$ is an increasing function when $F_H(x) \geq 0$: we need to check $F_H(x)$ is monotonically increasing when $x > a_1$, i.e.
$\frac{dF_H(x)}{dx} = \frac{(2\bar{v}_1 - \underline{v}_1 - \underline{v}_2)[2x^2 - 2(p\underline{v}_1 + \underline{v}_2)x + p\underline{v}_1(\bar{v}_1 + \underline{v}_2) - \bar{v}_1\underline{v}_2 + \underline{v}_2^2]}{2(p-1)(\underline{v}_2 - x)^2(\bar{v}_1 - x)^2} > 0$ when $x > a_1$. We need $2x^2 - 2(p\underline{v}_1 + \underline{v}_2)x + p\underline{v}_1(\bar{v}_1 + \underline{v}_2) - \bar{v}_1\underline{v}_2 + \underline{v}_2^2 < 0$ when $x > a_1$ given $p \in (0,1)$. Since $2x^2 - 2(p\underline{v}_1 + \underline{v}_2)x + p\underline{v}_1(\bar{v}_1 + \underline{v}_2) - \bar{v}_1\underline{v}_2 + \underline{v}_2^2$ is decreasing when $x < \frac{p\underline{v}_1 + \underline{v}_2}{2}$, we only need $2x^2 - 2(p\underline{v}_1 + \underline{v}_2)x + p\underline{v}_1(\bar{v}_1 + \underline{v}_2) - \bar{v}_1\underline{v}_2 + \underline{v}_2^2 < 0$ when $x = a_1$. Computation shows the condition we need is $p \in (\frac{\underline{v}_2}{2\bar{v}_1 - \underline{v}_1}, \frac{\underline{v}_2}{\underline{v}_1})$ for any $\bar{v}_1 > \underline{v}_1 > \underline{v}_2$. ■

Since we have perfectly correlated equilibrium on region R_3, we conclude that region R_2 will be at most zero-measure. We can omit R_2 and look at R_1:

Lemma II.12
When $\frac{\underline{v}_2}{2\bar{v}_1 - \underline{v}_1} < p < \frac{2\bar{v}_1^2\underline{v}_1 - \bar{v}_1(\underline{v}_1 + \underline{v}_2)^2 + \underline{v}_2(\underline{v}_1^2 + \underline{v}_2^2)}{-\underline{v}_2^3 + 2\bar{v}_1^2\underline{v}_2 + \underline{v}_1^2\underline{v}_2 + \underline{v}_1\underline{v}_2^2 + \underline{v}_1^3 + 2\bar{v}_1(\underline{v}_1^2 - 2\underline{v}_1\underline{v}_2 - \underline{v}_2^2)}$
$- \frac{\sqrt{(\bar{v}_1 - \underline{v}_2)^2(-2\bar{v}_1 + \underline{v}_1 + \underline{v}_2)^2(-\underline{v}_1^2 + \underline{v}_2^2)}}{-\underline{v}_1^3 + 2\bar{v}_1^2\underline{v}_2 + \underline{v}_1^2\underline{v}_2 + \underline{v}_1\underline{v}_2^2 + \underline{v}_1^3 + 2\bar{v}_1(\underline{v}_1^2 - 2\underline{v}_1\underline{v}_2 - \underline{v}_2^2)}$ *low type will bid according to* $G_{L1}(x) = G_{L2}(x) = \frac{(1-p)x}{p(\underline{v}_1 + \underline{v}_2 - 2x)}$ *in region* $R_1 = \{(b_{l1}, b_{l2}) : b_{l2} \leq b_{l1} \leq a_1\}$.

Proof. There is only low type with indifferent condition
$p[G_{L2}(b_{l1})(\underline{v}_1 - b_{l1}) + G_{L1}(b_{l2})(\underline{v}_2 - b_{l2})] + (1-p)(\underline{v}_1 - b_{l1}) = \underline{v}_1(1-p)$. If we assume $b_{l1} = b_{l2}$ (i.e. perfectly correlated), $G_{L1}(x) = G_{L2}(x) = \frac{(1-p)x}{p(\underline{v}_1 + \underline{v}_2 - 2x)}$. Then expected payment for first bid of low type is
$pG_{L2}(b_{l1})(\underline{v}_1 - b_{l1}) + (1-p)(\underline{v}_1 - b_{l1}) = \frac{(1-p)x(\underline{v}_1 - b_{l1})}{\underline{v}_1 + \underline{v}_2 - 2b_{l1}} + (1-p)(\underline{v}_1 - b_{l1})$
$= \frac{(1-p)(\underline{v}_1 - b_{l1})(\underline{v}_1 + \underline{v}_2 - b_{l1})}{\underline{v}_1 + \underline{v}_2 - 2b_{l1}}$. Derivative of payoff w.r.t. first bid is

$\frac{(p-1)(2b_{l1}^2-2(\underline{v}_1+\underline{v}_2)b_{l1}+\underline{v}_2(\underline{v}_1+\underline{v}_2))}{(\underline{v}_1+\underline{v}_2-2b_{l1})^2}$. On the other hand, expected payoff of second bid from low type is $pG_{L2}(b_{l2})(\underline{v}_2 - b_{l2}) = \frac{(1-p)b_{l2}(\underline{v}_2-b_{l2})}{\underline{v}_1+\underline{v}_2-2b_{l2}}$ with derivative $\frac{(1-p)(2b_{l2}^2-2(\underline{v}_1+\underline{v}_2)b_{l2}+\underline{v}_2(\underline{v}_1+\underline{v}_2))}{(\underline{v}_1+\underline{v}_2-2b_{l2})^2}$, which is exactly the opposite of derivative of expected payoff of first bid. It is straight forward to check that G functions in the lemma will coincide with G functions in the previous lemma at exactly a_1 as desired.

As long as payoff from first bid is decreasing, payoff from second bid will be increasing. The common term on numerator of those derivatives is $2x^2 - 2(\underline{v}_1 + \underline{v}_2)x + \underline{v}_2(\underline{v}_1 + \underline{v}_2)$ and $2x^2 - 2(\underline{v}_1 + \underline{v}_2)x + \underline{v}_2(\underline{v}_1 + \underline{v}_2) > 0$ is equivalent to $x < \frac{\underline{v}_1+\underline{v}_2}{2} - \frac{\sqrt{\underline{v}_1^2-\underline{v}_2^2}}{2}$. If $F_H(x) = 0$ at a_1, we want $a_1 \leq \frac{\underline{v}_1+\underline{v}_2}{2} - \frac{\sqrt{\underline{v}_1^2-\underline{v}_2^2}}{2}$ to support equilibrium bids in region R_1, which generates range of p to be $\frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1} < p < \frac{2\bar{v}_1^2\underline{v}_1-\bar{v}_1(\underline{v}_1+\underline{v}_2)^2+\underline{v}_2(\underline{v}_1^2+\underline{v}_2^2)}{-\underline{v}_1^3+2\bar{v}_1^2\underline{v}_2+\underline{v}_1^2\underline{v}_2+\underline{v}_1\underline{v}_2^2+\underline{v}_1^3+2\bar{v}_1(\underline{v}_1^2-2\underline{v}_1\underline{v}_2-\underline{v}_2^2)}$ $-\frac{\sqrt{(\bar{v}_1-\underline{v}_2)^2(-2\bar{v}_1+\underline{v}_1+\underline{v}_2)^2(-\underline{v}_1^2+\underline{v}_2^2)}}{-\underline{v}_1^3+2\bar{v}_1^2\underline{v}_2+\underline{v}_1^2\underline{v}_2+\underline{v}_1\underline{v}_2^2+\underline{v}_1^3+2\bar{v}_1(\underline{v}_1^2-2\underline{v}_1\underline{v}_2-\underline{v}_2^2)}$.

Given distributions on R_1 and R_3, we check high type will not deviate. If high type bids below a_1, she will get expected payoff $pG_{L2}(b_{h1})(\bar{v}_1 - b_{h1}) + (1-p)(\bar{v}_1 - b_{h1}) = \frac{(1-p)b_{h1}}{(\underline{v}_1+\underline{v}_2-2xb_{h1})}(\bar{v}_1 - b_{h1}) + (1-p)(\bar{v}_1 - b_{h1})$ $= \frac{(1-p)(\bar{v}_1-b_{h1})(\underline{v}_1+\underline{v}_2-b_{h1})}{\underline{v}_1+\underline{v}_2-2b_{h1}}$ with derivative $\frac{(p-1)(2b_{h1}^2-2(\underline{v}_1+\underline{v}_2)b_{h1}-(\bar{v}_1-\underline{v}_1-\underline{v}_2)(\underline{v}_1+\underline{v}_2))}{(\underline{v}_1+\underline{v}_2-2b_{h1})^2}$. We want $2b_{h1}^2 - 2(\underline{v}_1 + \underline{v}_2)b_{h1} - (\bar{v}_1 - \underline{v}_1 - \underline{v}_2)(\underline{v}_1 + \underline{v}_2)$ to be negative for a positive first order derivative. Note that $2b_{h1}^2 - 2(\underline{v}_1 + \underline{v}_2)b_{h1} - (\bar{v}_1 - \underline{v}_1 - \underline{v}_2)(\underline{v}_1 + \underline{v}_2)$ is decreasing when $b_{h1} < \frac{\underline{v}_1+\underline{v}_2}{2}$. If $\bar{v}_1 \geq \underline{v}_1 + \underline{v}_2$, we have a positive derivative: if we plug $b_{h1} = 0$ into $2b_{h1}^2 - 2(\underline{v}_1 + \underline{v}_2)b_{h1} - (\bar{v}_1 - \underline{v}_1 - \underline{v}_2)(\underline{v}_1 + \underline{v}_2)$, it will become $-(\bar{v}_1 - \underline{v}_1 - \underline{v}_2)(\underline{v}_1 + \underline{v}_2) < 0$. So we will see $(p-1)(2b_{h1}^2 - 2(\underline{v}_1 + \underline{v}_2)b_{h1} - (\bar{v}_1 - \underline{v}_1 - \underline{v}_2)(\underline{v}_1 + \underline{v}_2))$ is always positive.

We have argued in proof of lemma II.9 that a low type will not deviate within a region when she is bidding identical bids. We can now eliminate "across region deviation", which means low type deviates from bidding identically to put one bid in region R_1 and the other one in region R_3. We have established that expected payment for second bid for low type is increasing in region R_1 and R_3 and expected payment for first bid is decreasing in region R_1 and R_3. If a low type deviates to bid a (b'_{l1}, b'_{l2}) with $b'_{l2} < a_1 < b'_{l1}$, monotone condition for expected payoff for each bid will require the low type to increase her second bid and decrease first bid. ■

Remark II.6 *We assume that bids in region R_1 will be as low as 0. We can* *eliminate situations where lower bound of bids is strictly positive since we normalize $\bar{v}_2 = 0$. If lower bounds of both types' bids are strictly positive, high type will deviate to bid below b since bidding at the lower bound will generate a strictly lower payoff than bidding 0 given that distributions of bids from low type are atomless.*

In region R_1, so far we have assumed $b_{l1} = b_{l2}$ without any justification. Now we move to check availability of non-identical bids in region R_1:

Lemma II.13

When $\frac{2\bar{v}_1^2\underline{v}_1-\bar{v}_1(\underline{v}_1+\underline{v}_2)^2+\underline{v}_2(\underline{v}_1^2+\underline{v}_2^2)}{-\underline{v}_1^3+2\bar{v}_1^2\underline{v}_2+\underline{v}_1^2\underline{v}_2+\underline{v}_1\underline{v}_2^2+\underline{v}_1^3+2\bar{v}_1(\underline{v}_1^2-2\underline{v}_1\underline{v}_2-\underline{v}_2^2)} - \frac{\sqrt{(\bar{v}_1-\underline{v}_2)^2(-2\bar{v}_1+\underline{v}_1+\underline{v}_2)^2(-\underline{v}_1^2+\underline{v}_2^2)}}{-\underline{v}_1^3+2\bar{v}_1^2\underline{v}_2+\underline{v}_1^2\underline{v}_2+\underline{v}_1\underline{v}_2^2+\underline{v}_1^3+2\bar{v}_1(\underline{v}_1^2-2\underline{v}_1\underline{v}_2-\underline{v}_2^2)} < p < \frac{\underline{v}_2}{\underline{v}_1}$*, first bid of low type follows distribution* $G_{L1}(x) = \frac{C}{\underline{v}_2-x}$ *and second bid of low type follows* $G_{L2}(x) = \frac{(1-p)x-pC}{p(\underline{v}_1-x)}$ *in interval* $I = [a_3, a_4]$*,*

where $C = \frac{(\bar{v}_1-\underline{v}_1)^2+\underline{v}_2^2+\underline{v}_1\underline{v}_2-2\bar{v}_1\underline{v}_2+p(\bar{v}_1\underline{v}_1-\underline{v}_1^2+\bar{v}_1^2-\bar{v}_1\underline{v}_1)}{2p(-2\bar{v}_1+\underline{v}_1+\underline{v}_2)}$
$+\frac{1}{2}\frac{(\bar{v}_1-\underline{v}_1)\sqrt{\bar{v}_1^2(1+p)^2+\underline{v}_2^2(2-2p+p^2)+2\underline{v}_1\underline{v}_2(1-p+p^2)+\underline{v}_1^2(1-2p+2p^2)-2\bar{v}_1[\underline{v}_2(2-p+p^2)+\underline{v}_1(1-p+2p^2)]}}{p(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}$*. And low type will bid identically on* $[0, a_3]$ *by* $G_{L1}(x) = G_{L2}(x) = \frac{(1-p)x}{p(\underline{v}_1+\underline{v}_2-2x)}$*. What's more, endpoints of interval* $I = [a_3, a_4]$ *are determined by* $\frac{C}{\underline{v}_2-x} = \frac{(1-p)x-pC}{p(\underline{v}_1-x)}$ *with* a_4 *being equivalent to* a_1 *from lemma II.11.*

Proof. In the previous lemma we have studied what if we assume low type bids identically. We can now assume that for any given b_{l1}, we have an optimal $b_{l2} < b_{l1}$ such that (b_{l1}, b_{l2}) optimizes expected payoff for low type. We further assume I is the first non-trivial (i.e. positive measure) interval where $b_{l2} = h(b_{l1})$ (with $(h(b_{l1}) < b_{l1})$ solves the first order condition on interior of interval I. We denote $a_3 = \inf_{x \in I} I$ and $a_4 = \sup_{x \in I} I$ so $I = [a_3, a_4]$. By construction we have $h(a_3) = a_3$ and $h(a_4) = a_4$. Indifferent condition in region R_1 is
$p[G_{L2}(b_{l1})(\underline{v}_1 - b_{l1}) + G_{L1}(b_{l2})(\underline{v}_2 - b_{l2})] + (1-p)(\underline{v}_1 - b_{l1}) = \underline{v}_1(1-p)$. If we take derivative with respect to b_{l2} for any fixed b_{l1}, we get
$g_{L1}(b_{l2})(\underline{v}_2 - b_{l2}) - G_{L1}(b_{l2}) = 0$, with g being derivative of G functions. Solving the differential equation, we have $G_{L1}(x) = \frac{C}{\underline{v}_2-x}$ for some constant C, and hence $G_{L2}(y) = \frac{(1-p)y-pC}{p(\underline{v}_1-y)}$. We can formally define $a_3 < a_4$ to be solution to
$\frac{C}{\underline{v}_2-x} = \frac{(1-p)x-pC}{p(\underline{v}_1-x)}$.

Since $\frac{(1-p)x-pC}{p(\underline{v}_1-x)} = \frac{C}{\underline{v}_2-x}$ at a_3, a_4, we can rearrange the equation above to

$\frac{C}{\underline{v}_2-x} = \frac{(1-p)x}{p(\underline{v}_1+\underline{v}_2-2x)}$ at a_3, a_4. So we need to find out $a_3 < a_4$ such that $C = \frac{(1-p)a_3}{(\underline{v}_1+\underline{v}_2-2a_3)}(\underline{v}_2 - a_3) = \frac{(1-p)a_4}{(\underline{v}_1+\underline{v}_2-2a_4)}(\underline{v}_2 - a_4)$. But we know that function $\frac{(1-p)x}{(\underline{v}_1+\underline{v}_2-2x)}(\underline{v}_2 - x)$ is increasing when $x < \frac{\underline{v}_1+\underline{v}_2-\sqrt{\underline{v}_1-\underline{v}_2^2}}{2}$ and decreasing when $x > \frac{\underline{v}_1+\underline{v}_2-\sqrt{\underline{v}_1^2-\underline{v}_2^2}}{2}$ by proof in the previous lemma. So to make equation $\frac{(1-p)a_3}{(\underline{v}_1+\underline{v}_2-2a_3)}(\underline{v}_2 - a_3) = \frac{(1-p)a_4}{(\underline{v}_1+\underline{v}_2-2a_4)}(\underline{v}_2 - a_4)$ valid, we must make $a_3 < \frac{\underline{v}_1+\underline{v}_2-\sqrt{v_2^2-\underline{v}_2^2}}{2} < a_4$. What's more, for the right neighbourhood of a_4, we are in the perfectly correlated equilibrium by construction. To support such an equilibrium, our previous result requires that $2x^2 - 2(\underline{v}_1 + \underline{v}_2)x + \underline{v}_2(\underline{v}_1 + \underline{v}_2) > 0$, which is positive when $x < \frac{\underline{v}_1+\underline{v}_2-\sqrt{\underline{v}_1^2-\underline{v}_2^2}}{2}$ or $x > \frac{\underline{v}_1+\underline{v}_2-\sqrt{\underline{v}_1^2-\underline{v}_2^2}}{2}$. So right-neighbourhood of a_4 must be greater than $\frac{\underline{v}_1+\underline{v}_2+\sqrt{\underline{v}_1^2-\underline{v}_2^2}}{2}$, which is impossible since $\frac{\underline{v}_1+\underline{v}_2+\sqrt{\underline{v}_1^2-\underline{v}_2^2}}{2}$ is already greater than $\underline{v}_2$. So we conclude $a_4 = a_1$. Although we assume I to be the first interval where first and second bids of low type differ, it is actually the only interval since it ends at endpoint of region R_1.

Although we have a specific relation between first and second bid by $b_{l2} = h(b_{l1})$, it actually does not matter if low type deviates in the interval I, because payoff from first and second bid of low type are constructed to be constant at respectively $\underline{v}_1(1-p) - pC$ and pC [8]. If second bid of low type deviates downward to become smaller than a_3, the optimal deviating bid should be bidding at a_3 because we know that for values lower than a_3 low type is bidding identically. And in such a perfectly correlated equilibrium expected payment from second bid is strictly increasing. Similarly if first bid of low type deviates upward to be higher than a_4, the deviating bid better be $a_4 = a_1$ since in region R_3 low type will bid identically and first bid is strictly decreasing. If high type deviates to bid below a_1 in interval I, she will get $(1-p)(\bar{v}_1 - b_{l1}) + p\frac{(1-p)b_{l1}-pC}{p(\underline{v}_1-b_{l1})}(\bar{v}_1 - b_{l1}) = (\bar{v}_1 - b_{l1})[(1-p) + \frac{(1-p)b_{l1}-pC}{\underline{v}_1-b_{l1}}]$ with derivative $\frac{(\underline{v}_1-\bar{v}_1)[\underline{v}_1(-1+p)+pC]}{(\underline{v}_1-b_{l1})^2}$. We require $\underline{v}_1(-1+p) + pC < 0$ for a positive derivative so that a high type would rather bid a_1 instead of prices lower than a_1. If a high type further deviates to bid below a_3, we use the argument in proof of lemma II.12 to eliminate such a deviating possibility: derivative of high type's expected payoff will be increasing as long as her bid is lower than a_1 so high type will bid a_3 when she has to bid no greater than a_3. But high type will then immediately bid $a_4 = a_1$ since her deviating payoff is an increasing function on interval (a_3, a_4).

[8] $pG_{L2}(b_{l1})(\underline{v}_1 - b_{l1}) + (1-p)(\underline{v}_1 - b_{l1}) = \underline{v}_1(1-p) - pC$ and $pG_{L1}(b_{l2})(\underline{v}_2 - b_{l2}) = pC$

To make the distributions consistent, we have to let $\frac{C}{\underline{v}_2-x} = \frac{(1-p)x-pC}{p(\underline{v}_1-x)} = \frac{\bar{v}_1 p-a_2+(1-p)x}{p(\bar{v}_1-x)}$ when $x = a_4 = a_1$. The last expression is distribution of low type's bids on region R_3 (when $b_{l1} \geq b_{l2} \geq a_1$). Interpretation of the equalities above is that since $a_4 = a_1$ and $G_{L1}(a_1) = G_{L2}(a_1)$ on R_3, we should have the distribution at a_1 on interval I to be identical to the distribution at a_1 on region R_3. Condition satisfying equations above is $\frac{2\bar{v}_1^2\underline{v}_1-\bar{v}_1(\underline{v}_1+\underline{v}_2)^2+\underline{v}_2(\underline{v}_1^2+\underline{v}_2^2)}{-\underline{v}_1^3+2\bar{v}_1^2\underline{v}_2+\underline{v}_1^2\underline{v}_2+\underline{v}_1\underline{v}_2^2+\underline{v}_1^3+2\bar{v}_1(\underline{v}_1^2-2\underline{v}_1\underline{v}_2-\underline{v}_2^2)} - \frac{\sqrt{(\bar{v}_1-\underline{v}_2)^2(-2\bar{v}_1+\underline{v}_1+\underline{v}_2)^2(-\underline{v}_1^2+\underline{v}_2^2)}}{-\underline{v}_1^3+2\bar{v}_1^2\underline{v}_2+\underline{v}_1^2\underline{v}_2+\underline{v}_1\underline{v}_2^2+\underline{v}_1^3+2\bar{v}_1(\underline{v}_1^2-2\underline{v}_1\underline{v}_2-\underline{v}_2^2)} < p < \frac{\underline{v}_2}{\underline{v}_1}$. We actually have an expression for constant C by solving $\frac{C}{\underline{v}_2-a_1} = \frac{(1-p)a_1-pC}{p(\underline{v}_1-a_1)} = \frac{\bar{v}_1 p-a_1+(1-p)a_1}{p(\bar{v}_1-a_1)}$:

$C = \frac{(\bar{v}_1-\underline{v}_1)^2+\underline{v}_2^2+\underline{v}_1\underline{v}_2-2\bar{v}_1\underline{v}_2+p(\bar{v}_1\underline{v}_1-\underline{v}_1^2+\bar{v}_1^2-\bar{v}_1\underline{v}_2)}{2p(-2\bar{v}_1+\underline{v}_1+\underline{v}_2)}$

$+\frac{1}{2}\frac{(\bar{v}_1-\underline{v}_1)\sqrt{\bar{v}_1^2(1+p)^2+\underline{v}_2^2(2-2p+p^2)+2\underline{v}_1\underline{v}_2(1-p+p^2)+\underline{v}_1^2(1-2p+2p^2)-2\bar{v}_1[\underline{v}_2(2-p+p^2)+\underline{v}_1(1-p+2p^2)]}}{p(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}$.

Our last task is to check condition supporting perfectly correlated equilibrium holds when $b_{l1} < a_3$ and $b_{l2} < a_3$. Recall in proof of lemma II.12, we require $2x^2 - 2(\underline{v}_1 + \underline{v}_2)x + \underline{v}_2(\underline{v}_1 + \underline{v}_2) > 0$ for a perfectly correlated equilibrium. Note $2x^2 - 2(\underline{v}_1 + \underline{v}_2)x + \underline{v}_2(\underline{v}_1 + \underline{v}_2)$ is a decreasing function when $x < \frac{\underline{v}_1+\underline{v}_2}{2}$, and hence we need to guarantee that $2x^2 - 2(\underline{v}_1 + \underline{v}_2)x + \underline{v}_2(\underline{v}_1 + \underline{v}_2)$ is positive when $x = a_3$. Some computation will show that we need condition $\underline{v}_1 + \sqrt{\frac{(\underline{v}_1^2-\underline{v}_2^2)(1-p)^2}{p^2}} + 2C \leq \frac{\underline{v}_1}{p}$. Adding this condition into $\frac{C}{\underline{v}_2-a_1} = \frac{(1-p)a_1-pC}{p(\underline{v}_1-a_1)} = \frac{\bar{v}_1 p-a_2+(1-p)a_1}{p(\bar{v}_1-a_1)}$, we still get the same range of p and expression of C. What's more, we need $\frac{C}{\underline{v}_2-x} = \frac{(1-p)x}{p(\underline{v}_1+\underline{v}_2-2x)}$ at a_3 to support atomless distributions. And computation will show that solutions to this equation are just a_3 computed by solving $\frac{C}{\underline{v}_2-x} = \frac{\bar{v}_1 p-a_2+(1-p)a_1}{p(\bar{v}_1-x)}$. This should not be a surprising observation since we have argued in the second paragraph of this proof that $\frac{(1-p)x-pC}{p(\underline{v}_1-x)} = \frac{C}{\underline{v}_2-x}$ can be rearranged to $\frac{C}{\underline{v}_2-x} = \frac{(1-p)x}{p(\underline{v}_1+\underline{v}_2-2x)}$ when $x = a_3$. ■

Remark II.7 *We have two remarks to make:*

- $a_3 > 0$ *because otherwise* G_{L1} *will be just be* 0.
- *It is easy to exclude deviations above the common upper bound: all distributions are atomless at upper bounds. So bidding* $(\frac{p\underline{v}_1+\underline{v}_2}{2}, \frac{p\underline{v}_1+\underline{v}_2}{2})$ *(*$\frac{p\underline{v}_1+\underline{v}_2}{2}$ *is the upper bound in this scenario) will give low type two objects with certainty and bidding* $\frac{p\underline{v}_1+\underline{v}_2}{2}$ *will give high type one object with certainty. And hence bidding above the upper bound will only decrease the expected payoff for any type.*

For the two marginal distributions G_{L1}, G_{L2} introduced in lemma II.13, since we have computed distribution of second bid G_{L2} by solving first order condition to maximize expected payoff for any given first bid, we are actually able to characterize a functional relationship between distributions of first bid G_{L1} and second bid G_{L2}. We will compute a function h on interval $I = [a_3, a_4]$ introduced in lemma II.13 which relates G_{L1}, G_{L2} by $G_{L2}(h(x)) = G_{L1}(x)$. We are also able to prove that $h(x) < x$ in the interior of I:

Corollary II.2 *$h(x) < x$ in interval $I = (a_3, a_4)$ and $h(x)$ is an increasing function as long as $pC < \underline{v}_1(1-p)$.*

Proof. We assume x, y evolve according to $y = h(x)$ in interval I since we solve an optimal y for any given x to maximize the expected payoff for low type in interval $I = [a_3, a_4]$. We must have $G_{L2}(h(x)) = G_{L1}(x)$ for all x in interval I by change of variable technique. Using functional forms of G_{L1} and G_{L2}, we have $h(x) = \frac{Cp(\underline{v}_1+\underline{v}_2-x)}{Cp+(1-p)\underline{v}_2-(1-p)x}$, which will be an increasing function when $pC < \underline{v}_1(1-p)$. Note that this requirement is actually the identical condition to prevent high type from deviating below a_1 constructed in the proof of previous lemma.

If we want $h(x) < x$ in the interior of I, we must have $\frac{Cp(\underline{v}_1+\underline{v}_2-x)}{Cp+(1-p)\underline{v}_2-(1-p)x} < x$, which is equivalent to $C < \frac{(1-p)(\underline{v}_2-x)x}{p(\underline{v}_1+\underline{v}_2-2x)}$ in the interior of I. Note that $C = \frac{(1-p)a_3}{(\underline{v}_1+\underline{v}_2-2a_3)}(\underline{v}_2 - a_3) = \frac{(1-p)a_4}{(\underline{v}_1+\underline{v}_2-2a_4)}(\underline{v}_2 - a_4)$ with $a_3 < \frac{\underline{v}_1+\underline{v}_2-\sqrt{\underline{v}_1^2-\underline{v}_2^2}}{2} < a_4$. Function $\frac{(1-p)(\underline{v}_2-x)x}{p(\underline{v}_1+\underline{v}_2-2x)}$ is actually decreasing when $x \in (\frac{\underline{v}_1+\underline{v}_2-\sqrt{\underline{v}_1^2-\underline{v}_2^2}}{2}, \frac{\underline{v}_1+\underline{v}_2+\sqrt{\underline{v}_1^2-\underline{v}_2^2}}{2})$ and increasing when $x < \frac{\underline{v}_1+\underline{v}_2-\sqrt{\underline{v}_1^2-\underline{v}_2^2}}{2}$. So we conclude that $C < \frac{(1-p)(\underline{v}_2-x)x}{p(\underline{v}_1+\underline{v}_2-2x)}$ in the interior of I as desired. ■

Remark II.8 *We have two remarks to make:*

1. *By change of variable and $h(x) < x$, we now confirm that $G_{L2}(x) > G_{L2}(h(x)) = G_{L1}(x)$.*

2. *$b_{l2} = h(b_{l1})$ can be treated as an interior solution to the maximization problem $\max_{0 \le b_{l2} \le b_{l1}} p[G_{L2}(b_{l1})(\underline{v}_1 - b_{l1}) + G_{L1}(b_{l2})(\underline{v}_2 - b_{l2})] + (1-p)(\underline{v}_1 - b_{l1})$ since $b_{l2} = h(b_{l1})$ solves the first order condition: $g_{L1}(b_{l2})(\underline{v}_2 - b_{l2}) - G_{L1}(b_{l2}) = 0$. The next question to ask is do we have an interior solution where $b_{l2} = h(b_{l1}) = b_{l1}$? If so, we must have $G_{L1}(b_{l1}) = G_{L2}(b_{l1}) = \frac{(1-p)b_{l1}}{p(\underline{v}_1+\underline{v}_2-2b_{l1})}$.*

First order condition to the maximization problem will become $\frac{(1-p)[2b_{l1}^2-2(\underline{v}_1+\underline{v}_2)b_{l1}+(\underline{v}_1+\underline{v}_2)\underline{v}_2)]}{p(\underline{v}_1+\underline{v}_2-2b_{l1})^2}=0$. *However, this equation only achieves* 0 *at* 2 *specific values of* x*, which is contradictory to our assumption of an interior solution on an interval. So there is no interior solution generating the perfectly correlated equilibrium.*

Graphical Illustration

We graphically illustrate density functions proposed in lemma II.13 via:

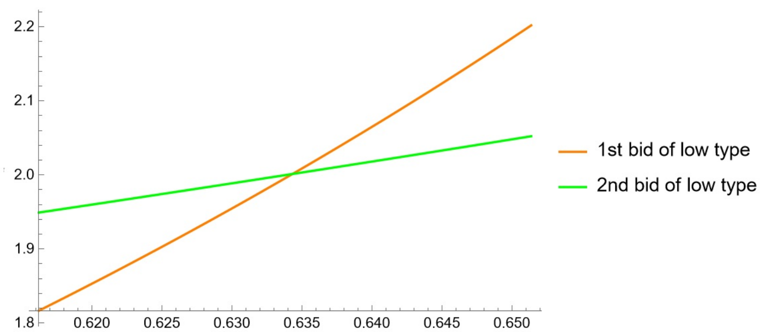

Figure 2.2: Illustration of lemma II.13

We pick $\bar{v}_1=3, \underline{v}_1=2, \underline{v}_2=1$ and $p=\frac{1}{3}$. Two bids from low type will be different in interval $[\frac{1}{12}(11-\sqrt{13}), \frac{\sqrt{13}-1}{4}]$. Note that corollary III.2 demonstrates that $G_{L1}(x)=G_{L2}(h(x))$ in interval $[\frac{1}{12}(11-\sqrt{13}), \frac{\sqrt{13}-1}{4}]$, where $h(x)<x$ for values in $(\frac{1}{12}(11-\sqrt{13}), \frac{\sqrt{13}-1}{4})$ and $h(x)=x$ for endpoints. The graph above reflects such a property by assigning G_{L1} a flatter slope when x is small and steeper slope when x is high.

We can also illustrate density functions of equilibrium distributions graphically:

We still select $\bar{v}_1=3, \underline{v}_1=2, \underline{v}_2=1$ and the first graph is when $p=\frac{1}{3}$, which covers points 1,2, 3.b and 4 of theorem II.5, when there is an interval where first and second bid of low type are different. Support for distributions of low type is $[0, \frac{5}{6}]$ and support for distribution of high type is $[\frac{\sqrt{13}-1}{4}, \frac{5}{6}]$. The second graph is when $p=0.3$ where two bids of low type are always identical, as shown in points 1, 2, 3.a and 4 of theorem II.5. Support for distributions of low type is $[0, \frac{4}{5}]$ and support for distribution of high type is $[0.527, \frac{4}{5}]$.

If we take derivative on the distribution functions and compute pdfs of low type's

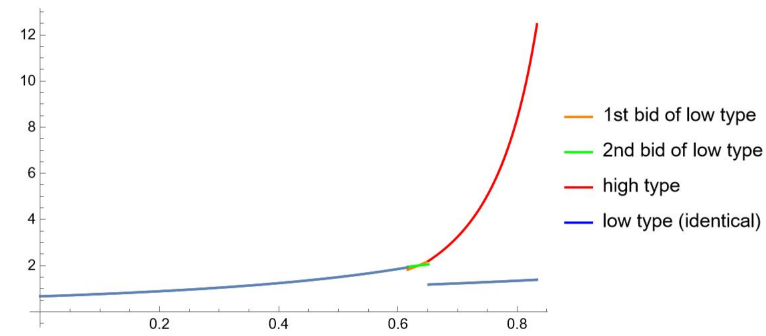

Figure 2.3: Illustration of points 1,2, 3.b and 4 of theorem II.5

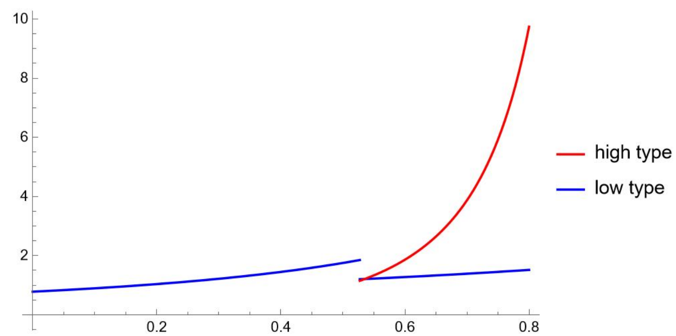

Figure 2.4: Illustration of points 1,2, 3.a and 4 of theorem II.5

each bid are, we will have the following result at a_1. We omit the proof since it is just direct computation.

Corollary II.3 *We have the following results for pdfs of low type's distributions:*

1. *When* $\frac{\underline{v}_2}{2\bar{v}_1 - \underline{v}_1} < p < p^*$*, left derivative of* G *functions in region* R_1 *at* a_1 *is greater than right derivative of* G *functions in region* R_3 *at* a_1*;*

2. *When* $p^* < p < \frac{\underline{v}_2}{\underline{v}_1}$*, left derivative will satisfy* $\frac{dG_{L1}(x)}{dx} > \frac{dG_{L2}(x)}{dx}$ *at* a_1 *and left derivative* $\frac{dG_{L2}(x)}{dx}$ *in region* R_1 *at* a_1 *will be greater than right derivative* $\frac{dG(x)}{dx}$ *in region* R_3 *at* a_1*.*

We can also illustrate density functions for only low type only:

The pdfs only differ for bids in $[\frac{1}{12}(11 - \sqrt{13}), \frac{\sqrt{13}-1}{4}]$.

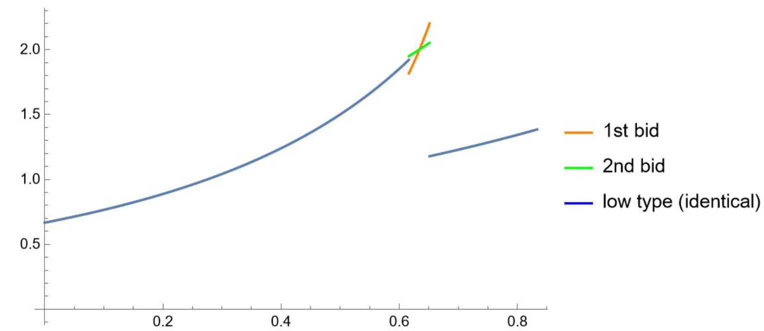

Figure 2.5: Low type's pdf

2.5.2 $\bar{v}_1 < \underline{v}_1 + \underline{v}_2$

With $F_H(x)$ in the form in lemma II.10, we should require $F_H(x)$ to be an increasing function on $(0, \bar{v}_1 p)$. $\frac{dF_H(x)}{dx} = \frac{(2\bar{v}_1 - \underline{v}_1 - \underline{v}_2)[x^2 - 2p\bar{v}_1 x + p\bar{v}_1\underline{v}_2 - \bar{v}_1\underline{v}_2 + p\bar{v}_1^2]}{(p-1)(\bar{v}_1 - x)^2(\underline{v}_2 - x)^2}$. To make $F_H(x)$ an increasing function, we need $x^2 - 2p\bar{v}_1 x + p\bar{v}_1\underline{v}_2 - \bar{v}_1\underline{v}_2 + p\bar{v}_1^2$ to be negative, which means maximum of $x^2 - 2p\bar{v}_1 x + p\bar{v}_1\underline{v}_2 - \bar{v}_1\underline{v}_2 + p\bar{v}_1^2$ is negative. And $x^2 - 2p\bar{v}_1 x + p\bar{v}_1\underline{v}_2 - \bar{v}_1\underline{v}_2 + p\bar{v}_1^2$ is decreasing on $(0, p\bar{v}_1)$ so we should require $p\bar{v}_1\underline{v}_2 + p\bar{v}_1^2 < \bar{v}_1\underline{v}_2 \iff p < \frac{\bar{v}_1\underline{v}_2}{\bar{v}_1(\bar{v}_1 + \underline{v}_2)} = \frac{\underline{v}_2}{\bar{v}_1 + \underline{v}_2}$. When $\bar{v}_1 \geq \underline{v}_1 + \underline{v}_2$, $\frac{\underline{v}_2}{\bar{v}_1 + \underline{v}_2} \geq \frac{\underline{v}_2}{2\bar{v}_1 - \underline{v}_1}$ and we get an increasing F_H for free. But if $\bar{v}_1 < \underline{v}_1 + \underline{v}_2$, $\frac{\underline{v}_2}{\bar{v}_1 + \underline{v}_2} < \frac{\underline{v}_2}{2\bar{v}_1 - \underline{v}_1}$. So computation above indicates that when $\bar{v}_1 < \underline{v}_1 + \underline{v}_2$ we are missing some range of p.

What's more, we can look at computation in lemma II.12 when $\bar{v}_1 < \underline{v}_1 + \underline{v}_2$ as well. Recall in lemma II.12, we compute derivative of expected payoff of high type if she deviates to bid below a_1. If $\bar{v}_1 < \underline{v}_1 + \underline{v}_2$, the derivative of high type's deviating payoff [9] is negative when $x < \frac{\underline{v}_1 + \underline{v}_2 - \sqrt{(2\bar{v}_1 - \underline{v}_1 - \underline{v}_2)(\underline{v}_1 + \underline{v}_2)}}{2}$ and positive when $x > \frac{\underline{v}_1 + \underline{v}_2 - \sqrt{(2\bar{v}_1 - \underline{v}_1 - \underline{v}_2)(\underline{v}_1 + \underline{v}_2)}}{2}$. Computation will show that $a_1 > \frac{\underline{v}_1 + \underline{v}_2 - \sqrt{(2\bar{v}_1 - \underline{v}_1 - \underline{v}_2)(\underline{v}_1 + \underline{v}_2)}}{2}$. So candidates for potential optimizers must be at the endpoints. We compare $\bar{v}_1 - a_2 \geq (1-p)(\bar{v}_1 - 0) \iff p \geq \frac{\underline{v}_2}{2\bar{v}_1 - \underline{v}_1}$. Computation above shows that when $\bar{v}_1 < \underline{v}_1 + \underline{v}_2$, to support results like lemma II.12, we just need $p > \frac{\underline{v}_2}{2\bar{v}_1 - \underline{v}_1}$.

Combining the previous two paragraphs, we miss to characterize equilibria when $p \in (\frac{\underline{v}_2}{\bar{v}_1 + \underline{v}_2}, \frac{\underline{v}_2}{2\bar{v}_1 - \underline{v}_1})$ with $\bar{v}_1 < \underline{v}_1 + \underline{v}_2$.

When $p \in (\frac{\underline{v}_2}{\bar{v}_1 + \underline{v}_2}, \frac{\underline{v}_2}{2\bar{v}_1 - \underline{v}_1})$, our conjecture is that F_H is mixture of the previous F_H

[9] $\frac{(p-1)(2x^2 - 2(\underline{v}_1 + \underline{v}_2)x - (\bar{v}_1 - \underline{v}_1 - \underline{v}_2)(\underline{v}_1 + \underline{v}_2))}{(\underline{v}_1 + \underline{v}_2 - 2x)^2}$

distributions for high type in theorem III.2 and III.3, i.e. support of F_H has an atom at 0, puts no probability on interval $(0, a_1)$ (i.e. gap on interval $(0, a_1)$) and put the remaining probability on interval $[a_1, a_2]$. As with the paragraph before lemma II.10, high type will choose to put an atom at 0 since range of p, the probability low type appears in the population, is still not that high. We will later show there is no incentive for high type to deviate from bidding according to F_H.

We still assume a to be the upper bound of distributions. Let R_1, R_2, R_3 still be defined as $R_1 = \{(b_{l1}, b_{l2}) : b_{l2} \leq b_{l1} \leq a_1\}$, $R_2 = \{(b_{l1}, b_{l2}) : b_{l2} \leq a_1 \leq b_{l1}\}$ and $R_3 = \{(b_{l1}, b_{l2}) : a_1 \leq b_{l2} \leq b_{l1}\}$.

As usual, we summarize our results into a theorem:

Theorem II.9 *Suppose $\bar{v}_1 < \underline{v}_1 + \underline{v}_2$ and $p \in (\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2}, \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1})$.*

1. *High type will bid by $F_H(x) = \frac{x+T\underline{v}_2}{\underline{v}_2-x} - \frac{\bar{v}_1 p - a_2 + (1-p)x}{(1-p)(\bar{v}_1-x)(\underline{v}_2-x)}(\underline{v}_1 + \underline{v}_2 - 2x)$ with support $\{0\} \cup [a_1, a_2]$. $a_1 = \frac{\underline{v}_1+\underline{v}_2-\bar{v}_1(1+T)}{1-T}$, $T = \frac{p\underline{v}_1+\underline{v}_2-2p\bar{v}_1}{(1-p)\underline{v}_2}$ and $a_2 = \bar{v}_1 p$.*
2. *In region $R_3 = \{(b_{l1}, b_{l2}) : a_1 \leq b_{l2} \leq b_{l1}\}$, low type will bid by $G_{L1}(x) = G_{L2}(x) = \frac{\bar{v}_1 p - a_2 + (1-p)x}{p(\bar{v}_1 - x)}$ with support $[a_1, a_2]$.*
3. *In region $R_1 = \{(b_{l1}, b_{l2}) : b_{l2} \leq b_{l1} \leq a_1\}$, letting $C = \frac{(\bar{v}_1-\underline{v}_1-\underline{v}_2)(-\underline{v}_2+\bar{v}_1 p)}{\underline{v}_2 p}$,*
   (a) *when $\underline{v}_1 + \underline{v}_2 + \sqrt{\underline{v}_1^2 - \underline{v}_2^2} \leq 2\bar{v}_1$, low type bids according to distribution $G_{L1}(x) = G_{L2}(x) = \frac{(1-p)x+T(1-p)x}{p(\underline{v}_1+\underline{v}_2-2x)}$ for all $p \in (\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2}, \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1})$ in the region;*
   (b) *When $\underline{v}_1 + \underline{v}_2 + \sqrt{\underline{v}_1^2 - \underline{v}_2^2} > 2\bar{v}_1$,*
      i. *when $\frac{\underline{v}_2(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}{2\bar{v}_1^2-2\bar{v}_1\underline{v}_1+\underline{v}_1^2-\underline{v}_2^2} < p < \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1}$, low type will bid her first bid according to $G_{L1}(x) = \frac{C}{\underline{v}_2-x} - \frac{1-p}{p}T$ and second bid by $G_{L2}(x) = \frac{(1-p)x-pC+(1-p)T\underline{v}_2}{p(\underline{v}_1-x)}$ in interval $[a_3, a_1]$ for C defined above. The two marginal distributions are related by $G_{L2}(h(x)) = G_{L1}(x)$ where $h(x) = \frac{-\underline{v}_2(\underline{v}_2-p(2\bar{v}_1+C))(\underline{v}_2-x)+\underline{v}_1^2 p(-\underline{v}_2+x)+\underline{v}_1(-\underline{v}_2^2(1+p)-2\bar{v}_1 px+\underline{v}_2(2\bar{v}_1 p+x+p(C+x)))}{p(2\bar{v}_1(\underline{v}_2-x)+b(-\underline{v}_2+x)+c(-\underline{v}_2+C+x))}$. And low type will bid by $G_{L1}(x) = G_{L2}(x) = \frac{(1-p)x+T(1-p)x}{p(\underline{v}_1+\underline{v}_2-2x)}$ in region $[0, a_3]$. $a_3 < a_1$ solve $\frac{C}{\underline{v}_2-x} - \frac{1-p}{p}T = \frac{(1-p)x-pC+(1-p)T\underline{v}_2}{p(\underline{v}_1-x)}$.*
      ii. *when $p < \frac{\underline{v}_2(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}{2\bar{v}_1^2-2\bar{v}_1\underline{v}_1+\underline{v}_1^2-\underline{v}_2^2}$, low type will bid according to*

$G_{L1}(x) = G_{L2}(x) = \frac{(1-p)x+T(1-p)x}{p(\underline{v}_1+\underline{v}_2-2x)}$ *in region* R_1.

4. *Region* $R_2 = \{(b_{l1}, b_{l2}) : b_{l2} \leq a_1 \leq b_{l1}\}$ *has zero probability under distributions of bids for low type.*

The first two points are illustrated in lemma II.14. 3.a and 3.b.ii come from lemma II.15 and 3.b.i is dealt in lemma II.16. Point 4 is a direct result of point 2. Similar to previous cases, there will be a positive probability that one low type gets both objects due to common support of mixed strategy equilibrium distributions. And hence the equilibrium strategy is not necessarily efficient.

Lemma II.14 *High type will bid by* $F_H(x) = \frac{x+T\underline{v}_2}{\underline{v}_2-x} - \frac{\bar{v}_1 p - a_2 + (1-p)x}{(1-p)(\bar{v}_1-x)(\underline{v}_2-x)}(\underline{v}_1 + \underline{v}_2 - 2x)$ *with support* $\{0\} \cup [a_1, a_2]$. *In region* $R_3 = \{(b_{l1}, b_{l2}) : a_1 \leq b_{l2} \leq b_{l1}\}$, *low type will bid by* $G_{L1}(x) = G_{L2}(x) = \frac{\bar{v}_1 p - a_2 + (1-p)x}{p(\bar{v}_1 - x)}$ *with support* $[a_1, a_2]$. *We express* $a_2 = \bar{v}_1 p$ *and* $a_1 = \frac{\underline{v}_1+\underline{v}_2-\bar{v}_1(1+T)}{1-T}$ *where* T *can be expressed as* $T = \frac{p\underline{v}_1+\underline{v}_2-2p\bar{v}_1}{(1-p)\underline{v}_2}$.

Proof. In region R_3, we still have $G_{L2}(x) = \frac{\bar{v}_1 p - a_2 + (1-p)x}{p(\bar{v}_1-x)}$ as in lemma II.11 by looking at high type's indifferent condition in the region. And by similar argument in region R_3 low type must be bidding identical prices. In region R_3 we have indifferent condition for low type:
$p[G_{L2}(b_{l1})(\underline{v}_1 - b_{l1}) + G_{L1}(b_{l2})(\underline{v}_2 - b_{l2})] + (1-p)[(\underline{v}_1 - b_{l1}) + F_H(b_{l2})(\underline{v}_2 - b_{l2})] = \underline{v}_1(1-p) + (1-p)T\underline{v}_2$. Plugging G_{L1}, G_{L2} we have
$(1-p)F_H(x)(\underline{v}_2 - b_{l1}) = (1-p)b_{l1} + (1-p)H\underline{v}_2 - \frac{\bar{v}_1 p - a_2 + (1-p)b_{l1}}{\bar{v}_1 - b_{l1}}(\underline{v}_1 + \underline{v}_2 - 2b_{l1})$.
And we solve $F_H(x) = \frac{x+T\underline{v}_2}{\underline{v}_2-x} - \frac{\bar{v}_1 p - a_2 + (1-p)x}{(1-p)(\bar{v}_1-x)(\underline{v}_2-x)}(\underline{v}_1 + \underline{v}_2 - 2x)$. Solving $F_H(a_2) = 1$ gives $a_2 = \frac{p\underline{v}_1+\underline{v}_2}{2} - \frac{T(1-p)\underline{v}_2}{2}$.

We should require $F_H(0) = T$, which is only valid when $a_2 = \bar{v}_1 p$. So $T = \frac{p\underline{v}_1+\underline{v}_2-2p\bar{v}_1}{(1-p)\underline{v}_2}$. T is an decreasing function of p. When $p = \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1}$, $T = 0$. This computation guarantees $T \in (0, \frac{\underline{v}_1+\underline{v}_2-\bar{v}_1}{\bar{v}_1})$ for $p \in (\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2}, \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1})$. Another requirement is $F_H(a_1) = T$, and we have $a_1 = \frac{\underline{v}_1+\underline{v}_2-\bar{v}_1(1+T)}{1-T}$. It is easy to compute expected payment for high type is $(\bar{v}_1 - a_1)(1-p) + p(\bar{v}_1 - a_1)G_{L2}(a_1) = \bar{v}_1 - \bar{v}_1 p$.

To make sure $F_H(x)$ is increasing when $x > a_1$, we need $\frac{dF_H(x)}{dx} > 0$ when $x \in (a_1, a_2)$. Plugging $a_2 = \bar{v}_1 p$, we have
$\frac{dF_H(x)}{dx} = \frac{x^2(T\underline{v}_2+\underline{v}_1-2\bar{v}_1)-2(T-1)\bar{v}_1\underline{v}_2 x+\bar{v}_1\underline{v}_2(T\bar{v}_1-\underline{v}_1-\underline{v}_2+\bar{v}_1)}{(\bar{v}_1-x)^2(\underline{v}_2-x)^2} > 0$ when $x \in (a_1, a_2)$. $\Longleftrightarrow$
$x^2(T\underline{v}_2 + \underline{v}_1 - 2\bar{v}_1) + 2(1-T)\bar{v}_1\underline{v}_2 x + \bar{v}_1\underline{v}_2(T\bar{v}_1 - \underline{v}_1 - \underline{v}_2 + \bar{v}_1) > 0$ when $x \in (a_1, a_2)$. And we have to guarantee

$x^2(T\underline{v}_2+\underline{v}_1-2\bar{v}_1)-2(T-1)\bar{v}_1\underline{v}_2x+\bar{v}_1\underline{v}_2(T\bar{v}_1-\underline{v}_1-\underline{v}_2+\bar{v}_1)>0$ when $x=a_1$. Plugging $a_1=\frac{\underline{v}_1+\underline{v}_2-\bar{v}_1(1+T)}{1-T}$ into the equation, we have
$x^2(T\underline{v}_2+\underline{v}_1-2\bar{v}_1)-2(T-1)\bar{v}_1\underline{v}_2x+\bar{v}_1\underline{v}_2(T\bar{v}_1-\underline{v}_1-\underline{v}_2+\bar{v}_1)=$
$a_1^2(T\underline{v}_2+\underline{v}_1-2\bar{v}_1)-(T-1)\bar{v}_1\underline{v}_2a_1$. And $a_1^2(T\underline{v}_2+\underline{v}_1-2\bar{v}_1)-(T-1)\bar{v}_1\underline{v}_2a_1>0$ can be achieved when $a_1\in(0,\frac{-\bar{v}_1\underline{v}_2+T\bar{v}_1\underline{v}_2}{-2\bar{v}_1+\underline{v}_1+T\underline{v}_2})$. Some computation will show that $a_1\in(0,\frac{-\bar{v}_1\underline{v}_2+T\bar{v}_1\underline{v}_2}{-2\bar{v}_1+\underline{v}_1+T\underline{v}_2})$ is satisfied as long as $T\in(-\frac{\bar{v}_1-\underline{v}_1}{\bar{v}_1-\underline{v}_2},\frac{\underline{v}_1+\underline{v}_2-\bar{v}_1}{\bar{v}_1})$, which contains $(0,\frac{\underline{v}_1+\underline{v}_2-\bar{v}_1}{\bar{v}_1})$. So we confirm that as long as $0<a_1<\frac{-\bar{v}_1\underline{v}_2+H\bar{v}_1\underline{v}_2}{-2\bar{v}_1+\underline{v}_1+H\underline{v}_2}$, derivative is positive when $x=a_1$.

Note that $x^2(T\underline{v}_2+\underline{v}_1-2\bar{v}_1)+2(1-T)\bar{v}_1\underline{v}_2x+\bar{v}_1\underline{v}_2(T\bar{v}_1-\underline{v}_1-\underline{v}_2+\bar{v}_1)$ is a quadratic function with a negative coefficient on x^2 term and positive coefficient on x term. So such an expression will be increasing when $x<\frac{(T-1)\bar{v}_1\underline{v}_2}{T\underline{v}_2-2\bar{v}_1+\underline{v}_1}$. Another fact is that if we solve $F_H(a_2)=1$ by plugging into $a_2=\bar{v}_1p$, we can get $a_2=\frac{-\bar{v}_1\underline{v}_2+T\bar{v}_1\underline{v}_2}{-2\bar{v}_1+\underline{v}_1+T\underline{v}_2}$ [10]. So results above imply that derivative at a_1 is positive and it is the minimal value $\frac{dF_H(x)}{dx}$ will achieve. And hence we can prove that $\frac{dF_H(x)}{dx}>0$ when $x>a_1$ by showing $\frac{dF_H(x)}{dx}>0$ when $x=a_1$. ■

When low type is bidding the same in region R_3, we are able to conclude that R_2 will at most be a zero-measure region. And hence we move on to look at region R_1 and we propose similar solutions to lemma II.12. But condition to support the lemma will be more complicated:

Lemma II.15 *In region $R_1=\{(b_{l1},b_{l2}):b_{l2}\le b_{l1}\le a_1\}$,*

1. *If $\underline{v}_1+\underline{v}_2+\sqrt{\underline{v}_1^2-\underline{v}_2^2}>2\bar{v}_1$ there is a perfectly correlated equilibrium where low type bids according to distribution $G_{L1}(x)=G_{L2}(x)=\frac{(1-p)x+T(1-p)x}{p(\underline{v}_1+\underline{v}_2-2x)}$ with support $[0,a_1]$ when $\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2}<p<\frac{\underline{v}_2(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}{2\bar{v}_1^2-2\bar{v}_1\underline{v}_1+\underline{v}_1^2-\underline{v}_2^2}$.*
2. *If $\underline{v}_1+\underline{v}_2+\sqrt{\underline{v}_1^2-\underline{v}_2^2}\le 2\bar{v}_1$, perfectly correlated equilibrium can be supported by all $p\in(\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2},\frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1})$ with $G_{L1}(x)=G_{L2}(x)=\frac{(1-p)x+T(1-p)x}{p(\underline{v}_1+\underline{v}_2-2x)}$ on support $[0,a_1]$.*

Proof. On region R_1 we have indifferent condition for low type:
$p[G_{L2}(b_{l1})(\underline{v}_1-x)+G_{L1}(b_{l2})(\underline{v}_2-b_{l2})]+(1-p)(\underline{v}_1-b_{l1})+(1-p)T(\underline{v}_2-b_{l2})=$

[10] Expressions $a_2=\bar{v}_1p=\frac{-\bar{v}_1\underline{v}_2+T\bar{v}_1\underline{v}_2}{-2\bar{v}_1+\underline{v}_1+T\underline{v}_2}=\frac{p\underline{v}_1+\underline{v}_2}{2}-\frac{T(1-p)\underline{v}_2}{2}$ are equivalent as long as $T=\frac{p\underline{v}_1+\underline{v}_2-2p\bar{v}_1}{(1-p)\underline{v}_2}$.

$\underline{v}_1(1-p)+(1-p)T\underline{v}_2$. Perfectly correlated equilibrium will give a result $G_{L1}(x)=G_{L2}(x)=\frac{(1-p)x+T(1-p)x}{p(\underline{v}_1+\underline{v}_2-2x)}$. Note that $G_{L1}(0)=G_{L2}(0)=0$. G functions on R_1 and G functions on R_3 will coincide when $x=a_1$. Expected payoff for first bid of low type is $p\frac{(1-p)x+T(1-p)b_{l1}}{p(\underline{v}_1+\underline{v}_2-2b_{l1})}(\underline{v}_1-b_{l1})+(1-p)(\underline{v}_1-b_{l1})$ with derivative $-\frac{(p-1)[2(T-1)b_{l1}^2-2(T-1)(\underline{v}_1+\underline{v}_2)b_{l1}+(\underline{v}_1+\underline{v}_2)(T\underline{v}_1-\underline{v}_2)]}{(\underline{v}_1+\underline{v}_2-2b_{l1})^2}$. Expected payoff for second bid of low type is $p\frac{(1-p)b_{l2}+T(1-p)b_{l2}}{p(\underline{v}_1+\underline{v}_2-2b_{l2})}(\underline{v}_2-b_{l2})+(1-p)T(\underline{v}_2-b_{l2})$ with derivative $\frac{(p-1)[2(T-1)b_{l2}^2-2(T-1)(\underline{v}_1+\underline{v}_2)b_{l2}+(\underline{v}_1+\underline{v}_2)(T\underline{v}_1-\underline{v}_2)]}{(\underline{v}_1+\underline{v}_2-2b_{l2})^2}$ which is exactly the negative of derivative of expected payoff for first bid of low type.

So condition to make payment from first bid to be decreasing is still the same condition to make payment from second bid to be increasing: $(p-1)[2(T-1)x^2-2(T-1)(\underline{v}_1+\underline{v}_2)x+(\underline{v}_1+\underline{v}_2)(T\underline{v}_1-\underline{v}_2)]>0$, which is equivalent to $2(1-T)x^2-2(1-T)(\underline{v}_1+\underline{v}_2)x-(\underline{v}_1+\underline{v}_2)(T\underline{v}_1-\underline{v}_2)>0$. We know that $2(1-T)x^2-2(1-T)(\underline{v}_1+\underline{v}_2)x-(\underline{v}_1+\underline{v}_2)(T\underline{v}_1-\underline{v}_2)$ is decreasing when $x<a_1<\frac{\underline{v}_1+\underline{v}_2}{2}$. We can compute that when $x=a_1=\frac{\underline{v}_1+\underline{v}_2-\bar{v}_1(1+T)}{1-T}$, the expression above is positive for 2 conditions: if $\underline{v}_1+\underline{v}_2+\sqrt{\underline{v}_1^2-\underline{v}_2^2}\le 2\bar{v}_1$, we have perfectly correlated equilibrium for all $p\in(\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2},\frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1})$; and if $\underline{v}_1+\underline{v}_2+\sqrt{\underline{v}_1^2-\underline{v}_2^2}>2\bar{v}_1$ perfectly correlated equilibrium exists as long as $\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2}<p<\frac{\underline{v}_2(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}{2\bar{v}_1^2-2\bar{v}_1\underline{v}_1+\underline{v}_1^2-\underline{v}_2^2}$.

Computation shows when $\underline{v}_1+\sqrt{2}\underline{v}_1\le 2\bar{v}_1$, $\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2}<p<\frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1}$ is enough;

On the other hand, if $\underline{v}_1+\sqrt{2}\underline{v}_1\ge 2\bar{v}_1$, computation generates $(\underline{v}_1+\sqrt{-4\bar{v}_1^2+4\bar{v}_1\underline{v}_1+\underline{v}_1^2}+2\underline{v}_2\le 2\bar{v}_1)\cup(2\bar{v}_1+\sqrt{-4\bar{v}_1^2+4\bar{v}_1\underline{v}_1+\underline{v}_1^2}\le\underline{v}_1+2\underline{v}_2)$ $\cup[(2\bar{v}_1+\sqrt{-4\bar{v}_1^2+4\bar{v}_1\underline{v}_1+\underline{v}_1^2}>\underline{v}_1+2\underline{v}_2)\cap(\underline{v}_1+\sqrt{-4\bar{v}_1^2+4\bar{v}_1\underline{v}_1+\underline{v}_1^2}+2\underline{v}_2>2\bar{v}_1)\cap(p<\frac{\underline{v}_2(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}{2\bar{v}_1^2-2\bar{v}_1\underline{v}_1+\underline{v}_1^2-\underline{v}_2^2})]$. To interpret this result, we denote $\mathbb{P}_1=(\underline{v}_1+\sqrt{-4\bar{v}_1^2+4\bar{v}_1\underline{v}_1+\underline{v}_1^2}+2\underline{v}_2\le 2\bar{v}_1)$, $\mathbb{P}_2=(2\bar{v}_1+\sqrt{-4\bar{v}_1^2+4\bar{v}_1\underline{v}_1+\underline{v}_1^2}\le\underline{v}_1+2\underline{v}_2)$ and $A=(p<\frac{\underline{v}_2(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}{2\bar{v}_1^2-2\bar{v}_1\underline{v}_1+\underline{v}_1^2-\underline{v}_2^2})$. So expression above is actually $\mathbb{P}_1\cup\mathbb{P}_2\cup[\mathbb{P}_1^c\cap\mathbb{P}_2^c\cap A]$.

$\mathbb{P}_1\cup\mathbb{P}_2\cup[\mathbb{P}_1^c\cap\mathbb{P}_2^c\cap A]=[(\mathbb{P}_1\cup\mathbb{P}_2)\cup(\mathbb{P}_1^c\cap\mathbb{P}_2^c)]\cap[(\mathbb{P}_1\cup\mathbb{P}_2)\cup A]$ by distributive law of set operations. Note that complement of $\mathbb{P}_1\cup\mathbb{P}_2$ is $\mathbb{P}_1^c\cap\mathbb{P}_2^c$, and accordingly $\mathbb{P}_1\cup\mathbb{P}_2\cup[\mathbb{P}_1^c\cap\mathbb{P}_2^c\cap A]=(\mathbb{P}_1\cup\mathbb{P}_2)\cup A$. To see what is $\mathbb{P}_1\cup\mathbb{P}_2$, we still compute its complement and it turns out complement of $\mathbb{P}_1\cup\mathbb{P}_2$ is $(2\bar{v}_1<\underline{v}_1+\underline{v}_2+\sqrt{\underline{v}_1^2-\underline{v}_2^2})$. So $\mathbb{P}_1\cup\mathbb{P}_2=(2\bar{v}_1\ge\underline{v}_1+\underline{v}_2+\sqrt{\underline{v}_1^2-\underline{v}_2^2})$. But it is straightforward to check $\underline{v}_1+\underline{v}_2+\sqrt{\underline{v}_1^2-\underline{v}_2^2}\le\underline{v}_1+\sqrt{2}\underline{v}_1$ since $\underline{v}_2<\underline{v}_1$, which indicates that

$\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2} < p < \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1}$ when $\underline{v}_1+\underline{v}_2+\sqrt{\underline{v}_1^2-\underline{v}_2^2} \leq 2\bar{v}_1 < \underline{v}_1+\sqrt{2}\underline{v}_1$ and $\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2} < p < \frac{\underline{v}_2(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}{2\bar{v}_1^2-2\bar{v}_1\underline{v}_1+\underline{v}_1^2-\underline{v}_2^2}$ when $2\bar{v}_1 < \underline{v}_1+\underline{v}_2+\sqrt{\underline{v}_1^2-\underline{v}_2^2}$ are both feasible solutions. (To be precise, the 2nd result should be $\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2} < p < \frac{\underline{v}_2(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}{2\bar{v}_1^2-2\bar{v}_1\underline{v}_1+\underline{v}_1^2-\underline{v}_2^2}$ when $2\bar{v}_1 < \underline{v}_1+\sqrt{2}\underline{v}_1$, but $\underline{v}_1+\underline{v}_2+\sqrt{\underline{v}_1^2-\underline{v}_2^2} \leq \underline{v}_1+\sqrt{2}\underline{v}_1$ implies $2\bar{v}_1 < \underline{v}_1+\underline{v}_2+\sqrt{\underline{v}_1^2-\underline{v}_2^2}$ is a subset of condition $2\bar{v}_1 < \underline{v}_1+\sqrt{2}\underline{v}_1$).

In conclusion, when $\underline{v}_1+\underline{v}_2+\sqrt{\underline{v}_1^2-\underline{v}_2^2} \leq 2\bar{v}_1$, perfectly correlated equilibrium exists for $\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2} < p < \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1}$ while when $2\bar{v}_1 < \underline{v}_1+\underline{v}_2+\sqrt{\underline{v}_1^2-\underline{v}_2^2}$, perfectly correlated equilibrium exists when $\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2} < p < \frac{\underline{v}_2(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}{2\bar{v}_1^2-2\bar{v}_1\underline{v}_1+\underline{v}_1^2-\underline{v}_2^2}$.

If high type bids below a_1, she will get $p\frac{(1-p)b_{h1}+T(1-p)b_{h1}}{p(\underline{v}_1+\underline{v}_2-2b_{h1})}(\bar{v}_1-b_{h1})+(1-p)(\bar{v}_1-b_{h1})$ $=\frac{(1-p)(\bar{v}_1-b_{h1})(\underline{v}_1+\underline{v}_2-b_{h1})}{\underline{v}_1+\underline{v}_2-2b_{h1}}+\frac{T(1-p)b_{h1}}{\underline{v}_1+\underline{v}_2-2b_{h1}}(\bar{v}_1-b_{h1})=\frac{\underline{v}_1+\underline{v}_2-b_{h1}+Tb_{h1}}{\underline{v}_1+\underline{v}_2-2b_{h1}}(1-p)(\bar{v}_1-b_{h1})$ with derivative $\frac{(1-p)[-2(1-T)b_{h1}^2+2(\underline{v}_1+\underline{v}_2)(1-T)x+(\underline{v}_1+\underline{v}_2)(T+1)\bar{v}_1-(\underline{v}_1+\underline{v}_2)^2]}{(\underline{v}_1+\underline{v}_2-2b_{h1})^2}$. When $b_{h1}=0$, the numerator is $(\underline{v}_1+\underline{v}_2)(T+1)\bar{v}_1-(\underline{v}_1+\underline{v}_2)^2 < (\underline{v}_1+\underline{v}_2)(\frac{\underline{v}_1+\underline{v}_2-\bar{v}_1}{\bar{v}_1}+1)\bar{v}_1-(\underline{v}_1+\underline{v}_2)^2$ $=0$. So the numerator of derivative (a quadratic function) of deviating payoff will be negative and may turn to positive afterwards since coefficient for term x^2 is negative while coefficient for term x is positive. In fact, if we plug $a_1=\frac{\underline{v}_1+\underline{v}_2-\bar{v}_1(1+T)}{1-T}$ into the derivative, the quadratic function in numerator is $[\frac{2\bar{v}_1(1+T)}{1-T}-(\underline{v}_1+\underline{v}_2)\frac{1+T}{1-T}][\underline{v}_1+\underline{v}_2-(1+T)\bar{v}_1]>0$. So we just need to compare deviating payments when high type bids 0 since computation above reveals that derivative below a_1 is initially negative and will eventually turn positive. We need $\bar{v}_1-a_2 \geq (1-p)(\bar{v}_1-0)$. And it is satisfied by an equality since $a_2=\bar{v}_1 p$. So we know that high type will not deviate to bid anything below a_1 unless she is bidding 0. ■

We propose lemma II.15 by simply assuming low type is bidding the same in region R_1. And we can check we have a result similar to lemma II.13 when $p \in (\frac{\underline{v}_2(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}{2\bar{v}_1^2-2\bar{v}_1\underline{v}_1+\underline{v}_1^2-\underline{v}_2^2}, \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1})$.

Lemma II.16 *When* $\frac{\underline{v}_2(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}{2\bar{v}_1^2-2\bar{v}_1\underline{v}_1+\underline{v}_1^2-\underline{v}_2^2} < p < \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1}$ *and* $\underline{v}_1+\underline{v}_2+\sqrt{\underline{v}_1^2-\underline{v}_2^2} > 2\bar{v}_1$, *low type will bid according to* $G_{L1}(x)=\frac{C}{\underline{v}_2-x}-\frac{1-p}{p}T$ *and* $G_{L2}(x)=\frac{(1-p)x-pC+(1-p)T\underline{v}_2}{p(\underline{v}_1-x)}$ *in interval* $I=[a_3,a_4]$, *with* $G_{L2}(h(x))=G_{L1}(x)$ *where* $h(x)=\frac{-\underline{v}_2(\underline{v}_2-p(2\bar{v}_1+C))(\underline{v}_2-x)+\underline{v}_1^2 p(-\underline{v}_2+x)+\underline{v}_1(-\underline{v}_2^2(1+p)-2\bar{v}_1 px+\underline{v}_2(2\bar{v}_1 p+x+p(C+x)))}{p(2\bar{v}_1(\underline{v}_2-x)+b(-\underline{v}_2+x)+c(-\underline{v}_2+C+x))}$. *And low type will bid by* $G_{L1}(x)=G_{L2}(x)=\frac{(1-p)x+T(1-p)x}{p(\underline{v}_1+\underline{v}_2-2x)}$ *in interval* $[0,a_3]$. *We express* $T=\frac{p\underline{v}_1+\underline{v}_2-2p\bar{v}_1}{(1-p)\underline{v}_2}$ *and* $C=\frac{(\bar{v}_1-\underline{v}_1-\underline{v}_2)(-\underline{v}_2+\bar{v}_1 p)}{\underline{v}_2 p}$. *What's more,* $a_4=a_1$ *introduced in* *lemma II.14.*

Proof. Similar to lemma II.13, we still define I as the first non-trivial (i.e. positive measure) interval where $b_{l2} = h(b_{l1})$ solves the first order condition and $h(b_{l1}) < b_{l1}$ on interior of interval I. We denote $a_3 = \inf_{x \in I} I$ and $a_4 = \sup_{x \in I} I$. By construction we have $h(a_3) = a_3$ and $h(a_4) = a_4$.

Recall expected payoff for low type in region R_1 is $p[G_{L2}(b_{l1})(\underline{v}_1 - b_{l1}) + G_{L1}(b_{l2})(\underline{v}_2 - b_{l2})] + (1-p)(\underline{v}_1 - b_{l1}) + (1-p)T(\underline{v}_2 - b_{l2})$ and first order condition with respect to b_{l2} will be $p[g_{L1}(b_{l2})(\underline{v}_2 - b_{l2}) - G_{L1}(b_{l2})] - (1-p)T = 0$. We express $G_{L1}(x) = \frac{C}{\underline{v}_2 - x} - \frac{1-p}{p}T$ with some constant C to be determined. Plugging $G_{L1}(b_{l2}) = \frac{C}{\underline{v}_2 - b_{l2}} - \frac{1-p}{p}T$ into the indifferent condition of low type, which is $p[G_{L2}(b_{l1})(\underline{v}_1 - b_{l1}) + G_{L1}(b_{l2})(\underline{v}_2 - b_{l2})] + (1-p)(\underline{v}_1 - b_{l1}) + (1-p)T(\underline{v}_2 - b_{l2}) = \underline{v}_1(1-p) + (1-p)T\underline{v}_2$, we can solve $G_{L2}(x) = \frac{(1-p)x - pC + (1-p)T\underline{v}_2}{p(\underline{v}_1 - x)}$. Similar to lemma II.13, $G_{L1}(x) = G_{L2}(x)$ at $a_3 < a_4$ so we have $\frac{C}{\underline{v}_2 - x} - \frac{1-p}{p}T = \frac{(1-p)x - pC + (1-p)T\underline{v}_2}{p(\underline{v}_1 - x)}$ when $x = a_3$ and $x = a_4$. Rearranging equation above presents $C = \frac{(1-p)x + (1-p)H\underline{v}_2}{p(\underline{v}_1 + \underline{v}_2 - 2x)}(\underline{v}_2 - x) + \frac{(1-p)T(\underline{v}_1 - x)(\underline{v}_2 - x)}{p(\underline{v}_1 + \underline{v}_2 - 2x)} = \frac{(1-p)x(\underline{v}_2 - x)}{p(\underline{v}_1 + \underline{v}_2 - 2x)} + \frac{(1-p)T(\underline{v}_2 - x)(\underline{v}_1 + \underline{v}_2 - x)}{p(\underline{v}_1 + \underline{v}_2 - 2x)}$ when $x = a_3$ or a_4. Taking derivative on C with respect to x will give us $(1-p)\frac{2(1-T)x^2 - 2(\underline{v}_1 + \underline{v}_2)(1-T)x - (\underline{v}_1 + \underline{v}_2)(T\underline{v}_1 - \underline{v}_2)}{p(\underline{v}_1 + \underline{v}_2 - 2x)^2}$. The derivative is positive when $x < \frac{1}{2}[\underline{v}_1 + \underline{v}_2 - \sqrt{\frac{(1+T)(\underline{v}_1^2 - \underline{v}_2^2)}{1-T}}]$ and $x > \frac{1}{2}[\underline{v}_1 + \underline{v}_2 + \sqrt{\frac{(1+T)(\underline{v}_1^2 - \underline{v}_2^2)}{1-T}}]$. By a similar argument from lemma II.13, if we still denote a_3 as left endpoint of I and a_4 as right endpoint of I, we must have $a_3 < \frac{1}{2}[\underline{v}_1 + \underline{v}_2 - \sqrt{\frac{(1+T)(\underline{v}_1^2 - \underline{v}_2^2)}{1-T}}] < a_4$ and that $a_4 = a_1$. So we have equations $\frac{C}{\underline{v}_2 - x} - \frac{1-p}{p}H = \frac{(1-p)x - pC + (1-p)H\underline{v}_2}{p(\underline{v}_1 - x)} = \frac{\bar{v}_1 p - a_2 + (1-p)x}{p(\bar{v}_1 - x)}$ when $x = a_4$, similar to lemma II.13. Solving this equation we get $\frac{\underline{v}_2(2\bar{v}_1 - \underline{v}_1 - \underline{v}_2)}{2\bar{v}_1^2 - 2\bar{v}_1\underline{v}_1 + \underline{v}_1^2 - \underline{v}_2^2} < p < \frac{\underline{v}_2}{2\bar{v}_1 - \underline{v}_1}$ and $C = \frac{(\bar{v}_1 - \underline{v}_1 - \underline{v}_2)(-\underline{v}_2 + \bar{v}_1 p)}{\underline{v}_2 p}$ as long as $2\bar{v}_1 < \underline{v}_1 + \underline{v}_2 + \sqrt{\underline{v}_1^2 - \underline{v}_2^2}$.

We can also confirm that $2(1-T)x^2 - 2(1-T)(\underline{v}_1 + \underline{v}_2)x - (\underline{v}_1 + \underline{v}_2)(T\underline{v}_1 - \underline{v}_2)$ is positive when $x \leq a_3$ given range of p and expression of C provided in the last paragraph, which shows existence of perfectly correlated equilibrium for low type when $x \in R_1 \setminus I = (0, a_3)$. What's more, solving $G_{L2}(h(x)) = G_{L1}(x)$ gives us $h(x) = \frac{-\underline{v}_2(\underline{v}_2 - p(2\bar{v}_1 + C))(\underline{v}_2 - x) + \underline{v}_1^2 p(-\underline{v}_2 + x) + \underline{v}_1(-\underline{v}_2^2(1+p) - 2\bar{v}_1 px + \underline{v}_2(2\bar{v}_1 p + x + p(C+x)))}{p(2\bar{v}_1(\underline{v}_2 - x) + b(-\underline{v}_2 + x) + c(-\underline{v}_2 + C + x))}$ with derivative $\frac{\underline{v}_2^2 C(\underline{v}_1 + \underline{v}_2 - p(2\bar{v}_1 + C))}{p((2\bar{v}_1 - \underline{v}_1)(\underline{v}_2 - x) + \underline{v}_2(-\underline{v}_2 + C + x))^2}$. We need $\underline{v}_1 + \underline{v}_2 - p(2\bar{v}_1 + C) > 0$ for an increasing function $h(x)$. And this condition is consistent with the p, C expressions computed last paragraph.

Similar to argument in lemma II.13, payoff from first and second bid of low type are constructed to be constant in interval I. If second bid of low type deviates downward to become smaller than a_3, the optimal deviating bid should be bidding at a_3 because we know that in perfectly correlated equilibrium payment from second bid is strictly increasing. Similarly if first bid of low type deviates upward to be higher than a_4, the deviating bid better be bidding $a_4 = a_1$ since in perfectly correlated equilibrium payment from first bid is strictly decreasing. If high type deviates to bid below a_1 and chooses to bid in I, her expected payoff will be $(1-p)(\bar{v}_1 - b_{h1}) + p\frac{(1-p)b_{h1} - pC + (1-p)T\underline{v}_2}{p(\underline{v}_1 - b_{h1})}(\bar{v}_1 - b_{h1})$ with derivative $-\frac{\bar{v}_1(\bar{v}_1 - \underline{v}_1)[\frac{\bar{v}_1 - \underline{v}_1 + \underline{v}_2}{\underline{v}_2}p - 1]}{(\underline{v}_1 - b_{h1})^2}$. Given range of p, $1 - \frac{\bar{v}_1 - \underline{v}_1 + \underline{v}_2}{\underline{v}_2}p > 1 - \frac{\bar{v}_1 - \underline{v}_1 + \underline{v}_2}{\underline{v}_2}\frac{\underline{v}_2}{2\bar{v}_1 - \underline{v}_1} = \frac{\bar{v}_1 - \underline{v}_2}{2\bar{v}_1 - \underline{v}_1} > 0$ and hence the derivative is positive. So bidding in interior of I will be dominated by bidding at a_1. Proof of lemma II.15 can be used to show that high type should not be bidding below a_3.

It is easy to exclude deviations above the common upper bound: all distributions are atomless at upper bounds. So bidding $(\bar{v}_1 p, \bar{v}_1 p)$ ($\bar{v}_1 p$ is the upper bound in this scenario) will give low type two objects with certainty and bidding $\bar{v}_1 p$ will give high type one object with certainty. And hence bidding above the upper bound will only decrease the expected payoff for any type. ■

Graphical Illustration

We will also demonstrate lemma II.16 separately since it shows situation where bids from low type are distinct. We pick $\bar{v}_1 = 7$, $\underline{v}_1 = 6$, $\underline{v}_2 = 3$ and $p = 0.37$. Two bids from low type will be different in interval $[1.8, 1.892]$. We still know that $G_{L1}(x) = G_{L2}(h(x))$ in interval $[1.8, 1.892]$ with $h(x) < x$ for $x \in (1.8, 1.892)$ and $h(x) = x$ at endpoints as in the graphical illustration in the last subsection.

We illustrate density functions of equilibrium distributions graphically:

The first graph is when $\bar{v}_1 = 4, \underline{v}_1 = 3, \underline{v}_2 = 2$ and $p = \frac{3}{8}$, which covers points 1,2, 3.a and 4 of theorem II.9. Support for distribution of high type is $[\frac{2}{3}, \frac{3}{2}]$ and support for distributions of low type is $[0, \frac{3}{2}]$. Note that there will be an atom of size $\frac{1}{10}$ for distribution of mixed strategy of high type at 0.

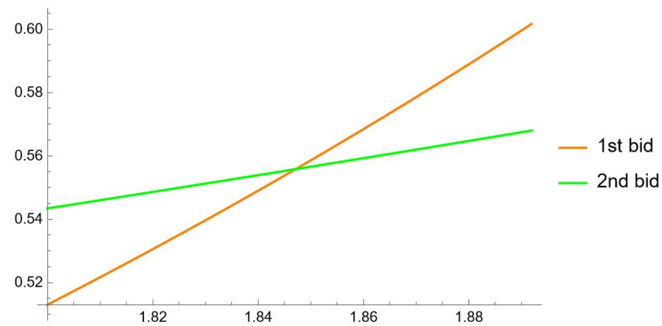

Figure 2.6: Illustration of lemma II.16

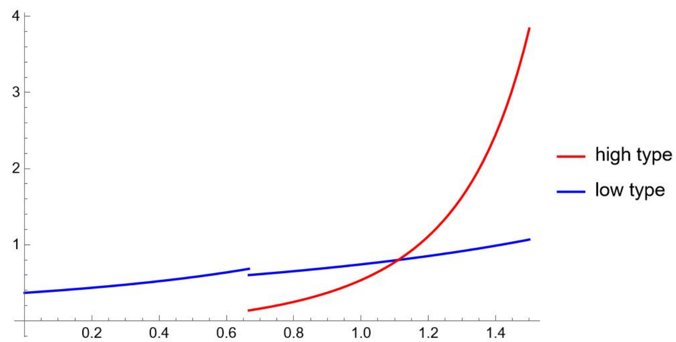

Figure 2.7: Iluustration of points 1,2, 3.a and 4 of theorem II.9

We continue to select $\bar{v}_1 = 7, \underline{v}_1 = 6, \underline{v}_2 = 3$ for the second and third graphs and the second graph is when $p = 0.37$, which covers points 1,2, 3.b.i and 4 of theorem II.9, when there is an interval where first and second bid of low type are different. Support for distributions of low type is $[0, 2.6]$ and support for distribution of high type is $[1.89, 2.6]$. The third graph is when $p = 0.35$ where two bids of low type are always identical, as shown in points 1,2 , 3.b.ii and 4 of theorem II.9. Support for distributions of low type is $[0, 2.45]$ and support for distribution of high type is $[1.43, 2.45]$. Note that there will be an atom of size 0.021 for distribution of mixed strategy of high type at 0.

We can also illustrate density functions for low type only:

The pdfs only differ with bids in $[1.8, 1.89]$.

Analogous to corollary II.3, we have a similar result regarding pdf at a_1:

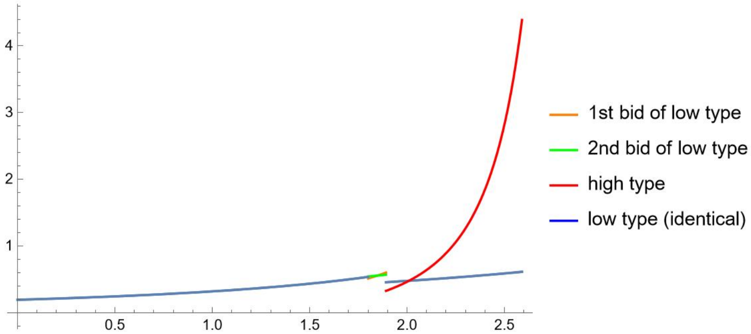

Figure 2.8: Illustration of points 1,2, 3.b.i and 4 of theorem II.9

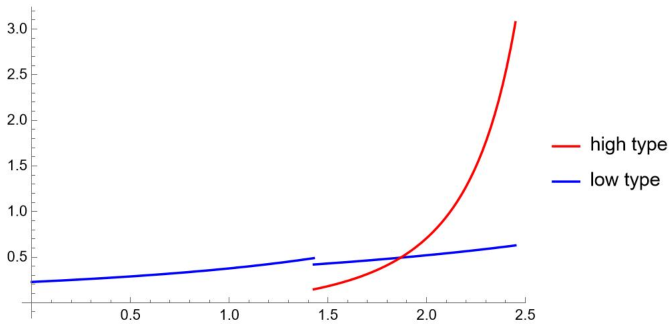

Figure 2.9: Illustration of points 1,2, 3.b.ii and 4 of theorem II.9

Corollary II.4 *For $p \in (\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2}, \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1})$, we have the following results:*

1. *When $\underline{v}_1 + \underline{v}_2 + \sqrt{\underline{v}_1^2 - \underline{v}_2^2} \leq 2\bar{v}_1$ or $\underline{v}_1 + \underline{v}_2 + \sqrt{\underline{v}_1^2 - \underline{v}_2^2} > 2\bar{v}_1$ but $p < \frac{\underline{v}_2(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}{2\bar{v}_1^2-2\bar{v}_1\underline{v}_1+\underline{v}_1^2-\underline{v}_2^2}$, left derivative of low type's distribution at a_1 will be greater than right derivative of low type at a_1*

2. *When $\underline{v}_1 + \underline{v}_2 + \sqrt{\underline{v}_1^2 - \underline{v}_2^2} > 2\bar{v}_1$ but $p > \frac{\underline{v}_2(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}{2\bar{v}_1^2-2\bar{v}_1\underline{v}_1+\underline{v}_1^2-\underline{v}_2^2}$, left derivative will satisfy $\frac{dG_{L1}(x)}{dx} > \frac{dG_{L2}(x)}{dx}$ at a_1 and left derivative $\frac{dG_{L2}(x)}{dx}$ in region R_1 at a_1 will be greater than right derivative $\frac{dG(x)}{dx}$ in region R_3 at a_1.*

Corollary II.5 *With $\bar{v}_1 < \underline{v}_1 + \underline{v}_2$, results in theorem III.2 are valid when $p < \frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2}$; results in theorem III.3 are valid when $\frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1} < p < \frac{\underline{v}_2}{\underline{v}_1}$. What's more, corollary* **??** *holds for $\bar{v}_1 < \underline{v}_1 + \underline{v}_2$ when $\frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1} < p < \frac{\underline{v}_2}{\underline{v}_1}$.*

2.5.3 When $p \geq \frac{\underline{v}_2}{\underline{v}_1}$

Theorem II.10 *When $p \geq \frac{\underline{v}_2}{\underline{v}_1}$, high type will bid $\underline{v}_2$ and low type will be bidding $\underline{v}_2$ and by distribution $G_{L2}(x) = \frac{(1-p)x+p\underline{v}_1-\underline{v}_2}{p(\underline{v}_1-x)}$ in interval $[0, \underline{v}_2)$ for her first and second*

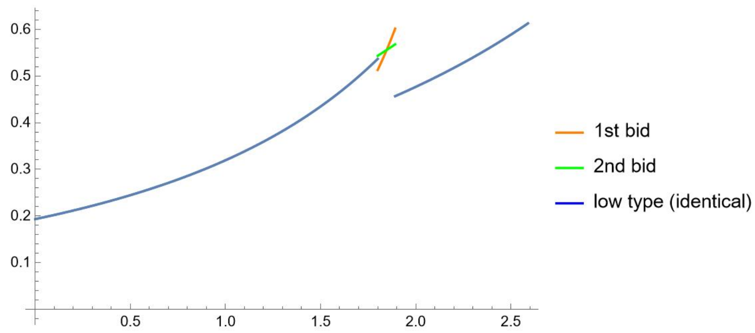

Figure 2.10: Low type's pdf

bid respectively.

Proof. Abusing notation, high type will not deviate to bid $b_{h1} < \underline{v}_2$ if $pG_{L2}(b_{h1})(\bar{v}_1 - b_{h1}) + (1-p)(\bar{v}_1 - b_{h1}) \leq \bar{v}_1 - \underline{v}_2 \iff G_{L2}(x) \leq \frac{(1-p)x + p\bar{v}_1 - \underline{v}_2}{p(\bar{v}_1 - x)}$ and low type will not deviate to make her first bid $b_{l1} < \underline{v}_2$ for her first bid if $pG_{L2}(b_{l1})(\underline{v}_1 - b_{l1}) + (1-p)(\underline{v}_1 - b_{l1}) \leq \underline{v}_1 - \underline{v}_2 \iff G_{L2}(x) \leq \frac{(1-p)x + p\underline{v}_1 - \underline{v}_2}{p(\underline{v}_1 - x)}$.

Some computation will show that $\frac{(1-p)x + p\underline{v}_1 - \underline{v}_2}{p(\underline{v}_1 - x)} < \frac{(1-p)x + p\bar{v}_1 - \underline{v}_2}{p(\bar{v}_1 - x)}$ as long as $x < \underline{v}_2$.

If we let $G_{L2}(x) = \frac{(1-p)x + p\underline{v}_1 - \underline{v}_2}{p(\underline{v}_1 - x)}$, with $G_{L2}(x) = 1$ when $x = \underline{v}_2$, we can successfully support bids from high type and first bid of low type to be degenerated on $\underline{v}_2$. When $x = 0$, $G_{L2}(x) = \frac{p\underline{v}_1 - \underline{v}_2}{p\underline{v}_1}$. So as long as $p \geq \frac{\underline{v}_2}{\underline{v}_1}$, distribution $G_{L2}(x)$ is valid. When $p > \frac{\underline{v}_2}{\underline{v}_1}$, the low type will put an atom with size $\frac{p\underline{v}_1 - \underline{v}_2}{p\underline{v}_1}$ for G_{L2} when $x = 0$.

The indifferent condition of first bid from low type is binding only when $p = \frac{\underline{v}_2}{\underline{v}_1}$, which implies that both types strictly prefer bidding $\underline{v}_2$ with higher p. What's more, when $p > \frac{\underline{v}_2}{\underline{v}_1}$ and high (low) type unilaterally deviates to bid 0, high (low) type will only get half object on average when second bid of low type is 0, (because deviating to bid 0 means first bid ties with opponent's second bid which is strictly smaller than opponent's first bid), which leads to payoff strictly smaller than $\bar{v}_1 - \underline{v}_2$ ($\underline{v}_1 - \underline{v}_2$). When high type and first bid of low type bid a pure strategy on $\underline{v}_2$, second bid of low type will not win and will not generate any positive expected payoff. A low type will be indifferent to any distribution on her second bid.

When $p \to 1$, $G_{L2}(x)$ will also be degenerated on $\underline{v}_2$. And the equilibrium result will converge to low type bidding 1 for both bids, and each bidder get exactly 1 object, i.e. the pure strategy equilibrium found in lemma II.7 ■

Remark II.11 *This equilibrium is obviously not unique. We can pick any $x > 0$ on support of G_{L2} and construct an atom at x by truncate the probability for values strictly below x to be exactly at x.*

2.6 Other Cases

We have studied case where high type has marginal valuation $(\bar{v}_1, \bar{v}_2)$ and low type has marginal valuation $(\underline{v}_1, \underline{v}_2)$ where $\bar{v}_1 > \underline{v}_1 > \underline{v}_2 > \bar{v}_2$. In this section, we will show mixed strategy equilibrium of the other two cases. We continue to assume that probability a low type appears in the population is p so probability a high type appears in the population is $1-p$. Additionally, we still assume that F_{H1}, F_{H2} are marginal distributions of high type's first and second bids respectively while G_{L1}, G_{L2} are marginal distributions of low type's first and second bids.

2.6.1 Value Ordering $\bar{v}_1 > \underline{v}_1 > \bar{v}_2 > \underline{v}_2$

We can state the mixed strategy equilibrium when value ordering is $\bar{v}_1 > \underline{v}_1 > \bar{v}_2 > \underline{v}_2$:

Theorem II.12 *When $p \leq 1 - \frac{\bar{v}_2}{\underline{v}_1}$, first bids of both type will be pure strategy at $\underline{v}_2$ and second bid of high type will follow $F_{H2}(x) = \frac{(\underline{v}_1 - \bar{v}_2) - p(\underline{v}_1 - x)}{(1-p)(\underline{v}_1 - x)}$.*

Proof. High type will not deviate her first bid to $b_{h1} < \bar{v}_2$ when $(1-p)F_{H2}(x)(\bar{v}_1 - b_{h1}) + p(\bar{v}_1 - b_{h1}) \leq \bar{v}_1 - \bar{v}_2$, which makes $F_{H2}(x) \leq \frac{(\bar{v}_1 - \bar{v}_2) - p(\bar{v}_1 - x)}{(1-p)(\bar{v}_1 - x)}$. Low type will not deviate her first bid to $b_{l1} < \bar{v}_2$ when $(1-p)F_2(b_{l1})(\underline{v}_1 - b_{l1}) + p(\underline{v}_1 - b_{l1}) \leq \underline{v}_1 - \bar{v}_2$, which makes $F_{H2}(x) \leq \frac{(\underline{v}_1 - \bar{v}_2) - p(\underline{v}_1 - x)}{(1-p)(\underline{v}_1 - x)}$. Computation will show that $\frac{(\underline{v}_1 - \bar{v}_2) - p(\underline{v}_1 - x)}{(1-p)(\underline{v}_1 - x)} \leq \frac{(\bar{v}_1 - \bar{v}_2) - p(\bar{v}_1 - x)}{(1-p)(\bar{v}_1 - x)}$. So similar to the previous result, we have a mixed strategy equilibrium where first bids of both type are degenerate at $\bar{v}_2$ and second bid of high type follows distribution $F_{H2}(x) = \frac{(\underline{v}_1 - \bar{v}_2) - p(\underline{v}_1 - x)}{(1-p)(\underline{v}_1 - x)}$ with support $[0, \bar{v}_2]$. When $x = 0$, $F_{H2}(x) = \frac{(1-p)\underline{v}_1 - \bar{v}_2}{(1-p)\underline{v}_1}$. So we should require $(1-p)\underline{v}_1 - \bar{v}_2 \geq 0$ or equivalently $p \leq 1 - \frac{\bar{v}_2}{\underline{v}_1}$.

We construct the distributions by making sure that first bids will not deviate to lower values. If first bids deviate to higher values, bidders just pay more to get lower payoff. When first bids are both at $\bar{v}_2$, second bid of high type will never get positive payoff. So high type will have no incentive to deviate her first bid. ■

Theorem II.13 *When* $p > 1 - \frac{\bar{v}_2}{\underline{v}_1}$*, low type will mix by distribution* $G_{L1}(x) = \frac{T\bar{v}_2}{\bar{v}_2 - x}$ *where* $T = \frac{-\bar{v}_1+\bar{v}_2+\bar{v}_2 p+\sqrt{\bar{v}_1^2+\bar{v}_2(-1+p)(\bar{v}_2(-1+p)-4\underline{v}_1 p)+2\bar{v}_1(-1+p)(\bar{v}_2+2\underline{v}_1 p)}}{2\bar{v}_2 p}$ *with support* $[0, a_1]$*. In interval* $[0, a_1]$ *second bid of high type will follow distribution* $F_{H2}(x) = \frac{px}{(1-p)(\underline{v}_1 - x)}$*. In interval* $[a_1, a_2]$ *first and second bid of high type will mix by distribution* $F_{H1}(x) = \frac{C}{\bar{v}_2 - x} - \frac{p}{1-p}$ *and* $F_{H2}(x) = \frac{\bar{v}_1+\bar{v}_2-2b-(1-p)C}{(1-p)(\bar{v}_1 - x)} - \frac{p}{1-p}$ *respectively. We are able to express* $C = \frac{-\bar{v}_1+\bar{v}_2+\bar{v}_2 p+\sqrt{\bar{v}_1^2+\bar{v}_2(-1+p)(\bar{v}_2(-1+p)-4\underline{v}_1 p)+2\bar{v}_1(-1+p)(\bar{v}_2+2\underline{v}_1 p)}}{2(1-p)}$, $a_1 = \bar{v}_2(1-H)$ *and* $a_2 = \bar{v}_2 - (1-p)C$*.*

Proof. We suppose that G_{L1} will have support $[0, a_1]$, F_{H1} will have support $[a_1, a_2]$ and F_{H2} will have support $[0, a_2]$. To be more precise, we require an $a_3 \in (a_1, a_2)$ so that when first bid of high type is bidding in interval (a_1, a_3), second bid of high type will be bidding in interval $(0, a_1)$. And when first bid of high type is in interval (a_3, a_2), second bid of high type will be in interval (a_1, a_2).

Consider indifferent condition for low type, which is $p(\underline{v}_1 - b_{l1}) + (1-p)F_{H2}(b_{l1})(\underline{v}_1 - b_{l1}) = p\underline{v}_1$. So we have $F_{H2}(x) = \frac{px}{(1-p)(\underline{v}_1 - b_{l1})}$ on $(0, a_1)$. Note that $\frac{px}{(1-p)(\underline{v}_1 - x)} = 1$ when $x = (1-p)\underline{v}_1$. We require $a_1 < (1-p)\underline{v}_1$ since upper bound of support for F_{H2} is $a_2 > a_1$.

Consider indifferent condition for high type when first bid of high type is in (a_3, a_2) and second bid is in (a_1, a_2):
$p(\bar{v}_1 + \bar{v}_2 - b_{l1} - b_{l2}) + (1-p)[F_{H2}(b_{l1})(\bar{v}_1 - b_{l1}) + F_{H1}(b_{l2})(\bar{v}_2 - b_{l2})] = \bar{v}_1 + \bar{v}_2 - 2a_2$. By construction it is unlikely that bids from high type are perfectly correlated (supports for first and second bids have different measure) and we need to construct separating equilibrium. We assume a bidder is maximizing her expected payoff by choosing the optimal second bid b_{l2} given any first bid b_{l1}, so we have $-p + (1-p)[f_{H1}(b_{l2})(\bar{v}_2 - b_{l2}) - F_{H1}(b_{l2})] = 0$ by taking first order derivative with respect to y. Solving the differential equation, we have $F_{H1}(y) = \frac{C}{\bar{v}_2 - y} - \frac{p}{1-p}$ on interval (a_1, a_2) with some constant C to be determined. (Note that we are looking at symmetric mixed strategies.) Note that expected payoff for second bid of high type is $C(1-p)$, which is constant. And we solve $F_{H2}(x) = \frac{\bar{v}_1+\bar{v}_2-2b-(1-p)C}{(1-p)(\bar{v}_1 - x)} - \frac{p}{1-p}$ on interval (a_1, a_2). The two distribution functions should match when $F_{H1}(a_2) = F_{H2}(a_2) = 1$ by construction. And we solve $a_2 = \bar{v}_2 + (-1+p)C$.

When first bid of high type is in (a_1, a_3) and second bid of high type is in $(0, a_1)$, indifferent condition for high type will become $p[\bar{v}_1 - b_{l1} + G_{L1}(b_{l2})(\bar{v}_2 - b_{l2})] + (1-p)F_{H2}(x)(\bar{v}_1 - b_{l1}) = \bar{v}_1 + \bar{v}_2 - 2a_2$. Note that expected payoff for second bid is $pG_{L1}(b_{l2})(\bar{v}_2 - b_{l2})$, which indicates that second bid of high type is only possible to win from low type. We argue high type is getting constant payoff from her second bid: if $pG_{L1}(b_{l2})(\bar{v}_2 - b_{l2})$ is not constant and high type can get higher payoff by bidding at b_{l2}^* than any other bids, high type will be always bidding such b_{l2}^* regardless of how she bids her first bid when she faces the indifferent condition mentioned above. And hence high type will not be randomizing in interval $(0, a_1)$. An additional requirement is that the constant payoff high type is getting for her second bid is positive, which requires low type to put an atom with size T at 0. So high type will get $pG_{L1}(b_{h2})(\bar{v}_2 - b_{h2}) = pT\bar{v}_2$ when her second bid b_{h2} is below a_1 and we better require $pT\bar{v}_2 = C(1-p)$ so that high type will not want to deviate her second bid. So $G_{L1}(x) = \frac{T\bar{v}_2}{\bar{v}_2 - x}$, which reaches 1 when $x = \bar{v}_2(1-T)$. And we require $a_1 = \bar{v}_2(1-T) < \underline{v}_1(1-p)$. We continue to solve $F_{H2}(x) = \frac{\bar{v}_1 + \bar{v}_2 - 2b - (1-p)C}{(1-p)(\bar{v}_1 - x)} - \frac{p}{1-p}$ on interval (a_1, a_3) since $pT\bar{v}_2 = C(1-p)$. So we have same expressions for F_{H2} on interval (a_1, a_3) and (a_3, a_2). When two expressions of F_{H2} match at a_1, we are able to solve another expression of a_1, i.e. $a_1 = \frac{\bar{v}_1\underline{v}_1 - \underline{v}_1\bar{v}_2 - \bar{v}_1\underline{v}_1 p + \underline{v}_1 C - \underline{v}_1 pC}{\bar{v}_1 - \bar{v}_2 - \underline{v}_1 p + C - pC}$ should also solve $\frac{\bar{v}_1 + \bar{v}_2 - 2b - (1-p)C}{(1-p)(\bar{v}_1 - x)} - \frac{p}{1-p} = \frac{px}{(1-p)(\underline{v}_1 - x)}$.

Additionally, since we know from indifferent condition of high type that $F_{H1}(x) = \frac{C}{\bar{v}_2 - x} - \frac{p}{1-p}$ on interval (a_1, a_2), we are able to generate another version of a_1, which is $a_1 = \frac{\bar{v}_2 p - C + pC}{p}$ by solving $F_{H1}(x) = 0$. So letting $\frac{\bar{v}_2 p - C + pC}{p} = \frac{\bar{v}_1\underline{v}_1 - \underline{v}_1\bar{v}_2 - \bar{v}_1\underline{v}_1 p + \underline{v}_1 C - \underline{v}_1 pC}{\bar{v}_1 - \bar{v}_2 - \underline{v}_1 p + C - pC}$, we solve an expression of the constant parameter $C = \frac{-\bar{v}_1 + \bar{v}_2 + \bar{v}_2 p + \sqrt{\bar{v}_1^2 + \bar{v}_2(-1+p)(\bar{v}_2(-1+p) - 4\underline{v}_1 p) + 2\bar{v}_1(-1+p)(\bar{v}_2 + 2\underline{v}_1 p)}}{2(1-p)}$, which is positive given $p > 1 - \frac{\bar{v}_2}{\underline{v}_1}$. We solve T via $pH\bar{v}_2 = C(1-p)$ and $T = \frac{-\bar{v}_1 + \bar{v}_2 + \bar{v}_2 p + \sqrt{\bar{v}_1^2 + \bar{v}_2(-1+p)(\bar{v}_2(-1+p) - 4\underline{v}_1 p) + 2\bar{v}_1(-1+p)(\bar{v}_2 + 2\underline{v}_1 p)}}{2\bar{v}_2 p}$. Similarly, as long as $p > 1 - \frac{\bar{v}_2}{\underline{v}_1}$, T is guaranteed to be positive but smaller than 1 [11]. Note that we are also able to express $a_1 = \bar{v}_2(1-T)$. Some computation will show that $\bar{v}_2(1-T) = \frac{\bar{v}_2 p - C + pC}{p}$ is equivalent to $pH\bar{v}_2 = C(1-p)$ and hence all expressions of a_1 are consistent. What's more, it is easy to see $a_1 = \bar{v}_2(1-T) < b = \bar{v}_2 + (-1+p)C < \bar{v}_2$ since $pT\bar{v}_2 = C(1-p)$ for $p \in (0,1)$. Note that we need $a_2 < \bar{v}_2$ in this scenario since b is constructed to be upper bound

[11] $H \to 1$ when $p \to 1$ and $T \to 0$ when $p \to 1 - \frac{\bar{v}_2}{\underline{v}_1}$.

of support for F_{H2} and a high type can not bid beyond $\bar{v}_2$ for her second bid.

By construction, expected payoffs for each bid of high type are constant, that is to say, first and second bid of high type are always getting expected payoffs $\bar{v}_1 + \bar{v}_2 - 2a_2 - (1-p)C$ and $C(1-p)$ respectively. So second bid of low type will not have particular incentive to deviate. Similarly, expected payoff for first bid of high type is constant, which makes a high type indifferent between bidding her first bid above or below a as long as her first bid is higher than a_1. If a high type puts both her bids b_{h1} below a_1 (i.e. first bid of low type is also below a_1), expected payoff for higher bid of high type is $p(\bar{v}_1 - b_{h1}) + (1-p)\frac{pb_{h1}}{(1-p)(\underline{v}_1 - b_{h1})}(\bar{v}_1 - x) = p(\bar{v}_1 - x)\frac{\underline{v}_1}{\underline{v}_1 - b_{h1}}$, which is an increasing function of b_{h1}. So high type would rather make her first bid at a_1. And we conclude high type will not deviate her bids. If low type deviates and bids b_{l1} higher than a_1, she gets expected payoff $p(\underline{v}_1 - b_{l1}) + (1-p)[\frac{\bar{v}_1 + \bar{v}_2 - 2b - (1-p)C}{(1-p)(\underline{v}_1 - b_{l1})} - \frac{p}{1-p}](\underline{v}_1 - b_{l1}) = \frac{\underline{v}_1 - b_{l1}}{\underline{v}_1 - b_{l1}}[\underline{v}_1 + \bar{v}_2 - 2a_2 - (1-p)C]$, which is decreasing in b_{l1}. So for a low type bidding above a_1 is dominated by bidding exactly at a_1. What's more, no type will bid higher than a_2 since all marginal distributions contain no atoms at upper bound of support and bidding (a_2, a_2) will guarantee high type two objects.

We introduce an a_3 in indifferent conditions of high type and our last task is to figure out what a_3 should be. It turns out we only need to place $a_3 \in (a_1, a_2)$ since both bids of high type are making the same constant payoffs under both indifferent conditions. So it actually does not matter which value we select as a_3 as long as it is strictly smaller than a_2 and strictly greater than a_1. In other words, we do not have the conditionally deterministic relationship between bids of the same type as in case when ordering valuation is $\bar{v}_1 > \underline{v}_1 > \underline{v}_2 > \bar{v}_2$. ■

Graphical Illustration

We demonstrate theorem II.13 by picking $\bar{v}_1 = 3$, $\underline{v}_1 = 2$, $\bar{v}_2 = 1$ and $p = \frac{2}{3}$:

Pdf of low type's mixed strategy is displayed in red, with support being $[0, 0.586]$. Second bid of high type has two parts: the blue curve in interval $[0, 0.586]$ and orange curve on interval $[0.586, 0.724]$. Pdf of first bid of high type is displayed by the green curve with support being $[0.586, 0.724]$. Note that there will be an atom of size 0.414 for distribution of mixed strategy of low type at 0.

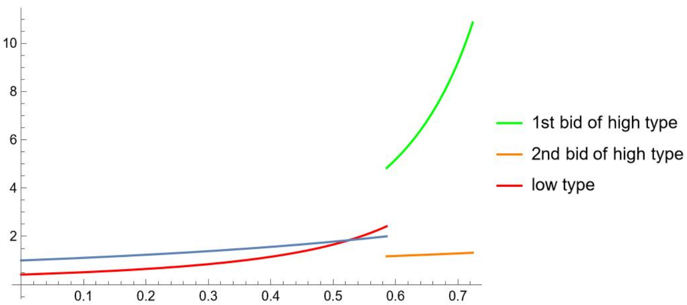

Figure 2.11: Illustration of theorem II.13

We plot an illustration of joint support of bids of high type in the graph above with the same numerical values of $\bar{v}_1, \bar{v}_2, \underline{v}_1$ and p. We pick a_3 at roughly 0.65 since by our theorem the intermediate cutoff value a_3 can be any real number between 0.586 and 0.724. When first bid is between a_1 and a_2, the second bid is below a_1, which is represented by the shaded rectangle in the plot. When first bid is between a_3 and a_2, second bid of high type will be in (a_1, a_2). But we always have an implicit condition that first bid should be no lower than second bid, so we introduce the dashed 45-degree line and denote the trapezoid as the joint support.

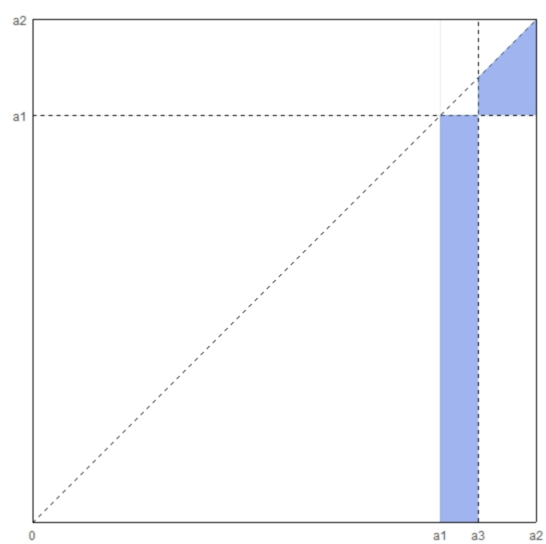

Figure 2.12: Illustration of joint support

2.6.2 Value Ordering $\bar{v}_1 > \bar{v}_2 > \underline{v}_1 > \underline{v}_2$

We will move on to show mixed strategy equilibria when value ordering becomes $\bar{v}_1 > \bar{v}_2 > \underline{v}_1 > \underline{v}_2$. And we have a result similar to theorem II.13.

Theorem II.14 *When $p \in (0,1)$, low type will mix by distribution $G_{L1}(x) = \frac{T\underline{v}_1}{\bar{v}_2 - x}$ where $T = -\frac{-\bar{v}_1+\bar{v}_2(1+p)+\sqrt{\bar{v}_1^2+(-1+p)(\bar{v}_2(\bar{v}_2(-1+p)-4\underline{v}_1 p)+2\bar{v}_1(\bar{v}_2+2\underline{v}_1 p)}}{2\bar{v}_2 p}$ with support $[0, a_1]$. In interval $[0, a_1]$, second bid of high type will follow distribution $F_{H2}(x) = \frac{px}{(1-p)(\underline{v}_1 - x)}$. In interval $[a_1, a_2]$, first and second bid of high type will mix by distribution $F_{H1}(x) = \frac{C}{\bar{v}_2 - x} - \frac{p}{1-p}$ and $F_{H2}(x) = \frac{\bar{v}_1 + \underline{v}_1 - 2b - (1-p)C}{(1-p)(\bar{v}_1 - x)} - \frac{p}{1-p}$ respectively. We are able to express $C = -\frac{-\bar{v}_1+\bar{v}_2(1+p)+\sqrt{\bar{v}_1^2+(-1+p)(\bar{v}_2(\bar{v}_2(-1+p)-4\underline{v}_1 p)+2\bar{v}_1(\bar{v}_2+2\underline{v}_1 p)}}{2(-1+p)}$, $a_1 = \bar{v}_2(1-H)$ and $a_2 = \bar{v}_2 - (1-p)C$.*

Proof. We continue to assume the same structure of support as marginal distributions in theorem II.13. So low type will face indifferent condition $p(\underline{v}_1 - b_{l1}) + (1-p)F_{H2}(b_{l1})(\underline{v}_1 - x) = p\underline{v}_1$. So we solve $F_{H2}(x) = \frac{px}{(1-p)(\underline{v}_1 - x)}$ on interval $[0, a_1]$ with $a_1 < (1-p)\underline{v}_1$.

Consider high type's indifferent condition when her first bid b_{h1} is in (a, b) and second bid b_{h2} is in (a_1, a_2):
$p(\bar{v}_1 + \bar{v}_2 - b_{h1} - b_{h2}) + (1-p)[F_{H2}(b_{h1})(\bar{v}_1 - b_{h1}) + F_{H1}(b_{h2})(\bar{v}_2 - b_{h2})] = \bar{v}_1 + \bar{v}_2 - 2a_2$. Similar to the previous theorem, we check separating equilibrium for high type: if we assume a bidder is maximizing her expected payoff by choosing the optimal second bid b_{h2} for any first bid b_{h1}, so we have $-p + (1-p)[f_{H1}(b_{h2})(\bar{v}_2 - b_{h2}) - F_{H1}(b_{h2})] = 0$ by taking first order derivative with respect to b_{h2}. Solving the differential equation, we have $F_{H1}(x) = \frac{C}{\bar{v}_2 - x} - \frac{p}{1-p}$ on interval (a_1, a_2) with some constant C to be determined. Expected payoff for second bid of high type is $p(\bar{v}_2 - b_{h2}) + (1-p)F_{H1}(b_{h2})(\bar{v}_2 - b_{h2}) = (1-p)C$ and we can solve $F_{H2}(x) = \frac{\bar{v}_1 + \bar{v}_2 - 2a_2 - (1-p)C}{(1-p)(\bar{v}_1 - x)} - \frac{p}{1-p}$ on interval (a_3, a_2). The two distribution functions should match when $x = a_2$ since we have $F_{H1}(a_2) = F_{H2}(a_2) = 1$ by construction. And we solve $a_2 = \bar{v}_2 + (-1+p)C$.

When second bid of high type b_{h2} is below a_1 and first bid b_{h1} is in interval (a_1, a_3), indifferent condition for high type will become $p[\bar{v}_1 - b_{h1} + G_{L1}(b_{h2})(\bar{v}_2 - b_{h2})] + (1-p)F_{H2}(b_{h1})(\bar{v}_1 - b_{h1}) = \bar{v}_1 + \bar{v}_2 - 2a_2$. We still conclude the constant payoff for second bid of high type is positive, which requires low type to put an atom with size T at 0. So high type will get $pG_{L1}(b_{h2})(\bar{v}_2 - b_{h2}) = pH\bar{v}_2$ when her second bid is below a_1 and we better require $pT\bar{v}_2 = C(1-p)$ so that high type will not want to deviate her second bid. So $G_{L1}(x) = \frac{T\bar{v}_2}{\bar{v}_2 - x}$, which reaches 1 when $x = \bar{v}_2(1-T)$. And we require $a_1 = \bar{v}_2(1-T) < \underline{v}_1(1-p)$. We continue to solve $F_{H2}(x) = \frac{\bar{v}_1+\bar{v}_2-2b-(1-p)C}{(1-p)(\bar{v}_1-x)} - \frac{p}{1-p}$ on interval (a_1, a_3) since $pT\bar{v}_2 = C(1-p)$. So we have same expressions for F_{H2} on interval (a_1, a_3) and (a_3, a_2). When two expressions of F_{H2} match, we solve another expression of a_1, i.e. $a_1 = \frac{\bar{v}_1\underline{v}_1 - \bar{v}_2\underline{v}_1 - \bar{v}_1\underline{v}_1 p + \underline{v}_1 C - \underline{v}_1 pC}{\bar{v}_1 + \bar{v}_2 - \underline{v}_1 p + C - pC}$ should also solve $\frac{\bar{v}_1+\bar{v}_2-2a_2-(1-p)C}{(1-p)(\bar{v}_1-x)} - \frac{p}{1-p} = \frac{px}{(1-p)(\underline{v}_1-x)}$.

Additionally, solving $F_{H1}(x) = 0$ we are able to generate another version of a_1, where $a_1 = \frac{\bar{v}_2 p - C + pC}{p}$. So letting $\frac{\bar{v}_2 p - C + pC}{p} = \frac{\bar{v}_1\underline{v}_1 - \bar{v}_2\underline{v}_1 - \bar{v}_1\underline{v}_1 p + \underline{v}_1 C - \underline{v}_1 pC}{\bar{v}_1 + \bar{v}_2 - \underline{v}_1 p + C - pC}$, we solve $C = -\frac{-\bar{v}_1+\bar{v}_2(1+p)+\sqrt{\bar{v}_1^2+\bar{v}_2(-1+p)((\bar{v}_2(-1+p)-4\underline{v}_1 p)+2\bar{v}_1(\bar{v}_2+2\underline{v}_1 p)}}{2(-1+p)}$, which is positive as long as $p \in (0,1)$. We solve T via $pT\bar{v}_2 = C(1-p)$ and $T = -\frac{-\bar{v}_1+\bar{v}_2(1+p)+\sqrt{\bar{v}_1^2+(-1+p)(\bar{v}_2(\bar{v}_2(-1+p)-4\underline{v}_1 p)+2\bar{v}_1(\bar{v}_2+2\underline{v}_1 p)}}{2\bar{v}_2 p}$. Similarly to C, T is guaranteed to be positive but smaller than 1 for all $p \in (0,1)$ [12].. What differs this theorem from theorem II.13 is that in this scenario we only require $a_2 = \bar{v}_2 + (-1+p)C < \bar{v}_2$ because marginal valuation of second object of high type is now $\bar{v}_2$ and low type never bids above a_1. (We will argue $a_1 < \underline{v}_1$ later.) We may move on to compute that the upper bound of support b is smaller than $\underline{v}_1$ *if and only if* $p \in (\frac{1}{2}[\frac{\bar{v}_1(-\bar{v}_2+\underline{v}_1)}{(\bar{v}_1-\bar{v}_2)\underline{v}_1} + \frac{\sqrt{(\bar{v}_2-\underline{v}_1)(4\bar{v}_2\underline{v}_1^2-4\bar{v}_1\underline{v}_1(\bar{v}_2+\underline{v}_1)+\bar{v}_1^2(\bar{v}_2+3\underline{v}_1)}}{(\bar{v}_1-\bar{v}_2)\underline{v}_1}], 1)$, which indicates a very intuitive result: when p, the probability of low type appearing in the population is relatively large, high type will focus on outbidding low type and hence a high type will not bid above $\underline{v}_1$, the highest marginal valuation a low type will have; when probability of low type appearing in the population is relatively small, a high type will focus on outbidding another high type, which indicates that bids for high type will surpass $\underline{v}_1$ but not $\bar{v}_2$ since second bid of high type will never be higher than $\bar{v}_2$. At the same time, to make sure that $a_1 = \bar{v}_2(1-T) < \underline{v}_1(1-p)$, we only need $p \in (0,1)$. Additionally, it is easy to see $a_1 < \underline{v}_1(1-p) < \underline{v}_1$ and $a_1 = \bar{v}_2(1-T) < b = \bar{v}_2 + (-1+p)C < \bar{v}_2$ since we construct $pT\bar{v}_2 = C(1-p)$.

By construction, expected payoffs for each bid of high type are constant, that is to say, first and second bid of high type are always getting expected payoffs

[12] $T \to 1$ when $p \to 1$ and $H \to 0$ when $p \to 0$.

$\bar{v}_1 + \underline{v}_1 - 2b - (1-p)C$ and $C(1-p)$ respectively. So second bid of low type will not have particular incentive to deviate. Similarly, expected payoff for first bid of high type is constant, which makes a high type between indifferent bidding first bid above or below a as long as her first bid is higher than a_1. If a high type puts both her bids below a_1 (i.e. first bid of low type is below a_1), expected payoff for higher bid of high type is $p(\bar{v}_1 - b_{h1}) + (1-p)\frac{pb_{h1}}{(1-p)(\underline{v}_1 - b_{h1})}(\bar{v}_1 - x) = p(\bar{v}_1 - b_{h1})\frac{\bar{v}_2}{\underline{v}_1 - b_{h1}}$, which is an increasing function of b_{h1}. So high type would rather make her first bid at a_1. And we conclude high type will not deviate her bids. If low type deviates and bid higher than a_1, she gets
$p(\underline{v}_1 - b_{l1}) + (1-p)[\frac{\bar{v}_1 + \underline{v}_1 - 2a_2 - (1-p)C}{(1-p)(\bar{v}_1 - b_{l1})} - \frac{p}{1-p}](\underline{v}_1 - b_{l1}) = \frac{\underline{v}_1 - b_{l1}}{\bar{v}_1 - b_{l1}}[\bar{v}_1 + \underline{v}_1 - 2a_2 - (1-p)C]$, which is decreasing in b_{l1}. So for a low type, bidding above a_1 is dominated by bidding exactly at a_1. What's more, no type will bid higher than a_2 since the marginal distributions will have no atoms at upper bound of support and bidding (a_2, a_2) will guarantee high type two objects.

Note that our two indifferent conditions for high type indicate that when first bid of high type is bidding below a threshold $a_3 < a_2$, second bid of high type will be no greater than a_1. But since we have established that F_{H1}, the marginal distribution of first bid of high type, will be following the same functional form in both scenarios and that both bids of high type are making the same constant payoffs in both scenarios, it actually does not matter which value we select as a_3 as long as it is strictly smaller than b and greater than a_1. ■

In this subsection, we do not have an analogous result to theorem II.12, since theorem II.12 in this scenario would require high type to bid $(\underline{v}_1, \underline{v}_1)$ for both bids and low type to mix in interval $[0, \underline{v}_1]$. However, in this scenario when a low type faces another low type, she will want to bid 0 instead of mixing in any interval.

Graphical Illustration

We demonstrate theorem II.14 by picking $\bar{v}_1 = 3$, $\bar{v}_2 = 2$, $\underline{v}_1 = 1$ and $p = \frac{2}{3}$:

Pdf of low type's mixed strategy is displayed in red, with support being $[0, 0.149]$. Second bid of high type has two parts: the blue curve in interval $[0, 0.586]$ and orange curve on interval $[0.149, 0.766]$. Pdf of first bid of high type is displayed by the green curve with support being $[0.149, 0.766]$. Note that there will be an atom of size 0.925 for distribution of mixed strategy of low type at 0, which is

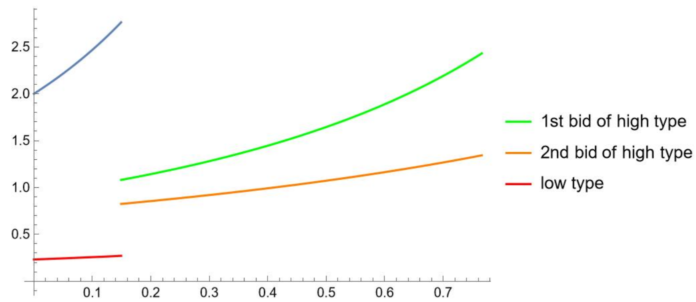

Figure 2.13: Illustration of theorem II.14

understandable since low type's marginal valuation is only $\underline{v}_1$.

2.6.3 Explaining Bidding Behaviour

One of the most prominent feature we have for theorem III.2 and III.3 is that realization of two bids for low type will typically be identical. Mathematically speaking, we have argued that this is because expected revenue of each marginal bid is monotone. And we can provide an intuitive explanation for the identical bidding behaviour we found for theorem III.2 and III.3. We establish in lemma II.1 that her second bid of any bidder is competing with her opponent's first bid and vice versa. So for results like theorem III.2 and III.3, where low type's both bids usually have overlapping of support with high type's bid, low type will understand that her first bid is competing against her opponent's second bid. We know that for any bidder, the first bid is the higher bid while the second bid is the lower bid. Low type will understand her first bid is competing against her opponent's lower bid, and her first bid will probably win. So low type will have incentive to decrease her first bid for a higher net payoff. On the other hand, low type also understands that her second bid is competing against her opponent's higher bid, and low type needs to raise her first bid in order to win and get some payoff. These two forces described above will keep being effective until low type's two bids are identical.

However, we do not have the identical bidding behaviour for theorem II.13 and II.14. And we may still provide an intuitive explanation using implication of lemma II.1. The most significant contrast between theorem II.13 and II.14 and theorem III.2 and III.3 is that for theorem II.13 and II.14, distribution of high type's two bids do not have overlapping of support with distribution of low type's bids. On the contrary, as pointed out by theorem II.13 and II.14, support of equilibrium strategy the low type will be using is $[0, a_1]$ while support of equilibrium strategy the first and second bid the high type is using are respectively $[a_1, a_2]$ and $[0, a_2]$. We can look at high type's behaviour on support $[a_1, a_2]$: by lemma II.1, high type understands her second bid is competing with her opponent's higher bid. But high type also realizes that low type's first bid is only distributed in interval $[0, a_1]$ and accordingly high type will have incentive to decrease her second bid since second bid of high type is guaranteed to win when distributing in interval $[a_1, a_2]$. On the other hand, when facing first bid of another high type on support $[a_1, a_2]$, high type will have incentive to raise her second bid. So for scenarios like theorem II.13 and II.14, we have contradicting forces for second bid of high type, and we can not tell which force is dominating. And accordingly, we do not have identical bidding behaviour for theorem II.13 and II.14.

2.7 More Than Two Units

In the Introduction, we highlight that bidding behavior in the Turkish Treasury auction takes the form of a step-function. That is, there are quantities $q_1 < \cdots < q_K$, so that bids jump downward at each quantity q_k but bids are constant at all quantities between q_k and q_{k+1}. (See Hortaçsu and McAdams, 2010.) This section shows, by way of example, that bidding behavior in a multi-unit auction can take the form of a step function.

To do so, we focus on the minimal environment that can distinguish a step function from either a cutoff rule or a strictly negatively sloped bid function: an environment with four units. We provide two examples in which the low type's bidding behavior is consistent with a step function. (The focus on the low type is only for tractability.) The two examples differ in the qualitative nature of the step functions: In the first example, the bids are same for the first two units; in the second example, the bids are the same for the middle two units.

There are four identical units and two bidders that are ex-ante identical. The high type ($\bar{v}$) has marginal valuations $(\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4)$ with $\bar{v}_1 > \bar{v}_2 > 0$ and $\bar{v}_3 = \bar{v}_4 = 0$. The low type has marginal valuations $(\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4)$, $\underline{v}_1 > \underline{v}_2 > \underline{v}_3 \geq \underline{v}_4 \geq 0$. The probability of the low type ($\underline{v}$) is p and the probability of the high type ($\bar{v}$) is $1 - p$.

Now a bid for i is a profile $(b_{i1}, b_{i2}, b_{i3}, b_{i4}) \in R_+^4$ with $b_{i1} \geq b_{i2} \geq b_{i3} \geq b_{i4}$. We refer to b_{in} as bidder i's n^{th} bid. For a given strategy of bidder i, the marginal distribution of the high type's ($\bar{v}$'s) n^{th} bid is F_{Hn} and the marginal distribution of the low type's ($\underline{v}$'s) n^{th} bid is G_{Ln}.

Assume $\bar{v}_1 > \bar{v}_2 > \underline{v}_1 > \underline{v}_2 > \underline{v}_3 = \underline{v}_4 > 0$. Moreover, assume that the probability of type $\underline{v}$ is some $p = \frac{\underline{v}_3}{\underline{v}_2}$. Then there exists an equilibrium that takes the following form: Each type bids $\underline{v}_3$ for their first and second bids. The high type bids 0 for their third and fourth bids. But, the low type mixes on the interval $(0, \underline{v}_3)$ for their third and fourth bids; in particular, the low type's mixture is different for the third and fourth bids. Under this equilibrium, if the pure-strategy (b_1, b_2, b_3, b_4) is in the support of the equilibrium for the low type, then the pure strategy is a step function that is constant on units 1 and 2, lower for the third unit and even lower for the fourth unit. Such a realized pure-strategy is illustrated in the following plot:

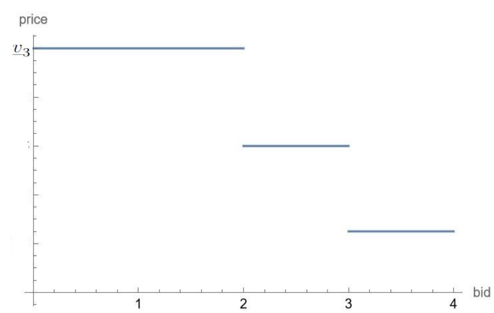

Figure 2.14: Higher-unit example: realization of identical bidding at first two units

To understand why this is an equilibrium, note that under this strategy profile both bidders receive 2 units for sure. There is no incentive to bid higher, since a higher bid can only serve to pay a higher price and potentially get a third unit at that higher price. But since $\underline{v}_3 > \bar{v}_3$, no bidder would want to get a third unit at a higher price. Likewise, no bidder has an incentive to bid lower. This is because the low type bids aggressively on the third and fourth bids—sufficiently aggressively to ensure that the high type would not deviate downward. In particular, choosing $G_{L3}(x) = \frac{(1-p)x}{p(\underline{v}_2 - x)}$ and $G_{L4}(y) = \frac{p\underline{v}_1 - \underline{v}_3 + (1-p)y}{p(\underline{v}_1 - y)}$ ensures that this constitutes an equilibrium. What's more, by restricting probability p to $p = \frac{\underline{v}_3}{\underline{v}_2}$, we are able to construct a correlation (functional relationship) between third and forth bids b_{l3}, b_{l4} of low type by $b_{l4} = h(b_{l3}) = \frac{\underline{v}_3(\underline{v}_2 - b_{l3}) + \underline{v}_1(-\underline{v}_3 + b_{l3})}{\underline{v}_2 - \underline{v}_3}$. This correlation guarantees realization of bids for low type will be of the shape shown in the previous paragraph with probability 1.

For the second set of examples, we assume the ordering of private valuation is $\underline{v}_1 > \bar{v}_1 > \bar{v}_2 > \underline{v}_2 > \underline{v}_3 > \underline{v}_4 > 0$. We will show an example when $p = \frac{\underline{v}_3}{2\bar{v}_2 - \underline{v}_2} > \frac{\underline{v}_4}{\bar{v}_1}$, $\bar{v}_2 > \underline{v}_2 + \underline{v}_3$ and $\bar{v}_2 = \frac{\underline{v}_2 \underline{v}_4}{2\underline{v}_4 - \underline{v}_3}$. In this example, each bidder will use pure strategy at $\underline{v}_4$ for first bids and accordingly is guaranteed to win her first unit. Observing this, a type-$\underline{v}$ bidder knows that she can not win the fourth unit and hence the fourth bid of type-$\underline{v}$ bidder will only mix to prevent first bids from deviating downward. Type-$\underline{v}$ bidders understand that their second (third) bids are competing with opponents' third (second) bids, and hence bidders will have incentive to bid lower (higher) prices for their second (third) bids, which makes second and third bids identical since the lowest feasible bids for bidders' to pick for second bids will be the third bids and vice versa. First and fourth bids in this example is similar to our result from theorem II.10, and argument for second and third bids are similar to scenario described in theorem III.2. Mathematically speaking, first bid of type-$\bar{v}$ will be a pure strategy at $\underline{v}_4$, and second bid of type-$\bar{v}$ will be mixing in interval $(0, \underline{v}_4)$ by distribution $F_{H2}(x) = \frac{x(x+\bar{v}_2-\underline{v}_2-\underline{v}_3)}{(\bar{v}_2-x)(\underline{v}_3-x)}$. First bid of type-$\underline{v}$ will be a pure strategy at $\underline{v}_4$, second and third bids of type-$\underline{v}$ will be identical in interval $(0, \underline{v}_4)$ by distribution $G_{L2}(x) = G_{L3}(x) = \frac{(1-p)x}{p(\bar{v}_2-x)}$ and fourth bid of type-$\underline{v}$ will follow distribution $G_{L4}(x) = \frac{p\bar{v}_1+(1-p)x-\underline{v}_4}{p(\bar{v}_1-x)}$ in interval $(0, \underline{v}_4)$. We particularly require $p = \frac{\underline{v}_3}{2\bar{v}_2-\underline{v}_2}$ and $\bar{v}_2 = \frac{\underline{v}_2\underline{v}_4}{2\underline{v}_4-\underline{v}_3}$ to make the support of distributions above to have identical endpoints. What's more, to make $G_{L4}(x)$ and $F_{H2}(x)$ non-negative over support $(0, \underline{v}_4)$, we impose conditions $p(= \frac{\underline{v}_3}{2\bar{v}_2-\underline{v}_2}) > \frac{\underline{v}_4}{\bar{v}_1}$ and $\bar{v}_2 > \underline{v}_2 + \underline{v}_3$ [13]. A feasible example of private valuations can be $\bar{v} = (6, 4, 0, 0)$ and $\underline{v} = (7, 2, 1.5, 1)$. We can plot a possible realization of bids for type-$\underline{v}$ as well [14]:

2.7.1 Construction of examples

Example 1 can be constructed in the following method:

[13] We pick the precise probability at $\frac{\underline{v}_3}{2\bar{v}_2-\underline{v}_2}$ for simplicity since it leads to an atomless $F_{H2}(x)$ distribution. and we may allow $\frac{\underline{v}_4}{\bar{v}_1} < p < \frac{\underline{v}_3}{2\bar{v}_2-\underline{v}_2}$ by putting an atom at 0 with $F_{H2}(x) = \frac{x(x+\bar{v}_2-\underline{v}_2-\underline{v}_3)}{(\bar{v}_2-x)(\underline{v}_3-x)} + \frac{\underline{v}_3-2\bar{v}_2 p+\underline{v}_2 p}{(1-p)(\underline{v}_3-x)}$, as with theorem III.2.

[14] There is a functional relationship h between third and fourth bids b_{l3}, b_{l4} of type-$\underline{v}$ by $b_{l4} = h(b_{l3}) = \frac{\underline{v}_4(\bar{v}_2-b_{l3})+\bar{v}_1(-\underline{v}_4+b_{l3})}{\bar{v}_2-\underline{v}_4}$.

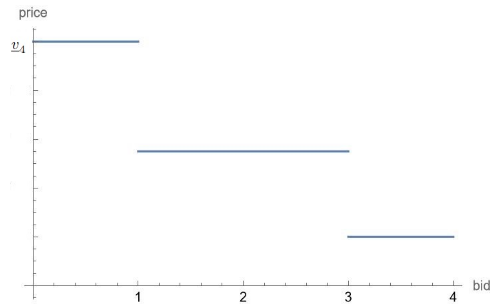

Figure 2.15: Higher-unit example: realization of identical bidding at the second and third units

Consider a new case where high type has private valuation $\bar{v}_1 > \bar{v}_2 > v_7 = v_8 = 0$ and low type has private valuation $\underline{v}_1 > \underline{v}_2 > \underline{v}_3 = \underline{v}_4$. We assume that bidders are competing four identical objects where $\bar{v}_1 > \bar{v}_2 > \underline{v}_1 > \underline{v}_2 > \underline{v}_3$.

High type will not make one of her highest two bids lower than $\underline{v}_3$ with her deviating bids denoted as x if
$\bar{v}_1 + \bar{v}_2 - 2\underline{v}_3 \geq (1-p)(\bar{v}_1 - \underline{v}_3 + \bar{v}_2 - x) + p[(\bar{v}_1 - \underline{v}_3) + (\bar{v}_2 - x)G_{L3}(x)]$
$\iff G_{L3}(x) \leq \frac{p\bar{v}_2 + (1-p)x - \underline{v}_3}{p(\bar{v}_2 - x)}$. Similarly, low type will not make one of her highest two bids lower than $\underline{v}_3$ with her deviating bids denoted as x if
$\underline{v}_1 + \underline{v}_2 - 2\underline{v}_3 \geq (1-p)(\underline{v}_1 - \underline{v}_3 + \underline{v}_2 - x) + p[(\underline{v}_1 - \underline{v}_3) + (\underline{v}_2 - x)G_{L3}(x)]$
$\iff G_{L3}(x) \leq \frac{p\underline{v}_2 + (1-p)x - \underline{v}_3}{p(\underline{v}_2 - x)}$. We conclude $G_{L3}(x) \leq \frac{p\underline{v}_2 + (1-p)x - \underline{v}_3}{p(\underline{v}_2 - x)}$ since $\frac{p\underline{v}_2 + (1-p)x - \underline{v}_3}{p(\underline{v}_2 - x)} \leq \frac{p\bar{v}_2 + (1-p)x - \underline{v}_3}{p(\bar{v}_2 - x)}$.

High type will not make both her highest two bids lower than $\underline{v}_3$ with her deviating bids denoted as $x \geq y$ if
$\bar{v}_1 + \bar{v}_2 - 2\underline{v}_3 \geq (1-p)(\bar{v}_1 - x + \bar{v}_2 - y) + p[G_{L4}(x)(\bar{v}_1 - x) + (\bar{v}_2 - y)G_{L3}(y)]$
$\iff G_{L4}(x) \leq \frac{p\bar{v}_1 + p\bar{v}_2 + (1-p)x - \underline{v}_3 - p\underline{v}_2 - (\bar{v}_2 - \underline{v}_2)\frac{p\underline{v}_2 + (1-p)y - \underline{v}_3}{\underline{v}_2 - y}}{p(\bar{v}_1 - x)}$.

Abusing notation, low type will not make both her highest two bids lower than $\underline{v}_3$ with her deviating bids denoted as $x \geq y$ if
$\underline{v}_1 + \underline{v}_2 - 2\underline{v}_3 \geq (1-p)(\underline{v}_1 - x + \underline{v}_2 - y) + p[G_{L4}(x)(\underline{v}_1 - x) + (\underline{v}_2 - y)G_{L3}(y)]$
$\iff G_{L4}(x) \leq \frac{p\underline{v}_1 - \underline{v}_3 + (1-p)x}{p(\underline{v}_1 - x)}$. Note that we can rewrite
$\frac{p\bar{v}_1 + p\bar{v}_2 + (1-p)x - \underline{v}_3 - p\underline{v}_2 - (\bar{v}_2 - \underline{v}_2)\frac{p\underline{v}_2 + (1-p)y - \underline{v}_3}{\underline{v}_2 - y}}{p(\bar{v}_1 - x)} = \frac{p(\bar{v}_1 - x) + p\bar{v}_2 + x - \underline{v}_3 - p\underline{v}_2 - (\bar{v}_2 - \underline{v}_2)\frac{p\underline{v}_2 + (1-p)y - \underline{v}_3}{\underline{v}_2 - y}}{p(\bar{v}_1 - x)}$
$= 1 + \frac{p\bar{v}_2 + x - \underline{v}_3 - p\underline{v}_2 - (\bar{v}_2 - \underline{v}_2)\frac{p\underline{v}_2 + (1-p)y - \underline{v}_3}{\underline{v}_2 - y}}{p(\bar{v}_1 - x)} = 1 + \frac{p(\bar{v}_2 - \underline{v}_2) - (\bar{v}_2 - \underline{v}_2)\frac{p\underline{v}_2 + (1-p)y - \underline{v}_3}{\underline{v}_2 - y} + x - \underline{v}_3}{p(\bar{v}_1 - x)}$

$= 1 + \frac{(\bar{v}_2 - \underline{v}_2)[p - \frac{p\underline{v}_2 + (1-p)y - \underline{v}_3}{\underline{v}_2 - y}] + x - \underline{v}_3}{p(\bar{v}_1 - x)} = 1 + \frac{x - \underline{v}_3}{p(\bar{v}_1 - x)} + \frac{(\bar{v}_2 - \underline{v}_2)[p - \frac{p\underline{v}_2 + (1-p)y - \underline{v}_3}{\underline{v}_2 - y}]}{p(\bar{v}_1 - x)}$
$= 1 + \frac{x - \underline{v}_3}{p(\bar{v}_1 - x)} + \frac{(\bar{v}_2 - \underline{v}_2)(\underline{v}_3 - y)}{p(\bar{v}_1 - x)(\underline{v}_2 - y)}$. Since both x, y satisfy $y \leq x \leq \underline{v}_3 \leq \underline{v}_2 \leq \bar{v}_2 \leq \bar{v}_1$, we conclude that $\frac{(\bar{v}_2 - \underline{v}_2)(\underline{v}_3 - y)}{p(\bar{v}_1 - x)(\underline{v}_2 - y)}$ is positive. We have
$\frac{p\underline{v}_1 - \underline{v}_3 + (1-p)x}{p(\underline{v}_1 - x)} = 1 + \frac{x - \underline{v}_3}{p(\underline{v}_1 - x)} \leq 1 + \frac{x - \underline{v}_3}{p(\bar{v}_1 - x)} \leq 1 + \frac{x - \underline{v}_3}{p(\bar{v}_1 - x)} + \frac{(\bar{v}_2 - \underline{v}_2)(\underline{v}_3 - y)}{p(\bar{v}_1 - x)(\underline{v}_2 - y)}$. So we conclude that $G_{L4}(x) \leq \frac{p\underline{v}_1 - \underline{v}_3 + (1-p)x}{p(\underline{v}_1 - x)}$. And it is not hard to check
$\frac{p\underline{v}_2 + (1-p)x - \underline{v}_3}{p(\underline{v}_2 - x)} \leq \frac{p\underline{v}_1 - \underline{v}_3 + (1-p)x}{p(\underline{v}_1 - x)}$. So if $G_{L3}(x) = \frac{p\underline{v}_2 + (1-p)x - \underline{v}_3}{p(\underline{v}_2 - x)}$, it is feasible to pick
$G_{L4}(y) = \frac{p\underline{v}_1 - \underline{v}_3 + (1-p)y}{p(\underline{v}_1 - y)}$.

Similarly, example 2 can be constructed in the following method:

We suppose high type is bidding $\underline{v}_4$ for her first bid so she will be getting constant payoff. We need to look at her indifferent condition:
$pG_{L3}(x)(\bar{v}_2 - x) + (1-p)(\bar{v}_2 - x) = (1-p)\bar{v}_2$. And hence we have $G_{L3}(x) = \frac{(1-p)x}{p(\bar{v}_2 - x)}$.
A high type will not deviate to bid $\underline{v}_4 > x \geq y$ if
$pG_{L3}(y)(\bar{v}_2 - y) + (1-p)(\bar{v}_2 - y) + pG_{L4}(x)(\bar{v}_1 - x) + (1-p)(\bar{v}_1 - x) \leq$
$pG_{L3}(x)(\bar{v}_2 - x) + (1-p)(\bar{v}_2 - x) + \bar{v}_1 - \underline{v}_4$. Note that
$pG_{L3}(y)(\bar{v}_2 - y) + (1-p)(\bar{v}_2 - y) = \frac{(1-p)y}{(\bar{v}_2 - y)}(\bar{v}_2 - y) + (1-p)(\bar{v}_2 - y) = (1-p)\bar{v}_2$, so we should have $G_{L4}(x) \leq \frac{\bar{v}_1 - \underline{v}_4 - (1-p)(\bar{v}_1 - x)}{p(\bar{v}_1 - x)} = \frac{p\bar{v}_1 - \underline{v}_4 + (1-p)x}{p(\bar{v}_1 - x)}$.

For a low type, we still assume her first bid is at $\underline{v}_4$. And a low type will not deviate her first bid downward if expected payoff for her first bid is non-increasing: i.e. $(1-p)(\underline{v}_1 - x) + pG_{L4}(x)(\underline{v}_1 - x)$ needs to have a non-decreasing derivative.
$\frac{d}{dx}[(1-p)(\underline{v}_1 - x) + pG_{L4}(x)(\underline{v}_1 - x)] = -\frac{(\bar{v}_1 - \underline{v}_1)(\bar{v}_1 - \underline{v}_4)}{(\bar{v}_1 - x)^2}$ and hence we need $\underline{v}_1 > \bar{v}_1$.
With first bids bidding a pure strategy at $\underline{v}_4$, fourth bid of low type will never win positive expected payoff and hence indifferent condition for low type can be simplified to
$(1-p)[(\underline{v}_2 - x) + F_{H2}(y)(\underline{v}_3 - y)] + p[G_{L3}(x)(\underline{v}_2 - x) + G_{L2}(y)(\underline{v}_3 - y)] = (1-p)\underline{v}_2$.
Expected payoff for second bid for low type is $(1-p)(\underline{v}_2 - x) + pG_3(x)(\underline{v}_2 - x)$ and will be a decreasing function for x. So second and third bid for low type should be identical. And we solve $F_{H2}(y) = \frac{y(y + \bar{v}_2 - \underline{v}_2 - \underline{v}_3)}{(\bar{v}_2 - y)(\underline{v}_3 - y)}$. To make right endpoints of the distributions established above identical, we need $p\bar{v}_2 = \underline{v}_4 = \frac{\bar{v}_2 \underline{v}_3}{2\bar{v}_2 - \underline{v}_2}$. We need
$p = \frac{\underline{v}_3}{2\bar{v}_2 - \underline{v}_2}$ and $\bar{v}_2 = \frac{\underline{v}_2 \underline{v}_4}{2\underline{v}_4 - \underline{v}_3}$. With identical endpoints, it is not hard to check
$G_2(x) \leq G_4(x)$. A low type will not deviate any single bid by construction.
Expected payoff for first bid of low type is an increasing function so making first bid at $\underline{v}_4$ is always a best response. Monotone conditions on second and third bids

will imply identical bids. At last, a low type is willing to mix her fourth bid in interval $(0, \underline{v}_2)$ since first bids of any bidder will be $\underline{v}_4$ so fourth bid of low type will never win.

2.8 Comparison of Expected Revenue

We are interested in comparing expected revenue for three common formats of multi-unit auctions: pay-as-bid auction, uniform-price auction and Vickrey auction. For uniform-price auction, winners in the auction will pay the highest losing price. It is straightforward to check that the strategy where bidders bid truthfully for their first units and bid 0 for their second units forms an equilibrium. So each bidder wins exactly one unit but always pays zero, which leads to an expected revenue of zero.

For Vickrey auction, any bidder i who wins k_i units will be paying the highest k_i losing bids among her rivals. And accordingly, one equilibrium for Vickrey auction in our multi-unit setting is that each bidder is bidding the marginal valuations truthfully for every unit [15]. So in this equilibrium, each bidder will win exactly one unit and be paying 0 when facing a high type and $\underline{v}_2$ when facing a low type. Expected payment for each bidder is $p\underline{v}_2$, which makes total revenue equivalent to $2p\underline{v}_2$.

For pay-as-bid auction, we restrict to situation where private valuation satisfies $\bar{v}_1 > \underline{v}_1 + \underline{v}_2$ and check all range of p. When $p < \frac{\underline{v}_2}{2\bar{v}_1 - \underline{v}_1}$, both bidders will mix in interval $(0, \bar{v}_1 p)$, as summarized by theorem III.2. There will be probability p^2 two low types meet, probability $2p(1-p)$ a high and a low type meet and probability $(1-p)^2$ two high types meet. Our approach for expected revenue is to establish is the order statistics for highest and second highest bid and compute the expected value.

First we consider the scenario when two high types meet. We denote B_1, B_2 as non-zero bids for those two bidders and B_1, B_2 are independent. Since high type's valuation toward second unit is normalized to 0,

[15] Actually, equilibrium strategy mentioned in the previous paragraph also forms an equilibrium in Vickrey auction, but it is traditional to look at the truthful reporting equilibrium for Vickrey auction.

$F_{(2)}(x) = \mathbb{P}(B_1 \leq x, B_2 \leq x) = F_H^2(x)$ and
$F_{(1)}(x) = 1 - \mathbb{P}(B_1 > x, B_2 > x) = 1 - (1 - F_H(x))^2 = 2F_H(x) - F_H^2(x)$. Expected value of highest two bids in this scenario will be
$\pi_{HH} = \int_0^{\bar{v}_1 p} x\, dF_H^2(x) + \int_0^{\bar{v}_1 p} y\, d[2F_H(y) - F_H^2(y)]]$. We now consider scenario when two low types meet. In this scenario, bidders will propose four bids. We assume the bids are $B_{11}, B_{12}, B_{21}, B_{22}$ with $B_{11} = B_{12}$ and $B_{21} = B_{22}$. What's more, B_1's and B_2's are independent. So $F_{(3)}(x) = F_{(4)}(x) = \mathbb{P}(B_{11} \leq x, B_{12} \leq x, B_{21} \leq x, B_{22} \leq x) = \mathbb{P}(B_{11} \leq x, B_{21} \leq x) = G_L^2(x)$ since $B_{11} = B_{12}$ and $B_{21} = B_{22}$. Expected value of highest two bids in this scenario will be $\pi_{LL} = 2\int_0^{\bar{v}_1 p} x\, d[G_L^2(x)]$.

Now we consider the scenario when a high and a low type meet. Assume that high type's bid is B_H, and low type's bids are B_{L1}, B_{L2} with $B_{L1} = B_{L2}$. It is clear that B_H and B_L's are independent since they comes from different bidders who bid independently. So the order statistics will be
$\mathbb{P}(B_{(1)} \leq x) = 1 - \mathbb{P}(B_H \geq x, B_{L1} \geq x, B_{L2} \geq x) = 1 - \mathbb{P}(B_H \geq x, B_{L1} \geq x)$ since we have $B_{L1} = B_{L2}$.
$\mathbb{P}(B_{(2)} \leq x) = \mathbb{P}(B_H \leq x, B_{L1} \leq x, B_{L2} \leq x) + \mathbb{P}(B_H > x, B_{L1} \leq x, B_{L2} \leq x)$
$= \mathbb{P}(B_{(3)} \leq x) + \mathbb{P}(B_{L1} \leq x < B_H)$.
$\mathbb{P}(B_{(3)} \leq x) = \mathbb{P}(B_H \leq x, B_{L1} \leq x, B_{L2} \leq x) = \mathbb{P}(B_H \leq x, B_{L1} \leq x)$ by $B_{L1} = B_{L2}$.
Note that $1 - \mathbb{P}(B_H \geq x, B_{L1} \geq x), \mathbb{P}(B_H \leq x, B_{L1} \leq x)$ are just expressions for order statistics when there are only two bids B_H and B_{L1}. And $\mathbb{P}(B_{(2)} \leq x)$ happens when all bids are smaller than x or when only bids from low type are smaller than x.

We can invoke the Bapat–Beg Theorem [16] to compute CDF of order statistics of non-identical distributions when we only have B_H, B_{L1}. If we use $F_{X_{(i)}}$ to denote distributions of order statistics when we have three bids B_{L1}, B_{L2}, B_H and $F_{(i)}$ to denote distributions of order statistics when we have two bids B_{L1}, B_H, our argument above shows that $F_{X_{(3)}}(x) = F_{(2)}(x) = \frac{\text{per}\begin{bmatrix} F_H(x) & F_H(x) \\ G_L(x) & G_L(x) \end{bmatrix}}{2!(2-2)!} = F_H(x)G_L(x)$ with per being permanent of the given block matrix. And accordingly,
$F_{X_{(2)}}(x) = F_H(x)G_L(x) + G_L(x)[1 - F_H(x)] = G_L(x)$. Expected value of highest two bids in this scenario will be $\pi_{HL} = \int_0^{\bar{v}_1 p} x\, d[F_H(x)G_L(x)] + \int_0^{\bar{v}_1 p} y\, d[G_L(y)]]$.

[16] Theorem 4.2 from Bapat, Beg (1989), proved in Hande (1994).

With order statistics established, we may compute expected valuation of the two distributions which is also the monetary payment for the first and second unit respectively, i.e. expected revenue of the pay-as-bid auction is $p^2\pi_{LL} + 2p(1-p)\pi_{HL} + (1-p)^2\pi_{HH}$. If we assume $\bar{v}_1 = 3, \underline{v}_1 = 2$, and $\underline{v}_2 = 1$, expected revenue from pay-as-bid auction will dominate Vickrey auction when $p \in [0.125, \frac{1}{4}]$ [17]. So our example indicates that revenue ranking between pay-as-bid auction and Vickrey auction is ambiguous. We conclude that pay-as-bid and Vickrey auction dominates uniform-price auction in terms of expected revenue but ranking between pay-as-bid and Vickrey auction is ambiguous.

If we raise probability of p to range covered by theorem III.3, where distribution function gets more complicated since for some subset of p low type may bid differently, we can instead compute expected value of bids from high and low type. Summation of any such two expected values should be no greater than the summation of expected value of highest and second highest bids by construction. However, we can report that expected value of any single bid from either type is greater than p, which makes summation of expected values of any two bids greater than $2p$, the expected revenue of Vickrey auction. When $p \in [\frac{\underline{v}_2}{\underline{v}_1}, 1]$, theorem II.9 indicates that each bidder will always bid $\underline{v}_2$ and accordingly expected revenue of auction under theorem II.9 will be $2\underline{v}_2$, which is higher than the expected revenue of Vickrey auction as well. In all, we conclude that if we assume $\bar{v}_1 = 3, \underline{v}_1 = 2$, and $\underline{v}_2 = 1$, Vickrey auction generates higher expected revenue when $p < 0.125$ and pay-as-bid auction generates higher expected revenue when $p > 0.125$.

The last interesting result to notice is that despite having identical allocations where each bidder wins one unit, our hypothetical uniform-price auction and Vickrey auction generate different expected revenue. To validate the revenue equivalence theorem for single-unit auction, one necessary condition is some type should get same expected payoff from different formats of auctions. But payment from Vickrey and uniform-price auctions are not identical as shown in the previous paragraph. Another obvious violation in the establishment of revenue equivalence theorem is that we need to integrate over the range from lowest type to some type to construct payment, but we do not have such integration due to our discrete type space.

[17] 0.125 is a decimal approximation of an irrational number starting with 0.1249595, not $\frac{1}{8}$.

2.9 Conclusion and Discussion

We study a multi-item auction where there are two discrete types of bidders and each type of bidder demands two objects. We always assume a high type will have marginal valuations $\bar{v}_1, \bar{v}_2$ and low type will have marginal valuations $\underline{v}_1, \underline{v}_2$. But we focus on case with ordering $\bar{v}_1 > \underline{v}_1 > \underline{v}_2 > \bar{v}_2$. After normalizing the smallest marginal valuation (i.e. $\bar{v}_2$) to 0, we look at symmetric pure and mixed strategy equilibria for different proportions of high and low types in the population. We find out that high type may put an atom at 0 for distribution of first bid when probability low type appears in the auction is small, and low type will bid identically for both units in most mixed strategy equilibria (i.e. perfectly correlated equilibrium). We find out empirical evidence which is consistent with the identical bidding behaviour from our theoretical results and are able to extend some of our results into higher-unit environment to show bidders would still bid identically for several units when they bid for more units. We will have pure strategy equilibrium when probability low type appears is relatively large and bidders are just bidding $\underline{v}_2$, the marginal valuation of low type's second object.

Given that private valuations in our auction are $\bar{v}_1 > \underline{v}_1 > \underline{v}_2 > \bar{v}_2$, an efficient allocation should let each bidder get one object since whenever a high/low type meets another high/low type, the highest two valuations always come from valuations of first marginal valuation from different bidder. But in majority of our results, we propose perfectly correlated equilibrium where low type bids identically. What's more, we have overlapping of supports when high and low types bid mixed strategy. All the features above indicate that our equilibrium allocation is likely to be inefficient by assigning both objects to one low-type bidder with positive probability (i.e. misallocation). This inefficiency arises from the fact that bidders understand their higher bids are competing with opponents' lower bids and they will have incentive to make their first bids lower for higher net payoff. But knowing first bids will be generally low in price, bidders will consequently bid higher second bids for a better chance of winning.

Using a terminology from auction literature, we conclude low types in our model are displaying *differential bid-shading* behaviour: when two bids from a low type are identical it must be that a low type is bid-shading more on her first bid. The differential bid-shading behaviour in our multi-item auction makes it impossible to know the true types of bidders from auction results when both high and low types share the identical support for their bids, as situations described in lemma II.11 and II.14.

Besides this inefficient allocation feature, our analysis finds a conditionally deterministic relationship between two bids for low type, i.e. if we know the range of p and what low type bids for her first bid b_{l1}, we can compute her second bid b_{l2}. The most common case in our result is when low type bids identically i.e. $b_{l2} = b_{l1}$. Previous literature like Anwar (2007) and Engelbrecht-Wiggans and Kahn (1998) also reported such type of pooling equilibrium. We also find out cases where first (b_{l2}) and second bid (b_{l2}) of low type follow a functional relationship $b_{l2} = h(b_{l1}) \leq b_{l1}$ for all (b_{l1}, b_{l2}) in support $[a_3, a_4]^2$, as displayed in lemma II.13 and II.16. We may treat low type's bids x, y as solution to an optimization problem where low type is trying to compute her optimal second bid b_{l2} given every possible first bid b_{l1} in the joint support of bids (b_{l1}, b_{l2}). And consequently situations where $b_{l2} = b_{l1}$ can be treated as corner solution to the optimization problem while $b_{l2} = h(b_{l1}) \leq b_{l1}$ is an interior solution.

We know that inefficiency comes from misallocation of objects since our symmetric equilibria propose identical bids for low type and overlapping of support for different types. To achieve efficiency under the private valuations in our model, each bidder should just get one object. Our results always imply a positive probability of inefficient allocations, although we have checked all possible combinations of high and low types. However, we do not establish uniqueness of our mixed strategies, and hence we can not exclude possibilities of efficient allocations through mixed strategy equilibrium. Other potential solutions to this issue and future questions to answer may include whether we can have efficient allocation if we implement simultaneous auctions with the same valuation distribution introduced in multi-item auction. It may be that in a simultaneous auction bidders propose their higher bids toward different objects and each ends up getting one object.

We also compare expected revenue of several common formats of multi-unit auction: pay-as-bid auction, uniform-price auction and Vickrey auction. We find that uniform-price auction would give the lowest expected revenue among the three while ranking between pay-as-bid and Vickrey auction is ambiguous. Our numerical example comparing revenue from pay-as-bid and Vickrey auction is weakly monotone in p, which indicates that there is a cutoff p^* so that pay-as-bid auction dominated Vickrey auction if and only if $p > p^*$.

Our results when valuation ordering is $\bar{v}_1 > \underline{v}_1 > \bar{v}_2 > \underline{v}_2$ or $\bar{v}_1 > \bar{v}_2 > \underline{v}_1 > \underline{v}_2$ differ from the main results discussed above in two features: we do not find perfectly correlated equilibrium for any type and we do not have the conditional deterministic relationship between two bids from same type. One common feature is that we are not guaranteed to have efficient allocations in these cases either since overlapping of support persists. which will lead to misallocation of objects.

CHAPTER III

Package Bidding with Distinct Objects

3.1 Introduction

Consider two real estate firms whose headquarters locate respectively in the western and eastern part of Michigan [1]. They are competing to purchase two pieces of lands, where each piece is located in a region near the headquarter of one firm. The lands are located sufficiently distant so that even if a firm can grasp both lands, it would be impossible to get any synergy from the real estates it can build. It would be a straightforward prediction that each firm will bid high prices for the land near to their headquarter since they both value such lands highly. But do there exist conditions (e.g. a firm who locates in the western part of the state values highly for land in the opposite side) or bidding strategies so that one firm can take ownership of both lands?

We will be looking at an auction with both pieces of land being offered with a permission of using package bidding by making the following assumptions: two distinct and indivisible objects A and B are being sold and two bidders with multi-unit demand are participating the auction; we always assume that bidder 1 values A higher than B and bidder 2 values B higher than A. To be more precise, we focus on case where "high" type of bidders has private valuations $\bar{v} = (\bar{v}_1, \bar{v}_2)$ and "low" type of bidders have private valuations $\underline{v} = (\underline{v}_1, \underline{v}_2)$, with value ordering

[1] e.g. The Platform and Rockford Construction. The former is a Detroit-based firm focusing on developing and redeveloping residential and mixed-use properties in Detroit while the latter puts its headquarters in Grand Rapids, who focuses heavily on Western Michigan with business ranging from urban redevelopment to new residential constructions.

$\bar{v}_1 > \underline{v}_1 > \bar{v}_2 > \underline{v}_2$ and normalizing $\underline{v}_2 = 0$. Bidders are risk-neutral and only care about monetary payoff. We further assume bidders have additive valuation, meaning each bidder's value for the two units as a whole is simply the summation of marginal values of the two units. Both bidders have a common prior that a high type opponent will appear with probability $p \in [0, 1]$ and a low type opponent will appear with probability $1 - p$. Each bidder will propose bids for both single objects with an extra bid for the bundle. And for each bidder, the bundle bid must be no smaller than summation of bid for single objects.

To make our analysis tractable, we focus on equilibrium in which bidders are using pure strategies for some of her bids for the single objects. We will propose two sets of equilibria where both types of bidders are bidding pure strategies: in the first equilibrium bidders of both types are bidding 0 for single objects while bundle bid of high type is mixing in interval $[0, p(\bar{v}_1 + \bar{v}_2)]$. This equilibrium will hold for all $p \in (0, 1)$. And when p reaches 1, the model yields the same outcome as symmetric Bertrand competition where each bidder just bids her marginal cost.

Our second set of equilibria is developed from the equilibrium where the high and low type are bidding pure bids equivalent to $\bar{v}_2 > 0$ for their favoured objects with high type mixing her bundle bid in interval $[0, \frac{\bar{v}_2(\bar{v}_2 + \underline{v}_1(-2+p)) + \bar{v}_1(\bar{v}_2 - \underline{v}_1 p)}{\bar{v}_2 - \underline{v}_1}]$. After showing this equilibrium, we return to examine no-deviating constraints and notice that many of the constraints is binding. We realize that if distributions for bundle bid make the equalities be strict inequalities and satisfy other indifferent conditions for mixed strategy equilibrium, we can create more equilibrium distributions for the bundle bid. What's more, we also invent a simplified class of equilibria by only using part of the no deviating conditions. Those equilibria satisfy properties that upper bound of bundle is precisely $2\bar{v}_2$, with an atom of positive size at $2\bar{v}_2$ as well.

Efficient allocation is each bidder win one object regardless of types. The second set of equilibria will perform better than the first set in terms of welfare since under the first set of equilibrium a high type will either win both objects or none, while when bundle bid is not too high each type will win one object in the second class of equilibria.

Literature Review

Literature reports package bidding or combinatorial auction might be a key factor for promoting welfare or achieving higher revenue. Chernomaz and Levin (2012) studied a multi-unit auction with one global bidder and two local bidders where each local bidder only cares about the object in local market and the global bidder is interested in both items. The auction is first-price auction, but they studied two auction rules where the first one only allows bids for single items and the second one allows the global bidder to propose not only bid for single item but also a bid for the bundle. Chernomaz and Levin (2012) showed through simulations of auctions that package bidding has a slight negative impact on efficiency when synergies are absent, but a significantly positive impact when synergies are present. What's more when package bidding is banned, synergies have a minor negative effect, whereas they have a markedly positive effect under package auction rules. Subramaniam and Venkatesh (2009) compared revenue from second-price auctions with two objects and multiple bidders under three types of mechanisms, selling separately, selling as bundle and combinatorial auctions. They found that combinatorial auctions perform better in terms of revenue than selling as bundle, and would be the optimal option when two objects are substitute, strong complements or weak to moderate complements when fewer than four bidders participate.

Many literature has studied combinatorial auction under numerous mechanisms. For example, Bernheim and Whinston (1986) studied a first-price auction where bidders propose a menu of actions to the auctioneer under the assumption that bidders have complete information. The paper showed that for a certain refinement of Nash Equilibrium, first-price menu auctions would implement efficient actions. Although this paper is about first price auction, it is about complete information, which is on the contrary to our incomplete information assumption. Ausubel and Milgrom (2002) studied package bidding for ascending price auction when bidders have quasi-linear utilities. The paper focused particularly on the ascending proxy auction in which bid increments are negligibly small and ascending proxy auction coule be treated as a version of "deferred acceptance algorithm" in the matching theory. The paper showed that sincere reporting constitutes a Nash equilibrium, which is efficient and in the core of the exchange game, when goods are substitutes. Additionally, the ascending proxy would overcome some of the shortcomings of Vickrey auction, such as generating higher revenues, and being more robust to shill bidding (i.e. bidders can profit by submitting additional bids under false identities) and collusion.

Synergies between objects in auctions are also studied in simultaneous auctions, where multiple single-unit auctions are run simultaneously. Szentes (2007) and Szentes and Rosenthal (2003) studied two identical bidders with three and two objects simultaneous auctions respectively. They considered complementarities or substitutes (sub-additive) among objects. What's more, Rosenthal and Wang (1996) considered a simulataneous auction similar to Chernomaz and Levin (2012) where there are both local bidders and global bidders. But in Rosenthal and Wang (1996), number of local and global bidders are both higher than 1. Contrary to Chernomaz and Levin (2012), Rosenthal and Wang (1996) assumed there would be a certain probability bidders receiving signal that the objects are of high quality. All the papers proposed symmetric mixed strategy equilibria. However, all equilibria constructed by papers in simultaneous auctions mentioned above can not guarantee efficient allocations.

The literature on simultaneous auctions with synergies has argued the exposure problem would arise when complementary goods are sold individually. For example, Goeree and Lindsay (2020) mentioned that in spectrum auction bidders typically want consecutive blocks of spectrum within a specific band or a combination of licenses that cover neighboring geographic areas. But bidders hesitate to factor synergies into their bids with the fear of only being able to win a portion of the desired combination if they participate into simultaneous auctions. Combinatorial auctions, on the other hand, allow bidders to propose bids for a combination of objects (i.e. package) in multi-unit auctions, which will alleviate such exposure problems and may lead to higher welfare and revenue. Cramton, Shoham and Steinberg (2006) mentioned another prominent example of package bidding, the Estate auction: each item will be auctioned off separately at the first place, which are followed by opportunities of auctioning off bundles of items. Items will be sold as packages only if price for the package exceeds summation of prices of individual items combined. In general, package bidding would be popular when there are obvious synergies between objects, since bidders will offer a higher price for the bundle when they can enjoy the complementarities from different objects. In this model, exposure problem will not arise even if objects are sold in simultaneous auctions since we do not assume synergies.

3.2 Model

We assume that two distinct objects A and B are being auctioned in a pay-as-bid auction. There are two bidders, bidder 1 and 2, and each of whom will demand up to 2 objects. And it is always the case that bidder 1 values object A higher than B, and bidder 2 values object B higher than A. We assume the type space is discrete, and bidders can be one of two types: high or low.

A a high type will have private valuations $\bar{v} = (\bar{v}_1, \bar{v}_2)$ and a low type will have private valuations $\underline{v} = (\underline{v}_1, \underline{v}_2)$. In this paper, we focus on value ordering $\bar{v}_1 > \underline{v}_1 > \bar{v}_2 > \underline{v}_2 = 0$. Let $V = \{\overline{v}, \underline{v}\}$ be the set of possible valuations (or types). The bidders' types are drawn independently from a common prior. And we denote $p \in (0, 1)$ for the probability that bidder i is a high type. We continue to assume the objects in the auction is additive-value, i.e. the value of the bundle is the summation of marginal values. This assumption is mainly to avoid inventing another function on the value of the bundle.

The objects are auctioned off in a multi-unit pay-as-bid auction: the bidders simultaneously submit bids for both objects and the bundle. For any bidder i, we assume she will bid a vector of three prices $b^i = (b^i_{12}, b^i_1, b^i_2)$ with b^i_{12} being her bid for the bundle and b^i_1, b^i_2 being her bid for each object, with requirement $b^i_{12} \geq b^i_1 + b^i_2$ since otherwise the summation of bid for each single object is the real bid for the bundle. [2] If we denote the two bidders‘ bid as (b^1_{12}, b^1_1, b^1_2) and (b^2_{12}, b^2_1, b^2_2), by our assumption above b^1_1 is the bid for bidder 1's the favoured object (i.e. object A), while b^2_2 is the bid for bidder 2's the favoured object (i.e. object B). We further require that $b^1_1 \geq b^1_2$ but $b^2_2 \geq b^2_1$, i.e. for any bidder, her bid for the favoured object is no smaller than her bid for the less favoured object.

The auction is a pay-as-bid auction, and the winning bid in this auction will be decided by $\max\{b^1_1 + b^2_2, b^2_1 + b^1_2, b^1_{12}, b^2_{12}\}$. That is to say, if $b^i_1 + b^j_2$ is the maximal, each bidder will win one object and pay their bid in the winning bid for the two objects. Note that the bundle bid is guaranteed to be no smaller than the summation of single bid from the same bidder, and hence if $b^i_1 + b^j_2$ is the maximal, it must come from different bidders, i.e. $i \neq j$. Otherwise, one bidder will win both

[2] Due to the nature of different types, we can simplify by assuring that the low type only proposes one single price for her favoured object.

objects and pay her bundle bid.

Tie-breaking Rules In this paper, we will always assume that when tie happens, the allocation will be the one leading to higher welfare.

Preliminary Analysis We will argue that given our assumption that for any bidder, her bid for the favoured object is no smaller than her bid for the less favoured object, $b_1^2 + b_2^1$ being the winning bid will not happen. Note that $(x_2 + y_1$ is the winning bid$) = (b_1^2 + b_2^1 \geq b_{12}^1, b_1^2 + b_2^1 \geq b_{12}^2, b_1^2 + b_2^1 \geq b_1^1 + b_2^2)$ is equivalent to event $(b_1^2 \geq b_1^1, b_2^1 \geq b_2^2)$. But with our assumption $b_1^1 \geq b_2^1, b_2^2 \geq b_1^2$, we have $b_1^2 \geq b_1^1 \geq b_2^1 \geq b_2^2 \geq b_1^2$, which indicates that bids for single objects from different bidders are identical, i.e. $b_1^2 = b_1^1 = b_2^1 = b_2^2$. When this event happens, actually $b_1^1 + b_2^2$ is also the winning bid. Since b_1^1, b_2^2 are bids for the favoured objects, our tie-breaking rule will give each bidder their favoured objects.

3.3 Bidding Equilibrium

The paper restricts to **symmetric** Bayesian Nash equilibria. In this section, we denote distribution of low type's bid as $G_1(\cdot)$ and distribution of high type's bundle bid as $F(\cdot)$. Abusing notation, in this section, we will denote the bundle bid bidder i uses as z_i for bids following proposed strategies or z for deviating bids, while her bid for favoured and less favoured object as x_i, y_i for bids following proposed strategies or x, y for deviating bids. We first propose one benchmark equilibrium which will work for full ranges of p.

3.3.1 Equilibrium with Bidding 0 on Single Objects

Theorem III.1 *The following strategies form a Bayesian Nash Equilibrium: the high and low type bid* 0 *for single object and the high type's bundle bid will follow distribution* $F(z_1) = \frac{(1-p)(\bar{v}_1+\bar{v}_2)}{p(\bar{v}_1+\bar{v}_2-z_1)} - \frac{1-p}{p}$ *on support* $[0, p(\bar{v}_1 + \bar{v}_2)]$.

Proof. The $F(\cdot)$ function is constructed through the indifferent condition of a high type: $(1-p)(\bar{v}_1 + \bar{v}_2 - z_1) + pF(z_1)(\bar{v}_1 + \bar{v}_2 - z_1) = (1-p)(\bar{v}_1 + \bar{v}_2)$.

We check no bidder will want to deviate: if the low type deviates and bid $x > 0$, her expected payoff will be $(1-p)(\underline{v}_1 - x) + pF(x)(\underline{v}_1 - x) = \frac{(1-p)\bar{v}_2(\underline{v}_1-x)}{\bar{v}_1+\bar{v}_2-x}(\frac{\bar{v}_1}{\bar{v}_2} + 1)$, which has a negative derivative w.r.t. x. So the low type should never deviate to bid higher prices.

The high type should not deviate her bids either. The high type can choose to deviate to bid positive prices x, y for the two objects respectively. When $x + y \leq z$, the bundle bid is still the winning bid for the high type. If $x + y > z$ instead, expected payoff for high type becomes $(1-p)(\bar{v}_1 + \bar{v}_2 - x - y) + pF(x+y)(\bar{v}_1 + \bar{v}_2 - x - y) = (1-p)(\bar{v}_1 + \bar{v}_2 - x - y) + p(\bar{v}_1 + \bar{v}_2 - x - y)[\frac{(1-p)(\bar{v}_1+\bar{v}_2)}{p(\bar{v}_1+\bar{v}_2-x-y)} - \frac{1-p}{p}] = (1-p)(\bar{v}_1 + \bar{v}_2)$. It is straightforward to see that the high type will not deviate to make $x + y > p(\bar{v}_1 + \bar{v}_2)$. ■

We can make some comments about this result: firstly this result is inefficient. The structure of the equilibrium guarantees that as long as a high type shows up, she will either get both objects or no objects. While the efficient allocation is always that each bidder win one object regardless of type. Secondly this result works for a full range of p. And we can point out its implication at the two extremes. When p approaches 0, i.e. hardly any high type shows up in the population, the equilibrium is that bidders will just bid 0 since the low type does not value her less favoured object. On the other hand, when p approaches to 1, i.e. almost every individual in the population is a high type, our equilibrium distribution F will converge to indicator function at $\bar{v}_1 + \bar{v}_2$. This scenario is comparable to the symmetric Bertrand competition where firms just price at their marginal cost.

Graphical Illustration of Theorem III.1

If we pick $\bar{v}_1 = 3, \bar{v}_2 = 1, \underline{v}_1 = 2$ and $p = \frac{1}{5}$, distribution used by high type will be

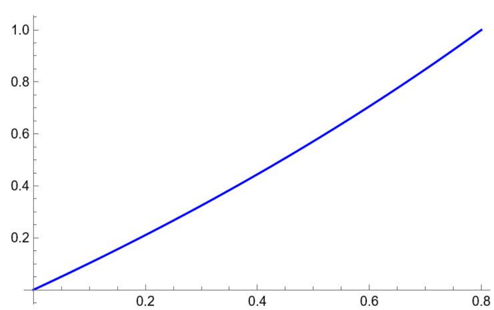

Figure 3.1: Illustration of theorem III.1

3.3.2 Equilibrium with Bidding Positive Pure Bids on Single Objects

We will report two more equilibria where achieving higher welfare is more accessible. We start with a simpler equilibrium where bid for single objects are still pure strategies:

Theorem III.2 *When $p \in (\frac{\bar{v}_2}{\underline{v}_1}, 1)$, the following strategies form a Bayesian Nash Equilibrium: high and low type use pure bids equivalent to $\bar{v}_2$ for their favoured objects and high type's bid for the bundle will follow distribution* $F(z_1) = \frac{1-p}{p}[\frac{(\bar{v}_1-\bar{v}_2)\underline{v}_1}{(\underline{v}_1-\bar{v}_2)(\bar{v}_1+\bar{v}_2-z_1)} - 1]$ *when* $z \in [2\bar{v}_2, \bar{z}]$ *and* $F(z_1) = \frac{1-p}{p}\frac{z_1-\bar{v}_2}{\underline{v}_1-z_1+\bar{v}_2}$ *when* $z \in [\bar{v}_2, 2\bar{v}_2]$*. For the notation,* $\bar{z} = \frac{\bar{v}_2(\bar{v}_2+\underline{v}_1(-2+p))+\bar{v}_1(\bar{v}_2-\underline{v}_1 p)}{\bar{v}_2-\underline{v}_1}$.

Proof. We check the no-deviating condition. First we look at the low type:

If low type deviates to $y < \bar{v}_2$, her expected payoff will be $(1-p)(\underline{v}_1 - y) + p(\underline{v}_1 - y)F(\bar{v}_2 + y) = (1-p)(\underline{v}_1 - y) + p(\underline{v}_1 - y)\frac{1-p}{p}\frac{y}{\underline{v}_1-y}$ $= (1-p)\underline{v}_1$. But notice that equilibrium expected payoff is $(\underline{v}_1 - \bar{v}_2)(1-p) + p(\underline{v}_1 - \bar{v}_2)F(2\bar{v}_2) = (\underline{v}_1 - \bar{v}_2)(1-p) + p(\underline{v}_1 - \bar{v}_2)\frac{1-p}{p}\frac{\bar{v}_2}{\underline{v}_1-\bar{v}_2}$ $= (1-p)(\underline{v}_1 - \bar{v}_2) + (\underline{v}_1 - \bar{v}_2)(1-p)\frac{\bar{v}_2}{\underline{v}_1-\bar{v}_2} = (1-p)(\underline{v}_1 - \bar{v}_2)\frac{\underline{v}_1}{\underline{v}_1-\bar{v}_2} = (1-p)\underline{v}_1$. So deviating downward generates the same expected payoff.

If low type deviates to $y > \bar{v}_2$, her expected payoff will be $(1-p)(\underline{v}_1 - y) + p(\underline{v}_1 - y)F(\bar{v}_2 + y)$ $= (1-p)(\underline{v}_1 - y) + p(\underline{v}_1 - y)\frac{1-p}{p}[\frac{(\bar{v}_1-\bar{v}_2)\underline{v}_1}{(\underline{v}_1-\bar{v}_2)(\bar{v}_1+\bar{v}_2-\bar{v}_2-y)} - 1]$ $= (1-p)(\underline{v}_1 - y) + (1-p)(\underline{v}_1 - y)[\frac{(\bar{v}_1-\bar{v}_2)\underline{v}_1}{(\underline{v}_1-\bar{v}_2)(\bar{v}_1-y)} - 1] = (1-p)(\underline{v}_1 - y)[\frac{(\bar{v}_1-\bar{v}_2)\underline{v}_1}{(\underline{v}_1-\bar{v}_2)(\bar{v}_1-y)}]$. The last expression is smaller than $(1-p)\underline{v}_1$ iff $(\bar{v}_1 - \underline{v}_1)\bar{v}_2 < (\bar{v}_1 - \underline{v}_1)y$, which is correct since $y > \bar{v}_2$.

Now we shift to look at the high type:

First suppose $z_1 > 2\bar{v}_2$. If a high type deviates her bid for the favoured object to $z_1 - \bar{v}_2 > x > \bar{v}_2$: the deviating expected payoff is $(1-p)(\bar{v}_1 + \bar{v}_2 - z_1) + p(\bar{v}_1 + \bar{v}_2 - z_1)F(z_1)$, i.e. the equilibrium expected payoff.

When $z_1 - y_1 > x > z_1 - \bar{v}_2$ [3], the deviating expected payoff is

[3] y_1 is the high type's bid for the less favoured object

$(1-p)(\bar{v}_1 - x) + p(\bar{v}_1 - x)F(x + \bar{v}_2)$
$= (1-p)(\bar{v}_1 - x) + p(\bar{v}_1 - x)\frac{1-p}{p}[\frac{(\bar{v}_1-\bar{v}_2)\underline{v}_1}{(\underline{v}_1-\bar{v}_2)(\bar{v}_1+\bar{v}_2-\bar{v}_2-x)} - 1]$
$= (1-p)(\bar{v}_1 - x)[\frac{(\bar{v}_1-\bar{v}_2)\underline{v}_1}{(\underline{v}_1-\bar{v}_2)(\bar{v}_1+\bar{v}_2-\bar{v}_2-x)}] = (1-p)\frac{(\bar{v}_1-\bar{v}_2)\underline{v}_1}{\underline{v}_1-\bar{v}_2}$, which is equivalent to the expected equilibrium payoff. When $x > z_1 - y_1$, the summation $x + y_1$ becomes the new bundle bid, and the deviating expected payoff will be equivalent to equilibrium expected payoff unless $x + y_1 > b$, which will be guaranteed to have strictly lower payoff. If the high type deviates to bid lower for her preferred object when $z > 2\bar{v}_2$, the deviation will not impact any winning bids.

When $z_1 \le 2\bar{v}_2$, but a high type deviates upward to $z_1 - y_1 > x > \bar{v}_2$: the deviating expected payoff is $(1-p)(\bar{v}_1 - x) + p(\bar{v}_1 - x)F(x + \bar{v}_2)$
$= (1-p)(\bar{v}_1 - x) + p(\bar{v}_1 - x)\frac{1-p}{p}[\frac{(\bar{v}_1-\bar{v}_2)\underline{v}_1}{(\underline{v}_1-\bar{v}_2)(\bar{v}_1+\bar{v}_2-\bar{v}_2-x)} - 1]$, an expression checked previously [4]. If a high type deviates downward to $z_1 - \bar{v}_2 < x < \bar{v}_2$, the deviating expected payoff is $(1-p)(\bar{v}_1 - x) + p(\bar{v}_1 - x)F(x + \bar{v}_2)$
$= (1-p)(\bar{v}_1 - x) + p(\bar{v}_1 - x)\frac{1-p}{p}\frac{x}{\underline{v}_1 - x} = (1-p)(\bar{v}_1 - x)\frac{\underline{v}_1}{\underline{v}_1 - x}$. But the last expression is maximized when $x = \bar{v}_2$. If a high type deviates to $x < z_1 - \bar{v}_2$, the expected deviating payoff will be
$(1-p)(\bar{v}_1+\bar{v}_2-z_1)+p(\bar{v}_1+\bar{v}_2-z_1)F(z_1) = (1-p)(\bar{v}_1+\bar{v}_2-z_1)+p(\bar{v}_1+\bar{v}_2-z_1)\frac{1-p}{p}\frac{z_1-\bar{v}_2}{\underline{v}_1-z_1+\bar{v}_2}$
$= (1-p)(\bar{v}_1 + \bar{v}_2 - z_1) + (1-p)(\bar{v}_1 + \bar{v}_2 - z_1)\frac{z_1-\bar{v}_2}{\underline{v}_1-z_1+\bar{v}_2}$
$= (1-p)(\bar{v}_1 + \bar{v}_2 - z_1)\frac{\underline{v}_1}{\underline{v}_1-z_1+\bar{v}_2}$. The last expression is an increasing function in z_1 and will be maximized when $z_1 = 2\bar{v}_2$. So we have deviating expected payoff bounded above by $(1-p)(\bar{v}_1 - \bar{v}_2)\frac{\underline{v}_1}{\underline{v}_1-\bar{v}_2}$, which is the equilibrium expected payoff.

The high type will have no incentive to deviate her bid for the less favoured object because she knows that her opponent's bid for that object is at least $\bar{v}_2$, the valuation of a high type has for her less favoured object. The restriction on p is needed to make sure that upper bound of the bundle bid lies strictly in interval $(2\bar{v}_2, \bar{v}_1 + \bar{v}_2)$.

If a high type deviates to bid a bundle bid smaller than $\bar{v}_2$, bid for single objects must also be smaller than $\bar{v}_2$. And hence deviating expected payoff will be
$pF(x + \bar{v}_2)(\bar{v}_1 - x) + (1-p)(\bar{v}_1 - x) = (\bar{v}_1 - x)\frac{(1-p)x}{(\underline{v}_1-x)} + (1-p)(\bar{v}_1 - x) =$
$(1-p)(\bar{v}_1 - x)\frac{\underline{v}_1}{\underline{v}_1-x} \le (1-p)(\bar{v}_1 - \bar{v}_2)\frac{\underline{v}_1}{\underline{v}_1-\bar{v}_2}$, i.e. the equilibrium expected payoff.

Lastly, if a high type deviates multiple bids, first we can restrict to the scenario

[4] If $x > z_1 - y_1$, $x + y_1$ is the new bundle bid, and such a scenario are also checked in the proof

where $z > x + y$ since otherwise summation of bids for single objects is the new bundle bid. When $z > x + \bar{v}_1$, the bundle bid is the winning bid for the deviating high type. But our computation in previous paragraphs reveal that deviating expected payoff in those scenario can not surpass the equilibrium expected payoff. When $z < x + \bar{v}_1$, expected payoff from deviating is $(1-p)(\bar{v}_1 - x) + p(\bar{v}_1 - x)F(x + \bar{v}_2)$. But we have checked all possible range of x and z regarding this expression in the previous cases and conclude that deviating expected payoff can not surpass equilibrium expected payoff. ■

Given this strategy, the allocation is efficient as long as bundle bid is below $2\bar{v}_2$. Inefficient allocation may happen but only if the high type bids a very high price for the bundle. The equilibrium constructed in this theorem is similar to the equilibrium in theorem III.1 in the sense that support is one interval whose lower bound can go down all the way to zero. What's more, upper bound for the bundle will approach $\bar{v}_1 + \bar{v}_2$ when p reaches 1, as indicated by theorem III.1. However, if we make some editions to theorem 2, we can construct a more complicated equilibrium where the support of distribution for the bundle contains a gap, while the distribution remains atomless.

We can actully make some modifications to theorem III.2 and generate a pure strategy equilibrium:

Corollary III.1 *When $p > \frac{\bar{v}_2}{\underline{v}_1}$, there exists a pure strategy equilibrium when high type bids $(2\bar{v}_2, \bar{v}_2, \bar{v}_2)$ and low type bids $\bar{v}_2$.*

Proof. Given the assumptions, the low type will always win one object and get $\underline{v}_1 - \bar{v}_2$ as final payoff. She will not deviate since $\underline{v}_1 - \bar{v}_2 > (1-p)\underline{v}_1$ given range of p.

For the high type, she also wins one object with payoff $\bar{v}_1 - \bar{v}_2$. If the high type increases her bid for the favoured object, she gets two objects since her bundle bid also increases to the summation of her new bids for single objects, but her payoff is strictly smaller than $\bar{v}_1 - \bar{v}_2$. If the high type decreases her bid for the favoured object, her payoff will only be $\frac{p}{2}(\bar{v}_1 - \bar{v}_2) + (1-p)(\bar{v}_1 - \bar{v}_2) < \bar{v}_1 - \bar{v}_2$ because she now wins with one-half probability by the tie of bundle bids when facing a high type. If the high type decreases her bundle bid, her payoff win not change. If the high type increases her bundle bid, her payoff decreases as argued previously.

If the high type decreases her bids for single objects and for the bundle simultaneously, the maximal payoff will be $(1-p)\bar{v}_1$ since she will not win when facing a high type, and she can get $(1-p)\bar{v}_1$ by making all her bids close to 0. But given range of p, we still have $(1-p)\bar{v}_1 < \bar{v}_1 - \bar{v}_2$. ■

Graphical Illustration of Theorem III.2

If we pick $\bar{v}_1 = 3, \bar{v}_2 = 1, \underline{v}_1 = 2$ and $p = \frac{3}{5}$, CDF used by high type will be

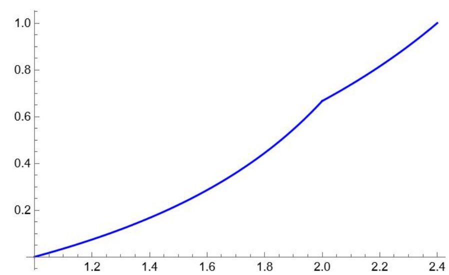

Figure 3.2: Illustration of theorem III.2

Wider class of Equilibria

Noticing that theorem III.2 and corollary III.1 seem similar but are not directly connected, we are curious about one extra question: whether there exists a class of equilibria which contains theorem III.2 so that corollary III.1 is actually an extreme or special case of such a class of equilibria.

Theorem III.3 *When $p \in (\frac{\bar{v}_2}{\underline{v}_1}, 1)$, strategies that satisfying the following conditions form a class of Bayesian Nash Equilibrium:*

- *High and low type bid a pure strategy at price $\bar{v}_2$;*
- *The distribution of bundle bid used by high type $F(\cdot)$ satisfy conditions*
  - $F(z) \leq \frac{(1-p)(z-\bar{v}_2)}{p(\underline{v}_1 - z + \bar{v}_2)}$ *for* $z \in (0, 2\bar{v}_2)$ *;*
  - $F(z) \leq \frac{(1-p)(\bar{v}_1-\bar{v}_2)\underline{v}_1}{p(\underline{v}_1-\bar{v}_2)(\bar{v}_1+\bar{v}_2-z)} - \frac{1-p}{p}$ *for* $z > 2\bar{v}_2$ *;*
  - *For $z > 2\bar{v}_2$, $F(2\bar{v}_2)$ and $F(z)$ should be related by equation* $p(\bar{v}_1-\bar{v}_2)F(2\bar{v}_2)+(1-p)(\bar{v}_1-\bar{v}_2) = (1-p)(\bar{v}_1+\bar{v}_2-z)+p(\bar{v}_1+\bar{v}_2-z)F(z)$.

Proof. We revisit the no-deviating conditions in theorem III.2: We denote low type's deviating bid as y and high type's deviating bids for the bundle and for her favoured objects and z, x respectively.

To prevent low type from deviating, we need conditions:
$(1-p)(\underline{v}_1 - y) + p(\underline{v}_1 - y)F(\bar{v}_2 + y) < (1-p)\underline{v}_1$ for all $y \neq \bar{v}_2$. So we should have $F(\bar{v}_2 + y) < \frac{(1-p)y}{p(\underline{v}_1 - y)}$ for all $y \neq \bar{v}_2$.

To prevent high type from deviating, we need the following conditions:

- When $z_1 > 2\bar{v}_2$ and $z_1 - y_1 > x > z_1 - \bar{v}_2, (1-p)(\bar{v}_1 - x) + p(\bar{v}_1 - x)F(x + \bar{v}_2) \leq (1-p)\frac{(\bar{v}_1 - \bar{v}_2)\underline{v}_1}{\underline{v}_1 - \bar{v}_2}$;
- When $z_1 \leq 2\bar{v}_2$ and $z_1 - y_1 > x > \bar{v}_2, (1-p)(\bar{v}_1 - x) + p(\bar{v}_1 - x)F(x + \bar{v}_2) \leq (1-p)\frac{(\bar{v}_1 - \bar{v}_2)\underline{v}_1}{\underline{v}_1 - \bar{v}_2}$;
- When $z_1 - \bar{v}_2 < x < \bar{v}_2$, $(1-p)(\bar{v}_1 - x) + p(\bar{v}_1 - x)F(x + \bar{v}_2) \leq (1-p)\frac{(\bar{v}_1 - \bar{v}_2)\underline{v}_1}{\underline{v}_1 - \bar{v}_2}$;
- When $x < z_1 - \bar{v}_2$, $(1-p)(\bar{v}_1 + \bar{v}_2 - z_1) + p(\bar{v}_1 + \bar{v}_2 - z_1)F(z_1) \leq (1-p)\frac{(\bar{v}_1 - \bar{v}_2)\underline{v}_1}{\underline{v}_1 - \bar{v}_2}$;
- When $z < \bar{v}_2, (1-p)(\bar{v}_1 - x) + pF(x + \bar{v}_2)(\bar{v}_1 - x) \leq (1-p)\frac{(\bar{v}_1 - \bar{v}_2)\underline{v}_1}{\underline{v}_1 - \bar{v}_2}$.

The first two conditions require $F(x + \bar{v}_2) \leq (1-p)\frac{(\bar{v}_1 - \bar{v}_2)\underline{v}_1}{p(\underline{v}_1 - \bar{v}_2)(\bar{v}_1 - x)} - \frac{1-p}{p}$ when $x > \bar{v}_2$. Requirements 3 and 5 are also identical: $F(x + \bar{v}_2) \leq (1-p)\frac{(\bar{v}_1 - \bar{v}_2)\underline{v}_1}{p(\underline{v}_1 - \bar{v}_2)(\bar{v}_1 - x)} - \frac{1-p}{p}$ when $x < \bar{v}_2$. In addition, we need $F(z_1) < (1-p)\frac{(\bar{v}_1 - \bar{v}_2)\underline{v}_1}{p(\underline{v}_1 - \bar{v}_2)(\bar{v}_1 + \bar{v}_2 - z_1)} - \frac{1-p}{p}$ for $z < 2\bar{v}_2$ according to requirement 4. An observation is that when we replace z by $x + \bar{v}_2$, the last two inequalities are equivalent. So we only need to care about the last inequality containing $F(z_1)$ since it is more general.

Direct computation shows that $\frac{(1-p)(z - \bar{v}_2)}{p(\underline{v}_1 - z + \bar{v}_2)} \leq (1-p)\frac{(\bar{v}_1 - \bar{v}_2)\underline{v}_1}{p(\underline{v}_1 - \bar{v}_2)(\bar{v}_1 + \bar{v}_2 - z)} - \frac{1-p}{p}$ for all $z < 2\bar{v}_2$ and $z > \bar{v}_2 + \underline{v}_1$. So we can claim that as long as distribution of bundle bid satisfies $F(z) \leq \frac{(1-p)(z - \bar{v}_2)}{p(\underline{v}_1 - z + \bar{v}_2)}$ for $z \in (0, 2\bar{v}_2) \cup (\bar{v}_2 + \underline{v}_1, \bar{v}_1 + \bar{v}_2)$ and $F(z) \leq (1-p)\frac{(\bar{v}_1 - \bar{v}_2)\underline{v}_1}{p(\underline{v}_1 - \bar{v}_2)(\bar{v}_1 + \bar{v}_2 - z)} - \frac{1-p}{p}$ for $z \in (2\bar{v}_2, \bar{v}_2 + \underline{v}_1)$ the no-deviating condition will continue to hold. However, notice that a low type will never propose a bid surpassing $\underline{v}_1$, which means we do not need to worry about low type bidding higher than $\underline{v}_1$ at the first place. So as long as distribution of bundle bid satisfies $F(z) \leq \frac{(1-p)(z - \bar{v}_2)}{p(\underline{v}_1 - z + \bar{v}_2)}$ for $z \in (0, 2\bar{v}_2)$ and $F(z) \leq (1-p)\frac{(\bar{v}_1 - \bar{v}_2)\underline{v}_1}{p(\underline{v}_1 - \bar{v}_2)(\bar{v}_1 + \bar{v}_2 - z)} - \frac{1-p}{p}$ for

$z \in (2\bar{v}_2, \bar{v}_2 + \bar{v}_1)$, there will be no deviation.

The last point we need to make sure is that high type gets identical expected payoff by making bundle bids mixed for values higher than $z_1 > 2\bar{v}_2$: i.e. $F(2\bar{v}_2)$ and $F(z_1)$ should be related by equation
$p(\bar{v}_1 - \bar{v}_2)F(2\bar{v}_2) + (1-p)(\bar{v}_1 - \bar{v}_2) = (1-p)(\bar{v}_1 + \bar{v}_2 - z_1) + p(\bar{v}_1 + \bar{v}_2 - z_1)F(z_1)$. Knowing $F(z_1) \leq (1-p)\frac{(\bar{v}_1 - \bar{v}_2)\underline{v}_1}{p(\underline{v}_1 - \bar{v}_2)(\bar{v}_1 + \bar{v}_2 - z)} - \frac{1-p}{p}$, $F(2\bar{v}_2) \leq \frac{(1-p)\bar{v}_2}{p(\underline{v}_1 - \bar{v}_2)}$, i.e. the requirement we developed previously. If z further goes beyond $\bar{v}_2 + \underline{v}_1$, the inequality relating $F(2\bar{v}_2)$ and $F(z_1)$ will generate
$F(2\bar{v}_2) \leq (1-p)(\bar{v}_1 + \bar{v}_2 - z)\frac{\underline{v}_1}{(\bar{v}_2 + \underline{v}_1 - z)(\bar{v}_1 + \bar{v}_2)} - \frac{1-p}{p}$, and this expression will be greater than $(1-p)\frac{(\bar{v}_1 - \bar{v}_2)\underline{v}_1}{p(\underline{v}_1 - \bar{v}_2)(\bar{v}_1 + \bar{v}_2 - z)} - \frac{1-p}{p}$. ■

If the high type only mixes her bundle bid in interval $(0, 2\bar{v}_2)$, we can also borrow part of the argument used in proof of theorem III.3 and provide a corollary:

Corollary III.2 *When $p \in (\frac{\bar{v}_2}{\underline{v}_1}, 1)$, strategies that satisfying the following conditions form a class of Bayesian Nash Equilibria:*

- *High and low type bid a pure strategy at price $\bar{v}_2$;*
- *The distribution of bundle bid used by high type $F(\cdot)$ satisfy conditions :*
  - *$F(z) \leq \frac{(1-p)(z-\bar{v}_2)}{p(\underline{v}_1 - z + \bar{v}_2)}$ for $z \in (0, 2\bar{v}_2)$*
  - *The remaining probability mass of $F(z)$ will be atom at $2\bar{v}_2$.*

Proof. Functional form of $F(\cdot)$ is determined via theorem III.3. $\frac{(1-p)(z-\bar{v}_2)}{p(\underline{v}_1 - z + \bar{v}_2)} = 1$ when $z = \bar{v}_2 + p\underline{v}_1$. And $\bar{v}_2 + p\underline{v}_1$ is greater than $2\bar{v}_2$. So there will be an atom at $2\bar{v}_2$ with size of at least $1 - \frac{(1-p)\bar{v}_2}{p(\underline{v}_1 - \bar{v}_2)}$. Our tie-breaking rule favors allocations that generate higher welfare, which means bundle bid will not win under this corollary. So high type will accept putting an atom at $2\bar{v}_2$. What's more, if the high type raise bundle bid, her payoff will be strictly smaller than $\bar{v}_1 - \bar{v}_2$.

Bound for p is set for the extreme case, where $F(z) = 0$ for $\forall z < 2\bar{v}_2$. And this corollary is studied in corollary III.1, i.e. the pure strategy equilibrium. It is transformed by making $F(z) = 0$ for $\forall z < 2\bar{v}_2$. ■

In conclusion, corollary III.1 becomes an extreme case described by corollary III.2.

3.4 Conclusion and Discussion

In the paper, we study a combinatorial auction with two bidders competing for two distinct objects. To make our computation tractable, we make the following assumptions on model and selection a certain criterion of equilibria: First we assume the type space is discrete. We always assume a high type will have marginal valuations $\bar{v}_1, \bar{v}_2$ and low type will have marginal valuations $\underline{v}_1, \underline{v}_2$. But we focus on case with ordering $\bar{v}_1 > \underline{v}_1 > \bar{v}_2 > \underline{v}_2$, and normalize $\underline{v}_2$ to 0. Unlike usual assumption of synergy for package bidding, we continue to assume the objects in auctions are neither super-additive (complements) nor sub-additive (substitutes). We display two different sets of equilibria, but all of the equilibria contains pure strategies on bid for the favoured object. Focusing on pure strategies is another method we apply to make our computation tractable. Given ordering of private values, efficient allocation is each bidder win one object regardless of types. Among the equilibria introduced in the paper, the second class of equilibria, i.e. theorem III.2 and theorem III.3 will perform better in terms of welfare since they lead to efficient allocations when bundle bid of high type is not too high. While for theorem III.1, it it unlikely we achieve efficient allocation since a high type will always get 0 or 2 objects.

We are also curious about the format of equilibria when more mixed strategies are used by bidders. A conjecture that will lead to a slightly complicated equilibrium would be that high type is using pure bid on favoured objects while low type is mixing in an interval. Actually, if we assume the pure bid high type uses for her favoured object (denoted as b_1^H) is higher than the upper bound of low type's mixed bids (denoted as $\underline{b}^L$) , we have to construct a gap in support of bundle bid since bundle bid in interval $(\underline{b}^L + b_1^H, 2b_1^H)$ is (weakly) dominated by bundle bid at precisely $\underline{b}^L + b_1^H$. However, this conjecture will have a free-riding problem: the high type will have incentive to decrease her bid for the favoured object to prices in interval $(\underline{b}^L, b_1^H)$ to free-ride the other high type's pure bid due to the gap on support of bundle bid. We can treat our second set of equilibria as an extreme case of this conjecture in the way that $b_1^H = \underline{b}^L$, and to adjust to the feature that $b_1^H = \underline{b}^L$ we must let low type to bid pure bid. So our current computation seems to suggest this conjecture is unlikely to be true. A more complicated conjecture is to assume that all bids are mixed strategies, but currently we can not exclude possibilities of such a type of equilibrium nor have we succeeded in constructing any equilibrium with just mixed strategies. One of our main task in the future is to thoroughly study mixed strategy equilibrium by providing equilibria or showing non-existence. Besides, it would also be a good practice if we can characterize what equilibria would look like for other rankings of private values.

CHAPTER IV

Smooth Ambiguity Averse Level-k Auction

4.1 Introduction

Auction theory predicts Bayesian Nash Equilibrium for 1st and 2nd price sealed bid auctions with independent values. For a 1st-price auction with uniform distribution of private values, the unique pure strategy Bayesian Nash equilibrium is half of private valuation when there are two bidders while bidding truthfully is a (weakly) dominant strategy in a 2nd-price auction. But experimental evidence presents results deviating from theoretical equilibria from both directions. For example, Goeree, Holt and Palfrey (2002) found that undergraduate student subjects tended to overbid in first price auctions: they used low and high value treatments for the bidders' valuation and only reported the last 10 periods of experiment sessions to avoid erratic behavior. GHP (2002) found that starting from period 5 subjects' bids would be above theoretical equilibrium (i.e. half of valuation) in both treatments. Underbidding is less common in literature but Cox, Smith, and Walker (1988) found underbidding for low valuations and overbidding in higher valuation in 1st-price auction in one of their experiments [1]. What's more, Garratt, Walker and Wooders (2011) invited highly experienced users in eBay to participate in a 2nd-price auction experiment where values range from 25 to 125 usd uniformly. 38% of the bids were overbids and 41% were underbids. What's more, if subjects had experience of being sellers in the platform, underbid rate would increase to 51% and overbids rate would decrease to 32%.

[1] Figure 8 in page 84 (a group of 4 bidders)

We might expect experienced bidders to make value bids in the 2nd-price auction and student subjects deviate from theoretical results due to lack of auction experience. And hence it might not be that surprising that undergraduate subjects in GHP(2002) were not bidding according to Bayesnian Nash Equilibrium. But result from GWW(2011) gave the result that over 80% of experienced bidders were deviating from Bayesnian Nash Equilibrium. Such deviation obviously indicates BNE strategies can not do a good job explaining the behaviors of real-life bidders. And hence we will look into auctions via the approach of elimination of implausible strategies: we assume bidders will eliminate prices that are implausible to bid given their beliefs about the auction and publicly available knowledge, in particular, we will assume all bidders know distribution of private values, full rationality of all participants and range of plausible bids. Bidders will accordingly bid prices that survive *Iterated Elimination of Implausible prices*.

One issue bidders could encounter when using iterated elimination approach is uncertainty over types of opponent and ambiguity on what bidding prices their opponents would be using. A plausible way to address these issues simultaneously is to use the smooth ambiguity averse model proposed in Klibanoff, Marinacci and Mukerji (2005), which enables bidders to impose subjective beliefs on how opponents' bid would distribute. We will look into a 2-individual first price auction where both bidders believe their opponents would only use pure bids. Smooth ambiguity averse model solves strategic ambiguity by allowing bidders to make subjective beliefs on how opponents' bid would distribute. For simplicity, we may assume that bidders believe their opponents' strategies are distributed uniformly from their current ranges of plausible bids.

We compare the use of smooth ambiguity averse model by making comparison with other approaches. The Δ-rationalizability approach by Battigalli and Siniscalchi can not eliminate close-to-zero bids since such a price can be a "best response" to an extreme optimistic scenario where a bidder believes her opponent is also bidding close to zero. So the most significant practical advantage of using ambiguity averse model is the capability of eliminating such extremely small bids: our assumption from the smooth ambiguity averse model where bidders believe the bids her opponents are using distribute uniformly is able to avoid putting all weight on any single (extreme) event. And accordingly we will not have a situation where a bidder believes her opponent is bidding a close-to-zero price with probability 1. What's more, Battigalli and Siniscalchi(2003) analyzed a first-price single-unit auction with two bidders by employing their Δ-rationalizability model with Δ being beliefs that bidding strategies are monotone w.r.t. private values and bidders believe that a positive bid wins with positive probability. They proved that the bids less than the BNE result are all rationalizable, which allows existence of extremely small bids. We conclude that their results are consistent with the empirical evidence, but do not make a strong prediction.

Another natural approach to address the strategic uncertainty would be maximizing the worst case scenario, i.e. using the maxmin utility function. However, individual with maxmin utility will bid very close to their true valuation, which is also in contrast with empirical data. Battigalli and Siniscalchi's Δ-rationalizability allows bidders to hold various belief on what their opponents may be doing and one particular bids can be justify as long as it is best responding to an allowed belief, regardless of how implausible that belief may seem to be. Same issue occurs when we use the maxmin approach, since a belief that opponents are bidding close to private value would also be quite implausible.

On the contrary, smooth ambiguity averse model is able to aggregate all possible cases a bidder may encounter evenly through bidders' subjective belief on how her opponents' bids might distribute, if we use a moderate range of ambiguity averse attitudes. KMM(2005) used a concave function $\phi(x) = -\frac{1}{\alpha}e^{-\alpha x}$ where α is the coefficient to measure ambiguity averse attitude, with 0 being ambiguity neutral, positive being ambiguity averse and negative being ambiguity loving. And hence the moderate range of ambiguity averse attitudes indicates a positive but relatively low value for α in function ϕ. KMM (2005)'s smooth ambiguity averse model showed that $E_\mu \phi(E_\pi u \circ b))$ could be used to measure the preferences over act b. According to KMM(2005)'s explanation, π is probability measure on act space and μ measure the bidder's subjective relevance of a particular π as the right probability. In our situation, π is a bidder's belief that her opponent will only bid pure strategies and μ is her subjective belief that her opponent's plausible bids are distributed uniformly over current range of plausible bids. Additionally, Denti and Pomatto (2022) followed an alternative interpretation of KMM (2005) where μ is "a prior over the true law" and π will denote as probability law that governs the true state. Our assumptions are a special subclass of these preferences. The ambiguity over opponents' bids can be identified since opponents' actual bids would reveal their bidding functions, thereby satisfying Denti and Pomatto (2022)'s identifiable condition.

Recall that bidder will believe that uniform distribution of value types, rationality for all participates in the auction as well as range of feasible bids are public knowledge and will aggregate her ambiguity over opponent's strategies based on such public knowledge. We can now introduce our solution concept: each bidder will construct new upper (lower) bounds by best responding to beliefs that her opponents' pure bids are distributed uniformly from the range of plausible bids when opponents' ambiguity averse attitude reaches the highest (smallest), since the more ambiguity averse a bidder is, the higher bidding price she is likely to use. If θ denotes the public belief that private value follows a uniform distribution when the type space is $[0, 1]$, our utility function should be $E_\theta(E_\mu \phi(E_\pi u \circ b)))$. So the maximizer of $E_\theta(E_\mu \phi(E_\pi u \circ b)))$ when plugging the largest (smallest) ambiguity averse coefficient α into ϕ will be new upper (lower) bounds of plausible bids. Any current plausible price greater (smaller) than the newly computed upper (lower) bound will be called implausible bids and hence will be eliminated. If it is public knowledge that each bidder in the auction knows range of ambiguity averse attitude across all types of bidders are identical, each bidder should be able to compute range of plausible bids in each round recursively. And hence each participant is able to repeat the elimination (optimization) process on the newly computed range of plausible bids until upper and lower bounds converge.

We have constructed our elimination process via dealing with ambiguity averse while we still need to solve some technical issue. Iterated Elimination of Implausible Strategies in a continuum support always has a major drawback: the impossibility to define the smallest increment/decrement. And hence for a 2-individual first price auction with independent values, although people always know that bidder should try to bid only "epsilon" higher than their opponents' highest feasible as upper bound or bid only "epsilon" higher than 0 as lower bound in the current round of elimination of dominated bidding prices, it is practically unfeasible to find such an increment in the bidding space. Battigalli and Siniscalchi (2003) is an example to consider rationalizable bids in first price auction in continuum space. Battigalli and

Siniscalchi (2003) assumed that bidders would construct upper bounds of feasible bids by best responding to the case where bidders assumed their opponents would bid their upper bounds from previous round of elimination. But they were only able to eliminate bids from upward and left lower bounds constant at 0. The most straightforward way to avoid the issue encountered in Battigalli and Siniscalchi (2003) is to work on a discrete support. Dekel and Wolinsky (2003) studied an k-individual first price auction with discrete bidding space. Incremental of available discrete bids is $d = \frac{1}{m}$ with m being number of grids in the bidding space. And they succeeded to eliminate all bids except $v_i - d$, due to their large number of bidders and discrete type space assumption. Similar to Dekel and Wolinsky (2003), we will simulate our computational process by computational software. We look at a 2-player first price single-unit auction and discretizing the type space $[0, 1]$ into n grid points with equal grid margin $\frac{1}{n}$. We further assume type and bidding spaces are identical. The available bids for participants in the auction is accordingly the n discrete grid points. We will construct a lower and upper bound for each grid point in the discretized space and compute the bids that survive iterated elimination. **??** Ahn, Choi, Gale and Kariv (2014) discovered that the range of ambiguity averse coefficient is usually $[0, 2]$, with more than 20% of the population being ambiguity neutral and the 95% percentile of ambiguity averse coefficient being only 1.9 or less. So we will be using 2 and 0 as our ambiguity averse coefficients for upper and lower bounds respectively. With selection of range of α to be $[0, 2]$, our model features underbidding relative to BNE prediction, which matches observation of Cox, Smith, and Walker (1988). We also try to increase the upper bound of ambiguity averse coefficient to larger integers, and the result is higher stable bounds.

The assumption of a uniformly distributed bidding strategy in our model is just for computational simplicity, but also makes our model very similar to the level-k theory. Level-k theory in the context of an auction assumes that in the 1st round, $L0$ participants will bid uniformly and randomly from a range between the highest and lowest prices. While in the next rounds of actions, $L1$ participants will believe that others will behave according to $L0$, and hence they will best respond to such beliefs. A future $Lk(k > 1)$ will iterate such type of best response k times, in particular, $L2$ will believe others behaving like $L1$ and best respond to such a belief. We say our model is extending the L1 response to L0 from the level-k theory. The extension we make from Crawford and Iriberri (2007b)'s level-k theory is that bidders in our model construct new ranges of plausible bids by plugging bounds of ambiguity averse coefficients into ϕ function in $E_\theta(E_\mu\phi(E_\pi u \circ b)))$. So each of our round is similar to how L1 is best responding to random L0 in level-k theory: bidders in our model construct upper and lower bounds simultaneously with subjective beliefs that opponents' pure bidding prices distribute uniformly, under the additional condition that bidders themselves have the highest and smallest plausible ambiguity averse coefficients. In a word, our model is eliminating implausible bids and leaving a range of plausible bids every round during the elimination process while after L1 the level-k theory's best response will just be singleton sets. We can name our model "Smooth Ambiguity Averse Level-k" since our elimination process is similar to how L1 responds to L0 in level-k theory and we use smooth ambiguity averse representation to study the elimination process. Crawford and Iriberri (2007b) applied level-k theory into only stage L0, L1 and L2 while ours will solve the whole elimination process.

Section 2 introduces our ambiguity averse version of elimination of dominated prices, and we may call it smooth ambiguity averse level-k to represent that it is a hybrid of the famous smooth ambiguity averse model and level-k theory. We present the numerical results in section and compare it with literature and BNE result in section 3. The last section shows bidding function if we expand the range of ambiguity averse coefficient and compares different solution concepts.

4.2 Model

We will formally introduce our model in this section. As mentioned, we consider a sealed-bid first price auction with independent private values. We allow 2 ex-ante identical participants in this auction and each player is informed with her private value, v_i, of an indivisible subject. Each bidder submits a price and the object is rewarded to the bidder who bids a higher price. In case of ties, the object will be rewarded equally with 50% probability between the 2 bidders. We further assume that bidding space and value space are equivalent to the discretized unit interval $[0, 1]$. The discretized values are from set $V = \{v_1, v_2, ...v_{n-1}, v_n\}$ and every 2 consecutive grid points v_k, v_{k+1} share the same grid margin $\frac{1}{n}$, i.e. $v_{k+1} - v_k = \frac{1}{n}$ for

$\forall k$. So $v_1 = \frac{1}{n}$ and $v_n = 1$. We assume it is public knowledge that private values have a uniform distribution on the discretized unit interval.

We employ and extend model from the smooth ambiguity averse model proposed in in Klibanoff, Marinacci and Mukerji (2005), where a double expectation $E_\mu \phi(E_\pi u \circ b)$ is used to measure preferences over acts. In our case, b is the pure bid a bidder is using, and u is the material payoff from the first price single unit auction. ϕ is an increasing transform which characterises altitude towards ambiguity. KMM(2005) derived $\phi(x) = \frac{1}{\alpha}e^{-\alpha x}$ as a function of constant ambiguity averse with α being the constant absolute ambiguity averse coefficient. The higher this α is, the higher ambiguity averse a bidder will become, which will indicate that a bidder will be more likely to bid a higher price to avoid loss in the auction. We will require the ambiguity averse coefficient $\alpha \in [0, \infty]$. If $\alpha = 0$, we say the bidder is ambiguity neutral while if $\alpha = \infty$ KMM(2005) pointed out a bidder would maxmin preference. We will modify our $\phi(x)$ to be $\phi(x) = \frac{1}{\alpha} - \frac{1}{\alpha}e^{-\alpha x}$ so as to normalize $\phi(0) = 0$. KMM(2005) defined μ to be a subjective probability over the set of probability measures π that a decision maker thought were possible given her subjective information. In terms of our construction of upper and lower bound, π reflects a bidder's belief that her opponent will be bidding a pure strategy and hence should be a degenerate measure on single prices. μ will assign any possible outcome a bidder finds possible a probability. We have assumed that bidders are believing bidders' pure strategy are distributing uniformly, which indicates that μ is assigning probability uniformly to plausible pure bidding prices over range of plausible bids. To be more precise, our model assumes that a bidder with private value v_i will believe her opponent whose private value is one element from the $\{v_1, v_2, ...v_{n-1}, v_n\}$ are bidding pure strategies which are uniformly distributed between the current upper and lower bounds. We extend KMM(2005)'s smooth ambiguity averse model by adding another expectation: π and μ only addresses the scenario when a bidder believes her opponent's valuation is one element from set V, but does not reflect how a certain value is selected from V. The expected utility function will be complete when we introduce another expectation sign outside $E_\mu \phi(E_\pi u \circ b)$. $E_\theta(E_\mu \phi(E_\pi u \circ b))$ with θ being the public belief that private valuations are distributed uniformly on value space. We can restate our model from the most external expectation to the most internal one: a bidder who believes that her opponent's private valuation is uniformly distributed from set V will believe her opponent's bids are pure strategy distributing uniformly from current round of plausible bids.

We should start the elimination process with each bidder (with private value v_i) treating v_1 as the initial lower bound and v_{i-1} as the initial upper bound since bidding the exact private value v_i will not bring positive payoff. The exception here is bidder with private value v_1 who is only able to bid v_1. We call this round 0. We construct upper and lower bounds of feasible in round 1 by solving the following question: a bidder with private v_i will construct the new upper (lower) bound by solving the maximizing question $\max_{b_i \in \{v_1, \ldots, v_{i-1}\}} \sum_{j=1}^{n} \frac{1}{n} \sum_{v_1}^{v_{j-1}} \phi(u(b_i, b_j, v_i, v_j) \frac{1}{j-1}$ when she plugs the highest (lowest) feasible ambiguity averse coefficient into her smooth ambiguity averse representation ϕ. Round 1 will end when bidder with each private value finishes solving these 2 questions. Problem to solve for remaining rounds will be similar to that in round 1 with some change in notations. If we use $lb_m(v_i)$ as lower bound for private value v_i after round m and $ub_m(v_i)$ as upper bound for private value v_i after round m, and denote number of the feasible bids between $lb_m(v_i)$ and $ub_m(v_i)$ as $gap_{m,j}$, (i.e. $gap_{m,j} = n(ub_m(v_j) - lb_m(v_j)) + 1$),

the maximization problem in round k will be looking like

$\max_{b_i \in \{lb_{k-1}(v_i), \ldots, ub_{k-1}(v_i)\}} \sum_{j=1}^{n} \frac{1}{n} \sum_{b_j = lb_{k-1}(v_j)}^{ub_{k-1}(v_j)} \phi(u(b_i, b_j, v_i, v_j) \frac{1}{gap_{k-1,j}}$. The remaining rounds will continue recursively for every type value until its upper and lower bounds coincide (or each type end up in stable interval of bids).

Ahn, Choi, Gale and Kariv (2014) discovered that the range of ambiguity averse coefficient is usually $[0, 2]$, and hence we will compute new upper (lower) bounds to be the maximizers of $E_\theta(E_\mu \phi(E_\pi u \circ b))$ with $\alpha = 2(0)$. When $\alpha = 0$, the $\phi(x)$ will just be x, which will simplify our work to some extent. Our goal is to construct the process of eliminating implausible bids. And we will construct the process with the help of computational software.

4.3 Results

4.3.1 Stable Bids

Starting from section, we will use terminology "**stable bids**" to describe the bids that upper and lower bounds of plausible bids converge to after rounds of elimination of implausible bids.

The graph of stable bids for $n = 1000$ is

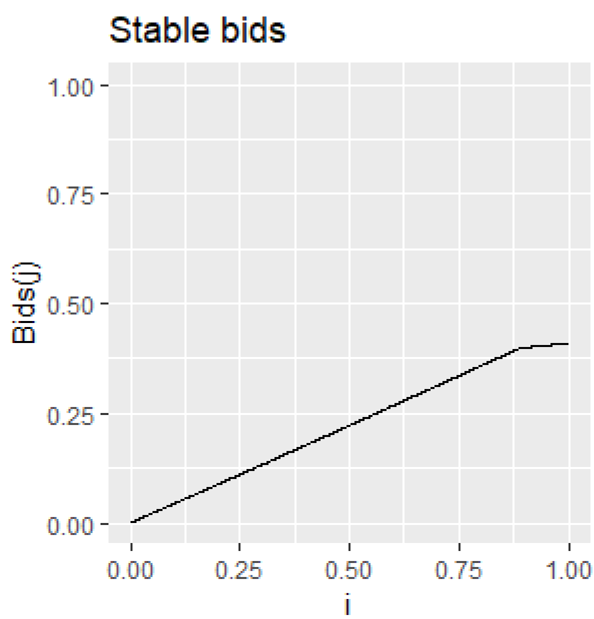

Figure 4.1: Stable bids when $n = 1000$

Here is how we interpret the graph: the horizontal axis represents value of each type while the vertical axis is the stable bids. A noticeable fact about the bidding function is its significantly concave tail, which indicates bidders with higher private values are underbidding much more than other types. Such fact is reasonable since bidders with higher private values may think themselves have very small probability of loss due to their high private valuations so they choose to decrease their bids to some extent to reflect such confidence as well as to guarantee themselves higher payoff when they win in the auction.

If we compare our results with BNE result (red line), it turns out that we find underbidding for all private values, which means our ambiguity averse reasoning will generate similar result in terms of final bids as GWW(2011). This result should not be surprising since the upper bound of α, 2, is not large. People with small ambiguity averse coefficients would prefer bidding lower with higher net payoff when winning the auction. And they will not weigh a lot on possible situations where they lose the auction by bidding relatively low:

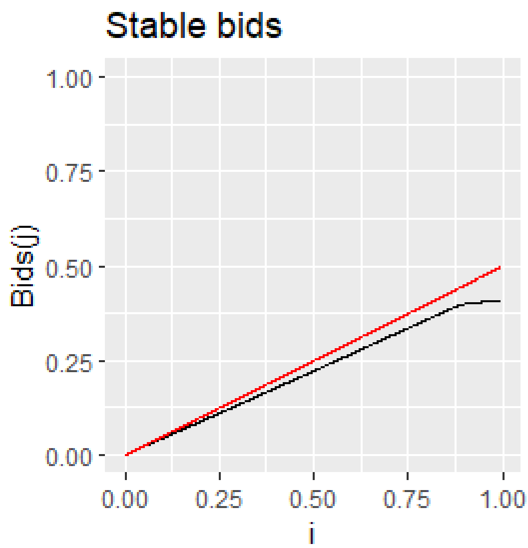

Figure 4.2: Comparing with BNE result

From the graph of stable bids when $\alpha \in [0, 2]$, it seems that we have a close-to-linear bidding function until i reaches 0.800. If we show the graph of stable bids when $\alpha \in [0, 6]$, we have:

It is clear from the graph when upper bound of α gets larger, the bidding function will very close to be linear at first and become concave eventually. And hence our conjecture regarding shape of bidding function is that the bidding function when $\alpha \in [0, 2]$ should have the same shape but the magnitude is relatively small. Such a guess can be checked by fitting the bidding function with polynomials and check the concavity and convexity of the fitted function. And we will check the fitness in the next subsection.

Finally we compare our model to the level-k theory. Our Round 1 is similar to Crawford and Iriberri (2007b)'s random L0, where the latter assumes bidders just

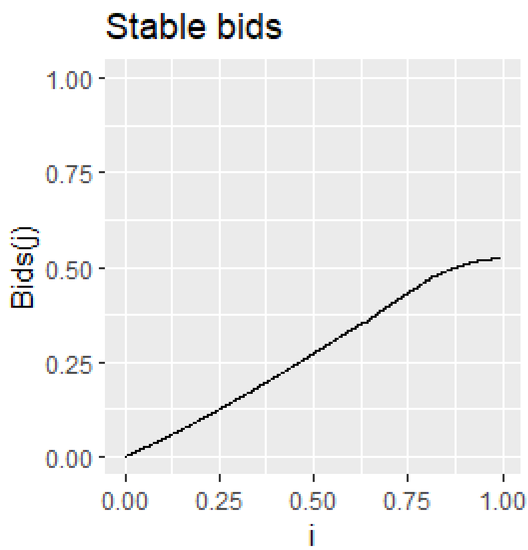

Figure 4.3: Another range of α

bid uniformly from the range of feasible bids. But starting from Round 2 our model drifts away from Crawford and Iriberri (2007b)'s setting where we continue to assume that bidders only know the distribution of plausible bids and they are only able to construct another range of feasible bids by assuming their opponents' ambiguity averse coefficients are the highest or lowest since the more ambiguity averse a bidder is, the more likely she is bidding higher prices. On the contrary, Crawford and Iriberri (2007b) assumed that bidders would choose a strategy which is best response to random L0. Our ambiguity averse elimination process ends after round 6, which is surprisingly short but still longer than L2 (2 rounds) considered in Crawford and Iriberri (2007b). Real participants of auction experiments may not have a deep understanding of the auction and hence they are not able to think their strategies beyond L2. Our model shows that the auctions with bidders who fully understand the mechanism of the auctions will endure longer rounds.

4.3.2 Fitness

We can run several regressions up to the third power of i to fit the graph for polynomials. The reason we only look at i up to the power of 3 is due to the small absolute value of estimation: the coefficient for i^3 is already 10^{-8} and will only get smaller when we raise the power.

Estimate	Scenario 1	Scenario 2	Scenario 3	Scenario 4
i	0.4372475 ***	4.811e-01***	4.679e-01***	3.585e-01***
i^2		-4.381e-05 ***		2.623e-04 ***
i^3			-3.402e-08***	-2.039e-07***
Constant	2.7686066	-4.554e+00	-4.059e+00	5.700e+00
Adjusted R-squared	0.9976	0.9982	0.9985	0.9992

Note: $^{*} p < 0.1$; $^{**} p < 0.05$; $^{***} p < 0.01$

Table 4.1: Approximation by polynomials

Despite the small absolute values in estimation, statistical significance makes us confident to assert the stable bidding function is not a linear function. And we can fit the bidding function by a polynomial which is slightly convex initially but becomes convex eventually as shown in scenario 3 and 4. The fitness practice seems to justify our conjecture in the last subsection regarding convexity and concavity of the bidding function. If we approximate the bidding function by a polynomial with coefficients computed from scenario 4, firstly it is straightforward to see that when private value is small the bidding function is close-to-linear (actually it is slightly convex). Secondly, we are able to conclude that the polynomial will have a negative second order derivative when private valuation is greater than 0.428 and the second order derivative becomes significantly larger as private valuations increase. This discovery reflects the significantly concave tail we observe in plots of bidding function.

What's more, if we only run regression to approximate bidding functions for types whose private valuations are smaller than 0.5, we have similar estimates for the constant and parameters for i as in scenario 1. And this result confirms our conjecture that the bidding function is close-to-linear when private valuation is small.

4.4 Discussion

4.4.1 Selection of Ambiguity Averse Coefficient

We have illustrated stable bidding functions when upepr bound of α is 2. We can now turn to comment cases with higher α's. If we return to graph when $\alpha \in [0, 6]$:

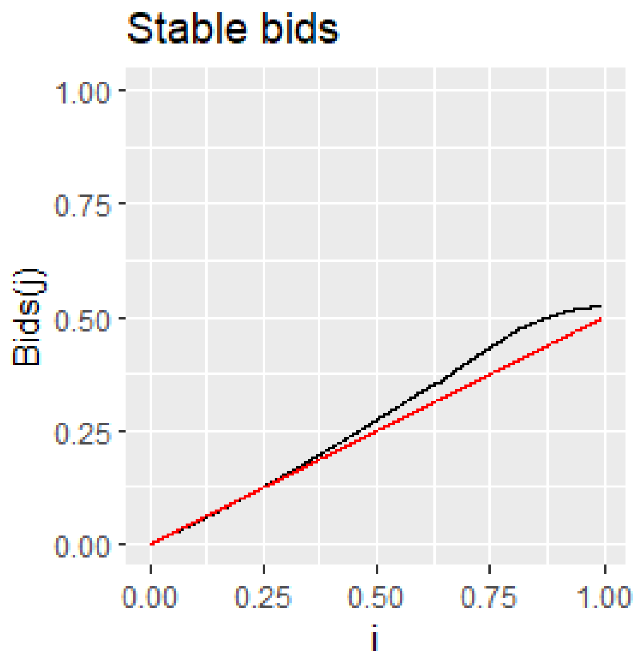

Figure 4.4: Comparing with BNE result

where the black line is the stable bids from our model and red line is what the Bayesian Nash equilibrium predicts.

We find underbidding for extremely low private values ($i < 0.24$) and overbidding for all other values. This result satisfies our intuitive understanding of the ambiguity averse coefficient: with large ambiguity averse coefficients, people tend to fear that they may lose the auction when they could have won by bidding a higher price. The underbidding case for extremely small private values are symmetric to the "significantly concave tail" phenomenon when α is small. Now bidders with extremely small private values will find themselves next to impossible of winning any auction and they would rather win with some higher payoff if their winning somehow happened and they believe themselves unable to win even if they increase their bids due to small private valuations. Such a finding is consistent with the majority of literature like GHP(2002). The "significantly concave tail" still exists when $\alpha \in [0, 6]$ but shrinks a lot in magnitude, which reflects that when bidders are more ambiguity averse they will not risk losing the auction by bidding relatively lower unless their private values are extremely high. What's more, our $\alpha \in [0, 6]$ setting replicates figure 8 from CSW(1988)'s result where they also found

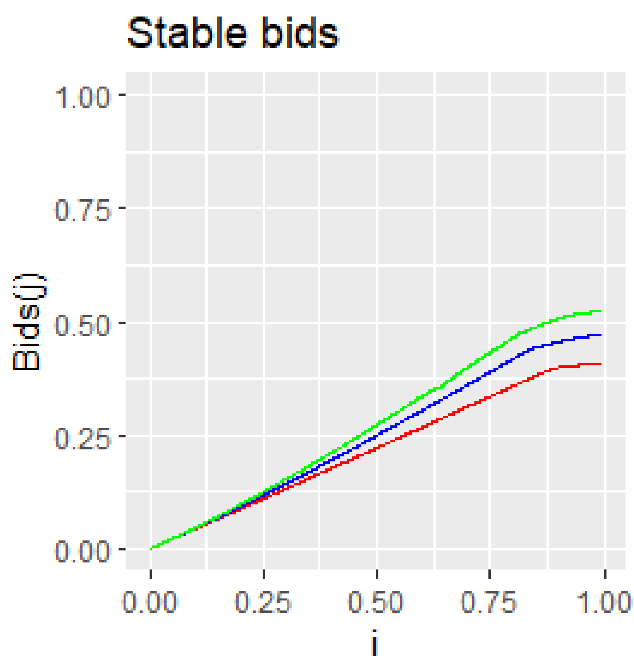

Figure 4.5: Different ranges of α

underbidding when private values are small and overbidding when private values are large, except that figure 8 from CSW(1988) was an experiment with 4 participants.

Furthermore, we can report bidding functions with upper bound of α being $2, 4$ and 6 where the green, blue and red lines are bidding function when $\alpha \in [0, 6], [0, 4], [0, 2]$ respectively.

Bidding function will increase for the same value type i as the highest feasible α increases. And hence bidding function goes from underbidding to overbidding as upper bound of α increases. The shape of bidding functions (firstly linear but eventually concave) can be witnessed clearly from the graph. Another observation is that the pivotal point for bidding function turning into concavity from convexity decreases as upper bound of α increases.

Kirchkamp and Reiß (2004) and (2019) studied bidders' behavior in a 2-bidder first-price auction via experiments and one of the auctions had private values distribute uniformly between $[0, 50]$ and only permitted non-negative bids, which was called as "0+" treatment. The "0+" treatment is the traditional first-price auction, which is also consistent with our setting except for the continuum support.

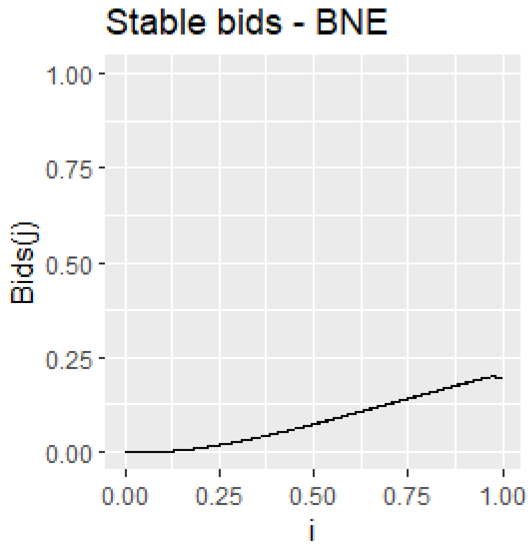

Figure 4.6: Difference between stable bids and BNE

"0+" experiment [2] found overbidding for all types of private values and the highest type would overbid the BNE result by 10, which is about 20% of the total range of private values. We can roughly approximate such a result using our model by restricting the ambiguity averse coefficient α to be $[8, 12]$. We can show the difference between stable bidding prices and BNE result in the graph below:

The lower bound $\alpha = 8$ will make sure the low private value types are not underbidding and the upper bound $\alpha = 12$ makes sure the highest overbidding percentage is only 20% of the total range of values.

4.4.2 Comparison with other solution concepts

In this subsection, we want to re-illustrate and emphasize the differences and connections between our model and some existing solution concepts in the literature. The most common solution concept used to study auction is (symmetric) Bayesian Nash Equilibrium. Our approach is obviously not the BNE approach since bids in our model reach stable state (equilibrium) after bidders gradually eliminate bidding prices that are implausible to be best responses, which makes our model very similar to Iterated Elimination of Dominated Strategies

[2] We look at figure 6 in Kirchkamp and Reiß (2004) and 50% quantile line of 0+ treatment (median amount of overbidding) in Fig. 7 in Kirchkamp and Reiß (2019)

under some restrictions on beliefs like Dekel and Wolinsky (2003). But the main difference is that we define upper (lower) bound of plausible strategies as best response to subjective belief that opponents' pure bids are distributing uniformly when having the highest (smallest) ambiguity averse coefficient while the latter theory eliminates strategies that are never best responses to any belief. Our model and iterated elimination of dominated strategies justify beliefs differently since the latter essentially allows existence of beliefs that assigns probability 1 to opponent bidding extremely small bids, which is rare and implausible. On the other hand, our approach aggregates every possible scenario evenly by imposing an uniform subjective belief on what opponents' bids could be. It would be much easier to eliminate extremely small (and high) bids in our model since to support such bids as best response bidders usually need to come up with a rare event with probability 1.

Maxmin approach is to maximize the smooth ambiguity averse representation under the worst possible scenario and in terms of auction the worst scenario usually means bidders believe that opponents are bidding their highest feasible bids. Interestingly, KMM(2005) pointed out that the maxmin preference is a special case of ambiguity averse model where bidders' ambiguity averse attitudes rise to infinity. Intuitively speaking, the higher the ambiguity averse coefficients, the more likely it is for bidders to focus on cases where where they could have won if they had increased their bids. (Bidders with small or mild ambiguity averse coefficients do not fear of the case above and hence they will bid smaller prices than bidders who are more ambiguity averse.) And hence such bidders will tend to bid close to private values to avoid potential losses when they are extremely ambiguity averse. If we simulate maxmin preference in smooth ambiguity averse model, we can plug very high ambiguity averse coefficients α into the ϕ function and let bidders believe opponents are bidding the upper bounds of plausible bids. We should expect to see extremely high bidding prices. For example, the highest bid will be 88% of private values if we pick $\alpha \in [40, 50]$ and be higher than 90% if we pick $\alpha \in [90, 100]$. Majority (more than 75%) of stable bids will be higher than 75% of private value if $\alpha \in [40, 50]$ and higher than 80% of private value if $\alpha \in [90, 100]$.

We show the two bidding functions in graphs below, The first graph is when $\alpha \in [40, 50]$ and the second is when $\alpha \in [90, 100]$. The black lines represent stable bids predicted by our model while the red lines represent bidding 75% and 80% of private valuations respectively in each plot.

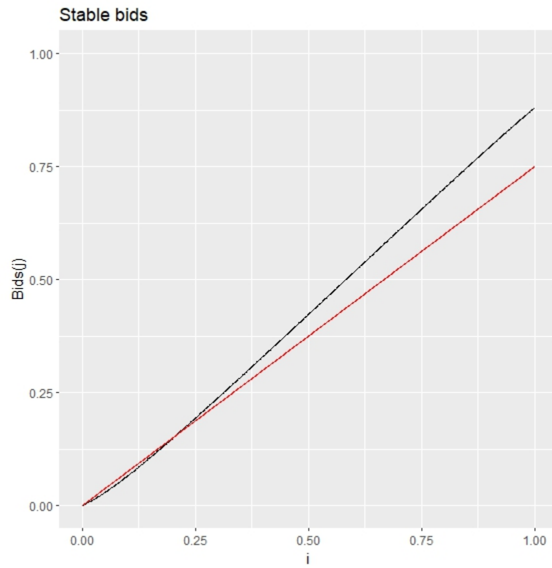

Figure 4.7: High range of α

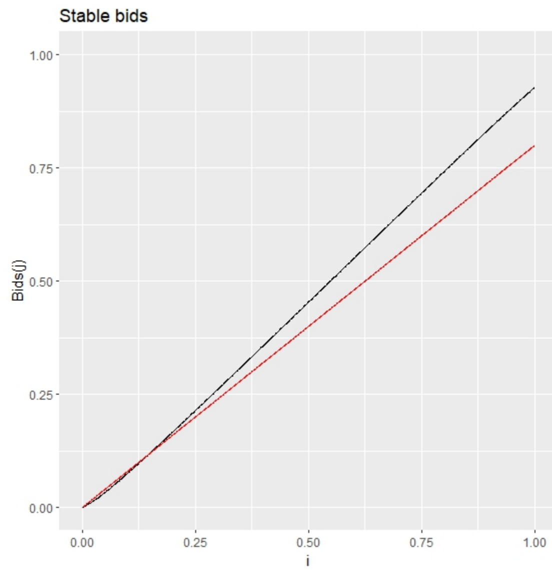

Figure 4.8: Even Higher range of α

But such close-to-private-value bidding prices are never observed in experiments and neither do participants in real experiments own extremely high ambiguity averse attitudes . Ahn, Choi, Gale and Kariv (2014) discovered that the range of ambiguity averse coefficient is only $[0, 2]$. In conclusion, we do not select maxmin utility since literature finds it very rare for real individuals to have extreme ambiguity averse attitudes.

The last solution concept we want to compare is level-k theory. We have mentioned in the introduction section that the main difference between our model and level-k is that we introduce a range of ambiguity averse coefficients so that we are able to construct upper and lower bounds by using the highest and lowest ambiguity averse coefficient. We can accordingly treat Crawford and Iriberri (2007b)'s level-k theory as an extreme case of our smooth ambiguity averse level-k model where the upper and lower bound of ambiguity averse coefficients are set to be identical at 0. If $\alpha = 0$, $\phi(x)$ is easily proved to be identity function, which makes $\phi \circ u$ the material payoff function used in Crawford and Iriberri (2007b). An L0 bidder is defined to bid uniformly from the plausible set of prices and L1 is best responding to L0, which is similar to our first round of elimination of implausible bids where bidders believe opponents' definitive bids are distributing uniformly. But Lk's best response to L(k-1) for any $k \geq 1$ will only be a singleton set of bidding prices since level-k theory is essentially that the upper and lower bound of ambiguity averse coefficient are both 0. According to the description above, we may view level-k theory as a very specific case of our smooth ambiguity averse level-k model. A difference between our model and level-k theory in Crawford and Iriberri (2007b), however, is that they stopped their study at L2, which is only 2nd round since they thought that experiment subjects may not be able to think beyond that level. Ours will not stop until equilibrium is reached.

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