Quantum Anatomy of Supersymmetric Black Holes in AdS Spacetimes

by

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Abstract

The entropy of the universe might decrease if black holes did not have entropy. Hawking's derivation of black hole temperature and the first law of thermodynamics suggest that black holes indeed have entropy. However, Einstein's classical gravity does not allow black holes to have internal degrees of freedom that entropy implies. Thus it is the central mission of quantum gravity to uncover the many quantum microstates of black holes. Via the AdS/CFT correspondence, black holes in Anti-de-Sitter (AdS) spacetimes are dual to ensembles of quantum states in conformal field theories (CFTs). Recently, the number of supersymmetric quantum states in CFTs has been counted to successfully account for the supersymmetric AdS black hole entropy. We take a step forward by studying properties of such supersymmetric quantum states dual to black holes. First, a supersymmetric AdS black hole may exist only if its charges obey a certain relation. We reproduce the same relation from a certain ensemble of the supersymmetric states in CFT. This gives a heuristic derivation of the supersymmetric black hole charge constraint for AdS black holes in 3, 4, 5, and 7 dimensions from the respective microscopic theories. Second, we find explicit expressions for the black hole cohomologies in the weakly coupled 4d maximal Super-Yang-Mills theory with gauge groups $SU(2)$ and $SU(3)$. These are connected to the actual microstates of quantum black holes in the 5-dimensional AdS spacetime.

Chapter 1

Introduction

It took around a century from the beginning of quantum physics to complete the standard model [1, 2], that provides a fairly accurate theoretical framework for electromagnetic, weak and strong interactions. Many valuable questions remain, including the dark sectors, neutrinos, strongly coupled QCD, precision of the electroweak sector, baryogenesis and naturalness to name a few, but a major goal of fundamental understanding of Nature has moved towards quantum gravity in the last several decades.

Unfortunately, Einstein's theory of general relativity is not directly compatible with the framework of quantum field theory. Gravity is not renormalizable, meaning that nonsensical divergences that occur at very high energies cannot be separated from physics of ordinary energy scale that we experience. Superstring theory has been developed and established itself as an improved and plausible theory of quantum gravity, but it also poses many difficulties, both conceptually and technically.

The AdS/CFT correspondence [3] opened a new window towards understanding gravity. Of many versions, it states that the quantum gravity theory in the Anti-de-Sitter space of $d + 1$ dimensions is 'dual' to a conformal field theory without gravity in d dimensions. The Anti-de-Sitter space, AdS in short, is the maximally symmetric space with constant negative curvature. It can be considered the vacuum spacetime for a negative cosmological constant. The AdS/CFT correspondence is not proven, but is widely accepted throughout the physics community. In fact, it is hard to imagine how one would prove it without completely understanding theories on both sides. We refer to [4–7] for reviews of the subject.

One important arena of the AdS/CFT correspondence pertains to the black hole solutions of the gravity theories and the conformal field theories with finite temperature. It has been known that a black hole is a thermal system that has temperature [8], so naturally it is dual to a finite temperature 'solution' of the conformal field theory. This statement is made precise by identification of the partition functions of both sides of the duality:

$$Z_{\text{AdS}} \;=\; Z_{\text{CFT}} \,. \tag{1.1}$$

In fact, the black holes are believed to be extremely well suited for a window to new physics, because of the no-hair theorem [9–12]. According to the no-hair theorem, which is again not a proved theorem but a conjecture that represents conventional wisdom about gravity, black holes are completely described by a few macroscopic quantities such as the energy, angular momenta and charges, and do not possess any internal degrees of freedom. Provided the no-hair theorem, black holes allow physicists to study gravity via a model with few parameters, akin to a toy model. Therefore, it is a very approachable but interesting goal to understand black holes in AdS spacetimes through CFTs.

Meanwhile, black holes are extremely interesting objects that intermingle all areas of physics. This fact is concisely symbolized by the Hawking temperature of a black hole [8]

$$T \;=\; \frac{\hbar c^3}{8\pi G_N k_B M} \,, \tag{1.2}$$

that puts together

- $\hbar$: Planck's constant, the fundamental constant of quantum mechanics,
- c : Speed of light, the fundamental constant of special theory of relativity,
- G_N : Newton's constant, the fundamental constant for general theory of relativity, and
- k_B : Boltzmann's constant, the fundamental constant of statistical mechanics.

Black holes originated as the first non-trivial solution to the classical Einstein gravity [13], but many mysteries regarding the event horizon have been properly raised and addressed several decades later. For example, if one throws in a cup of hot tea into the black hole, the black hole grows slightly larger but the entropy of the universe seems to decrease. It was then realized that if one attributes to a black hole a temperature proportional to its surface gravity and an entropy proportional to the area of its event horizon [14–16], then all laws of thermodynamics including the first and the second fit perfectly [17–21].

Hawking showed that the attribution of temperature is not merely an analogy to the laws of thermodynamics, but that the black hole in fact radiates as if it were a blackbody with that temperature [8, 22].

Similarly, if the Bekenstein-Hawking entropy of the black hole is indeed the entropy that is known from thermodynamics, then the black hole must be an ensemble that consists of a corresponding — in fact, an enormous — number of microscopic degrees of freedom. This is contrary to the no-hair theorem of classical gravity, that the black hole is completely described by several macroscopic parameters and does not have any internal structure. This is what a successful theory of quantum gravity, if any, must address [23–26].

One of the biggest successes of superstring theory is exactly this. In the seminal work [27], the authors considered a particular setup of string theory, namely the type-IIB string theory compactified on $K3 \times S^1$. It yields supersymmetric black hole solutions in 5 non-compact dimensions with finite horizon area, and thus finite Bekenstein-Hawking entropy. These solutions are realized by D-branes that source the charge of the black hole, and the number of their bound states was matched with the entropy.

With the advent of the AdS/CFT correspondence, it looked hopeful that microscopic accounting of the entropy of AdS black holes from the dual CFTs, which are arguably more thoroughly understood than systems of strings and branes, may shed brighter light on quantum gravity. Supergravity solutions for AdS black holes were found in various dimensions, including [28–33] for 5 dimensions. However, despite various attempts including [34–41], the success had to wait until recently.

The difficulties encountered in the early attempts and overcome recently, are strongly tied to a property of the AdS/CFT correspondence that weakly coupled theory on one side is dual to strongly coupled theory on the other. The black holes as supergravity solutions are valid in the quantum gravity side of the duality, namely the superstring theory, when the string coupling constant g_s is small and the string length scale ℓ_s is smaller than the length scale of the curvature. These limits translate in the CFT side of the correspondence to large gauge group $N \to \infty$ and large 't Hooft coupling $\lambda \to \infty$. Strongly coupled gauge theories are much harder to work on analytically than the weakly coupled. In order to circumvent this difficulty, an index [35,36], which is a coupling independent function of a CFT, has been devised. One can compute the index in the more approachable weakly coupled field theory but still argue that it counts the same number of states as in the strongly coupled theory. However, the index only contains information about the number of bosonic states minus the number of fermionic states, while the total number of microstates that accounts for the black hole entropy should be a sum over both. It was realized only during the recent advances that it is possible to faithfully count the total number of microstates using the index by complexifying the chemical potentials, the variables that the index depends on. Based on various approaches, entropies of various supersymmetric black holes in different dimensions of AdS space with different limits and precisions have been matched with enumerations of microscopic states in the dual conformal field theories, see [42–69] among the vast sea of literature.

Extending the remarkable match between the black hole entropy and the number of microstates in the dual conformal field theory, we now aim to anatomize the microstates beyond counting them.

First, we study how the quantum numbers of the field theory microstates match the conserved quantities of the black hole. All supersymmetric AdS black holes that we study have a property that their conserved quantities, also known as charges, obey a certain relation between themselves. This relation is non-linear and quite non-trivial except in AdS_3, and its interpretation is not yet clear. It is sometimes linked to the absence of closed timelike curves, but one clear and simple way to put it is that no regular, supersymmetric black hole solutions are known away from the constraint.

If the black hole is dual to an ensemble of black hole microstates in the field theory via (1.1), then the charges, or quantum numbers, of the microstates must reproduce the black hole charge constraint. Apparently, the supersymmetric states in the dual field theories that are believed to be the black hole microstates according to the counting, exist all over the charge configuration space and do not obey any particular constraint. We interpret the supersymmetric black hole charge constraint as a property of the ensemble, rather than of individual black hole microstates, and present a heuristic derivation of the charge constraint from an ensemble of supersymmetric states in the dual field theory.

Second, we look for explicit expressions of black hole microstates in the field theory language. Although supergravity black holes are dual to microstates in the large-N, strongly coupled limit of the field theory, the fact that the black hole entropy is counted by a coupling-independent quantity suggests that there are as many analogous states in the weakly coupled field theory as there are black hole microstates in the strongly coupled theory. From a different point of view, one may argue that microstates in the finite-N, weakly coupled regime of the field theory hint towards black hole microstates in the quantum gravity theory, as opposed to its supergravity approximation.

In this light, we explore the Hilbert space of supersymmetric states in the 4-dimensional $\mathcal{N} = 4$ Yang-Mills theory with finite gauge group $SU(N)$, dual to black holes in the 5-dimensional AdS space, and identify supersymmetric states that corresponds to the black holes. In the weakly coupled, or perturbative field theories, there is an established way of assembling basic elements of the theory to compose the Hilbert space. In particular, the supersymmetric states can be represented by cohomologies with respect to the preserved supercharge. We shall find some of these cohomologies that are dual to the supersymmetric black hole in AdS_5, but not to the gas of super-graviton particles that are more trivial.

1.1 Overview of the Dissertation

This dissertation is organized as follows.

In chapter 2 that belongs to the introductory part, we introduce black holes in AdS_3 and AdS_5 spacetimes in gravity perspective. We will focus on the macroscopic quantities including the energy, charges and the entropy that describe the black holes, and thermodynamic relations between these quantities. We also illustrate the supersymmetric limits of the black holes, which will reveal two properties, namely the entropy and the charge constraint, that will be the target of the next part. For AdS_3 black holes we also discuss its classical stability property. The AdS_3 part of this chapter is largely based on [65].

The rest of the dissertation is divided into two main parts. The first part is about the microscopic accounting of the entropy and the charge constraint of supersymmetric black holes. The second part is about the black hole cohomology problem, an attempt to find explicit expressions for supersymmetric black hole microstates in the language of perturbative field theory.

The first part consists of two chapters: chapter 3 on the entropy and chapter 4 on the charge constraint.

We start in section 3.1 by reviewing the early attempts, including the introduction of the index that will be the core concept throughout the dissertation. As explained in the introduction, complexifying the chemical potentials in the index has been the key to the recent success. In section 3.2 we demonstrate how the complex chemical potentials overcome the issue of boson/fermion cancellation, in a simplified setup with the $U(1)$ gauge group. Then in section 3.3 we present a derivation of the black hole entropy given the index, for AdS_3 and AdS_5 black holes. The AdS_3 part of this section is based on [65].

In section 4.1 we present a microscopic argument for the supersymmetric charge constraint of AdS_3 black holes. This section is based on [65]. In section 4.2 we develop the argument for the AdS_3 black holes into a generic prescription for deriving the supersymmetric charge constraints of higher dimensional AdS black holes heuristically. Then in sections 4.3 through 4.5, we apply the generic prescription to black holes in AdS_5, AdS_4 and AdS_7. In section 4.6, we discuss some future directions that may reinforce our derivation to be more complete and rigorous. The sections on higher dimensions are based on [70].

The second part consists of three chapters: chapter 5 on formulation of the black hole cohomology problem, chapter 6 on computing the non-graviton index, and chapter 7 on constructing the black hole cohomologies.

We start in section 5.1 by arguing how the writing of black hole microstates can be turned into a problem of finding cohomologies with respect to the preserved supercharge. In section

5.2, we introduce the BMN sector where the computations can be done more easily while allowing nearly as powerful answers as in the full sector. We define the index as a tool for counting the cohomologies in section 5.3, define the graviton cohomologies to be ruled out from the search of black hole cohomologies in section 5.4, and lay out the strategy for solving this problem in section 5.5.

Then we explain some ideas for computing the index over graviton coholomogies in section 6.1. Using these ideas, we compute the graviton index, and therefore the non-graviton index, for the BMN sector of the $SU(2)$ theory, the full $SU(2)$ theory, the BMN sector of the $SU(3)$ theory and the BMN sector of the $SU(4)$ theory in sections 6.2 through 6.5.

In section 7.1, we construct the expressions for all core black hole cohomologies detected by the index in the BMN sector of the $SU(2)$ theory. In section 7.2, we show that there must be a new black hole cohomology in the $SU(2)$ theory that is not in the BMN sector. We also discuss the partial no-hair behavior of the black hole cohomologies. Finally in section 7.3, we construct the expression for the threshold black hole cohomology in the $SU(3)$ theory.

The second part is based on two papers [71, 72].

We conclude in chapter 8 with a brief summary and future directions.

Chapter 2

Black Holes in AdS Spacetime

In this chapter, we review black holes in asymptotically AdS_3 and AdS_5 spacetimes as gravitational objects. The focus will be on their thermodynamic properties, including energy, temperature, charges, chemical potentials and entropy. The supersymmetric limits of the black holes will be also important, as most parts of this thesis will take advantage of the supersymmetry to study quantum aspects of the black holes. With the AdS_3 black hole as an example, we also comment on the stability of black holes.

2.1 AdS_3 Black Holes

In this section we consider asymptotically AdS_3 black hole solutions and review its semi-classical properties. This section is largely based on [65] in collaboration with Finn Larsen.

The AdS_3 black hole derives from rotating black hole solutions [73] to the 5-dimensional $\mathcal{N}=4$ or $\mathcal{N}=8$ supergravity, by interpreting it as a system of rotating black strings in 6 dimensions and taking the decoupling limit [74]. The local geometry of the black hole is a direct product between a BTZ black hole solution [75, 76] to the $2+1$-dimensional gravity and a three-sphere S^3 with equal but opposite constant curvatures. The global structure enables rotation of the S^3 with respect to the time in AdS_3.

Regardless of its string theoretical origin or the structure of the transverse space, it is important for our purposes that the solution can be understood as a black hole in the AdS_3 spacetime, and that it is described by four conserved quantities. There are energy, or mass, E and angular momentum J from the isometry $SO(2,2)$ of AdS_3, and there are two charges Q_L and Q_R from the isometry $SO(4) \sim SU(2)_L \times SU(2)_R$ of S^3. We consider the isometry of the transverse space as the internal symmetry, and therefore name the corresponding conserved quantities as charges. Often throughout this thesis, we will use the term 'charges' to collectively refer to the charges and the angular momenta.

This black hole is known to be dual to the 2-dimensional conformal field theory with $(4,4)$ supersymmetry [3]. The symmetry algebra of this theory consists of two copies of the small $\mathcal{N}=4$ Virasoro algebra $\mathfrak{su}(2|1,1)$ [77, 78]. Each copy includes two Cartans: one related to $\mathfrak{su}(1,1)$, a half of the 2-dimensional conformal algebra $\mathfrak{so}(2,2)$, and one related to the $\mathfrak{su}(2)$ R-symmetry. The four Cartans from both copies are identified with the four Cartans of the isometry of $\mathrm{AdS}_3 \times S^3$, or with the four conserved quantities of the black hole. In particular, $E_L \equiv E - J$ and Q_L are identified with one copy of the algebra, while $E_R \equiv E + J$ and Q_R are with the other. More precisely, the eigenvalues of Virasoro generators are introduced through

$$L_0 - \frac{k_R}{4} = \frac{\epsilon + j}{2} \ , \qquad \tilde{L}_0 - \frac{k_L}{4} = \frac{\epsilon - j}{2} \ . \tag{2.1}$$

The constants $k_{L,R}$ are levels of the $SU(2)$ R-currents. They are related to the central charges as $c_{L,R} = \frac{1}{6}k_{L,R}$ by $\mathcal{N}=4$ supersymmetry. In the absence of gravitational anomaly, that is when $k = k_L = k_R$, they are related to the 3-dimensional Newton's constant by [79]

$$6k \ = \ c \ = \ \frac{3R}{2G_3} \ . \tag{2.2}$$

The quantum numbers ϵ, j, q_R, q_L characterize individual states. The corresponding macroscopic charges, evaluated as averages over many states, are denoted E, J, Q_R, Q_L. The unique $SL(2) \times SL(2)$ invariant ground state annihilated by L_0 and $\tilde{L}_0$ has strictly negative energy $E_{\text{vac}} = -\frac{1}{4}(k_R + k_L)$ and corresponds to the AdS_3 vacuum. It is separated by a gap from the black holes which have non-negative energy in the CFT_2 terminology.

2.1.1 Thermodynamics

The entropy of the black hole as a function of the charges, or the microcanonical density of states, contains all essential information about the black hole as a thermal system:

$$S \ = \ 2\pi\sqrt{\frac{1}{2}k_R(E+J) - \frac{1}{4}Q_R^2} + 2\pi\sqrt{\frac{1}{2}k_L(E-J) - \frac{1}{4}Q_L^2} \ . \tag{2.3}$$

From the entropy (2.3), the chemical potentials conjugate to the charges and the temperature conjugate to the energy can be derived using the first law of thermodynamics,

$$TdS = dE - \mu dJ - \omega_L dQ_L - \omega_R dQ_R \ . \tag{2.4}$$

The potentials written as functions of charges (E, J, Q_L, Q_R) are

$$\begin{aligned}
\beta &= \frac{\partial S}{\partial E} = \frac{1}{2}\sqrt{\frac{2\pi^2 k_L}{E-J-\frac{Q_L^2}{2k_L}}} + \frac{1}{2}\sqrt{\frac{2\pi^2 k_R}{E+J-\frac{Q_R^2}{2k_R}}} \ , \\
\mu &= -\frac{1}{\beta}\frac{\partial S}{\partial J} = \frac{-\sqrt{\frac{E-J}{k_L}-\frac{Q_L^2}{2k_L^2}}+\sqrt{\frac{E+J}{k_R}-\frac{Q_R^2}{2k_R^2}}}{\sqrt{\frac{E-J}{k_L}-\frac{Q_L^2}{2k_L^2}}+\sqrt{\frac{E+J}{k_R}-\frac{Q_R^2}{2k_R^2}}} \ , \\
\omega_R &= -\frac{1}{\beta}\frac{\partial S}{\partial Q_R} = \frac{\sqrt{\frac{E-J}{k_L}-\frac{Q_L^2}{2k_L^2}}}{\sqrt{\frac{E-J}{k_L}-\frac{Q_L^2}{2k_L^2}}+\sqrt{\frac{E+J}{k_R}-\frac{Q_R^2}{2k_R^2}}} \cdot \frac{Q_R}{k_R} \ , \\
\omega_L &= -\frac{1}{\beta}\frac{\partial S}{\partial Q_L} = \frac{\sqrt{\frac{E+J}{k_R}-\frac{Q_R^2}{2k_R^2}}}{\sqrt{\frac{E-J}{k_L}-\frac{Q_L^2}{2k_L^2}}+\sqrt{\frac{E+J}{k_R}-\frac{Q_R^2}{2k_R^2}}} \cdot \frac{Q_L}{k_L} \ .
\end{aligned} \tag{2.5}$$

One can invert these relations to write the charges in terms of the potentials:

$$\begin{aligned}
E_R \equiv E+J &= \frac{2k_R}{\beta^2(1-\mu)^2}\left(\pi^2+\beta^2\omega_R^2\right) \ , \\
E_L \equiv E-J &= \frac{2k_L}{\beta^2(1+\mu)^2}\left(\pi^2+\beta^2\omega_L^2\right) \ , \\
Q_R &= \frac{2k_R}{1-\mu}\omega_R \ , \\
Q_L &= \frac{2k_L}{1+\mu}\omega_L \ .
\end{aligned} \tag{2.6}$$

The grand canonical partition function in thermodynamics is defined from the microcanonical ensemble by a weighted sum over microstates:

$$Z(\beta,\, \mu,\, \omega_R,\, \omega_L) = \mathrm{Tr}\left[e^{-\beta(E-\mu J-\omega_R Q_R-\omega_L Q_L)}\right] \ . \tag{2.7}$$

It is a function of the inverse temperature β and chemical potentials μ, ω_R and ω_L, which determine the weight with which each microstate with certain charges E, J, Q_R and Q_L contributes to the partition function. It is possible to write the grand canonical partition function from the information about the thermodynamic system presented thus far:

$$\begin{aligned}
\log Z &= S-\beta\left(E-\mu J-\omega_R Q_R-\omega_L Q_L\right) \\
&= \frac{k_R}{\beta(1-\mu)}\left(\pi^2+\beta^2\omega_R^2\right)+\frac{k_L}{\beta(1+\mu)}\left(\pi^2+\beta^2\omega_L^2\right) \ .
\end{aligned} \tag{2.8}$$

Although we have not presented in that order, one may alternatively take (2.8) as the starting point for describing the thermodynamic system and derive all quantities from it. For example, the macroscopic energy and charges as ensemble averages are

$$
\begin{aligned}
E-\mu J-\omega_R Q_R-\omega_L Q_L &= -\frac{\partial \log Z}{\partial \beta} \\
&= \frac{k_R}{\beta^2(1-\mu)}\left(\pi^2-\beta^2\omega_R^2\right)+\frac{k_L}{\beta^2(1+\mu)}\left(\pi^2-\beta^2\omega_L^2\right) \ , \\
J &= \frac{1}{\beta}\frac{\partial \log Z}{\partial \mu} \\
&= \frac{k_R}{\beta^2(1-\mu)^2}\left(\pi^2+\beta^2\omega_R^2\right)-\frac{k_L}{\beta^2(1+\mu)^2}\left(\pi^2+\beta^2\omega_L^2\right) \ , \\
Q_{L,R} &= \frac{1}{\beta}\frac{\partial \log Z}{\partial \omega_{L,R}} = \frac{2k_{L,R}}{1\pm\mu}\omega_{L,R} \ .
\end{aligned}
\tag{2.9}
$$

These are equivalent to (2.6).

Note that (2.3) does not make sense unless [74]

$$
\begin{aligned}
E-J-\frac{1}{2k_L}Q_L^2 &\geq 0 \ , \\
E+J-\frac{1}{2k_R}Q_R^2 &\geq 0 \ .
\end{aligned}
\tag{2.10}
$$

It implies that there is simply no black hole solution for energy and charges that violate either of (2.10), and all of the thermodynamic formulae above have assumed these inequalities. Saturation of both inequalities corresponds to $\beta \to \infty$ with generic $-1 < \mu < 1$, but leads to zero entropy. Saturation of only one of the two corresponds to $\beta \to \infty$ with either $\beta(1 \pm \mu)$ kept finite, as we will elaborate in the next subsection.

2.1.2 Supersymmetry

Up to this point we did not impose any conditions on the black hole parameters. We now impose supersymmetry and show that the resulting BPS black holes satisfy *two* conditions.

In the 2d superconformal theory with $(4,4)$ supersymmetry, there are four $\frac{1}{4}$-BPS sectors. Each sector preserves two real supersymmetries that are either holomorphic (R) or anti-holomorphic (L), and that either raise or lower the corresponding R-charge. We focus without loss of generality on the $\frac{1}{4}$-BPS sector which preserves supersymmetries that are anti-holomorphic (L) and raise the R-charge. Then the unitarity bound from the anticommutator of the supercharges on individual CFT states in the NS sector is:

$$
\epsilon - j + \frac{1}{2}k_L \geq q_L \ , \tag{2.11}
$$

from which a bound for black hole energy and charges follows:

$$E - J + \frac{1}{2}k_L \geq Q_L \ . \tag{2.12}$$

Microscopic states whose quantum numbers saturate the inequality (2.11) are called chiral primaries. Unitarity further requires that chiral primaries have $0 \leq q_L \leq 2k_L$ [80, 81].

Saturation of the inequality (2.12) is a necessary condition for a supersymmetric black hole but it is not sufficient. Recall that the black hole charges must obey (2.10). A hypothetical black hole solution that violates this inequality would have event horizon with imaginary area. Such geometries are not regular so black holes with these quantum numbers simply do not exist. This regularity condition is variously referred to as the cosmic censorship bound or the condition for absence of closed timelike curves.

The BPS condition demands that the inequality (2.12) be saturated but then compatibility with regularity (2.10) gives

$$Q_L = k_L \ . \tag{2.13}$$

This is the charge constraint on BPS black holes in AdS_3. Thus BPS black holes have the same quantum numbers as the particular chiral primaries situated in the middle of the interval $0 \leq q_L \leq 2k_L$ allowed by unitarity.

We established that BPS black holes in AdS_3 are co-dimension 2 in parameter space: saturation of *two* inequalities (2.10) and (2.12) introduces *two* relations between the four parameters E, J, and $Q_{R,L}$. Now recall the formulae (2.6) that relate the quantum numbers to potentials, reproduced here for convenience:

$$E = \frac{k_R}{\beta^2(1-\mu)^2}\left(\pi^2 + \beta^2\omega_R^2\right) + \frac{k_L}{\beta^2(1+\mu)^2}\left(\pi^2 + \beta^2\omega_L^2\right) \ , \tag{2.14a}$$

$$J = \frac{k_R}{\beta^2(1-\mu)^2}\left(\pi^2 + \beta^2\omega_R^2\right) - \frac{k_L}{\beta^2(1+\mu)^2}\left(\pi^2 + \beta^2\omega_L^2\right) \ , \tag{2.14b}$$

$$Q_{L,R} = \frac{2k_{L,R}}{1 \pm \mu}\omega_{L,R} \ . \tag{2.14c}$$

In the canonical ensemble the extremal limit amounts to vanishing temperature $\beta \to \infty$. However, we must be careful with what remains finite in this limit.

Consider a pair of particular combinations of these charges:

$$E + J - \frac{Q_R^2}{2k_R} = \frac{2k_R\pi^2}{\beta^2(1-\mu)^2} \geq 0 \ , \tag{2.15a}$$

$$E - J - \frac{Q_L^2}{2k_L} = \frac{2k_L\pi^2}{\beta^2(1+\mu)^2} \geq 0 \ . \tag{2.15b}$$

If one naïvely takes $\beta \to \infty$ with the chemical potential μ finite and generic, both of these inequalities will be saturated. However, when the expressions on the left hand sides of both equations in (2.15) vanish, the black hole entropy (2.3) will be zero as well. Therefore, the limit taken this way yields an extremal "black hole" with an event horizon that has vanishing area. Such a geometry is singular, it is not a black hole solution.

In order to circumvent this obstacle, we need to saturate only one of the inequalities (2.15). We pick the latter without loss of generality. To avoid also saturating (2.15a), we take $\beta \to \infty$ while rescaling μ so that $\tilde{\mu} \equiv \beta(\mu - 1)$ remains finite. Note that $\tilde{\mu} \leq 0$ because $\mu \leq 1$. It further follows from (2.14c) that, in order to describe black holes with generic values of Q_R, the limit must also take $\omega_R \to 0$ with $\tilde{\omega}_R \equiv \beta\omega_R$ kept finite. In contrast, ω_L does not require any rescaling, it can be kept finite by itself.

In summary, the extremal limit of a general AdS_3 black hole is:

$$\text{Extremal limit:} \quad \begin{cases} \beta \to \infty \ , & \\ \mu \to 1 & \text{with } \tilde{\mu} \equiv \beta(\mu - 1) \text{ finite,} \\ \omega_R \to 0 & \text{with } \tilde{\omega}_R \equiv \beta\omega_R \text{ finite,} \\ \omega_L \text{ finite.} & \end{cases} \tag{2.16}$$

This limit was designed so that (2.14) gives expressions that are finite:

$$E = \frac{k_R}{\tilde{\mu}^2}\left(\pi^2 + \tilde{\omega}_R^2\right) + \frac{k_L}{4}\omega_L^2 \ , \tag{2.17a}$$

$$J = \frac{k_R}{\tilde{\mu}^2}\left(\pi^2 + \tilde{\omega}_R^2\right) - \frac{k_L}{4}\omega_L^2 \ , \tag{2.17b}$$

$$Q_R = -\frac{2k_R}{\tilde{\mu}}\tilde{\omega}_R \ , \tag{2.17c}$$

$$Q_L = k_L\omega_L \ . \tag{2.17d}$$

The explicit sign in the formula for Q_R compensates $\tilde{\mu} < 0$ so that the angular momentum Q_R has the same sign as the rescaled angular velocity $\tilde{\omega}_R$, as expected. These formulae for the conserved charges give the energy as a function of the charges

$$E_{\rm ext} = J + \frac{1}{2k_L} Q_L^2 \ . \tag{2.18}$$

This is the ground state energy for these conserved charges. It saturates (2.10) and is identified with the extremal black hole mass. The extremal entropy becomes

$$\begin{aligned} S_{\rm ext} &= -\frac{2k_R\pi^2}{\tilde{\mu}} = 2\pi\sqrt{\frac{1}{2}k_R(E_{\rm ext}+J) - \frac{1}{4}Q_R^2} \\ &= 2\pi\sqrt{k_R J + \frac{k_R}{4k_L}Q_L^2 - \frac{1}{4}Q_R^2} \ . \end{aligned} \tag{2.19}$$

The last equation eliminated the energy using the extremality condition (2.18).

As we have stressed, the extremal black holes are not necessarily supersymmetric. As the second and last step of implementing the BPS limit, we now examine supersymmetry. Recall from (2.12) that charges of supersymmetric black holes must saturate the inequality

$$E - J - Q_L + \frac{1}{2}k_L \geq 0 \ .$$

The left hand side can be recast as a sum of two squares

$$E - J - Q_L + \frac{1}{2}k_L = \frac{2k_L\pi^2}{\beta^2(1+\mu)^2} + \frac{k_L}{2}\left(1 - \frac{2\omega_L}{1+\mu}\right)^2 \ , \tag{2.20}$$

using (2.14). The first square is precisely (2.15b) so it vanishes in the extremal limit. In order to saturate the BPS bound (2.12) the second square must vanish as well so we demand that the potentials satisfy

$$\varphi \equiv 1 + \mu - 2\omega_L = 0 \ , \tag{2.21}$$

in addition to conditions for extremality. We defined the parameter φ for future use. Since $\mu = 1$ at extremality we must have $\omega_L = 1$ in the BPS limit. However, just as the extremal limit is taken with $\tilde{\mu} \equiv \beta(\mu - 1)$ kept finite there is no obstacle to taking the BPS limit $\omega_L \to 1$ so $\tilde{\omega}_L \equiv \beta(\omega_L - 1)$ remains finite. The value of $\tilde{\omega}_L$ is, like $\tilde{\mu}$ and $\tilde{\omega}_R$, not constrained.

To summarize, the BPS AdS_3 black holes are limits of generic AdS_3 black holes as

$$T = \beta^{-1} \to 0 \ , \tag{2.22}$$

while the potentials

$$\tilde{\mu} = \beta(\mu - 1)\ , \quad \tilde{\omega}_R = \beta\omega_R\ , \quad \tilde{\omega}_L = \beta(\omega_L - 1)\ , \tag{2.23}$$

are kept finite. In this limit two inequalities (2.12) and (2.10) are saturated.

The definition of the grand canonical partition function can be adapted to the BPS limit (2.22)-(2.23) as

$$\begin{aligned} Z(\beta,\, \mu,\, \omega_R,\, \omega_L) &= \mathrm{Tr}\left[e^{-\beta(E-\mu J-\omega_R Q_R-\omega_L Q_L)}\right] \\ &= e^{\frac{1}{2}\beta k_L}\mathrm{Tr}\left[e^{-\beta\left(E-J-Q_L+\frac{k_L}{2}\right)+\tilde{\mu}J+\tilde{\omega}_R Q_R+\tilde{\omega}_L Q_L}\right]\ . \end{aligned} \tag{2.24}$$

The second line manifests that in the BPS limit (2.22)-(2.23), contribution to the partition function from non-BPS states such that $E - J - Q_L + \frac{1}{2}k_L > 0$ will be suppressed, and the partition function will have an overall divergent factor $e^{\frac{1}{2}\beta k_L}$ which is its sole dependence on $\beta \to \infty$. This factor can be interpreted as the supersymmetric Casimir energy [82]

$$E_{\mathrm{SUSY}} \;=\; -\frac{1}{2}k_L\ , \tag{2.25}$$

that is common to all states. Note that it is not the conventional Casimir energy $E_C = -\frac{1}{4}(k_L + k_R)$ that enters here and the two notions of Casimir energy agree only when the levels $k_L = k_R$. The Casimir energy appears explicitly because we study the partition function *defined as a path integral* rather than as a trace over a Hilbert space normalized such that the vacuum contributes unity.

Therefore, the BPS partition function as the limit (2.22)-(2.23) of the grand canonical partition function, can be written as

$$\begin{aligned} Z_{\mathrm{BPS}}(\beta,\, \tilde{\mu},\, \tilde{\omega}_R,\, \tilde{\omega}_L) &= \lim_{\beta\to\infty} e^{\frac{1}{2}\beta k_L}\mathrm{Tr}\left[e^{-\beta\left(E-J-Q_L+\frac{k_L}{2}\right)+\tilde{\mu}J+\tilde{\omega}_R Q_R+\tilde{\omega}_L Q_L}\right] \\ &= \left(\lim_{\beta\to\infty} e^{\frac{1}{2}\beta k_L}\right)\mathrm{Tr}_{\mathrm{BPS}}\left[e^{\tilde{\mu}J+\tilde{\omega}_R Q_R+\tilde{\omega}_L Q_L}\right]\ . \end{aligned} \tag{2.26}$$

In the second line, we have restricted the trace to BPS states only, as non-BPS states are suppressed by $\beta \to \infty$.

For the black hole system, the BPS partition function is obtained by taking the limit (2.22)-(2.23) on the grand canonical partition function (2.8):

$$\log Z_{\mathrm{BPS}} \;=\; \frac{1}{2}\beta k_L - \frac{k_R}{\tilde{\mu}}\left(\pi^2 + \tilde{\omega}_R^2\right) + k_L\left(\tilde{\omega}_L - \frac{1}{4}\tilde{\mu}\right)\ . \tag{2.27}$$

The BPS limit of the macroscopic energy and charges can be obtained by differentiating (2.27) as in (2.9), or by taking the BPS limit of the potentials (2.22)-(2.23) from (2.9):

$$E = \frac{k_R}{\tilde{\mu}^2}\left(\pi^2 + \tilde{\omega}_R^2\right) + \frac{k_L}{4} , \tag{2.28a}$$

$$J = \frac{k_R}{\tilde{\mu}^2}\left(\pi^2 + \tilde{\omega}_R^2\right) - \frac{k_L}{4} , \tag{2.28b}$$

$$Q_R = -\frac{2k_R}{\tilde{\mu}}\tilde{\omega}_R , \tag{2.28c}$$

and notably,

$$Q_L = k_L . \tag{2.29}$$

The extremal black hole entropy (2.19) also simplifies further in the BPS limit

$$S_{\text{BPS}} = 2\pi\sqrt{k_R\left(J + \frac{1}{4}k_L\right) - \frac{1}{4}Q_R^2} . \tag{2.30}$$

In the BPS limit, the four macroscopic quantities $E, J, Q_{L,R}$ are parametrized by only two potentials $\tilde{\mu}$ and $\tilde{\omega}_R$, they are independent of the third potential $\tilde{\omega}_L$. This confirms the expectation that the parameters of a BPS black hole form a co-dimension 2 surface in the space of all possible charges. On the other hand, there really are three independent rescaled potentials $\tilde{\mu}, \tilde{\omega}_{L,R}$. This is possible because $\tilde{\omega}_L$ parametrizes a flat direction along which the BPS black hole does not change.

2.1.3 Stability

So far we have discussed the AdS_3 black hole solutions that are described by 4 charges (E, J, Q_L, Q_R) or equivalently by 4 chemical potentials $(\beta, \mu, \omega_L, \omega_R)$. The charges must obey the extremality bound (2.10) as well as the unitarity bound (2.12), and we have discussed in detail the saturation of these bounds. In this subsection we touch on another issue of semi-classical black holes, namely stability.

A quick way to determine the stability condition is from the first law of thermodynamics (2.4):

$$TdS = dE - \mu dJ - \omega_R dQ_R - \omega_L dQ_L . \tag{2.31}$$

Consider a particle with generic quantum numbers (ϵ, j, q_R, q_L). We assume that these quantum numbers are infinitesimal compared to corresponding macroscopic charges of the black hole. Change of black hole entropy under the emission of such a particle is proportional to

$$TdS = -\epsilon + \mu j + \omega_R q_R + \omega_L q_L \ . \tag{2.32}$$

If this quantity is positive, the black hole gains entropy by emitting this particle, and thus it is unstable against decaying into this particle.

There are several candidate stability bounds, depending on the quantum numbers of particles to which the black hole may emit.

- First, consider particles with $(\epsilon, j, q_R, q_L) \propto (1, 0, 1, 0)$ or $(\epsilon, j, q_R, q_L) \propto (1, 0, 0, 1)$. The stability bounds against these particles, as obtained from (2.31), are

$$\omega_R \leq 1 \ , \qquad \omega_L \leq 1 \ , \tag{2.33}$$

  respectively. These stability bounds are analogous to those for black holes in higher dimensions [83, 84].

- Next, consider particles with $(\epsilon, j, q_R, q_L) \propto (1/2, 1/2, 1, 0)$. In CFT_2, these correspond to chiral primaries, because they have $(L_0, q_R) \propto (1/2, 1)$ and $\bar{L}_0 = q_L = 0$. The black hole stability bound against chiral primaries is

$$0 \geq \omega_R - \frac{1-\mu}{2} \qquad \Leftrightarrow \qquad Q_R \leq k_R \ . \tag{2.34}$$

- Similarly, consider particles with $(\epsilon, j, q_R, q_L) \propto (1/2, -1/2, 0, 1)$. In CFT_2, these correspond to anti-chiral primaries, because they have $(\bar{L}_0, q_L) \propto (1/2, 1)$ and $L_0 = q_R = 0$. The black hole stability bound against anti-chiral primaries is

$$0 \geq \omega_L - \frac{1+\mu}{2} \qquad \Leftrightarrow \qquad Q_L \leq k_L \ . \tag{2.35}$$

The question of stability can be rephrased as follows. Given a black hole with given total charges, it is unstable if a system with another black hole and a gas of particles but with same total charges has bigger entropy than the original system. Therefore, given a microcanonical system with given total charges, it is important to find a configuration of a black hole and particles that maximizes the entropy. We neglect the entropy of the gas of the particles, so the problem reduces to maximizing the entropy of the black hole piece. Phrased in this way, the microcanonical system in question does not have to be realized by a black hole alone. For example, one may consider a system with total charges that violate the extremality bound (2.10). Then there is no such system that contains only a black hole and no others, but it is still meaningful to ask what is the most entropic configuration of a black hole and particles.

In the rest of this subsection, we address this question while assuming existence of chiral and anti-chiral primaries in the theory.

For our purposes in this subsection, it is advantageous to use $E_R = E + J$, $E_L = E - J$, Q_R and Q_L as the four charges of a system, emphasizing chirality. Suppose that a system has total charges $E_{R,\text{tot}}$, $E_{L,\text{tot}}$, $Q_{R,\text{tot}}$ and $Q_{L,\text{tot}}$. These need to satisfy the unitarity bound for both holomorphic and antiholomorphic sectors:

$$E - J + \frac{1}{2}k_L \geq Q_L \ , \qquad E + J + \frac{1}{2}k_R \geq Q_R \ , \tag{2.36}$$

but the extremality (2.10) is not imposed.

The system consists of a black hole (bh) and gas of particles (gp), so $E_{R,\text{tot}} = E_{R,\text{bh}} + E_{R,\text{gp}}$ and similarly for the other charges. We assume that the entropy of the gas of particles is negligible to that of the black hole. Therefore, the entropy of the system is

$$S_{\text{bh}} = 2\pi\sqrt{\frac{1}{2}k_R E_{R,\text{bh}} - \frac{1}{4}Q_{R,\text{bh}}^2} + 2\pi\sqrt{\frac{1}{2}k_L E_{L,\text{bh}} - \frac{1}{4}Q_{L,\text{bh}}^2} \ . \tag{2.37}$$

Meanwhile, each chiral and anti-chiral primary carries charges $(\epsilon_R, \epsilon_L, q_R, q_L) \propto (1, 0, 1, 0)$ and $(\epsilon_R, \epsilon_L, q_R, q_L) \propto (0, 1, 0, 1)$, respectively. We use the proportionality sign because the macroscopic charges of the black hole scale differently from those of the microscopic particles. In fact, we expect the macroscopic charges to scale with the large central charge. It follows that $E_{R,\text{gp}} = Q_{R,\text{gp}}$ and $E_{L,\text{gp}} = Q_{L,\text{gp}}$. Thus, the contribution to the entropy (2.37) from the right (holomorphic) sector is

$$\begin{aligned} S_{R,\text{bh}} &= 2\pi\sqrt{\frac{1}{2}k_R E_{R,\text{bh}} - \frac{1}{4}Q_{R,\text{bh}}^2} \\ &= 2\pi\sqrt{\frac{1}{2}k_R (E_{R,\text{tot}} - E_{R,\text{gp}}) - \frac{1}{4}(Q_{R,\text{tot}} - E_{R,\text{gp}})^2} \\ &= \pi\sqrt{2k_L\left(E_{R,\text{tot}} - Q_{R,\text{tot}} + \frac{k_R}{2}\right) - (Q_{R,\text{tot}} - k_R - E_{R,\text{gp}})^2} \ . \end{aligned} \tag{2.38}$$

For the entropy to be maximized,

$$E_{R,\text{gp}} = \begin{cases} Q_{R,\text{tot}} - k_R \\ 0 \end{cases} \quad \Rightarrow \quad S_{R,\text{bh}} = \begin{cases} 2\pi\sqrt{\frac{1}{2}k_L\left(E_{R,\text{tot}} - Q_{R,\text{tot}} + \frac{k_R}{2}\right)} & (Q_{R,\text{tot}} > k_R) \\ 2\pi\sqrt{\frac{1}{2}k_R E_{R,\text{tot}} - \frac{1}{4}Q_{R,\text{tot}}^2} & (Q_{R,\text{tot}} \leq k_R) \end{cases}$$

The upshot is that if the total charges are such that they violate the stability bound $Q_{R,\text{tot}} > k_R$, the excess Q_R must be taken up by the particles so that the black hole sits at the threshold of the stability bound: $E_{R,\text{bh}} = k_R$. The same logic applies to the anti-holomorphic (left) sector independently.

We have only considered the chiral and anti-chiral primaries. However, the descendants do not play a role even if they are included. This is because emission of a descendant necessarily deprives the black hole of more energy (either left or right) than a primary with same Q_R or Q_L would, which necessarily results in the smaller entropy.

2.2 AdS$_5$ Black Holes

In this section we consider asymptotically AdS$_5$ black hole solutions and review its semi-classical properties.

Asymptotically AdS$_5$ black holes arise as solutions to type-IIB supergravity in AdS$_5 \times S^5$ [28–33]. They carry the mass, or energy E and two angular momenta $J_{1,2}$ for the isometry $SO(2,4)$ of AdS$_5$, and three charges $Q_{1,2,3}$ for the isometry $SO(6)$ of S^5. The black hole solution with all 5 charges independent is known. Similarly to the AdS$_3$ black holes, we consider the isometry of the transverse space S^5 as an internal symmetry for the AdS$_5$ black hole and therefore name the corresponding conserved quantities as charges.

The type-IIB theory in AdS$_5 \times S^5$ is known to be dual to the 4-dimensional maximally supersymmetric ($\mathcal{N} = 4$) Yang-Mills theory, in fact it is the original and most well understood case of the AdS/CFT correspondence [3]. The symmetry group of the $\mathcal{N} = 4$ SYM is the 4d $\mathcal{N} = 4$ superconformal group $PSU(2,2|4)$, whose maximal bosonic subalgebra is the 4d conformal group $SU(2,2) \sim SO(4,2)$ times the R-symmetry $SU(4)$. The three Cartans of the conformal group correspond to the mass and the two angular momenta of the black hole, while the three Cartans of the R-symmetry group correspond to the three charges of the black hole.

Thermodynamic quantities such as energy, temperature, charges, potentials and entropy and their relations are algebraically complicated. We shall present only some of those with a simplification by restricting to the case of equal charges $Q_1 = Q_2 = Q_3$, following [85]. The goal of this section is to illustrate the followings.

- Supersymmetric AdS$_5$ black holes are co-dimension 2 in the space of AdS$_5$ black holes.
- The two conditions for supersymmetry translates to vanishing temperature and a relation between the charges, that we shall refer to as the supersymmetric charge constraint.

- The formulae for the entropy of the black hole, as well as for the supersymmetric charge constraint, are known without restriction to $Q_1 = Q_2 = Q_3$.

2.2.1 Thermodynamics

As anticipated, we restrict to the AdS$_5$ black holes with equal charges $Q \equiv Q_1 = Q_2 = Q_3$. This does not qualitatively alter the main arguments of this section. It is algebraically convenient to express the mass and the three charges (two angular momenta J_1 and J_2, and one synchronized charges Q) of the AdS$_5$ black holes using four auxiliary parameters (r_+, q, a, b) as

$$
\begin{aligned}
E &= \frac{\pi}{4G_5}\frac{m(2(1-a^2)+2(1-b^2)-(1-a^2)(1-b^2))+2qab((1-a^2)+(1-b^2))}{(1-a^2)^2(1-b^2)^2} \ , \\
Q &= \frac{\pi}{4G_5}\frac{q}{(1-a^2)(1-b^2)} \ , \\
J_1 &= \frac{\pi}{4G_5}\frac{2ma+qb(1+a^2g^2)}{(1-a^2)^2(1-b^2)} \ , \\
J_2 &= \frac{\pi}{4G_5}\frac{2mb+qa(1+b^2g^2)}{(1-a^2)(1-b^2)^2} \ .
\end{aligned}
\tag{2.39}
$$

Here G_5 is five-dimensional Newton's gravitational constant, we have set $g = \ell_5^{-1} = 1$ where g is the coupling of gauged supergravity and ℓ_5 is the AdS$_5$ radius, and

$$
2m \quad = \quad \frac{(r_+^2+a^2)(r_+^2+b^2)(1+g^2r_+^2)+q^2+2abq}{r_+^2} \ . \tag{2.40}
$$

The entropy of the black hole can be expressed using the same parameters:

$$
S \quad = \quad 2\pi \cdot \frac{\pi}{4G_5}\frac{(r_+^2+a^2)(r_+^2+b^2)+abq}{(1-a^2)(1-b^2)r_+} \ . \tag{2.41}
$$

Implicitly via the auxiliary parameters, (2.39)-(2.41) yield the entropy as a function of the mass and the charges. This defines the microcanonical density of states, which contains all essential information about the black hole as a thermal system. From the entropy (2.41), the chemical potentials conjugate to the charges and the temperature conjugate to the energy can be derived using the first law of thermodynamics as in (2.5):

$$
TdS = dE - \Phi dQ - \Omega_1 dJ_1 - \Omega_2 dJ_2 \ . \tag{2.42}
$$

For example, the temperature can be derived as

$$T = \left(\frac{\partial S}{\partial E}\right)^{-1} = \frac{r_+^4[1+(2r_+^2+a^2+b^2)]-(ab+q)^2}{2\pi r_+[(r_+^2+a^2)(r_+^2+b^2)+abq]} \ . \tag{2.43}$$

2.2.2 Supersymmetry

The general AdS_5 black holes introduced in the last subsection have independent thermodynamic quantities (E, Q, J_1, J_2). Now we discuss the conditions that these black holes become supersymmetric.

Unitarity guarantees that their mass and the charges satisfy

$$E - (3Q + J_1 + J_2) \geq 0 \ , \tag{2.44}$$

where the coefficient 3 stands for the three redundant charges $Q \equiv Q_1 = Q_2 = Q_3$. The black hole is supersymmetric, i.e. it is $\frac{1}{16}$-BPS when its charges saturate this inequality:

$$E^* - 3Q^* - J_1^* - J_2^* = 0 \ . \tag{2.45}$$

We use the starred symbols (E^*, Q^*, J_1^*, J_2^*) instead of (E, Q, J_1, J_2) when we stress that the variables refer to the BPS case.

Collecting (2.39), the left hand side of (2.44) can be written in terms of the auxiliary parameters as

$$E - (3Q + J_1 + J_2) = \frac{\pi}{4G_5}\frac{3+(a+b)-ab}{(1-a)(1+a)^2(1-b)(1+b)^2}\left[m - q(1+a+b)\right] \ , \tag{2.46}$$

where m is a placeholder for the expression (2.40) in terms of r_+. The coefficient in front of the square bracket is always positive, so the BPS condition (2.45) is equivalent to the following relation between the BPS (starred) quantities:

$$q^* = \frac{m^*}{1+a+b} \ . \tag{2.47}$$

A very interesting result of [85] is that the factor in the square bracket of (2.46), when m is replaced by the corresponding expression in terms of r_+ via (2.40), can be reorganized as

$$m - q(1+a+b) = \frac{r_+^2(q-q^*)^2 + \left(((1+a+b)^2+r_+^2)(r_+^2-r^{*2}) - (1+a+b)(q-q^*)\right)^2}{2r_+^2\left((1+a+b)^2+r_+^2\right)} \ , \tag{2.48}$$

where

$$q^* = (a+b)(1+a)(1+b) \ , \qquad r^* \equiv r_+ = \sqrt{a+b+ab} \ . \tag{2.49}$$

Note that the right hand side of (2.48) is a sum of two squares. It thus amplifies the BPS condition (2.45), which is a single relation between real auxiliary parameters, into two relations $q = q^*$ and $r_+ = r^*$.

As a result of the two relations between the four auxiliary parameters that describe the AdS_5 black holes, the BPS black hole is parametrized by only two remaining auxiliary parameters (a, b). The energy and the charges (2.39) of BPS black holes are

$$\begin{aligned} E^* &= \frac{\pi}{4G_5}\frac{(3(a+b)-(a^3+b^3)-ab(a+b)^2)}{(1-a)^2(1-b)^2} \ , \\ Q^* &= \frac{\pi}{4G_5}\frac{a+b}{(1-a)(1-b)} \ , \\ J_1^* &= \frac{\pi}{4G_5}\frac{(a+b)(2a+b+ab)}{(1-a)^2(1-b)} \ , \\ J_2^* &= \frac{\pi}{4G_5}\frac{(a+b)(a+2b+ab)}{(1-a)(1-b)^2} \ . \end{aligned} \tag{2.50}$$

These expressions satisfy the BPS condition (2.45) for any (a, b), as they must.

We highlight two consequences for the BPS black holes.

First, the temperature of a BPS black hole vanishes. The temperature of the black hole (2.43) can be rewritten using the BPS values (2.49) of parameters q^* and r^* as

$$T = \frac{\left[1+3(a+b)+(a^2+b^2+3ab)\right](r_+^2-r^{*2})-(1+a+b)(q-q^*)}{\pi r^* q^*} \ . \tag{2.51}$$

This expression makes it clear that the temperature vanishes for BPS black holes, for which $q = q^*$ and $r_+ = r^*$.

Second, the charges Q^*, J_1^* and J_2^* of a BPS black hole obeys a constraint among themselves. This can be seen from the fact that the three charges (2.50) are parametrized by only two variables. For future reference, we present a more general charge constraint where $Q_{1,2,3}$ are not identified.

$$\begin{aligned} &\left(Q_1Q_2Q_3 + \frac{\pi}{4G_5}J_1J_2\right) \\ =\ &\left(Q_1+Q_2+Q_3+\frac{\pi}{4G_5}\right)\left(Q_1Q_2+Q_2Q_3+Q_3Q_1-\frac{\pi}{4G_5}(J_1+J_2)\right) \ . \end{aligned} \tag{2.52}$$

The relation between the three charges in (2.50) is obtained by simply setting $Q = Q_1 = Q_2 = Q_3$ in (2.52).

The BPS limit of AdS_5 black holes is in complete analogy with that of AdS_3 black holes introduced in the previous section. Supersymmetry requires saturation of a linear bound between the energy and the charges that derives from unitarity, namely (2.12) and (2.44). However, parametrization of the black hole solutions is non-linear in such a way that the saturation of the unitary bound translates to a vanishing sum of two squares, namely (2.20) and (2.48), when written in terms of black hole parameters. Therefore the supersymmetry of the black hole is amplified into two relations: vanishing of temperature and charge constraints (2.13) and (2.52).

The entropy of the BPS black hole can be also obtained by substituting the BPS values (2.49) for the auxiliary parameters in (2.41):

$$S^* \quad = \quad 2\pi \cdot \frac{\pi}{4G_5} \frac{a+b}{(1-a)(1-b)} \sqrt{a+b+ab} \; . \tag{2.53}$$

Ideally, we would like to have a formula for the entropy as a function of its charges, which was not feasible in (2.41) where we did so only implicitly via the auxiliary parameters. For the BPS black holes, this can be done, i.e. a and b in (2.53) can be replaced by the charges Q^*, J_1^* and J_2^* via (2.50). Note that there is not a unique way to do so, because the charges parametrize a and b redundantly, up to the relation (2.52). A particularly nice expression has been found in [86], which in fact applies to a more general BPS black holes whose $Q_{1,2,3}$ are not identified:

$$S^* \quad = \quad 2\pi \sqrt{Q_1^* Q_2^* + Q_2^* Q_3^* + Q_3^* Q_1^* - \frac{\pi}{4G_5} \left(J_1^* + J_2^*\right)} \; . \tag{2.54}$$

Part I

Entropy and Charges of Supersymmetric Black Holes

Chapter 3

Black Hole Entropy from the Index

In the first main part of this thesis that consists of this and the next chapter, we derive two important properties of the supersymmetric black holes in AdS space, namely the entropy and the charge constraint, from the dual conformal field theories.

As we reviewed in chapter 2 for AdS_3 and AdS_5 black holes, the supersymmetric AdS black holes have large entropy, and their charges cannot take arbitrary values but must obey one constraint. The asymptotically AdS black holes in gravity theories are known to be dual to ensembles of quantum states in superconformal field theories in one fewer dimensions [3]. Therefore, reproducing the properties of the black hole from the microstates in the field theory will shed light on understanding the gravity through quantum theories.

In this chapter, we address that the entropy of the black holes can be accounted for by degeneracy of quantum states in the field theory. We will first introduce the index, a powerful tool for enumerating quantum states in supersymmetric field theories. We will review some early attempts on using the index to count the black hole microstates, then demonstrate how the difficulties encountered were overcome recently. This chapter will conclude with a review of the entropy extremization principle, which derives the black hole entropy by treating the index as a partition function and performing Legendre transformation.

3.1 Early Attempts

Given the central position of the 5-dimensional type-IIB supergravity and the 4-dimensional maximally supersymmetric Yang-Mills theory in the AdS/CFT correspondence [3], it is not surprising that there have been many attempts to account for the entropy of the AdS_5 black holes from the gauge theory. In this section, we review some important developments [34–36] that have paved the way for the later progress.

3.1.1 The Superconformal Index

Perhaps the most important development towards microscopic accounting of the AdS black hole entropy is the introduction of the index, also referred to as the superconformal index [35, 36]. It can be understood as a special case of the grand canonical partition function for the Hilbert space of the theory, that has the remarkable property of being invariant under continuous deformations of the theory, thus allowing one to learn about the strongly coupled theory by studying the weakly coupled theory.

Consider the $\mathcal{N} = 4$ Super-Yang-Mills theory in 4 dimensions. Its symmetry group is the 4d $\mathcal{N} = 4$ superconformal group $PSU(2, 2|4)$, that consists of the 4d conformal group $SO(4, 2) \sim SU(2, 2)$, the R-symmetry group $SU(4)$, and fermionic generators that transform under both bosonic groups and complete the graded Lie group.

As we have mentioned in the context of AdS_5 black holes in section 2.2, we define the three Cartans of the conformal group as E, J_1 and J_2. E corresponds to the timelike part and therefore plays the role of energy, and $J_{1,2}$ corresponds to each factor of $SU(2)$ in the $SU(2)\times SU(2) \sim SO(4)$ Lorentz group. We also define the three Cartans of the R-symmetry group as $Q_{1,2,3}$, in such a way that they correspond to rotations within orthogonal 2-planes among 6-dimensional rotations $SO(6) \sim SU(4)$.

In group theoretic contexts, it is often useful to use the Dynkin basis instead of the orthogonal bases introduced in the previous paragraph. The Dynkin basis $(E, j_1, j_2, R_1, R_2, R_3)$ is linearly related to the orthogonal basis $(E, J_1, J_2, Q_1, Q_2, Q_3)$ above by

$$\begin{aligned} J_1 &= \frac{j + \bar{j}}{2} \ , & J_2 &= \frac{j - \bar{j}}{2} \ , & & \\ Q_1 &= R_2 + \frac{R_1 + R_3}{2} \ , & Q_2 &= \frac{R_1 + R_3}{2} \ , & Q_3 &= \frac{R_1 - R_3}{2} \ . \end{aligned} \tag{3.1}$$

The energy E is common to the two bases.

Every state in the Hilbert space, or every local operator of the theory, must be grouped into representations of the symmetry algebra. Therefore, it is always possible to find a basis of the states/operators that diagonalize all 6 Cartans of the symmetry algebra. Since we will always assume this diagonalization, we do not distinguish notations for the symmetry operators and for the corresponding eigenvalues. So every state/operator has definite values of six quantum numbers $(E, J_1, J_2, Q_1, Q_2, Q_3)$, or equivalently $(E, j_1, j_2, R_1, R_2, R_3)$ in the Dynkin basis, and it is possible to define the grand canonical partition function of the theory as the following trace over the Hilbert space:

$$Z(\beta,\, \Delta_I,\, \omega_i) \ \equiv \ \mathrm{Tr}\left[e^{-\beta E} e^{\Delta_I Q_I + \omega_i J_i}\right] \ . \tag{3.2}$$

The partition function is a function of 6 variables: β which is usually understood as the inverse temperature, and five chemical potentials $\Delta_{1,2,3}$ and $\omega_{1,2}$. The role of the chemical potentials, as well as the inverse temperature, is to weigh different states within the partition function. We shall often refer to the factors $e^{-\beta}$, e^{Δ_I} and e^{ω_i} as fugacities.

The supersymmetric AdS_5 black holes discussed in section 2.2.2 preserve $\frac{1}{16}$ of the supersymmetries, so are referred to as $\frac{1}{16}$-BPS. Similarly, we expect the dual microstates in the gauge theory to preserve the same amount of supersymmetries. There are 32 Hermitian supersymmetry generators in $PSU(2,2|4)$: 16 Poincaré supercharges Q^i_α, $\overline{Q}_{i\dot{\alpha}}$ and 16 conformal supercharges $S_{i\alpha}$, $\overline{S}^i_{\dot{\alpha}}$, that are Hermitian conjugates of the Poincaré supercharges in a radially quantized theory. $i = 1, 2, 3, 4$ is the fundamental or anti-fundamental index for the $SU(4)$ R-symmetry and α and $\dot{\alpha}$ are the doublet indices for the Lorentz group $SU(2)_L \times SU(2)_R \sim SO(4)$. We choose 2 of them, $Q \equiv Q^4_-$ and $S = Q^\dagger \equiv S_4^-$, as the preserved supercharges. An important commutation relation among the $PSU(2,2|4)$ algebra is

$$2\{Q, Q^\dagger\} = E - (Q_1 + Q_2 + Q_3 + J_1 + J_2) \ , \tag{3.3}$$

for our choice of the preserved supercharges.

For any state $|\psi\rangle$ of the SYM, the norm of $Q|\psi\rangle$ must be non-negative. It follows that the eigenvalues of the $\frac{1}{16}$-BPS states must obey

$$E \ \geq \ Q_1 + Q_2 + Q_3 + J_1 + J_2 \ . \tag{3.4}$$

Moreover, the $\frac{1}{16}$-BPS states $|\psi_{\mathrm{BPS}}\rangle$ of the SYM are annihilated by the chosen supercharge: $Q|\psi_{\mathrm{BPS}}\rangle = 0$. It follows that the eigenvalues of the $\frac{1}{16}$-BPS states saturate (3.4):

$$E \ = \ Q_1 + Q_2 + Q_3 + J_1 + J_2 \ . \tag{3.5}$$

We can adapt the grand canonical partition function (3.2) to the BPS states that satisfy (3.5). First rewrite (3.2) as

$$Z(\beta,\, \Delta_I,\, \omega_i) \ = \ \mathrm{Tr}\left[e^{-\beta(E-Q_1-Q_2-Q_3-J_1-J_2)}e^{\tilde{\Delta}_I Q_I + \tilde{\omega}_i J_i}\right] \ . \tag{3.6}$$

where $\tilde{\Delta}_I \equiv \Delta_I - \beta$ and $\tilde{\omega}_i \equiv \omega_i - \beta$. Then, take the limit $\beta \to \infty$ while keeping the redefined chemical potentials $\tilde{\Delta}_I$ and $\tilde{\omega}_i$ finite. As a result, non-BPS states that do not saturate (3.4) will be suppressed, and effectively the trace will sum only over the $\frac{1}{16}$-BPS states.

$$
\begin{aligned}
Z_{\rm BPS}(\tilde{\Delta}_I, \tilde{\omega}_i) &= \lim_{\beta\to\infty} {\rm Tr}\left[e^{-\beta(E-Q_1-Q_2-Q_3-J_1-J_2)} e^{\tilde{\Delta}_I Q_I + \tilde{\omega}_i J_i}\right] \\
&= {\rm Tr}_{\rm BPS}\left[e^{\tilde{\Delta}_I Q_I + \tilde{\omega}_i J_i}\right] \qquad (3.7)
\end{aligned}
$$

This BPS partition function depends only on 5 chemical potentials. Dependence on one fewer variables reflects that the $\frac{1}{16}$-BPS states have only 5 independent charges due to (3.5).

Now let us introduce another way to restrict the trace to the BPS states. Consider a generic state $|\psi\rangle$ that has certain eigenvalues E, Q_I and J_i. Since all states of the theory must organize into representations of the symmetry algebra, it follows that another state $Q|\psi\rangle$ must also exist in the Hilbert space. Our choice of Q is such that the eigenvalues of $Q|\psi\rangle$ is $E+\frac{1}{2}$, $Q_I+\frac{1}{2}$ and $J_i-\frac{1}{2}$. Note from (3.4) that if $|\psi\rangle$ does not saturate the BPS bound, nor does $Q|\psi\rangle$: the value of $E-Q_1-Q_2-Q_3-J_1-J_2$ is the same for both states. Due to the nilpotency of the fermionic operator Q, Q may be applied to $|\psi\rangle$ only once. All non-BPS states in the theory must appear in such a pair: $|\psi\rangle$ and $Q|\psi\rangle$, while the $\frac{1}{16}$-BPS states appears alone because $Q|\psi_{\rm BPS}\rangle = 0$. Therefore, if one tunes the chemical potentials in (3.6) such that

$$
e^{\frac{\tilde{\Delta}_1+\tilde{\Delta}_2+\tilde{\Delta}_3-\tilde{\omega}_1-\tilde{\omega}_2}{2}} = -1 \ , \qquad (3.8)
$$

then contributions to (3.6) from $|\psi\rangle$ and from $Q|\psi\rangle$ exactly cancel each other, so (3.6) receives contributions only from $\frac{1}{16}$-BPS states and the β-dependence automatically vanishes. The grand canonical partition function defined as such is the index, also known as the superconformal index to emphasize that this is an adaptation of the Witten index [87] to the superconformal field theory.

$$
\begin{aligned}
\mathcal{I}(\tilde{\Delta}_I, \tilde{\omega}_i) &= {\rm Tr}\left[e^{-\beta(E-Q_1-Q_2-Q_3-J_1-J_2)} e^{\tilde{\Delta}_I Q_I + \tilde{\omega}_i J_i}\right] \\
&= {\rm Tr}_{\rm BPS}\left[e^{\tilde{\Delta}_I Q_I + \tilde{\omega}_i J_i}\right] \ , \qquad \text{where} \quad e^{\frac{\tilde{\Delta}_1+\tilde{\Delta}_2+\tilde{\Delta}_3-\tilde{\omega}_1-\tilde{\omega}_2}{2}} = -1 \ . \qquad (3.9)
\end{aligned}
$$

It is important to notice that, due to the condition (3.8), the index is a function of only 4 independent chemical potentials, one fewer than the partition function (3.7) with a similar definition, so the index contains less information than the partition function. For example, suppose there are two BPS states in the Hilbert space whose charges Q_I differ by $\frac{n}{2}$ and J_i by $-\frac{n}{2}$, where n is an integer. If n is even, the index will not distinguish contributions from both states and only tell us that there are two states along a line in the 5-dimensional charge space. If n is odd, contributions from both states will cancel each other and the index will not notice these states in any way even though they are BPS states.

Despite this loss of information, the index has a remarkable advantage over the partition function, in that it is coupling independent. A superconformal field theory may receive continuous deformations due to interactions, and as a result the spectrum of the Hilbert space is shifted. In this process, it is possible that a BPS state is lifted by an anomalous dimension and become non-BPS. The grand canonical partition function (3.7) will change under this process, as a state suddenly drops out of the range of summation. However, such a process may only happen in a limited manner. Since all states in the Hilbert space must organize into representations of $PSU(2,2|4)$ in any case, such transitions between BPS and non-BPS states may occur only if there is a set of representations that contain a BPS state that is continuously isomorphic to a set of representations that do not. Such limited relations between sets of representations are called recombination rules, see [88] for an extensive list of them.

The index (3.9), unlike the partition function (3.7), is invariant under the recombinations. If a BPS state may be lifted, i.e. it is not protected under recombination rules, then it must be that there is another BPS state whose charges differ from the former as in the example of the previous paragraph with odd n, and that they must participate in the recombination rule together. This is to guarantee that a lift from the BPS state $|\psi\rangle$ has its superpartner $Q|\psi\rangle$ somewhere in the Hilbert space. The index had not depended on these two BPS states even though they were BPS states, so it does not change under the recombination process. From this, it is clear that the loss of information for the index is precisely what gives it the powerful property of coupling independence.

Historically, the minus sign on the right hand side of (3.8) has been replaced with an explicit factor $(-1)^F$ where F is a fermion number operator. That is, the index was originally defined as

$$\mathcal{I}(\tilde{\Delta}_I, \tilde{\omega}_i) \;=\; \mathrm{Tr}\left[(-1)^F e^{\tilde{\Delta}_I Q_I + \tilde{\omega}_i J_i}\right] \;, \qquad \text{where} \quad e^{\frac{\tilde{\Delta}_1+\tilde{\Delta}_2+\tilde{\Delta}_3-\tilde{\omega}_1-\tilde{\omega}_2}{2}} = 1 \;. \quad (3.10)$$

The two definitions are ultimately equivalent because the $(-1)^F$ factor can be replaced by $e^{2\pi i J_1}$ among many other possible choices [64, 89], effectively shifting $\tilde{\omega}_1$ by $2\pi i$ and flipping the sign on the right hand side of (3.8). However, the modern definition (3.9) is more suggestive that the chemical potentials may be complex numbers, which has been the key to the recent advance as will be reviewed later in this chapter.

3.1.2 The Index as a Matrix Integral

Taking advantage of the coupling independence of the index, attempts have been made to count the number of microstates of the weakly coupled $\mathcal{N}=4$ SYM to account for the AdS$_5$ black hole entropy, which is dual to the strongly coupled $\mathcal{N}=4$ SYM with a large-N gauge group $SU(N)$. We review an early attempt of [35] where the index has been computed using a unitary matrix model.

The free $\mathcal{N}=4$ SYM consists of six real scalars, eight fermions and a gauge field. Each of them transforms in the adjoint representation of the gauge group $SU(N)$:

$$\begin{aligned} \text{vector} &: \; A_\mu \sim A_{\alpha\dot\beta} \; , && (\mu = 1,2,3,4 \; , \; \alpha = \pm \; , \; \dot\beta = \dot\pm) \\ \text{scalar} &: \; \Phi_{ij}(=-\Phi_{ji}) \; , \; \overline{\Phi}^{ij} \sim \tfrac{1}{2}\epsilon^{ijkl}\Phi_{kl} \; , && (i,j,k,l = 1,2,3,4) \\ \text{fermion} &: \; \Psi_{i\alpha} \; , \; \overline{\Psi}^i_{\dot\alpha} \; . && \end{aligned} \tag{3.11}$$

As explained above (3.3), $\alpha, \dot\alpha$ are the doublet indices for the Lorentz group, μ is the vector index, superscripts i,j are for the fundamental representation of the $SU(4)$ R-symmetry, while the subscripts are for the anti-fundamental representation. Of these, 3 scalars $\bar\phi^m = \overline{\Phi}^{4m}$ where $m = 1,2,3$, 3 chiralini $\psi_{m+} = -i\Psi_{m+}$, 2 gaugini $\bar\lambda_{\dot\alpha} = \overline{\Psi}^4_{\dot\alpha}$ and the gauge field $f_{++} = (\sigma^{\mu\nu}{}_{++})F_{\mu\nu}$ are $\frac{1}{16}$-BPS at the free, i.e. zero-loop $\mathcal{O}(g^0_{\rm YM})$, level. The $\frac{1}{16}$-BPS states are those whose charges satisfy (3.5).

Each field also has a spacetime argument, and a field localized at different points in the spacetime are considered separate degrees of freedom. Equivalently, a field and its derivatives at the origin are separate degrees of freedom. Therefore, any numbers of 4 derivatives $\partial_{\alpha\dot\alpha}$ in the 4d spacetime may act on each field. Of these, only 2 derivatives $\partial_{+\dot\alpha}$ preserve the $\frac{1}{16}$-BPSness, i.e. commute with the preserved supercharge Q.

Finally, the free fields obey equations of motion. Of these, only one equation of motion concerning gaugini:

$$\partial_{+\dot\alpha}\bar\lambda^{\dot\alpha} = 0 \;\Leftrightarrow\; \partial_{+[\dot\alpha}\bar\lambda_{\dot\beta]} = 0 \; , \tag{3.12}$$

is $\frac{1}{16}$-BPS.

These define the $\frac{1}{16}$-BPS letters: the 9 free fields and any numbers of 2 derivatives acting on them, modulo the gaugino equation of motion.

The free fields belong to a single super-representation of $PSU(2,2|4)$, known as the free vector multiplet. It is a representation that has 6 real scalars as its superconformal primaries, and consists of descendants that can be obtained by acting the $PSU(2,2|4)$ generators on the primary. The descendants include other free fields, by action of supercharges, and their derivatives, by action of momentum operators, modulo the equation of motions which

	Bosonic Rep.	E	j	$\bar{j}$	R_1	R_2	R_3	J_1	J_2	Q_1	Q_2	Q_3
Free fields	$[0;0]_1^{[0,1,0]}$	1	0	0	0	1	0	0	0	1	0	0
		1	0	0	1	-1	1	0	0	0	1	0
		1	0	0	1	0	-1	0	0	0	0	1
	$[1;0]_{\frac{3}{2}}^{[0,0,1]}$	$\frac{3}{2}$	1	0	0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
		$\frac{3}{2}$	1	0	0	1	-1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
		$\frac{3}{2}$	1	0	1	-1	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	$[0;1]_{\frac{3}{2}}^{[1,0,0]}$	$\frac{3}{2}$	0	1	1	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
		$\frac{3}{2}$	0	-1	1	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	$[2;0]_2^{[0,0,0]}$	2	2	0	0	0	0	1	1	0	0	0
Eq. of motion	$[1;0]_{\frac{5}{2}}^{[1,0,0]}$	$\frac{5}{2}$	1	0	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
Derivatives	$[1;1]_1^{[0,0,0]}$	1	1	1	0	0	0	1	0	0	0	0
		1	1	-1	0	0	0	0	1	0	0	0

Table 3.1: Components of the BPS operators in the free vector multiplet $B_1\bar{B}_1[0;0]_1^{[0,1,0]}$. The first 9 rows are free fields, followed by the equation of motion and 2 derivatives.

is derived from the symmetry algebra. This representation has the name $B_1\overline{B}_1[0;0]_1^{[0,1,0]}$ following notation of [88], to which we refer for extensive information on representations of superconformal algebras. $[0;0]$ indicates that the superconformal primary is a singlet under the Lorentz group, $[0,1,0]$ indicates that it is a representation **6** of the R-symmetry group $SU(4) \sim SO(6)$, and the subscript 1 indicates its conformal dimension. The letters and the subscripts B_1 and $\overline{B}_1$ indicate the structure of the representation. We summarize the $\frac{1}{16}$-BPS contents of the representation $B_1\overline{B}_1[0;0]_1^{[0,1,0]}$, as well as their charges in both bases, in Table 3.1. In the Table, the charges of each entry were displayed in two bases: the Dynkin basis $(j, \bar{j}, R_1, R_2, R_3)$ and the orthogonal basis $(J_1, J_2, Q_1, Q_2, Q_3)$. They are related by

$$\begin{aligned} J_1 &= \frac{j+\bar{j}}{2} , & J_2 &= \frac{j-\bar{j}}{2} , \\ Q_1 &= R_2 + \frac{R_1+R_3}{2} , & Q_2 &= \frac{R_1+R_3}{2} , & Q_3 &= \frac{R_1-R_3}{2} , \end{aligned} \tag{3.13}$$

and E is common in both bases.

The BPS letters freely generate the Fock space. That is, a product of arbitrary numbers of each bosonic BPS letter, times a product of either 0 or 1 of each fermionic BPS letter is a BPS operator included in the Fock space. Note that there are infinite numbers of BPS letters: corresponding to each free field, any of its derivatives is a new BPS letter. Furthermore, each field transforms as an adjoint representation of the gauge group $SU(N)$, so a field should be understood as a set of N^2-1 independent degrees of freedom each with its own gauge charges. Finally, only the gauge singlets are considered the physical degrees of freedom. Therefore, the Hilbert space of the $\mathcal{N}=4$ SYM is a projection onto gauge singlets of the BPS Fock space.

Now, let us translate the structure of the Hilbert space into the index (3.10) defined in the previous subsection.

Suppose there is a bosonic BPS letter. Let its contribution to the trace be $x_B \equiv e^{\frac{\tilde{\Delta}_1+\tilde{\Delta}_2+\tilde{\Delta}_3-\tilde{\omega}_1-\tilde{\omega}_2}{2}}$. The contribution from an arbitrary number of the letter is

$$\begin{aligned} 1+x_B+x_B^2+\cdots &= \frac{1}{1-x_B} \\ &= \exp[-\log(1-x_B)] \\ &= \exp\left[\sum_{n=1}^{\infty}\frac{x_B^n}{n}\right] \\ &\equiv \mathrm{PE}[x_B]\ . \end{aligned} \tag{3.14}$$

In the last line we have defined the Plethystic exponential. Instead, suppose there is a fermionic BPS letter, and let its contribution to the trace be x_F. Here we pull out the $(-1)^F$ factor explicitly, using the definition of the index as in (3.10). Therefore, the contribution from all allowed number of the letter is

$$\begin{aligned} 1-x_F &= \exp[\log(1-x_F)] \\ &= \exp\left[-\sum_{n=1}^{\infty}\frac{x_F^n}{n}\right] \\ &\equiv \mathrm{PE}[-x_F]\ . \end{aligned} \tag{3.15}$$

Note that the plethystic exponential follows the rule of ordinary exponentials: $\mathrm{PE}[-x_F] = (\mathrm{PE}[x_F])^{-1}$ and $\mathrm{PE}[x_1+x_2] = \mathrm{PE}[x_1]\times\mathrm{PE}[x_2]$. Therefore, the index over the BPS Fock space will be a Plethystic exponential of the sum over all bosonic BPS letters minus the sum over all fermionic BPS letters:

$$\mathcal{I}_{\text{Fock}} = \mathrm{PE}\left[\sum_{\text{bosonic letters}} x_B - \sum_{\text{fermionic letters}} x_F\right] \tag{3.16}$$

Let us temporarily introduce the gauge fugacities $e^{i\alpha_a}$ $(a=1,\cdots,N)$ conjugate to the gauge charges ζ_a into the index, that we will shortly project away. Furthermore, let us replace the chemical potentials $e^{\tilde{\Delta}_I}$ and $e^{\tilde{\omega}_i}$ in favor of the fugacities

$$x^2 = e^{\tilde{\Delta}_1}\ , \quad y^2 = e^{\tilde{\Delta}_2}, \quad z^2 = e^{\tilde{\Delta}_3}\ , \quad p^2 = e^{\tilde{\omega}_1}\ , \quad q^2 = e^{\tilde{\omega}_2}\ . \tag{3.17}$$

So we define the index over the Fock space as

$$\mathcal{I}_{\text{Fock}}(x,y,z,p,q;\zeta_a) = \text{Tr}_{\text{Fock}}\left[(-1)^F x^{2Q_1} y^{2Q_2} z^{2Q_3} p^{2J_1} q^{2J_2} \prod_a e^{i\alpha_a \zeta_a}\right] , \qquad (3.18)$$

where $\frac{xyz}{pq} = 1$. From Table 3.1, we can read off the appropriate factors that correspond to x_B or to x_F of the BPS letters. Also including the gauge factor, we have

$$\mathcal{I}_{\text{Fock}}(x,y,z,p,q;\zeta_a) = \text{PE}\left[f(x,y,z,p,q) \cdot \chi^{adj.}(\alpha_a)\right] , \qquad (3.19)$$

where

$$\begin{aligned} f(x,y,z,p,q) &= \frac{x^2+y^2+z^2 - xyzpq\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{p^2}+\frac{1}{q^2}-1\right)+p^2q^2}{(1-p^2)(1-q^2)} \\ &= 1 - \frac{(1-x^2)(1-y^2)(1-z^2)}{(1-p^2)(1-q^2)} \end{aligned} \qquad (3.20)$$

is the single particle index, and

$$\chi^{adj.}(\alpha_a) = \sum_{a,b=1}^{N} e^{i(\alpha_a - \alpha_b)} \qquad (3.21)$$

is the character of the $U(N)$ adjoint representation. Note the role of the equation of motion and the derivatives in (3.20): the -1 inside the parenthesis and the geometric series of p^2 and q^2. For the second line of (3.20), we used $xyz = pq$.

Finally, the index is a projection of (3.19) onto gauge singlets. The projection can be done by integrating over the gauge fugacities with the Haar measure:

$$\mathcal{I}(x,y,z,p,q) = \oint d\mu[\alpha_a]\, \mathcal{I}_{\text{Fock}}(x,y,z,p,q;\alpha_a) , \qquad (3.22)$$

where

$$\begin{aligned} \oint d\mu[\alpha_a] &= \frac{1}{N!} \cdot \int_0^{2\pi} \prod_{a=1}^{N} \frac{d\alpha_a}{2\pi} \cdot \prod_{a,b=1}^{N} \left(1 - e^{i(\alpha_a-\alpha_b)}\right) \\ &= \frac{1}{N!} \cdot \int_0^{2\pi} \prod_{a=1}^{N} \frac{d\alpha_a}{2\pi} \cdot \text{PE}\left[-\sum_{a,b=1}^{N} e^{i(\alpha_a-\alpha_b)}\right] \end{aligned} \qquad (3.23)$$

Collecting (3.19)-(3.23), one obtains the following matrix integral formula for the index of the $\mathcal{N}=4$ SYM:

$$\begin{aligned}\mathcal{I}(x,y,z,p,q) &= \frac{1}{N!}\int_0^{2\pi}\prod_{a=1}^{N}\frac{d\alpha_a}{2\pi}\cdot \mathrm{PE}\left[-(1-f(x,y,z,p,q))\cdot\sum_{a,b=1}^{N}e^{i(\alpha_a-\alpha_b)}\right] , \qquad (3.24)\\ &= \frac{1}{N!}\int_0^{2\pi}\prod_{a=1}^{N}\frac{d\alpha_a}{2\pi}\cdot \exp\left[-\sum_{n=1}^{\infty}\sum_{a,b=1}^{N}\frac{1}{n}\left(1-f(x^n,y^n,z^n,p^n,q^n)\right)\cdot e^{in(\alpha_a-\alpha_b)}\right] .\end{aligned}$$

(3.24) is an integral over N angles, but we take N to be very large because the validity of supergravity approximation of the string theory, i.e. the small string length scale limit, is related to the large-N limit via the AdS/CFT correspondence. Since the large number of integration variables α_a appear symmetrically in the integrand, it is useful to think of a distribution of N variables within the range $[0,2\pi)$ instead of their individual values. Let $\rho(\alpha)$ be the distribution function of the angles α_a, normalized such that $\int_0^{2\pi} d\theta\rho(\alpha) = 1$. (3.24) becomes a functional integral over the distribution function:

$$\mathcal{I}(x,y,z,p,q) = \frac{1}{N!}\int [d\rho] e^{-S[\rho(\alpha)]} , \qquad (3.25)$$

where $S[\rho(\alpha)]$ can be thought of as an effective action functional:

$$\begin{aligned}S[\rho(\alpha)] &= N^2\sum_{n=1}^{\infty}\int_0^{2\pi} d\alpha_1 d\alpha_2 \rho(\alpha_1)\rho(\alpha_2)e^{in(\alpha_1-\alpha_2)}\cdot\frac{1}{n}\left(1-f(x^n,y^n,z^n,p^n,q^n)\right)\\ &= N^2\sum_{n=1}^{\infty}\frac{1}{n}\left(1-f(x^n,y^n,z^n,p^n,q^n)\right)\cdot|\rho_n|^2 , \qquad (3.26)\end{aligned}$$

where

$$\rho_n \equiv \int_0^{2\pi} d\alpha\rho(\alpha)e^{in\alpha} , \qquad (3.27)$$

is a Fourier coefficient of the distribution function $\rho(\alpha)$.

Since $\rho(\alpha)$ has been normalized so that it is independent of N, $S[\rho(\alpha)]$ is proportional to N^2. Thus, as $N\to\infty$, the functional integral (3.25) will be strongly dominated by a function $\rho(\alpha)$ that minimizes the effective action $S[\rho(\alpha)]$.

Meanwhile, note from (3.20) that

$$1-f(x^n,y^n,z^n,p^n,q^n) = \frac{(1-x^{2n})(1-y^{2n})(1-z^{2n})}{(1-p^{2n})(1-q^{2n})} > 0 , \qquad (3.28)$$

as long as $0<x,y,z,p,q<1$. The assumption that each fugacity is smaller than 1 is natural, since otherwise the index defined as a trace over infinite dimensional Hilbert space would be divergent. For example, suppose $x > 1$. Among the bosonic BPS letters (3.20), there is one that contributes x^2, then the contribution from arbitrary numbers of this letter would be divergent, $x^2 + x^4 + x^6 + \cdots$.

Therefore, the minimum of the action $S[\rho(\alpha)]$ corresponds to all Fourier coefficients vanishing: $\rho_1 = \cdots = 0$. This indicates a constant function $\rho(\alpha)$, or the uniform distribution of the eigenvalues α_a, also known as the confined phase [90–92]. Importantly, $S[\rho(\alpha)] = 0$ for such a distribution, and therefore the index scales as

$$\mathcal{I}(x, y, z, p, q) \;=\; e^{\mathcal{O}(N^0)} \; . \tag{3.29}$$

Recall from (2.50) that the charges Q_I and J_i of the AdS_5 black holes all scale as $\frac{\pi}{4G_5}$, and therefore the entropy (2.54) also scales as such $\frac{\pi}{4G_5}$. This factor is related to the rank of the gauge group N via the AdS/CFT correspondence:

$$\frac{\pi}{4G_5} = \frac{N^2}{2} \; . \tag{3.30}$$

Therefore, it is expected for a successful microscopic accounting of the black hole entropy, that the index scales as $e^{\mathcal{O}(N^2)}$. In this sense (3.29) does not account for the black hole entropy.

[35] went further to evaluate the right hand side of (3.29), and showed that the result corresponds to a gas of supergravitons. The gas of supergravitons will be discussed in some detail in section 5.4. However, both here and there, the important point is that the gas of supergravitons do not exhibit a large enough degeneracy to account for the microscopics of the black hole entropy, and our goal is to find contributions other than the supergravitons.

As noted in [35], it is not a contradiction that the index (3.29) does not capture the black hole entropy. The index counts the BPS states only up to cancellations between bosons and fermions whose charges differ in certain direction. For this reason, there have been various efforts to study the BPS operators themselves — not just counting them — that are counted by the index, for example [37, 39–41], but those have not been fully successful either. The last part of this thesis addresses progress in this direction.

3.2 Amplifying the Index by Complex Chemical Potentials

The AdS black hole entropy has finally been accounted for by the number of microstates in the gauge theories only very recently. Initiated by successes in magnetically charged black holes in $\text{AdS}_4 \times S^7$ from the topologically twisted index and supersymmetric localization [42–46] and hinted by [47], a major success for the AdS_5 black holes has been made in [48–50], followed by many contributions for AdS black holes in various dimensions. See [51–69] among many others.

Although different methods have been explored, the key difference that distinguishes the modern approach from the early attempts introduced in section 3.1 was to let the chemical potentials take complex values. It turns out that, by attributing appropriate complex values to the chemical potentials in (3.9) while still respecting the condition $e^{\frac{\tilde{\Delta}_1+\tilde{\Delta}_2+\tilde{\Delta}_3-\tilde{\omega}_1-\tilde{\omega}_2}{2}} = -1$ for the index, it is possible to reduce the effect of cancellations between bosonic and fermionic pairs dramatically, so much that the index scales as $e^{\mathcal{O}(N^2)}$. It is not just the scaling of the index, but its actual value in some subleading orders in various limits and approximations that were matched with the black hole entropy, for example [60, 67, 68].

In this section, we illustrate how a complex fugacity can 'amplify' the index by reducing the effect of cancellations. We consider the index for an abelian theory with $U(1)$ gauge group, and unrefine the fugacities as far as possible so the index becomes a function of a single variable. Coefficients of the series expansion of the index oscillate, in a pattern that has essentially been observed in [62]. We show that they can be made to add up constructively by giving the fugacity a resonating phase.

For this section, we take the following simplified definition of the index:

$$\mathcal{I}(x) \quad = \quad \text{Tr}\left[(-1)^F x^{2Q_1+2Q_2+2Q_3+3J_1+3J_2}\right] \;\; \equiv \;\; \text{Tr}\left[(-1)^F x^{\mathcal{J}}\right] \tag{3.31}$$

where we have defined the 'overall' charge

$$\mathcal{J} \equiv 2Q_1 + 2Q_2 + 2Q_3 + 3J_1 + 3J_2 \ , \tag{3.32}$$

for the last equality. (3.31) is nothing more than the index (3.10) where we have taken $x^2 = e^{\Delta_1} = e^{\Delta_2} = e^{\Delta_3}$ and $x^3 = e^{\omega_1} = e^{\omega_2}$. Note that this is compatible with the condition $e^{\frac{\tilde{\Delta}_1+\tilde{\Delta}_2+\tilde{\Delta}_3-\tilde{\omega}_1-\tilde{\omega}_2}{2}} = 1$.

One can compute the index on a computer using the formula (3.24), except that there is no matrix integral since we take the $U(1)$ gauge group, so simply

$$\mathcal{I}(x) \;=\; \exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}\left(1-f(x^n)\right)\right] \;,$$

where

$$\begin{aligned} f(x) &= \frac{3x^2-2x^3-3x^4+2x^6}{(1-x^3)^2} \\ &= 1-\frac{(1-x^2)^3}{(1-x^3)^2} \;. \end{aligned} \tag{3.33}$$

For generic N, even for moderate values of N, the matrix integral for projecting out the gauge singlets is the bottleneck for computing the index as a series expansion [61, 62]. So for $N = 1$, the computation simplifies greatly and one can series expand $\mathcal{I}(x)$ until very high orders of x. For discussions in this section, we have computed until x^{10000}. Also, as can be anticipated from (3.33), the series expansion is regular: there is no fractional powers of x. The fractional powers of x and their analogues in various setups have confused many researchers. For example, in [64] the authors used our x^3 as their x, and as a result their $\log x$ acquired a period $6\pi i$, causing confusions with branch cuts.

Let us denote the coefficients of the series expansion by $\Omega_{\mathcal{J}}$:

$$\mathcal{I}(x) \;=\; \sum_{j=0}^{\infty}\Omega_{\mathcal{J}}x^{\mathcal{J}} \;. \tag{3.34}$$

All $\Omega_{\mathcal{J}}$ are integers, not necessarily non-negative because fermions contribute negatively. We plot the growth of $|\Omega_{\mathcal{J}}|$ in the left panel of Figure 3.1, in logarithmic scale. It is concave down in the logarithmic scale, indicating that the growth of $|\Omega_{\mathcal{J}}|$ with $\mathcal{J}$ is sub-exponential. As a result, for any given $0 < x < 1$, $|\Omega_{\mathcal{J}}\, x^{\mathcal{J}}|$ initially grows with $\mathcal{J}$ but after a certain point where the slope of $\log\Omega_{\mathcal{J}}$ compensates the negative $\log x^{\mathcal{J}} = -\mathcal{J}\log(1/x)$, it starts to attenuate. Therefore, for each $0 < x < 1$ there exists some value $\mathcal{J}_{\max}(x)$ that maximizes $|\Omega_{\mathcal{J}}\, x^{\mathcal{J}}|$. This value of $\mathcal{J}$, as a function of x, is plotted in the right panel of Figure 3.1, again in logarithmic scale. For $x \lesssim 0.79$, $\Omega_2\, x^2 = 3x^2$ is the largest contribution to the index. Such small values of x yield non-generic situations and are uninformative. For higher values of x, $\mathcal{J}_{\max}(x)$ grows super-exponentially, and for $x \gtrsim 0.96$, $\mathcal{J}_{\max}(x)$ is larger than 10^4, so it is not captured in Figure 3.1.

Now let us take $x = 0.95$. As one can read from Figure 3.1, $|\Omega_{\mathcal{J}}\, 0.95^{\mathcal{J}}|$ is maximized around $\mathcal{J} \approx 4500$. We plot $\Omega_{\mathcal{J}}\, 0.95^{\mathcal{J}}$, not its magnitude but with sign, in Figure 3.2. At its maximum which is around $\mathcal{J} \approx 4500$, the magnitude $|\Omega_{\mathcal{J}}\, 0.95^{\mathcal{J}}| \approx 2\times 10^{44}$. However, if one

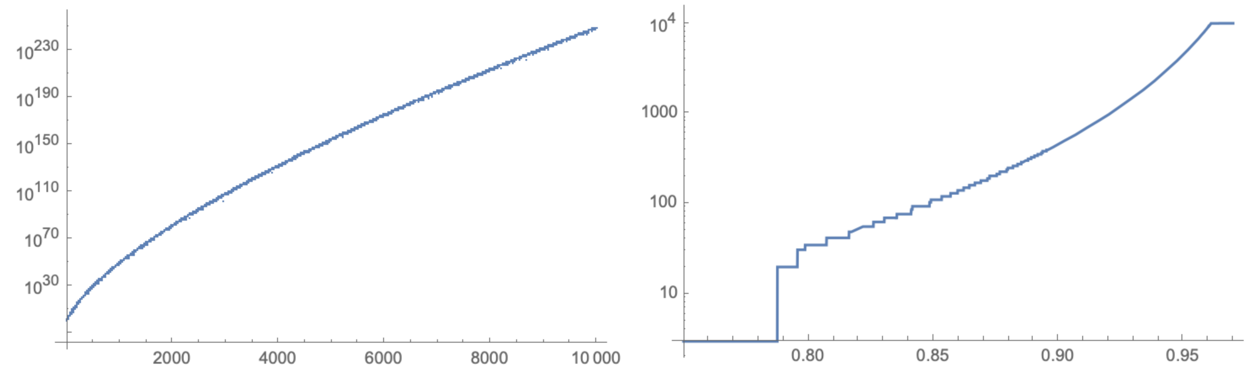

Figure 3.1: Index coefficients $|\Omega_{\mathcal{J}}|$ for each $0 \leq \mathcal{J} \leq 10000$ (left), $\mathcal{J}_{\max}(x)$ that maximizes $|\Omega_{\mathcal{J}}\, x^{\mathcal{J}}|$ for given $0.75 < x < 0.97$ (right). Both drawn in log scale.

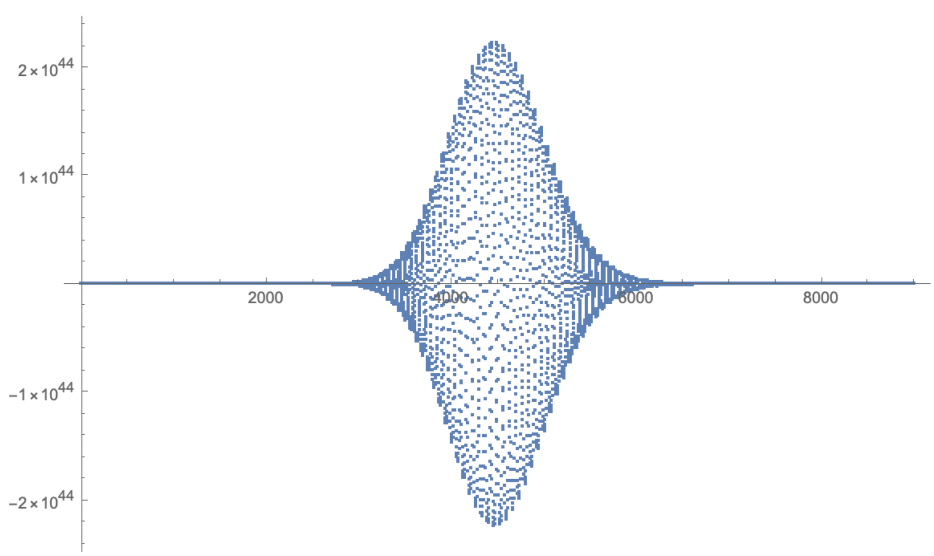

Figure 3.2: $\Omega_{\mathcal{J}}\, x^{\mathcal{J}}$ where $x = 0.95$, for each $\mathcal{J}$, in linear scale.

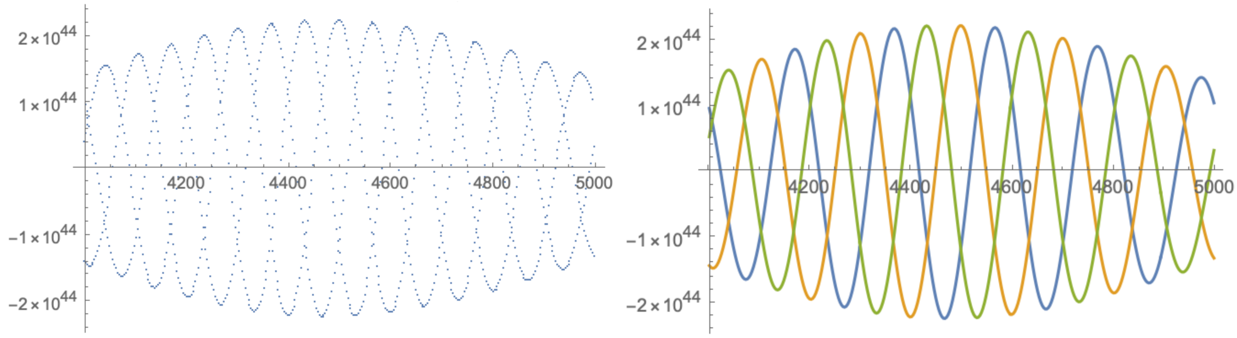

Figure 3.3: Figure 3.2 zoomed into $4000 < \mathcal{J} < 5000$ (left), the same figure color-coded according to $\mathcal{J}$ mod 3 (right).

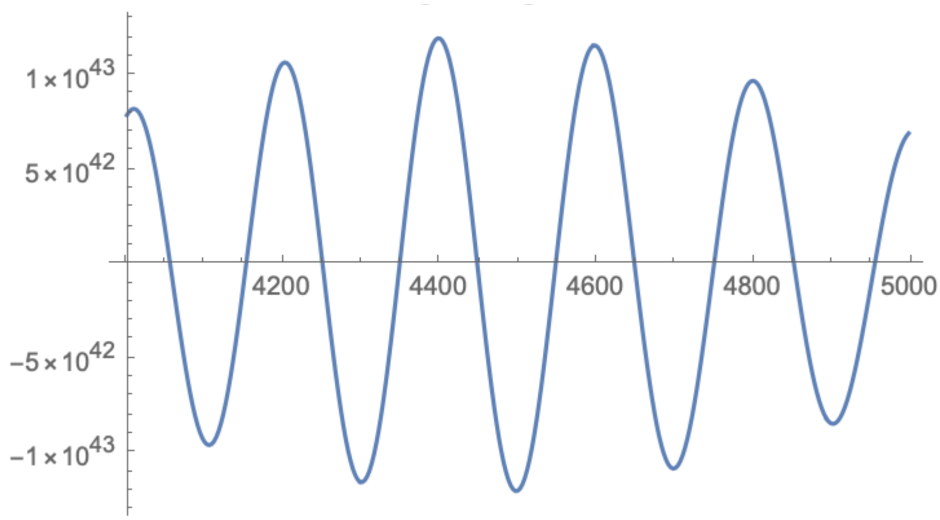

Figure 3.4: $\Omega_{\mathcal{J}}\, x^{\mathcal{J}} + \Omega_{\mathcal{J}+1}\, x^{\mathcal{J}+1} + \Omega_{\mathcal{J}+2}\, x^{\mathcal{J}+2}$ where $x = 0.95$, for each $\mathcal{J}$.

looks at the individual coefficients around $\mathcal{J} \approx 4500$, the sign and even the magnitude of each $\Omega_{\mathcal{J}}\, 0.95^{\mathcal{J}}$ oscillates wildly. If one sums over the contributions from different $\mathcal{J}$ simply by $\sum_{\mathcal{J}} \Omega_{\mathcal{J}}\, 0.95^{\mathcal{J}}$ to compute the index as a single number, there will be massive cancellations between adjacent terms.

However, if one zooms in closely into around $\mathcal{J} \approx 4500$, a clear pattern of oscillation arises, see the left panel of Figure 3.3. The pattern is such that three sine waves are superimposed, equally spaced between each other. The right panel of Figure 3.3 is designed to illustrate this pattern. It is the same plot as the left, but dots corresponding to different values of $\mathcal{J}$ mod 3 are color-coded respectively. Clearly, the values of $\Omega_{\mathcal{J}}\, 0.95^{\mathcal{J}}$ for same values of $\mathcal{J}$ mod 3 form a sine wave.

This pattern of oscillation has a direct implication. For any $\mathcal{J}$, consider the sum of the

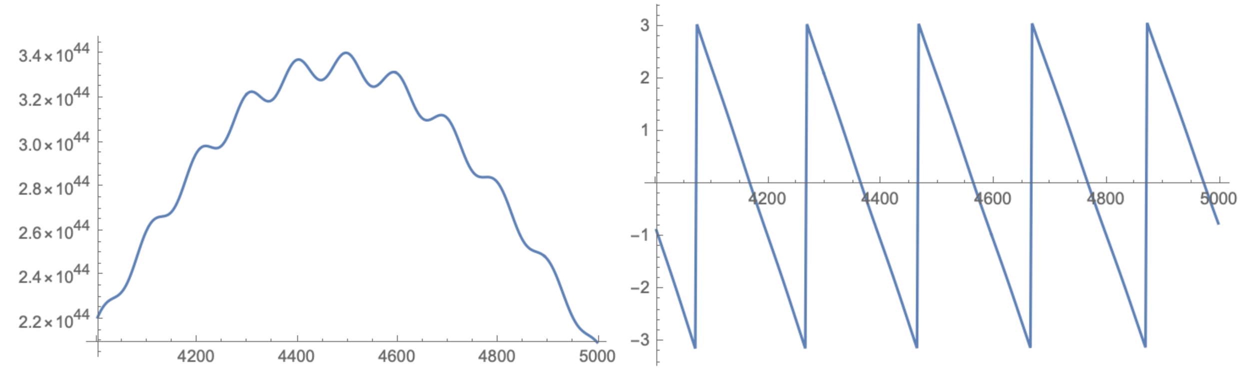

Figure 3.5: $\Omega_{\mathcal{J}}\, x^{\mathcal{J}} + \Omega_{\mathcal{J}+1}\, x^{\mathcal{J}+1} + \Omega_{\mathcal{J}+2}\, x^{\mathcal{J}+2}$ where $x = 0.95 \cdot e^{i\left(\frac{2\pi}{3}\right)}$, for each $\mathcal{J}$. The magnitude (left) and the phase (right).

three adjacent contributions to the index: $\Omega_{\mathcal{J}}\, 0.95^{\mathcal{J}} + \Omega_{\mathcal{J}+1}\, 0.95^{\mathcal{J}+1} + \Omega_{\mathcal{J}+2}\, 0.95^{\mathcal{J}+2}$. The three terms are basically the three points of an equilateral triangle centered at the origin in the complex plane, then projected onto one axis. Therefore, the sum over the three terms will be much smaller than each individual term. In fact, Figure 3.4 illustrates this fact: the sum is indeed smaller by more than an order of magnitude than individual terms.

There is a simple way to turn this destructive sum into a constructive one. If we give x a complex phase of $\frac{2\pi}{3}$, the three points of the triangle will gather around one of them. We plot the magnitude and the phase of the sum

$$\Omega_{\mathcal{J}}\, x^{\mathcal{J}} + \Omega_{\mathcal{J}+1}\, x^{\mathcal{J}+1} + \Omega_{\mathcal{J}+2}\, x^{\mathcal{J}+2}\ , \quad \text{where}\ \ x = 0.95 \cdot \exp\left(\frac{2\pi i}{3}\right)\ , \tag{3.35}$$

in Figure 3.5. The magnitude of this sum is now consistently larger than individual term, indicating a constructive sum of the three terms.

This is not the end, however. The right panel of Figure 3.5 shows that the sum (3.35) over the three terms, while being a constructive sum over three adjacent terms, also slowly oscillate in its phase. This is because of the sine wave pattern shown in Fig 3.3: the equilateral triangle slowly rotates as a whole. Therefore, although the phase $\frac{2\pi}{3}$ ensures constructive sum over 3 adjacent terms, such sums may destruct each other between a more distant values of $\mathcal{J}$.

Fortunately, the rate of rotation of the phase of (3.35) is close to a constant. So it is possible to cancel this rotation effect by adjusting the phase of x slightly from $\frac{2\pi}{3}$. For $x = 0.95$, the ideal phase is found to be

$$x = 0.95 \cdot \exp i\left(\frac{2\pi}{3} + 0.0313458\right)\ . \tag{3.36}$$

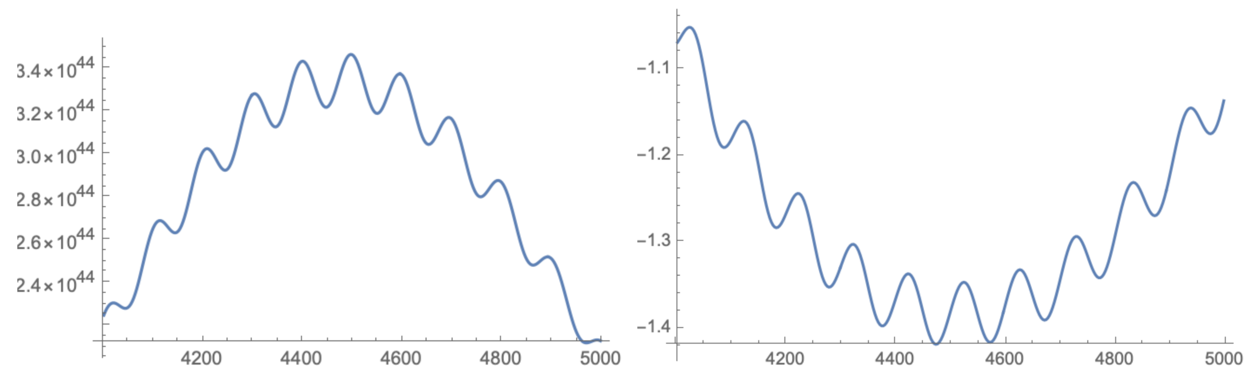

Figure 3.6: $\Omega_{\mathcal{J}}\, x^{\mathcal{J}} + \Omega_{\mathcal{J}+1}\, x^{\mathcal{J}+1} + \Omega_{\mathcal{J}+2}\, x^{\mathcal{J}+2}$ where $x = 0.95 \cdot e^{i\left(\frac{2\pi}{3}+0.0313458\right)}$, for each $\mathcal{J}$. The magnitude (left) and the phase (right).

(3.35) with this ideal phase of (3.36), both its magnitude and its phase, is plotted in Figure 3.6. The magnitudes are similar to that in Figure 3.5, because both are a constructive sum over 3 adjacent terms, but in Figure 3.6 the phase of the sum is also close to being stationary. Therefore, with the optimal phase of x given by (3.36), finally the sum over all contributions $\sum_{\mathcal{J}} \Omega_{\mathcal{J}} x^{\mathcal{J}}$ to the index will add up constructively.

As a result of this careful tuning of the phase of x given $|x| = 0.95$, the numerical value of the index evaluates to

$$\mathcal{I}\left(0.95 \cdot e^{i\left(\frac{2\pi}{3}+0.0313458\right)}\right) \quad = \quad 1.33 \times 10^{47} \; . \tag{3.37}$$

This is a very sensible result considering that each $\mathcal{J}$ around 4500 contributes $\sim 10^{44}$ in magnitude, and there are $\sim 10^3$ orders that contribute to the index significantly, recall Figure 3.2. This rough comparison that the sum is close to a magnitude of each term times the number of significant terms, supports that the sum in the index has been completely constructive. For comparison, we note the numerical value of the index for real $x = 0.95$:

$$\mathcal{I}\,(0.95) \quad = \quad 2.1 \times 10^{31} \; . \tag{3.38}$$

Its smallness compared to (3.37) shows that for real x, the destructiveness of the sum is extremely thorough.

The phase of x needed to optimize the sum, which was $\frac{2\pi}{3} + 0.0313458$ for $|x| = 0.95$, is a function of $|x|$. In fact, for higher $\mathcal{J}$, the phase approaches $\frac{2\pi}{3}$. We expect that the phase indeed converge to $\frac{2\pi}{3}$ as $\mathcal{J} \to \infty$. This expectation is aligned with the new Cardy limit discussed in [93]. In the rest of this section, let us make a quantitative comparison.

It was found in [93] that as $x \to 1 \cdot e^{i\left(\frac{2\pi}{3}\right)}$, what is called the new Cardy limit, the

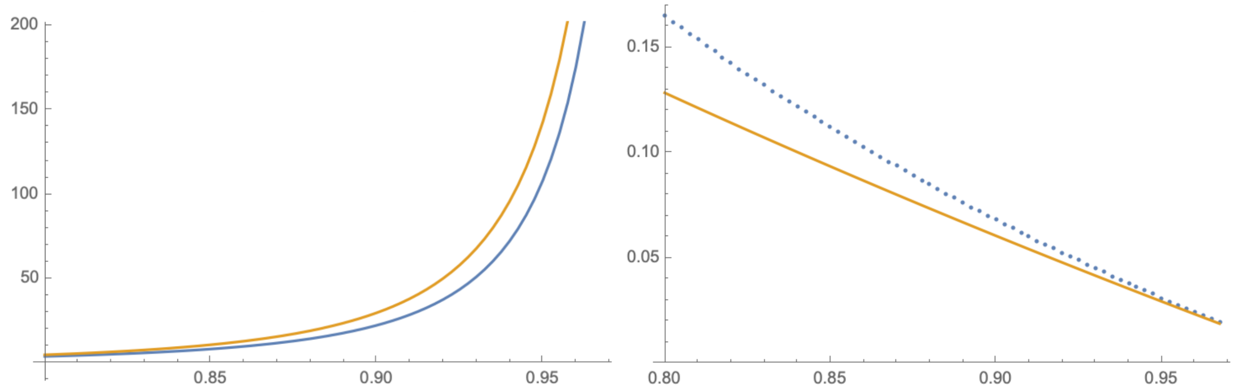

Figure 3.7: Left: $\log \mathcal{I}_{U(1)}$ for the optimal phase of x given $|x|$. Right: the optimal phase of x in excess of $\frac{2\pi}{3}$ given $|x|$. Blue lines represent the numerical $U(1)$ index and yellow lines represent corresponding expectations (3.41) and (3.40).

asymptotic behavior of the index is given by

$$\log \mathcal{I} = \frac{8(5a-3c)}{27\omega^2}(-\pi i+\omega)^3 + \frac{8\pi^2(a-c)}{3\omega^2}(-\pi i+\omega)\ , \tag{3.39}$$

where a and c are the central charges of the 4d superconformal field theory, and ω translates to our x via $x^3 = e^{-\omega}$, so it is a small parameter. It was also found that the index (3.39) is extremized, for a given $|x|$, when the phase of x is such that

$$\mathrm{Re}\,\omega = \sqrt{3}\cdot \mathrm{Im}\,\omega \quad \leftrightarrow \quad -\log|x| = \sqrt{3}\cdot\left(\arg(x) - \frac{2\pi}{3}\right) \tag{3.40}$$

Under this condition, the magnitude of the extremized index is given by

$$\mathrm{Re}\,(\log \mathcal{I}) \quad = \quad \frac{2\pi^2(3c-2a)}{9\sqrt{3}(\mathrm{Im}\,\omega)^2} \tag{3.41}$$

In Figure 3.7, we plot the expected results (3.41) for the value of the index (left) and (3.40) for the extra phase from $\frac{2\pi}{3}$ (right) for $3c-2a=1$. In blue solid and dotted lines, we plot the corresponding results from the numerical analysis of the $U(1)$ index. The comparison on the left panel is not perfect, but we believe that it demonstrates the $\frac{1}{\omega^2}$ behavior that was expected in the new Cardy limit [93] as opposed to the $\frac{1}{\omega}$ behavior that had been expected from the 'old' Cardy limit [94].

3.3 The Entropy Extremization Principle

It was demonstrated in the previous section that it is possible to overcome the cancellations between bosons and fermions to obtain the microcanonical degeneracy faithfully from the index. One way or another, the index is obtained as a function of complex chemical potentials. In this section, we review the derivation of the black hole entropy from the index, often referred to as the entropy extremization principle. This derivation has universality across dimensions, including AdS_3 [65], AdS_4 [43, 44], AdS_5 [47, 49] and AdS_7 [49, 51], but we only illustrate it in AdS_3 and AdS_5, in respective subsections.

3.3.1 AdS_3

We first present the entropy extremization principle for AdS_3 black holes. AdS_3 is special from higher dimensions, in line with CFT_2 being special from CFTs in higher dimensions. For the AdS_3 black holes, the grand canonical partition function, as opposed to the index, is known, as reviewed in section 2.1. Recall that grand canonical partition function is more general than the index in that it depends on one more chemical potentials. Furthermore, the partition function is known beyond its BPS limit, whereas the index only contains information about the BPS states. However, in this subsection we shall focus on the index and reproduce the entropy of BPS black holes (2.30) in AdS_3 spacetime from the index. This subsection is based on [65].

The grand canonical partition function was defined in (2.7), as a trace over all states:

$$\begin{aligned} Z(\beta,\, \mu,\, \omega_R,\, \omega_L) &= \mathrm{Tr}\left[e^{-\beta(E-\mu J-\omega_R Q_R-\omega_L Q_L)}\right] \\ &= e^{\frac{1}{2}\beta k_L}\,\mathrm{Tr}\left[e^{-\beta\left(E-J-Q_L+\frac{k_L}{2}\right)+\tilde{\mu}J+\tilde{\omega}_R Q_R+\tilde{\omega}_L Q_L}\right]\,. \end{aligned} \tag{3.42}$$

In the second line we have simply replaced the chemical potentials with their rescaled versions with tildes using (2.23). In subsection 2.1.2 we isolated the BPS states by taking $\beta\to\infty$ with the rescaled potentials kept finite. This gave the BPS partition function (2.26):

$$Z_{\rm BPS}(\beta,\, \tilde{\mu},\, \tilde{\omega}_R,\, \tilde{\omega}_L) = \left(\lim_{\beta\to\infty} e^{\frac{1}{2}\beta k_L}\right)\mathrm{Tr}_{\rm BPS}\left[e^{\tilde{\mu}J+\tilde{\omega}_R Q_R+\tilde{\omega}_L Q_L}\right]\,. \tag{3.43}$$

In this subsection, we study the supersymmetric index, also known as the elliptic genus in CFT_2, rather than the partition function. Recall from section 3.1.1, in particular around (3.8), that the index is a special case of the partition function where the chemical potentials are such that a pair $|\psi\rangle$ and $Q|\psi\rangle$ contributes equal but opposite weights. We choose the supercharge Q in the anti-holomorphic (left) sector of CFT_2, in line with saturation of (2.12).

This condition for the index is equivalent to the complex constraint

$$\tilde{\mu} - 2\tilde{\omega}_L = 2\pi i \ , \tag{3.44}$$

on the potentials. Thus, the index is the grand canonical partition function (3.42) under the condition (3.44). We also pull out the Casimir energy factor $e^{\frac{1}{2}\beta k_L}$ from the index. It is conventional to omit this overall factor from the definition of the supersymmetric index, or of the elliptic genus.

To summarize, the index is

$$\begin{aligned} \mathcal{I} &\equiv e^{\beta E_{\rm SUSY}} Z\big|_{\tilde{\omega}_L = \frac{\tilde{\mu}}{2} - i\pi} \\ &= {\rm Tr}_{\rm BPS}\left[e^{\tilde{\mu}J + \tilde{\omega}_R Q_R + \tilde{\omega}_L Q_L}\right]\big|_{\tilde{\omega}_L = \frac{\tilde{\mu}}{2} - i\pi} \ , \end{aligned} \tag{3.45}$$

where $E_{\rm SUSY} = -\frac{1}{2}k_L$ was given in (2.25). Going from the first to the second line, we used the aforementioned cancellation for non-BPS states to get rid of the β-dependent term in the trace of the second line of (3.42) and restrict the trace to BPS states only, then cancelled the Casimir energy factor. Note that $\beta \to \infty$ is not needed in the definition of the index, but β-dependence was eliminated by the cancellation that is enabled by (3.44).

The BPS partition function $Z_{\rm BPS}$ (3.43) depends on *three* independent potentials: $\tilde{\mu}$ and $\tilde{\omega}_{L,R}$, apart from the formal $e^{-\beta E_{\rm SUSY}}\big|_{\beta\to\infty}$ factor. On the other hand the index $\mathcal{I}$ depends on only *two* independent parameters due to (3.44) which we take as $\tilde{\mu}$ and $\tilde{\omega}_R$.

We can compute the index for supersymmetric black holes in AdS$_3$ explicitly by starting from the general partition function (2.8), introducing tilde potentials through (2.23), and then imposing the index constraint (3.44):

$$\begin{aligned} \log \mathcal{I} &= -\frac{k_L}{2}\beta + \frac{k_R}{\beta(1-\mu)}\left(\pi^2 + \beta^2\omega_R^2\right) + \frac{k_L}{\beta(1+\mu)}\left(\pi^2 + \beta^2\omega_L^2\right) \\ &= -\frac{k_L}{2}\beta - \frac{k_R}{\tilde{\mu}}\left(\pi^2 + \tilde{\omega}_R^2\right) + \frac{k_L}{\tilde{\mu} + 2\beta}\left(\pi^2 + (\tilde{\omega}_L + \beta)^2\right) \\ &= -\frac{k_R}{\tilde{\mu}}\left(\pi^2 + \tilde{\omega}_R^2\right) + \frac{k_L}{4}(\tilde{\mu} - 4\pi i) \\ &= -\frac{k_R}{\tilde{\mu}}\left(\pi^2 + \tilde{\omega}_R^2\right) + \frac{k_L}{\tilde{\mu}}(\pi^2 + \tilde{\omega}_L^2) \ . \end{aligned} \tag{3.46}$$

We present the manipulations in detail to highlight that they are exact, the dependence on β disappears without any limit taken, as anticipated. The final expression with the constraint (3.44) implied agrees with the BPS partition function (2.27), again as anticipated. A simpler but less illuminating route to the formula for the index given in the last line of (3.46) is to evaluate the partition function and take the *high* temperature limit $\beta \to 0$ with the tilde variables kept fixed. In other words, the last line of (3.46) follows from the second line by taking $\beta = 0$. This uses the β-independence of the index rather than showing it.

The computation illustrates how the index (3.45) and the BPS partition function (3.43) are closely related, yet they are different in significant ways such that they complement one another:

- The BPS partition function restricts the trace to the chiral primary states by an explicit limit $\beta \to \infty$. In contrast, the index is independent of β, the limit $\beta \to \infty$ is possible but not mandatory. This is one aspect of the index being protected under continuous deformations of the theory, while the BPS partition function is not.

- The supersymmetric index is defined with chemical potentials constrained by (3.44) or else it is not protected under continuous deformations. In contrast, the BPS partition function keeps all three potentials $\tilde{\mu}$ and $\tilde{\omega}_{R,L}$ independent. It is possible to focus on variables that satisfy the constraint, but the general case incorporates more information about the theory.

- The supersymmetric index is defined with the supersymmetric Casimir energy stripped off, while the partition function retains it.

In the non-chiral case $k_L = k_R = k$ in the absence of gravitational anomaly, we can recast our result for the index (3.46) as

$$\log \mathcal{I} = k \frac{\tilde{\omega}_1 \tilde{\omega}_2}{\tilde{\mu}} \ , \tag{3.47}$$

by choosing the basis $\tilde{\omega}_{L,R} = \frac{1}{2}(\tilde{\omega}_1 \pm \tilde{\omega}_2)$ for the potentials. This result is aligned with the form of the index that plays a central role in discussions of black hole entropy in higher dimensional AdS spaces, as we will see in (3.63) for AdS_5.

Whereas we have derived the supersymmetric index (3.46) for AdS_3 black holes by imposing a complex condition (3.44) on the more general BPS partition function (2.8), in higher dimensional AdS spaces it is only the index that can be reliably computed. In that context a procedure to extract the entropy and the charge constraint of supersymmetric black holes directly from the index has been developed [47]. We now apply this procedure to the AdS_3 case and show that it reproduces the results derived from the BPS partition function in section 2.1.

The claim that is now standard in higher dimensional AdS spaces is that we can process the index as if it was an ordinary partition function. According to this prescription [47], the black hole entropy is given by the Legendre transform of the index (3.46), subject to the complex constraint (3.44). That is, the entropy function is defined by

$$S(\tilde{\mu}, \tilde{\omega}_R, \tilde{\omega}_L) \equiv \log \mathcal{I} - \tilde{\omega}_L Q_L - \tilde{\omega}_R Q_R - \tilde{\mu} J \ , \tag{3.48}$$

and we extremize this function subject to (3.44). This can be done efficiently by introducing the Lagrange multiplier Λ that enforces the condition (3.44), thus extremizing

$$\begin{aligned} S &= S(\tilde{\mu}, \tilde{\omega}_R, \tilde{\omega}_L) - \Lambda(\tilde{\mu} - 2\tilde{\omega}_L - 2\pi i) \\ &= \frac{k_L(\tilde{\omega}_L^2 + \pi^2) - k_R(\tilde{\omega}_R^2 + \pi^2)}{\tilde{\mu}} - \tilde{\omega}_L Q_L - \tilde{\omega}_R Q_R - \tilde{\mu} J - \Lambda(\tilde{\mu} - 2\tilde{\omega}_L - 2\pi i) \ , \end{aligned} \tag{3.49}$$

with respect to the potentials $\tilde{\mu}$, $\tilde{\omega}_{R,L}$ and the Lagrange multiplier Λ.

S is homogeneous of degree one in the potentials $\tilde{\mu}$, $\tilde{\omega}_{R,L}$, except for $2\pi i\Lambda$ which is constant, and for the terms proportional to π^2 which are homogeneous of degree minus one. Keeping track of the inhomogeneous terms, the extremization conditions give

$$0 = (\tilde{\omega}_L \partial_{\tilde{\omega}_L} + \tilde{\omega}_R \partial_{\tilde{\omega}_R} + \tilde{\mu}\partial_{\tilde{\mu}})\, S = S - 2\pi i\Lambda + \frac{2\pi^2(k_R - k_L)}{\tilde{\mu}} \ , \tag{3.50}$$

so that

$$S = 2\pi i\Lambda - \frac{2\pi^2(k_R - k_L)}{\tilde{\mu}} \ . \tag{3.51}$$

The second term vanishes only when $k_R = k_L$. It represents a novel refinement when compared to analogous computations in higher dimensional AdS spaces.

The individual entropy extremization conditions are

$$\partial_{\tilde{\omega}_L} S = k_L \frac{2\tilde{\omega}_L}{\tilde{\mu}} + (2\Lambda - Q_L) = 0 \ , \tag{3.52a}$$

$$\partial_{\tilde{\omega}_R} S = -k_R \frac{2\tilde{\omega}_R}{\tilde{\mu}} - Q_R = 0 \ , \tag{3.52b}$$

$$\partial_{\tilde{\mu}} S = -\frac{k_L(\tilde{\omega}_L^2 + \pi^2) - k_R(\tilde{\omega}_R^2 + \pi^2)}{\tilde{\mu}^2} - (\Lambda + J) = 0 \ . \tag{3.52c}$$

Using the constraint (3.44), the first equation gives

$$k_L \frac{\tilde{\mu} - 2\pi i}{\tilde{\mu}} = Q_L - 2\Lambda \quad \Rightarrow \quad \frac{\pi i k_L}{\tilde{\mu}} = \Lambda - \frac{1}{2}(Q_L - k_L) \ . \tag{3.53}$$

The entropy function therefore becomes

$$S = 2\pi i \left[\Lambda + \frac{i\pi}{\tilde{\mu}}(k_R - k_L)\right] = 2\pi i \left[\frac{k_R}{k_L}\Lambda - \frac{1}{2k_L}(k_R - k_L)(J_L - k_L)\right] \equiv 2\pi i \Lambda_{\text{eff}} \ , \tag{3.54}$$

where we defined

$$\Lambda_{\text{eff}} = \frac{k_R}{k_L}\Lambda - \frac{1}{2k_L}(k_R - k_L)(Q_L - k_L) \ . \tag{3.55}$$

Rewriting the last extremization condition (3.52c) using the others (3.52a-3.52b) and the expression for $\tilde{\mu}$ (3.53) we find

$$-\frac{1}{k_L}(\Lambda - \frac{1}{2}Q_L)^2 + \frac{1}{4k_R}Q_R^2 - (\Lambda + J) - \frac{1}{k_L^2}(k_R - k_L)\left(\Lambda - \frac{1}{2}(Q_L - k_L)\right)^2 = 0 \ , \tag{3.56}$$

which we reorganize into a quadratic equation for Λ_{eff}:

$$\Lambda_{\text{eff}}^2 - (Q_L - k_L)\Lambda_{\text{eff}} + \frac{1}{4}(Q_L - k_L)^2 + k_R\left(J + \frac{Q_L}{2} - \frac{k_L}{4}\right) - \frac{1}{4}Q_R^2 = 0 \ . \tag{3.57}$$

Selecting the root with negative imaginary part we find the extremized entropy function in terms of charges:

$$S = 2\pi i\Lambda_{\text{eff}} = 2\pi\sqrt{k_R\left(J + \frac{Q_L}{2} - \frac{k_L}{4}\right) - \frac{Q_R^2}{4}} + \pi i(Q_L - k_L) \ . \tag{3.58}$$

For BPS black holes in higher dimensional AdS the standard prescription posits that charges must be constrained such that the extremized entropy function is real [47, 49]. Applying this rule in AdS_3 as well, we find

$$Q_L = k_L \ , \tag{3.59}$$

in agreement with the charge constraint (2.29) that we inferred from gravitational considerations. Only after fixing the charges this way, the entropy function (3.58) is real with the value

$$S_{\text{BPS}} = 2\pi\sqrt{k_R\left(J + \frac{1}{4}k_L\right) - \frac{1}{4}Q_R^2} \ , \tag{3.60}$$

in agreement with the entropy (2.30) of a BPS black hole in AdS_3 .

In summary, in this subsection we defined the supersymmetric index for the AdS_3 black holes, and applied the entropy extremization procedure to recover thermodynamic properties from the index (3.46). The computation is novel in that the index (3.46) used here is more refined than the version (3.47) that is directly analogous to higher dimensional cases.

3.3.2 AdS_5

In this subsection, we demonstrate the entropy extremization principle for AdS_5 black holes [47, 49]. In fact, in this context was the principle first established [47] based on a similar extremization principle used for AdS_4 black hole entropy [43, 44]. Shortly after, the index required for this principle to reproduce the AdS_5 black hole entropy was computed [48–50].

As we reviewed in section 3.1.1, the AdS_5 black holes and the dual 4d $\mathcal{N} = 4$ SYM are described by six quantum numbers, namely the energy or the scaling dimension E, two angular momenta $J_{1,2}$, and three charges $Q_{1,2,3}$. The index of the 4d $\mathcal{N} = 4$ SYM is defined by (3.9):

$$\begin{aligned} \mathcal{I}(\tilde{\Delta}_I, \tilde{\omega}_i) &= \mathrm{Tr}\left[e^{-\beta(E-Q_1-Q_2-Q_3-J_1-J_2)}e^{\tilde{\Delta}_I Q_I+\tilde{\omega}_i J_i}\right] \\ &= \mathrm{Tr}_{\mathrm{BPS}}\left[e^{\tilde{\Delta}_I Q_I+\tilde{\omega}_i J_i}\right] \ , \end{aligned} \tag{3.61}$$

where $\tilde{\Delta}_I \equiv \Delta_I - \beta$ and $\tilde{\omega}_i \equiv \omega_i - \beta$ satisfy $e^{\frac{\tilde{\Delta}_1+\tilde{\Delta}_2+\tilde{\Delta}_3-\tilde{\omega}_1-\tilde{\omega}_2}{2}} = -1$, which is the condition that ensures independence on β as well as the coupling independence of the index. This condition is realized by

$$\tilde{\Delta}_1 + \tilde{\Delta}_2 + \tilde{\Delta}_3 - \tilde{\omega}_1 - \tilde{\omega}_2 = 2\pi i \ . \tag{3.62}$$

The index (3.61) with complex chemical potentials subject to (3.62) has been computed in various limits and approximations. For example, [49] took the Cardy-like limit $|\omega_i| \ll 1$ while [50] took two equal angular momenta for the Bethe Ansatz approach. The regime of applicability has broaden in [66] but not completely. In one way or another, the leading term in large N of the index is evaluated to be

$$\log \mathcal{I} = -\frac{N^2}{2}\frac{\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\Delta}_3}{\tilde{\omega}_1\tilde{\omega}_2} \ . \tag{3.63}$$

Let us now derive the AdS_5 black hole entropy from this index, following the entropy extremization principle [47], in which the index is treated as an ordinary grand canonical partition function that yields the entropy via Legendre transformation.

First, the entropy function is defined by

$$S(\tilde{\Delta}_I, \tilde{\omega}_i) \ \equiv \ \log \mathcal{I} - \sum_I \tilde{\Delta}_I Q_I - \sum_i \tilde{\omega}_i J_i \ , \tag{3.64}$$

and we extremize this function subject to (3.62). This can be done efficiently by introducing the Lagrange multiplier Λ that enforces the condition (3.62), thus extremizing

$$\begin{aligned} S &= S(\tilde{\Delta}_I, \tilde{\omega}_i) - \Lambda \left(\sum_I \tilde{\Delta}_I - \sum_i \tilde{\omega}_i - 2\pi i \right) \\ &= -\frac{N^2}{2} \frac{\tilde{\Delta}_1 \tilde{\Delta}_2 \tilde{\Delta}_3}{\tilde{\omega}_1 \tilde{\omega}_2} - \sum_I \tilde{\Delta}_I Q_I - \sum_i \tilde{\omega}_i J_i - \Lambda \left(\sum_I \tilde{\Delta}_I - \sum_i \tilde{\omega}_i - 2\pi i \right) , \end{aligned} \tag{3.65}$$

with respect to the potentials $\tilde{\Delta}_I$, $\tilde{\omega}_i$ and the Lagrange multiplier Λ.

The entropy function is homogeneous of degree one in the potentials except for the $2\pi i \Lambda$ which is constant. Therefore,

$$0 = \left(\sum_I \tilde{\Delta}_I \partial_{\tilde{\Delta}_I} + \sum_i \tilde{\omega}_i \partial_{\tilde{\omega}_i} \right) S = S - 2\pi i \Lambda , \tag{3.66}$$

so that

$$S = 2\pi i \Lambda . \tag{3.67}$$

This is analogous to (3.51), where the $k_R - k_L$ term does not have a higher dimensional analogue.

The individual entropy extremization conditions are

$$\begin{aligned} -\tilde{\Delta}_I \partial_{\tilde{\Delta}_I} S &= \frac{N^2}{2} \frac{\tilde{\Delta}_1 \tilde{\Delta}_2 \tilde{\Delta}_3}{\tilde{\omega}_1 \tilde{\omega}_2} + \tilde{\Delta}_I (Q_I + \Lambda) = 0 , \\ \tilde{\omega}_i \partial_{\tilde{\omega}_i} S &= \frac{N^2}{2} \frac{\tilde{\Delta}_1 \tilde{\Delta}_2 \tilde{\Delta}_3}{\tilde{\omega}_1 \tilde{\omega}_2} + \tilde{\omega}_i (-J_i + \Lambda) = 0 , \end{aligned} \tag{3.68}$$

It follows that

$$\begin{aligned} \tilde{\Delta}_I &= -\frac{1}{Q_I + \Lambda} \cdot \frac{N^2}{2} \frac{\tilde{\Delta}_1 \tilde{\Delta}_2 \tilde{\Delta}_3}{\tilde{\omega}_1 \tilde{\omega}_2} , \\ \tilde{\omega}_i &= -\frac{1}{-J_i + \Lambda} \cdot \frac{N^2}{2} \frac{\tilde{\Delta}_1 \tilde{\Delta}_2 \tilde{\Delta}_3}{\tilde{\omega}_1 \tilde{\omega}_2} , \end{aligned} \tag{3.69}$$

so

$$\begin{aligned} \frac{N^2}{2} \frac{\tilde{\Delta}_1 \tilde{\Delta}_2 \tilde{\Delta}_3}{\tilde{\omega}_1 \tilde{\omega}_2} &= \frac{N^2}{2} \cdot \left(-\frac{N^2}{2} \frac{\tilde{\Delta}_1 \tilde{\Delta}_2 \tilde{\Delta}_3}{\tilde{\omega}_1 \tilde{\omega}_2} \right)^{3-2} \cdot \frac{(-J_1 + \Lambda)(-J_2 + \Lambda)}{(Q_1 + \Lambda)(Q_2 + \Lambda)(Q_3 + \Lambda)} , \\ \Rightarrow \quad 0 &= (Q_1 + \Lambda)(Q_2 + \Lambda)(Q_3 + \Lambda) + \frac{N^2}{2} (-J_1 + \Lambda)(-J_2 + \Lambda) . \end{aligned} \tag{3.70}$$

This is a cubic equation on Λ.

Now, the entropy extremization principle posits that charges must be constrained such that the extremized entropy function is real [47, 49]. This requires, via (3.67), that Λ must be purely imaginary. Since the coefficients of the cubic equation (3.70) are all real, it must be that terms in orders Λ^3 and Λ^1, and terms in orders Λ^2 and Λ^0, must separately add up to zero. Therefore,

$$\Lambda^3 + \left(Q_1Q_2 + Q_2Q_3 + Q_3Q_1 - \frac{N^2}{2}(J_1 + J_2)\right)\Lambda \;=\; 0 \;, \tag{3.71}$$

$$\left(Q_1 + Q_2 + Q_3 + \frac{N^2}{2}\right)\Lambda^2 + \left(Q_1Q_2Q_3 + \frac{N^2}{2}J_1J_2\right) \;=\; 0 \;. \tag{3.72}$$

One consequence of both equations of (3.71) is

$$\begin{aligned} & Q_1Q_2Q_3 + \frac{N^2}{2}J_1J_2 \\ = \;& \left(Q_1Q_2 + Q_2Q_3 + Q_3Q_1 - \frac{N^2}{2}(J_1 + J_2)\right)\left(Q_1 + Q_2 + Q_3 + \frac{N^2}{2}\right) \;, \end{aligned} \tag{3.73}$$

and taking one of the solutions of the first equation to make $S = 2\pi i\Lambda$ positive, we have for the extremized entropy function,

$$S \;=\; 2\pi\sqrt{Q_1Q_2 + Q_2Q_3 + Q_3Q_1 - \frac{N^2}{2}(J_1 + J_2)}\;. \tag{3.74}$$

The rank (plus one) of the gauge group N in the 4d $\mathcal{N} = 4$ SYM is related to the 5-dimensional Newton's gravitational constant by [3]

$$N^2 = \frac{\pi}{2G_5}\;. \tag{3.75}$$

Through this dictionary, (3.73) is the constraint (2.52) between the charges Q_I and J_i that a supersymmetric black hole must satisfy, and (3.74) is the entropy (2.54) of the supersymmetric AdS_5 black hole.

To summarize, the entropy extremization principle directs us to extremize the entropy function (3.64) determined from the index, and demand that the complex result be real. As a result, one obtains the value of the extremized entropy function as well as a constraint between the charges. The former gives the entropy of the supersymmetric black hole, while the latter gives its charge constraint.

Chapter 4

The Supersymmetric Charge Constraints

In this chapter we turn to another important property of supersymmetric AdS black holes, the charge constraint. We present a heuristic derivation of the constraint from the dual field theories as a relation between macroscopic charges of an ensemble that is classically generated by free fields, up to a uniform rescaling of all charges. After a brief introduction, we shall first present the derivation for AdS_3 black holes. Motivated by this example, we develop a generic prescription to derive the charge constraint from the field theories and apply to the AdS_5, AdS_4 and AdS_7 black holes in the subsequent sections. We obtain the correct functional form of the fully refined charge constraint in each of the dimensions. We conclude this chapter by discussing various shortcomings and implications of our arguments. Section 4.1 is based on [65] and the rest of this chapter is based on [70], both in collaboration with Finn Larsen.

As we have addressed in the previous section, the entropies of supersymmetric AdS black holes in different dimensions have been matched with the number of supersymmetric states in their dual conformal field theories. The supersymmetric states are counted using the index, which can be understood as a special case of the grand canonical partition function for the ensemble of supersymmetric microstates.

Importantly, the supersymmetric index inevitably depends on one less chemical potentials than there are independent charges, recall the discussion after (3.9). In other words, the index as a grand canonical partition function does not distinguish microstates along the direction in the space of conserved charges generated by the preserved supercharge. This fundamentally prevents the index from addressing the charge constraint, which contains information about the location in the space of charges. Indeed, the charge constraint is surprising from the CFT side of the duality, because numerous local supersymmetric operators exist also for charges that violate the constraint.

Curiously, the charge constraint emerges in the microscopic accounting of the supersymmetric black hole entropy from the condition that the extremum of the complex entropy function is real [47, 49, 52]. This is suggestive, but given the intrinsic shortcoming of the index, it does not provide a satisfying microscopic explanation.

In section 4.1, we make a proposal for the microscopic origin of the charge constraints for supersymmetric AdS_3 black holes. In any unitary supermultiplet of the small $\mathcal{N}=4$ super-Virasoro algebra, whether short or long, all weights appear in pairs. The two weights in each pair are separated in the charge configuration space along the direction of the preserved supercharge Q, and the R-charges of the two weights average to k, the level of the $SU(2)_R$ algebra. This is precisely the condition that an extremal BTZ black hole is supersymmetric. This proposal goes beyond the scope of the index, in that the index does not see both states in a pair individually.

Generalization of this argument to higher dimensions is not straightforward. Superconformal algebras in higher dimensions are not as large and constraining as the super-Virasoro algebra in CFT_2, so they are consistent with more diverse multiplet structures. Moreover, in AdS_{d+1} with $d>2$, the constraints on conserved charges that we want to illuminate are non-linear and highly non-trivial.

In sections 4.2 through 4.5, we offer a heuristic derivation of the charge constraints for higher dimensions. In each dimension, we start from the free multiplet of the corresponding superconformal algebra. We then construct a grand canonical partition function that depends on as many chemical potentials as there are charges, thereby overcoming the fundamental limitation of the index. We define a supersymmetric ensemble that gives equal weight to all states along the direction generated by the supercharge, and compute the macroscopic charges of the ensemble. This procedure gives the correct functional form of the fully refined charge constraint in AdS_5, AdS_4, and AdS_7. The major heuristic element of our computation is the number of free multiplets in the theory, which we simply put in by hand. For example, for the $SU(N)$ SYM in $d=4$, we need $\frac{1}{2}N^2$ free multiplets, compared with N^2 in a genuinely free theory. This number sets the scale of all conserved charges.

4.1 AdS_3

In this section we discuss BTZ black holes in $\mathrm{AdS}_3 \times S^3$ that are dual to 2d CFT with $(4,4)$ superconformal symmetry. This section is based on section 5 of [65].

4.1.1 The BTZ Black Hole and its Charge Constraint

Thermodynamic properties of the AdS_3 black holes as well as their BPS limit have been reviewed in section 2.1. Let us briefly recall relevant information. The BTZ black holes in $AdS_3 \times S^3$ carry an energy E and an angular momentum J that both arise from the isometry of AdS_3, as well as two charges Q_L and Q_R associated with the isometry $SU(2)_L \times SU(2)_R$ of S^3.

One choice of $\frac{1}{4}$-BPS sector in this theory corresponds to energy that saturates the unitarity bound:

$$E \geq J + Q_L - \frac{k_L}{2} . \tag{4.1}$$

In this formula k_L is the level of the $SU(2)_L$ current which, because of $\mathcal{N} = 4$ supersymmetry, is related to the central charge as $c_L = 6k_L$. On the other hand, all black hole solutions in $AdS_3 \times S^3$ satisfy the extremality bound,

$$E \geq J - \frac{Q_L^2}{2k_L} , \tag{4.2}$$

which is saturated at vanishing temperature. This formula is entirely gravitational, but we have simply expressed Newton's constant G_3 in terms of the level k_L using the Brown-Henneaux formula for the central charge [79].

A BTZ black hole can only be $\frac{1}{4}$-BPS if it saturates both of (4.1) and (4.2). That is only possible if

$$Q_L = k_L . \tag{4.3}$$

This is the charge constraint on supersymmetric AdS_3 black holes. In a charge sector that violates (4.3) there are no supersymmetric black holes.

4.1.2 Multiplets of CFT_2 with $(4, 4)$ Supersymmetry

The dual 2d CFT has $(4, 4)$ supersymmetry. Its superconformal algebra factorizes into two independent copies of super-Virasoro algebra, and includes a bosonic subgroup $SO(2, 2) \times SU(2)_L \times SU(2)_R$ that matches the isometry of $AdS_3 \times S^3$. To make progress, we first review representations of one chiral copy of the small $\mathcal{N} = 4$ super-Virasoro algebra with $SU(2)$ R-symmetry [77, 80, 81, 95].

The maximal bosonic subalgebra of the small $\mathcal{N} = 4$ super-Virasoro algebra has two Cartans: L_0 of the Virasoro algebra and Q_L of the $SU(2)$ R-symmetry. Every weight in a representation of the super-Virasoro algebra can be chosen to diagonalize the Cartans of the bosonic subalgebra, so it is described by its L_0 and Q_L eigenvalues (h, q_L). Each unitary representation of the super-Virasoro algebra is labeled by the L_0 and Q_L eigenvalues (h, q_L) of its superconformal primary. Given a superconformal primary, the entire contents of the multiplet is determined, as the descendants are obtained by applying various operators of the algebra to the primary. We can focus on states that satisfy NS boundary conditions, because representations in the Ramond sector are isomorphic through spectral flow by a half-integral unit.

There are two qualitatively different types of representations: long multiplets whose primary has $h > q_L$ and short multiplets whose primary has $h = q_L$. The allowed values for q_L of the primary are $0, 1, \cdots, k_L - 1$ for long multiplets, and $0, 1, \cdots, k_L$ for short multiplets. The content of either type of representation can be summarized by its character defined by $\text{Tr}\, q^{L_0} y^{Q_L}$, a function of two fugacities q and y. The characters of the two types of representations are [81]:

$$\begin{aligned}
\text{Long}: \ \text{ch}_{h,q_L}(q,y) &= q^h F^{NS} \sum_{m=-\infty}^{\infty} \left(y^{2(k_L+1)m+q_L+1} - y^{-2(k_L+1)m-q_L-1} \right) \frac{q^{(k_L+1)m^2+(q_L+1)m}}{y - y^{-1}} \ , \\
\text{Short}: \quad \chi_{q_L}(q,y) &= q^{\frac{q_L}{2}} F^{NS} \sum_{m=-\infty}^{\infty} \left(\frac{y^{2(k_L+1)m+q_L+1}}{(1+yq^{m+\frac{1}{2}})^2} - \frac{y^{-2(k_L+1)m-q_L-1}}{(1+y^{-1}q^{m+\frac{1}{2}})^2} \right) \frac{q^{(k_L+1)m^2+(q_L+1)m}}{y-y^{-1}} \ ,
\end{aligned} \tag{4.4}$$

where

$$F^{NS} = \prod_{n \geq 1} \frac{\left(1 + yq^{n-\frac{1}{2}}\right)^2 \left(1 + y^{-1}q^{n-\frac{1}{2}}\right)^2}{(1-y^2q^n)(1-q^n)^2(1-y^{-2}q^n)} \ , \tag{4.5}$$

accounts for the action of creation operators, i.e. the negative frequency modes $\{G_{r<0}\}$ and $\{L_{n<0}, J^i_{n<0}\}$ of the four fermionic and four bosonic generators.

4.1.3 The Supersymmetric Ensemble and the Charge Constraint

The long and short multiplets discussed in the previous subsection are the only unitary representations of the small $\mathcal{N} = 4$ super-Virasoro algebra. Since the supersymmetry algebra of the 2d $(4, 4)$ theory is a direct sum of two copies of the small $\mathcal{N} = 4$ super-Virasoro algebra, any representation thereof is a direct product between two representations of the small $\mathcal{N} = 4$ super-Virasoro algebra. Therefore, the microscopic duals of AdS_3 black holes must also organize themselves into such representations.

The long and short multiplet characters (4.4) both exhibit the following property:

$$\begin{aligned} \chi_{q_L}(q,y) &= \chi_{q_L}(q,q^{-1}y^{-1}) \cdot q^{k_L} y^{2k_L} \ , \\ \mathrm{ch}_{h,q_L}(q,y) &= \mathrm{ch}_{h,q_L}(q,q^{-1}y^{-1}) \cdot q^{k_L} y^{2k_L} \ . \end{aligned} \tag{4.6}$$

This shows that, within any representation, a weight with (h, q_L) is always paired with another one with $(h + k_L - q_L, 2k_L - q_L)$. To see this, suppose that there is a weight with (h, q_L) either in the short or the long multiplet. This contributes to the right hand side of (4.6) by

$$q^h (q^{-1}y^{-1})^{q_L} \cdot q^k y^{2k_L} = q^{h+k_L-q_L} y^{2k_L-q_L} \ . \tag{4.7}$$

Then it follows from the equation that the character must contain a term $q^{h+k_L-q_L}y^{2k_L-q_L}$, which can only be true if a weight $(h + k_L - q_L, 2k_L - q_L)$ belonged to the multiplet.

The pair is characterized by the R-charges being mirrored about k_L and the conformal weights *in excess of* the unitarity bound $L_0 - \frac{1}{2}Q_L$ being the same:

$$h - \frac{1}{2}q_L = (h + k_L - q_L) - \frac{1}{2}(2k_L - q_L) \ . \tag{4.8}$$

Therefore, if both weights contribute equally to the grand canonical partition function, then the macroscopic charge Q_L obtained as a statistical average over the ensemble will necessarily be k_L.

The condition that the weights (h, q_L) and $(h + k_L - q_L, 2k_L - q_L)$ in the pair contribute equally to the grand canonical partition function is

$$yq^{1/2} = 1 \ . \tag{4.9}$$

Therefore, when this condition is satisfied, the average $\langle Q_L \rangle = k_L$. Indeed, explicit computation shows that

$$\langle Q_L \rangle = y \frac{\partial \log Z}{\partial y}\Big|_{y=q^{-\frac{1}{2}}} = k_L \ , \tag{4.10}$$

for any partition function that is a product of characters satisfying (4.6). We refer to grand canonical partition function with $yq^{1/2} = 1$ as the supersymmetric ensemble.

Geometrically, the condition $yq^{1/2} = 1$ defining the supersymmetric ensemble means all quantum states along a straight line in the (h, q_L) plane are counted equally. This is precisely the direction generated by the preserved supercharge, corresponding to one of the factors in the numerator of (4.5).

The definition of the supersymmetric ensemble is reminiscent of imposing $yq^{1/2} = -1$, the substitution that turns the grand canonical partition function into the index, or the elliptic genus. With the condition $yq^{1/2} = -1$, two microstates related by the supercharge Q contribute equal magnitude, but with opposite signs. Therefore, the only non-vanishing contributions are from short multiplets where the primary is annihilated by the supercharge. Moreover, combinations of short multiplets along the direction of the supercharge combine to a long multiplet

$$\chi_{h,q_L-1}(q,y) + 2\chi_{h,q_L}(q,y) + \chi_{h,q_L+1}(q,y) = \mathrm{ch}_{h,q_L}(q,y)\Big|_{h=\frac{1}{2}q_L} , \tag{4.11}$$

and also cancel automatically in the index. With these cancellations, the index is unable to assign relative probabilities to the charges in this direction, and so it cannot account for the constraint. In contrast, the supersymmetric ensemble avoids massive cancellations and controls the direction generated by the supercharge by taking the average over all configurations. This prescription reproduces the charge constraint $\langle Q_L \rangle = k_L$ (4.3) that is satisfied for all supersymmetric black holes in $\mathrm{AdS}_3 \times S^3$.

4.2 Prescription for Higher Dimensions and Summary

We now turn to the charge constraints in higher dimensions. Before we present specific examples, let us briefly outline the generic prescription for a heuristic derivation of the charge constraints from dual CFTs across dimensions. We shall follow this prescription in sections 4.3 through 4.5 to obtain charge constraints of AdS_5, AdS_4 and AdS_7 black holes from respective CFTs.

AdS black holes are dual to ensembles of quantum states in a superconformal field theory in one fewer dimensions that all preserve the same amount of supersymmetry as the black holes. The local operators in the dual theory organize themselves into representations of the applicable superconformal algebra. Our starting point is the field content of the free representation, which provides the basic building blocks of the CFTs. It consists of free fields, both bosons and fermions, as well as derivatives that generate conformal descendants, and equations of motion that impose physical conditions. In a free CFT the particle number operator is well-defined, and so the free fields correspond to single particle states. There are infinitely many, because an arbitrary number of derivatives may act on the fields.

Every single particle state can be chosen as eigenstates of the Cartan generators of the bosonic subalgebra. The corresponding eigenvalues are the conformal dimension E, angular momenta J_i, and the R-symmetry charges Q_I, where the ranges of i and I depend on the dimension and on the amount of supersymmetry. The totals of the microscopic quantum numbers for the entire ensemble give the E, J_i, and Q_I that we identify with the black hole charges. We only pick microscopic states that individually preserve the same supersymmetries as the black hole. These single particle BPS states are referred to as BPS letters. Since they are annihilated by the chosen supercharges, $Q|\psi\rangle = 0$, the BPS states must saturate the unitarity bound $\{Q, Q^\dagger\} \geq 0$. The superalgebra expresses the left hand side as a sum over the bosonic Cartan operators, so, schematically,

$$\text{BPS}: \quad \{Q, Q^\dagger\} = E - \sum_i J_i - \sum_I Q_I = 0 \ . \tag{4.12}$$

The quantum numbers of the BPS letters must satisfy the equality on the right, giving a linear BPS relation between the energy and the other conserved charges.[1]

The grand canonical partition function is the trace over all quantum states, with weights assigned to each state by chemical potentials that couple to the conserved charges. We define it with an explicit restriction to BPS states:

$$\begin{aligned} Z &\equiv \text{Tr}_{\text{BPS}}\left[e^{-\beta\{Q,Q^\dagger\}} e^{\sum_i \omega_i J_i + \sum_I \Delta_I Q_I}\right] \\ &= \text{Tr}_{\text{BPS}}\left[e^{\sum_i \omega_i J_i + \sum_I \Delta_I Q_I}\right] \ . \end{aligned} \tag{4.13}$$

The second line is because the superalgebra (4.12) gives $\{Q, Q^\dagger\} = 0$. This removes the dependence on conformal dimension, but the partition function retains dependence on all chemical potentials ω_i and Δ_I, there are as many of them as there are charges. Therefore, it is sensitive to the distribution of microstates along all directions in the charge space.

It is useful to define the grand canonical partition function over the BPS letters only. This gives the single particle BPS partition function Z_{sp}. However, in a quantum field theory, general states belong to a multiparticle Fock space that is generated by the single particle states in the usual way, with occupation numbers restricted by fermion or boson statistics. In the free theory any quantum number of a multiparticle state, including its energy, is the sum over the corresponding single particle quantum numbers. Therefore, the BPS partition function Z over the entire BPS Hilbert space can be derived from the single particle BPS partition function Z_{sp}, by taking combinatorics into account.

For example, for a single particle bosonic or fermionic BPS state that yields the single particle partition function x_B or x_F, the partition function for the full Fock space is

$$1 + x_B + x_B^2 + \cdots \ = \ \frac{1}{1 - x_B} \ , \tag{4.14}$$

[1] Depending on normalization of the charges, one or more terms in the sum (4.12) may contain numerical coefficients that differ from one. An example is (4.51) for the 6d (2, 0) superconformal algebra. Throughout this paper, we use the notation of [88], to which we refer for details on the algebra and representations.

D	Charges	Constraint	SCA	Free multiplet	N	G_D
4	$J,\ Q_{1,2,3,4}$	(4.35)	3d $\mathcal{N}=8$	$B_1[0]^{[0,0,1,0]}_{1/2}$	$\frac{\sqrt{2}}{3}N^{\frac{3}{2}}=1$	$G_4=\frac{1}{2}$
5	$J_{1,2},\ Q_{1,2,3}$	(4.19)	4d $\mathcal{N}=4$	$B_1\bar{B}_1[0;0]^{[0,1,0]}_1$	$\frac{1}{2}N^2=1$	$G_5=\frac{\pi}{4}$
7	$J_{1,2,3},\ Q_{1,2}$	(4.52)	6d (2, 0)	$D_1[0,0,0]^{[1,0]}_2$	$\frac{2}{3}N^3=1$	$G_7=\frac{\pi^2}{8}$

Table 4.1: Summary of sections 4.3–4.5.

and $1+x_F$, respectively. If there are N_B bosonic and N_F single particle BPS states, each of which yields the single particle partition function $x_{B,i}$ and $x_{F,j}$, the full partition function becomes

$$Z = \frac{\prod_{j=1}^{N_F}(1+x_{F,j})}{\prod_{i=1}^{N_B}(1-x_{B,i})} \ . \tag{4.15}$$

These formulae are simply the standard Bose-Einstein and Fermi-Dirac distributions from elementary statistical physics, but expressed in a notation commonly used when discussing supersymmetric indices. In our prescription, we compute the multiparticle partition function as a simple exponential of the single particle partition function:

$$Z \ = \ e^{Z_{\rm sp}} = \exp\left(\sum_{i=1}^{N_B} x_{B,i} + \sum_{j=1}^{N_F} x_{F,j}\right) \ . \tag{4.16}$$

This is the limit of classical statistical physics. It is justified when the occupation number for any single particle state is so small that it is likely to be either 0 or 1. This assumption may be realized by the large number of gauge degrees of freedom for each single particle state. An improved treatment of such gauge degrees of freedom would project onto gauge singlets at the end, and that we do not do.

Given the grand canonical partition function Z for the full Hilbert space, we can derive the macroscopic charges as ensemble averages in a standard manner. (4.13) gives

$$\begin{aligned} Q_J \ &= \ \frac{\mathrm{Tr}_{\rm BPS}\left[Q_J \cdot e^{\sum_i \omega_i J_i + \sum_I \Delta_I Q_I}\right]}{\mathrm{Tr}_{\rm BPS}\left[e^{\sum_i \omega_i J_i + \sum_I \Delta_I Q_I}\right]} = \frac{\partial}{\partial \Delta_J}\log Z \ , \\ J_j \ &= \ \frac{\mathrm{Tr}_{\rm BPS}\left[J_j \cdot e^{\sum_i \omega_i J_i + \sum_I \Delta_I Q_I}\right]}{\mathrm{Tr}_{\rm BPS}\left[e^{\sum_i \omega_i J_i + \sum_I \Delta_I Q_I}\right]} = \frac{\partial}{\partial \omega_j}\log Z \ . \end{aligned} \tag{4.17}$$

These formulae express all the charges in terms of an equal number of chemical potentials.

Denoting by Q the supercharge that is preserved by the black hole and by the dual BPS states, we now impose a linear relation between the chemical potentials such that quantum states that differ by the charges of Q are given the same weight. The statistical computations of macroscopic charges (4.17) were done prior to this stage, and included the gradient of the partition function in the direction along the constraint between the potentials. Therefore, the computation reflects the dependence of the partition function on all chemical potentials.

After the constraint on the chemical potentials is imposed, the statistical formulae (4.17) express all macroscopic charges in terms of one variable less than there are charges. Equivalently, the charges that are realized form a co-dimension one surface in the space of all charges, i.e. they satisfy a constraint. We find that the constraint on charges arrived at this way, from the field content of the microscopic theory, has the same highly non-trivial form as the non-linear charge constraint of the supersymmetric black holes.

The black hole charges that satisfy the non-linear constraint in the gravitational theory, are in units of Newton's gravitational coupling constant. In contrast, in its simplest form, the microscopic computation considers a single free field. Our results are incomplete, because we do not determine the relative scale of the charges in the two computations. Comparison between the computations gives a value for Newton's constant or, equivalently, for the effective number of free fields. The summary in Table 4.1 records these values.

4.3 AdS_5

In this section we discuss how the charge constraint of supersymmetric, rotating and charged black holes in AdS_5, emerges from its dual $\mathcal{N}=4$ Super-Yang-Mills theory in 4d. We follow the prescription outlined in section 4.2.

4.3.1 The Black Hole and the Charge Constraint

Asymptotically AdS_5 black holes arise as solutions to type-IIB supergravity in $\mathrm{AdS}_5 \times S^5$ [28–33]. They carry the mass E and two angular momenta $J_{1,2}$ for the isometry $SO(2,4)$ of AdS_5, and three charges $Q_{1,2,3}$ for the isometry $SO(6)$ of S^5. The black hole solution with all 6 conserved quantities independent is known.

The black hole is supersymmetric when the unitarity bound between the mass and the charges

$$E \geq J_1 + J_2 + Q_1 + Q_2 + Q_3 \ , \tag{4.18}$$

is saturated. We have set the AdS_5 radius $\ell_5 = 1$. Importantly, saturation is possible only when the charges obey an additional relation [49, 85, 86]

$$
\begin{aligned}
&\left(Q_1 Q_2 Q_3 + \frac{N^2}{2} J_1 J_2\right) \\
= \ &\left(Q_1 + Q_2 + Q_3 + \frac{N^2}{2}\right)\left(Q_1 Q_2 + Q_2 Q_3 + Q_3 Q_1 - \frac{N^2}{2}(J_1 + J_2)\right) \ . \qquad (4.19)
\end{aligned}
$$

We have traded the 5d Newton's constant G_5 into the field theory variable N via

$$
\frac{1}{2} N^2 = \frac{\pi \ell_5^3}{4 G_5} \ ,
$$

for future convenience, but we stress that the origin of the charge constraint (4.19) is purely gravitational. (4.19) is the charge constraint for supersymmetric AdS_5 black holes.

We also present an unrefined ($J_1 = J_2 = J$ and $Q_1 = Q_2 = Q_3 = Q$, where Q should not be confused with the preserved supercharge) version of (4.19) that is more approachable, but still quite non-trivial:

$$
\left(Q^3 + \frac{N^2}{2} J^2\right) - \left(3Q + \frac{N^2}{2}\right)\left(3Q^2 - N^2 J\right) \ = \ 0 \ . \qquad (4.20)
$$

4.3.2 The 4d $\mathcal{N} = 4$ Free Vector Multiplet

The charged, rotating AdS_5 black holes introduced in the previous subsection are dual to quantum states in the $\mathcal{N} = 4$ Super-Yang-Mills theory in 4d. In this subsection we introduce the free vector multiplet of the 4d $\mathcal{N} = 4$ superconformal algebra $\mathfrak{psu}(2, 2|4)$ that generates the single particle states.

Local operators can be organized into super-representations of the 4d $\mathcal{N} = 4$ superconformal algebra. A super-representation consists of a superconformal primary and its descendants. Following the notation of [88], we identify representations by the Dynkin labels of the superconformal primary under the maximal bosonic subalgebra:

$$
[j; \bar{j}]_E^{[R_1, R_2, R_3]} \ .
$$

Here E is the conformal weight, j, $\bar{j}$ are the integer-quantized Dynkin labels for the $SU(2) \times SU(2)$ Lorentz group, and $R_{1,2,3}$ are the Dynkin labels for the $SU(4)$ R-symmetry group.

The black hole charges used in section 4.3.1 refer to the $SO(2, 4) \times SO(6)$ isometry group of the $\mathrm{AdS}_5 \times S^5$ geometry. They are charges of $SO(2)$ rotations in orthogonal 2-planes. The orthogonal basis are related to the Dynkin basis as:

$$\begin{aligned} J_1 &= \frac{j+\bar{j}}{2} , \qquad & J_2 &= \frac{j-\bar{j}}{2} , \\ Q_1 &= R_2 + \frac{R_1+R_3}{2} , \qquad & Q_2 &= \frac{R_1+R_3}{2} , \qquad & Q_3 = \frac{R_1-R_3}{2} . \end{aligned} \tag{4.21}$$

The energy E is common to the two bases. We further note that $[R_1, R_2, R_3]$ are $SU(4)$ Dynkin labels, not to be confused with $SO(6)$ Dynkin labels that are related via $R_1 \leftrightarrow R_2$. In our conventions $[1,0,0]$ is **4** (fundamental of $SU(4)$ but spinor of $SO(6)$) and $[0,1,0]$ is **6** (fundamental of $SO(6)$ but antisymmetric tensor of $SU(4)$).

The supersymmetric black holes discussed in section 4.3.1 preserve $\frac{1}{16}$ of the supersymmetry, so they correspond to BPS states that are annihilated by 2 out of 32 Hermitian supercharges. We choose Q and $Q^\dagger$ that obey the algebra

$$\begin{aligned} 2\{Q, Q^\dagger\} &= E - \left(j + \frac{3}{2}R_1 + R_2 + \frac{1}{2}R_3\right) \\ &= E - (Q_1 + Q_2 + Q_3 + J_1 + J_2) \geq 0 , \end{aligned} \tag{4.22}$$

which plays the role of (4.12) in the generic prescription. As explained in section 4.2, any field component can be identified with a weight in a representation, and so it is an eigenstate with respect to the operators E, Q_I and J_i. It is BPS if the corresponding eigenvalues saturate (4.22).

In 4d superconformal theories, a field $[j;\bar{j}]_E^{[R_1,R_2,R_3]}$ is a free field if at least one of j and $\bar{j}$ is zero and, in addition, $E = 1 + \frac{j+\bar{j}}{2}$. There is one multiplet of the 4d $\mathcal{N}=4$ superconformal algebra that contains a free field: the free vector multiplet, $B_1\bar{B}_1[0;0]_1^{[0,1,0]}$. All that we need is Table 3.1, where we summarize the BPS content of the free vector multiplet, i.e. all weights in the multiplet that saturate the unitarity bound (4.22).

There are 9 field components that satisfy the BPS condition. The BPS bosons are 3 of the 6 scalars in the theory, and 1 of the 2 gauge field components. The fermions are, in the language of $\mathcal{N}=1$ supersymmetry, 3 chiralini and 2 gaugini. The entry below the first double line is an equation of motion that relates the two gaugini. It should be counted as a "negative" field that serves to cancel some gaugini operators with derivatives acting on them. There are equations of motion for other free fields as well, but this component of the gaugino equation of motion is the only one that is consistent with the BPS condition. The last two entries in Table 3.1 are derivatives that may act on any of the fields, and on the equation of motion, to produce BPS descendants. The gradient operator has 4 components in 4 dimensions, but only 2 preserve the BPS-ness of the field. The $9-1=8$ free fields and their derivatives generate the entire list of supersymmetric operators in the free vector multiplet. From a bulk point of view, these are the single particle BPS states.

4.3.3 The Supersymmetric Ensemble

Given the exhaustive list of single particle BPS states generated by the supersymmetric operators in Table 3.1, we can now define a grand canonical partition function $Z_{\rm sp}$ over the single particle states. Rather than the chemical potentials as in (4.13), we use fugacities (p, q, x, y, z) that are related by

$$e^{\omega_1} = p^2 \ , \qquad e^{\omega_2} = q^2 \ , \qquad e^{\Delta_1} = x^2 \ , \qquad e^{\Delta_2} = y^2 \ , \qquad e^{\Delta_3} = z^2 \ , \tag{4.23}$$

and so define the single particle BPS partition function by

$$Z_{\rm sp} \ \equiv \ {\rm Tr}_{\rm BPS}\left[p^{2J_1} q^{2J_2} x^{2Q_1} y^{2Q_2} z^{2Q_3}\right] \ . \tag{4.24}$$

The maneuver doubling the exponents avoids fractional powers, although the subtle feature of non-analyticity and "second sheet" [89] is not relevant to our purpose.

We read off the single particle partition function $Z_{\rm sp}$ from Table 3.1. The sum over the weights of the $8 = 9 - 1$ free fields gives

$$x^2 + y^2 + z^2 + xyzpq\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} + \frac{1}{p^2} + \frac{1}{q^2} - 1\right) + p^2q^2 \ . \tag{4.25}$$

Any number of the two derivatives that preserve the BPS condition can act on each of the free fields, and on the equation of motion. Each derivative contributes a factor of p^2 or q^2, so we need a geometric sum over these. We then find the single particle BPS partition function

$$Z_{\rm sp} \ = \ \frac{x^2 + y^2 + z^2 + xyzpq\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} + \frac{1}{p^2} + \frac{1}{q^2} - 1\right) + p^2q^2}{(1-p^2)(1-q^2)} \ . \tag{4.26}$$

According to our prescription discussed in section 4.2, the full partition function is equal to the exponential of the single particle partition function (4.26):

$$Z \ \equiv \ \exp[Z_{\rm sp}] \ . \tag{4.27}$$

From this grand canonical partition function, we obtain the macroscopic charges as ensemble averages in the standard manner. Changing variables $(\Delta_I, \omega_i) \rightarrow (p, q, x, y, z)$, (4.17) becomes

$$2Q_1 = x\frac{\partial}{\partial x}\log Z \ , \qquad 2J_1 = p\frac{\partial}{\partial p}\log Z \ , \tag{4.28}$$

and analogously for the charges with different indices. The charges obtained from (4.27) in this way are

$$\begin{aligned} Q_1 &= \frac{2x^2 + xyzpq\left(-\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{p^2}+\frac{1}{q^2}-1\right)}{2(1-p^2)(1-q^2)} \ , \\ J_1 &= \frac{2p^2(q^2+x^2+y^2+z^2) + xyzpq(1+p^2)\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}-\frac{1}{p^2}+\frac{1}{q^2}-1\right) + 4xyzpq}{2(1-p^2)^2(1-q^2)} \ , \end{aligned} \tag{4.29}$$

and similarly for the permutations. (4.29) express the 5 average charges of the ensemble in terms of 5 potentials. BPS states are populated throughout the five-dimensional charge space, not just on some specific hypersurface thereof. Thus, the 5 average charges may take generic values without any particular constraint as well, as the 5 potentials are varied.

We now define the supersymmetric ensemble as a grand canonical ensemble where the operators that are separated in the charge space along the direction of the preserved supercharge are weighed equally. The preserved supercharge Q carries quantum numbers $(E, J_1, J_2, Q_1, Q_2, Q_3) = \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, so the supersymmetric ensemble corresponds to the relation

$$\frac{xyz}{pq} = 1 \ , \tag{4.30}$$

between the fugacities. For the supersymmetric ensemble satisfying (4.30), there is one relation between the five charges (4.29):

$$(Q_1Q_2Q_3 + J_1J_2) - (Q_1+Q_2+Q_3+1)(Q_1Q_2+Q_2Q_3+Q_3Q_1-J_1-J_2) = 0 \ . \tag{4.31}$$

This is precisely the supersymmetric AdS_5 black hole charge constraint (4.19) with

$$\frac{1}{2}N^2 = 1 \quad \leftrightarrow \quad \frac{\pi\ell_5^3}{4G_5} = 1 \ . \tag{4.32}$$

Equivalently, the statistical constraint (4.31) agrees with the macroscopic constraint (4.19) if macroscopic charges are in units of $\frac{1}{2}N^2$. A truly free $SU(N)$ theory would have N^2 identical copies of the free fields. We interpret the remaining relative factor $\frac{1}{2}$ as a reduction that is due to interactions, but we claim no quantitative understanding of this factor. This feature non-withstanding, our computation establishes the functional dependence on charges of the constraint (4.19) from the combinatorics of free fields.

The unrefined charges are defined by taking $x = y = z$ and $p = q$ in (4.29):

$$\begin{aligned} Q &= \frac{2x^2 + 2x^3 + xp^2 - x^3p^2}{2(1-p^2)^2} , \\ J &= \frac{3x(1+x)^2p^2 + (2+3x-x^3)p^4}{2(1-p^2)^3} . \end{aligned} \tag{4.33}$$

It follows automatically that, upon picking the supersymmetric ensemble $x^3 = p^2$, these charges satisfy the unrefined charge constraint (4.20) with $\frac{1}{2}N^2 = 1$.

4.4 AdS_4

In this section we derive the charge constraint for the supersymmetric AdS_4 black holes. The AdS_4 theory and its dual CFT_3 have features that are absent in $\mathrm{AdS}_5/\mathrm{CFT}_4$, such as magnetic charges and the Chern-Simons term. Such complications are not directly relevant to our computation. We find the charge constraint of the supersymmetric, rotating and electrically charged black holes in AdS_4 from the free hypermultiplet of the 3d $\mathcal{N} = 8$ superconformal algebra.

4.4.1 The Black Hole and the Charge Constraint

Asymptotically AdS_4 black holes arise as solutions to the 4d gauged supergravity theories [96, 97]. They carry the mass E and an angular momentum J for the isometry $SO(2,3)$ of AdS_4, and four electric charges $Q_{1,2,3,4}$ for the isometry $SO(8)$ of S^7. The solution with the four electric charges pairwise equal ($Q_1 = Q_3$ and $Q_2 = Q_4$) was found in [96], and the most general solution with all four electric charges independent was found in [97].

The black hole is supersymmetric when the unitarity bound between the mass and the charges

$$E \geq J + \frac{1}{2}\left(Q_1 + Q_2 + Q_3 + Q_4\right) , \tag{4.34}$$

is saturated. However, the saturation is possible only when the charges obey the additional relation [52, 97][2]

$$(\mathbb{Q}_3)^2 - (\mathbb{Q}_1)(\mathbb{Q}_2)(\mathbb{Q}_3) + (\mathbb{Q}_1)^2(\mathbb{Q}_4) \quad = \quad 0 \ , \tag{4.35}$$

where we have used the shorthand notation

$$\begin{aligned}
(\mathbb{Q}_1) &\equiv Q_1 + Q_2 + Q_3 + Q_4 \ , \\
(\mathbb{Q}_2) &\equiv Q_1Q_2 + Q_1Q_3 + Q_1Q_4 + Q_2Q_3 + Q_2Q_4 + Q_3Q_4 + \frac{2N^3}{9} \ , \\
(\mathbb{Q}_3) &\equiv Q_1Q_2Q_3 + Q_1Q_2Q_4 + Q_1Q_3Q_4 + Q_2Q_3Q_4 - \frac{4N^3}{9}J \ , \\
(\mathbb{Q}_4) &\equiv Q_1Q_2Q_3Q_4 + \frac{2N^3}{9}J^2 \ .
\end{aligned} \tag{4.36}$$

In the formulae above, we set the AdS_4 radius $\ell_4 = 1$. We traded the 4d Newton's constant for the field theory variable N via

$$N^{\frac{3}{2}} = \frac{3}{2\sqrt{2}G_4} \ ,$$

for future convenience, but we stress that the origin of the charge constraint (4.35) is purely gravitational. (4.35) is the supersymmetric AdS_4 black hole charge constraint.

To make the formulae more approachable and to make the connection to the literature, we also present the charge constraint (4.35) with pairwise equal electric charges (see e.g. [52])

$$Q_1Q_2(Q_1 + Q_2)^2 - (Q_1 + Q_2) \cdot \frac{2N^3}{9}J - \frac{2N^3}{9}J^2 \quad = \quad 0 \ . \tag{4.37}$$

as well as the version with all four electric charges equal (see e.g. [98]):

$$4Q^4 - 2Q \cdot \frac{2N^3}{9}J - \frac{2N^3}{9}J^2 \quad = \quad 0 \ . \tag{4.38}$$

The formulae simplify greatly, but they remain quite nontrivial. The unrefined charge Q in this formula should not be confused with the preserved supercharge.

[2] Although the solution with all four electric charges independent was found in [97], its charge constraint had been correctly conjectured earlier [52], based on the solution with pairwise equal charges [96] and the structure of the entropy function.

4.4.2 The 3d $\mathcal{N}=8$ Free Hypermultiplet

In this subsection we present the free hypermultiplet of the 3d $\mathcal{N}=8$ superconformal algebra, from which the AdS$_4$ charge constraint (4.35) will be derived in the next subsection.

The 3d $\mathcal{N}=8$ superconformal algebra has maximal bosonic subalgebra $\mathfrak{so}(2,3)\oplus\mathfrak{so}(8)$, matching the isometry of AdS$_4 \times S^7$. Local operators in the theory are organized into representations of this subalgebra. A super-representation of the 3d $\mathcal{N}=8$ superconformal algebra is uniquely specified by the Dynkin labels of its superconformal primary. Following the notation of [88], we write representations of the bosonic subalgebra as

$$[j]_E^{[R_1,R_2,R_3,R_4]} \ ,$$

where E is the conformal weight, j is the integer-quantized $SO(3)$ Dynkin label, and $[R_1,R_2,R_3,R_4]$ are the $SO(8)$ Dynkin labels so that $[1,0,0,0]$ is the vector $\mathbf{8}$.

The black hole charges used in section 4.4.1 refer to the orthogonal basis that is related to the Dynkin basis as

$$\begin{aligned}
&J=\frac{j}{2}\ , \\
&Q_1=R_3+R_2+\frac{R_1+R_4}{2}\ ,\quad Q_2=R_2+\frac{R_1+R_4}{2}\ ,\quad Q_3=\frac{R_1+R_4}{2}\ ,\quad Q_4=\frac{R_1-R_4}{2}\ .
\end{aligned} \tag{4.39}$$

This relation between the orthogonal and the Dynkin bases of $SO(8)$ differs from the more conventional one by $R_1\leftrightarrow R_3$. We have exploited the S_3 outer automorphism of $SO(8)$ to match the convention (4.34) with that of [88].

A (not necessarily the highest) weight $[j]_E^{[R_1,R_2,R_3,R_4]}$ is annihilated by our choice of supercharge Q if it saturates the unitarity bound

$$\begin{aligned}
E &\geq \frac{1}{2}j+R_1+R_2+\frac{1}{2}R_3+\frac{1}{2}R_4 \\
&= J+\frac{1}{2}\left(Q_1+Q_2+Q_3+Q_4\right)\ ,
\end{aligned} \tag{4.40}$$

that every weight must satisfy. Such weights correspond to local BPS operators.

In 3d superconformal theories, a field $[j]_E^{[R_1,R_2,R_3,R_4]}$ is free if $j\leq 1$ and $E=\frac{j+1}{2}$. There are two multiplets of the 3d $\mathcal{N}=8$ superconformal algebra that contain a free field [88]. The free hypermultiplets $B_1[0]_{\frac{1}{2}}^{[0,0,1,0]}$ and $B_1[0]_{\frac{1}{2}}^{[0,0,0,1]}$ are related by a Z_2 subgroup of the outer automorphism of $SO(8)$, so we can choose $B_1[0]_{\frac{1}{2}}^{[0,0,1,0]}$ without loss of generality. The rest of this section would be reproduced with minimal relabeling had we chosen otherwise.

In Table 4.2 we summarize all weights in this free hypermultiplet that saturate the uni-

	Bosonic Rep.	E	j	R_1	R_2	R_3	R_4	J	Q_1	Q_2	Q_3	Q_4
Free fields	$[0]^{[0,0,1,0]}_{\frac{1}{2}}$	$\frac{1}{2}$	0	0	0	1	0	0	1	0	0	0
		$\frac{1}{2}$	0	0	1	-1	0	0	0	1	0	0
		$\frac{1}{2}$	0	1	-1	0	1	0	0	0	1	0
		$\frac{1}{2}$	0	1	0	0	-1	0	0	0	0	1
	$[1]^{[0,0,0,1]}_{1}$	1	1	0	0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
		1	1	0	1	0	-1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
		1	1	1	-1	1	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
		1	1	1	0	-1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
Derivative	$[2]^{[0,0,0,0]}_{1}$	1	2	0	0	0	0	1	0	0	0	0

Table 4.2: Components of the BPS operators in the free hypermultiplet $B_1[0]^{[0,0,1,0]}_{\frac{1}{2}}$. The first 8 rows are free fields, followed by one derivative that preserves BPS.

tarity bound (4.40). There are 8 free fields: 4 scalars and 4 spinors. There is no equation of motion that is compatible with the BPS condition. The last entry is a derivative that can act on any of the fields and so produce its BPS descendants. Note that out of 3 derivatives in 3 dimensions, only 1 preserves the BPS-ness of the field. The 8 free fields and their derivatives are the exhaustive list of supersymmetric operators in the free hypermultiplet.

4.4.3 The Supersymmetric Ensemble

We now compute the single particle BPS partition function as a trace over the free hypermultiplet states given in Table 4.2, with fugacities (p, x, y, z, w) conjugate to each charge:

$$\begin{aligned} Z_{\text{sp}} &\equiv \text{Tr}_{\text{BPS}}\left[p^{2J} x^{2Q_1} y^{2Q_2} z^{2Q_3} w^{2Q_4}\right] \\ &= \frac{x^2 + y^2 + z^2 + w^2 + pxyzw\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} + \frac{1}{w^2}\right)}{1 - p^2}\ . \end{aligned} \tag{4.41}$$

It is the derivative that gives rise to the geometric series in p^2. (4.41) is the single particle partition function.

The grand canonical partition function over the full Hilbert space is given by the ordinary exponential of the single particle partition function: $Z \equiv \exp[Z_{\text{sp}}]$. We then compute the macroscopic charges as statistical averages. They are

$$\begin{aligned}
Q_1 &= \frac{pxyzw\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{w^2}\right)+2x^2-\frac{2pxyzw}{x^2}}{2(1-p^2)} , \qquad (4.42)\\
J &= \frac{pxyzw\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{w^2}\right)}{2\left(1-p^2\right)}+\frac{p^2\left(x^2+y^2+z^2+w^2+pxyzw\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{w^2}\right)\right)}{\left(1-p^2\right)^2} ,
\end{aligned}$$

Analogous expressions for Q_2, Q_3, and Q_4 follow by simple permutations of indices.

Finally, we define the supersymmetric ensemble as a grand canonical ensemble where the operators that are separated in the charge space along the direction of the preserved supercharge are weighed equally. The preserved supercharge Q carries quantum numbers $(E, J, Q_1, Q_2, Q_3, Q_4) = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, so the supersymmetric ensemble corresponds to imposing the relation

$$\frac{xyzw}{p} = 1 , \qquad (4.43)$$

between the fugacities.

Because of the relation (4.43) between the 5 potentials, in the supersymmetric ensemble the 5 charges $Q_{1,2,3,4}$ and J are not independent. The expressions (4.42) give the relation:

$$(\mathbb{Q}_3)^2_{\frac{9}{2}} - (\mathbb{Q}_1)_{\frac{9}{2}}(\mathbb{Q}_2)_{\frac{9}{2}}(\mathbb{Q}_3)_{\frac{9}{2}} + (\mathbb{Q}_1)^2_{\frac{9}{2}}(\mathbb{Q}_4)_{\frac{9}{2}} = 0 , \qquad (4.44)$$

where

$$\begin{aligned}
(\mathbb{Q}_1)_{\frac{9}{2}} &\equiv Q_1+Q_2+Q_3+Q_4 ,\\
(\mathbb{Q}_2)_{\frac{9}{2}} &\equiv Q_1Q_2+Q_1Q_3+Q_1Q_4+Q_2Q_3+Q_2Q_4+Q_3Q_4+1 ,\\
(\mathbb{Q}_3)_{\frac{9}{2}} &\equiv Q_1Q_2Q_3+Q_1Q_2Q_4+Q_1Q_3Q_4+Q_2Q_3Q_4-2J ,\\
(\mathbb{Q}_4)_{\frac{9}{2}} &\equiv Q_1Q_2Q_3Q_4+J^2 . \qquad (4.45)
\end{aligned}$$

It is precisely the supersymmetric AdS_4 black hole charge constraint (4.35) with the numerical values

$$\frac{\sqrt{2}}{3}N^{\frac{3}{2}} = 1 \quad \leftrightarrow \quad G_4 = \frac{1}{2} . \qquad (4.46)$$

We interpret this relative scale of all charges as the effective number of free multiplets needed to account for the constraint.

The formulae simplify significantly when we do not distinguish between all 4 electric charges. First, let $z = x$ and $w = y$ in (4.42):

$$\begin{aligned} Q_1 = Q_3 &= \frac{x^2}{1-p} , \\ Q_2 = Q_4 &= \frac{y^2}{1-p} , \\ J &= \frac{p(x^2+y^2)}{(1-p)^2} . \end{aligned} \tag{4.47}$$

The definition of the supersymmetric ensemble (4.43) simplifies to $p = x^2y^2$, and then the charges (4.47) satisfy

$$Q_1Q_2(Q_1+Q_2)^2 - (Q_1+Q_2)J - J^2 = 0 . \tag{4.48}$$

This is the pairwise unrefined version of the charge constraint (4.37) with $\frac{\sqrt{2}}{3}N^{\frac{3}{2}} = 1$.

To treat all 4 electric charges as identical, we further let $x = y$ in (4.47):

$$\begin{aligned} Q \equiv Q_{1,2,3,4} &= \frac{x^2}{1-p} , \\ J &= \frac{2px^2}{(1-p)^2} . \end{aligned} \tag{4.49}$$

These charges, with the equation $p = x^4$ defining the supersymmetric ensemble, satisfy

$$4Q^4 - 2QJ - J^2 = 0 . \tag{4.50}$$

This is the fully unrefined version of the charge constraint (4.38) with $\frac{\sqrt{2}}{3}N^{\frac{3}{2}} = 1$.

4.5 AdS$_7$

In this section we derive the charge constraint for the supersymmetric, rotating and charged black holes in AdS$_7$. from the dual $(2,0)$ theory in 6d.

4.5.1 The Black Hole and the Charge Constraint

Asymptotically AdS$_7$ black holes arise as solutions to a consistent truncation of the 11d supergravity on S^4. They carry the mass E and three angular momenta $J_{1,2,3}$ for the isometry $SO(2,6)$ of AdS$_7$, and two charges $Q_{1,2}$ for the isometry $SO(5) \sim Sp(4)$ of S^4. Particular solutions with equal angular momenta [99], those with equal charges [100] and those with two vanishing angular momenta and two independent charges [101, 102] were constructed some time ago, but the solution with all angular momenta and charges independent was found only recently in [103].

These black holes are supersymmetric when the unitarity bound between the mass and the charges

$$E \geq J_1 + J_2 + J_3 + Q_1 + Q_2 \ , \tag{4.51}$$

is saturated. However, the saturation is possible only when the charges obey the additional relation [98, 103][3]

$$\begin{aligned} &\frac{1}{2}(Q_1^2 + Q_2^2 + 4Q_1Q_2) + \frac{N^3}{3}(J_1 + J_2 + J_3) - \frac{Q_1Q_2(Q_1+Q_2) - \frac{N^3}{3}(J_1J_2 + J_2J_3 + J_3J_1)}{Q_1 + Q_2 - \frac{N^3}{3}} \\ = &\sqrt{\left(\frac{1}{2}(Q_1^2 + Q_2^2 + 4Q_1Q_2) + \frac{N^3}{3}(J_1 + J_2 + J_3)\right)^2 - \left(Q_1^2Q_2^2 + \frac{2N^3}{3}J_1J_2J_3\right)} \ . \end{aligned} \tag{4.52}$$

In the formulae above, we set the AdS$_7$ radius $\ell_7 = 1$. We traded the 7d Newton's constant for the field theory variable N via

$$N^3 = \frac{3\pi^2}{16G_7} \ ,$$

for future convenience, but we stress that the origin of the charge constraint (4.52) is purely gravitational. (4.52) is the supersymmetric AdS$_7$ black hole charge constraint. We also present an unrefined ($J_1 = J_2 = J_3 = J$ and $Q_1 = Q_2 = Q$, where Q should not be confused with the preserved supercharge) version of (4.52) to make the formula more approachable:

$$\begin{aligned} &\left(Q^4 + \frac{2N^3}{3}J^3\right)\left(Q - \frac{N^3}{6}\right)^2 \\ = \ &2(3Q^2 + N^3J)\left(Q^3 - \frac{N^3}{2}J^2\right)\left(Q - \frac{N^3}{6}\right) - \left(Q^3 - \frac{N^3}{2}J^2\right)^2 \ . \end{aligned} \tag{4.53}$$

4.5.2 The 6d (2, 0) Free Tensor Multiplet

The charged, rotating AdS$_7$ black holes introduced in the previous subsection are dual to the 6d (2, 0) theory. In this subsection we present the free tensor multiplet of the (2, 0) superconformal algebra needed to construct the single particle partition function.

The 6d (2, 0) superconformal algebra has maximal bosonic subalgebra $\mathfrak{so}(2, 6) \oplus \mathfrak{sp}(4)$,

[3]The convention for charges differ from that of [103] by $J_i^{here} = J_i^{there}$ and $Q_i^{here} = \frac{Q_i^{there}}{2}$.

matching the isometry of $\mathrm{AdS}_7 \times S^4$. Local operators in the theory are organized into representations of this subalgebra. A super-representation of the 6d $(2,0)$ superconformal algebra is uniquely specified by the Dynkin labels of its superconformal primary. For easy comparison with black hole spacetimes, we use $SO(6)$ for the Lorentz group and $SO(5)$ for the R-symmetry group, instead of $SU(4)$ for Lorentz and $Sp(4)$ for R-symmetry used in [88].[4] So we write representations of the bosonic subalgebra as

$$[j_1, j_2, j_3]_E^{[R_1,R_2]} \ ,$$

where E is the conformal weight, $[j_1, j_2, j_3]$ are the $SO(6)$ Dynkin labels so that $[1,0,0]$ is the vector **6**, and $[R_1, R_2]$ are the $SO(5)$ Dynkin labels so that $[1,0]$ is the vector **5**.

The black hole charges used in section 4.5.1 refer to the orthogonal basis that is related to the Dynkin basis as

$$\begin{aligned} J_1 &= j_1 + \frac{j_2+j_3}{2} \ , \qquad J_2 = \frac{j_2+j_3}{2} \ , \qquad J_3 = \frac{-j_2+j_3}{2} \ , \\ Q_1 &= R_1 + \frac{R_2}{2} \ , \qquad Q_2 = \frac{R_2}{2} \ . \end{aligned} \tag{4.54}$$

A (not necessarily the highest) weight $[j_1, j_2, j_3]_E^{[R_1,R_2]}$, is annihilated by our choice of a supercharge Q if it saturates the unitarity bound

$$\begin{aligned} E &\geq j_1 + \frac{1}{2}j_2 + \frac{3}{2}j_3 + 2R_1 + 2R_2 \\ &= J_1 + J_2 + J_3 + 2Q_1 + 2Q_2 \ , \end{aligned} \tag{4.55}$$

that every weight must satisfy. Such weights correspond to local BPS operators.

In 6d superconformal theories, a field $[j_1, j_2, j_3]_E^{[R_1,R_2]}$ is free if $j_1 = 0$, at least one of j_2 and j_3 is zero, and $E = 2 + \frac{j_2+j_3}{2}$. There is only one multiplet of the 6d (2, 0) superconformal algebra that contains a free field: the free tensor multiplet $D_1[0,0,0]_2^{[1,0]}$ [88]. In Table 4.3 we summarize all weights in the free tensor multiplet that saturate the unitarity bound (4.55).

In Table 4.3, we have listed 5 free fields: 2 scalars and 3 spinors. The entry below is an equation of motion that implements a relation between two spinors, so it can be counted as a negative field. The three last entries are derivatives that may act on any of the fields and on the equation of motion, to produce their BPS descendants. The gradient in 6 dimensions has 6 components but only 3 preserve the BPS-ness of the field. The 5 free fields, modulo the equation of motion, and with possible derivatives taken into account, are the exhaustive list of supersymmetric operators in the free tensor multiplet.

[4] This amounts to the interchanges $j_1 \leftrightarrow j_2$ and $R_1 \leftrightarrow R_2$.

	Bosonic Rep.	E	j_1	j_2	j_3	R_1	R_2	J_1	J_2	J_3	Q_1	Q_2
Free fields	$[0,0,0]^{[1,0]}_{2}$	2	0	0	0	1	0	0	0	0	1	0
		2	0	0	0	-1	2	0	0	0	0	1
	$[0,1,0]^{[0,1]}_{\frac{5}{2}}$	$\frac{5}{2}$	0	1	0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
		$\frac{5}{2}$	1	-1	0	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
		$\frac{5}{2}$	-1	0	1	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
Eq. of motion	$[0,0,1]^{[0,1]}_{\frac{7}{2}}$	$\frac{7}{2}$	0	0	1	0	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
Derivatives	$[1,0,0]^{[0,0]}_{1}$	1	1	0	0	0	0	1	0	0	0	0
		1	-1	1	1	0	0	0	1	0	0	0
		1	0	-1	1	0	0	0	0	1	0	0

Table 4.3: Components of the BPS operators in the free tensor multiplet $D_1[0,0,0]^{[1,0]}_2$. The first 5 rows are free fields, followed by the equation of motion and 3 derivatives.

4.5.3 The Supersymmetric Ensemble

We now compute the single particle BPS partition function as a trace over the free tensor multiplet states given in Table 4.3, with fugacities (p,q,r,x,y) conjugate to each charge:

$$\begin{aligned} Z_{\text{sp}} &\equiv \text{Tr}_{\text{BPS}}\left[p^{2J_1}q^{2J_2}r^{2J_3}x^{2Q_1}y^{2Q_2}\right] \\ &= \frac{x^2+y^2+xypqr\left(\frac{1}{p^2}+\frac{1}{q^2}+\frac{1}{r^2}-1\right)}{(1-p^2)(1-q^2)(1-r^2)}\ . \end{aligned} \tag{4.56}$$

The -1 inside the parenthesis in the numerator is due to the equation of motion. The geometric series in p^2, q^2, and r^2 are from the derivatives. (4.56) is the single particle partition function.

The grand canonical partition function over the full Hilbert space is given by the ordinary exponential of the single particle partition function: $Z \equiv \exp[Z_{\text{sp}}]$. We use it to compute the macroscopic charges as statistical averages:

$$\begin{aligned} Q_1 &= \frac{2x^2+xypqr\left(\frac{1}{p^2}+\frac{1}{q^2}+\frac{1}{r^2}-1\right)}{2(1-p^2)(1-q^2)(1-r^2)}\ , \\ J_1 &= \frac{2p^2(x^2+y^2)+xypqr(1+p^2)\left(-\frac{1}{p^2}+\frac{1}{q^2}+\frac{1}{r^2}-1\right)+4xypqr}{2(1-p^2)^2(1-q^2)(1-r^2)}\ , \end{aligned} \tag{4.57}$$

Analogous expressions for Q_2, J_2 and J_3 follow by permutations of indices.

Finally, we define the supersymmetric ensemble as a grand canonical ensemble where the operators that are separated in the charge space along the direction of the preserved supercharge are weighed equally. The preserved supercharge Q carries quantum numbers $(E, J_1, J_2, J_3, Q_1, Q_2) = \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, so the supersymmetric ensemble corresponds to imposing the relation

$$\frac{xy}{pqr} = 1 \ , \tag{4.58}$$

between the fugacities.

In the supersymmetric ensemble defined by (4.58), there is one relation between the 5 charges (4.57):

$$\begin{aligned} &\frac{1}{2}(J_1+J_2+J_3) + \frac{1}{2}(Q_1^2+Q_2^2+4Q_1Q_2) - \frac{Q_1Q_2(Q_1+Q_2) - \frac{1}{2}(J_1J_2+J_2J_3+J_3J_1)}{Q_1+Q_2-\frac{1}{2}} \\ = &\sqrt{\left(\frac{1}{2}(J_1+J_2+J_3)+\frac{1}{2}(Q_1^2+Q_2^2+4Q_1Q_2)\right)^2 - (J_1J_2J_3+Q_1^2Q_2^2)} \ . \end{aligned} \tag{4.59}$$

It is precisely the supersymmetric AdS_7 black hole charge constraint (4.52) with the numerical values

$$\frac{2}{3}N^3 = 1 \quad \leftrightarrow \quad \frac{\pi^2}{8G_7} = 1 \ . \tag{4.60}$$

We interpret this relative scale of all charges as the effective number of free multiplets needed to account for the constraint. It is satisfying that the numerical factor $\frac{2}{3} < 1$ since the interpolation from weak to strong coupling is expected to decrease the effective number of degrees of freedom.

The formulae simplify significantly when we do not distinguish between the 3 angular momenta and between the 2 electric charges. Let $x = y$ and $p = q = r$ in (4.57)):

$$\begin{aligned} Q &= \frac{2x^2+3x^2p-x^2p^3}{2(1-p^2)^3} \ , \\ J &= \frac{x^2p+4x^2p^2+4x^2p^3-x^2p^5}{2(1-p^2)^4} \ . \end{aligned} \tag{4.61}$$

The definition of the supersymmetric ensemble (4.58) simplifies to $x^2 = p^3$, and then the charges (4.61) satisfy the unrefined charge constraint (4.53) with $\frac{2}{3}N^3 = 1$.

4.6 Discussions

In this chapter, we derived the supersymmetric charge constraint for the $AdS_{4,5,7}$ black holes using the simple prescription given in section 4.2. We think the computations are illuminating, especially because they are so simple. However, we acknowledge that, in its current form, the argument is heuristic and subject to significant concerns. These challenges are the subject of this final section. We divide them into four issues, even though their possible resolutions are interrelated:

1) *Coupling dependence.* Unlike the index, the partition function depends on the coupling $g_{\rm YM}$. We study an ensemble of states generated by free fields and, even so, we compare the result to black holes that correspond to strong coupling.

2) *Gauge dynamics.* In each case, we consider a single free field, rather than the dynamics due to gauge degrees of freedom.

   The dependence on Newton's constant is determined by dimensional analysis in gravity, while the dependence on the rank in the dual CFT is reproduced by assuming that it is in its deconfined phase. However, a numerical constant of $\mathcal{O}(1)$ is put in by hand.

3) *Classical statistics.* We consider a classical gas of BPS particles. Technically, we take the multiparticle partition function to be the simple exponential of the single particle partition function, rather than the plethystic exponential. We did not justify why this approximation is sufficient.

4) *The supersymmetric ensemble*: is defined so states that differ by the charges of the preserved supercharge Q are given equal weight. This is motivated by the real part of the supersymmetry constraint on complex fugacities, which are well established in supersymmetric black hole spacetimes [47–49, 65, 85]. We did not provide a self-contained justification of this ensemble in the CFT, except for the CFT_2 argument presented in section 4.1.

The numerical factor mentioned in 2) presents a concrete goal that involves several of these issues. If all fields were genuinely free, the number of independent multiplets would be N^2 in AdS_5/CFT_4, from the dimension of the $SU(N)$ gauge group of $\mathcal{N}=4$ SYM, and similarly in other dimensions. This type of a naïve count of multiplets would not even take the projection onto gauge singlets for the physical Hilbert space into account. This can in principle be addressed by upgrading to a matrix model and, in particular, confronting 3) [34]. However, this still leaves 1), the dependence on the coupling constant: some of the BPS states in the free theory may gain anomalous dimensions and be lifted from being

BPS. The projection onto singlets and the dependence on the coupling both suggest that the naïve scaling in N overcounts the microscopic states rather than undercounts. It is therefore encouraging that all the needed $\mathcal{O}(1)$ adjustments are smaller than 1. Table 4.1 records the rescaling factors $\frac{\sqrt{2}}{3}N^{3/2}$, $\frac{1}{2}N^2$ and $\frac{2}{3}N^3$ for the AdS_4, AdS_5, and AdS_7 charge constraints, respectively. The situation is reminiscent of the famous 3/4-renormalization of high temperature D3-brane entropy as the coupling is taken from weak to strong [104].

Our discussion of supersymmetric black holes in AdS_3 is on more solid footing than in the higher dimensions. That is because the superalgebra is much stronger, it gives a complete basis of characters for both short and long supermultiplets of the $\mathcal{N} = 4$ super-Virasoro algebra, and so no free field assumption is needed. In this context the supersymmetric ensemble is justified by a symmetry, and the constraint we find agrees precisely with the black hole side, with no numerical factor put in by hand. These results offer a template for higher dimensions that we have pursued, especially when addressing 4), but it is possible that other lessons remain hidden in plain sight.

Our approach is fundamentally limited by us studying the partition function, rather than the supersymmetric index. Therefore, our computation is unavoidably subject to dependence on the coupling constant that is beyond our control. On the other hand, although the index is an invaluable tool for circumventing the coupling dependence, it has its own structural limitations. Because it is insensitive to many quantum states, it can at best provide a lower bound on the black hole entropy, and so any agreement is only genuinely successful if it is understood why cancellations are subleading. The limitations of the index are especially pertinent in our context, the constraint on charges that is satisfied by all supersymmetric black holes in AdS spacetimes. That is because the index is independent of the relevant physical variable, to the best of our understanding.

For the future, the vision ultimately is that all the various contributions to the partition function, in gravity and in CFT, whether boundary conditions correspond to an index or not, can be disentangled. Significant strides have been taken towards this goal in the most favorable circumstances, such as asymptotically flat spacetimes with at least $\frac{1}{8}$ of the supersymmetries [105–109]. For asymptotically AdS spacetimes with maximal supersymmetry, the setting we have studied, the current research frontier is at a lower level of understanding, but recent years have witnessed much progress, using a variety of techniques [110–112]. The work presented in this chapter, including the challenges discussed in this section, is a contribution to these developments.

Part II

Towards Quantum Black Hole Microstates

Chapter 5

The Black Hole Cohomology Problem

In the second part starting from this chapter, we work on constructing the explicit expressions for supersymmetric black hole microstates in the language of the dual weakly coupled superconformal field theory.

As mentioned in the Introduction, the AdS_5 black holes introduced in section 2.2 are solutions of the supergravity theory in 5 dimensions, and therefore the dual black hole microstates live in the strongly coupled 4-dimensional CFT with a large gauge group $SU(N)$ with $N \to \infty$. However, since the black hole microstates are correctly counted by a coupling independent quantity, namely the index, there should be as many analogous states in the weakly coupled field theory. See [113] for the connection between states in the weakly coupled and the strongly coupled theories. In a different point of view, one may argue that microstates in the finite-N, weakly coupled regime of the field theory are dual to black hole microstates in the full quantum gravity theory, rather than its supergravity approximation. With these arguments in mind, we study the supersymmetric quantum states, or local BPS operators, in the weakly coupled, finite-N field theory.

We focus on $\frac{1}{16}$-BPS states of the 4d $\mathcal{N} = 4$ Yang-Mills theory with $SU(N)$ gauge group, dual to type IIB string theory in $AdS_5 \times S^5$. The BPS states can be reformulated as classical cohomologies with respect to a nilpotent supercharge Q. Our goal in this part is to construct such cohomologies for finite values of $N = 2, 3, 4$ that are not of the graviton type, and therefore potentially represent the black hole microstates.

This part is based on [71, 72] in collaboration with Jaehyeok Choi, Sunjin Choi, Seok Kim, Eunwoo Lee, Jehyun Lee and Jaemo Park.

5.1 Formulation of the Problem

In this section, we reformulate the problem of listing local BPS operators in the weakly coupled 4d $\mathcal{N} = 4$ Yang-Mills theory on $\mathbb{R}^4$ into that of finding classical cohomologies with respect to a supercharge Q. We also review how to systematically construct such cohomologies, using the BPS letters as building blocks, partly repeating 3.1.2.

The $\mathcal{N} = 4$ Yang-Mills theory with $SU(N)$ gauge group carries a continuous real marginal coupling constant $g_{\rm YM}$, and enjoys $\mathcal{N} = 4$ superconformal symmetry $PSU(2,2|4)$ at any value of $g_{\rm YM}$. The theta angle will not be relevant in our discussions.

The theory includes six real scalars, eight fermions and the gauge field, all in the $SU(N)$ adjoint representation. To repeat (3.11), we denote them as

$$\begin{aligned}
\text{vector} &: A_\mu \sim A_{\alpha\dot\beta} \ , && (\mu = 1,2,3,4 \ , \ \alpha = \pm \ , \ \dot\beta = \dot\pm) \\
\text{scalar} &: \Phi_{ij}(= -\Phi_{ji}) \ , \ \overline{\Phi}^{ij} \sim \tfrac{1}{2}\epsilon^{ijkl}\Phi_{kl} \ , && (i,j,k,l = 1,2,3,4) \\
\text{fermion} &: \Psi_{i\alpha} \ , \ \overline{\Psi}^i_{\dot\alpha} \ . &&
\end{aligned} \tag{5.1}$$

$\alpha, \dot\alpha$ are the doublet indices of the Lorentz group $SU(2)_L \times SU(2)_R \sim SO(4)$ which rotate the S^3, and μ is the vector index. Superscripts i, j are for the fundamental representation of the $SU(4)$ R-symmetry, while the subscripts are for the anti-fundamental representation. For later convenience, we arrange these fields into $\mathcal{N} = 1$ supermultiplets as follows, with manifest covariance only for the $SU(3) \subset SU(4)$ part of R-symmetry,

$$\begin{aligned}
\text{vector multiplet} &: A_{\alpha\dot\beta} \ , \ \lambda_\alpha = \Psi_{4\alpha} \ , \ \bar\lambda_{\dot\alpha} = \overline{\Psi}^4_{\dot\alpha} \ , \\
\text{3 chiral multiplets} &: \phi_m = \Phi_{4m} \ , \ \bar\phi^m = \overline{\Phi}^{4m} \ , \ \psi_{m\alpha} = -i\Psi_{m\alpha} \ , \ \bar\psi^m_{\dot\alpha} = i\overline{\Psi}^m_{\dot\alpha} \ ,
\end{aligned} \tag{5.2}$$

where $m = 1,2,3$ is the index for the $SU(3)$ subset of the R-symmetry and labels the chiral multiplets.

We consider the Euclidean CFT on $\mathbb{R}^4$, related to the Lorentzian CFT on $S^3 \times \mathbb{R}$ by radial quantization, which regards the radius of $\mathbb{R}^4$ as the exponential of the Euclidean time τ and makes a Wick rotation $\tau = it$. Here we note the operator-state map, in which the local operators at the origin of $\mathbb{R}^4$ map to the states propagating in $S^3 \times \mathbb{R}$. We will omit the spacetime arguments of the local operators.

The CFT is invariant under 32 supersymmetries, represented by the 16 Poincaré supercharges Q^i_α, $\overline{Q}_{i\dot\alpha}$ and the 16 conformal supercharges $S_{i\alpha}$, $\overline{S}^i_{\dot\alpha}$. In the radially quantized theory, S's are Hermitian conjugates of Q's: $S_i^\alpha = (Q^i_\alpha)^\dagger$, $\overline{S}^{i\dot\alpha} = (\overline{Q}_{i\dot\alpha})^\dagger$. Together with other symmetry generators, these supercharges form the $PSU(2,2|4)$ superconformal algebra. The most important part of the algebra for our discussion is [35]

$$\{Q^i_\alpha, S^\beta_j\} = \tfrac{1}{2}E\delta^i_j\delta^\beta_\alpha + R^i{}_j\delta^\beta_\alpha + J_\alpha{}^\beta\delta^i_j \ , \tag{5.3}$$

where E is the dilatation operator (or the Hamiltonian on $S^3 \times \mathbb{R}$ multiplied by the radius of S^3), $R^i{}_j$ is the $SU(4)$ R-charges, and $J_\alpha{}^\beta$ is the left $SU(2) \subset SO(4)$ angular momenta. We choose two of the supercharges to be preserved: $Q \equiv Q^4_-$ and $S = Q^\dagger \equiv S^-_4$. These two supercharges satisfy $Q^2 = 0$, $(Q^\dagger)^2 = 0$, and from (5.3) one obtains

$$2\{Q, Q^\dagger\} = E - (Q_1 + Q_2 + Q_3 + J_1 + J_2) \ . \tag{5.4}$$

This is the identical choice of supercharges as in (3.3). On the right hand side, we expressed $2R^4{}_4 = -Q_1 - Q_2 - Q_3$ and $2J_-{}^- = -J_1 - J_2$ in terms of the five charges which rotate the mutually orthogonal 2-planes on $\mathbb{R}^6 \supset S^5$ and $\mathbb{R}^4 \supset S^3$, respectively, all normalized to have $\pm\frac{1}{2}$ values for spinors.

The $\frac{1}{16}$-BPS states/operators of our interest preserve these 2 Hermitian supercharges. Thus, we are interested in gauge-invariant local operators O that are annihilated by Q:

$$[Q, O\} = 0 \ , \ [Q^\dagger, O\} = 0 \ . \tag{5.5}$$

It follows from (5.4) that the BPS operators of our interest can be arranged to be the eigenstates of H, R_I and J_i, with respective eigenvalues E, R_I and J_i that satisfy

$$E = Q_1 + Q_2 + Q_3 + J_1 + J_2 \ . \tag{5.6}$$

The charges Q_I, J_i on the right hand side are part of the non-Abelian charges and cannot depend on the coupling $g_{\rm YM}$. However, E is in general a function of $g_{\rm YM}$, so that a BPS state may become anomalous as $g_{\rm YM}$ changes.

Let us first consider local BPS operators of the free ($g_{\rm YM} = 0$) theory. In the free theory, the operators satisfying the BPS relation (5.6) can be easily constructed using the BPS elementary fields. The BPS elementary fields are the members of the free vector multiplet $B_1\overline{B}_1[0;0]^{[0,1,0]}_1$ that satisfy the BPS relation (5.6), and they have been summarized in Table 3.1. We give the following names to the nine free fields and two derivatives:

$$\phi^m \ , \quad \psi_m \ , \quad f \ , \quad \lambda_{\dot\alpha} \ , \quad \partial_{\dot\alpha} \ . \tag{5.7}$$

Note that these are a BPS subset of (5.1), but with bars and some indices that are common to BPS fields stripped off because non-BPS fields will not appear any more. Also note that

$m = 1, 2, 3$ and $\alpha = \pm$. With these, we construct independent 'letters' for the gauge invariant operators. Basically, acting any numbers of two derivatives $\partial_{\dot\alpha}$ on a BPS field forms a letter. In the free theory, the derivatives $\partial_{\dot\alpha}$ acting on the same field commute, so all $SU(2)_R$ indices appearing in a letter should be symmetrized. However, the equation of motion operator is null and should not be included. The only equation of motion constructed using (5.7) is

$$\partial_{\dot\alpha}\lambda^{\dot\alpha} = 0 \quad \Leftrightarrow \quad \partial_{[\dot\alpha}\lambda_{\dot\beta]} = 0 \ . \tag{5.8}$$

The equation of motion can be imposed by requiring the $SU(2)_R$ indices carried by the derivatives and the gaugino $\lambda_{\dot\alpha}$ to be symmetrized within a letter. So for example,

$$\begin{aligned} &\partial_{(\dot\alpha_1}\cdots\partial_{\dot\alpha_n)}\phi^m \ , \qquad \partial_{(\dot\alpha_1}\cdots\partial_{\dot\alpha_n)}\psi_m \ , \qquad \partial_{(\dot\alpha_1}\cdots\partial_{\dot\alpha_n)}f \ , \\ &\partial_{(\dot\alpha_1}\cdots\partial_{\dot\alpha_{n-1}}\lambda_{\dot\alpha_n)} \end{aligned} \tag{5.9}$$

are the BPS letters. Multiplying these letters and contracting all $SU(N)$ indices that are omitted in (5.9), one can construct general gauge-invariant BPS operators in the free theory.

Moving away from the free theory, we want to study how many of these operators remain BPS at the 1-loop level, i.e. at the order $\mathcal{O}(g_{\rm YM}^2)$. The dilatation operator $H(g_{\rm YM})$ can be expanded in $g_{\rm YM}^2$, $H(g_{\rm YM}) = \sum_{L=0}^{\infty} g_{\rm YM}^{2L} H_{(L)}$. At least in perturbation theory, this operator can be diagonalized within the subspace of free BPS operators.[1] Within this subspace, $H_{(0)}$ is equal to $\sum_I R_I + \sum_i J_i$. We want to find the subset of free BPS operators that satisfy (5.6) in the next order, so they must be annihilated by $H_{(1)}$. Within the free BPS sector, one finds that

$$\{Q(g_{\rm YM}), Q^\dagger(g_{\rm YM})\} = H(g_{\rm YM}) - \sum_I R_I - \sum_i J_i = \sum_{L=1}^{\infty} g_{\rm YM}^{2L} H_{(L)} \ . \tag{5.10}$$

Q and $Q^\dagger$ also depend on $g_{\rm YM}$. Since the free BPS fields are annihilated by Q and $Q^\dagger$ at the leading $\mathcal{O}(g_{\rm YM}^0)$ order, their coupling expansions start from the $\mathcal{O}(g_{\rm YM}^1)$ 'half-loop' order. Therefore, the leading 1-loop Hamiltonian $H_{(1)}$ in (5.10) is given by the anticommutator of Q and $Q^\dagger$ at the half-loop order. In particular, $Q_{(\frac{1}{2})}$ at $\mathcal{O}(g_{\rm YM}^1)$ is precisely the supercharge of the classical interacting field theory. So the 1-loop BPS operators should be annihilated by both Q and $Q^\dagger$ at the classical half-loop order.

The local BPS operators annihilated by Q and $Q^\dagger$ are in 1-1 map with cohomology classes of Q. The cohomology class is defined by the set of operators O built from the BPS letters

[1] More precisely, for the gauge invariance in the interacting theory, the subsector is defined at $g_{\rm YM} \neq 0$ by promoting the derivatives $\partial_{\dot\alpha}$ appearing in the operators to the covariant derivatives $D_{\dot\alpha} \equiv \partial_{\dot\alpha} - i[A_{+\dot\alpha}, \ ]$.

(5.9) that are closed under the action of Q, i.e. $[Q, O\} = 0$, with the equivalence relation $O \sim O + [Q, \Lambda\}$, where Λ is also an operator constructed from the BPS letters (5.9). We can call this a cohomology because of the nilpotency $Q^2 = 0$. These cohomology classes are in 1-to-1 map to the BPS operators O_{BPS} that satisfy $[Q, O_{\text{BPS}}\} = 0$ and $[Q^\dagger, O_{\text{BPS}}\} = 0$, because the latter can be understood as harmonic forms [39]. Therefore, we shall construct and study the representatives of the cohomologies of the classical half-loop supercharge Q, which map to the 1-loop BPS operators. The actions of classical (half-loop) Q on the free BPS fields are given by

$$\begin{aligned} &Q\phi^m = 0 \ , \quad Q\lambda_{\dot{\alpha}} = 0 \ , \quad Q\psi_m = -\tfrac{i}{2}\epsilon_{mnp}[\phi^n, \phi^p] \ , \\ &Qf = -i[\phi^m, \psi_m] \ , \quad [Q, D_{\dot{\alpha}}] = -i[\lambda_{\dot{\alpha}}\} \ , \end{aligned} \tag{5.11}$$

where we absorbed the g_{YM} factors on the right hand sides into the normalization of fields.

It is well known that there are fewer BPS states at the 1-loop level than in the free theory. It has been conjectured (for instance, explicitly in [114]) that the 1-loop BPS states remain BPS at general non-zero coupling. Some perturbative evidence of this conjecture was discussed in [115]. We will assume this conjecture.

Let us summarize this section. Equivalently to listing 1-loop BPS local operators of the $\mathcal{N} = 4$ SYM, we shall find classical cohomology classes with respect to the supercharge Q. The cohomology classes are defined as gauge invariant operators constructed using the BPS letters (5.9) by multiplying them and contracting the gauge indices, that are annihilated by Q under the rule (5.11), up to identification of operators that differ by Q-exact operators.

5.2 The BMN Sector

Combinatorial possibilities of gauge invariant operators constructed using the BPS letters (5.9) grow rapidly with the number of letters allowed and with the gauge rank N, and the cohomology problem quickly becomes computationally complicated. In order to reach meaningful results with limited computing power, we will restrict to a subset of the operators, that we call the BMN sector. This restriction has been motivated by the observation that the smallest black hole cohomology found for $N = 2$ can be expressed without any gauginos and derivatives, as we shall show in section 7.2. In this section we introduce the BMN sector, or truncation.

The radially quantized QFT lives on $S^3 \times \mathbb{R}$. The fields are expanded in spherical harmonics of the Lorentz group $SO(4)$. It was shown in [116] that the *classical* $\mathcal{N} = 4$ Yang-Mills theory has a consistent truncation which keeps finite degrees of freedom, described by the

BMN matrix model [117]. The modes kept after the truncation are given by: (1) s-wave modes $\phi_m(t)$, $\phi^m(t)$ of the scalars, (2) lowest spinor harmonics modes $\psi_{m\alpha}(t)$, $\lambda_\alpha(t)$ (the spinor indices are defined using the labels of Killing spinor fields [116]), (3) vector potential 1-form restricted to $A = A_0(t)dt + A_i(t)\sigma_i$ where σ_i with $i = 1, 2, 3$ are the right-invariant 1-forms on S^3 in our convention. This is a consistent truncation of the nonlinear equations of motion, and not a quantum reduction in any sense. So the full quantum BMN theory is a priori unrelated to the 4d Yang-Mills theory. However, since our 1-loop cohomology problem uses classical supercharge Q only, it can be truncated to the BMN model. If the conjecture of [114] is true, the whole BPS cohomology problem would have a quantum truncation to this model.

In general, the BMN theory and the full Yang-Mills theory behave differently in many ways. The difference starts from the number of ground states. The Yang-Mills theory on $S^3 \times \mathbb{R}$ has a unique vacuum, while the BMN model has many ground states labeled semiclassically by the discrete values of A_i. In the quantum BMN theory, viewed as an M-theory in the plane wave background, these ground states describe various M2/M5-brane configurations with zero lightcone energies [118]. In the Yang-Mills theory, however, there are large gauge transformations on S^3 which can gauge away these ground states to $A_i = 0$. So if one wishes to study the Yang-Mills theory using this matrix model, it suffices to consider the physics around $A_i = 0$.

Recall that our cohomology problem is completely classical, using the classical supercharge Q at the half-loop order. Therefore, this problem should have a truncation to the BMN matrix model. This turns out to be the cohomology problem defined using

$$\phi^m \ , \quad \psi_m \ , \quad f \ , \tag{5.12}$$

without using any gauginos $\lambda_{\dot\alpha}$ or derivatives $D_{\dot\alpha}$. These operators close under the action of Q: $[Q, \phi^m] = 0$, $\{Q, \psi_m\} = -i\epsilon_{mnp}[\phi^m, \phi^n]$, $[Q, f] = -i[\psi_m, \phi^m]$. So it is possible to restrict the cohomology problem by using operators constructed only using the BMN letters (5.12). Note that the truncation is also applied to the operator Λ when one identifies two operators O_1 and O_2 related as $O_2 - O_1 = [Q, \Lambda\}$. This is why the gauginos $\lambda_{\dot\alpha}$ cannot be included in this truncation. Although it is Q-closed by itself, $\lambda_{\dot\alpha}$ can be obtained by acting Q on the covariant derivative, $[Q, D_{\dot\alpha}] = -i[\lambda_{\dot\alpha},\ \}$. So if one had tried to include $\lambda_{\dot\alpha}$ into the truncation and construct operators like O_1, O_2, Λ, one may incorrectly conclude that certain O_1 and O_2 are different by not including derivatives in Λ. This truncation of the cohomology problem was known in [115, 119], although the relation to the BMN truncation was not explicitly addressed.[2] Notice also that this truncation is not kinematic, i.e. cannot be inferred without knowing the dynamical information of the classical theory.

The BMN truncation is the $SU(2)_R$ invariant truncation. In our cohomology problem, this means that no ingredients include the $\dot{\alpha}$ indices for $SU(2)_R$. This is why $\lambda_{\dot{\alpha}}$ and $D_{\dot{\alpha}}$ are excluded. Similarly, in the representation theory, only a small subset of $PSU(1,2|3)$ generators can be used to generate a multiplet. Among the $PSU(1,2|3)$ generators Q^m_+, $\overline{Q}_{m\dot{\alpha}}$ and $P_{+\dot{\alpha}}$, only the three supercharges Q^m_+ which belong to $SU(1|3)$ act within BMN cohomologies. It serves as a great combinatorial advantage that the derivatives are disallowed, because it only allows a finite number of BPS letters (5.9) to be used for construction of the BPS operators.

5.3 The Index over Cohomologies

Now that we have defined cohomologies in the full and the BMN sector of the 4d $\mathcal{N}=4$ Yang-Mills theory, it is useful to introduce a tool to count them. As always, it is the index.

As we have argued in section 5.1, the cohomologies with respect to the supercharge Q are in 1-1 map with the local BPS operators, or the $\frac{1}{16}$-BPS states in the 1-loop level of the field theory. That being said, an index over the cohomologies is identical to that over the $\frac{1}{16}$-BPS states that have been defined in section 3.1.1. We repeat the definition (3.10) while getting rid of tildes. Also, in this section we use the letter Z to indicate the index.

$$Z(\Delta_I,\,\omega_i) \;=\; \mathrm{Tr}\left[(-1)^F e^{\Delta_I Q_I+\omega_i J_i}\right]\,, \qquad \text{where} \quad e^{\frac{\Delta_1+\Delta_2+\Delta_3-\omega_1-\omega_2}{2}}=1\,. \tag{5.13}$$

Note from (5.13) and from discussions in section 3.1.1 that due to the relations between five chemical potentials, the index is only able to distinguish cohomologies in 4 directions in the 5-dimensional charge space. Specifically, the index does not distinguish two cohomologies whose charges Q_I differ by $\frac{n}{2}$ and J_i by $-\frac{n}{2}$, where n is an integer.

It is often useful to unrefine the chemical potentials to make the index a function of only one variable. We have done this unrefinement in section 3.2, see (3.31):

$$\begin{aligned} & e^{\Delta_1}=e^{\Delta_2}=e^{\Delta_3}=t^2\,, \qquad e^{\omega_1}=e^{\omega_2}=t^3\,, \\ \Rightarrow \quad & Z(t)=\mathrm{Tr}\left[(-1)^F t^{2(Q_1+Q_2+Q_3)+3(J_1+J_2)}\right]\equiv \mathrm{Tr}\left[(-1)^F x^{\mathcal{J}}\right]\,, \end{aligned} \tag{5.14}$$

[2] We thank Nakwoo Kim for first pointing this out to us.

where we defined a combination of the charges

$$\mathcal{J} \equiv 2Q_1 + 2Q_2 + 2Q_3 + 3J_1 + 3J_2 \ . \tag{5.15}$$

This $\mathcal{J}$ can be thought of as an 'overall' charge, and we shall expand in t, in powers of this overall charge to truncate results throughout this part of the dissertation.

The index as defined (5.13) can be taken over all cohomologies, but it is possible at will to take it only over cohomologies in the BMN sector. However, in the BMN sector all cohomologies have $J = J_1 = J_2$, so it is allowed yet redundant to keep both chemical potentials $\omega_{1,2}$. Therefore, we take $\omega = \omega_1 = \omega_2$ for the BMN index. Then, we substitute $\omega = \frac{\Delta_1+\Delta_2+\Delta_3}{2}$ to satisfy the condition in (5.13). As a result, the BMN index is a function of 3 chemical potentials:

$$Z_{\text{BMN}}(\Delta_I) \ = \ \text{Tr}_{\text{BMN}} \left[(-1)^F e^{\Delta_I(Q_I+J)}\right] \ , \tag{5.16}$$

that only distinguishes 3 combinations of 4 charges $Q_I + J$ where $I = 1, 2, 3$. We often denote these three charges as

$$q_I \equiv Q_I + J \ . \tag{5.17}$$

Similarly to the full index, we can unrefine the BMN index via $e^{\Delta_1} = e^{\Delta_2} = e^{\Delta_3} = t^2$, resulting in the unrefined BMN index as a function of t only:

$$Z_{\text{BMN}}(t) \ = \ \text{Tr}_{\text{BMN}} \left[(-1)^F t^{2(Q_1+Q_2+Q_3)+6J}\right] \ . \tag{5.18}$$

Since the BMN sector is a restriction of the BPS cohomologies, the entropy of BMN cohomologies will be smaller than the entropy of all cohomologies. Despite, the large N BMN entropy will still exhibit the black hole like growth. Taking j (schematically) to be the charges, the black hole like entropy growth is

$$S(j, N) = N^2 f(\tfrac{j}{N^2}) \ , \tag{5.19}$$

where $f(x)$ is a generic function that does not explicitly depend on N, $N \gg 1$, $j \gg 1$ and the ratio $\epsilon \equiv \frac{j}{N^2}$ does not scale in N. Roughly, the scaled charge parameter ϵ measures the size of the black hole in the AdS unit. In the rest of this section, we show that the BMN entropy scales as (5.19) when ϵ is parametrically small (but not scaling in N), i.e. for small black holes. We expect without a proof the same to be true at general ϵ, see [72] for some comments on the BMN entropy of large black holes, and [120] for a more general recent work on the BMN matrix model.

Recall the matrix integral expression (3.24) for the index (5.13). From this the BMN index can be obtained via truncation:

$$\begin{aligned} Z_{\rm BMN}(\Delta_I) &= \frac{1}{N!}\int_0^{2\pi}\prod_{a=1}^{N}\frac{d\alpha_a}{2\pi}\frac{\prod_{a\neq b}(1-e^{i\alpha_{ab}})\prod_{a,b=1}^{N}\prod_{I<J}(1-e^{\Delta_I+\Delta_J}e^{i\alpha_{ab}})}{\prod_{a,b=1}^{N}\left[(1-e^{\Delta_1+\Delta_2+\Delta_3}e^{i\alpha_{ab}})\prod_{I=1}^{3}(1-e^{\Delta_I}e^{i\alpha_{ab}})\right]} \\ &\quad\times\frac{(1-e^{\Delta_1+\Delta_2+\Delta_3})\prod_{I=1}^{3}(1-e^{\Delta_I})}{\prod_{I<J}(1-e^{\Delta_I+\Delta_J})}\ , \end{aligned} \tag{5.20}$$

where the second line (inverse of the $U(1)$ index) is multiplied to make it an $SU(N)$ index rather than $U(N)$. This integral can be computed either exactly using the residue sum or in a series expansion in t defined by $(e^{\Delta_1},e^{\Delta_2},e^{\Delta_3}) = t^2(x,y^{-1},x^{-1}y)$.

Using (5.20), let us compute the large N entropy in the small black hole regime: $j\gg 1$, $N\gg 1$ and $\epsilon\equiv\frac{j}{N^2}$ fixed and much smaller than 1 (but not scaling in N).[3] This regime is reached by taking all Δ_I's to be small. The approximate large N calculation of the entropy can be done by following all the calculations in section 5.3 of [122] with minor changes in the setup. In particular, the calculations from (5.88) to (5.91) there can be repeated by simply replacing all $2-(-e^\gamma)^n-(-e^\gamma)^{-n}$ by 1 (which are the denominators of the letter indices in the two setups) and remembering that β_I there are $-\frac{\Delta_I}{2}$ here. The resulting eigenvalue distribution is along the interval $\alpha\in(-\pi,\pi)$ on the real axis (the gap closes in the small black hole limit), with the distribution function

$$\rho(\alpha) = \frac{3}{4\pi^3}(\pi^2-\alpha^2)\ . \tag{5.21}$$

The free energy $\log Z$ of this saddle point is given by

$$\log Z = \frac{3N^2}{\pi^2}\Delta_1\Delta_2\Delta_3\ . \tag{5.22}$$

(For small black holes with negative susceptibility, the grand canonical index is not well defined. Whenever we address $\log Z$, a Laplace transformation to the micro-canonical ensemble is assumed.) The entropy at given charges $q_I\equiv R_I+J$ is given by extremizing

$$S_{\rm BMN}(q_I;\Delta_I) = \log Z - \sum_I q_I\Delta_I\ , \tag{5.23}$$

[3] The term 'small black hole' has at least three different meanings in the literature. It sometimes denotes string scale black holes, for which 2-derivative gravity description breaks down near the horizon. In our example, since ϵ does not scale in N, the 2-derivative gravity is reliable everywhere. Also, small black holes sometimes mean AdS black holes with negative specific heat or susceptibility. What we call 'small black holes' belong to this class, but are more specific. Our notion is precisely the same as [121, 122].

in Δ_I's, which is given by

$$S_{\rm BMN}(q_I) = 2\pi\sqrt{\frac{q_1q_2q_3}{3N^2}}\ . \tag{5.24}$$

This expression is valid when $q_I = N^2\epsilon_I$ with $\epsilon_I \ll 1$. The entropy $\sim N^2\sqrt{\epsilon_1\epsilon_2\epsilon_3} \propto N^2$ exhibits a black hole like scaling (5.19). $S_{\rm BMN}$ is smaller than the full entropy $S(q_I) = 2\pi\sqrt{\frac{2q_1q_2q_3}{N^2}}$ in the small black hole regime [122] by a factor of $\frac{1}{\sqrt{6}}$. This is natural since the BMN truncation loses cohomologies. However, the fact that $S_{\rm BMN}(q_I)$ scales like $S(q_I)$ implies that the truncation provides a good simplified model for black holes, at least for the small black holes $\epsilon_I \ll 1$.

5.4 Graviton Cohomologies

There is a well known family of cohomology classes in $\mathcal{N} = 4$ SYM, known as the multi-graviton cohomologies. These graviton-type cohomologies are well-defined and completely classified [35], but it was shown that there are not enough of them to account for the black hole entropy. We want to exclude them in our discussions, and instead find cohomologies that are not of the graviton type. We use the terms non-graviton cohomology and black hole cohomology interchangably, simply because the non-graviton cohomologies at least include the black hole cohomologies for sure (from counting) and we are not aware of any further classification among the non-graviton cohomologies that is applicable. We also refer to a recent work [123] in this direction. In this section, we review the notion of graviton cohomologies, especially at finite N, and also explain how to list and count them.

We first distinguish the multi-graviton and single-graviton cohomologies. The multi-graviton cohomologies are the same as non-graviton cohomologies and they include the single-graviton cohomologies.

The multi-graviton cohomologies are defined to be the polynomials of single-graviton cohomologies. (This definition naturally yields the familiar large N cohomologies for the supergravitons.) Single-graviton cohomologies are completely understood [35, 38, 41], as we shall review in a moment: they are nontrivial cohomologies at arbitrary N by definition, in the sense that no trace relations exist between single-graviton operators. Polynomials of these single-trace cohomologies define the multi-trace cohomologies. Some polynomials may be trivial, i.e. Q-exact at finite N. However, they are Q-closed at arbitrary N without using any trace relations. This will be in contrast to the black hole cohomologies, which should become Q-closed only after applying trace relations at particular N.

When N is larger than the energy, the multi-graviton operators defined above are all nontrivial cohomologies since no trace relations can be applied to make them Q-exact. So in this setup, the 'graviton cohomologies' defined abstractly in the previous paragraph actually map to the familiar $\frac{1}{16}$-BPS graviton states in $AdS_5 \times S^5$. Trace number of the operator is regarded as the particle number.

At finite N, all the multi-trace operators mentioned in the previous paragraphs are still Q-closed. However, some of their linear combinations may be zero or Q-exact only when N takes a particular value smaller than their energies, due to the trace relations. So the independent graviton cohomologies reduce at finite N. Such reductions of states are a well known finite N effect in the gravity dual. It is called the stringy exclusion principle [124], which happens because gravitons polarize into D-brane giant gravitons [125–127]. The reduction/exclusion mechanism is the same for any N in QFT, making it natural to call them 'finite N gravitons' at general finite N.

Now we concretely explain the list of the graviton cohomologies. One starts by listing the single-trace graviton cohomologies. These are completely found and collected into supermultiplets. The relevant algebra for these multiplets is the $PSU(1,2|3)$ subset of the superconformal symmetry $PSU(2,2|4)$ that commutes with $Q, Q^\dagger$. The multiplets for single-trace graviton cohomologies are called S_n with $n = 2, 3, \cdots$ [35]. S_n is obtained by acting the Poincaré supercharges Q^m_+, $\overline{Q}_{m\dot{\alpha}}$ and the translations $P_{+\dot{\alpha}}$ in $PSU(1,2|3)$ on the following primary operators

$$u^{i_1 i_2 \cdots i_n} = \mathrm{tr}(\phi^{(i_1}\phi^{i_2} \cdots \phi^{i_n)}) \ . \tag{5.25}$$

See [35] for more details. In the notation of [88], it is also a subrepresentation of the short representation $B_1\bar{B}_1[0;0]^{[0,n,0]}$. At large N, multiplying the operators in S_n's yields independent multi-trace cohomologies. At finite N, trace relations reduce the independent single-trace and multi-trace operators. Following [71], we first identify the dependent single-trace operators as follows. Using the Cayley-Hamilton identity, one can show that all single-trace operators in $S_{n\geq N+1}$ can be expressed as polynomials of operators in $S_{n\leq N}$ [71]. So it suffices to use only the operators in $S_{n\leq N}$ to generate graviton cohomologies. The remaining single-trace generators in $S_{n\leq N}$ are not independent when we multiply them. In other words, there are further trace relations for gravitons within $S_{n\leq N}$. The last trace relations are not systematically understood, to the best of our knowledge.

To simplify the discussions, let us temporarily consider the BMN sector. The subset of $PSU(2,2|4)$ that acts within the BMN sector is $SU(2|4)$. The subset $SU(1|3) \subset SU(2|4)$ commutes with $Q, Q^\dagger$ and generates the supermultiplets of BMN cohomologies. In each S_n, there is a finite number of single-trace generators in the BMN sector. They are given by

$$\begin{aligned} (u_n)^{i_1\cdots i_n} &= \mathrm{tr}(\phi^{(i_1}\cdots\phi^{i_n)}) \\ (v_n)^{i_1\cdots i_{n-1}}{}_j &= \mathrm{tr}(\phi^{(i_1}\cdots\phi^{i_{n-1})}\psi_j) - \text{`trace'} \\ (w_n)^{i_1\cdots i_{n-1}} &= \mathrm{tr}(\phi^{(i_1\cdots i_{n-1})}f + \tfrac{1}{2}\epsilon^{jk(i_p}\textstyle\sum_{p=1}^{n-1}\phi^{i_1}\cdots\phi^{i_{p-1}}\psi_j\phi^{i_{p+1}}\cdots\phi^{i_{n-1})}\psi_k)\ . \end{aligned} \tag{5.26}$$

Here, 'trace' denotes the terms to be subtracted to ensure that the contractions of the upper/lower $SU(3)$ indices are zero. They are completely determined by this condition from the first term, but the general expression is cumbersome to write. The BMN multi-graviton cohomologies are polynomials of u_n, v_n, w_n. These polynomials are subject to trace relations. These trace relations hold up to Q-exact terms.[4] For instance, the lowest trace relations for $N = 2$ are

$$R_{ij} \equiv \epsilon_{ikm}\epsilon_{jln}(u_2)^{kl}(u_2)^{mn} = Q\left[-i\epsilon_{a_1a_2(i}\mathrm{tr}(\psi_{j)}\phi^{a_1}\phi^{a_2})\right]\ . \tag{5.27}$$

More concretely, some components of these relations are

$$\mathrm{tr}(X^2)\mathrm{tr}(Y^2) - [\mathrm{tr}(XY)]^2 \sim 0\ ,\ \ \mathrm{tr}(XY)\mathrm{tr}(XZ) - \mathrm{tr}(X^2)\mathrm{tr}(YZ) \sim 0\ , \tag{5.28}$$

where $\sim$ hold up to Q-exact terms. Such Q-exact combinations are zeros in cohomology. Of course multiplying gravitons to such relations yields further relations. Trace relations cannot be seen if one does not know that the 'meson' or 'glueball' operators u_n, v_n, w_n are made of 'gluons' ϕ, ψ, f. To enumerate graviton cohomologies without overcounting, we first consider the Fock space made by the operators $\{u_n, v_n, w_n\}$ with $n = 2, \cdots, N$ and then take care of the trace relations to eliminate the dependent states.

It is important to find all fundamental trace relations of the polynomials of u_n, v_n, w_n, which cannot be decomposed into linear combinations of smaller relations. Let us denote by $R_a(\{u_n, v_n, w_n\})$ the fundamental trace relations, with a being the label. Non-fundamental trace relations are obtained by linear combinations of R_a's,

$$\sum_a f_a(\{u_n, v_n, w_n\})R_a(\{u_n, v_n, w_n\})\ . \tag{5.29}$$

In general, (5.29) is nonzero and Q-exact. However, for some choices of f_a's, the combination (5.29) may be exactly zero. If (5.29) exactly vanishes, this yields a 'relation of relations.' In terms of the mesonic variables u_n, v_n, w_n, they are trivial expressions, meaning that various

[4]In principle there might be relations which hold without any Q-exact terms. In practice, with extensive studies of the $SU(2)$ and $SU(3)$ graviton operators in the BMN sector, all trace relations of this sort that we found have nontrivial Q-exact terms.

$(-1)^F E'$	J'	R_1'	R_2'	construction
n	0	n	0	$\|n\rangle$
$-(n+\frac{1}{2})$	$\frac{1}{2}$	$n-1$	0	$\overline{Q}_{m\dot{\alpha}}\|n\rangle$
$n+1$	0	$n-2$	0	$\overline{Q}_{m\dot{+}}\overline{Q}_{n\dot{-}}\|n\rangle$
$-(n+1)$	0	$n-1$	1	$Q^m_+\|n\rangle$
$n+\frac{3}{2}$	$\frac{1}{2}$	$n-2$	1	$Q^m_+\overline{Q}_{n\dot{\alpha}}\|n\rangle$
$-(n+2)$	0	$n-3$	1	$Q^m_+\overline{Q}_{n\dot{+}}\overline{Q}_{p\dot{-}}\|n\rangle$
$n+2$	0	$n-1$	0	$Q^m_+Q^n_+\|n\rangle$
$-(n+\frac{5}{2})$	$\frac{1}{2}$	$n-2$	0	$Q^m_+Q^n_+\overline{Q}_{p\dot{\alpha}}\|n\rangle$
$n+3$	0	$n-3$	0	$Q^m_+Q^n_+\overline{Q}_{p\dot{+}}\overline{Q}_{q\dot{-}}\|n\rangle$

Table 5.1: The state contents of the $PSU(1,2|3)$ supergraviton multiplet S_n. For low n's, the rows with negative R_1' are absent. $|n\rangle$ schematically denotes the superconformal primaries.

terms just cancel to zero. They just represent the ways in which fundamental relations R_a can be redundant at higher orders. For example, consider the relations R_{ij} of (5.27) in the $SU(2)$ gauge theory. Some relations of these relations are given by

$$u^{ik}R_{jk}(u_2) - \tfrac{1}{3}\delta^i_j u^{kl}R_{kl}(u_2) = 0 \ , \tag{5.30}$$

in the $[1,1]$ representation. For instance, one can immediately see for $i=1$, $j=2$ that

$$u^{1i}R_{2i} = u^{11}[u^{23}u^{13} - u^{12}u^{33}] + u^{12}[u^{33}u^{11} - (u^{13})^2] + u^{13}[u^{12}u^{13} - u^{11}u^{23}] = 0 \ . \tag{5.31}$$

This is a trivial identity if expanded in mesons. $u^{11}R_{21}$ and $-u^{12}R_{22} - u^{13}R_{23}$ represent same constraint $u^{11}(u^{23}u^{13} - u^{12}u^{33}) = Q[\cdots]$, implying that R_{ij}'s are not independent.

The graviton cohomologies in the full theory, instead of restricting to the BMN sector, can be understood similarly. In (5.26), we have listed the finite number of single-trace generators, which are the BPS operators in the multiplet S_n, in the BMN sector. The full list of the single-trace generators in the multiplet S_n is shown in Table 5.1. They consist of the superconformal primary $|n\rangle = \mathrm{tr}(\phi^{(i_1}\cdots\phi^{i_n)})$ in the first row, actions of supercharges that commute with Q on the primary as displayed in the other rows, and any number of derivatives $\partial_{\dot{\alpha}}$ of them. Polynomials of these single-trace generators constitute the multi-graviton cohomologies in the full $SU(N)$ theory.

Interestingly, trace relations described so far will be used in section 7 to construct the ansatz for the non-graviton cohomologies. In the meantime, we shall exploit a more practical way of enumerating the graviton cohomologies, as we explain in section 6.

5.5 Strategies for Finding Black Hole Cohomologies

At this point, the problem has been well-defined. Our goal is to list cohomologies with respect to Q in the $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ where N is finite, that do not belong to the graviton-type cohomologies as defined in section 5.4. For computational advantage, we will often restrict to the BMN sector of the cohomologies, as introduced in section 5.2. In this last section of the chapter, let us outline our strategy for solving this problem.

We summarize our strategy for solving this problem as follows:

1. Compute the non-graviton index to locate non-graviton cohomologies.
2. List non-graviton Q-closed operators in the target charge sector.
3. Find a subset of operators found in step 1 which are not Q-exact.

First, we compute the index over the non-graviton cohomologies at finite N, with or without restriction to the BMN sector. This is done by computing the index over the graviton cohomologies, and subtracting from the index over all cohomologies. With chemical potentials refining the charges, one can identify the charge sectors that contain non-graviton states. Because of the property of the index that bosons and fermions are counted with opposite signs, one could miss pairs of non-graviton cohomologies which cancel in the index. Due to this limitation, we give up finding all cohomologies and search only for those captured by the index.

Exactly computing the full index is relatively easy at not too large N, especially in the BMN sector, using the matrix integral. More difficult is to count the finite N gravitons to be subtracted, taking into account the trace relations between finite N matrices which produce extremely nontrivial linear dependence between the multi-graviton operators. To make the computation of the graviton index more feasible, we use the property of the graviton cohomologies that they are faithfully counted by substituting each elementary field with corresponding diagonal matrix, rather than a more general traceless matrix. Using this property, we compute the graviton index in the BMN sector of $SU(2)$ theory analytically by hand, and that of the full $SU(2)$ theory manually on a computer.

For the BMN sectors of $SU(3)$ and $SU(4)$ theories, we boost the computation of the graviton index using Gröbner basis. As we will explain in section 6.4, counting finite N gravitons reduces to counting certain class of polynomials, whose generators are subject to certain relations. In principle, these relations can be systematically studied using the Gröbner basis. However in practice, finding Gröbner basis can be computationally very difficult. So we use a hybrid method of the Gröbner basis (in a subsector in which this basis can be found easily) and a more brutal counting of independent polynomials by computer, order by order in the charges. The results for the non-graviton indices are presented in chapter 6.

With charge sectors where non-graviton cohomologies exist are located using the index, we would now like to construct the non-graviton cohomologies. For the $SU(2)$ theory, it was possible to find the expression for the smallest non-graviton cohomology [115, 119], and for all non-graviton cohomologies within the BMN sector [71], by trials and errors and extensive searches. This result will be presented in section 7.2.

For the $SU(3)$ theory, even with restriction to the BMN sector, the naive approach has proved to be infeasible due to combinatorial complexity. Therefore we take a more streamlined approach of steps 2 and 3 described above.

We present a class of ansatz for the Q-closed operators. In order for the final cohomology not to be of graviton type, Q acting on the operator should vanish by trace relations. We find a method of constructing a class of operators which become Q-closed only after imposing trace relations. Our ansatz uses the trace relations of the graviton cohomologies that we detected while computing the index. Trace relations of gravitons mean that certain polynomials of single-graviton cohomologies are Q-exact. These relations satisfy 'relations of relations', i.e. certain linear combinations of trace relations (with the coefficients being graviton cohomologies) are identically zero. In other words, relations of relations are linear combinations of Q-exact terms which vanish. So they provide operators which become Q-closed thanks to the trace relations, which validate our ansatz for the non-graviton cohomologies.

Some Q-closed operators mentioned in the previous paragraph are not Q-exact, providing new cohomologies, while others are Q-exact. Determining whether a Q-closed operator is Q-exact or not is very hard. We developed a numerics-assisted approach to make this step affordable on the computer, by ordering the Grassmann variables and then inserting many random integers to the matrix elements. As a result, we construct the smallest non-graviton cohomology in the BMN sector of the $SU(3)$ theory in 7.3. Furthermore, by extending the numerics-assisted approach, we prove that the smallest non-graviton cohomology that we construct is the only one in its charge sector, denying the possibility that the index may have missed a boson-fermion pair of non-graviton cohomologies in the charge sector.

Chapter 6

The Non-Graviton Index

As the first step of solving the cohomology problem, we compute the index over graviton cohomologies and subtract from the full index to obtain the index over non-graviton cohomologies. It identifies the charges of some of the non-graviton cohomologies.

6.1 Methods for the Graviton Index

Recall from section 5.4 that graviton cohomologies are polynomials of single-trace graviton cohomologies, which are elements of supermultiplets $S_{n\leq N}$. They are listed in Table 5.1, and in particular those in the BMN sector are the mesons listed in (5.26). We wish to enumerate linearly independent operators among these, i.e. we wish to mod out by linear relations between them. There are two main strategies that we exploit to ease this computation: the eigenvalue counting and the Gröbner basis. We only employ the first strategy for computations in $SU(2)$, while for $SU(3)$ and $SU(4)$ we employ both strategies. In this section we explain these two strategies.

Let us explain the first idea, the eigenvalue counting. We first review how the multi-gravitons made only of the chiral primaries u_n of (5.25) are enumerated. Based on rather physical arguments, [35] proposed to count them by taking all three scalars ϕ^m to be diagonal matrices.[1] With this restriction, the problem of enumerating independent gauge-invariant operators, which are multi-trace operators of the matrices u_n, reduces to enumerating independent Weyl-invariant polynomials of the eigenvalues.

Our interest is in counting the finite N graviton cohomologies involving all the descendants in S_n, not only the chiral primaries u_n. The descendants are obtained from u_n by

[1] The argument is often dubbed 'quantizing the moduli space' of the QFT. For exact quantum states, it relies on the protection of the moduli space against quantum corrections. At the level of classical cohomologies, its proof should be elementary, although we do not pursue it here.

acting the supercharges in $PSU(1,2|3)$. Since the single-graviton states belong to absolutely protected multiplets S_n, and since their multiplications trivially remain in cohomology both for free and 1-loop calculations, we can generate the descendants by acting the supercharges of the strictly free theory [71]. The actions of free supercharges are linear so that diagonal ϕ^m's transform to diagonal ψ_m, f, and other descendants. The covariant derivatives on the fields also reduce to ordinary derivatives since $g_{\rm YM} = 0$. Therefore, the descendant BPS letters can be taken to be diagonal matrices as well, for the purpose of enumerating graviton operators. This is an especially big advantage for the $SU(2)$ theory, where each elementary field is represented by only one eigenvalue.

So the counting of graviton operators is reduced to the counting of certain polynomials of the eigenvalues. We have $N-1$ eigenvalues for each field ϕ^m, ψ_m, $\lambda_{\dot\alpha}$, f and their derivatives. As we truncate by the overall order $\mathcal{J}$ of operators, only a finite number of derivatives are allowed and thus the number of variables that are needed to describe graviton operators is also finite. In the BMN sector where there are no gauginos nor derivatives, $7(N-1)$ variables are needed to describe graviton operators. Let us denote these eigenvalues collectively as λ_I, not to be confused with the gauginos. Let us also denote the single-trace graviton operators collectively as g_i's. They are the members of Table 5.1 and their descendants with $n = 2, \cdots, N$ for the full sector, and the 'mesonic generators' $\{u_n, v_n, w_n\}$ (5.26) for the BMN sector. These are now regarded as polynomials $g_i(\lambda_I)$ of the eigenvalues λ_I. Then, we want to count the polynomials $p(g_i)$ of the mesons g_i, which can be written as polynomials $p(g_i(\lambda_I))$ of eigenvalues λ_I.

These polynomials are not all independent because certain polynomials $p(g_i)$ of g_i's may be zero when written as polynomials $p(g_i(\lambda_I))$ of λ_I. Such polynomials can be thought of as constraints on the space of polynomials. These are remnants of the trace relations of the $N \times N$ matrices. Had we been keeping all the $N \times N$ matrix elements, trace relation would have been zero up to a Q-exact term. Since the action of Q yields a commutator, the Q-exact term vanishes when the fields are diagonal. So general trace relations up to Q-exact terms reduce to exact polynomial constraints.

For the $SU(2)$ theory, it was possible to count the number of linearly independent polynomials given a set of polynomials $p(g_i(\lambda_I))$ of λ_I. However, as the number of variables grows, a more systematic treatment became inevitable. Therefore, we further develop the strategy for enumerating independent graviton cohomologies in the BMN sectors of the $SU(3)$ and $SU(4)$ theories, as we now explain.

Counting constrained polynomials is a classic mathematical problem, with known solution. This brings us to the second strategy that we exploit: the Gröbner basis. See e.g. [128]. Let us briefly explain a flavor of its properties and how it is used to solve the enumeration problem.

Recall that the multi-graviton operators are given by the set of all polynomials $p(g_i)$ of g_i's. However, this set is overcomplete and therefore not suitable for the counting purpose, because of the constraints. That is, some of the polynomials are zero and consequently some of the polynomials are equivalent to each other.

We want to better understand the constraints, i.e. polynomials of g_i that are zero. The constraints appear because each meson g_i is not an independent variable but instead made of the gluons λ_I, i.e. $g_i = g_i(\lambda_I)$ where the right hand side is a polynomial of λ_I that corresponds to the meson g_i. All constraints are derived from the fact that

$$G_i(g_i, \lambda_I) \equiv g_i - g_i(\lambda_I) = 0 \ , \tag{6.1}$$

for each meson labeled by i. Therefore, the set of all polynomials of the mesons g_i and the eigenvalues λ_I that are zero (also known as the ideal) is *generated* by (6.1), in the sense that any element of this set can be written as

$$\sum_i q_i(g_i, \lambda_I) G_i(g_i, \lambda_I) \ , \tag{6.2}$$

where $q_i(g_i, \lambda)$ are polynomials of g_i and λ_I. If we restrict to elements of this 'set of zeroes' that only involve g_i but not λ_I, those will be precisely the constraints that mod out the set of all polynomials $p(g_i)$.

Although (6.1) is the most intuitive basis that generates the set of zeroes like (6.2), it is often not the most convenient basis. The same set of zeroes can be generated by many different choices of the basis, possibly with different numbers of generators. Gröbner basis is one of these choices with the following special property. Let $\{G_a(g_i, \lambda_I)\}$ be a basis of the set of zero polynomials of (g_i, λ_I). Then, for any polynomial $p(g_i, \lambda_I)$, suppose one tries to 'divide' this polynomial by the basis $\{G_a(g_i, \lambda_I)\}$. This is a process of writing the polynomial as

$$p(g_i, \lambda_I) = \sum_a q_a(g_i, \lambda_I) G_a(g_i, \lambda_I) + r(g_i, \lambda_I) \ , \tag{6.3}$$

where $r(g_i, \lambda_I)$ can no longer be 'divided by' $\{G_a(g_i, \lambda_I)\}$, which can be well-defined by setting an ordering scheme between variables and their monomials. Naturally, $r(g_i, \lambda_I)$ can be thought of as the remainder of the division. In general, there can be multiple ways — with different q_a and r — to write $p(g_i, \lambda_I)$ as (6.3). The special property of the Gröbner basis is that if $\{G_a(g_i, \lambda_I)\}$ were the Gröbner basis of the set of zeroes, then the remainder $r(g_i, \lambda_I)$ is unique for each given $p(g_i, \lambda_I)$. Note that since $\{G_a(g_i, \lambda_I)\}$ generates the set of zeroes, (6.3) implies that the polynomial $p(g_i, \lambda_I)$ is equivalent to its remainder $r(g_i, \lambda_I)$. It follows that the set of all polynomials $p(g_i, \lambda_I)$ is identical to the set of all possible remainders $r(g_i, \lambda_I)$ under division by the Gröbner basis. However, unlike in the set of all polynomials $p(g_i, \lambda_I)$, there are no polynomials in the set of all remainders that are equivalent due to the constraints, because otherwise one of them should have been divided once more to yield the other as the remainder. Therefore, the set of remainders can be used to count the number of independent polynomials of (g_i, λ_I) under constraints.

There is a canonical procedure to find the Gröbner basis of the set of zeroes given one choice of basis (6.1), known as *Buchberger's algorithm*. Many computer algebra softwares implement this algorithm or its improved versions. The Gröbner basis depends wildly on the ordering scheme between variables and monomials, so it is important to choose a nice ordering scheme which eases the calculations. This ordering is difficult to know in advance, so some amount of trials and errors is involved in finding the Gröbner basis.

By setting an appropriate ordering scheme, it is possible to consistently truncate the Gröbner basis for zero polynomials of (g_i, λ_I), into that for zero polynomials of g_i only. Then, the set of all possible remainders $r(g_i)$ under division by the truncated Gröbner basis form a faithful — complete but not overcomplete — set of all independent polynomials of g_i, and therefore the set of all independent graviton operators. Moreover, one can easily construct a monomial basis for this set of remainders, from which it is straightforward to compute both the partition function and the index over graviton operators.

Although the graviton index for the BMN sector of the $SU(2)$ theory can be computed analytically by hand using the first strategy of eigenvalue counting, as we will show in section 6.2, we easily reproduce this result by employing both strategies — the eigenvalue counting and the Gröbner basis — explained so far. This is done by finding a Gröbner basis of relations between $SU(2)$ BMN gravitons that consists of 66 generators (after truncation), and counting the set of all possible remainders under division by those.

Unfortunately, the computation of the Gröbner basis quickly becomes very cumbersome if the generators of the constraints $\{g_i - g_i(\lambda_I)\}$ are numerous and complicated. For relations between a subset of $SU(3)$ BMN gravitons that do not involve f, i.e. u_n and v_n in (5.26), we found the Gröbner basis with 1170 generators (after truncation) after several hours of computation on a computer. For the complete set of $SU(3)$ BMN gravitons including w_n, we were unable to find the Gröbner basis due to lack of computing resources: it takes months at least and it is tricky to parallelize. Therefore, we have devised a hybrid method to take maximal advantage of the Gröbner basis obtained for the non-f subsector as we now describe.

We first list the complete and independent monomial basis of graviton operators, i.e. set of monomials of the mesons g_i, that consist of u_n, v_n but not of w_n ($n = 2, 3$), up to the charge order $\mathcal{J} = 54$. This can be done for any order $\mathcal{J}$ because the Gröbner basis for the non-f subsector has been obtained. Then, one can construct an overcomplete set of all graviton operators by multiplying each basis from the previous step by arbitrary numbers of w_2 and w_3, again up to $\mathcal{J} = 54$. Note that w_2 and w_3 include 3 and 6 different species of single-graviton operators, respectively, so the size of the overcomplete set grows quickly.

It is helpful to fragment the problem by classifying the operators according to their charges. Namely, each charge sector is specified by 4 non-negative integers $2J$ and $q_I = Q_I + J$ (where $I = 1, 2, 3$). The overall order $\mathcal{J} = 2(q_1 + q_2 + q_3)$, defined in (5.15), is always even in the BMN sector. This classification is useful because all single-graviton operators u_n, v_n, w_n and therefore all multi-graviton operators have definite charges, and operators with different sets of charges can never have a linear relation between them. Moreover, different charge sectors with merely permuted charges (q_1, q_2, q_3) should contain the same number of independent graviton operators. Therefore, we separately consider the overcomplete basis of gravitons in each charge sector with $q_1 \leq q_2 \leq q_3$.

In order to count linearly independent operators among the overcomplete set in any charge sector, we rewrite each operator as a polynomial of the eigenvalues. This is done by substituting the mesons with corresponding eigenvalue polynomials $u_n(\lambda_I)$, $v_n(\lambda_I)$ and $w_n(\lambda_I)$, which are obtained by writing the gluons in terms of their eigenvalues. For the eigenvalues of the $SU(3)$ traceless elementary fields, we use the convention

$$f = \begin{pmatrix} f_1 & 0 & 0 \\ 0 & f_2 & 0 \\ 0 & 0 & -f_1 - f_2 \end{pmatrix} , \tag{6.4}$$

and likes.

The number of independent polynomials within each charge sector is determined as the rank of their coefficient matrix. We have used the software `Singular` [129] for finding the Gröbner basis, writing each operator as an eigenvalue polynomial, and extracting the coefficient matrix within each charge sector, and `numpy` for computing the rank of the matrix.

The computation of indices for the $SU(3)$ theory have been performed up to $\mathcal{J} = 54$ on personal computers. For example, the computation for the charge sector $(2J, q_1, q_2, q_3) = (7, 9, 9, 9)$, which turns out to be the largest, the coefficient matrix was 31026×20940 with rank 3242.

For the counting of $SU(4)$ BMN gravitons, we take a similar hybrid approach. Separation into charge sectors works identically to the $SU(3)$ theory. However, computation of the Gröbner basis is even more heavy, both time-wise and memory-wise, so we were only able to obtain the Gröbner basis for a subsector of $SU(4)$ BMN gravitons involving u_n $(n = 2, 3, 4)$, i.e. the chiral primaries. We first list the complete and independent monomial basis of the chiral primaries u_n using the Gröbner basis, up to the order $\mathcal{J} = 30$. Then we construct an overcomplete set of all multi-graviton operators within each charge sector by multiplying each independent basis by appropriate numbers of v_2, v_3, v_4, w_2, w_3 and w_4, again up to $\mathcal{J} = 30$.

We write each operator in the overcomplete basis as a polynomial of the eigenvalues. For the traceless elementary fields in the $SU(4)$ theory, we used the following convention for the diagonal entries:

$$f = \begin{pmatrix} f_1 & 0 & 0 & 0 \\ 0 & f_2 & 0 & 0 \\ 0 & 0 & f_3 - f_1 & 0 \\ 0 & 0 & 0 & -f_2 - f_3 \end{pmatrix} , \tag{6.5}$$

which slightly simplifies the polynomials compared to the more canonical convention $f = \mathrm{diag}(f_1,\ f_2,\ f_3,\ -f_1 - f_2 - f_3)$.

The computation of indices for the $SU(4)$ theory have been performed up to $\mathcal{J} = 30$ on personal computers. For example, the computation for the charge sector $(2J, q_1, q_2, q_3) = (3, 5, 5, 5)$, which turns out to be the largest, the coefficient matrix was 12079×116042 with rank 3788.

6.2 $SU(2)$, BMN Sector

In this section, we compute the graviton index, and thus the non-graviton index, for the BMN sector of the $SU(2)$ theory. This is done by employing the first of two strategies explained above, namely the eigenvalue counting. We represent each of the seven elementary fields by a single eigenvalue, so all graviton operators can be written as polynomials of 7 variables, 3 of which are Grassmannian.

In terms of eigenvalues, BPS graviton polynomials are arbitrary products of the following single-gravitons:

$$\begin{aligned} \mathbf{6} \ &: \ x^2 \ , \ y^2 \ , \ z^2 \ , \ xy \ , \ yz \ , \ zx \ , \\ \mathbf{8} \ &: \ \psi_1 \cdot (y,z) \ , \ \psi_2 \cdot (z,x) \ , \ \psi_3 \cdot (x,y) \ , \ \psi_1 x - \psi_2 y \ , \ \psi_2 y - \psi_3 z \ , \\ \mathbf{3} \ &: \ xf - \tfrac{1}{2}\psi_2\psi_3 \ , \ yf - \tfrac{1}{2}\psi_3\psi_1 \ , \ zf - \tfrac{1}{2}\psi_1\psi_2 \ . \end{aligned} \tag{6.6}$$

and the goal of this section is to count independent polynomials among them. $\psi_{1,2,3}$ are Grassmann variables while x, y, z, f are bosonic.

In the third line of (6.6), xf, yf, zf are accompanied by two-fermion terms, but for the purpose of counting independent graviton polynomials, these terms can be omitted, as we prove now.

Let $\mathfrak{V}$ be the infinite set of all possible products of $6 + 8 = 14$ polynomials in the first two lines of (6.6). Define two series of vector spaces V_k and $\tilde{V}_k$ as

$$\begin{aligned} V_k &= \text{span}\left\{\mathfrak{v} \times (xf)^a (yf)^b (zf)^c \mid \mathfrak{v} \in \mathfrak{V}\,,\ a+b+c \leq k\right\}\,, \\ \tilde{V}_k &= \text{span}\left\{\mathfrak{v} \times (xf - \tfrac{1}{2}\psi_2\psi_3)^a (yf - \tfrac{1}{2}\psi_3\psi_1)^b (zf - \tfrac{1}{2}\psi_1\psi_2)^c \mid \mathfrak{v} \in \mathfrak{V}\,,\ a+b+c \leq k\right\}\,. \end{aligned} \tag{6.7}$$

We want to show that rank of V_∞ and rank of $\tilde{V}_\infty$ are equal. We do this by induction. Clearly $\text{rank}(V_0) = \text{rank}(\tilde{V}_0)$. Now, suppose that $\text{rank}(V_{k-1}) = \text{rank}(\tilde{V}_{k-1})$ and let us show that $\text{rank}(V_k) = \text{rank}(\tilde{V}_k)$. The equivalent statement is the following:

- Consider a pair of polynomials

  $$\begin{aligned} v &= \sum_{i=1}^{n} r_i (xf)^{a_i} (yf)^{b_i} (zf)^{c_i}\,, \\ \tilde{v} &= \sum_{i=1}^{n} r_i (xf - \tfrac{1}{2}\psi_2\psi_3)^{a_i} (yf - \tfrac{1}{2}\psi_3\psi_1)^{b_i} (zf - \tfrac{1}{2}\psi_1\psi_2)^{c_i}\,, \end{aligned}$$

  where $r_i \in V_0 = \tilde{V}_0$ and $a_i + b_i + c_i = k$ for all i so that $v \in V_k$ and $\tilde{v} \in \tilde{V}_k$.

  Then $v \in V_{k-1}$ if and only if $\tilde{v} \in \tilde{V}_{k-1}$.

The $\leftarrow$ part is easy. If $\tilde{v} \in \tilde{V}_{k-1}$, then $\tilde{v}$ equals a linear combination of polynomials that are at most of degree $k-1$ in $xf - \frac{1}{2}\psi_2\psi_3$ and the likes. Collecting terms with degree k in f, the equality becomes $v = 0 \in V_{k-1}$.

To show the $\rightarrow$ part, first note that $v \in V_{k-1}$ implies $v = 0$, since v is homogeneous in f with degree k. Now,

$$\begin{aligned} \tilde{v} = -\frac{1}{2}\sum_{i}^{n} r_i \Big[& a_i\psi_2\psi_3 (xf - \tfrac{1}{2}\psi_2\psi_3)^{a_i-1} (yf - \tfrac{1}{2}\psi_3\psi_1)^{b_i} (zf - \tfrac{1}{2}\psi_1\psi_2)^{c_i} \\ & + b_i\psi_3\psi_1 (xf - \tfrac{1}{2}\psi_2\psi_3)^{a_i} (yf - \tfrac{1}{2}\psi_3\psi_1)^{b_i-1} (zf - \tfrac{1}{2}\psi_1\psi_2)^{c_i} \\ & + c_i\psi_1\psi_2 (xf - \tfrac{1}{2}\psi_2\psi_3)^{a_i} (yf - \tfrac{1}{2}\psi_3\psi_1)^{b_i} (zf - \tfrac{1}{2}\psi_1\psi_2)^{c_i-1} \Big]\,. \end{aligned} \tag{6.8}$$

If $r_i\psi_j\psi_{j+1}$ for all i and $j = 1, 2, 3$ all belong to $V_0 = \tilde{V}_0$, it will establish $\tilde{v} = \tilde{V}_{k-1}$. Indeed, if r_i, which is a product of 14 polynomials in the first two lines of (6.6), contains any of the **6** in the first line, this factor can combine with two ψ's and $r_i\psi_j\psi_{j+1} \in V_0$. For example, $y^2\psi_2\psi_3 = (\psi_2 y - \psi_3 z)(\psi_3 y)$. On the other hand, if r_i contains two or more factors of the **8** in the second line, after multiplication by two ψ's it will vanish due to Grassmannian nature of ψ, so automatically $r_i\psi_j\psi_{j+1} = 0 \in V_0$.

Therefore the only possibility that remains in concern is when r_i is precisely one of the **8**. This leaves only a finite number of exceptions that one can explicitly work out. That is, if $v = 0$ with the eight r_i (they cannot mix with other r_i due to homogeneity) with appropriate numerical coefficients α_i:

$$r_1 = \alpha_1\psi_1 y \ , \ r_2 = \alpha_2\psi_1 z \ , \ \cdots \ , \ r_8 = \alpha_8(\psi_2 y - \psi_3 z) \ ,$$

it follows that $\tilde{v} = 0$ as well. This completes the proof that $\text{rank}(V_k) = \text{rank}(\tilde{V}_k)$ given $\text{rank}(V_{k-1}) = \text{rank}(\tilde{V}_{k-1})$, and by induction the number of independent products of (6.6) is not affected by the $\psi\psi$ terms in the third line.

With this rule established, we now count the number of independent graviton polynomials in the BMN sector. This task is greatly simplified by the fact that all $6 + 8 + 3 = 17$ but only two single-graviton generators are monomials, because linear independence between monomials is rather transparent. Our strategy will be to order the counting problem carefully so that we can work with the monomial basis as far as possible, and treat the contribution from the two polynomial generators later.

Since there are 3 Grassmann variables, it is convenient to classify the graviton operators into $2^3 = 8$ sectors according to their Grassmannian contents.

0-fermion sector

We first focus on the 0-fermion sector: graviton operators that do not contain any ψ's. It is clear that such operators are created by multiplying bosonic single-gravitons on the first and third lines of (6.6). Since all of them are monomials, we may simply write down a list of distinct monomials that can be obtained by multiplying bosonic single-gravitons, then their linear independence is guaranteed. The first six single-gravitons can be used to create any monomial $x^a y^b z^c$, where a, b, c are non-negative integers and $a + b + c$ is even. Including xf, yf, zf, an eligible monomial may contain any number of f as long as it is supported by at least as many x, y, or z. Therefore, multi-gravitons in the 0-fermion sector are precisely described as

$$G_0 \quad = \quad \{x^a y^b z^c f^d \mid a,b,c,d \in \mathbb{Z}^{\geq 0} \ , a+b+c \geq d \ , \ a+b+c+d = 0 \ (\text{mod } 2)\} \ . \quad (6.9)$$

Because we can attribute to each of x, y, z and f a unit of their own quantum numbers, the partition function for G_0 can be simply defined by the sum over monomials,

$$Z_0(x,y,z,f) = \sum_{g\in G_0} g \ . \tag{6.10}$$

It can be computed as follows. If there were no restrictions to a,b,c,d except being non-negative integers, the generating function would be $\frac{1}{(1-x)(1-y)(1-z)(1-f)}$. From this, we subtract the sum of monomials for which $d > a+b+c$, which is

$$\frac{1}{(1-xf)(1-yf)(1-zf)} \cdot \frac{f}{1-f} \ . \tag{6.11}$$

Then we project to the even part under $(x,y,z,f) \to (-x,-y,-z,-f)$, obtaining

$$\begin{aligned} Z_0 &= \left[\frac{1}{(1-x)(1-y)(1-z)(1-f)} - \frac{1}{(1-xf)(1-yf)(1-zf)} \cdot \frac{f}{1-f}\right]_{\rm even} \\ &= \left[\frac{1-f(xy+yz+zx-xyz)+f^2xyz}{(1-x)(1-y)(1-z)(1-xf)(1-yf)(1-zf)}\right]_{\rm even} \\ &= \frac{1+\chi_2+f(\chi_3-\chi_1\chi_2)+f^2(\chi_3^2+\chi_1\chi_3)}{(1-x^2)(1-y^2)(1-z^2)(1-xf)(1-yf)(1-zf)} \ . \end{aligned} \tag{6.12}$$

Abbreviations for cyclic polynomials

$$\begin{aligned} \chi_1 &= x+y+z \ , \\ \chi_2 &= xy+yz+zx \ , \\ \chi_3 &= xyz \ , \end{aligned} \tag{6.13}$$

will be used from now on.

1-fermion sector

Now we list (independent) operators with one fermion, either ψ_1, ψ_2 or ψ_3. These are obtained by multiplying any operator in 0-fermion sector G_0 by a generator on the second line of (6.6). As mentioned earlier, the last two of these may create non-monomial operators, so let us first proceed without them.

Operators with one ψ_1 can only be obtained by multiplying operators in G_0 by either $y\psi_1$ or $z\psi_1$. As a result, the list of such operators is simply the following monomials:

$$\{x^a y^b z^c f^d \psi_1 \,|\, a,b,c,d \in \mathbb{Z}^{\geq 0} \ , \ b+c \geq 1 \ , \ a+b+c-1 \geq d \ , \ a+b+c+d = 1 \ (\text{mod } 2)\} \ . \tag{6.14}$$

Operators containing one ψ_2 or one ψ_3 can be listed by cyclic permutations of letters.

Next, we ask what new operators arise when multiplying an operator in the 0-fermion sector G_0 by $x\psi_1 - y\psi_2$. If $x\psi_1 - y\psi_2$ multiplies $x^a y^b z^c f^d \in G_0$ such that (i) $c \geq 1$ or (ii) $a \geq 1$ and $b \geq 1$, both monomials $x^{a+1} y^b z^c f^d \psi_1$ and $x^a y^{b+1} z^c f^d \psi_2$ that appear in the product are already counted in (6.14) and corresponding ψ_2 sector respectively. So no new independent operators arise. Therefore, new operators that are obtained using $x\psi_1 - y\psi_2$ are classified as follows:

1. $(x^{a\geq 1} y^0 z^0 f^d) \cdot (x\psi_1 - y\psi_2)$: In this case, the second monomial $x^a y^1 z^0 f^d \psi_2$ is already counted in ψ_2 sector corresponding to (6.14), while the first monomial is not counted in the ψ_1 sector. Therefore, these can be regarded new monomials $x^{a+1} y^0 z^0 f^d \psi_1$ in ψ_1 sector.

2. $(x^0 y^{b\geq 1} z^0 f^d) \cdot (x\psi_1 - y\psi_2)$: In this case, the first monomial $x^1 y^b z^0 f^d \psi_1$ is already counted in ψ_1 sector (6.14), while the second monomial is not counted in the ψ_2 sector. Therefore, these can be regarded new monomials $x^0 y^{b+1} z^0 f^d \psi_2$ in ψ_2 sector.

3. $(1) \cdot (x\psi_1 - y\psi_2)$: In this case, both monomials $x\psi_1$ and $y\psi_2$ have not been counted in respective sectors. Therefore, this cannot be regarded as a new monomial in one of ψ_1 or ψ_2 sector. Instead, this should be understood as an exceptional non-monomial operator.

Similar arguments can be made for multiplication by $y\psi_2 - z\psi_3$.

As a result, the list of monomials in ψ_1 sector is now extended to

$$G_{\psi_1} = \{x^a y^b z^c f^d \psi_1 \mid a,b,c,d \in \mathbb{Z}^{\geq 0} \ , a+b+c-1 \geq d \ , a+b+c+d = 1 \ (\mathrm{mod}\ 2)\} \backslash \{x\psi_1\} \ . \tag{6.15}$$

List of monomials G_{ψ_2} in ψ_2 sector and G_{ψ_3} in ψ_3 sector are defined by cyclicity. In addition, there are two exceptional operators $x\psi_1 - y\psi_2$ and $y\psi_2 - z\psi_3$ that are not monomials and do not belong to any of G_{ψ_m}. So the whole set G_1 of 1-fermion BPS gravitons is given by

$$G_1 \quad = \quad G_{\psi_1} \cup G_{\psi_2} \cup G_{\psi_3} \cup \{x\psi_1 - y\psi_2 \ , \ y\psi_2 - z\psi_3\} \ . \tag{6.16}$$

Alternatively, one can take G_{ψ_1} to *not* exclude $x\psi_1$, similarly G_{ψ_2} and G_{ψ_3} to *not* exclude $y\psi_2$ and $z\psi_3$ respectively, but instead exclude just $x\psi_1 + y\psi_2 + z\psi_3$ at the end.

The existence of such non-monomial operators forbids us from attributing individual quantum numbers to ψ's. Instead, they carry a negative unit of respective scalar quantum numbers, and a positive unit of overall ψ-number:

$$x \to [x] \ , y \to [y] \ , z \to [z] \ , f \to [f] \ , \psi_1 \to \frac{[\psi]}{[x]} \ , \psi_2 \to \frac{[\psi]}{[y]} \ , \psi_3 \to \frac{[\psi]}{[z]} \ . \tag{6.17}$$

The partition function of the 1-fermion sector is given by a function of x, y, z, f and ψ. The partition function in ψ_1 sector (and of the rest of the 1-fermion sector) can be computed analogously to the 0-fermion sector. Starting from $\frac{1}{(1-x)(1-y)(1-z)(1-f)} \cdot \frac{\psi}{x}$, we implement the restriction $a+b+c-1 \geq d$ by subtracting its complement, extract the odd part under $(x,y,z,f) \to (-x,-y,-z,-f)$, and further subtract $x\psi_1 \to \psi$.

$$\begin{aligned} Z_{\psi_1} &= \left[\frac{1}{(1-x)(1-y)(1-z)(1-f)} - \frac{1}{(1-xf)(1-yf)(1-zf)} \cdot \frac{1}{1-f}\right]_{\text{odd}} \cdot \frac{\psi}{x} - \psi \\ &= \left[\frac{x+y+z-(xy+yz+zx)(1+f)+xyz(1+f+f^2)}{(1-x)(1-y)(1-z)(1-xf)(1-yf)(1-zf)}\right]_{\text{odd}} \cdot \frac{\psi}{x} - \psi \\ &= \frac{\chi_1+\chi_3-f(\chi_2+\chi_2^2-\chi_1\chi_3-\chi_3^2)+f^2\chi_3(1+\chi_2)}{(1-x^2)(1-y^2)(1-z^2)(1-xf)(1-yf)(1-zf)} \cdot \frac{\psi}{x} - \psi \; . \end{aligned} \tag{6.18}$$

Note that Z_{ψ_2} and Z_{ψ_3} can be computed similarly. Further including $x\psi_1 - y\psi_2$, $y\psi_2 - z\psi_3$, one obtains the following partition function for G_1:

$$Z_1 = \frac{\chi_1+\chi_3-f(\chi_2+\chi_2^2-\chi_1\chi_3-\chi_3^2)+f^2\chi_3(1+\chi_2)}{(1-x^2)(1-y^2)(1-z^2)(1-xf)(1-yf)(1-zf)} \cdot \frac{\chi_2}{\chi_3} \cdot \psi - \psi \; . \tag{6.19}$$

2-fermion sector

We consider operators that contain two of three ψ's. These are obtained by multiplying a generator on the second line of (6.6) to an operator in G_1. Focusing on the $\psi_1\psi_2$ sector, we first note there are three ways to obtain an operator in this sector.

1. Multiply either $x\psi_2$ or $z\psi_2$ to an operator in G_{ψ_1} (6.15). Such a set of operators are

$$\{x^a y^b z^c f^d \psi_1\psi_2 \mid a,b,c,d \in \mathbb{Z}^{\geq 0} \; , \; a+c \geq 1 \; , \; a+b+c-2 \geq d \; , \\ a+b+c+d = 0 \; (\text{mod } 2)\} \;\; \backslash \;\; \{x^2\psi_1\psi_2\} \; . \tag{6.20}$$

2. Multiply either $y\psi_1$ or $z\psi_1$ to an operator in G_{ψ_2}, analogous to (6.15):

$$\{x^a y^b z^c f^d \psi_1\psi_2 \mid a,b,c,d \in \mathbb{Z}^{\geq 0} \; , \; b+c \geq 1 \; , \; a+b+c-2 \geq d \; , \\ a+b+c+d = 0 \; (\text{mod } 2)\} \;\; \backslash \;\; \{y^2\psi_1\psi_2\} \; . \tag{6.21}$$

3. Multiply $x\psi_2$, $z\psi_2$, $y\psi_1$ or $z\psi_1$ to $x\psi_1 - y\psi_2$. These supplement $x^2\psi_1\psi_2$ and $y^2\psi_1\psi_2$ excluded in (6.20) and (6.21).

Taking the union of the three sets above, we arrive at

$$G_{\psi_1\psi_2} = \{x^a y^b z^c f^d \psi_1\psi_2 \mid a,b,c,d \in \mathbb{Z}^{\geq 0} \ , \ a+b+c-2 \geq d \ , \ a+b+c+d = 0 \ (\mathrm{mod}\ 2)\} \ , \quad (6.22)$$

and similarly for $\psi_2\psi_3$ and $\psi_3\psi_1$ sectors.

Note that we have not explicitly considered multiplying, for example, $x\psi_3$ or $y\psi_3$ to $x\psi_1 - y\psi_2$. Both monomials obtained this way are already included in $G_{\psi_3\psi_1}$ and $G_{\psi_2\psi_3}$, so they do not add any new independent operators. Furthermore, there is a possibility of multiplying $x\psi_1 - y\psi_2$ or $y\psi_2 - z\psi_3$ to the operators in the 1-fermion sector. These may give rise to

$$\begin{aligned}
(x\psi_1 - y\psi_2)(x\psi_1 - y\psi_2) &\sim xy\psi_1\psi_2 \ , \\
(y\psi_2 - z\psi_3)(y\psi_2 - z\psi_3) &\sim yz\psi_2\psi_3 \ , \\
(x\psi_1 - y\psi_2)(y\psi_2 - z\psi_3) &\sim xy\psi_1\psi_2 + yz\psi_2\psi_3 + zx\psi_3\psi_1 \ ,
\end{aligned} \quad (6.23)$$

but again, all of the monomials are already counted in respective 2-fermion sectors. Therefore, we conclude that the 2-fermion sectors can be written completely in monomial basis, by (6.22) and its cyclic versions:

$$G_2 = G_{\psi_1\psi_2} \cup G_{\psi_2\psi_3} \cup G_{\psi_3\psi_1} \ . \quad (6.24)$$

The partition function of 2-fermion sector can be computed as before. The result is:

$$Z_{\psi_1\psi_2} = \frac{\chi_1^2 - \chi_2 - \chi_2^2 + 2\chi_1\chi_3 + \chi_3^2 + f(\chi_3 - \chi_1\chi_2) + f^2\chi_3(\chi_1 + \chi_3)}{(1-x^2)(1-y^2)(1-z^2)(1-xf)(1-yf)(1-zf)} \cdot \frac{\psi^2}{xy} \ , \quad (6.25)$$

for the individual sector, and

$$\begin{aligned}
Z_2 &= Z_{\psi_1\psi_2} + Z_{\psi_2\psi_3} + Z_{\psi_3\psi_1} \\
&= \frac{\chi_1^2 - \chi_2 - \chi_2^2 + 2\chi_1\chi_3 + \chi_3^2 + f(\chi_3 - \chi_1\chi_2) + f^2\chi_3(\chi_1 + \chi_3)}{(1-x^2)(1-y^2)(1-z^2)(1-xf)(1-yf)(1-zf)} \cdot \frac{\chi_1}{\chi_3} \cdot \psi^2 \ ,
\end{aligned} \quad (6.26)$$

for the entire 2-fermion sector.

3-fermion sector

We finally investigate the 3-fermion sector, i.e. operators that contain all ψ_1, ψ_2 and ψ_3. One way to obtain 3-fermion operators is to multiply $x\psi_3$ or $y\psi_3$ to the $\psi_1\psi_2$-sector (6.22).

Set of such operators is

$$\{x^a y^b z^c f^d \psi_1\psi_2\psi_3 \mid a,b,c,d \in \mathbb{Z}^{\geq 0}\ ,\ a+b \geq 1\ ,\ a+b+c-3 \geq d\ ,\ a+b+c+d = 1 \ (\mathrm{mod}\ 2)\}\ . \tag{6.27}$$

By cyclicity, there are two more sets of 3-fermion operators that are obtained by $x \to y \to z \to x$ from (6.27). Their union is,

$$G_{\psi_1\psi_2\psi_3} = \{x^a y^b z^c f^d \psi_1\psi_2\psi_3 \mid a,b,c,d \in \mathbb{Z}^{\geq 0}\ ,\ a+b+c-3 \geq d\ ,\ a+b+c+d = 1 \ (\mathrm{mod}\ 2)\}\ . \tag{6.28}$$

One can easily check that multiplying non-monomial blocks $x\psi_1 - y\psi_2$ or $y\psi_2 - z\psi_3$ to 2-fermion sector does not produce any new operator.

Partition function of the 3-fermion sector (6.27) is

$$Z_3 = \left[\frac{\begin{array}{c} -1 + \chi_1^2 - 2\chi_2 - \chi_2^2 + 2\chi_1\chi_3 + \chi_3^2 + f(\chi_1 + \chi_3) \\ - f^2(\chi_2 + \chi_2^2 - \chi_1\chi_3 - \chi_3^2) + f^3\chi_3(1 + \chi_2) \end{array}}{(1-x^2)(1-y^2)(1-z^2)(1-xf)(1-yf)(1-zf)} + 1 \right] \cdot \frac{\psi^3}{f\chi_3}\ . \tag{6.29}$$

<u>The index</u>

The complete list of BPS multi-graviton operators in BMN sector of the $SU(2)$ theory is given by (6.9), (6.16), (6.24) and (6.28). Corresponding partition function is $Z_0+Z_1+Z_2+Z_3$, each of which is presented in (6.12), (6.19), (6.26) and (6.29). Attributing minus sign to the fermion number ψ in the partition function and further setting $\psi, f \to xyz$ will yield the index, where $(x,y,z) = (e^{\Delta_1}, e^{\Delta_2}, e^{\Delta_3})$.

To facilitate comparison with the other parts of this paper, we compute the unrefined index of the graviton partition function. This is obtained simply by substituting

$$x, y, z \to t^2\ , \qquad f \to t^6\ , \qquad \psi \to -t^6\ . \tag{6.30}$$

in to the partition function. The result is

$$Z_{\mathrm{grav}} = \frac{\begin{array}{c} 1 + 3t^4 - 8t^6 - 6t^{10} + 10t^{12} + 9t^{14} - 9t^{16} + 16t^{18} \\ - 18t^{20} - 3t^{22} + t^{24} - 3t^{26} + 9t^{28} - 2t^{30} + 3t^{32} - 3t^{34} \end{array}}{(1-t^4)^3(1-t^8)^3}\ . \tag{6.31}$$

Meanwhile, the full index over all cohomologies in the BMN sector of the $SU(2)$ theory can be computed via residue sum of the matrix integral (5.20). We only present the unrefined

($e^{\Delta_1} = e^{\Delta_2} = e^{\Delta_3} \equiv t^2$) version, as our main focus will be on the non-graviton index.

$$\begin{aligned} Z \;=\; \Big[& \; 1 + 3t^2 + 12t^4 + 20t^6 + 42t^8 + 48t^{10} + 75t^{12} + 66t^{14} + 81t^{16} \\ & + 55t^{18} + 54t^{20} + 27t^{22} + 19t^{24} + 6t^{26} + 3t^{28} \;\Big] \frac{(1-t^2)^3}{(1-t^{12})(1-t^8)^3} \; . \end{aligned} \tag{6.32}$$

The difference $Z - Z_{\text{grav}}$ will be the index that counts non-graviton operators. We find a simple analytic formula for the difference:

$$Z - Z_{\text{grav}} = \left[-\frac{e^{4(\Delta_1+\Delta_2+\Delta_3)}}{1 - e^{2(\Delta_1+\Delta_2+\Delta_3)}} \right] \cdot \left[\prod_{I=1}^{3} (1 - e^{\Delta_I}) \right] \cdot \left[\prod_{I=1}^{3} \frac{1}{1 - e^{\Delta_I} e^{\Delta_1+\Delta_2+\Delta_3}} \right] \; . \tag{6.33}$$

Its unrefined version ($e^{\Delta_1} = e^{\Delta_2} = e^{\Delta_3} \equiv t^2$) is also informative:

$$Z - Z_{\text{grav}} \;=\; -\frac{t^{24}}{1-t^{12}} \cdot \frac{(1-t^2)^3}{(1-t^8)^3} \; . \tag{6.34}$$

From this formula, one finds the first black hole cohomology at $j = 24$. This 'threshold' black hole cohomology was already identified in [115, 119], as we shall review and rewrite in a more compact form in the next chapter. It may look like there are many black hole states beyond this threshold, but most of them are rather trivial. To make this point clear, we would like to first interpret various factors of (6.33), which will be extensively justified later.

(6.33) is a multiplication of three factors. We interpret the first factor as the 'core' black hole primary operators. Constructing this part of the cohomologies will be the goal of section 7.2. The second factor comes from the $SU(1|3)$ descendants obtained from the first factor by acting Q^m_+. The supercharge Q^m_+ carries charges $Q_I = \delta_{I,m} - \frac{1}{2}$ and $J = \frac{1}{2}$, so is weighted by e^{Δ_I}. So the second factor comes from the Fock space obtained by acting three Q^m_+'s. Finally, the third factor comes from multiplying certain multi-gravitons to the core black hole cohomologies. Among the 17 graviton states listed in (6.6), only 3 types on the third line can contribute. The remaining 14 gravitons multiplying the core black hole operators do not appear in the index. This aspect too will be discussed further in section 7.2.

6.3 $SU(2)$

For the $SU(2)$ theory but without restriction to the BMN sector, it is not possible to compute the non-graviton index analytically, partly because the graviton operators are polynomials of an infinite number of variables. Note that derivatives of the eigenvalues should be considered as different variables in the algebraic point of view. Therefore, truncation by the order of the operator is inevitable.

Counting the graviton cohomologies with a computer using the eigenvalue setup explained in section 6.1 , we have obtained the graviton index $Z_{\rm grav}$ for the $SU(2)$ theory until t^{40} order. Substracting from the full index, the non-graviton index for the $SU(2)$ theory is given by

$$\begin{aligned} Z - Z_{\rm grav} \ &= \ \big[-t^{24} - \chi_{(1,3)}t^{32} - (\chi_{(1,\bar{3})} + \chi_{(3,6)})t^{34} - \chi_{(2,3)}t^{35} + (\chi_{(3,1)} + \chi_{(3,8)})t^{36} \\ &\quad -(\chi_{(2,\bar{3})} + \chi_{(4,6)})t^{37} + \chi_{(5,3)}t^{38} + (\chi_{(2,1)} + 2\chi_{(4,1)} + \chi_{(4,8)})t^{39} \\ &\quad -(2\chi_{(1,6)} + \chi_{(3,\bar{3})} + \chi_{(5,\bar{3})} + \chi_{(5,6)})t^{40} \big] \, \chi_{\rm D} + \mathcal{O}(t^{41}) \ . \end{aligned} \tag{6.35}$$

We have organized the result into $SU(2)_R \times SU(3) \subset PSU(1,2|3)$ characters. We have also factored out by χ_D which is given by

$$\begin{aligned} \chi_{(2J'+1,{\rm R})} \ &\equiv \ \chi^{SU(2)_R}_{J'}(p)\chi^{SU(3)}_{\rm R}(x,y) \ , \qquad (6.36) \\ \chi_D \ &\equiv \ \frac{(1-t^2z_1)(1-\frac{t^2}{z_2})(1-\frac{t^2z_2}{z_1})(1-\frac{tp}{z_1})(1-\frac{t}{pz_1})(1-tz_2p)(1-\frac{tz_2}{p})(1-\frac{tz_1p}{z_2})(1-\frac{tz_1}{z_2p})}{(1-t^3p)(1-\frac{t^3}{p})} \ , \end{aligned}$$

where $t^6 = e^{\Delta_1+\Delta_2+\Delta_3} = e^{\omega_1+\omega_2}$, $z_1 = e^{\frac{2\Delta_1-\Delta_2-\Delta_3}{3}}$, $z_2^{-1} = e^{\frac{-\Delta_1+2\Delta_2-\Delta_3}{3}}$, $p = e^{\frac{\omega_1-\omega_2}{2}}$. This is a factor for superconformal descendants. Since the non-gravitons should appear in representations of $PSU(1,2|3)$, the subset of the $\mathcal{N} = 4$ superconformal group $PSU(2,2|4)$ that commutes with the supercharge Q, it is economical to write only the superconformal primaries from which the descendants automatically follow. It is very likely that all non-graviton operators belong to the $A_1\bar{L}$-type supermultiplets in the notation of [88], and χ_D is the factor that yields the character of the supermultiplet when multiplied to the character of the superconformal primary. So each term in the square bracket of (6.35) should represent a $PSU(1,2|3)$ supermultiplet whose superconformal primary transforms under the denoted representations under the bosonic subalgebra.

6.4 $SU(3)$, BMN Sector

Following the computational procedures explained earlier in this chapter, including the eigenvalue counting and the Gröbner basis, we have computed the $SU(3)$ graviton index $Z_{\rm grav}$ until t^{54} order. We write the difference $Z - Z_{\rm grav}$ with the full index Z, which is the index over non-graviton cohomologies or the 'black hole' cohomologies, in the form of

$$Z - Z_{\rm grav} = Z_{\rm core}(\Delta_I) \cdot \prod_{I=1}^{3} \frac{1}{1 - e^{\Delta_I}e^{\Delta_1+\Delta_2+\Delta_3}} \cdot \prod_{I<J}(1 - e^{\Delta_I+\Delta_J}) \ . \tag{6.37}$$

j	F_0	F_1	F_2	F_3	F_4	F_{exc}	B_1	B_2	B_3	B_{exc}
24	[0, 0]									
26										
28										
30	[0, 0]	[3, 0]								
32		[4, 0]								
34		[5, 0]					[3, 1]			
36	[0, 0]	[6, 0]					[4, 1]			[3, 0]
38		[7, 0]				[1, 0]	[5, 1]			
40		[8, 0]	[5, 0]		[3, 1]		[6, 1]			
42	[0, 0]	[9, 0]	[6, 0]		[4, 1]		[7, 1]			[1, 1]
44		[10, 0]	[7, 0]		[5, 1]		[8, 1]	[5, 1]		
46		[11, 0]	[8, 0]		[6, 1]	[2, 0]	[9, 1]	[6, 1]		[5, 0]
48	[0, 0]	[12, 0]	[9, 0]		[7, 1]	[3, 0]	[10, 1]	[7, 1]		[4, 1]
50		[13, 0]	[10, 0]	[7, 0]	[8, 1]		[11, 1]	[8, 1]		[4, 0]
52		[14,0]	[11,0]	[8,0]	[9,1]	[2,0]	[12, 1]	[9,1]		[3,1]
54		[15,0]	[12,0]	[9,0]	[10,1]	[4,1]	[13, 1]	[10,1]	[7,1]	

Table 6.1: $SU(3)$ Dynkin labels of fermionic/bosonic black hole cohomologies after factoring out the descendants and the conjectured graviton hairs of w_2, organized into towers by empirical reasons.

The factors that dress the index over *core* non-graviton cohomologies will be explained shortly. $Z_{\text{core}}(\Delta_I) \equiv f(t,x,y)$ with $e^{\Delta_1} = t^2 x$, $e^{\Delta_2} = t^2 y^{-1}$, $e^{\Delta_3} = t^2 x^{-1} y$ can be expanded as

$$f(t,x,y) = \sum_{\mathcal{J}=0}^{54} \sum_{\mathbf{R}_j} (-1)^{F(\mathbf{R}_\mathcal{J})} \chi_{\mathbf{R}_\mathcal{J}}(x,y) t^\mathcal{J} + \mathcal{O}(t^{56}) \ , \tag{6.38}$$

where $\mathbf{R}_\mathcal{J}$ runs over the $SU(3)$ irreducible representations which appear at $t^\mathcal{J}$ order ($\mathcal{J}$ is even in the BMN sector), $\chi_{\mathbf{R}_\mathcal{J}}(x,y)$ is its character, and $F(\mathbf{R}_\mathcal{J})$ is its fermion number. The representations $\mathbf{R}_\mathcal{J}$ appearing in the expansion of f, together with their bosonic/fermionic natures, are shown in Table 6.1. We have classified the representations into several groups, i.e. what we suspect to be the fermionic towers $F_0, ..., F_4$, the bosonic towers $B_1, ..., B_3$, and the remainders F_{exc}, B_{exc} for which we do not see particular patterns (thus named 'exceptional'). Entries that appear in cyan may be related to the towers F_1 and B_1 of core primaries by dressing of w_3 gravitons. We will comment on the dressings later. Entries in gray are not observed in the non-graviton index, but we included them because if we assume that they appear in boson/fermion pairs, then the tower structure is reinforced.

We comment on the factors which we have taken out in (6.37). The factor $\prod_{I<J}(1 - e^{\Delta_1+\Delta_2})$ accounts for $SU(1|3)$ descendants. For each non-graviton cohomology in $\mathbf{R}_j$ that

contributes to Z_{core}, the entire $SU(1|3)$ multiplet obtained by acting the three fermionic generators Q_+^m must also be non-graviton cohomologies. Every such multiplet is a long multiplet of the $SU(1|3)$, so the corresponding character is simply the contribution from the primary times the factor $\prod_{I<J}(1-e^{\Delta_1+\Delta_2})$. This fact can be argued using the embedding supergroup $PSU(2,2|4)$ of the 4d $\mathcal{N}=4$ theory. For any of the three generators Q_+^m to annihilate the $SU(1|3)$ primary, the primary of a bigger representation of $PSU(2,2|4)$ that includes the $SU(1|3)$ multiplet must be annihilated by Q_+^4 *and* by the $SU(4)_R$ lowering operator that is not part of the $SU(3) \subset SU(4)_R$. The only $PSU(2,2|4)$ representations that satisfy this property are $B_1\bar{B}_1[0;0]^{[0,n,0]}$, namely the graviton operators, or the identity. For details on the relevant representation theory, we refer to [88], particularly its section 2.2.4, or to appendix B of [71].

The second factor of (6.37) was taken out for an empirical reason, with an expectation that they come from the graviton hairs of w_2's in (5.26). Namely, we conjecture that w_2 gravitons multiplying the core black hole cohomologies represented by Z_{core} provide nontrivial product cohomologies. Although we have little logical justification of the last claim (except that similar hairs are allowed in the $SU(2)$ theory), we think that the phenomenological evidence of this claim is compelling since various simple patterns in Table 6.1 are clear only after factoring it out.

We refer to section 3.1 of [72] for discussions on the tower structures. Various scenarios and suggestions are presented there, as to how and which graviton cohomologies may multiply to some of the black hole cohomologies displayed in Table 6.1 to yield other cohomologies also displayed in Table 6.1. The discussion on partial no-hair behavior that will be explained in section 7.2 extends with various complications to Table 6.1.

6.5 $SU(4)$, BMN Sector

In the $SU(4)$ case, using similar strategies as for the $SU(3)$ case, we computed Z_{grav} until $j=30$ level. The index $Z - Z_{\text{grav}}$ over non-graviton cohomologies is given by

$$Z - Z_{\text{grav}} = \left[-\chi_{[2,0]}(x,y)t^{28} - \chi_{[3,0]}(x,y)t^{30} + \mathcal{O}(t^{32})\right] \cdot \prod_{I<J}(1-e^{\Delta_I+\Delta_J}) \ . \tag{6.39}$$

The second factor generates the Fock space of each $SU(1|3)$ multiplet, while the first factor in the square parenthesis represents the primary non-gravitons. One finds that the BMN index predicts an apparent threshold of non-graviton cohomologies at $\mathcal{J} = 2(Q_1+Q_2+Q_3)+6J = 28$. Again, conservatively, this is an upper bound for the threshold for two different reasons: first because the index may miss a pair of canceling threshold cohomologies at lower charges, and also because the true threshold might lie outside the BMN sector (carrying nonzero $SU(2)_R$ spin $J_1 - J_2$). Anyway, the above apparent threshold is higher than the $SU(3)$ threshold. So it is natural to expect that it was an exception that the $SU(2)$ and $SU(3)$ thresholds were the same: the (apparent) thresholds for $\mathcal{J}$ are $24, 24, 28, \cdots$ for $N = 2, 3, 4 \cdots$. To obtain the threshold level in terms of energy $E = \sum_I Q_I + \sum_i J_i$, one should construct the actual cohomologies which account for the t^{28} term. This will not be done in this thesis.

Chapter 7

Constructing the Cohomologies

The non-graviton indices computed in the previous chapter guide us to focus on certain charge sectors to construct the simplest non-graviton cohomologies. For example, the indices for BMN sectors of the $SU(2)$ and $SU(3)$ theories suggest that we attempt to construct a fermionic black hole cohomology only using the BMN letters, that is a singlet under the $SU(3)$ subgroup of the R-symmetry group at the order $\mathcal{J} = 24$, equivalently $q_1 = q_2 = q_3 = 4$.

For the $SU(2)$ theory, such a cohomology will indeed be the threshold black hole cohomology, i.e. one with the lowest order. It has been shown in [115] through an extensive search in the space of all cohomologies that in the $SU(2)$ theory, the fermionic singlet cohomology at $\mathcal{J} = 24$ is the first and the only one until the order $\mathcal{J} = 25$. For the $SU(3)$ theory, the extensive study of [115] has been performed only until $\mathcal{J} = 19$, so the possibility that a black hole cohomology exists between $\mathcal{J} = 20$ and $\mathcal{J} = 24$ but outside of the BMN sector, or the possibility that a boson-fermion pair of black hole cohomologies exists between the same order, are not ruled out. However, it is unlikely that the threshold cohomology for the $SU(3)$ theory appears at a lower order than for the $SU(2)$ theory, so we are somewhat confident that the $\mathcal{J} = 24$ black hole cohomology that we shall present is indeed the threshold black hole cohomology of the $SU(3)$ theory.

7.1 $SU(2)$, BMN Sector

The threshold black hole cohomology for the $SU(2)$ theory was shown to exist at the order $\mathcal{J} = 24$ through an extensive search in the space of all cohomologies [115], and its explicit form was written down shortly after in [119]. Meanwhile, our result on the BMN non-graviton index of the $SU(2)$ theory, in particular the first factor of (6.33),

$$-\frac{t^{24}}{1-t^{12}} = -t^{24} - t^{36} - t^{48} - t^{60} - \cdots \ , \tag{7.1}$$

suggests that there is one fermionic black hole cohomology at every 12 value of $\mathcal{J}$ starting from the threshold at $\mathcal{J} = 24$. In this section, we present the explicit form (7.35) of these core black hole cohomologies. These, together with the factors in (6.33) who interpretation was given below the equation, account for all black hole cohomologies detected by the index in the BMN sector of the $SU(2)$ theory.

The index, in particular the core factor (7.1), predicts unique fermionic cohomology at each order $\mathcal{J} = 24 + 12n$ ($n = 0, 1, 2, \cdots$), all singlets of $SU(3) \subset SU(4)$. For the $SU(2)$ gauge group, we use the 3-dimensional vector notation for the adjoint fields. In the remaining part of this section, $\phi^m = (X, Y, Z)$, ψ_m ,f will denote 3 dimensional vectors, and inner/outer products will replace the trace/commutators. The Q-transformations of these 3-vectors are given by

$$Q\phi^m = 0 \ , \quad Q\psi_m = \tfrac{1}{2}\epsilon_{mnp}\phi^n \times \phi^p \ , \ Qf = \phi^m \times \psi_m \ . \tag{7.2}$$

<u>O_0 **operator at** t^{24}</u>

This operator has charges $E = \frac{19}{2}$, $Q_1 = Q_2 = Q_3 = \frac{3}{2}$, $J_1 = J_2 = \frac{5}{2}$. A representative of this cohomology [119] is given by

$$\begin{aligned} O_0' \ &= \ (X \cdot \psi_1 - Y \cdot \psi_2)(X \cdot \psi_3)(\psi_2 \cdot \psi_1 \times \psi_1) + (Y \cdot \psi_2 - Z \cdot \psi_3)(Y \cdot \psi_1)(\psi_3 \cdot \psi_2 \times \psi_2) \\ &\quad +(Z \cdot \psi_3 - X \cdot \psi_1)(Z \cdot \psi_2)(\psi_1 \cdot \psi_3 \times \psi_3) \ . \end{aligned} \tag{7.3}$$

Note that the second and third terms are obtained by making cyclic permutations of (X, ψ_1), (Y, ψ_2), (Z, ψ_3) on the first term. The cyclic permutations are part of the $SU(3)$ symmetry, thus symmetries of the cohomology problem, On the other hand, odd permutations accompanied by the sign flips of all ψ_m's and ϕ^m's are part of $SU(4) \times SU(2)_L$ symmetry which leave Q invariant, thus being symmetries of the cohomology problem. To construct a better representative of this cohomology, consider the following operator obtained by permuting $(X, \psi_1) \leftrightarrow (Y, \psi_2)$ and flipping signs of all ϕ^m, ψ_m on (7.3):

$$\begin{aligned} O_0'' \ &= \ (X \cdot \psi_1 - Y \cdot \psi_2)(Y \cdot \psi_3)(\psi_1 \cdot \psi_2 \times \psi_2) + (Y \cdot \psi_2 - Z \cdot \psi_3)(Z \cdot \psi_1)(\psi_2 \cdot \psi_3 \times \psi_3) \\ &\quad +(Z \cdot \psi_3 - X \cdot \psi_1)(X \cdot \psi_2)(\psi_3 \cdot \psi_1 \times \psi_1) \ . \end{aligned} \tag{7.4}$$

One can show

$$\begin{aligned} O_0' - O_0'' &= -2Q[(\psi_1 \cdot \psi_2)(\psi_2 \cdot \psi_3)(\psi_3 \cdot \psi_1)] \ , \\ O_0 &\equiv -5(O_0' + O_0'') = \epsilon^{p_1 p_2 p_3} v^m{}_{p_1} v^n{}_{p_2} (\psi_m \cdot \psi_n \times \psi_{p_3}) \ , \end{aligned} \tag{7.5}$$

where

$$v^m{}_n \equiv (\phi^m \cdot \psi_n) - \tfrac{1}{3}\delta^m_n (\phi^p \cdot \psi_p) \tag{7.6}$$

are graviton cohomologies in the S_2 multiplet. O_0 is manifestly an $SU(3)$ singlet. Note that the second term of v proportional to δ^m_n drops out when v is inserted into (7.5), because of the symmetry of $\psi_m \cdot \psi_n \times \psi_{p_3}$ and the antisymmetry of $\epsilon^{p_1 p_2 p_3}$. So we can write

$$O_0 = \epsilon^{p_1 p_2 p_3} (\phi^m \cdot \psi_{p_1})(\phi^n \cdot \psi_{p_2})(\psi_m \cdot \psi_n \times \psi_{p_3}) \ . \tag{7.7}$$

To show that O_0 is a black hole cohomology, one should check that it is Q-closed, not Q-exact, and not of graviton type. The first and third are trivial. O_0 is not graviton-like because it consists of seven (odd) letters: since $SU(2)$ gravitons are made of operators in S_2, they always have an even number of letters. To check Q-closedness, first note that Q acts only on $\psi_m \cdot \psi_n \times \psi_{p_3}$ because $v^m{}_n$ are Q-closed. One finds

$$Q(\psi_m \cdot \psi_n \times \psi_p) = \tfrac{3}{2}\epsilon_{(m|qr}(\phi^q \times \phi^r) \cdot (\psi_{|n} \times \psi_{p)}) = 3\epsilon_{(m|qr}(\phi^q \cdot \psi_{|n})(\phi^r \cdot \psi_{p)}) = 3\epsilon_{(m|qr} v^q{}_{|n} v^r{}_{p)} \ . \tag{7.8}$$

At the last step, the second term of $\phi^q \cdot \psi_n = v^q{}_n + \delta^q_n(\cdots)$ etc. does not survive after the index contractions. Inserting it to QO_0 and replacing the product of two ϵ's by three δ's, QO_0 is given by various row/column contractions of four 3×3 traceless matrices $v^m{}_n$. Possible terms are $\mathrm{tr}(v^4)$ and $\mathrm{tr}(v^2)\mathrm{tr}(v^2)$, but the fermionic nature of v and the cyclicity of trace ensure that they are all zero. So QO_0 is zero because there are no nonzero terms that can contribute.

The non-Q-exactness was originally shown after a calculation using computer [115, 119]. Here we provide an analytic argument. We assume Q-exactness, narrow down the possible Q-exact terms and then show that no combination of them works. O_0 is at the $\mathcal{O}(\phi^2\psi^5)$ order. If this is Q-exact, the schematic structure should be as follows:

$$\phi^2\psi^5 = Q_f[f\phi\psi^4] + Q[\psi^6] \ . \tag{7.9}$$

Q_f means the part of Q acting on f. Q may also act on ψ in this term to produce a term at $\mathcal{O}(f\phi^3\psi^3)$ order, and if O_0 is completely Q-exact, $Q_\psi(f\phi\psi^4)$ should cancel $Q_f[f^2\phi^2\psi^2]$. We shall only consider the Q-exactness of O_0 within the $\phi^2\psi^5$ order and find a contradiction.

The terms on the right hand side should respect all the $SU(3) \times SU(2)$ tensor structures of the left hand side. There might be terms violating some of these structures separately on the first and second terms, but they should cancel by themselves and we do not care about this part. We consider the terms which respect them and the equation should hold within this sector separately if (7.9) is generally true. (7.7) is given by multiplying the following scalar and fermion factors,

$$\begin{aligned} \phi^{(m}_{(i}\phi^{n)}_{j)} &\;:\; (\mathbf{6}, \mathbf{5}+\mathbf{1}) \in SU(3) \times SU(2) \\ \psi_{p_1(i|}\psi_{p_2|j)}(\psi_m \cdot \psi_n \times \psi_{p_3}) &\;:\; (\mathbf{3}, \mathbf{5}+\mathbf{1}) \otimes (\overline{\mathbf{10}}, \mathbf{1}) \ , \end{aligned} \tag{7.10}$$

where $i, j = 1, 2, 3$ are $SU(2) \sim SO(3)$ indices. The operators on the right hand side of (7.9) should respect these structures.

We shall first write down all possible terms on the right hand side satisfying several consequences of (7.10) after contracting all the indices, obtaining only a small number of terms. Some useful requirements are: (1) $SU(3)$ singlet condition of O_0, (2) exchange symmetry of the two $SU(3)$ indices carried by the scalars. We first consider the term $Q[\psi^6]$. Q acting on any ψ produces a term of the form $\phi^m \times \phi^n$, violating the condition (2). So there are no terms of the form $Q[\psi^6]$ that we can write down. Now we try to write down all the gauge-invariant operators at $f\phi\psi^4$ order which can appear inside Q_f in (7.9). Since it consists of six letters, we take three pairwise inner products. (Contractions by two ϵ tensors can also be written as three inner products.) The possible terms are

$$(f \cdot \phi^m)(\psi_{[n_1} \cdot \psi_{n_2]})(\psi_{[p_1} \cdot \psi_{p_2]}) \ \ , \ \ (\phi^m \cdot \psi_{n_1})(f \cdot \psi_{n_2})(\psi_{[p_1} \cdot \psi_{p_2]}) \ . \tag{7.11}$$

Q_f transformation of the first term violates the condition (2) since $Q[f \cdot \phi^m] = (\phi^n \times \psi_n) \cdot \phi^m = (\phi^m \times \phi^n) \cdot \psi_n$. Now imposing the condition (1) on the second term, one should contract the $SU(3)$ indices to form singlets. One finds

$$(\mathbf{8} \oplus \mathbf{1}) \otimes \mathbf{3} \otimes \overline{\mathbf{3}} \to \mathbf{27} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1} \oplus \mathbf{1}$$

so there are two possible singlets. They are

$$(\phi^m \cdot \psi_m)\epsilon^{npq}(f \cdot \psi_n)(\psi_p \cdot \psi_q) \ \ , \ \ (\phi^m \cdot \psi_n)(f \cdot \psi_m)\epsilon^{npq}(\psi_p \cdot \psi_q) \ . \tag{7.12}$$

Acting Q_f on them and separating the ϕ^2 and ψ^5 parts as we did in (7.10), we obtain a part consistent with (7.10) and the rest. Focusing on the former part, they are given by $\phi^{(m}_{(i}\phi^{r)}_{j)}$

times

$$\psi^{(i}_{(m}(\psi_{r)}\times\psi_n)^{j)}\epsilon^{npq}(\psi_p\cdot\psi_q)\quad,\quad \psi^{(i}_n(\psi_r\times\psi_m)^{j)}\epsilon^{npq}(\psi_p\cdot\psi_q) \tag{7.13}$$

respectively. If O_0 is Q-exact, a suitable linear combination of these two terms should yield O_0. The agreement should happen for every coefficient of $\phi^{(m}_{(i}\phi^{r)}_{j)}$ separately, demanding

$$\epsilon^{pqr}\psi^{(i}_p\psi^{j)}_q(\psi_m\cdot\psi_n\times\psi_r) = A\psi^{(i}_{(m}(\psi_{n)}\times\psi_r)^{j)}\epsilon^{pqr}(\psi_p\cdot\psi_q)+B\psi^{(i}_r(\psi_n\times\psi_m)^{j)}\epsilon^{pqr}(\psi_p\cdot\psi_q) \tag{7.14}$$

for suitable A, B. Inserting two different sets of m, r, i, j, we found that there are no solutions for A and B. This proves that O_0 is not Q-exact.

One can also easily show the non-Q-exactness by studying the $SU(1|3)$ descendants obtained by acting $Q^a_+Q^b_+$. For instance, one obtains

$$\begin{aligned}
&Q^2_+Q^1_+O'_0 = \\
&-(Y\cdot f+\psi_3\cdot\psi_1)^2\psi_3\cdot(\psi_2\times\psi_2)-(X\cdot f+\psi_2\cdot\psi_3)(Z\cdot f+\psi_1\cdot\psi_2)\psi_1\cdot(\psi_3\times\psi_3)\\
&-(X\cdot f+\psi_2\cdot\psi_3)(X\cdot\psi_3)f\cdot(\psi_1\times\psi_1)+2(Y\cdot\psi_2-Z\cdot\psi_3)(Y\cdot f+\psi_3\cdot\psi_1)\psi_3\cdot(\psi_2\times f)\\
&-2(Y\cdot f+\psi_3\cdot\psi_1)(X\cdot\psi_3)\psi_2\cdot(\psi_1\times f)-(Z\cdot\psi_3-X\cdot\psi_1)(Z\cdot f+\psi_1\cdot\psi_2)f\cdot(\psi_3\times\psi_3)
\end{aligned} \tag{7.15}$$

which contains uncanceled $\phi^0\psi^7$ terms on the second line. Since acting Q always creates one or more ϕ factors, these terms cannot be Q-exact. Since a descendant of O'_0 is not Q-exact, O_0 cannot be Q-exact either, providing a simpler proof. Or alternatively, one can prove non-Q-exactness by acting three Q_+'s to O'_0 and check that it contains nonzero term at $f\psi^6$ order,

$$\begin{aligned}
&Q^1_+Q^2_+Q^3_+O'_0 = Q^1_+Q^2_+Q^3_+O''_0 = \\
&(X\cdot f+\psi_2\cdot\psi_3)^2 f\cdot(\psi_1\times\psi_1)+2(X\cdot f+\psi_2\cdot\psi_3)(Y\cdot f+\psi_3\cdot\psi_1)f\cdot(\psi_1\times\psi_2)\\
&+(1,2,3\to 2,3,1)+(1,2,3\to 3,1,2) = G^mG^n f\cdot(\psi_m\times\psi_n)
\end{aligned} \tag{7.16}$$

where $G^m \equiv \phi^m\cdot f+\frac{1}{2}\epsilon^{mnp}\psi_n\cdot\psi_p$. Proof of this sort will sometimes be useful later. For instance, one can show that $(Z\cdot f+\psi_1\cdot\psi_2)O'_0$ is not Q-exact, since its descendant

$$Q^2_+Q^1_+\left[(Z\cdot f+\psi_1\cdot\psi_2)O'_0\right] = (Z\cdot f+\psi_1\cdot\psi_2)Q^2_+Q^1_+O'_0 \tag{7.17}$$

contains a term at $\phi^0\psi^9$ order.

O_1 operator at t^{36}

Now we construct the cohomology which accounts for the $-t^{36}$ term of (7.1). It should be fermionic, has charge $\mathcal{J} = 2(Q_1+Q_2+Q_3)+6J = 36$, and should be an $SU(2)_R\times SU(3)$

singlet because we expect unique cohomology (unless there is a cancellation at this order which obscures the true degeneracy). We call this operator O_1. From the last condition, we set three Q_I equal and two J_i equal. Still, we do not know the individual Q and J so we should make a guess. Our first guess was to add extra $\Delta J = 2$ to the charges $Q = \frac{3}{2}$, $J = \frac{5}{2}$ of O_0. We listed all operators in this sector and found the cohomology by computer. Then we made several trials until we found the following $SU(3)$-invariant representative:

$$\begin{aligned} O_1 &= (f \cdot f)\epsilon^{c_1c_2c_3}(\phi^a \cdot \psi_{c_1})(\phi^b \cdot \psi_{c_2})(\psi_a \cdot \psi_b \times \psi_{c_3}) \\ &\quad +\epsilon^{b_1b_2b_3}\epsilon^{c_1c_2c_3}(f \cdot \psi_{b_1})(\phi^a \cdot \psi_{c_1})(\psi_{b_2} \cdot \psi_{c_2})(\psi_a \cdot \psi_{b_3} \times \psi_{c_3}) \\ &\quad -\tfrac{1}{72}\epsilon^{a_1a_2a_3}\epsilon^{b_1b_2b_3}\epsilon^{c_1c_2c_3}(\psi_{a_1} \cdot \psi_{b_1} \times \psi_{c_1})(\psi_{a_2} \cdot \psi_{b_2} \times \psi_{c_2})(\psi_{a_3} \cdot \psi_{b_3} \times \psi_{c_3}) \ . \end{aligned} \tag{7.18}$$

It is not graviton type since it is made of nine (odd) letters. One can also easily check that it is not Q-exact. This is because the last term contains no scalars. Since Q transformations (7.2) always yield scalars, the last term cannot be made Q-exact. So O_1 is not Q-exact.

Now we discuss the Q-closedness. O_1 takes the form of

$$O_1 = (f \cdot f)O_0 + f \cdot \xi + \chi \ , \tag{7.19}$$

where the $SU(2)$ triplet $\vec{\xi}$ and the singlet χ are given by

$$\begin{aligned} \vec{\xi} &= \epsilon^{b_1b_2b_3}\epsilon^{c_1c_2c_3}\vec{\psi}_{b_1}(\phi^a \cdot \psi_{c_1})(\psi_{b_2} \cdot \psi_{c_2})(\psi_a \cdot \psi_{b_3} \times \psi_{c_3}) \\ \chi &= -\tfrac{1}{72}\epsilon^{a_1a_2a_3}\epsilon^{b_1b_2b_3}\epsilon^{c_1c_2c_3}(\psi_{a_1} \cdot \psi_{b_1} \times \psi_{c_1})(\psi_{a_2} \cdot \psi_{b_2} \times \psi_{c_2})(\psi_{a_3} \cdot \psi_{b_3} \times \psi_{c_3}) \\ &= -120\psi_1^1\psi_1^2\psi_1^3\psi_2^1\psi_2^2\psi_2^3\psi_3^1\psi_3^2\psi_3^3 \ . \end{aligned} \tag{7.20}$$

Q-closedness is equivalent to the following equations:

$$2(\vec{\phi}^m \times \vec{\psi}_m)O_0 + Q_\psi\vec{\xi} = 0 \quad , \quad \vec{\phi}^m \cdot (\vec{\psi}_m \times \vec{\xi}) + Q_\psi\chi = 0 \ . \tag{7.21}$$

Note that $\vec{\xi}$ is related to O_0 by

$$\vec{\xi} = -\tfrac{1}{2}\epsilon^{mnp}\vec{\psi}_m\psi_n \cdot \tfrac{\partial}{\partial\phi^p}O_0 \ . \tag{7.22}$$

So the first equation can be written as the following equations of O_0:

$$\begin{aligned} &4(\vec{\phi}^m \times \vec{\psi}_m)O_0 \\ &= \left[(\vec{\phi}^a \times \vec{\phi}^b)\psi_a \cdot \tfrac{\partial}{\partial\phi^b} + \vec{\psi}_a(\phi^a \times \phi^b) \cdot \tfrac{\partial}{\partial\phi^b} - \vec{\psi}_a(\psi_b \times \phi^a) \cdot \tfrac{\partial}{\partial\psi_b} + \vec{\psi}_b(\psi_a \times \phi^a) \cdot \tfrac{\partial}{\partial\psi_b}\right] O_0 \ . \end{aligned} \tag{7.23}$$

This is a property of O_0. The second/third terms cancel due to $\left(\phi^b \times \frac{\partial}{\partial\phi^b} + \psi_b \times \frac{\partial}{\partial\psi_b}\right) O_0 = 0$, which holds because it is the $SU(2)$ gauge transformation on a gauge invariant operator O_0. One can further simplify (7.23) using various properties of O_0. Obvious ones are

$$\begin{aligned}
&\phi^m \cdot \tfrac{\partial}{\partial\phi^m} O_0 = n_{\rm B} O_0 \quad , \quad \psi_m \cdot \tfrac{\partial}{\partial\psi_m} O_0 = n_{\rm F} O_0 \quad (n_{\rm B}, n_{\rm F}) = (2,5) \\
&\vec{\varepsilon}_a^{\ b} \left[\phi^a \cdot \tfrac{\partial}{\partial\phi^b} - \psi_b \cdot \tfrac{\partial}{\partial\psi_a}\right] O_0 = 0 \quad (\vec{\varepsilon}_a^{\ a} = 0) \ .
\end{aligned} \tag{7.24}$$

The first two equations count the numbers of bosonic/fermionic fields in O_0. The last equation is the $SU(3)$ invariance of O_0, which holds for any ε. Equivalently, one obtains

$$\left[\phi^a \cdot \tfrac{\partial}{\partial\phi^b} - \psi_b \cdot \tfrac{\partial}{\partial\psi_a}\right] O_0 = \tfrac{1}{3}(n_{\rm B} - n_{\rm F}) \delta^a_b O_0 \ . \tag{7.25}$$

Finally, note that δ_{ij} contracts the $SU(2)$ gauge triplet indices only between boson-fermion pairs in O_0, while fermion indices are contracted only with ϵ_{ijk}. This effectively promotes $SU(2) \sim SO(3)$ to $SL(3)$ within O_0, where bosons/fermions transform in the fundamental and anti-fundamental representations, respectively. This leads to the following property:

$$\left[\phi^a_i \cdot \tfrac{\partial}{\partial\phi^a_j} - \psi^j_a \cdot \tfrac{\partial}{\partial\psi^i_a}\right] O_0 = \tfrac{1}{3}(n_{\rm B} - n_{\rm F}) \delta^j_i O_0 \ . \tag{7.26}$$

Using these properties, (7.23) can be written as

$$(\vec{\psi}_a \times \vec{\phi}^b)(\phi^a \cdot \tfrac{\partial}{\partial\phi^b}) O_0 = (4 - \tfrac{n_{\rm F} + 2n_{\rm B}}{3})(\vec{\phi}^m \times \vec{\psi}_m) O_0 = (\vec{\phi}^m \times \vec{\psi}_m) O_0 \ . \tag{7.27}$$

Both (7.27) and the second equation of (7.21) can be easily checked on a computer. We have no extra analytic insights on why (7.27) this holds, except that using complicated representation analysis of $SU(2) \times SU(3)$ should provide the analytic proof. (We tried to simplify the equation for O_0 as much as possible since they might provide insights on the generalization to higher N's in the future.) On the other hand, one can easily prove the second equation of (7.21). First note that $\psi_m \times \xi$ is an $SU(2)$ vector involving 8 ψ's. There are nine independent operators involving eight ψ's, depending on which of the 9 components is lacking. So it is proportional to $\frac{\partial}{\partial\psi^i_m}\chi$. Since it has to form a gauge-invariant by contracting with two scalars ϕ^m_i, ϕ^a_j, one should be able to write $\frac{\delta}{\delta\psi^i_m}\chi$ as an object with two $SU(3)$ antifundamental and two $SU(2)$ triplet indices by multiplying invariant tensors. The only possible term is $\epsilon_{man}\epsilon_{ijk}\frac{\partial}{\partial\psi^k_n}\chi$. One can compute the proportionality constant by computing a term, e.g. at $m = 1, a = 2, i = 1, j = 2$, finding $-\frac{1}{2}$. So one obtains

$$\phi^m \cdot (\psi_m \times \xi) = -\tfrac{1}{2}\phi^m_i \phi^a_j \epsilon_{man}\epsilon_{ijk}\tfrac{\partial\chi}{\partial\psi^k_n} = -\tfrac{1}{2}\epsilon_{man}(\phi^m \times \phi^a) \cdot \tfrac{\partial\chi}{\partial\psi_n} = -Q_\psi \chi \ , \tag{7.28}$$

proving the second equation of (7.21).

One may wonder if O_1 is a descendant of O_0, or a lower black hole operator times graviton operators appearing in (6.33). Since O_1 is at t^{36} order, the only possible way of getting operators at this order from O_0 is $(Q_+^m Q_+^n O_0)(\phi^p \cdot f + \frac{1}{2}\epsilon^{pqr}\psi_q \cdot \psi_r)$. However, during our numerical construction of the cohomologies at this order, we separately constructed the last operator which is not cohomologous to O_1. See also the end of this subsection for an analytic proof (applicable to all O_n's with $n \geq 1$).

O_n operator at t^{24+12n} ($n \geq 2$)

We can use the structures of the operators O_0 and O_1 to analytically construct an infinite tower of cohomologies O_n accounting for (7.1). Consider

$$O_n \equiv (f \cdot f)^n O_0 + n(f \cdot f)^{n-1} f \cdot \xi + \tfrac{2n^2+n}{3}(f \cdot f)^{n-1}\chi \tag{7.29}$$

for $n \geq 2$. At $n = 1$, this is just O_1 that we discussed above. We will now show that these are new black hole like cohomologies at t^{24+12n} order. It is again easy to show that these are not graviton type because they are made of odd letters. It is not Q-exact because the last term does not contain scalars.

Now we derive the Q-closedness. Its Q-action is given by

$$\begin{aligned} QO_n \;=\; & (f \cdot f)^{n-1}\left[\vec{f} \cdot \left(2n(\vec{\phi}^m \times \vec{\psi}_m)O_0 + nQ_\psi\vec{\xi}\right) + n(\vec{\phi}^m \times \vec{\psi}_m) \cdot \vec{\xi} + \tfrac{2n^2+n}{3}Q_\psi\chi\right] \\ & +2(n^2-n)(f \cdot f)^{n-2}\vec{f} \cdot (\vec{\phi}^m \times \vec{\psi}_m)(f \cdot \xi) + \tfrac{2n(n-1)(2n+1)}{3}(f \cdot f)^{n-2}\vec{f} \cdot (\vec{\phi}^m \times \vec{\psi}_m)\chi \;. \end{aligned} \tag{7.30}$$

The first two terms on the first line cancel due to the first equation of (7.21). The last term on the second line is zero because it includes 10 fermions. Inserting the second equation of (7.21) to the last term on the first line, one obtains

$$QO_n = \tfrac{2(n^2-n)}{3}(f \cdot f)^{n-2}\left[-(f \cdot f)(\phi^m \times \psi_m) \cdot \xi + 3(f \times \phi^m) \cdot \psi_m (f \cdot \xi)\right] \;. \tag{7.31}$$

The second term contains 8 fermions, where the fermions carry ma indices for $SU(3)$ and three $SU(2)$ triplet indices to be contracted with $(f \times \phi^m)_k$, f_i, ϕ_j^a. From the contraction structures of ξ, one finds that b_1, c_1 are antisymmetric so the corresponding i, j indices should be symmetric. The only possible 8-fermion terms satisfying these conditions are

$$\epsilon_{man}\delta_{ij}\frac{\partial}{\partial\psi_n^k}\chi \quad , \quad \epsilon_{man}\delta_{k(i}\frac{\partial}{\partial\psi_n^{j)}}\chi \;. \tag{7.32}$$

Explicitly computing two components in the second term of (7.31), one finds that the linear combination is

$$\epsilon_{man}\left[\delta_{ij}\frac{\partial}{\partial\psi_n^k}-\delta_{k(i}\frac{\partial}{\partial\psi_n^{j)}}\right]\chi\ . \tag{7.33}$$

Contracting this with f_i, ϕ_j^a, $(f\times\phi^m)_k$, one obtains

$$\begin{aligned}
&\epsilon_{man}\left[(f\cdot\phi^a)(f\times\phi^m)\cdot\frac{\partial}{\partial\psi_n}-\frac{1}{2}[(f\times\phi^m)\cdot\phi^a]f\cdot\frac{\partial}{\partial\psi_n}\right]\chi \\
&=\frac{1}{2}\epsilon_{man}\left[[f\times(f\times(\phi^m\times\phi^a))]\cdot\frac{\partial}{\partial\psi_n}-[(f\times\phi^m)\cdot\phi^a]f\cdot\frac{\partial}{\partial\psi_n}\right]\chi \\
&=-\frac{1}{2}\epsilon_{man}(f\cdot f)(\phi^m\times\phi^a)\cdot\frac{\partial}{\partial\psi_n}\chi=-(f\cdot f)Q_\psi\chi=(f\cdot f)(\phi^m\times\psi_m)\cdot\xi\ .
\end{aligned} \tag{7.34}$$

So the second term of (7.31) cancels the first term, ensuring that O_n is Q-closed. So we have shown that the operator

$$\begin{aligned}
O_n \ =\ & (f\cdot f)^n\epsilon^{c_1c_2c_3}(\phi^a\cdot\psi_{c_1})(\phi^b\cdot\psi_{c_2})(\psi_a\cdot\psi_b\times\psi_{c_3}) \\
&+n(f\cdot f)^{n-1}\epsilon^{b_1b_2b_3}\epsilon^{c_1c_2c_3}(f\cdot\psi_{b_1})(\phi^a\cdot\psi_{c_1})(\psi_{b_2}\cdot\psi_{c_2})(\psi_a\cdot\psi_{b_3}\times\psi_{c_3}) \\
&-\left(\frac{n}{72}+\frac{n^2-n}{108}\right)(f\cdot f)^{n-1}\epsilon^{a_1a_2a_3}\epsilon^{b_1b_2b_3}\epsilon^{c_1c_2c_3} \\
&\qquad\cdot(\psi_{a_1}\cdot\psi_{b_1}\times\psi_{c_1})(\psi_{a_2}\cdot\psi_{b_2}\times\psi_{c_2})(\psi_{a_3}\cdot\psi_{b_3}\times\psi_{c_3})
\end{aligned} \tag{7.35}$$

at t^{24+12n} order is a black hole cohomology.

One may wonder if these are primaries captured in the first factor of (6.33), or if they are related to other $O_{n'}$ with $n'<n$ by acting some Q_+^m's and/or gravitons on the third factor. One can show that the latter possibilities are all impossible. Suppose O_n is obtained by acting acting p Q's on $O_{n'}$ and multiplying q gravitons. Then p,q should satisfy

$$2p+8q=12(n-n')\ ,\quad p=0,1,2,3\ ,\ q\geq 0\ . \tag{7.36}$$

Possible solutions are

$$(p,q,n-n')=(2,1,1)\ ,\ (0,3,2)\ ,\ (2,4,3)\ ,\ (0,6,4)\ ,\ (2,7,5)\ ,\ (0,9,6)\ ,\ \cdots\ . \tag{7.37}$$

The cases with even $n-n'$ and $p=0$ yield operators at t^{24+12n} order obtained by multiplying $O_{n'}$ and $\frac{3}{2}(n-n')$ graviton operators of the form $\phi^m\cdot f+\frac{1}{2}\epsilon^{mnp}\psi_n\psi_p$. However, these cannot be cohomologous to O_n because they do not have a term at $\mathcal{O}(f^{2n-2}\phi^0\psi^9)$ order that O_n has, which cannot be changed by adding Q-exact terms. Now we consider the cases with odd $n-n'$ and $p=2$, $q=\frac{3}{2}(n-n')-\frac{1}{2}$, and again consider whether the operator $(Q_+^aQ_+^bO_{n'})(\phi\cdot f+\psi\cdot\psi)^q$ has a term at $f^{2n-2}\phi^0\psi^9$ order. Let us first study how the actions of Q_+^a and Q_+^b on $O_{n'}$ can produce a term with no scalars. Q_+^a either act as $\phi\to\psi$ or $\psi\to f$, so there are following possibilities:

$$f^{2n'}\phi^2\psi^5 \to f^{2n'}\psi^7 \ , \ f^{2n'-1}\phi\psi^7 \to f^{2n'}\psi^7 \ , \ \ f^{2n'-2}\psi^9 \to f^{2n'}\psi^7 \ . \tag{7.38}$$

In all three cases, we multiply gravitons of the form $(\phi \cdot f + \psi \cdot \psi)^q$ and see whether there can be a term at $f^{2n-2}\phi^0\psi^9$ order. This is possible only if $n = n' + 1$, $p = 2$, $q = 1$. That is, the only possible relations between different O_n's are

$$O_n \overset{?}{\sim} \epsilon_{abc}(Q_+^a Q_+^b O_{n-1})(\phi^c \cdot f + \tfrac{1}{2}\epsilon^{cde}\psi_d \cdot \psi_e) \ , \tag{7.39}$$

where $\sim$ means up to a multiplicative factor and addition of Q-exact terms. We act three Q_+^a's on (7.39) and show that this equation cannot hold. Acting $Q_+^1 Q_+^2 Q_+^3$ on the right hand side yields zero, so if this equation is true, $Q_+^1 Q_+^2 Q_+^3 O_n$ should be Q-exact. However, this cannot be the case since it contains a term at $f^{2n+1}\psi^6$ order, which does not contain scalars so cannot be Q-exact. More concretely, one starts from

$$\begin{aligned} O_n &= (f\cdot f)^n O_0 + \frac{20n}{3}(f\cdot f)^{n-1}\sum_{\text{cyclic}}(f\cdot\psi_3)(\psi_3\cdot\psi_2)(X\cdot\psi_2)(\psi_1\cdot\psi_1\times\psi_1) \\ &\quad -\frac{10}{3}\left(\frac{n}{6}+\frac{n^2-n}{9}\right)(f\cdot f)^{n-1}(\psi_1\cdot\psi_1\times\psi_1)(\psi_2\cdot\psi_2\times\psi_2)(\psi_3\cdot\psi_3\times\psi_3) \end{aligned} \tag{7.40}$$

where $\sum_{\text{cyclic}}$ means summation over the cyclic permutations of (X,ψ_1), (Y,ψ_2), (Z,ψ_3). Acting $Q_+^1 Q_+^2 Q_+^3$, one obtains the following terms without scalars,

$$\begin{aligned} &Q_1^+ Q_2^+ Q_3^+ O_0 \\ &\quad = -10(X\cdot f + \psi_2\cdot\psi_3)^2 f\cdot(\psi_1\times\psi_1) \\ &\qquad -20(X\cdot f+\psi_2\cdot\psi_3)(Y\cdot f+\psi_3\cdot\psi_1)f\cdot(\psi_1\times\psi_2) + \text{cyclic}, \\ &\quad \to -20(\psi_2\cdot\psi_3)^2(f\cdot\psi_1\times\psi_1) + \text{cyclic} \ , \\ &Q_1^+ Q_2^+ Q_3^+ (f\cdot\psi_3)(\psi_3\cdot\psi_2)(X\cdot\psi_2)(\psi_1\cdot\psi_1\times\psi_1) + \text{cyclic} \\ &\quad \to -3(f\cdot f)(\psi_2\cdot\psi_3)^2(f\cdot\psi_1\times\psi_1) + 6(f\cdot\psi_2)(f\cdot\psi_3)(\psi_2\cdot\psi_3)(f\cdot\psi_1\times\psi_1) + \text{cyclic}, \\ &Q_1^+ Q_2^+ Q_3^+ (\psi_1\cdot\psi_1\times\psi_1)(\psi_2\cdot\psi_2\times\psi_2)(\psi_3\cdot\psi_3\times\psi_3) \\ &\quad = -27(f\cdot\psi_1\times\psi_1)(f\cdot\psi_2\times\psi_2)(f\cdot\psi_3\times\psi_3) \\ &\quad = 18((f\cdot f)(\psi_2\cdot\psi_3)^2(f\cdot\psi_1\times\psi_1)) - 2(f\cdot\psi_2)(f\cdot\psi_3)(\psi_2\cdot\psi_3)(f\cdot\psi_1\times\psi_1) + \text{cyclic}) \ . \end{aligned} \tag{7.41}$$

These terms at $f^{2n+1}\phi^0\psi^6$ order do not cancel, implying that $Q_+^1 Q_+^2 Q_+^3 O_0$ cannot be Q-exact. So at least among the possibilities visible in the index (6.33), we have checked that different O_n's are not related in trivial manners.

Note also that the product of two O_n's vanishes, $O_m O_n = 0$. This is because each operator includes 5 or more ψ's, so the product involves 10 or more ψ's which vanishes by Fermi statistics.

7.2 $SU(2)$ and Partial No-Hair Behavior

The non-graviton index (6.35) for the full $SU(2)$ theory shows that there are many more black hole cohomologies outside of the BMN sector. We are interested in the BPS cohomologies contained in the square bracket of (6.35), because we believe that the χ_D factor addresses the $PSU(1,2|3)$ descendants of those in the square bracket which are not really new. In this section, we shall see that many of the cohomologies in the square bracket are actually the threshold cohomology at $\mathcal{J} = 24$ multiplied by some graviton operators. Furthermore, we show that multiplication by many other graviton operators lead to Q-exact operators and therefore do not create a new cohomology. We refer to this phenomenon as the partial no-hair behavior.

In principle, constructing all the cohomologies order by order as done in [115] will confirm that the χ_D factor indeed stands for the $PSU(1,2|3)$ descendants, but we shall not comprehensively do this job in this thesis. Rather, we shall proceed by considering possible superconformal representation structures of the $\frac{1}{16}$-BPS states compatible with this index, finding many illuminating structures. As emphasized, we may miss some BPS states in case their multiplets completely cancel in the index.

The index can be written as a sum over the short $\mathcal{N} = 4$ representations. Equivalently, it can be written as a sum over $\frac{1}{16}$-BPS multiplets of $PSU(1,2|3) \subset PSU(2,2|4)$. The last multiplets are embedded in the short representations of $PSU(2,2|4)$ in canonical manners: see appendix B of [71]. Knowing this representation sum is equivalent to knowing the primary contents. We will study this expansion order by order in t. As already mentioned in section 7.1, there are two classes of black hole cohomologies: those which can be written as products of other black hole cohomologies and gravitons which we call 'hairy' and the rest which we call 'core.'

We start by studying the black hole cohomologies in the BMN sector that we identified in section 7.1. Among these, two of them O_0, O_1 appear within the t^{40} order. We can show that all O_n's are core black hole primaries of the $\frac{1}{16}$-BPS multiplets. The coreness of O_n is already shown in section 7.1, at least within the states visible in the index (6.33), since it suffices to show this within the BMN sector. We only need to show that they are $\frac{1}{16}$-BPS primaries in their full $PSU(1,2|3)$ representations. O_0 is clearly a $\frac{1}{16}$-BPS primary since it is the lowest black hole cohomology. Since $j \equiv J_1 + J_2 = 5$ is too large, O_0 can only belong to the $\mathcal{N}=4$ multiplet $A_1\overline{L}[4;0]_9^{[2,0,0]}$. The primary O_0 of the $\frac{1}{16}$-BPS multiplet is obtained by acting $Q' \equiv Q_+^4$ on a primary of this $\mathcal{N}=4$ multiplet. The index over this multiplet is

$$\chi_{24} \equiv -t^{24}\chi_D(t,x,y,p) \ , \tag{7.42}$$

where χ_D is defined in (6.36). So the first term $-t^{24}$ in the square bracket of (6.35) corresponds to the contribution of this multiplet.

Next we consider other O_n's. We can prove that they are also primaries by showing that acting any of the nine Q's in $PSU(1,2|3)$ yields nontrivial and independent cohomologies. (This is because O_n does not contain derivatives and cannot be a conformal descendant.) We have shown in section 7.1 that the action of any Q_+^m on O_n is nontrivial and independent because acting all three of them yields a nontrivial cohomology. One can also show that $\overline{Q}_{m\dot{\alpha}}O_n$ are all nontrivial and independent. It suffices to show that the six $\overline{Q}_{m\dot{\alpha}}$'s acting on $Q_+^1Q_+^2Q_+^3O_n$ are independent. This is easily shown by studying the terms obtained by acting $\overline{Q}_{m\dot{\alpha}}$ on the $\mathcal{O}(f^{2n+1}\phi^0\psi^6)$ order terms of $Q_+^1Q_+^2Q_+^3O_n$ in (7.41). In particular, one obtains terms at $f^{2n}\phi^0\psi^6 D\psi$ by acting $\overline{Q}_{m\dot{\alpha}}$ on f. These terms cannot be Q-exact since it involves neither ϕ^m or $\lambda_{\dot{\alpha}}$. This proves that all 6 operators $\overline{Q}_{m\dot{\alpha}}Q_+^1Q_+^2Q_+^3O_n$ are nontrivial. They are also independent since their $SU(2)_R\times SU(3)$ quantum numbers are different. This shows that $O_{n\geq 1}$ are $\frac{1}{16}$-BPS primaries. O_n belongs to the $\mathcal{N}=4$ multiplet $A_1\overline{L}[4+4n;0]_{9+4n}^{[2,0,0]}$, which contributes to to the index as $-t^{24+12n}\chi_D(t,x,y,p)$.

Now with the nature of O_n understood, we come back to study the series (6.35) until t^{40} order, trying to better characterize other cohomologies order by order in t. Once the lowest operator O_0 is identified, all the states in its $\frac{1}{16}$-BPS multiplet are not really new operators. So we subtract χ_{24} from $Z - Z_{\rm grav}$ and see what are left:

$$\begin{aligned} Z - Z_{\rm grav} - \chi_{24} \;\; = \;\; & \big[-\chi_{(1,3)}t^{32} - (\chi_{(1,\bar{3})}+\chi_{(3,6)})t^{34} - \chi_{(2,3)}t^{35} + (\chi_{(3,1)}+\chi_{(3,8)})t^{36} \\ & -(\chi_{(2,\bar{3})}+\chi_{(4,6)})t^{37} + \chi_{(5,3)}t^{38} + (\chi_{(2,1)}+2\chi_{(4,1)}+\chi_{(4,8)})t^{39} \\ & -(2\chi_{(1,6)}+\chi_{(3,\bar{3})}+\chi_{(5,\bar{3})}+\chi_{(5,6)})t^{40}\big]\,\chi_D + \mathcal{O}(t^{41}) \ . \end{aligned} \tag{7.43}$$

Somewhat surprisingly, after subtracting the multiplet of O_0, one finds that the remaining index starts from t^{32} order. Namely, in the range $t^{25}\sim t^{31}$, the index does not capture any new black hole cohomologies except the trivial descendants of O_0. At first sight this may look like a boring result, but the triviality of the index in this range has a nontrivial implication.

Recall that cohomologies multiply to yield new cohomologies. This is because of the Leibniz rule of the classical Q acting on product operators. So apparently, one can multiply light graviton cohomologies to O_0 or its descendants to obtain many new cohomologies in the range $t^{25} \sim t^{31}$. The possible product cohomologies of O_0 and gravitons below t^{32} order are

$$
\begin{aligned}
& O_0 \cdot (\phi^{(m} \cdot \phi^{n)}) \ , \quad O_0 \cdot (\phi^m \cdot \lambda_{\dot\alpha}) \ , \quad O_0 \cdot (\lambda_{\dot +} \cdot \lambda_{\dot -}) \ , \\
& O_0 \cdot (\phi^m \cdot \psi_n - \tfrac{1}{3}\delta^m_n \phi^p \cdot \psi_p) \ , \quad O_0 \cdot (\lambda_{\dot\alpha} \cdot \psi_m - \tfrac{1}{2}\epsilon_{mnp}\phi^n \cdot D_{\dot\alpha}\phi^p) \ , \\
& O_0 \cdot \partial_{\dot\alpha}(\phi^{(m} \cdot \phi^{n)}) \ .
\end{aligned} \tag{7.44}
$$

Other possible products below t^{32} involving the descendants of O_0 are

$$
\begin{aligned}
\overline{Q}O_0 \quad & \times \quad (\phi^m \cdot \phi^n \ , \ \phi^m \cdot \lambda_{\dot\alpha} \ , \ \lambda_{\dot +} \cdot \lambda_{\dot -} \ , \ \phi^m \cdot \psi_n - \tfrac{1}{3}\delta^m_n \phi^p \cdot \psi_p) \ , \\
(Q, \overline{QQ})O_0 \quad & \times \quad (\phi^m \cdot \phi^n \ , \ \phi^m \cdot \lambda_{\dot\alpha}) \ , \\
(Q\overline{Q}, \overline{QQQ}, \partial)O_0 \quad & \times \quad (\phi^m \cdot \phi^n) \ .
\end{aligned} \tag{7.45}
$$

The triviality of the index (7.43) in this range implies two possibilities for these product cohomologies. The first possibility is that these product cohomologies are Q-exact, i.e. absent in the BPS spectrum. Another possibility is that these product cohomologies are nontrivial but there are cancellations in the index, either among themselves or with new core black hole cohomologies.[1] Among (7.44) and (7.45), we explicitly show that

$$
O_0 \cdot (\phi^m \cdot \phi^n) \ , \ O_0 \cdot (\phi^m \cdot \lambda_{\dot\alpha}) \ , \ O_0 \cdot (\phi^m \cdot \psi_n - \tfrac{1}{3}\delta^m_n \phi^p \cdot \psi_p) \tag{7.46}
$$

are all Q-exact.

Six operators $O_0(\phi^{(m} \cdot \phi^{n)})$ at t^{28} order are all Q-exact. An $SU(3)$ covariant expression is

$$
\begin{aligned}
O_0 \cdot (\phi^{(m} \cdot \phi^{n)}) = -\frac{1}{14} Q[& 20\epsilon^{rs(m}(\phi^{n)} \cdot \psi_p)(\phi^p \cdot \psi_r)(\phi^q \cdot \psi_q)(f \cdot \psi_s) \\
& -20\epsilon^{prs}(\phi^{(m} \cdot \psi_p)(\phi^{n)} \cdot \psi_r)(\phi^q \cdot \psi_q)(f \cdot \psi_s) \\
& +30\epsilon^{prs}(\phi^{(m} \cdot \psi_p)(\phi^{n)} \cdot \psi_r)(\phi^q \cdot \psi_s)(f \cdot \psi_q) \\
& -7\epsilon^{a_1 a_2 p}\epsilon^{b_1 b_2 (m}(\phi^{n)} \cdot \psi_p)(\phi^q \cdot \psi_q)(\psi_{a_1} \cdot \psi_{a_2})(\psi_{b_1} \cdot \psi_{b_2}) \\
& +18\epsilon^{a_1 a_2 p}\epsilon^{b_1 b_2 (m}(\phi^{n)} \cdot \psi_q)(\phi^q \cdot \psi_p)(\psi_{a_1} \cdot \psi_{a_2})(\psi_{b_1} \cdot \psi_{b_2})] \ .
\end{aligned} \tag{7.47}
$$

Six operators $O_0 \cdot (\phi^m \cdot \lambda_{\dot\alpha})$ at t^{29} order are also all Q-exact. An $SU(2)_R \times SU(3)$ covariant

[1] We have checked that cancellations cannot happen within the product cohomologies listed above. It is logically possible (although a bit unnatural) that some new core black hole primaries appear in this range, precisely canceling with some of the product operators above if they are not Q-exact. Although in different contexts, certain black holes are known not to appear in the index. For instance, asymptotically flat multi-center BPS black holes or BPS black rings are not captured by the index [130].

expression is

$$O_0\cdot(\phi^m\cdot\lambda_{\dot\alpha})=\frac{1}{8}Q[40\epsilon^{mnp}(f\cdot\psi_q)(\lambda_{\dot\alpha}\cdot\psi_r)(\phi^q\cdot\psi_n)(\phi^r\cdot\psi_p)$$
$$-4\epsilon^{ma_1a_2}\epsilon^{nb_1b_2}(\lambda_{\dot\alpha}\cdot\psi_n)(\phi^p\cdot\psi_p)(\psi_{a_1}\cdot\psi_{a_2})(\psi_{b_1}\cdot\psi_{b_2})$$
$$+6\epsilon^{ma_1a_2}\epsilon^{nb_1b_2}(\lambda_{\dot\alpha}\cdot\psi_p)(\phi^p\cdot\psi_n)(\psi_{a_1}\cdot\psi_{a_2})(\psi_{b_1}\cdot\psi_{b_2})$$
$$+\epsilon^{na_1a_2}\epsilon^{pb_1b_2}(\lambda_{\dot\alpha}\cdot\psi_n)(\phi^m\cdot\psi_p)(\psi_{a_1}\cdot\psi_{a_2})(\psi_{b_1}\cdot\psi_{b_2})]\ . \tag{7.48}$$

Eight operators $O_0\cdot\left(\phi^m\cdot\psi_n-\frac{1}{3}\delta^m_n\phi^p\cdot\psi_p\right)$ at t^{30} order are all Q-exact. An $SU(3)$ covariant expression is

$$O_0\cdot\left(\phi^m\cdot\psi_n-\tfrac{1}{3}\delta^m_n\phi^p\cdot\psi_p\right) \tag{7.49}$$
$$=\frac{1}{4}Q\left[\epsilon_{npq}\epsilon^{ra_1a_2}\epsilon^{qb_1b_2}\epsilon^{mc_1c_2}(\phi^p\cdot\psi_r)(\psi_{a_1}\cdot\psi_{a_2})(\psi_{b_1}\cdot\psi_{b_2})(\psi_{c_1}\cdot\psi_{c_2})\right]\ .$$

We did not manage to prove the Q-exactness of operators other than (7.46). Since these operators do not appear at all in the index, all of them may be Q-exact until t^{31} order. More robustly/modestly, we can say that our index exhibits a no-hair behavior for O_0 until t^{31} order. It will be interesting to clarify this issue in the future.

The Q-exactness of these product operators implies that O_0 abhors the dressings by certain gravitons, reminiscent of the black hole no-hair theorem. Especially, $(\phi^m\cdot\phi^n)\sim\mathrm{tr}(\phi^m\phi^n)$ multiplied to O_0 are Q-exact. This is interesting because these operators correspond to bulk scalar fields which have been discussed in the context of hairy AdS_5 black holes [121, 131, 132]. More precisely, it is the 's-wave' modes of these scalars that have been used to construct hairy black holes, precisely dual to the conformal primary operator $\mathrm{tr}(\phi^m\phi^n)$. Here, note that the BPS limits of the hairy black holes constructed this way all exhibit substantial back reactions to the core black holes, at least near the horizon, no matter how small the hair parameter is [131, 132].

Now we consider the lowest term $-\chi_{(1,3)}t^{32}$ of (7.43). In fact, this term comes from the following product of O_0 and gravitons:

$$O_0\cdot(\phi^m\cdot f+\tfrac{1}{2}\epsilon^{mnp}\psi_n\cdot\psi_p)\ . \tag{7.50}$$

It is easy to show that this is not Q-exact, e.g. by acting two Q^m_+ as shown in (7.17). These operators contain terms at $f^0\phi^0\psi^9$ order, which cannot be Q-exact. So the operators (7.50) themselves are not Q-exact either. Therefore, the no-hair interpretation that we made so far holds only for certain low-lying gravitons, at best. Among the conformal primaries of S_2 (see Table 5.1), these three gravitons are the only ones which explicitly appear in the index when multiplied to O_0. At this stage, it may seem that two more gravitons $f \cdot \lambda_{\dot\alpha} + \frac{2}{3}\psi_m \cdot D_{\dot\alpha}\phi^m - \frac{1}{3}\phi^m \cdot D_{\dot\alpha}\psi_m$ at $\mathcal{O}(t^9)$ might multiply O_0 to show up at t^{33} order, but we will see below that the index does not capture them. Therefore, out of the 32 particle species of conformal primary particles in the S_2 multiplet, 29 gravitons except $\phi^m \cdot f + \frac{1}{2}\epsilon^{mnp}\psi_n \cdot \psi_p$ do not appear in the index when they multiply O_0. In the BMN sector, our studies in section 3.1 imply a similar theorem for all O_n, at least as seen by the index. Among the 17 particle species of gravitons in the BMN sector, all 14 particles except $\phi^m \cdot f + \frac{1}{2}\epsilon^{mnp}\psi_n \cdot \psi_p$ do not appear in the index when they multiply any O_n.

The 3 product coholomologies at t^{32} order violating the no-hair theorem should be the primaries of $PSU(1,2|3)$. This is again contained in a short multiplet of $A_1\overline{L}$ type, whose contribution to the index is given by $\chi_{32} = t^{32}\chi_{(1,3)}\chi_D$. We subtract this from $Z - Z_{\text{grav}} - \chi_{24}$, and study the remaining cohomologies. We can then try to interpret the lowest order term of the remainder and judge whether it comes from new core black hole primaries or products of already known core primaries and gravitons. If one can clarify the nature of the cohomologies at this lowest order, one can again subtract the characters of their supermultiplets and keep exploring even higher orders. Since it becomes more and more difficult to judge the Q-exactness of the possible product operators, we shall only make much simpler and structural studies until the t^{40} order. Namely, we shall try to see if the surviving index can be explained as the products of known gravitons and core primaries O_n, without the need of any new core black hole primaries. Studies we made so far showed that this is possible until t^{32} order. Namely, the index until this order is compatible with having no more new core primaries and only three more product cohomologies (7.50). We shall show that the graviton spectrum is such that new core black hole primaries should appear at t^{39} order at the latest. This not only proves from the index the existence of new core black hole primaries, but will also show scenarios of possible hairy black holes.

After eliminating the contribution of the multiplet χ_{32} to the index, the remaining index vanishes at t^{33} order. In principle, there are two possible product operators at this order that completely cancel each other in the index even if they are not Q-exact. They are

$$O_0 \cdot \partial_{\dot\alpha}(\lambda_{\dot\beta} \cdot \lambda^{\dot\beta}) \quad , \quad O_0 \cdot \left(f \cdot \lambda_{\dot\alpha} + \tfrac{2}{3}\psi_m \cdot D_{\dot\alpha}\phi^m - \tfrac{1}{3}\phi^m \cdot D_{\dot\alpha}\psi_m\right) \ . \tag{7.51}$$

So these product operators, even if they exist, do not appear in the index.

The lowest nonzero term of $Z - Z_{\text{grav}} - \chi_{24} - \chi_{32}$ is $-(\chi_{(1,\bar{3})} + \chi_{(3,6)})t^{34}$. The only possible product operators at this order which may account for these two terms, unless they are

Q-exact, are

$$O_0 \cdot \partial^{\dot\alpha} \left(\lambda_{\dot\alpha} \cdot \psi_m - \tfrac{1}{2}\epsilon_{mnp}\phi^n \cdot D_{\dot\alpha}\phi^p\right) \quad , \quad O_0 \cdot \partial_{\dot\alpha}\partial_{\dot\beta}(\phi^m \cdot \phi^n) \ . \tag{7.52}$$

If they are nontrivial, they are in the $\mathcal{N}=4$ representations $A_1\overline{L}[6;2]^{[2,2,0]}_{13}$ and $A_1\overline{L}[6;0]^{[3,0,1]}_{13}$, respectively. Assuming that they are both non-Q-exact, the order t^{34} is accounted for by these hairy black hole operators. Their multiplets will contribute $-(\chi_{(1,\bar{3})} + \chi_{(3,6)})t^{34}\chi_D$ to the index, because both are of type $A_1\overline{L}$.

Subtracting them, now the leading term is $-\chi_{(2,3)}t^{35}$. The only possible product cohomologies which can account for this term are

$$O_0 \cdot \partial_{\dot\alpha}(f \cdot \phi^m + \tfrac{1}{2}\epsilon^{mnp}\psi_n \cdot \psi_p) \ , \tag{7.53}$$

provided they are not Q-exact. In this case, its multiplet is again $A_1\overline{L}$ type and contributes $-\chi_{(2,3)}t^{35}\chi_D$ to the index.

Subtracting this, now the leading term is $+(\chi_{(3,1)} + \chi_{(3,8)})t^{36}$. Since there is one fermionic black hole primary O_1 that we now from section 7.1, we study whether the product cohomologies may account for $+(1 + \chi_{(3,1)} + \chi_{(3,8)})t^{36}$. The only possible set is

$$\begin{aligned} &O_0 \cdot \partial_{\dot\alpha}(f \cdot \lambda_{\dot\beta} + \tfrac{2}{3}\psi_m \cdot D_{\dot\beta}\phi^m - \tfrac{1}{3}\phi^m \cdot D_{\dot\beta}\psi_m) \ , \\ &O_0 \cdot \partial_{\dot\alpha}\partial_{\dot\beta}(\phi^m \cdot \psi_n - \tfrac{1}{3}\delta^m_n\phi^p \cdot \psi_p) \ . \end{aligned} \tag{7.54}$$

Provided they are not Q-exact, they are again the primaries of $A_1\overline{L}$ type multiplets, so they contribute to the index by (7.54) times χ_D.

Subtracting the contributions of these multiplets, the leading term is $-(\chi_{(2,\bar{3})} + \chi_{(4,6)})t^{37}$. The only possible product cohomologies that can account for this term are

$$O_0 \cdot \partial_{\dot\alpha}\partial^{\ \dot\beta}(\lambda_{\dot\beta} \cdot \psi_m - \tfrac{1}{2}\epsilon_{mnp}\phi^n \cdot D_{\dot\beta}\phi^p) \quad , \quad O_0 \cdot \partial_{\dot\alpha}\partial_{\dot\beta}\partial_{\dot\gamma}(\phi^m \cdot \phi^n) \ . \tag{7.55}$$

Further processing to subtract the contributions of their multiplets, again $A_1\overline{L}$ type, the lowest term is $+\chi_{(5,3)}t^{38}$. one possible set of product cohomologies which can account for this is

$$O_0 \cdot \partial_{(\dot\alpha}\partial_{\dot\beta}\partial_{\dot\gamma}(\lambda_{\dot\delta)} \cdot \phi^m) \ . \tag{7.56}$$

Apart from these, the following two sets of product cohomologies

$$O_0 \cdot \partial_{\dot\alpha}\partial_{\dot\beta}\partial^{\dot\gamma}(\lambda_{\dot\gamma} \cdot \phi^m) \quad , \quad O_0 \cdot \partial_{\dot\alpha}\partial_{\dot\beta}(f \cdot \phi^m + \tfrac{1}{2}\epsilon^{mnp}\psi_n \cdot \psi_p) \tag{7.57}$$

exactly cancel in the index, so there are two possible ways in which product hairy cohomologies can account for this order. In either case, they are all in the $A_1\overline{L}$ type multiplets, so their contribution to the index is again just $\chi_{(5,3)}t^{38} \cdot \chi_D$.

Subtracting the last multiplets, the lowest term is $+(\chi_{(2,1)}+2\chi_{(4,1)}+\chi_{(4,8)})t^{39}$. All possible product cohomologies at this order are

$$\begin{aligned}
(\mathbf{4},\mathbf{1})^{\mathrm{F}} &: \quad O_0\partial_{\dot\alpha}\partial_{\dot\beta}\partial_{\dot\gamma}(\lambda_{\dot\delta}\cdot\lambda^{\dot\delta})\ , \\
(\mathbf{2},\mathbf{1})^{\mathrm{B}} &: \quad O_0\partial_{\dot\alpha}\partial^{\ \dot\beta}(f\cdot\lambda_{\dot\beta}+\tfrac{2}{3}\psi_m\cdot D_{\dot\beta}\phi^m-\tfrac{1}{3}\phi^m\cdot D_{\dot\beta}\psi_m)\ , \\
(\mathbf{4},\mathbf{1})^{\mathrm{B}} &: \quad O_0\partial_{(\dot\alpha}\partial_{\dot\beta}(f\cdot\lambda_{\dot\gamma)}+\tfrac{2}{3}\psi_m\cdot D_{\dot\gamma)}\phi^m-\tfrac{1}{3}\phi^m\cdot D_{\dot\gamma)}\psi_m)\ , \\
(\mathbf{4},\mathbf{8})^{\mathrm{B}} &: \quad O_0\partial_{\dot\alpha}\partial_{\dot\beta}\partial_{\dot\gamma}(\phi^m\cdot\psi_n-\tfrac{1}{3}\delta^m_n\phi^p\cdot\psi_p)\ .
\end{aligned} \tag{7.58}$$

We used the superscripts B/F to mark their bosonic/fermionic statistics, respectively. With these candidates, we find that the closest one can get to the index at this order is the case in which all three classes of bosonic operators are nontrivial while the fermionic operators are Q-exact. In this case, their contribution at this order is maximal and becomes $+(\chi_{(2,1)}+\chi_{(4,1)}+\chi_{(4,8)})t^{39}$. There is still one factor of $\chi_{(4,1)}\cdot t^{39}$ remaining to be addressed. Therefore, there should be at least 4 core black hole primaries in the $SU(2)_R$ representation (4,1), to account for the remaining $+\chi_{(4,1)}t^{39}$. Of course this is only the latest order in which new core black hole primaries should appear, because it may as well appear at lower orders due to some non-Q-exactness assumptions we made for product cohomologies being invalid.

So we have shown that, from the index data until t^{40} order, there should exist more core primary operators other than O_n in the BMN sector. This conclusion is obtained by supposing otherwise, and trying to explain the index as product cohomologies of O_n and gravitons but finding a contradiction at t^{39}. We should also emphasize that the structure of the index admits natural explanations in terms of hairy product operators in a wide range $t^{33}\sim t^{38}$. Note also that most of the gravitons appearing in this range are conformal descendants in the S_2 multiplet.

7.3 $SU(3)$, BMN Sector

We turn to constructing the threshold black hole cohomology of the $SU(3)$ theory. From the index we computed in section 6.4, we know that it is a singlet under the $SU(3)$ subgroup of the R-symmetry group $SU(4)$, has the order $\mathcal{J}=24$, and is fermionic. Whereas the analogous threshold black hole cohomology presented in section 7.1 could be found by some clever trials and errors in [119], such an approach is not viable for the $SU(3)$ theory where the elementary fields are represented by larger matrices. Therefore we take a more strategic approach, that we organize into four subsections to explain.

In subsection 7.3.1, we will introduce an ansatz for constructing Q-closed non-graviton operators, that takes advantage of various trace relations that were obtained as byproducts of the graviton index computation of section 6. Based on the ansatz, we will present in subsection 7.3.2 many Q-closed non-graviton operators in the target charge sector, i.e. $q_1 = q_2 = q_3 = 4$ leading to $\mathcal{J} = 24$. Among these, we comment that all but (7.66) are Q-exact. Then in subsection 7.3.3, we explain the numerics-assisted method that we have used to determine the Q-exactness of the Q-closed operators. Utilizing our check of Q-exactness, it is possible to prove that (7.66) is in fact the only cohomology in the target sector, denying the possibility that the index may have missed a boson-fermion pair of non-graviton cohomologies in the target sector. This is explained in subsection 7.3.4

7.3.1 An Ansatz for Closed Non-Graviton Operators

The cohomologies we would like to construct should be, by definition, Q-closed and not Q-exact. Unlike gravitons, the Q-closedness of the black hole cohomologies should be ensured by the trace relations. (Otherwise, that is if it is a cohomology at given energy and at arbitrary values of N, it is a graviton cohomology.) So it is important to know what kind of nontrivial trace relations are available for $N \times N$ matrices when the number of fields is larger than N.

It seems to be widely believed that all $SU(N)$ trace relations are derived from the Cayley-Hamilton identity. For instance, see [133] (p.7, below eqn.(19)) and [134]. But in practice it is inefficient to search for the trace relations that we need just from this identity. Fortunately, we already implicitly know many trace relations from the calculations reported in section 6. Namely, when enumerating finite N gravitons, we have counted multi-graviton operators subject to various trace relations between the generators g_i. So one can take advantage of these trace relations to construct black hole cohomologies. This leads to our 'ansatz' for black hole cohomologies, which we explain now.

We can motivate the ideas with a simple example in the $SU(2)$ theory [71, 115, 119]. A representative of the threshold non-graviton cohomology in $SU(2)$ can be written using the BMN mesons (5.26) by

$$O_0 \equiv \epsilon^{abc}(v_2)^m{}_a(v_2)^n{}_b \mathrm{tr}(\psi_{(c}\psi_m\psi_{n)}) \tag{7.59}$$

where v_2 is the graviton operator in the S_2 multiplet. Let us see how this operator becomes Q-closed. Acting Q on O_0, Q acts only on $\mathrm{tr}(\psi_{(c}\psi_m\psi_{n)})$ since v_2 is Q-closed. One obtains

$$Q\mathrm{tr}(\psi_{(c}\psi_m\psi_{n)}) \propto \epsilon_{ab(c}(v_2)^a{}_m(v_2)^b{}_{n)} \equiv R(v_2)_{cmn} \tag{7.60}$$

after using $SU(2)$ trace relations. Plugging this into QO_0, one obtains

$$QO_0 \propto \epsilon^{abc}(v_2)^m{}_a(v_2)^n{}_b R(v_2)_{cmn} = 0 \ . \tag{7.61}$$

At the last step, one can show that the quartic mesonic polynomial $\epsilon^{abc}(v_2)^m{}_a(v_2)^n{}_b R(v_2)_{cmn}$ is identically zero [71]. From the viewpoint of section 6, (7.60) are graviton trace relations and the last step of (7.61) is a relation of relations. So the operator O_0 is shown to be Q-closed by using the trace relations and a relation of relations of the finite N graviton operators.

This idea can be extended to construct operators which become Q-closed only after using trace relations. Namely, for each relation of relations such as (7.61), we can construct a Q-closed operator such as (7.59). We still need to check that they are not Q-exact for them to represent nontrivial Q-cohomologies, which we will do in section 7.3.3. Also, there are non-graviton cohomologies which are not constructed in this way [71]. For these reasons, the Q-closed operators constructed in this way are mere ansätze for the non-graviton cohomologies.

In appendix A, we have collected all $SU(3)$ fundamental trace relations that involve u_n, v_n only, and manifestly wrote them in Q-exact forms. We have found trace relations involving u_n, v_n, w_n until $\mathcal{J} = 20$ order. We have also found all relations between the fundamental graviton trace relations at $\mathcal{J} = 24$ and some more at $\mathcal{J} = 30$ orders in the $SU(3) \subset SO(6)_R$ singlet sector, where the index predicts non-graviton cohomologies (see Table 6.1). In other charge sectors, one can immediately write down Q-closed operators if one finds new relations of the fundamental trace relations.

When we write a fundamental trace relation R_a in a Q-exact form as $R_a \sim Qr_a$, there is an ambiguity in r_a by addition of arbitrary Q-closed operators. We partly fix it so that r_a vanishes when all the letters are restricted to diagonal matrices. Since the Q-closed operators constructed from relations of relations are linear combinations of r_a's, they vanish with diagonal letters. This makes it impossible for our ansatz to be gravitons. So our ansatz is guaranteed to yield a non-graviton cohomology unless it is Q-exact.

7.3.2 Q-Closed Non-Graviton Operators

Based on the ansatz, we now list the non-graviton Q-closed operators at the threshold level $\mathcal{J} = 24$, which are singlets under the $SU(3) \subset SU(4)_R$ global symmetry, in the BMN sector of the $SU(3)$ gauge theory.

At $\mathcal{J} \equiv 2(Q_1 + Q_2 + Q_3) + 6J = 24$, operators are further distinguished by the overall R-charge $R \equiv \frac{Q_1+Q_2+Q_3}{3}$. The BMN operators which are $SU(3) \subset SU(4)_R$ singlets satisfy $Q_1 = Q_2 = Q_3$ and $J_1 = J_2$. Then the possible charges of the operators are $(R, J) = (\frac{n}{2}, \frac{8-n}{2})$

where $n = 0, \cdots, 8$. In each charge sector, the number of letters is fixed to $n+4$. However, our ansatz further restricts the charges since acting Q on our ansatz should become a polynomial of $u_{2,3}, v_{2,3}, w_{2,3}$. As a result, there exist in total 7 possible charge sectors within our ansatz: $(R, J) = (\frac{n}{2}, \frac{8-n}{2})$ where $n = 1, \cdots, 7$.

When $(R, J) = (\frac{1}{2}, \frac{7}{2})$ or $(1, 3)$, there are no Q-closed operators within our ansatz using the trace relations in the appendix. One can understand it heuristically as follows. At these charges, R is so small that only a small number of scalars is admitted. As the graviton generators contain at least one scalar field, only few types of graviton polynomials exist in these sectors, which are not enough to host relations of relations. Therefore, these charge sectors are incompatible with our ansatz. The other 5 charge sectors host Q-closed operators in our ansatz, whose explicit forms will be presented below.

We now present the Q-closed non-graviton operators in each of the five charge sectors, $(R, J) = (\frac{n}{2}, \frac{8-n}{2})$ where $n = 3, \cdots, 7$. For convenience, we rewrite here the definition of the single-trace generators of the $SU(3)$ BMN gravitons $u_{2,3}, v_{2,3}, w_{2,3}$:

$$\begin{aligned}
&u^{ij} \equiv \mathrm{tr}\left(\phi^{(i}\phi^{j)}\right)\ ,\ u^{ijk} \equiv\ \mathrm{tr}\left(\phi^{(i}\phi^{j}\phi^{k)}\right)\ ,\\
&v^{i}{}_{j} \equiv \mathrm{tr}\left(\phi^{i}\psi_{j}\right) - \tfrac{1}{3}\delta^{i}_{j}\mathrm{tr}\left(\phi^{a}\psi_{a}\right)\ ,\ v^{ij}{}_{k} \equiv\ \mathrm{tr}\left(\phi^{(i}\phi^{j)}\psi_{k}\right) - \tfrac{1}{4}\delta^{i}_{k}\mathrm{tr}\left(\phi^{(j}\phi^{a)}\psi_{a}\right) - \tfrac{1}{4}\delta^{j}_{k}\mathrm{tr}\left(\phi^{(i}\phi^{a)}\psi_{a}\right)\ ,\\
&w^{i} \equiv \mathrm{tr}\left(f\phi^{i} + \tfrac{1}{2}\epsilon^{ia_1a_2}\psi_{a_1}\psi_{a_2}\right)\ ,\ w^{ij} \equiv\ \mathrm{tr}\left(f\phi^{(i}\phi^{j)} + \epsilon^{a_1a_2(i}\phi^{j)}\psi_{a_1}\psi_{a_2}\right)\ .
\end{aligned} \tag{7.62}$$

i) $(R, J) = (\frac{3}{2}, \frac{5}{2})$**.** The operators in this sector are made of 7 letters. The possible numbers (n_ϕ, n_ψ, n_f) of scalars, fermions and f in each term are $(n_\phi, n_\psi, n_f) = (4, 1, 2)$, $(3, 3, 1)$ and $(2, 5, 0)$. We find one Q-closed operator in this sector from the trace relations and a relation of relations in appendix A. This Q-closed operator is given by

$$\begin{aligned}
O^{(2,1)} \ \equiv\ & 65u^{ij}(r^{(2,1)}_{20})_{ij} - 39w^{ij}(r^{(1,1)}_{14})_{ij} + 5w^{i}(r^{(1,1)}_{16})_{i}\\
&+312v^{jk}{}_{i}(r^{(1,2)}_{16})^{i}_{jk} + 26v^{j}{}_{i}(r^{(1,2)}_{18})^{i}_{j} + 6w^{i}(r^{(0,3)}_{16})_{i}\ .
\end{aligned} \tag{7.63}$$

The superscripts denote (n_f, n_ψ) of the terms with maximal n_f in the operator. $r^{(n_f,n_\psi)}_{j}$'s are given in (A.6), (A.7) where $R^{(n_f,n_\psi-1)}_{j} \equiv i\,Q\,r^{(n_f,n_\psi)}_{j}$'s are the fundamental trace relations. The Q-closed operator (7.63) turns out to be Q-exact. In fact, (7.63) is even under the parity transformation of [135]. It is already known that all such even operators in this charge sector are Q-exact for all $N \geq 3$ [113], which we confirm.

ii) $(R, J) = (2, 2)$**.** The operators in this sector are made of 8 letters. Allowed (n_ϕ, n_ψ, n_f) are $(6, 0, 2), (5, 2, 1), (4, 4, 0)$. We find 4 Q-closed operators in this sector given by

$$
\begin{aligned}
O_1^{(1,2)} \equiv & -3v^{(j}{}_i w^{k)} (r_{10}^{(0,1)})^i_{jk} - 3u^{(ij} w^{k)} (r_{12}^{(0,2)})_{ijk} + \epsilon_{a_1 a_2 i} u^{a_1 j} w^{a_2} (r_{12}^{(0,2)})^i_j \ , \\
O_2^{(1,2)} \equiv & -9u^{a(i} v^{j)}{}_a (r_{14}^{(1,1)})_{ij} + 10\epsilon_{a_1 a_2 (i} u^{a_1 k} v^{a_2}{}_{j)} (r_{14}^{(1,1)})^{ij}_k \\
& + 30 v^{(j}{}_i w^{k)} (r_{10}^{(0,1)})^i_{jk} + 60 u^{(jk} v^{l)}{}_i (r_{14}^{(0,3)})^i_{jkl} \ , \\
O_3^{(1,2)} \equiv & -3u^{a(i} v^{j)}{}_a (r_{14}^{(1,1)})_{ij} + 6\epsilon_{a_1 a_2 (i} u^{a_1 k} v^{a_2}{}_{j)} (r_{14}^{(1,1)})^{ij}_k + 4u^{ijk} (r_{18}^{(1,2)})_{ijk} + 14 v^{(j}{}_i w^{k)} (r_{10}^{(0,1)})^i_{jk} \\
& - 6w^{ij} (r_{14}^{(0,2)})_{ij} - 12\epsilon^{a_1 a_2 (i} v^j{}_{a_1} v^{k)}{}_{a_2} (r_{12}^{(0,2)})_{ijk} - 4v^j{}_a v^a{}_i (r_{12}^{(0,2)})^i_j \ , \\
O_4^{(1,2)} \equiv & -3u^{a(i} v^{j)}{}_a (r_{14}^{(1,1)})_{ij} + 14\epsilon_{a_1 a_2 (i} u^{a_1 k} v^{a_2}{}_{j)} (r_{14}^{(1,1)})^{ij}_k + 8v^{jk}{}_i (r_{16}^{(1,1)})^i_{jk} + 42 v^{(j}{}_i w^{k)} (r_{10}^{(0,1)})^i_{jk} \\
& + 12u^{(ij} w^{k)} (r_{12}^{(0,2)})_{ijk} - 24 w^{ij} (r_{14}^{(0,2)})_{ij} - 36\epsilon^{a_1 a_2 (i} v^j{}_{a_1} v^{k)}{}_{a_2} (r_{12}^{(0,2)})_{ijk} - 8v^{jk}{}_i (r_{16}^{(0,3)})^i_{jk} \ .
\end{aligned}
\tag{7.64}
$$

All operators in (7.64) are Q-exact.

iii) $(R, J) = (\frac{5}{2}, \frac{3}{2})$ The operators in this sector are made of 9 letters. Allowed (n_ϕ, n_ψ, n_f) are $(7, 1, 1), (6, 3, 0)$. We find 13 Q-closed operators in this sector given by

$$
\begin{aligned}
O_1^{(1,1)} \equiv & \ \epsilon_{a_1 a_2 i} u^{a_1 (j} w^{k) a_2} (r_{10}^{(0,1)})^i_{jk} \ , \\
O_2^{(1,1)} \equiv & \ \epsilon_{a_1 a_2 i} u^{a_1 jk} w^{a_2} (r_{10}^{(0,1)})^i_{jk} \ , \\
O_3^{(1,1)} \equiv & \ \epsilon_{a_1 a_2 i} \epsilon_{b_1 b_2 j} u^{a_1 b_1} u^{a_2 b_2 k} (r_{14}^{(1,1)})^{ij}_k + 5v^a{}_i v^{jk}{}_a (r_{10}^{(0,1)})^i_{jk} - 2v^{(j}{}_a v^{k)a}{}_i (r_{10}^{(0,1)})^i_{jk} \ , \\
O_1^{(0,3)} = & -\epsilon_{i a_1 a_2} \left(4u^{a_1 b} v^{j a_2}{}_b + 3u^{j a_1 b} v^{a_2}{}_b\right) (r_{12}^{(0,2)})^i_j = \frac{1}{2} i \, Q((r_{12}^{(0,2)})^i_j (r_{12}^{(0,2)})^j_i) \ , \\
O_2^{(0,3)} = & -\epsilon_{a_1 a_2 (i} \left(u^{a_1 (k} v^{l) a_2}{}_{j)} + u^{k l a_1} v^{a_2}{}_{j)}\right) (r_{12}^{(0,2)})^{ij}_{kl} = \frac{1}{2} i \, Q((r_{12}^{(0,2)})^{kl}_{ij} (r_{12}^{(0,2)})^{ij}_{kl}) \ , \\
O_3^{(0,3)} \equiv & -u^{a(i} v^{jk)}{}_a (r_{12}^{(0,2)})_{ijk} \ , \\
O_4^{(0,3)} \equiv & -\epsilon_{a_1 a_2 i} u^{a_1 b} v^{a_2}{}_b (r_{14}^{(0,2)})^i \ , \\
O_5^{(0,3)} \equiv & \ 6v^a{}_i v^{jk}{}_a (r_{10}^{(0,1)})^i_{jk} + 6u^{a(ij} v^{k)}{}_a (r_{12}^{(0,2)})_{ijk} + \epsilon_{a_1 a_2 i} u^{a_1 b j} v^{a_2}{}_b (r_{12}^{(0,2)})^i_j \ , \\
O_6^{(0,3)} \equiv & \ 24 v^{(j}{}_a v^{k)a}{}_i (r_{10}^{(0,1)})^i_{jk} + 6u^{a(i} v^{j)}{}_a (r_{14}^{(0,2)})_{ij} - \epsilon_{a_1 a_2 (i} u^{a_1 k} v^{a_2}{}_{j)} (r_{14}^{(0,2)})^{ij}_k \ , \\
O_7^{(0,3)} \equiv & \ v^a{}_i v^{jk}{}_a (r_{10}^{(0,1)})^i_{jk} - 10 v^{(j}{}_a v^{k)a}{}_i (r_{10}^{(0,1)})^i_{jk} + 6u^{a(ij} v^{k)}_a (r_{12}^{(0,2)})_{ijk} + 10\epsilon_{a_1 a_2 (i} u^{a_1 kl} v^{a_2}_{j)} (r_{12}^{(0,2)})^{ij}_{kl} \ , \\
O_8^{(0,3)} \equiv & \ 5v^a{}_i v^{jk}{}_a (r_{10}^{(0,1)})^i_{jk} - 2v^{(j}{}_a v^{k)a}{}_i (r_{10}^{(0,1)})^i_{jk} + 9u^{a(ij} v^{k)}_a (r_{12}^{(0,2)})_{ijk} + 6\epsilon_{a_1 a_2 i} u^{a_1 (j} u^{kl) a_2} (r_{14}^{(0,3)})^i_{jkl} \ , \\
O_9^{(0,3)} \equiv & \ 6v^a{}_i v^{jk}{}_a (r_{10}^{(0,1)})^i_{jk} + 12 v^{(j}{}_a v^{k)a}{}_i (r_{10}^{(0,1)})^i_{jk} + 18u^{a(ij} v^{k)}_a (r_{12}^{(0,2)})_{ijk} - \epsilon_{a_1 a_2 (i} u^{a_1 k} v^{a_2}{}_{j)} (r_{14}^{(0,2)})^{ij}_k \ , \\
O_{10}^{(0,3)} \equiv & \ 38 v^a{}_i v^{jk}{}_a (r_{10}^{(0,1)})^i_{jk} + 4v^{(j}{}_a v^{k)a}{}_i (r_{10}^{(0,1)})^i_{jk} + 24u^{a(ij} v^{k)}{}_a (r_{12}^{(0,2)})_{ijk} + 5u^{(jk} v^{l)}{}_i (r_{14}^{(0,2)})^i_{jkl} \ .
\end{aligned}
\tag{7.65}
$$

All except for $O_6^{(0,3)}$ in (7.65) are Q-exact. Therefore, a representative of the cohomology in this sector can be written as

$$\begin{aligned} O \;\equiv\;& -6O_6^{(0,3)} \qquad (7.66)\\ =\;& 288v^j{}_a v^{ka}{}_i\epsilon_{c_1c_2(j}\,\mathrm{tr}\left(\phi^{c_1}\phi^{c_2}\phi^i\psi_{k)}\right) - 72v^a{}_b v^{bk}{}_a\epsilon_{c_1c_2(k}\,\mathrm{tr}\left(\phi^{c_1}\phi^{c_2}\phi^d\psi_{d)}\right)\\ &+36\epsilon_{a_1a_2i}u^{a_1k}v^{a_2}{}_j\left[2\,\mathrm{tr}\left(\phi^{(i}\phi^c\phi^{j)}\psi_{(c}\psi_{k)}\right) + 2\,\mathrm{tr}\left(\phi^{(i|}\phi^c\phi^{|j)}\psi_{(c}\psi_{k)}\right)\right.\\ &\left.\qquad +9\,\mathrm{tr}\left(\phi^{(i}\phi^j\psi_{(c}\phi^{c)}\psi_{k)}\right) - 6\,\mathrm{tr}\left(\phi^{(i}\phi^{j)}\psi_{(c}\phi^c\psi_{k)}\right)\right]\\ &-9\epsilon_{a_1a_2j}u^{a_1b}v^{a_2}{}_b\left[2\,\mathrm{tr}\left(\phi^{(j}\phi^c\phi^{d)}\psi_{(c}\psi_{d)}\right) + 2\,\mathrm{tr}\left(\phi^{(j|}\phi^c\phi^{|d)}\psi_{(c}\psi_{d)}\right)\right.\\ &\left.\qquad +9\,\mathrm{tr}\left(\phi^{(j}\phi^d\psi_{(c}\phi^{c)}\psi_{d)}\right) - 6\,\mathrm{tr}\left(\phi^{(j}\phi^{d)}\psi_{(c}\phi^c\psi_{d)}\right)\right]\\ &-20u^{ai}v^j{}_a\epsilon_{b_1b_2b_3}\left[2\,\mathrm{tr}\left(\psi_{(i}\psi_{j)}\phi^{b_1}\phi^{b_2}\phi^{b_3}\right) + \mathrm{tr}\left(\psi_{(i}\phi^{b_1}\psi_{j)}\phi^{b_2}\phi^{b_3}\right)\right]\\ &-36u^{ai}v^j{}_a\epsilon_{b_1b_2(i}\left[\mathrm{tr}\left(\psi_{j)}\psi_c\phi^{b_1}\phi^{b_2}\phi^c\right) + \mathrm{tr}\left(\psi_{j)}\psi_c\phi^{b_1}\phi^c\phi^{b_2}\right) + \mathrm{tr}\left(\psi_{j)}\psi_c\phi^c\phi^{b_1}\phi^{b_2}\right)\right]\\ &-36u^{ai}v^j{}_a\epsilon_{b_1b_2(i}\left[\mathrm{tr}\left(\psi_{j)}\phi^{b_1}\psi_c\phi^{b_2}\phi^c\right) + \mathrm{tr}\left(\psi_{j)}\phi^{b_1}\psi_c\phi^c\phi^{b_2}\right) + \mathrm{tr}\left(\psi_{j)}\phi^c\psi_c\phi^{b_1}\phi^{b_2}\right)\right]\\ &-36u^{ai}v^j{}_a\epsilon_{b_1b_2(i}\left[\mathrm{tr}\left(\psi_{j)}\phi^{b_1}\phi^{b_2}\psi_c\phi^c\right) + \mathrm{tr}\left(\psi_{j)}\phi^{b_1}\phi^c\psi_c\phi^{b_2}\right) + \mathrm{tr}\left(\psi_{j)}\phi^c\phi^{b_1}\psi_c\phi^{b_2}\right)\right]\\ &-36u^{ai}v^j{}_a\epsilon_{b_1b_2(i}\left[\mathrm{tr}\left(\psi_{j)}\phi^{b_1}\phi^{b_2}\phi^c\psi_c\right) + \mathrm{tr}\left(\psi_{j)}\phi^{b_1}\phi^c\phi^{b_2}\psi_c\right) + \mathrm{tr}\left(\psi_{j)}\phi^c\phi^{b_1}\phi^{b_2}\psi_c\right)\right]\\ &+12u^{ai}v^j{}_a\epsilon_{b_1b_2(i}\left[5\,\mathrm{tr}\left(\psi_{j)}\phi^{b_1}\phi^{b_2}\right)\mathrm{tr}\left(\psi_c\phi^c\right) + 2\,\mathrm{tr}\left(\psi_{j)}\phi^{(b_1}\phi^{c)}\right)\mathrm{tr}\left(\psi_c\phi^{b_2}\right)\right.\\ &\left.\qquad -2\,\mathrm{tr}\left(\psi_{j)}\phi^{b_2}\right)\mathrm{tr}\left(\psi_c\phi^{(b_1}\phi^{c)}\right)\right]\;. \end{aligned}$$

The scaling dimension of this cohomology O is $E = 3R + 2J = \frac{21}{2}$. Note that the representative found above does not contain the letter f.

iv) $(R, J) = (3, 1)$**.** The operators in this sector are made of 10 letters. Allowed (n_ϕ, n_ψ, n_f) are $(9, 0, 1)$ and $(8, 2, 0)$. We find 6 Q-closed operators in this sector given by

$$\begin{aligned} O_1^{(0,2)} &\equiv -\epsilon_{a_1a_2i}u^{a_1b}u^{jk}v^{a_2}{}_b(r_{10}^{(0,1)})^i_{jk} + 2\epsilon_{a_1a_2i}u^{a_1b}u^{a_2(j}v^{k)}{}_b(r_{10}^{(0,1)})^i_{jk}\;,\\ O_2^{(0,2)} &\equiv -6\epsilon_{a_1a_2i}u^{a_1b(j}v^{k)a_2}{}_b(r_{10}^{(0,1)})^i_{jk} - \epsilon_{a_1a_2(i}u^{a_1(k}v^{l)a_2}{}_{j)}(r_{12}^{(0,1)})^{ij}_{kl}\;,\\ O_3^{(0,2)} &\equiv -\epsilon_{a_1a_2i}u^{a_1b}u^{jk}v^{a_2}{}_b(r_{10}^{(0,1)})^i_{jk} - \epsilon_{a_1a_2(i}u^{a_1kl}v^{a_2}{}_{j)}(r_{12}^{(0,1)})^{ij}_{kl}\;,\\ O_4^{(0,2)} &\equiv -\epsilon_{a_1a_2i}u^{a_1b}u^{jk}v^{a_2}{}_b(r_{10}^{(0,1)})^i_{jk} + \epsilon_{a_1a_2(i}\epsilon_{j)b_1b_2}u^{a_1b_1}u^{a_2b_2}u^{kl}(r_{12}^{(0,2)})^{ij}_{kl}\;,\\ O_5^{(0,2)} &\equiv -4\epsilon_{a_1a_2i}u^{a_1b}u^{jk}v^{a_2}{}_b(r_{10}^{(0,1)})^i_{jk} - 24\epsilon_{a_1a_2i}u^{a_1b(j}v^{k)a_2}{}_b(r_{10}^{(0,1)})^i_{jk}\\ &\quad - \epsilon_{a_1a_2(i}\epsilon_{j)b_1b_2}u^{a_1b_1}u^{a_2b_2k}(r_{14}^{(0,2)})^{ij}_k\;,\\ O_6^{(0,2)} &\equiv -\epsilon_{a_1a_2i}u^{a_1b}u^{jk}v^{a_2}{}_b(r_{10}^{(0,1)})^i_{jk} + 12\epsilon_{a_1a_2i}u^{a_1b(j}v^{k)a_2}{}_b(r_{10}^{(0,1)})^i_{jk} + 3\epsilon_{a_1a_2i}u^{a_1(j}u^{kl)a_2}(r_{14}^{(0,2)})^i_{jkl}\;. \end{aligned} \qquad (7.67)$$

All the operators in (7.67) are Q-exact.

v) $(R,J) = (\frac{7}{2}, \frac{1}{2})$**.** The operators in this sector are made of 11 letters. The only allowed (n_ϕ, n_ψ, n_f) is $(9,1,0)$. We find 1 Q-closed operator in this sector given by

$$O^{(0,1)} \equiv 36\epsilon_{a_1a_2a_3}\epsilon_{b_1b_2i}u^{a_1b_1}u^{a_2b_2}u^{a_3jk}(r_{10}^{(0,1)})^i_{jk} + 5\epsilon_{a_1a_2a_3}\epsilon_{b_1b_2b_3}u^{a_1b_1}u^{a_2b_2}u^{a_3b_3}r_{12}^{(0,1)} \\ -6\epsilon_{a_1a_2(i}\epsilon_{j)b_1b_2}u^{a_1b_1}u^{a_2b_2}u^{kl}(r_{12}^{(0,1)})^{ij}_{kl} \ . \tag{7.68}$$

The operator (7.68) is Q-exact.

In summary, we have found 1 fermionic black hole cohomology using our ansatz, which is a singlet under $SU(3) \subset SU(4)_R$ at order $\mathcal{J} = 24$. It is represented by (7.66). Its charges and scaling dimension are given by $(R,J,E) = \left(\frac{5}{2}, \frac{3}{2}, \frac{21}{2}\right)$.

7.3.3 Filtering Exact Operators

In this subsection, we sketch how to determine Q-exactness of various Q-closed operators listed in the previous subsection.

To check whether a given operator is Q-exact or not, especially to check non-Q-exactness, one has to rule out all possible ways of writing the operator as Q of 'something'. That being said, one needs to construct all possible operators that can participate in 'something' (the meaning of which will be made clear shortly) and show that the *target* operator is linearly independent of Q-actions of them. More specifically, we divide the check of Q-exactness into 4 steps, that we summarize as follows.

1. Construct all gauge-invariant operators whose Q-action may participate in reproducing the *target*.

2. Count the number of linearly independent operators from step 1, and extract the maximal subset of linearly independent operators. This is called the *basis*.

3. Act Q on the basis operators, then again count and extract the maximal subset of linearly independent ones between them.

4. Check if the target is linearly independent of the result of step 3.

Now we explain what operators 'may participate in reproducing the target' in step 1. This consists of two criteria: the charges and the parity under permutation.

First, the charges of the target operator constrain the charges, thus the letter contents of the basis operators. Note that the action of Q increases $Q_{I=1,2,3}$ by $\frac{1}{2}$ and decreases $J = J_1 = J_2$ by $\frac{1}{2}$. Therefore, the basis operators must have the set of charges that differ by the corresponding amount from the target, otherwise their Q-actions are disjoint from the target. Note that all of our targets are $SU(3)$ singlets, so we always have $R = Q_1 = Q_2 = Q_3$.

Second, all of our targets being singlets under the $SU(3)$ subgroup of the $SU(4)$ R-symmetry group, imposes a stronger constraint than just restricting to the charge sectors with $Q_1 = Q_2 = Q_3$. Each basis operator must be invariant under cyclic permutation $\phi^i \to \phi^{i+1}$ and simultaneously $\psi_i \to \psi_{i+1}$, where $i = 1, 2, 3 \mod 3$. Moreover, if there are even/odd number of ϕ's and ψ's combined, which carries one $SU(3)$ index each, it requires even/odd number of Levi-Civita symbols to write the operator covariantly while contracting all indices. Therefore, we may restrict to i) operators with even number of ϕ's and ψ's combined, that are even under all 3! permutations of $SU(3)$ indices, and ii) operators with odd number of ϕ's and ψ's combined, that are even under even (cyclic) permutations of $SU(3)$ indices and odd under odd (swap) permutations of $SU(3)$ indices. Also note that this permutation property commutes with the action of Q, so that Q of a non-trivial operator satisfies this property if and only if the original operator does. This permutation property is necessary but not sufficient for an operator to be an $SU(3)$ singlet. However, we impose this property on the basis instead of requiring $SU(3)$ singlets, because the latter requires many sums over dummy indices and thus the former is computationally more efficient. Our conclusions on the singlet sector will be valid despite.

For example, suppose that the target operator is (7.66), which has charges $(R, J) = (\frac{5}{2}, \frac{3}{2})$. Operators whose Q-action may reproduce this target operator must then have $(R, J) = (2, 2)$. Possible choices of letter contents are $(n_\phi, n_\psi, n_f) = (6, 0, 2), (5, 2, 1)$, or $(4, 4, 0)$, and numbers of ϕ^i minus numbers of ψ_i must be equal between $i = 1, 2, 3$. Further taking into account the permutation property, the basis operators whose Q-action 'may participate in reproducing the target' (7.66) can be classified into the following 7 subsectors. ($(-1)^\epsilon$ in subsectors 5 and 6 indicates minus sign for odd permutations, because there are odd number of ϕ's and ψ's in those subsectors.)

- Subsector 1: $(\phi^1)^4(\psi_1)^4+$ (permutations)
- Subsector 2: $(\phi^1)^3(\phi^2)^1(\psi_1)^3(\psi_2)^1+$ (permutations)
- Subsector 3: $(\phi^1)^2(\phi^2)^2(\psi_1)^2(\psi_2)^2+$ (permutations)
- Subsector 4: $(\phi^1)^2(\phi^2)^1(\phi^3)^1(\psi_1)^2(\psi_2)^1(\psi_3)^1+$ (permutations)
- Subsector 5: $(\phi^1)^3(\phi^2)^1(\phi^3)^1(\psi_1)^2f^1 + (-1)^\epsilon$ (permutations)
- Subsector 6: $(\phi^1)^2(\phi^2)^2(\phi^3)^1(\psi_1)^1(\psi_2)^1f^1 + (-1)^\epsilon$ (permutations)
- Subsector 7: $(\phi^1)^2(\phi^2)^2(\phi^3)^2f^2+$ (permutations)

Appropriate sums over permutations of single- and multi-trace operators in each of these subsectors are the result of step 1, some of which we write down below to help visualize:

$$
\begin{aligned}
&\mathrm{tr}(\phi^1\phi^1\psi_1\phi^1\psi_1\psi_1\phi^1\psi_1)+\mathrm{tr}(\phi^2\phi^2\psi_2\phi^2\psi_2\psi_2\phi^2\psi_2)+\mathrm{tr}(\phi^3\phi^3\psi_3\phi^3\psi_3\psi_3\phi^3\psi_3)\ ,\\
&\mathrm{tr}(\phi^1\phi^1\phi^2\psi_2)\mathrm{tr}(\psi_1\psi_2)\mathrm{tr}(\phi^2\psi_1)+\mathrm{tr}(\phi^2\phi^2\phi^3\psi_3)\mathrm{tr}(\psi_2\psi_3)\mathrm{tr}(\phi^3\psi_2)\\
&\qquad+\mathrm{tr}(\phi^3\phi^3\phi^1\psi_1)\mathrm{tr}(\psi_3\psi_1)\mathrm{tr}(\phi^1\psi_3)+\mathrm{tr}(\phi^3\phi^3\phi^2\psi_2)\mathrm{tr}(\psi_3\psi_2)\mathrm{tr}(\phi^2\psi_3)\\
&\qquad+\mathrm{tr}(\phi^1\phi^1\phi^3\psi_3)\mathrm{tr}(\psi_1\psi_3)\mathrm{tr}(\phi^3\psi_1)+\mathrm{tr}(\phi^2\phi^2\phi^1\psi_1)\mathrm{tr}(\psi_2\psi_1)\mathrm{tr}(\phi^1\psi_2)\ ,\\
&\mathrm{tr}(\phi^2\phi^2\psi_1\psi_2\phi^3)\mathrm{tr}(f\phi^1\phi^1)+\mathrm{tr}(\phi^3\phi^3\psi_2\psi_3\phi^1)\mathrm{tr}(f\phi^2\phi^2)\\
&\qquad+\mathrm{tr}(\phi^1\phi^1\psi_3\psi_1\phi^2)\mathrm{tr}(f\phi^3\phi^3)-\mathrm{tr}(\phi^3\phi^3\psi_1\psi_3\phi^2)\mathrm{tr}(f\phi^1\phi^1)\\
&\qquad-\mathrm{tr}(\phi^1\phi^1\psi_2\psi_1\phi^3)\mathrm{tr}(f\phi^2\phi^2)-\mathrm{tr}(\phi^2\phi^2\psi_3\psi_2\phi^1)\mathrm{tr}(f\phi^3\phi^3)\ .
\end{aligned}
\tag{7.69}
$$

Given the operators from step 1, the rest is relatively straightforward, at least conceptually. There are non-trivial trace relations between operators from step 1, so in step 2 we extract linearly independent basis operators. Then in step 3, we consider Q-actions of the basis operators, and again count the number of linearly independent ones among them. These should form a complete basis of all Q-exact operators in the target charge sector and with the aforementioned permutation property. Therefore, the target operator is Q-exact if and only if it is a linear combination of the Q-actions of the basis operators. More generally, if there are multiple target operators, the number of cohomologies among them would be equal to the number of linearly independent ones among the basis *and* all target operators, minus the number of linearly independent ones among the basis only.

Each of step 2-4 involves counting and/or finding linearly independent operators among a given set of gauge-invariant operators. Each operator is a sum over single- and multi-trace operators written in terms of seven species of fields ϕ^m, ψ_m and f. To completely account for trace relations between them, we first convert the operators written in terms of adjoint fields into polynomials of their matrix elements, by substituting

$$
f = \begin{pmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & -f_1-f_5 \end{pmatrix}\ , \tag{7.70}
$$

and likes for 6 other fields. In this way, every operator is now written as a polynomial of $8\times 7 = 56$ variables, 24 of which are Grassmannian. So the problem boils down to finding linear dependence between a set of polynomials. Although this is the same problem that was encountered while computing the graviton index in section 3, the same method of extracting the coefficient matrix is extremely unpractical here. It is because there are four times as many variables (recall that for counting gravitons, we substituted each field with a diagonal matrix), and therefore exponentially larger number of monomials appear in polynomials. As a result, the coefficient matrix will have a huge number of columns that is not viable for computers.

For this reason, we have devised a numerics-assisted approach to find linear dependence between the polynomials with large number of variables. The approach stems from the basic fact that if some linear combination of certain polynomials vanishes, it must also be zero if we attribute any specific number to each variable. So let us represent each polynomial by an array of numbers, i.e. a row vector, by substituting each variable with a set of randomly chosen integers. Then we examine the linear dependence between vectors, instead of polynomials.

The substitution can be repeated for arbitrarily many sets of integers, so the row vector can be made arbitrarily long. Obviously, the length of the row vectors, i.e. number of columns, must be at least as many as there are independent polynomials. Otherwise, it will be always possible to find a relation between the row vectors even if the polynomials they represent are independent. On the other hand, the length of the row vectors need not be much more than the number of independent polynomials, as we will explain shortly.

This makes it clear why this method is efficient. It naturally realizes the basic principle that in order to distinguish n different entities, one needs at least n data for each entity, whereas extracting the coefficient matrix for the polynomials with so many variables will equivalently convert each polynomial into an unnecessarily long vector.

There are two issues with this approach that we need to address. The first is that 24 of 56 variables are Grassmannian, which cannot be properly substituted with c-numbers. The second is that randomness is involved in this approach, and it may lead to errors albeit unlikely.

The issue with Grassmann variables can be easily addressed by ordering them in a definite manner within each monomial. That is, we fully expand each polynomial (which includes eliminating squares of Grassmann variables), and let variables be multiplied only in a certain order within each monomial. During this process the coefficients may flip signs, but the result of this process is unique for each polynomial. Once we have done this, none of the Grassmann properties will be used when finding linear relations between the polynomials, because each monomial is now compared verbatim with monomials in other polynomials. Therefore, it is now safe to substitute Grassmann variables with c-numbers. This principle was also implied while extracting the coefficient matrix of graviton operators in section 3.

As for the randomness, first note that substituting (sufficiently many sets of) random integers never miss the true dependence between polynomials. If there is a true linear dependence between polynomials, i.e. a linear combination that vanishes, the same linear combination must be zero for whatever numbers are put in, so the row vectors corresponding to the polynomials must be linearly dependent. Note that all polynomials have rational coefficients and we put in random integers, so there is no issue with machine precision.

However, the converse is possible: this method may find false linear dependence between polynomials. This is simply because a non-vanishing polynomial may evaluate to zero when certain values are put into variables. That is, the randomly chosen values could miraculously be the roots of the polynomial. This type of error can be made arbitrarily more unlikely by increasing the number of columns, i.e. number of sets of random integers that are put in. Let us roughly estimate the unlikelihood.

Suppose that the number of columns is $m+n$ where m is the true number of independent polynomials. For this method to find a false dependence, both of the followings must happen: i) there exists a non-trivial linear combination of the polynomials that vanishes for the first m sets of random integers, and ii) this polynomial further vanishes for the additional n sets of random integers. The probability of i) is relatively difficult to estimate, since it involves intricate tuning of $m-1$ coefficients in a linear combination of the polynomials. Therefore we only estimate the probability of ii) as follows. A typical basis polynomial such as Q-action of those in (7.69) [2] evaluates to $\sim 10^{28}$ when a random integer between 1 and 1000 are substituted into each variable. (See Fig. 7.1. for an example.) This is a natural scale considering that the typical polynomial is a sum over $\sim 10^6$ monomials (with both signs) that each consists of 9 letters, so for example $10^6 \times (10^{2.5})^9 \sim 10^{28}$. This value is far smaller than the number of all possible random choices — which is $(10^3)^{56}$ if all 7 gluons, thus $7 \times (3^2-1)$ variables, are involved — so each integer value within magnitude $\sim 10^{28}$ will be sufficiently populated. Furthermore, since a typical polynomial consists of many$\sim 10^6$ monomials, we assume that the evaluation of the polynomial is like a random walk with sufficient iterations, and thus the factorization property of integers is blurred. For these reasons, let us assume that the distribution of the evaluated values is continuous. Then, the probability that this value falls within $O(1)$ is estimated to be $\sim 10^{-28}$, even accounting for the shape of the distribution. For ii), this must happen for n independent sets of random variables, so the probability of ii) is estimated to be 10^{-28n}. In step 3, n was taken to be 175, so the estimated probability of ii) is 10^{-4900}.

This method of detecting linear dependence was used between numerous sets of polynomials while determining Q-exactness of various operators in different charge sectors. Numbers that appeared in the previous paragraph slightly differ between occasions. Typical values

[2] These are used in step 3 and 4 of determining Q-exactness of Q-closed operators in the charge sector $(R, J) = (\frac{5}{2}, \frac{3}{2})$, of which one is the non-graviton cohomology (7.66)

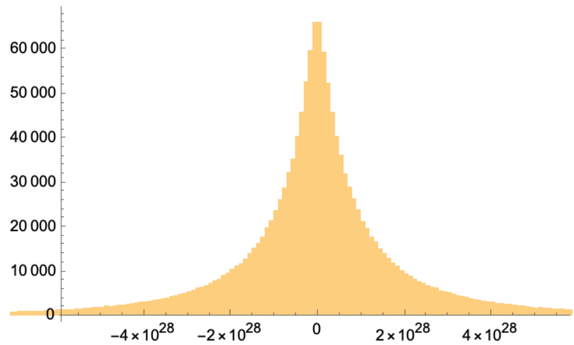

Figure 7.1: An example distribution of 1.5×10^6 evaluated values of Q-actions of basis polynomials for (7.65). Width of each bin is 10^{27}.

of the polynomials differ because they consist of different numbers of letters, and n is inevitably different because the number of columns are set before we know the number of true independent polynomials. However, in any case, we use at least $n \geq 30$ and the estimated probability of ii) has order of magnitude of a few negative hundreds at the worst. Furthermore, when a Q-closed operator is determined to be Q-exact, we checked analytically the relation between the target and basis polynomials to further eradicate the margin for error.

Employing the method explained so far, we have constructed the basis operators in each and all charge sectors with $Q_1 = Q_2 = Q_3$ at the order $\mathcal{J} = 24$, with the aforementioned permutation property. We have also evaluated the Q-actions of the bases, that should form the basis of Q-exact operators. Then we have determined Q-exactness of all Q-closed non-graviton operators obtained from our ansatz in the previous subsection. The result is that all operators in section 7.3.2 except for the fermionic (7.66) are Q-exact.

7.3.4 Ansatz-Independent Studies

From the fact that we have constructed and counted all operators and their Q-actions in the $Q_1 = Q_2 = Q_3$ charge sectors at $\mathcal{J} = 24$ order with the permutation property, we can also prove the non-existence of any other $SU(3)$ singlet non-graviton cohomology at $\mathcal{J} = 24$ order. Recall that the result of step 2 in the previous subsection is a complete basis of all operators, in a given charge sector (R, J) and with permutation property that is designed to include all $SU(3)$ singlets. There are further linear relations between Q-actions of these basis operators, reducing the number of independent Q-exact operators at charge sector $(R + \frac{1}{2}, J - \frac{1}{2})$ in step 3. The reduced operators correspond to the Q-closed operators at charge sector (R, J):

$$(\#\text{closed})_{(R,J)} = (\#\text{basis})_{(R,J)} - (\#\text{exact})_{(R+\frac{1}{2},J-\frac{1}{2})}\ .$$

R	J	#letters	#basis	#closed	#exact	#coh.	#gravitons	#BH coh.
0	4	4	1	0	0	0	0	0
$\frac{1}{2}$	$\frac{7}{2}$	5	9	1	1	0	0	0
1	3	6	91	8	8	0	0	0
$\frac{3}{2}$	$\frac{5}{2}$	7	511	85	83	2	2	0
2	2	8	1369	445	426	19	19	0
$\frac{5}{2}$	$\frac{3}{2}$	9	1898	953	924	29	28	1
3	1	10	1456	961	945	16	16	0
$\frac{7}{2}$	$\frac{1}{2}$	11	633	505	495	10	10	0
4	0	12	136	136	128	8	8	0

Table 7.1: For each charge sector $R = Q_1 = Q_2 = Q_3$ and J at level $\mathcal{J} = 24$, we present the numbers of operators discussed in the text. The last column shows that (7.66) is the only black hole cohomology in the target charge sector.

Then the number of Q-cohomologies is given by

$$(\#\text{coh.})_{(R,J)} = (\#\text{closed})_{(R,J)} - (\#\text{exact})_{(R,J)} \ .$$

Meanwhile, we can also count the number of independent graviton cohomologies in these charge sectors and with the same permutation property, like we counted the full set of gravitons in subsection 3.1. The number of non-graviton cohomologies is given by

$$(\#\text{BH coh.})_{(R,J)} = (\#\text{coh.})_{(R,J)} - (\#\text{gravitons})_{(R,J)} \ .$$

We present all the numbers mentioned in this paragraph in Table 7.1. We find only one non-graviton cohomology in the $(R, J) = (\frac{5}{2}, \frac{3}{2})$ sector, which is the fermionic cohomology presented in (7.66). Since the operators with the permutation property in the $Q_1 = Q_2 = Q_3$ charge sectors include all $SU(3)$ singlets, we conclude that (7.66) is the only non-graviton cohomology that is an $SU(3)$ singlet at order $\mathcal{J} = 24$.

The computation presented in this subsection, of constructing the basis operators and counting independent ones between them and their Q-actions, is essentially the sort of computation that was performed in [115], although we find our numerics-assisted approach to be more efficient. Moreover, we have only performed this computation in the $Q_1 = Q_2 = Q_3$ charge sectors at $\mathcal{J} = 24$ order in the BMN sector, and further restricted to operators with certain permutation property. This is because we focused on the $SU(3)$ singlet sector at order $\mathcal{J} = 24$, where the non-graviton index indicated the existence of a non-graviton cohomology.

Chapter 8

Conclusion

Throughout this dissertation, we have discussed various properties of the supersymmetric states, or local BPS operators, of superconformal field theories dual to AdS black holes in different dimensions including $\text{AdS}_{3,4,5,7}$. In particular we focussed on the AdS_3 black holes and the dual $(4,4)$ SCFT_2 with $\frac{1}{4}$ of the supersymmetries, and on the AdS_5 black holes and the dual 4d $\mathcal{N}=4$ Super-Yang-Mills theory with $\frac{1}{16}$ of the supersymmetries. The supersymmetric states subject to our study have been enumerated using the index to account for the Bekenstein-Hawking entropy of the dual black holes, which is a thermodynamic quantity, in a statistical sense à la Boltzmann. This dissertation went beyond the enumeration to study the macroscopic charges of the supersymmetric ensemble, and to identify the black hole states.

In part I, we have done the followings.

- We argued that the AdS_3 black holes may be unstable under decay into some particles.
- We demonstrated clearly in a simple example with the $U(1)$ gauge group for the 4d $\mathcal{N}=4$ Super-Yang-Mills theory, how complexification of chemical potentials may overcome the boson/fermion cancellations in the index.
- We addressed that the derivation of the black hole entropy by treating the index as a BPS limit of the partition function can be applied to AdS_3 black holes under intricate limiting procedures.
- We gave a heuristic derivation of the supersymmetric charge constraints on the $\text{AdS}_{3,4,5,7}$ black holes as the relations between macroscopic charges of the supersymmetric ensembles of the free field theories.

In part II, we have done the followings. Recall that the black hole cohomologies i) are related to the BPS states in the strongly coupled field theory, which are the dual black hole microstates, or ii) are dual states to the smallest, the most quantum black holes in the quantum gravity theory.

- We wrote explicit expressions for possibly all black hole cohomologies in the BMN sector of the 4d $\mathcal{N} = 4$ Super-Yang-Mills theory with the gauge group $SU(2)$.

- We showed that there must be another set of four black hole cohomologies in the $SU(2)$ theory at the charge $\mathcal{J} = 39$.

- We observed the black hole partial no-hair behavior, that the black holes abhor dressing by gravitons, especially by chiral primary black holes with little rotation.

- We constructed a fermionic singlet black hole cohomology in the $SU(3)$ theory at $\mathcal{J} = 24$, which is likely the smallest one in the theory.

- We found that a fermionic triplet black hole cohomology in the $SU(4)$ theory must appear at $\mathcal{J} = 28$.

Related either directly or indirectly to this dissertation, we suggest several future research directions.

- Our heuristic derivation of the supersymmetric charge constraints on the $\text{AdS}_{4,5,7}$ black holes may be made more rigorous. We have already discussed several directions in section 4.6. For examples, we expect that inclusion of the gauge degrees of freedom in a principled manner may explain the scaling of the charges by appropriate powers of N. Furthermore, it is possible that the $O(1)$ numerical factors appear while connecting the free field theory results to the strongly coupled theory.

- It was observed recently [136–139] that the index for the $\frac{1}{16}$-BPS states of the 4d $\mathcal{N} = 4$ Yang-Mills theory admits the structure of an expansion of expansions. There has been progress [140–146] in interpreting this expansion in terms of giant gravitons, or D3-branes in string theory [125–127]. Given the coupling independence of the index, this opens a new window to understand the structure of the black holes in the quantum gravity theory, not limited to its supergravity approximation.

- To complete our argument on the instability of the AdS_3 black hole, we may need to address existence and abundance of particles with the assumed charges. Giant gravitons in the AdS_3 black hole background [147, 148] can potentially play the role of these particles. This is a work in progress with Finn Larsen.

- It is interesting to slightly lift the focus from strictly supersymmetric black holes. There has been considerable progress [110–112] regarding the spectrum of nearly extremal and nearly BPS black holes. These works are based on the effective theory of the near-extremal black holes, a gravity theory in the near-horizon geometry AdS_2 that is being slightly broken by a dilaton [149–154]. This theory is approximated by the Jackiw-Teitelboim (JT) gravity [155, 156] and its supersymmetric generalizations [157–159] leading to the (super-)Schwarzian actions. The fact that the (super-)JT gravity is solved at quantum level [160, 161] is leveraged into quantum corrections to the low energy spectrum of the near-extremal and near-BPS black holes. See [162] for a review. It will be valuable to work out the details of the effective theory, including scrutinizing the validity of approximations used in the process and examining the higher order effects, and sketching out the pattern of symmetry breaking. This is a work in progress with Sangmin Choi and Finn Larsen.

- One may consider extending the results of Part II into gauge groups with higher ranks and/or into higher levels of the charge, to detect and construct more examples of black hole cohomologies. This is certainly a desired progress, but limitations on the computing power strongly suggests that one take a completely different approach.

- Since the black hole cohomologies represent BPS states in the weakly coupled gauge theory, it is important to make connections between operators in the weakly and strongly coupled theories. It is one of the major goals of the field of integrability. For an example, see [113] that followed our work.

We hope that through the enormous effort from generations of physicists that this dissertation joins, black holes shed bright light to quantum gravity.

Appendix A

Graviton Trace Relations

In this appendix that is related to section 7.3 of the main part, we first list the trace relations between the graviton cohomologies in the BMN sector of the $SU(3)$ theory. Then we construct the relations of relations at $\mathcal{J} = 24$ which are singlets under the $SU(3) \subset SU(4)_R$ global symmetry. These are the two sectors in which the index predicted fermionic cohomologies in the $SU(3)$ singlet. The results at $\mathcal{J} = 24$ are used in section 7.3.2 to construct the threshold cohomology.

The trace relations are the linear dependence between the multi-trace operators, up to Q-exact operators, due to the finite size of the matrices. In this appendix, we shall only consider the trace relations between gravitons. Let us first arrange the trace relations by their level $\mathcal{J}$ and distinguish them into two types; fundamental ones and the others. The fundamental trace relations at level $\mathcal{J}$ cannot be written as linear combinations of the trace relations at lower levels $\mathcal{J}'(< \mathcal{J})$, multiplied by the gravitons at level $\mathcal{J} - \mathcal{J}'$. All trace relations of gravitons can be expressed as linear combinations of the fundamental trace relations with the coefficients being graviton cohomologies. We explicitly constructed the fundamental trace relations until certain levels, which will be presented below.

The single-trace generators of the $SU(3)$ BMN gravitons are given by

$$\begin{aligned}
&u^{ij} \equiv \mathrm{tr}\left(\phi^{(i}\phi^{j)}\right) \ , \ u^{ijk} \equiv \ \mathrm{tr}\left(\phi^{(i}\phi^{j}\phi^{k)}\right) \ , \\
&v^{i}{}_{j} \equiv \mathrm{tr}\left(\phi^{i}\psi_{j}\right) - \tfrac{1}{3}\delta^{i}_{j}\mathrm{tr}\left(\phi^{a}\psi_{a}\right) \ , \ v^{ij}{}_{k} \equiv \ \mathrm{tr}\left(\phi^{(i}\phi^{j)}\psi_{k}\right) - \tfrac{1}{4}\delta^{i}_{k}\mathrm{tr}\left(\phi^{(j}\phi^{a)}\psi_{a}\right) - \tfrac{1}{4}\delta^{j}_{k}\mathrm{tr}\left(\phi^{(i}\phi^{a)}\psi_{a}\right) \ , \\
&w^{i} \equiv \mathrm{tr}\left(f\phi^{i} + \tfrac{1}{2}\epsilon^{ia_1a_2}\psi_{a_1}\psi_{a_2}\right) \ , \ w^{ij} \equiv \ \mathrm{tr}\left(f\phi^{(i}\phi^{j)} + \epsilon^{a_1a_2(i}\phi^{j)}\psi_{a_1}\psi_{a_2}\right) \ ,
\end{aligned} \tag{A.1}$$

where we suppressed the subscript of u_n, v_n, w_n since it can be easily read off from the number of the indices. Note that the Q-actions on ϕ, ψ, f are given by

$$Q\phi^m = 0 \ , \quad Q\psi_m = -\frac{i}{2}\epsilon_{mnp}[\phi^n, \phi^p] \ , \quad Qf = -i[\phi^m, \psi_m] \ . \tag{A.2}$$

We would like to find the fundamental trace relations of (A.1).

It is helpful to start from the Gröbner basis for the trace relations. The Gröbner basis contains all fundamental trace relations. In general, the Gröbner basis also contains some non-fundamental trace relations. We shall obtain the fundamental trace relations from the Gröbner basis by induction.

At the lowest level of the trace relations, all of them are fundamental. Namely, every generator of the Gröbner basis at such level are the fundamental relations. For the $SU(3)$ theory, the lowest level is $\mathcal{J} = 10$. In order to organize them into covariant forms in the $SU(3)$ global symmetry, we use the following computational strategy (which also proves useful at higher orders). We list the polynomials of (A.1) which have the same representations as the lowest fundamental trace relations at $\mathcal{J} = 10$. Among them, we should find particular linear combinations which vanish when all off-diagonal elements of ϕ^m, ψ_m, f are turned off, since the graviton trace relations vanish with diagonal fields. Once such combinations are identified, keeping ϕ, ψ, f general in this combination will yield the Q-exact operators for the lowest fundamental trace relations. This way, we can find the fundamental trace relations at the lowest level.[1]

Now, suppose that we found all fundamental trace relations until the level $\mathcal{J}$. We can construct the fundamental trace relations at $\mathcal{J} + 2$ as follows. We first construct all non-fundamental trace relations at level $\mathcal{J} + 2$ by multiplying suitable graviton cohomologies to the fundamental ones below the level $\mathcal{J}$. Not all of them are linearly independent so we should extract a linearly independent set among them. This lets us to compute the $SU(3)$ character of the non-fundamental trace relations at level $\mathcal{J} + 2$. Next, we consider a union of the non-fundamental trace relations and the Gröbner bases at level $\mathcal{J} + 2$. Note that the Gröbner basis will contain all fundamental trace relations and some non-fundamental ones. We extract a linearly independent set among such union, which contains all fundamental and non-fundamental relations. We also compute the $SU(3)$ character over them. Finally, we subtract the former character from the latter, which yields the $SU(3)$ character of the fundamental trace relations at level $\mathcal{J}+2$. Then we list the multi-trace operators using (A.1) which can account for it as before. Among them, we find particular linear combinations which vanish when all off-diagonal elements of ϕ, ψ, f are turned off, and which are linearly independent from the non-fundamental trace relations we constructed above. The final results are the fundamental trace relations at level $\mathcal{J}+2$. In this way, one can construct the fundamental trace relations inductively.

[1] There can be linear combinations which vanish even when the off-diagonal elements are turned on. In principle, they can also be the trace relations but most of them are just the identities that hold at arbitrary N. In practice, we only find them as mesonic identities between (A.1) rather than the trace relations.

In principle, one can obtain all fundamental trace relations of gravitons from the above induction. For the $SU(2)$ theory, it can be easily done. We found a 66-dimensional Gröbner basis, and there exist 48 fundamental trace relations among them. However, for the $SU(3)$ theory, we could not do a similar calculation since the construction of the Gröbner basis is time-consuming. We constructed it only in two subsectors: (1) all trace relations between u_2, u_3, v_2, v_3, and (2) trace relations between $u_2, u_3, v_2, v_3, w_2, w_3$ until $\mathcal{J} \leq 20$. From the subsector (1), which has 1170 generators, we obtained all fundamental trace relations between u_2, u_3, v_2, v_3, i.e. the relations which do not involve f's. There are in total 287 relations whose lowest level is $\mathcal{J} = 10$ and the highest level is $\mathcal{J} = 30$. On the other hand, from the subsector (2), we could generate the fundamental trace relations involving f's until $\mathcal{J} \leq 20$. There are in total 130 relations involving f's between $14 \leq \mathcal{J} \leq 20$. These are enough to construct relations of relations at $\mathcal{J} = 24$.

Before presenting their explicit forms, we first explain our notation. When we write down certain operator in the irreducible representation $\mathbf{R}$ under $SU(3) \subset SU(4)_R$ as $O^{i_1 i_2 i_3 \cdots}_{j_1 j_2 j_3 \cdots}$, the actual form of such an operator should be understood as $O^{i_1 i_2 i_3 \cdots}_{j_1 j_2 j_3 \cdots}$ subtracted by its trace part to make it traceless, like

$$
\begin{aligned}
&[n,0] : O^{i_1 i_2 i_3 \cdots i_n} \to O^{i_1 i_2 i_3 \cdots i_n} \ , \qquad [0,n] : O_{i_1 i_2 i_3 \cdots i_n} \to O_{i_1 i_2 i_3 \cdots i_n} \ , \\
&[1,1] : O^i_j \to O^i_j - \tfrac{1}{3}\delta^i_j O^a_a \ , \\
&[2,1] : O^{ij}_k \to O^{ij}_k - \tfrac{1}{2}\delta^{(i}_k O^{j)a}_a \ , \qquad [1,2] : O^i_{jk} \to O^i_{jk} - \tfrac{1}{2}\delta^i_{(j} O^a_{k)a} \ , \\
&[3,1] : O^{ijk}_l \to O^{ijk}_l - \tfrac{3}{5}\delta^{(i}_l O^{jk)a}_a \ , \qquad [1,3] : O^i_{jkl} \to O^i_{jkl} - \tfrac{3}{5}\delta^i_{(j} O^a_{kl)a} \ , \\
&[2,2] : O^{ij}_{kl} \to O^{ij}_{kl} - \tfrac{4}{5}\delta^{(i}_{(k} O^{j)a}_{l)a} + \tfrac{1}{10}\delta^{(i}_{(k}\delta^{j)}_{l)} O^{a_1 a_2}_{a_1 a_2} \ ,
\end{aligned}
\tag{A.3}
$$

and so on. Here, $[\cdot,\cdot]$ are the Dynkin labels for $SU(3)$.

Below, we list the explicit forms of the fundamental trace relations according to their level $\mathcal{J}$ and representation under $SU(3) \subset SU(4)_R$ as $t^{\mathcal{J}}[R'_1, R'_2]$. The relations which do not involve f's are given as follows:

$$
\begin{aligned}
&t^{10}[1,2](u_2 u_3) : (R^{(0,0)}_{10})^i_{jk} = \epsilon_{a_1 a_2 (j}\epsilon_{k) b_1 b_2} u^{a_1 b_1} u^{i a_2 b_2} \\
&t^{12}[0,0](u_2 u_2 u_2) : R^{(0,0)}_{12} = \epsilon_{a_1 a_2 a_3}\epsilon_{b_1 b_2 b_3} u^{a_1 b_1} u^{a_2 b_2} u^{a_3 b_3} \\
&t^{12}[2,2](u_2 u_2 u_2, u_3 u_3) : (R^{(0,0)}_{12})^{ij}_{kl} = \epsilon_{a_1 a_2 (k}\epsilon_{l) b_1 b_2} \left(u^{a_1 b_1} u^{a_2 b_2} u^{ij} + 6 u^{a_1 b_1 (i} u^{j) a_2 b_2}\right) \\
&t^{12}[0,3](u_2 v_3) : (R^{(0,1)}_{12})_{ijk} = \epsilon_{(i|a_1 a_2}\epsilon_{|j|b_1 b_2} u^{a_1 b_1} v^{a_2 b_2}{}_{|k)} \\
&t^{12}[1,1](u_2 v_3, u_3 v_2) : (R^{(0,1)}_{12})^i_j = \epsilon_{j a_1 a_2}\left(4 u^{a_1 b} v^{i a_2}{}_b + 3 u^{i a_1 b} v^{a_2}{}_b\right) \\
&t^{12}[2,2](u_2 v_3, u_3 v_2) : (R^{(0,1)}_{12})^{ij}_{kl} = \epsilon_{a_1 a_2 (k}\left(u^{a_1 (i} v^{j) a_2}{}_{l)} + u^{ij a_1} v^{a_2}{}_{l)}\right) \\
&t^{14}[1,0](u_2 u_2 v_2) : (R^{(0,1)}_{14})^i = \epsilon_{a_1 a_2 a_3} u^{i a_1} u^{b a_2} v^{a_3}{}_b
\end{aligned}
$$

$$
\begin{aligned}
&t^{14}[0,2](u_2u_2v_2,u_3v_3):(R^{(0,1)}_{14})_{ij}=\epsilon_{a_1a_2(i|}\left(\epsilon_{b_1b_2b_3}u^{a_1b_1}u^{a_2b_2}v^{b_3}{}_{|j)}-2\epsilon_{|j)b_1b_2}u^{a_1b_1c}v^{a_2b_2}{}_c\right)\\
&t^{14}[2,1](u_2u_2v_2,u_3v_3):(R^{(0,1)}_{14})^{ij}_k=\epsilon_{ka_1a_2}\left(3u^{(a_1b}u^{ij)}v^{a_2}{}_b+4u^{a_1b}u^{a_2(i}v^{j)}{}_b+24u^{a_1b(i}v^{j)a_2}{}_b\right)\\
&t^{14}[1,3](u_2u_2v_2,u_3v_3):(R^{(0,1)}_{14})^i_{jkl}=\epsilon_{(j|a_1a_2}\epsilon_{|k|b_1b_2}\left(u^{a_1b_1}u^{a_2b_2}v^i{}_{|l)}+6u^{ia_1b_1}v^{a_2b_2}{}_{|l)}\right)\\
&t^{14}[3,2](u_2u_2v_2,u_3v_3):(R^{(0,1)}_{14})^{ijk}_{lm}=\epsilon_{a_1a_2(l}\left(u^{(a_1i}u^{jk)}v^{a_2}{}_{m)}+6u^{a_1(ij}v^{k)a_2}{}_{m)}\right)\\
&t^{14}[1,3](v_2v_3):(R^{(0,2)}_{14})^i_{jkl}=\epsilon_{a_1a_2(j}v^{a_1}{}_kv^{ia_2}{}_{l)}\\
&t^{16}[0,1](u_2v_2v_2,v_3v_3):(R^{(0,2)}_{16})_i=\epsilon_{ia_1a_2}\left(12u^{bc}v^{a_1}{}_bv^{a_2}{}_c+13u^{a_1b}v^{a_2}{}_cv^c{}_b+12v^{a_1b}{}_cv^{a_2c}{}_b\right)\\
&t^{16}[1,2](u_2v_2v_2,v_3v_3):(R^{(0,2)}_{16})^i_{jk}=\epsilon_{a_1a_2(j}\Big(3u^{ib}v^{a_1}{}_{k)}v^{a_2}{}_b-7u^{ia_1}v^b{}_{k)}v^{a_2}{}_b\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+6u^{a_1b}v^i{}_{k)}v^{a_2}{}_b+24v^{a_1b}{}_{k)}v^{ia_2}{}_b\Big)\\
&t^{16}[2,3](u_2v_2v_2,v_3v_3):(R^{(0,2)}_{16})^{ij}_{klm}=\epsilon_{a_1a_2(k}\left(u^{a_1(i}v^{j)}{}_lv^{a_2}{}_{m)}+3v^{a_1(i}{}_lv^{j)a_2}{}_{m)}\right)\\
&t^{18}[0,0](u_3v_2v_2):R^{(0,2)}_{18}=\epsilon_{a_1a_2a_3}u^{a_1bc}v^{a_2}{}_bv^{a_3}{}_c\\
&t^{20}[1,0](v_2v_2v_3):(R^{(0,3)}_{20})^i=2v^a{}_cv^b{}_av^{ic}{}_b-3v^i{}_av^c{}_bv^{ab}{}_c\\
&t^{22}[2,0](u_2v_2v_2v_2):(R^{(0,3)}_{22})^{ij}=u^{ij}v^a{}_bv^b{}_cv^c{}_a-3u^{a(i}v^{j)}{}_bv^b{}_cv^c{}_a+3u^{ab}v^{(i}{}_av^{j)}{}_cv^c{}_b\\
&t^{24}[0,0](u_2v_2v_2v_3):R^{(0,3)}_{24}=\epsilon_{a_1a_2a_3}u^{a_1b}v^{a_2}{}_bv^{a_3c}{}_dv^d{}_c\\
&t^{26}[1,0](v_2v_2v_2v_3):(R^{(0,4)}_{26})^i=v^i{}_av^a{}_bv^d{}_cv^{bc}{}_d\\
&t^{30}[0,0](v_2v_2v_2v_2v_2):R^{(0,5)}_{30}=v^a{}_bv^b{}_cv^c{}_dv^d{}_ev^e{}_a\\
&t^{30}[3,0](v_2v_2v_2v_2v_2):(R^{(0,5)}_{30})^{ijk}=\epsilon^{a_1a_2(i}v^j{}_{a_1}v^{k)}{}_{a_2}v^b{}_cv^c{}_dv^d{}_b\ .
\end{aligned}
\tag{A.4}
$$

Here, the superscripts of R denote (n_f,n_ψ) of the terms with maximal n_f in the trace relations and the subscripts denote their $\mathcal{J}$. Their $SU(3)$ representations can be read off from the number of upper and lower indices. The listed trace relations vanish up to Q-exact operators whose explicit form will be discussed below. As explained before, this is the exhaustive set of the fundamental trace relations of gravitons which do not involve f's. The fundamental trace relations involving f's until $\mathcal{J}=20$ are given by

$$
\begin{aligned}
&t^{14}[0,2](v_2v_3,u_2w_3):(R^{(1,0)}_{14})_{ij}=\epsilon_{a_1a_2(i}\left(8v^{a_1b}{}_{j)}v^{a_2}{}_b+5\epsilon_{j)b_1b_2}u^{a_1b_1}w^{a_2b_2}\right)\\
&t^{14}[2,1](v_2v_3,u_2w_3,u_3w_2):\\
&\quad(R^{(1,0)}_{14})^{ij}_k=2v^{(i}{}_av^{j)a}{}_k-5v^a{}_kv^{ij}{}_a+3\epsilon_{ka_1a_2}u^{a_1(i}w^{j)a_2}+3\epsilon_{ka_1a_2}u^{ija_1}w^{a_2}\\
&t^{16}[0,1](v_3v_3,u_2v_2v_2,u_2u_2w_2):\\
&\quad(R^{(1,0)}_{16})_i=\epsilon_{ia_1a_2}\left(48v^{a_1b_1}{}_{b_2}v^{a_2b_2}{}_{b_1}+9u^{b_1b_2}v^{a_1}{}_{b_1}v^{a_2}{}_{b_2}-13\epsilon_{b_1b_2b_3}u^{a_1b_1}u^{a_2b_2}w^{b_3}\right)\\
&t^{16}[1,2](v_3v_3,u_2v_2v_2,u_3w_3,u_2u_2w_2):\\
&\quad(R^{(1,0)}_{16})^i_{jk}=\epsilon_{a_1a_2(j|}\Big(24v^{ia_1}{}_bv^{ba_2}{}_{|k)}+2u^{ia_1}v^{a_2}{}_bv^b{}_{|k)}-6u^{a_1b}v^{a_2}{}_bv^i{}_{|k)}\\
&\qquad\qquad+6\epsilon_{|k)b_1b_2}u^{ia_1b_1}w^{a_2b_2}+\epsilon_{|k)b_1b_2}u^{a_1b_1}u^{a_2b_2}w^i\Big)
\end{aligned}
$$

$$
\begin{aligned}
&t^{16}[3,1](v_3v_3, u_2v_2v_2, u_3w_3, u_2u_2w_2):\\
&\quad (R_{16}^{(1,0)})_l^{ijk} = 24v^{(ij}{}_a v^{k)a}{}_l + 7u^{(ij}v^{k)}{}_a v^a{}_l - 6u^{a(i}v^j{}_a v^{k)}{}_l\\
&\qquad\qquad + 18\epsilon_{la_1a_2}u^{a_1(ij}w^{k)a_2} + 3\epsilon_{la_1a_2}u^{(ij}u^{k)a_1}w^{a_2}\\
&t^{16}[1,2](v_2w_3, v_3w_2) : (R_{16}^{(1,1)})^i_{jk} = \epsilon_{a_1a_2(j}\left(v^{a_1}{}_{k)}w^{a_2i} + v^{a_1i}{}_{k)}w^{a_2}\right)\\
&t^{18}[0,0](v_2v_2v_2, u_2v_2w_2) : R_{18}^{(1,1)} = v^{a_1}{}_{a_2}v^{a_2}{}_{a_3}v^{a_3}{}_{a_1} - 3\epsilon_{a_1a_2a_3}u^{a_1b}v^{a_2}{}_b w^{a_3}\\
&t^{18}[1,1](v_2v_2v_2, v_3w_3, u_2v_2w_2):\\
&\quad (R_{18}^{(1,1)})^i_j = 9v^i{}_{a_1}v^{a_1}{}_{a_2}v^{a_2}{}_j - 24\epsilon_{ja_1a_2}v^{ia_1}{}_b w^{ba_2}\\
&\qquad\qquad - 13\epsilon_{ja_1a_2}u^{ia_1}v^{a_2}{}_b w^b - 16\epsilon_{ja_1a_2}u^{ib}v^{a_1}{}_b w^{a_2} + 5\epsilon_{ja_1a_2}u^{a_1b}v^i{}_b w^{a_2}\\
&t^{18}[0,3](v_2v_2v_2, v_3w_3, u_2v_2w_2):\\
&\quad (R_{18}^{(1,1)})_{ijk} = \epsilon_{a_1a_2(i|}\big(3v^{a_1}{}_{|j|}v^{a_2}{}_b v^b{}_{|k)} - 3\epsilon_{b_1b_2|j}v^{a_1b_1}{}_{k)}w^{a_2b_2} - \epsilon_{b_1b_2|j}u^{a_1b_1}v^{a_2}{}_{k)}w^{b_2}\big)\\
&t^{18}[2,2](v_2v_2v_2, v_3w_3, u_2v_2w_2):\\
&\quad (R_{18}^{(1,1)})^{kl}_{ij} = 2v^{(i}{}_a v^{j)}{}_{(k}v^a{}_{l)} - 6\epsilon_{a_1a_2(k}v^{a_1(i}{}_{l)}w^{j)a_2} - \epsilon_{a_1a_2(k}u^{ij}v^{a_1}{}_{l)}w^{a_2}\\
&t^{20}[0,2](v_2v_2w_2, u_2w_2w_2, w_3w_3):\\
&\quad (R_{20}^{(2,0)})_{ij} = 2\epsilon_{a_1a_2(i|}v^{a_1}{}_b v^b{}_{|j)}w^{a_2} - 3\epsilon_{a_1a_2a_3}v^{a_1}{}_i v^{a_2}{}_j w^{a_3}\\
&\qquad\qquad + \epsilon_{ia_1a_2}\epsilon_{jb_1b_2}u^{a_1b_1}w^{a_2}w^{b_2} + 3\epsilon_{ia_1a_2}\epsilon_{jb_1b_2}w^{a_1b_1}w^{a_2b_2}\ .
\end{aligned}
\tag{A.5}
$$

The relations involving one f appear from $\mathcal{J} = 14$ and those involving two f's appear from $\mathcal{J} = 20$. We do not find any relations involving three f's until $\mathcal{J} = 20$.

As explained before, the trace relations (A.4), (A.5) vanish up to Q-exact operators, which we now construct explicitly. In principle, one should first construct the complete basis of the Q-exact operators, which have the same level $\mathcal{J}$ and $SU(3)$ representation with the target trace relation. (The Q-action does not change $\mathcal{J}$ and $SU(3)$ representation.) However, in practice, we can make some ansätze for the Q-exact form to reduce the dimension of Q-exact basis. One of our working assumptions is that the maximal number of f's appearing before the Q-action is the same as that of the trace relation. There is a priori no reason to assume that but it turns out to be true for our examples. After imposing this assumption (and a couple of extra practical assumptions), we find a particular linear combination of the Q-exact operators in our basis for the target trace relation. In general, when we write $R_I \sim Qr_I$ for a trace relation R_I, there exist ambiguities of r_I since we can add arbitrary Q-closed operators to r_I. We partly fix them by requiring r_I to vanish when ϕ, ψ, f are restricted to diagonal matrices. We do not know whether such a requirement can be satisfied in general, but it does work for our examples. The purpose of this requirement will be explained later. The other ambiguities are fixed by hand to get compact expressions.

Below, we list the operators $r_j^{(n_f,n_\psi)}$ related to the fundamental trace relations $R_j^{(n_f,n_\psi-1)}$

by $i\,Qr_j^{(n_f,n_\psi)} = R_j^{(n_f,n_\psi-1)}$. We will not list all $r_j^{(n_f,n_\psi)}$'s, but only those which are used in section 7.3. For the relations without f's in (A.4), we obtain

$$
\begin{aligned}
(r_{10}^{(0,1)})^i_{jk} &= -2\,\epsilon_{a_1a_2(j}\,\mathrm{tr}\left(\phi^{a_1}\phi^{a_2}\phi^i\psi_{k)}\right) \ , \\
r_{12}^{(0,1)} &= \epsilon_{a_1a_2a_3}\left[6\,\mathrm{tr}\left(\psi_b\phi^{a_1}\right)\mathrm{tr}\left(\phi^b\phi^{a_2}\phi^{a_3}\right) - \mathrm{tr}\left(\psi_b\phi^{a_1}\phi^{a_2}\right)\mathrm{tr}\left(\phi^b\phi^{a_3}\right)\right] \\
&\quad - 3\epsilon_{a_1a_2a_3}\Big[\,\mathrm{tr}\left(\psi_b\phi^b\phi^{a_1}\phi^{a_2}\phi^{a_3}\right) + \mathrm{tr}\left(\psi_b\phi^{a_1}\phi^b\phi^{a_2}\phi^{a_3}\right) \\
&\qquad + \mathrm{tr}\left(\psi_b\phi^{a_1}\phi^{a_2}\phi^b\phi^{a_3}\right) + \mathrm{tr}\left(\psi_b\phi^{a_1}\phi^{a_2}\phi^{a_3}\phi^b\right)\,\Big] \ , \\
(r_{12}^{(0,1)})^{ij}_{kl} &= -2\epsilon_{a_1a_2(k}\left[\mathrm{tr}\left(\psi_{l)}\phi^{(i}\phi^{j)}\phi^{a_1}\phi^{a_2}\right) + 7\,\mathrm{tr}\left(\psi_{l)}\phi^{(i|}\phi^{a_1}\phi^{|j)}\phi^{a_2}\right)\right] \ , \\
(r_{12}^{(0,2)})_{ijk} &= \frac{1}{2}\epsilon_{a_1a_2(i}\,\mathrm{tr}\left(\phi^{a_1}\psi_j\phi^{a_2}\psi_{k)}\right) \ , \\
(r_{12}^{(0,2)})^i_j &= 6\,\mathrm{tr}\left(\phi^{(i}\phi^{a)}\psi_{(a}\psi_{j)}\right) - 5\,\mathrm{tr}\left(\phi^{[i}\psi_a\phi^{a]}\psi_j\right) \ , \\
(r_{12}^{(0,2)})^{ij}_{kl} &= \mathrm{tr}\left(\phi^{(i}\phi^{j)}\psi_{(k}\psi_{l)}\right) \ , \\
(r_{14}^{(0,2)})^i &= 3\ \mathrm{tr}\left(\phi^i\psi_{a_1}\phi^{a_1}\phi^{a_2}\psi_{a_2}\right) + 2\ \mathrm{tr}\left(\phi^i\phi^{a_1}\right)\mathrm{tr}\left(\phi^{a_2}\psi_{(a_1}\psi_{a_2)}\right) \\
&\qquad - 6\ \mathrm{tr}\left(\phi^i\psi_{a_1}\right)\mathrm{tr}\left(\phi^{[a_1}\phi^{a_2]}\psi_{a_2}\right) - \ \mathrm{tr}\left(\phi^i\psi_{a_1}\psi_{a_2}\right)\mathrm{tr}\left(\phi^{a_1}\phi^{a_2}\right) \ , \\
(r_{14}^{(0,2)})_{ij} &= \frac{5}{9}\epsilon_{a_1a_2a_3}\left[2\,\mathrm{tr}\left(\psi_{(i}\psi_{j)}\phi^{a_1}\phi^{a_2}\phi^{a_3}\right) + \mathrm{tr}\left(\psi_{(i}\phi^{a_1}\psi_{j)}\phi^{a_2}\phi^{a_3}\right)\right] \\
&\quad + \epsilon_{a_1a_2(i}\left[\mathrm{tr}\left(\psi_{j)}\psi_{a_3}\phi^{a_1}\phi^{a_2}\phi^{a_3}\right) + \mathrm{tr}\left(\psi_{j)}\psi_{a_3}\phi^{a_1}\phi^{a_3}\phi^{a_2}\right) + \mathrm{tr}\left(\psi_{j)}\psi_{a_3}\phi^{a_3}\phi^{a_1}\phi^{a_2}\right)\right] \\
&\quad + \epsilon_{a_1a_2(i}\left[\mathrm{tr}\left(\psi_{j)}\phi^{a_1}\psi_{a_3}\phi^{a_2}\phi^{a_3}\right) + \mathrm{tr}\left(\psi_{j)}\phi^{a_1}\psi_{a_3}\phi^{a_3}\phi^{a_2}\right) + \mathrm{tr}\left(\psi_{j)}\phi^{a_3}\psi_{a_3}\phi^{a_1}\phi^{a_2}\right)\right] \\
&\quad + \epsilon_{a_1a_2(i}\left[\mathrm{tr}\left(\psi_{j)}\phi^{a_1}\phi^{a_2}\psi_{a_3}\phi^{a_3}\right) + \mathrm{tr}\left(\psi_{j)}\phi^{a_1}\phi^{a_3}\psi_{a_3}\phi^{a_2}\right) + \mathrm{tr}\left(\psi_{j)}\phi^{a_3}\phi^{a_1}\psi_{a_3}\phi^{a_2}\right)\right] \\
&\quad + \epsilon_{a_1a_2(i}\left[\mathrm{tr}\left(\psi_{j)}\phi^{a_1}\phi^{a_2}\phi^{a_3}\psi_{a_3}\right) + \mathrm{tr}\left(\psi_{j)}\phi^{a_1}\phi^{a_3}\phi^{a_2}\psi_{a_3}\right) + \mathrm{tr}\left(\psi_{j)}\phi^{a_3}\phi^{a_1}\phi^{a_2}\psi_{a_3}\right)\right] \\
&\quad - \frac{1}{3}\epsilon_{a_1a_2(i}\Big[5\,\mathrm{tr}\left(\psi_{j)}\phi^{a_1}\phi^{a_2}\right)\mathrm{tr}\left(\psi_{a_3}\phi^{a_3}\right) + 2\,\mathrm{tr}\left(\psi_{j)}\phi^{(a_1}\phi^{a_3)}\right)\mathrm{tr}\left(\psi_{a_3}\phi^{a_2}\right) \\
&\qquad - 2\,\mathrm{tr}\left(\psi_{j)}\phi^{a_2}\right)\mathrm{tr}\left(\psi_{a_3}\phi^{(a_1}\phi^{a_3)}\right)\,\Big] \ , \\
(r_{14}^{(0,2)})^{ij}_k &= 12\,\mathrm{tr}\left(\phi^{(i}\phi^a\phi^{j)}\psi_{(a}\psi_{k)}\right) + 12\,\mathrm{tr}\left(\phi^{(i|}\phi^a\phi^{|j)}\psi_{(a}\psi_{k)}\right) \\
&\qquad + 54\,\mathrm{tr}\left(\phi^{(i}\phi^j\psi_{(a}\phi^{a)}\psi_{k)}\right) - 36\,\mathrm{tr}\left(\phi^{(i}\phi^{j)}\psi_{(a}\phi^a\psi_{k)}\right) \ , \\
(r_{14}^{(0,2)})^i_{jkl} &= 2\epsilon_{a_1a_2(j}\left[\mathrm{tr}\left(\phi^i\phi^{a_1}\phi^{a_2}\psi_k\psi_{l)}\right) + 3\,\mathrm{tr}\left(\phi^i\phi^{a_1}\psi_k\phi^{a_2}\psi_{l)}\right) - 2\,\mathrm{tr}\left(\phi^i\psi_k\phi^{a_1}\phi^{a_2}\psi_{l)}\right)\right] \ , \\
(r_{14}^{(0,3)})^i_{jkl} &= -\frac{1}{2}\,\mathrm{tr}\left(\phi^i\psi_{(j}\psi_k\psi_{l)}\right) \ , \\
(r_{16}^{(0,3)})_i &= \frac{39}{4}\,\mathrm{tr}\left(\psi_i\{\psi_{b_1}\psi_{b_2},\phi^{b_1}\phi^{b_2}\}\right) + 2\,\mathrm{tr}\left(\psi_i\psi_{b_1}\phi^{b_1}\psi_{b_2}\phi^{b_2}\right) - \frac{61}{4}\,\mathrm{tr}\left(\psi_i\psi_{b_1}\phi^{b_2}\psi_{b_2}\phi^{b_1}\right) \\
&\quad + \frac{97}{4}\,\mathrm{tr}\left(\psi_i\phi^{b_1}\psi_{b_1}\psi_{b_2}\phi^{b_2}\right) - \frac{41}{4}\,\mathrm{tr}\left(\psi_i\phi^{b_2}\psi_{b_1}\psi_{b_2}\phi^{b_1}\right) - 5\,\mathrm{tr}\left(\psi_i\psi_{b_1}\phi^{b_1}\phi^{b_2}\psi_{b_2}\right) \\
&\quad - \frac{25}{2}\,\mathrm{tr}\left(\psi_i\psi_{b_1}\phi^{b_2}\phi^{b_1}\psi_{b_2}\right) + 2\,\mathrm{tr}\left(\psi_i\phi^{b_1}\psi_{b_1}\phi^{b_2}\psi_{b_2}\right) - \frac{61}{4}\,\mathrm{tr}\left(\psi_i\phi^{b_2}\psi_{b_1}\phi^{b_1}\psi_{b_2}\right) \\
&\quad - \frac{11}{4}\,\mathrm{tr}\left(\phi^{b_1}\phi^{b_2}\right)\mathrm{tr}\left(\psi_i\psi_{b_1}\psi_{b_2}\right) - \frac{27}{2}\,\mathrm{tr}\left(\psi_{b_1}\psi_{b_2}\right)\mathrm{tr}\left(\psi_i\phi^{b_1}\phi^{b_2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{29}{4}\operatorname{tr}\left(\phi^{b_2}\psi_{b_2}\right)\operatorname{tr}\left(\psi_i[\psi_{b_1},\phi^{b_1}]\right) \quad ,\\
(r_{16}^{(0,3)})^i_{jk} = {}& 2\operatorname{tr}\left(\psi_{(j}\psi_{k)}\psi_b\phi^b\phi^i\right) - 4\operatorname{tr}\left(\psi_{(j}\psi_{k)}\psi_b\phi^i\phi^b\right) - \operatorname{tr}\left(\psi_{(j|}\psi_b\psi_{|k)}\{\phi^b,\phi^i\}\right)\\
&- 4\operatorname{tr}\left(\psi_{(j}\psi_{k)}\phi^{(b}\psi_b\phi^{i)}\right) + 7\operatorname{tr}\left(\psi_{(j|}\{\psi_b,\phi^b\}\psi_{|k)}\phi^i\right) - 11\operatorname{tr}\left(\psi_{(j|}\{\psi_b,\phi^i\}\psi_{|k)}\phi^b\right)\\
&- 4\operatorname{tr}\left(\psi_{(j}\psi_{k)}\phi^b\phi^i\psi_b\right) + 2\operatorname{tr}\left(\psi_{(j}\psi_{k)}\phi^i\phi^b\psi_b\right)\\
&+ 3\operatorname{tr}\left(\psi_{(j|}\psi_b\right)\operatorname{tr}\left(\psi_{|k)}[\phi^b,\phi^i]\right) + 6\operatorname{tr}\left(\psi_{(j}\phi^{[b}\right)\operatorname{tr}\left(\{\psi_{k)},\psi_b\}\phi^{i]}\right) \quad . \qquad \text{(A.6)}
\end{aligned}
$$

For the relations involving f's in (A.5), we find

$$
\begin{aligned}
(r_{14}^{(1,1)})_{ij} ={}& 5\epsilon_{a_1a_2(i}\operatorname{tr}\left(f\phi^{a_1}\psi_{j)}\phi^{a_2}\right) + \operatorname{tr}\left(\phi^a\left\{\psi_a,\psi_{(i}\psi_{j)}\right\}\right) - 4\operatorname{tr}\left(\phi^a\psi_{(i|}\psi_a\psi_{|j)}\right) \quad ,\\
(r_{14}^{(1,1)})^{ij}_k ={}& 3\operatorname{tr}\left(f\phi^{(i}\phi^{j)}\psi_k\right) - 3\operatorname{tr}\left(f\psi_k\phi^{(i}\phi^{j)}\right)\\
&+ \epsilon^{a_1a_2(i}\operatorname{tr}\left(\phi^{j)}\psi_k\psi_{a_1}\psi_{a_2}\right) - \epsilon^{a_1a_2(i}\operatorname{tr}\left(\phi^{j)}\psi_{a_1}\psi_{a_2}\psi_k\right) \quad ,\\
(r_{16}^{(1,1)})_i ={}& 13\epsilon_{a_1a_2a_3}\operatorname{tr}\left(f\psi_i\right)\operatorname{tr}\left(\phi^{a_1}\phi^{a_2}\phi^{a_3}\right) + \frac{10}{3}\epsilon_{a_1a_2a_3}\operatorname{tr}\left(f\phi^{a_1}\right)\operatorname{tr}\left(\psi_i\phi^{a_2}\phi^{a_3}\right)\\
&+ \frac{10}{3}\epsilon_{a_1a_2a_3}\operatorname{tr}\left(f\phi^{a_1}\phi^{a_2}\right)\operatorname{tr}\left(\psi_i\phi^{a_3}\right) + 46\epsilon_{ia_1a_2}\operatorname{tr}\left(f\phi^b\right)\operatorname{tr}\left(\psi_b\phi^{a_1}\phi^{a_2}\right)\\
&- 7\epsilon_{ia_1a_2}\operatorname{tr}\left(f\phi^{a_1}\right)\operatorname{tr}\left(\psi_b\phi^{a_2}\phi^b\right) - 7\epsilon_{ia_1a_2}\operatorname{tr}\left(f\phi^b\phi^{a_1}\right)\operatorname{tr}\left(\psi_b\phi^{a_2}\right)\\
&+ 6\epsilon_{ia_1a_2}\operatorname{tr}\left(f\phi^{a_1}\phi^{a_2}\right)\operatorname{tr}\left(\psi_b\phi^b\right) - \frac{115}{3}\epsilon_{a_1a_2a_3}\operatorname{tr}\left(f\psi_i\phi^{a_1}\phi^{a_2}\phi^{a_3}\right)\\
&- \frac{95}{3}\epsilon_{a_1a_2a_3}\operatorname{tr}\left(f\phi^{a_1}\psi_i\phi^{a_2}\phi^{a_3}\right) + 5\epsilon_{a_1a_2a_3}\operatorname{tr}\left(f\phi^{a_1}\phi^{a_2}\psi_i\phi^{a_3}\right)\\
&+ 36\epsilon_{ia_1a_2}\operatorname{tr}\left(f\psi_b\phi^{a_1}\phi^{a_2}\phi^b\right) - 43\epsilon_{ia_1a_2}\operatorname{tr}\left(f\psi_b\phi^{a_1}\phi^b\phi^{a_2}\right)\\
&+ 39\epsilon_{ia_1a_2}\operatorname{tr}\left(f\phi^{a_1}\psi_b\phi^{a_2}\phi^b\right) - 68\epsilon_{ia_1a_2}\operatorname{tr}\left(f\phi^{a_1}\phi^{a_2}\psi_b\phi^b\right)\\
&+ 39\epsilon_{ia_1a_2}\operatorname{tr}\left(f\phi^{a_1}\phi^b\psi_b\phi^{a_2}\right) + 13\operatorname{tr}\left(\psi_i\{\psi_{b_1}\psi_{b_2},\phi^{b_1}\phi^{b_2}\}\right)\\
&- 31\operatorname{tr}\left(\psi_i\{\psi_{b_1}\psi_{b_2},\phi^{b_2}\phi^{b_1}\}\right) + 14\operatorname{tr}\left(\psi_i\psi_{b_1}\phi^{b_1}\psi_{b_2}\phi^{b_2}\right)\\
&- 22\operatorname{tr}\left(\psi_i\psi_{b_1}\phi^{b_2}\phi^{b_1}\psi_{b_2}\right) + 14\operatorname{tr}\left(\psi_i\phi^{b_1}\psi_{b_1}\phi^{b_2}\psi_{b_2}\right) \quad ,\\
(r_{16}^{(1,1)})^i_{jk} ={}& \epsilon_{a_1a_2(j}\left[-4\operatorname{tr}\left(f\phi^i\right)\operatorname{tr}\left(\psi_{k)}\phi^{a_1}\phi^{a_2}\right) - \operatorname{tr}\left(\phi^i\phi^{a_2}\right)\operatorname{tr}\left(f[\psi_{k)},\phi^{a_1}]\right)\right]\\
&+ \epsilon_{a_1a_2(j}\Big[3\operatorname{tr}\left(f\phi^{a_1}\{\psi_{k)},\phi^i\}\phi^{a_2}\right) + 5\operatorname{tr}\left(f\{\psi_{k)},\phi^{a_1}\phi^i\phi^{a_2}\}\right)\\
&\qquad - 4\operatorname{tr}\left(f\psi_{k)}\phi^i\phi^{a_1}\phi^{a_2}\right) - 4\operatorname{tr}\left(f\phi^{a_1}\phi^{a_2}\phi^i\psi_{k)}\right)\Big]\\
&+ 2\operatorname{tr}\left(\psi_{(j}\psi_{k)}\psi_b[\phi^b,\phi^i]\right) - 3\operatorname{tr}\left(\psi_{(j|}\psi_b\psi_{|k)}\{\phi^b,\phi^i\}\right) + 6\operatorname{tr}\left(\psi_{(j|}\{\psi_b,\phi^b\}\psi_{|k)}\phi^i\right)\\
&- 9\operatorname{tr}\left(\psi_{(j|}\{\psi_b,\phi^i\}\psi_{|k)}\phi^b\right) - 2\operatorname{tr}\left(\psi_{(j}\psi_{k)}\left[\phi^b,\phi^i\right]\psi_b\right)\\
&+ \operatorname{tr}\left(\psi_{(j|}\psi_b\right)\operatorname{tr}\left(\psi_{|k)}[\phi^b,\phi^i]\right) + \operatorname{tr}\left(\psi_{(j|}\phi^b\right)\operatorname{tr}\left(\{\psi_{|k)},\psi_b\}\phi^i\right) \quad ,\\
(r_{16}^{(1,2)})^i_{jk} ={}& -\frac{1}{2}\operatorname{tr}\left(f\phi^i\psi_{(j}\psi_{k)}\right) - \frac{1}{2}\operatorname{tr}\left(f\psi_{(j}\phi^i\psi_{k)}\right)\\
&- \frac{1}{2}\operatorname{tr}\left(f\psi_{(j}\psi_{k)}\phi^i\right) - \frac{1}{4}\epsilon^{ia_1a_2}\operatorname{tr}\left(\psi_{a_1}\psi_{a_2}\psi_{(j}\psi_{k)}\right) \quad ,
\end{aligned}
$$

$$
\begin{aligned}
(r_{18}^{(1,2)})^i_j =& - 4\,\mathrm{tr}\left(f\phi^i\phi^a\right)\mathrm{tr}\left(\psi_j\psi_a\right) - 5\,\mathrm{tr}\left(f\phi^a\phi^i\right)\mathrm{tr}\left(\psi_j\psi_a\right) - \frac{53}{2}\,\mathrm{tr}\left(f\phi^i\psi_j\right)\mathrm{tr}\left(\phi^a\psi_a\right)\\
&+ 7\,\mathrm{tr}\left(f\phi^i\psi_a\right)\mathrm{tr}\left(\phi^a\psi_j\right) + \frac{15}{2}\,\mathrm{tr}\left(f\phi^a\psi_j\right)\mathrm{tr}\left(\phi^i\psi_a\right) + 12\,\mathrm{tr}\left(f\phi^a\psi_a\right)\mathrm{tr}\left(\phi^i\psi_j\right)\\
&+ 2\,\mathrm{tr}\left(f\psi_j\phi^i\right)\mathrm{tr}\left(\phi^a\psi_a\right) - 13\,\mathrm{tr}\left(f\psi_a\phi^i\right)\mathrm{tr}\left(\phi^a\psi_j\right) + 4\,\mathrm{tr}\left(f\psi_j\phi^a\right)\mathrm{tr}\left(\phi^i\psi_a\right)\\
&+ 6\,\mathrm{tr}\left(f\psi_j\psi_a\right)\mathrm{tr}\left(\phi^i\phi^a\right) + \frac{13}{2}\,\mathrm{tr}\left(f\psi_a\psi_j\right)\mathrm{tr}\left(\phi^i\phi^a\right) - 4\,\mathrm{tr}\left(f\phi^i\right)\mathrm{tr}\left(\phi^a\psi_j\psi_a\right)\\
&+ 14\,\mathrm{tr}\left(f\phi^i\right)\mathrm{tr}\left(\phi^a\psi_a\psi_j\right) - 8\,\mathrm{tr}\left(f\phi^a\right)\mathrm{tr}\left(\phi^i\psi_j\psi_a\right) - 8\,\mathrm{tr}\left(f\phi^a\right)\mathrm{tr}\left(\phi^i\psi_a\psi_j\right)\\
&- 4\,\mathrm{tr}\left(f\psi_j\right)\mathrm{tr}\left(\psi_a\phi^i\phi^a\right) - 9\,\mathrm{tr}\left(f\psi_a\right)\mathrm{tr}\left(\psi_j\phi^i\phi^a\right) + 6\,\mathrm{tr}\left(f\psi_a\right)\mathrm{tr}\left(\psi_j\phi^a\phi^i\right)\\
&+ 3\,\mathrm{tr}\left(f\phi^i\phi^a\psi_j\psi_a\right) - \frac{31}{2}\,\mathrm{tr}\left(f\phi^i\phi^a\psi_a\psi_j\right) + 3\,\mathrm{tr}\left(f\phi^a\phi^i\psi_j\psi_a\right)\\
&+ \frac{5}{2}\,\mathrm{tr}\left(f\phi^a\phi^i\psi_a\psi_j\right) + 12\,\mathrm{tr}\left(f\phi^i\psi_j\phi^a\psi_a\right) - \frac{13}{2}\,\mathrm{tr}\left(f\phi^i\psi_a\phi^a\psi_j\right)\\
&- 6\,\mathrm{tr}\left(f\phi^a\psi_j\phi^i\psi_a\right) - \frac{13}{2}\,\mathrm{tr}\left(f\phi^a\psi_a\phi^i\psi_j\right) + 18\,\mathrm{tr}\left(f\phi^i\psi_j\psi_a\phi^a\right)\\
&- 12\,\mathrm{tr}\left(f\psi_j\phi^i\phi^a\psi_a\right) + \frac{17}{2}\,\mathrm{tr}\left(f\psi_a\phi^i\phi^a\psi_j\right) - \frac{43}{2}\,\mathrm{tr}\left(f\psi_a\phi^a\phi^i\psi_j\right)\\
&+ \frac{1}{3}\epsilon^{a_1a_2a_3}\,\mathrm{tr}\left(\phi^i\psi_j\right)\mathrm{tr}\left(\psi_{a_1}\psi_{a_2}\psi_{a_3}\right) - 2\epsilon^{a_1a_2i}\,\mathrm{tr}\left(\phi^b\psi_{a_1}\right)\mathrm{tr}\left(\psi_b\psi_j\psi_{a_2}\right)\\
&- 10\epsilon^{a_1a_2a_3}\,\mathrm{tr}\left(\phi^i\psi_j\psi_{a_1}\psi_{a_2}\psi_{a_3}\right) + 8\epsilon^{a_1a_2a_3}\,\mathrm{tr}\left(\phi^i\psi_{a_1}\psi_j\psi_{a_2}\psi_{a_3}\right)\\
&- 2\epsilon^{a_1a_2a_3}\,\mathrm{tr}\left(\phi^i\psi_{a_1}\psi_{a_2}\psi_j\psi_{a_3}\right)\ ,\\
(r_{18}^{(1,2)})_{ijk} =& - \epsilon_{a_1a_2(i}\Big[\mathrm{tr}\left(f\phi^{a_1}\right)\mathrm{tr}\left(\phi^{a_2}\psi_j\psi_{k)}\right) - \frac{3}{2}\,\mathrm{tr}\left(f\psi_j\right)\mathrm{tr}\left(\psi_{k)}\phi^{a_1}\phi^{a_2}\right)\\
&\qquad + 3\,\mathrm{tr}\left(f\phi^{a_1}\psi_j\phi^{a_2}\psi_{k)}\right) - 3\,\mathrm{tr}\left(f\psi_j\phi^{a_1}\psi_{k)}\phi^{a_2}\right)\Big]\\
&- \frac{1}{2}\,\mathrm{tr}\left(\phi^a\psi_a\right)\mathrm{tr}\left(\psi_{(i}\psi_j\psi_{k)}\right) + \frac{3}{2}\,\mathrm{tr}\left(\phi^a\psi_{(i|}\right)\mathrm{tr}\left(\psi_a\psi_{|j}\psi_{k)}\right)\\
&+ \frac{1}{2}\,\mathrm{tr}\left(\phi^a\psi_{(i}\psi_{j|}\right)\mathrm{tr}\left(\psi_a\psi_{|k)}\right) + \frac{3}{2}\,\mathrm{tr}\left(\phi^a\psi_{(i|}\psi_a\psi_{|j}\psi_{k)}\right) - \frac{3}{2}\,\mathrm{tr}\left(\phi^a\psi_{(i}\psi_{j|}\psi_a\psi_{|k)}\right)\ ,\\
(r_{20}^{(2,1)})_{ij} =& - \epsilon_{a_1a_2(i}\Big[\mathrm{tr}\left(ff\right)\mathrm{tr}\left(\phi^{a_1}\phi^{a_2}\psi_{j)}\right) + \frac{1}{2}\,\mathrm{tr}\left(f\psi_{j)}\right)\mathrm{tr}\left(f\phi^{a_1}\phi^{a_2}\right)\\
&\qquad + 2\,\mathrm{tr}\left(f\phi^{a_1}\right)\mathrm{tr}\left(f[\phi^{a_2},\psi_{j)}]\right)\Big]\\
&+ \epsilon_{a_1a_2(i}\left[4\,\mathrm{tr}\left(ff\phi^{a_1}\phi^{a_2}\psi_{j)}\right) - \mathrm{tr}\left(f\phi^{a_1}\phi^{a_2}f\psi_{j)}\right)\right] + 2\,\mathrm{tr}\left(f\phi^a\psi_{(i}\right)\mathrm{tr}\left(\psi_{j)}\psi_a\right)\\
&- 4\,\mathrm{tr}\left(f\psi_{(i}\phi^a\right)\mathrm{tr}\left(\psi_{j)}\psi_a\right) - \frac{1}{2}\,\mathrm{tr}\left(f\psi_{(i}\right)\left(\phi^a\psi_{j)}\psi_a\right) - \frac{5}{2}\,\mathrm{tr}\left(f\psi_{(i|}\right)\left(\phi^a\psi_a\psi_{|j)}\right)\\
&+ 2\,\mathrm{tr}\left(f\psi_a\right)\left(\phi^a\psi_{(i}\psi_{j)}\right) - 4\,\mathrm{tr}\left(f\psi_{(i|}\psi_a\right)\left(\phi^a\psi_{|j)}\right) + 2\,\mathrm{tr}\left(f\phi^a\psi_{(i}[\psi_{j)},\psi_a]\right)\\
&+ 4\,\mathrm{tr}\left(f\phi^a\psi_a\psi_{(i}\psi_{j)}\right) + 4\,\mathrm{tr}\left(f\psi_{(i}\phi^a\psi_{j)}\psi_a\right) - 3\,\mathrm{tr}\left(f\psi_{(i|}\phi^a\psi_a\psi_{|j)}\right)\\
&- 2\,\mathrm{tr}\left(f\psi_a\phi^a\psi_{(i}\psi_{j)}\right) - \mathrm{tr}\left(f\psi_{(i|}\psi_a\phi^a\psi_{|j)}\right) + 4\,\mathrm{tr}\left(f\psi_a\psi_{(i}\phi^a\psi_{j)}\right)
\end{aligned}
$$

$$+\frac{2}{5}\epsilon^{a_1a_2a_3}\left[2\,\mathrm{tr}\left(\psi_{a_1}\psi_{a_2}\right)\mathrm{tr}\left(\psi_{a_3}\psi_{(i}\psi_{j)}\right)-3\,\mathrm{tr}\left(\psi_{(i|}\psi_{a_1}\psi_{|j)}\psi_{a_2}\psi_{a_3}\right)\right] \ . \tag{A.7}$$

Finally, we construct relations of these trace relations. Consider a linear combination of the trace relations with coefficients being the graviton cohomologies. If it vanishes identically, we call it a relation of relations. While the trace relations are identities that can be seen at the level of 'gluons' ϕ, ψ, f, the relations of relations are the identities of mesons $u_2, u_3, v_2, v_3, w_2, w_3$. We do not need to know how $u_2, u_3, v_2, v_3, w_2, w_3$ are made of ϕ, ψ, f to obtain the relations of relations. After constructing relations of relations, one can write them as the Q-action on certain operators using (A.6), (A.7). They are the Q-closed operators since their Q-actions vanish due to the relations of relations. This is the way we obtain the Q-closed operators in section 7.3.2. They can be either Q-exact or not and there is no trivial way to judge it easily. If they are not Q-exact, they are the non-graviton cohomologies since they are made of the linear combinations of r_I's, which vanish with diagonal ϕ, ψ, f. For the check of the (non-)Q-exactness, refer to section 7.3.3.

Now we will construct relations of relations at the threshold level $\mathcal{J} = 24$ which are singlets under $SU(3) \subset SU(4)_R$, from the trace relations (A.4), (A.5). There are 5 choices of (R, J) in this sector in which relations of relations exist.

i) $(R, J) = (2, 2)$**.** Let us first enumerate all $SU(3) \subset SU(4)_R$ singlets in this sector made by the product of the trace relations in (A.4), (A.5) and the graviton cohomologies. There are following 6 singlets:

$$\begin{aligned}
&s_1^{(2,0)} = u^{ij}(R_{20}^{(2,0)})_{ij} \ , \quad s_2^{(2,0)} = w^{ij}(R_{14}^{(1,0)})_{ij} \ , \quad s_3^{(2,0)} = w^{i}(R_{16}^{(1,0)})_{i} \ , \\
&s_1^{(1,2)} = v^{jk}{}_i(R_{16}^{(1,1)})^i_{jk} \ , \quad s_2^{(1,2)} = v^{j}{}_i(R_{18}^{(1,1)})^i_{j} \ , \quad s_3^{(1,2)} = w^{i}(R_{16}^{(0,2)})_{i} \ .
\end{aligned} \tag{A.8}$$

The superscripts denote (n_f, n_ψ) of the terms with maximal n_f in the operator, as before. There is one relation of these relations given by

$$i\,QO^{(2,1)} \equiv 65s_1^{(2,0)} - 39s_2^{(2,0)} + 5s_3^{(2,0)} - 312s_1^{(1,2)} - 26s_2^{(1,2)} + 6s_3^{(1,2)} = 0 \ . \tag{A.9}$$

This is the Q-action on the Q-closed operator (7.63).

ii) $(R,J)=(\frac{5}{2},\frac{3}{2})$**.** There exist 12 $SU(3)$ singlets in this sector given by

$$\begin{aligned}
&s_1^{(1,1)}=u^{a(i}v^{j)}{}_a(R_{14}^{(1,0)})_{ij}\ ,\quad s_2^{(1,1)}=\epsilon_{a_1a_2(i}u^{a_1k}v^{a_2}{}_{j)}(R_{14}^{(1,0)})^{ij}_k\ ,\quad s_3^{(1,1)}=v^{jk}{}_i(R_{16}^{(1,0)})^i_{jk}\ ,\\
&s_4^{(1,1)}=u^{ijk}(R_{18}^{(1,1)})_{ijk}\ ,\quad s_5^{(1,1)}=v^{(j}{}_iw^{k)}(R_{10}^{(0,0)})^i_{jk}\ ,\quad s_6^{(1,1)}=u^{(ij}w^{k)}(R_{12}^{(0,1)})_{ijk}\ ,\\
&s_7^{(1,1)}=\epsilon_{a_1a_2i}u^{a_1j}w^{a_2}(R_{12}^{(0,1)})^i_j\ ,\quad s_8^{(1,1)}=w^{ij}(R_{14}^{(0,1)})_{ij}\ ,\\
&s_1^{(0,3)}=\epsilon^{a_1a_2(i}v^j{}_{a_1}v^{k)}{}_{a_2}(R_{12}^{(0,1)})_{ijk}\ ,\quad s_2^{(0,3)}=v^j{}_av^a{}_i(R_{12}^{(0,1)})^i_j\ ,\\
&s_3^{(0,3)}=u^{(jk}v^{k)}{}_i(R_{14}^{(0,2)})^i_{jkl}\ ,\quad s_4^{(0,3)}=v^{jk}{}_i(R_{16}^{(0,2)})^i_{jk}\ .
\end{aligned}\tag{A.10}$$

There are 4 relations of these relations, given by

$$\begin{aligned}
&i\,QO_1^{(1,2)}\equiv 3s_5^{(1,1)}-3s_6^{(1,1)}+s_7^{(1,1)}=0\ ,\\
&i\,QO_2^{(1,2)}\equiv 9s_1^{(1,1)}-10s_2^{(1,1)}-30s_5^{(1,1)}-60s_3^{(0,3)}=0\ ,\\
&i\,QO_3^{(1,2)}\equiv 3s_1^{(1,1)}-6s_2^{(1,1)}+4s_4^{(1,1)}-14s_5^{(1,1)}-6s_8^{(1,1)}-12s_1^{(0,3)}-4s_2^{(0,3)}=0\ ,\\
&i\,QO_4^{(1,2)}\equiv 3s_1^{(1,1)}-14s_2^{(1,1)}-8s_3^{(1,1)}-42s_5^{(1,1)}+12s_6^{(1,1)}-24s_8^{(1,1)}-36s_1^{(0,3)}+8s_4^{(0,3)}=0\ .
\end{aligned}\tag{A.11}$$

They are the Q-actions on (7.64).

iii) $(R,J)=(3,1)$**.** There exist 16 $SU(3)$ singlets in this sector given by

$$\begin{aligned}
s_1^{(1,0)}&=\epsilon_{a_1a_2i}\epsilon_{b_1b_2j}u^{a_1b_1}u^{a_2b_2k}(R_{14}^{(1,0)})^{ij}_k\ ,\\
s_2^{(1,0)}&=\epsilon_{a_1a_2i}u^{a_1(j}w^{k)a_2}(R_{10}^{(0,0)})^i_{jk}\ ,\\
s_3^{(1,0)}&=\epsilon_{a_1a_2i}u^{a_1jk}w^{a_2}(R_{10}^{(0,0)})^i_{jk}\ ,\\
s_1^{(0,2)}&=v^a{}_iv^{jk}{}_a(R_{10}^{(0,0)})^i_{jk}\ ,\\
s_2^{(0,2)}&=v^{(j}{}_av^{k)a}{}_i(R_{10}^{(0,0)})^i_{jk}\ ,\\
s_3^{(0,2)}&=u^{a(i}v^{jk)}{}_a(R_{12}^{(0,1)})_{ijk}\ ,\\
s_4^{(0,2)}&=u^{a(ij}v^{k)}{}_a(R_{12}^{(0,1)})_{ijk}\ ,\\
s_5^{(0,2)}&=\epsilon_{a_1a_2i}u^{a_1b}v^{a_2j}{}_b(R_{12}^{(0,1)})^i_j\ ,\\
s_6^{(0,2)}&=\epsilon_{a_1a_2i}u^{a_1bj}v^{a_2}{}_b(R_{12}^{(0,1)})^i_j\ ,\\
s_7^{(0,2)}&=\epsilon_{a_1a_2(i}u^{a_1(k}v^{l)a_2}{}_{j)}(R_{12}^{(0,1)})^{ij}_{kl}\ ,\\
s_8^{(0,2)}&=\epsilon_{a_1a_2(i}u^{a_1kl}v^{a_2}{}_{j)}(R_{12}^{(0,1)})^{ij}_{kl}\ ,\\
s_9^{(0,2)}&=\epsilon_{a_1a_2i}u^{a_1(j}u^{kl)a_2}(R_{14}^{(0,2)})^i_{jkl}\ ,\\
s_{10}^{(0,2)}&=\epsilon_{a_1a_2i}u^{a_1b}v^{a_2}{}_b(R_{14}^{(0,1)})^i\ ,\\
s_{11}^{(0,2)}&=u^{a(i}v^{j)}{}_a(R_{14}^{(0,1)})_{ij}\ ,\\
s_{12}^{(0,2)}&=\epsilon_{a_1a_2(i}u^{a_1k}v^{a_2}{}_{j)}(R_{14}^{(0,1)})^{ij}_k\ ,
\end{aligned}$$

$$s_{13}^{(0,2)} = u^{(jk}v^{l)}{}_i(R_{14}^{(0,1)})^i_{jkl} \ . \tag{A.12}$$

There are 13 relations of these relations, given by

$$\begin{aligned}
&i\,QO_1^{(1,1)} \equiv s_2^{(1,0)} = 0 \ ,\\
&i\,QO_2^{(1,1)} \equiv s_3^{(1,0)} = 0 \ ,\\
&i\,QO_3^{(1,1)} \equiv s_1^{(1,0)} + 5s_1^{(0,2)} - 2s_2^{(0,2)} = 0 \ ,\\
&i\,QO_1^{(0,3)} \equiv 4s_5^{(0,2)} + 3s_6^{(0,2)} = (R_{12}^{(0,1)})^i_j(R_{12}^{(0,1)})^j_i = i\,Q\left[\frac{1}{2}i\,Q((r_{12}^{(0,2)})^i_j(r_{12}^{(0,2)})^j_i)\right] = 0 \ ,\\
&i\,QO_2^{(0,3)} \equiv s_7^{(0,2)} + s_8^{(0,2)} = (R_{12}^{(0,1)})^{ij}_{kl}(R_{12}^{(0,1)})^{kl}_{ij} = i\,Q\left[\frac{1}{2}i\,Q((r_{12}^{(0,2)})^{ij}_{kl}(r_{12}^{(0,2)})^{kl}_{ij})\right] = 0 \ ,\\
&i\,QO_3^{(0,3)} \equiv s_3^{(0,2)} = 0 \ ,\\
&i\,QO_4^{(0,3)} \equiv s_{10}^{(0,2)} = 0 \ ,\\
&i\,QO_5^{(0,3)} \equiv 6s_1^{(0,2)} - 6s_4^{(0,2)} - s_6^{(0,2)} = 0 \ ,\\
&i\,QO_6^{(0,3)} \equiv 24s_2^{(0,2)} - 6s_{11}^{(0,2)} + s_{12}^{(0,2)} = 0 \ ,\\
&i\,QO_7^{(0,3)} \equiv s_1^{(0,2)} - 10s_2^{(0,2)} - 6s_4^{(0,2)} - 10s_8^{(0,2)} = 0 \ ,\\
&i\,QO_8^{(0,3)} \equiv 5s_1^{(0,2)} - 2s_2^{(0,2)} - 9s_4^{(0,2)} + 6s_9^{(0,2)} = 0 \ ,\\
&i\,QO_9^{(0,3)} \equiv 6s_1^{(0,2)} + 12s_2^{(0,2)} - 18s_4^{(0,2)} + s_{12}^{(0,2)} = 0 \ ,\\
&i\,QO_{10}^{(0,3)} \equiv 38s_1^{(0,2)} + 4s_2^{(0,2)} - 24s_4^{(0,2)} - 5s_{13}^{(0,2)} = 0 \ .
\end{aligned} \tag{A.13}$$

They are the Q-actions on (7.65). Here, $O_1^{(0,3)}$ and $O_2^{(0,3)}$ are explicitly shown to be Q-exact.

iv) $(R,J) = (\frac{7}{2},\frac{1}{2})$**.** There exist 8 $SU(3)$ singlets in this sector given by

$$\begin{aligned}
s_1^{(0,1)} &= \epsilon_{a_1a_2i}u^{a_1b}u^{jk}v^{a_2}{}_b(R_{10}^{(0,0)})^i_{jk} \ , \quad s_2^{(0,1)} = \epsilon_{a_1a_2i}u^{a_1b}u^{a_2(j}v^{k)}{}_b(R_{10}^{(0,0)})^i_{jk} \ ,\\
s_3^{(0,1)} &= \epsilon_{a_1a_2i}u^{a_1b(j}v^{k)a_2}{}_b(R_{10}^{(0,0)})^i_{jk} \ , \quad s_4^{(0,1)} = \epsilon_{a_1a_2(i}u^{a_1(k}v^{l)a_2}{}_{j)}(R_{12}^{(0,0)})^{ij}_{kl} \ ,\\
s_5^{(0,1)} &= \epsilon_{a_1a_2(i}u^{a_1kl}v^{a_2}{}_{j)}(R_{12}^{(0,0)})^{ij}_{kl} \ , \quad s_6^{(0,1)} = \epsilon_{a_1a_2(i}\epsilon_{j)b_1b_2}u^{a_1b_1}u^{a_2b_2}u^{kl}(R_{12}^{(0,1)})^{ij}_{kl} \ ,\\
s_7^{(0,1)} &= \epsilon_{a_1a_2(i}\epsilon_{j)b_1b_2}u^{a_1b_1}u^{a_2b_2k}(R_{14}^{(0,1)})^{ij}_k \ , \quad s_8^{(0,1)} = \epsilon_{a_1a_2i}u^{a_1(j}u^{kl)a_2}(R_{14}^{(0,1)})^i_{jkl} \ .
\end{aligned} \tag{A.14}$$

There are 6 relations of these relations, given by

$$\begin{aligned}
i\,QO_1^{(0,2)} &\equiv s_1^{(0,1)} - 2s_2^{(0,1)} = 0 \ , \\
i\,QO_2^{(0,2)} &\equiv 6s_3^{(0,1)} + s_4^{(0,1)} = 0 \ , \\
i\,QO_3^{(0,2)} &\equiv s_1^{(0,1)} + s_5^{(0,1)} = 0 \ , \\
i\,QO_4^{(0,2)} &\equiv s_1^{(0,1)} + s_6^{(0,1)} = 0 \ , \\
i\,QO_5^{(0,2)} &\equiv 4s_1^{(0,1)} + 24s_3^{(0,1)} - s_7^{(0,1)} = 0 \ , \\
i\,QO_6^{(0,2)} &\equiv s_1^{(0,1)} - 12s_3^{(0,1)} + 3s_8^{(0,1)} = 0 \ .
\end{aligned} \tag{A.15}$$

They are the Q-actions on (7.67).

v) $(R, J) = (4, 0)$ There exist 4 $SU(3)$ singlets in this sector given by

$$\begin{aligned}
s_1^{(0,0)} &= \epsilon_{a_1a_2a_3}\epsilon_{b_1b_2i}u^{a_1b_1}u^{a_2b_2}u^{a_3jk}(R_{10}^{(0,0)})^i_{jk} \ , \\
s_2^{(0,0)} &= R_{12}^{(0,0)}R_{12}^{(0,0)} \ , \\
s_3^{(0,0)} &= \epsilon_{a_1a_2(i}\epsilon_{j)b_1b_2}u^{a_1b_1}u^{a_2b_2}u^{kl}(R_{12}^{(0,0)})^{ij}_{kl} \ , \\
s_4^{(0,0)} &= \epsilon_{a_1a_2(i}\epsilon_{j)b_1b_2}u^{a_1b_1(k}u^{l)a_2b_2}(R_{12}^{(0,0)})^{ij}_{kl} \ .
\end{aligned} \tag{A.16}$$

There is 1 relation of these relations, given by

$$i\,QO^{(0,1)} \equiv 36s_1^{(0,0)} + 5s_2^{(0,0)} - 6s_3^{(0,0)} = 0 \ . \tag{A.17}$$

This is the Q-action on (7.68).

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