RL 201

A NOTE ON IMPEDANCE BOUNDARY CONDITIONS

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In recent years the application of impedance boundary conditions has received considerable attention in the literature. The conditions are now the crux of most analyses of surface wave phenomena and are widely employed in diffraction problems in which it is desired to take into account the material constitution and/or surface characteristics of the body.

In its simplest form the impedance (or Leontovich) boundary condition can be written as

$$\underline{E} - (\hat{n} \cdot \underline{E})\, \hat{n} = \eta\, Z\, \hat{n} \wedge \underline{H} \tag{1}$$

where $(\underline{E}, \underline{H})$ is the total field in the region surrounding the body, which region is assumed for convenience to be free space, $Z = 1/Y$ is the intrinsic impedance of free space, η is the surface impedance and $\hat{n}$ is a unit vector normal drawn outwards as regards the surface. If the body is composed of a material of large refractive index, η is proportional to the reciprocal of the complex refractive index of the material relative to free space and is, in fact,

$$\eta = \frac{1}{\sqrt{\frac{\mu_0}{\mu}\left(\frac{\epsilon}{\epsilon_0} + i\frac{\sigma}{\omega\epsilon_0}\right)}} \qquad (2)$$

so that η is zero for a perfectly conducting body. The condition (1) is then a valid approximation to the true condition if the radii of curvature are everywhere large compared with the wavelength, and can even be justified (Senior, 1961 a) when η varies from point to point providing the variation is sufficiently slow. A further instance in which (1) can be employed is when the surface is perfectly conducting but rough, and in this case the impedance is directly related to the roughness characteristics (Senior, 1961 b). In both cases it will be observed that the function of the boundary condition is to perform in one fell swoop a sequence of perturbations about the solution for a perfectly conducting (or smooth) body, and thereby account for the effect of either the fields inside the body or the roughness of the surface without considering these explicitly.

Apart from the question of its physical justification, the boundary condition has the following interesting mathematical ramification.

THEOREM

If the scattered field for a field $(\underline{E}^i, \underline{H}^i) = (\underline{F}, Y\underline{G})$ incident on the body is $(\underline{E}^s, \underline{H}^s) = \left(\underline{f}(\eta), Y\underline{g}(\eta)\right)$, the scattered field for a field $(\underline{E}^i, \underline{H}^i) = (-Z\underline{G}, \underline{F})$, is $(\underline{E}^s, \underline{H}^s) = \left(-Z\underline{g}(1/\eta), \underline{f}(1/\eta)\right)$.

PROOF

Whereas the scalar product of (1) with respect to $\hat{n}$ yields an identity, the vector product gives

$$\underline{H} - (\hat{n} \cdot \underline{H}) \hat{n} = -\frac{1}{\eta} Y (\hat{n}_\wedge \underline{E}) . \qquad (3)$$

This is merely an alternative statement of (1) and corresponds to (1) under the transformation $\underline{E} \to \underline{H}$, $Z\underline{H} \to -Y\underline{E}$, $\eta \to 1/\eta$. Since the boundary condition (1) (or (3)) is the only place where η enters into the problem, and since the incident field change is identical to the first two parts of this transformation, the proof now follows immediately.

It will be observed that the theorem is true independently of the shape of the body in either two or three dimensions, and is independent of the physical interpretation associated with the parameter η. At first sight the result is quite surprising and enables the exact solution of the boundary value problem for one polarization to be deduced from the exact solution for an incident field of the 'complementary' type. As such the theorem has affinities with Babinet's principle and underlines one of the advantages attached to a mathematically exact solution rather than a simple perturbation result. Solutions of specific boundary value problems illustrating this are given in (Senior, 1952, 1959).

On the other hand, if the parameter η is attributed to the material constituents of the body, the origins of the theorem are more apparent.

Whereas Maxwell's equations in free space are invariant under the transformation $\underline{E} \rightarrow \underline{H}$, $Z\underline{H} \rightarrow -Y\underline{E}$, those within the body are invariant under the transformation

$$\underline{E}' \rightarrow \underline{H}', \quad \eta Z\underline{H}' \rightarrow -\frac{Y}{\eta}\underline{E}',$$

and the theorem is a natural consequence of forcing the external field to contain this additional symmetry.

REFERENCES

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