The Closed Stratum of a Parahoric Deligne-Lusztig Variety

by

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To my family

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TABLE OF CONTENTS

DEDICATION ii

ACKNOWLEDGEMENTS iii

ABSTRACT vi

CHAPTER 1: Introductions 1

CHAPTER 2: Definitions 3

2.1: Quotients of Parahoric Models of G 5

CHAPTER 3: Preliminaries on Root Systems 9

3.1: Conjectural Generalization 10

CHAPTER 4: The Geometry of Y_h 12

CHAPTER 5: The Lang Torsor and Rank One Multiplicative Local Systems 21

CHAPTER 6: Computing Cohomology 25

CHAPTER 7: Final Computations 35

CHAPTER 8: Nonelliptic Elements 38

CHAPTER 9: The Example of GL_4 42

ABSTRACT

In 1976, Deligne and Lusztig constructed irreducible representations for the rational points of a reductive algebraic group over a finite field by identifying the representations within the cohomology of a particular affine variety. We study an analogy of this construction to the case of an inner form of an unramified group over a local field, which replaces the single affine variety with an inverse system of affine varieties of increasing dimension defined over the residue field. Each of these varieties is equipped with a natural stratification, and the minimal strata, also called the closed strata, has particularly accessible structure.

We study the geometry of the closed strata. When the inner twist is given by a coxeter element we prove that the varieties in the inverse system are all maximal, in the sense that the variety has as many points as are permitted by the Weil bound for its Betti numbers. We also show that the torus weight spaces of the cohomology are all supported in one degree. Despite the relevance to representation theory, our methods almost entirely algebro-geometric, and rely on a detailed description of the variety in terms of Lie-theoretic data.

CHAPTER 1

Introductions

In [8], Deligne and Lusztig associate to a reductive algebraic group G over a finite field $\mathbb{F}_q$ certain affine varieties, now known as Deligne-Lusztig varieties, whose cohomology realizes the irreducible representations of the group $G(\mathbb{F}_q)$. Chan and Ivanov, in [5], introduce a generalization of this to reductive groups over a local field. The construction mimics the finite field case, but takes place in the context of parahoric subgroups and produces an infinite sequence of varieties over the residue field. This thesis studies the geometry of these varieties.

In sections 2 and 3 we recall the construction of these parahoric Deligne-Lusztig varieties, and a certain subscheme referred to as X_h^T, the minimal strata of a natural stratification, then state the precise requirements for the paper's main result:

Theorem 1.1. *Let $\mathbb{F}_q^n$ be the minimal field of definition for X_h^T. Then X_h^T is a maximal scheme over this field in the sense that the action of F^n on the group $H_c^i(Z_h, \overline{\mathbb{Q}}_\ell)$ is diagonalizable and has all eigenvalues equal to $(-1)^i q^{in/2}$.*

This property is referred to as being maximal, because maximal varieties over a finite field have the greatest number of rational points allowed by the Weil bounds and their Betti numbers. In sections 4 through 7 we prove the main theorem. Then in section 8, we discuss how the result can be partially extended to a broader class of parahoric Deligne-Lusztig varieties. Section 9 overviews the main argument as it applies to the specific reductive group GL_4. It can be read independently, and will make the rest of the paper clearer.

Despite the relevance of Deligne-Lusztig varieties to representation theory, this paper takes most of its techniques from finite characteristic algebraic geometry. We produce a fairly explicit description of the varieties that allows us to relate the cohomology of X_h^T to progressively simpler and simpler varieties, and eventually determine the groups directly.

CHAPTER 2

Definitions

Let k be a nonarchimedean local field with residue field $\mathbb{F}_q$ and ring of integers $\mathcal{O}_k$. Denote the completion of the maximal unramified extension of k by $\breve{k}$, and the residue field of $\breve{k}$ by $\overline{\mathbb{F}_q}$.

Let H be a split reductive group over k, S a split maximal torus of H and $\sigma \in \mathrm{Gal}(\breve{k}/k)$ be a Frobenius element which induces qth power automorphisms on $\overline{\mathbb{F}}_q$.

Choose an element c in the Weyl group of S and an element $\widetilde{c}$ in the normalizer of S so that the action of c is realized by conjugating by $\widetilde{c}$.

Define a group scheme G over k with $G_{\breve{k}} \simeq H_{\breve{k}}$ and the rational structure on G coming from the endomorphism $Fg = \mathrm{adj}_c\, \sigma(g)$.. Let T be the maximal torus in G corresponding to S, and B a Borel subgroup containing T with unipotent radical U. T descends to a torus over k, but B and U may have a field of definition larger than k.

In [5] the authors construct for any nonnegative integer h smooth affine group schemes $\mathbb{G}_h$ and $\mathbb{T}_h$ over $\mathbb{F}_q$ and $\mathbb{U}_h$ over $\overline{\mathbb{F}}_q$. We will recall the construction in more detail below. For now, consider them as finite-type approximations of the schemes G, T, and U, and the first two, just like their local field counterparts, are equipped with an action by F, such that $\mathbb{G}_h(\mathbb{F}_q) = \mathbb{G}_h(\overline{\mathbb{F}}_q)^F$ and similarly for $\mathbb{T}$. There are natural maps from $\mathbb{G}_h$ to $\mathbb{G}_i$ for $h > i$ and we write $\mathbb{G}_h^i$ for the kernel of this map.

Definition 2.1. *The* parahoric Deligne-Lusztig variety *is the* $\overline{\mathbb{F}}_q$ *scheme*

$$X_h := \{g \in \mathbb{G}_h \ |g^{-1}Fg \in \mathbb{U}_h\}/(\mathbb{U}_h \cap F^{-1}\mathbb{U}_h).$$

Every F-stable Levi subgroup of H corresponds to a k-rational twisted Levi subgroup of G. For $L \subset G$ such a subgroup, we may construct an associated subgroup $\mathbb{L}_h$ of $\mathbb{G}_h$. Define $\mathbb{U}_{h,L}$ as $(\mathbb{L}_h \cap \mathbb{U}_h)\mathbb{U}_h^1$, and *The L-strata of the parahoric Deligne-Lusztig variety* X_h^L as

$$X_h^L := \{g \in \mathbb{G}_h \ ||g^{-1}Fg \in \mathbb{U}_{h,L}\}/(\mathbb{U}_{h,L} \cap F^{-1}\mathbb{U}_{h,L}).$$

The following result is [6, Lemma 3.3.3]:

Lemma 2.2. *Set* $Y_h^L = X_h^L \cap \mathbb{L}_h\mathbb{G}_h^1$. *Then*

$$X_h^L = \bigsqcup_{\gamma \in \mathbb{G}_h(\mathbb{F}_q)/\mathbb{L}_h(\mathbb{F}_q)\mathbb{G}_h^1(\mathbb{F}_q)} \gamma \cdot Y_h^L$$

Proof. Lang's Theorem states that for Z an algebraic group and F_Z a surjective endomorphism with a finite number of fixed points, the Lang map

$$L(x) = x^{-1} \cdot F_Z x$$

is a surjection from Z to itself. Since $L_h\mathbb{G}_h^1$ and F satisfy these hypotheses, the restriction of the Lang map to Y_h^L must be a surjection onto $\mathbb{U}_{h,L}$. For any elements x and y in $\mathbb{G}_h$, $L(x) = L(y)$ if and only if $y = \gamma x$ for γ an element in $\mathbb{G}_h(\mathbb{F}_q)$. Therefore

$$X_h^L = \bigcup_{\gamma \in \mathbb{G}_h(\mathbb{F}_q)} \gamma \cdot Y_h^L$$

and the disjoint union in the lemma easily follows. □

The torus S is a minimal F-stable Levi of H, and it corresponds to the subgroup T in

G. For the rest of the paper, we will be focused on this strata. We may now state the main result:

Theorem 2.3. *Assume that X_h is constructed from a hyperspecial model of G (see below for more details) and further suppose that the Weyl group element c is U-balanced (see Definition 3.1), and let $\mathbb{F}_q^n$ be the field of definition for $\mathbb{U}_h$ (and therefore X_h^T). Then X_h^T is a maximal scheme over this field in the sense that the action of F^n on the group $H_c^i(Z_h, \overline{\mathbb{Q}}_\ell)$ is diagonalizable and has all eigenvalues equal to $(-1)^i q^{in/2}$.*

Per Lemma 2.2, it suffices to show this for Y_h^T, which we will refer to simply as Y_h. The balanced hypothesis is a bit awkward, but is not a highly restrictive criteria, and the set of acceptable elements c includes all coxeter elements, assuming U is chosen appropriately. When c is non-elliptic, there is no U for which it is U-balanced, see Section 8 for a discussion of what happens in those cases.

In the course of proving this theorem, we will also provide an algorithm for computing the ranks of the cohomology groups $H_c^i(Y_h, \overline{\mathbb{Q}}_\ell)$.

2.1: Quotients of Parahoric Models of G

Here we will detail the construction of $\mathbb{G}_h$ and list some relevant properties. First, we recall the Bruhat and Tits construction of parahoric subgroups, simplifying matters greatly because we only use the construction in the split case.

We may take a finite unramified extension K of k such that T_K, and therefoe G_K, is split. Let $\omega : K \to \mathbb{R}$ be a valuation map with $\omega(\mathcal{O}_K) = \mathbb{Z}^+$. Let Φ be the root system of G_K, and for a root α write U_α for the root subgroup of G_K corresponding to α normalized by T_K.

U_α is noncanonically isomorphic to the additive group $\mathbb{G}_a$ over K. Given an isomorphism $\phi : \mathbb{G}_a \to U_\alpha$. we define the subgroup $U_{\alpha,r}$ of $U_\alpha(K)$ by $U_{\alpha,r} := \phi(\omega^{-1}([r, \infty))$. A *Chevalley system* is a set of isomorphisms $\phi_\alpha : \mathbb{G}_a \to U_\alpha$ with the property that for two roots $\alpha \neq -\beta$

the commutator group $[U_{\alpha,r}, U_{\beta,s}]$ is contained in the group generated by $U_{i\alpha+j\beta,r+s}$ for all positive integers i and j with $i\alpha + j\beta$ a root.

Define

$$T_0 = \{t \in T(K) \mid \omega(\chi(t)) = 0 \text{ for all } \chi \in X^*(T) := \mathrm{Hom}_K(T, \mathbb{G}_m)\}$$

and for a positive real number r

$$T_r = \{t \in T_0 \mid \omega(\chi(t) - 1) \geq 0 \text{ for all } \chi \in X^*(T)\}.$$

The apartment of T, written as $\mathcal{A}(T)$ is the affine space under $X_*(T) \otimes \mathbb{R}$. In [9] Tits describes how the points in $\mathcal{A}(T)$ are in bijection with the set of Chevalley systems. The building $\mathscr{B}(G, K)$ is a quotient of the disjoint union of the apartments of every maximal torus in G, and inherits an action by F. Let x be a vertex in $\mathscr{A}(T) \cap \mathscr{B}(G, K)^F$. This intersection is always nonempty and contains a vertex, and if c is chosen to be elliptic, as it usually is in this paper, x is uniquely determined. Define $G_{x,r}$ to be the group generated by T_r and $U_{\alpha,r}$ and

$$G_{x,r+} := \bigcup_{s>r} G_{x,s}.$$

We may find ϵ, depending on x but not r such that $G_{x,r+} = G_{x,r+\epsilon}$. These are group schemes over $\mathcal{O}_K$, and inherit an action of F, and therefore descend to group schemes over $\mathcal{O}_k$.

For A an algebra over $\mathbb{F}_q$ let $\mathbb{W}(A)$ denote the ring of Witt vectors of A if $\mathrm{char}(k) = 0$ or the power series ring $A[[t]]$ if $\mathrm{char}(k) \neq 0$. Now we can present a definition for $\mathbb{G}_h$.

Definition 2.4. *For A an $\mathbb{F}_q$ algebra,*

$$\mathbb{G}_h(A) := G_{x,0}(\mathbb{W}(A))/G_{x,(h-1)+}(\mathbb{W}(A)).$$

T_0 is a subscheme of $G_{x,0}$ and we define $\mathbb{T}_h$ as its image in $\mathbb{G}_h$. $\mathbb{T}_h$ is a group scheme over $\mathbb{F}_q$. For α a root in Φ, write T_α for the image of the coroot $\alpha^\vee$, $T_{\alpha,0} := T_\alpha \cap T_0$, and

$\mathbb{T}_{\alpha,h}$ the image of T_{α_0} in $\mathbb{G}_h$.

Let Ψ be a subset of Φ for which there exists a hyperplane $P \subset \Phi \otimes \mathbb{R}$ with all elements of Ψ on one side of the plane. Then define U_Ψ to be the group generated by the $U_{\alpha,0}$ for α in Ψ. This is a subgroup of $G_{x,0}$ and define $\mathbb{U}_{\Psi,h}$ as its image in $\mathbb{G}_h$. This new scheme is not necessarily defined over $\mathbb{F}_q$, but is defined over the residue field of K.

The scheme $\mathbb{G}_1$ is a reductive group over $\mathbb{F}_q$, and split over $\overline{\mathbb{F}}_q$, and $\mathbb{T}_1$ is a maximal torus inside it. For $h_1 < h_2$ there is a clear quotient map from $\mathbb{G}_{h_2}$ to $\mathbb{G}_{h_1}$ and we write $\mathbb{G}_{h_2}^{h_1}$ for the kernel of this map.

We will now assume that c is an elliptic element. As mentioned, this implies that there is a unique point x in $\mathscr{A}(T) \cap \mathscr{B}^F$. We then further assume that the point x will be *hyperspecial*, so $G_{x,r+} = G_{x,r+1}$ and $U_{\alpha,r+} = U_{\alpha,r+1}$ for all r. With this assumption, w eprove the following statements:

Proposition 2.5. *For two roots $\alpha \neq -\beta$ the commutator subgroup $[\mathbb{U}^r_{\alpha,h}, \mathbb{U}^s_{\beta,h}]$ is contained in the group generated by the $\mathbb{U}^{r+s}_{i\alpha+j\beta}$ for all positive integers i and j with $i\alpha + j\beta$ a root.*

For x in $\mathbb{U}^r_{\alpha,h} \backslash \mathbb{U}^{r+1}_{\alpha,h}$ the map $y \mapsto [x, y]$ is an isomorphism from $\mathbb{U}^s_{-\alpha,h}/\mathbb{U}^{s+1}_{-\alpha,h}$ to $\mathbb{T}^{r+s}_{\alpha,h}/\mathbb{T}^{r+s+1}_{\alpha,h}$.

Proof. This is [5, Lemma 2.8]. We summarize the argument here. $\mathbb{U}_{\alpha,h}$ is the image of $U_{\alpha,0}$ after quotienting by $G_{x,(h-1)+}$. $\mathbb{U}^r_{\alpha,h}$ is the kernel of the map from $\mathbb{U}_{\alpha,h}$ to $\mathbb{U}_{\alpha,r}$, so it is isomorphic to the image of $U_{\alpha,(r-1)+} \simeq U_{\alpha,r}$ after quotienting by $G_{x,h-1+}$. The first claim then follows from the properties of a Chevalley system. The second claim can be reduced to the case of $G = \mathrm{SL}_2$ by restricting to the subgroup generated by U_α and $U_{-\alpha}$, where it is a straightforward computation. □

Proposition 2.6. *For $1 \leq r < h$, the quotient $\mathbb{G}^r_h/\mathbb{G}^{r+1}_h$ is isomorphic over the residue field of K to $\mathfrak{g}$, the Lie algebra of $\mathbb{G}_1$.*

Proof. The scheme $\mathbb{G}^r_h$ is the kernel of the map from $\mathbb{G}_h$ to $\mathbb{G}_r$, which we can we can see from the definitions is equal to $G_{x,r-1+}/G_{x,(h-1)+}$. Applying this for $\mathbb{B}\mathbb{G}^{r+1}_h$ as well we can see the quotient $\mathbb{G}^r_h/\mathbb{G}^{r+1}_h$ is to be equal to $G_{x,r}/G_{x,r+1}$. Since $r \geq 1$, this group is commutative. It is generated by $U_{\alpha,r}/U_{\alpha,r+1}$ and T_r/T_{r+1}. The first groups are each equal $\mathcal{O}_K/\mathfrak{m}$ or $\mathfrak{m}$ the maximal ideal of $\mathcal{O}_K$, by our choice of normalization of ω. T is split over K, and we may choose an isomorphism $T_K \simeq (\mathrm{GL}_1^n$ which makes $T_r = (1+\mathfrak{m}^r)^n$. Since $(1+m^r)/(1+m^{r+1}) \simeq \mathcal{O}_K/\mathfrak{m}$, we have that $G_{x,r}/G_{x,r+1}$ is a vector space with basis in bijection with a set that is a union of the root system Φ and a spanning set of cocharacters in $X^*(T_K)$. We can identify the latter vector space with the Lie Algebra of $\mathbb{G}_1$ □

Note that because $\mathbb{G}_h^h$ is trivial, $\mathbb{G}_h^{h-1}$ is isomorphic to $\mathfrak{g}$.

Finally, for ψ as above, there are noncanonical isomorphisms from U_ψ to affine space over K, and also from $\mathbb{U}_{\Psi,h}$ to affine space over the residue field of K.

CHAPTER 3

Preliminaries on Root Systems

Let Φ be an irreducible root system and with Weyl Group W, and Δ a base of Φ, with corresponding positive roots Φ^+.

Definition 3.1. *An element c of W is said to be Δ-balanced if*

- *Every orbit of c has the same size*
- *Every orbit of c contains a unique element of $\Phi^+ \cap c\Phi^-$*

Being Δ-balanced depends only on the choice of positive roots, not the base Δ. The root systems we will encounter in this paper come to us as the roots of a maximal torus T in a reductive group, so choosing a set of positive roots is equivalent to choosing a borel B containing T. In a situation where we have chosen a borel containing T with unipotent radical U we will talk about c being Δ-balanced if it is balanced for the set of positive roots determined by U.

In the majority of the rest of this paper we will be working with a unipotent subgroup U and a U-balanced element of the Weyl group c. Let us first observe that there are many such elements.

Let c be a coxeter element of an irreducible Weyl group W, and let n be its order. Then there is a base Δ in Φ consisting of roots α_1 through α_r with associated reflections s_1 through s_r such that

$$s_1 \cdot s_2 \cdots s_r = c \tag{3.1}$$

and this is a minimal expression. We let Φ^+ denote the set of positive roots with respect to Δ. The following is [1, p. VI.I.33]:

Theorem 3.2. *Every orbit of the action of c on Φ has the same size. For $1 \leq i \leq r$ set*

$$\theta_i = s_1 \cdots s_{i+1} \alpha_i. \tag{3.2}$$

The θ_i are all positive, $c^{-1}\theta_i$ is always negative, and for any root β with $\beta > 0$ and $c^{-1}\beta < 0$ $\beta = \theta_i$ for some i. Every orbit of c contains a unique θ_i

Corollary 3.3. *The coxeter element c is Δ-balanced.*

An arbitrary element of W can not balanced if it fixes some root. But noncoxeter elliptic elements may be balanced.

Definition 3.4. *Suppose c is Δ-balanced. Let n be the order of c. Negation acts by involution on the orbits of c and therefore induces an involution on the set $\Phi^+ \cap c\Phi^-$ which we will denote by ι. Then for $\theta \in \Phi^+ \cap c\Phi^-$ the root $-\theta$ is in the c-orbit of the root $\iota(\theta)$ and there is a unique positive integer $\kappa(\theta)$ less than n such that*

$$c^{\kappa(\theta)} \iota(\theta) = -\theta \tag{3.3}$$

Note that these definitions immediately imply $\kappa(\iota(\theta)) = n - \kappa(\theta)$.

3.1: Conjectural Generalization

The above definitions unfortunately depend on the choice of U as well as c. Further, given an elliptic c, it is unclear how to tell if there is a Δ for which c is Δ-balanced, and c may not be Δ-balanced for many Δ.

We suggest a potential improvement with the following definition

Definition 3.5. *An element c of W is said to be Δ-shifting if it is not trivial, and for any integer k the intersection*

$$c^k \left(\Phi^+ \cap c\Phi^-\right) \cap \left(\Phi^+ \cap c\Phi^-\right)$$

is either empty or the entirety of $\Phi^+ \cap c\Phi^-$.

This is clearly a generalization of Δ-balanced. The proof to follow of the maximality of $H^i_c(Y_h, \overline{\mathbb{Q}}_\ell)$ can be modified without much trouble to show the result for any Δ-shifting c. The only major change required is a new definiition for ι and κ, since there is no longer a bijection of $\Phi^+ \cap c\Phi^-$ with the orbits of c. The change was omitted in the paper because it adds complexity for no gain in the strength of the theorem. However, while most c are Δ-balanced for only some Δ, small computational evidence suggests that every elliptic element may be Δ-shifting for every Δ. If this conjecture is true, then Theorem 2.3 could be reworked to remove any dependence on U.

CHAPTER 4

The Geometry of Y_h

In this section we will study the geometry of Y_h. Our main result will be a structural theorem that establishes an isomorphism between the groups $H^i_c(Y_h, \overline{\mathbb{Q}}_\ell)$ and the cohomology of a different sheaf on an affine space.

First, a note about lifting: There are natural projection maps from $\mathbb{G}_h$ to $\mathbb{G}_{h-1}$. There is not, in general, a morphism of schemes from $\mathbb{G}^1_{h-1}$ to $\mathbb{G}^1_h$ that provides a section to this projection, or a similar section of the projection map from Y_h to Y_{h-1}. The work in this chapter can be seen as precisely identifying the obstruction to such a section.

There are, however, lifts from $\mathbb{W}_{h-1}$ to $\mathbb{W}_h$. For a k-algebra A, we parametrize $\mathbb{W}_h(A)$ as a sequence of $h+1$ elements of A, and so we can construct one such lift by appending 0 to the end of our sequences. We emphasize that this is not a ring homomorphism, but does produce a map of schemes.

The groups $\mathbb{U}_h$, $\mathbb{U}_h \cap F\overline{\mathbb{U}}_h$, $\mathbb{U}_h \cap F^{-1}\mathbb{U}_h$, or similar variants are all abstractly isomorphic to $\mathbb{W}^I_h$ for some integer I (depending on which group, of course). So there is a section of the quotient map $\mathbb{U}_h \to \mathbb{U}_{h-1}$, or any of the other quotients. In the rest of this paper, we will make use of such sections, sometimes by constructing specific maps, but often just relying on the existence of some map.

Note that $\mathbb{G}^1_2$ and $\mathbb{G}^{h-1}_h$ are both isomorphic to the Lie algebra $\mathfrak{g}$, and $\mathbb{G}^{h-1}_h$ is central in $\mathbb{G}^1_h$. The group operation of $\mathbb{G}_h$ restricts to vector space addition. When discussing $\mathfrak{g}$, we will use the decomposition $\mathfrak{g} \simeq \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_-$, where $\mathfrak{g}_+$ (respectively $\mathfrak{g}_-$) is the space spanned by the positive (resp. negative) root subgroups corresponding to our choice of U, and $\mathfrak{h}$ is the Cartan subalgebra corresponding to our choice of T.

Theorem 4.1.

$$Y_h \simeq \{x \in \mathbb{G}_h^1 \mid x^{-1}Fx \in \mathbb{U}_h^1 \cap F\overline{\mathbb{U}}_h^1\} \tag{4.1}$$

Proof. First, we show this is true for $h = 2$. We can argue on the level of Lie algebras. There is a Lang map $L(v) = Fv - v$. We need to show

$$L^{-1}(\mathfrak{g}_+)/(\mathfrak{g}_+ \cap F^{-1}\mathfrak{g}_+) \simeq L^{-1}(\mathfrak{g} \cap F\mathfrak{g}_-). \tag{4.2}$$

Define an action of $\mathfrak{g}_+ \cap F^{-1}\mathfrak{g}_+$ on $\mathfrak{g}_+$ given by

$$v \cdot w = -v + w + Fv = w + L(v). \tag{4.3}$$

Since $\mathfrak{g}$ is commutative, L is an endomorphism. It suffices to show that the quotient of $\mathfrak{g}_+$ by the above action is isomorphic to $\mathfrak{g}_+ \cap F\mathfrak{g}_-$. For $\mathfrak{g}_\alpha$ a root space, the image $F\mathfrak{g}_\alpha$ is contained in a root space $\mathfrak{g}_\beta$, and F induces an isomorphism of group schemes from $\mathfrak{g}_\alpha$ to $\mathfrak{g}_\beta$. Assume α and β are both positive. Then, for any x in $\mathfrak{g}_\beta$ there is a unique y in $\mathfrak{g}_\alpha$ with $Fy = -x$. By assumption, y is in $\mathfrak{g}_+ \cap F^{-1}\mathfrak{g}_+$, so x is equivalent to $x - y + Fy = -y$ under the action of $\mathfrak{g}_+ \cap F^{-1}\mathfrak{g}_+$. Hence every element of $\mathfrak{g}_\beta$ is equivalent to some element of $\mathfrak{g}_\alpha \simeq F^{-1}\mathfrak{g}_\beta$. Iterating upon this, we can see

$$\mathfrak{g}_+/(\mathfrak{g}_+ \cap F^{-1}\mathfrak{g}_+) \simeq \bigoplus_{\mathfrak{g}_\alpha \in \mathfrak{g}_+, \mathfrak{g}_\alpha \notin F\mathfrak{g}_+} \mathfrak{g}_\alpha \simeq \mathfrak{g}_+ \cap F\mathfrak{g}_-. \tag{4.4}$$

Now, assume that the theorem holds for Y_{h-1}. Let y be an element in Y_h and $\overline{y}$ be its projection to Y_{h-1}. By inductive assumption there exists an element $\overline{u}$ in $\mathbb{U}_{h-1}^1 \cap F^{-1}\mathbb{U}_{h-1}^1$

such that

$$(\overline{yu})^{-1}F(\overline{yu}) \in \mathbb{U}^1_{h-1} \cap F\overline{\mathbb{U}}^1_{h-1}. \tag{4.5}$$

Then if u is any lift of $\overline{u}$ to $\mathbb{U}^1_h \cap F^{-1}\mathbb{U}^1_h$ we have

$$(yu)^{-1}F(yu) \in \left(\mathbb{U}^1_h \cap F\overline{\mathbb{U}}^1_h\right) \cdot \mathbb{U}^{h-1}_h. \tag{4.6}$$

But our base case argument can be repurposed to show that for any element v in $\mathbb{U}^{h-1}_h$, there is an element w in $\mathbb{U}^{h-1}_h \cap F^{-1}\mathbb{U}^{h-1}_h$ such that $-w + v + Fw$ is contained in $\mathbb{U}^{h-1}_h \cap F\overline{\mathbb{U}}^{h-1}_h$. Then we can find a w in $\mathbb{U}^{h-1}_h \cap F^{-1}\mathbb{U}^{h-1}_h$ so that

$$(yuw)^{-1}F(yuw) \in (\mathbb{U}^1_h \cap F\overline{\mathbb{U}}^1_h). \tag{4.7}$$

This completes the argument.

□

The following definitions simplify our notation to avoid clutter

Definition 4.2. *Define*

$$V_h := \mathbb{U}^1_h \cap F\overline{\mathbb{U}}^1_h \tag{4.8}$$

and

$$\mathfrak{v}_h := \mathbb{U}^1_h \cap F\overline{\mathbb{U}}^1_h \cap \mathbb{G}^{h-1}_h. \tag{4.9}$$

In terms of the Lie algebra, we have $\mathfrak{v}_h \simeq \mathfrak{g}_+ \cap F\mathfrak{g}_-$. We will often drop the subscript on $\mathfrak{v}_h$ when thinking of it as living inside $\mathfrak{g}$.

Definition 4.3. *Let n be the order of c. On any group where F is defined we define the map (not necessarily a homomorphism)*

$$N(g) := F^{n-1}g \cdot F^{n-2}g \cdots g \tag{4.10}$$

This is an endomorphism if the group is abelian.

Since $F = \mathrm{adj}_c \circ \sigma$, $F^n = \sigma^n$ and we have that $\mathbb{U}_h$ is a $\mathbb{F}_{q^n}$ rational subgroup of $\mathbb{G}_h$.

Lemma 4.4. *Define the map ν from $\mathfrak{h}^F \times \mathfrak{v}$ to $\mathfrak{g}$ by*

$$\nu(s, v) := s \cdot N(v). \tag{4.11}$$

ν is an isomorphism from $\mathfrak{h}^F \times \mathfrak{v}$ onto $L^{-1}\mathfrak{v}$

Proof. On any commutative group we have

$$L \circ N(v) = F^n v - v.$$

Since F^n stabilizes $\mathfrak{v}$, and $L(\mathfrak{h}^F) = 0$ the image of ν is contained in $L^{-1}\mathfrak{v}$. Further, by Lang's theorem, the map $v \mapsto F^n v - v$ provides a surjection from $\mathfrak{v}$ onto itself. It follows that

$$L^{-1}\mathfrak{v} = \mathfrak{g}^F + \mathfrak{h}^F + N(\mathfrak{v}) \tag{4.12}$$

since any two vectors with the same image under the Lang map differ only by an element of $\mathfrak{g}^F$.

Since the group operation of $\mathfrak{g}$ is vector space addition, and therefore commutative, we may rewrite this as

$$L^{-1}\mathfrak{v} = \mathfrak{g}^F + N(\mathfrak{h}^{F^n}) + N(\mathfrak{v}). \tag{4.13}$$

Obviously $N(\mathfrak{v})$ contains $N(\mathfrak{v}^{F^n})$. We will show that $\mathfrak{g}^F$ is contained in $N(\mathfrak{h}^{F^n})+N(\mathfrak{v}^{F^n})$ and the surjectivity of ν will follow.

The action of F permutes the root spaces of $\mathfrak{g}$. Write O for the set of orbits of this action. If for every orbit o in O we choose one root space $\mathfrak{g}_o$ in o and set $Z = \bigoplus_O \mathfrak{g}_o$ then $N(\mathfrak{h}^{F^n}+Z^{F^n}) = N(\mathfrak{g}^{F^n}) = \mathfrak{g}^F$. Our assumptions on c guarantee every orbit contains exactly one root space in $\mathfrak{v}$. This also guarantees that the intersection $\mathfrak{v} \cap F^i\mathfrak{v} = 0$ for $1 \leq i < n$. Therefore $\mathrm{Ker}(N) \cap \mathfrak{v} = 0$, and we have shown ν is a bijection. □

Now we prove the main result of this section.

Theorem 4.5. *There is a map φ_h from V_h to $\mathbb{T}_h^1$ such that in the pushout*

$$\begin{array}{ccc} W_h & \xrightarrow{p_1} & V_h \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \varphi_h} \\ \mathbb{T}_h^1 & \xrightarrow{L} & \mathbb{T}_h^1. \end{array}$$

W_h is isomorphic to Y_h

To do this, we will first show that Y_h is *almost* a quotient of $\mathbb{G}_h^1$.

Lemma 4.6. *There is a morphism $\rho_h : \mathbb{G}_h^1 \to Y_h\mathbb{T}_h^1$ for which the natural inclusion $Y_h\mathbb{T}_h \hookrightarrow \mathbb{G}_h^1$ is a section.*

Proof. The construction proceeds inductively. We start with $h = 2$ and identify $\mathbb{G}_2^1$ with $\mathfrak{g}$. Let v be an element of $\mathfrak{g}$ and write $Fv - v = w_+ + w_0 + w_-$ according to the decomposition $\mathfrak{g} \simeq \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_-$. We have the following:

- The Lie algebra argument in Theorem 4.1 provides us with a unique u_1 in $\mathfrak{g}_- \cap F^{-1}\mathfrak{g}_-$ such that $Fu_1 - u_1 + w_-$ is contained in $\mathfrak{g}_- \cap F\mathfrak{g}_+$.

- The Lie algebra argument in Lemma 4.4 provides us with a unique u_2 in $\mathfrak{g}_- \cap F\mathfrak{g}_+$ such that $FN(u_2) - N(u_2) = -(Fu_2 - u_2 + w_-)$. We now have that $L(v + u_1 + N(u_2))$ is contained in $\mathfrak{g}_+ \oplus \mathfrak{h}$.

- Just as in the first bullet point, we can construct a unique u_3 in $\mathfrak{g}_+ \cap F^{-1}\mathfrak{g}_+$ such that $Fu_3 - u_3 + w_+$ is contained in $\mathfrak{v}$.

Then $v + u_1 + N(u_2) + u_3$ is contained in $L^{-1}(\mathfrak{v} \oplus \mathfrak{h})$ as desired. Since our choices were all unique we have a well defined map $\rho_2(v) = v + u_1 + T(u_2) + u_3$. The inclusion of $L^{-1}(\mathfrak{v})$ into $\mathfrak{g}$ is clearly a section of ρ_2.

Suppose the construction works for $h-1$. Let g be an element of $\mathbb{G}_h^1$, and $\overline{g}$ be its projection to $\mathbb{G}_{h-1}^1$. By assumption we have unique elements

- $u_1 \in \overline{\mathbb{U}}_{h-1}^1 \cap F^{-1}\overline{\mathbb{U}}_{h-1}^1$,
- $u_2 \in \overline{\mathbb{U}}_{h-1}^1 \cap F\mathbb{U}_{h-1}^1$,
- $u_3 \in \mathbb{U}_{h-1}^1 \cap F^{-1}\mathbb{U}_{h-1}^1$

such that

$$\overline{g}u_1 N(u_2) u_3 \in Y_{h-1}\mathbb{T}_{h-1}^1. \tag{4.14}$$

For $i = 1, 2$ and 3 let $\widetilde{u}_i$ be a lift of u_i to $\mathbb{G}_h^1$ within the same unipotent subgroup as u_i. Then

$$g\widetilde{u}_1 N(\widetilde{u}_2)\widetilde{u}_3 \in Y_h \mathbb{T}_h^1 \mathbb{G}_h^{h-1} \simeq Y_h \mathbb{T}_h^1 \mathfrak{g}. \tag{4.15}$$

Then our previous analysis in the Lie algebra case shows that we can find unique v_1, v_2 and v_3 in $\mathfrak{g}_- \cap F^{-1}\mathfrak{g}_-$, $\mathfrak{g}_- \cap F\mathfrak{g}_+$, and $\mathfrak{g}_+ \cap F^{-1}\mathfrak{g}_+$ respectively such that right multiplication by $v_1 + N(v_2) + v_3$ corrects the above expression to lie in $Y_h\mathbb{T}_h^1$. Setting $\widetilde{\widetilde{u}}_i = \widetilde{u}_i + v_i$ we have found unique elements of the relevant unipotent subgroups such that right multiplication sends g into $Y_h\mathbb{T}_h^1$. This completes the construction of ρ_h. □

Now we prove the theorem

Proof of Theorem 4.5. We construct φ_h as a series of compositions. The map $N_h := \rho_h \circ N$ sends V_h to $Y_h\mathbb{T}_h^1$. Let x be an element of $Y_h\mathbb{T}_h^1$ and write $x = ys$ with y in Y_h and s in $\mathbb{T}_h^1$, then

$$x^{-1}Fx = (ys)^{-1}F(ys) = s^{-1}y^{-1}FyFs = s^{-1}uFs = (s^{-1}us)s^{-1}Fs. \tag{4.16}$$

So the lang map L sends $Y_h\mathbb{T}_h^1$ to $V_h\mathbb{T}_h^1$. Since $\mathbb{T}_h^1$ normalizes V_h, this is a group, and V_h is a normal subgroup, so there is a quotient map $\mathrm{proj}_T : V_h\mathbb{T}_h^1$ to $\mathbb{T}_h$. Finally, take inv to be the inversion map on $\mathbb{T}_h^1$. We will then define φ_h as the composition.

$$\varphi_h := \mathrm{inv} \circ \mathrm{proj}_T \circ L \circ \rho_h \circ N. \tag{4.17}$$

Then from the definition of W_h as a pushout, we have a map $\widetilde{N}_h$ from W_h to $\mathbb{G}_h^1$ given by

$$\widetilde{N}_h(w) := N_h(p_1(w)) \cdot p_2(w). \tag{4.18}$$

Let us confirm that this map has image contained in Y_h. Take $v = p_1(w)$ and $s = p_2(w)$. We know $L(s) = \varphi_h(v)$. Then $L(N_h(v) \cdot s) = s^{-1}L(N_h(v))F(s)$ is an element in $V_h\mathbb{T}_h^1$. Applying the quotient q we see

$$\mathrm{proj}_T\left(s^{-1}L(N_h(v))F(s)\right) = s^{-1}\varphi_h(v)^{-1}F(s) = \varphi_h(v)^{-1} \cdot \varphi_h(v) = 1. \tag{4.19}$$

Therefore $L(\widetilde{N}_h(w))$ is contained in V_h so $\widetilde{N}_h(w)$ is contained in Y_h. We will show this map is an isomorphism by showing it is injective and surjective. First observe that N_h is an injection. When $h = 2$, φ_2 is trivial and $\widetilde{N}_2$ is the map η from Lemma 4.4, and therefore an isomorphism. If N_{h-1} is injective, then $N_h(v_1) = N_h(v_2)$ implies that $\overline{v}_1 = \overline{v_2}$, and so $v_1 = v_2w$ for $w \in V_h^{h-1}$. But $N_h(v_2w) = N_h(v_2)N(w)$, so we must have $N(w) = 0$ and therefore $w = 0$.

Therfore if $\widetilde{N}_h(x_1) = \widetilde{N}_h(x_2)$ then $p_1(x_1) = p_1(x_2)$. It quickly follows from the formula for $\widetilde{N}_h$ that $p_2(x_1) = p_2(x_2)$, so $x_1 = x_2$.

Now we prove surjectivity. Once again, proceed injectively and note that we have already proven $\widetilde{N}_2 = \nu$ is surjective in Lemma 4.4. Assume $\widetilde{N}_{h-1}$ is also surjective. For y in Y_h we may choose a $\overline{w}$ in W_{h-1} with $\widetilde{N}_{h-1}(\overline{w}) = \overline{y}$. Then let v be a lift of $p_1(\overline{w})$ to V_h. There must be an s in the Lang preimage of $\varphi_h(v)$ and a corresponding element w in W_h with $p_1(w) = v$ and $p_2(w) = s$ such that

$$\widetilde{N}_h(w) = ya$$

for a in $Y_h \mathbb{T}_h^1 \cap \mathbb{G}_h^{h-1}$. We can write a uniquely as $s_0 T(v_0)$ for s_0 in $\mathfrak{h}$ and v_0 in $\mathfrak{v}_h$. The element w' in W_h corresponding to ss_0 and vv_0 then has $\widetilde{T}_h(w') = y$. □

Then, applying the smooth base change theorem to the pushout diagram in the definition of W_h, we have

Corollary 4.7.

$$H_c^i(W_h, \overline{\mathbb{Q}}_\ell) \simeq H_c^i(V_h, \varphi_h^* \circ L_* \overline{\mathbb{Q}}_\ell). \tag{4.20}$$

There is a natural action of $\mathbb{G}_h^1(\mathbb{F}_q)$ on Y_h by left multiplication, which means there is an action on $\mathbb{W}_h$ as well, such that $g \cdot w$ is the unique element of W_h with $\widetilde{N}_h(g \cdot w) = g \cdot \widetilde{N}_h(w)$. The equality $p_1(w_1) = p_1(w_2)$ is true if and only if $\widetilde{N}_h(w_1) = \widetilde{N}_h(w_2)s$ for $s \in \mathbb{T}_h^1$. This means that if $p_1(w_1) = p_1(w_2)$ then $p_1(g \cdot w_1) = p_1(g \cdot w_2)$, so the action of $\mathbb{G}_h^1(\mathbb{F}_q)$ on W_h descends to an action on V_h.

Despite the simple description of V_h, this action is not easy to describe via a formula. However, since the group action must be compatible with projection we can paritally describe it in some cases.

Let g be an element of $\mathbb{G}_h^j(\mathbb{F}_q)$, and let $\overline{g}$ be its projection to $\mathbb{G}_{j+1}^j$. From Lemma 4.4 we have $\overline{g} = N(v) + s$ for $v \in \mathbb{V}_{j+1}^j$ and s in $\mathbb{T}_{j+1}^j$. Since $\mathbb{G}_{j+1}^j$ is central in $\mathbb{G}_{j+1}^1$, the action of $\overline{g}$

on V_{j+1} is $\overline{g} \cdot u = u + v$. Therefore for u an element of V_h, the projection of $g \cdot u$ to V_{j+1} is $\overline{u} + v$.

CHAPTER 5

The Lang Torsor and Rank One Multiplicative Local Systems

We have constructed in Theorem ?? an isomorphism $Y_h \simeq W_h$ and related in Corollary 4.7 the cohomology of W_h with a constant sheaf to the cohomology on a certain affine space of $\varphi_h^* L_* \overline{\mathbb{Q}}_\ell$ we collect here some results about sheaves of the form $f^* L_* \overline{\mathbb{Q}}_\ell$ for various functions f. This section owes a great deal to Boyarchenko's paper *Deligne-Lusztig constructions for unipotent and p-adic groups*, [3]. The results here are generalizations of ones found in Section 6 of that paper.

Definition 5.1. *Let A be an algebraic group, a rank one $\overline{\mathbb{Q}}_\ell$ local system $\mathscr{L}$ is said to be* multiplicative *when there is an isomorphism*

$$\mu^* \mathscr{L} \simeq \mathrm{pr}_1^* \mathscr{L} \otimes \mathrm{pr}_2^* \mathscr{L}$$

where $\mu : A \times A \to A$ is the group operation and the pr_i are the obvious projections.

Now suppose A is defined over $\mathbb{F}_q$, with Frobenius ζ. For m a positive integer define the mth power Lang Map

$$L_m(x) = x^{-1} \zeta^m(x). \tag{5.1}$$

This map realizes A as an $A(\mathbb{F}_{q^m})$ torsor over itself. We call this the mth Lang Torsor, and write it as $\mathscr{L}_m$. For χ a $\overline{\mathbb{Q}}_\ell$-character of $A(\mathbb{F}_{q^m})$ we may apply χ to $\mathscr{L}_m$ to get a rank one local system which we write as $\mathscr{L}_\chi$. It is easy to see that $\mathscr{L}_\chi$ is the constant local system if and only if χ is the trivial character.

We recall some facts about the $\mathscr{L}_\chi$. First, let $A(\mathbb{F}_{q^m})^\vee$ be the group of $\overline{\mathbb{Q}}_\ell$ characters of $A(\mathbb{F}_{q^m})$.

Proposition 5.2.

$$(L_m)_*\overline{\mathbb{Q}}_\ell \simeq \bigoplus_{\chi\in A(\mathbb{F}_{q^m})^\vee} \mathscr{L}_\chi \tag{5.2}$$

Let A and B both be algebraic groups defined over the same field $\mathbb{F}_{q^m}$, χ a character of $B(\mathbb{F}_{q^m})$, $\mathscr{L}_\chi$ the associated local multiplicative system, and $f : A \to B$ a homomorphism also defined over $\mathbb{F}_{q^m}$. Then f induces a group homomorphism from $A(\mathbb{F}_{q^m})$ to $B(\mathbb{F}_{q^m})$ and

$$f^*\mathscr{L}_\chi \simeq \mathscr{L}_{\chi\circ f}. \tag{5.3}$$

When A is commutative there is a group homomorphism $N_{m/1}$ from $A(\mathbb{F}_{q^m})$ to $A(\mathbb{F}_q)$ given by

$$Nm/1(g) = g\cdot\zeta(g)\cdots\zeta^{m-1}(g).$$

Let χ a character of $A(\mathbb{F}_q)$. In [7, p. 1.7.7] Deligne shows that

$$(\mathscr{L}_\chi)_{\mathbb{F}_{q^m}} \simeq \mathscr{L}_{\chi\circ Nm_m}$$

is an isomorphism of sheaves over the base changes $A_{\mathbb{F}_q}$

Now, we state and prove the main result of the section, a generalization of [3, Proposition 2.10].

Lemma 5.3. *Let S_1 be a scheme of finite type over $\mathbb{F}_q$, put $S = S_1 \times \mathbb{G}_a$. Let R be an* *algebraic group over $\mathbb{F}_q$ and $\mathscr{F}$ be a multiplicative rank one local sysem on R. Finally, let f be a morphism $S \to R$ that sends a point $(x, y) \in S_1 \times \mathbb{A}^1$ to*

$$f(x, y) = f_1(x, y) \cdot f_2(x) \tag{5.4}$$

and such that at each point x in S_1 the restriction to the fiber $f_1|_{S_x} : \mathbb{G}_a \to R$ is a homomorphism.

Define S_2 to be the subscheme of points x in S_1 such that $f_1^*\mathscr{F}|_{S_x}$ is trivial. Then

$$H_c^i(S, f^*\mathscr{F}) \simeq H_c^i(S_2 \times \mathbb{G}_a, f_2^*\mathscr{F}). \tag{5.5}$$

Proof. Let pr denote the projection map from S to S_1. Since $\mathscr{F}$ is multiplicative, $f^*\mathscr{F} \simeq f_1^*\mathscr{F} \otimes \mathrm{pr}^* f_2^*\mathscr{F}$. Then by the projection formula weh have the isomorphism

$$\mathrm{Rpr}_! f^*\mathscr{F} \simeq f_2^*\mathscr{F} \otimes \mathrm{Rpr}_! \circ f_1^*\mathscr{F} \tag{5.6}$$

in the bounded derived category of complexes of constructible sheaves on S_1. Let ι be the closed embedding from S_2 into S_1. We want to show

$$\mathrm{Rpr}_! f_1^*\mathscr{F} \simeq \iota_*\overline{\mathbb{Q}}_\ell[2](-1). \tag{5.7}$$

For x in S_2 we have the pullback $f_1^*\mathscr{F}|_{S_x}$ is trivial and so $\mathrm{Rpr}_! \circ f_1^*\mathscr{F}$ is isomorphic to $\overline{\mathbb{Q}}_\ell[2]$ when restricted to S_2.

Now choose a point z away from S_2. Then we have an isomorphism of stalks

$$\left(\mathrm{R}^j \mathrm{pr}_! \circ f_1^*\mathscr{F}\right)_z \simeq H_c^j(\mathrm{pr}^{-1}(z), f_1^*\mathscr{F}) \simeq H_c^j(\mathbb{G}_a, f_1^*\mathscr{F}). \tag{5.8}$$

Since f_1 restricts to a homomorphism on the fiber, $f_1^*\mathscr{F}$ is a nontrivial rank one multiplicative local system on $\mathbb{G}_a$. But, as shown by Boyarchenko in [2, Lemma 9.4], $H_c^j(\mathbb{G}_a, \mathscr{G}) = 0$ for all j and for any nontrivial multiplicative local system $\mathscr{G}$.. This establishes the isomorphism in (5.7). Since $\mathrm{Rpr}_! f^*\mathscr{F}$ computes the cohomology of $f^*\mathscr{F}$, this allows us to conclude our result. □

CHAPTER 6

Computing Cohomology

Combining Corollary 4.7 and Proposition 5.2 we see that we need to understand the cohomology groups

$$H_c^i(V_h, \varphi_h^* L_* \overline{\mathbb{Q}}_\ell) \simeq H_c^i(V_h, \varphi_h^* \oplus \mathscr{L}_\chi) \simeq \oplus H_c^i(V_h, \varphi_h^* \mathscr{L}_\chi) \tag{6.1}$$

with the sums taken of the set of all $\overline{\mathbb{Q}}_\ell$ characters of $\mathbb{T}_h^1(\mathbb{F}_q)$. In this section, we will study the map φ_h. Since V_h is isomorphic to affine space, there are many ways of writing it as a product of one dimensional affine space and a smaller scheme in order to apply Lemma 5.3.

Recall that $\mathbb{G}_h^{h-1}$ is central in $\mathbb{G}_h^1$, so if a is an element of V_h and b an element of V_h^{h-1} we have $N(ab) = N(a)N(b)$. Furthermore, $N(b)$ is in Y_h, so $\varphi_h(ab) = \varphi_h(a)$. Therefore, we may choose a splitting $V_h \simeq V_{h-1} \times V_h^{h-1}$ satisfying the hypotheses of Lemma 5.3, though in this case the lemma is not very interesting, because the restriction of the map to the fibers is trivial, and therefore the result is vacuously true. We will need to understand φ_h explicitly in a larger context.

Let a be an element of $V_h \cap \mathbb{G}_h^i$ and b be an element of $V_h \cap \mathbb{G}_h^j$ for i and j positive integers with $i + j = h - 1$ and $j > \frac{h-1}{2}$. We list several facts about this set up that will be relevant to the proceeding computation:

- The commutator $[a, b]$ is contained in $\mathbb{G}_h^{h-1}$, which is central in $\mathbb{G}_h^1$.
- Furthermore, this commutator only depends on $\overline{a}$, the image of a under projection to

$\mathbb{G}^i_{i+1}$ and $\bar{b}$, the of b image under projection to $\mathbb{G}^j_{j+1}$, so we will write $[a,b] = [\bar{a}, \bar{b}]$.

- Because $j > \frac{h-1}{2}$, $\mathbb{G}^j_h$ is a commutative group. Therefore $L \circ N(b) = b^{-1}F^n(b)$, which is contained in V_h, so $N_h(b) = \rho_h \circ N(b) = N(b)$.

- $N_h(ab) = \rho_h(N(ab)) = N(ab)u$ where u is the element of $\mathbb{G}^1_h$ coming from our construction of ρ_h. Importantly, since $N(ab)$ is contained in $\mathbb{G}^i_h$, the element u will be contained in $\mathbb{G}^{i+1}_h$. This implies u commutes with b, and any other element of $\mathbb{G}^j_h$

Now we may determine $\varphi_h(ab)$. We begin by computing $L(N(ab)u)$, which we will do by grouping all the terms involving b together, and keeping track of commutators we introduce. This will be broken into steps to make it readable.

Expand $L(N(ab)u)$ as $u^{-1}N(ab)^{-1}FN(ab)Fu$. Moving left to right we have

$$\begin{aligned}
N(ab)^{-1} &= F^{n-1}(b^{-1}a^{-1}) \cdot F^{n-2}(b^{-1}a^{-1}) \cdots (b^{-1}a^{-1}) \\
&= \left(F^{n-1}a^{-1} \cdot F^{n-2}a^{-1} \cdots a^{-1}\right) \cdot \left(F^{n-1}b^{-1} \cdot F^{n-2}b^{-1} \cdots b^{-1}\right) \cdot \left(\sum_{0 \le i \le j < n} [-F^j\bar{b}, -F^i\bar{a}]\right) \\
&= N(a)^{-1} \cdot N(b)^{-1} \left(\sum_{0 \le i \le j < n} [F^j\bar{b}, F^i\bar{a}]\right).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
FN(ab) &= F(ab) \cdot F^2(ab) \cdots F^n(ab) \\
&= \left(Fa \cdot F^2a \cdots F^n a \cdot\right) \left(Fb \cdot F^2 b \cdots F^n b\right) \left(\sum_{1 \le k < l \le n} [F^k\bar{b}, F^l\bar{a}]\right) \\
&= FN(a) \cdot FN(b) \left(\sum_{1 \le k < l \le n} [F^k\bar{b}, F^l\bar{a}]\right).
\end{aligned}$$

Finally,

$$\begin{aligned}
N(b)^{-1} \cdot FN(a) &= (F^{n-1}b^{-1} \cdot F^{n-2}b^{-1} \cdots b^{-1})(Fa \cdot F^2 a \cdots F^n a) \\
&= (Fa \cdot F^2 a \cdots F^n a)(F^{n-1}b^{-1} \cdot F^{n-2}b^{-1} \cdots b^{-1}) \left(\sum_{m=0}^{n-1} \sum_{r=1}^{n} [-F^m \overline{b}, F^r \overline{a}] \right) \\
&= FN(a) \cdot N(b)^{-1} \left(\sum_{m=0}^{n-1} \sum_{r=1}^{n} [-F^m \overline{b}, F^r \overline{a}] \right)
\end{aligned}$$

Putting this all together, we have

$$\begin{aligned}
L(N(ab)u) &= u^{-1} \cdot N(ab)^{-1} \cdot FN(ab) \cdot Fu \\
&= u^{-1} \cdot LN(a) \cdot Fu \cdot LN(b) \\
&\quad \cdot \left(\sum_{0 \le i \le j < n} [F^j \overline{b}, F^i \overline{a}] + \sum_{1 \le k < l \le n} [F^k \overline{b}, F^l \overline{a}] + \sum_{m=0}^{n-1} \sum_{r=1}^{n} [-F^m \overline{b}, F^r \overline{a}] \right) \\
&= u^{-1} \cdot LN(a) \cdot FuLN(b) \left(\sum_{i=0}^{n-1} [F^i \overline{b}, \overline{a}] - \sum_{j=1}^{n} [\overline{b}, F^j \overline{a}] \right)
\end{aligned}$$

From the uniqueness of u we may write $u = u_1 u_2$ with u_2 in $\mathbb{G}_h^{h-1}$ so that $\rho_h \circ N(a) = N(a)u_1$ and

$$-u_2 + \left(\sum_{i=0}^{n-1} [F^i \overline{b}, \overline{a}] - \sum_{j=1}^{n} [\overline{b}, F^j \overline{a}] \right) + Fu_2 \in \mathfrak{v}_h \oplus \mathfrak{h}.$$

Recall $\varphi_h(x) = \operatorname{inv} \circ q \circ L \circ N_h(x)$ where proj_T is the projection map from $V_h \cdot \mathbb{T}_h^1$ to $\mathbb{T}_h^1$, and inv is inversion. Applying $\operatorname{inv} \circ \operatorname{proj}_T$ to our expression fro $L \circ N_h(ab)$ we see

$$\varphi_h(ab) = \varphi_h(a) \cdot \operatorname{proj}_T \left(-\sum_{i=0}^{n-1} [F^i \overline{b}, \overline{a}] + \sum_{j=1}^{n} [\overline{b}, F^j \overline{a}] \right). \tag{6.2}$$

We will abbreviate the rightmost term as

$$\overline{\varphi}_h := q\left(\sum_{i=0}^{n-1}[\overline{a}, F^i\overline{b}, \overline{a}] - \sum_{j=1}^{n}[F^j\overline{a}, \overline{b}]\right).$$

Identifying $\mathbb{G}_{i+1}^{i}$ and $\mathbb{G}_{j+1}^{j}$ with $\mathfrak{g}$, we can write

$$\overline{a} = \sum_{\theta\in\Phi^+\cap c\Phi^-} \overline{a}_\theta \quad \text{and} \quad \overline{b} = \sum_{\psi\in\Phi^+\cap c\Phi^-} \overline{b}_\psi$$

for $\overline{a}_\theta$ and $\overline{b}_\psi$ elements of the roots space of θ or ψ. We can expand a commutator $[F^i\overline{b}, F^j\overline{a}]$ as a sum of commutators of the form $[F^i\overline{b}_\psi, F^j\overline{a}_\theta]$. the element $F^i\overline{b}_\psi$ is contained in the root space of $c^k\psi$. Per Proposition 2.5, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta]$ is nonzero and projects nontrivially onto $\mathfrak{h}$ only when $\alpha = -\beta$, and in this case it is contained in $\mathfrak{h}$. Recall the functions ι and κ from Definition 3.4, satisfying $c^{\kappa(\theta)}\iota(\theta) = -\theta$ for θ in $\Phi^+ \cap c\Phi^-$. Since

$$\mathfrak{v} \simeq \bigoplus_{\theta\in\Phi^+\cap c\Phi^-} \mathfrak{g}_\theta,$$

we will write a_θ for the projection of $\overline{a}$ onto the root space $\mathfrak{g}_\theta$ and similarly for b. Then we have

$$\text{(6.3)}\quad \overline{\varphi}_h(a,b) = q\left(\sum_{i=0}^{n-1}[\overline{a}, F^i\overline{b}, \overline{a}] - \sum_{j=1}^{n}[F^j\overline{a}, \overline{b}]\right) = \sum_{\theta\in\Phi^+\cap c\Phi^-} [a_{\iota(\theta)}, F^{n-\kappa(\theta)}b_\theta] - [F^{\kappa(\theta)}a_{\iota(\theta)}, b_\theta].$$

Taking a in $\mathbb{G}_h^1$ and $b \in \mathbb{G}_h^{h-2}$, it is clear that for a in V_h, the function $\overline{\varphi}_h(a, -)$ is a *homomorphism* from V_h^{h-2} to $\mathbb{T}_h^{h-1}$. Therefore the hypotheses of lemma 5.3 hold. Then for a character χ of $\mathbb{T}_h^1(\mathbb{F}_q)$, we need to determine the subvariety of points a in V_{h-2} where $\overline{\varphi}_v(a, -)^*\mathscr{L}_\chi$ is trivial.

Let's consider an extreme case first. χ restricts to a character of $\mathbb{T}_h^{h-1}(\mathbb{F}_q)$. If the restriction of χ to $\mathbb{T}_h^{h-1}(\mathbb{F}_q)$ is a trivial character, then there is a unique character χ_1 of $\mathbb{T}_{h-1}^1(\mathbb{F}_q)$ such that the following commutes:

$$\begin{array}{ccccc} V_h & \xrightarrow{\varphi_h} & \mathbb{T}_h^1 & & \\ \downarrow & & \downarrow & \searrow^{\chi} & \\ V_{h-1} & \xrightarrow{\varphi_{h-1}} & \mathbb{T}_{h-1}^1 & \xrightarrow{\chi_1} & \overline{\mathbb{Q}}_\ell \end{array}$$

In this case we then have

$$H_c^i(V_h, \varphi_h^* \mathscr{L}_\chi) \simeq H_c^{i-2\dim(\mathfrak{v})}(V_{h-1}, \varphi_{h-1}^* \mathscr{L}_{\chi_1}). \tag{6.4}$$

If χ_1 is trivial when restricted to $\mathbb{T}_{h-1}^{h-2}$ we repeat this reduction step. Either we find some level where the character is no longer trivial, in which case the analysis proceeds, or we discover that χ was the trivial character and

$$H_c^i(V_h, \varphi_h^* \overline{\mathbb{Q}}_\ell) \simeq H_c^i(\mathbb{A}^{(h-1)\dim(\mathfrak{v})}, \overline{\mathbb{Q}}_\ell) \simeq \begin{cases} 0 & i \neq 2(h-1)\dim(\mathfrak{v}) \\ \overline{\mathbb{Q}}_\ell & i = 2(h-1)\dim(\mathfrak{v}) \end{cases}. \tag{6.5}$$

Now we suppose the restriction of χ to $\mathbb{T}_h^{h-1}$ is not the trivial character. The scheme V_{h-2} has a $\mathbb{F}_{q^n}$-rational structure since it is stabilized by F^n. Let a be a point in $V_{h-2}(\mathbb{F}_{q^{nm}})$. We apply the results of section 5 to see

$$\overline{\varphi}_v(a,-)^* \mathscr{L}_\chi \simeq \mathscr{L}_{\chi \circ \mathrm{N}_{nm/1} \circ \overline{\varphi}_v(a,-)} \tag{6.6}$$

The norm $N_{nm/1}$ is a map from $\mathbb{T}_h^1(\mathbb{F}_{q^{nm}})$ to $\mathbb{T}_h^1(\mathbb{F}_q)$. We can factor this map as $N_{nm/1} = N_{n/1} \circ N_{nm/n}$, with $N_{n/1}$ being the same map N from Definition 4.3. Then we have

$$\overline{\varphi}_v(a,-)^* \mathscr{L}_\chi \simeq= \mathscr{L}_{\chi \circ N \circ \mathrm{N}_{nm/n} \circ \overline{\varphi}_v(a,-)}. \tag{6.7}$$

The restriction of the Lie bracket to $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}$ is a nondegenerate bilinear form whose image, $\mathfrak{t}_\alpha$, is also the image of the coroot $\alpha^\vee$ in $\mathfrak{h}$.

The restriction of $\overline{\varphi}_h(a,-)$ to the subspace $\mathfrak{g}_\theta$ spanned by b_θ depends only on the value of $a_{\iota(\theta)}$. We will abuse notation and refer to this restriction as $\overline{\varphi}(a_{\iota(\theta)},-)$, a map that sends $\mathfrak{g}_\theta$ to either $\mathfrak{t}_\theta$ or $\mathfrak{t}_\theta \oplus \mathfrak{t}_{\iota(\theta)}$ depending on whether or not $\iota(\theta)=\theta$.

Lemma 6.1. *If $a_{\iota(\theta)}$ is in $\mathbb{F}_{q^n}$, then the map $N\circ\overline{\varphi}(a_{\iota(\theta)},-)$ from $\mathfrak{g}_\theta(\mathbb{F}_{q^n})$ to $\mathfrak{h}(\mathbb{F}_q)$ is trivial. If $a_{\iota(\theta)}$ is contained in some extension $\mathbb{F}_{q^{mn}}$ but not $\mathbb{F}_{q^n}$, the map $\mathrm{N}_{mn/1}\circ\overline{\varphi}(x_{\iota(i)},-)$ is a surjection from $\mathfrak{g}_\theta(\mathbb{F}_{q^{mn}})$ onto $N_{mn/1}(\theta^\vee(\mathbb{F}_{q^{mn}}))$.*

Proof. Abbreviate $a_{\iota(\theta)}$ as a_0, and $\kappa(\theta)$ as κ. Let $\mathbb{F}_{q^{mn}}$ be an extension of $\mathbb{F}_{q^n}$ Say a_0 is in $\mathfrak{g}_{\iota(\theta)}(\mathbb{F}_{q^{mn}})$ and b_0 is a point in $\mathfrak{g}_\theta(\mathbb{F}_{q^{mn}})$ then

$$\begin{aligned}
N_{mn/1}\circ\overline{\varphi}_h(a_0,b_0) &= N_{mn/1}\left([a_0,F^{n-\kappa}b_0]-[F^\kappa a_0,b_0]\right)\\
&= N_{mn/1}\left([a_0,F^{n-\kappa}b_0]\right)-N_{mn/1}\left([F^\kappa a_0,b_0]\right)\\
&= N_{mn/1}\left(F^{n-\kappa}[F^{\kappa-n}a_0,b_0]\right)-N_{mn/1}\left([F^\kappa a_0,b_0]\right)\\
&= N_{mn/1}\left([b_0,F^{\kappa-n}a_0]\right)-N_{mn/1}\left([F^\kappa a_0,b_0]\right)\\
&= N_{mn/1}\left([F^{\kappa-n}a_0-F^\kappa a_0,b_0]\right).
\end{aligned}$$

Then the above expression as a function of b_0 is 0 if $F^n a_0 = a_0$ and a surjection onto $N(\theta^\vee(\mathbb{F}_{q^{mn}}))$ if $F^n a_0 \neq a_0$. $\square$

Now take the opposite extreme from earlier and suppose the restriction of χ to $\mathbb{T}_h^1$ is nontrivial on $N\theta^\vee(\mathbb{F}_q^n)$ for every root θ. Then, applying Lemma 5.3 we see

$$H_c^i(V_h,\varphi_h^*\mathscr{L}_\chi)\simeq H_c^i(V_h(\mathbb{F}_{q^n})\cdot V_h^2,\varphi_h^*\mathscr{L}_\chi). \tag{6.8}$$

From our discussion about how $\mathbb{G}_h^1(\mathbb{F}_q)$ acts on V_h, we see that

$$V_h(\mathbb{F}_{q^n}) \cdot V_h^2 \simeq \bigcup_{g \in \mathbb{G}_h^1(\mathbb{F}_q)} g \cdot V_h^2 \tag{6.9}$$

Since this action doesn't affect φ_h, and has exactly $q^n(\dim(\mathfrak{v}))$ orbits, we conclude

$$H_c^i(V_h(\mathbb{F}_{q^n}) \cdot V_h^2, \varphi_h^* \mathscr{L}_\chi) \simeq H_c^i(V_h^2, \varphi_h^* \mathscr{L}_\chi)^{\oplus q^n(\dim(\mathfrak{v}))}. \tag{6.10}$$

Generalizing, for $j < \frac{h-1}{2}$ take a in V_h^j and b in V_h^{h-1-j}. Then $\varphi_h(a,b) = \varphi_h(a)\overline{\varphi}_h(a,b)$, for the exact same map $\overline{\varphi}_h$ from (in essence) $V_{j+1}^j \times V_{h-j}^{h-1-j}$ to $\mathbb{T}_h^{h-1}$. Then, since χ has not changed and remains trivial, we apply Lemma 5.3 to see

$$H_c^i(V_h^j, \varphi_h^* \mathscr{L}_\chi) \simeq H_c^i(V_h^j(\mathbb{F}_{q^n}) \cdot V_h^{j+1}, \varphi_h^* \mathscr{L}_\chi), \tag{6.11}$$

and use the action of $\mathbb{G}_h^j(\mathbb{F}_q)$ to conclude

$$H_c^i(V_h^j(\mathbb{F}_{q^n}) \cdot V_h^{j+1}, \varphi_h^* \mathscr{L}_\chi) \simeq H_c^i(V_h^{j+1}, \varphi_h^* \mathscr{L}_\chi)^{\oplus q^n(\dim(\mathfrak{v}))}. \tag{6.12}$$

We must analyze two situations: h even and h odd. If h is even, we reduce to computing the groups $H_c^i(V_h^{h/2}, \varphi_h^* \mathscr{L}_\chi)$. But $\mathbb{G}_h^{h/2}$ is commutative, so $N(V_h^{h/2})$ is contained in Y_h and φ_h is the constant map to the identity. Therefore

$$H_c^i(V_h^{h/2}, \varphi_h^* \mathscr{L}_\chi) \simeq H_c^i(V_h^{h/2}, \overline{\mathbb{Q}}_\ell) \simeq H_c^i(\mathbb{A}^{(h/2)\dim(\mathfrak{v})}, \overline{\mathbb{Q}}_\ell) \simeq \begin{cases} 0 & i \neq h\dim(\mathfrak{v}) \\ \overline{\mathbb{Q}}_\ell & i = h\dim(\mathfrak{v}) \end{cases}. \tag{6.13}$$

If h is odd, we are reduced to computing $H_c^i(V_h^{(h-1)/2}, \varphi_h^* \mathscr{L}_\chi)$. We may write $V_h^{(h-1)/2} \simeq V_{(h+1)/2}^{(h-1)/2} \times V_h^{(h+1)/2}$ in such a way that φ_h factors through projection onto the first term.

φ_h restricted to $V_{(h+1)/2}^{(h-1)/2}$ takes the form $\varphi_h(a) = \overline{\varphi}_h(a,a)$. We relate the cohomology group $H^i_c(V_{(h-1)/2}^{(h+1)/2}, \varphi_h^*\mathscr{L}_\chi)$ to character sums and compute it in the next section.

Now we handle the last case, when the restriction of χ to $\mathbb{T}_h^1$ is not trivial, but vanishes on $N(\theta^\vee(\mathbb{F}_{q^n}))$ for some θ. Let Λ be the set of roots α in Φ such that the restriction of χ to $\mathbb{T}_h^{h-1}$ vanishes on $N(\mathfrak{t}_\alpha)$. If α is in Λ, so are $c\alpha$ and $-\alpha$. If α and β are in Λ so is $\alpha+\beta$, if it is a root, and therefore so is $r_\alpha\beta$, the reflection of β over the hyperplane orthogonal to α.. Therefore Λ is a c-stabilized subroot system of Φ. The roots α are exactly the ones whose root spaces centralize the torus in G with cartan algebra equal to the kernel of χ on $\mathbb{T}_h^{h-1}$, so the roots generate to an F-stable subgroup of G. Denote the subgroup by L^χ and its Lie subalgebra of $\mathfrak{g}$ by $\mathfrak{l}$. In most cases, but not always, L^χ is a Levi subgroup of $G_{\breve{k}}$.

We further have a sequence of $\mathbb{G}_h$ subschemes $\mathbb{L}_h^\chi$. Proceeding in a similar manner to our discussion of when L^χ is trivial, we see that

$$H^i_c(V_h, \varphi_h^*\mathscr{L}_\chi) \simeq H^i_c(V_h(\mathbb{F}_{q^n}) \cdot (V_h \cap \mathbb{L}_h^\chi) \cdot V_h^2, \varphi_h^*\mathscr{L}_\chi). \tag{6.14}$$

Now we can generalize this to all levels.

Definition 6.2. *Set* $V_h^\chi := V_h \cap \mathbb{L}_h^\chi$ *and define*

$$V_h^{\chi,j-} := V_h^{j-1}(\mathbb{F}_q^n) \cdot V_h^j \cdot V_h^\chi \quad \text{and} \quad V_h^{\chi,j} := V_h^j \cdot V_h^\chi$$

We have

$$V_h^{\chi,j-} = \bigcup_{g \in \mathbb{G}_h^{j-2}} g \cdot V_{h^{\chi,j}} \tag{6.15}$$

which, just like before, yields

$$H^i_c(V_h^{\chi,j-}, \varphi_h^*\mathscr{L}_\chi) \simeq H^i_c(V_h^{\chi,j}, \varphi_h^*\mathscr{L}_\chi)^M. \tag{6.16}$$

In this case $M = q^n\,(\dim \mathfrak{v}) - \dim(\mathfrak{l})$).

Take $j < \frac{h-1}{2}$, a in $V_h^{\chi,j}$, a root θ in $\Phi^+ \cap c\Phi^-$ but not in L^χ and $b \in V_h^{h-1-j}$ contained in the root space of θ. If we write $\varphi_h(ab) = \varphi_h(a)\overline{\varphi}_h(a,b)$ the second factor depends only on the projection $\overline{a}$ of a to V_{j+1} and the projection $\overline{b}$ of b to V_{h-j}^{h-1-j}. We may write $\overline{a}$ uniquely as $\overline{a}_L \cdot \overline{a}_0$, an element $\overline{a}_L$ of $\mathbb{L}^1_{j+1} \cap V_{j+1}$ multiplied by an element $\overline{a}_0$ of $\mathfrak{v}/\mathfrak{l}$ in $\mathbb{G}^j_{j+1}$.

For α and β roots, recall that the commutator subgroup $[\mathbb{U}^r_{h,\alpha}, \mathbb{U}^s_{h,\beta}]$ is contained in the group generated by $\mathbb{U}^{r+s}_{h,i\alpha+j\beta}$ for all positive integers i and j with $i\alpha+j\beta$ a root. For any root β write $\mathbb{U}^r_{h,L+\beta}$ for the subgroup generated by all $\mathbb{U}^r_{h,i\alpha+j\beta}$ with α a root in L^χ and $i\alpha + j\beta$ a root.

By rearranging terms at the cost of adding in commutators, we relate $L \circ N(ab)$ to $L \circ N(a) \cdot L \circ N(b)$. Observe that

$$F^i b F^j a = F^j a F^i b U[\overline{b}, \overline{a}_0]$$

for $[\overline{b}, \overline{a}_0]$ an element in $\mathbb{G}_h^{h-1}$, as before, and U an element in $\mathbb{U}^{j+1}_{h,L+c^i\theta}$. Further, we have that $F^k b$ commutes with U for any k and $F^i a U_1 = U_2 F^i a$ for U_1 and U_2 elements of $\mathbb{U}^{j+1}_{h,L+c^k\theta}$ for any integer k. Therefore there are a collection of elements U_i in $\mathbb{U}^{j+1}_{h,L+c^i\theta}$ such that

$$L \circ N(ab) = L \circ N(a) \cdot L \circ N(b) \cdot \left(\prod U_i\right) \left(\sum_{i=0}^{n-1} [F^i\overline{b}, \overline{a}_0] - \sum_{j=1}^{n} [\overline{b}, F^j \overline{a}_0]\right) \tag{6.17}$$

Since θ is not a root in L^χ, the projection of each U_i to $V_h \mathbb{T}^1_h$ is contained in V_h, and therefore, for the exact same formula for $\overline{\varphi}_h$ we have

$$\varphi_h(ab) = \varphi_h(a)\overline{\varphi}_h(\overline{a}_0, \overline{b}).$$

Therefore

$$H_c^i(V_h^{\chi,j}, \varphi_h^* \mathscr{L}_\chi) \simeq H_c^i(V_h^{\chi,(j+1)-}, \varphi_h^* \mathscr{L}_\chi). \tag{6.18}$$

We may repeat this process until all that remains is to compute the cohomology of $\mathscr{L}_\chi$ on $V_h^\chi \cdot V_h^{h/2}$ or $V_h^\chi \cdot V_h^{(h-1)/2}$, depending on the parity of h. Either space may be written as a product of $\mathbb{L}_h^1$ and an affine space $\mathbb{A}^M$ corresponding to the root subgroups $\mathbb{U}_{h,\theta}^{h/2}$ or $\mathbb{U}_{h,\theta}^{(h-1)/2}$ for θ a root in $\Phi^+ \cap c\Phi^-$ that is not in L^χ.

The map φ_h now factors as a product of maps, one from $\mathbb{L}_h^1$ to $\mathbb{T}_h^1$ and the other from the affine space. If h is even, the map on the affine space factor is trivial, and we have

$$H_c^i(V_h^\chi \cdot V_h^{h/2}, \varphi_h^* \mathscr{L}_\chi) \simeq H_c^{i-2M}(V_h^\chi, \varphi_h^* \mathscr{L}_\chi).$$

If h is odd, the map on the affine factor is not trivial, but in the next section, we will show that $H_c^*(\mathbb{A}^M, \varphi_h^* \mathscr{L}_\chi)$ is supported in only middle dimension, so we have

$$H_c^i(V_h^\chi \cdot V_h^{(h-1)/2}, \varphi_h^* \mathscr{L}_\chi) \simeq H_c^{i-M}(V_h^\chi, \varphi_h^* \mathscr{L}_\chi).$$

In either case, it suffices to compute the cohomology groups of $\mathbb{L}_h^1$. Since $\chi \circ N$ is trivial on the intersection of $\mathbb{L}_h$ with $\mathbb{T}_h^{h-1}$ there is a unique character χ' on $\mathbb{T}_{h-1}^1$ such that we have

$$\begin{array}{ccccc} V_h^\chi & \xrightarrow{\varphi_h} & \mathbb{T}_h^1 & & \\ \downarrow & & \downarrow & \overset{\chi}{\searrow} & \\ V_{h-1}^\chi & \xrightarrow{\varphi_{h-1}} & \mathbb{T}_{h-1}^1 & \xrightarrow{\chi'} & \overline{\mathbb{Q}}_\ell \end{array}$$

and can relate $H_c^i(V_h^\chi, \varphi_h^* \mathscr{L}_\chi) \simeq H_c^{i-2\dim \mathfrak{l}}(V_{h-1}^\chi, \varphi_{h-1}^* \mathscr{L}_{\chi'})$. If χ' is trivial at the deepest level, we can reduce to $h-2$. If χ is nontrivial on part of the torus, we can find $L^{\chi'} \subset L^\chi$ and repeat the process. All that remains is to compute the cohomology of the various $\varphi_h^* \mathscr{L}_\chi$ on the affine spaces we have constructed, which we will do in the next section.

CHAPTER 7

Final Computations

We have now reduced our computations to the following problem:

Let θ be a root in $\Phi^+ \cap c\Phi^-$, $\mathfrak{g}_\theta \subset \mathfrak{v}$ the associated root space in $\mathfrak{g}$ rand $\mathfrak{h}_\theta$ the image of the coroot $\theta^\vee$ in $\mathfrak{h}$. Let χ be a character on $\mathfrak{h}(\mathbb{F}_q)$ such that $\chi \circ N$ is not trivial on $\mathfrak{h}_\theta$. Set $R := \mathfrak{g}_\theta \times \mathfrak{g}_{\iota(\theta)}$ if $\iota(\theta) \neq \theta$ and $R =: \mathfrak{g}_\theta$ if not. We have a map

$$\varphi : R \to \mathfrak{h}$$

We need to compute the cohomology groups $H^i_c(R, \varphi^* \mathscr{L}_\chi)$

This can be done explicitly, and in this section we will show that the resulting cohomology groups are supported in only one dimension, and have maximal eigenvalues.

Since we are interested in the pullback character $\chi \circ N \circ \varphi$, we may modify φ so long as the composition is unchanged. As originally presented, φ sends

$$(a, b) \to [F^{n-\kappa}a, b] - [a, F^\kappa b]$$

which is contained in $\mathfrak{h}_\theta + \mathfrak{h}_{\iota(\theta)}$. But this has the same image under N as the function

$$(a, b) \to F^{\kappa-n}[F^{n-\kappa}a, b] - [a, F^\kappa b] = [a, F^{\kappa-n}b - F^\kappa b]$$

which has image contained in $\mathfrak{h}_\theta$. Furthermore we may choose isomorphisms from $\mathfrak{g}_\theta$, $\mathfrak{g}_{\iota(\theta)}$

and $\theta^\vee$ to $\mathbb{A}^1$ such that this map simplifies to

$$(x, y) \mapsto x(y^{q^{\kappa-n}} - y^{q^\kappa}).$$

First, consider the case where $\theta \neq \iota(\theta)$.

Write ψ for $\chi \circ N$, a nontrivial character of $\mathbb{A}^1(\mathbb{F}_{q^n})$. The trace formula [7][1.9] tells us for X a separated scheme of finite type over $\mathbb{F}_{q^{mn}}$ and f a morphism $f : X \to \mathbb{A}^1$

$$\sum (-1)^i \operatorname{Tr}\left(F^{mn} | H^i_c(X, f^*\mathscr{L}_\psi)\right) = \sum_{x \in X(\mathbb{F}_{q^{mn}})} \psi \circ \mathrm{Nm}_{nm/n} \circ f(x)$$

Then

$$\begin{aligned}
\sum_{i=0}^{4} (-1)^i \operatorname{Tr}\left((F^{mn} | H^i_c(\mathbb{A}^2, \varphi^*\mathscr{L}_\psi)\right) &= \sum_{y \in \mathbb{F}_{q^{nm}}} \sum x \in \mathbb{F}_{q^{nm}} \psi\left(\mathrm{Nm}_{mn/n}(x(y^{q^{\kappa-n}} - y^{q^\kappa}))\right) \\
&= \sum_{y \in \mathbb{F}_{q^n}} \sum xy \in \mathbb{F}_{q^{nm}} \psi(0) \\
&\quad + \sum_{y \in \mathbb{F}_{q^{nm}} \setminus \mathbb{F}_{q^n}} \sum x \in \mathbb{F}_{q^{mn}} \psi\left(\mathrm{Nm}_{mn/n}(x(y^{q^{\kappa-n}} - y^{q^\kappa})0\right) \\
&= q^n q^{nm} + \sum_{y \in \mathbb{F}_{q^{nm}} \setminus \mathbb{F}_{q^n}} \sum z \in \mathbb{F}_{q^{mn}} \psi(\mathrm{Nm}_{mn/n}(z)) \\
&= q^n q^{nm}
\end{aligned}$$

Deligne's theorem on the Riemann Hypothesis [**DL80**][3.3.1] constrains the eigenvalues of F^{nm} acting on the cohomology groups and allows us to conclude $\dim\left(H^i_c(\mathbb{A}^2, \varphi^*\mathscr{L}_{\widetilde{\chi}})\right) = 0$ unless $i = 2$, in which case the dimension is q^n and the action of F^n has all eigenvalues equal to q^n.

Now consider the case $\theta = \iota(\theta)$. This immediately implies n is even and $2\kappa = n$. The space $\mathfrak{h}_\theta$ is not F-stable, but it is stabilized by F^κ, which per our isomorphism sends x to $-x^{q^\kappa}$. Since $N(x - F^\kappa x)$ must equal zero, we must have $\psi(x + x^{q^\kappa}) = 0$, so for ψ to be nontrivial it must not have conductor dividing κ. In this case we do not need to adjust φ

for it to land in $\mathfrak{h}_\theta$ and one choice for the formula sends[1]

$$\varphi(x) = 2xx^{q^k}.$$

In [4, Proposition 6.2] Boyarchenko and Weinstein show

Proposition 7.1. *Let $f : \mathbb{A}^1 \to \mathbb{A}^1$ be the function $x \mapsto x^{q^\kappa+1}$ and ψ a nontrivial character of $\mathbb{F}_{q^{2\kappa}}$ that is trivial on $\mathbb{F}_{q^\kappa}$. Then*

$$\dim H_c^j(\mathbb{A}^1, f^*\mathscr{L}_\psi)) = \begin{cases} 0 & j \neq 1 \\ q^\kappa & j = 1 \end{cases}$$

And q^n Frobenius acts on the first cohomology group by $-q^\kappa$.

This exactly describes our situation, so we have computed the cohomology groups. Note that in all cases the cohomology groups we found were maximal, in the sense that the eigenvalues of F^n on the ith cohomology group were equal to $(-1)^i q^{in/2}$. We have related every cohomology group in $H_c^i(Y_h, \overline{\mathbb{Q}}_\ell)$ to these groups, or the trivial group $H_c^0(*, \overline{\mathbb{Q}}_\ell)$ by a combination of summing, shifting degree, and tensoring, so we conclude that Y_h is a maximal variety.

[1] Here is the only place where we need the assumption that the characteristic of the residue field is odd

CHAPTER 8

Nonelliptic Elements

The paper has so far been concerned with the closed strata Y_h for c ellptic. Here, we return to the context of X_h, not any strata, and general c, and we show that the non-elliptic case can be related back to the elliptic one on smaller root systems.

As in section 2, let H be a connected split reductive group, S a split maximal torus in H and c an element of the Weyl group of S. Define $F = \mathrm{adj}_c \circ \sigma$, the group G and maximal torus T just as before. Let L be the Levi subgroup of $G_{\breve{k}}$ that centralizes $(T^c)^\circ$. L is preserved by F therefore there is a subgroup L_0 subgroup of G defined over k such that $(L_0)_{\breve{k}} \simeq L$. We write $\mathbb{L}_h$ for the associated subschemes of $\mathbb{G}_h$.

Theorem 8.1. *We may choose a borel B containing T with unipotent radical U such that*

$$X_h = \{x \in \mathbb{G}_h | x^{-1}Fx \in \mathbb{L}_h \cap \mathbb{U}_h\} / \left((\mathbb{L}_h \cap \mathbb{U}_h) \cap F^{-1}(\mathbb{L}_h \cap \mathbb{U}_h) \right). \tag{8.1}$$

Note that when c is elliptic this theorem is vacuously true becuase $(T^c)^\circ$ is in the center of G so L is the entirety of G.

Proof. To prove this we will find a U with the property that for any $u \in U$ there is a v in $U \cap F^{-1}(U)$ such that $v^{-1}uFv \in L \cap U$.

For our theorem we will need the following lemma

Lemma 8.2. *There is a unipotent U which can be decomposed as a product $(L \cap U) \cdot J$ where $J = U \cap FU \cap F^2U \cap \cdots$ is the largest subgroup of U stabilized by F.*

Granting the lemma, let us prove the theorem. Given u in U we want to show

$$u = v^{-1} m F(v) \tag{8.2}$$

for some $v \in U \cap F^{-1}U$ and m in L. By the lemma, we may write $u = m_0 j_0$ for some m_0 in $L \cap U$ and j_0 in J. Take $m = m_0$. Define the map $F_m(g) = F(mgm^{-1})$. J is a normal subgroup of U so F_m is an isogeny and by Lang's theorem $j \mapsto j^{-1}F_m(j)$ yields a surjection from J to itself. We can then find a j_1 such that $j_1^{-1}F_m(j_1) = j_0$. If we take $v = mj_1m^{-1}$ then

$$v^{-1}mF(v) = mm^{-1}v^{-1}mF(v) = mj_1^{-1}F_m(j_1) = mj_0 = u \tag{8.3}$$

□

Now we need to prove the lemma.

Proof. For any choice of borel B containing T, its unipotent radical U can be written as a product of root subgroups. Such a root subgroup is in L if it corresponds to a root α with $\langle \gamma, \alpha \rangle = 0$ for every cocharacter γ of T whose image is contained in T^c. Let Φ be the root system of T, Φ' be the subset of roots contained in L, and let $V = \Phi \otimes \mathbb{R}$ and $V' = \Phi' \otimes \mathbb{R}$ be the vector spaces spanned by these sets. Since c is in the Weyl group of T, it acts on Φ and V.

Choosing a U is equivalent to choosing a hyperplane H in V containing no roots and a positive side H^+. The group $U \cap FU \cap F^2U \cdots$ is generated by the roots in H^+ that are never sent to H^- under the repeated action of c.

Then our lemma can be rephrased in the following way:

There is a hyperplane H in V containing no roots such that for any root r in H^+ either r is in V' or cr is also in H^+

Now we construct such an H. Choose a nonzero cocharacter γ fixed by c. (If there are no nonzero cocharacters then $V' = V$ and the theorem is clearly true.) Let $H' = \gamma^{\perp}$ be the hyperplane in V orthogonal to γ. This is not an acceptable choice for H because it contains roots; it contains all of Φ'. (If Φ' is empty, then w is trivial and the theorem is once again clearly true). Nonetheless, let $(H')^+$ be the set of vectors in V with $\langle \gamma, v \rangle > 0$ for a root α in $(H')^+$ we have

$$\langle \gamma, w \cdot \alpha \rangle = \langle w^{-1} \cdot \gamma, \alpha \rangle = \langle \gamma, \alpha \rangle > 0. \tag{8.4}$$

We can therefore perturb H' slightly to no longer contain any elements of Φ' and the result will be a hyperplane H with the desired properties. □

This proves the lemma, and therefore Theorem 8.1.

Definition 8.3. *For an F-rational Levi subgroup L of G define*

$$X_h(L) = \{x \in \mathbb{L}_h | x^{-1}Fx \in \mathbb{L}_h \cap \mathbb{U}_h\} / \left((\mathbb{L}_h \cap \mathbb{U}_h) \cap F^{-1}(\mathbb{L}_h \cap \mathbb{U}_h) \right). \tag{8.5}$$

$$X_h = \bigsqcup_{g \in \mathbb{G}_h^F / \mathbb{L}_h^F} g \cdot X_h(L) \tag{8.6}$$

Any lift of c to $G_{\breve{k}}$ will centralize T^c and therefore be contained in $L \cap N(T)$, so c is an element of the Weyl group of T in L. It must be an elliptic element of the Weyl group of T in L, because its fixed torus is central. L is isomorphic to a product of reductive groups $G_1 \times G_2 \times \cdot G_n$ with irreducible Weyl groups. We also have $c = c_1 c_2 \cdots c_n$ where the c_i are pairwise commuting lifts of Weyl group elements such that $c_i|_{G_i}$ is an elliptic element of the Weyl group of $T_i \subset G_i$. and $c_i|_{G_j}$ is trivial for $i \neq j$.

Therefore $X_h(L)$ can be written as a product of schemes $X_{h,1} \times X_{h,2} \cdots \times X_{h,n}$ where $X_{h,i}$ recreates our construction at the start of this paper for the group G_i, a twist of a split reductive group H_i by an elliptic element. So understanding the cohomology of the X_h construction for elliptic c suffices to cover the construction for all c.

Note that this does not imply maximality results for the closed strata. For one, we currently require the c_i to all be balanced, but even if that were the case, the scheme Y_h would not necessarily be maximal. The Y_{h_i}, even if maximal, would be maximal over separate fields, and their product might not be maximal at all.

As an example, take $H = \mathrm{GL}_6$ and

$$c = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Then L is a twist of $\mathrm{GL}_2 \times \mathrm{GL}_4$. Let Y_h^2 and Y_h^4 be the corresponding schemes for GL_2 and GL_4 respectively. F^2 acts on $H_c^i(Y_h^2, \overline{\mathbb{Q}}_\ell)$ with eigenvalue $(-1)^i q^i$ and F^4 acts on $H_c^j(Y_h^2, \overline{\mathbb{Q}}_\ell)$ with eigenvalue $(-1)^j q^{2j}$. The product $Y_h^2 \times Y_h^4$ has cohomology groups

$$H_c^k(Y_h, \overline{\mathbb{Q}}) \simeq \oplus_{i+j=k} H_c^i(Y_h^2, \overline{\mathbb{Q}}_\ell) \otimes H_c^j(Y_h^4, \overline{\mathbb{Q}}_\ell).$$

Y_h is defined over $\mathbb{F}_{q^4}$, but we can see that the action of F^4 on its cohomology groups will not have constant eigienvalues, and in fact there is no extension field over which Y_h is maximal.

CHAPTER 9

The Example of GL_4

In this section we will walk through the arguments presented elsewhere in this thesis, but refer throughout to the specific example of $H = \mathrm{GL}_4$. We will prove very little, and instead cite the relevant theorems that occur earlier in the paper. GL_4 was chosen as an example because in this case it will be very easy to write down elements of $\mathbb{G}_h$. The results for semisimple groups, though they imply the result for GL_n were modeled on work originally done for GL_n.

LEt k be a local field of characteristic p[1], and write $\breve{k}$ for the completion of its maximal unramified extension. Let $\sigma \in \mathrm{Gal}(\breve{k}/k)$ be the frobenius which acts on the residue field by raising to the qth power for some $q = p^i$. We define a σ action on $H(\breve{k})$ by acting separately on each matrix entry. We let S be the torus of diagonal matrices in H and U be the unipotent group of lower triangular matrices[2]. Finally, we write $\overline{U}$ for the unipotent opposite U

The Lie algebra $\mathfrak{gl}_4$ has root system A_3. We can model A_3 as the set of vectors in $\mathbb{R}^4$ with length $\sqrt{2}$, integer coordinates, and sum of all coordinates equal to 0. As a base for the root system we may choose $\alpha_i = e_{i+1} - e_i$ for $1 \leq i \leq 3$. Let s_i be the reflection about α_i, and take $c = s_1 s_2 s_3$ as our choice of elliptic element in the Weyl group of S.

[1] The results in this paper, and even the below construction, work equally well in the case where k is characteristic 0. We have assumed characteristic p here because under this assumption the schemes we work with have the simplest description

[2] Though a somewhat nonstandard choice, this example works most cleanly if we choose these instead of upper triangular matrices.

If we identify $X_*(S)$ with the cocharacter lattice of A_4 via

$$\alpha_1^\vee(t) = \begin{bmatrix} t^{-1} & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and similarly for α_2 and α_3, then we see that U is generated by all of the positive root subgroups. This identification allows us to choose an element of $N(S)$, so that the action of c is realized by conjugation in SL_4. We define the endomorphism adj_c of H by conjugation by

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Now define the morphism $F := \mathrm{adj}_c \circ \sigma(g)$. Let G be the group scheme whose R-points, for any k-algebra R are

$$G(R) = \{g \in \mathrm{SL}_4(R \otimes_k \breve{k}) \mid Fg = g\}.$$

G is an inner twist of H. Let T be the group of diagonal matrices in G, a twist of the torus S. Note that the group $T(k)$ is isomorphic to the multiplicative group of the unique unramified degree four extension of k.

To get a parahoric model for G, we take a point in the intersection of the apartment of T with the F-fixed locus in the building $\mathscr{B}(G, \breve{k})$. In this case, the intersection is only one point, x. The attached parahoric model has

$$G_{x,0}(\mathcal{O}) \simeq \mathrm{SL}_4(\mathcal{O}) \quad \text{and} \quad G_{x,0}(\mathcal{O}_k) = \mathrm{SL}_4(\mathcal{O})^F$$

Then, following [5] we have for any integer h an affine group scheme $\mathbb{G}_h$ over $\mathbb{F}_q$ with

$$\mathbb{G}_h(\overline{\mathbb{F}}_q) = G_{x,0}(\mathcal{O})/G_{x,(h-1)+}(\mathcal{O}) \quad \text{and} \quad \mathbb{G}_h(\mathbb{F}_q) = G_{x,0}(\mathcal{O}_k)/G_{x,(h-1)+}(\mathcal{O}_k).$$

We may write this more concretely with the following definition adapted from [5]. Define $\mathbb{W} = \mathrm{Spec}(\mathbb{F}_q[[t]])$. The Frobenius σ acts on $\mathbb{W}(A)$ by $\sigma(a_0 + a_1 t + \cdots) = a_0^q + a_1^q t + \cdots$. We let V be the shift operator: $V(a_0 + a_1 t + \cdots) = a_0^t + a_1 t^2 + \cdots$. and define $\mathbb{W}_h := \mathbb{W}/V^h\mathbb{W}$. Then we can write elements of $\mathbb{G}_h$ explicitly with this construction, and we have

$$\mathbb{G}_h(\overline{\mathbb{F}_q}) \simeq \mathrm{GL}_4(\mathbb{W}_{h-1}(\overline{\mathbb{F}}_q)) \quad \text{and} \quad \mathbb{G}_h(\mathbb{F}_q) \simeq \mathrm{GL}_4(\mathbb{W}_{h-1}(\overline{\mathbb{F}}_q))^F.$$

Though F is defined as an endomorphism of $\mathrm{GL}_4(\breve{k})$, it descends naturally to an endomorphism of $\mathrm{GL}_4(\mathbb{W}_{h-1})$, since adj_c is well defined over $\mathbb{F}_q$. We also define group schemes $\mathbb{T}_h$ and $\mathbb{U}_h$ over $\overline{\mathbb{F}}_q$ as

$$\mathbb{T}_h(\overline{\mathbb{F}}_q) = T(\mathbb{W}_{h-1}(\overline{\mathbb{F}}_q)) \quad \text{and} \quad \mathbb{U}_h(\overline{\mathbb{F}}_q) = U(\mathbb{W}_{h-1}(\overline{\mathbb{F}}_q))$$

F stabilizes T so it gives $\mathbb{T}_h$ the structure of a group scheme over $\mathbb{F}_q$ with

$$\mathbb{T}_h(\mathbb{F}_q) = T(\mathbb{W}_{h-1})^F.$$

For $i > j$ there is a quotient map from $\mathbb{W}_i$ to $\mathbb{W}_j$, and thus a map from $\mathbb{G}_i$ to $\mathbb{G}_j$, and similar maps for $\mathbb{T}_i$ and $\mathbb{U}_i$. We write $\mathbb{G}_i^j$ (or $\mathbb{U}_i^j$ or $\mathbb{T}_i^j$) for the kernels of these maps.

Definition 9.1. *The* Parahoric Deligne-Lusztig Varieties *are schemes X_h defined over $\overline{\mathbb{F}}_q$ by*

$$X_h = \{g \in \mathbb{G}_h | g^{-1}Fg \in \mathbb{U}_h \cap F(\overline{\mathbb{U}}_h)\}$$

The coxeter element c has order 4. For every divisor d of 4 (so d=1,2,4) let $S^{(d)}$ be the subtorus of S that is fixed by adj_c^d. Now define $M^{(d)}$ as the centralizer of $S^{(d)}$. These are Levi subgroups of H. Since adj_c and F commute, $M^{(d)}$ is stabilized by F and yields a k-rational subscheme of G, and an associated sequence of group schemes $\mathbb{M}_h^{(d)}$. Note that $M^{(4)} = T$ and $M^{(1)} = G$ so only $M^{(2)}$ is a new scheme. We then define

$$\mathbb{U}_h^{(d)} := (\mathbb{M}_h^{(d)}\mathbb{U}_h^1) \cap \mathbb{U}_h$$

and

$$X_h^{(d)} = \{g \in \mathbb{G}_h | g^{-1}Fg \in \mathbb{U}_h^{(d)} \cap F(\overline{\mathbb{U}}_h^{(d)})\}.$$

Our main focus of interest will be $X_h^{(4)}$. Since $L^{(4)} = T$ and the intersection of T and U is trivial $\mathbb{U}_h^{(4)} = \mathbb{U}_h^1$. We can reduce even further to studying the scheme

$$Y_h = \{g \in \mathbb{G}_h^1 | g^{-1}Fg \in \mathbb{U}_h^1 \cap F(\overline{\mathbb{U}}_h^1)\}$$

by showing that $X_h^{(4)}$ is a disjoint union of copies of Y_h.

Now we look at the sturcture of Y_h. To simplify notation, abbreviate

$$V_h := \mathbb{U}_h^1 \cap F\overline{\mathbb{U}}_h^1.$$

The group consists of matrices of the form

$$v = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \vec{x_1} & 1 & 0 & 0 \\ \vec{x_2} & 0 & 1 & 0 \\ \vec{x_3} & 0 & 0 & 1 \end{bmatrix}$$

for $\vec{x}_i$ an element of $\mathbb{W}_{h-1}$ with constant term 0. We will refer to its elements simply as vectors $[\vec{x_1}, \vec{x_2}, \vec{x_3}]^T$. We want to make elements of Y_h from the set of such v. We define

$$N_h(v) = \begin{bmatrix} 1 & \sigma\vec{x_3} & \sigma^2\vec{x_2} & \sigma^3\vec{x_1} \\ \vec{x_1} & 1 & \sigma^2\vec{x_3} & \sigma^3\vec{x_2} \\ \vec{x_2} & \sigma\vec{x_1} & 1 & \sigma^3\vec{x_3} \\ \vec{x_3} & \sigma\vec{x_2} & \sigma^2\vec{x_1} & 1 \end{bmatrix}$$

Since we are able to write the original element v as $\mathrm{Id}+M$, then $N_h(v) = \mathrm{Id}+M+FM+F^2M+F^3M$. Of course, we can only write elements of $\mathbb{G}_h^1$ as $\mathrm{Id}+M$ because we are exploiting the explicit presentation we chose for GL_4 as our example. Even an element of SL_4 becomes much more difficult to write down so explicitly.

We could hope that $N_h(v)$ is contained in Y_h, but that is not quite true. It is easy to check, however, that

$$L\left(N_h(v)\right) = N_h(v)^{-1}FN_h(v) = \begin{bmatrix} 1+\vec{s} & 0 & 0 & 0 \\ \vec{y_1} & 1 & 0 & 0 \\ \vec{y_2} & 0 & 1 & 0 \\ \vec{y_3} & 0 & 0 & 1 \end{bmatrix}$$

where s and the $\vec{y_i}$ are elements of $\mathbb{W}_{h-1}$ with the constant terms equal to 0. This means that $L(N_h(v))$ is equal to an element of V_h mutiplied by an element of $\mathbb{T}_h^1$. We can write it uniquely as tu.

Let g be an element of $\mathbb{G}_h^1$ with $g^{-1}Fg = tu$. For s a second element of $\mathbb{T}_h^1$ we have

$$(gs)^{-1}F(gs) = s^{-1}g^{-1}FgFs = s^{-1}tuFs = \left(s^{-1}Fst\right)\left(Fs^{-1}uFs\right)$$

Since $\mathbb{T}_h^1$ normalizes V_h, the conjugation $Fs^{-1}uFs$ remains in V_h, so we see that sg is contained in Y_h if and only if $s^{-1}Fs = s^{-1}$.

We define a map φ_h from V_h to $\mathbb{T}_h^1$ that sends v to t^{-1}, and define a scheme W_h via the pushout

$$\begin{array}{ccc} W_h & \xrightarrow{\mathrm{pr}_1} & V_h \\ \Big\downarrow{\scriptstyle \mathrm{pr}_2} & & \Big\downarrow{\scriptstyle \varphi_h} \\ \mathbb{T}_h^1 & \xrightarrow{L} & \mathbb{T}_h^1 \end{array}$$

Then there is a map from W_h to Y_h. Given w in W_h, we can send it to $N_h(\mathrm{pr}_1(w))\cdot\mathrm{pr}_2(w)$. The resulting map is in fact an isomorphism from W_h to Y_h. See 4.5 for more details.

The cartesian diagram lets us relate

$$H_c^i(W_h, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\chi} H_c^i(V_h, \varphi_h^* \mathscr{L}_\chi)$$

where the sum on the right is taken over $\overline{\mathbb{Q}}_\ell$ characters of $\mathbb{T}_h^1(\mathbb{F}_q)$, and $\mathscr{L}_\chi$ is the rank one multiplicative local system associated to χ, defined in section 5. We can then compute the cohomology separately for each character.

We group the characters of $\mathbb{T}_h^1(\mathbb{F}_q)$ into 3 separate classes. Any character restricts to a character of $\mathbb{T}_h^{h-1}(\mathbb{F}_q) \simeq \mathbb{F}_{q^4}$. We define the conductor of a character of $\mathbb{F}_{q^4}$ to be the smallest integer d such that the character factors through the trace map from $\mathbb{F}_{q^4}$ to $\mathbb{F}_{q^d}$. Clearly the only conductors we see are $d = 1, 2$ or 4, and we group the characters accordingly.

If χ has conductor 1, analysis of the formula for φ_h allows us to find a character χ_1 of $\mathbb{T}_{h-1}^1(\mathbb{F}_q)$ satisfying the following commutative diagram

$$\begin{array}{ccccc} V_h & \xrightarrow{\varphi_h} & \mathbb{T}_h^1 & \overset{\chi}{\searrow} & \\ \Big\downarrow & & \Big\downarrow & & \\ V_{h-1} & \xrightarrow{\varphi_{h-1}} & \mathbb{T}_{h-1}^1 & \xrightarrow{\chi_1} & \overline{\mathbb{Q}}_\ell \end{array}$$

So the cohomology groups $H_c^\bullet(V_h, \varphi_h^* \mathscr{L}_\chi)$ are a shift of the groups $H_c^\bullet(V_{h-1}, \varphi_{h-1}^* \mathscr{L}_{\chi_1})$. The latter groups occur in the cohomology of Y_{h-1} so they may be determined by induction.

For the other characters we must do more work. Lemma 5.3 says the following

Theorem (5.3)**.** *Let S_1 be a scheme of finite type over $\mathbb{F}_q$, put $S = S_1 \times \mathbb{G}_a$. Let R be an algebraic group over $\mathbb{F}_q$ and $\mathscr{F}$ be a multiplicative rank one local sysem on R. Finally let f*

be a morphism $S \to R$ that sends a point $(x,y) \in S_1 \times \mathbb{A}^1$ to

$$f(x,y) = f_1(x,y) \cdot f_2(x) \tag{9.1}$$

and such that at each point x in S_1 the restriction to the fiber $f_1|_{S_x} : \mathbb{G}_a \to R$ is a homomorphism.

Define S_2 to be the subscheme of points x in S_1 such that $f_1^*\mathscr{F}|_{S_x}$ is trivial. Then

$$H^i_c(S, f^*\mathscr{F}) \simeq H^i_c(S_2 \times \mathbb{G}_a, f_2^*\mathscr{F}). \tag{9.2}$$

Since V_h is isomorphic as an $\overline{\mathbb{F}}_q$ scheme to the affine space $\mathbb{A}^{3(h-1)}$ there are many ways to write it as a product of a smaller scheme and $\mathbb{G}_a$. For $i = 1,2,3$ expand

$$\vec{x_i} = x_{i,1}t + \cdots + x_{i,h-1}t^{h-1}.$$

Then for $i = 1,2,3$ and $1 \leq j \leq h-1$ we may write

$$V_h \simeq V_h^{i,j} \times \operatorname{Spec}(\overline{\mathbb{F}}_q[x_{i,j}])$$

for $V_h^{i,j}$ a scheme isomorphic to affine $3(h-1)-1$ space. We think of elements of $V_h^{i,k}$ as a collection of coefficients $x_{a,b}$ for $a = 1,2,3$ and $1 \leq b \leq h-1$ except the pair $a = i$ and $b = j$. There are many choices for such isomorphisms, but we will need to take care to find ones such that the restriction of φ_h to the fibers is a group homomorphism, so that we satisfy the hypotheses of lemma 9.

When $j = h-1$ we can choose such an isomorphism. In this case, regardless of the conductor of χ, we find the homomorphism from $\mathbb{G}_a \to \mathbb{T}_h^1$ is always trivial.

This does not much help us compute cohomology, but it is a necessary step for us to find the right splitting of V_h into a smaller scheme times an affine space when $j = h-2$. In this case we see a few options. Let x be a point in $V_h^{i,h-2}$.

- If χ has conductor 4, the induced character on $\mathrm{Spec}(\overline{\mathbb{F}}_q[x_{i,h-2}])$ is trivial if and only if $x_{4-i,1}$ is fixed by σ^4.

- If χ has conductor 2, the induced character on $\mathrm{Spec}(\overline{\mathbb{F}}_q[x_{2,h-2}])$ is always trivial and the inducted character on $\mathrm{Spec}(\overline{\mathbb{F}}_q[x_{i,h-2}])$ for odd i is trivial if and only if $x_{4-i,1}$ is fixed by σ^4.

Then if χ has conductor 4 we may compute the cohomology of $\varphi_h^*\mathscr{L}_\chi$ on the subscheme of V_h consisting of vectors

$$v = \begin{bmatrix} a_{1,1}t + x_{1,2}t^2 + \cdots x_{1,h-1}t^{h-1} \\ a_{2,1}t + x_{2,2}t^2 + \cdots x_{2,h-1}t^{h-1} \\ a_{3,1}t + x_{3,2}t^2 + \cdots x_{3,h-1}t^{h-1} \end{bmatrix}$$

with the as fixed by σ^4. A further reductive step lets us compute cohomology on the subscheme of vectors

$$v = \begin{bmatrix} x_{1,2}t^2 + x_{1,3}t^3 + \cdots x_{1,h-1}t^{h-1} \\ x_{2,2}t^2 + x_{2,3}t^3 + \cdots x_{2,h-1}t^{h-1} \\ x_{3,2}t^2 + x_{3,3}t^3 + \cdots x_{3,h-1}t^{h-1} \end{bmatrix}$$

On this subscheme, we can once again find a product decomposition as in Lemma 9, this time taking $\mathrm{Spec}(\overline{\mathbb{F}}_q[x_{i,h-3}])$ as the affine space. When we do this, we see once again that the pullback of the character χ is trivial if and only if $x_{4-i,2}$ is fixed by σ^4.

We may repeat this process over and over until we have cleared away the $x_{i,k}$ for $k < \frac{h-1}{2}$. We have now reduced to two cases. If h is even, we are left with the subscheme consisting of

$$v = \begin{bmatrix} x_{1,h/2}t^{h/2} + \cdots x_{1,h-1}t^{h-1} \\ x_{2,h/2}t^{h/2} + \cdots x_{2,h-1}t^{h-1} \\ x_{3,h/2}t^{h/2} + \cdots x_{3,h-1}t^{h-1} \end{bmatrix}.$$

Observe that $\mathbb{G}_h^{h/2}$ is a *commutative* group, which implies that for v as above, $N_h(v)$ is contained in Y_h, not $Y_h\mathbb{T}_h^1$. Then φ_h restricted to this scheme is the constant map to 0, so the pullback of $\mathscr{L}_\chi$ is trivial and the desired cohomology groups are exactly those of affine $3(h-\frac{h}{2})$ space with the constant sheaf.

If h is odd, we are left with the subscheme consisting of

$$v = \begin{bmatrix} x_{1,(h-1)/2}t^{(h-1)/2} + \cdots x_{1,h-1}t^{h-1} \\ x_{2,(h-1)/2}t^{(h-1)/2} + \cdots x_{2,h-1}t^{h-1} \\ x_{3,(h-1)/2}t^{(h-1)/2} + \cdots x_{3,h-1}t^{h-1} \end{bmatrix}.$$

Note that φ_h is not trivial when restricted to this subscheme. However, we may write the scheme as a product

$$\mathrm{Spec}(\overline{\mathbb{F}}_q[x_{1,(h-1)/2}, x_{2,(h-1)/2}, x_{3,(h-1)/2}]) \times \mathbb{A}^{3(h-3)/2}$$

and then see that φ_h factors through projection onto the first factor. We are reduced to computing $H_c^i(\mathbb{A}^3, \varphi_h^*\mathscr{L}_\chi)$, and the dimensions of these cohomology groups can be written out explicitly as some simple character sums.

The process is similar if χ has conductor 2. However, this time in our first reduction step we may only truncate rows 1 and 3, arriving at the subscheme consiting of

$$v = \begin{bmatrix} 0 + x_{1,2}t^2 + \cdots + x_{1,m}t^m + \cdots x_{1,h-1}t^{h-1} \\ x_{2,1}t + x_{2,2}t^2 + \cdots x_{2,h-1}t^{h-1} \\ 0 + x_{3,2}t^2 + \cdots + x_{3,m}t^m + \cdots x_{3,h-1}t^{h-1} \end{bmatrix}$$

In this situation, we can continue with our truncation process, but only for the first and third entries. We are left with the scheme of vectors of the form

$$v = \begin{bmatrix} 0 + \cdots + x_{1,m}t^m + \cdots + x_{1,h-1}t^{h-1} \\ x_{2,1}t + x_{2,2}t^2 + \cdots x_{2,h-1}t^{h-1} \\ 0 + \cdots + x_{3,m}t^m + \cdots x_{3,h-1}t^{h-1} \end{bmatrix}$$

with m equal to $\frac{h}{2}$ or $\frac{h-1}{2}$ depending on the parity of h, just like before. This scheme can be decomposed as a product

$$(\mathbb{M}_h^{(d)} \cap V_h) \times \mathbb{A}^{2(h-m)}.$$

On this scheme, φ_h can be expressed as a product of separate maps from each of the factors to $\mathbb{T}_h^1$, and we may compute the cohomology of $\varphi_h^* \mathscr{L}_\chi$ separately on each factor.The computation on the right factor is handled exactly the the same way as in the conductor 4 case: If h is even the pullback is trivial and if h is odd it reduces to the exact same character sums. For the left factor we have a commutative diagram

$$\begin{array}{ccccc} \mathbb{M}_h^{(2)} \cap V_h & \xrightarrow{\varphi_h} & \mathbb{T}_h^1 & & \\ \downarrow & & \downarrow & \searrow^{\chi} & \\ \mathbb{M}_{h-1}^{(2)} \cap V_{h-1} & \xrightarrow{\varphi_{h-1}} & \mathbb{T}_{h-1}^1 & \xrightarrow{\chi_1} & \overline{\mathbb{Q}}_\ell \end{array}$$

for a unique character χ_1. If the restriction of χ_1 to $\mathbb{T}_{h-1}^{h-2}$ has conductor 4, the cohomology groups $H_c^i(\mathbb{M}_{h-1}^{(2)} \cap V_{h-1}, \varphi^* \mathscr{L}_{\chi_1})$ can be expressed via character sum. If the conductor is 1 or 2, we will be able to project down to V_{h-2} and repeat the process. If we make it all the way down to $\mathbb{T}_1^1$, we will know the cohomology is trivial.

Though this process is lengthy, it is not overly difficult to keep track of which cohomology groups occur in $H_c^i(Y_h, \overline{\mathbb{Q}}_\ell)$, and the eigenvalues of Frobenius on these groups are determined by explicit character sums.

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