SCATTERING BY A TORUS

by

Pushpamala Laurin

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Doctoral Committee:

Professor Otto Laporte, Chairman
Professor Chiao-Min Chu
Doctor Ralph E. Kleinman
Associate Professor C. R. Worthington
Associate Professor Alfred C. T. Wu

ABSTRACT

Solutions of low frequency scattering of acoustic and electromagnetic waves by a torus are derived in this work using a "quasi-static" approach based on a method due to Stevenson (1953) and Kleinman (1965). The solutions are in power series, in ascending powers of k, the wave number. These series are also called the Rayleigh series and are valid for small k.

Since the method is such that the solution is constructed from the solution of the potential equation, Laplace's equation is solved in toroidal coordinates for both the Dirichlet and Neumann boundary conditions. Green's function is derived for the Dirichlet case and particular problems are solved for the Neumann case. These results also have applications in fluid dynamics.

Two non-zero terms in the low frequency solution expansion are explicitly derived for the cases of acoustic scattering by soft and rigid tori. Two terms in the low frequency expansions for both the electric and the magnetic fields are derived for the scattering of a normally incident plane electromagnetic wave. For this case the torus is assumed to be perfectly conducting.

The far field is calculated for a small torus for the acoustic problem, with normal incidence on a soft forus, and compared with the known results for the corresponding problem of a sphere and of a disc. The radii of these bodies which give equivalent scattered far fields are calculated as a function of the radius of the torus.

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INTRODUCTION

Scattering theory deals with the interaction of an obstacle in a wave field and the best known and most often studied wave motions are those associated with acoustics and electromagnetism.

A standard problem in acoustic scattering involves the determination of a function $\Phi(\bar{r})$ which is the field scattered by a body. In particular we seek a solution of the Helmholtz equation

$$(\nabla^2 + k^2)\,\Phi = 0$$

such that

$$\Phi(\bar{r}_B) = -\Phi^{inc}(\bar{r}_B)$$

or

$$\left.\frac{\partial \Phi(\bar{r})}{\partial n}\right|_{\bar{r}=\bar{r}_B} = -\left.\frac{\partial \Phi^{inc}}{\partial n}\right|_{\bar{r}=\bar{r}_B}$$

and Φ satisfies the Sommerfeld radiation condition (assuming harmonic time dependence, $e^{-i\omega t}$).

$$\lim_{r \to \infty} r\left(\frac{\partial}{\partial r}\Phi - ik\Phi\right) = 0$$

where Φ^{inc} is the incident field, $\vec{r}$ is the radius vector (employing spherical coordinates) and $\vec{r}_B$ is a vector from the origin to a point on the boundary, B, of the scatterer and $\hat{n}$ is the outward normal from the scatterer. Correspondingly, in a vector problem we seek solutions of the Maxwells equations in vacuum,

$$\nabla \times \bar{E} = -\frac{\partial B}{\partial t} \qquad \nabla \cdot \bar{B} = 0$$

$$\nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t} + J \qquad \nabla \cdot \bar{D} = 0$$

where $\bar{E}$ and $\bar{H}$ denote the electric and magnetic field strengths, while

$$\bar{B} = \mu\bar{H} \quad \text{and} \quad \bar{D} = \epsilon\bar{E}$$

where ϵ and μ are the permittivity and permeability of the medium, assumed to be constants.

It is of particular interest to solve for $\bar{E}$ and $\bar{H}$, exterior to a body of infinite conductivity and in a region of space containing no free charges and electric currents. If we assume harmonic time dependence of the wave motion, our problem can be stated as follows:

Determine $\bar{E}$ and $\bar{H}$, such that

$$\nabla \times (\nabla \times \bar{E}) - k^2 \cdot \bar{E} = 0, \quad \nabla \times (\nabla \times \bar{H}) - k^2 \bar{H} = 0$$

$$\hat{n} \times \bar{E}\Big|_{\bar{r} = \bar{r}_B} = 0$$

$$\hat{n} \times \bar{H}\Big|_{\bar{r} = \bar{r}_B} = \bar{k}$$

$$\hat{n} \cdot \bar{B}\Big|_{\bar{r} = \bar{r}_B} = 0$$

$$\hat{n} \cdot \bar{D} = \delta$$

where $\bar{k}$ represents the surface current density and δ is the surface charge density. The scattered fields must, in addition, satisfy the radiation conditions:

$$\lim_{r \to \infty} \bar{r} \times (\nabla \times \bar{E}^{scat}) + ikr\bar{E}^{scat} = \bar{r} \times (\nabla \times H^{scat}) + ikr\bar{H}^{scat} = 0$$

The scalar Helmholtz equation in three dimensions has only been solved in eleven coordinate systems. These eleven coordinates systems have the property of separability; viz., the wave equation is split into three second order differential equations, each equation being a function of only one variable. Then the field Φ can be written as a linear combination of terms of the form

$$\Phi = U(u)V(v)W(w)$$

where $u, v,$ and w represent the coordinates in an orthogonal curvilinear coordinate system.

On the other hand, the vector wave equation in three dimensions is separable only in rectangular coordinates and spherical coordinates. In general the vector problem is considerably more difficult than the scalar, since one has to solve for six scalar components of the fields E and H.

Efforts have been made to extend the method of separation of variables to certain other coordinate systems where one of the coordinates can be separated, leaving a non-separable second order differential equation. This, then is reduced to a recurrence set of ordinary differential equations in one variable. In particular, this method has been applied to the wave equation in toroidal coordinates by Weston (1956). Although the solution of the equation is written in terms of toroidal wave functions, the application of boundary conditions poses some problems because the wave functions do not form a complete set. But in the limit of a very thin ring results have been obtained by Weston for the scattering of a plane electromagnetic wave.

These problems have led researchers into attacking the scattering by non-separable bodies by techniques, where separability of the wave equation is not involved. In the low frequency limit (when the wavelength of the incident radiation is larger than the dimension of the scatterer), advantage has been taken of the solution of Laplace's equation, which is a limiting form of the wave equation when the wave number, k, is zero. This method of treating the scattering problem as a perturbation ot the potential equation is, of course, valid only when k is small. A detailed exposition of various methods of treating the low frequency scattering problems for both scalar and vector problems is given by Kleinman (1966).

The correspondence between low frequency solutions and the static problem was recognized by Lord Rayleigh as early as 1897. He seems to be the first to have observed that the solution of Laplace's equation constitutes the first term in an expansion for the scattered field in powers of k, when k is small. This was not pursued any further until the 1950's, when interest was revived in this subject to obtain systematic series expansion valid for small k. The advantage of such a method lies in that the potential problems, though formidable at times, still are simpler than problems in wave phenomenon. The major contributions in this area have come from Stevenson (1953), Noble (1962), and Kleinman (1965).

Stevenson's method holds good for both scalar and vector problems. The method is fairly straight forward and has been applied successfully to certain shapes including an ellipsoid of revolution. But the major disadvantage of this method comes from the fact that one has to solve a static problem at every stage of the expansion. Every term is derived in terms of the previous term and a static problem. In spite of it, it has proved to be a powerful method for low frequency scattering problems.

Noble (1962) formulates the problem in terms of integral equations and a solution for a scattering problem for a general boundary is obtained as the perturbation of the solution of the corresponding potential problem. Each term in the low frequency expansion is the solution of an integral equation differing from term to term in the inhomogeneous part. In general the solution is obtained only as a formal inverse for successive terms and does not yield an explicit representation. Difficulties arise in carrying out the scheme except for some simple shapes.

The technique proposed by Kleinman is a very elegant one and one can obtain the n th iteration in the expansion of the scattered field systematically. The method is limited to scalar problems as yet and it is limited in another sense that one would have to know the static Green's function. Details of the method are given in his paper (1965). It was originally applied to problems with Dirichlet boundary conditions, but it has recently been extended to scattering problems with Neumann boundary conditions by Ar (1966).

In this thesis a quasi-static approach based on Stevenson's method will be used to study the scattering of plane acoustic and electromagnetic waves by a torus. This brings us to the subject of solution of Laplace's equation. For separable coordinates systems this is a fairly simple matter, but the Laplace's equation in toroidal coordinates is not separable. By removing a factor called the R-factor, the equation can be made simply separable and the solution of the equation can be written as

$$\Phi(\bar{r}) = \frac{U(u)\ V(v)\ W(w)}{R(u, v, w)}$$

where R is a function of the coordinates: and is not a constant. The separability of Laplace's and Helmholtz equations are discussed in great length by Moon and Spencer (1961).

The work reported in the literature on the subject of toroidal coordinates have dealt only with the solution of Laplace's equation and in particular for the Dirichlet case. The earliest one reported is by Hicks * (1881), who has developed many interesting results involving toroidal functions with applications to some potential problems. It was followed by another paper by the same author in 1884, where he focussed his attention on the motion of a hollow vortex, where cyclic motion exists in a fluid. His work is by far one of the most detailed and valuable. The papers by Basset (1893) and Dyson (1893) are mainly concerned with the toroidal functions as such and serve as a good supplement to Hick's work.

Recently, S. Loh and his coworkers have done extensive numerical work on the toroidal functions which are published in a series of papers 1959a, 1959b, 1961). Most of the theoretical work in their papers is already contained in Hick's papers.

Although a good deal of work has been done on the Dirichlet or the electrostatic problems involving tores surprisingly little has been reported on the Neumann boundary problem. In the preparation of this thesis it was found that there are great difficulties in solving the Neumann boundary problems, while the Dirichlet problems can be solved in a fairly straight forward way as has been reported by Hicks (1881), Moon and Spencer (1961), Morse and Feshbach (1953), Hobson (1955), Loh (1960) etc., just to name a few. But with the exception of Hicks' work, no mention has been made by any of the authors about the other problem where the normal derivative rather than the potential itself is specified on the boundary. Fluid dynamicists have been interested in the Neumann problem in connection with the vortex rings and Basset (1893) and Lamb (1932) have reported some results for the above mentioned problem for the particular case of the fluid flowing in the direction parallel to the axis of symmetry of the torus. This special case can be handled elegantly by means of a vector potential which satisfies Stoke's equation and enables one to obtain the stream-lines. But unfortunately this method is not

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* Hicks gives credit to Neumann (1864) for introducing the toroidal functions for the first time to study the temperatures in a shell bounded by non-concentric spheres.

applicable to studying the related problems when the fluid is flowing in any other direction. Hicks also has studied the problem of a torus moving parallel to its axis of symmetry in an infinite fluid. This was done in terms of a scalar velocity potential and considerable simplification resulted due to symmetry in the azimuthal variable.

The above mentioned symmetry does not necessarily exist in the scattering problems as such, and as a first step the solution of the potential equation with non-symmetric Neumann boundary condition is derived in this thesis. But still much remains to be done in potential problems for a torus.

For example, it has only been found possible so far to solve the Neumann boundary value potential problems when the torus is immersed in uniform fields, at any angle. But for arbitrary sources the results are yet not suitable, because the Green's function of the second kind is known only up to a set of constants.

It is worthwhile to mention at this outset that very few authors have considered the solution of wave equation in toroidal coordinates. The two papers known so far are the ones of Weston (1956) and Bond (1955). Both solutions hold good for thin rings. The latter, obtained by the method of local separation, being valid only in a limited region.

Chapter II of the present work deals with the toroidal coordinates and the solution of Laplace's equation. Chapter III describes the method of low frequency expansion for the scattered fields and the derivation of the far fields. Scattering of acoustic waves by both soft and hard tores are considered and explicit results have been obtained up to the third order terms using the quasi-static approach in Chapter IV. In Chapter V the zeroeth and first order terms in the expansion of the scattered field are derived for scattering of an electromagnetic wave incident normally on a perfectly conducting torus. The derivation and discussion of Helmholtz formula is included in an appendix for the sake of completeness.

II

SOLUTION OF LAPLACE'S EQUATION

2.1 TOROIDAL COORDINATE SYSTEM

Toroidal coordinates are generated by rotating the bipolar coordinates obtained by the familiar transformation

$$z* = a \frac{(e^w + 1)}{e^w - 1}$$

about the y - axis. (See Fig. 2-1), where

$$z = x + iy \qquad\qquad w = u + iv \quad .$$

These orthogonal curvilinear coordinates η, θ, ψ, are defined by the equations:

$$x = a \frac{\sinh \eta \cos \psi}{\cosh \eta - \cos \theta}$$

$$y = a \frac{\sinh \eta \sin \psi}{\cosh \eta - \cos \theta}$$

$$z = a \frac{\sin \theta}{\cosh \eta - \cos \theta}$$

where η ranges from 0 to ∞, θ from 0 to 2π and ψ from 0 to 2π. The surfaces η = constant are tores or anchor-rings with an axial circle in the x - y plane centered at the origin and of radius a coth η, having a circular cross section of radius a csch η. The surface $\eta = \eta_o$ defines a torus. (See Fig. 2-2).

$$z^2 + (\rho - a \coth \eta_o)^2 = a^2 \operatorname{csch}^2 \eta_o$$

and $\theta = \theta_o$ defines a spherical bowl,

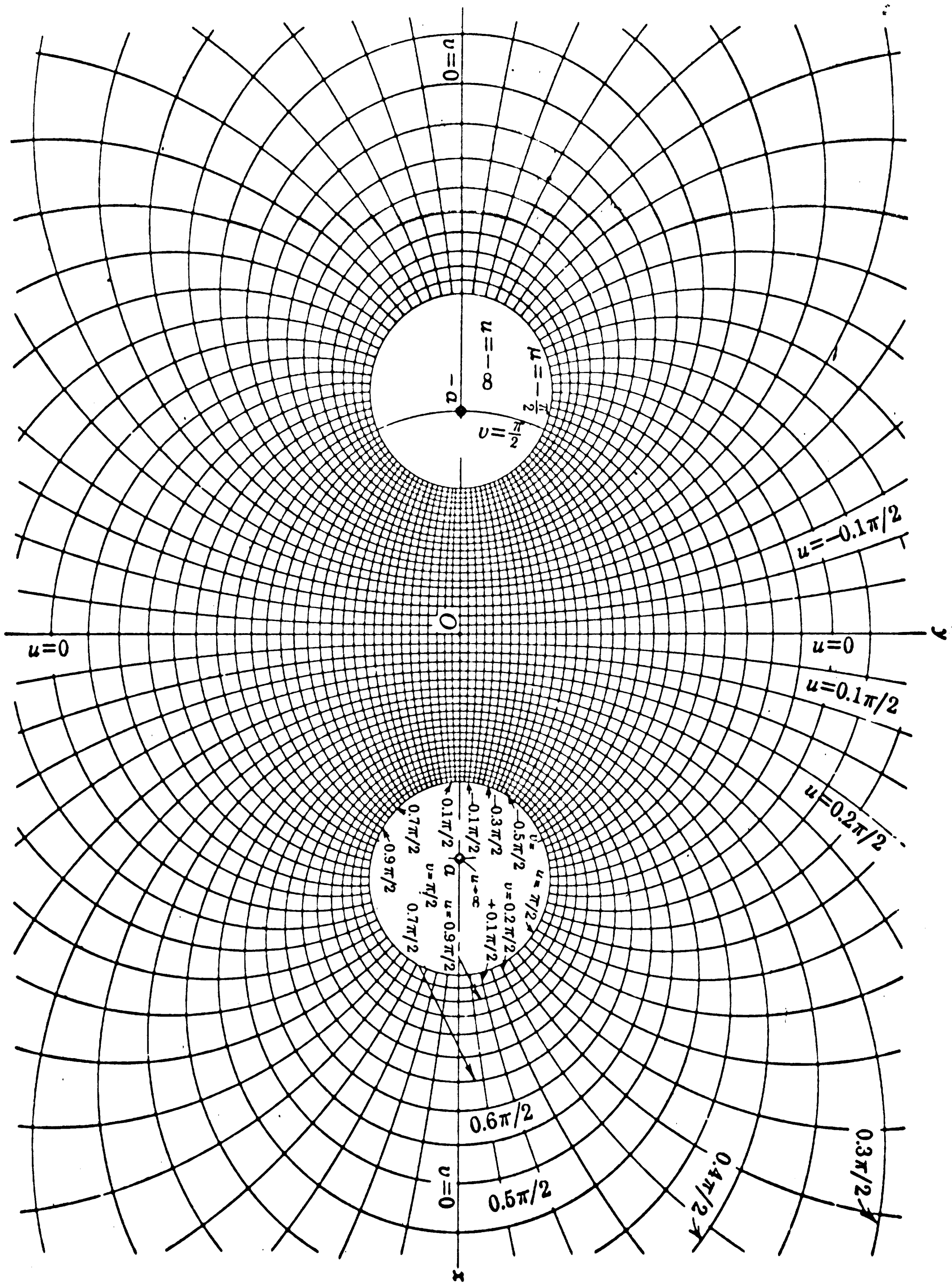

FIG. 2-1: BIPOLAR CIRCLES, $z^* = \dfrac{a(e^W + 1)}{e^W - 1}$

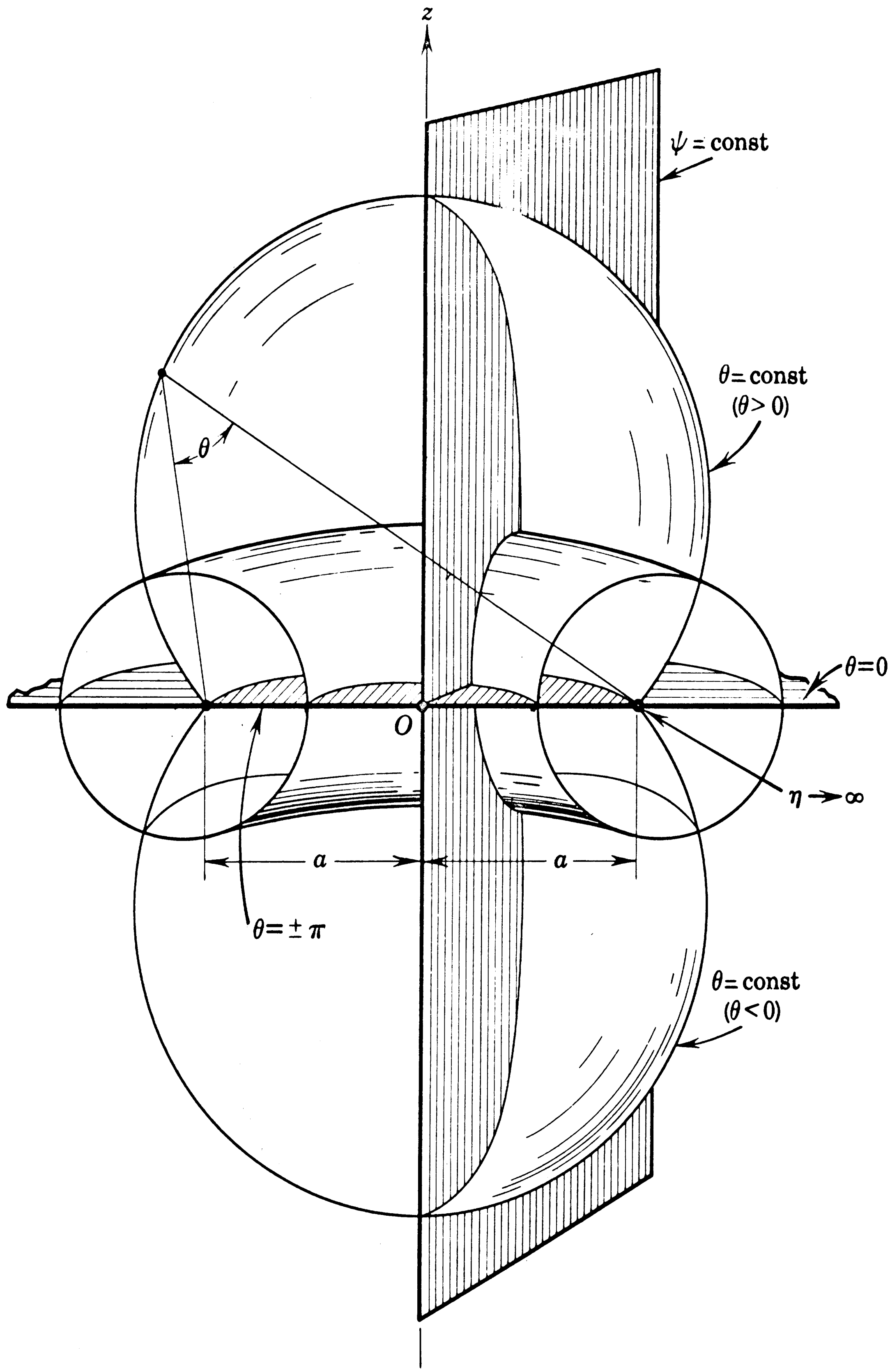

FIG. 2- 2: TOROIDAL COORDINATES (η, θ, ψ). COORDINATE SURFACES ARE TOROIDS (η = constant), SPHERICAL BOWLS (θ = constant), AND HALF-PLANES (ψ = constant).

$$(z - a\cot\eta_o)^2 + \rho^2 = a^2 \csc^2 \theta_o$$

where

$$\rho = \sqrt{x^2 + y^2} = \frac{a \sinh\eta}{\cosh\eta - \cos\theta} \quad .$$

The metric coefficients are given by ,

$$h_\eta = h_\theta = \frac{a}{\cosh\eta - \cos\theta}$$

$$h_\psi = \frac{a \sinh\eta}{\cosh\eta - \cos\theta} \quad .$$

If we define $a \coth \eta_o = R_o$ and $a \operatorname{csch} \eta_o = r_o$ then

$$a = \sqrt{R_o^2 - r_o^2}$$

and

$$\cosh\eta_o = \frac{R_o}{r_o} \quad .$$

The z - axis corresponds to $\eta = o$ and $\eta = \infty$ corresponds to an infinitely thin ring of radius a .

The Laplacian in toroidal coordinates can be written as follows:

$$\nabla^2 \Phi = \frac{(\cosh\eta - \cos\theta)^3}{a^2 \sinh\eta} \left\{ \frac{\partial}{\partial\eta} \left[\frac{\sinh\eta}{\cosh\eta - \cos\theta} \frac{\partial\Phi}{\partial\eta} \right] + \frac{\partial}{\partial\theta} \left[\frac{\sinh\eta}{\cosh\eta - \cos\theta} \frac{\partial\Phi}{\partial\theta} \right] + \frac{1}{\sinh\eta\left[\cosh\eta - \cos\theta\right]} \frac{\partial^2\Phi}{\partial\psi^2} \right\}$$

2.2 GENERAL SOLUTION OF LAPLACE'S EQUATION

The Laplace's equation $\nabla^2 \Phi = 0$ can be written in toroidal coordinates as,

$$\frac{\partial}{\partial \eta}\left[\frac{\sinh \eta}{\cosh \eta - \cos\theta}\frac{\partial \Phi}{\partial \eta}\right] + \frac{\partial}{\partial \theta}\left[\frac{\sinh \eta}{\cosh \eta - \cos\theta}\frac{\partial \Phi}{\partial \theta}\right] + \frac{1}{\sinh \eta\left[\cosh \eta - \cos\theta\right]}\frac{\partial^2 \Phi}{\partial \psi^2} = 0 \quad . \tag{2.1}$$

This equation is not simply separable, but if we set

$$\Phi = \sqrt{\cosh \eta - \cos\theta}\; F(\eta, \theta, \psi)$$

the Eq. (2.1) reduces to

$$\frac{1}{\sinh \eta}\frac{\partial}{\partial \eta}\left[\sinh \eta \frac{\partial F}{\partial \eta}\right] + \frac{\partial^2 F}{\partial \theta^2} + \frac{1}{\sinh^2 \eta}\frac{\partial^2 F}{\partial \psi^2} + \frac{1}{4} F = 0 \quad . \tag{2.2}$$

Equation (2.2) can be separated into three second order differential equations given by,

$$\frac{1}{\sinh \eta}\frac{d}{d\eta}\left(\sinh \eta \frac{dE}{d\eta}\right) + \left[\frac{1}{4} - \alpha_2 - \frac{\alpha_3}{\sinh^2 \eta}\right] E = 0 \tag{2.3}$$

$$\frac{d^2 \Theta}{d\theta^2} + \alpha_2 \Theta = 0 \tag{2.4}$$

$$\frac{d^2 \Psi}{d\psi^2} + \alpha_3 \Psi = 0 \quad . \tag{2.5}$$

Following the usual procedure

$$F = E(\eta)\; \Theta(\theta)\; \Psi(\psi)$$

and letting $\alpha_2 = n^2$ and $\alpha_3 = m^2$, the solutions of the above set of equations are, (Morse and Feshbach, 1952)

$$E = \alpha_{nm} P^m_{n-1/2}(\cosh \eta) + \beta_{nm} Q^m_{n-1/2}(\cosh \eta)$$

$$\Theta(\theta) = C_n \cos n\theta + D_n \sin n\theta$$

$$\Psi(\psi) = A_m \cos m\psi + B_m \sin m\psi$$

where α_{nm}, β_{nm}, C_n, D_n, A_m and B_m represent constants.

$P^m_{n-1/2}(\cosh \eta)$ and $Q^m_{n-1/2}(\cosh \eta)$ are called tesseral toroidal functions. They are Legendre functions of the first and second kind respectively, of order m and degree $n - 1/2$. These functions are defined by:*

$$P^m_{n-1/2}(\cosh \eta) = \frac{1}{\pi} \frac{(-1)^m \Gamma(n+\frac{1}{2})}{\Gamma(n-m+\frac{1}{2})} \int_0^{\pi} \frac{\cos mu \, du}{\left[\cosh \eta + \sinh \eta \cos u\right]^{n+1/2}}$$

$$Q^m_{n-1/2}(\cosh \eta) = \frac{(-1)^m \Gamma(n+\frac{1}{2})}{\Gamma(n-m+\frac{1}{2})} \int_0^{\infty} \frac{\cosh mu \, du}{\left[\cosh \eta + \sinh \eta \cosh u\right]^{n+1/2}} .$$

Now, we can write the complete solution of the Laplace's equations in toroidal coordinates.

$$\Phi(\eta, \theta, \psi) = \sqrt{\cosh \eta - \cos \theta} \sum_{m=0}^{\infty} \sum_{m=0}^{\infty} \left[A_m \cos m\psi + B_m \sin m\psi\right]$$

$$\left[C_n \cos n\theta + D_n \sin n\theta\right] \left[\alpha_{nm} P^m_{n-1/2}(\cosh \eta) + \beta_{nm} Q^m_{n-1/2}(\cosh \eta)\right] . \tag{2.6}$$

The toroidal functions were first introduced by C. Neumann (1864), and they have been studied in detail by Hicks (1881, 1884), Basset (1893), and briefly

* Alternate definitions of these functions can be found in Hobson (1955) (see Appendix C).

by Heine (1881).

It should be noted that neither the space interior nor exterior to the torus is simply connected, hence the direct application of these functions for potential problems involving circulation is often not possible as the potential in such cases are not always uniquely determined by their values on the surface of the torus.

Equation (2.6) is the general solution of Laplace's equation, but now we have to consider the values of these functions in the space interior and exterior to the torus. $0 \leq \eta \leq \eta_0$ constitutes the exterior to the torus and $\eta_0 \leq \eta_1 \leq \infty$, the interior. The outer space contains the plane surface $\eta = 0$.

Investigation of $P^m_{n-1/2}(\cosh \eta)$ and $Q^m_{n-1/2}(\cosh \eta)$ in these regions reveal that as $\eta \longrightarrow 0$, the function $Q^m_{n-1/2}(\cosh \eta)$ goes to infinity and as $\eta \longrightarrow \infty$, $P^m_{n-1/2}(\cosh \eta)$ goes to infinity. Therefore the potential function suitable to the exterior region of the torus, in which we will be primarily interested, is given by

$$\Phi = \sqrt{\cosh \eta - \cos \theta} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[A_m \cos m\psi + B_m \sin m\psi \right] \times$$

$$\times \left[C_n \cos n\theta + D_n \sin n\theta \right] \quad P^m_{n-1/2}(\cosh \eta) \quad 0 \leq \eta < \eta_0 \quad . \quad (2.7)$$

Similarly for the interior region,

$$\Phi = \sqrt{\cosh \eta - \cos \theta} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[A'_m \cos m\psi + B'_m \sin m\psi \right] \times$$

$$\times \left[C'_n \cos n\theta + D'_n \sin n\theta \right] \quad Q^m_{n-1/2}(\cosh \eta) \quad \eta_0 < \eta \leq \infty \quad . \quad (2.8)$$

Two of the most important types of potential problems encountered in physics are, to find the potential everywhere,

(i) when the value of the potential is specified on the surface

(ii) the value of the normal derivative of the potential is specified on the surface.

These are commonly referred to as the Dirichlet and the Neumann boundary value problems respectively.

2.3 GREEN'S FUNCTION FOR THE EXTERIOR DIRICHLET PROBLEM FOR A TORUS

The Green's function for the Dirichlet problem can be derived easily, but apparently, it has not been done before. So we proceed to derive this in the following way. Let G_{DE} represent the Dirichlet Green's function.

The free space Green's function for the Laplace's equation is $\frac{1}{R}$ where R is the distance between the source point $(\eta_o, \theta_o, \psi_o)$ and the field point (η, θ, ψ).

$$\frac{1}{R} = \frac{1}{a\sqrt{2}} \frac{\left[\cosh\eta - \cos\theta\right]^{1/2} \left[\cosh\eta_o - \cos\theta_o\right]^{1/2}}{\left[\cosh\eta\cosh\eta_o - \sinh\eta\sinh\eta_o\cos(\psi-\psi_o) - \cos(\theta-\theta_o)\right]^{1/2}}$$

and $\frac{1}{R}$ can be expanded in toroidal coordinates (Hobson, 1955):

$$\frac{1}{R} = \frac{\sqrt{\cosh\eta_o - \cos\theta_o}\sqrt{\cosh\eta - \cos\theta}}{a\pi} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \epsilon_m \epsilon_n (-1)^m \frac{\Gamma(n-m+\frac{1}{2})}{\Gamma(n+m+\frac{1}{2})}$$

$$\cos m(\psi-\psi_o)\cos n(\theta-\theta_o) \begin{cases} P^m_{n-1/2}(\cosh\eta)\, Q^m_{n-1/2}(\cosh\eta_o) & \eta < \eta_o \\ P^m_{n-1/2}(\cosh\eta_o)\, Q^m_{n-1/2}(\cosh\eta) & \eta > \eta_o \end{cases} \tag{2.9}$$

ϵ_n and ϵ_m are the Neumann numbers given by

$$\epsilon_n = \begin{matrix} 1 & n = 0 \\ 2 & n \neq 0 \end{matrix}$$

We can now find a function G_{DE}

$$G_{DE} = -\frac{1}{4\pi R} + g_{DE}$$

where

$$G_{DE} = 0 \Big|_{\eta = \eta_s}$$

and g_{DE} is such that

$$\nabla^2 g_{DE} = 0$$

$$g_{DE}\Big|_{\eta = \eta_s} = \frac{1}{4\pi R}\Big|_{\eta=\eta_s}$$

g_{DE} regular at ∞.

in the sense of Kellogg (1929), i.e.,

$$\lim_{r \to \infty} \left| r\, g_{DE} \right| < \infty \qquad \text{and} \qquad \lim_{r \to \infty} \left| r^2 \frac{\partial}{\partial r} g_{DE} \right| < \infty$$

($\eta = \eta_s$ represents the surface of the torus).

Matching boundary conditions we get

$$g_{DE} = \frac{\sqrt{\cosh\eta - \cos\theta}\sqrt{\cosh\eta_0 - \cos\theta_0}}{4\pi a^2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \epsilon_n \epsilon_m (-1)^m \frac{\Gamma(n-m+\frac{1}{2})}{\Gamma(n+m+\frac{1}{2})}$$

$$\cos m(\psi-\psi_o) \cos n(\theta-\theta_o) \frac{P^m_{n-1/2}(\cosh\eta_o)\, Q^m_{n-1/2}(\cosh\eta_s)}{P^m_{n-1/2}(\cosh\eta_s)} P^m_{n-1/2}(\cosh\eta) \qquad (2.10)$$

The Green's function for the Dirichlet problem is then given by:

$$G_{DE} = + \frac{\sqrt{\cosh\eta_o - \cos\theta_o}\sqrt{\cosh\eta - \cos\theta}}{4\pi^2 a} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \epsilon_m \epsilon_n (-1)^m$$

$$\frac{\Gamma(n-m+\frac{1}{2})}{\Gamma(n+m+\frac{1}{2})} \cos m(\psi-\psi_o) \cos n(\theta-\theta_o) \left[\frac{P^m_{n-1/2}(\cosh\eta_o)\, Q^m_{n-1/2}(\cosh\eta_s)}{P^m_{n-1/2}(\cosh\eta_s)} - Q^m_{n-1/2}(\cosh\eta_o) \right] P^m_{n-1/2}(\cosh\eta) \quad . \qquad (2.11)$$

With the help of this Green's function, one can solve any Dirichlet potential problem (i.e., whenever the potential value is specified on the surface) by using the Green's function as the kernel of an integral representation.

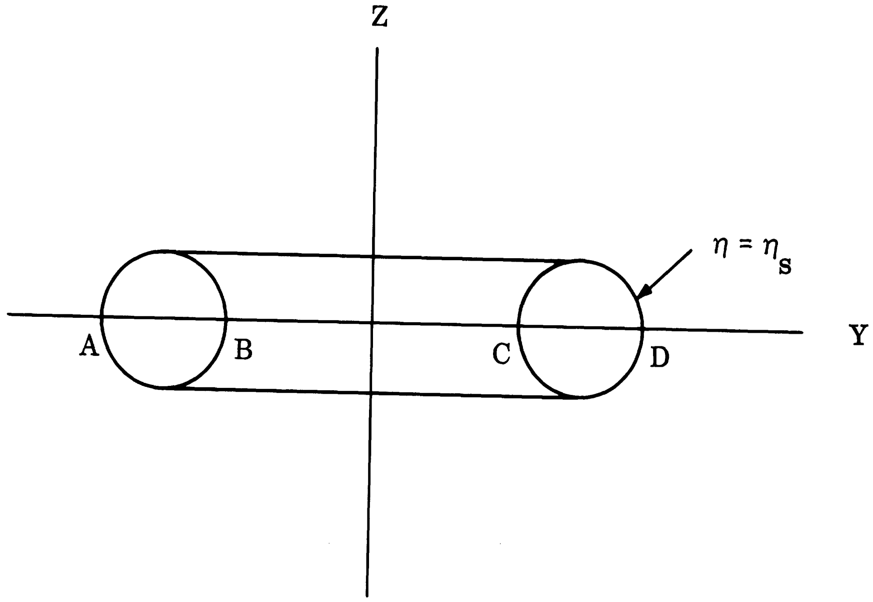

FIG. 2-3: CROSS-SECTION OF A TORUS.

2.4 EXTERIOR NEUMANN BOUNDARY VALUE PROBLEM

Physical problems which involve such boundary conditions frequently occur in studying fluid dynamics, magnetostatics and some steady - state heat conduction problems. However, as stated in the introduction only special case can be solved explicitly as follows and Green's function can be written up to a set of constants and in the following we treat the two cases when the torus is moving parallel to the fluid and when the flow is perpendicular to the axis of the torus. The former has been studied by Hicks (1881), but a summary of it is included here.

2.4.1 Torus Immersed in an Infinite Incompressible Fluid Flowing Parallel to the Axis of the Torus with a Uniform Velocity v

It is quite clear that there is complete symmetry in the angular variable ψ and hence the velocity potential V is independent of ψ. We seek a solution of the problem

$$\nabla^2 V = 0 \tag{2.12}$$

$$\left.\frac{\partial V^{total}}{\partial n}\right|_{\eta=\eta_s} = 0 \quad \text{i.e.,} \quad \left.\frac{\partial V}{\partial n}\right|_{\eta=\eta_s} = -\left.\frac{\partial V^i}{\partial n}\right|_{\eta=\eta_s} \tag{2.13}$$

and V is regular at infinity in the sense of Kellogg. The incident velocity potential V^i can be obtained from the velocity vector $\bar{v}$

$$\bar{v} = v\hat{z} \; ; \; V^i = vz + c = v\,\frac{a\sin\theta}{\cosh\eta - \cos\theta} + c \quad .$$

where c is an integration constant.

Expanding V^i in an orthogonal expansion

$$V^i = \frac{v\sqrt{8}\,a}{\pi}\sqrt{\cosh\eta - \cos\theta}\sum_{n=0}^{\infty} n \sin n\theta \; Q_{n-1/2}(\cosh\eta) \;. \tag{2.14}$$

The solution of (2.12) can be written as

$$V = \sqrt{\cosh\eta - \cos\theta}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[A_m \cos m\psi + B_m \sin m\psi\right]\left[C_n \cos n\theta + D_n \sin n\theta\right] P^m_{n-1/2}(\cosh\eta) \tag{2.15}$$

in the exterior of the torus (cf. Eq. (2.7)) .

Because the problem is independent of ψ (2.15) can be written as

$$V = \sqrt{\cosh\eta - \cos\theta}\sum_{n=0}^{\infty}\left[C_n \cos n\theta + D_n \sin n\theta\right] P_{n-1/2}(\cosh\eta) \;. \tag{2.16}$$

Applying the boundary condition (2.13) , we get

$$\sum_{n=0}^{\infty}\frac{\sinh\eta_s}{2\sqrt{\cosh\eta_s - \cos\theta}}\left[C_n \cos n\theta + D_n \sin n\theta\right] P_{n-1/2}(\cosh\eta_s) \; +$$

$$+ \sqrt{\cosh\eta_s - \cos\theta}\sum_{n=0}^{\infty}(C_n \cos n\theta + D_n \sin n\theta)\, P'_{n-1/2}(\cosh\eta_s)$$

$$= -\frac{v\sqrt{8}\,a}{\pi}\,\frac{\sinh\eta_s}{2\sqrt{\cosh\eta_s - \cos\theta}}\sum_{n=0}^{\infty} n\, Q_{n-1/2}(\cosh\eta_s)\sin n\theta \; -$$

$$-\frac{v\sqrt{8}\,a}{\pi}\sqrt{\cosh\eta_s - \cos\theta}\sum_{n=0}^{\infty} n\, Q'_{n-1/2}(\cosh\eta_s)\sin n\theta \;. \tag{2.17}$$

The primes denote differentiation with respect to η. Since the right hand side is only a function of $\sin n\theta$, all C_n's must equal zero. Dividing throughout

(2.17) by $\frac{v\sqrt{8}\,a}{\pi}$ and rearranging

$$\sum_{n=0}^{\infty}\left\{\sinh\eta_s\left[B_n P_{n-1/2}(\cosh\eta_s) - nQ_{n-1/2}(\cosh\eta_s)\right] + 2(\cosh\eta_s - \cos\theta)\right.$$

$$\left.\left[B_n P'_{n-1/2}(\cosh\eta_s) - nQ'_{n-1/2}(\cosh\eta_s)\right]\right\}\sin n\theta = 0 \qquad (2.18)$$

where

$$B_n = \frac{D_n\pi}{v\sqrt{8}\,a} \quad .$$

Rearranging further,

$$B_{n+1}P'_{n+1/2} + B_{n-1}P'_{n-3/2} - B_n\left[\sinh\eta_s P_{n-1/2} + 2\cosh\eta_s P'_{n-1/2}\right]$$

$$= (n+1)Q'_{n+1/2} + (n-1)Q'_{n-3/2} - n\left[\sinh\eta_s Q_{n-1/2} + 2\cosh\eta_s Q'_{n-1/2}\right] \cdot \qquad (2.19)$$

The arguments of the Legendre functions are omitted for convenience, but they must be understood to be $\cosh\eta_s$. But

$$P'_{n+1/2} + P'_{n-3/2} - \left[\sinh\eta_s\ P_{n-1/2} + 2\cosh\eta_s P'_{n-1/2}\right] = 0 \qquad (2.20)$$

and similarly for the Q-functions. Introducing this simplification in (2.19), we obtain

$$(B_{n+1} - B_n)P'_{n+1/2} - (B_n - B_{n-1})P'_{n-3/2} = Q'_{n+1/2} - Q'_{n-3/2} \qquad (2.21)$$

true for $n > 1$, . . with

$$(B_2 - B_1)P'_{3/2} - B_1 P'_{1/2} = Q'_{3/2} - Q'_{1/2} \quad . \qquad (2.22)$$

for $n = 1$.

If we write the successive equations in order and multiply the equations containing $P'_{n-3/2}$ and $P'_{n+1/2}$ by $P'_{n-1/2}$ and add, we get

$$\left[B_{n+1} - B_n\right] P'_{n+1/2}P'_{n-1/2} - B_1 P'_{1/2}P'_{-1/2} = P'_{n-1/2}Q'_{n+1/2} - P'_{1/2}Q'_{-1/2}$$

$$+ \sum_{r=1}^{n-1} \left[P'_{r-1/2}Q'_{r+1/2} - Q'_{r-1/2}P'_{r+1/2}\right]. \tag{2.23}$$

But

$$P'_{r-1/2}Q'_{r+1/2} - Q'_{r-1/2}P'_{r+1/2} = \frac{2r+1}{2}. \tag{2.24}$$

Using (2.24) in (2.23) gives

$$\left[B_{n+1} - B_n\right] P'_{n+1/2}P'_{n-1/2} - B_1 P'_{1/2}P'_{-1/2} = P'_{n-1/2}Q'_{n+1/2} - P'_{-1/2}Q'_{+1/2} +$$

$$+ \sum_{r=0}^{n-1} \frac{2r+1}{2}. \tag{2.25}$$

Summing the series in (2.25) and rearranging, we get

$$B_{n+1} - B_n = \frac{Q'_{n+1/2}}{P'_{n+1/2}} + \frac{P'_{-1/2}\left[B_1 P'_{+1/2} - Q'_{+1/2}\right]}{P'_{n+1/2}P'_{n-1/2}} + \frac{1}{2}\,\frac{n^2}{P'_{n+1/2}P'_{n-1/2}}. \tag{2.26}$$

Let

$$2\,P'_{-1/2}\left[B_1 P'_{1/2} - Q'_{1/2}\right] = \alpha \tag{2.26a}$$

then (2.26) can be written as

$$B_{n+1} - B_n = \frac{Q'_{n+1/2}}{P'_{n+1/2}} + \frac{n^2+\alpha}{2n+1}\left[\frac{Q'_{n+1/2}}{P'_{n+1/2}} - \frac{Q'_{n-1/2}}{P'_{n-1/2}}\right]. \tag{2.27}$$

Rearranging we obtain

$$B_{n+1} - B_n = \frac{1}{2n+1}\left\{\left[(n+1)^2 + \alpha\right]\frac{Q'_{n+1/2}}{P'_{n+1/2}} - (n^2+\alpha)\,\frac{Q'_{n-1/2}}{P'_{n-1/2}}\right\} \tag{2.28}$$

writing n more equations in succession, and adding,

$$B_{n+1} - B_1 = \frac{(n+1)^2+\alpha}{2n+1} \frac{Q'_{n+1/2}}{P'_{n+1/2}} + 2 \sum_{r=2}^{n} \frac{r^2+\alpha}{4r^2-1} \frac{Q'_{r-1/2}}{P'_{r-1/2}} - \frac{1+\alpha}{3} \frac{Q'_{1/2}}{P'_{1/2}} . \tag{2.29}$$

B_n is now determined up to the extent of (α or B_1) and to determine α (2.26a), follow the argument due to Hicks (1881). Since we know that the velocity potential must be finite everywhere, we can choose an α such that $\sum_0^\infty B_n \sin n\theta P_{n-1/2}(\cosh\eta)$ is convergent. A necessary condition for this series to be convergent is that the nth term of this series goes to zero as n goes to ∞. So we proceed as follows:

(i) It must be proved that B_n is finite when n is large.

(ii) α must be chosen such that B_n vanishes when n is infinite, by making the limit of B_∞ go to zero.

From (2.29) we can see that for large n, the term that should be considered is

$$\sum_{r=2}^{\infty} \frac{r^2+\alpha}{4r^2-1} \frac{Q'_{r-1/2}}{P'_{r-1/2}}$$

and for large n (Hobson, 1955)

$$Q^m_{n-1/2} \approx e^{im\pi} \left(\frac{\pi}{n}\right)^{1/2} \frac{e^{-n\eta}}{\left[2\sinh\eta\right]^{1/2}} \frac{\Gamma(n+m+1)}{\Gamma(n+1)}$$
$$P^m_{n-1/2} \approx \frac{1}{(n\pi)^{1/2}} \frac{e^{n\eta}}{(2\sinh\eta)^{1/2}} \frac{\Gamma(n+1)}{\Gamma(n-m+1)} \tag{2.30}$$

hence, for large n

$$\frac{Q'_{n-1/2}}{P'_{n-1/2}} \sim e^{-2n\eta}$$

which is highly convergent. So,

$$\sum_{r=1}^{\infty} \frac{r^2+\alpha}{4r^2-1} \frac{Q'_{r-1/2}}{P'_{r-1/2}}$$

is also convergent.

Thus B_n lends to a finite limit for increasing n

$$\lim_{n\to\infty} B_n \approx 2 \sum_{r=1}^{\infty} \frac{r^2+\alpha}{4r^2-1} \frac{Q'_{r-1/2}}{P'_{r-1/2}} - \alpha \frac{Q'_{-1/2}}{P'_{-1/2}} . \tag{2.31}$$

This limit must be set equal to zero, when we get a vlue for α *

$$\alpha = \frac{-2\sum_{r=1}^{\infty} \frac{r^2}{4r^2-1} \frac{Q'_{r-1/2}}{P'_{r-1/2}}}{\left[2\sum_{r=1}^{\infty} \frac{1}{4r^2-1} \frac{Q'_{r-1/2}}{P'_{r-1/2}} - \frac{Q'_{-1/2}}{P'_{-1/2}}\right]} \tag{2.32}$$

and hence

$$B_n = \frac{n^2+\alpha}{2n-1} - 2\sum_{r=n}^{\infty} \frac{r^2+\alpha}{4r^2-1} \frac{Q'_{r-1/2}}{P'_{r-1/2}} \tag{2.33}$$

and the velocity potential is given by

$$V = \frac{a\sqrt{8}\,v}{\pi} \sqrt{\cosh\eta - \cos\theta} \sum_{1}^{\infty} B_n \sin n\theta \, P_{n-1/2}(\cosh\eta)$$

where B_n is given by (2.33).

---

* This value for α is unique as it is establised by Hicks, by verifying the convergence of the series $\sum_{0}^{\infty} B_n \sin n\theta \, P_{n-1/2}(\cosh\eta)$, with this value. It must also be mentioned that such a method of determining the coefficients has not been necessary in any other diffraction problems in the author's knowledge and seems to be rather peculiar, being necessary for this case.

2.4.2 Torus Immersed in a Fluid Flowing Perpendicular to the Axis of Torus with a Uniform Velocity v

The boundary value problem in this case is very similar to the previous case, except that we no longer have symmetry in ψ.

We seek a solution for the velocity potential V,

$$\nabla^2 V = 0$$

$$\left.\frac{\partial V}{\partial n}\right|_{\eta = \eta_s} = -\left.\frac{\partial V^i}{\partial n}\right|_{\eta = \eta_s}$$

V regular at infinity in the sense of Kellogg. The velocity potential of the uniform flow is

$$V^i = vy = va\,\frac{\sinh\eta\,\sin\psi}{\cosh\eta - \cos\theta}$$

$$= -va\sqrt{\cosh\eta - \cos\theta}\sum_{n=0}^{\infty}\frac{2\sqrt{2}}{\pi}\sin\psi\cos n\theta\, Q^{(1)}_{n-1/2}(\cosh\eta) \quad . \tag{2.34}$$

Employing the boundary condition, we obtain

$$\frac{\partial}{\partial\eta}\left\{\sqrt{\cosh\eta - \cos\theta}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[A_m\cos m\psi + B_m\sin m\psi\right]\left[C_n\cos n\theta + D_n\sin n\theta\right]\right.$$

$$\left.\left. P^m_{n-1/2}(\cosh\eta)\right\}\right|_{\eta=\eta_s} = \frac{\partial}{\partial\eta}\left\{\frac{va\,2\sqrt{2}}{\pi}\sqrt{\cosh\eta - \cos\theta}\right.$$

$$\left.\left.\sum_{n=0}^{\infty}\sin\psi\cos n\theta\, Q^{(1)}_{n-1/2}(\cosh\eta)\right\}\right|_{\eta=\eta_s} \quad . \tag{2.35}$$

This yields

$$A_m = 0 \qquad \text{for all } m$$

$$B_m = \begin{matrix} 1 & \text{for } m = 1 \\ 0 & \text{for } m \neq 1 \end{matrix} \quad .$$

Rearranging (2.35), we obtain

$$\sum_{n=0}^{\infty}\left[\sinh\eta_s P^{(1)}_{n-1/2} + 2\cosh\eta_s P^{(1)'}_{n-1/2}\right] C_n \cos n\theta - \sum_{n=-1}^{\infty} C_{n+1} P^{(1)'}_{n+1/2} \cos n\theta -$$

$$- \sum_{n=1}^{\infty} C_{n-1} P^{(1)'}_{n-3/2} \cos n\theta$$

$$= \frac{v\, a\, 2\sqrt{2}}{\pi}\left\{\sum_{n=0}^{\infty}\left[\sinh\eta_s Q^{(1)}_{n-1/2} + 2\cosh\eta_s Q^{(1)'}_{n-1/2}\right]\cos n\theta - \sum_{n=-1}^{\infty} Q^{(1)'}_{n+1/2} \cos n\theta -\right.$$

$$\left. - \sum_{n=1}^{\infty} Q^{(1)'}_{n-3/2} \cos n\theta \right\} \quad . \tag{2.36}$$

The arguments of the Legendre functions are understood to be $\cosh\eta_s$ and are omitted here for convenience. Equation (2.36), in turn gives rise to a set of recurrence relations for the coefficients C_n.

$$\left\{\sinh\eta_s P^{(1)}_{n-1/2} + 2\cosh\eta_s P^{(1)'}_{n-1/2}\right\} C_n - C_{n+1} P^{(1)'}_{n+1/2} - C_{n-1} P^{(1)'}_{n-3/2}$$

$$= \frac{v\, a\, 2\sqrt{2}}{\pi}\left\{\left[\sinh\eta_s Q^{(1)}_{n-1/2} + 2\cosh\eta_s Q^{(1)'}_{n-1/2}\right] - Q^{(1)'}_{n+1/2} - Q^{(1)'}_{n-3/2}\right\} \tag{2.37}$$

for $n = 2, 3, 4 \ldots$

with the initial equations,

$$C_0\left\{\sinh\eta_s P^{(1)}_{-1/2} + 2\cosh\eta_s P^{(1)'}_{-1/2}\right\} - C_1 P^{(1)'}_{1/2} =$$

$$= \frac{v\, a\, 2\sqrt{2}}{\pi}\left\{\left[\sinh\eta_s Q^{(1)}_{-1/2} + 2\cosh\eta_s Q^{(1)'}_{-1/2}\right] - Q^{(1)'}_{1/2}\right\} \tag{2.38}$$

and

$$- 2 C_0 P^{(1)'}_{-1/2} + C_1 \left\{ \sinh\eta_s P^{(1)}_{1/2} + 2\cosh\eta_s P^{(1)'}_{1/2} \right\} - C_2 P^{(1)'}_{3/2} =$$

$$= \frac{v\, a\, 2\sqrt{2}}{\pi} \left\{ -2Q^{(1)'}_{-1/2} + \left[\sinh\eta_s Q^{(1)}_{1/2} + 2\cosh\eta_s Q^{(1)'}_{1/2} \right] - Q^{(1)'}_{3/2} \right\} . \tag{2.39}$$

C_n is determined to the extent of C_0. (We <u>cannot</u> employ the method of the previous section to determine C_0). But it has not been found possible to express C_n explicitly in terms of C_0 or to get a sufficiently simple expression to test for convergence. Hence a different approach has to be taken.

Although we do not have a symmetry in the angular variable ψ, the symmetry about the plane $z = 0$ brings a simplification. Since the flow of fluid is parallel to the y - axis, the flow has 'stagnation points' at the points A, B, C, D shown in Fig. 2-3.

These four points correspond to $\eta = \eta_s$, $\theta = 0$, $\psi = \pi$; $\eta = \eta_s$, $\theta = \pi$, $\psi = \pi$; $\eta = \eta_s$, $\theta = \pi$, $\psi = 0$; and $\eta = \eta_s$, $\theta = 0$, $\psi = 0$ respectively. At these points the total velocity should be zero. This can be seen as follows. We know that the normal component of total velocity is zero everwhere on the surface. Due to the symmetry about $z = 0$ plane, the tangential velocity components must be zero at these points or this would give rise to circulation. There is a cancellation of the tangential components at these points thus making the total velocity zero.

This can be written as,

$$v + v^i \Big|_{\eta = \eta_s \;\; \theta = {\pi \atop 0},\; \psi = {\pi \atop 0}} = 0$$

i.e.,

$$\nabla (V + V^i) \Big|_{\eta = \eta_s,\;\; \theta = {\pi \atop 0},\; \psi = {\pi \atop 0}} = 0 .$$

We obtain the two equations

$$\frac{(\cosh\eta_s + 1)^{3/2}}{\sinh\eta_s\, a} \sum_0^\infty C_n(-1)^n P^{(1)}_{n-1/2}(\cosh\eta_s) \quad +$$

$$+ \quad \frac{(\cosh\eta_s + 1)^{3/2}}{\sinh\eta_o\, a} \; \frac{v\,a\,2\sqrt{2}}{\pi} \sum_0^\infty (-1)^n Q^{(1)}_{n-1/2}(\cosh\eta_s) = 0 \qquad (2.40)$$

and

$$\frac{(\cosh\eta_s - 1)^{3/2}}{\sinh\eta_s\, a} \sum_0^\infty C_n P^{(1)}_{n-1/2}(\cosh\eta_s) \quad +$$

$$+ \quad \frac{(\cosh\eta_s - 1)^{3/2}}{\sinh\eta\; a} \; \frac{v\,a\,2\sqrt{2}}{\pi} \sum_0^\infty Q^{(1)}_{n-1/2}(\cosh\eta_s) = 0\;. \qquad (2.41)$$

These two equations will be consistant if it is possible to find a set of C_n's that satisfy both the equations. This set of C_n's is then given by

$$\sum_{n=0}^\infty C_{2n} P^{(1)}_{2n-1/2}(\cosh\eta_s) = -\frac{v\,a\,2\sqrt{2}}{\pi} \sum_{n=0}^\infty Q^{(1)}_{2n-1/2}(\cosh\eta_s) \qquad (2.42)$$

and

$$\sum_{n=0}^\infty C_{2n+1} P^{(1)}_{2n+1/2}(\cosh\eta_s) = -\frac{v\,a\,2\sqrt{2}}{\pi} \sum_{n=0}^\infty Q^{(1)}_{2n+1/2}(\cosh\eta_s)\;. \qquad (2.43)$$

Equations (2.42) and (2.43) seem to be the additional equations necessary to determine all the coefficients C_n. Although this still does not provide us with an explicit value for C_o, Eqs. (2.42) and (2.43) implicitly give the information needed. These conditions come from the symmetry of the problem involved and are connected with the circulation, and hence serve as additional conditions to the boundary conditions.

2.5 GREEN'S FUNCTION

We can follow the method of the previous section to write the Green's function as follows

$$G_{NE} = -\frac{1}{4\pi R} + g_{NE}$$

where

$$\nabla^2 g_{NE} = 0$$

$$\left.\frac{\partial g_{NE}}{\partial n}\right|_{\eta=\eta_s} = \left.\frac{\partial}{\partial n}\frac{1}{4\pi R}\right|_{\eta=\eta_s}$$

g_{NE} is regular at infinity. Using (2.9) and (2.7) we get

$$g_{NE} = \sqrt{\cosh\eta - \cos\theta}\ \sqrt{\cosh\eta_o - \cos\theta}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\epsilon_m \cos m(\psi-\psi_o)$$

$$\left[C_n^m \cos n\theta + D_n^m \sin n\theta\right] P^m_{n-1/2}(\cosh\eta)$$

where C_n^m satisfy the recursion relation

$$C_n^m\left\{\sinh\eta_s P^m_{n-1/2}(\cosh\eta_s) + 2\cosh\eta_s P^{m'}_{n-1/2}(\cosh\eta_s)\right\} -$$

$$- C^m_{n+1} P^{m'}_{n+1/2}(\cosh\eta_s) - C^m_{n-1} P^{m'}_{n-3/2}(\cosh\eta_s)$$

$$= \frac{2\,\Gamma(n-m+1/2)}{\Gamma(n+m+1/2)}\left[\left\{(\sinh\eta_s Q^m_{n-1/2}(\cosh\eta_s) + 2\cosh\eta_s Q^{m'}_{n-1/2}(\cosh\eta_s)\right\}\right.$$

$$\cos n\theta_o$$

$$\left. - Q^{m'}_{n+1/2}(\cosh\eta_s)\cos(n+1)\theta_o - Q^{m'}_{n-1/2}(\cosh\eta_s)\cos(n-1)\theta_o\right]$$

for $n = 2, 3, \ldots$

$$C_o^m \left\{ \sinh\eta_s P^m_{-1/2}(\cosh\eta_s) + 2\cosh\eta_s P^{m'}_{-1/2}(\cosh\eta_s) \right\} - C_1^m P^{m'}_{1/2}(\cosh\eta_s)$$

$$= \left\{ \sinh\eta_s Q^m_{-1/2}(\cosh\eta_s) + 2\cosh\eta_s Q^m_{-1/2}(\cosh\eta_s) \right\} - Q^{m'}_{1/2}(\cosh\eta_s)\cos\theta_o$$

and

$$C_1^m \left\{ \sinh\eta_s P^m_{+1/2}(\cosh\eta_s) + 2\cosh\eta_s P^{m'}_{1/2}(\cosh\eta_s) \right\} - C_2^n P^{m'}_{3/2}(\cosh\eta_s)$$

$$- 2C_o^m P^{m'}_{-1/2}(\cosh\eta_s) = \frac{2\Gamma(\frac{3}{2}-m)}{\Gamma(\frac{3}{2}+m)} \Bigg[\Big\{ \sinh\eta_s Q^m_{1/2}(\cosh\eta_s) + 2\cosh\eta_s$$

$$P^{m'}_{1/2}(\cosh\eta_s) \Big\} \cos\theta_o$$

$$- 2Q^{m'}_{-1/2}(\cosh\eta_s) - Q^{m'}_{3/2}(\cosh\eta_s)\cos 2\theta_o \Bigg] .$$

Similarly D_n's satisfy the recurrence relations

$$D_n^m \left\{ \sinh\eta_s P^m_{n-1/2}(\cosh\eta_s) + 2\cosh\eta_s P^{m'}_{n-1/2}(\cosh\eta_s) \right\} - D^m_{n+1} P^{m'}_{n+1/2}(\cosh\eta_s)$$

$$- D^m_{n-1} P^{m'}_{n-3/2}(\cosh\eta_s) = \frac{2\Gamma(n-m+1/2)}{\Gamma(n+m+1/2)} \Bigg[\Big\{ (\sinh\eta_s Q^m_{n-1/2}(\cosh\eta_s) +$$

$$+ 2\cosh\eta_s Q^{m'}_{n-1/2}(\cosh\eta_s) \Big\} \sin n\theta_o$$

$$- Q^{m'}_{n+1/2}(\cosh\eta_s)\sin(n+1)\theta_o$$

$$- Q^{m'}_{n-1/2}(\cosh\eta_s)\sin(n-1)\theta_o \Bigg]$$

for $n = 2, 3, 4 \ldots$

with the initial equation

$$D_1^m \left\{ \sinh\eta_s P_{1/2}^m(\cosh\eta_s) + 2\cosh\eta_s P_{1/2}^{m'}(\cosh\eta_s) \right\} - D_2^m P_{3/2}^{m'}(\cosh\eta_s)$$

$$= \frac{2\,\Gamma(\frac{3}{2}-m)}{\Gamma(\frac{3}{2}+m)} \left[\left\{ \sinh\eta_s P_{1/2}^m(\cosh\eta_s) + 2\cosh\eta_s P_{1/2}^{m'}(\cosh\eta_s) \right\} \sin\theta_o - Q_{3/2}^{m'}(\cosh\eta_s) \sin 2\theta_o \right].$$

These equations, as before, determine C_n's and D_n's up to C_o^m and D_1^m. At this point, it has not been possible to obtain C_o^m and D_1^m for all values of m, although for $m = 0$ and $m = 1$, the constants have been determined laboriously in the previous sections. For $m = 0$, D_1 is given by Eq. (2. 32) explicitly where

$$\alpha = 2 P'_{-1/2} \left[D_1 P'_{1/2} - Q_{1/2} \right] .$$

and C_o for $m = 0$ can be determined in a similar way. For $m = 1$, $C_o^{(1)}$ is obtained from Eqs. (2. 38) through (2. 43), and analogously $D_1^{(1)}$ can be obtained. In spite of many efforts it has not yet been successful to determine the coefficients C_o and D_o for all values of m.

III

QUASI-STATIC APPROXIMATION FOR LOW FREQUENCY SCATTERING

The basic assumption underlying this method is that when an acoustic or electromagnetic wave is incident on a scattering surface, the scattered field can be expanded in a convergent series in powers of k, the wave number.* The problem of determining the coefficients is then reduced to a succession of 'standard' potential problems. The problem can be formulated using a method due to Kleinman (1966):

3.1 THE SCALAR PROBLEM

Let a small amplitude sound-wave propagating with a constant velocity v_e be incident on the body, whose boundary is denoted by B. (The density and compressibility of the external medium is denoted by ρ_e and m_e respectively.) The resulting disturbance, as a function of position and time, can be calculated in terms of a velocity potential Φ. Figure 3-1 shows the arrangement. We are concerned only with the exterior problem and the velocity potential Φ_{ext} satisfies the equation

$$\nabla^2 \Phi_{ext} - \frac{1}{v_e^2} \frac{\partial^2 \Phi_{ext}}{\partial t^2} = 0 \tag{3.1}$$

$v_e = \frac{m_e}{\rho_e}$ is the velocity of propagation. Assuming Φ_{ext} has a harmonic time dependence $e^{-i\omega t}$, the Eq. (3.1) becomes,

$$\nabla^2 \Phi_{ext} + \frac{\omega^2}{v_e^2} \Phi_{ext} = 0 \quad .$$

*Such an expansion has been proved to exist (see Werner, 1962, Kleinman, 1965) for sufficiently small k.

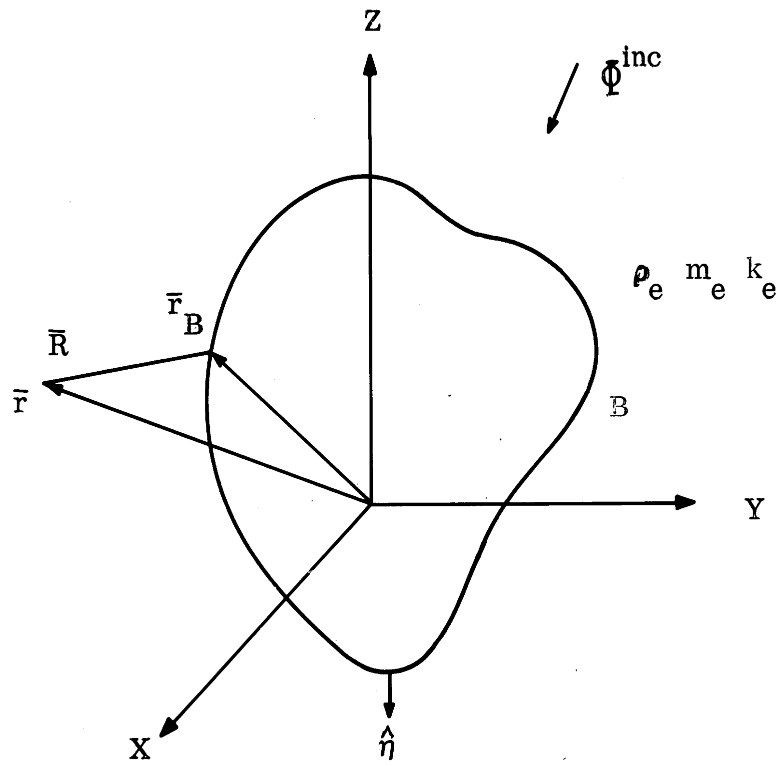

FIG. 3-1: GEOMETRY FOR SCATTERING BY AN ARBITRARY BODY.

The acoustically soft and hard boundaries are generally represented by the boundary conditions

$$\Phi_{ext}\Big|_{\bar{r}=\bar{r}_B} = 0$$

$$\frac{\partial\Phi}{\partial n}\Big|_{\bar{r}=\bar{r}_B} = 0$$

respectively.

Φ_{ext} represents the total field in the exterior, i.e., $\Phi_{ext} = \Phi_e^{inc} + \Phi_e^{scat}$.

Hence if we consider a plane wave incident on the body, our problem becomes one of finding a Φ_e^{scat}, such that

$$(\nabla^2 + k_e^2)\,\Phi_e^{scat} = 0 \tag{3.2}$$

where

$$k_e = \frac{\omega}{v_e}$$

$$\lim_{r\to\infty} r\left(\frac{\partial\,\Phi_e^{scat}}{\partial r} - ik\,\Phi_e^{scat}\right)^{*} = 0 \quad .$$

The latter is the Sommerfeld radiation condition implying that the outgoing waves look like $\frac{e^{ikr}}{r}\,f(\theta, \phi)^{**}$ for large r. The subscripts and superscripts on Φ_e^{scat} and k_e will be dropped henceforth, because we are only concerned with the exterior and we can refer to the above quantities as Φ and k, without any cause for confusion.

As usual, one begins to solve the problem with the Helmholtz integral representation, (Baker and Copson, 1950), which expresses the scalar solution

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* See Appendix A for a discussion of Sommerfeld's radiation condition.

** r, θ, ϕ here represent spherical coordinates.

of the Helmholtz equation in terms of its values and those of its normal derivatives on a closed surface.*

$$\Phi(\bar{r}) = \frac{1}{4\pi} \int_B \left[\Phi \frac{\partial}{\partial n} \frac{e^{ikR}}{R} - \frac{e^{ikR}}{R} \frac{\partial}{\partial n} \Phi \right] dB \quad . \tag{3.3}$$

The integration is carried over the surface of the scatterer and $\frac{\partial}{\partial n}$ refers to $\hat{n} \cdot \vec{\nabla}$, and dB refers to a surface element of area. We now introduce the expansion for the incident and scattered fields, viz.,

$$\begin{aligned} \Phi(\bar{r}) &= \sum_{m=0}^{\infty} \Phi_m(\bar{r})\,(ik)^m \\ \Phi^{inc}(\bar{r}) &= \sum_{m=0}^{\infty} \Phi_m^{inc}(\bar{r})\,(ik)^m \quad . \end{aligned} \tag{3.4}$$

The factor i is included in the expansion parameter for the scattered field because it appears in the expansion for the incident field. e^{ikR} is an entire function which can be expanded as follows,

$$e^{ikR} = \sum_{\ell=0}^{\infty} \frac{(ik)^{\ell} R^{\ell}}{\ell!} \quad . \tag{3.5}$$

Substituting (3.4) and (3.5) in (3.3), we get

$$\begin{aligned} \sum_{m=0}^{\infty} \Phi_m(\bar{r})(ik)^m = \frac{1}{4\pi} \int_B dB \Bigg\{ & \sum_{m=0}^{\infty} \Phi_m(\bar{r})(ik)^m \frac{\partial}{\partial n} \sum_{\ell=0}^{\infty} \frac{(ik)^{\ell} R^{\ell}}{\ell!} - \\ & \sum_{\ell=0}^{\infty} \frac{(ik)^{\ell} R^{\ell-1}}{\ell!} \frac{\partial}{\partial n} \sum_{0}^{\infty} \Phi_m(\bar{r})\,(ik)^m \Bigg\} \quad . \end{aligned} \tag{3.6}$$

Interchanging the order of summation and integration and rearranging terms, we get

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*See Appendix B for the derivation and the physical significance of the Helmholtz formula.

$$\sum_{m=0}^{\infty}\Phi_m(\bar{r})(ik)^m = \frac{1}{4\pi}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\frac{(ik)^\ell}{(\ell-m)!}\int_B\left\{\Phi_m\frac{\partial}{\partial n}R^{\ell-m-1} - R^{\ell-m-1}\frac{\partial}{\partial n}\Phi_m\right\}. \tag{3.7}$$

Matching coefficients of k on both sides of the equation, we obtain

$$\Phi_\ell(\bar{r}) = \frac{1}{4\pi}\sum_{m=0}^{\ell}\frac{1}{(\ell-m)!}\int_B dB\left\{\Phi_m\frac{\partial}{\partial n}R^{\ell-m-1} - R^{\ell-m-1}\frac{\partial}{\partial n}\Phi_m\right\}$$

$$\ell = 0, 1, 2 \ldots \tag{3.8}$$

We can now introduce the boundary conditions

$$\Phi_\ell(\bar{r}_B) = -\Phi_\ell^{inc}(\bar{r}_B) \tag{3.9}$$

$$\left.\frac{\partial\Phi_\ell}{\partial n}\right|_{\bar{r}=\bar{r}_B} = -\left.\frac{\partial\Phi_\ell^{inc}}{\partial n}\right|_{\bar{r}=\bar{r}_B}. \tag{3.10}$$

Φ_ℓ^{inc} is a known quantity and is given by (3.4).

Hence,

$$\Phi_\ell(\bar{r}) = -\frac{1}{4\pi}\sum_{m=0}^{\ell}\frac{1}{(\ell-m)!}\int_B dB\left\{\Phi_m^{inc}\frac{\partial}{\partial n}R^{\ell-m-1} + R^{\ell-m-1}\frac{\partial\Phi_m}{\partial n}\right\}$$

$$\ell = 0, 1, 2 \ldots \tag{3.11}$$

Let us first treat the term $\ell = 0$ in Eq. (3.9).

Considering the case of a soft boundary, we substitute (3.9) in (3.8) to get

$$\Phi_0(\bar{r}) = -\frac{1}{4\pi}\int_B dB\,\Phi_0^{inc}\frac{\partial}{\partial n}\frac{1}{R} - \frac{1}{4\pi}\int_B \frac{1}{R}\frac{\partial\Phi_0}{\partial n}\,dB \quad . \tag{3.12}$$

The first term on the right hand side is known and the second term represents an exterior potential function (for e.g. see Kellogg, 1929). In other words, if we let

$$U_o(\bar{r}) = -\frac{1}{4\pi}\int_B \frac{1}{R}\frac{\partial \Phi_o}{\partial n}\, dB$$

then $U_o(\bar{r})$ is a solution of

$$\nabla^2 U_o(\bar{r}) = 0$$

and $U_o(\bar{r})$ is regular in the sense of Kellogg. viz.,

$$\lim_{r \to \infty} |r U_o| < \infty \quad \text{and} \quad \lim_{r \to \infty} \left| r^2 \frac{\partial U_o}{\partial r} \right| < \infty$$

the boundary condition for $U_o(\bar{r})$ is specified by

$$U_o(\bar{r}_B) = -\Phi_o^{inc}(\bar{r}_B) + \lim_{r \to \bar{r}_B} \frac{1}{4\pi}\int_B \Phi_o^{inc} \frac{\partial}{\partial n}\frac{1}{R}\, dB \quad . \tag{3.13}$$

Equations (3.9) to (3.13) constitute a standard Dirichlet potential problem with a unique solution. Therefore $U_o(\bar{r})$ is determined completely in terms of the incident field. (The integral on the right hand side must be evaluated before the limit is taken.) Hence $\Phi_o(\bar{r})$ is known . The succeeding terms $\Phi_\ell(r)$ can be determined as follows:

Let us assume all the terms $\Phi_0, \Phi_1, \Phi_2 \ldots$ up to and including $\Phi_{\ell-1}(r)$ are known. If we write (3.8) as

$$\Phi_\ell(r) = U_\ell(r) + F_\ell(r) \tag{3.14}$$

where,

$$F_\ell(r) = -\frac{1}{4\pi}\sum_{m=0}^{\infty} \frac{1}{(\ell - m)!}\int_B \Phi_m^{inc} \frac{\partial}{\partial n} R^{\ell - m - 1}\, dB$$

$$-\frac{1}{4\pi}\sum_{m=0}^{\ell - 1} \frac{1}{(\ell - m)!}\int_B R^{\ell - m - 1} \frac{\partial \Phi_m}{\partial n}\, dB \tag{3.15}$$

and

$$U_\ell(\bar{r}) = -\frac{1}{4\pi}\int \frac{1}{R}\frac{\partial \Phi_\ell}{\partial n}\, dB \tag{3.16}$$

$F_\ell(r)$ is a known quantity since all the Φ's up to and not including Φ_ℓ are known. Φ_ℓ^{inc} is known and so is R. $U_\ell(r)$ is again a single layer distribution which obeys the Laplace's equation and the regularity conditions in the sense of Kellogg, and whose boundary value is given by

$$U_\ell(\bar{r}_B) = -\Phi_\ell^{inc}(\bar{r}_B) - F_\ell(\bar{r}_B) \quad .$$

Thus the determination of U_ℓ is reduced to a standard potential problem and U_ℓ is known in terms of incident field and the previous term. Then Φ_ℓ is given by Eq. (3.14).

An analogous procedure holds good for the Neumann boundary condition (the hard boundary). Substituting the boundary condition (3.10) in (3.18), we get

$$\Phi_\ell(\bar{r}) = \frac{1}{4\pi}\sum_{m=0}^{\ell}\frac{1}{(\ell-m)!}\int_B dB\left\{R^{\ell-m-1}\frac{\partial}{\partial n}\Phi_m^{inc} + \Phi_m\frac{\partial}{\partial n}R^{\ell-m-1}\right\} \quad . \tag{3.17}$$

Following the same procedure as in the previous case, we can write the velocity potential Φ_ℓ as follows:

$$\Phi_\ell(\bar{r}) = G_\ell(\bar{r}) + V_\ell(\bar{r}) \tag{3.18}$$

where

$$G_\ell(r) = \frac{1}{4\pi}\sum_{m=0}^{\ell}\frac{1}{(\ell-m)!}\int_B dB\left[R^{\ell-m-1}\frac{\partial}{\partial n}\Phi_m^{inc}\right]$$

$$+\frac{1}{4\pi}\sum_{m=0}^{\ell-1}\frac{1}{(\ell-m)!}\int_B dB\left[\Phi_m(r)\frac{\partial}{\partial n}R^{\ell-m-1}\right] \tag{3.19}$$

and

$$V_\ell(r) = \frac{1}{4\pi} \int_B \Phi_\ell \frac{\partial}{\partial n} \frac{1}{R} \, dB \qquad . \tag{3.20}$$

If we assume all the Φ's are known up to and not including Φ_ℓ, then $G_\ell(\bar{r})$ is a known function. $V_\ell(\bar{r})$ is a double layer distribution, which implies $V_\ell(\bar{r})$ is the solution of a standard exterior Neumann potential problem, satisfying the equation

$$\nabla^2 V_\ell(\bar{r}) = 0 \quad .$$

$V_\ell(\bar{r})$ is regular in the sense of Kellogg, namely,

$$\lim_{r \to \infty} \left| r V_\ell \right| < \infty \quad , \quad \lim_{r \to \infty} \left| r^2 \frac{\partial V_\ell}{\partial r} \right| < \infty$$

and whose boundary value is given by

$$\left. \frac{\partial V_\ell}{\partial n} \right|_{\bar{r} = \bar{r}_B} = - \left. \frac{\partial \Phi_\ell^{inc}}{\partial n} \right|_{\bar{r} = \bar{r}_B} - \left. \frac{\partial G_\ell}{\partial n} \right|_{\bar{r} = \bar{r}_B} \quad .$$

Thus $V_\ell(\bar{r})$ is uniquely determined and hence $\Phi_\ell(\bar{r})$. This method permits the evaluation of any number of terms in the low frequency expansion of the scattered field in terms of a potential solution. But the explicit determinations of the terms grow immensely in complexity and limits the ease of calculation to bodies with very simple geometry.

3.2 FAR-ZONE FIELD

The far-zone scattered field, i.e. when $kr >> ka$, provides interesting information in practical applications. In particular, while studying the scattering cross section of objects one is primarily concerned with the far field scattered. Generally, the far field is written as

$$\Phi = \frac{e^{ikr}}{R} f(\theta, \phi)$$

where $f(\theta, \phi)$ is called the far field amplitude.

One can calculate readily the far scattered field when all the terms in the expansion for the scattered field are known. But when only a finite number of terms are available one has to proceed taking the far field approximation at the initial stages.

$$\frac{e^{ikR}}{R} \sim \frac{e^{ikr - ik\frac{\bar{r}\cdot\bar{r}_B}{r}}}{r} = \frac{e^{ikr - ik\hat{r}\cdot\bar{r}_B}}{r} \tag{3.21}$$

$$\nabla \frac{e^{ikR}}{R} \sim ik\hat{r}\, \frac{e^{ik\hat{r} - ik\hat{r}\cdot\bar{r}_B}}{r} \quad . \tag{3.22}$$

Substituting these in Eq. (2.6), we get

$$\Phi(r) \sim \frac{e^{ikR}}{r} \int_B dB \left\{ \Phi\, ik\, \hat{n}\cdot\hat{r}\, e^{-ik\hat{r}\cdot\bar{r}_B} - e^{ik\hat{r}\cdot\bar{r}_B} \frac{\partial\Phi}{\partial n} \right\} . \tag{3.23}$$

But

$$\Phi(r) = \sum_{m=0}^{\infty} \Phi_m(\bar{r})\, (ik)^m$$

$$e^{-ik\hat{r}\cdot\bar{r}_B} = \sum_{m=0}^{\infty} \frac{(-ik)^m\, (\hat{r}\cdot\bar{r}_B)^m}{m!} \quad .$$

Using these, Eq. (3.23) becomes

$$\Phi(\bar{r}) = \frac{e^{ikr}}{r} \int_B \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^n \frac{(r\cdot\bar{r}_B)^n}{n!} \left[(ik)^{n+m} \frac{\partial\Phi_m}{\partial n} + (ik)^{n+m+1} \hat{n}\cdot\hat{r}\,\Phi_n \right] dB \ .$$

Rearranging the terms, we have

$$\Phi(r) \sim \frac{e^{ikr}}{4\pi r} \sum_{n=0}^{\infty} (ik)^n \sum_{m=0}^{n} \frac{(-1)^{n-m}}{(n-m)!} \int_B (\hat{r}\cdot\bar{r}_B)^{n-m} \left\{ \hat{n}\cdot\hat{r}\,\Phi_{m-1} - \frac{\partial\Phi_m}{\partial n} \right\} dB \tag{3.25}$$

where

$$\Phi_{-1} = 0 \quad .$$

Equation (2.34) indicates that the knowledge of a finite number of terms in the near field will determine the same number of terms in the far field.

3.3 THE VECTOR PROBLEM

A plane electromagnetic wave is assumed to be incident on a perfectly conducting body. $\overline{E}^{inc}$ and $\overline{H}^{inc}$ represent the incident fields vectors. We seek a solution of the Maxwell's equations

$$\begin{aligned} \nabla \times \overline{E} &= ik\overline{H} \qquad \nabla \cdot \overline{E} = 0 \\ \nabla \times \overline{H} &= -ik\overline{E} \qquad \nabla \cdot \overline{H} = 0 \end{aligned} \tag{3.26}$$

in the region exterior to the scatterer and subject to the boundary conditions

$$\begin{aligned} \hat{n} \times \overline{E} \Big|_{\overline{r} = \overline{r}_B} &= - \hat{n} \times \overline{E}^{inc} \Big|_{\overline{r} = \overline{r}_B} \\ \hat{n} \cdot \overline{H} \Big|_{\overline{r} = \overline{r}_B} &= - \hat{n} \cdot \overline{H}^{inc} \Big|_{\overline{r} = \overline{r}_B} \end{aligned} \tag{3.27}$$

and the Sommerfeld radiation condition

$$\lim_{r \to \infty} \overline{r} \times (\nabla \times \overline{E}) + ikr\overline{E} = 0$$

$$\lim_{r \to \infty} \overline{r} \times (\nabla \times \overline{H}) + ikr\overline{H} = 0$$

The procedure for reducing the solution of (3.26) to a set of potential problems is analogous to that of the scalar problem. The starting point is the vector analogue of the Helmholtz representation viz., the Stratton-Chu formula (Stratton, 1941), relating the field at an exterior point to their values on the surface.

$$\overline{E}(\overline{r}) = \frac{1}{4\pi}\nabla x \int_B \frac{e^{ikR}}{R}\hat{n} \times \overline{E}\ dB + \frac{ik}{4\pi}\int_B \frac{e^{ikR}}{R}\hat{n} \times \overline{H}\ dB - \frac{1}{4\pi}\nabla \int_B \frac{e^{ikR}}{R}\hat{n}\cdot \overline{E}\, dB \tag{3.29}$$

$$\overline{H}(\overline{r}) = \frac{1}{4\pi}\nabla x \int_B \frac{e^{ikR}}{R}\hat{n} \times \overline{H}\, dB - \frac{ik}{4\pi}\int_B \frac{e^{ikR}}{R}\hat{n} \times \overline{E}\ dB - \frac{1}{4\pi}\nabla \int_B \frac{e^{ikR}}{R}\hat{n}\cdot \overline{H}\, dB \tag{3.30}$$

where the quantities R, $\hat{n}$, B have the same meaning as in the previous section. $\overline{\nabla}$ operates on $\overline{r}$ and the corresponding operator on $\overline{r}_B$ will be denoted by ∇_B in the future.

Expanding e^{ikR}, $\overline{E}$, $\overline{H}$, $\overline{E}^{inc}$ and $\overline{H}^{inc}$ in power series, we get

$$\frac{e^{ikR}}{R} = \sum_{\ell=0}^{\infty} \frac{(ik)^{\ell} R^{\ell-1}}{\ell!}$$

$$\overline{E} = \sum_{\ell=0}^{\infty} \overline{E}_{\ell}(r)\,(ik)^{\ell} \qquad \ell = 0, 1, 2 \ldots \tag{3.31}$$

$$\overline{E}^{inc} = \sum_{\ell=0}^{\infty} \overline{E}_{\ell}^{inc}\,(ik)^{\ell}$$

and similarly for $\overline{H}$. Now the boundary conditions can be rewritten as,

$$\hat{n} \times \overline{E}_{\ell}\Big|_{\overline{r}=\overline{r}_B} = -\hat{n} \times \overline{E}_{\ell}^{inc}\Big|_{\overline{r}=\overline{r}_B}$$

$$\hat{n}\cdot \overline{H}_{\ell}\Big|_{\overline{r}=\overline{r}_B} = -\hat{n}\cdot \overline{H}_{\ell}^{inc}\Big|_{\overline{r}=\overline{r}_B} \quad .$$

In addition to this,

$$\int_B \hat{n}\cdot \overline{E}_{\ell}\, dB = 0 \qquad \text{and} \qquad \int_B \hat{n}\cdot \overline{H}_{\ell}\, dB = 0 \quad .$$

Incorporating all these in Eqs. (3.29) and (3.30), we can write

$$\bar{E}_\ell(\bar{r}) = \bar{F}_\ell(\bar{r}) - \frac{1}{4\pi} \nabla \int \frac{\hat{n} \cdot \bar{E}_\ell}{R} dB \tag{3.32}$$

$$\bar{H}_\ell(\bar{r}) = G_\ell(\bar{r}) + \frac{1}{4\pi} \nabla x \int \frac{\hat{n} x \bar{H}_\ell}{R} dB \tag{3.33}$$

where

$$F_\ell(\bar{r}) = -\frac{1}{4\pi} \nabla x \sum_{m=0}^{\ell} \frac{1}{m!} \int_B \hat{n} x \bar{E}^{inc}_{\ell-m} R^{m-1} dB$$

$$+ \frac{1}{4\pi} \sum_{m=0}^{\ell-1} \frac{1}{m!} \int_B \hat{n} x \bar{H}_{\ell-m-1} R^{m-1} dB$$

$$- \frac{1}{4\pi} \sum_{m=1}^{\ell} \frac{1}{m!} \nabla \int_B \hat{n} \cdot \bar{E}_{\ell-m} R^{m-1} dB \tag{3.34}$$

$$G_\ell(\bar{r}) = \frac{1}{4\pi} \nabla x \sum_{m=1}^{\ell} \frac{1}{m!} \int_B \hat{n} x \bar{H}_{\ell-m} R^{m-1} dB$$

$$+ \frac{1}{4\pi} \sum_{m=0}^{\ell-1} \frac{1}{m!} \int_B \hat{n} x \bar{E}^{inc}_{\ell-m-1} R^{m-1} dB$$

$$+ \frac{1}{4\pi} \sum_{m=0}^{\ell} \frac{1}{m!} \nabla \int_B \hat{n} \cdot \bar{H}^{inc}_{\ell-m} R^{m-1} dB \quad . \tag{3.35}$$

It must be noted that $\sum_{m=0}^{\ell-1}$ and $\sum_{m=0}^{\ell}$ are identically zero for $\ell = 0$. $F_\ell(\bar{r})$ and $G_\ell(\bar{r})$ are expressed only in terms of known quantities, in view of the fact that we know $E_0, E_1 \ldots E_{\ell-1}$ and $\bar{H}_0, \bar{H}_1, \ldots \bar{H}_{\ell-1}$.

The unknown term in the right hand side of (3.32) is the gradient of a function which we know to be an exterior potential function. Let

$$U_\ell = -\frac{1}{4\pi}\int_\infty \frac{\hat{n}\cdot\overline{E}_\ell}{R}\, dB \tag{3.36}$$

then

$$\nabla^2 U_\ell = 0$$

$U_\ell \sim$ regular at infinity in the sense of Kellog

$$\hat{n} \times \nabla U_\ell \Big|_{\overline{r}=\overline{r}_B} = -\hat{n} \times \left[\overline{E}_\ell^{inc} + \overline{F}_\ell\right]\Big|_{\overline{r}=\overline{r}_B}$$

which can be solved in a conventional way. Hence $\overline{E}_\ell$ can be determined.

But the determination of the $\overline{H}_\ell$ is a considerably more difficult task, because it is not clear that the unknown term in the right hand side of (3.33) is the gradient of a potential function. In order to make it possible to solve for this function in terms of a potential function, we introduce a function $g_\ell(r)$ such that

$$\frac{1}{4\pi}\nabla \times \int \frac{\hat{n}\times\overline{H}_\ell}{R}\, dB + g_\ell(\overline{r}) = \nabla V_\ell \quad . \tag{3.37}$$

To solve for $g_\ell(\overline{r})$ which satisfies (3.37), we proceed as follows:

$$\frac{1}{4\pi}\nabla \times \nabla \times \int_B \frac{\hat{n}\times\overline{H}_\ell}{R}\, dB + \nabla \times g_\ell(\overline{r}) = 0 \quad . \tag{3.38}$$

Using the vector identity

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla\cdot\overline{A}) - \nabla^2\overline{A}$$

and

$$\nabla\frac{1}{R} = -\nabla_B\frac{1}{R}$$

(3.38) can be rearranged to give

$$\nabla \times g_\ell(\bar{r}) = \frac{1}{4\pi} \nabla \int \frac{\hat{n}}{R} \cdot \nabla_B \times \bar{H}_\ell \, dB \tag{3.39}$$

$$= -\frac{1}{4\pi} \nabla \int_B \frac{\hat{n} \cdot \bar{E}_{\ell-1}}{R} \, dB \qquad \ell > 0$$

$$= 0 \qquad \ell = 0 \tag{3.40}$$

Stevenson (1953) has developed a method for finding particular solutions of equations of the type

$$\nabla \times \bar{F} = \bar{f}$$

when $\bar{f}$ is a gradient of a scalar potential function. We have from (3.40)

$$\nabla \times \bar{g}_\ell = \nabla U_\ell^e$$

where

$$U_\ell^e = -\frac{1}{4\pi} \int_B \frac{\hat{n} \cdot \bar{E}_{\ell-1}}{R} \, dB \quad . \tag{3.41}$$

Stevenson, at this point, introduces an interior potential function $U_\ell^i(\bar{r})$ defined when $\bar{r}$ is interior to B, (This function is defined purely for analytical convenience and does not have a physical significance in this problem) such that

$$\nabla^2 U_\ell^i(\bar{r}) = 0$$

$\bar{r}$ interior to B.

$$\hat{n} \cdot \nabla U_\ell^i(\bar{r}) \Big|_{\bar{r} = \bar{r}_B} = \hat{n} \cdot \nabla U_\ell^e(\bar{r}) \Big|_{\bar{r} = \bar{r}_B} \tag{3.42}$$

This can be solved for in a conventional way. It is also necessary to have

$$\int_B \hat{n} \cdot \nabla U_\ell^i \, dB = 0 \quad .$$

But this, clearly, is true.

The particular solution of $g_\ell(\bar{r})$, then is

$$g_\ell(\bar{r}) = \frac{1}{4\pi} \nabla \times \left[\int \frac{\nabla_V U_\ell^e(\bar{r})}{R} dV + \int \frac{\nabla_V U_\ell^i(\bar{r}_V)}{R} dV \right] \tag{3.43}$$

which can be rearranged to give

$$g_\ell(\bar{r}) = -\frac{1}{4\pi} \nabla \times \int_B \frac{U_\ell^e(\bar{r}_B) - U_\ell^i(\bar{r}_B)}{R} dB \quad . \tag{3.43}$$

Now we can determine $V_\ell(\bar{r})$ by solving a potential problem because

$$\frac{1}{4\pi} \nabla \times \int_B \frac{n \times \bar{H}_\ell}{R} dB + \bar{g}_\ell = \nabla V_\ell$$

and $\nabla^2 V_\ell = 0$, with the value for V_ℓ on the surface given by

$$\hat{n} \cdot \nabla V_\ell \Big|_{\bar{r}=\bar{r}_B} = \left\{ -\hat{n} \cdot H_\ell^{inc} - \hat{n} \cdot \bar{G}_\ell + \frac{1}{4\pi} \hat{n} \cdot \int \frac{\hat{n} \times \nabla_B (U_\ell^e - U_\ell^i)}{R} dB \right\} \Big|_{\bar{r}=\bar{r}_B}$$

from this we get H_ℓ.

Although the above method is a systematic way of obtaining terms of all orders, certain simplification result when one solves for the first term.

From Maxwell's equations using the expansions (3.31), it is easy to see

$$\nabla \times \bar{E}_p = \bar{H}_{p-1} \quad \text{and} \quad \nabla \times \bar{H}_p = -\bar{E}_{p-1} \qquad p > 0 \; .$$

In particular for $p = 1$,

$$\nabla \times \bar{E}_1 = \bar{H}_o \qquad \nabla \times \bar{H}_1 = -\bar{E}_o \; .$$

But H_o and E_o are known to be gradients of scalar functions and hence we have new equations of the type

$$\nabla \times \bar{E}_1 = \nabla U_o \qquad \nabla \times \bar{H}_1 = -\nabla V_o$$

and particular solutions of these equations can be obtained by using (3.43) which is a solution of equation of the type $\nabla \times F = \nabla U$. Since arbitrary gradient functions can be added to these solutions, they are required to satisfy the boundary conditions, viz.,

$$\hat{n} \times \bar{E}_1 \Big|_{\bar{r} = \bar{r}_B} - \hat{n} \times \bar{E}_1^{inc} \Big|_{\bar{r} = \bar{r}_B} \quad \text{and} \quad \hat{n} \times \bar{H}_1 \Big|_{\bar{r} = \bar{r}_B} - \hat{n} \times \bar{H}_1^{inc} \Big|_{\bar{r} = \bar{r}_B} .$$

In this chapter the theory of approximating a diffraction problem, at low frequencies, in terms of a series of potential problems was discussed. No assumption as to the shape of the diffracting body was made. In the next chapter we apply this method to the problem of diffraction by a torus.

IV

ACOUSTIC SCATTERING BY A TORUS

In this chapter we shall apply the method described in Chapter III to the scattering of a plane acoustic wave incident normally on a torus.

4.1 PLANE WAVE INCIDENT NORMALLY ON A RIGID TORUS

The incident wave is propagating down the negative z - axis, and taken to be of unit amplitude.

$$\Phi^{inc} = \sum_{\ell=0}^{\infty} \Phi_{\ell}^{inc} (ik)^{\ell} = e^{-ikz} = \sum_{\ell=0}^{\infty} \frac{(-z)^{\ell}(ik)^{\ell}}{\ell !}$$

and hence

$$\Phi_{\ell}^{inc} = \frac{(-z)^{\ell}}{\ell !} \tag{4.1}$$

we now proceed to obtain the first three terms, directly from the Eqs. (3.17) to (3.19) of the previous section.

4.1.1 Zero'th Order Term

$$\Phi_o(\bar{r}) = G_o(\bar{r}) + V_o(\bar{r}) \tag{4.2}$$

$$G_o(\bar{r}) = \frac{1}{4\pi}\int_B \frac{1}{R} \frac{\partial}{\partial n} \Phi_o^{inc} . \tag{4.3}$$

But $\Phi_o^{inc} = 1$ and $\dfrac{\partial \Phi_o^{inc}}{\partial n} = 0$. Therefore $G_o(\bar{r}) = 0$.

$$V_o(\bar{r}) = \frac{1}{4\pi}\int_B \Phi_o \frac{\partial}{\partial n} \frac{1}{R} \, dB$$

is a single layer distribution and a solution of Laplace's equation, given by

$$V_o(\bar{r}) = \sqrt{\cosh\eta - \cos\theta} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[A_m \cos m\psi + B_m \sin m\psi\right]$$

$$\left[C_n \cos n\theta + D_n \sin n\theta\right] \quad P^m_{n-1/2}(\cosh\eta) \tag{4.4}$$

A_m, B_m, C_n and D_n are constants to be determined by the boundary conditions. $V_o(\bar{r})$ is regular in the sense of Kellogg can be easily verified.

Since

$$r = a \left[\frac{\cosh\eta + \cos\theta}{\cosh\eta - \cos\theta}\right]^{1/2} ,$$

$$\frac{\partial}{\partial r} = -\frac{1}{a} \frac{(\cosh\eta - \cos\theta)^{1/2}}{(\cosh\eta + \cos\theta)^{1/2}} \sinh\eta \cos\theta \frac{\partial}{\partial\eta} - \frac{1}{a} \frac{(\cosh\eta - \cos\theta)^{1/2}}{(\cosh\eta + \cos\theta)^{1/2}} \cosh\eta \sin\theta \frac{\partial}{\partial\eta} .$$

Substituting these along with (4.4) gives

$$\lim_{r \to \infty} \left|r V_o\right| = \lim_{\substack{\eta \to 0 \\ \theta \to 0}} \left|r V_o\right| < \infty$$

$$\lim_{r \to \infty} \left|r^2 \frac{\partial V_o}{\partial r}\right| = \lim_{\substack{\eta \to 0 \\ \theta \to 0}} \left|r^2 \frac{\partial V_o}{\partial r}\right| < \infty . \tag{4.5}$$

Boundary condition on $V_o(r)$ is

$$\left.\frac{\partial V_o}{\partial n}\right|_{\eta = \eta_s} = -\left.\frac{\partial \Phi_o^{inc}}{\partial n}\right|_{\eta = \eta_s} - \left.\frac{\partial G_o}{\partial n}\right|_{\eta = \eta_s} . \tag{4.6}$$

But

$$\Phi_o^{inc} = 1$$

and hence

$$\frac{\partial \Phi_o^{inc}}{\partial n} = 0 \quad \text{and also} \quad G_o = 0 .$$

This gives $V_o = 0$.

Therefore, the first term in the expansion for the scattered field

$$\Phi_o = 0 \quad .$$

4.1.2 First Order Term

This term Φ_1 is given by

$$\Phi_1(r) = G_1(r) + V_1(r)$$

where

$$G_1(\bar{r}) = \frac{1}{4\pi} \int_B \frac{\partial}{\partial n} \Phi_o^{inc} \, dB + \frac{1}{4\pi} \int_B \frac{1}{R} \frac{\partial}{\partial n} \Phi_1^{inc} \, dB$$

But

$$\frac{\partial \Phi_o^{inc}}{\partial n} = 0$$

and

$$G_1(\bar{r}) = \frac{1}{4\pi} \int_B \frac{1}{R} \frac{\partial}{\partial n} \Phi_1^{inc} \, dB \quad . \tag{4.8}$$

From (4.1) we have

$$\Phi_1^{inc} = -z = -\frac{a \sin\theta}{\cosh\eta - \cos\theta}$$

and

$$\frac{\partial \Phi_1^{inc}}{\partial n} = -\frac{\cosh\eta - \cos\theta}{a} \frac{\partial}{\partial \eta} \Phi_1^{inc}$$

$$= -\frac{\sinh\eta \sin\theta}{\cosh\eta - \cos\theta} \quad .$$

The surface element of area dB is given by

$$dB = \frac{a^2 \sinh\eta_s}{(\cosh\eta_s - \cos\theta_B)^2}\, d\theta_B\, d\psi_B$$

and $\frac{1}{R}$ is given by (2.9). Substituting for all these quantities in (4.8) and integrating over ψ_B gives a value of 2π for the case $m=0$ and vanishes for all values $m \neq 0$.

$$G_1(\bar{r}) = -\frac{a}{\pi}\sqrt{\cosh\eta - \cos\theta}\sum_{n=0}^{\infty} \epsilon_n P_{n-1/2}(\cosh\eta) Q_{n-1/2}(\cosh\eta_s)\sinh^2\eta_s$$

$$\int_0^{2\pi} \frac{\sin\theta_B \cos n(\theta-\theta_B)}{(\cosh\eta_s - \cos\theta_B)^{5/2}}\, d\theta_B \quad . \qquad (4.9)$$

But

$$\frac{\sin\theta_B}{(\cosh\eta_s - \cos\theta_B)^{5/2}} = -\frac{4\sqrt{2}}{3\pi}\,\frac{1}{\sinh\eta_s}\sum_{p=0}^{\infty} p \sin p\theta_B\, Q'_{p-1/2}(\cosh\eta_s) . \qquad (4.10)$$

(the prime on $Q'_{p-1/2}$ refers to a differentiation with respect to η).
Finally using (4.10) in (4.9) and integrating, yields

$$G_1(r) = a\frac{2\sqrt{2}}{3\pi}\sqrt{\cosh\eta - \cos\theta}\,\sinh\eta_s \sum_{n=0}^{\infty} n\,\epsilon_n P_{n-1/2}(\cosh\eta) Q_{n-1/2}(\cosh\eta_s)$$

$$Q'_{n-1/2}(\cosh\eta_s)\sin n\theta \qquad (4.11)$$

$V_1(\bar{r})$, again, is given by

$$V_1(\bar{r}) = \sqrt{\cosh\eta - \cos\theta}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(A_m \cos m\psi + B_m \sin m\psi)(C_n \cos n\theta + D_n \sin n\theta)$$

$$P^m_{n-1/2}(\cosh\eta) \qquad (4.12)$$

with the condition

$$\frac{\partial V_1}{\partial n}(\bar{r}) \Bigg|_{\eta=\eta_s} = -\frac{\partial \Phi_1^{inc}}{\partial n} \Bigg|_{\eta=\eta_s} - \frac{\partial G_1(\bar{r})}{\partial n} \Bigg|_{\eta=\eta_s} \tag{4.13}$$

$$\Phi_1^{inc} = -\frac{a \sin\theta}{\cosh\eta - \cos\theta} = -a\sqrt{\cosh\eta - \cos\theta}\,\frac{2\sqrt{2}}{\pi}\sum n \sin n\theta\, Q_{n-1/2}(\cosh\eta) \ . \tag{4.14}$$

Examining (4.11) and (4.14), we see clearly that the expressions are independent of ψ and therefore the only non zero coefficient of $\cos m\psi$ is for $m = 0$, i.e., $A_o = 1$ and all the B_m's are identically zero. By inspection we can also see that all the C_ns are zero, since $G_1(\bar{r})$ and $V_1(\bar{r})$ are only series of $\sin n\theta$ and not $\cos n\theta$. Hence,

$$\frac{\sinh\eta_s}{2\sqrt{\cosh\eta_s - \cos\theta}} \sum_{n=1}^{\infty} D_n \sin n\theta\, P_{n-1/2}(\cosh\eta_s) \ +$$

$$+ \sqrt{\cosh\eta_s - \cos\theta} \sum_{n=1}^{\infty} D_n \sin n\theta\, P'_{n-1/2}(\cosh\eta_s)$$

$$= \frac{\sinh\eta_s}{2\sqrt{\cosh\eta_s - \cos\theta}} \sum_{n=1}^{\infty} n \sin n\theta \Bigg[\frac{a\,2\sqrt{2}}{\pi} Q_{n-1/2}(\cosh\eta_s) \ -$$

$$- \frac{a\,4\sqrt{2}}{3\pi} \sinh\eta_s\, P_{n-1/2}(\cosh\eta_s) Q_{n-1/2}(\cosh\eta_o) Q'_{n-1/2}(\cosh\eta_s)\Bigg]$$

$$+ \sqrt{\cosh\eta_s - \cos\theta} \sum_{n=1}^{\infty} n \sin n\theta \Bigg[\frac{a\,2\sqrt{2}}{\pi} Q'_{n-1/2}(\cosh\eta_s) \ -$$

$$- \frac{a\,4\sqrt{2}}{\pi} \sinh\eta_s\, P'_{n-1/2}(\cosh\eta_s) Q_{n-1/2}(\cosh\eta_s) Q'_{n-1/2}(\cosh\eta_s)\Bigg] . \tag{4.15}$$

After simplification we obtain (the arguments of the Legendre functions are omitted, but should be understood to be $\cosh\eta_s$)

$$D_1\left[P'_{3/2} - P'_{-1/2}\right] - D_2 P'_{3/2} = -\frac{a\,2\sqrt{2}}{\pi}\left(Q'_{3/2} - Q'_{-1/2}\right) -$$

$$-\frac{a\,4\sqrt{2}}{3\pi}\sinh\eta_s\left[Q_{1/2}Q'_{1/2}P'_{3/2} + Q_{1/2}Q'_{1/2}P'_{-1/2} - 2P'_{1/2}Q'_{3/2}Q_{3/2}\right].$$

$$\text{for } n = 1 \qquad (4.16)$$

$$P'_{n+1/2}\left[D_{n+1} - D_n\right] - P'_{n-3/2}\left[D_n - D_{n-1}\right] = \frac{a\,2\sqrt{2}}{3\pi}\left[Q'_{n+1/2} - Q'_{n-3/2}\right]$$

$$+\frac{a\,4\sqrt{2}}{3\pi}\,n\sinh\eta_s\left[P'_{n+1/2}Q_{n-1/2}Q'_{n-1/2} - P'_{n-3/2}Q_{n-1/2}Q'_{n-1/2}\right]$$

$$-\frac{a\,4\sqrt{2}}{3\pi}(n+1)\sinh\eta_s P'_{n+1/2}Q_{n+1/2}Q'_{n+1/2}$$

$$-\frac{a\,4\sqrt{2}}{3\pi}(n-1)\sinh\eta_s P'_{n-3/2}Q_{n-3/2}Q'_{n-3/2} \quad \text{for } n > 1 \qquad (4.17)$$

This can be summed to become:

$$D_{n+1} - D_n = \frac{a\,2\sqrt{2}}{\pi(2n+1)}\left(n^2 + \beta\right)\left\{\frac{Q'_{n+1/2}}{P'_{n+1/2}} - \frac{Q'_{n-1/2}}{P'_{n-1/2}}\right\} + \frac{a\,2\sqrt{2}}{\pi}\frac{Q'_{n+1/2}}{P'_{n+1/2}}$$

$$+\frac{a\,4\sqrt{2}}{3\pi}\sinh\eta_s\left\{nQ_{n-1/2}Q'_{n-1/2} - (n+1)\,Q'_{n+1/2}Q_{n+1/2}\right\}$$

$$+\frac{a\,4\sqrt{2}}{3\pi}\,\frac{\sinh\eta_s\,P_{-1/2}P'_{1/2}Q_{1/2}Q'_{1/2}}{P'_{n+1/2}P'_{n-1/2}} \qquad (4.18)$$

where

$$\beta = \frac{\pi}{a\sqrt{2}}\left[D_1 P'_{1/2} - \frac{a\,2\sqrt{2}}{\pi}Q'_{1/2}\right]P'_{-1/2}\,. \qquad (4.19)$$

Writing successive equations from $n = n, n-1 \ldots 1$ and adding together, we get, using (4.19)

$$D_{n+1} = \frac{a\,2\sqrt{2}}{\pi}\left\{\frac{(n+1)^2+\beta}{2n+1}\,\frac{Q'_{n+1/2}}{P'_{n+1/2}} + 2\sum_{r=1}^{n}\frac{r^2+\beta}{4r^2-1}\,\frac{Q'_{r-1/2}}{P'_{r-1/2}} - \alpha\,\frac{Q'_{-1/2}}{P'_{-1/2}}\right\}$$

$$+ \frac{a\,4\sqrt{2}}{3\pi}\sinh\eta_s\left\{Q_{1/2}Q'_{1/2} - (n+1)Q'_{n+1/2}Q_{n+1/2} + \right.$$

$$\left. + P_{-1/2}P'_{1/2}Q_{1/2}Q'_{1/2}\sum_{r=1}^{n}\frac{1}{P'_{r+1/2}P'_{r-1/2}}\right\} \quad . \qquad (4.20)$$

In order to find β, (i.e., D_1) we employ the same method as in (2.41) due to Hicks, i.e., we find the limit of D_n as $n \to \infty$ and set the limit equal to zero in order to have $V_1(r)$ finite everywhere. The series

$$\sum_{r=1}^{\infty}\frac{r^2+\beta}{4r^2-1}\,\frac{Q'_{r-1/2}}{P'_{r-1/2}} \qquad (4.21)$$

converges and the series

$$\sum_{r=1}^{\infty}\frac{1}{P'_{r+1/2}P'_{r-1/2}}$$

also tends to a finite limit. Hence β is given by

$$\beta = \frac{-\left[2\sinh\eta_s Q_{1/2}Q'_{1/2} + \sum_{r=1}^{\infty}\frac{P_{-1/2}P'_{1/2}Q_{1/2}Q'_{1/2}}{P'_{r+1/2}P'_{r-1/2}}\right]}{\left[2\sum_{1}^{\infty}\frac{1}{4r^2-1}\,\frac{Q'_{r-1/2}}{P'_{r-1/2}} - \frac{Q'_{-1/2}}{P'_{-1/2}}\right]} \qquad (4.22)$$

and D_n is then

$$D_n = \frac{a\,2\sqrt{2}}{\pi}\,\frac{n^2+\beta}{2n+1}\,\frac{Q'_{n-1/2}}{P'_{n-1/2}} - \frac{a\,4\sqrt{2}}{\pi}\sum_{r=n}^{\infty}\frac{r^2+\beta}{4r^2-1}\,\frac{Q'_{r-1/2}}{P'_{r-1/2}}$$

$$+ \frac{a\,4\sqrt{2}}{\pi}\left\{ n\,Q_{n-1/2}Q'_{n-1/2} + \sum_{n}^{\infty}\frac{P_{-1/2}P'_{1/2}Q_{1/2}Q'_{1/2}}{P'_{r+1/2}P'_{r-1/2}}\right\} . \qquad (4.23)$$

Finally, we obtain from

$$\Phi_1(r) = G_1(r) + V_1(r) \ ,$$

$$\Phi_1(r) = \frac{a\,2\sqrt{2}}{3\pi}\sqrt{\cosh\eta - \cos\theta}\ \sinh\eta_s \sum_{n=0}^{\infty} n\,\epsilon_n P_{n-1/2}(\cosh\eta)\, Q_{n-1/2}(\cosh\eta_s)\ \times$$

$$\times\ Q'_{n-1/2}(\cosh\eta_s)\sin n\theta$$

$$+ \sqrt{\cosh\eta - \cos\theta}\sum_{n=0}^{\infty} D_n \sin n\theta\, P_{n-1/2}(\cosh\eta) \qquad (4.24)$$

where D_n is given by (4.23) and (4.22).

4.1.3 Second Order Terms

We shall proceed in an exactly similar manner as we did in Section 4.1.2 to determine Φ_2, the third term coefficient in the expansion of the scattered field.

$$\Phi_2(\bar{r}) = G_2(\bar{r}) + V_2(\bar{r})$$

$$G_2(\bar{r}) = \frac{1}{4\pi}\,\frac{1}{2!}\int_B R\,\frac{\partial}{\partial n}\Phi_o^{inc}\,dB + \frac{1}{4\pi}\int_B \frac{\partial}{\partial n}\,\Phi_1^{inc}\,dB + \frac{1}{4\pi}\int \frac{1}{R}\,\frac{\partial}{\partial n}\Phi_2^{inc}\,dB$$

$$+\ \frac{1}{4\pi}\,\frac{1}{2!}\int \Phi_o\,\frac{\partial}{\partial n}\,R\,dB \quad . \qquad (4.25)$$

Since

$$\Phi_o = 0 \text{ and } \Phi_o^{inc} = 1$$

we get

$$G_2(\bar{r}) = \frac{1}{4\pi}\int \frac{\partial}{\partial n}\,\Phi_1^{inc}\,dB + \frac{1}{4\pi}\int \frac{1}{R}\,\frac{\partial}{\partial n}\,\Phi_2^{inc}\,dB \qquad (4.26)$$

$$\Phi_2^{inc} = \frac{z^2}{2} = \frac{a^2\sin^2\theta}{(\cosh\eta - \cos\theta)^2}$$

$$\frac{\partial\Phi_2^{inc}}{\partial n} = a\,\frac{\sin^2\theta\,\sinh\eta}{(\cosh\eta - \cos\theta)^2} \quad ; \quad \frac{\partial\Phi_1^{inc}}{\partial n} = -\,\frac{\sin\theta\,\sinh\eta}{\cosh\eta - \cos\theta}$$

$\frac{1}{R}$ is given by (2.9).

Substituting these in (4.26) we get

$$G_2(\bar{r}) = \frac{a^2}{2\pi}\sqrt{\cosh\eta - \cos\theta}\;\sum_{n=0}^{\infty}\epsilon_n\,\frac{P_{n-1/2}(\cosh\eta)\,Q_{n-1/2}(\cosh\eta_s)}{\sinh\eta_s}\cos n\theta \;\; x$$

$$x \left\{ -\sqrt{2}\,\frac{\Gamma(n+7/2)}{\Gamma(n-5/2)}\,Q^{(3)}_{n-1/2}(\cosh\eta_s) + 2\sqrt{2}\,\frac{\Gamma(n+11/2)}{\Gamma(n-1/2)}\,Q^{(3)}_{n+3/2}(\cosh\eta_s)\right.$$

$$\left. +\ 2\sqrt{2}\,\frac{\Gamma(n+3/2)}{\Gamma(n-9/2)}\,Q^{(3)}_{n-5/2}(\cosh\eta_s)\right\} \qquad (4.27)$$

Also

$$\frac{\partial \Phi_2^{inc}}{\partial n} = - \frac{a^2\sqrt{2}}{2\pi} \frac{\sqrt{\cosh\eta - \cos\theta}}{\sinh^2\eta} \sum_{n=0}^{\infty} \left\{ - \frac{\Gamma(n+7/2)}{\Gamma(n-5/2)} Q^{(3)}_{n-1/2}(\cosh\eta) \right.$$

$$\left. + 2 Q^{(3)}_{n+3/2}(\cosh\eta) \frac{\Gamma(n+11/2)}{\Gamma(n-1/2)} + 2 Q^{(3)}_{n-5/2}(\cosh\eta) \frac{\Gamma(n+3/2)}{\Gamma(n-9/2)} \right\} \cos n\theta . \tag{4.28}$$

$V_2(\bar{r})$ once again is a solution of Laplace's equation which satisfies the regularity conditions and the boundary conditions

$$\left.\frac{\partial V_2(\bar{r})}{\partial n}\right|_{\eta=\eta_s} = - \left.\frac{\partial \Phi_2^{inc}}{\partial n}\right|_{\eta=\eta_s} - \left.\frac{\partial G_2(\bar{r})}{\partial n}\right|_{\eta=\eta_s} . \tag{4.29}$$

Let us abbreviate

$$\frac{a^2\sqrt{2}}{2\pi\sinh\eta_s} \left[- \frac{\Gamma(n+7/2)}{\Gamma(n-5/2)} Q^{(3)}_{n-1/2}(\cosh\eta_s) + 2Q^{(3)}_{n+3/2}(\cosh\eta_s) \frac{\Gamma(n+1/2)}{\Gamma(n-1/2)} + \right.$$

$$\left. + 2 Q^{(3)}_{n-5/2}(\cosh\eta_s) \frac{\Gamma(n+3/2)}{\Gamma(n-9/2)} \right] = K_n . \tag{4.30}$$

Equation (4.29) takes the form

$$\sqrt{\cosh\eta_s - \cos\theta} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (A_m \cos m\psi + B_m \sin m\psi)(C_n \cos n\theta + D_n \sin n\theta)$$

$$P^{m'}_{n-1/2}(\cosh\eta_s)$$

$$+ \frac{\sinh\eta_s}{2\sqrt{\cosh\eta_s - \cos\theta}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (A_m \cos m\psi + B_m \sin m\psi)(C_n \cos n\theta + D_n \sin n\theta)$$

$$P^{m}_{n-1/2}(\cosh\eta_s)$$

$$= -\sqrt{\cosh\eta_s - \cos\theta}\sum_{n=0}^{\infty} \epsilon_n K_n P_{n-1/2}(\cosh\eta_s) Q'_{n-1/2}(\cosh\eta_s)\cos n\theta$$

$$- \frac{\sinh\eta_s}{2\sqrt{\cosh\eta_s - \cos\theta}} \sum_{n=0}^{\infty} \epsilon_n K_n P_{n-1/2}(\cosh\eta_s) Q_{n-1/2}(\cosh\eta_s)\cos n\theta. \tag{4.31}$$

We shall omit the details of getting the coefficients since the arguments and procedures are identical to the previous case . After rearrangement and simplification we get

$$C_{n+1} = C_o + P_{-1/2}\left[C_o P'_{-3/2} - P_{1/2}Q_{1/2}K_1\right] \sum_{r=1}^{n} \frac{1}{P'_{r+1/2}P'_{r-1/2}}$$

$$+ \sum_{r=0}^{n} \frac{1}{P'_{r+1/2}P'_{r-1/2}} \sum_{s=1}^{r} (r-s+1) K_s P'_{s-1/2} P_{s-1/2}(Q'_{s-1/2} + Q'_{s+3/2})$$

$$- \sum_{r=0}^{n} \frac{1}{P'_{r+1/2}P'_{r-1/2}} \sum_{s=1}^{r} (r-s+1) P'_{s-1/2} P_{s+1/2} Q'_{s+1/2} K_{s+1} -$$

$$- \sum_{r=0}^{n} \frac{1}{P'_{r+1/2}P'_{r-1/2}} \sum_{s=1}^{r} (r-s+1) P'_{s-1/2} P_{s-3/2} Q'_{s-3/2} K_{s-1} \tag{4.32}$$

with C_o given by

$$C_o + P_{-1/2}\left[C_o P'_{-3/2} - P_{1/2}Q_{1/2}K_1\right] \sum_{r=1}^{\infty} \frac{1}{P'_{r+1/2}P'_{r-1/2}}$$

$$+ \sum_{r=1}^{\infty} \frac{1}{P'_{r+1/2}P'_{r-1/2}} \left[\sum_{s=1}^{r} K_s P_{s-1/2} P'_{s-1/2}(Q'_{s+1/2} + Q'_{s-3/2}) - \right.$$

$$\left. - \sum_{s=1}^{r} P_{s-1/2} P'_{s+1/2} Q'_{s+1/2} K_{s+1} - \sum_{s=1}^{r} P_{s-1/2} P_{s-3/2} Q'_{s-3/2} K_{s-1}\right] = 0. \tag{4.33}$$

Thus $\Phi_2(\bar{r})$ is given by

$$\Phi_2(\bar{r}) = \sqrt{\cosh\eta - \cos\theta}\sum_{n=0}^{\infty} \epsilon_n \, P_{n-1/2}(\cosh\eta) Q_{n-1/2}(\cosh\eta_s) K_n \cos n\theta$$

$$+ \sqrt{\cosh\eta - \cos\theta}\sum_{n=0}^{\infty} C_n \cos n\theta \, P_{n-1/2}(\cosh\eta) \tag{4.34}$$

where K_n and C_n are given by (4.30) and (4.32), respectively.

4.2 PLANE WAVE INCIDENT NORMALLY ON A SOFT TORUS

We start with Eq. (3.14) and (3.15) to obtain the zeroeth and the first order terms in the expansion of the scattered field when a plane acoustic wave is incident normally on a soft torus.

4.2.1 Zero'th Order Term

$$\Phi_o(\bar{r}) = F_o(\bar{r}) + U_o(\bar{r}) \tag{4.35}$$

where

$$F_o(\bar{r}) = -\frac{1}{4\pi}\int_B \frac{\partial}{\partial n}\frac{1}{R}\, dB \tag{4.36}$$

$$U_o(\bar{r}) = -\frac{1}{4\pi}\int_B \frac{1}{R}\frac{\partial \Phi_o}{\partial n}\, dB$$

and

$U_o(\bar{r})$

is such that

$$\nabla^2 U_o(\bar{r}) = 0 \tag{4.37}$$

$U_o(\bar{r})$ regular at infinity

$$U_o(\bar{r})\Big|_{\eta=\eta_s} = -F_o(\bar{r})\Big|_{\eta=\eta_s} - \Phi_o^{inc}(\bar{r})\Big|_{\eta=\eta_s} . \tag{4.38}$$

Solving (4.37) we get for $U_o(\bar{r})$, and $F_o(\bar{r})$

$$U_o(\bar{r}) = \sqrt{\cosh\eta - \cos\theta}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(A_m \cos m\psi + B_m \sin m\psi)(C_n \cos n\theta + D_n \sin n\theta)\, P^m_{n-1/2}(\cosh\eta)$$

and

$$F_o(\bar{r}) = 0$$

Also

$$\Phi_o^{inc} = 1 \quad .$$

Matching boundary conditions,

$$\sqrt{\cosh\eta_s - \cos\theta}\sum_{n=0}^{\infty}\sum_{n=0}^{\infty}\left[A_m \cos m\psi + B_m \sin m\psi\right]\left[C_n \cos n\theta + D_n \sin n\theta\right] P^m_{n-1/2}(\cosh\eta_s) = -1$$

and

$$\frac{1}{\sqrt{\cosh\eta_s - \cos\theta}} = \frac{\sqrt{2}}{\pi}\sum_{n=0}^{\infty}\cos n\theta\, Q_{n-1/2}(\cosh\eta_s)$$

giving thereby

$$A_m = -1 \qquad m = 0$$

$$\quad = 0 \qquad m \neq 0$$

$$B_m = 0 \text{ for all } m \text{ and } C_n = \frac{\sqrt{2}}{\pi}\,\frac{Q_{n-1/2}(\cosh\eta_s)}{P_{n-1/2}(\cosh\eta_s)} \quad .$$

Hence

$$U_o(\bar{r}) = -\frac{\sqrt{2(\cosh\eta - \cos\theta)}}{\pi}\sum_{n=0}^{\infty}\frac{Q_{n-1/2}(\cosh\eta_s)}{P_{n-1/2}(\cosh\eta_s)}P_{n-1/2}(\cosh\eta)\cos n\theta \quad .$$

Substituting for this in (4.35), the first term is found to be

$$\Phi_o(\bar{r}) = -\frac{\sqrt{2(\cosh\eta - \cos\theta)}}{\pi}\sum_{n=0}^{\infty}\cos n\theta\,\frac{Q_{n-1/2}(\cosh\eta_s)}{P_{n-1/2}(\cosh\eta_s)}\,P_{n-1/2}(\cosh\eta) \tag{4.39}$$

4.2.2 First Order Terms

$$\Phi_1(\bar{r}) = F_1(\bar{r}) + U_1(\bar{r}) \tag{4.40}$$

where

$$F_1(\bar{r}) = -\frac{1}{4\pi}\int_B \Phi_o^{inc}\frac{\partial}{\partial n}\frac{1}{R}\,dB - \frac{1}{4\pi}\int\frac{\partial\Phi_o}{\partial n}\,dB \tag{4.41}$$

$$U_1(\bar{r}) = -\frac{1}{4\pi}\int\frac{1}{R}\,\frac{\partial\Phi_1}{\partial n}\,dB \quad . \tag{4.42}$$

The incident part is

$$\Phi_1^{inc} = -z = \frac{a\sin\theta}{\cosh\eta - \cos\theta} \quad \text{and} \quad \Phi_o^{inc} = 1 \quad .$$

This may also be expanded as follows:

$$\Phi_1^{inc} = \frac{-2\sqrt{2}\,a}{\pi}\sqrt{\cosh\eta - \cos\theta}\sum_{n=1}^{\infty} Q_{n-1/2}(\cosh\eta)\,n\sin n\theta \quad .$$

Substituting these in (4.41) one obtains for $F_1(\bar{r})$

$$F_1(\bar{r}) = \frac{4\sqrt{2}}{3\pi}\,a\sqrt{\cosh\eta - \cos\theta}\sum_{n=1}^{\infty} n\,P_{n-1/2}(\cosh\eta)\sinh\eta_s\,Q_{n-1/2}(\cosh\eta_s)$$

$$Q'_{n-1/2}(\cosh\eta_s)\,\sin n\theta$$

$$-\frac{a}{\pi}\sum_{n=0}^{\infty}\epsilon_n\,\frac{Q_{n-1/2}(\cosh\eta_s)}{P_{n-1/2}(\cosh\eta_s)} \quad . \tag{4.43}$$

$U_1(r)$ is once again a solution of $\nabla^2 U_1(\bar{r}) = 0$, whose boundary conditions are

$$U_1(\bar{r})\Big|_{\eta=\eta_s} = \Phi_1(\bar{r})\Big|_{\eta=\eta_s} - F_1(\bar{r})\Big|_{\eta=\eta_s}$$

The quantity

$$\frac{a}{\pi} \sum_{0}^{\infty} \epsilon_n \frac{Q_{n-1/2}(\cosh \eta_s)}{P_{n-1/2}(\cosh \eta_s)}$$

is actually the measure of the capacity of the torus, since the capacity K_T of the torus with respect to infinity is defined by

$$K_T = \frac{\epsilon}{V} \int_B \frac{\partial \phi}{\partial n}\, dB$$

where V is the constant potential on the torus and ϵ is the free space permittivity. Substituting for these values, we obtain for the capacity K_T, the value

$$K_T = 8a\epsilon \sum_{n=0}^{\infty} \epsilon_n \frac{Q_{n-1/2}(\cosh \eta_s)}{P_{n-1/2}(\cosh \eta_s)} .$$

It is interesting to compare this with the capacity of a sphere whose radius is a, and is carrying a total charge distributed uniformly,

$$K_{Sph} = 4\pi\epsilon a .$$

Then the ratio is given by

$$\frac{2}{\pi} \sum_{n=0}^{\infty} \epsilon_n \frac{Q_{n-1/2}(\cosh \eta_s)}{P_{n-1/2}(\cosh \eta_s)}$$

and is plotted in Fig. 4-1 as a function of $\operatorname{sech} \eta_s$.

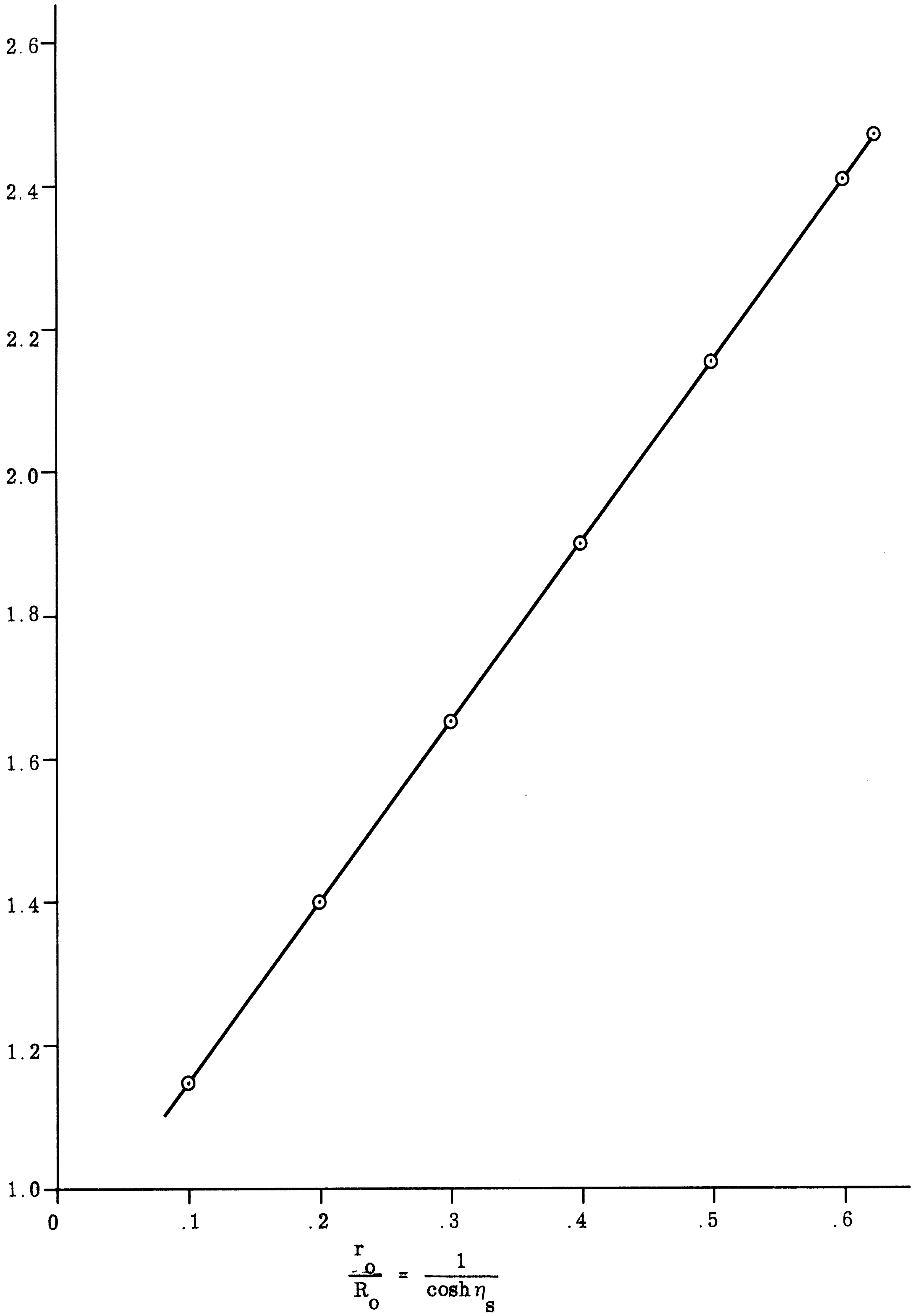

FIG. 4-1: CAPACITY OF A TORUS / CAPACITY OF A SPHERE

Matching boundary conditions

$$\sqrt{\cosh\eta_s - \cos\theta}\sum_{n=0}^{\infty}(A_m\cos m\psi + B_m\sin m\psi)(C_n\cos n\theta + D_n\sin n\theta)\, P^m_{n-1/2}(\cosh\eta_s)$$

$$= \frac{\pi K_T}{8} - \frac{4\sqrt{2}}{3}\frac{a}{\pi}\sqrt{\cosh\eta_s - \cos\theta}\ \mathrm{x}$$

$$\mathrm{x}\sum_{n=1}^{\infty} n P_{n-1/2}(\cosh\eta_s)\sinh\eta_s Q_{n-1/2}(\cosh\eta_s)Q'_{n-1/2}(\cosh\eta_s)\sin n\theta$$

$$+ \frac{2\sqrt{2}\,a}{\pi}\ \cosh\eta_s - \cos\theta\sum_{n=1}^{\infty} Q_{n-1/2}(\cosh\eta_s) n\sin n\theta\ . \qquad (4.44)$$

Solving for the coefficients A_m, B_m, C_n and D_n we get

$$A_m = 1 \quad \text{for } m = 0 \qquad B_m = 0 \text{ for } m = 0, 1, 2 \ldots$$
$$= 0 \quad\quad m \neq 0$$

$$C_n = \frac{K_T}{4\sqrt{2}}\,\frac{Q_{n-1/2}(\cosh\eta_s)}{P_{n-1/2}(\cosh\eta_s)} \qquad n = 0, 1, 2 \ldots$$

$$D_n = n\left\{\frac{2\sqrt{2}}{\pi}\,\frac{Q_{n-1/2}(\cosh\eta_s)}{P_{n-1/2}(\cosh\eta_s)} - \frac{4\sqrt{2}}{3\pi}\, a \sinh\eta_s Q_{n-1/2}(\cosh\eta_s)\, Q'_{n-1/2}(\cosh\eta_s)\right\}.$$

$$n = 0, 1, 2 \ldots$$

Thus the solution for $U_1(\bar{r})$ is given by

$$U_1(\bar{r}) = \sqrt{\cosh\eta - \cos\theta}\; \sum_{n=0}^{\infty} \left[\frac{K_T}{4\sqrt{2}} \frac{Q_{n-1/2}(\cosh\eta_s)}{P_{n-1/2}(\cosh\eta_s)} \cos n\theta \; + \right.$$

$$+ \left(\frac{n\,2\sqrt{2}}{\pi} \frac{Q_{n-1/2}(\cosh\eta_s)}{P_{n-1/2}(\cosh\eta_s)} - \frac{n4\sqrt{2}}{3\pi} a \sinh\eta_s Q'_{n-1/2}(\cosh\eta_s) Q_{n-1/2}(\cosh\eta_s) \right)$$

$$\left. \sin n\theta \right] P_{n-1/2}(\cosh\eta) \qquad (4.45)$$

$$\Phi_1(\bar{r}) = \sqrt{\cosh\eta - \cos\theta}\; \sum_{0}^{\infty} \left\{ \frac{K_T}{4\sqrt{2}} \cdot \frac{Q_{n-1/2}(\cosh\eta_s)}{P_{n-1/2}(\cosh\eta_s)} \cos n\theta \; + \right.$$

$$\left. + \frac{n\,2\sqrt{2}}{\pi} \frac{Q_{n-1/2}(\cosh\eta_s)}{P_{n-1/2}(\cosh\eta_s)} \sin n\theta \right\} P_{n-1/2}(\cosh\eta) - \frac{\pi K_T}{8} \qquad (4.46)$$

We can continue to obtain higher order terms this way, but the complexity in the functions involved not only make it formidable, but also it becomes more difficult to derive meaningful results from them.

4.3 FAR-ZONE FIELD

The expressions obtained for the scattered fields for the scattering of an acoustic wave by a hard and a soft torus are in a rather complicated form but can be readily utilized for computer calculations. Since it is beyond the scope of this work to do numerical computations we shall look at the analytical expressions and try to compare this with some known results say, for a sphere and for a disc. In order to do this we shall look at the far-zone field for a particular case, viz., the soft torus. The soft torus is chosen because the results in this case are relatively simpler to deal with.

Equation (3.25) can be directly used to obtain the far zone field i.e.,

$$\Phi(\bar{r}) \sim \frac{e^{ikr}}{4\pi r} \sum_{n=0}^{\infty} (ik)^n \sum_{m=0}^{n} \frac{(-1)^{n-m}}{(n-m)!} \int_B (\hat{r}\cdot \bar{r}_B)^{n-m} \left[\hat{n}\cdot \hat{r}\,\Phi_{m-1} - \frac{\partial \Phi_m}{\partial n} \right] dB$$

when $\Phi_{-1} = 0$ and all the quantities have the same meaning as in Chapter III. Examining the term $n = 0$

$$\Phi(\bar{r}) \sim \frac{e^{ikr}}{4\pi r} \int_B -\frac{\partial \Phi_o}{\partial n}\, dB \, .$$

From (4.39), we know

$$\Phi_o(\bar{r}) = -\frac{\sqrt{2(\cosh\eta - \cos\theta)}}{\pi} \sum_{n=0}^{\infty} \frac{Q_{n-1/2}(\cosh\eta_s)}{P_{n-1/2}(\cosh\eta_s)} P_{n-1/2}(\cosh\eta)\cos n\theta \, .$$

Therefore,

$$\Phi(\bar{r}) \sim \frac{e^{ikr}}{r} \frac{a}{\pi} \sum_0^{\infty} \epsilon_n \frac{Q_{n-1/2}(\cosh\eta_s)}{P_{n-1/2}(\cosh\eta_s)} = \frac{e^{ikr}}{r} \frac{K_T}{8\pi} \tag{4.47}$$

where

$$a^2 = R_o^2 - r_o^2 \, .$$

A plot of the quantity

$$K_T = \sum_0^{\infty} \epsilon_n \frac{Q_{n-1/2}(\cosh\eta_s)}{P_{n-1/2}(\cosh\eta_s)} \, .$$

against $\operatorname{sech}\eta_s$ is shown in Fig. 4-1.

This quantity is already shown to correspond to the capacity of a torus. So, the first terms in the far scattered field for a soft torus, in terms of the first terms in the near scattered field is always a measure of the capacity of the body under consideration. It is interesting to compare this term with that of a disc and a sphere (Senior, 1960).

$$\Phi_{disc}(\bar{r}) \sim \frac{e^{ikr}}{r} \frac{2}{\pi} a_d$$

$$\Phi_{sphere}(\bar{r}) \sim \frac{e^{ikr}}{r} a_s$$

where a_d and a_s are the radii of the disc and the sphere respectively.

Figure 4-2 enables us to estimate the equivalent sizes of the discs and spheres which would give size to the same far zone scattered field. Figure 4-2 is a plot of $\frac{\text{radius}}{R_o}$ against $\frac{r_o}{R_o}$. To illustrate this, let us take a disc whose radius is $.6 R_o$, thus the size of the torus which gives the same far scattered field has $\frac{r_o}{R_o} = .125$. Similarly for a sphere whose radius is $.6 R_o$, the size of the torus which gives the same return is $\frac{r_o}{R_o} = 0.5$. But for large values of $\frac{r_o}{R_o}$, the relative sizes do not make much difference, as is expected.

Unfortunately, no results are available for scattering of an acoustic wave by a torus. But it is expected that at a future time numerical computations done with these analytical expressions will give systematic information (for both the soft and the rigid torus) for scattering as a function of r_o and/or R_o.

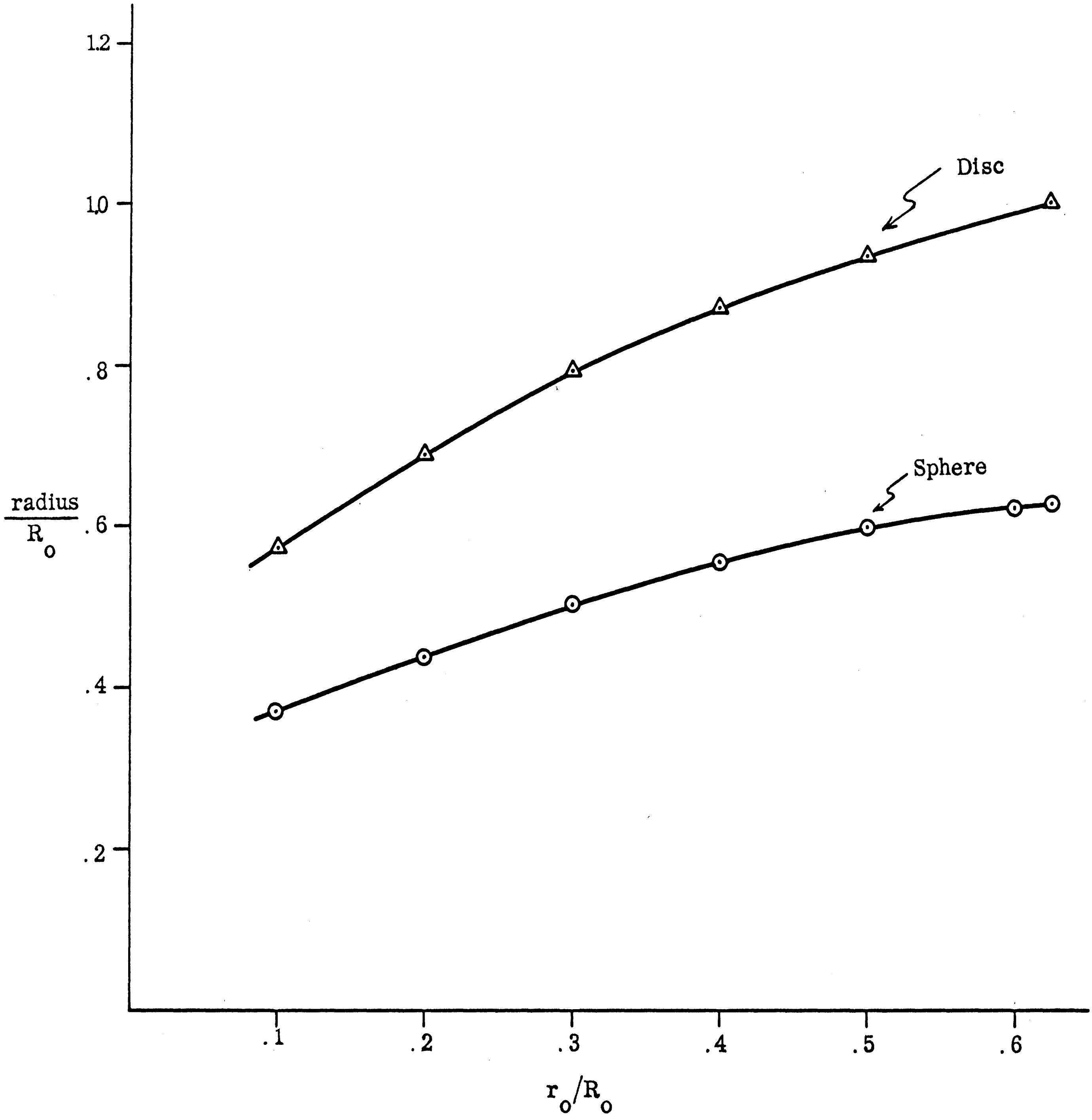

FIG. 4-2: EQUIVALENT RADII FOR SPHERE AND DISC IN TERMS OF THE RADIUS OF TORUS.

V

ELECTROMAGNETIC SCATTERING FROM A TORUS

Let us consider now the problem of scattering of a linearly polarized plane electromagnetic wave by a perfectly conducting torus. The incident wave is propagating down the z-axis, with the electric field polarized parallel to x-axis and the magnetic field parallel to the y-axis.

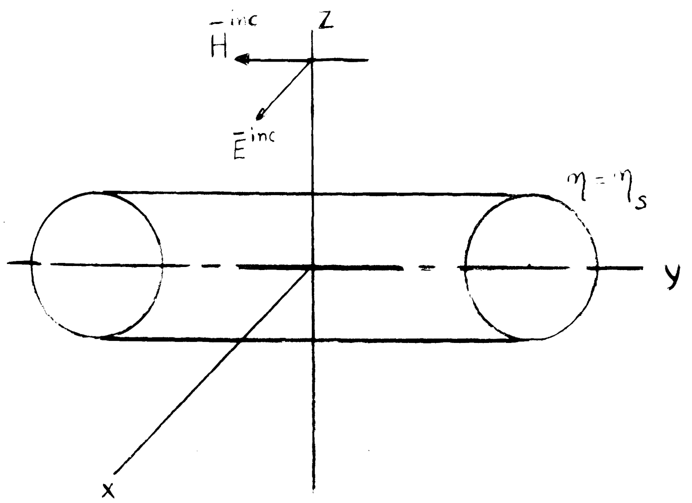

FIG. 5-1: ELECTROMAGNETIC WAVE INCIDENT ON A TORUS.

$$\bar{E}^{inc} = \hat{i}_x e^{-ikz} = \sum_{\ell=0}^{\infty} (ik)^{\ell} \bar{E}_{\ell}^{inc} \; ; \; \bar{E}_{\ell}^{inc} = \frac{(-z)^{\ell}}{\ell!} \hat{i}_x \tag{5.1}$$

$$\bar{H}^{inc} = -\hat{i}_y e^{-ikz} = \sum_{\ell=0}^{\infty} (ik)^{\ell} \bar{H}^{inc} \; ; \; H_{\ell}^{inc} = -\frac{(-z)^{\ell}}{\ell!} \hat{i}_y \tag{5.2}$$

We shall follow the method in Chapter III to derive the zeroeth and first order terms in the expansion for the scattered field in powers of k.

5.1 ZERO'TH ORDER TERMS

$$\bar{E}_o^{inc} = \hat{i}_x = \hat{i}_{\eta} \frac{1-\cos\theta\cosh\eta}{\cosh\eta-\cos\theta} \cos\psi - \hat{i}_{\theta} \frac{\sinh\eta\sin\theta}{\cosh\eta-\cos\theta} \cos\psi - \hat{i}_{\psi} \sin\psi \tag{5.3}$$

The scattered electric field to this order is

$$\bar{E}_o = \nabla V_o$$

where V_o is an external harmonic function to be determined under the conditions

$$\nabla^2 V_o = 0 \tag{5.4}$$

$$\hat{n} \times \nabla V_o \Big|_{\eta = \eta_s} = - \hat{n} \times \bar{E}_o^{inc} \Big|_{\eta = \eta_s} . \tag{5.5}$$

V_o regular at infinity and $\displaystyle\int \frac{\partial V_o}{\partial n} \, dS = 0$.

Solution of (5.4) is given by (2.6) and (5.5) can then be written as

$$-\hat{i}_\psi \Bigg[\frac{\sqrt{\cosh \eta_s - \cos\theta}}{2a} \sin\theta \sum \sum (A_m \cos m\psi + B_m \sin m\psi)(C_n \cos n\theta + D_n \sin n\theta) P_{n-1/2}(\cosh \eta_s)$$

$$+ \frac{\left[\cosh \eta_s - \cos\theta\right]^{3/2}}{a} \sum \sum (A_m \cos m\psi + B_m \sin m\psi)(-nC_n \sin n\theta + n D_n \cos n\theta) P_{n-1/2}(\cosh \eta_s) \Bigg]$$

$$+ \hat{i}_\theta \Bigg[\frac{(\cosh \eta_s - \cos\theta)^{3/2}}{a \sinh \eta_s} \sum \sum (-mA_m \sin m\psi + mB_m \cos m\psi)(C_n \cos n\theta + D_n \sin n\theta) P_{n-1/2}(\cosh \eta_s) \Bigg]$$

$$= -\hat{i}_\psi \frac{\sinh \eta_s \sin\theta}{\cosh \eta_s - \cos\theta} \cos\psi - \hat{i}_\theta \sin\psi \tag{5.6}$$

which gives

$$A_m = -1 \quad \text{for} \quad m = 1$$

$$= 0 \quad \text{for} \quad m \neq 1$$

$$C_n = \frac{a\,2\sqrt{2}}{\pi}\,\frac{Q'_{n-1/2}(\cosh\eta_s)}{P'_{n-1/2}(\cosh\eta_s)}$$

$$D_n = 0 \qquad \text{for all} \quad n$$

Thus

$$V_o = -\sqrt{\cosh\eta - \cos\theta}\sum_{n=0}^{\infty}\frac{a2\sqrt{2}}{\pi}\,\frac{Q'_{n-1/2}(\cosh\eta_s)}{P'_{n-1/2}(\cosh\eta_s)}\cos\psi\cos n\theta\, P^{(1)}_{n-1/2}(\cosh\eta) \tag{5.7}$$

and $E_o = \nabla V_o$ gives the first electric field term.

To determine the first term in the series for the scattered magnetic field, we write

$$\overline{H}_o = \nabla U_o \quad .$$

The excitation is given by $\overline{H}_o^{inc} = -\nabla U_o^{inc}$

$$-\nabla U_o^{inc} = -\hat{i}_y = \hat{i}_y\,\frac{1-\cosh\eta\cos\theta}{\cosh\eta - \cos\theta}\sin\psi - \hat{i}_\theta\,\frac{\sinh\eta\sin\theta}{\cosh\eta-\cos\theta}\,\sin\psi + \hat{i}_\psi\cos\psi \tag{5.8}$$

U_o satisfies $\nabla^2 U_o = 0$

$$\hat{n}\cdot\nabla U_o\Big|_{\eta=\eta_s} = -\hat{n}\cdot\overline{H}_o^{inc}\Big|_{\eta=\eta_s}$$

and the radiation condition.

From the boundary conditions we obtain

$$-\frac{\sinh\eta_s}{2a}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(A_m\cos m\psi + B_m\sin m\psi)(C_n\cos n\theta + D_n\sin n\theta)\,P^m_{n-1/2}(\cosh\eta_s)$$

$$+\frac{(\cosh\eta_s - \cos\theta)}{a}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(A_m\cos m\psi + B_m\sin m\psi)(C_n\cos n\theta + D_n\sin n\theta)\;P^{m'}_{n-1/2}(\cosh\eta)$$

$$= \frac{1-\cosh\eta_s\cos\theta}{\cosh\eta_s - \cos\theta}\,\sin\psi \quad . \tag{5.9}$$

Expanding the right hand side of (5.9) in a series over toroidal functions, we solve for the coefficients,

$$A_m = 0 \quad \text{for all } m$$

$$B_m = \begin{matrix} 1 & \text{for } m = 1 \\ 0 & \text{for } m \neq 0 \end{matrix}$$

$$D_n = 0 \quad \text{for all } n$$

and C_n is given by

$$C_n \left\{ \sinh\eta_s P^{(1)}_{n-1/2}(\cosh\eta_s) + 2\cosh\eta_s P^{(1)'}_{n-1/2}(\cosh\eta_s) \right\} - C_{n+1} P^{(1)'}_{n+1/2}(\cosh\eta_s)$$

$$- C_{n-1} P^{(1)'}_{n-3/2}(\cosh\eta_s) = \frac{2\sqrt{2}a}{\pi} \left\{ \sinh\eta_s Q^{(1)}_{n-1/2}(\cosh\eta_s) + 2\cosh\eta_s Q^{(1)'}_{n-1/2}(\cosh\eta_s) \right\} \tag{5.10}$$

$$- \frac{2\sqrt{2}}{\pi} Q^{(1)'}_{n+1/2} - \frac{2\sqrt{2}a}{\pi} Q^{(1)'}_{n-3/2} \qquad n = 2, 3, 4. \ . \ .$$

with the initial equations

$$C_o \left[\sinh\eta_s P^{(1)}_{-1/2} + 2\cosh\eta_s P^{(1)'}_{-1/2} \right] - C_1 P^{(1)'}_{1/2} =$$

$$= \frac{2\sqrt{2}a}{\pi} \left\{ \left[\sinh\eta_s Q^{(1)}_{-1/2} + 2\cosh\eta_s Q^{(1)'}_{-1/2} \right] - Q^{(1)'}_{1/2} \right\} \tag{5.11}$$

and

$$C_1 \left[\sinh\eta_s P^{(1)}_{1/2} + 2\cosh\eta_s P^{(1)'}_{1/2} \right] - 2C_o P^{(1)'}_{-1/2} - C_2 P^{(1)'}_{3/2} =$$

$$= \frac{2\sqrt{2}a}{\pi} \left[(\sinh\eta_s Q^{(1)}_{1/2} + 2\cosh\eta_s Q^{(1)'}_{1/2}) + 2Q^{(1)'}_{-1/2} - Q^{(1)'}_{3/2} \right]. \tag{5.12}$$

Now C_n is determined up to C_o. To determine C_o, we proceed in a way analogous to the fluid flow problem in section 2.4, since the flow of a fluid around a rigid torus is analogous to the magnetostatic problem for a perfectly conducting body.

By virtue of this, the magnetic field at the points A, B, C, D, shown in Fig. 2-3, must be zero, which gives rise to a condition on the C's viz.,

$$\sum_0^\infty C_n P_{n-1/2}(\cosh\eta_s) = \sum_0^\infty \frac{2\sqrt{2}\,a}{\pi} Q_{n-1/2}(\cosh\eta_s) \quad . \tag{5.13}$$

Thus C_o can be determined from Eq. (5.13) in addition to Eqs. (5.10) through (5.12). Therefore, U_o and hence H_o is completely known.

5.2 FIRST ORDER TERMS

The next higher order terms in k can be solved for by using the equations

$$\nabla \times \overline{E}_p = \overline{H}_{p-1} \quad \text{and} \quad \nabla \times \overline{H}_p = -\overline{E}_{p-1} \quad p > 0 \tag{5.14}$$

thus $\nabla \times \overline{E}_1 = \overline{H}_o$ and $\nabla \times \overline{H}_1 = -\overline{E}_o$ and H_o and E_o are both gradients of scalar functions and therefore Eq. (3.43) can be utilized in solving for E_1 and H_1 directly.

$$\nabla \times \overline{E}_1 = \overline{H}_o = \nabla \left[\sqrt{\cosh\eta - \cos\theta}\sum_{n=0}^{\infty} C_n \sin\psi \cos n\theta \, P^{(1)}_{n-1/2}(\cosh\eta)\right]$$

where C_n is given by (5.10) through (5.13).

Using

$$g_\ell = -\frac{1}{4\pi} \nabla \times \int_B \frac{(U_\ell^e - U_\ell^i)\,\hat{n}}{R} \, dB$$

where U_ℓ^e and $U_\ell^{(i)}$ refer to the exterior and interior potential functions the same as in Section (3.3).

Omitting all the cumbersome details. we obtain the quantities E_1 and H_1 given by

$$\bar{E}_1 = \hat{i}_\psi \sqrt{\cosh\eta - \cos\theta} \sum_{n=0}^{\infty} \left[C_n P^{(1)}_{n-1/2}(\cosh\eta_s) - \frac{2\sqrt{2}a}{\pi} Q^{(1)}_{n-1/2}(\cosh\eta_s) \right]$$

$$\epsilon_n \frac{\Gamma(n-1/2)}{\Gamma(n+3/2)} Q^{(1)}_{n-1/2}(\cosh\eta_s)\pi \Bigg\{ n\sinh\eta_s +$$

$$+ \frac{1}{4} \left[(\sinh\eta_s - \cosh\eta_s)^{|1-n|} - (\sinh\eta_s - \cosh\eta_s)^{1+n} \right] \Bigg\} \sin\psi \sin n\theta P^{(1)}_{n-1/2}(\cosh\eta)$$

$$+ \hat{i}_\theta \sqrt{\cosh\eta - \cos\theta}\ \pi \sum_{n=0}^{\infty} \left[C_n\ P^{(1)}_{p-1/2}(\cosh\eta_s) - \frac{2\sqrt{2}}{\pi} a\, Q^{(1)}_{p-1/2}(\cosh\eta_s) \right]$$

$$\epsilon_n \frac{\Gamma(n-1/2)}{\Gamma(n+3/2)} P^{(1)}_{n-1/2}(\cosh\eta) Q^{(1)}_{n-1/2}(\cosh\eta_s) \cos\psi\cos n\theta \tag{5.15}$$

and similarly

$$\nabla \mathbf{x}\, H_1 = -E_0 = +\nabla \left[\sqrt{\cosh\eta - \cos\theta} \sum_{0}^{\infty} \frac{2\sqrt{2}a}{\pi} \frac{Q'_{n-1/2}(\cosh\eta_s)}{P'_{n-1/2}(\cosh\eta_s)} \cos\psi \cos n\theta\, P^{(1)}_{n-1/2}(\cosh\eta) \right]$$

and using (3.43)

$$\bar{H}_1 = \hat{i}_\psi \sqrt{\cosh\eta - \cos\theta} \sum_{n=0}^{\infty} \left[\frac{2\sqrt{2}a}{\pi} + A_n \right] Q'_{n-1/2}(\cosh\eta_s)$$

$$\epsilon_n \frac{\Gamma(n-1/2)}{\Gamma(n+3/2)} Q^{(1)}_{n-1/2}(\cosh\eta_s)\ \pi \Bigg\{ -n\sinh\eta_s$$

$$+ \frac{1}{4} \left[(\sinh\eta_s - \cosh\eta_s)^{|1-n|} - (\sinh\eta_s - \cosh\eta_s)^{1+n} \right] \Bigg\}$$

$$\cos\psi\cos n\theta\, P^{(1)}_{n-1/2}(\cosh\eta)$$

$$+ \hat{i}_\theta \sqrt{\cosh\eta - \cos\theta}\ \pi \sum_{n=0}^{\infty} \left[\frac{2\sqrt{2}}{\pi} + A_n \right] Q'_{n-1/2}(\cosh\eta_s) \tag{5.16}$$

$$\epsilon_n \frac{\Gamma(n-1/2)}{\Gamma(n+3/2)} \frac{Q'_{n-1/2}(\cosh\eta_s)}{P_{n-1/2}(\cosh\eta_s)} \sin\psi \sin n\theta\, P^{(1)}_{n-1/2}(\cosh\eta)$$

where A_n is given by:

$$A_n\left[2\cosh\eta_s Q^{(1)}_{n-1/2} + \sinh\eta_s Q^{(1)}_{n-1/2}\right] - A_{n+1}Q^{(1)'}_{n+1/2} - A_{n+1}Q^{(1)'}_{n+1/2}$$

$$= \frac{a 2\sqrt{2}}{\pi}\left[2\cosh\eta_s \frac{Q'_{n-1/2}}{P'_{n-1/2}} P^{(1)'}_{n-1/2} + \sinh\eta_s \frac{Q'_{n-1/2}}{P'_{n-1/2}} P^{(1)}_{n-1/2}\right]$$

$$- \frac{a 2\sqrt{2}}{\pi}\frac{Q'_{n+1/2}}{P'_{n+1/2}} P^{(1)'}_{n-1/2}(\cosh\eta_s) - \frac{a 2\sqrt{2}}{\pi}\frac{Q'_{n-3/2}}{P'_{n-3/2}} P^{(1)'}_{n-3/2}(\cosh\eta_s) \, .$$

This can be solved, as usual, up to A_0, and A_0 remains to be determined. In order to obtain this, the static Neumann Green's function must be obtained complete with all the coefficients. (5.15) and (5.16) represent general solutions of $\bar{E}_1$ and $\bar{H}_1$ that satsify Eq.(5.14). Now any arbitrary gradient of a scalar (say ∇S_1 and ∇S_2) can be added to (5.15) and (5.16) and still have them satisfy (5.14). These arbitrary functions then should be fixed by the boundary values, which $\bar{E}_1$ and $\bar{H}_1$ should satisfy viz.,

$$\hat{n} \times (\bar{E}_1 + \bar{E}_1^{inc})\Big|_{\eta=\eta_s} = 0 \tag{5.17}$$

$$\hat{n} \times (\bar{H}_1 + \bar{H}^{inc})\Big|_{\eta=\eta_s} = 0 \tag{5.18}$$

$$E_1^{inc} = -z\hat{i}_x = -\hat{i}_\eta \frac{(1-\cos\theta\cosh\eta)\sin\theta}{(\cosh\eta-\cos\theta)^2} + \hat{i}_\theta \frac{a\sinh^2\eta\sin^2\theta}{(\cosh\eta-\cos\theta)^2}\cos\psi + \hat{i}_\psi \frac{a\sin\psi\sin\theta}{(\cosh\eta-\cos\theta)} \tag{5.19}$$

$$H_1^{inc} = z\hat{i}_y = -\hat{i}_\eta \frac{a\sin\theta(1-\cos\theta\cosh\eta)}{(\cosh\eta-\cos\theta)^2} + \hat{i}_\theta \frac{a\sinh\eta\sin^2\theta}{(\cosh\eta-\cos\theta)^2} - \hat{i}_\psi \frac{a\sin\theta\cos\psi}{(\cosh\eta-\cos\theta} \tag{5.20}$$

If we let

$$\alpha_n = \pi\left[C_n P^{(1)}_{n-1/2}(\cosh\eta_s) - \frac{2\sqrt{2}a}{\pi} Q^{(1)}_{n-1/2}(\cosh\eta_s)\right] \epsilon \frac{(n-1/2)}{(n+3/2)} Q^{(1)}_{n-1/2}(\cosh\eta_s) \tag{5.21}$$

$$\beta_n = \left[n \sinh \eta_s + \frac{1}{4} (\sinh \eta_s - \cosh \eta_s)^{(1-n)} - \frac{1}{4} (\sinh \eta_s - \cosh \eta_s)^{1+n} \right] \tag{5.22}$$

$$E_1 = \hat{i}_\psi \sum_{n=0}^{\infty} \alpha_n \left[\beta_n\right] \sin\psi \sin n\theta \, P^{(1)}_{n-1/2} (\cosh \eta)$$

$$+ \hat{i}_\theta \sum_{n=0}^{\infty} \alpha_n \cos\psi \cos n\theta \, P^{(1)}_{n-1/2} (\cosh \eta_s) + \hat{i}_\eta \nabla_\eta S_1 + \hat{i}_\theta \nabla_\theta S_1 + \hat{i}_\psi \nabla_\psi S_1 \quad . \tag{5.23}$$

where

$$\nabla_1 = \hat{i}_\eta \nabla_\eta S_1 + \hat{i}_\theta \nabla_\theta S_1 + \hat{i}_\psi \nabla_\psi S_1$$

is the arbitrary gradient of a scalar that is added to E_1.

Hence

$$\hat{n} \times (\bar{E}_1 + \bar{E}_1^{inc}) \Big|_{\eta = \eta_s} = 0 \quad .$$

$$\nabla_\eta S_1 = 0$$

$$\nabla_\psi S_1 = \frac{a \sinh \eta_s \sin\theta \cos\psi}{(\cosh \eta_s - \cos\theta)^2} - \sqrt{\cosh \eta_s - \cos\theta} \sum_{0}^{\infty} \alpha_n \cos\psi \cos n\theta \, P^{(1)}_{n-1/2} (\cosh \eta_s)$$

$$\nabla_\theta S_1 = - \frac{a \sinh \eta_s \sin\psi \sin\theta}{(\cosh \eta_s - \cos\theta)} - \sqrt{\cosh \eta_s - \cos\theta} \sum_{0}^{\infty} \alpha_n (\beta_n) \sin\psi \sin n\theta \, P^{(1)}_{n-1/2} (\cosh \eta_s) \quad . \tag{5.24}$$

Similarly for $\bar{H}_1$ we can add ∇S_2, and solve for it using

$$\hat{n} \cdot (\bar{H} + \bar{H}^{inc}) \Big|_{\eta = \eta_s} = 0$$

we get

$$\nabla_\eta S_2 = - \frac{a \sin\theta (1 - \cosh\eta \cos\theta)}{(\cosh\eta - \cos\theta)^2} \quad , \tag{5.25}$$

$$\nabla_\theta S_2 = 0$$

$$\nabla_\psi S_2 = 0$$

This technique of obtaining higher order terms in a low frequency expansion can be continued, but at this point it is considered to be not so important to get the higher order terms, but rather to derive some meaning from these complicated expressions.

The only result known for the scatterin of an electromagnetic wave by a perfectly conducting torus is that due to Weston (1956). He has derived the far-zone scattered field for a very thin ring. In order to compare our result with his, we first have to derive the far-zone field for our case. We use a form for this field, given by Kleinman (1966):

$$\bar{E}^{scat} \sim \frac{e^{ikr}}{4\pi r} \omega^2 \hat{r} \times \sum_{n=0}^{\infty} (i\omega)^n \sum_{m=0}^{n} \frac{(\sqrt{\epsilon_o \mu_o})^{n-m}}{(n-m)!} \int_B (-\hat{r} \cdot \bar{r}_B)^{n-m}$$

$$-\hat{r} \times \bar{r}_B \mu_o \sqrt{\epsilon_o \mu_o}\, \hat{r} \cdot \hat{n} \times H_m^B + \hat{r} \times \bar{r}_B \epsilon_o \mu_o \hat{n} \cdot \bar{E}_m^B + \epsilon_o \mu_o \hat{r} \cdot \hat{n} \times E_m^B$$

$$+ \mu_o \sqrt{\epsilon_o \mu_o}\, \bar{r}_B \hat{n} \cdot H_m^B \quad dB \quad . \tag{5.26}$$

This expression is a power series expansion for the far field in powers of ω(i.e., k) in terms of the near field terms. This also gives n far field terms when n near field terms are known. If we are interested only in the first term (i.e., $n = 0$)

$$\bar{E}^{scat} \sim \frac{e^{ikr}}{4\pi r} \omega^2 \hat{r} \times \int_B \Big\{ -\hat{r} \times \bar{r}_B \mu_o \sqrt{\epsilon_o \mu_o}\, \hat{r} \cdot \hat{n} \times H_o^B + \hat{r} \times \bar{r}_B \epsilon_o \mu_o \hat{n} \cdot E_o^B +$$

$$+ \epsilon_o \mu_o \hat{r} \cdot \hat{n} \times E_o^B + \mu_o \sqrt{\epsilon_o \mu_o}\, \bar{r}_B \hat{n} \cdot H_o^B \Big\} \, dB \tag{5.27}$$

Since (5.27) involves $\bar{E}_o$ and $\bar{H}_o$ and in as one $\bar{E}_o$ is known explicitly, but H_o is known only implicitly, we cannot yet obtain a suitable far-zone scattered field to compare with Weston. It is expected that some numerical work will be done to be able to compare with experimental results for a ring.

The low frequency expansion terms is carried out here for the case of the electric field polarized in the plane of incidence, but the other polarization can be used without any difficulty. The scattering cross-section is also a quantity of interest which can be obtained from the far field. The scattering cross-section is defined by

$$\sigma = \lim_{r \to \infty} 4\pi r^2 \frac{W_s}{W_o} .$$

when W_s is the scattered power density and W_o, the incident power density at the scatterer.

VI

SUMMARY AND RECOMMENDATIONS FOR FUTURE WORK

The main results of this work contain the derivation of the solution of the potential equation for a torus placed in a uniform field in directions parallel to and perpendicular to the axis of the torus. From this one can solve the corresponding problem for a torus placed in a uniform field in an arbitrary direction. Further these results have been used to construct the solutions of scattering of acoustic and electromagnetic waves using the techniques proposed by Kleinman (1965) and Stevenson (1953). The solutions are in a power series, in ascending powers of k, the wave number. This series is often called the Rayleigh series and is valid for small k or low frequencies.

Two non-zero terms are derived in the scattered field expansion for the acoustic problem for rigid and soft toroids and for the electromagnetic problem for a perfectly conducting torus. The process could be carried to higher order, but the complexity in the forms of the terms increase progressively thus making it more difficult to derive meaningful results from them. The far field is calculated for the simpler case of the acoustic problem i.e., for an acoustic wave incident normally on a soft torus, and compared with the known results for the corresponding problem of a disc and a sphere. The equivalent radii for the latter are plotted in terms of the radius of the torus, to obtain the same scattered field.

No attempt has been made to do any numerical computation, but the formulae can readily be used to get results from an electronic digital computer. Also, no attempt has been made to estimate the radius of convergence of these solutions.*

The exterior Neumann problem for a point source excitation can only be solved yet up to a set of constants, which still need to be determined. The

*The series for the scattered field converges for $|k|$ sufficiently small, that is, there exists some number $|k_0| > 0$, such that the series converges for $|k| < |k_0|$. $|k_0|$ in turn determines the radius of convergence. This has been estimated only for very special surfaces so far (e.g., Senior and Darling, 1965).

procedure of determining the scattered fields will apply directly to all R - separable bodies (for definition see Chapter I), for e.g., flat ring, washer, special bowl, ogive etc. In contrast to a torus which is a smooth body, these bodies have edges, and it is expected that this might help in solving for the unknown set of constants, because in addition to boundary and radiation conditions the scattered fields should also satisfy edge condition.

Once the Green's function of the second kind is completely explicitly found, one could use a method to find the scattered fields by an iteration technique due to Kleinman; however, it is hard to predict whether the method will render itself convenient at this stage, since when applied to ogive (Ar, 1966) the results could only be found in terms of integrals which could not be evaluated easily.

Torus is typical of a class of R-separable bodies and any work done on this will certainly lead a way to solve the problems involving other R-separable bodies.

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APPENDIX
A

Sommerfeld, in 1912, was the first to discuss the conditions , viz.,

$$\left| R\, \Phi \right| < k \qquad R \longrightarrow \infty \tag{A.1}$$

$$\left(R\, \frac{\partial u}{\partial R} - ik\, \Phi \right) \longrightarrow 0 \qquad R \rightarrow \infty \quad . \tag{A.2}$$

Equations (A. 1) is called the "condition of finiteness" (Endlich keitsbedingung) while (A. 2) is the important " radiation condition" (Ausstrahlungsbedingung). From the mathematical point of view, these equations are important because this enables one to find a unique solution to the Helmholtz equation i.e., a solution of a Helmholtz equation which itself or whose normal derivative takes on prescribed value on a surface and which satisfies (A. 1) and (A. 2), is necessarily the only solution. These conditions specify the behavior at infinity of the wave function and in particular the condition of finiteness states that $\Phi \rightarrow 0$ as $R \rightarrow \infty$ and the radiation condition implies that Φ must behave like outgoing spherical wave at infinity and the sources written cannot give rise to incoming waves at infinity. All physical fields generated by finite source distributions must satisfy these conditions.

APPENDIX
B
HELMHOLTZ FORMULA

Helmholtz formula expresses a scalar solution of the reduced wave equation

$$(\nabla^2 + k^2)u = 0 \tag{B.1}$$

in terms of the boundary values of u or $\frac{\partial u}{\partial n}$. In order to derive this formula, let us suppose u is a solution of (B.1) in a domain V bounded by a closed surface S, and internally by S_1, and u together with its first and second derivatives is continuous in V. Let v be another function defined throughout in V with the same continuity properties as u. Now we can use Green's identity, which states

$$\int_V (u\nabla^2 u - u\nabla^2 v)\, dv = \int_{S+S_1} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS \tag{B.2}$$

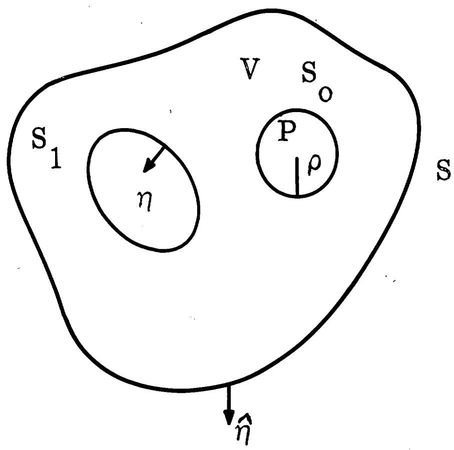

FIG. B-1: REGION OF APPLICATION OF GREEN'S THEOREM

$\frac{\partial}{\partial n}$ denotes a differentiation along an normal drawn outward from V.

Let v satisfy (B.1) except at a point P_o within V, where it is singular. If we choose v, in particular, to be spherically symmetric, given by

$$v = -\frac{1}{4\pi}\frac{e^{ikR(P,P_o)}}{R(P,P_o)} \tag{B.3}$$

where P and P_o are both inside V, then v corresponds to spherical waves emanating from P_o since (B.2) is not valid unless P_o is excluded from V, and let us define the new volume by V' which represents the volume bounded externally by S and internally by S_1 and S_o, which is the surface of a small sphere bounding P.

The Green's identity becomes

$$\int_{S+S_o+S_1} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\right) dS = 0 \quad . \tag{B.4}$$

S_o is a sphere of radius ρ and $\frac{\partial}{\partial n} = -\frac{\partial}{\partial \rho}$. Also $dS_o = \rho^2 d\,\Omega$ where $d\Omega$ is the solid angle subtended by dS_o at P_o.

Also $\frac{\partial v}{\partial n} = \frac{e^{ik\rho}}{4\pi}(ik - \rho^{-1})/\rho$ and is of the order $\frac{1}{\rho^2}$ as $\rho \longrightarrow 0$. But dS_o is of the order ρ^2 as $\rho \rightarrow 0$ and the integral

$$\int_{S_o} u \frac{\partial v}{\partial n} dS \quad ,$$

is independent or ρ. Hence ρ can be made very small i.e., we can choose $\rho \longrightarrow 0$. But

$$\int_{S_o} v \frac{\partial u}{\partial n} dS_o = 0(\rho)$$

as $\rho \longrightarrow 0$ and constitutes nothing. However,

$$\lim_{\rho \rightarrow 0} \int_{S_o} u \frac{\partial v}{\partial n} d\rho_o = -\frac{u(P_o)}{4\pi} \int d\Omega = -u(P_o) \tag{B.5}$$

and so Eq. (B.4) gives

$$u(P_o) = \int_S \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\right) dS \tag{B.6}$$

We can drop the subscripts on P_0 and we get

$$U(P) = \int_{S+S_1} \left[-u(P_s) \frac{e^{ikR(P,P_s)}}{4\pi R(P,P_s)} + \frac{e^{ikR(P,P_s)}}{4\pi R(P,P_s)} \frac{\partial u(P_s)}{\partial n} \right] dS$$

where P_s is a point on the bounding surface. Thus Eq. (B.6) express u at P in V in terms of u and $\frac{\partial u}{\partial n}$ on the bounding surfaces of v. A similar formula could be easily derived if there are source distributions within a small volume. Equation (B.6) is termed the Helmholtz formula.

In many problems, we are interested in the field outside the surface S_1 which contain the sources. Then there are no sources outside S and we expect the surface integral over S to vanish. In order to verify this let us use the Helmholtz formula on two functions u and v with no singularities outside S, satisfying the wave equation σ the same continuity conditions as Sefore.

If we suppose S is a large sphere of radius R, centre P and has S_1 interior to S.

If

$$v = \frac{e^{ikR}}{4\pi R}$$

on S, we get

$$u(P) = \int_{S_1} \left[v(P,P_s) \frac{\partial}{\partial n} u(P_s) - u(P_s) \frac{\partial}{\partial n} v(P,P_s) \right] dS$$

$$+ \int_S \left[u(P_s) \frac{\partial}{\partial R} v(P,P_s) - v(P,P_s) \frac{\partial}{\partial R} u(P_s) \right] dS$$

then the integral over S takes the form

$$\int_0^{\pi} d\theta_s \int_0^{2\pi} d\phi_s \sin\theta_s \left\{ R_s u(P_s) R_s \left[\frac{\partial v}{\partial R_s} - ikv \right] - R_s v R_s \left[\frac{\partial u}{\partial R_s} - iku \right] \right\}.$$

In order to have this integral vanish, we have to impose restrictions dictated by physical considerations and hence u must process the property

$$\lim_{R \to \infty} |rR| \quad \text{bounded}$$

$$\lim_{R \to \infty} R \left[\frac{\partial u}{\partial R} - iku \right] \to 0$$

in order to be a physically meaningful solution if there are no sources at infinity.

(That these conditions are 'stronger' than necessary has been shown by some authors who have termed the first condition superfluous).

Thus,

$$u(P) = \int_{S_1} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS$$

P outside S_1 .

APPENDIX C

Some useful relations involving the toroidal harmonics:

$$P^{-m}_{n-1/2}(\cosh\eta) = \frac{2^{-m}}{\Gamma(m+1)}(1-e^{-2\eta})^m e^{-(n+1/2)\eta} \times$$

$$\times {}_2F_1(\tfrac{1}{2}+m+n, \tfrac{1}{2}+m; 2m+1; 1-e^{-2\eta})$$

$$Q^{m}_{n-1/2}(\cosh\eta) = \frac{(-1)^m 2^m \Gamma(n+m+1/2)\sqrt{\pi}}{\Gamma(n+1)}(\sinh\eta)^m e^{-(n+m+1/2)\eta} \times$$

$$\times {}_2F_1(\tfrac{1}{2}+m, n+m+\tfrac{1}{2}; n+1; e^{-2\eta})$$

$$P^{m}_{n-1/2}(z) = (-1)^m(z^2-1)^{m/2}\frac{d^m}{dz^m}P_{n-1/2}(z)$$

$$Q^{m}_{n-1/2}(z) = (z^2-1)^{m/2}\frac{d^m}{dz^m}Q_{n-1/2}(z)$$

$$P^{\mu}_{-\nu-1}(z) = P^{\mu}_{\nu}(z)$$

$$P^{-\mu}_{\nu}(z) = \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu-1)}\left[P^{\mu}_{\nu}(z) - \frac{2}{\pi}e^{-\mu\pi i}\sin\mu\pi i\, Q^{m}_{\nu}(z)\right]$$

$$W(P,Q) = P^{m}_{n-1/2}\frac{d}{d\eta}(\cosh\eta) - Q^{m}_{n-1/2}\frac{d}{d\eta}P^{m}_{n-1/2}(\cosh\eta) =$$

$$= -\frac{(-1)^m\Gamma(n+m+1/2)}{\Gamma(n-m+1/2)\sinh\eta}.$$