THE CURRENTS ON STRIP ANTENNAS

by T.B.A. Senior*

Summary

An exact expression is obtained for the longitudinal distribution of current excited on a perfectly conducting strip by a normally incident plane wave, and computations are carried out for quarter and half wave antennas. By using the known transverse variation of the current on strips of small width, the complete surface distribution is determined, leading to an expression for the total current carried by the antenna. This is compared with the current distribution for a thin wire, but little agreement is found. Some reasons for the differences are given.

1. Introduction

It is frequently assumed that the currents excited on a strip antenna have a longitudinal distribution which is similar to that for a thin wire. As a result, the variation of the current as a function of position can be closely represented by a cosine term. This appears a reasonable assumption for narrow strips when the incident field is 'edge-on', but at normal incidence the analogy with the wire is not quite so obvious, suggesting that further consideration be given to this case.

In the following, attention is confined to an idealized antenna consisting of a perfectly conducting infinitely thin strip of arbitrary length and width. As such, the strip can be likened to some types of indoor TV aerials. The excitation is by means of a normally incident plane wave and no account is taken of antenna connections or impedance losses.

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*The University of Michigan, Radiation Laboratory

In Section 2 it is shown that the longitudinal distribution of the surface current on a strip of length 2a can be deduced from the transverse distribution on a strip of width 2a and infinite length. This last can be expressed as a series of Mathieu functions, and computations are carried out for $a = \lambda/8$ and $a = \lambda/4$, corresponding to quarter and half wave antennas.

For strips of relatively small width the total current distribution can be obtained using the known variation across the width of the strip, and this is done in Section 4. The resulting distribution is compared with that for a thin wire. If the length of the strip is not large compared with the wavelength, the two distributions do not agree, and cannot be brought into agreement whatever the radius of the wire. Although the differences are not necessarily significant, it is clear that for practical strip antennas of small length, the new distribution will represent a better initial approximation in any iterative scheme for finding the true current.

2. The Method

An expression for the current is most easily obtained by considering the current which would be induced in an infinite strip by an incident plane wave.

In terms of Cartesian coordinates x, y, z the infinite strip will be assumed to occupy the region $-a < x < a$, $-\infty < z < \infty$ of the plane $y = 0$. A plane wave is incident in the direction of the negative y axis and if its magnetic vector is entirely in the z direction

$$\underline{H}^i = (0, 0, e^{-iky}), \qquad (1)$$

where the affix 'i' is used to denote the incident field and $k = 2\pi/\lambda$ is the propagation constant. M.k.s. units are employed and a time factor $e^{-i\omega t}$ is suppressed throughout.

[FIG. 1]

From symmetry considerations it is clear that the magnetic vector in the scattered field must also be confined to the z direction and since the induced current is determined by the component of the total magnetic field parallel to the surface of the strip, the current vector $\underline{I}$ must be entirely in the x direction. This implies that the only current flow is across the strip. Moreover, the fact that the whole problem has two-dimensional symmetry requires the current to be the same at all points along the length of the strip and hence $\underline{I}$ can only be a function of x. Since $\underline{I} = \underline{n} \wedge \underline{H}$, where $\underline{n}$ is a unit vector normal to the surface, we now have

$$\underline{I} = (I, 0, 0)$$

and on the upper surface ($y = +0$), $I = I_+(x)$ with

$$I_+(x) = \left[H_z\right]_{y=+0}$$

$$= 1 + \left[H_z^s\right]_{y=+0} \quad . \qquad (2)$$

On the lower surface of the strip the sign of I is reversed.

A strip antenna of the type under consideration can be obtained by chopping up an infinite strip with cuts parallel to the x axis, thereby producing a rectangular surface of length 2a and width 2d (say). In general d will be small compared with a. The new edges which are formed in this way cannot generate any transverse current, nor can they affect the x dependence of the longitudinal current which was present on the larger surface. This fact enables us to identify the current on the antenna with that on the infinite strip. The only difference is that the actual antenna current is a function of z as well as x, and this other dependence has been discussed by, for example, Moullin & Phillips (1952). The variation with z is required in any calculation of the total (integrated) current carried by the antenna and will be referred to again later.

3. The Analysis

It is now a simple matter to determine the antenna current using the exact expression for the current excited on an infinite strip by a normally incident plane wave.

From McLachlan (1947) we have

$$H_z^s \Big|_{y=+0} = 2ib \sum_{n=0}^{\infty} B_1^{(2n+1)} \frac{Ne_{2n+1}^{(1)}(0)}{Ne_{2n+1}^{(1)\prime}(0)} se_{2n+1}(\eta)$$

where $b = \frac{ka}{2}$ and $\eta = \cos^{-1}\left(\frac{x}{a}\right)$. The Mathieu function coefficients are denoted by $B_{2m+1}^{(2n+1)}$ and in terms of these

$$se_{2n+1}(\eta) = \sum_{m=0}^{\infty} B_{2m+1}^{(2n+1)} \sin(2m+1)\eta\ ,$$

$$\frac{Ne_{2n+1}^{(1)}(0)}{Ne_{2n+1}^{(1)\prime}(0)} = -\frac{\sum_{m=0}^{\infty} (-1)^m B_{2m+1}^{(2n+1)}}{\sum_{m=0}^{\infty} (-1)^m (2m+1) B_{2m+1}^{(2n+1)} + \pi b^2 \sum_{m=0}^{\infty} (-1)^m B_{2m+1}^{(2n+1)} (P_m - iQ_m)}$$

where P_m and Q_m are given by the Bessel function formulae

$$P_m = J_m(b)Y_m(b) + J_{m+1}(b)Y_{m+1}(b) - \frac{2m+1}{b} J_{m+1}(b)Y_m(b),$$

$$Q_m = \left\{J_m(b)\right\}^2 + \left\{J_{m+1}(b)\right\}^2 - \frac{2m+1}{b} J_{m+1}(b)\, J_m(b).$$

Hence,

$$I_+(x) = 1 + 2ib \sum_{n=0}^{\infty} B_1^{(2n+1)} \frac{Ne_{2n+1}^{(1)}(0)}{Ne_{2n+1}^{(1)\prime}(0)} se_{2n+1}(\eta). \qquad (3)$$

For the lower surface ($y = -0$) the sign of I has to be reversed, but since η must also be replaced by $2\pi - \eta$,

$$I_-(x) = I_+(x) - 2. \qquad (4)$$

The coefficients $B_{2m+1}^{(2n+1)}$ are, of course, functions of b and their values have been tabulated by the Computation Laboratory, National Bureau of Standards (1951). For large b the series for H_z^s is only slowly convergent, but when b is less than unity the convergence is sufficiently rapid for the series to be cut off after the first few terms.

Two examples will be considered, corresponding to quarter and half wave antennas ($a = \lambda/8$ and $\lambda/4$ respectively). Using tabulated values of $B_{2m+1}^{(2n+1)}$ and of the Bessel functions (the latter being supplemented, where necessary, by direct calculation of $J_n(b)$ and $Y_n(b)$ from their series expansions), the following results are obtained:

$a = \lambda/8$

$$I_+(x) = 1 - (0.32212 - i\ 1.01278) \sin \eta + (0.00636 - i\ 0.01485) \sin 3\eta$$

$$- (0.00005 - i\ 0.00021) \sin 5\eta \qquad (5)$$

$a = \lambda/4$

$$I_+(x) = 1 + (1.80385 + i\ 0.88454) \sin \eta - (0.14997 + i\ 0.03026) \sin 3\eta$$

$$+ (0.00396 + i\ 0.00061) \sin 5\eta - (0.00005 + i\ 0.00002) \sin 7\eta . \qquad (6)$$

The coefficients of the higher trigonometrical functions are zero to the first 5 places of decimals.

The amplitudes and phases of $I_+(x)$ are plotted as functions of x in Figs. 2 and 3 respectively. Taking first the quarter wave antenna, it is seen that the phase behaves in a perfectly regular manner, increasing monotonically as x increases from 0 to a, but the amplitude curve shows a minimum at about $x = 0.95a$. This would be explained if there were an appreciable build up of charge at the ends, giving rise to a capacitive effect. The antenna would then resonate at a wavelength which is greater than that predicted by its physical dimensions.

[FIGS. 2 & 3]

No such minimum is found in the amplitude curve for $a = \lambda/4$ and this can be attributed to the fact that the first harmonic (which is zero at the ends) now dominates the current distribution. Since the constant current is mainly responsible for any charge appearing at the ends, the capacitive effect is no longer important.

4. Comparison with Thin Wire Currents

The discussion so far has been devoted to the current $I_+(x)$ which would be measured if a probe were placed on the upper surface of the antenna. Such measurements are entirely feasible in spite of the experimental difficulties involved, and some results obtained with rectangular and triangular surfaces have been described by Hey & Senior (1958).

For the lower surface of the antenna the expression for the current differs from that of equation (3) in the sign of the constant term. This discontinuity between the currents on the illuminated and non-illuminated sides of the surface is to be expected and is the same as that occuring in the case of a half-plane (see, for example, Clemmow, 1951).

To determine the total current carried by the antenna it is necessary to multiply each of the surface currents by a factor which takes into account the variation across the width of the strip. This dependence has been considered at length by Moullin & Phillips (1952) and they have shown that for narrow strips ($kd \ll 1$) a close approximation to the transverse distribution is provided by the function

$$\left(1 - z^2/d^2\right)^{-1/2} .$$

This is precisely the z dependence which would be arrived at by a study of the current near to the edge of a half-plane, and the singularities at $z = \pm d$ are those which are required in order to satisfy the edge conditions in diffraction theory (Jones, 1952).

The distribution of current over the upper surface of the strip can now be represented as a function of x and z by

$$\frac{I_+(x)}{(1 - z^2/d^2)^{1/2}}$$

and to obtain the total current flowing in the x direction it is only necessary to integrate with respect to z from $z = -d$ to $z = d$. Since

$$\int_{-d}^{d} (1 - z^2/d^2)^{-1/2}\, dz = \pi d,$$

the current carried by both the upper and lower surfaces is

$$I_{tot}(x) = \pi d \left\{ I_+(x) + I_-(x) \right\}$$

$$= 2\pi d \left\{ I_+(x) - 1 \right\}, \qquad (8)$$

and the magnitude of the current is plotted in Fig. 4 for $a = \lambda/8$ [illegible]. It will be observed that the curve for $a = \lambda/8$ does not show the [illegible] occured with $I_+(x)$.

The above expression for $I_{tot}(x)$ is certainly valid when d is [illegible] fact suggests that the distribution should be comparable with [illegible] of suitably chosen radius. The longitudinal distribution [illegible] wire has been considered by King & Harrison (1944) and [illegible] tion is a normally incident plane wave, a general formula [illegible] is

$$I_{tot}(x) = A \left\{ \cos kx - \text{[illegible]} \right.$$

where f(x) is a complicated function [illegible] a normalizing constant. The [illegible]

[illegible]

where r is the radius of the [illegible] that $\Omega \gg 1$.

It is immediately apparent that the distributions for a strip and a wire are entirely different in character, but this does not rule out the possibility of finding an alternative expression for the strip current which will bring the two into agreement. The natural expansion for $I_+(x)$, and hence $I_{tot}(x)$, is based upon the functions $\sin (2n+1)\eta$, $n=0, 1, 2, \ldots$, where $\eta = \cos^{-1} \frac{x}{a}$, and since

$$\sin \eta = \left(1 - \frac{x^2}{a^2}\right)^{1/2} ,$$

$$\sin 3\eta = 3\left(1 - \frac{x^2}{a^2}\right)^{1/2} - 4\left(1 - \frac{x^2}{a^2}\right)^{3/2} ,$$

etc., the series can also be written in terms of the functions $\left(1 - \frac{x^2}{a^2}\right)^{n+1/2}$, $n=0, 1, 2, \ldots$. Both of these expansions are rapidly convergent for values of ka of order unity and a reasonable approximation to the current distribution can be had by neglecting all harmonics above the first. Thus, for $a = \lambda/8$,

$$I_{tot}(x) \simeq - 2\pi d\, (0.32212-i\, 1.01278)\, (1 - \frac{x^2}{a^2})^{1/2} \tag{10}$$

and for $a = \lambda/4$,

$$I_{tot}(x) \simeq 2\pi d\, (1.80385 + i\, 0.88454)\, (1 - \frac{x^2}{a^2})^{1/2} . \tag{11}$$

The function $\sin \eta$ can also be expressed in a Fourier series of the form

$$\sin \eta = \sum_{m=0}^{\infty} \frac{2}{2m+1} J_2 \left\{(m + \frac{1}{2})\pi\right\} \cos (2m+1)kx,$$

where J_2 is the second order Bessel function, and similarly

$$\sin 3\gamma = \sum_{m=0}^{\infty} \frac{6}{2m+1} \left\{ 1 - \frac{8}{(2m+1)\pi} \right\} J_2 \left\{ (m + \frac{1}{2})\pi \right\} \cos (2m+1)kx.$$

Such expansions, however, converge only slowly and if they are inserted into the equation for the total current, a large number of terms are required in order to reproduce the accuracy represented by the single terms in, for example, equations (10) and (11). The first term involving cos kx in no sense dominates the series when ka is small and hence, if an attempt is made to write the current distribution for a strip in a form analogous to that of equation (9), the correction term corresponding to $\frac{1}{\Omega} f(x)$ will be at least as important as cos kx. As a result the differences between the distributions for a strip and a wire must be regarded as fundamental and, indeed, if a numerical comparison is made, no practicable value of Ω exists which will bring them into even approximate agreement.

5. Conclusions

In the preceding section it has been shown that the longitudinal distribution of current on a strip of finite length a can be derived from the exact Mathieu function expansion for the current on an infinite strip. The resulting expression is valid for all a.

If the width of the strip is small, the known variation of the current in the transverse direction may be used to predict the entire surface distribution, leading to an expression for the total current $I_{tot}(x)$ carried by the strip. For small ka it is found that

$$I_{tot}(x) \simeq 2\pi d \alpha \left(1 - \frac{x^2}{a^2}\right)^{1/2}, \qquad (12)$$

where α is a complex numerical constant depending on the length a and given by a rapidly convergent Mathieu function series. In reality the term on the right of equation (12) is the first in a series of ascending powers of $\left(1 - \frac{x^2}{a^2}\right)$, but for ka < 1 the coefficients of the higher powers are negligible.

Although it is freely admitted that the discussion has been confined to the most idealized type of strip antenna, equation (12) does suggest that any iterative scheme for finding the current on a practical antenna should be based on an initial approximation of the form $\left(1 - \frac{x^2}{a^2}\right)^{1/2}$. Providing ka is not large compared with unity, this will certainly be better than the usual assumption of a cosine variation, since the end effects are a major factor in determining the nature of the distribution. When ka >> 1, however, the end effects are less important and in this case it may well be that a cosine dependence is a more accurate approximation.

6. References

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Jones, D. S. 1952 Quart. J. Mech. Appl. Math. 5, 363.

McLachlan, N. W. 1947 Theory and Application of Mathieu [illegible] Oxford: Clarendon Press.

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National Bureau of Standards 1951 Tables relating to [illegible] New York: Columbia University Press.

Legends for Figures

Fig. 1

Fig. 2 Amplitude of surface current $I_+(x)$ on quarter wave ($a = \lambda/8$) and half wave ($a = \lambda/4$) strips.

Fig. 3 Phase of surface current $I_+(x)$ on quarter wave ($a = \lambda/8$) and half wave ($a = \lambda/4$) strips.

Fig. 4 Longitudinal distribution of total current amplitude.

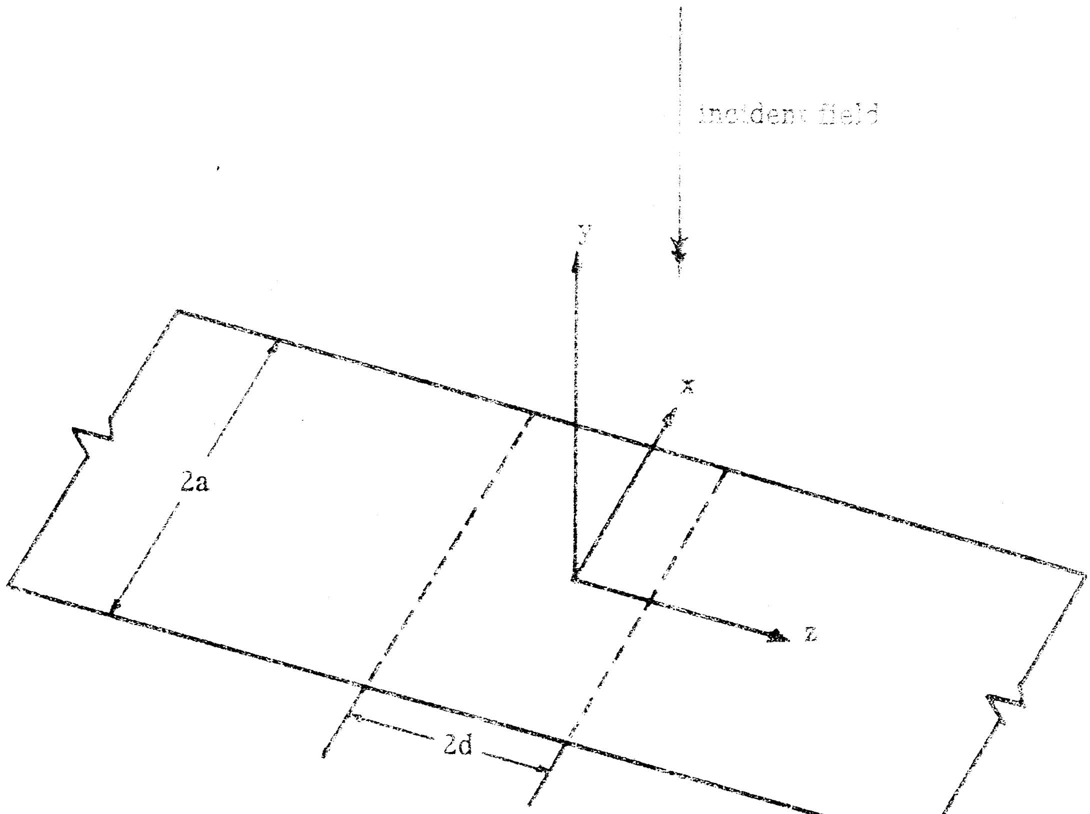


Fig. 1

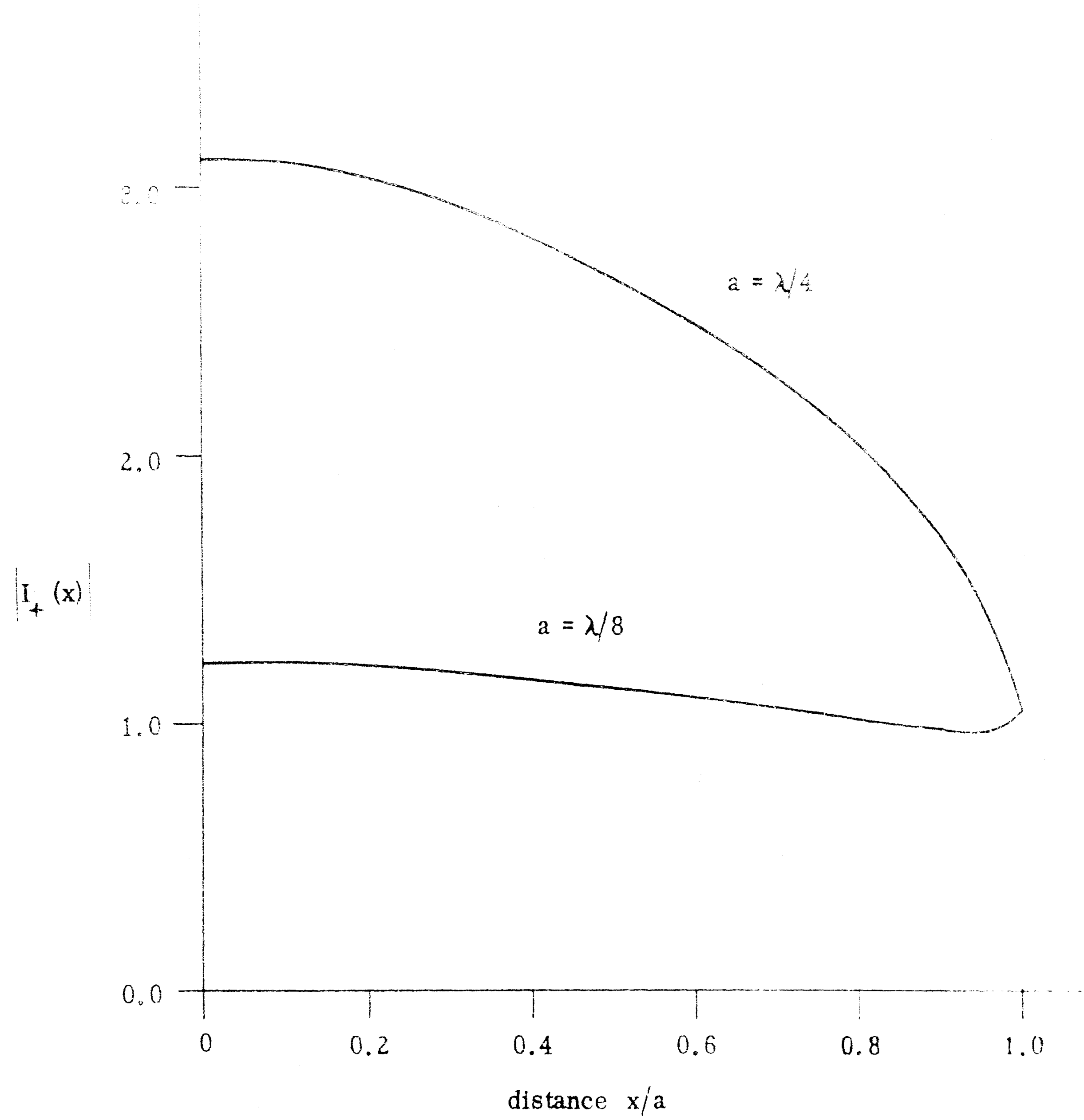
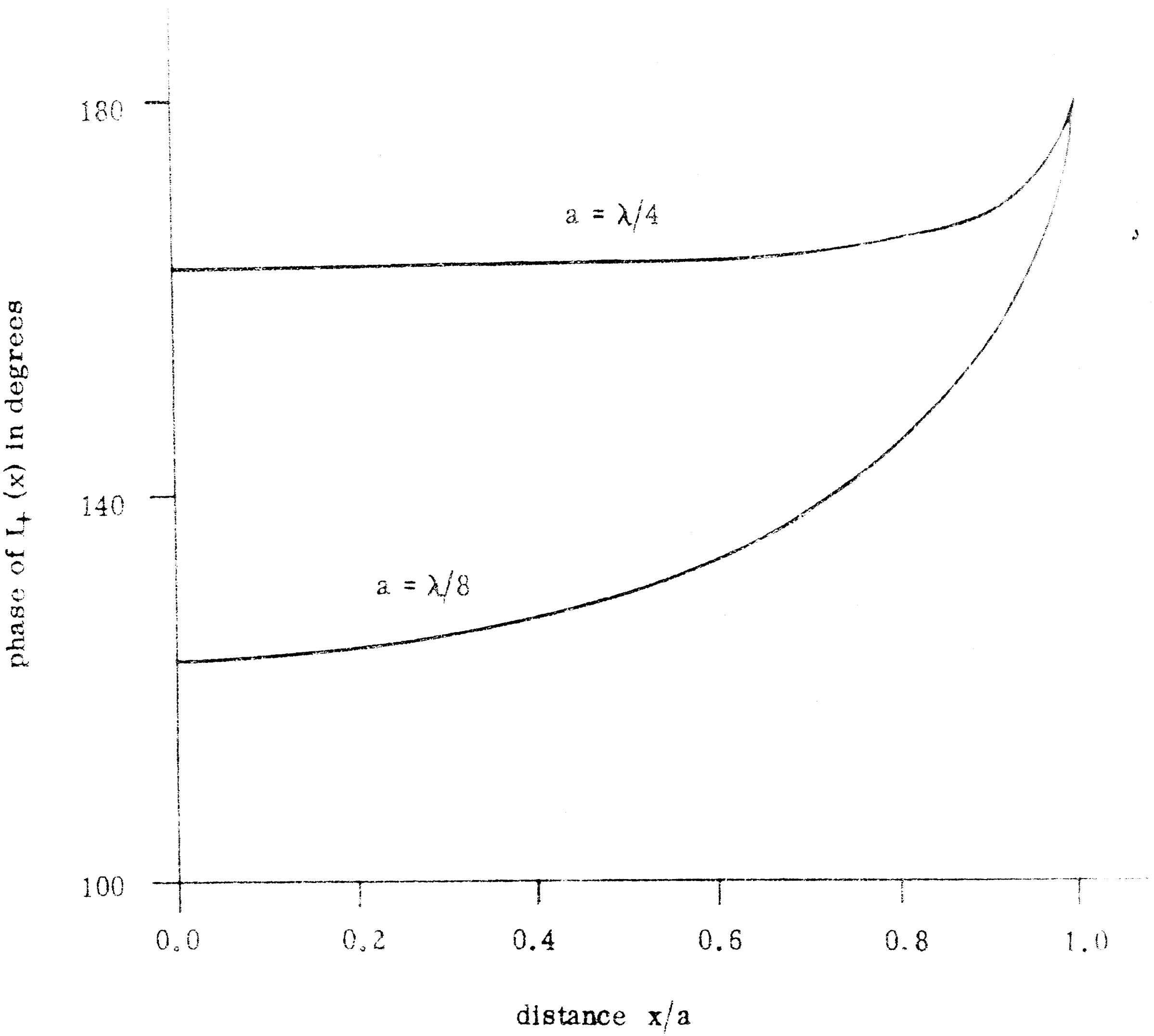


Fig. 2



phase of I_+ (x) in degrees
180
140
100
a = λ/4
a = λ/8
0.0
0.2
0.4
0.6
0.8
1.0
distance x/a

Fig. 3

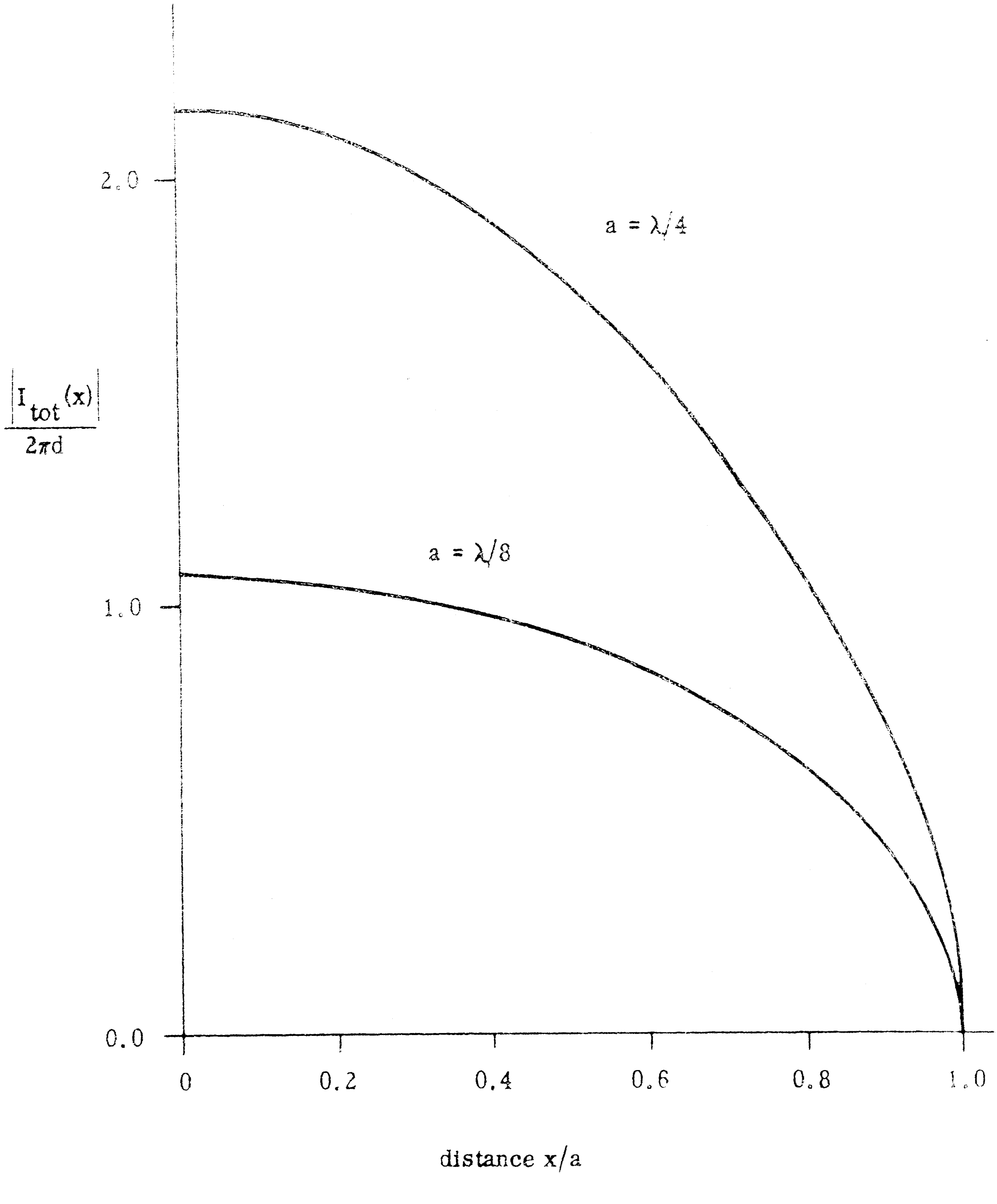


Fig. 4