334

ON A CLASS OF INTEGRAL EQUATIONS AND ITS APPLICATIONS TO THE THEORY OF LINEAR ANTENNAS

by
Dario Castellanos

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in the
University of Michigan
1964

Doctoral Committee:

Professor Chiao-Min Chu, Chairman
Professor Lamberto Cesari
Professor Nicholas D. Kazarinoff
Professor Charles B. Sharpe
Associate Professor Herschel Weil

RL-334 = RL-334

ADDENDUM TO

ON A CLASS OF INTEGRAL EQUATIONS AND ITS APPLICATIONS TO THE THEORY OF LINEAR ANTENNAS.

PREFACE

An exact solution of Hallén's integral equation has long been a goal of many researchers in the theory of linear antennas. This thesis presents a method of solution for a class of integral equations which, in particular, yields a closed form solution to Hallén's equation.

The germ of the idea leading to the solution is to be found in P. M. Morse and H. Feshbach's "Methods of Theoretical Physics" in the chapter on integral equations. Professor Feshbach was most kind in communicating his solution of the infinite system of equations, and his assistance is kindly acknowledged.

The contents of Professor E. D. Rainville's course "Special Functions" form most of the foundation of this thesis, and to him I express my most sincere gratitude.

To Professor C. M. Chu, chairman of the doctoral committee, I am especially indebted. His constant encouragement and guidance provided the continuing impetus for the completion of this work.

I also wish to acknowledge the assistance of Professors L. Cesari and N. D. Kazarinoff of the Mathematics Department, and C. B. Sharpe and H. Weil of the Electrical Engineering Department. From them I obtained many valuable suggestions during the preparation of the manuscript.

Finally I wish to thank Dr. S. G. Gallo and The Creole Foundation for the financial assistance they provided during my completion of the doctor's degree.

Ann Arbor, Michigan
July 1964.

An error has been brought to my attention by Dr. Olov Einarsson of the Radiation Laboratory, who has pointed out that one cannot hope to solve the system of equations (3.1-7) of Chapter I when a or b is infinite.

If one proceeds as indicated in the thesis it will always be found that

$$C_m = \int_a^b \exp(-y^2) H_m(y) g(y) dy \,, \tag{3.1-6}$$

is independent of the limits a and b. However, this equation gives us the m'th coefficient in the expansion of the function

$$\begin{cases} g(y) & a \leqslant x \leqslant b \\ 0 & \text{elsewhere} \end{cases}$$

in series of Hermite polynomials; and

$$C_m = \int_{-\infty}^{\infty} \exp(-y^2) H_m(y) h(y) dy \tag{3.1-12}$$

is the m'th coefficient in the expansion of

$$h(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-isy) \frac{\tilde{f}(s)}{\bar{K}(s)} ds \tag{3.1-11}$$

in series of Hermite polynomials. Since the Hermite polynomials form a complete set it would follow that g(y)=h(y) almost everywhere, which is certainly incorrect.

This shortcoming of the procedure can, however, be overcome by operating directly on the system of equations (3.1-7) and the corresponding one for the equation with infinite limits without making any attempt to eliminate the C's.

That is, one has

$$f^{(k)}(0) = \sum_{n=o}^{\infty} (-)^n a_n C_{n+k}$$

where

$$C_m = \int_a^b e^{-y^2} H_m(y) g(y) dy$$

and

$$f^{(k)}(0) = \sum_{n=o}^{\infty} (-)^n a_n C'_{n+k}$$

where

$$C'_m = \int_{-\infty}^{\infty} e^{-y^2} H_m(y) h(y) dy \ .$$

From this one can write

$$\sum_{n=o}^{\infty} (-)^n a_n C_{n+k} = \sum_{n=o}^{\infty} (-)^n a_n C'_{n+k} \ ,$$

or explicitly

$$C_0 - a_1 C_1 + a_2 C_2 - \ldots = C'_0 - a_1 C'_1 + a_2 C'_2 - \ldots$$
$$C_1 - a_1 C_2 + a_2 C_3 - \ldots = C'_1 - a_1 C'_2 + a_2 C'_3 - \ldots$$
$$C_2 - a_1 C_3 + a_2 C_4 - \ldots = C'_2 - a_1 C'_3 + a_2 C'_4 - \ldots \ .$$

If one multiplies the second equation by

$$a_1 + \frac{(-in\pi)}{(b-a)1!} \ ,$$

and adds it to the first equation, this gives

$$C_0 + \frac{(-in\pi)}{(b-a)1!} C_1 + a'_2 C_2 - \ldots = C'_0 + \frac{(-in\pi)}{(b-a)1!} C'_1 + a'_2 C'_2 \ldots$$

where a'_2, $a'_3, \ldots$ are the new coefficients. We can now multiply the third equation by

$$-a'_2 + \frac{(-in\pi)^2}{(b-a)^2 2!}$$

and add it to the first to give

$$C_0 + \frac{(-in\pi)}{(b-a)1!} C_1 + \frac{(-in\pi)^2}{(b-a)^2 2!} C_2 - a''_3 C_3 + \ldots$$

$$= C'_0 + \frac{(-in\pi)}{(b-a)!} C'_1 + \frac{(-in\pi)^2}{(b-a)^2 2!} C'_2 - a''_3 C'_3 + \ldots$$

It is obvious that by continuing this procedure one can again recover Equation (3.1-18)

$$\frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-s^2/4)\exp(-\frac{n\pi s}{b-a}) \frac{\tilde{f}(s)}{\bar{K}(s)}\, ds$$

$$= \int_a^b \exp(-y^2) g(y) \exp(-2n\frac{i\pi y}{b-a} + \frac{n^2\pi^2}{(b-a)^2})\, dy \; .$$

It should be noted that from an infinite set of relations between the C's and the C' 's we have obtained one relation at the best. It would seem a more satisfactory procedure to begin the above scheme with the k'th equation, i.e. multiply the (k+1)th equation by a certain constant and add it to the k'th . Multiply the (k+2)th equation by a certain constant and add it to the k' th, etc. In this manner one would obtain one such relation for each k and hope to determine a g(y) which would satisfy each of them. I must confess that if I proceed in this fashion I find myself unable to solve the resulting set.

It is my belief that because of the form of the solution

$$g(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-isy) \frac{\tilde{f}(s)}{\bar{K}(s)}\, \theta_3 \left[(b-a)(\frac{s}{2} - iy), e^{-(b-a)^2}\right] ds$$

and its limiting behavior as $b \to \infty$, $a \to -\infty$ giving h(y), that this g(y) is at least a part of the complete solution. If an independent proof were to be found, the above expression should appear naturally in the complete solution.

Furthermore when applied to the problem of the radiating antenna the solution yields an expansion for the current whose structure is in agreement with the travelling wave argument advanced by Hallén. Quantitatively, however, the solution falls short of being in agreement with experiment.

Recently P. C. Waterman in a Technical Memorandum entitled "Exact Theory of Scattering by Conducting Strips" (AVCO Research and Development Division, Wilmington, Mass.) has shown that the wave scattered by a strip $\psi(\alpha_0;r)$ is determined by the equation

$$\psi(\alpha_0;r)=-(-1/2)\,|\sin\alpha_0|\int_{-a}^{a} dkx'e^{ikx'\cos\alpha_0}H_0(kR')-S_\alpha\psi(\alpha;r),$$

where S_α is a linear operator which depends only in the solution for the infinite plane

$$-(1/2)\,|\sin\alpha_0|\int_{-\infty}^{\infty} kdx'e^{ikx'\cos\alpha_0}H_0(kR')\ .$$

It is seen that in this equation the scattering by the finite strip is described as the solution of an operator equation involving the solution of the infinite plane. This is a confirmation of our general belief that the solution of the equation with finite limits can be expressed in terms of the solution of the equation with infinite limits, inasmuch as the scattering by a strip is determined by a solution of the equation

$$\int_{-a}^{a} H_0^{(1)}(k|x-\xi|)J_z(\xi) = A\,E_z^i \qquad a\leqslant x\leqslant a\ .$$

....

DC/cfw

Dario Castellanos
The University of Michigan
Radiation Laboratory
30 March 1965

ABSTRACT

In this study a systematic method of solution is presented for a class of Fredholm integral equations. The most significant result is a closed form solution, valid under very weak restrictions, for the equations

$$f(x) = \int_a^b K(x-y)\, g(y)\, dy\,, \qquad a < x < b$$

and

$$f(x) + g(x) = \int_a^b K(x-y)\, g(y)\, dy\,, \qquad a < x < b$$

if the interval (a,b) is finite.

If the interval (a,b) is infinite the solution is given in series of orthogonal polynomials with explicit coefficients.

Integral equations of the first kind whose kernels are generating functions for polynomial sets are also treated,

$$f(x) = \int_a^b K(x,y)\, g(y)\, dy\,, \qquad a < x < b$$

$$K(x,y) = \sum_{n=0}^{\infty} \phi_n(y)\, x^n$$

where $\phi_n(y)$ is a polynomial of degree n in y.

A general solution is also obtained for the equation

$$f(x) = \int_a^b e^{-\lambda xy}\, g(y)\, dy$$

with λ generally complex. The special choice $a = 0$, $b = \infty$, and $\lambda = 1$ leading to Laplace's integral equation is illustrated by two examples.

A striking application of these results is a closed form solution of Hallén's integral equation for a radiating antenna. The current wave is obtained as the superposition of traveling waves predicted by Hallén. The current consists of two parts: a series of TM waves representing waveguide modes inside a hollow tube antenna and a superposition of traveling waves on the surface of the antenna.

One further application of the method concerns the problem of electromagnetic back-scattering from a cylindrical wire. The solution to this problem possesses all the qualitative features one would naturally expect, but contrary to common belief the end surfaces are seen to play a major role in the determination of the current distribution. The solution, consequently, affords a very poor representation of the physical facts.

The difficulties created by the presence of the end surfaces for the scattering problem do not exist for the radiating antenna when the antenna is assumed to be a hollow tube.

Table of Contents

List of Illustrations

Chapter One

Chapter Two

Chapter Three

CHAPTER ONE

A CLASS OF INTEGRAL EQUATIONS.

Section 1. Statement of the problem.

We wish to consider a class of Fredholm integral equations of the first kind

$$f(x) = \int_a^b K(x,y)g(y)dy \qquad a<x<b \qquad (1\text{-}1)$$

where f(x) and K(x,y) are given functions satisfying appropriate conditions and g(y) is to be determined. f(x) is defined in the interval a<x<b and K(x,y) in the square a<x,y<b.

As is well known, the central difficulty in the study of equations of this type is the following: if K(x,y) is continuous, Eq.(1-1) maps the set of all piecewise continuous functions into a more restrictive set, since all functions f(x) obtained in this manner are certainly continuous. If K(x,y) is differentiable every piecewise continuous function and in fact every integrable function is mapped into a differentiable function. Hence, in general, the integral equation with continuous f(x) cannot be solved by a continuous function g(y). If f(x) belongs to a more general class of functions we may expect Eq.(1-1) to be solvable only if K(x,y) deviates from continuity in some way.

In light of the above remarks, the method to be expounded in the sequel possesses a definite advantage. For instance, when the kernel is of the form K(x-y) it is only necessary in order for the method to be applicable that g(x) be integrable and bounded, provided that f and K satisfy suitable conditions. For other types of kernels the conditions on g(x) become more severe, but it is never necessary that the kernel be continuous.

The techniques to be presented were developed in order to solve certain integral equations which are of importance in the study of distribution of currents in linear antennas.

Section 2. Some results from the theory of distributions.

In what follows we shall have occasion to use some results of the theory of distributions. The pertinent lemmas are stated in this section and are proved in the appendix.

Lemma 1. In the sense of convergence of distributions

$$\lim_{\nu\to\infty} \int_{-\nu}^{\nu} \exp(ix\xi)\, d\xi = 2\pi\, \delta(x)$$

Lemma 2. If

$$f_t(x) = \frac{1}{2\sqrt{\pi t}} \exp(-x^2/4t), \qquad t>0$$

Then in the sense of convergence of distributions

$$\lim_{t\to 0} f_t(x) = \delta(x)$$

<u>Lemma 3</u>. If f(x) * g(x) denotes the convolution of two functions f(x) and g(x)

$$f(x) * g(x) = \int f(\xi)\, g(x-\xi)\, d\xi$$

and if D be a differential operator, then

$$D(f*g) = Df*g = f*Dg$$

<u>Lemma 4</u>. If $f_\nu \to f$, then $f_\nu * g \to f*g$ in each of the following cases:

a) The functionals f_ν are concentrated in one and the same bounded set.

b) g is concentrated in a bounded set.

c) The supports of the functionals f_ν and g are bounded on the same side in a manner independent of ν.

<u>Lemma 7</u>. With the Fourier transform of the functional f denoted by the symbols $\tilde{f}$ or F[f] we have

$$\widetilde{\exp(bx)} = 2\pi\, \delta(s-ib)$$

for any complex b.

<u>Lemma 8</u>. The Fourier transform of the convolution of two functions

$$f(x) * g(x) = \int f(\xi)\, g(x-\xi)\, d\xi$$

is given by

$$F\left[f(x) * g(x)\right] = \widetilde{f(\sigma)}\widetilde{g(\sigma)}.$$

Section 3. The kernel $K(x-y)$.

To obtain a solution of the equation

$$f(x) = \int_a^b K(x-y)g(y)dy \qquad a<x<b \qquad (3-1)$$

we will consider two separate cases: the interval (a,b) is finite and the interval (a,b) is infinite.

Section 3.1. The interval (a,b) is finite.

Section 3.11. Expression of the solution as an infinite integral.

We will construct a solution of Eq.(3-1) as a function of the solution of the same equation with both limits infinite. The essence of the method will be the determination of an entity related to Eq.(3-1), which is independent of the limits of the integral equation, this quantity being then evaluated in terms of the solution of the equation with infinite limits.

We will assume that f, K and g satisfy the following conditions

(i-1) $f(x)$ has a Maclaurin expansion

(ii-1) $f(x)$ can be continued analytically for all x and is such that its analytic continuation belongs to $L^2(-\infty,\infty)$

(iii-1) $K(x-y)$ has a formal (not necessarily convergent) series expansion

$$K(x-y) = \sum_{n=0}^{\infty} k_n(y)x^n$$

(iv-1) K(x) can be continued analytically for all x and is such that its analytic continuation is $L(-\infty,\infty)$.

(v-1) K(x) = $o(|x|^{-n})$ as $|x|\to\infty$ for all n>0.

(vi-1) $\tilde{f}(s)/\tilde{K}(s)$ belongs to $L^2(-\infty,\infty)$.

(vii-1) g(x) is bounded in (a,b), and the interval can be broken up into a finite number of open partial intervals, in each of which g(x) is monotonic.

(viii-1) $\exp(-s^2/4)\tilde{f}/\tilde{K}$ is bounded and integrable in every interval (a,b).

Conditions (i-1) to (viii-1) will later be relaxed by using distributions.

In Eq.(3-1) let us expand $\exp(t^2)K(t)$ in a series of Hermite polynomials(1)

$$\exp(t^2)K(t) = \sum_{n=0}^{\infty} a_n H_n(t) \qquad (3.1\text{-}1)$$

The coefficients a_n in this expansion are given by

$$a_n = \frac{1}{2^n n!\sqrt{\pi}} \int_{-\infty}^{\infty} H_n(t)K(t)dt \qquad (3.1\text{-}2)$$

The convergence of this integral is guaranteed by conditions (iv-1) and (v-1).

Replacing t by x-y in Eq.(3.1-1), there follows

$$K(x-y) = \exp(-y^2) \sum_{n=0}^{\infty} a_n \exp(2xy-x^2) H_n(x-y) \qquad (3.1\text{-}3)$$

By use of the generating function(2)

$$\exp(2xy-x^2)H_n(x-y) = \sum_{k=0}^{\infty} \frac{(-)^n H_{n+k}(y)\, x^k}{k!} \qquad (3.1\text{-}4)$$

Eq.(3.1-3) becomes

$$K(x-y) = \exp(-y^2) \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_n \frac{(-)^n H_{n+k}(y)\, x^k}{k!}$$

Introducing this expansion into Eq.(3-1), and making use of condition (i-1), there follows

$$f^{(k)}(0) = \sum_{n=0}^{\infty} (-)^n a_n \int_a^b \exp(-y^2) H_{n+k}(y) g(y) dy \quad , \quad (3.1\text{-}5)$$

Some words are necessary regarding the validity of Eq.(3.1-5), which was obtained by means of formal series; especially, since similar arguments are to be used again. It is necessary when studying the summation of series to satisfy requirements of absolute convergence for rearrangement of infinite series and uniform convergence for term by term integration. It is to be remembered, however, that such arguments are only pertinent when one is concerned with the sum of the series. Our only need was for establishing the relationship among the coefficients expressed by Eq.(3.1-5), and the sum of the series did not enter into the discussion. To justify these remarks one can replace the infinite series by finite sums which agree with the infinite expressions through the term in x^s to obtain

$$\sum_{k=0}^{s} f^{(k)}(0) \frac{x^k}{k!} = \sum_{k=0}^{s} \frac{x^k}{k!} \sum_{n=0}^{\infty} a_n (-)^n \int_a^b \exp(-y^2) H_{n+k}(y) g(y) dy.$$

Hence

$$f^{(k)}(0) = \sum_{n=0}^{\infty} a_n(-)^n \int_a^b \exp(-y^2)H_{n+k}(y)g(y)dy, \quad k=0,1,2,\ldots,s.$$

As s is arbitrary it follows that the above relation is true for all k.

Since the use of formal series leads to correct results in the case considered above, we shall continue, when necessary, to use them in what follows leaving the arguments in the present section as a guide to any further proofs.

Returning now to Eq.(3.1-5), we let

$$C_m = \int_a^b \exp(-y^2)H_m(y)g(y)dy \qquad (3.1\text{-}6)$$

and Eq.(3.1-5) takes the form

$$f^{(k)}(0) = \sum_{n=0}^{\infty} (-)^n a_n C_{n+k}; \text{ for each } k. \qquad (3.1\text{-}7)$$

Eqs.(3.1-7) are a set of equations determining C_n. This system of equations was solved by Professor Feshbach in order to solve Eq.(3-1) with $a = -\infty$, $b = \infty$[3]. His solution is reproduced here by permission. Let us write out equations (3.1-7)

$$f(0) = C_o - a_1C_1 + a_2C_2 - \ldots$$

$$f^{(1)}(0) = C_1 - a_1C_2 + a_2C_3 - \ldots$$

$$f^{(2)}(0) = C_2 - a_1C_3 + a_2C_4 - \ldots$$

$$\vdots$$

In the above a_o was set equal to one.

To find C_o one multiplies the expression for $f^{(1)}(0)$ by a_1 and adds the first two equations. This automatically eliminates C_1. This gives a new coefficient for C_2 which can be eliminated by multiplying the expression for $f^{(2)}(0)$ by a suitable coefficient. In this manner one can successively eliminate all the C's except C_o. A similar procedure obviously works for any C_n. Each C_n is thus obtained as a linear combination of $f^{(s)}(0)$

$$C_n = \sum_{s=0}^{\infty} f^{(n+s)}(0)\, T_s \qquad (3.1\text{-}8)$$

where the coefficients T_s are given by

$$T_s = \Sigma\, (-)^{r_2+r_4+\ldots} \frac{(r_1+r_2+r_3+\ldots)!}{r_1!\, r_2!\, r_3! \ldots} a_1^{r_1} a_2^{r_2} a_3^{r_3} \ldots \qquad (3.1\text{-}9)$$

summed over all combinations of the integers r_i such that

$$r_1 + 2r_2 + 3r_3 + 4r_4 + \ldots = s.$$

It is seen that the C's are determined only by the values of the derivatives of f(x) at the origin and the coefficients a_n of Eq.(3.1-1), but neither of these depends on the limits of the integral equation. In other words, we will obtain the system of equations (3.1-7) regardless of the limits in Eq.(3-1).

Because of conditions (ii-1) and (iv-1) we see that we can determine C_n without carrying out the difficult inversion (3.1-8) to (3.1-9), provided that we can solve the integral equation

$$f(x) = \int_{-\infty}^{\infty} K(x-y)h(y)dy \qquad -\infty<x<\infty \qquad (3.1\text{-}10)$$

This equation has been considered by Titchmarsh[4]. Titchmarsh proves the following theorem

Theorem 1. Let f(x) belong to $L^2(-\infty,\infty)$, and K(x) to $L(-\infty,\infty)$. Then in order that there should be a solution h(x) of $L^2(-\infty,\infty)$, it is necessary and sufficient that $\tilde{f}/\tilde{K}$ should belong to $L^2(-\infty,\infty)$.

Since the conditions of theorem 1 are satisfied by hypothesis it follows that h(x) is given by

$$h(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-isx) \frac{\tilde{f}(s)}{\tilde{K}(s)} \, ds \qquad (3.1\text{-}11)$$

and

$$C_m = \int_{-\infty}^{\infty} \exp(-y^2) H_m(y) h(y) dy. \tag{3.1-12}$$

In order to express C_m as a single integral let us expand exp(-isx) in a series of Hermite polynomials. We have

$$\exp(-isx) = \sum_{n=0}^{\infty} \frac{(-)^n i^n s^n x^n}{n!} \tag{3.1-13}$$

but[5]

$$x^n = \sum_{k=0}^{[n/2]} \frac{n!\ H_{n-2k}(x)}{k!\ 2^n (n-2k)!} \tag{3.1-14}$$

where [n/2] denotes the greatest integer in n/2. Replacing (3.1-14) in (3.1-13) we have

$$\exp(-isx) = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \frac{(-)^n i^n s^n n!\ H_{n-2k}(x)}{n!\ 2^n\ k!(n-2k)!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-)^n i^n (-)^k s^{n+2k}\ H_n(x)}{2^{n+2k}\ k!\ n!},$$

$$= \exp(-s^2/4) \sum_{n=0}^{\infty} \frac{(-)^n\ i^n\ s^n\ H_n\ (x)}{2^n\ n!} \tag{3.1-15}$$

and from Eq.(3.1-11)

$$h(x) = \sum_{n=0}^{\infty} \frac{(-)^n i^n H_n(x)}{2^n\, n!\, 2\pi} \int_{-\infty}^{\infty} \exp(-s^2/4) s^n \frac{\tilde{f}(s)}{\tilde{k}(s)}\, ds$$

so that, using the orthogonality of the Hermite polynomials

$$C_m = \frac{(-)^m i^m}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-s^2/4) s^m \frac{\tilde{f}(s)}{\tilde{k}(s)}\, ds \qquad (3.1\text{-}16)$$

This solution is, of course, equivalent to (3.1-8).

Combining Eqs. (3.1-6) and (3.1-16) we have

$$\frac{(-)^m i^m}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-s^2/4) s^m \frac{\tilde{f}(s)}{\tilde{k}(s)}\, ds = \int_a^b \exp(-y^2) H_m(y) g(y)\, dy \qquad (3.1\text{-}17)$$

Let us multiply both sides of Eq.(3.1-17) by $(-in\pi/(b-a))^m/m!$ and sum over m. This gives

$$\frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-s^2/4) \exp(-\frac{n\pi s}{b-a}) \frac{\tilde{f}(s)}{\tilde{k}(s)}\, ds$$

$$= \int_a^b \exp(-y^2) g(y) \exp(-2n\frac{i\pi y}{b-a} + \frac{n^2\pi^2}{(b-a)^2})\, dy \qquad (3.1\text{-}18)$$

where we made use of Eq.(3.1-4) with n = 0.

The interchange of the order of summation and integration can be justified without difficulty. For the right hand side it follows by condition (vii-1) that g(x) is integrable[(6)] and bounded. The series (3.1-4) converges uniformly for all y. The interchange is then permissible by a well known theorem of analysis [(7)]. Similarly for the left hand side it suffices to recall that the exponential series converges uniformly for all values of its argument. Hence by condition (viii-1) and a pertinent theorem[8], term by term integration is permissible.

Equation (3.1-18) can be written in the form

$$\frac{1}{b-a}\int_a^b \exp\left(-2i\,\frac{n\pi y}{b-a}\right)\exp(-y^2)\,g(y)\,dy$$

$$= \frac{\exp\left(\frac{-n^2\pi^2}{(b-a)^2}\right)}{2\sqrt{\pi}\,(b-a)}\int_{-\infty}^{\infty}\exp(-s^2/4)\exp\left(-\frac{n\pi s}{b-a}\right)\frac{\widetilde{f(s)}}{\widetilde{K(s)}}\,ds.$$

This is just the Fourier coefficient in the expansion of $\exp(-y^2)g(y)$ in a Fourier series in the interval (a,b)

$$g(y)=\frac{\exp(y^2)}{2\sqrt{\pi}(b-a)}\sum_{n=-\infty}^{\infty}\exp\left(i\,\frac{2n\pi y}{b-a}-\frac{n^2\pi^2}{(b-a)^2}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{s^2}{4}-\frac{n\pi s}{b-a}\right)\frac{\widetilde{f(s)}}{\widetilde{K(s)}}\,ds$$

Since condition (vii-1) corresponds to Dirichlet conditions[9], the above series is uniformly convergent and it follows that

$$g(y) = \frac{\exp(y^2)}{2\sqrt{\pi}(b-a)}\int_{-\infty}^{\infty}\exp(-s^2/4)\,\frac{\widetilde{f(s)}}{\widetilde{K(s)}}$$

$$\left[\sum_{n=-\infty}^{\infty}\exp\left(-\frac{n^2\pi^2}{(b-a)^2}\right)\exp\, i2n\left(\frac{\pi y}{b-a}+i\frac{\pi s}{2(b-a)}\right)\right]\,ds$$

The expression in brackets will be recognized as the third theta function defined by[10]

$$\theta_3(z,q) = \sum_{n=-\infty}^{\infty} q^{n^2}\exp(i2nz)$$

hence

$$g(y)=\frac{\exp(y^2)}{2\sqrt{\pi}(b-a)}\int_{-\infty}^{\infty}\exp(-s^2/4)\frac{\widetilde{f(s)}}{\widetilde{K(s)}}\,\theta_3\left(\frac{\pi y}{b-a}+i\frac{\pi s}{2(b-a)},\ e^{-\pi^2/(b-a)^2}\right)ds \qquad (3.1\text{-}19)$$

With the notation

$$q = \exp(\pi i \tau), \quad \mathrm{Im}(\tau) > 0, \qquad (3.1\text{-}20)$$

and

$$\theta_3(z, q) = \theta_3(z|\tau)$$

the third theta function is known to satisfy the functional equation(11)

$$\theta_3(z|\tau) = (-i\tau)^{-1/2} \exp\left(\frac{z^2}{\pi i \tau}\right)\theta_3\left(\frac{z}{\tau}\Big| -\frac{1}{\tau}\right) \qquad (3.1\text{-}21)$$

Use of equation (3.1-21) in equation (3.1-19) yields

$$g(y) = \frac{1}{2\pi}\int_{\infty}^{\infty} \exp(-iys)\frac{\tilde{f}(s)}{K(s)}\,\theta_3\left((b-a)\left(\frac{s}{2} - iy\right), e^{-(b-a)^2}\right) ds$$

(3.1-22)

which is the desired solution.

We wish now to show that by appealing to the theory of distributions it is possible to relax considerably conditions i-1 to viii-1 and still preserve Eq.(3.1-22) as a solution.

We will assume, then, that f, K and g of Eq.(3-1) satisfy the following conditions

(i-2) f(x) can be continued analytically for all x and is such that its analytic continuation belongs to $L(-\infty, \infty)$

(ii-2) K(x-y) has a formal expression

$$K(x-y) = \sum_{n=0} k_n(y)x^n$$

(iii-2) K(x) can be continued analytically for all x and is such that its analytic continuation is $L(-\infty, \infty)$.

(iv-2) $K(x) = o(|x|^{-n})$ as $|x| \to \infty$ for all $n > 0$.

(v-2) G(y,s) of Eq. (3.1-23) is integrable and bounded in the interval $a < y < b$.

Consider the equation

$$\frac{1}{2\pi} \widetilde{f(s)} \exp(-isx) = \int_a^b K(x-y)G(y,s)dy, \quad a < x < b \qquad (3.1\text{-}23)$$

where $\widetilde{f(s)}$ denotes the Fourier transform of $f(x)$ and s is to be interpreted as a parameter.

Since the left hand side of Eq.(3.1-23) has a Maclaurin expansion it is possible to proceed as before to Eq.(3.1-7) with

$$C_m = \int_a^b \exp(-y^2) H_m(y)G(y,s)dy \, . \qquad (3.1\text{-}24)$$

Since $\exp(-isx)$ can be continued analytically for all x, we will determine C_m by solving the equation

$$\frac{1}{2\pi} \widetilde{f(s)} \exp(-isx) = \int_{-\infty}^{\infty} K(x-y)h(y)dy \qquad (3.1\text{-}25)$$

By use of lemmas 7 and 8 of section 2, we have

$$h(x) = F^{-1}\left[\frac{\widetilde{f(s)}\,\delta(\sigma-s)}{\widetilde{K(\sigma)}}\right] \qquad (3.1\text{-}26)$$

where F^{-1} denotes the inverse Fourier transform. It follows then that

$$\left(F^{-1}\left[\frac{\delta(\sigma-s)\widetilde{f(s)}}{\widetilde{K(\sigma)}}\right], \varphi\right) = \frac{1}{2\pi}\left(\frac{\delta(\sigma-s)\widetilde{f(s)}}{\widetilde{K(\sigma)}}, \psi\right)$$

$$= \frac{1}{2\pi}\left(\delta(\sigma - s), \frac{\overline{\widetilde{f(\sigma)}}}{\overline{\widetilde{K(\sigma)}}}\psi\right) = \frac{1}{2\pi}\frac{\overline{\widetilde{f(s)}}}{\overline{\widetilde{K(s)}}}\psi(s) =$$

$$\int_{-\infty}^{\infty} \frac{1}{2\pi}\frac{\overline{\widetilde{f(s)}}}{\overline{\widetilde{K(s)}}} e^{isx}\,\varphi(x)\,dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi}\,\overline{\frac{\widetilde{f(s)}}{\widetilde{K(s)}}\,\bar{e}^{isx}}\;\varphi(x)\,dx = \left(\frac{1}{2\pi}\frac{\widetilde{f(s)}}{\widetilde{K(s)}}\,\bar{e}^{isx}, \varphi(x)\right),$$

so that

$$F^{-1}\left[\frac{\widetilde{f(s)}\,\delta(\sigma - s)}{\widetilde{K(\sigma)}}\right] = \frac{1}{2\pi}\frac{\widetilde{f(s)}}{\widetilde{K(s)}}\exp(-isx). \qquad (3.1\text{-}27)$$

The above derivation of Eq.(3.1-27) is based on Parseval's formula for distributions

$$(g,\psi) = 2\pi(f,\varphi)$$

which is discussed in the appendix.

Hence

$$h(x) = \frac{1}{2\pi}\frac{\widetilde{f(s)}}{\widetilde{K(s)}}\exp(-isx).$$

By use of Eq.(3.1-15) it follows at once that

$$C_m = \frac{\widetilde{f(s)}\exp(-s^2/4)(-)^m\, i^m\, s^m}{2\sqrt{\pi}\,\widetilde{K(s)}}$$

and from Eq.(3.1-24)

$$\frac{\widetilde{f(s)}\exp(-s^2/4)(-)^m\, i^m\, s^m}{2\sqrt{\pi}\,\widetilde{K(s)}} = \int_a^b \exp(-y^2)H_m(y)G(y,s)\,dy$$

~~Proceeding as in~~ Eq.(3.1-17) we obtain

$$G(y,s)=\frac{1}{2\pi}\exp(-iys)\frac{\tilde{f}(s)}{\tilde{K}(s)}\;\theta_3\left((b-a)\left(\frac{s}{2}-iy\right),\,e^{-(b-a)^2}\right) \qquad (3.1\text{-}28)$$

as the solution of Eq.(3.1-23).

Let us now integrate both sides of Eq.(3.1-23) with respect to s from $-\nu$ to ν

$$\frac{1}{2\pi}\int_{-\nu}^{\nu}\tilde{f}(s)\exp(-isx)\,ds=\int_a^b K(x-y)\int_{-\nu}^{\nu}G(y,s)\,ds\,dy \qquad (3.1\text{-}29)$$

The interchange of the order of integration follows by Fubini's theorem since the left hand side is integrable.

Letting

$$g_\nu=\int_{-\nu}^{\nu}G(y,s)\,ds$$

and assuming that the infinite integral

$$\int_{-\infty}^{\infty}G(y,s)\,ds$$

converges, we have by virtue of lemma 4, since the interval (a,b) is finite

$$f(x)=\int_a^b k(x-y)\int_{-\infty}^{\infty}G(y,s)\,ds\,dy$$

In other words the solution of Eq.(3-1) is

$$g(y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-iys)\frac{\widetilde{f(s)}}{K(s)}\quad \theta_3\Big((b-a)(\frac{s}{2}-iy),\exp(-(b-a)^2)\Big)\,ds$$

which is equation (3.1-22).

It is not difficult to extend the previous results to equations of the second kind

$$f(x)+g(x)=\int_a^b K(x-y)g(y)dy,\quad a<x<b. \qquad (3.1\text{-}30)$$

Considering g(x) as a testing function vanishing outside the interval (a,b) we can write

$$f(x)+\big(\delta(x-y),\,g(y)\big)=\big(K(x-y),g(y)\big),$$

that is

$$f(x)=\big(K(x-y)-\delta(x-y),\ g(y)\big) \qquad (3.1\text{-}31)$$

Let us consider the related equation

$$f(x)=\int_a^b \Big(K(x-y)-\frac{1}{2\sqrt{\pi t}}\exp(-(x-y)^2/4t\Big)G(y,t)\,dy \qquad (3.1\text{-}32)$$

where t is a parameter greater than zero.

Letting

$$K'(x-y)=K(x-y)-\frac{1}{2\sqrt{\pi t}}\exp\big(-(x-y)^2/4t\big) \qquad (3.1\text{-}33)$$

and assuming that f, K, and G satisfy conditions i-2 to v-2, we have from Eq. (3.1-22)

$$G(y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-iys)\,\frac{\widetilde{f(s)}}{\widetilde{K}(s)}\,\theta_3\left((b-a)\left(\frac{s}{2} - iy\right), e^{-(b-a)^2}\right)ds. \tag{3.1-34}$$

~~Use of~~ the result

$$\exp(-s^2t) = \int_{-\infty}^{\infty}\exp(isx)\left(\frac{1}{2\sqrt{\pi t}}\ \exp(-x^2/4t)\right)\ dx$$

in Eq.(3.1-34) gives

$$G(y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-iys)\,\frac{\widetilde{f(s)}}{\widetilde{K(s)}-\exp(s^2t)}\,\theta_3\left((b-a)\left(\frac{s}{2} - iy\right), e^{-(b-a)^2}\right)ds \tag{3.1-35}$$

as the solution of Eq.(3.1-32).

Taking the limit of Eq. (3.1-32) as t approaches zero gives, in view of lemma 2

$$f(x) = \Big(K(x-y) - \delta(x-y), g(y)\Big)$$

where

$$g(y) = \lim_{t\to 0} G(y,t)$$

It follows then from Eq.(3.1-35) that the solution of Eq.(3.1-30) is

$$g(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-iys)\,\frac{\widetilde{f(s)}}{\widetilde{K(s)}-1}\,\theta_3\left((b-a)\left(\frac{s}{2} - iy\right), e^{-(b-a)^2}\right)ds \tag{3.1-36}$$

which is the required result

Section 3.12. Expression of the solution in series of orthogonal polynomials.

It is possible to obtain the solution of Eq.(3-1) in series of orthogonal polynomials. This form of the solution has the advantage that it is not necessary to obtain the analytic continuation of f(x) in order to determine g(x). One other point in its favor will be seen in the next section.

f, K, and g will be assumed to satisfy the following conditions

(i-3) f(x) has a Maclaurin expansion

(ii-3) K(x-y) has a formal expansion

$$K(x-y) = \sum_{n=0}^{\infty} k_n(y)x^n$$

(iii-3) K(x) can be continued analytically for all x and is such that its analytic continuation is $L(-\infty,\infty)$. (iv-3) $K(x) = o(|x|^{-n})$ as $|x| \to \infty$ for all n>0.

(v-3) $\exp(-y^2)\ g(y)/w(y)$ can be expanded in a uniformly convergent series

$$\sum_{n=0}^{\infty} b_n \mathcal{M}_n(y)$$

where $\{\mathcal{M}_n\}$ is a set of polynomials orthogonal in the interval (a,b) with respect to the weighting function $w(y) > 0$.

Conditions (i-3) to (iv-3) are sufficient to go from Eq.(3.1) to Eq.(3.1-5). With C_m defined by Eq.(3.1-6) we have

$$C_m = \int_a^b \exp(-y^2)\, H_m(y) g(y) dy \qquad (3.1\text{-}6)$$

Multiplying both sides of this equation by $(-t)^m/2^m m!$ and summing from $m = 0$ to ∞, we obtain

$$\sum_{m=0}^{\infty} \frac{C_m(-)^m t^m}{2^m\, m!} = \int_a^b \exp(-y^2)\exp\left(-yt - \frac{t^2}{4}\right) g(y) dy$$

or

$$\exp\left(\frac{t^2}{4}\right) \sum_{m=0}^{\infty} \frac{C_m(-)^m\, t^m}{2^m\, m!} = \int_a^b \exp(-yt)\,(\exp(-y^2) g(y))dy$$

Expanding the exponentials in t in a power series and collecting powers of t we have

$$\sum_{m=0}^{\infty} \sum_{n=0}^{[m/2]} \frac{C_{m-2n}(-)^m t^m}{2^m (m-2n)!\,n!} = \sum_{m=0}^{\infty} \frac{(-)^m t^m}{m!} \int_a^b y^m \exp(-y^2) g(y) dy$$

comparing coefficients

$$\sum_{n=0}^{[m/2]} \frac{C_{m-2n}\; m!}{2^m (m-2n)!\,n!} = \int_a^b y^m \exp(-y^2)\; g(y) dy \qquad (3.1\text{-}37)$$

We now expand $\exp(-y^2)g(y)/w(y)$ in a uniformly convergent series

$$\exp(-y^2)g(y)/w(y) = \sum_{n=0}^{\infty} b_n \Psi_n(y) \qquad (3.1\text{-}38)$$

where $\Psi_n(y)$ are polynomials orthogonal in the interval (a,b) with respect to the weighting function $w(y) > 0$. The existence of such a set is guaranteed by application of the Gram-Schmidt orthogonalization process to the sequence of functions $(w(y))^{1/2}1$, $(w(y))^{1/2}y$, $(w(y))^{1/2}y^2$,... in the interval (a,b). We shall assume that the set $\{\Psi_n(y)\}$ is complete.

We will assume that the polynomials $\Psi_n(y)$ are given by an equation of the form

$$\Psi_k(y) = \sum_{m=0}^{K} \alpha(k,m) y^m \qquad (3.1\text{-}39)$$

Multiplying both sides of Eq. (31.-37) by $\alpha(k,m)$ and summing over m from 0 to k we find

$$\sum_{m=0}^{K} \sum_{n=0}^{[m/2]} \frac{\alpha(k,m) C_{m-2n}\, m!}{2^m (m-2n)!\, n!} = \int_a^b \Psi_k(y) \exp(-y^2) g(y)\, dy \qquad (3.1\text{-}40)$$

substitution of Eq. (3.1-38) in Eq. (3.1-40) yields

$$b_k = \frac{1}{g_k} \sum_{m=0}^{K} \sum_{n=0}^{[m/2]} \frac{\alpha(k,m) C_{m-2n}\, m!}{2^m (m-2n)!\, n!} \qquad (3.1\text{-}41)$$

where

$$g_k = \int_a^b w(y)\mathcal{H}_k^2(y)\,dy \qquad (3.1\text{-}42)$$

~~If the~~ polynomials $\mathcal{H}_n(y)$ are such that $\mathcal{H}_n(y)$ is an even function of y when n is even and odd when n is odd, then $\mathcal{H}_n(y)$ is given by an equation of the form

$$\mathcal{H}_n(y) = \sum_{m=0}^{[n/2]} \beta(n,m)\, y^{n-2m} \qquad (3.1\text{-}43)$$

In this case we have from Eq. (3.1-37)

$$\sum_{m=0}^{[k/2]} \sum_{n=0}^{[k/2-m]} \frac{\beta(k,m)\, C_{k-2m-2n}\,(k-2m)!}{2^{k-2m}(k-2m-2n)!\; n!} = \int_a^b \mathcal{H}_k(y)\exp(-y^2)\,g(y)\,dy \qquad (3.1\text{-}44)$$

and substituting, as before, Eq. (3.1-38) in Eq. (3.1-44) we find

$$b_k = \frac{1}{g_k} \sum_{m=0}^{[k/2]} \sum_{n=0}^{[k/2-m]} \frac{\beta(k,m)\, C_{k-2m-2n}\,(k-2m)!}{2^{k-2m}(k-2m-2n)!\; n!} \qquad (3.1\text{-}45)$$

with g_k given by Eq. (3.1-42).

We have shown then that the solution of Eq. (3-1) is given by

$$g(y) = \exp(y^2)\, w(y) \sum_{k=0}^{\infty} \sum_{m=0}^{k} \sum_{n=0}^{[m/2]} \frac{\alpha(k,m)\, m!\; C_{m-2n}\,\mathcal{H}_k(y)}{g_k 2^m\,(m-2n)!\,n!} \qquad (3.1\text{-}46)$$

where

$$\mathcal{H}_k(y) = \sum_{m=0}^{k} \alpha(k,m)\, y^m \tag{3.1-39}$$

are polynomials orthogonal in the interval (a,b) with respect to the weighting function $w(y) > 0$. C_m is given by Eq.(3.1-8).

If the polynomial set is such that $\mathcal{H}_n(y)$ is an even function of y when n is even and odd when n is odd, then with

$$\mathcal{H}_n(y) = \sum_{m=0}^{[n/2]} \beta(n,m)\, y^{n-2m} \tag{3.1-43}$$

the solution takes the form

$$g(y) = \exp(y^2)\, w(y) \sum_{k=0}^{\infty} \sum_{m=0}^{[k/2]} \sum_{n=0}^{[k/2-m]} \frac{\beta(k,m)\, C_{k-2m-2n}\, (k-2m)!\, \mathcal{H}_k(y)}{g_k\, 2^{k-2m}\, (k-2m-2n)!\, n!} \tag{3.1-47}$$

If instead of conditions (i-3) to (iv-3) f, K and g satisfy conditions (i-1) to (vi-1) and also condition (v-3), then C_m is given by Eq.(3.1-16)

$$C_m = \frac{(-)^m i^m}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-s^2)\,(2s)^m\, \frac{\tilde{f}(2s)}{\tilde{K}(2s)}\, ds \tag{3.1-16}$$

where we replaced s by 2s. Substitution of this result in Eq. (3.1-46) gives

$$g(y)=\exp(y^2)w(y)\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{\alpha(k,m)m!\ (-)^m i^m \mathcal{H}_k(y)}{g_k}$$

$$\frac{1}{2^m m!\sqrt{\pi}}\int_{-\infty}^{\infty}\sum_{n=0}^{[m/2]}\frac{(-)^n m!\,(2s)^{m-2n}}{n!\,(m-2n)!}\exp(-s^2)\frac{\widetilde{f(2s)}}{\widetilde{K(2s)}}\,ds \qquad (3.1\text{-}48)$$

but (12)

$$H_m(s) = \sum_{n=0}^{[m/2]}\frac{(-)^n m!\,(2s)^{m-2n}}{n!\,(m-2n)!} \qquad (3.1\text{-}49)$$

and it follows immediately that

$$\frac{1}{2^m m!\sqrt{\pi}}\int_{-\infty}^{\infty}\exp(-s^2)H_m(s)\frac{\widetilde{f(2s)}}{\widetilde{K(2s)}}\,ds$$

is the mth coefficient in the expansion of $\widetilde{f(2s)}/\widetilde{K(2s)}$ in a series of Hermite polynomials

$$\frac{\widetilde{f(2s)}}{\widetilde{k(2s)}} = \sum_{m=0}^{\infty} h_m H_m(s) \qquad (3.1\text{-}50)$$

and Eq. (3.1-48) becomes

$$g(y)=\exp(y^2)w(y)\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{\alpha(k,m)m!(-)^m i^m h_m \mathcal{H}_k(y)}{g_k} \quad (3.1\text{-}51)$$

Similarly, by substituting Eq.(3.1-16) in Eq.(3.1-47) we find

$$g(y)=\exp(y^2)w(y)\sum_{k=0}^{\infty}\sum_{m=0}^{[k/2]}\frac{\beta(k,m)(k-2m)!\,\mathcal{H}_k(y)(-)^{m+k}i^k}{g_k}$$

$$\frac{1}{2^{k-2m}(k-2m)!\sqrt{\pi}}\int_{-\infty}^{\infty}\sum_{n=0}^{[k/2-m]}\frac{(-)^n(k-2m)!(2s)^{k-2m-2n}}{n!(k-2m-2n)!}\exp(-s^2)\frac{\tilde{f}(2s)}{\tilde{K}(2s)}ds$$

$$(3.1\text{-}52)$$

but again

$$H_{k-2m}(s) = \sum_{n=0}^{[(k-2m)/2]}\frac{(-)^n(k-2m)!(2s)^{k-2m-2n}}{n!(k-2m-2n)!}$$

and Eq. (3.1-52) becomes

$$g(y)=\exp(y^2)w(y)\sum_{k=0}^{\infty}\sum_{m=0}^{[k/2]}\frac{\beta(k,m)(k-2m)!(-)^{m+k}i^k h_{k-2m}\mathcal{H}_k(y)}{g_k}$$

$$(3.1\text{-}53)$$

where h_m is given by Eq.(3.1-50). The possibility of the expansion (3.1-50) is guaranteed by condition (vi-1).[13]

In the event that f(x) does not have a Maclaurin expansion as assumed by condition (i-1), one can, at least in principle, replace f(x) by a sequence $\{f_\nu\}$ of C^∞ functions converging to f; solve the corresponding integral equation for f_ν and take the limit of the resulting solution.

As illustration let a = -1, b = +1. One possible choice for $\Psi_n(y)$ are the Legendre polynomials[14].

$$P_n(y) = \sum_{k=0}^{[n/2]} \frac{(-)^k \left(\frac{1}{2}\right)_{n-k} (2y)^{n-2k}}{k!\ (n-2k)!} \qquad (3.1\text{-}54)$$

The factorial function $(a)_n$ used in Eq. (3.1-54) is defined as

$$(a)_n = a(a+1)(a+2)\ldots(a+n-1),\ n \geq 1$$

and

$$(a)_0 = 1 \qquad a \neq 0.$$

For the Legendre polynomials we have

$$g_k = \frac{2}{2k+1} \quad , \quad w(y) = 1$$

$$\beta(k,m) = \frac{(-)^m \left(\frac{1}{2}\right)_{k-m} 2^{k-2m}}{m!\ (k-2m)!}$$

and Eq.(3.1-46) becomes

$$g(y)=\exp(y^2)\sum_{k=0}^{\infty}\sum_{m=0}^{[k/2]}\frac{(-)^k i^k \left(k+\frac{1}{2}\right)\left(\frac{1}{2}\right)_{k-m} 2^{k-2m} h_{k-2m}}{m!}P_k(y) \qquad (3.1\text{-}55)$$

corresponding to the solution of the equation

$$f(x) = \int_{-1}^{1} k(x-y)g(y)dy. \qquad (3.1\text{-}56)$$

Solutions in series of orthogonal polynomials can also be obtained for equations of the second kind by using the limit process employed to derive Eq.(3.1-36)

Section 3.13. <u>Expression of the solution as a contour integral</u>.

In many cases it is possible to transform the solution in series of orthogonal polynomials into a contour integral We do this following a method due to Watson.[15]

The solution of Eq.(3.1) in series of polynomials is given in its most general form by Eq.(3.1-38)

$$g(y) = w(y)\exp(y^2)\sum_{n=0}^{\infty} b_n \Psi_n(y). \qquad (3.1\text{-}57)$$

b_n will be assumed such that if n is allowed to vary continuously, then b_n is an analytic function of n.

Also $\Psi_n(y)$ will be assumed defined when the index n varies continuously and to be an analytic function of n. We will show that in this case $g(y)$ can be represented by the following integral

$$g(y) = w(y)\exp(y^2)\,\frac{i}{2}\int_C \frac{b_z \exp(i\pi z)\,\Psi_z(y)\,dz}{\sin \pi z} \qquad (3.1\text{-}58)$$

where C is a contour that starts at $\infty - i\delta$ in the z-plane, goes below the real axis to $z = -\frac{1}{2}$ and then above the real axis to $\infty + i\delta$.

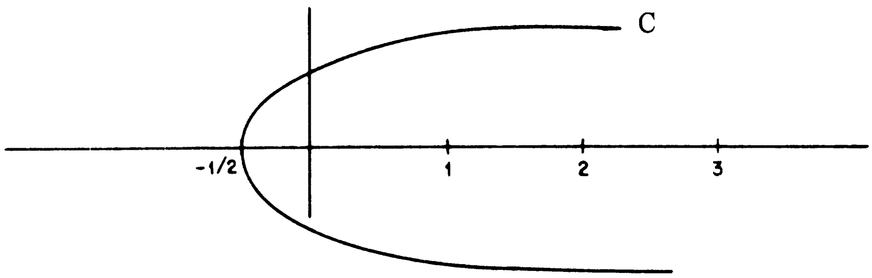

FIGURE 1.1: CONTOUR C

Since b_z and $\Psi_z(y)$ are analytic functions of z, the only singularities of the integrand are poles at those values of z inside C for which $\sin \pi z = 0$, those are $z = 0, 1, 2, \ldots$. Since the contour C is described in the clockwise direction, the integral (3.1-58) is equal to the negative sum of the residues at the poles, so that

$$g(y) = w(y)\exp(y^2)\frac{i}{2}(-2\pi i)\sum_{n=0}^{\infty} \frac{b_n(-)^n\,\Psi_n(y)}{\pi\cos \pi n}$$

$$= w(y)\exp(y^2) \sum_{n=0}^{\infty} b_n \Psi_n(y)$$

which is Eq.(3.1-57).

Equation (3.1-58) can be used to advantage with the principle of deformation of contours to transform the solution into another one more directly accessible to numerical computation.

Section 3.2. The interval (a,b) is infinite.

Equations (3.1-46) and (3.1-47) are, of course, also valid if the interval (a,b) is infinite, provided that g(y) is a sufficiently well behaved function.

Condition (v-3) is no longer sufficient to guarantee term by term integration when the interval is infinite, but we will assume g(y) to be such that term by term integration is permissible. One example of the conditions g(y) may satisfy is given in reference (6), page 178, Theorem III.

When $a = 0$, $b = \infty$ we may select for $\Psi_n(y)$ the Laguerre polynomials $L_n^{(\alpha)}(y)$,[16] orthogonal in the interval $(0, \infty)$ with respect to the weighting function $y^{\alpha}\exp(-y)$, $\mathrm{Re}(\alpha) > -1$.

$$L_n^{(\alpha)}(y) = \sum_{k=0}^{n} \frac{(-)^k (1+\alpha)_n \, y^k}{k!\,(n-k)!\,(1+\alpha)_k} \qquad (3.2\text{-}1)$$

For this set we have[17]

$$\gamma_n = \frac{\Gamma(1+\alpha+n)}{n!} = \frac{\Gamma(1+\alpha)(1+\alpha)_n}{n!}$$

$$w(y) = y^{\alpha}\exp(-y)\ , \quad Re(\alpha) > -1$$

$$\alpha(k,m) = \frac{(-)^m\ (1+\alpha)_k}{m!\,(k-m)!\,(1+\alpha)_m}$$

and Eq.(3.1-51) becomes

$$g(y) = \frac{\exp(y^2-y)y^{\alpha}}{\Gamma(1+\alpha)} \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{i^m\ k!\ h_m\ L_k^{(\alpha)}(y)}{(k-m)!\,(1+\alpha)_m} \tag{3.2-2}$$

corresponding to the solution of the equation

$$f(x) = \int_0^{\infty} K\,(x-y)g(y)dy \qquad 0<x<\infty \tag{3.2-3}$$

In particular for $\alpha = 0$, Eq.(3.2-2) takes the form

$$g(y) = \exp(y^2-y) \sum_{k=0}^{\infty} \sum_{m=0}^{K} i^m \binom{k}{m} h_m L_k\ (y) \tag{3.2-4}$$

where $\binom{k}{m}$ is the binomial coefficient.

In both Eqs.(3.2-2) and (3.2-4) h_m is given in Eq.(3.1-50).

An equation of the type

$$f(x) = \int_a^b K(xy)\, g(y)\, dy \qquad (3.2\text{-}5)$$

can be reduced to the form (3-1) by putting $x = \exp(\xi)$, $y = \exp(-\eta)$, and writing $f(\exp \xi) = \phi(\xi)$, $-\exp(-\eta) g(\exp(-\eta)) = \psi(\eta)$, and $K(\exp(\xi - \eta)) = K'(\xi - \eta)$ to obtain

$$\phi(\xi) = \int_{a'}^{b'} K'(\xi - \eta)\psi(\eta) d\eta$$

where $a' = -\ln a$, $b' = -\ln b$.

A similar set of substitutions can be used for the equation

$$f(x) = \int_a^b K(x/y) g(y) dy. \qquad (3.2\text{-}6)$$

Section 4. Kernels which are generating functions for polynomial sets.

A class of equations directly amenable to the previous techniques consists of those integral equations whose kernels are generating functions for a set of polynomials.

$$k(x,y) = \sum_{n=0}^{\infty} \varphi_n(y) \frac{x^n}{n!} \qquad (4\text{-}1)$$

where $\varphi_n(y)$ is a polynomial of degree n in y.

If we substitute Eq. (4-1) in the equation

$$f(x) = \int_a^b K(x,y)g(y)dy \qquad (4\text{-}2)$$

we obtain

$$f^{(n)}(0) = \int_a^b \varphi_n(y)g(y)dy \qquad (4\text{-}3)$$

where we have assumed that f(x) possesses a Maclaurin expansion.

Let us also assume that g(y)/w(y) can be expanded in a series of polynomials $\psi_n(y)$ orthogonal in the interval (a,b) with respect to the weighting function $w(y) > 0$.

$$g(y) = w(y) \sum_{m=0}^{\infty} b_m \psi_m(y) \qquad (4\text{-}4)$$

Substituting Eq.(4-4) in Eq.(4-3) and assuming term by term integration permissible, we have

$$f^n(0) = \sum_{m=0}^{\infty} b_m \int_a^b w(y)\ \varphi_n(y)\ \psi_m(y)dy \qquad (4\text{-}5)$$

We now appeal to a theorem in the theory of polynomial sets.

Theorem 2. Let $\psi_n(x)$ be a polynomial of degree n in x and assume there is one such polynomial for each n. Let $w(x) > 0$ on $a < x < b$. Then a necessary and sufficient condition that the set $\{\psi_n(x)\}$ be orthogonal with respect to w(x) over the interval $a<x<b$ is that

$$\int_a^b w(x)x^k\psi_n(x)dx = 0, \quad k=0,1,\ldots,(n-1).$$

This is a well know result.(18)

Since $\varphi_n(y)$ is a polynomial of degree n, it follows from theorem 2 that the integral in Eq.(4-5) vanishes identically for $m>n$,

$$f^{(n)}(0) = \sum_{m=0}^{n} b_m \int_a^b w(y)\,\varphi_n(y)\,\psi_m(y)\,dy \qquad (4\text{-}6)$$

This relation can be written in the form

$$b_0 = \frac{f(0)}{\int_a^b w(y)\,\varphi_0(y)\,\psi_0(y)dy}$$

$$b_n = \frac{f^{(n)}(0) - \sum_{m=0}^{n-1} b_m \int_a^b w(y)\,\varphi_n(y)\,\psi_m(y)dy}{\int_a^b w(y)\,\varphi_n(y)\,\psi_n(y)\,dy}, \quad n\geq 1 \qquad (4\text{-}7)$$

Giving successive values to n, Eq.(4-7) provides an iterative scheme for determining the b's of Eq.(4-4).

If in Eq. (4-6) we let

$$f^{(n)}(0) = \int_a^b w(y)\,\varphi_n(y)\,\psi_0(y)dy \qquad (4\text{-}8)$$

Then as is obvious by inspection of Eq.(4-7) $b_0 = 1$, $b_m = 0$ $m \geq 1$. Hence if we have iterated Eq.(4-7) N times and have obtained a set of relations

$$b_m = \sum_{n=0}^{m} a_{nm} f^{(n)}(0), \quad m = 0, 1, \ldots, N. \qquad (4\text{-}9)$$

then by letting $f^{(n)}(0)$ take the values specified by Eq.(4-8) we must obtain $b_0 = 1$, $b_m = 0$, $m \geq 1$. If this result is not obtained, a mistake was made in the computations. This device can be used to check the numerical accuracy of Eqs.(4-9).

It should be noticed that the integral in Eq.(4-6)

$$\int_a^b w(y)\, \varphi_n(y)\, \Psi_m(y)\, dy \qquad (4\text{-}10)$$

is g_m^{-1} times the mth coefficient in the expansion of $\varphi_n(y)$ in a series of polynomials $\Psi_m(y)$, where g_m is given by Eq.(3.1-42). Hence in order to do the integral (4-10) it is, perhaps, the simplest procedure, to expand $\varphi_n(y)$ in series of $\Psi_m(y)$ and pick out the coefficients. Such expansions are given for a number of classical polynomials in Professor Rainville's "Special Functions." (See References).

We shall see now that it is quite easy to solve the system of equations (4-6) when the polynomials $\varphi_n(y)$ possess a generating function of the type

$$A(t)G(xt) = \sum_{n=0}^{\infty} \lambda(n)\, \varphi_n(x) t^n , \qquad (4\text{-}11)$$

where λ is some function of n. We shall assume that A(t) and G(t) have the formal expansions

$$A(t) = \sum_{n=0}^{\infty} \alpha_n t^n , \qquad \left[A(t)\right]^{-1} = \sum_{n=0}^{\infty} \beta_n t^n \qquad (4\text{-}12)$$

$$G(t) = \sum_{n=0}^{\infty} \gamma_n t^n . \qquad (4\text{-}13)$$

Consider the integral equation

$$f(x) = \int_a^b G(xy) g(y) dy \qquad (4\text{-}14)$$

Multiply both sides of this equation by A(x)

$$A(x)f(x) = \int_a^b A(x)G(xy)g(y)dy \qquad (4\text{-}15)$$

Let

$$A(x)f(x) = \sum_{m=0}^{\infty} \lambda(m)\, c_m\, x^m \qquad (4\text{-}16)$$

then from Eqs.(4-11) and (4-16)

$$c_m = \int_a^b \varphi_m(y) g(y) dy \qquad (4\text{-}17)$$

Let now

$$g(y) = w(y) \sum_{n=0}^{\infty} b_n \psi_n(y) \qquad (4\text{-}18)$$

Substitution of Eq.(4-18) in Eq.(4-17) gives in view of theorem 2

$$c_m = \sum_{n=0}^{m} b_n \int_a^b w(y) \varphi_m(y) \psi_n(y) dy$$

which are equations (4-6).

Now from Eqs.(4-13) and (4-14) there follows

$$\frac{f^{(n)}(0)}{\gamma_n \, n!} = \int_a^b y^n g(y) dy \qquad (4\text{-}19)$$

Let the polynomials $\psi_k(y)$ be of the form

$$\psi_k(y) = \sum_{n=0}^{k} \alpha(k,n) \, y^n \qquad (4\text{-}20)$$

Multiply both sides of Eq.(4-19) by $\alpha(k,n)$ and sum over n

$$\sum_{n=0}^{k} \frac{\alpha(k,n) f^{(n)}(0)}{\gamma_n \, n!} = \int_a^b \psi_k(y) g(y) dy \qquad (4\text{-}21)$$

Inserting Eq.(4-18) in Eq.(4-21) we have by virtue of the orthogonality of the polynomials $\Psi_n(y)$

$$b_k = \frac{1}{g_k} \sum_{n=0}^{k} \frac{\alpha(k,n) f^{(n)}(0)}{\gamma_n \; n!} \tag{4-22}$$

where

$$g_k = \int_a^b w(y) \Psi_k^2(y) dy. \tag{4-23}$$

From Eqs.(4-16) and (4-12) we have

$$\begin{aligned} f(x) &= [A(x)]^{-1} \sum_{m=0}^{\infty} \lambda(m) c_m x^m \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \lambda(m) \; c_m \beta^n x^{n+m} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \lambda(m) \; c_m \beta_{n-m} \; x^n \end{aligned}$$

so that

$$f^{(n)}(0) = \sum_{m=0}^{n} \lambda(m) c_m \;\; n! \; \beta_{n-m} \tag{4-24}$$

and Eq.(4-22) becomes

$$b_k = \frac{1}{g_k} \sum_{n=0}^{k} \sum_{m=0}^{n} \frac{\lambda(m) \; \beta_{n-m} \alpha(k,n) c_m}{\gamma_n} \tag{4-25}$$

which is the solution of Eqs.(4-6).

If the polynomials $\Psi_k(y)$ are of the form

$$\psi_k(y) = \sum_{n=0}^{[k/2]} \beta(k,n)\; y^{k-2n} \qquad (4\text{-}26)$$

then we have from Eq.(4-19)

$$\sum_{n=0}^{[k/2]} \frac{\beta(k,n)\; f^{(k-2n)}(0)}{\gamma_{k-2n}\;(k-2n)!} = \int_a^b \psi_k(y) g(y)\,dy$$

Use of Eq.(4-18) gives as before

$$b_k = \frac{1}{g_k} \sum_{n=0}^{[k/2]} \frac{\beta(k,n) f^{(k-2n)}(0)}{\gamma_{k-2n}\;(k-2n)!} \qquad (4\text{-}27)$$

with g_k given by Eq.(4-23).

Finally, from Eq.(4-24)

$$b_k = \frac{1}{g_k} \sum_{n=0}^{[k/2]} \sum_{m=0}^{k-2n} \frac{\beta(k,n)\; \lambda(m)\; \beta_{k-2n-m}C_m}{\gamma_{k-2n}} \qquad (4\text{-}28)$$

We have thus shown that the solution of the equation

$$f(x) = \int_a^b K(x,y) g(y)\,dy \qquad (4\text{-}29)$$

with

$$K(x,y) = \sum_{n=0}^{\infty} \varphi_n(y) \frac{x^n}{n!} \qquad (4\text{-}30)$$

and $\varphi_n(y)$ satisfying Eqs.(4-11) to (4-13) is given by

$$g(y) = w(y) \sum_{n=0}^{\infty} b_n \mathcal{M}_n(y) \tag{4-31}$$

where the $\mathcal{M}_k(y)$ are polynomials orthogonal in the interval (a,b) with respect to the weighting function $w(y) > 0$, and b_k is given by

$$b_k = \frac{1}{g_k} \sum_{n=0}^{k} \sum_{m=0}^{n} \frac{\lambda(m)\, \beta_{n-m}\, \alpha(k,n) f^{(m)}(0)}{\gamma_n} \tag{4-32}$$

if

$$\mathcal{M}_k(y) = \sum_{n=0}^{k} \alpha(k,n) y^n \tag{4-33}$$

or

$$b_k = \frac{1}{g_k} \sum_{n=0}^{[k/2]} \sum_{m=0}^{k-2n} \frac{\beta(k,n)\, \lambda(m)\, \beta_{k-2n-m}\, f^{(m)}(0)}{\gamma_{k-2n}} \tag{4-34}$$

if

$$\mathcal{M}_k(y) = \sum_{n=0}^{[k/2]} \beta(k,n)\, y^{k-2n} . \tag{4-35}$$

Many polynomial sets have generating functions of the form (4-11). Thus for example, the Hermite polynomials

$$\exp(-t^2)\exp(2xt) = \exp(2xt - t^2) = \sum_{n=0}^{\infty} \frac{H_n(x)\, t^n}{n!} \tag{4-36}$$

and

$$\lambda(n) = 1/n!, \quad \beta_{2n} = 1/n!, \quad \beta_{2n+1} = 0, \quad \gamma_n = 2^n/n!$$

In Eq.(4-25) replace m by n-2m, then

$$b_k = \frac{1}{g_k} \sum_{n=0}^{k} \sum_{m=0}^{[n/2]} \frac{\lambda(n-2m)\ \beta_{2m}\ \alpha(k,m)\ c_{n-2m}}{\gamma_n}$$

since $\beta_{2m+1} \equiv 0$

Hence for this case

$$b_k = \frac{\Gamma}{g_k} \sum_{n=0}^{k} \sum_{m=0}^{[n/2]} \frac{\alpha(k,n) n!\ c_{n-2m}}{(n-2m)!\ m!\ 2^n} \qquad (4\text{-}37)$$

which is, of course, Eq.(3.1-41).

A similar substitution can be made for Eq.(4-34).

For the Laguerre polynomials[19]

$$\exp(t)\,{}_0F_1(-;\ 1+\alpha;\ -xt) = \sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(t)\ t^n}{(1+\alpha)_n} \qquad (4\text{-}38)$$

$$\beta_n = (-)^n/n!\ ,\quad \gamma_n = (-)^n/(1+\alpha)_n\ n!\ ,\quad \lambda(n) = 1/(1+\alpha)_n$$

and from Eq.(4-32)

$$b_k = \frac{1}{g_k} \sum_{n=0}^{k} \sum_{m=0}^{n} \frac{(-)^m\ \alpha(k,n)\,(1+\alpha)_n\ n!\ c_m}{(1+\alpha)_m\ (n-m)!} \qquad (4\text{-}39)$$

The substitution could just as easily have been made in Eq.(4-34).

Other polynomial sets with generating functions of the form (4-11) are

Bernoulli polynomials,[20]

$$\frac{t}{\exp(t)-1}\exp(xt) = \sum_{n=0}^{\infty} \frac{B_n(x)\ t^n}{n!} \qquad (4\text{-}40)$$

Euler polynomials,[21]

$$\frac{2}{\exp(t)+1}\exp(xt) = \sum_{n=0}^{\infty} \frac{E_n(x)\ t^n}{n!} \qquad (4\text{-}41)$$

etc.

Section 5. The kernel $\exp(-\lambda xy)$.

As an incidental result to the derivation of Eqs. (3.1-41) and (3.1-45) we have found an interesting way to solve the integral equation

$$f(x) = \int_a^b \exp(-\lambda xy) g(y) dy \qquad (5\text{-}1)$$

with the parameter λ generally complex.

We shall assume that f(x) has a Taylor series expansion about some point x_0.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n. \qquad (5\text{-}2)$$

Also g(y) will be assumed to have the uniformly convergent series expansion

$$g(y) = w(y)\exp(\lambda x_0 y) \sum_{n=0}^{\infty} b_n \Psi_n(y) \qquad (5\text{-}3)$$

where $\Psi_n(y)$ are polynomials orthogonal in the interval (a,b) with respect to the weighting function $w(y) > 0$.

In Eq.(5-1) replace x by $x + x_0$

$$f(x + x_0) = \int_a^b \exp(-\lambda xy)\exp(-\lambda x_0 y) g(y)\,dy \qquad (5\text{-}4)$$

Expanding the exponential in x in a power series and making use of Eq.(5-2) we have

$$\frac{(-)^n f^{(n)}(x_0)}{\lambda^n} = \int_a^b y^n \exp(-\lambda x_0 y) g(y)\,dy \qquad (5\text{-}5)$$

We shall assume the polynomials $\Psi_k(y)$ to be given by an equation of the form

$$\Psi_k(y) = \sum_{n=0}^{k} \alpha(k,n)\, y^n \qquad (5\text{-}6)$$

Multiplying Eq.(5-5) by $\alpha(k,n)$ and summing over n from 0 to k, gives

$$\sum_{n=0}^{k} \frac{(-)^n \alpha(k,n)\, f^{(n)}(x_0)}{\lambda^n} = \int_a^b \Psi_k(y)\exp(-\lambda x_0 y) g(y)\,dy \qquad (5\text{-}7)$$

Use of Eq.(5-3) in Eq.(5-7) gives finally

$$b_k = \frac{1}{g_k} \sum_{n=0}^{k} \frac{(-)^n \alpha(k,n)\, f^{(n)}(x_0)}{\lambda^n} \tag{5-8}$$

so that

$$g(y) = w(y)\exp(\lambda x_0 y) \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\alpha(n,k)\,(-)^k f^{(k)}(x_0)\, \mathcal{H}_n(y)}{g_n \lambda^k} \tag{5-9}$$

is the solution of Eq.(5-1)

If the polynomial set is such that $\mathcal{H}_k(y)$ is given by an equation of the form

$$\mathcal{H}_k(y) = \sum_{n=0}^{[k/2]} \beta(k,n)\, y^{k-2n} \tag{5-10}$$

then we modify Eq.(5-5) to read

$$\frac{(-)^k f^{(k-2n)}(x_0)}{\lambda^{k-2n}} = \int_a^b y^{k-2n} \exp(-\lambda x_0 y) g(y) dy$$

and an identical procedure leads to

$$b_k = \frac{1}{g_k} \sum_{n=0}^{[k/2]} \frac{(-)^k f^{(k-2n)}(x_0)\, \beta(k,n)}{\lambda^{k-2n}} \tag{5-11}$$

The solution of Eq.(5-1) is then

$$g(y) = w(y)\exp(\lambda x_0 y) \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \frac{(-)^n \beta(n,k)\, f^{(n-2k)}(x_0)\, \mathcal{H}_n(y)}{g_n \lambda^{n-2k}} \tag{5-12}$$

A special case of Eq.(5-1) corresponding to $\lambda = i$ and the interval (a,b) finite

$$f(x) = \int_a^b \exp(-ixy) g(y)\,dy \qquad (5\text{-}13)$$

is of central importance in the study of antenna synthesis.[22]

Another special case is provided by the choice $\lambda = 1$, $a = 0$, $b = \infty$, which gives Laplace's intergral equation[23]

$$f(s) = \int_0^\infty \exp(-st) F(t)\,dt \qquad (5\text{-}14)$$

Selecting as the set $\{\Psi_n(y)\}$ the Laguerre polynomials $L_n^{(\alpha)}(y)$ we have from Eq.(5-9) and the results on page 30

$$F(t) = \frac{t^{\alpha} \exp(-(1-s_0)t)}{\Gamma(1+\alpha)} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{f^{(k)}(s_0)}{(1+\alpha)_k} L_n^{(\alpha)}(t) \qquad (5\text{-}15)$$

as the solution of Eq.(5-14)

Suppose for example that

$$f(s) = \frac{s^m}{(s+a)^{m+1}} \qquad (5\text{-}16)$$

then

$$\frac{s^m}{(s+a)^{m+1}} = \sum_{k=0}^{\infty} \frac{(m+1)_k \; (-)^k \; s^{k+m}}{a^{k+m+1} \; k!}$$

and

$$f^{(k+m)}(0) = \frac{(m+1)_k \; (-)^k \; (1)_{k+m}}{a^{k+m+1} \; k!} \tag{5-17}$$

Then from Eq.(5-15) with $\alpha = 0$, $s_0 = 0$

$$F(t)=\exp(-t) \sum_{n=0}^{\infty} \sum_{k=m}^{\infty} \frac{(1)_{n+k} \; f^{(k)}(0) \; L_{n+k}(t)}{(k!)^2 \; n!}$$

$$= \exp(-t) \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(1)_{n+k+m} \; f^{(k+m)}(0) \; L_{n+k+m}(t)}{(1)_{k+m}^2 \; n!}$$

$$=\exp(-t) \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(1)_{n+m} \; f^{(k+m)}(0) \; L_{n+m}(t)}{(1)_{k+m}^2 \; (n-k)!}$$

With Eq.(5-17), F(t) takes the form

$$F(t) = \exp(-t) \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(1)_{n+m} \; (m+1)_k \; (-)^k \; L_{n+m}(t)}{a^{k+m+1} \; k! \; (1)_{k+m} \; (n-k)!} \tag{5-18}$$

but

$$(1)_{k+m} = (1)_m \; (1+m)_k$$

and Eq.(5-18) becomes

$$F(t)=\exp(-t)\frac{1}{a^{m+1}} \sum_{n=0}^{\infty} \frac{(1)_{n+m} \; L_{n+m}(t)}{m! \; n!} \sum_{k=0}^{n} \binom{n}{k} (-)^k \; a^{-k}$$

$$= \exp(-t)\,\frac{1}{a^{m+1}} \sum_{n=0}^{\infty} \frac{(1)_{n+m}\, L_{n+m}(t)\,(1 - 1/a)^n}{m!\; n!}$$

Use of the result[24]

$$\sum_{n=0}^{\infty} \frac{(n+m)!\; L_{n+m}(t)\; x^n}{m!\; n!} = (1-x)^{-1-m} \exp\left(-\frac{tx}{1-x}\right) L_m\left(\frac{t}{1-x}\right)$$

gives

$$F(t) = \exp(-t)\,\frac{1}{a^{m+1}}\, a^{m+1} \exp(-t(a-1))\; L_m(at)$$

$$= \exp(-at)\; L_m(at) = \exp(-at)\,{}_1F_1(-m;1;at) \tag{5-19}$$

or by use of Kummer's first formula[25]

$$\exp(-z)\,{}_1F_1(a;b;z) = {}_1F_1(b-a;b;-z)$$

equation (5-19) becomes

$$F(t) = {}_1F_1\,(1 + m;\; 1;\; -at).$$

Thus if $\mathcal{L}^{-1}$ denotes the inverse Laplace transform we have

$$\mathcal{L}^{-1}\left\{\frac{s^m}{(s + a)^{m+1}}\right\} = {}_1F_1(1 + m;\; 1;\; -at) \tag{5-20}$$

This result is well known.[26]

Suppose now that

$$f(s) = \frac{1}{s}\left(1 - \frac{1}{s}\right)^n \qquad (5\text{-}21)$$

then with $s_0 = 1$

$$f(s + 1) = \frac{s^n}{(s + 1)^n},$$

but this is the example we just considered with a = 1. It follows then from Eqs. (5-15) and (5-19) that

$$\exp(-t)\, F(t) = \exp(-t)\, L_n(t)$$

that is

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\left(1 - \frac{1}{s}\right)^n\right\} = L_n(t), \qquad (5\text{-}22)$$

also a known result.[27]

We now proceed to consider some applications of the above methods to physical problems.

CHAPTER TWO

RADIATION FROM A LINEAR ANTENNA.

Section 1. Formulation of Hallén's integral equation.

We will consider the problem of electromagnetic radiation from a perfectly conducting cylindrical antenna excited across a small air gap by an external a-c source.

It is well known that the electric field vector can be derived from a scalar and a vector potential according to the relation

$$\bar{E} = - \nabla \phi - i\omega \bar{A} \qquad (1\text{-}1)$$

If the potentials are subject to the Lorentz condition

$$\nabla \cdot \bar{A} + i\omega\mu\epsilon\phi = 0 \qquad (1\text{-}2)$$

then $\bar{E}$ can be derived from a vector potential alone

$$\bar{E} = \frac{c^2}{i\omega} \left(\nabla (\nabla \cdot \bar{A}) + \frac{\omega^2}{c^2} \bar{A} \right) \qquad (1\text{-}3)$$

where $c = 1/\sqrt{\mu\epsilon}$.

For the radiated field $\bar{E}^{(r)}$ the vector potential $\bar{A}^{(r)}$ is related to the induced current density by means of the equation

$$\bar{A}^{(r)} = \frac{\mu}{4\pi} \int \frac{\bar{J}\left(|\bar{r} - \bar{r}'|,\, t - \frac{|\bar{r} - \bar{r}'|}{c}\right)}{|\bar{r} - \bar{r}'|} dv \qquad (1\text{-}4)$$

r being the distance from the origin of coordinates to the field point, and r' the corresponding distance to the source point.

We shall assume the z-axis chosen along the axis of the antenna, the length of which extends from -l to +l (Fig. 2.1). At z = 0 the antenna is fed across a small air gap by an external a-c voltage $2V_o \exp(i\omega t)$. Because of the assumption of infinite conductivity, the current only flows along the surface of the antenna, and we may assume that it is symmetrically distributed with respect to the axis of the cylinder. The current density vector has thus only a z-component, and the vector potential is parallel to the z-axis at all points. The end surfaces of the cylinder make an exception and should consequently be assumed small. We can, however, surmount the difficulty by assuming that the antenna consists of a tube thus eliminating the end surfaces.

On the surface of the antenna $E_z = 0$ because of the assumption of infinite conductivity. Furthermore, for a straight antenna $\nabla \cdot \bar{A} = \partial A_z / \partial z$. We have then for the z-component of Eq.(1-3)

$$\frac{\partial^2 A_z}{\partial z^2} + k^2 A_z = 0 , \qquad (1\text{-}5)$$

where $k = \omega \sqrt{u\epsilon}$.

The solutions of Eq.(1-5) are of the type coskz and sin kz.

As the antenna we have considered consists of two halves between which there exists a difference of potential we should not expect the coefficients of cos kz and sin kz terms to be the same for both halves of the antenna. Instead it is the vector potential which is to be symmetrical with respect to the air gap when the antenna is fed at the middle. Consequently we shall put

$$A_z = (A_1 \cos kz + A_2 \sin k|z|)\exp(i\omega t),$$

where A_1 and A_2 are constants. For the scalar potential of the antenna we have from Eq.(1-2) $i\omega\phi = - c^2 \partial A_z/\partial z$ i.e.,

$$i\omega\phi = c^2 k(A_1 \sin kz \mp A_2 \cos kz)\exp(i\omega t)$$

where the upper sign corresponds to $z > 0$ and the lower one to $z < 0$. Hence the potential has two values $\phi = \pm icA_z \exp(i\omega t)$ at $z = 0$. We know already, however, that there $\phi = \pm V_o \exp(i\omega t)$ and A_2 is thus defined. The vector potential obtains then the form

$$A_z = (A_1 \cos kz - \frac{i}{c} V_o \sin k|z|)\exp(i\omega t).$$

It is possible to complement the last term in the parenthesis by a corresponding real cosine term to give an exponential function, the cosine part of which is taken from the first term. A_z obtains then the alternative form

$$A_z = (A' \cos\ kz + \frac{V_0}{c} \exp(-ik\,|z|))\ \exp(i\omega t) \quad (1\text{-}6)$$

where A' is a new constant.

The vector potential A_z may be expressed in terms of the antenna current with the help of Eq.(1-4). Let ρ be the radial coordinate in a cylindrical coordinate system whose polar axis coincides with the axis of the antenna and with origin in the mid point of the antenna. Let "a" be the radius of the antenna. The surface current I(z) is then given by the limit of $(J_z 2\pi a)\Delta\rho$ as $\Delta\rho \to 0$ and $J \to \infty$.

Equation (1-4) becomes a surface integral upon replacing $J_z dv$ by $I(\xi)d\xi\ a\ d\phi/2\pi\ a$, where the integration over ρ has, so to speak, already been performed. For the time dependence $\exp(i\omega t)$ the retardation factor takes the form $\exp\ i\omega(t - \frac{|\bar{r} - \bar{r}'|}{c})$. The magnitude of the vector potential $\bar{A}$ <u>on the surface of the antenna</u> is given by

$$A_z = \frac{\mu}{4\pi} \int_{-\ell}^{\ell} I(\xi)\ \frac{1}{2\pi} \int_{0}^{2\pi} \frac{\exp(-ikr)}{r}\ d\phi\ d\xi \quad (1\text{-}7)$$

where r is the distance from the source point on the surface of the antenna, to the field point also on the surface of the antenna (Fig.2.1)

$$r = \sqrt{(z - \xi)^2 + 4a^2 \sin^2 \frac{1}{2}\phi} \quad (1\text{-}8)$$

From Eqs.(1-6) and (1-7), the integral equation of a cylindrical transmitting antenna fed at the middle is obtained in the form

$$\frac{4\pi}{\eta_0}(V_0 \exp(-ik|z|) + A\cos kz) = \int_{-\ell}^{\ell} d\xi\, I(\xi)\frac{1}{2\pi}\int_0^{2\pi}\frac{\exp(-ikr)}{r}\,d\phi$$

where A is a new constant and $\eta_0 = \sqrt{\mu/\epsilon}$.

This is Hallén's integral equation.

It has been argued by Hallén[28] that to Eq.(1-9) one must add the boundary condition $I(\ell) = I(-\ell) = 0$ to specify the constant A. We shall not follow his suggestion for, as we shall see, it is the constant A that is determined by the solution and not the other way around; that is $I(\xi)$ is independent of A.

Section 2. Solution of Hallén's integral equation.

In Eq.(1-9) replace z by lx and ξ by ly, then

$$\frac{4\pi}{\eta_0}(V_0 \exp(-ikl|x|) + A\cos klx) = \int_{-1}^{1} dy\, I(ly)\frac{1}{2\pi}\int_0^{2\pi}\frac{\exp(-ikrl)}{r}\,d\phi \tag{2-1}$$

where now

$$r = \sqrt{(x-y)^2 + 4\frac{a^2}{\ell^2}\sin^2\frac{1}{2}\phi} \tag{2-2}$$

Multiply both sides of Eq.(2-1) by $\frac{d^2}{dx^2} + (kl)^2$, to obtain

$$\frac{4\pi V_0}{\eta_0}(-2ikl)\delta(x) = \left(\frac{d^2}{dx^2} + (kl)^2\right)\int_{-1}^{1} dy\, I(ly)\frac{1}{2\pi}\int_0^{2\pi}\frac{\exp(-iklr)}{r}\,d\phi \tag{2-3}$$

where $\delta(x)$ is the delta function. The factor (-2ikl) accounts for the jump discontinuity in the first derivative of $\exp(-ikl|x|)$ at x = 0.

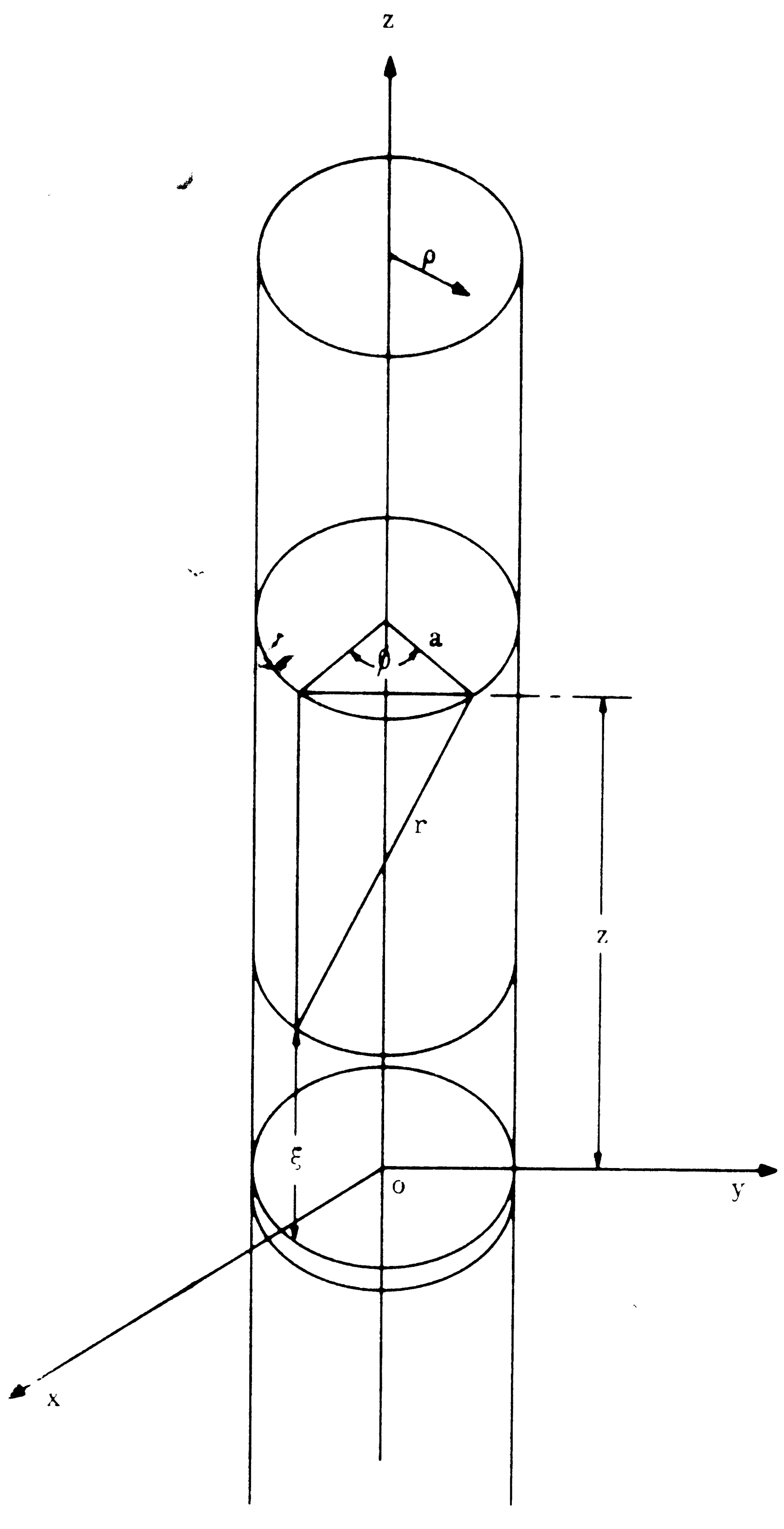

FIGURE 2.1 CYLINDRICAL ANTENNA

Let us as in page 14, consider the related equation

$$\exp(isx) = \frac{i\eta_o}{4V_o k\ell}\left(\frac{d^2}{dx^2} + (k\ell)^2\right)\int_{-1}^{1} dy G(y,s)\,\frac{1}{2\pi}\int_0^{2\pi}\frac{\exp(-ik\ell r)}{r}\,d\phi \qquad (2\text{-}4)$$

where s is a parameter. By letting $G(y,s) = 0$ for $|y| > 1$ we have by virtue of lemma 3 of chapter one

$$\exp(isx) = \frac{in_o}{4V_o k\ell}\int_{-1}^{1} dy G(y,s)\left(\frac{\partial^2}{\partial x^2} + (k\ell)^2\right)\frac{1}{2\pi}\int_0^{2\pi}\frac{\exp(-ik\ell r)}{r}\,d\phi \qquad (2\text{-}5)$$

but this is Eq.(3.1-23) with $\widetilde{f(s)} = 2\pi$, $a = -1$, $b = 1$, s replaced by -s and

$$K(x-y) = \frac{i\eta_o}{4V_o k\ell}\left(\frac{\partial^2}{\partial x^2} + (k\ell)^2\right)\frac{1}{2\pi}\int_0^{2\pi}\frac{\exp\left(ik\ell\sqrt{(x-y)^2 + 4\frac{a^2}{\ell^2}\sin^2\frac{1}{2}\phi}\right)}{\sqrt{(x-y)^2 + 4\frac{a^2}{\ell^2}\sin^2\frac{1}{2}\phi}}\,d\phi \qquad (2\text{-}6)$$

In order to satisfy the requirement that $K(x) = o(|x|^{-n})$ as $|x| \to \infty$ for all $n > 0$, we require k to have a small negative imaginary part, corresponding to the medium having finite non-zero conductivity

$$k = p - iq \qquad p,q > 0. \qquad (2\text{-}7)$$

for substitution in Eq.(3.1-28) we need the Fourier transform of Eq.(2-6).

To obtain this consider the following integral [29]

$$\int_0^{\infty} J\mu(bt)\,\frac{K_\nu\left\{a\sqrt{(t^2+z^2)}\right\}}{(t^2+z^2)^{1/2\nu}}\,t^{m+1}\,dt = \frac{b^\mu}{a^\nu}\left\{\frac{\sqrt{a^2+b^2}}{z}\right\}^{\nu-\mu-1} K_{\nu-\mu-1}\left\{z\sqrt{a^2+b^2}\right\}$$

In this equation let $\nu = \frac{1}{2}$, $\mu = -\frac{1}{2}$, $a = ikl$, $b = s$, $z = a$ and since $J_{-\frac{1}{2}}(z) = (2/\pi\, z)\cos z$, and $K_{\frac{1}{2}}(z) = (\pi/2z)\exp(-z)$, there follows

$$\int_0^\infty \cos st \; \frac{\exp(-ik\ell\sqrt{t^2+a^2})}{\sqrt{t^2+a^2}}\, dt = K_o \left\{ a\sqrt{s^2-(k\ell)^2} \right\}$$

or

$$\int_{-\infty}^{\infty} \exp(ist) \; \frac{\exp(-ik\ell\sqrt{t^2+a^2})}{\sqrt{t^2+a^2}}\, dt = 2\, K_o \left\{ a\sqrt{s^2-(k\ell)^2} \right\}$$

Hence by Fourier's theorem

$$\frac{\exp(-ik\ell\sqrt{t^2+a^2})}{\sqrt{t^2+a^2}} = \frac{1}{\pi}\int_{-\infty}^{\infty} \exp(-ist) K_o \left\{ a\sqrt{s^2-(k\ell)^2} \right\} ds$$

Replace t by x-y, a by $2(a/l)\sin\frac{1}{2}\phi$ and integra with respect to ϕ from 0 to 2π

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\exp\left(-ik\ell\sqrt{(x-y)^2+4\frac{a^2}{\ell^2}\sin^2\frac{1}{2}\phi}\right)}{\sqrt{(x-y)^2+4\frac{a^2}{\ell^2}\sin^2\frac{1}{2}\phi}}\, d\phi$$

$$= \frac{1}{2\pi^2}\int_{-\infty}^{\infty} \exp(-is(x-y)) \int_0^{2\pi} K_o\left\{ 2\frac{a}{\ell}\sin\frac{1}{2}\phi\sqrt{s^2-(k\ell)^2} \right\} d\phi ds \qquad (2\text{-}8)$$

Consider now the following integral[30]

$$I_o(z)K_\nu(z) = \frac{2}{\pi}\int_0^{\pi/2} K_\nu(2z\cos\theta)\cos\nu\theta\, d\theta$$

that is

$$I_o(z)\, K_\nu(z) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} K_\nu(2z\cos\theta)\cos\nu\theta\, d\theta \qquad (2\text{-}9)$$

Let $\nu = 0$, $\theta = \frac{1}{2}\phi - \frac{\pi}{2}$, $z = (a/\ell)(s^2-(k\ell)^2)^{\frac{1}{2}}$, and Eq.(2-9) becomes

$$I_o\left(\frac{a}{\ell}\sqrt{s^2-(k\ell)^2}\right) K_o\left(\frac{a}{\ell}\sqrt{s^2-(k\ell)^2}\right) =$$

$$\frac{1}{2\pi}\int_0^{2\pi} K_o\left(2\frac{a}{\ell}\sin\frac{1}{2}\phi\sqrt{s^2-(k\ell)^2}\right) d\phi \qquad (2\text{-}10)$$

Substitution of this result in Eq.(2-8) yields

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\exp\left(-ik\ell\sqrt{(x-y)^2+4\frac{a^2}{\ell^2}\sin^2\frac{1}{2}\phi}\right)}{\sqrt{(x-y)^2+4\frac{a^2}{\ell^2}\sin^2\frac{1}{2}\phi}}\,d\phi$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-is(x-y))\;2I_o\left(\frac{a}{\ell}\sqrt{s^2-(k\ell)^2}\right)K_o\left(\frac{a}{\ell}\sqrt{s^2-(k\ell)^2}\right)ds \tag{2-11}$$

By use of the result, proved in the appendix, that

$$P\left(\frac{d}{ds}\right)\tilde{f}=F\left[P(ix)f\right]$$

where P(x) is a polynomial in x, we have

$$\left(\frac{\partial^2}{\partial x^2}+(k\ell)^2\right)\frac{1}{2\pi}\int_0^{2\pi}\frac{\exp\left(-ik\ell\sqrt{(x-y)^2+4\frac{a^2}{\ell^2}\sin^2\frac{1}{2}\phi}\right)}{\sqrt{(x-y)^2+4\frac{a^2}{\ell^2}\sin^2\frac{1}{2}\phi}}\,d\phi$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}2\exp(-is(x-y))((k\ell)^2-s^2)I_o\left(\frac{a}{\ell}\sqrt{s^2-(k\ell)^2}\right)K_o\left(\frac{a}{\ell}\sqrt{s^2-(k\ell)^2}\right)ds \tag{2-12}$$

Hence from Eq.(2-6) we have

$$\tilde{K}(s)=\frac{i\eta_o((k\ell)^2-s^2)}{2V_ok\ell}\,I_o\left(\frac{a}{\ell}\sqrt{s^2-(k\ell)^2}\right)K_o\left(\frac{a}{\ell}\sqrt{s^2-(k\ell)^2}\right) \tag{2-13}$$

which is the desired transform.

From Eq.(3.1-28) there follows

$$G(y,s)=\exp(iys)\,\frac{2V_ok\ell\,\theta_3(s+2iy,\exp(-4))}{i\eta_o((k\ell)^2-s^2)\,I_o\left(\frac{a}{\ell}\sqrt{s^2-(k\ell)^2}\right)K_o\left(\frac{a}{\ell}\sqrt{s^2-(k\ell)^2}\right)} \tag{2-14}$$

where we made use of the fact that the third theta function is an even function of its argument.

Let us now integrate both sides of Eq.(2-4) with respect to s from $-\nu$ to ν and take the limit as $\nu \to \infty$. Then by use of lemma 1, we have

$$\delta(x) = \frac{i\eta_o}{8 V_o \pi k\ell} \left(\frac{d^2}{dx^2} + (k\ell)^2\right) \int_{-1}^{1} dy \int_{-\infty}^{\infty} G(y,s)ds \frac{1}{2\pi} \int_{0}^{2\pi} \frac{\exp(-ik\ell r)}{r} d\phi$$

which is equation (2-3). Hence

$$I(\xi) = \frac{i2kV_o}{\eta_o} \int_{-\infty}^{\infty} \frac{\exp(is\xi)\, \theta_3\left(2i\frac{\xi}{\ell} + s\ell,\ \exp(-4)\right)ds}{(s^2 - k^2)I_o(a\sqrt{s^2 - k^2})K_o(a\sqrt{s^2 - k^2})} \tag{2-15}$$

which is the desired result.

Section 3. Nature of the solution.

We now examine, in some detail, the nature of the results given in section 2.

The solution of Hallén's integral equation is given from Eq.(2-15) by

$$I(z) = \frac{i2kV_o}{\eta_o} \int_{-\infty}^{\infty} \frac{\exp(isz)\, \theta_3\left(s\ell + 2i\, z/\ell,\ \exp(-4)\right)ds}{(s^2 - k^2)I_o(a\sqrt{s^2 - k^2})K_o(a\sqrt{s^2 - k^2})} \tag{3-1}$$

Let us first note that because of the complex nature of k (Eq.(2-7)), the integrand in (3-1) has no singularities in the real axis. The singularities of the integrand are located in the s-plane at $s = k$ and $s = -k$.

Use of the relation

$$\theta_3\left(s\ell + 2i\frac{z}{\ell},\ \exp(-4)\right) = 1 + 2\sum_{n=1}^{\infty} \exp(-4n^2)\cos 2n\left(2i\frac{z}{\ell} + s\ell\right)$$

in Eq.(3-1), gives

$$I(z) = \frac{i2kV_o}{\eta_o}\int_{-\infty}^{\infty} \frac{\exp(isz)\,ds}{(s^2-k^2)I_o(a\sqrt{s^2-k^2})K_o(a\sqrt{s^2-k^2})}$$

$$+ \frac{iV_o k}{\eta_o}\sum_{n=1}^{\infty}\exp(-4n(n-1))\int_{-\infty}^{\infty}\frac{\exp(is(\ell+z))\exp(-4n\frac{1}{\ell}(\ell+z))\exp(is(2n-1)\ell)\,ds}{(s^2-k^2)I_o(a\sqrt{s^2-k^2})K_o(a\sqrt{s^2-k^2})}$$

$$+ \frac{iV_o k}{\eta_o}\sum_{n=1}^{\infty}\exp(-4n(n-1))\int_{-\infty}^{\infty}\frac{\exp(is(\ell-z))\exp(-4n\frac{1}{\ell}(\ell-z))\exp(is(2n-1)\ell)\,ds}{(s^2-k^2)I_o(a\sqrt{s^2-k^2})K_o(a\sqrt{s^2-k^2})}$$

This expression is seen to be of the form

$$I(z) = i_o(z) + \sum_{n=1}^{\infty}\left[i_n(1+z) + i_n(1-z)\right] \qquad (3\text{-}3)$$

where

$$i_o(z) = \frac{i2kV_o}{\eta_o}\int_{-\infty}^{\infty}\frac{\exp(isz)\,ds}{(s^2-k^2)I_o(a\sqrt{s^2-k^2})K_o(a\sqrt{s^2-k^2})} \qquad (3\text{-}4)$$

$$i_n(1\pm z) = \frac{iV_o k}{\eta_o}\exp(-4n(n-1))\int_{-\infty}^{\infty}\frac{\exp(is(1\pm z))\exp(-4n\frac{1}{\ell}(1\pm z))\exp(is(2n-1)\ell)\,ds}{(s^2-k^2)I_o(a\sqrt{s^2-k^2})K_o(a\sqrt{s^2-k^2})} \qquad (3\text{-}5)$$

That is $i_o(z) = i_o(-z)$ is the outgoing current wave and the i_n are reflected traveling current waves of different orders as functions of the distances $1 + z$, $1 - z$ respectively from the end points, $i_n(1 + z)$ traveling in the positive and $i_n(1 - z)$ in the negative z-direction.

The expression for $i_o(z)$ was obtained independently by Hallén by considering the infinitely long antenna.

Note from Eqs.(3-4) and (3-5) that

$$i_n(\xi) = \frac{\exp(-4n(n-1))\exp(-4n\frac{\xi}{\ell})\, i2V_o k}{2\,\eta_o} \int_{-\infty}^{\infty} \frac{\exp(is(\xi + (2n-1)\ell))\, ds}{(s^2-k^2) I_o(a\sqrt{s^2-k^2}) K_o(a\sqrt{s^2-k^2})}$$

that is

$$i_n(\xi) = \frac{1}{2} \exp(-4n(\frac{\xi}{\ell} + n - 1))\, i_o(\xi + (2n-1)1) \qquad (3\text{-}6)$$

Hence it suffices to study $i_o(z)$ and to use Eq.(3-6) to obtain corresponding results for $i_n(1 \pm z)$.

The outgoing current wave $i_o(z)$ has been extensively studied by Hallén.(31) We shall illustrate Hallén's approach.

Since Eq.(3-4) does not change when s is replaced by $-s$ and z by $-z$, it is convenient to replace z by $-|z|$ in the exponential term

$$i_o(z) = \frac{i\,2V_o}{\eta_o} \int_{-\infty}^{\infty} \frac{k \exp(-is|z|)\, ds}{(s^2 - k^2) I_o(a\sqrt{s^2 - k^2}) K_o(a\sqrt{s^2 - k^2})} \qquad (3\text{-}7)$$

Let us now replace the path of integration by another path (Fig.2.2)

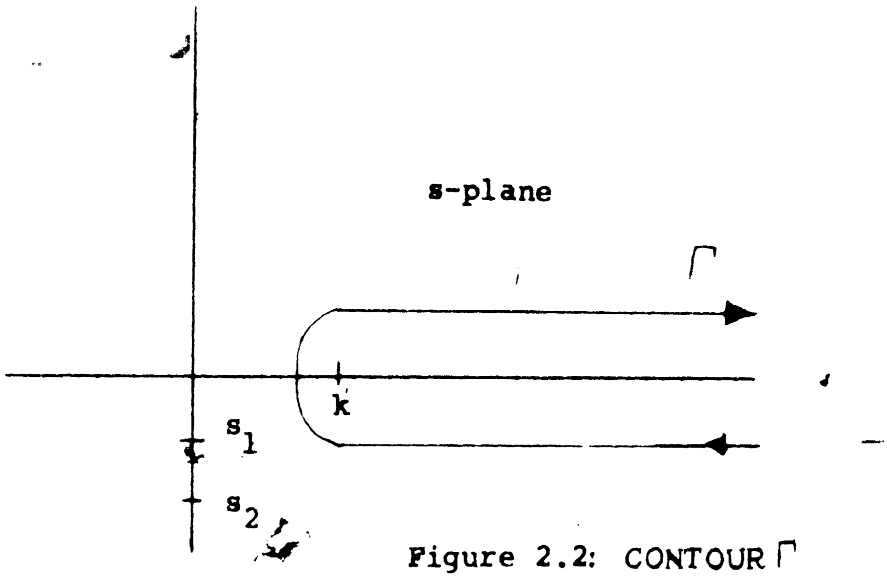

Figure 2.2: CONTOUR Γ

The rotation of the left half of the path of integration in Eq.(3-7) is permissible because of the factor $\exp(-is|z|)$ in the numerator, and the asymptotic behavior of I_oK_o,[32] when residues are provided for the poles. Since $I_o(z)K_1(z) + I_1(z)K_o(z) = 1/z$[33] there follows

$$\frac{1}{(s^2 - k^2)I_oK_o} = \frac{a}{\sqrt{s^2 - k^2}} \left(\frac{I_1}{I_o} + \frac{K_1}{K_o} \right) \Bigg|_{a\sqrt{s^2 - k^2}} \qquad (3\text{-}8)$$

K_o has no zeros in the lower half of the complex plane. The zeros of $I_o(a\sqrt{s^2 - k^2})$ are located at $a\sqrt{s_m^2 - k^2} = i\xi_{om}$, i.e.,

$$s_m = \sqrt{k^2 - \frac{\xi_{om}^2}{a^2}} \qquad (3\text{-}9)$$

where ξ_{om} are the zeros of $J_o(z)$. Now[34] $I_o'(z) = I_1(z)$, and so in the immediate neighborhood of the poles

$$\frac{aI_1}{\sqrt{s^2 - k^2}\, I_o} = \frac{1}{s_m(s - s_m)}$$

Consequently if the contribution of the poles is determined by the calculus of residues, the expression for the outgoing wave takes the form

$$i_o(z) = \sum_{m=1}^{\infty} \frac{4\pi}{\eta_o} V_o \frac{k}{s_m} \exp(-is_m|z|)$$

$$+ \frac{2iV_o}{\eta_o} \int_{\Gamma} \frac{k \exp(-is|z|)\, ds}{(s^2 - k^2) I_o(a\sqrt{s^2 - k^2}) K_o(a\sqrt{s^2 - k^2})} \qquad (3\text{-}10)$$

It is a simple matter to determine the physical significance of the infinite series above. When $k < \xi_{om}/a$, then s_m in Eq.(3.9) is imaginary (Fig.2). The terms of the series are then

$$i_{om} = \frac{4\pi V_o}{\eta_o}\, iak \frac{\exp\left(-\frac{|z|}{a}\sqrt{\xi_{om}^2 - a^2k^2}\right)}{\sqrt{\xi_{om}^2 - a^2k^2}} \qquad (3\text{-}11)$$

and are exponentially attenuated with distance. When $k > \xi_{om}/a$ then

$$s_m = \sqrt{\frac{\omega^2}{c^2} - \frac{\xi_{om}^2}{a^2}} = \frac{1}{c}\sqrt{\omega^2 - \omega_c^2} \qquad (3\text{-}12)$$

where $\omega_c = c\,\xi_{om}/a$. Also $\frac{k}{s_m} = \frac{\omega}{\sqrt{\omega^2 - \omega_c^2}}$. The poles in question then lie on the real axis between k and the origin. The terms of the series then are of the form

$$i_{om}(z) = \frac{V_o}{\eta} \exp(-i \frac{|z|}{c} \sqrt{\omega^2 - \omega_c^2}) \qquad (3\text{-}13)$$

where

$$\eta = \frac{\eta_o}{4\pi} \sqrt{1 - (\frac{\omega_c}{\omega})^2} \qquad (3\text{-}14)$$

The series terms in both cases represent guided waves within the hollow cylindrical antenna, and ω_c has the meaning of a cutoff frequency. Equation (3-10) gives the traveling wave partly as an external antenna current expressed by the last term, and partly as a series of internal waves in wave-quides. It was pointed out before, that Hallén's integral equation was rigorously valid when the antenna consisted of a thin metal tube. It is for this reason that the sum of both the internal and the external solution is obtained as the answer to the problem.

Hallén has shown that the integral in Eq.(3-10) can be put in the form

$$\frac{4\pi}{\eta_o} V_o \hat{\Psi}(kz) \exp(-ik|z|) \qquad (3\text{-}15)$$

where $\hat{\Psi}(kz)$ is a non-periodic complex amplitude function given by

$$\Psi(kz) = \frac{2}{\pi^2} \int_0^1 \frac{du}{u\sqrt{1-u^2}\,\left|H_o^{(1)}(aku)\right|^2}$$

$$+ \frac{2}{\pi^2} \exp(ik|z|) \left\{ \int_0^{\pi/2} \frac{\exp(-ik|z| \cos\phi) - \exp(-ik|z|)}{\sin\phi \left|H_o^{(1)}(ak\sin\phi)\right|^2} d\phi \right.$$

$$\left. + i \int_0^{\pi/2} \frac{\exp(-k|z|\cot\phi)}{\left|H_o^{(1)}(ak/\sin\phi)\right|^2} d\phi \right\} \qquad (3\text{-}16)$$

where $H_o^{(1)}(z)$ is a Hankel function of the first kind.

The last two integrals have both finite ranges of integration and finite integrands and are suitable for numerical evaluation as functions of kz. The first term which is a constant has an important physical significance. It forms the limiting value of the real part of Ψ when $z \rightarrow 0$, and this term alone in Eq.(3-15) gives the conductance at the feeding point of an infinitely long antenna, i.e., the characteristic wave conductance of an antenna of arbitrary length,

$$G_o = \frac{2\pi}{n_o} R_e \left\{ \Psi(0) \right\} = \frac{4}{n_o \pi} \int_0^1 \frac{du}{u\sqrt{1-u^2}\,\left|H_o^{(1)}(aku)\right|^2} \qquad (3\text{-}17)$$

The imaginary part of Ψ, according to Eq.(3-16) does not approach a finite limit as z approaches zero. The reason for this lies in the fact that as the gap between the two halves of the antenna is reduced, the mutual capacitance increases without bound. In reality the air gap never vanishes since even when the separation equals a the capacitance is still very small. The absence of a finite limit when the antenna is considered without the feeding line makes it impossible to assign to the antenna a well defined susceptance B_o whereas it may be ascribed a well defined conductance G_a.

From Eqs.(3-3), (3-6), (3-10) and (3-16) it follows that the solution of Hallén's integral equation is

$$I(z) = \sum_{m=1}^{\infty} \frac{4\pi}{\eta_o} V_o \frac{k}{s_m} \exp(-is_m|z|)$$

$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \exp\left\{-4n\left[\frac{1}{\ell}(\ell+z)+n-1\right]\right\} \frac{2\pi}{\eta_o} V_o \frac{k}{s_m} \exp\left\{-is_m\left[(\ell+z)+(2n-1)\ell\right]\right\}$$

$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \exp\left\{-4n\left[\frac{1}{\ell}(\ell-z)+n-1\right]\right\} \frac{2\pi}{\eta_o} V_o \frac{k}{s_m} \exp\left\{-is_m\left[(\ell-z)+(2n-1)\ell\right]\right\}$$

$$+ \frac{4\pi}{\eta_o} V_o \Psi(kz) \exp(-ik|z|)$$

$$+ \sum_{n=1}^{\infty} \frac{2\pi}{\eta_o} V_o \exp\left\{-4n\left[\frac{1}{\ell}(\ell+z)+n-1\right]\right\} \Psi(k(\ell+z)+k\ell(2n-1)) \exp\left\{-ik\left[(\ell+z)+\ell(2n-1)\right]\right\}$$

$$+ \sum_{n=1}^{\infty} \frac{2\pi}{n_o} V_o \exp\left\{-4n\left[\frac{1}{\ell}(\ell-z)+n-1\right]\right\} \Psi(k(\ell-z)+k\ell(2n-1)) \exp\left\{-ik\left[(\ell-z)+\ell(2n-1)\right]\right\}$$

(3-18)

where

$$s_m = \sqrt{k^2 - \frac{\xi_{om}^2}{a^2}}$$

and ξ_{om} are the zeros of $J_o(z)$.

The first term corresponds, as we have seen, to an outgoing internal wave in the tube antenna. The second and third terms given by the doubly infinite sums represent a series of reflected wave-guide currents of different orders as functions of the distance $\ell + z$, $\ell - z$ respectively from the end points, the former traveling in the positive and the latter in the negative z-direction. The fourth term represents the outgoing external current wave in the antenna. The fifth and sixth terms represent reflected external currents of different orders as functions of the distances $\ell + z$, $\ell - z$ respectively from the end points, one set traveling in the positive and one in the negative z-direction. The factor $\exp(-ik\ell(2n-1))$ in the last two terms corresponds to the total distance $\ell(2n-1)$ over which the wave has traveled before the last reflection.

CHAPTER THREE

SCATTERING BY A THIN WIRE.

Section 1. Formulation of the integral equation.

We wish now to consider the problem of electromagnetic back scattering from a perfectly conducting cylindrical wire excited by a plane electromagnetic wave with harmonic time dependence.

From Eq.(1-7) of chapter two we have for the magnitude of the vector potential on the surface of the wire

$$A_z = \frac{\mu}{4\pi} \int_{-\ell}^{\ell} I(\xi) \frac{1}{2\pi} \int_{0}^{2\pi} \frac{\exp(ikr)}{r} d\phi \, d\xi \qquad (1-1)$$

where as before $k = \omega/c$ and r is the distance from the source point on the surface of the wire, to the field point also on the surface of the wire

$$r = \sqrt{(z - \xi)^2 + 4a^2 \sin^2 \frac{1}{2} \phi} \qquad (1-2)$$

The wire must be assumed thin enough so that the assumption, used in arriving at Eq.(1-1), that the current is symmetrically distributed with respect to the axis of the cylinder is satisfied.

On the surface of the wire, the tangential component of the scattered field $\bar{E}^{(s)}$ must be equal to the negative

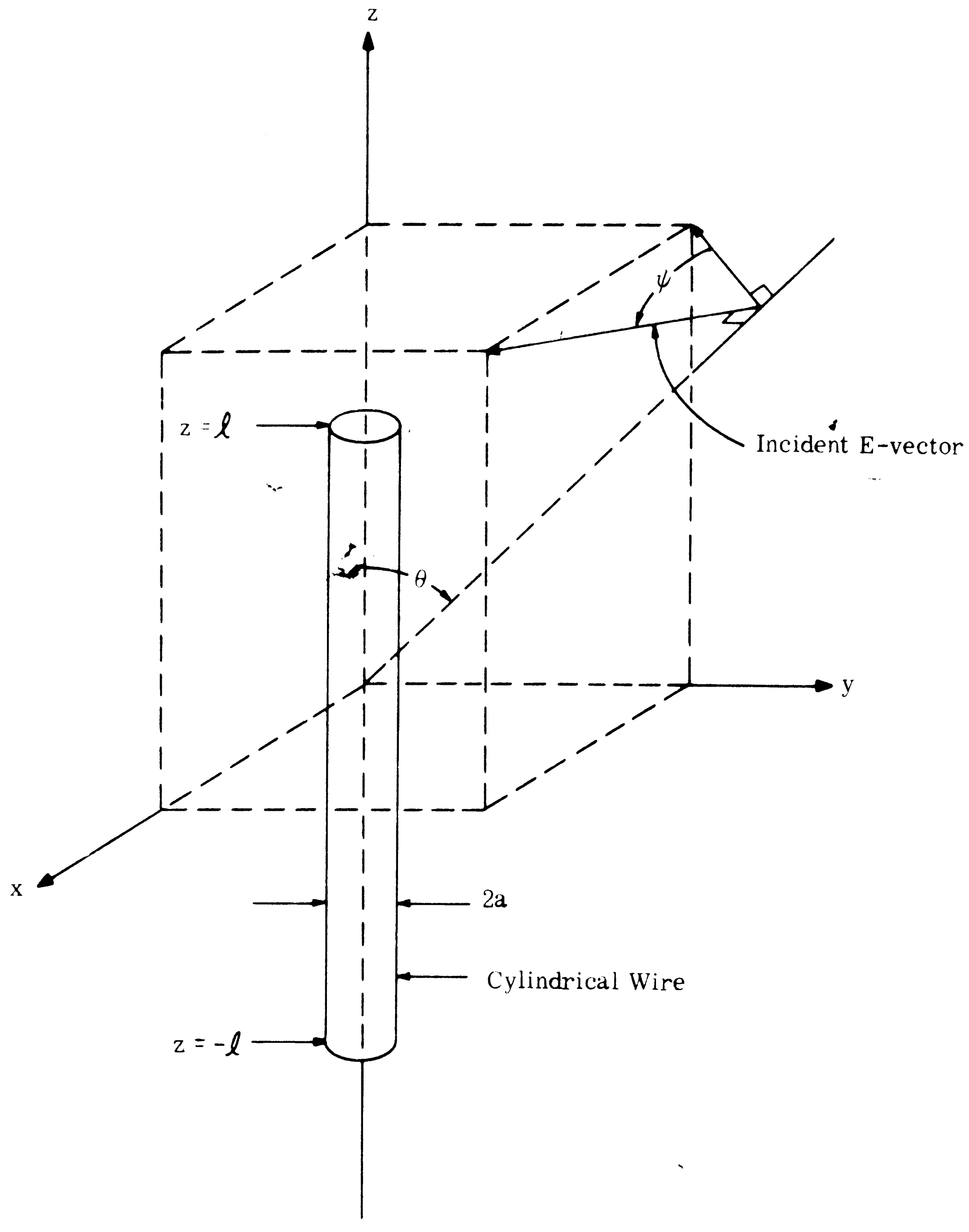

FIGURE 3.1 ORIENTATION OF THE INCIDENT E-VECTOR AND THE POSITION OF THE CYLINDRICAL WIRE

of the tangential component of the incident field $E^{(i)}$, that is

$$-E_z^{(i)} = \frac{c^2}{i\omega}\left(\frac{d^2}{dz^2} + k^2\right)\frac{\mu}{2\pi}\int_{-\ell}^{\ell} I(\xi)\,\frac{1}{2\pi}\int_0^{2\pi}\frac{\exp(-ikr)}{r}\,d\phi\,d\xi \tag{1-3}$$

With the direction of the incident E-vector indicated in Fig. (3.1) we obtain at once

$$E_o \sin\theta\cos\Upsilon\exp(ikz\cos\theta) = \frac{i\eta_o}{4\pi k}\left(\frac{d^2}{dz^2}+k^2\right)\int_{-\ell}^{\ell} I(\xi)\frac{1}{2\pi}\int_0^{2\pi}\frac{\exp(-ikr)}{r}\,d\phi\,d\xi \tag{1-4}$$

where $\eta_o = \sqrt{u/\epsilon}$ and E_o is the amplitude of the incident field.

Eq.(1-4) was derived on the assumption that the end surfaces play a negligible role and its solution will, accordingly, bear the same limitation.

Section 2. Solution of the integral equation.

In Eq.(1-4) replace z by lx and ξ by ly

$$E_o \sin\theta\cos\Upsilon\exp(ik\ell\, x\cos\theta) = \frac{i\eta_o}{4\pi k\ell^2}\left(\frac{d^2}{dx^2}+k^2\ell^2\right)\int_{-1}^{1} I(\ell y)\frac{1}{2\pi}\int_0^{2\pi}\frac{\exp(-ik\ell r)}{r}\,d\phi\,dy \tag{2-1}$$

$$r = \sqrt{(x-y)^2 + 4\,\frac{a^2}{\ell^2}\sin^2\frac{1}{2}\phi}$$

We have already solved this equation in section 2 of chapter two. In fact it follows from Eq.(2-4) that I(z) is obtained from G(y,s) in Eq.(2-14) upon replacing s by kl cos θ, V_o by $\pi E_o l$ sin θ cos Υ, and y by z/l; that is

$$I(z) = \frac{4E_o \cos\Upsilon \exp(ikz\cos\theta)\,\theta_3(k\ell\cos\theta + 2i\,\frac{z}{\ell}\,,\ \exp(-4))}{\eta_o k \sin\theta\, J_o\,(ak\sin\theta)\, H_o^{(2)}(ak\sin\theta)} \tag{2-2}$$

where we made use of the relations

$$I_o(iak\sin\theta) = J_o(ak\sin\theta),\quad K_o(iak\sin\theta) = \frac{1}{2}\pi i^{-1}\, H_o^{(2)}(ak\sin\theta).$$

Eq.(2-2) is the desired solution.

Section 3. Nature of the solution.

For the scattering problem we have obtained for the current

$$I(z) = \frac{4E_o \cos\Upsilon \exp(ikz\cos\theta)\ \theta_3\ (k\ell\cos\theta + 2i\,\frac{z}{\ell},\ \exp(-4))}{\eta_o\ k\sin\theta\ J_o\ (ak\sin\theta)\ H_o^{(2)}(ak\sin\theta)} \tag{3-1}$$

The imaginary part of k will henceforth be taken equal to zero.

Use of the relation

$$\theta_3(kl\cos\theta + 2i\,\frac{z}{\ell}, \exp(-4)) = 1 + 2\sum_{n=1}^{\infty} \exp(-4n^2)\cos 2n(k\ell\cos\theta + 2i$$

in Eq.(3-1) gives

$$I(z) = \frac{4E_o \cos\Upsilon \exp(ikz\cos\theta)}{\eta_o k \sin\theta J_o(aksin\theta) H_o^{(2)}(aksin\theta)}$$

$$+ \frac{4E_o \cos\Upsilon \sum_{n=1}^{\infty} \exp(-4n(n-1)\exp(-4n\frac{1}{\ell}(\ell+z))\exp(ik(\ell+z+(2n-1)\ell))\cos\theta}{\eta_o k \sin\theta J_o(aksin\theta) H_o^{(2)}(aksin\theta)}$$

$$+ \frac{4E_o \cos\Upsilon \sum_{n=1}^{\infty} \exp(-4n(n-1))\exp(-4n\frac{1}{\ell}(\ell-z))\exp(-ik(\ell-z+(2n-1)\ell))\cos\theta}{\eta_o k \sin\theta J_o(aksin\theta) H_o^{(2)}(aksin\theta)} \quad (3\text{-}2)$$

This expression is also of the form

$$I(z) = i_o(z) + \sum_{n=1}^{\infty}\left[i_n(1+z) + i_n(1-z)\right] \quad (3\text{-}3)$$

where

$$i_o(z) = \frac{4E_o \cos\Upsilon \exp(ikz\cos\theta)}{\eta_o k \sin\theta J_o(aksin\theta) H_o^{(2)}(aksin\theta)} \quad (3\text{-}4)$$

$$i_n(\ell+z) = \frac{4E_o \cos\Upsilon \exp(-4n(n-1))\exp(-4n\frac{1}{\ell}(\ell+z))\exp(ik(\ell+Z+(2n-1)\ell)\cos\theta)}{\eta_o k \sin\theta J_o(aksin\theta) H_o^{(2)}(aksin\theta)} \quad (3\text{-}5)$$

$$i_n(1-z) = \frac{4E_o \cos\Upsilon \exp(-4n(n-1))\exp(-4n\frac{1}{\ell}(\ell-z))\exp(-ik(\ell-z+2n-1)\ell)\cos\theta)}{\eta_o k \sin\theta J_o(aksin\theta) H_o^{(2)}(aksin\theta)} \quad (3\text{-}6)$$

That is, $i_o(z)$ is the induced primary current wave and the i_n are the reflected traveling currents of different orders as functions of the distances $l + z$, $l - z$ respectively from the end points.

The expression for $I(z)$ is given to a very good approximation by keeping terms to order $n = 1$ only,

$$I(z) = \frac{4E_o \cos \Upsilon}{\eta_o k \sin\theta \, J_o(ak\sin\theta) \, H_o^{(2)}(ak\sin\theta)} \Big\{ \exp(ikz\cos\theta)$$

$$+ \exp\left[-4\left(1+\frac{z}{\ell}\right)\right] \cos\left[k(z+2\ell)\cos\theta\right] + i\exp\left[-4\left(1+\frac{z}{\ell}\right)\right] \sin\left[k(z+2\ell)\cos\theta\right]$$

$$+ \exp\left[-4\left(1-\frac{z}{\ell}\right)\right] \cos\left[k(z-2\ell)\cos\theta\right] + i\exp\left[-4\left(1-\frac{z}{\ell}\right)\right] \sin\left[k(z-2\ell)\cos\theta\right] \quad (3\text{-}7)$$

This solution possesses the following features: $I(z)$ is composed of a traveling wave plus standing waves. Away from the ends the standing waves are nearly space harmonic; but contrary to what is commonly assumed, for normal incidence, $\theta = \pi/2$, the standing waves are no longer harmonic. In fact, for normal incidence the z-component of the impinging field is uniform in space and there is, indeed, no a priori reason for the belief that the standing waves in this case must be space harmonic.

One more fact of this solution is yet to be discussed, namely the behavior of the current at the ends of the wire. For this we return to Eq.(3-1)

$$I(z) = \frac{4E_o \cos\Upsilon \exp(ikz\cos\theta)\theta_3(k\ell\cos\theta + 2i\frac{z}{\ell}, \exp(-4))}{\eta_g k \sin\theta J_o(ak\sin\theta) H_o^{(2)}(ak\sin\theta)}$$

The zeros of the third theta function $\theta_3(z)$ are located at[35] $z = \frac{1}{2}\pi + \frac{1}{2}\pi\tau - m\pi - n\pi\tau$, where m and n are any integers and τ is defined in Eq.(3.1-20) of chapter one.

For Eq.(3-1) we have $\tau = i4/\pi$ and I(z) vanishes at

$$z = \frac{1}{2}l(2-4n-i\pi\ell(\frac{1}{2} - m + k\ell\cos\theta)) \tag{3-8}$$

since the range of variation of z is the interval $(-\ell,\ell)$, n can only take the values 0 and 1, and m must be such that

$$2\pi\cos\theta = (2m-1)\frac{1}{2}\lambda/l. \tag{3-9}$$

Equation (3-9) assures the vanishing of the imaginary part of Eq.(3-8). For angles such that Eq.(3-9) is satisfied I(z) vanishes at -l and +l.

For other angles the non-vanishing of the current at the ends indicates the presence there of a displacement current in order to satisfy the equation of continuity. In fact it is well known that the radiation field of an electric dipole vanishes along the axis of the dipole, but in the near zone there is an appreciable longitudinal component of $\bar{E}$; and indeed it is this component, and not the radiation field, which determines the distribution of current in a vertical antenna above the earth.[36]

On the other hand, experiment has shown that the current at the ends of the wire is always very small contrary to the large values predicted by Eq.(3-7).

This emphasizes the fact that although Eq.(3-1) is the exact solution of Eq.(1-4), the neglecting of the end surfaces affords a very poor representation of the physical situation. This point was emphasized by L. Brillouin.[37] Brillouin alleged that the end surfaces played a fundamental role in the scattering by a thin cylinder and consequently should be taken into account in the formulation. When this is done, the analysis leads to coupled integal equations which are considerably more difficult to solve.

Other researchers have compensated for the limitations of Eq.(1-4) by assuming current distributions suggested by the experimental results.

Section 4. The expression for the back-scattering cross section.

According to definition, the back-scattering cross section $\sigma(\theta, \Upsilon)$ is given by

$$\sigma(\theta, \Upsilon) = \frac{4\pi R_o^2 \left| E_\theta \cos \Upsilon \right|^2}{E_o^2} \tag{4-1}$$

where E_θ denotes the field produced by the induced current $I(z)$ at a large distance R_o in the direction opposite that of the incident wave; that is,

$$E_\theta = \frac{i\eta_o k \sin\theta \exp(-ik R_o)}{4\pi R_o} \int_{-\ell}^{\ell} I(z) \exp(ikz\cos\theta)\, dz \tag{4-2}$$

Hence

$$\sigma(\theta, \Upsilon) = \frac{\eta_o^2 k^2 \sin^2\theta \cos^2\Upsilon}{4\pi E_o^2} \left[\int_{-\ell}^{\ell} I(z) \exp(ikz\cos\theta)\, dz \right]^2 \tag{4-3}$$

For obtaining numerical values of $\sigma(\theta, \Upsilon)$, it is convenient to eliminate the Υ-dependence by introducing the average value of $\sigma(\theta, \Upsilon)$ over all values of Υ, corresponding to a random distribution of dipoles. Denoting the latter by $\sigma(\theta)$, we have

$$\sigma(\theta) = \frac{1}{\pi} \int_{0}^{\pi} \sigma(\theta, \Upsilon)\, d\Upsilon \tag{4-4}$$

From Eqs.(4-3) and (4-4) there follows

$$\frac{\sigma(\theta)}{\lambda^2} = \frac{3\ell^2}{2\pi\ \lambda^2 J_o^2(ak\sin\theta)\left|H_o^{(2)}(ak\sin\theta)\right|^2}$$

$$\left[\int_{-l}^{l} \exp(2ik\ell\, x\cos\theta)\ \theta_3\ (k\ell\cos\theta + 2ix,\ \exp(-4))dx\right]^2 \qquad (4\text{-}5)$$

Use of Eq.(3-7) gives to a very good approximation

$$\frac{\sigma(\theta)}{\lambda^2} = \frac{3\ell^2}{2\pi\ \lambda^2 J_o^2(ak\sin\theta)\left|H_o^{(2)}(ak\sin\theta)\right|^2} \qquad (4\text{-}6)$$

$$\left[\frac{\sin(2k\ell\cos\theta)}{k\ell\cos\theta} + \frac{2}{4 + k^2\ell^2\cos^2\theta}\right]^2$$

which is the expression sought.

If $ak \ll 1$, then $J_o(ak\sin\theta)$ is nearly unity and for the Hankel function we have for small values of z[38]

$$H_o^{(2)}(z) = 1 - 2\frac{i}{\pi}\log\frac{1}{2}z\gamma \qquad (4\text{-}7)$$

where $\log\gamma = 0.5772...$ is the Euler-Mascheroni constant. The expression for the scattering cross section takes the form

$$\frac{\sigma(\theta)}{\lambda^2} = \frac{3\pi \ell^2}{8\ \lambda^2 \left[(\frac{\pi}{2})^2 + (\log \frac{\gamma\, ak\, \sin\theta}{2})^2 \right]} \left[\frac{\sin(2k\ell\cos\theta)}{k\ell\cos\theta} + \frac{2}{4 + k^2 \ell^2 \cos^2 \theta} \right]^2 \qquad (4\text{-}8)$$

This expression is similar to that obtained by other research workers,[39,40] but it predicts values much smaller than those indicated by experiment. This is not a surprising fact since we have seen that the end surfaces play a dominant role in the shaping of the current distribution.

APPENDIX

FUNDAMENTALS OF THE THEORY OF DISTRIBUTIONS

In the main body of the thesis we had occasion to use several results from the theory of distributions. It is the purpose of this appendix to give rigorous justification to these assertions. Our approach is patterned after the Gelfand-Shilov theory as found in their book <u>Les Distributions</u>.[41]

By a class K of testing functions is meant a set of finite functions $\varphi_n(x)$, i.e., functions that vanish outside a bounded set (corresponding to each function $\varphi(x)$) and which have continuous derivatives of all orders.

Adding the testing functions, or multiplying them by complex numbers, we will always obtain testing functions; in other words K is a vector space.

The sequence of testing functions

$$\varphi_1(x), \quad \varphi_2(x), \quad \ldots, \quad \varphi_\nu(x), \quad \ldots$$

is said to converge to zero in K if they all vanish outside the same bounded set, and if they, together with their derivatives of all orders, converge uniformly to zero in the ordinary sense.

By a continuous linear functional f in the space K is meant a definite rule that assigns to each testing function a certain complex number (f, φ) satisfying the following conditions:

a) For any two complex numbers α_1, α_2 and any two testing functions $\varphi_1(x)$, $\varphi_2(x)$ we have the equality

$$(f, \alpha_1 \varphi_1 + \alpha_2 \varphi_2) = \alpha_1 (f, \varphi_1) + \alpha_2 (f, \varphi_2)$$

(linearity of the functional f);

b) If the sequence of testing functions $\varphi_1, \varphi_2, \ldots, \varphi_\nu, \ldots$ tends toward zero in K, the sequence of numbers

$$(f, \varphi_1), (f, \varphi_2), \ldots, (f, \varphi_\nu), \ldots$$

tends toward zero. (Continuity of the linear functional f).

From here on we will designate as distributions every continuous linear functional in the space of testing functions.

If a function f(x) is alsolutely integrable in every finite interval this function is said to be locally integrable. With each such function we can make correspond to each testing function $\varphi(x)$ the number

$$(f, \varphi) = \int_{-\infty}^{\infty} \overline{f(x)}\, \varphi(x) dx \qquad \text{(A-1)}$$

where $\bar{f}$ denotes the complex conjugate of f and the integration ranges over the bounded set outside of which $\varphi(x)$ vanishes. It is easy to show that the conditions (a) and (b) above are satisfied. Distributions of this type are known as regular distributions. Any other type is known as a singular distribution (in particular the δ-function defined by $(\delta(x), \varphi(x)) = \varphi(0)$ for any testing function).

A distribution f is said to vanish in the neighborhood U of the point x_o if for each testing function $\varphi(x)$ vanishing outside U the equality $(f, \varphi(x)) = 0$ holds. Hence, the distribution f, corresponding to the ordinary function f, vanishes in the neighborhood U of x_o if the function f(x) itself vanishes almost everywhere in this neighborhood. The singular distribution $\delta(x-x_1)$ vanishes in a certain neighborhood of any point $x_o \neq x_1$.

If the distribution f does not vanish in any neighborhood of the point x_o, x_o is called an essential point of the functional f. The union of all the essential points is called the support of the distribution. The support of a distribution f, corresponding to an ordinary function (continuous or piecewise continuous) is the closure of the set over which $f(x) \neq 0$. The support of the distribution $\delta(x-x_o)$ is made of the single point x_o. If the set F contains the support of the functional f, the functional f is said to be concentrated in the set F.

By definition, a distribution f vanishes in the open set G if it vanishes in a certain neighborhood of each point of this set.

Given two distributions f and g, we define their sum as the linear functional defined by

$$(f + g, \varphi) = (f, \varphi) + (g, \varphi).$$

The product of a distribution by a number α is defined by

$$(\alpha f, \varphi) = \bar{\alpha}(f, \varphi) = (f, \bar{\alpha}\varphi).$$

The multiplication by an infinitely differentiable function a(x) is defined by

$$(a(x)f, \varphi) = (f, \overline{a(x)}\varphi). \tag{A-2}$$

By definition, the sequence of distributions f_1, f_2, $f_3, \ldots, f_\nu, \ldots$ converges to the distribution f if for any testing function $\varphi(x)$

$$\lim_{\nu \to \infty} (f_\nu, \varphi) = (f, \varphi).$$

In the same fashion, the series of distributions $\sum_{n=0}^{\infty} h_n$ is said to converge to the distribution g if the sequence of partial sums $g_\nu = \sum_{n=0}^{\nu} h_n$ converges to the distribution g in the sense indicated above.

Let f be a continuous linear functional over the space K of testing functions: the distribution g given by the formula $(g, \varphi) = (f, -\varphi')$ is called the derivative of the functional f and is designated by the notation f' or df/dx.

Lemma 1. In the sense of convergence of distributions

$$\lim_{\nu \to \infty} \int_{-\nu}^{\nu} \exp(ix\xi)\,d\xi = 2\pi\,\delta(x).$$

Proof.

It is well known that

$$\int_{-\infty}^{\infty} f_\nu(x)\,dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin \nu x}{x}\,dx = 1$$

Also for $b > a > 0$

$$\int_a^b f_\nu(x)\,dx = \frac{1}{\pi}\int_a^b \frac{\sin \nu x}{x}\,dx = \frac{1}{\pi}\int_{a\nu}^{b\nu} \frac{\sin y}{y}\,dy \to 0 \text{ as } \nu \to \infty$$

The same result being valid for $a < b < 0$. Finally, the magnitude

$$\left| \frac{1}{\pi} \int_a^b \frac{\sin \nu x}{x} dx \right| = \left| \frac{1}{\pi} \int_{a\nu}^{b\nu} \frac{\sin y}{y} dy \right|$$

is uniformly bounded with respect to a and b, for any ν.

Let us now consider the sequence of primitives

$$F_\nu(x) = \int_{-1}^{x} f_\nu(\xi) d\xi$$

It follows immediately from the definition of the sequence $f_\nu(x)$, that the function $F_\nu(x)$ has for limit as $\nu \to \infty$ a constant equal to zero for $x < 0$, and unity for $x > 0$, and is also uniformly bounded with respect to ν in every finite interval. It follows then that the sequence $F_\nu(x)$ has for limit, in the sense of distributions, the step function $\alpha(x)$, 0 for $x < 0$ and 1 for $x > 0$, hence the sequence of functions $f_\nu(x) = F'_\nu(x)$ has for limit, in the sense of distributions, the function $\alpha'(x)$

$$(\alpha'(x), \varphi(x)) = (\alpha(x), -\varphi'(x)) = -\int_0^\infty \varphi'(x) dx = \varphi(0)$$

and by definition of the delta function

$$\alpha'(x) = \delta(x)$$

so that

$$\lim_{\nu \to \infty} \frac{1}{\pi} \frac{\sin \nu x}{x} = \delta(x).$$

The function $\frac{1}{\pi}\frac{\sin \nu x}{x}$ can in turn be represented as the result of integrating with respect to ξ, from $-\nu$ to ν, the function $(1/2\pi)\exp(i\xi x)$. We have then: In the sense of convergence of distributions

$$\lim_{\nu \to \infty} \int_{-\nu}^{\nu} \exp(ix\xi)\,d\xi = 2\pi\delta(x).$$

<u>Lemma 2</u>. If $f_t(x) = \frac{1}{2\sqrt{\pi t}} \exp(-x^2/4t)$, $t > 0$, then in the sense of convergence of distributions

$$\lim_{t \to 0} f_t(x) = \delta(x)$$

<u>Proof</u>.

Since $f_t(x) > 0$, then for any a and b

$$\int_a^b f_t(x)dx \leq \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \exp(-x^2/4t)dx = 1$$

Making the change of variables $x/\sqrt{t} = y$, we have for $a < 0 < b$

$$\lim_{t \to 0} \int_a^b f_t(x)dx = \lim_{t \to 0} \frac{1}{2\sqrt{\pi}} \int_{a/\sqrt{t}}^{b/\sqrt{t}} \exp(-y^2/4)dy = 1$$

also, for any $b > 0$:

$$\frac{1}{2\sqrt{\pi t}} \int_b^\infty \exp(-x^2/4t)dx < \frac{1}{2\sqrt{\pi t}} \int_b^\infty \exp(-x^2/4t) \frac{x}{2t} \frac{2t}{b} dx$$

$$= \frac{t}{b\sqrt{\pi}} \exp(-b^2/4t) \longrightarrow 0,$$

as $t \to 0$. The integrals over the intervals (b, ∞), $b > 0$ tend to zero, and an analogous result holds for the intervals $(-\infty, a)$, $a < 0$. Constructing as in lemma 1 the sequence of primitives

$$F_t(x) = \int_{-1}^{x} f_t(\xi)d$$

shows, in view of the results above, that the functions $f_t(x)$ form a sequence converging to the delta function. That is

$$\frac{1}{2\sqrt{\pi t}} \exp\ (-x^2/4t) \longrightarrow \delta(x).$$

<u>Convolution of distributions.</u>

In classical analysis one frequently uses the operation of convolution of two functions $f(x)$ and $g(x)$

$$f(x)*g(x) = \int f(\xi)g(x-\xi)d\xi. \qquad \text{(A-3)}$$

The definition of convolution in the domain of distributions is made starting with the concept of direct product of two distributions which is now introduced.

Let a distribution f(x) be defined over the space of testing functions K_x of one variable x and g(y) a distribution defined over the space of testing functions K_y of one variable y. Starting from these distributions we wish to define a distribution h(z) over the space of testing functions K_z of two independent variables x and y. We proceed in the following way: We denote the testing functions for the distribution h(z) by $\varphi(x,y)$. Let us fix x and consider $\varphi(x,y)$ as a function of y. This is obviously a testing function in the space K_y. Let us apply to it the functional g(y) to obtain a certain function $\psi(x)$. This function is infinitely differentiable, since

$$\frac{\psi(x+\Delta x)-\psi(x)}{\Delta x} = \left(g(y),\ \frac{\varphi(x+\Delta x,y)-\varphi(x,y)}{\Delta x}\right) \rightarrow \left(g(y),\ \frac{\partial\varphi(x,y)}{\partial x}\right)$$

Where we used the fact that the sequence

$$\frac{\varphi(x+\Delta x,y)-\varphi(x,y)}{\Delta x}$$

converges to $\partial\varphi(x,y)/\partial x$ in the sense of convergence in the space K_y and g(y) is continuous. It is evident also that

$\psi(x)$ is finite. Hence, $\psi(x)$ is a testing function in the space K_x and we can apply to it the functional $f(x)$. Therefore the expression

$$(f(x), (g(y), \varphi(x,y))) \qquad \text{(A-4)}$$

is well defined. This is a certain functional over the space K_z. From the continuity of the functionals $f(x)$ and $g(y)$ one can deduce the continuity of this functional. This functional is designated by

$$h(z) = f(x) \times g(y)$$

and is called the direct product of the functional $f(x)$ by the functional $g(y)$.

The direct product has a particularly simple appearance when it is applied to a testing function $\varphi(x,y)$, product of two testing functions $\varphi_1(x)$ and $\varphi_2(y)$. In this case, according to the definition

$$(f(x) \times g(y), \varphi_1(x)\,\varphi_2(y)) = (f(x), (g(y), \varphi_1(x)\,\varphi_2(y))) \qquad \text{(A-5)}$$

$$= (f(x), \varphi_1(x)(g(y), \varphi_2(y))) = (f(x), \varphi_1(x))(g(y), \varphi_2(y)).$$

If $f(x)$ and $g(x)$ are two absolutely integrable functions over the line and $h(x) = f(x)*g(x)$ their convolution, the functional defined by the function $h(x)$ (absolutely integrable) can be expressed in the following form

$$(h(x), \varphi(x)) = \int h(x)\varphi(x)dx = \int \left\{ \int f(\xi)g(x-\xi)d\xi \right\} \varphi(x)dx$$

$$= \iint f(\xi)\, g(\eta)\, \varphi(\xi + \eta)d\xi d\eta \ .$$

In other words, the desired result is no other than the result of the application of the functional $f(x)g(y)$, which one can consider as the direct product of the functions $f(x)$ and $g(y)$, over the function $\varphi(x + y)$.

Quite naturally, the general definition of the convolution of distributions f and g is given by

$$(f*g, \varphi) = (f(x) \times g(y),\ \varphi(x + y)) \ . \qquad \text{(A-6)}$$

We now show that the direct product of two distributions is commutative,

$$f(x) \times g(y) = g(y) \times f(x) \ . \qquad \text{(A-7)}$$

For the proof let us note that by virtue of the continuity of both members of Eq.(A-7), it is sufficient to prove the equality over a space of testing functions which is everywhere dense. We will consider then the dense set of functions of the form

$$\sum_{j=1}^{\nu} \varphi_j(x)\, \psi_j(y)$$

where $\varphi_j(x)$, $\psi_j(y)$ $(j = 1,2,..,\nu;\ \nu = 1,2,...)$ are testing functions in their corresponding variables. We have then

$$(f(x) \times g(y), \sum \varphi_j(x)\psi_j(y)) = \sum (f(x) \times g(y), \varphi_j(x)\psi_j(y))$$

$$= \sum (f(x), \varphi_j(x))(g(y), \psi_j(y)).$$

In the same fashion

$$(g(y) \times f(x), \sum \varphi_j(x)\psi_j(y)) = \sum (g(y), \psi_j(y))(f(x), \varphi_j(x)).$$

and the result follows at once.

Lemma 3. If D be a differential operator then

$$D(f*g) = Df*g = f*Dg.$$

Proof.

We have

$$(D(f*g), \varphi) = (f*g, D* \varphi)$$

where, for example

$$D* = (-)^{\nu} D$$

if D is a homogeneous differential operator of order ν. Furthermore, according to the definition of convolution

$$(f*g, D*\varphi) = (g(y), (f(x), D*\varphi(x+y)))$$

$$= (g(y), (Df(x), \varphi(x+y))) = (Df*g, \varphi)$$

hence

$$D(f*g) = Df*g$$

and by virtue of the commutativity of the convolution

$$D(f*g) = D(g*f) = Dg*f = f*Dg.$$

Let us now establish the following lemma, relating to the continuity of the convolution.

<u>Lemma 4</u>. If $f_\nu \to f$ then $f_\nu * g \to f*g$ in each of the following cases:

a) The functionals f_ν are concentrated in one and the same bounded set.

b) g is concentrated in a bounded set.

c) The supports of the functionals f_ν and g are bounded on one and the same side in a manner independent of ν

Proof.

According to the definition of convolution, for any testing function φ:

$$(f_\nu * g, \varphi) = (f_\nu(y), (g(x), \varphi(x + y)))$$

For the case (a) the function $(g(x), \varphi(x + y))$ can be modified to a function vanishing outside the set where all the $f_\nu(y)$ are concentrated. Hence

$$(f_\nu * g, \varphi) = (f_\nu(y), \psi(y)) \longrightarrow (f, \psi) = (f * g, \varphi),$$

and

$$f_\nu * g \longrightarrow f * g.$$

For the case (b),

$$\psi(y) = (g(x), \varphi(x + y))$$

is a testing function and the proof proceeds as above. For case (c), if we suppose, to fix the ideas, that the support of the functionals f_ν and g are bounded on the left, then the function

$$\psi(y) = (g(x), \varphi(x + y))$$

has a support bounded on the right. We can then change it to a testing function vanishing outside the set where the functionals $f'_{\nu}(x)$ are concentrated and the proof proceeds as before.

<u>Fourier transforms of distributions</u>.

In classical analysis there exists a one-to-one correspondence between the class of functions possessing a Fourier transform and their corresponding transforms. This correspondence preserves linear operations and the properties of convergence. An analogous coreespondence can be established among the linear functionals defined over these spaces. This correspondence is established in such a way that it makes correspond the classical Fourier transform to functionals corresponding to absolutely integrable functions.

Let f(x) be an absolutely integrable function and $g(\sigma)$ its Fourier transform. For each testing function $\varphi(x)$ and its Fourier transform $\psi(\sigma)$ we have Parseval's formula

$$(f, \varphi) = \int_{-\infty}^{\infty} \overline{f(x)}\varphi(x)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{f(x)} \left\{ \int_{-\infty}^{\infty} \psi(\sigma) \exp(-ix\sigma)d\sigma \right\} dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(\sigma) \left\{ \int_{-\infty}^{\infty} \overline{f(x)\exp(ix\sigma)dx} \right\} d\sigma = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{g(\sigma)}\psi(\sigma)d\sigma = \frac{1}{2\pi}(g, \psi).$$

This formula is valid when $f(x)$ and $\varphi(x)$, and hence their Fourier transforms $g(\sigma)$ and $\psi(\sigma)$, are of integrable square. Parseval's formula shows that $g(\sigma)$ considered as a distribution, acts on a testing function according to the formula

$$(g, \psi) = 2\pi(f, \varphi), \qquad \text{(A-8)}$$

This formula is used to definethe distribution g over the space Z of testing functions ψ corresponding to any given distribution f over the space K. The functional g defined by Eq.(A-8), is called the Fourier transform of the functional f and is denoted by the symbols $\tilde{f}$ or F[f].

The usual rules of differentiation of Fourier transforms carry over for Fourier transforms of distributions:

$$P\left(\frac{d}{ds}\right)\tilde{f} = F[P(ix)f] \qquad \text{(A-9)}$$

$$F\left[P\left(\frac{d}{dx}\right)f\right] = P(-is)\tilde{f} \qquad \text{(A-10)}$$

where P(x) is a polynomial in x.

For the proof it is sufficient to consider the case

$$P\left(\frac{d}{dx}\right) = \frac{d}{dx}$$

We have then

$$(\widetilde{(ixf)}, \widetilde{\varphi}) = 2\pi(ixf, \varphi) = 2\pi(f, -ix\varphi) = (\widetilde{f}, \widetilde{(-ix\varphi)})$$

$$= (\widetilde{f}, -\frac{d}{ds}\widetilde{\varphi}) = (\frac{d}{ds}\widetilde{f}, \widetilde{\varphi}),$$

which gives equation (A-9). Equation (A-10) is established in an analogous manner.

<u>Lemma</u> <u>5</u>.

$$\widetilde{\delta} = 1.$$

<u>Proof</u>.

According to the definition

$$(\widetilde{\delta}, \widetilde{\varphi}) = 2\pi(\delta, \varphi) = 2\pi\,\varphi(0) = \int_{-\infty}^{\infty} \widetilde{\varphi}(\sigma)\,d\sigma = (1, \widetilde{\varphi})$$

hence

$$\widetilde{\delta} = 1.$$

<u>Lemma 6</u>. If P(x) is a polynomial

$$\widetilde{P(x)} = 2\pi P(-i \frac{d}{ds})\delta(s).$$

<u>Proof</u>.

$$\widetilde{P(x)} = \widetilde{[P(x)\cdot 1]} = P(-i \frac{d}{ds})\tilde{1},$$

and

$$(\tilde{1},\tilde{\varphi}) = 2\pi(1,\varphi) = 2\pi \int_{-\infty}^{\infty} \varphi(x)dx = 2\pi \int_{-\infty}^{\infty} \varphi(x)\exp(-ix0)dx$$

$$= 2\pi \,\psi(0) = 2\pi(\delta, \psi)$$

hence

$$\widetilde{P(x)} = 2\pi\; P(-i \frac{d}{ds})\delta(s).$$

<u>Lemma 7</u>. The Fourier transform of the exponential function exp(bx) is $2\pi\delta(s-ib)$.

<u>Proof</u>.

We will make use of the convergence in the complex plane of the series

$$\exp(bx) = \sum_{n=0}^{\infty} \frac{b^n x^n}{n!}$$

applying the Fourier transform operator term by term to this series, we obtain

$$\widetilde{\exp(bx)} = \sum_{n=0}^{\infty} \frac{b^n}{n!} \widetilde{x^n} = 2\pi \sum_{n=0}^{\infty} \frac{b^n}{n!} \left(-i\frac{d}{ds}\right)^n \delta(s)$$

and

$$\left(2\pi \sum_{n=0}^{\infty} \frac{b^n}{n!} \left(-i\frac{d}{ds}\right)^n \delta(s),\ \psi(s)\right) = 2\pi \sum_{n=0}^{\infty} \left(\frac{b^n}{n!} \left(-i\frac{d}{ds}\right)^n \delta(s),\ \psi(s)\right)$$

$$= 2\pi \sum_{n=0}^{\infty} \left(\delta(s),\ \frac{i^n b^n}{n!} \psi^{(n)}(s)\right) = 2\pi \left(\delta(s),\ \sum_{n=0}^{\infty} \frac{i^n b^n}{n!} \psi^{(n)}(s)\right)$$

and the series

$$\sum_{n=0}^{\infty} \frac{i^n b^n}{n!} \psi^{(n)}(s)$$

converges in the functional sense to the function $\psi(s + ib)$.

Hence

$$\left(2\pi \sum_{n=0}^{\infty} \frac{b^n}{n!} \left(-i\frac{d}{ds}\right)^n \delta(s),\ \psi(s)\right) = 2\pi(\delta(s),\ \psi(s + ib))$$

$$= (2\pi\delta(s - ib),\ \psi(s))$$

where the translated δ-function, $\delta(s + h)$, is defined by the equality

$$(\delta\ (s + h), \psi(s)) = (\delta\ (s), \psi\ (s - h)) = \psi(-h) \quad \text{(A-11)}$$

for all (complex) h.

Hence

$$\widetilde{\exp(bx)} = 2\pi\ \delta\ (s-ib) \quad \text{(A-12)}$$

for any complex b.

<u>Fourier transform of a direct product</u>.

The Fourier transform of the functional f, acting over the space K of testing functions $\varphi(x)$ of several independent variables $x = (x_1, x_2, \ldots, x_n)$, is defined as the functional g acting over the space Z of testing functions $\psi(x)$, $s = (s_1, s_2, \ldots, s_n)$, by the formula

$$(g, \psi\ ) = (2\pi)^n (f, \varphi) \quad \text{(A-13)}$$

where $\psi = \widetilde{\varphi}$ is the Fourier transform of the function $\varphi(x)$. The functional g is also designated as $\tilde{f}$ or F[f].

Let f(x) and g(y) be given distributions over the variables x and y respectively, and let $\widetilde{f(\xi)}$ and $\widetilde{g(\eta)}$ be their Fourier transforms. The Fourier transform of the direct product $f(x) \times g(y)$ is given by the formula

$$\widetilde{f \times g} = \widetilde{f(\xi)} \times \widetilde{g(\eta)} \tag{A-14}$$

that is, it is equal to the direct product of the Fourier transforms of the functionals f and g.

For the proof it is sufficient to consider testing functions $\varphi(x,y)$ of the form

$$\sum_{j=1}^{n} \varphi_j(x) \, \psi_j(y).$$

In this case

$$(\widetilde{f \times g}, \widetilde{\sum \varphi_j \psi_j}) = (2\pi)^2 (f \times g, \sum \varphi_j \psi_j) = (2\pi)^2 \sum (f, \varphi_j)(g, \psi_j)$$

$$= (\tilde{f} \times \tilde{g}, \sum \tilde{\varphi}_j \tilde{\psi}_j) = (\tilde{f} \times \tilde{g}, \sum \widetilde{\varphi_j \psi_j}).$$

When the convolution is given by an expression of the form

$$f(x) * g(x) = \int f(\xi) g(x-\xi) d\xi \tag{A-15}$$

then the ordinary Fourier transform

$$\int_{-\infty}^{\infty} f(x) * g(x) \exp(ix\sigma)\, dx$$

can be considered as a distribution depending on a parameter σ defined over the space of testing functions $\exp(ix\sigma)$. According to Eq.(A-6) we have then

$$(f*g, \exp(ix\sigma)) = (f(x) \times g(y), \exp(i\sigma(x+y)))$$

$$= (f(x) \times g(y), \exp(ix\sigma)\exp(iy\sigma)) = (f(x), \exp(ix\sigma))(g(x), \exp(ix\sigma$$

$$= \widetilde{f(\sigma)}\widetilde{g(\sigma)},$$

where we used Eq.(A-5).

We have then

<u>Lemma</u> <u>8</u>. The Fourier transform of the convolution

$$f(x)*g(x) = \int f(\xi)g(x-\xi)d\xi$$

is given by

$$F[f(x)*g(x)] = \widetilde{f(\sigma)}\widetilde{g(\sigma)}.$$

This result, commonly used in classical analysis, is known as the convolution theorem for Fourier transforms.

BIBLIOGRAPHY

(1) E. D. Rainville, Special Functions. The Macmillan Company. New York. p. 187.

(2) E. D. Rainville. loc. cit. p. 197.

(3) P. M. Morse and H. Feshbach, Methods of Theoretical Physics. McGraw-Hill. New York. p. 936, and private communication.

(4) E. C. Titchmarsh, Introduction to the Theory of Fourier Integrals. Oxford. p. 314.

(5) E. D. Rainville. loc. cit. p. 194.

(6) H. S. Carslaw, Introduction to the Theory of Fourier Series and Integrals. Dover Publications. p. 99.

(7) H. S. Carslaw. loc. cit. p. 158, corollary 2.

(8) H. S. Carslaw loc. cit. p. 178, theorem III.

(9) H. S. Carslaw. loc. cit. p. 226.

(10) E. D. Rainville. loc. cit. p. 314.

(11) E. T. Whittaker and G. N. Watson, A Course of Modern Analysis. Cambridge. London. p. 475.

(12) E. D. Rainville. loc. cit. p. 187.

(13) E. C. Titchmarsh. loc. cit. p. 79, theorem 55.

(14) E. D. Rainville. loc. cit. p. 157.

(15) G. N. Watson. The diffraction of electrical waves by the earth. Proceedings of the Royal Society of London. V. A95, 1918-1919, p. 83.

(16) E. D. Rainville. loc. cit. p. 200.

(17) E. D. Rainville. loc. cit. p. 206.

(18) E. D. Rainville. loc. cit. p. 148.

(19) E. D. Rainville. loc. cit. p. 201.

(20) E. D. Rainville. loc. cit. p. 299.

(21) E. D. Rainville. loc. cit. p. 300.

(22) Ya. N. Fel'd and L. D. Bakhrakh. Present State of Antenna Synthesis Theory. Radio Engineering and Electronic Physics. Academy of Sciences. U.S.S.R. Number 2. Feb. 1963. pp. 163-169.

(23) E. C. Titchmarsh. loc. cit. p. 316.

(24) E. D. Rainville. loc. cit. p. 215. Eq. 25.

(25) E. D. Rainville. loc. cit. p. 125, theorem 42.

(26) E. D. Rainville. loc. cit. p. 106. exercise 11.

(27) E. D. Rainville. loc. cit. p. 216. exercise 10.

(28) E. Hallén, Electromagnetic Theory. John Wiley & Sons Inc. New York. p. 447.

(29) G. N. Watson, A Treatise on the Theory of Bessel Functions. Cambridge. p. 416.

(30) G. N. Watson. loc. cit. p. 441.

(31) E. Hallén. loc. cit. p. 448.

(32) G. N. Watson. loc. cit. pp. 202-203. Eqs. 1 and 2.

(33) G. N. Watson. loc. cit. p. 80. Eq. 20.

(34) G. N. Watson. loc. cit. p. 79. Eq. 7.

(35) E. D. Rainville. loc. cit. p. 320.

(36) J. A. Stratton, Electromagnetic Theory. McGraw-Hill. New York. p. 436.

(37) L. Brillouin. The Antenna Problem. Quarterly of Applied Mathematics. Oct. 1943. V. 1. Number 3. pp. 301-314.

(38) A. Sommerfeld, Electrodynamics. Academic Press Inc. New York. p. 179. Eq. 5.

(39) C. T. Tai, Electromagnetic Back-Scattering from Cylindrical Wires. Journal of Applied Physics. V. 23. Number 8. 1952. p. 909.

(40) Van Vleck, Bloch, and Hamermesh, Journal of Applied Physics. V. 18. 1947. p. 274.

(41) I. M. Gelfand and G. R. Shilov, Les Distributions. Dunod. Paris.