LOOP EXCITATION OF TRAVELLING WAVES

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1. Introduction

One aspect of diffraction theory which is of practical importance at the present time is the study of bodies which are characterized by a low back-scattering cross section over a range of incidence angles. Quite frequently the basic shape is a long thin body of revolution, and when this is viewed at or near nose-on, some of the major features of the return can be attributed to the travelling waves which are excited on the surface (Peters, 1958; Goodrich and Kazarinoff, 1962). In many instances these represent the dominant portion of the current distribution, and the extent to which it is possible to reduce the cross section by, for example, small changes in shape is then determined by the degree to which the travelling waves can be reduced or, hopefully, suppressed entirely. It is therefore desirable to give some attention to the manner in which travelling waves are excited, with particular reference to the influence of any surface 'singularities' such as discontinuities in the slope or derivatives thereof.

As a contribution to this end we consider the launching of travelling waves by the effective field singularity represented by the curve on the body separating the lit region from the shadow. Since the incident field is here moving parallel to the surface, intuitive reasoning suggests that the neighbourhood of the shadow boundary could act as a source of travelling waves, but because of the inherent complication of the problem it is difficult to calculate the power going into the travelling wave as opposed to the power in the radiated field. In order to estimate the efficiency of the shadow boundary as a source of these waves, we shall therefore restrict ourselves to the simpler problem in which the primary source of energy is itself placed on the surface of the body, and is so chosen as to excite only the fundamental travelling wave. The body can then be approximated by an infinite cylinder of circular cross section and large (but finite) conductivity, allowing a direct comparison of the powers via their integral expressions.

2. Formulation of the Problem

Consider an infinitely long circular cylinder of radius a which is excited by a circumferential ring current located at a distance $r_o > a$ from the axis. If the surface impedance of the cylinder is η, the boundary conditions on the total field can be written as

$$E_\theta = - \eta Z H_z \qquad (1)$$

$$E_z = \eta Z H_\theta \qquad (2)$$

at $r = a$, where (r, θ, z) are cylindrical polar coordinates with the z axis coincident with the axis of the cylinder, and Z is the intrinsic impedance of free space.

The fundamental travelling wave is the one with lowest attenuation in the direction of propagation, and this is consequently the wave of most interest in practical applications. Since its magnetic vector is entirely transverse with no variation in the θ direction it is convenient to choose as the source of excitation a magnetic ring current of constant amplitude and phase. Without loss of generality, the ring can be chosen to lie in the plane $z = 0$, and the incident field can then be represented by the single component (electric) Hertz vector

$$\underline{\pi} = (0,\ 0,\ U^i)$$

where

$$U^i = \int_0^{2\pi} \frac{e^{iu}}{u}\, d\theta_o\ , \tag{3}$$

with

$$u = k\sqrt{r^2 + r_o^2 - 2\, r r_o \cos(\theta - \theta_o) + z^2}\ . \tag{4}$$

The time convention is here $e^{-i\omega t}$ and the coordinates of a variable point on the ring source are denoted by the suffix 'o'.

In terms of the Hertz vector $\underline{\pi}$ the components of the incident field are

$$\underline{E}^i = \left(\frac{\partial^2 U^i}{\partial r \partial z},\ 0,\ k^2 U^i + \frac{\partial^2 U^i}{\partial z^2} \right), \tag{5}$$

$$\underline{H}^i = ik\, Y \left(0,\ \frac{\partial U^i}{\partial r},\ 0\right), \tag{6}$$

and this is a transverse magnetic field as required. From the boundary conditions it now follows that the scattered field must also be of similar type, so that only the second of the two conditions is relevant, and by introducing a single component

Hertz vector to represent the scattered field, the condition (2) becomes

$$\left\{ k^2 + \frac{\partial^2}{\partial z^2} - ik\eta \frac{\partial}{\partial r} \right\} (U^i + U^s) = 0, \tag{7}$$

which can be written alternatively as

$$\left\{ \frac{\partial}{\partial r} + ik\eta \left(r \frac{\partial}{\partial r} \right) \right\} (U^i + U^s) = 0 \tag{8}$$

at r = a. This can be satisfied by inserting a suitable expression for U^s, but in order to decide what is the appropriate form it is necessary to examine in more detail the structure of the incident field.

3. <u>The Incident Field</u>

Let us consider first the power radiated by the source. If R is the (spherical) radial variable defined as $R = \sqrt{r^2 + z^2}$, then at large distances from the current loop

$$u \sim kR \left\{ 1 + \frac{r_o^2 - 2 r r_o \cos(\theta - \theta_o)}{2 R^2} \right\},$$

giving

$$U^i \sim \frac{1}{kR} e^{ik(R + r_o^2/2R)} \int_0^{2\pi} \exp\left\{ -i \frac{krr_o}{R} \cos(\theta - \theta_o) \right\} d\theta_o$$

i.e.

$$U^i \sim 2\pi \, J_o(krr_o/R) \cdot \frac{1}{kR} e^{ik(R + r_o^2/2R)} \tag{9}$$

where $J_o(x)$ is the Bessel function of zero order. For $R >> r_o$ the incident field is therefore

$$\underline{E}^i = -\frac{k^2 r}{R}\left(\frac{z}{R},\ 0,\ -\frac{r}{R}\right) U^i$$

$$\underline{H}^i = -\frac{k^2 r}{R}(0,\ 1,\ 0)\, Y\, U^i$$

and the resulting Poynting vector is

$$\frac{1}{2}\underline{E}^i \wedge \tilde{\underline{H}}^i = \left(\frac{r}{R},\ 0,\ \frac{z}{R}\right) 2Y \left\{\frac{\pi k r}{R^2}\ J_o(krr_o/R)\right\}^2 .$$

From this it follows immediately that the radial flow of power is

$$2Y\left\{\frac{\pi k}{R}\ \sin\phi\ J_o(kr_o \sin\phi)\right\}^2 ,$$

where ϕ is defined by the relations

$$r = R\sin\phi , \qquad z = R\cos\phi ,$$

and consequently the total power radiated by the current loop is

$$P^i = 8\, Y\, \pi^3 k^2 \int_0^{\pi/2} \sin^3\phi \left\{J_o(kr_o \sin\phi)\right\}^2 d\phi . \qquad (10)$$

When $kr_o = 0$ the integral in (10) is clearly 2/3, but for more general values of kr_o no precise analytical evaluation is possible. Nevertheless, numerical results can be found for the smaller values of kr_o by introducing the series expansion of $J_o(x)$, and for larger values by numerical integration, and the data obtained in this way is plotted in Figure 1. It will be observed that as kr_o increases the total power radiated decreases in an oscillatory manner, with the first minimum occurring for kr_o approximately[+] 2.6.

For sufficiently large kr_o an alternative approach to the integral is to apply the method of steepest descents. Since the dominant contribution comes from values of ϕ in the neighbourhood of the upper limit, the Bessel function can be replaced by its asymptotic formula for large arguments to give

$$P^i \sim 8 Y \pi^3 k^2 \cdot \frac{1}{2\pi kr_o} \int_0^{\pi} \sin^2\phi \left\{ 1 + \sin(2kr_o \sin\phi) \right\} d\phi.$$

Hence

$$P^i \sim 8 Y \pi^3 k^2 \cdot \frac{1}{4kr_o} \left\{ 1 - \frac{2}{\pi \sqrt{kr_o}} \cos\left(2kr_o + \frac{\pi}{4}\right) \right\} \quad (11)$$

showing a decreasing amplitude of oscillation about the mean value $\frac{1}{4kr_o}$. This is in excellent agreement with the computed points (see Figure 1.)

Having calculated the power radiated by the loop, we now turn to the question of the incident field structure. For this purpose the original expression for U^i is not convenient, and it is necessary to seek an alternative form which will bring out the dependence on the coordinate z.

+ The positions of at least the first few minima are similar to the zeros of $J_o(kr_o)$.

To begin with we observe that

$$\frac{e^{iu}}{u} = \frac{i}{2k}\int_C e^{iz\tau} H_o\left\{(k^2-\tau^2)^{1/2}\left(r^2+r_o^2-2rr_o\cos(\theta-\theta_o)\right)^{1/2}\right\} d\tau \qquad (12)$$

(see, for example, Campbell and Foster, 1948) where $H_o(x)$ is the Hankel function of the first kind of order zero; the path C extends from $-\infty$ to ∞ passing above the branch point at $\tau = -k$ and below the branch point at $\tau = k$, and the chosen branch of $(k^2-\tau^2)^{1/2}$ is that which reduces to k at $\tau = 0$. But

$$H_o\left\{(k^2-\tau^2)^{1/2}\left(r^2+r_o^2-2rr_o\cos(\theta-\theta_o)\right)^{1/2}\right\} =$$

$$= \sum_{-\infty}^{\infty} H_n(r\sqrt{k^2-\tau^2})\, J_n(r_o\sqrt{k^2-\tau^2})\, e^{in(\theta-\theta_o)} \qquad \text{for } r > r_o \qquad (13)$$

$$= \sum_{-\infty}^{\infty} J_n(r\sqrt{k^2-\tau^2})\, H_n(r_o\sqrt{k^2-\tau^2})\, e^{in(\theta-\theta_o)} \qquad \text{for } r < r_o\,; \qquad (14)$$

moreover,

$$\int_0^{2\pi} e^{in(\theta-\theta_o)}\, d\theta_o = 2\pi\,\delta(n)$$

where δ is the standard delta function, and using this in conjunction with equations (3), (12), (13) and (14) the formula for U^i becomes

$$U^i = \frac{i\pi}{k} \int_C e^{iz\tau} J_o(r\sqrt{k^2-\tau^2}) H_o(r_o\sqrt{k^2-\tau^2}) \, d\tau , \qquad r \leq r_o, \tag{15}$$

$$= \frac{i\pi}{k} \int_C e^{iz\tau} H_o(r\sqrt{k^2-\tau^2}) J_o(r_o\sqrt{k^2-\tau^2}) \, d\tau , \qquad r \geq r_o \tag{16}$$

The dependence on the variable z is here made explicit.

4. The Scattered Field

Of the above expressions only the first is required to satisfy the boundary condition at $r = a$, and its form suggests that for the scattered field we take

$$U^s = \frac{i\pi}{k} \int_C e^{iz\tau} H_o(r\sqrt{k^2-\tau^2}) H_o(r_o\sqrt{k^2-\tau^2}) \, f(\tau) \, d\tau \tag{17}$$

where $f(\tau)$ is to be determined. Since the radiation condition is now satisfied automatically by virtue of the Hankel function dependence on r, it only remains to calculate $f(\tau)$ using the boundary condition, and if (15) and (17) are substituted into (7) we have

$$f(\tau) = - \frac{\sqrt{k^2-\tau^2} \, J_o(a\sqrt{k^2-\tau^2}) - ik\eta \, J_o'(a\sqrt{k^2-\tau^2})}{\sqrt{k^2-\tau^2} \, H_o(a\sqrt{k^2-\tau^2}) - ik\eta \, H_o'(a\sqrt{k^2-\tau^2})} \tag{18}$$

in which the prime denotes differentiation with respect to the whole argument. The Hertz vector for the scattered field is therefore

$$\underline{\pi} = (0,\ 0,\ U^s),$$

$$U^s = -i\frac{\pi}{k}\int_C e^{iz\tau}\,\frac{\sqrt{k^2-\tau^2}\,J_o(a\sqrt{k^2-\tau^2}) - ik\eta\,J_o'(a\sqrt{k^2-\tau^2})}{\sqrt{k^2-\tau^2}\,H_o(a\sqrt{k^2-\tau^2}) - ik\eta\,H_o'(a\sqrt{k^2-\tau^2})}\,H_o(r\sqrt{k^2-\tau^2})H_o(r_o\sqrt{k^2-\tau^2})\,d\tau, \tag{19}$$

which represents the formal solution of the problem, and from this the field components can be obtained by carrying out the differentiations indicated in equations (5) and (6).

For practical purposes the above solution is of little value as it stands and the main characteristics of the scattered field are by no means evident from (19). Some simplification is therefore necessary and in the course of this one of our primary objectives is to bring out the contribution of the travelling wave; however, the separation into travelling wave and radiated field appears as a natural consequence if the path of integration is deformed into one of steepest descents, and to this end we introduce the new variable α, where

$$\tau = k\cos\alpha.$$

The equation for U^s then becomes

$$U^s = -i\pi\int_{S(\pi/2)} e^{ikz\cos\alpha}\,\frac{\sin\alpha\,J_o(ka\sin\alpha) - i\eta\,J_o'(ka\sin\alpha)}{\sin\alpha\,H_o(ka\sin\alpha) - i\eta\,H_o'(ka\sin\alpha)}\,H_o(kr\sin\alpha) \tag{20}$$

$$\cdot\,H_o(kr_o\sin\alpha)\sin\alpha\,d\alpha$$

with $S(\pi/2)$ as a steepest descents contour passing through the real angle $\pi/2$.

Due to the logarithmic singularity of the Hankel function at the zero of its argument, branch points exist at $\alpha = 0$ and π, and for convenience the branch cuts are taken as shown in Figure 2. In addition the integrand has poles arising from the zeros of

$$H_0(ka \sin\alpha) - \frac{i\eta}{\sin\alpha} H_0'(ka \sin\alpha) \tag{21}$$

as a function of α, and these are in fact the sources of the travelling waves. A detailed discussion is given later, and for the moment it suffices to say that if $ka|\eta| < 0.1$, say, the relevant zero has $ka|\sin\alpha| = 0(10^{-1})$ or less, with arg $\sin\alpha \simeq 110^{o}$. In terms of α we now have the two zeros α_o and $\pi - \alpha_o$ outside but adjacent to the strip $0 \leq \mathrm{Re}\,\alpha \leq \pi$, and these are indicated in Figure 2.

To evaluate the integral in (20) the obvious approach is to apply a steepest descents analysis. If $ka = 0(10)$ or greater (as is true in most cases of practical interest), it follows that $kr \sin\alpha \gg 1$ providing only that the dominant contribution to the integral does not come from values of α in the neighbourhood of zero or π. Proceeding on the assumption that this requirement is fulfilled, the function $H_o(kr \sin\alpha)$ can be replaced by the leading term of its asymptotic expansion for large arguments to give

$$U^s \sim \sqrt{\frac{2\pi}{kR \sin\phi}}\; e^{i\pi/4} \int_{S(\pi/2)} e^{ikR\cos(\alpha-\phi)} f(k\cos\alpha)\, H_o(kr_o \sin\alpha)\,(\sin\alpha)^{1/2}\, d\alpha$$

where R and ϕ are as previously defined, and inasmuch as the saddle point is now $\alpha = \phi$, the substitution of the asymptotic formula for $H_o(kr \sin \alpha)$ is justified if $kr^2 >> R$. It will be observed that this condition is independent of the surface impedance η and consequently it is not entirely a statement of the minimum distance from the cylinder at which the influence of the travelling wave can be ignored. It is therefore feasible that in a displacement of the path of integration so as to pass through the saddle point a pole of the travelling wave could be included even though the above condition is still fulfilled. Practically, however, this is unlikely, and in the cases under investigation here the magnitude of α_o is such that the pole can be included only by violating the condition.

If $kr^2 >> R$ a simple displacement of the path of integration in (20) gives

$$U^s \sim \sqrt{\frac{2\pi}{kR \sin \phi}}\, e^{i\pi/4} \int_{S(\phi)} e^{ikR \cos(\alpha - \phi)} f(k \cos \alpha)\, H_o(kr_o \sin \alpha)(\sin \alpha)^{\frac{1}{2}}\, d\alpha$$

and since the non-exponential portion of the integrand is slowly varying in the neighbourhood of the saddle point, we have immediately that

$$U^s \sim 2\pi\, f(k \cos \phi)\, H_o(kr_o \sin \phi)\, \frac{e^{ikR}}{kR}, \tag{22}$$

which is entirely a radiating field. The polar diagram is 'spikey' with a continuous succession of peaks and near-zeros, but the average level shows little variation with ϕ.

This is clearly seen if ka is large enough to allow $J_o(ka \sin \phi)$ and $H_o(ka \sin \phi)$ to be replaced by their asymptotic forms, in which event

$$f(k \cos \phi) \sim -\frac{1}{2} \left[1 - i \frac{\gamma - \sin \phi}{+ \sin \phi} e^{-2ika \sin \phi} \right]$$

$$\simeq e^{-i(ka \sin \phi - \frac{\pi}{4})} \cos (ka \sin \phi - \frac{\pi}{4}),$$

and since $r_o \geqslant a$ it follows that

$$U^s \sim 2 \sqrt{\frac{2\pi}{kr_o \sin \phi}} \; e^{ik(r_o - a)\sin \phi} \cos(ka \sin \phi - \frac{\pi}{4}) \frac{e^{ikR}}{kR} . \qquad (23)$$

This is independent of γ implying that away from the cylinder the radiated field is unaffected by the surface impedance and is the same as if the cylinder had been perfectly conducting. Even if $r_o > a$ the polar diagram is equivalent to that of a source on the surface apart from a phase factor determined by the projected distance between the real and image currents.

5. <u>The Travelling Wave</u>

When $kR \gg 1$ but $(kr)^2$ <u>not</u> much greater than kR the above analysis fails and a detailed evaluation of the integral in (20) is no longer possible unless $kr \sin \alpha$ is small in the neighbourhood of the saddle point. In this case the Hankel function can be replaced by its logarithmic approximation, viz.

$$H_o(kr \sin \alpha) \sim \frac{2i}{\pi} \left\{ \log \frac{kr \sin \alpha}{2} + \gamma \right\} \qquad (24)$$

where γ is Euler's constant (0.5772157.....), and since $r \gg a$

$$f(k \cos \alpha) \sim \frac{ka}{2\gamma} (1 + i \frac{\eta ka}{2}) \sin^2 \alpha ,$$

leading to the following expression for the integrand in (20):

$$i \frac{ka}{\pi \eta} (1 + i \frac{\eta ka}{2}) e^{ikz \cos \alpha} (\log \frac{kr \sin \alpha}{2} + \gamma) H_o(kr_o \sin \alpha) \sin^3 \alpha.$$

The restrictions on kR and kr imply $k|z| >> 1$ and consequently we can again think in terms of a steepest descents evaluation with kz being the large parameter. The saddle point is now $\alpha = 0$ for ϕ small or $\alpha = \pi$ for $\pi - \phi$ small, and since the integrand vanishes at least as rapidly as $(\log \sin \alpha)^2 \sin^3 \alpha$ in either case, we have for the radiated field

$$U^s \sim 0 \qquad (25)$$

in the immediate vicinity of the cylinder (i.e. for $\sin \phi$ sufficiently small).

On the other hand, in a displacement of the path of integration to pass through this new saddle point the pole at $\alpha = \alpha_o$ or $\pi - \alpha_o$ will be included, leading to a residue contribution which is, in fact, the travelling wave. The residues at $\alpha = \alpha_o$ and $\pi - \alpha_o$ differ only in the sign of kz and obviously correspond to waves travelling in opposite directions. If α_o is the pole adjacent to the saddle point $\alpha = 0$ this will be included in a negative sense in any displacement of the path to pass through the origin, whereas the residue at $\pi - \alpha_o$ will have a positive sign associated with it, but these things apart the two waves are identical in all respects and it is sufficient to consider only the wave which travels in the positive z direction. In effect, therefore, we are restricting attention to the case in which ϕ is small.

The pole from which the travelling wave originates is provided by the function f(k cos α) and is given by the smallest root of

$$H_o(ka \sin \alpha) = \frac{i\eta}{\sin \alpha} H_o'(ka \sin \alpha) . \tag{26}$$

Unfortunately, a complete analytical solution of this equation is not possible and in order to proceed on a numerical basis it is necessary to set some bounds on the values of η and ka to be considered. Inasmuch as our purpose is to investigate travelling waves as they appear in radar scattering problems, η can be regarded as the surface impedance of a highly conducting metal. A typical value for $|\eta|$ is then 10^{-4}, corresponding to the conductivity of copper at a frequency of order 10 KMc, and since the complex refractive index is now dominated by the conduction current term,

$$\arg \eta = - \pi/4.$$

Under these conditions it is a relatively straightforward matter to determine $\sin \alpha_o$, and if $ka|\eta|$ is not greater than (say) 0.1, the solution of equation (26) can be found by inserting the logarithmic approximation (24) for the Hankel function. The details of the derivation are given in Goubau (1950) and it is there shown that

$$\sin \alpha_o = \frac{2\sqrt{b}}{ka} e^{-\gamma + i(5\pi/8 - \beta/2)}$$

where the real quantities b and β are related by the equations

$$b \log b = - \frac{ka}{2} |\eta| \cos \beta,$$

$$\tan \beta = \frac{\pi/4 - \beta}{\log b} .$$

These can be solved numerically, and in Figures 3 and 4 the resulting values of b and β are plotted as functions of ka $|\eta|$ for $10^{-6} \leq ka|\eta| \leq 10^{-1}$. It now only remains to specify η to determine α_o for different ka, and taking $\eta = 10^{-4}$ some values for α_o are as follows:

ka	10^{-1}	10	10^3
$\lvert\alpha_o\rvert$	6.5×10^{-3}	7.9×10^{-4}	1.2×10^{-4}
$\arg \alpha_o$	110.9^o	110.0^o	106.1^o

At the pole $\alpha = \alpha_o$ the residue of the integrand in equation (20) is

$$e^{ikz \cos \alpha_o} \frac{\sin \alpha_o J_o(ka \sin \alpha_o) - i\eta J_o'(ka \sin \alpha_o)}{\left[\frac{\partial}{\partial \alpha}\left\{\sin \alpha \, H_o(ka \sin \alpha) - i\eta \, H_o'(ka \sin \alpha)\right\}\right]_{\alpha=\alpha_o}}$$

$$\cdot H_o(kr \sin \alpha_o) \, H_o(kr_o \sin \alpha_o) \sin \alpha_o \cdot$$

Using equation (26) and the differential equation for the Hankel function the denominator becomes

$$\frac{\cos \alpha_o}{\eta} \left\{ 2\eta + ika\,(\eta^2 - \sin^2\alpha_o) \right\} H_o(ka \sin \alpha_o)$$

and since $J_o(ka \sin \alpha_o)$ can also be replaced by the first term of its series expansion, the residue can be written as

$$\frac{\eta \alpha_o^2}{2\gamma + ika(\eta^2 - \alpha_o^2)} \frac{H_o(kr_o \alpha_o)}{H_o(ka \alpha_o)} e^{ikz\sqrt{1-\alpha_o^2}} H_o(kr \alpha_o)$$

which has the general form characteristic of a travelling wave. For small ϕ we therefore have

$$U^s \sim -\frac{2\pi^2 \eta \alpha_o^2}{2\gamma + ika(\eta^2 - \alpha_o^2)} \frac{H_o(kr_o \alpha_o)}{H_o(ka \alpha_o)} e^{ikz\sqrt{1-\alpha_o^2}} H_o(kr \alpha_o), \qquad (27)$$

which reduces to

$$U^s \sim -\frac{2\pi^2 \eta \alpha_o^2}{2\gamma + ika(\eta^2 - \alpha_o^2)} e^{ikz\sqrt{1-\alpha_o^2}} H_o(kr \alpha_o) \qquad (28)$$

when the ring source is on the surface of the cylinder. The amplitude factor in (28) decreases uniformly from 2.0×10^{-2} to 1.0×10^{-6} as ka increases from 10^{-1} to 10^3, implying that the 'launching efficiency' of the source decreases with increasing size of cylinder. In fact the amplitude is almost proportional to 1/ka, but this must not be interpreted as indicating a total power which is independent of the radius of the cylinder.

6. Discussion

The above derivation of the travelling wave contribution has been carried out under the assumption that $ka|\eta| << 1$. This is certainly the situation which is likely to be encountered in practical cross section problems, but it is worth pointing out that if ka is so large that $ka|\eta| >> 1$, the solution of equation (26) is trivial. Substituting the asymptotic expansion of the Hankel function for large argument we obtain immediately

$$\sin \alpha_o = -\eta \tag{29}$$

which is comparable in magnitude to the results found in Section 5, but the most interesting feature is that now the travelling wave is nothing more than the Sommerfeld surface wave which a plane can support, with an amplitude which is exponentially attenuated away from the cylinder according to a factor $e^{-i\eta kr}$. This is in contrast to the behavior if $ka|\eta| << 1$ when it is necessary to go out to a distance of many tens of radii before the exponential decay sets in. Consequently, only with the very largest cylinders is it true that the travelling wave energy is confined to the immediate vicinity of the body, and even if $ka = 10^3$ the energy density may initially increase on moving away from the surface.

To make a direct calculation of the power in the travelling wave it is necessary to determine the field components and then integrate the z component of the Poynting vector over a plane normal to the axis of the cylinder. In the simplest case of a ring source actually on the cylinder the power contained is given approximately by the expression

$$P^t = -2Y\pi^3 k^2 \left| \frac{2\eta\,\alpha_0^3}{2\eta + ika(\eta^2 - \alpha_0^2)} \right|^2 \int_{ka}^{\infty} H_0^{(1)\prime}(x\,\alpha_0) H_0^{(2)\prime}(x\,\tilde{\alpha}_0)\, x\, dx, \qquad (30)$$

but we are now stuck by our inability to evaluate the integral, and even a numerical treatment would be a difficult undertaking in view of the complex nature of α_0. Only if ka is so large that ka $|\alpha_0| >> 1$ is any real progress possible, and bearing in mind that $\alpha_0 = -\eta$, the asymptotic expansion of the Hankel function can be inserted into (30) to give

$$P^t \sim 2\sqrt{2}\, Y\, \pi^2 k^2 \frac{|\eta|^4}{ka} e^{-|\eta| ka\sqrt{2}}. \qquad (31)$$

This is extremely small and although (31) does not necessarily provide any indication of the powers which would be obtained with cylinders of more practical size, the fact that the smallness is due in part to the modulus in (30) would appear to mitigate against P^t ever achieving a reasonable magnitude. Indeed, it seems likely that only in the case of very thin wires (ka << 1) can P^t become a significant fraction of P^i.

Further information in support of this conclusion can be obtained by considering the radiated field. If the ring source is on the surface so that $r_0 = a$, the Hertz vector for the radiated field reduces to

$$U^s \sim 2\sqrt{\frac{2\pi}{ka\sin\phi}} \cos(ka \sin\phi - \frac{\pi}{4}) \frac{e^{ikR}}{kR} \qquad (32)$$

(see equation 23), and this is identical to the expression for the incident Hertz vector (see equation 9) when approximated under the same condition $ka \sin \phi >> 1$. It follows that to this order of approximation the power in the radiated field is the same as that in the incident field when the source is placed in vacuo, and only an infinitesimal amount can be contained in the travelling wave; but since both U^i and U^s decrease with increasing r_o whereas the travelling wave amplitude is independent of r_o, the smaller the radius of the ring source the larger the power available for the travelling wave. As a result a source on the surface of the cylinder is the most effective in launching travelling waves, but it is still far from efficient as a launching mechanism.

7. Conclusion

Although the integrals representing the powers contained in the radiated field and the travelling wave are not amenable to computation, an examination of their form suggests that the latter is insignificant in comparison with the former. Inasmuch as the ring current was chosen in order to facilitate the coupling to the travelling wave, it would appear that in an actual diffraction problem where (for example) a plane wave is incident, the shadow boundary will not serve to launch the wave, and we must look elsewhere for the source. In effect, therefore, we are left with the nose of the body as the only possibility.

8. References

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9. Acknowledgement

The work described in this paper was carried out for the Air Force Ballistic Missile Division under subcontract from the Conductron Corporation to The University of Michigan as a part of the program being carried out under Chrysler Corporation Prime Contract AF 04(647)-912.

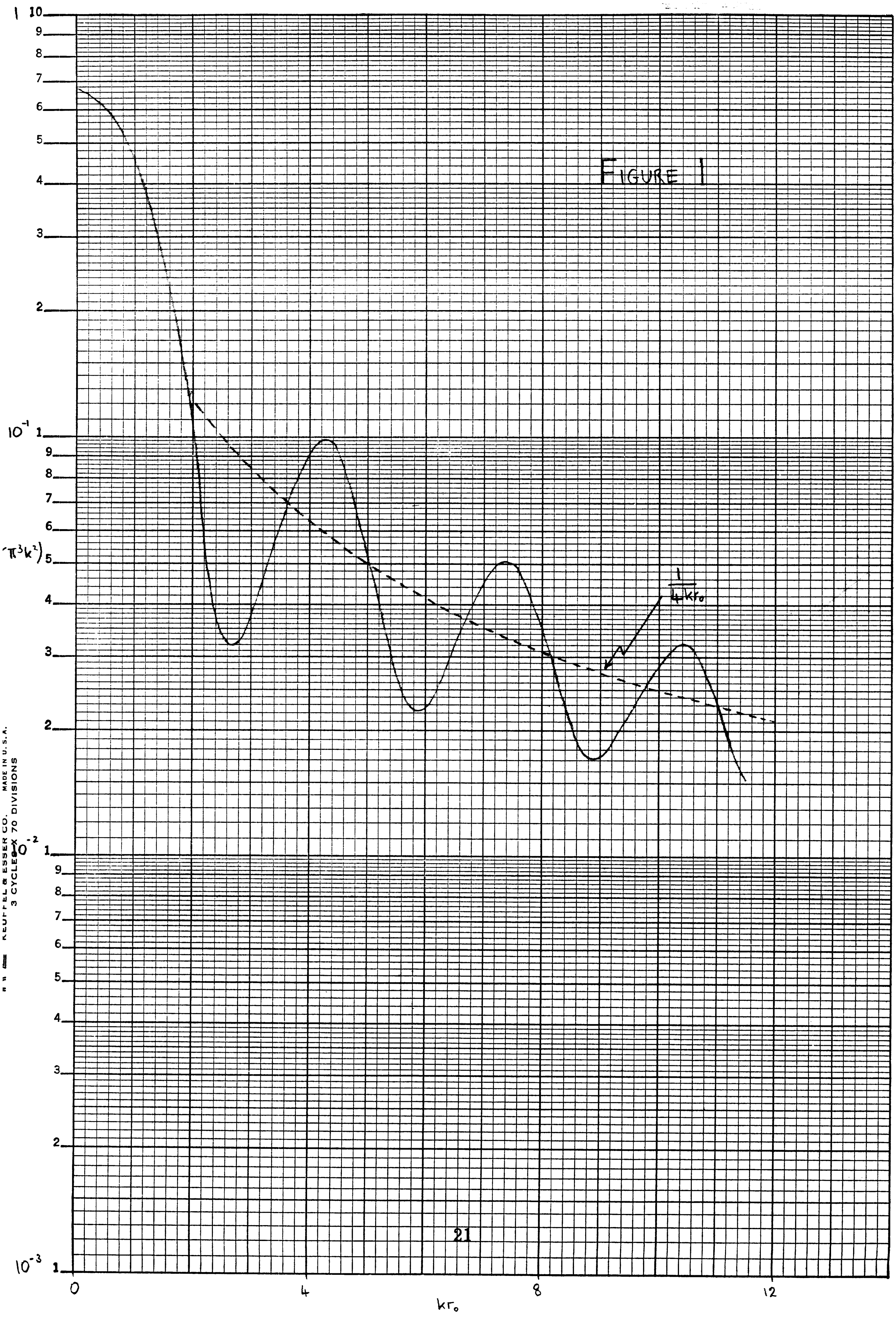


FIGURE 1
1/4kr₀
kr₀
(π³k²)
10⁻¹
10⁻²
10⁻³
0
4
8
12
KEUFFEL & ESSER CO. MADE IN U.S.A.
3 CYCLES X 70 DIVISIONS

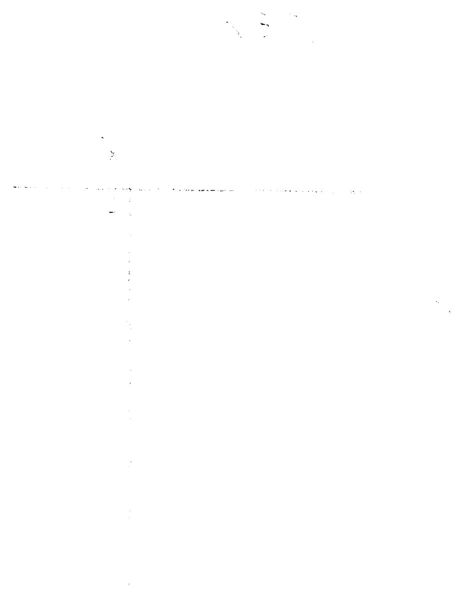
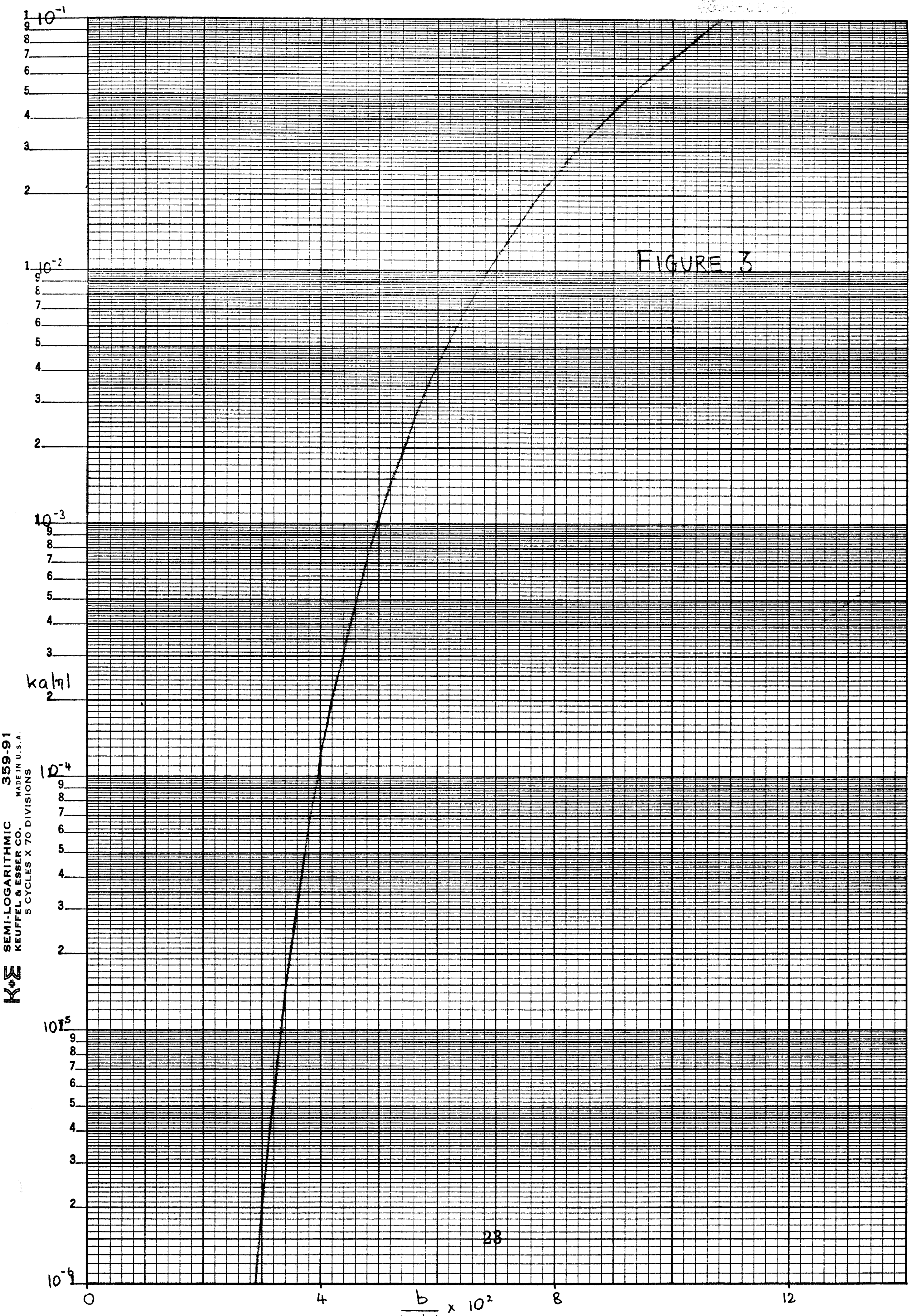


FIGURE 2 - STEEPEST DESCENTS PATH IN COMPLEX α PLANE



10-1
10-2
10-3
10-4
10-5
10-6
ka|η|
0
4
8
12
b/ka|η| x 10²
SEMI-LOGARITHMIC 359-91
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5 CYCLES X 70 DIVISIONS

FIGURE 3

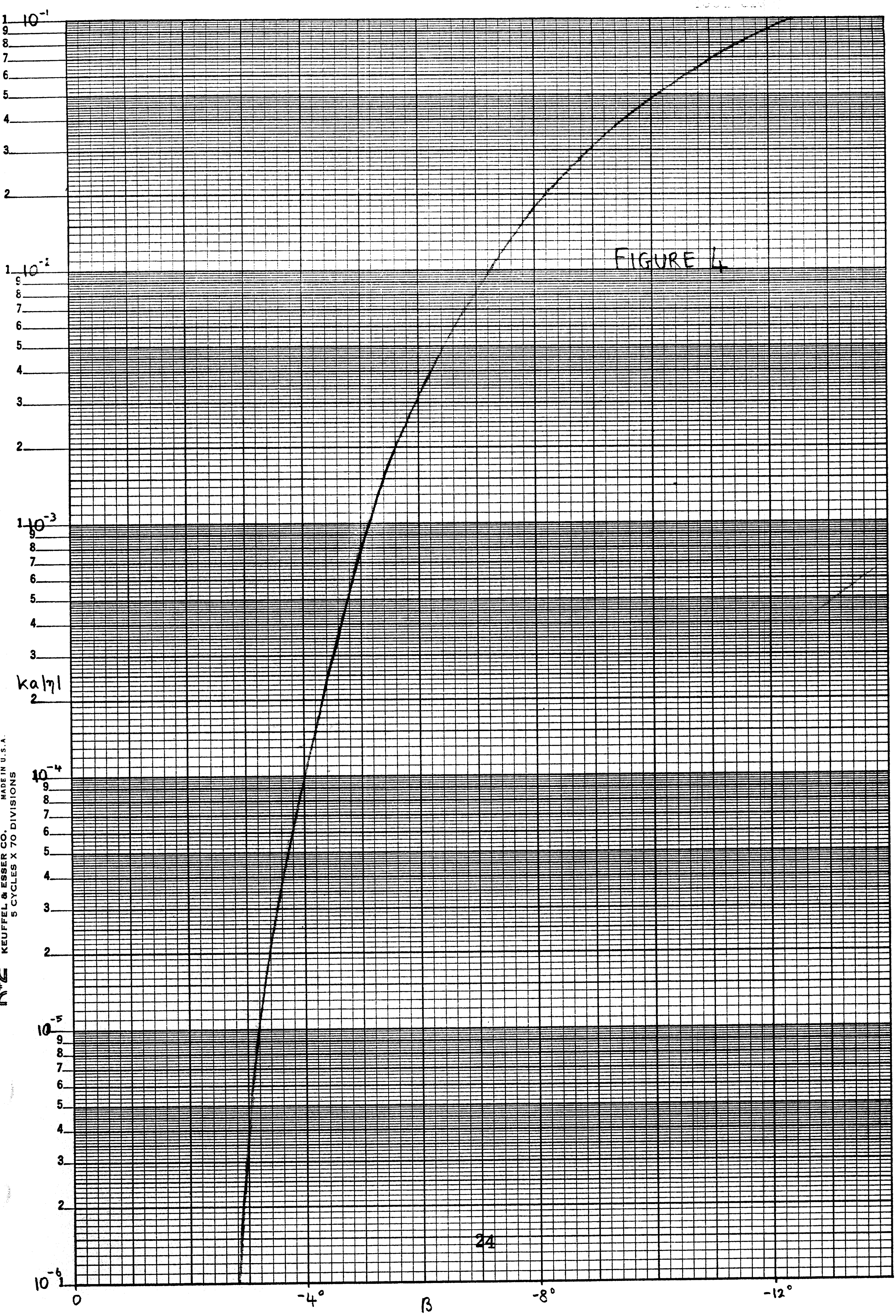
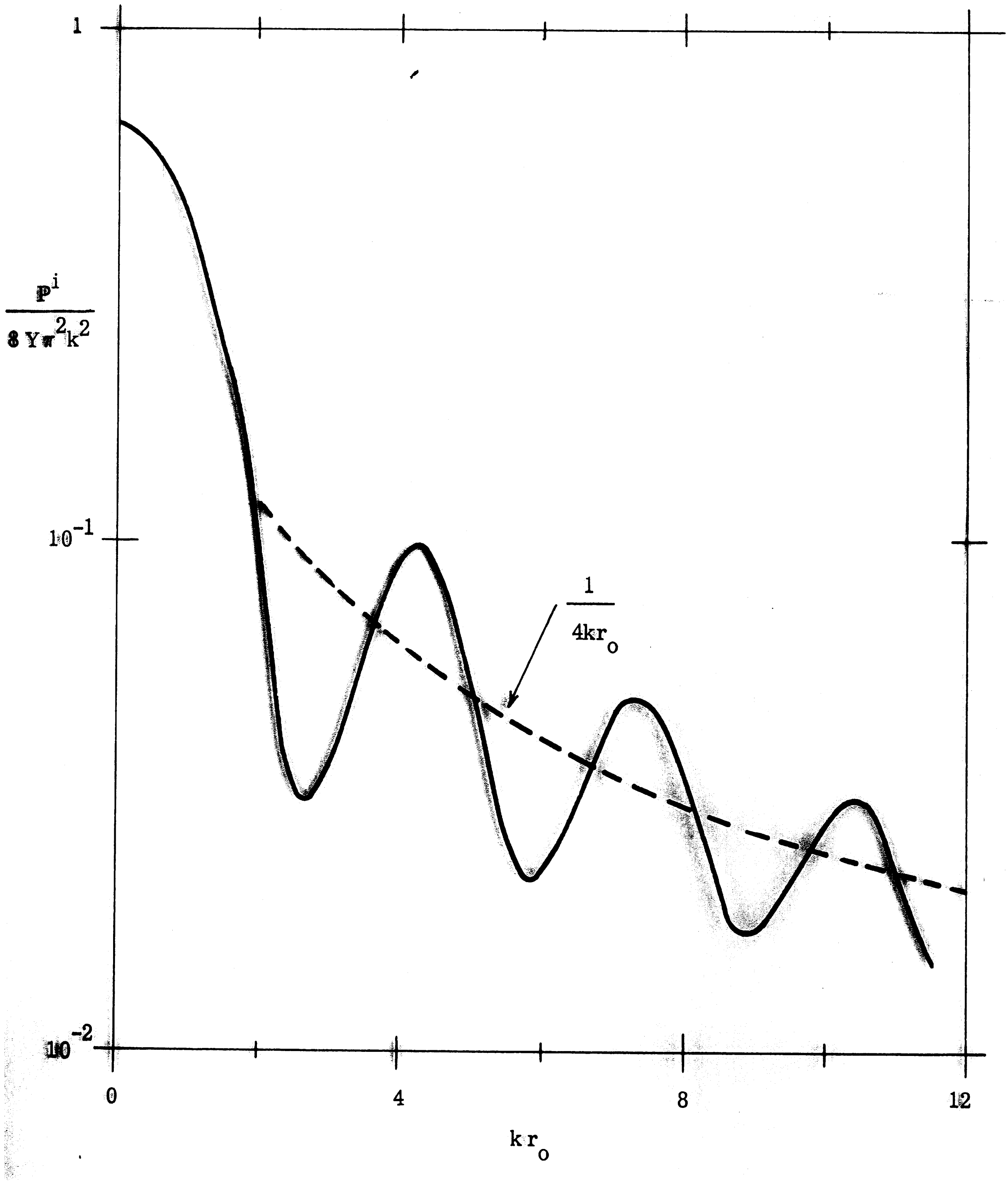


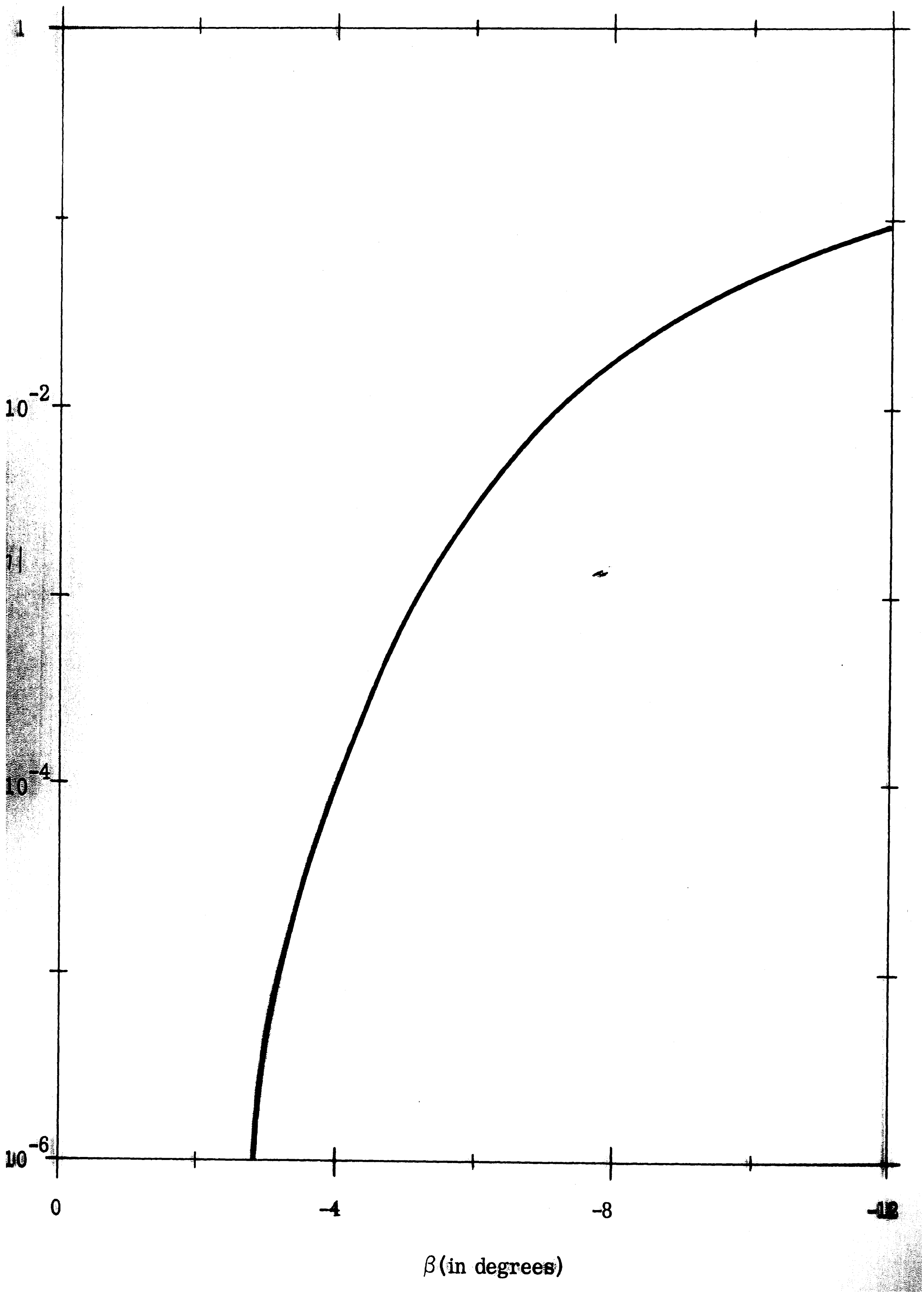
FIGURE 4
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
ka|η|
0
-4°
-8°
-12°
β

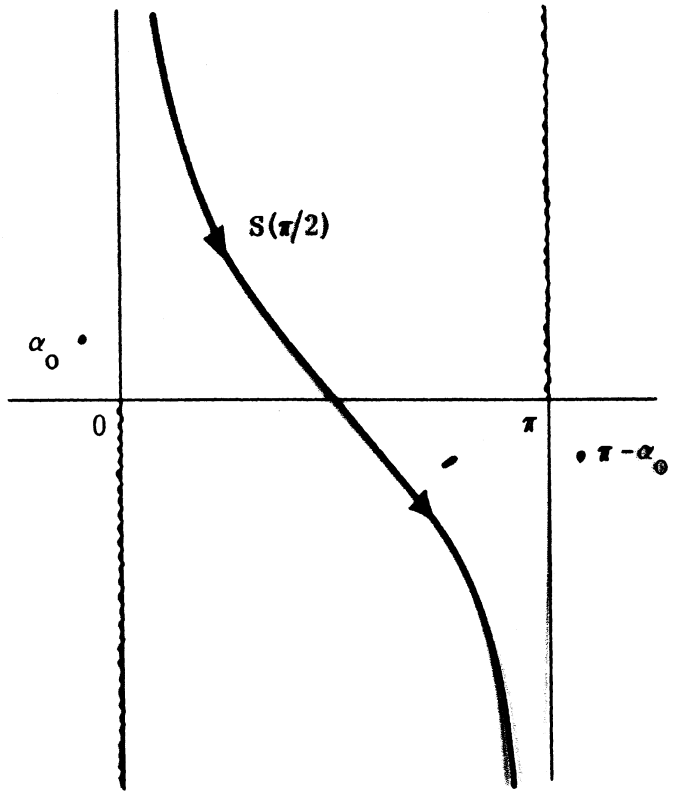
Legends for Figures

Fig. 1	Radiated power as function of loop radius
Fig. 2	Steepest descents path in complex α plane
Fig. 3	
Fig. 4	

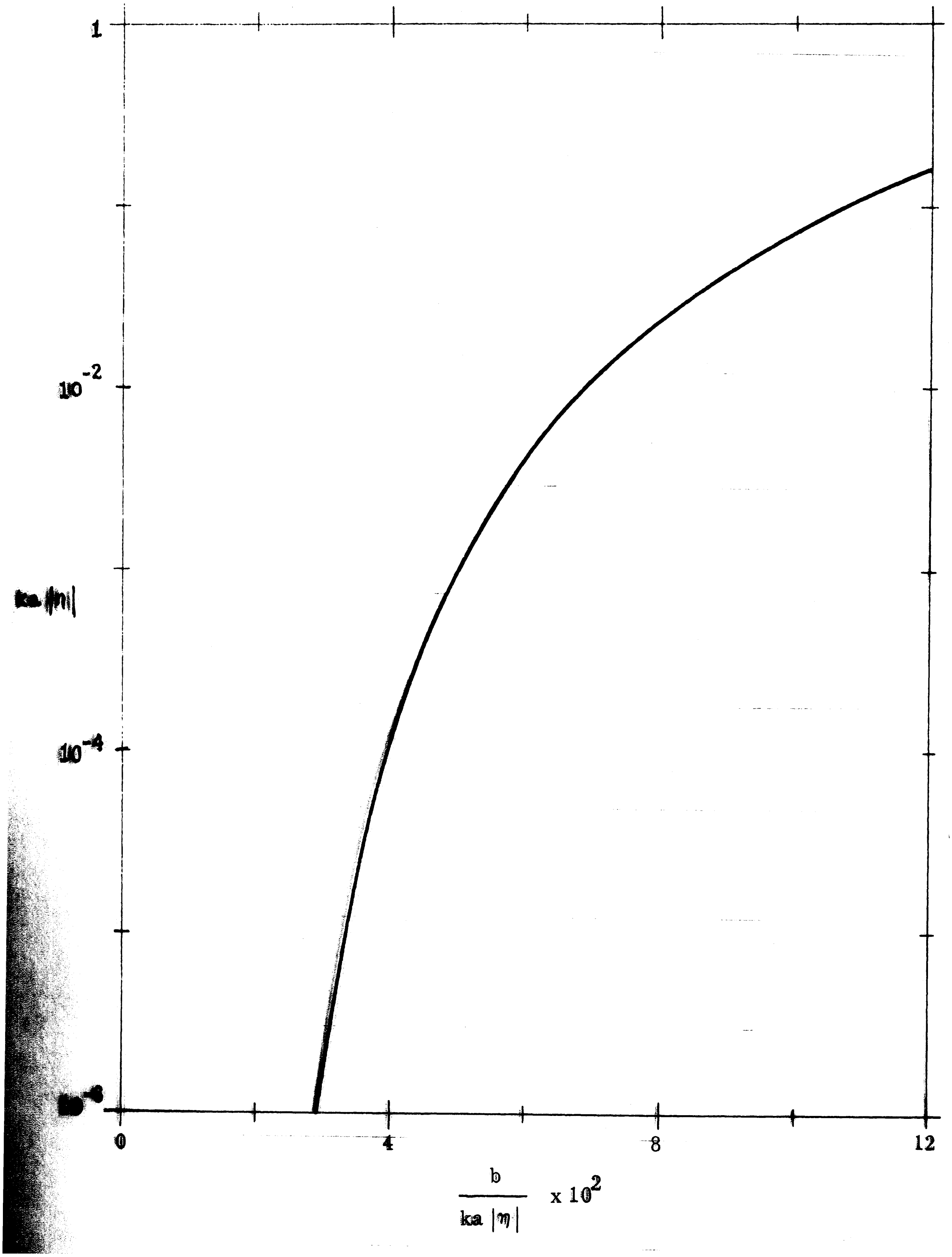


1
P^i
8 Yπ²k²
10^-1
10^-2
1
4kr_o
0
4
8
12
kr_o


1
10^{-2}
10^{-4}
10^{-6}
0
-4
-8
-12
β (in degrees)



S(π/2)
α_0
0
π
π - α_0



1
10^{-2}
10^{-4}
10^{-6}
$ka\,|\eta|$
0
4
8
12
$\frac{b}{ka\,|\eta|} \times 10^2$