ELECTROMAGNETIC SCATTERING BY MOVING BODIES

by

Robert Charles Restrick III

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The purpose of this work is to investigate the nature of electromagnetic scattering by moving objects by means of the Theory of Relativity. The object of such a study is to learn what may be anticipated in the case of relativistic velocities and to justify the low velocity approximations now being used.

Since the general problem allowing the scattering body to possess an arbitrary shape, motion, and material composition is highly difficult, this work considers two specific problems both involving perfect conductors scattering electromagnetic plane waves.

The first problem considered is that of a uniformly moving sphere. The incident plane wave is allowed to have arbitrary polarization and direction of propagation relative to the motion of the sphere. Because of the choice of a uniform motion and of a spherical geometry the problem can be transformed to an inertial system moving with the sphere where the scattered field can be found. In this moving system the scattered field is obtained by application of Mie's Theory. A straightforward transformation of this result to the inertial system in which the problem was stated would give a correct but extremely involved solution. It is found that a simpler and more meaningful solution is obtained by using the far-zone scattered field and expressing the solution in a "retarded" coordinate system whose coordinate surfaces turn out to be characteristics of the hyperbolic differential equation.

The resulting solution is seen to possess such expected behavior as a Doppler effect and an aberration of light effect. It is used to calculate the scattering cross section and the energy exchange processes pertaining to this problem. The solution may also be directly used to determine the far-zone

scattered field for other moving finite bodies provided only that their stationary far-zone solutions be known.

The second problem considered is that of an infinite conducting sheet moving with an accelerated motion, called hyperbolic motion, which is the relativistically correct motion of a body being acted upon by a constant force. The problem is attacked by two methods.

In the first, a transformation of the space-time coordinates, which in the case of accelerated motion is nonlinear, is made. This is accomplished by using the invariant formalism described by Post. The transformed differential equation is found to be separable in terms of exponential and modified Bessel functions. The incident field is transformed to the new space-time coordinate system and expanded in terms of an infinite series of the above functions. By means of the boundary condition and an argument based on causality the scattered field is determined in the form of an infinite series.

From another point of view a second solution analogous to the ray-optics solution pertaining to stationary scattering problems is developed and applied to the hyperbolically moving sheet. In the case where the incident wave is normally incident upon the accelerating sheet the series solution and the ray-optics type of solution are found to be identical.

For the case of oblique incidence the series solution obtained by the first method is written as a contour integral following the work of Watson. The contour integral is regarded as the sum of two terms. One of these terms for the case of small acceleration and wavelength is approximated by the stationary phase technique resulting in an expression equivalent to the solution obtained by the second method. Little meaning is given to the second term. It is suggested that this term results from the fact that at one time the motion of the sheet was such that its velocity equaled the velocity of light.

To Wanda, Bobby, and the author's
Mother and Father

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Chapter One

INTRODUCTION

Upon the revelation of the Special Theory of Relativity by Einstein and the further enhancement of it by Minkowski, science was given a means of introducing motion into the realm of classical electromagnetics. For not only did the theory give a new concept of the physical world, it also gave a convenient analytic technique for solving electrodynamic problems involving motion.

For example, Einstein's concept of the invariance of the free space Maxwell equations for an arbitrary inertial system allows an electromagnetics problem involving a uniformly moving body to be regarded from the viewpoint of an observer moving with the body to which it appears that of an electromagnetics problem involving a stationary body. For a material medium Minkowski wrote the equations of electromagnetics in terms of four indefinite equations that remain invariant for arbitrary intertial systems plus a set of constitutive equations that describe the material medium. The situation of a body moving with uniform motion through a material medium described by a set of constitutive relations can again be reduced to that of a stationary body, in this case in a material medium described by new constitutive relations. This powerful method has several important applications. One of the most famous of these is the treatment of charges moving through a dielectric predicting Čerenkov radiation (Nag and Sayied, 1956; Tai, 1965; Lee and Papas, 1964).

It was recognized by Weyl (1951) and Cartan (1924) that this important concept of invariance exists even in the case of accelerated motion. They were able to write the equations of electromagnetics in the form of a set of indefinite equations that remain invariant for a quite general set of non-linear space-time transformations and a set of constitutive relations which contain the effect of the motion (Post, 1962). There is a fundamental difference between this formalism and the Maxwell-Minkowski theory concerning uniform motion. The Maxwell-Minkowski theory is based on the Theory of Relativity while for arbitrary acceleration there exists no equivalent theory. However, this concept of invariance has been successfully applied to rotating electromagnetic cavities and in particular the Sagnac ring interferometer (Yildiz and Tang, 1966).

This present work is concerned with the scattering of an electromagnetic wave by a perfectly conducting body undergoing motion. Clearly a solution to the general problem can not be expected since even in the case of electromagnetic scattering by stationary conductors no general solution is known. Two specific problems are considered here as a means of obtaining more knowledge about the complex problem. The first problem is that of a perfectly conducting sphere moving with uniform motion scattering an incident electromagnetic plane wave. The second is that of a perfectly conducting infinite sheet moving with hyperbolic motion (Møller, 1952) also scattering an incident plane wave. The first was chosen since the sphere problem is one of the few problems possessing an exact solution in the stationary case. The motion in the second problem was chosen because it possesses a space-time symmetry quite compatible with the hyperbolic partial differential equation.

The earliest application of the Theory of Relativity to a problem involving scattering by a moving conductor was to that of a plane wave incident upon a moving mirror. This problem has been discussed by Sommerfeld (1964b) and by Pauli (1958). Problems involving more complicated geometry do not seem to have been treated until recently when the case of the uniformly moving cylinder was attacked by Lee and Mittra (1967) and that of an expanding sphere was treated by Lam (1967). Lewis and Pressman (1966) in discussing radiation by moving sources have suggested a coordinate system equivalent to the one used here in the sphere problem that is well suited to problems such as these.

The justification for a study such as this is twofold. First from an academic viewpoint to obtain more knowledge about relativity and electromagnetics, secondly from a practical viewpoint to justify the low velocity approximations now being used and to find their domain of validity.

In chapter two of this work the invariant formalism of the laws of electrodynamics is presented and the special case of uniform motion is discussed. A far field solution to the moving conducting sphere problem is obtained in chapter three. This solution is used in chapter four to calculate the energy exchange processes that occur. In chapter five an expression for the electromagnetic wave scattered by the hyperbolically moving sheet is found. The meaning of this solution is considered from several viewpoints in chapter six. The main body of this work is ended in chapter seven by a discussion of the results of the problems and some suggestions for future research to clarify some questions that arise.

Chapter Two

THE EQUATIONS OF ELECTRODYNAMICS IN GENERALIZED SPACE-TIME COORDINATES

2.1 Introduction

The problem of the electromagnetic scattering of an incident wave by a body undergoing motion consists of finding a scattered wave such that:

1. The scattered wave satisfies the equations of electrodynamics.
2. The scattered wave satisfies a causality or radiation condition.
3. The sum of the scattered and the incident waves satisfies a boundary condition on the surface of the scattering body—in four-dimensional space a hypersurface, $f(x, y, z, t) = 0$.

The process used in this paper entails the introduction of new four-dimensional, space-time, curvilinear coordinates. That is, with the original coordinates x, y, z, t written respectively x^1, x^2, x^3, x^4 or simply x^α, new coordinates $x^{\alpha'} = x^{\alpha'}(x^\alpha)$ are chosen and conditions 1, 2, and 3 above are transformed into this new coordinate system.

In this chapter, the transformation of the electrodynamic equations, condition 1, is discussed. The other two conditions will be considered later.

2.2 The Method of Natural Invariance

This presentation follows that of Post (1962). First in a stationary, free-space, environment the indefinite Maxwell-Minkowski equations are

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \tag{2.1}$$

$$\nabla \cdot \bar{B} = 0 \tag{2.2}$$

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} \tag{2.3}$$

$$\nabla \cdot \bar{D} = \rho \tag{2.4}$$

and the constitutive relations are

$$\bar{H} = \frac{1}{\mu_o} \bar{B} \tag{2.5}$$

$$\bar{D} = \epsilon_o \bar{E} \ . \tag{2.6}$$

A formalism for transforming the quantities $\bar{E}$, $\bar{B}$, $\bar{D}$, $\bar{H}$, $\bar{J}$, and ρ to a new system, such that the indefinite Maxwell-Minkowski relations remain invariant and the effect of the transformation is incorporated into new constitutive relations, exists and is given in the following.

First, equations (2.1) through (2.6) can be written in four-dimensional tensor notation as

$$\frac{\partial F_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial F_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial F_{\gamma\alpha}}{\partial x^\beta} = 0 \tag{2.7}$$

$$\frac{\partial f^{\alpha\beta}}{\partial x^\beta} = c^\alpha \tag{2.8}$$

$$f^{\alpha\beta} = \frac{1}{2} \chi^{\alpha\beta\gamma\delta} F_{\gamma\delta} \tag{2.9}$$

where

$$F_{\alpha\beta} = \begin{pmatrix} 0 & B_z & -B_y & E_x \\ -B_z & 0 & B_x & E_y \\ B_y & -B_x & 0 & E_z \\ -E_x & -E_y & -E_z & 0 \end{pmatrix} \tag{2.10}$$

$$f^{\alpha\beta} = \begin{pmatrix} 0 & H_z & -H_y & -D_x \\ -H_z & 0 & H_x & -D_y \\ H_y & -H_x & 0 & -D_z \\ D_x & D_y & D_z & 0 \end{pmatrix} \tag{2.11}$$

and $c^\alpha = (J_x, J_y, J_z, \rho)$. (2.12)

The nonzero components of the constitutive tensor $\chi^{\alpha\beta\gamma\delta}$ are

$$\chi^{1212} = \chi^{1313} = \chi^{2323} = 1/\mu_o \tag{2.13}$$

$$\chi^{1414} = \chi^{2424} = \chi^{3434} = -\epsilon_o \tag{2.14}$$

and

$$\chi^{\alpha\beta\gamma\delta} = -\chi^{\alpha\beta\delta\gamma} = -\chi^{\beta\alpha\gamma\delta} = \chi^{\gamma\delta\alpha\beta} \quad . \tag{2.15}$$

(Post, 1962, shows that (2.15) is a general property of any constitutive tensor not just the free-space stationary one.)

New coordinates $\chi^{\alpha'}(\chi^{\alpha})$ are now introduced. For the following presentation to be valid, only holonomic transformations can be considered, that is the transformation satisfies

$$\frac{\partial}{\partial x^{\alpha}}\left(\frac{\partial x^{\alpha'}}{\partial x^{\beta}}\right) = \frac{\partial}{\partial x^{\beta}}\left(\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\right) \quad . \tag{2.16}$$

Then, let the quantities $F_{\alpha\beta}$, $f^{\alpha\beta}$, c^{α}, and $\chi^{\alpha\beta\gamma\delta}$ transform to the new coordinate system as

$$F_{\alpha'\beta'} = \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \frac{\partial x^{\beta}}{\partial x^{\beta'}} F_{\alpha\beta} \tag{2.17}$$

$$f^{\alpha'\beta'} = |\Delta|^{-1} \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial x^{\beta'}}{\partial x^{\beta}} f^{\alpha\beta} \tag{2.18}$$

$$c^{\alpha'} = |\Delta|^{-1} \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} c^{\alpha} \tag{2.19}$$

$$\chi^{\alpha'\beta'\gamma'\delta'} = |\Delta|^{-1} \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial x^{\beta'}}{\partial x^{\beta}} \frac{\partial x^{\gamma'}}{\partial x^{\gamma}} \frac{\partial x^{\delta'}}{\partial x^{\delta}} \chi^{\alpha\beta\gamma\delta} \tag{2.20}$$

where Δ is the determinant of $\partial x^{\alpha'}/\partial x^{\alpha}$.

Thus $F_{\alpha\beta}$ is a second order covariant tensor of weight zero while $f^{\alpha\beta}$, c^{α}, and $\chi^{\alpha\beta\gamma\delta}$ are each contravariant tensors of weight +1 and order two, one, and four respectively. By direct substitution of these transformations, using

(2.16), (2.7) to (2.9) become

$$\frac{\partial F_{\beta'\gamma'}}{\partial x^{\alpha'}} + \frac{\partial F_{\alpha'\beta'}}{\partial x^{\gamma'}} + \frac{\partial F_{\gamma'\alpha'}}{\partial x^{\beta'}} = 0 \tag{2.21}$$

$$\frac{\partial f^{\alpha'\beta'}}{\partial x^{\beta'}} = c^{\alpha'} \tag{2.22}$$

and

$$f^{\alpha'\beta'} = \frac{1}{2} \chi^{\alpha'\beta'\gamma'\delta'} F_{\gamma'\delta'} \ . \tag{2.23}$$

If one defines $\bar{B}'$, $\bar{E}'$, $\bar{H}'$, $\bar{D}'$, $\bar{J}'$ and ρ' in terms of $F_{\alpha'\beta'}$, $f^{\alpha'\beta'}$, and $c^{\alpha'}$ by (2.10), (2.11), and (2.12), the equations of electrodynamics become

$$\nabla' \times \bar{E}' = -\frac{\partial \bar{B}'}{\partial t'} \tag{2.24}$$

$$\nabla' \cdot \bar{B}' = 0 \tag{2.25}$$

$$\nabla' \times \bar{H}' = \bar{J}' + \frac{\partial \bar{D}'}{\partial t'} \tag{2.26}$$

$$\nabla' \cdot \bar{D}' = \rho' \tag{2.27}$$

$$\bar{H}' = \bar{H}'(\bar{B}', \bar{E}') \tag{2.28}$$

$$\bar{D}' = \bar{D}'(\bar{B}', \bar{E}') \ . \tag{2.29}$$

It should be pointed out, however, that the quantities $\bar{B}'$, $\bar{E}'$, etc. were introduced in a purely mathematical manner and in the general case do not have the physical meaning of magnetic flux density vector, electric field strength vector, etc. They do have physical meaning and in fact are real physical quantities in the case of the Lorentz transformation representing translation with a constant velocity.

2.3 The Case of Uniform Motion

For uniform motion in the z-direction with velocity v the Lorentz transformation is

$$x' = x \tag{2.30}$$

$$y' = y \tag{2.31}$$

$$z' = \frac{z - vt}{\sqrt{1-\beta^2}} \tag{2.32}$$

$$t' = \frac{t - \frac{v}{c^2} z}{\sqrt{1-\beta^2}} \quad ; \qquad \beta = v/c \ . \tag{2.33}$$

In this case the new, primed, coordinates are the actual coordinates seen by an observer moving with the velocity v in the z direction. The quantities $\bar{E}'$, $\bar{B}'$, etc. obtained by the method of section 2.2 are the actual electric field, magnetic field, etc. seen by the observer.

The Lorentz transformation has two important properties:

1. A motion with the velocity of light, c, in the unprimed system is given by $x^2 + y^2 + z^2 - c^2t^2 = 0$. The same motion satisfies $x'^2 + y'^2 + z'^2 - c^2t'^2 = 0$ in the primed system and hence moves in this system with the same velocity c.
2. The electrodynamic equations including constitutive relations are completely invariant under this transformation.

The first statement is equivalent to Einstein's principle of the constancy of the velocity of light and the second to his principle of relativity (Einstein et al, 1923, p. 41). The principle of relativity implies that the application of the Lorentz transformation to a problem involving motion with constant velocity reduces the problem to that of scattering by stationary objects.

The transformation of the field quantities from one system to another moving with a constant velocity v relative to it can be written in the useful form given below. In these the field quantities are resolved into two components; one, denoted by the subscript $\parallel$, is parallel to the direction of motion and the other, denoted by the subscript $\perp$, is perpendicular to the motion. Then (Sommerfeld, 1964a)

$$\bar{E}'_{\parallel} = (\bar{E} + \bar{v} \times \bar{B})_{\parallel} \quad , \qquad \bar{E}'_{\perp} = \left(\frac{\bar{E} + \bar{v} \times \bar{B}}{\sqrt{1-\beta^2}}\right)_{\perp}$$

$$\bar{B}'_{\parallel} = \left(\bar{B} - \frac{1}{c^2}\,\bar{v} \times \bar{E}\right)_{\parallel} \quad , \qquad \bar{B}'_{\perp} = \left(\frac{\bar{B} - \frac{1}{c^2}\,\bar{v} \times \bar{E}}{\sqrt{1-\beta^2}}\right)_{\perp} \tag{2.34}$$

$$\bar{D}' = \epsilon_o \bar{E}' \quad , \qquad \bar{H}' = \frac{1}{\mu_o}\,\bar{B}'$$

These results will be used in the following chapter.

Chapter Three

THE SCATTERING OF A PLANE WAVE BY A MOVING PERFECTLY CONDUCTING SPHERE

3.1 Statement of the Problem

It was pointed out in chapter two that the principle of relativity implies that a situation involving electromagnetic scattering by a uniformly moving body can be reduced by the Lorentz transformation to one involving scattering by a stationary body. In this chapter the case of a uniformly moving sphere is considered. For a plane wave incident upon a stationary sphere the method of eigenfunction expansion can be used to find the scattered wave. This results in the well-known Mie series (Stratton, 1941). This problem could then be solved by making a Lorentz transformation to a system moving with the sphere; obtaining in this system the correct Mie solution, and then making a Lorentz transformation back to the system in which the problem was stated. However, it will be seen that the result of doing this directly would be an extremely complicated expression that although allowing numerical computation of the field components would give little insight into the problem. By examining the behavior of the far field an expression for the field scattered by the moving sphere can be found in a useful and physically meaningful form.

The specific problem considered here consists of a perfectly conducting sphere of radius a moving with the velocity v. For convenience several coordinate systems will be used (Fig. 3-1). Oriented with the motion of the sphere is the x, y, z system. Here the positive z direction is the direction of the motion. The incident plane wave is specified by the ξ, η, ζ system. Its direction of propagation is the negative ζ direction and its polarization is designated by the angle ψ between the η axis and the $\bar{E}$ vector. No generality is lost by making the y and η axes common so that the transformation between the x, y, z system and the ξ, η, ζ system is a rotation about the common y or η axes through an angle $\theta^{(i)}$.

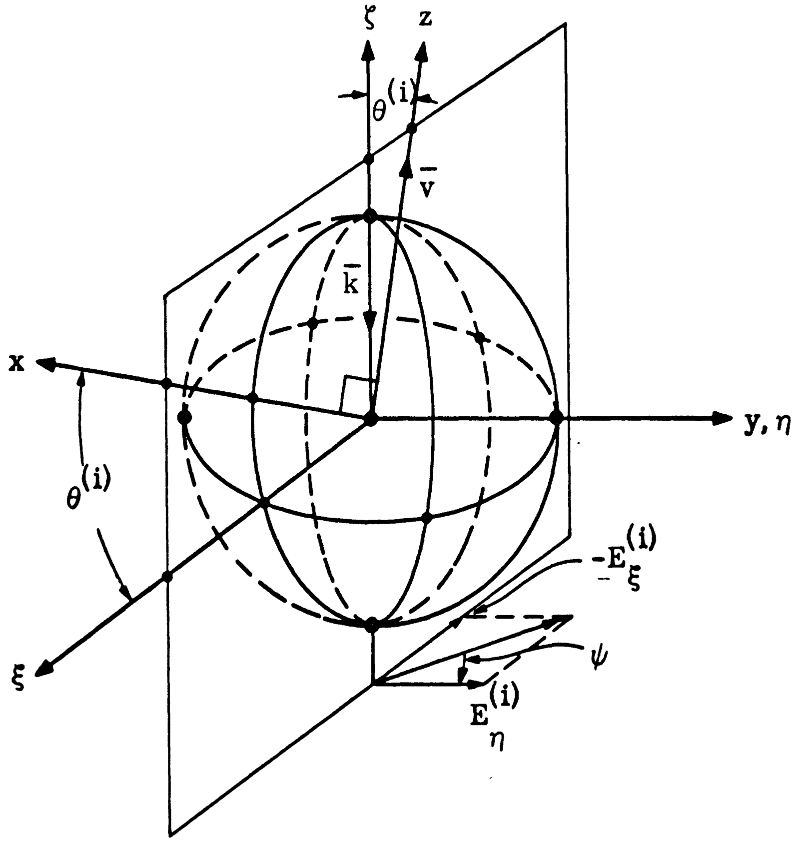

FIG. 3-1: STATEMENT OF THE PROBLEM FOR SCATTERING BY A MOVING SPHERE.

The incident field may be expressed

$$\bar{E}^{(i)} = \frac{E^{(i)}}{2}\left\{\cos\psi\,\hat{\eta} - \sin\psi\,\hat{\xi}\right\}e^{-i(k\zeta+\omega t)} + c.c. \tag{3.1}$$

$$c\bar{B}^{(i)} = -\hat{\zeta}\times\bar{E}^{(i)} . \tag{3.2}$$

Here the fields are written as real quantities by adding to each complex expression its complex conjugate, c.c. The symbols $\hat{\xi}$, $\hat{\eta}$, and $\hat{\zeta}$ represent unit vectors in the ξ, η and ζ directions respectively.

3.2 The Transformation to a System Moving with the Sphere

The problem is now transformed to a system moving with the sphere. To do this the Lorentz transformation presented in section 2.3 is used. Primes will be used to designate all quantities in this system. It is found that in this system, also, the incident wave is a plane wave; however, it has a new frequency, wave number, magnitude, and direction of propagation. Two new spatial coordinate systems will be defined in a manner analogous to the definitions of the x, y, z and ξ, η, ζ systems in the unprimed coordinates.

Oriented with the direction of motion will be the x', y', z' system while the ξ', η', ζ' system is oriented with the incident plane wave. In this coordinate system the incident wave is given by

$$\bar{E}'^{(i)} = \frac{E^{(i)}}{2}\left(\frac{1+\beta\cos\theta^{(i)}}{\sqrt{1-\beta^2}}\right)\left\{\cos\psi\,\hat{\eta}' - \sin\psi\,\hat{\xi}'\right\}e^{-i(k'\zeta'+\omega' t')} + c.c. \tag{3.3}$$

$$c\bar{B}'^{(i)} = -\hat{\zeta}'\times\bar{E}'^{(i)} \tag{3.4}$$

where

$$\beta = v/c\,; \qquad \omega' = \omega\,\frac{1+\beta\cos\theta^{(i)}}{\sqrt{1-\beta^2}} ,$$

$$k' = \frac{1+\beta\cos\theta^{(i)}}{\sqrt{1-\beta^2}}\,k , \qquad \cos\theta'^{(i)} = \frac{\cos\theta^{(i)}+\beta}{1+\beta\cos\theta^{(i)}} . \tag{3.5}$$

In the primed system the problem appears that of an incident plane wave scattered by a stationary sphere whose solution is the Mie series.

3.3 The Scattered Wave in the Primed System

The Mie solution assumes its simplest form in a spherical coordinate system having its polar axis parallel to the direction of the propagation of the incident wave. Therefore let the spherical coordinate system R', θ', ϕ' be defined so that the polar axis is the ζ' axis and the dihedral angle ϕ' is measured from the η' - ζ' plane. Then the scattered wave pertaining to this incident wave is

$$\bar{E}'^{(s)} = -E^{(i)} \frac{1+\beta\cos\theta^{(i)}}{\sqrt{1-\beta^2}} \sum_{n=1}^{\infty} (2n+1)(-i)^n$$

$$\cdot\Bigg\{\Bigg[\left(i\beta_n \frac{h_n^{(1)}(k'R')}{k'R'} P_n^{(1)}(\cos\theta')\right)\cos(\phi'-\psi)\hat{R}' + \frac{1}{n(n+1)}\left(\alpha_n h_n^{(1)}(k'R') \frac{P_n^{(1)}(\cos\theta')}{\sin\theta'}\right.$$

$$\left. + i\beta_n \left[\frac{1}{u}\frac{d}{du}\left[u\,h_n^{(1)}(u)\right]\right]_{u=k'R'} \frac{d}{d\theta'} P_n^{(1)}(\cos\theta')\right)\cos(\phi'-\psi)\hat{\theta}'$$

$$-\frac{1}{n(n+1)}\left(\alpha_n h_n^{(1)}(k'R')\frac{d}{d\theta'}P_n^{(1)}(\cos\theta')\right.$$

$$\left. + i\beta_n\left[\frac{1}{u}\frac{d}{du}\left[u\,h_n^{(1)}(u)\right]\right]_{u=k'R'} \frac{P_n^{(1)}(\cos\theta')}{\sin\theta'}\right)\sin(\phi'-\psi)\hat{\phi}'\Bigg]\Bigg\} + \text{c.c.} \qquad (3.6)$$

In this expression $h_n^{(1)}$ is the spherical Hankel function of the first kind, $P_n^{(1)}$ is the associated Legendre function of the first kind,

$$\alpha_n = \left.\frac{j_n(u)}{h_n^{(1)}(u)}\right|_{u=k'a} , \qquad (3.7)$$

and

$$\beta_n = \left.\frac{\frac{d}{du}\left[u j_n(u)\right]}{\frac{d}{du}\left[u h_n^{(1)}(u)\right]}\right|_{u=k'a} . \qquad (3.8)$$

Now $\bar{B}'^{(s)}$ can be calculated, then both $\bar{B}'^{(s)}$ and $\bar{E}'^{(s)}$ can be written in x', y', and z' components, and the results transformed back to the unprimed system by using the inverse of (2.34). Clearly the result will be complicated. The far field expression, $k'R' \gg 1$, is simpler and it can be transformed back to the unprimed system to give a result allowing physical interpretation. The far field expression is

$$\bar{E}'^{(s)} = -i\frac{E^{(i)}}{2}\left(\frac{1+\beta\cos\theta^{(i)}}{\sqrt{1-\beta^2}}\right)\left\{\cos(\phi'-\psi)S_1(\theta')\hat{\theta}' - \sin(\phi'-\psi)S_2(\theta')\hat{\phi}'\right\}\frac{e^{ik'(R'-ct')}}{k'R'} + c.c. \tag{3.9}$$

$$c\bar{B}'^{(s)} = \hat{R}' \times \bar{E}'^{(s)} \tag{3.10}$$

$$S_1(\theta') = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)}(-1)^n\left(\beta_n \frac{dP_n^{(1)}(\cos\theta')}{d\theta'} - \alpha_n \frac{P_n^{(1)}(\cos\theta')}{\sin\theta'}\right) \tag{3.11}$$

$$S_2(\theta') = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)}(-1)^n\left(\beta_n \frac{P_n^{(1)}(\cos\theta')}{\sin\theta'} - \alpha_n \frac{dP_n^{(1)}(\cos\theta')}{d\theta'}\right) \tag{3.12}$$

To obtain an insight on how to transform this field the surfaces of constant phase defined by setting $R' - ct'$ equal to a constant are observed in the unprimed system.

3.4 The Behavior of the Scattered Far Field in the Unprimed System

The exponent of equation (3.9) is written in terms of the unprimed variables by substituting the Lorentz transformation and the result is equated to a constant depending on a quantity τ which will be seen to have a physical meaning. That is

$$ik'(R'-ct') = ik'\left\{\left[x^2+y^2+\left(\frac{z-vt}{\sqrt{1-\beta^2}}\right)^2\right]^{1/2} - c\left(\frac{t-(vz/c^2)}{\sqrt{1-\beta^2}}\right)\right\}$$

$$= -ik'c\tau\sqrt{1-\beta^2} \quad . \tag{3.13}$$

Equation (3.13) is squared and the result simplified to give the following equation representing the motion of a surface of constant phase.

$$x^2 + y^2 + (z - v\tau)^2 = c^2(t - \tau)^2 \tag{3.14}$$

This represents a spherical wave front propagating with the velocity of light. The wave front appears to have been emitted at the time $t = \tau$ at which time the sphere's position was $x = 0$, $y = 0$, $z = v\tau$.

Alternately for an observer at the position x, y, z and time t, $v\tau$ would correspond to the retarded position of the sphere and τ the retarded time. The solution of (3.14) for given values of x, y, z, and t such that $t - \tau$ is positive gives an expression for the retarded time as a function of the observer's coordinates.

A certain amount can be said about the behavior of the scattered field by examining the motion of these surfaces of constant phase. As τ goes from τ_1 to τ_2 such that

$$\tau_2 - \tau_1 = \frac{2\pi}{k'c\sqrt{1-\beta^2}} \tag{3.15}$$

the exponent of (3.9) changes by 2π and the function goes through one period of oscillation.

An effective wavelength λ_e may be defined as the distance between two wave fronts emitted respectively at τ_1 and τ_2. Let an observer's position be given by the angle Θ shown in Fig. 3-2. Then using the far field assumption

$$\lambda_e = c(t - \tau_1) - c(t - \tau_2) - v(\tau_2 - \tau_1)\cos\Theta$$

$$= \frac{2\pi(1 - \beta\cos\Theta)}{k'\sqrt{1-\beta^2}} \quad . \tag{3.16}$$

Associated with this will be an effective scattered frequency $\omega^{(s)}$ and wave number $k^{(s)}$ where

$$\omega^{(s)} = \frac{2\pi c}{\lambda} = \omega\,\frac{1 + \beta\cos\theta^{(i)}}{1 - \beta\cos\Theta} \tag{3.17}$$

and

$$k^{(s)} = \omega^{(s)}/c \quad .$$

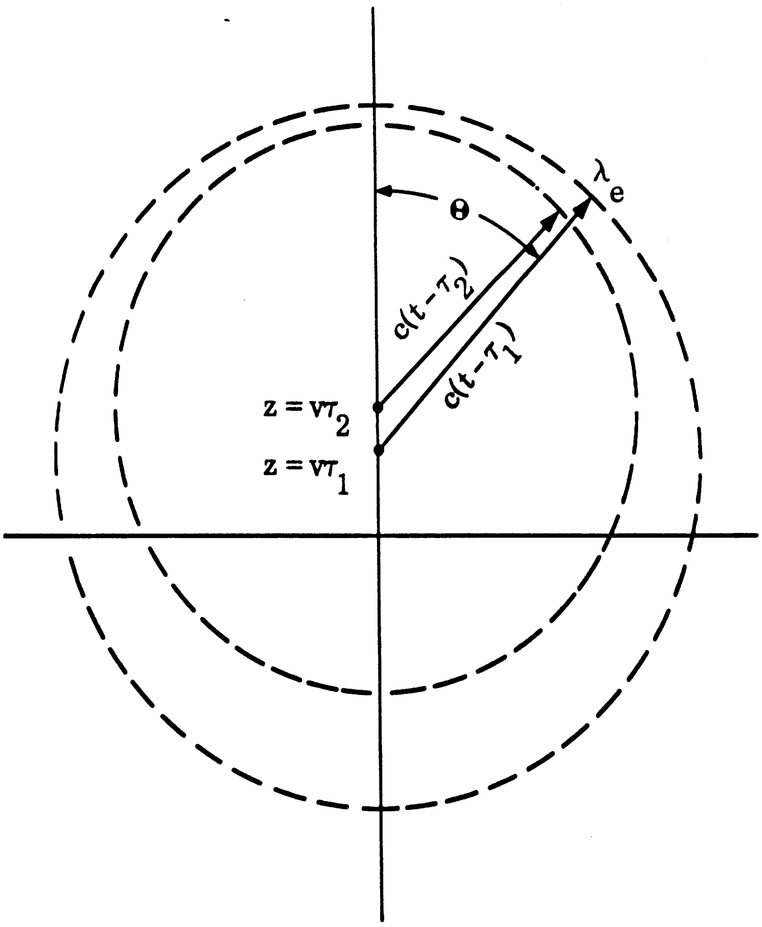

FIG. 3-2: TWO SCATTERED WAVE FRONTS.

Strictly speaking the scattered field is spatially and temporally harmonic only if θ equals 0 or π. Elsewhere such quantities as wavelength and frequency are only approximations valid over a certain interval of time and a certain volume of space.

3.5 The Coordinate Systems to be Used

The surfaces described in the previous section are seen to be important as far as the behavior of the scattered field is concerned. They are closely connected to the hyperbolic differential equations. The function $\tau(x, y, z, ct)$ satisfies

$$\left(\frac{\partial\tau}{\partial x}\right)^2 + \left(\frac{\partial\tau}{\partial y}\right)^2 + \left(\frac{\partial\tau}{\partial z}\right)^2 - \left(\frac{\partial\tau}{\partial ct}\right)^2 = 0 \quad (3.18)$$

implying that the surfaces,

$$\tau(x, y, z, ct) = \tau_o$$

where τ_o is a constant, are characteristics (Courant and Hilbert, 1962). These facts led to using the following coordinate systems in representing the scattered wave.

In addition to the R', θ', ϕ', t' coordinates in the primed system moving with the sphere, the R', Θ', Φ', t' spherical coordinate system will be used. This system has for its polar axis the z' axis and its dihedral angle Φ' is measured from the $z'-\eta'$ plane. The transformation from the R', Θ', Φ', t' system to the x', y', z', t' system is given by

$$\begin{aligned} z' &= R'\cos\Theta' \\ y' &= R'\sin\Theta'\cos\Phi' \\ x' &= -R'\sin\Theta'\sin\Phi' \\ t' &= t' \end{aligned} \quad (3.19)$$

In the stationary system two spherical coordinate systems will be defined. Both have their origins at the retarded position of the sphere, $x = 0$, $y = 0$, $z = v\tau$. The R, θ, ϕ, τ system has its polar axis in the positive ζ direction with its dihedral angle measured from a plane parallel to the $\eta - \zeta$ plane while the R, Θ, Φ, τ system has its polar axis in the z direction and its dihedral angle measured from the $y - z$ plane. The transformation from the x, y, z, t system to the R, Θ, Φ, τ system is

$$\begin{aligned} z &= R\cos\Theta + v\tau \\ y &= R\sin\Theta\cos\Phi \\ x &= -R\sin\Theta\sin\Phi \\ t &= \frac{R}{c} + \tau \quad . \end{aligned} \quad (3.20)$$

The transformation between the R', θ', ϕ', t' system and the R', Θ', Φ', t' system and that between the R, θ, ϕ, τ system and the R, Θ, Φ, τ system are both found from geometry. The transformation between the R', Θ', Φ', t' system and the R, Θ, Φ, τ system is found by substituting (3.20) and (3.19) into the Lorentz transformations. The result is that R', θ', and ϕ' may be expressed as functions of R, θ, and ϕ by

$$\cos\Theta = \cos\theta\cos\theta^{(i)} + \sin\theta\sin\theta^{(i)}\sin\phi\ ; \quad \cos\Phi = \frac{\sin\theta\cos\phi}{\sin\Theta} \tag{3.21}$$

$$\cos\Theta' = \frac{\cos\Theta - \beta}{1 - \beta\cos\Theta}\ ; \quad \Phi' = \Phi \tag{3.22}$$

$$\cos\theta' = \cos\Theta'\cos\theta'^{(i)} - \sin\Theta'\sin\Phi'\sin\theta'^{(i)}; \quad \cos\phi' = \frac{\sin\Theta'\cos\Phi'}{\sin\theta'} \tag{3.23}$$

and

$$R' = \frac{1 - \beta\cos\Theta}{\sqrt{1 - \beta^2}}\, R . \tag{3.24}$$

The exponent is seen to be

$$R' - ct' = -c\tau\sqrt{1 - \beta^2} \ . \tag{3.25}$$

The statement of the **solution** in the unprimed system is now straightforward.

3.6 The Scattered Field in the Stationary System

The next step is to find the transformation of the field components from the R', θ', ϕ', t' system to the R, θ, ϕ, τ system. This will be done in three stages. The first transformation is from the R', θ', ϕ', t' system to the R', Θ', Φ', t' system, then to the R, Θ, Φ, τ system, and finally to the R, θ, ϕ, τ system.

Both ordered sets of unit vectors, $\hat{R}', \hat{\Theta}', \hat{\Phi}'$ and $\hat{R}', \hat{\theta}', \hat{\phi}'$, are right-handed orthogonal systems with the $\hat{R}'$ vector an element of each set. This implies that the transformation between the $\hat{\Theta}', \hat{\Phi}'$ vectors and the $\hat{\theta}', \hat{\phi}'$ vectors is a rotation about the common $\hat{R}'$ vector. Let χ' be the angle between $\hat{\phi}'$ and $\hat{\Phi}'$ defined so that

$$\cos\chi' = \sin\Phi'\sin\phi' + \cos\Phi'\cos\phi'\cos\theta'^{(i)} \tag{3.26}$$

and $\chi' > 0$ if a rotation in the right-hand sense about $\hat{R}'$ takes $\hat{\phi}'$ to $\hat{\Phi}'$. Then the Θ' and Φ' components of the electric field are

$$E'^{(s)}_{\Theta} = E'^{(s)}_{\theta}\cos\chi' + E'^{(s)}_{\phi}\sin\chi' \ ;$$

and

$$E'^{(s)}_{\Phi} = -E'^{(s)}_{\theta}\sin\chi' + E'^{(s)}_{\phi}\cos\chi' \ . \tag{3.27}$$

For the $\hat{R}, \hat{\Theta}, \hat{\Phi}$ and the $\hat{R}, \hat{\theta}, \hat{\phi}$ systems there exists a χ satisfying a relation similar to (3.26) only written in terms of the unprimed coordinates. In this case

$$E^{(s)}_{\theta} = E^{(s)}_{\Theta}\cos\chi - E^{(s)}_{\Phi}\sin\chi$$

and

$$E^{(s)}_{\phi} = E^{(s)}_{\Theta}\sin\chi + E^{(s)}_{\Phi}\cos\chi \ . \tag{3.28}$$

Now in both the R, Θ, Φ and R', Θ', Φ' systems $\bar{E}$ and $\bar{B}$ can be written in terms of their perpendicular and parallel components defined in section 2.3 and the result substituted into (2.34) to give

$$E^{(s)}_{\Theta} = \frac{1+\beta\cos\Theta'}{\sqrt{1-\beta^2}}\,E'^{(s)}_{\Theta} = \frac{\sqrt{1-\beta^2}}{1-\beta\cos\Theta}\,E'^{(s)}_{\Theta}$$

and

$$E^{(s)}_{\Phi} = \frac{\sqrt{1-\beta^2}}{1-\beta\cos\Theta}\,E'^{(s)}_{\Phi} \quad . \tag{3.29}$$

Equations (3.21) to (3.29) are sufficient for the transformation of the field in the primed system to the unprimed system so that

$$E^{(s)}_{\theta} = -i\,\frac{E^{(i)}}{2}\,\frac{1-\beta^2}{(1-\beta\cos\Theta)^2}\left\{\sin(\chi-\chi')\sin(\phi'-\psi)S_2(\theta') + \cos(\chi-\chi')\cos(\phi'-\psi)S_1(\theta')\right\}\frac{e^{-ikc\tau(1+\beta\cos\theta^{(i)})}}{kR} + \text{c.c.} \tag{3.30}$$

and

$$E_{\phi}^{(s)} = i\frac{E^{(i)}}{2}\frac{1-\beta^2}{(1-\beta\cos\Theta)^2}\Bigg\{\cos(\chi-\chi')\sin(\phi'-\psi)S_2(\theta')$$

$$-\sin(\chi-\chi')\cos(\phi'-\psi)S_1(\theta')\Bigg\}\frac{e^{-ikc\tau(1+\beta\cos\theta^{(i)})}}{kR} + c.c. \tag{3.31}$$

The magnetic field can be found from

$$c\bar{B}^{(s)} = \hat{R}\times\bar{E}^{(s)} \quad . \tag{3.32}$$

For an observer whose position is fixed in the x, y, z system, θ, ϕ, and R are each functions of τ which in turn is a function of time. However, the change in θ, ϕ, and R is small over several time periods of oscillation and several wavelengths in space if $\beta \ll kR$ which is equivalent to the far field condition $kR \gg 1$. This implies that the scattered far field appears as a quasi-plane wave whose magnitude, direction of propagation, frequency, wavelength, and polarization are changing with time.

3.7 Conclusion

An expression for the far field scattered by a perfectly conducting sphere moving with uniform velocity through an electromagnetic plane wave has been obtained in this chapter. It is seen to exhibit such effects as a Doppler shift and an aberration causing the sphere to appear in a retarded position due to the finite velocity of propagation of light.

The far field was expressed in terms of the infinite series $S_1(\theta')$ and $S_2(\theta')$ defined by (3.11) and (3.12). For $k'a \ll 1$ these series converge rapidly, the terms being interpreted as representing radiation by moving magnetic and electric multipoles. For $k'a > 1$ the series converge slowly, but a Watson transformation may be applied. The result of the transformation may be expressed as the sum of two types of scattered waves. One of these found by a contour integration is the optics solution. The other a residue series can be interpreted as the sum of so-called creeping waves. Expressions for both the optics and creeping-wave solutions have been worked out in detail (see for instance, Senior and Goodrich, 1964).

In the next chapter such quantities as differential and total scattering cross sections, electromagnetic forces, and the electromechanical energy exchange processes that occur will be discussed.

Chapter Four

SOME ENERGY CALCULATIONS CONCERNING THE MOVING SPHERE PROBLEM

4.1 The Differential Scattering Cross Section

The instantaneous Poynting vector has the form

$$\left|\bar{E}^{(s)} \times \bar{H}^{(s)}\right| = f(\theta, \phi, R)\, e^{2iC\tau} + f^{*} e^{-2iC\tau} + g(\theta, \phi, R) \ . \tag{4.1}$$

This can be verified by substitution of the expression developed in chapter 3. If it is assumed that θ, ϕ, and R remain constant over several periods of oscillation, then since τ is an increasing function of time, the average,

$$<\left|\bar{E}^{(s)} \times \bar{H}^{(s)}\right|> = \lim_{(t_2 - t_1) \to \infty} \int_{t_1}^{t_2} \left|\bar{E}^{(s)} \times \bar{H}^{(s)}\right| dt \Big/ (t_2 - t_1) \ ,$$

is approximately

$$<\left|\bar{E}^{(s)} \times \bar{H}^{(s)}\right|> = g(\theta, \phi, R) \ .$$

In this sense substitution of $\bar{E}^{(s)}$ and $\bar{H}^{(s)}$ shows

$$<\bar{E}^{(s)} \times \bar{H}^{(s)}> = \frac{I_o}{k^2 R^2} \left(\frac{\sqrt{1-\beta^2}}{1-\beta\cos\Theta}\right)^4 \left\{\sin^2(\phi' - \psi) S_2 S_2^{*} + \cos^2(\phi' - \psi) S_1 S_1^{*}\right\} \hat{R} \tag{4.2}$$

where

$$I_o = \eta_o \frac{E^{(i)} E^{*(i)}}{2} \quad \text{and} \quad \eta_o = \sqrt{\epsilon_o/\mu_o} \ .$$

In the case where $k'a \ll 1$ this expression may be approximated by

$$<\bar{E}^{(s)} \times \bar{H}^{(s)}> \sim \frac{I_o k^4 a^6}{4R^2} \frac{(1+\beta\cos\theta^{(i)})^6}{(1-\beta^2)(1-\beta\cos\Theta)^4} \left\{4 - 3\cos^2(\phi' - \psi) + 4\cos\theta' + \left[1 + 3\cos^2(\phi' - \psi)\right]\cos^2\theta'\right\} \hat{R} \ . \tag{4.3}$$

The results of some calculations using (4.3) showing the differential scattering cross section (Jackson, 1962) defined as

$$\frac{d\sigma}{d\Omega} = \frac{\text{energy scattered per unit time per solid angle}}{\text{incident energy per unit time per unit area}}$$

$$= \frac{R^2}{I_o} <|\bar{E}^{(s)} \times \bar{H}^{(s)}|> \tag{4.4}$$

as a function of the angular position of the observer are given in Appendix A.

4.2 The Total Cross Section

The total scattering cross section is defined (Jackson, 1962) as

$$\sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega = \frac{P_{out}^{(s)}}{I_o} \tag{4.5}$$

where the integral is over a solid angle of 4π steradians. The quantity $P_{out}^{(s)}$ is the average power passing out of the surface of a stationary sphere of large radius due to the scattered field. Substitution of $\bar{E}^{(s)}$ and $\bar{H}^{(s)}$ shows that

$$\sigma = \frac{1}{I_o} \int <\bar{E}^{(s)} \times \bar{H}^{(s)}> d\bar{S}$$

$$= \frac{\pi}{k^2(1-\beta^2)} \left\{ \left(1 + \frac{\beta^2}{2} \sin^2\theta'^{(i)}\right) I_1 \right.$$

$$\left. + 2\beta \cos\theta'^{(i)} I_2 + \beta^2 \left(\cos^2\theta'^{(i)} - \frac{1}{2}\sin^2\theta'^{(i)}\right) I_3 + \frac{\beta^2}{4} \sin^2\theta'^{(i)} \cos 2\psi\, I_4 \right\} . \tag{4.6}$$

The terms I_1, I_2, I_3, and I_4 are given below.

α_n and β_n represent magnetic and electric multipole fields of order n. The fact that the series I_1 contains products of $\alpha_n\alpha_n^*$ and $\beta_n\beta_n^*$ only implies that stationary magnetic and electric multipoles radiate independently and do not interact with each other.

However, in the case of uniform motion it appears from I_2 that electric multipoles of order n and n+1 interact, magnetic multipoles of order n and n+1 interact, and electric and magnetic multipoles of the same order interact. From (4.6) it is seen that these effects are of first order in β. From I_3 electric multipoles of order n and n+2 interact, magnetic multipoles of order n and n+2 interact, and multipoles of one kind of order n interact with multipoles of the other kind of order n + 1. These effects are seen to be of second order in β.

The same type of interactions appear in I_4 and these are also of second order in β but the effect of this term is seen to depend on the polarization angle ψ.

That such interactions should occur is indicated in the following way. Consider a dipole source in the primed system given by

$$J'_z = \sin\alpha\,\delta(x')\delta(y')\delta(z')e^{i\omega' t'} ,$$

$$J'_y = \cos\alpha\,\delta(x')\delta(y')\delta(z')e^{i\omega' t'} ,$$

and

$$\rho' = -\left[\sin\alpha\,\delta(x')\delta(y')\left.\frac{d\delta(u)}{du}\right|_{u=z'} + \cos\alpha\,\delta(x')\delta(z')\left.\frac{d\delta(u)}{du}\right|_{u=y'}\right]\frac{e^{i\omega' t'}}{i\omega'} . \tag{4.11}$$

In the unprimed system this distribution appears as

$$J_z = \frac{J'_z + v\rho'}{\sqrt{1-\beta^2}}$$

$$= \sin\alpha\,\delta(x)\delta(y)\,\delta(z-vt)e^{i\omega t} - v\sqrt{1-\beta^2}\left[\sin\alpha\,\delta(x)\delta(y)\left.\frac{d\delta(u)}{du}\right|_{u=z-vt}\right]\frac{e^{i\omega t}}{i\omega'}$$

$$-v\left[\cos\alpha\,\delta(x)\delta(z-vt)\left.\frac{d\delta(u)}{du}\right|_{u=y}\right]\frac{e^{i\omega t}}{i\omega'} ,$$

$$I_1 = \int_0^{\pi} \left[S_1(\theta')S_1^*(\theta') + S_2(\theta')S_2^*(\theta') \right] \sin\theta' d\theta'$$

$$= 2 \sum_{n=1}^{\infty} (2n+1)(\alpha_n^* \alpha_n + \beta_n \beta_n^*) \quad . \tag{4.7}$$

$$I_2 = \int_0^{\pi} \left[S_1(\theta')S_1^*(\theta') + S_2(\theta')S_2^*(\theta') \right] \cos\theta' \sin\theta' d\theta'$$

$$= -2 \sum_{n=1}^{\infty} \left\{ (\alpha_n^* \beta_n + \text{c.c.}) \frac{2n+1}{n(n+1)} + (\alpha_{n+1}^* \alpha_n + \beta_{n+1} \beta_n^* + \text{c.c.}) \frac{n(n+2)}{n+1} \right\} \tag{4.8}$$

$$I_3 = \int_0^{\pi} \left[S_1(\theta')S_1^*(\theta') + S_2(\theta')S_2^*(\theta') \right] \cos^2\theta' \sin\theta' d\theta'$$

$$= 2 \sum_{n=1}^{\infty} \left\{ (\alpha_n^* \alpha_{n+2} + \beta_{n+2} \beta_n^* + \text{c.c.}) \frac{n(n+3)}{2n+3} + (\alpha_n^* \beta_{n+1} + \alpha_{n+1} \beta_n^* + \text{c.c.}) \frac{2}{n+1} \right.$$

$$\left. + (\alpha_n^* \alpha_n + \beta_n \beta_n^*)(2n+1) \frac{(2n+1)^4 - 12(2n+1)^2 + 59}{8n(n+1)(2n+3)(2n-1)} \right\} \tag{4.9}$$

$$I_4 = \int_0^{\pi} \left[S_2(\theta')S_2^*(\theta') - S_1(\theta')S_1^*(\theta') \right] \sin^3\theta' d\theta'$$

$$= \sum_{n=1}^{\infty} (\alpha_n \alpha_n^* - \beta_n \beta_n^*)(2n+1) \frac{(2n+1)^2 - 13}{(2n+1)^2 - 4}$$

$$-2(\alpha_n \alpha_{n+2}^* - \beta_n \beta_{n+2}^* + \text{c.c.}) \frac{n(n+3)}{2n+3} + 4(\alpha_n \beta_{n+1}^* - \beta_n \alpha_{n+1}^* + \text{c.c.}) \quad . \tag{4.10}$$

The series I_1 is recognized to be the same series that appears in the expression for the total cross section for a stationary sphere. The coefficients

$$J_y = J'_y$$
$$= \sqrt{1-\beta^2}\ \cos\alpha\,\delta(x)\delta(y)\delta(z-vt)\,e^{i\omega t}\ ,$$

and

$$\rho = \frac{\rho' + vJ'_z/c^2}{\sqrt{1-\beta^2}}$$
$$= -\sqrt{1-\beta^2}\left[\sin\alpha\,\delta(x)\delta(y)\left.\frac{d\delta(u)}{du}\right|_{u=z-vt}\right]\frac{e^{i\omega t}}{i\omega'}$$
$$-\left[\cos\alpha\,\delta(x)\delta(z-vt)\left.\frac{d\delta(u)}{du}\right|_{u=y}\right]\frac{e^{i\omega t}}{i\omega'} + \frac{v}{c^2}\sin\alpha\,\delta(x)\delta(y)\delta(z-vt)\,e^{i\omega t}\ , \tag{4.12}$$

where $\omega = \omega'\sqrt{1-\beta^2}$. Now the first terms in each of these expressions represent dipole sources undergoing uniform translation. However, the second and third terms in the expression for J_z represent moving quadrapole moments and the final term in the expression for ρ represents a moving monopole moment. In general a moving multipole of order n appears as a multipole of order n, n+1, and n-1. The magnitude of the n+1 and n-1 terms is proportional to the velocity making it first order in β. This then gives some physical justification for the various interactions appearing in the expression for the total cross section of the moving sphere.

4.3 The Rate of Change of Energy Stored in the Scattered Electromagnetic Field

In the last section an expression for the rate of flow of energy due to the scattered field through the surface of a large stationary volume of space containing the sphere was obtained. Now the rate of change of stored energy of the scattered field in this volume will be found.

The stored energy density

$$w^{(s)} = \left\langle \frac{\bar{E}^{(s)}\cdot\bar{E}^{(s)}\epsilon_o}{2} + \frac{\bar{B}^{(s)}\cdot\bar{B}^{(s)}}{2\mu_o} \right\rangle \tag{2.13}$$

is seen to have the property that

$$w^{(s)}(x, y, z, t) = w^{(s)}(x, y, z+a, t+\frac{a}{v}) \tag{4.14}$$

indicating that the stored energy moves with a velocity, v. The time rate of change of the total energy within this large sphere is then

$$\frac{dW^{(s)}}{dt} = \frac{d}{dt}\iiint w^{(s)}dV = \iint w^{(s)}\bar{v}\cdot d\bar{S} . \tag{4.15}$$

The integration can be performed and is seen to be

$$\frac{dW^{(s)}}{dt} = -\frac{\pi I_o}{k^2(1-\beta^2)}\left\{\left[\left(\beta^2+\frac{\beta^2}{2}\sin^2\theta'^{(i)}\right)I_1+(\beta+\beta^3)\cos\theta'^{(i)}I_2 + \beta^2\left(\cos^2\theta'^{(i)}-\frac{1}{2}\sin^2\theta'^{(i)}\right)I_3+\frac{\beta^2}{4}\sin^2\theta'^{(i)}\cos 2\psi I_4\right]\right\} \tag{4.16}$$

where I_1, I_2, I_3, and I_4 are the same quantities calculated in section 4.2.

From these two expressions the rate of flow of energy from the sphere to the scattered field is seen to be

$$P_{out}^{(s)} + \frac{dW^{(s)}}{dt} = \frac{\pi I_o}{k^2}\left\{\left[I_1+\beta\cos\theta'^{(i)}I_2\right]\right\} \tag{4.17}$$

In the next section the rate of flow of energy from the total field to the sphere is calculated and found to agree with (4.17).

4.4 The Rate of Flow of Energy from the Electromagnetic Field to the Sphere

In the case of a stationary perfectly conducting body, no energy can flow from the total (incident plus scattered) electromagnetic field to the body. However, in the case of a moving perfect conductor energy can be exchanged between the electromagnetic field and the body. This energy appears in the form of mechanical energy. It is expected that this should occur since there is an electromagnetic force acting on the body and the velocity of the body is nonzero.

The rate of flow of energy from the field to the sphere may be found in the following way. First the quantity $\bar{E} \cdot \bar{J}$ from the Poynting theorem has the meaning of the rate of flow of energy per unit volume from the field to the charges which constitute the current $\bar{J}$ in this case the material of the sphere. It is not necessary that this energy appear as Joule heat as in the case of an imperfect conductor or a lossy dielectric as far as this theorem is concerned.

The time averaged rate of flow of energy from the field to the sphere is

$$\lim_{(t_2-t_1)\to\infty} \int_{t_1}^{t_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\bar{E}\cdot\bar{J})dxdydzdt \Big/ (t_2-t_1) \,. \tag{4.18}$$

Now the variables of integration may be changed to the primed variables. The Jacobian of the transformation is unity. By direct substitution it is seen that

$$\bar{E}\cdot\bar{J} = \frac{\bar{E}'\cdot\bar{J}'+vf'_z}{\sqrt{1-\beta^2}} \tag{4.19}$$

where f'_z is the z' component of the volume force density. In the primed system $\bar{E}'\cdot\bar{J}' = 0$ since within the sphere $\bar{E}' = 0$, on the surface $\bar{E}'$ and $\bar{J}'$ are perpendicular, and outside of the surface $\bar{J}' = 0$. The domain of integration is transformed to the primed system so that (4.18) becomes

$$\lim_{(t_2-t_1)\to\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{t_1\sqrt{1-\beta^2}-vz'/c^2}^{t_2\sqrt{1-\beta^2}-vz'/c^2} \frac{vf'_z}{\sqrt{1-\beta^2}}\, dt'dz'dy'dx' \Big/ (t_2-t_1) \,. \tag{4.20}$$

Now in the primed system the sphere is stationary and a quadratic function of the field quantities such as f'_z varies as

$$f(x',y',z')e^{2i\omega't'} + f^{*}e^{-2i\omega't'} + g(x',y',z') \quad . \tag{4.21}$$

When (4.21) is substituted into (4.20) expression (4.18) becomes

$$v\langle F'_z\rangle = v\langle F'_\zeta\rangle \cos\theta'^{(i)} \tag{4.22}$$

where $\langle F'_z \rangle$ is the time averaged total electromagnetic force acting in the z' direction on the sphere. The form of (4.22) justifies the assumption that this quantity represents mechanical power. The total force acting on a stationary sphere due to an incident plane wave appears in the direction of propagation of the wave, $-\zeta$ direction, and its magnitude has been calculated by Debye (1909). Using his expression shows that the rate of flow of energy from the electromagnetic field to mechanical energy is

$$-\frac{\beta I_o \pi}{k^2}\left[I_5 + I_2\right]\cos\theta'^{(i)} \quad . \tag{4.23}$$

where I_2 is the same as the I_2 presented in section 4.2 and

$$I_5 = \sum_1^{\infty} (2n+1)(\alpha_n + \beta_n + c.c.) \quad . \tag{4.24}$$

Compare (4.23) with (4.17). It appears that

$$\frac{\pi I_o}{k^2}\left\{I_1 - \beta I_5 \cos\theta'^{(i)}\right\} \tag{4.25}$$

represents the rate of flow of energy from the incident wave to the sphere, while

$$\frac{\pi I_o}{k^2}\left\{I_1 + \beta\cos\theta'^{(i)} I_2\right\} \tag{4.26}$$

represents the rate of flow of energy from the sphere to the scattered field. The difference of these two expressions is (4.23), the mechanical power obtained from the total electromagnetic field.

Notice that for $0 < \theta'^{(i)} < \pi/2$, (4.23) is negative, implying a conversion of mechanical to electromagnetic energy, while for $\pi/2 < \theta'^{(i)} < \pi$ there is a conversion of electromagnetic energy to mechanical energy. The first terms in (4.25) and (4.26) are seen to appear in the case of scattering by a stationary sphere (Stratton, 1941, p. 569). The quantity I_2 is from (4.8) the integration of the ζ'

component of the Poynting vector over the surface of a large spherical volume containing the conducting sphere. Now, the scattered electromagnetic field may be regarded as containing a volume momentum density proportional to the Poynting vector which is being transported radially with the velocity c. This implies that I_2 is proportional to the rate of flow of the ζ' component of linear momentum from the sphere to the scattered electromagnetic field. In a similar manner I_5 is found to represent the rate of flow of momentum in the ζ' direction from the incident field to the sphere. The difference between (4. 25) and (4. 26) is proportional to the rate of flow of momentum from the total field to the sphere in the ζ' direction.

At this point the far field scattered by a perfectly conducting sphere moving through an incident plane wave has been obtained. This expression has been used to calculate the angular distribution of scattered energy for several cases and to discuss the energy exchanges that occur in such situations. Finally, the result of some numerical calculations using this solution is presented and discussed in Appendix A. Now this problem will be left and a problem concerning electromagnetic scattering by an accelerating body will be considered.

Chapter Five

AN EXPRESSION FOR THE FIELD SCATTERED BY A CONDUCTING SHEET MOVING WITH HYPERBOLIC MOTION

5.1 Statement of the Problem and Discussion

The problem considered in this and the following chapter consists of a plane wave incident upon a moving infinite sheet. The sheet is perfectly conducting, infinite in extent in the y and z directions, and moves in the x direction hyperbolically with time, t, satisfying

$$\left(x+\frac{c^2}{a}\right)^2 - c^2t^2 = \frac{c^4}{a^2} \quad . \tag{5.1}$$

This motion is the relativistically correct solution for the motion of an object under the influence of a constant force (Møller, 1952, pp 74-5). As a function of t, the position, velocity, and acceleration of the sheet are respectively

$$x = \frac{a t^2}{\left\{1+\frac{a^2t^2}{c^2}\right\}^{1/2} + 1} \quad , \tag{5.2}$$

$$\frac{dx}{dt} = \frac{at}{\left\{1+\frac{a^2t^2}{c^2}\right\}^{1/2}} \quad , \tag{5.3}$$

and

$$\frac{d^2x}{dt^2} = \frac{a}{\left\{1+\frac{a^2t^2}{c^2}\right\}^{1/2}} \quad . \tag{5.4}$$

For $|t| \ll c/a$ the motion corresponds to the case of constant acceleration, a, as predicted by classical mechanics.

For $|t| \gg c/a$,

$$x = c|t| - \frac{c^2}{a} + O(1/t) , \tag{5.5}$$

$$\frac{dx}{dt} = c\frac{|t|}{t} + O(1/t^2) , \tag{5.6}$$

and

$$\frac{d^2x}{dt^2} = 0 + O(1/t^3) , \tag{5.7}$$

showing that the motion asymptotically becomes that of constant translation with the velocity of light, c.

Initially, $t = -\infty$, the infinite sheet is at $x = \infty$, moving in the negative x direction with the velocity of light, with all space being permeated by a plane electromagnetic wave. The propagation of this wave is given by its wave vector $\bar{k}$ which is taken to be in the x-y plane making an angle ϕ with the x axis. Its magnitude is E_o and the following two polarizations are considered.

Case I:

$$E_z^{(i)} = E_o e^{ik(x\cos\phi + y\sin\phi - ct)} \tag{5.8}$$

$$B_x^{(i)} = \frac{1}{c} E_z^{(i)} \sin\phi \tag{5.9}$$

$$B_y^{(i)} = -\frac{1}{c} E_z^{(i)} \cos\phi . \tag{5.10}$$

Case II:

$$H_z^{(i)} = \eta_o E_o e^{ik(x\cos\phi + y\sin\phi - ct)} \tag{5.11}$$

$$D_x^{(i)} = -\frac{1}{c} H_z^{(i)} \sin\phi \tag{5.12}$$

$$D_y^{(i)} = \frac{1}{c} H_z^{(i)} \cos\phi \tag{5.13}$$

In these equations $\eta_o = \sqrt{\epsilon_o/\mu_o}$ and $\omega = ck$.

In the discussions to follow it is convenient to consider the x, t behavior for constant z and y of the electromagnetic waves. Some properties of a plane wave in this respect will now be considered.

First, to an observer at a point x_2, the field of a plane wave appears to lag the field seen by an observer at x_1 by a time interval $\Delta t = \frac{x_2 - x_1}{c} \cos \phi$. In this sense the plane wave appears to move with a velocity $v_{px} = \frac{\omega}{k_x} = \frac{c}{\cos \phi}$. However, the ratio of the flux of energy in the x direction to the volume energy density is, for a plane wave, $c \cos \phi$. In terms of energy transport a velocity $v_{gx} = \frac{d\omega}{dk_x} = c \cos \phi$ may be associated with the wave. These definitions are recognized to be respectively the phase and group velocities associated with the concept of propagation through a dispersive medium. Here also these concepts are meaningful.

In the x, ct plane a motion with the group velocity will be a straight line trajectory with a slope greater than or equal to unity and that with the phase velocity will be a straight line also, but with a slope less than or equal to unity. The two slopes are reciprocal.

For

$$t = \frac{c}{a} \cot \phi \tag{5.14}$$

the velocity of the sheet, (5.3), and the group velocity of the incident wave in the x direction, v_{gx}, are equal. For $t > \frac{c}{a} \cot \phi$ the sheet appears to be moving away from the incident wave and it is expected that the behavior of the solution will change as t goes from $-\infty < t < \frac{c}{a} \cot \phi$ to $\frac{c}{a} \cot \phi < t < \infty$.

5.2 The Transformation of the Problem

There exists for hyperbolic motion the following transformation (Møller, 1952, p. 255)

$$x^1 = x = x' \cosh \frac{at'}{c} - \frac{c^2}{a} \tag{5.15}$$

$$x^2 = y = y' \tag{5.16}$$

$$x^3 = z = z' \tag{5.17}$$

$$x^4 = t = \frac{x'}{c} \sinh \frac{at'}{c} \quad . \tag{5.18}$$

Its inverse is

$$x^{1'} = x' = \left\{ \left[\left(x + \frac{c^2}{a} \right)^2 - c^2 t^2 \right] \right\}^{1/2} \tag{5.19}$$

$$x^{2'} = y' = y \tag{5.20}$$

$$x^{3'} = z' = z \tag{5.21}$$

$$x^{4'} = t' = \frac{c}{a} \coth^{-1} \left\{ \left[\frac{x + \frac{c^2}{a}}{ct} \right] \right\} \quad . \tag{5.22}$$

The value $x' = x'_o = c^2/a$ describes the motion of the sheet

$$\left(x + \frac{c^2}{a} \right)^2 - c^2 t^2 = \frac{c^4}{a^2} \quad . \tag{5.1}$$

Within the **Minkowski** light cone given by

$$-\left(x + \frac{c^2}{a} \right) < ct < \left(x + \frac{c^2}{a} \right)$$

the transformation is one-to-one. From (5.19) curves of constant x' are hyperbolae approaching the light cone as x' goes to 0. Curves of constant t' are straight lines through $t = 0$, $x = c^2/a$. They also approach the light cone as $|t'|$ goes to infinity.

By differentiation of (5.15) and (5.18)

$$dx = \cosh \frac{at'}{c} dx' + \frac{ax'}{c} \sinh \frac{at'}{c} dt' \tag{5.23}$$

$$dt = \frac{1}{c} \sinh \frac{at'}{c} dx' + \frac{ax'}{c^2} \cosh \frac{at'}{c} dt' \quad . \tag{5.24}$$

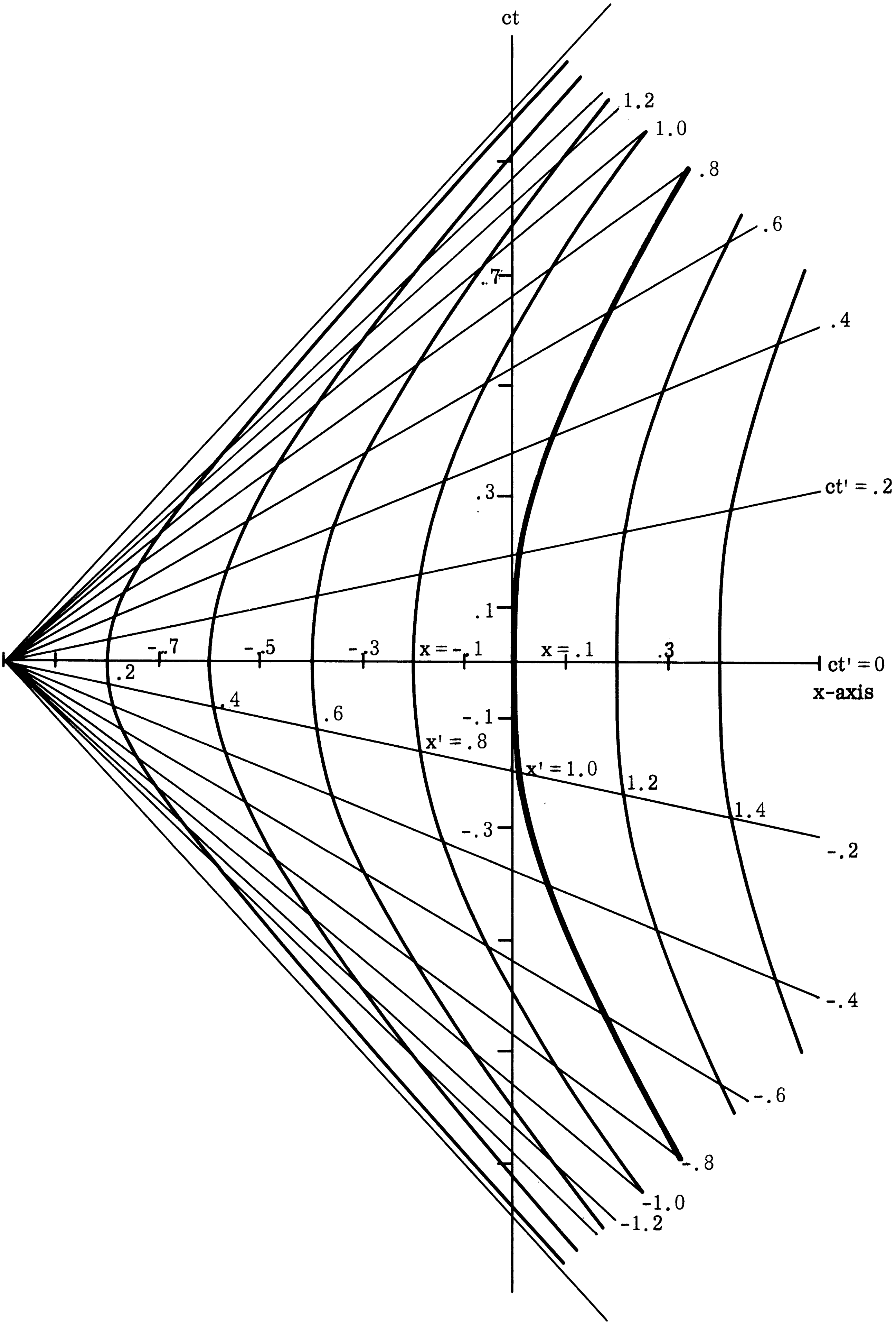

FIG. 5-1: THE COORDINATES x' AND t', $x'_o = 1$.

A point x' appears to move in the x, t system with the velocity

$$v = \left.\frac{dx}{dt}\right|_{dx'=0} = c \tanh \frac{at'}{c} \quad . \tag{5.25}$$

An interval of "length" dx' appears in the x, t system to have the length

$$\left. dx \right|_{dt=0} = \frac{dx'}{\cosh \frac{at'}{c}} = \sqrt{1 + \frac{v^2}{c^2}}\, dx' \ . \tag{5.26}$$

Infinitesimal lengths are contracted in accordance with the Lorentz-FitzGerald contraction. However, an interval of "time" dt' appears as an interval of time

$$\begin{aligned} \left. dt \right|_{dx'=0} &= \frac{ax'}{c^2} \cosh \frac{at'}{c}\, dt' \\ &= \frac{ax'}{c^2} \frac{dt'}{\sqrt{1 - \frac{v^2}{c^2}}} \end{aligned} \tag{5.27}$$

which agrees with the Einstein time dilation <u>only</u> if $x' = x'_0$.

A special property of this transformation is that on the surface of the conducting sheet, $x' = x'_0$, the transformation of infinitesimal quantities reduces to the Lorentz transformation,

$$dx = \frac{dx' + v\,dt'}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{5.28}$$

$$dt = \frac{dt' + \frac{v}{c^2} dx'}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{5.29}$$

where v is given by (5.25) and is a function of time. It is because of this that, on the surface of the sheet, the field quantities calculated by the method of section 2.2 are the actual physical quantities measured by an observer moving with constant velocity such that at the instant of the measurement, the motion of the sheet and the observer are coincident. The boundary condition as seen in the primed system may then be taken to be *

$$E_{z'} = 0 \tag{5.30}$$

$$E_{y'} = 0 \tag{5.31}$$

for

$$x' = x'_o \ . \tag{5.32}$$

The transformations between the primed and the unprimed systems can now be calculated. They are found by substitution of

$$\frac{\partial x^{\alpha'}}{\partial x^{\alpha}} = \begin{pmatrix} \cosh\frac{at'}{c} & 0 & 0 & -c\sinh\frac{at'}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\Delta}{c}\sinh\frac{at'}{c} & 0 & 0 & \Delta\cosh\frac{at'}{c} \end{pmatrix} , \tag{5.33}$$

$$\frac{\partial x^{\alpha}}{\partial x^{\alpha'}} = \begin{pmatrix} \cosh\frac{at'}{c} & 0 & 0 & \frac{c}{\Delta}\sinh\frac{at'}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{c}\sinh\frac{at'}{c} & 0 & 0 & \frac{1}{\Delta}\cosh\frac{at'}{c} \end{pmatrix} , \tag{5.34}$$

and the determinant of (5. 33),

$$\Delta = \frac{c^2}{ax'} = \frac{x'_o}{x'} , \tag{5.35}$$

into the equations of section 2.2. The transformations that will be needed are listed below.

* In this work it is assumed that the acceleration forces do not affect the conducting medium.

$$E_{z'} = \frac{1}{\Delta}\left(E_z \cosh\frac{at'}{c} + c\,B_y \sinh\frac{at'}{c}\right) \tag{5.36}$$

$$H_{z'} = \frac{1}{\Delta}\left(H_z \cosh\frac{at'}{c} - c\,D_y \sinh\frac{at'}{c}\right) \tag{5.37}$$

$$E_x = \Delta E_{x'} \tag{5.38}$$

$$E_y = \Delta E_{y'} \cosh\frac{at'}{c} + cB_{z'} \sinh\frac{at'}{c} \tag{5.39}$$

$$E_z = \Delta E_{z'} \cosh\frac{at'}{c} - cB_{y'} \sinh\frac{at'}{c} \tag{5.40}$$

$$B_x = B_{x'} \tag{5.41}$$

$$B_y = B_{y'} \cosh\frac{at'}{c} - \frac{\Delta E_{z'}}{c} \sinh\frac{at'}{c} \tag{5.42}$$

$$B_z = B_{z'} \cosh\frac{at'}{c} + \frac{\Delta E_{y'}}{c} \sinh\frac{at'}{c} \tag{5.43}$$

$$D_x = D_{x'} \tag{5.44}$$

$$D_y = D_{y'} \cosh\frac{at'}{c} + \frac{\Delta H_{z'}}{c} \sinh\frac{at'}{c} \tag{5.45}$$

$$D_z = D_{z'} \cosh\frac{at'}{c} - \frac{\Delta H_{y'}}{c} \sinh\frac{at'}{c} \tag{5.46}$$

$$H_x = \Delta H_{x'} \tag{5.47}$$

$$H_y = \Delta H_{y'} \cosh\frac{at'}{c} - cD_{z'} \sinh\frac{at'}{c} \tag{5.48}$$

$$H_z = \Delta H_{z'} \cosh\frac{at'}{c} + cD_{y'} \sinh\frac{at'}{c} \tag{5.49}$$

The differential equations may now be written in the primed system.

5.3 The Equations of Electrodynamics in the Primed System and Their Solutions

By equation (2.20) the non-zero elements of the constitutive tensor in the primed system are given by

$$\chi^{1'2'1'2'} = \chi^{1'3'1'3'} = \chi^{2'3'2'3'} = \frac{1}{\Delta\mu_o} , \tag{5.50}$$

$$\chi^{1'4'1'4'} = \chi^{2'4'2'4'} = \chi^{3'4'3'4'} = -\Delta\epsilon_o , \tag{5.51}$$

and the symmetry relation (2.15).

The resulting electrodynamic equations are

$$\nabla' \times \bar{E}' = -\frac{\partial \bar{B}'}{\partial t'} \tag{2.24}$$

$$\nabla' \cdot \bar{B}' = 0 \tag{2.25}$$

$$\nabla' \times \bar{H}' = \bar{J}' + \frac{\partial \bar{D}'}{\partial t'} \tag{2.26}$$

$$\nabla' \cdot \bar{D}' = \rho' \tag{2.27}$$

$$\bar{H}' = \frac{1}{\mu_o \Delta} \bar{B}' \tag{5.52}$$

$$\bar{D}' = \Delta\epsilon_o \bar{E}' . \tag{5.53}$$

The problem is invarient under an arbitrary translation in the z' direction; hence, it is anticipated that the scattered wave will not depend on z' —all z' derivatives vanish. This condition allows the above equations to be reduced to the following two sets of uncoupled equations valid in the source-free domain $x' \neq x'_o$.

$$x' \frac{\partial}{\partial x'}\left(x' \frac{\partial E_{z'}}{\partial x'}\right) + x'^2 \frac{\partial^2 E_{z'}}{\partial y'^2} - (x'_o/c)^2 \frac{\partial^2 E_{z'}}{\partial t'^2} = 0 \tag{5.54}$$

$$\frac{\partial B_{y'}}{\partial t'} = \frac{\partial E_{z'}}{\partial x'} \tag{5.55}$$

$$\frac{\partial B_{x'}}{\partial t'} = - \frac{\partial E_{z'}}{\partial y'} \tag{5.56}$$

and

$$x' \frac{\partial}{\partial x'}\left(x' \frac{\partial H_{z'}}{\partial x'}\right) + x'^2 \frac{\partial^2 H_{z'}}{\partial y'^2} - (x'_o/c)^2 \frac{\partial^2 H_{z'}}{\partial t'^2} = 0 \tag{5.57}$$

$$\frac{\partial D_{x'}}{\partial t'} = \frac{\partial H_{z'}}{\partial y'} \tag{5.58}$$

$$\frac{\partial D_{y'}}{\partial t'} = - \frac{\partial H_{z'}}{\partial x'} \quad . \tag{5.59}$$

If either $E_{z'}$ or $H_{z'}$ is written as the product $X(x')Y(y')T(t')$, equations (5.54) and (5.57) can be separated into three ordinary differential equations,

$$(x'_o/c)^2 \frac{\partial^2 T}{\partial t'^2} - q^2 T = 0 \ , \tag{5.60}$$

$$\frac{\partial^2 Y}{\partial y'^2} + p^2 Y = 0 \ , \tag{5.61}$$

and

$$x' \frac{\partial}{\partial x'}\left(x' \frac{\partial X}{\partial x'}\right) - (p^2 x'^2 + q^2) X = 0 \quad . \tag{5.62}$$

The solutions of (5.60) and (5.61) are exponentials,

$$T = e^{\pm qct'/x'_o} \tag{5.63}$$

and

$$Y = e^{\pm ipy'} \quad . \tag{5.64}$$

For $p \neq 0$, equation (5.62) is recognized to be a modified Bessel's equation and for $p = 0$ an Euler equation. Its solutions are for $p \neq 0$

$$X = I_q(px') = e^{-iq\pi/2} J_q(ipx') = e^{iq\pi/2} J_q(-ipx') \tag{5.65}$$

or

$$X = K_q(px') = \frac{\pi}{2} e^{i(q+1)\pi/2} H_q^{(1)}(ipx') = \frac{\pi}{2} e^{-i(q+1)\pi/2} H_q^{(2)}(-ipx') \tag{5.66}$$

and for $p = 0$

$$X = x'^{\pm q} \, . \tag{5.67}$$

In (5.65) and (5.66) I_q and K_q are the modified Bessel and Hankel functions. J_q, $H_q^{(1)}$, and $H_q^{(2)}$ are respectively the Bessel function and the Hankel functions of the first and second kinds; each of which has been extended to an imaginary argument. For integer order q and real x, both I_q and K_q are real quantities with I_q monotonically increasing with x and decreasing with q while K_q monotonically decreases with x and increases with q (Gray and Mathews, 1931).

Now the incident wave is transformed to the primed system where it is expanded in terms of the above solutions.

5.4 The Incident Field in the Primed System

The incident wave is transformed to the primed system by replacing the variables, x, y, z, t, by x(x', y', z', t'), etc. and then invoking expressions (5.36) and (5.37). The result obtained in this way is expanded in an infinite series involving the solutions presented in section 5.3. The resulting forms will be different depending on whether $\phi = 0$, normal incidence, or $\phi \neq 0$, oblique incidence. The two situations will be handled separately.

First oblique incidence is considered. The exponent of (5.8),

$$ik(x \cos\phi + y \sin\phi - ct) \, , \tag{5.68}$$

becomes in terms of the primed variables

$$ik\left(x'\left[\cos\phi \cosh\frac{at'}{c} - \sinh\frac{at'}{c}\right] - x_o' \cos\phi + y' \sin\phi\right) \, . \tag{5.69}$$

Define t_o' and k_o' so that

$$\sinh\frac{at'_0}{c} = \cot\phi \tag{5.70}$$

and

$$k'_0 = k\sin\phi \ . \tag{5.71}$$

Then (5.69) assumes the following form:

$$-ik'_0x'\sinh\frac{a}{c}(t'-t'_0) - ik'_0x'_0\sinh\frac{at'_0}{c} + ik'_0y' \ . \tag{5.72}$$

For the polarization of case I, from (5.36),

$$E^{(i)}_{z'} = \frac{x'}{x'_0}\left(\cosh\frac{at'}{c} - \cos\phi\sinh\frac{at'}{c}\right)E^{(i)}_z$$

$$= \frac{x'}{x'_0}\sin\phi\cosh\frac{a}{c}(t'-t'_0)E^{(i)}_z \ . \tag{5.73}$$

So that

$$E^{(i)}_{z'} = E_0\frac{x'}{x'_0}\sin\phi\cosh\frac{a}{c}(t'-t'_0)\exp\left\{ik'_0(y'-x'_0\sinh\frac{at'_0}{c}\right\} \cdot$$

$$\cdot\exp\left\{-ik'_0x'\sinh\frac{a}{c}(t'-t'_0)\right\}$$

$$= \frac{iE_0}{ck}\exp\left\{ik'_0\ y'-x'_0\sinh\frac{at'_0}{c}\right\}\frac{d}{dt'}\left[\exp\left\{-ik'_0x'\sinh\frac{a}{c}(t'-t'_0)\right\}\right] \ . \tag{5.74}$$

Now

$$e^{\frac{1}{2}u(w-w^{-1})} = \sum_{n=-\infty}^{\infty} J_n(u)w^n \ , \tag{5.75}$$

is an identity that can be verified by expanding the Bessel function and the exponential in their respective power series (Gray and Mathews, 1931, p. 31). The resultant power series are seen to converge for arbitrary complex but finite u and w. This allows u and w to be written $u = -ik_0x'$ and $w = \exp\left\{a(t'-t'_0)/c\right\}$

which results in the following identity

$$\exp\left\{-ik_o' x' \sinh \frac{a}{c}(t' - t_o')\right\} = \sum_{n=-\infty}^{\infty} J_n(-ik_o' x')\, e^{nc(t' - t_o')/x_o'} \quad . \qquad (5.76)$$

The substitution of (5.76) into (5.74) shows that for case I

$$E_{z'}^{(i)} = \frac{iE_o}{kx_o} \exp\left\{-ik_o' x_o' \sinh \frac{at_o'}{c}\right\} \sum_{n=-\infty}^{\infty} n\, e^{-nct_o'/x_o'} \left\{ e^{ik_o' y'} J_n(-ik_o' x') e^{nct'/x_o'} \right\} \quad . \qquad (5.77)$$

The quantity in the brackets is one of the solutions presented in section 5.3 with $q = n$, $p = k_o'$. A similar result is obtained in case II,

$$H_z^{(i)} = \frac{i\eta_o E_o}{kx_o'} \exp\left\{-ik_o' x_o' \sinh \frac{at_o'}{c}\right\} \sum_{n=-\infty}^{\infty} n\, e^{-nct_o'/x_o'} \left\{ e^{ik_o' y'} J_n(-ik_o' x') e^{nct'/x_o'} \right\} \quad . \qquad (5.78)$$

Little physical meaning as far as the behavior of the scattered wave is concerned can be given to the quantity k_o' other than the fact that it is the y component of the incident wave's propagation vector. It was introduced primarily for the purpose of simplifying the form of the resultant equations.

The quantity t_o', however, has a large importance in the behavior of the scattered wave. The curve $t' = t_o'$ is a straight line through $x = -x_o'$, $t = 0$. It intersects the trajectory of the accelerating sheet, $x' = x_o'$, at the point

$$t = \frac{c}{a} \cot \phi \qquad (5.79)$$

$$x = \frac{c^2}{a \sin \phi} - x_o' \quad . \qquad (5.80)$$

The time, (5.79), is the same critical value discussed in section 5.1.

Finally the case of normal incidence is considered. For $\phi = 0$ the exponent of (5.8) becomes

$$ikx' e^{-at'/c} - ikx'_o \tag{5.81}$$

and

$$E_{z'}^{(i)} = E_o \frac{x'}{x'_o} e^{-at'/c} e^{-ikx'_o} \exp\left\{ikx' e^{-at'/c}\right\}$$

$$= \frac{iE_o}{ck} e^{-ikx'_o} \frac{d}{dt'} \exp\left\{ikx' e^{-at'/c}\right\} \quad . \tag{5.82}$$

By using

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad , \tag{5.83}$$

the expression

$$E_{z'}^{(i)} = - \frac{iE_o}{x'_o k} e^{-ikx'_o} \sum_{n=1}^{\infty} \frac{(ik)^n}{(n-1)!} \left\{x'^n e^{-nct'/x'_o}\right\} \tag{5.84}$$

is obtained.

For the other polarization

$$H_{z'}^{(i)} = \eta_o E_{z'}^{(i)} \quad . \tag{5.85}$$

Now the choice of which of the solutions presented in section 5.3 may be used to represent the scattered wave must be made. For this a causality or radiation condition is necessary.

5.5 The Representation of the Scattered Wave

The representation of the scattered wave in the region $0 < x' < x'_o$ will now be determined. As this is a wave produced by currents and charges on the accelerating sheet, a solution valid in the above region must appear to have its sources in the region $x' \geqslant x'_o$.

The scattered wave is to be represented by an infinite series of the form

$$\sum_n E^{(s)}_{z'n} \qquad \text{or} \qquad \sum_n H^{(s)}_{z'n} \; ;$$

however, it is possible to consider a single term at a time. This is true since the substitution of a sum of solutions whose respective sources are in the region $x' \geqslant x'_o$ into the linear equations (2.24) and (2.27) shows that the sum also has its sources in the region $x' \geqslant x'_o$.

Again the fact that on the surface of the sheet the transformation to the primed system is an instantaneous Lorentz transformation allows the results of the special theory of relativity to be extended to this problem. Here, at $x' = x'_o$, the quantity $\bar{E}' \times \bar{H}'$ is the actual Poynting vector representing the flux of energy seen by a uniformly moving observer whose motion coincides with that of the sheet at the time of observation.

For the case of oblique incidence, examination of (5.77) and (5.78) shows that the $E^{(s)}_{z'n}$ and $H^{(s)}_{z'n}$ defined above are the sum of such quantities as

$$\alpha_n I_n(k'_o x') e^{ik'_o y'} e^{\pm nct'/x'_o} \tag{5.86}$$

and

$$\beta_n K_n(k'_o x') e^{ik'_o y'} e^{\pm nct'/x'_o} \; . \tag{5.87}$$

By using (5.55) and (5.59) the corresponding $B^{(s)}_{y'n}$ and $-D^{(s)}_{y'n}$ are the sum of

$$\pm \alpha_n \frac{k'_o x'_o}{nc} \frac{dI_n}{d(k'_o x')} e^{ik'_o y'} e^{\pm nct'/x'_o} \tag{5.88}$$

and

$$\pm \beta_n \frac{x'_o k'_o}{nc} \frac{dK_n}{d(k'_o x')} e^{ik'_o y'} e^{\pm nct'/x'_o} \; . \tag{5.89}$$

Also $B^{(s)}_{x'n}$ and $D^{(s)}_{x'n}$ are obtained from

$$\pm\alpha_n \frac{ik'x'_o}{nc} I_n e^{ik'_o y'} e^{\pm nct'/x'_o} \tag{5.90}$$

and

$$\pm\beta_n \frac{ik'x'_o}{nc} K_n e^{ik'_o y'} e^{\pm nct'/x'_o} \quad . \tag{5.91}$$

At $x' = x'_o$ equation (5.52) and (5.53) reduce to $\bar{H}' = \frac{1}{\mu_o}\bar{B}'$ and $\bar{D}' = \epsilon_o \bar{E}'$ so that the x-component of the Poynting vector may be written alternately

$$S_{x'} = (\bar{E}' \times \bar{H}')_{x'} = \frac{1}{\mu_o}(\bar{E}' \times \bar{B}')_{x'} = \frac{1}{\epsilon_o}(\bar{D}' \times \bar{H}')_{x'} \tag{5.92}$$

Now the field quantities are written as real quantities in the usual way—one half the sum of the complex field and its complex conjugate—and substituted into (5.92) which is averaged over y' by the operation

$$\lim_{Y \to \infty} \int_{-Y}^{Y} dy' \Big/ 2Y \; . \tag{5.93}$$

The result, using only the partial field involving the I_n term is

$$\langle S_{x'} \rangle = \mp A_n \frac{I_n}{n} \frac{dI_n}{dx'} e^{\pm 2nct'/x'_o} \tag{5.94}$$

and that involving only the K_n term is

$$\langle S_{x'} \rangle = \mp B_n \frac{K_n}{n} \frac{dK_n}{dx'} e^{\pm 2nct'/x'_o} \tag{5.95}$$

where A_n and B_n are positive and real. Both I_n and K_n are positive while dI_n/dx' is greater than zero and dK_n/dx' is less than zero.

The I_n term with the upper sign or the K_n term with the lower sign represent fields that always radiate energy from the sheet to the space $x' < x'_o$. The sources of these fields appear to be in the region $x'_o \lessdot x'$, an acceptable condition for a scattered wave. On the other hand, the fields represented by I_n with the lower sign and K_n with the upper sign show a constant absorption of energy by the sheet from the space $x' < x'_o$. The sources of these fields appear to be outside the region $x' > x'_o$. This is unacceptable in the sense of the above argument. This suggests that

$$\alpha_n = 0 \quad \text{for the lower sign} \tag{5.96}$$

and

$$\beta_n = 0 \quad \text{for the upper sign.} \tag{5.97}$$

This same result may be obtained from another point of view. This is done by examining the flow of energy as seen in the unprimed system when the lower part of the light cone given by $x + x'_o + ct = 0$ is approached. Since energy can never flow with a velocity greater than c, the only possible way that energy can propagate from the sheet to a point $x = -(x'_o + ct)$ is if the velocity of energy transport approach the velocity of light in the negative x-direction as the lower part of the light cone is approached. That is a causal wave must satisfy

$$\lim_{x \to -(x'_o + ct)} v_{gx} = -c \tag{5.98}$$

where v_{gx} is the group velocity defined earlier.

For the polarization given by case I or case II respectively, define

$$A = \frac{1}{2}(\Delta E_{z'} - cB_{y'}) e^{ct'/x'_o}$$

or

$$= \frac{1}{2}(\Delta H_{z'} + cD_{y'}) e^{ct'/x'_o} , \tag{5.99}$$

$$B = \frac{1}{2}(\Delta E_{z'} + cB_{y'}) e^{-ct'/x'_o}$$

or

$$= \frac{1}{2}(\Delta H_{z'} - cD_{y'}) e^{-ct'/x'_o} \tag{5.100}$$

and

$$D = cB_{x'}$$

or

$$= cD_{y'} \quad . \tag{5.101}$$

Then by using (5.38) to (5.49) the x-component of the Poynting vector averaged over y by the operation (5.93) is

$$<S_x> = <\bar{E} \times \bar{H}>_x = \frac{1}{c\mu_o} <A^2 - B^2>$$

or

$$= \frac{1}{c\epsilon_o} <A^2 - B^2> \quad . \tag{5.102}$$

The stored electromagnetic energy density averaged over y is

$$<w> = \epsilon_o \left(A^2 + B^2 + \frac{D^2}{2}\right)$$

or

$$= \mu_o \left(A^2 + B^2 + \frac{D^2}{2}\right) \tag{5.103}$$

so that in either case the group velocity is

$$<v_{gx}> = c \frac{<A^2> - <B^2>}{<A^2> + <B^2> + \frac{1}{2} <D^2>} \quad . \tag{5.104}$$

Clearly, its magnitude may never exceed the velocity of light, c.

For the case of oblique incidence there are four possibilities for either $E^{(s)}_{z'n}$ or $H^{(s)}_{z'n}$ given by (5.86) and (5.87). These, along with the associated field components, (5.88) to (5.91), are substituted into (5.104) and simplified by using the following properties of the modified Bessel functions (Gray and Mathews, 1931).

$$I_{n+1}(u) = \frac{dI_n}{du} - \frac{n}{u} I_n , \tag{5.105}$$

$$I_{n-1}(u) = \frac{dI_n}{du} + \frac{n}{u} I_n , \tag{5.106}$$

$$K_{n+1}(u) = -\frac{dK_n}{du} + \frac{n}{u} K_n , \tag{5.107}$$

and

$$K_{n-1}(u) = -\frac{dK_n}{du} - \frac{n}{u} K_n , \tag{5.108}$$

If $E^{(s)}_{z'n}$ or $H^{(s)}_{z'n}$ depends on I_n the result is

$$<v_{gx}> = \pm c \frac{I^2_{n+1} e^{\pm 2ct'/x'_o} - I^2_{n-1} e^{\mp 2ct'/x'_o}}{I^2_{n+1} e^{\pm 2ct'/x'_o} + I^2_{n-1} e^{\mp 2ct'/x'_o} + 2I^2_n} \tag{5.109}$$

and if they depend on K_n it is

$$<v_{gx}> = \pm c \frac{K^2_{n+1} e^{\pm 2ct'/x'_o} - K^2_{n-1} e^{\mp 2ct'/x'_o}}{K^2_{n+1} e^{\pm 2ct'/x'_o} + K^2_{n-1} e^{\mp 2ct'/x'_o} + 2K^2_n} \quad . \tag{5.110}$$

The various cases will be grouped as before. First expression (5.109) with the upper signs and expression (5.110) with the lower signs will be considered. The expansions of K_n and I_n about $x' = 0$ show

$$K_{n+1} \gg K_n \gg K_{n-1} \tag{5.111}$$

and

$$I_{n-1} \gg I_n \gg I_{n+1} \tag{5.112}$$

for $k'_o x' \ll 1$. For this case as the lower part of the light cone is approached, $t' < 0$, $x' \ll 1$,

$$<v_{gx}> \longrightarrow -c \tag{5.113}$$

satisfying condition (5.98).

Now the other case where the lower signs in (5.109) and the upper signs in (5.110) are used will be considered. For a given x' there exists a value of t', $t' < 0$, such that (5.109) or (5.110) will vanish. As the light cone is approached with t' greater than this value

$$<v_{gx}> \rightarrow c \quad , \tag{5.114}$$

while with t' much less than this value

$$<v_{gx}> \longrightarrow -c \quad . \tag{5.115}$$

It can be shown that in the limit as x approaches $-(x'_o+ct)$, $\langle v_{gx}\rangle$ goes continuously from c to -c, as t goes from 0 to $-\infty$. Clearly the solutions in this case violate condition (5.98).

Finally, the results of the previous discussion, the fact that $I_n = I_{-n}$ and $K_n = K_{-n}$, the expressions (5.65) and (5.66), and an examination of expression (5.77) or (5.78) imply the scattered wave, $E^{(s)}_{z'}$ or $H^{(s)}_{z'}$, can be written

$$\sum_{n=-\infty}^{\infty}\left[\alpha_n J_n(-ik'_o x')+\beta_n H^{(2)}_n(-ik'_o x')\right] e^{ik'_o y'} e^{nct'/x'_o} \tag{5.116}$$

where $\alpha_n = 0$ if $n \leqslant 0$

and $\beta_n = 0$ if $n \geqslant 0$.

The analysis of the case of normal incidence may be handled in the same manner. The expression for $E^{(s)}_{z'}$ and $H^{(s)}_{z'}$ is found to be

$$\sum_{n=1}^{\infty}\alpha_n x'^{-n} e^{-nct'/x'_o} . \tag{5.117}$$

In these expressions the unknown constants will now be evaluated by invoking the boundary conditions (5.30) to (5.32).

5.6 The Scattered Wave

Using (5.30) to (5.32), the boundary condition becomes in case I

$$E_{z'} = E^{(i)}_{z'} + E^{(s)}_{z'} \quad \text{for} \quad x' = x'_o \tag{5.118}$$

and in case II

$$\frac{\partial H_{z'}}{\partial x'} = \frac{\partial H^{(i)}_{z'}}{\partial x'} + \frac{\partial H^{(s)}_{z'}}{\partial x'} = 0 \quad \text{for} \quad x' = x'_o . \tag{5.119}$$

The constants α_n and β_n are determined by substituting the series expressions for the incident and scattered waves into the above boundary conditions. The resultant solutions are for oblique incidence for case I,

$$E_{z'}^{(s)} = -\frac{iE_o}{kx_o'} \exp\left\{ik_o'\left(y' - x_o' \sinh\frac{at_o'}{c}\right)\right\} \cdot$$

$$\cdot \left\{\sum_{n=1}^{\infty} n J_n(ik_o'x') e^{nc(t' - t_o')/x_o'}\right.$$

$$\left. + \sum_{n=-\infty}^{-1} n \frac{J_n(-ik_o'x_o')}{H_n^{(2)}(-ik_o'x_o')} H_n^{(2)}(-ik_o'x') e^{nc(t' - t_o')/x_o'}\right\}, \qquad (5.120)$$

and for case II,

$$H_{z'}^{(s)} = -\frac{i\eta_o E_o}{kx_o'} \exp\left\{ik_o'\left(y' - x_o' \sinh\frac{at_o'}{c}\right)\right\} \cdot$$

$$\cdot \left\{\sum_{n=1}^{\infty} n J_n(-ik_o'x') e^{nc(t' - t_o')/x_o'}\right.$$

$$\left. + \sum_{n=-\infty}^{-1} \left.\frac{\frac{d}{dx'} J_n(-ik_o'x')}{\frac{d}{dx'} H_n^{(2)}(-ik_o'x')}\right|_{x'=x_o'} H_n^{(2)}(-ik_o'x') e^{nc(t' - t_o')/x_o'}\right\}. \qquad (5.121)$$

For normal incidence for case I,

$$E_{z'}^{(s)} = \frac{iE_o}{x_o'k} e^{-ikx_o'} \sum_{n=1}^{\infty} \frac{(ik)^n}{(n-1)!} \left\{(x'/x_o'^2)^{-n} e^{-nct'/x_o'}\right\}, \qquad (5.122)$$

and for case II,

$$H_{z'}^{(s)} = -\frac{iE_o\eta_o}{x'_o k} e^{-ikx'_o} \sum_{n=1}^{\infty} \frac{(ik)^n}{(n-1)!} \left\{ (x'/x'^2_o)^{-n} e^{-nct'/x'_o} \right\}. \tag{5.123}$$

A formal solution to this problem that satisfies the Maxwell equations, the boundary condition, and that is causal in the sense of section 5.5 has now been obtained. Given a point x, y, z, t, the corresponding components of the electromagnetic field can be calculated by the results of this chapter. The solutions are given in the form of infinite series which can be shown to converge; however, in general the convergence is very slow.

In the next chapter an attempt is made to express the solutions in a more physically meaningful form. It will be seen that in the case of normal incidence the series sums to an exponential function which is easily transformed to the stationary system.

In the case of oblique incidence a simple closed form does not seem feasible. However, the series may be written as a contour integral after the Watson technique. One part of this integral is approximated by a saddle point integration, which results in a solution that can be interpreted by extending the solution of the problem of plane wave scattering by a uniformly moving mirror. Unfortunately little meaning has been given to the remaining portion of the integral.

Chapter Six

A DISCUSSION OF THE FIELD SCATTERED BY THE HYPERBOLICALLY MOVING SHEET

6.1 An Intuitive Analysis of the Problem

A solution of this problem analogous to the ray-optics solution of problems involving scattering by stationary bodies is presented here. The presentation will be intuitive with little justification being given at this time. In the following sections it will be shown that in the case of normal incidence such a **solution is** exact while in the case of oblique incidence a solution of this nature seems to be incorporated in the exact solution.

In this attack a particle nature is given to the electromagnetic field.

A plane wave of infinite extent whose behavior with x and t is given by

$$E_z^{(i)} = E_o(y)\exp\left\{ik\left(x/\beta_{px}^{(i)} - ct\right)\right\} \tag{6.1}$$

is regarded as a distribution or sum over a parameter $C^{(i)}$ of particles or wavelets each of whose motions are given by the trajectories

$$ct = x/\beta_{gx}^{(i)} + C^{(i)} \tag{6.2}$$

while the value of each particle varies with (6.1). Here the velocities are normalized to the speed of light— $\beta_{gx}^{(i)} = v_{gx}^{(i)}/c$, $v_{gx}^{(i)}$ is the group velocity, and $\beta_{px}^{(i)} = v_{px}^{(i)}/c$, $v_{px}^{(i)}$ is the phase velocity. For free space, $\beta_{gx}\beta_{px} = 1$. It will be assumed such a representation is valid for the scattered wave, also.

The motion of the scattering sheet is represented parametrically by $x_s = x_s(u)$ and $ct_s = ct_s(u)$. The normalized velocity of the sheet is then

$$\beta_s = \frac{dx_s/du}{dct_s/du} \quad . \tag{6.3}$$

The relativistic law of addition of velocities (Møller, 1952, p. 52) implies the velocity of the incident wave particle relative to the sheet will be

$$\beta^{(i)}_{gx'} = \frac{\beta^{(i)}_{gx} - \beta_s}{1 - \beta_s \beta^{(i)}_{gx}} \quad . \tag{6.4}$$

By observing the problem relative to the moving sheet at the time of reflection the problem appears that of reflection by a stationary sheet. The solution of plane wave reflection by a stationary mirror or, since a particle nature is being given to the field, an argument based on conservation of momentum and energy shows that the scattered wave velocity relative to the sheet is $\beta^{(s)}_{gx'} = -\beta^{(i)}_{gx'}$. Relative to the stationary system this velocity is

$$\beta^{(s)}_{gx} = \frac{\beta_s - \beta^{(i)}_{gx'}}{1 - \beta_s \beta^{(i)}_{gx'}} \quad . \tag{6.5}$$

The scattered wave particle reflected from the point on the sheet given by $u = u_o$ will move with a trajectory

$$x - x_{so} = \beta^{(s)}_{gx} c(t - t_{so}) \tag{6.6}$$

and this reflected wave particle with vary with

$$E^{(s)}_z \propto \exp\left\{ ik^{(s)}(x/\beta^{(s)}_{px} - ct + \Phi)\right\}. \tag{6.7}$$

Drawing from the behavior of the plane wave let

$$\beta^{(s)}_{gx} \beta^{(s)}_{px} = 1 \quad .$$

The quantities $k^{(s)}$ and Φ will be determined by making the exponents of (6.7) and (6.1) equal on the surface of the sheet about the instance of scattering u_o. About u_o

$$x_s \sim x_{so} + \left.\frac{dx_s}{du}\right|_{u_o} (u - u_o)$$

and

$$t_s \sim t_{so} + \left.\frac{dt_s}{du}\right|_{u_o} (u - u_o) .$$

Substitution of these into (6.1) and (6.7) shows

$$\Phi = \frac{k}{k^{(s)}} (x_{so}/\beta_{px}^{(i)} - ct_{so}) - (x_{so}/\beta_{px}^{(s)} - ct_{so}) \tag{6.8}$$

and

$$k^{(s)} = k\left\{\frac{\beta_s/\beta_{px}^{(i)} - 1}{\beta_s/\beta_{px}^{(s)} - 1}\right\} . \tag{6.9}$$

The boundary condition that must be satisfied is taken to be

$$(\bar{E} + \bar{v} \times \bar{B}) \times \hat{n} = 0 ,$$

where $\hat{n}$ is a vector normal to the surface of the sheet. Assuming that the behavior of the incident and scattered waves is dominated by the exponentials of (6.1) and (6.7),

$$B_y^{(i)} = -\frac{1}{c\beta_{px}^{(i)}} E_z^{(i)}$$

and

$$B_y^{(s)} = -\frac{1}{c\beta_{px}^{(s)}} E_z^{(s)} .$$

The boundary condition is satisfied if

$$E_z^{(s)} = -E_z^{(i)} \left\{\frac{\beta_s/\beta_{px}^{(i)} - 1}{\beta_s/\beta_{px}^{(s)} - 1}\right\} \tag{6.10}$$

at the instance of reflection u_o.

Finally if, for two scattered wave trajectories reflected from the points $u = u_o$ and $u = u_o + \Delta u$, it is seen that the x distance between them for constant t

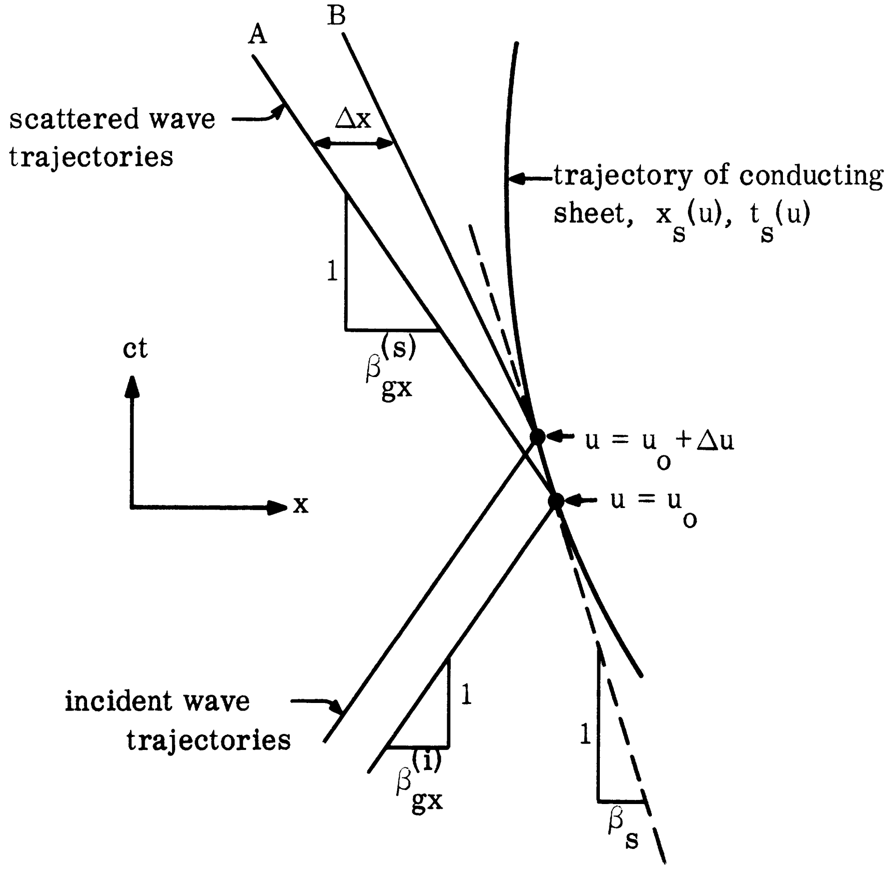

FIG. 6-1: TWO INCIDENT WAVE TRAJECTORIES SCATTERED BY A MOVING CONDUCTING SHEET.

changes as t increases; then, it is expected that the magnitude of the electromagnetic field will vary inversely with the square root of this x distance. A so-called divergence factor must be found. Let x^A and ct^A be the points on one of the above trajectories and x^B and ct^B the points on the other. Then

$$x^A - x_{so} = \beta_{gx}^{(s)} c(t^A - t_{so})$$

and to first order

$$x^B - x_{so} - \left.\frac{dx_s}{du}\right|_{u_o} \Delta u = \beta_{gx}^{(s)} c\left(t^B - t_{so} - \left.\frac{dt_s}{du}\right|_{u_o} \Delta u\right) + c(t^B - t_{so}) \left.\frac{d\beta_{gx}^{(s)}}{du}\right|_{u_o} \Delta u \ .$$

Subtract these two, let $x^B - x^A = \Delta x$, and let $t^B = t^A = t$ so that Δx is the x distance between the two trajectories for constant t. This results in

$$\Delta x = \left\{ \left.\frac{dx_s}{du}\right|_{u_o} - \beta^{(s)}_{gx}\, c \left.\frac{dt_s}{du}\right|_{u_o} + c(t - t_{so}) \left.\frac{d\beta^{(s)}_{gx}}{du}\right|_{u_o} \right\} \Delta u \ .$$

The spacing at the instance of reflection, $t = t_{so}$, is

$$\Delta x_o = \left\{ \left.\frac{dx_s}{du}\right|_{u_o} - \beta^{(s)}_{gs}\, c \left.\frac{dt_s}{du}\right|_{u_o} \right\} \Delta u$$

so that the divergence factor is

$$\sqrt{\Delta x_o / \Delta x} \quad .$$

The results of this discussion may be summarized as follows. The wave scattered from the point $x_s(u)$, $t_s(u)$ moves along the trajectory

$$x - x_s(u) = \beta^{(s)}_{gx}(u)\, c\left[t - t_s(u)\right] \qquad (6.6)$$

where $\beta^{(s)}_{gx}(u)$ is given by (6.5) and (6.4). Along this trajectory

$$E_z^{(s)} = -E_o(y) \left\{ \frac{\beta_s/\beta^{(i)}_{px} - 1}{\beta_s/\beta^{(s)}_{px} - 1} \right\} \left\{ 1 + \frac{c(t - t_s)\dfrac{d\beta^{(s)}_{gs}}{du}}{\dfrac{dx_s}{du} - \beta^{(s)}_{gs}\dfrac{dct_s}{du}} \right\}^{-1/2} \exp\left\{ ik^{(s)}(x/\beta^{(s)}_{px} - ct + \Phi) \right\} \ .$$

(6.11)

This method will now be applied to the problem of the sheet moving with hyperbolic motion.

6.2 Application of the Particle Trajectory Solutions to the Hyperbolically Moving Sheet

The method presented in the previous section may be applied to the hyperbolically moving sheet in a straightforward way. Here again the case of normal incidence and that of oblique incidence will be seen to be quite different, requiring that the two situations be handled separately.

For the case of oblique incidence as the time, t, becomes greater than the value $c \cos \phi / a$, mentioned in section 5.1, the solution obtained by the method of section 6.1 no longer has meaning. It will be seen that in this situation the scattered wave calculated in this way appears to be reflected from the side of the infinite sheet away from the source of the plane wave. Also it will be seen that there is no way for an incident wave particle moving along a trajectory given by (6.2) to encounter the surface of the sheet after this value of time.

What appears to be happening is that the sheet is moving away from the incident wave and beyond this time the total field attenuates to zero in a manner somewhat similar to the behavior of the electromagnetic field at a shadow boundary.

In this section only the polarization given by case I will be considered. The motion of the sheet may be expressed parametrically by

$$x_s = x_o' \cosh u - x_o' ,$$

$$ct_s = x_o' \sinh u ,$$

and

$$\beta_s = \tanh u . \tag{6.12}$$

For the case of normal incidence

$$E_o(y) = E_o$$

and

$$\beta_{gx}^{(i)} = \beta_{px}^{(i)} = 1 , \tag{6.13}$$

while for the case of oblique incidence

$$E_o(y) = E_o e^{iky \sin\phi}$$

and

$$\beta_{gx}^{(i)} = 1/\beta_{px}^{(i)} = \cos\phi = \tanh \frac{ct'_o}{x'_o} \quad . \tag{6.14}$$

In these, t'_o is the same quantity defined by equation (5.70). By direct application of the discussion of section 6.1, expressions for the scattered field can be found.

For the case of normal incidence the scattered field varies as

$$E_z^{(s)} = -E_o e^{-2u} \exp\left\{-ik e^{-2u}\left(x+ct-x'_o(2e^u - e^{2u} - 1)\right)\right\} \tag{6.15}$$

along the trajectory

$$x = x'_o(e^u - 1) - ct. \tag{6.16}$$

In this case the result may be simplified by eliminating the parameter u from (6.15) and (6.16), producing

$$E_z^{(s)} = -\frac{x_o'^2 E_o}{(x+x'_o+ct)^2} \exp\left\{ikx'_o\left(\frac{x'_o}{x+x'_o+ct} - 1\right)\right\} \tag{6.17}$$

which as will be shown in section 6.3, is identical to the result, (5.122) obtained in chapter five. In the case of normal incidence the solution described in the previous section is exact.

The solution in the case of oblique incidence assumes a more complicated form. Substitution of (6.14) into the equations of section 6.1 results in

$$E_z^{(s)} = -E_o \sin\phi \cosh\left(2u - \frac{ct'_o}{x'_o}\right)\left\{1 + \frac{2(ct - x'_o \sinh u)}{x'_o \sinh\left(\frac{ct'_o}{x'_o} - u\right)\cosh\left(2u - \frac{ct'_o}{x'_o}\right)}\right\}^{-1/2}$$

$$\cdot \exp\left\{ik \sin\phi\left[(x+x'_o)\sinh\left(2u - \frac{ct'_o}{x'_o}\right) - ct\cosh\left(2u - \frac{ct'_o}{x'_o}\right) + y\right.\right.$$

$$\left.\left. - x'_o \sinh\frac{ct'_o}{x'_o} + 2x'_o \sinh\left(\frac{ct'_o}{x'_o} - u\right)\right]\right\} \tag{6.18}$$

along the trajectory

$$(x+x'_o)\cosh\left(2u - \frac{ct'_o}{x'_o}\right) - ct\sinh\left(2u - \frac{ct'_o}{x'_o}\right) = x'_o\cosh\left(u - \frac{ct'_o}{x'_o}\right) \ . \qquad (6.19)$$

As u goes from $-\infty$ to ct'_o/x'_o the magnitude of $E_z^{(s)}$ goes from ∞ to 0 while its x component of group velocity goes from $-c$ to $\tanh(ct_o/x'_o)$ where its velocity equals that of the sheet. As u becomes greater than ct'_o/x'_o the wave's velocity becomes greater than the velocity of the sheet, a situation not physically acceptable. Examination of Fig. 6-2 shows that no incident wave trajectories reach the

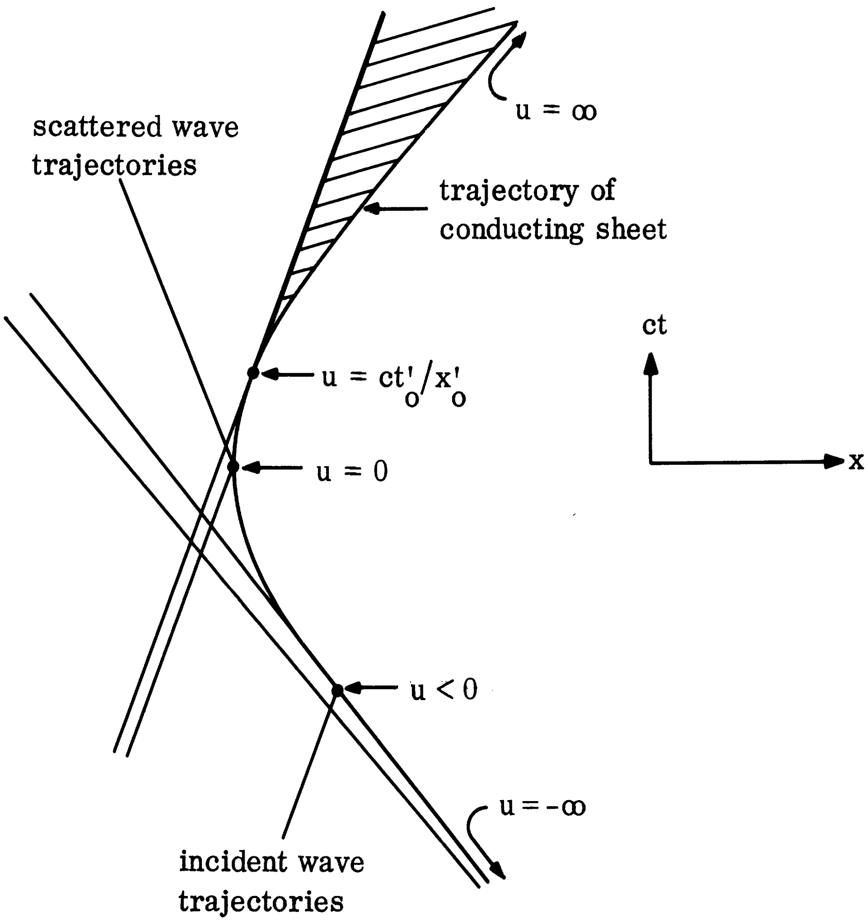

FIG. 6-2: INCIDENT WAVE TRAJECTORIES SCATTERED BY HYPERBOLICALLY MOVING SHEET.

surface of the scattering sheet in the shaded region. It appears that if this solution is valid anywhere, it certainly is not as the boundary of the shaded region is approached since an electromagnetic field can not be discontinuous along such a line, as implied by this analysis.

6.3 The Case of Normal Incidence

The case of normal incidence is particularly simple since the series (5.122) can be recognized as a series expansion of the exponential function. That is

$$E_{z'}^{(s)} = - \frac{x'_o E_o e^{-ikx'_o}}{x' e^{ct'/x'_o}} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \left\{ \frac{ikx'^2_o}{x'} e^{-ct'/x'_o} \right\}^{n-1}$$

$$= - \frac{x'_o E_o}{x' e^{ct'/x'_o}} \exp\left\{ ikx'_o \left(\frac{x'_o}{x' e^{ct'/x'_o}} - 1 \right) \right\} \qquad (6.20)$$

and by (5.55)

$$B_{y'}^{(s)} = - \frac{x'^2_o E_o}{cx'^2 e^{ct'/x'_o}} \exp\left\{ ikx'_o \left(\frac{x'_o}{x' e^{ct'/x'_o}} - 1 \right) \right\} . \qquad (6.21)$$

In the stationary system

$$E_{z}^{(s)} = - \frac{x'^2_o E_o}{x'^2 e^{2ct'/x'_o}} \exp\left\{ ikx'_o \left(\frac{x'_o}{x' e^{ct'/x'_o}} - 1 \right) \right\} . \qquad (6.22)$$

Finally from (5.15) and (5.18)

$$x' e^{ct'/x'_o} = x + x'_o + ct$$

so that

$$E_z^{(s)} = -\frac{x_o'^2 E_o}{(x+x_o'+ct)^2} \exp\left\{ ikx_o'\left(\frac{x_o'}{x+x_o'+ct} - 1\right)\right\} \tag{6.17}$$

which is the same as that obtained in section 6.2.

6.4 The Case of Oblique Incidence for Small Acceleration

For the case of oblique incidence the total field, incident plus scattered, is, for case I

$$E_{z'} = \frac{iE_o}{ck} \exp\left\{ ik_o'\left(y' - x_o' \sinh \frac{ct_o'}{x_o'}\right)\right\} \frac{d}{dt'}\left[\sum_{n=1}^{\infty} e^{in\pi} \left\{ J_n(-ik_o'x') - \frac{J_n(-ik_o'x_o')}{H_n^{(2)}(-ik_o'x_o')} H_n^{(2)}(-ik_o'x')\right\} e^{-nc(t'-t_o')/x_o'}\right], \tag{6.23}$$

and for case II

$$H_{z'} = \eta_o \frac{iE_o}{ck} \exp\left\{ ik_o'\left(y' - x_o' \sinh \frac{ct_o'}{x_o'}\right)\right\} \frac{d}{dt'}\left[\sum_{n=1}^{\infty} e^{in\pi} \left\{ J_n(-ik_o'x') - \left.\frac{\frac{d}{dx'} J_n(-ik_o'x')}{\frac{d}{dx'} H_n^{(2)}(-ik_o'x')}\right|_{x'=x_o'} H_n^{(2)}(-ik_o'x')\right\} e^{-nc(t'-t_o')/x_o'}\right]. \tag{6.24}$$

The behavior of these series is to be examined for the situation where $k_o'x' \gg 1$ and $k_o'x_o' \gg 1$. The second statement implies that ϕ must be away from zero while $kx_o' = kc^2/a = \omega c/a \gg 1$, a reasonable condition since in problems involving radiation $\omega \gg 1$ and the acceleration will for a real problem be much less than the speed of light. The first condition requires that a domain close to the light cone must be excluded. The analysis to follow will be simplified if only the quantities in the square brackets are considered.

These quantities may be written as contour integrals by using the method of residues which shows that

$$\sum_{n=1}^{\infty} A_n = \frac{1}{2i} \int_C e^{-i\nu\pi} \frac{A_\nu d\nu}{\sin \nu\pi} \tag{6.25}$$

where the contour C is shown in Fig. 6-3. It is assumed that A_ν is an analytic function in the domain being considered.

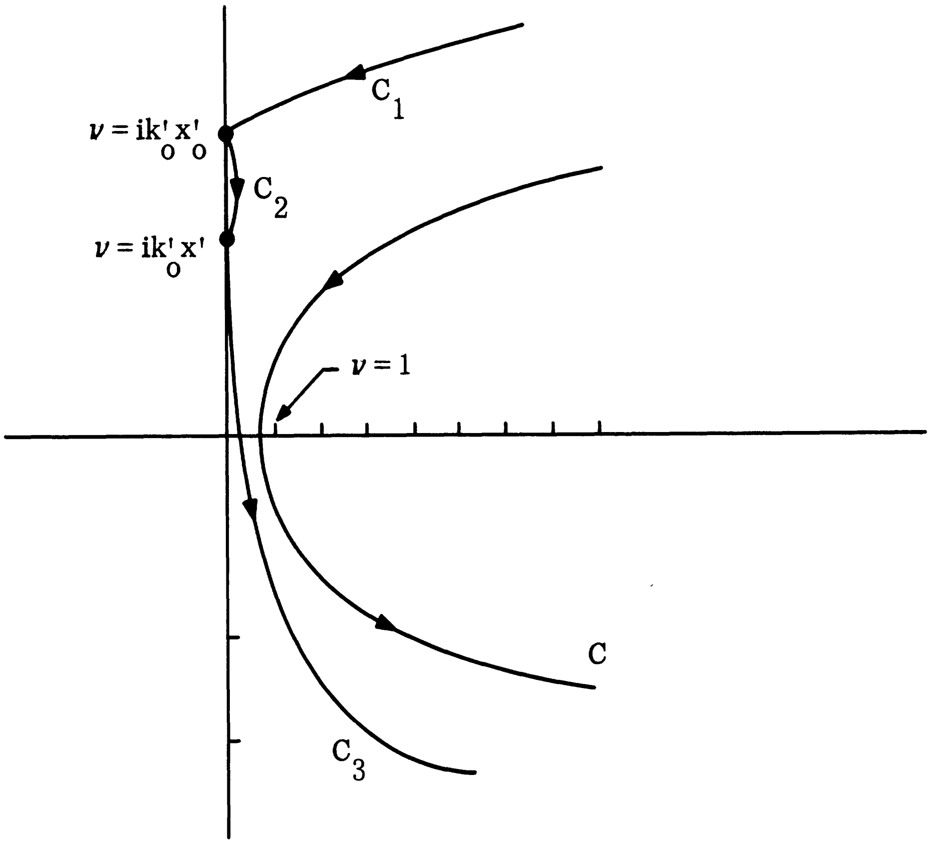

FIG. 6-3: THE CONTOURS C_1, C_2, C_3, AND C IN THE ν-PLANE.

The Bessel and Hankel functions have the following asymptotic approximations in the region of integration (Appendix B).

$$J_\nu(-iu) \sim \frac{e^{-i\nu\pi}}{\sqrt{2\pi u \cos W}}\, e^{f} \tag{6.26}$$

$$H_\nu^{(2)}(-iu) \sim \frac{2ie^{i\nu\pi}}{\sqrt{2\pi u \cos W}}\, e^{-f} \tag{6.27}$$

$$\frac{\frac{d}{du} J_\nu(-iu)}{\frac{d}{du} H_\nu^{(2)}(-iu)} \sim - \frac{J_\nu(-iu)}{H_\nu^{(2)}(-iu)} \tag{6.28}$$

where

$$\sin W = \frac{i\nu}{u}$$

$$\begin{aligned} &\mathrm{Im}(W) \geqslant 0 \quad &\text{for}\quad &-\pi/2 \leqslant \mathrm{Re}(W) \leqslant \pi/2 \\ &\mathrm{Im}(W) > 0 \quad &\text{for}\quad &-\pi \leqslant \mathrm{Re}(W) < -\pi/2 \\ &\mathrm{Im}(W) > 0 \quad &\text{for}\quad &\pi/2 < \mathrm{Re}(W) < \pi \end{aligned} \tag{6.29}$$

and

$$f = u\left(\cos W + (W + \frac{\pi}{2}) \sin W\right) \quad . \tag{6.30}$$

The series in the square brackets are converted to contour integrals and the above approximations are substituted giving

$$I = \frac{1}{2i} \int_C e^{-i\nu\pi} \frac{e^{f_2} \mp e^{2f_1 - f_2}}{\sin \nu\pi (2\pi k'_o x' \cos W_2)^{1/2}}\, e^{-\nu c(t' - t'_o)/x'_o}\, d\nu \ . \tag{6.31}$$

In this the upper sign is for case I and the lower for case II. The function

$$f_2 = k'_o x'\left(\cos W_2 + (W_2 + \frac{\pi}{2}) \sin W_2\right) \tag{6.32}$$

and

$$f_1 = k'_o x'_o\left(\cos W_1 + (W_1 + \frac{\pi}{2}) \sin W_1\right) \tag{6.33}$$

where

$$\sin W_2 = \frac{i\nu}{k'_o x'} \tag{6.34}$$

and

$$\sin W_1 = \frac{i\nu}{k'_o x'_o} \quad . \tag{6.35}$$

The functional relationship between W_1 and W_2 and ν is made one-to-one by restricting W_1 and W_2 in the same manner as was W.

The contour is deformed to pass through a vicinity about the point $\nu = ik'_o x'_o$ and about the point $\nu = ik'_o x'$. Let C_1 be the part of the contour C extending from ∞ in the upper half ν-plane to point $\nu = ik'_o x'_o$; C_2 be the part extending from $\nu = ik'_o x'_o$ to $\nu = ik'_o x'$; and C_3 be the part extending from $\nu = ik'_o x'$ down into the lower half ν-plane to infinity. By examining the behavior of these functions it is seen that along C_1 and C_2 the first term of the integrand is well suited to analysis by the method of stationary phase while along C_1 this is true for the second term. On the remaining portions of C such an analysis is not possible although these integrals do converge.

Let I be written $I = I_1 + I_2$ where

$$I_1 = \frac{1}{2i}\left\{\int_{C_1+C_2} e^{-i\nu\pi} \frac{e^{f_2}}{\sin\nu\pi\,(2\pi k'_o x' \cos W_2)^{1/2}}\, e^{-\nu c(t'-t'_o)/x'_o}\, d\nu\right.$$

$$+ i\int_{-C_2} e^{-i\nu\pi} \frac{e^{-f_2}}{\sin\nu\pi\,(2\pi k'_o x' \cos W_2)^{1/2}}\, e^{-\nu c(t'-t'_o)/x'_o}\, d\nu$$

$$\left.\pm \int_{-C_1} e^{-i\nu\pi} \frac{e^{2f_1-f_2}}{\sin\nu\pi\,(2\pi k'_o x' \cos W_2)^{1/2}}\, e^{-\nu c(t'-t'_o)/x'_o}\, d\nu\right\} \tag{6.36}$$

and

$$I_2 = \frac{1}{2i}\left\{\int_{C_3} e^{-i\nu\pi} \frac{e^{f_2}}{\sin\nu\pi(2\pi k_o' x' \cos W_2)^{1/2}} e^{-\nu c(t'-t_o')/x_o'} d\nu\right.$$

$$+ i\int_{C_2} e^{-i\nu\pi} \frac{e^{-f_2}}{\sin\nu\pi(2\pi k_o' x' \cos W_2)^{1/2}} e^{-\nu c(t'-t_o')/x_o'} d\nu$$

$$\left.\pm \int_{-C_1} e^{-i\nu\pi} \frac{e^{2f_1-f_2}}{\sin\nu\pi(2\pi k_o' x' \cos W_2)^{1/2}} e^{-\nu c(t'-t_o')/x_o'} d\nu\right\} . \qquad (6.37)$$

The term I_1 consists of the two parts of I that lend themselves to asymptotic evaluation and in addition a term resulting from the first integral of I being extended analytically around the branch point $\nu = ik_o'x'$ and up to $\nu = ik_o'x_o'$ making the total path of integration of I_1 a continuous curve starting at infinity, going to $\nu = ik_o'x'$, to $\nu = ik_o'x_o'$, and terminating at infinity.

I_1 may now be approximated. First since on the contour $\mathrm{Im}(\nu) \geqslant k_o'x' >> 1$, the following approximation is made

$$\frac{e^{-i\nu\pi}}{\sin\nu\pi} \sim -2i .$$

The contour is deformed to coincide with the imaginary axis. In the first integral of I_1 write $W = \frac{\pi}{2} - i\alpha$, $-\infty < \alpha < 0$, and in the second and third $W = \frac{\pi}{2} + i\alpha$, $0 < \alpha < \infty$. This results in

$$I_1 = -\sqrt{\frac{k_o'x'}{2\pi}}\left\{e^{-i3\pi/4}\int_{-\infty}^{0} \exp\left\{-ik_o'x'\left(\sinh\alpha - \left[\alpha - c(t'-t_o')/x_o'\right]\cosh\alpha\right)\right\}\right.$$

$$\cdot \left|\sinh\alpha\right|^{1/2} d\alpha$$

$$+ e^{+i3\pi/4}\int_{0}^{\alpha_o} \exp\left\{-ik_o'x'\left(\sinh\alpha - \left[\alpha - c(t'-t_o')/x_o'\right]\cosh\alpha\right)\right\} \left|\sinh\alpha\right|^{1/2} d\alpha$$

(cont'd)

$$\pm e^{i\pi/4} \int_{\alpha_o}^{\infty} \exp\left\{-ik_o' x'\left(\sinh\alpha - 2\,\frac{x_o'}{x'}\sinh\alpha_1 - \left[\alpha - 2\alpha_1 - c(t' - t_o')/x_o'\right]\cosh\alpha\right)\right\} \left|\sinh\alpha\right|^{1/2} d\alpha \; . \tag{6.38}$$

In these integrals

$$\cosh\alpha_1 = \frac{x'}{x_o'}\cosh\alpha \; , \qquad 0 < \alpha_1 < \infty \tag{6.39}$$

and

$$\cosh\alpha_o = \frac{x_o'}{x'} \; , \qquad 0 < \alpha_o < \infty \; . \tag{6.40}$$

The first integral has a point of stationary phase at

$$\alpha_s = \frac{c(t' - t_o')}{x_o'} \qquad \text{if} \qquad -\infty < ct' < ct_o' \; ;$$

the second at

$$\alpha_s = \frac{c(t' - t_o')}{x_o'} \qquad \text{if} \qquad 0 < ct' < \alpha_o x_o' + ct_o' \; ;$$

and the third at

$$\alpha_s = 2\alpha_1 + \frac{c(t' - t_o')}{x_o'} \qquad \text{if} \qquad -\infty < ct' < \alpha_o x_o' + ct_o' \; .$$

There are no stationary points if

$$ct' > \alpha_o x_1' + ct_o' \; .$$

The contribution to I_1 due to the stationary points is

$$I_1 = I_1^{(i)} + I_1^{(s)} = \exp\left\{-ik_o' x' \sinh \frac{c(t'-t_o')}{x_o'}\right\}$$

$$\mp \left\{ \frac{x_o' \sinh \alpha_1}{2x' \sinh\left[\frac{c(t'-t_o')}{x_o'} + 2\alpha_1\right] - x_o' \sinh \alpha_1} \right\}^{1/2}$$

$$\cdot \exp\left\{-ik_o'\left(x' \sinh\left[\frac{c(t'-t_o')}{x_o'} + 2\alpha_1\right] - 2x_o' \sinh \alpha_1\right)\right\},$$

$$\text{if } -\infty < ct' < \alpha_o x' + ct_o' \qquad (6.41)$$

By calculating $\partial I_1/\partial t'$ and $\partial I_1/\partial x'$ and ignoring all terms but those of the highest order in k_o' the asymptotic contribution of I_1 to $E_{z'}$, $B_{y'}$, $H_{z'}$ and $D_{y'}$ can be found. The part due to the first term, $I^{(i)}$, is the original expression for the incident wave. The result for the second term, $I^{(s)}$, is

$E_{z'}^{(s)}$ or $-H_{z'}^{(s)}/\eta_o$

$$= -\frac{x' E_o \sin\phi}{x_o'} \cosh\left(\frac{c(t'-t_o')}{x_o'} + 2\alpha_1\right) I_1^{(s)} \exp\left\{ik_o'\left(y' - x_o' \sinh \frac{ct_o'}{x_o'}\right)\right\} \qquad (6.42)$$

and

$B_{y'}^{(s)}$ or $D_{y'}^{(s)}/\eta_o$

$$= -\frac{E_o}{c} \sin\phi \sinh\left(\frac{c(t'-t_o')}{x_o'} + 2\alpha_1\right) I_1^{(s)} \exp\left\{ik_o'\left(y' - x_o' \sinh \frac{ct_o'}{x_o'}\right)\right\}. \qquad (6.43)$$

In the unprimed system

$E_z^{(s)}$ or $-H_z^{(s)}/\eta_o$

$$= -E_o \sin\phi \cosh\left(2\alpha_1 - \frac{ct'_o}{x'_o}\right)\left\{\frac{2x'\sinh\left(\frac{c(t'-t'_o)}{x'_o} + 2\alpha_1\right)}{x'_o\sinh\alpha_1} - 1\right\}^{-1/2}$$

$$\exp\left\{-ik\sin\phi\left[x'\sinh\left(\frac{c(t'-t'_o)}{x'_o} + 2\alpha_1\right) - y' + x'_o\sinh\frac{ct'_o}{x'_o} - 2x'_o\sinh\alpha_1\right]\right\} \tag{6.44}$$

By writing (6.44) and (6.39) in terms of x, y, and t and letting $\alpha_1 = \frac{ct'_o}{x'_o} - u$ and $\alpha = c\left(\frac{t'+t'_o}{x'_o}\right) - 2u$ it is found that (6.44) and (6.39) are equivalent to (6.18) and (6.19). The stationary phase approximation fails as α_1 approaches zero. The condition $\alpha_1 = 0$ implies

$$x - \left(\frac{c^2}{a\sin\phi} - x'_o\right) = \cos\phi\left(ct - \frac{c}{a}\cot\phi\right) \quad , \tag{6.45}$$

which is the line representing the boundary of the shaded area in Fig. 6-2. The quantities in parenthesis are those values of x and t given by (5.79) and (5.80) in section 5.4. As this line is crossed the contour integral I_1 has no points of stationary phase and so attenuates rapidly.

6.5 A Discussion of the Case of Oblique Incidence

In this section it will be shown that either the particle-trajectory solution or the solution obtained from the integral I_1, since the two are equivalent, satisfies time dependent forms of the Eikonal and the transport equations. First it is seen that

$$E_z^{(s)} = A(x, ct)e^{ik\sin\phi y}\, e^{-ik\sin\phi S(x,ct)} \tag{6.46}$$

where

$$S(x, ct) = (x+x'_o)\sinh\left(\frac{ct'_o}{x'_o} - 2u\right) + ct\cosh\left(\frac{ct'_o}{x'_o} - 2u\right) + x'_o\sinh\frac{ct'_o}{x'_o} - 2x'_o\sinh\left(\frac{ct'_o}{x'_o} - u\right) \quad , \tag{6.47}$$

$$A(x, ct) = -E_o\sin\phi\cosh\left(\frac{ct'_o}{x'_o} - 2u\right) \cdot \left\{2\,\frac{(x+x'_o)\sinh\left(\frac{ct'_o}{x'_o} - 2u\right) + ct\cosh\left(\frac{ct'_o}{x'_o} - 2u\right)}{x'_o\sinh\left(\frac{ct'_o}{x'_o} - u\right)} - 1\right\}^{-1/2} \tag{6.48}$$

and

$$(x+x'_o)\cosh\left(\frac{ct'_o}{x'_o} - 2u\right) + ct\sinh\left(\frac{ct'_o}{x'_o} - 2u\right) = x'_o\cosh\left(\frac{ct'_o}{x'_o} - u\right). \tag{6.49}$$

This solution will be substituted into the wave equation (Sommerfeld, 1964b). It is seen that

$$\nabla^2 E_z^{(s)} - \frac{\partial^2 E_z^{(s)}}{\partial(ct)^2} = k^2\sin^2\phi\left\{\left(\frac{\partial S}{\partial ct}\right)^2 - \left(\frac{\partial S}{\partial x}\right)^2 - 1\right\} E_z^{(s)}$$
$$- ik\sin\phi\left\{\left(\frac{\partial^2 S}{\partial x^2} - \frac{\partial^2 S}{\partial(ct)^2}\right) + \frac{2}{A}\left(\frac{\partial A}{\partial x}\cdot\frac{\partial S}{\partial x} - \frac{\partial A}{\partial ct}\cdot\frac{\partial S}{\partial ct}\right)\right\} E_{z1}^{(s)}$$
$$+ \frac{1}{A}\left\{\frac{\partial^2 A}{\partial x^2} - \frac{\partial^2 A}{\partial(ct)^2}\right\} E_z^{(s)} \tag{6.50}$$

Substitution of A and S shows

$$\left\{ \left(\frac{\partial^2 S}{\partial ct}\right)^2 - \left(\frac{\partial S}{\partial x}\right)^2 - 1 \right\} = 0 \tag{6.51}$$

and

$$\left\{ \left(\frac{\partial^2 S}{\partial x^2} - \frac{\partial^2 S}{\partial (ct)^2}\right) + \frac{2}{A} \left(\frac{\partial A}{\partial x} \cdot \frac{\partial S}{\partial x} - \frac{\partial A}{\partial ct} \cdot \frac{\partial S}{\partial ct}\right) \right\} = 0 \quad . \tag{6.52}$$

Equation (6.51) is recognized to be a time dependent form of the Eikonal equation while (6.52) is a time dependent form of the transport equation.

So, on one hand, drawing from the classical results of geometric optics it appears that the above solution should be valid in the limit as k goes to to infinity; while, on the other hand, the analysis of section 6.4 shows the existence of a second term I_2 . The two above statements would be in direct agreement, of course, if I_2 were seen to vanish for large k ; however it does not appear that this is the case. This work has not determined a meaning for that part of the electromagnetic field obtained from I_2 and it will terminate this question not completely answered. To the final chapter a possible reason for the existence of this term is given along with some suggestions on how this reason may be tested.

Chapter Seven

CONCLUSIONS

7.1 Some Observations on the Sphere Problem

The analysis of the scattering of a plane wave by a moving perfectly conducting sphere presented in this work leads to some conclusions which are pertinent to the sphere problem itself and also to ones which apply to more general problems.

Pertaining directly to the sphere problem an expression that is relativistically correct and valid for the far scattered field was obtained. This solution leaves the polarization and direction of the incident wave arbitrary and can be used to calculate the scattered field for any position and time provided only that the far field condition is fulfilled. In addition the form of this solution is such that the behavior expected from observation of other problems is apparent. That is the scattered wave appears locally that of a plane wave. Its direction of propagation is away from the retarded position of the sphere exhibiting the well known aberration of light effect. Its wave length and frequency are explained as a Doppler effect.

This solution was seen to lend itself to the calculation of the various forms of energy giving results that although of a complex nature permitted physical interpretation.

More generally the method that was used in obtaining this solution depended only on the fact that in the stationary case the far scattered field appeared as a spherical wave, a property that by the Sommerfeld radiation condition all finite bodies must possess. The far field scattered by an arbitrarily shaped finite body whose stationary far field solution is known in the form of a product of one function depending only on the angular coordinates and the other varying with the radial coordinate and time in the manner of a spherical wave can be found by direct application of (3.30) and (3.31). This is done by replacing the various spherical harmonics representing the angular variation of the scattered field components pertaining to the sphere in (3. 30) and (3. 31) by those pertaining to the body being considered. Also, the far field of any moving source may be determined this way.

7.2 The Accelerating Sheet Problem

A solution in the form of an infinite series was obtained for the problem of a hyperbolically moving, infinite, perfectly conducting sheet scattering an incident plane wave. In the case of the plane being normally incident upon the sheet the resultant solution was seen to be identical to a solution analogous to the ray-optics solution pertaining to stationary scattering problems.

In the case of oblique incidence it was possible to show that for small wavelength and small acceleration that the scattered wave could be approximated by the sum of two terms. One of these terms was equivalent to a scattered wave predicted by the same ray-optics type argument used in discussing the case of normal incidence. On the other hand this research was able to find little about the meaning of the second term.

It is suspected that in a more realistic problem in the limit of small acceleration and wavelength that this first solution is a valid representation of the electromagnetic field and that the appearance of the second term was due to the fact that the motion assumed in this problem allowed the sheet to be initially moving with the velocity of light. This could possibly be tested by;

1) solving other problems involving motions that do not approach the velocity of light,

or

2) a study of the behavior of I_2 including numerical calculations.

A justification of the ray-optics type of analysis would lead to some interesting effects worthy of future study. For instance, such an analysis of the case of electromagnetic scattering by an oscillating sheet shows the scattered wave trajectories intersecting resulting in an effect similar to the situation of caustics.

A second effect that is suggested for study is that which occurs when, in the direction of the sheets motion, the velocities of the incident wave and of the sheet are equal, resulting in an effect analogous to that of a shadow boundary in ray-optics problems (see Fig. 6-2). Perhaps an extension of the technique used by Fock (1946) would be applicable to this situation.

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Appendix A

NUMERICAL RESULTS OF THE SPHERE PROBLEM

Polar plots of the differential scattering cross section as a function of the observer's position for a uniformly moving, perfectly conducting, sphere satisfying the small sphere condition, $k'a << 1$, are given in the following figures. The calculations were based on Eqs. (4.3) and (4.4). The coordinate system used is described in Fig. 3-1.

The following observations can be made:

1. Let the ζ-ξ plane which contains both the direction of motion and the direction of the incident wave be called the plane of incidence. From Figs. A-1 and A-2 it is seen that the patterns are symmetrical about the plane of incidence.

2. For the case of a stationary sphere a rotation of both the polarization of the incident wave and of the plane of the observer about the axis of the incident wave's propagation vector by the same angle leaves the scattered wave pattern unchanged. Applying the symmetry in statement (1) to Fig. A-3 shows this isotropy is lost in the case of motion except for the case where the velocity vector and the incident wave propagation vector are parallel. This effect may also be observed by reference to the last term of equation (4.6).

3. From Figs. A-4 and A-5 it is observed that the energy scattered into the direction of motion is increased while that scattered into a direction opposite the direction of motion is reduced.

4. Figures A-1 through A-5 show that both the backscattering and total cross sections are increased when the motion is into the incident wave while they are decreased when the motion is away from the incident wave. This agrees with the results of section 4.4. In the first case there is a conversion of mechanical energy to electromagnetic energy while in the second there is a conversion from electromagnetic to mechanical.

5. Figures A-6 and A-7 show the nonlinear dependence on β so that as β approaches 1 the above effects become exaggerated.

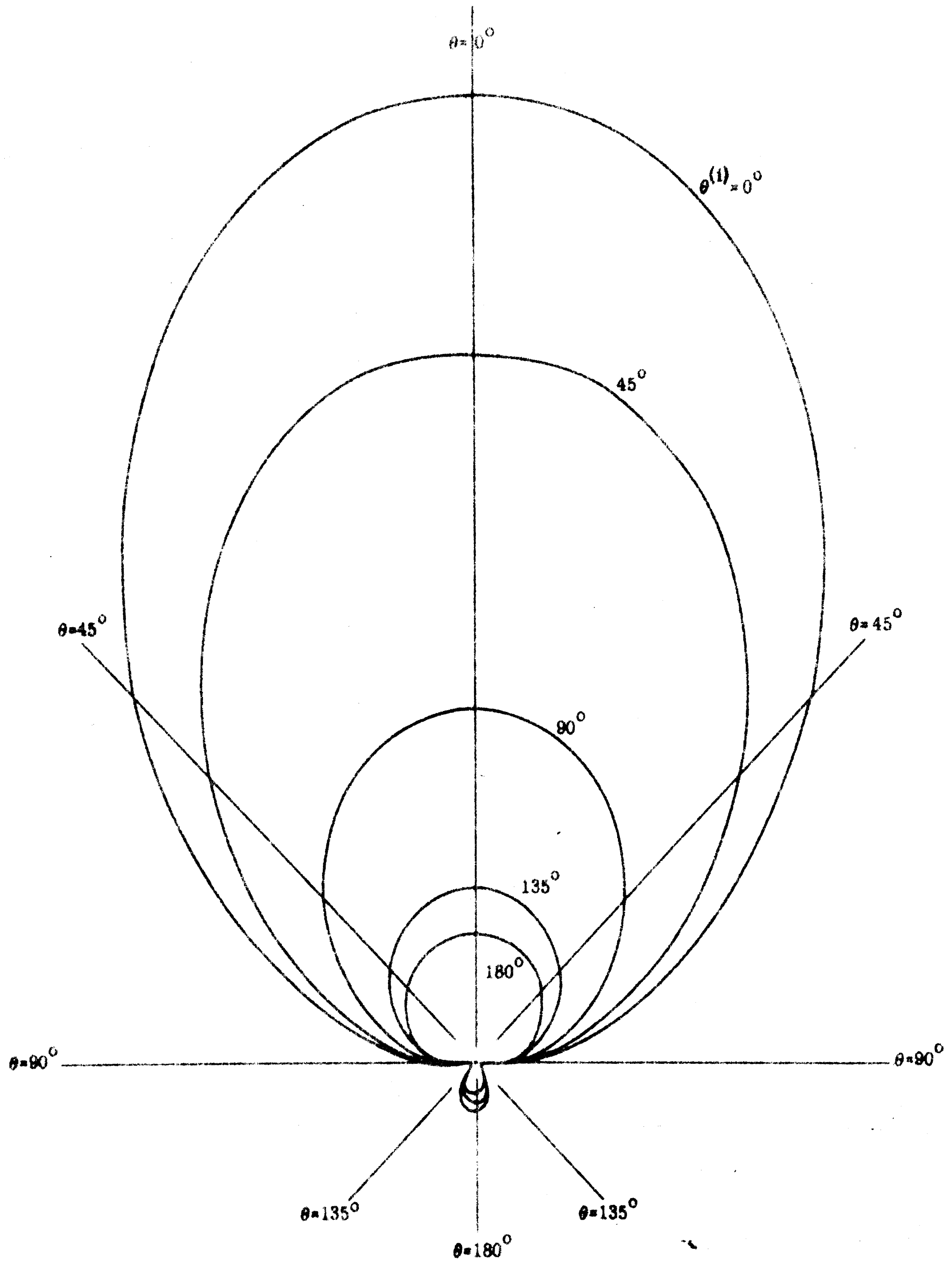

FIG. A-1: $\frac{d\sigma}{d\Omega}$ IN THE $\zeta - \eta$ PLANE FOR $\beta = 0.1$; $\psi = 0$; AND SEVERAL ANGLES OF INCIDENCE, $\theta^{(1)}$.

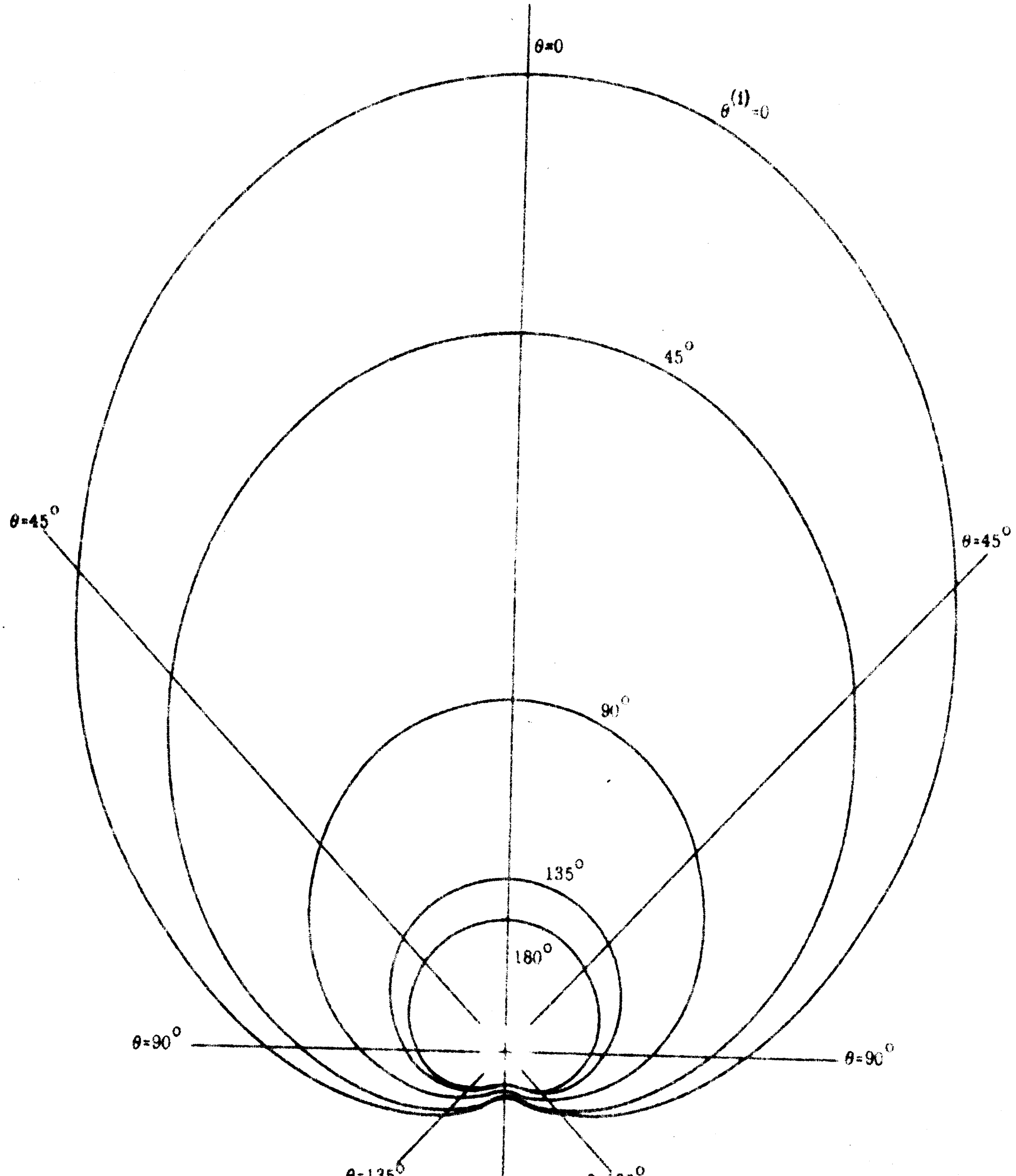

FIG. A-2: $\frac{d\sigma}{d\Omega}$ IN THE $\zeta - \eta$ PLANE FOR $\beta = 0.1$; $\psi =$ [illegible]$^\circ$; AND SEVERAL ANGLES OF INCIDENCE, $\theta^{(1)}$.

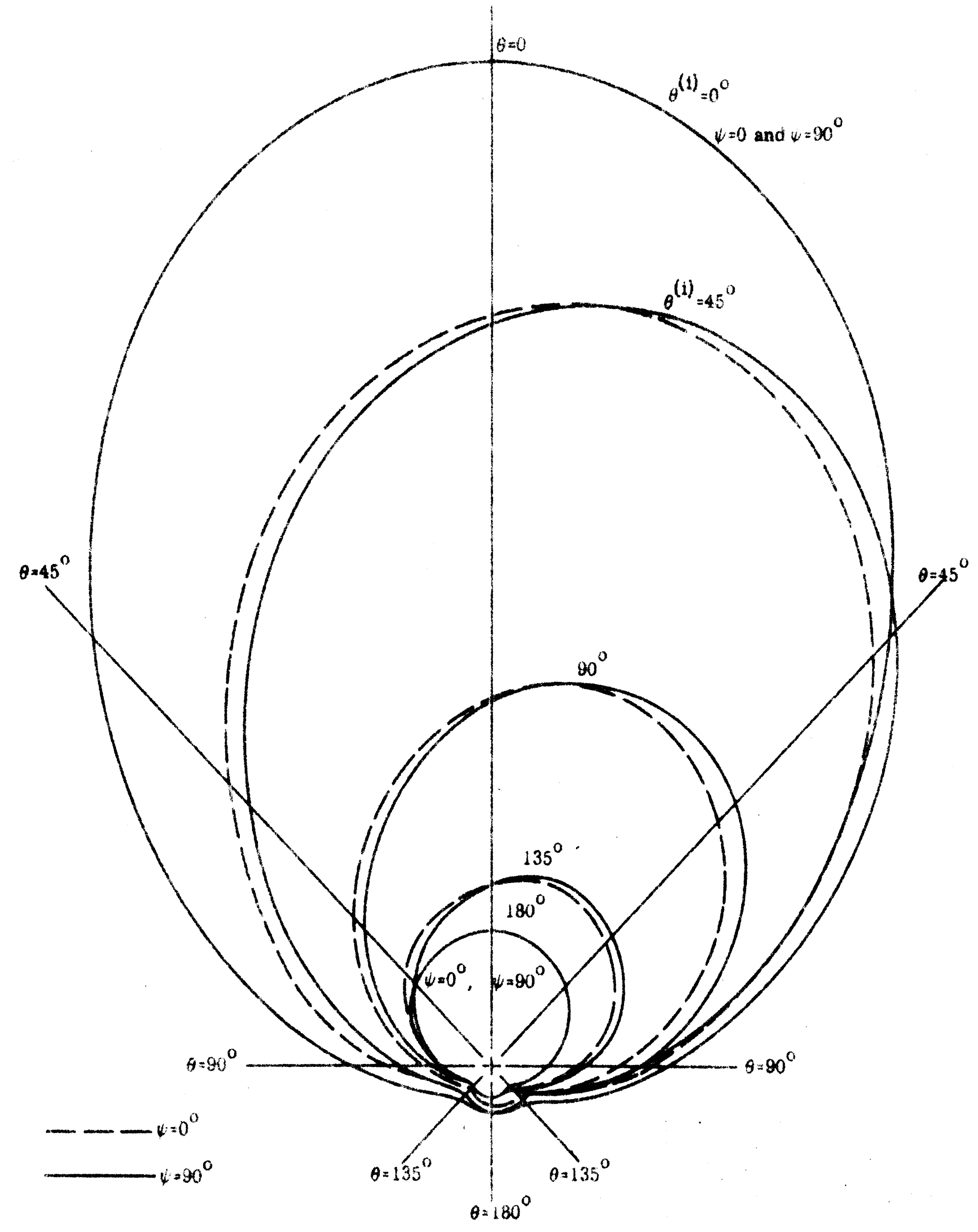

FIG. A-3: $\frac{d\sigma}{d\Omega}$ IN THE ϕ = 45° PLANE FOR β = 0.1; BOTH POLARIZATIONS; AND SEVERAL ANGLES OF INCIDENCE, $\theta^{(1)}$.

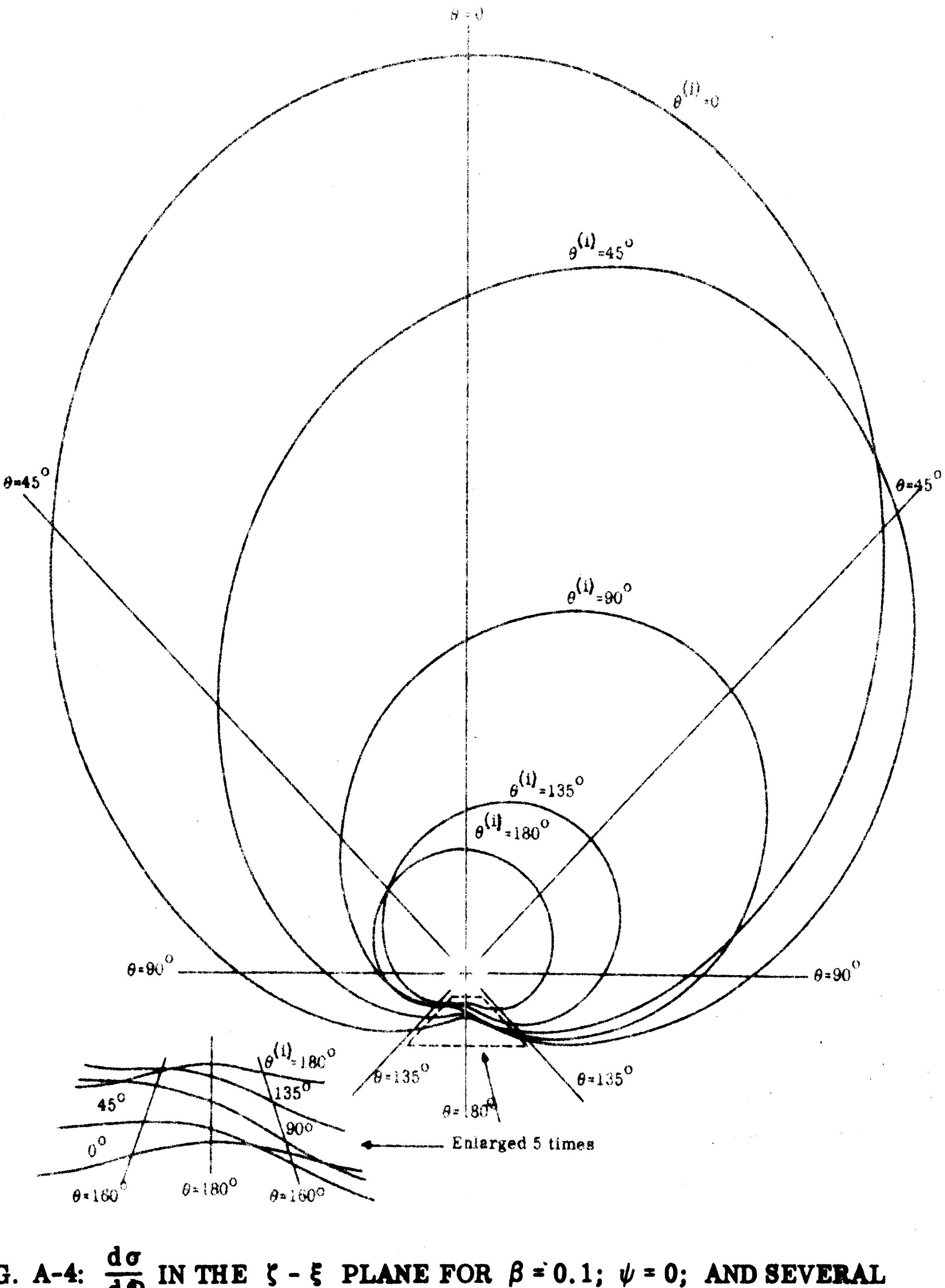

FIG. A-4: $\frac{d\sigma}{d\Omega}$ IN THE ζ - ξ PLANE FOR $\beta = 0.1$; $\psi = 0$; AND SEVERAL ANGLES OF INCIDENCE, $\theta^{(i)}$.

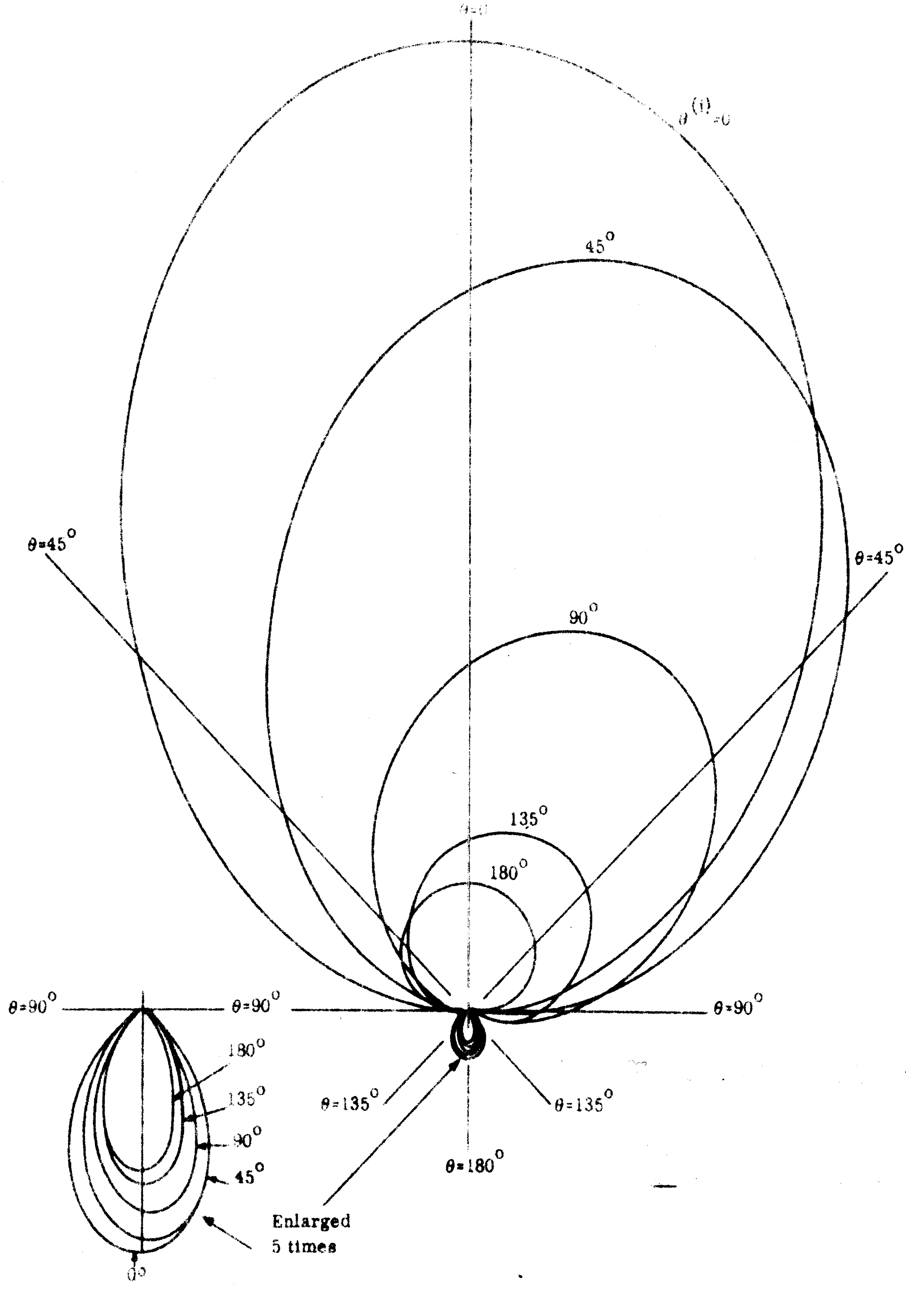

FIG. A-5: $\frac{d\sigma}{d\Omega}$ IN THE ζ - ξ PLANE FOR $\beta = 0.1$; $\psi = 90^{\circ}$; AND SEVERAL ANGLES OF INCIDENCE, $\theta^{(i)}$.

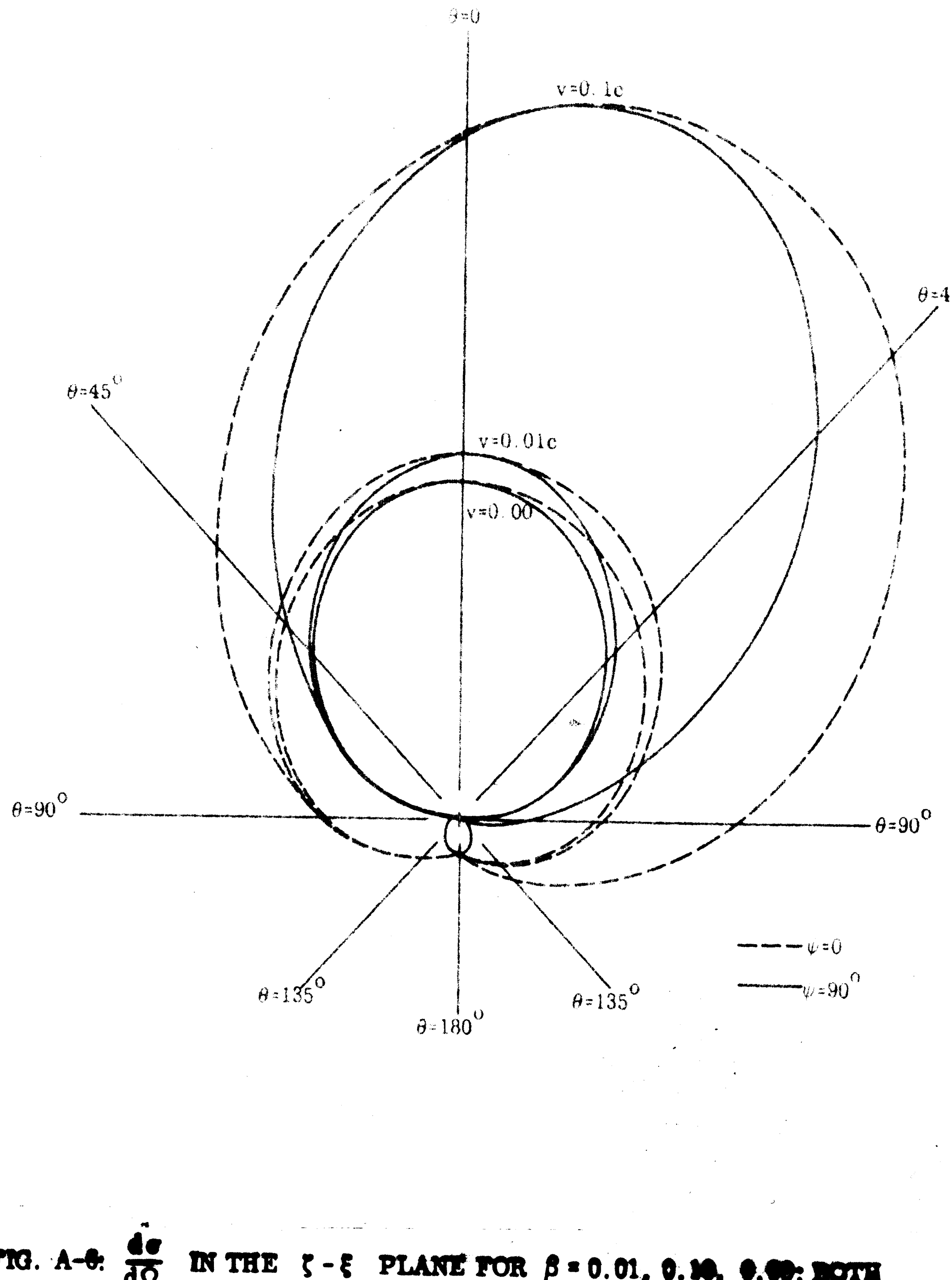

FIG. A-6: $\frac{d\sigma}{d\Omega}$ IN THE $\zeta - \xi$ PLANE FOR $\beta = 0.01, 0.10, 0.00$; BOTH POLARIZATIONS; AND $\theta^{(1)} = 45^{\circ}$.

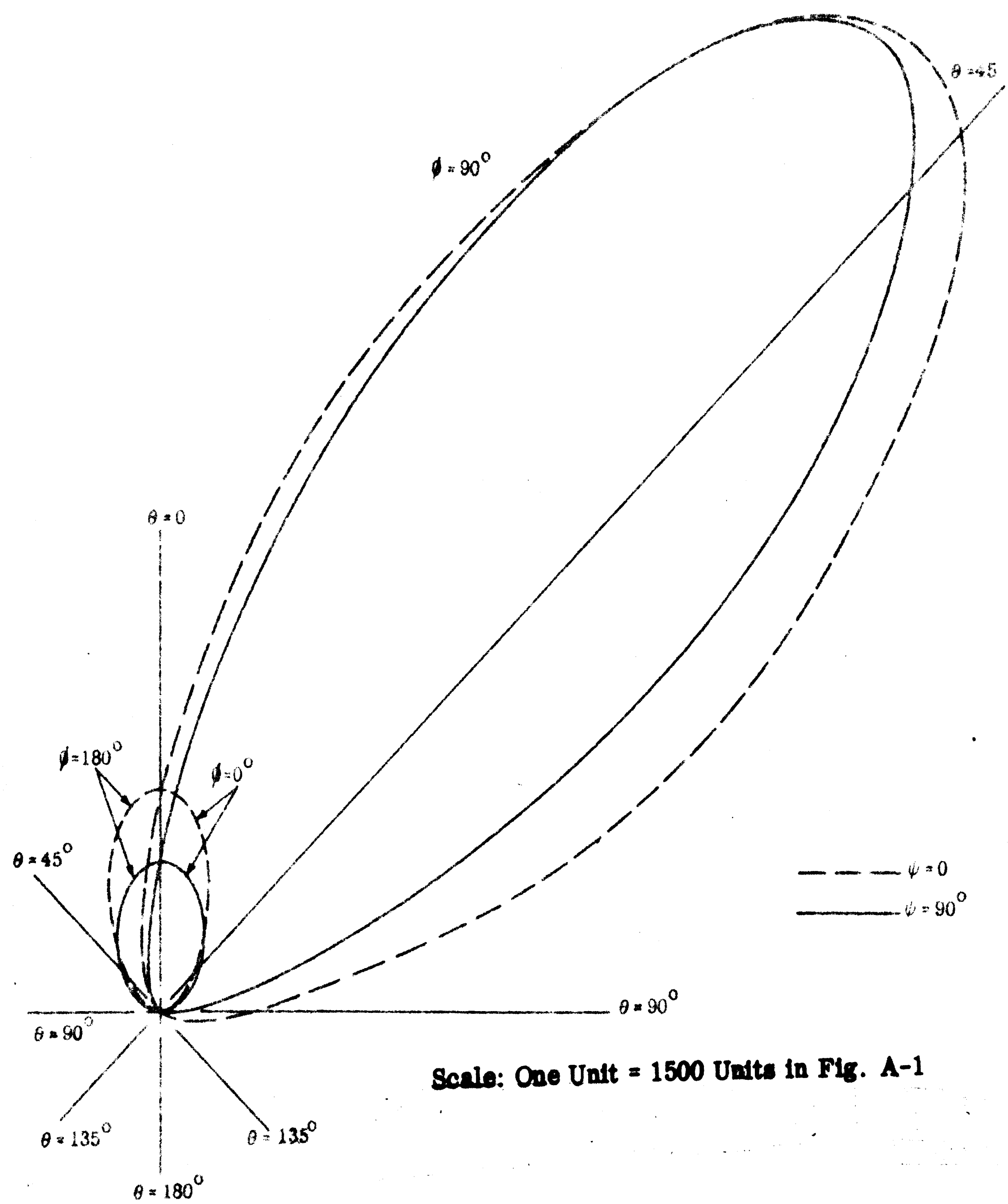

FIG. A-7: $\frac{d\sigma}{d\Omega}$ IN THE ξ - ζ AND ζ - η PLANES FOR $\beta = 0.60$; BOTH POLARIZATIONS; AND $\theta^{(i)} = 45^\circ$.

Appendix B

ASYMPTOTIC EXPRESSIONS FOR BESSEL FUNCTIONS OF LARGE NEGATIVE IMAGINARY ARGUMENT AND ARBITRARY COMPLEX ORDER

The integral representations of the Bessel functions given by Sommerfeld (1964c) are extended to the case where the argument is a negative imaginary number. They become

$$J_\nu(-iu) = \frac{1}{2\pi}\int_A e^{u\cos W + i\nu(W-\frac{\pi}{2})}\,dW \tag{B.1}$$

and

$$H_\nu^{(2)}(-iu) = \frac{1}{\pi}\int_B e^{u\cos W + i\nu(W-\frac{\pi}{2})}\,dW \tag{B.2}$$

where the paths A and B are shown in Fig. B-1. In these u is real and positive and ν is complex with $-\pi < \arg\nu \leqslant \pi$.

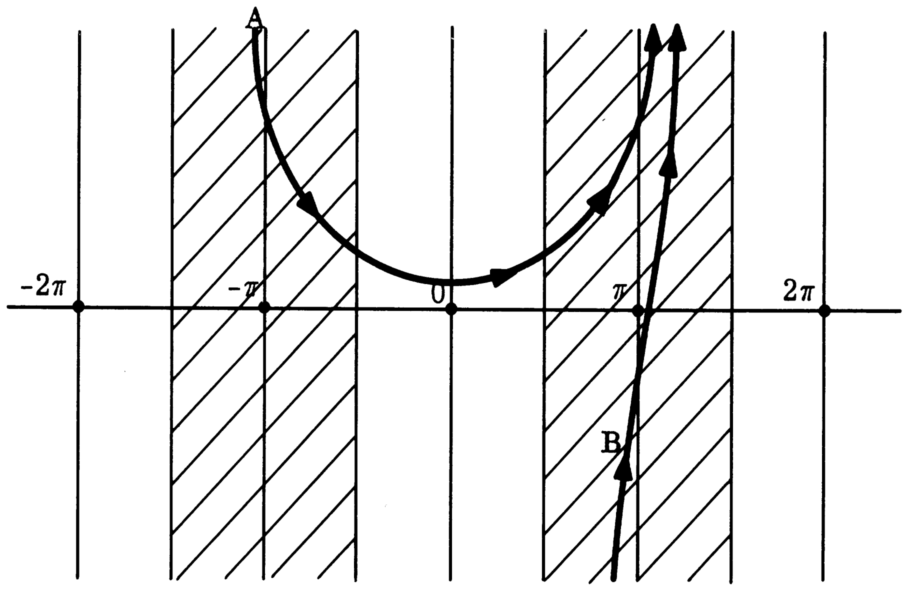

FIG. B-1: THE PATHS A AND B IN THE ν-PLANE.

The integral is of the form $e^{uf(W)}$ which in some cases can be approximated by the saddle point method. The path of integration is deformed to pass through the pertinent saddle points, W_n, such that* $f'(W_n) = 0$. The approximation depends on u being large and $f''(W_n) > \epsilon$ where $\epsilon > 0$. In this case the second condition excludes values of ν about the turning points of the Bessel equation, $\nu = \pm iu$. Away from these points the contribution to the integral due to the saddle point W_n is approximately

$$\sqrt{\frac{2\pi}{-uf''(W_n)}}\; e^{uf(W_n)} \quad . \tag{B.3}$$

When ν is along the positive real axis paths of steepest ascent of $f(W)$ are shown in Fig. B-2 with the arrows pointing in the direction of maximum increase.

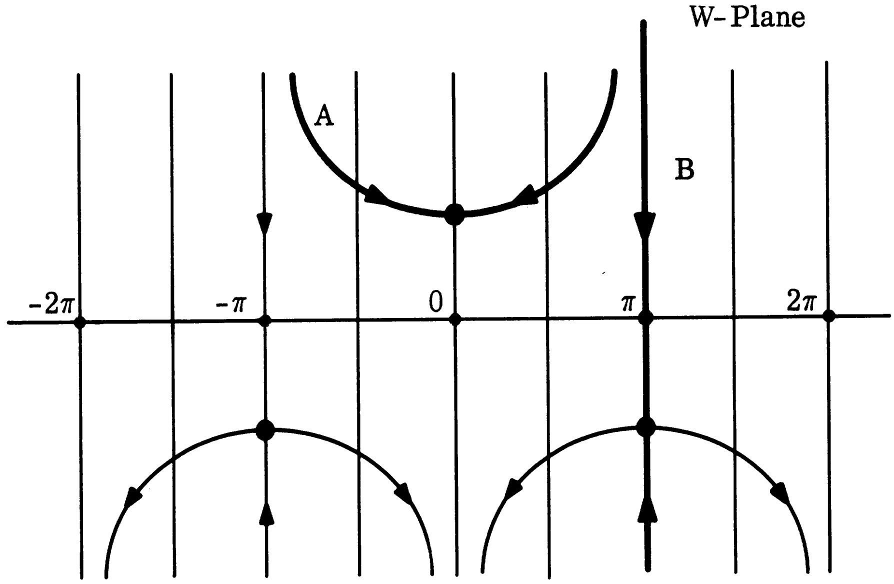

FIG. B-2: SADDLE POINTS FOR ν REAL AND POSITIVE.

* In this appendix the prime is used to denote differentiation.

The choice of which saddle points represent the Bessel and Hankel function is apparent in this case -

for $\quad J_\nu(-iu), \qquad W_n = i\sinh^{-1}\nu/u$,

and

for $\quad H_\nu^{(2)}(-iu), \qquad W_n = \pi - i\sinh^{-1}\nu/u$.

Let W_o be defined so that

$$\sin W_o = i\nu/u$$

and

$$\begin{aligned} &\mathrm{Im}(W_o) \geqslant 0 \quad \text{for} \quad -\pi/2 \leqslant \mathrm{Re}(W_o) \leqslant \pi/2 \\ &\mathrm{Im}(W_o) > 0 \quad \text{for} \quad -\pi \leqslant \mathrm{Re}(W_o) < -\pi/2 \\ &\mathrm{Im}(W_o) > 0 \quad \text{for} \quad \pi/2 < \mathrm{Re}(W_o) < \pi \end{aligned} \tag{B.4}$$

making the relation between W_o and ν one-to-one with branch points in the ν plane at $\nu = \pm iu$ and a branch line along the negative real axis from $\nu = -\infty$ to $\nu = 0$, and another along the imaginary ν axis from $\nu = -iu$ to $\nu = iu$. Then by carrying out the saddle point integration and writing the results in terms of W_o it is seen that for ν real and positive

$$J_\nu(-iu) = \frac{1}{\sqrt{2\pi u \cos W_o}} \exp\left\{u\left[\cos W_o + (W_o - \tfrac{\pi}{2})\sin W_o\right]\right\} \tag{B.5}$$

and

$$H_\nu^{(2)}(-iu) = i\sqrt{\frac{2}{\pi u \cos W_o}} \exp\left\{-u\left[\cos W_o + (W_o - \tfrac{\pi}{2})\sin W_o\right]\right\} \tag{B.6}$$

Along the negative real ν axis the lines of steepest ascent are shown in Fig. B-3 The saddle points to be used in this case are for $J_\nu(-iu)$

$$W_n = -\pi + i\sinh\nu/u \ ; \quad -i\sinh\nu/u \ ; \quad \text{and } \pi + i\sinh\nu/u$$

and for $H_\nu^{(2)}(-iu)$

$$W_n = \pi + i\sinh\nu/u$$

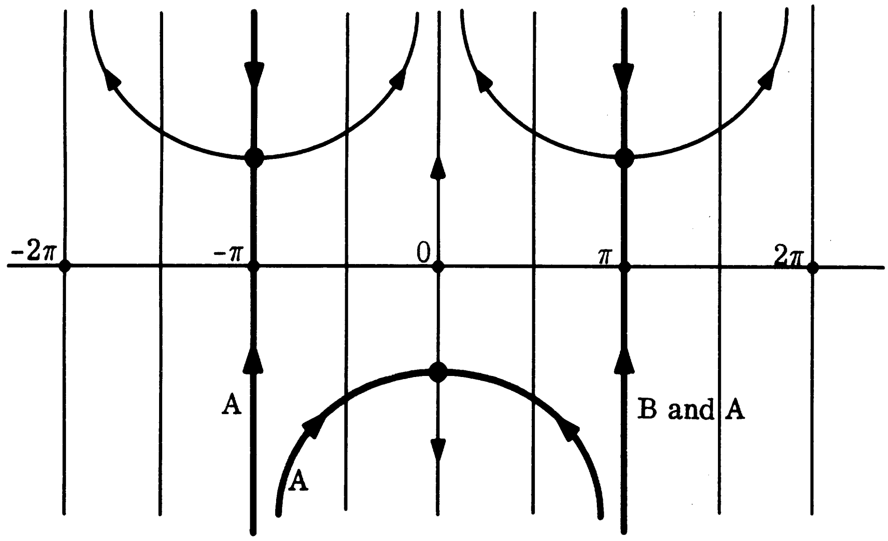

FIG. B-3: SADDLE POINTS FOR ν NEGATIVE AND REAL.

The saddle point integration then gives

$$J_\nu(-iu) = \frac{1}{\sqrt{2\pi u \cos W_0}} \left\{ (i e^{-i\nu\pi}) \exp\left\{-u\left[\cos W_0 + (W_0 + \tfrac{\pi}{2}) \sin W_0\right]\right\} - (e^{i\nu\pi} - e^{-i\nu\pi}) \exp\left\{u\left[\cos W_0 + (W_0 + \tfrac{\pi}{2}) \sin W_0\right]\right\}\right\} \quad \text{(B. 7)}$$

and

$$H_\nu^{(2)}(-iu) = \frac{-2}{\sqrt{2\pi u \cos W_0}}\, e^{i\nu\pi} \cdot \exp\left\{u\left[\cos W_0 + (W_0 + \tfrac{\pi}{2}) \sin W_0\right]\right\} . \quad \text{(B. 8)}$$

Now both $J_\nu(-iu)$ and $H_\nu^{(2)}(-iu)$ are analytic functions of ν so they must be continuous as ν goes from $\nu = -a + i\epsilon$ to $\nu = -a - i\epsilon$, $a > 0$, $\epsilon > 0$, i.e. as the negative real axis is crossed. However, the relationship between W_0 and ν has a branch cut along the negative real axis implying that W_o is not continuous across this line. To make the Bessel and Hankel functions continuous, just above the real axis use (B.7) and (B.8), while just below this axis use

$$J_\nu(-iu) = \frac{1}{\sqrt{2\pi u \cos W_o}} \left\{ (1 - e^{-i2\pi\nu}) \exp\left\{u\left[\cos W_o + (W_o - \frac{\pi}{2})\sin W_o\right]\right\} - i \exp\left\{-u\left[\cos W_o + (W_o - \frac{\pi}{2})\sin W_o\right]\right\}\right\} \qquad \text{(B.9)}$$

and

$$H_\nu^{(2)}(-iu) = \frac{2}{\sqrt{2\pi u \cos W_o}} \exp\left\{u\left[\cos W_o + (W_o - \frac{\pi}{2})\sin W_o\right]\right\} \qquad \text{(B.10)}$$

Now let

$$f = u\left[\cos W_o + (W_o + \frac{\pi}{2})\sin W_o\right] \quad . \qquad \text{(B.11)}$$

Its behavior in the upper half ν-plane is shown in Fig. B-4. Its magnitude is zero

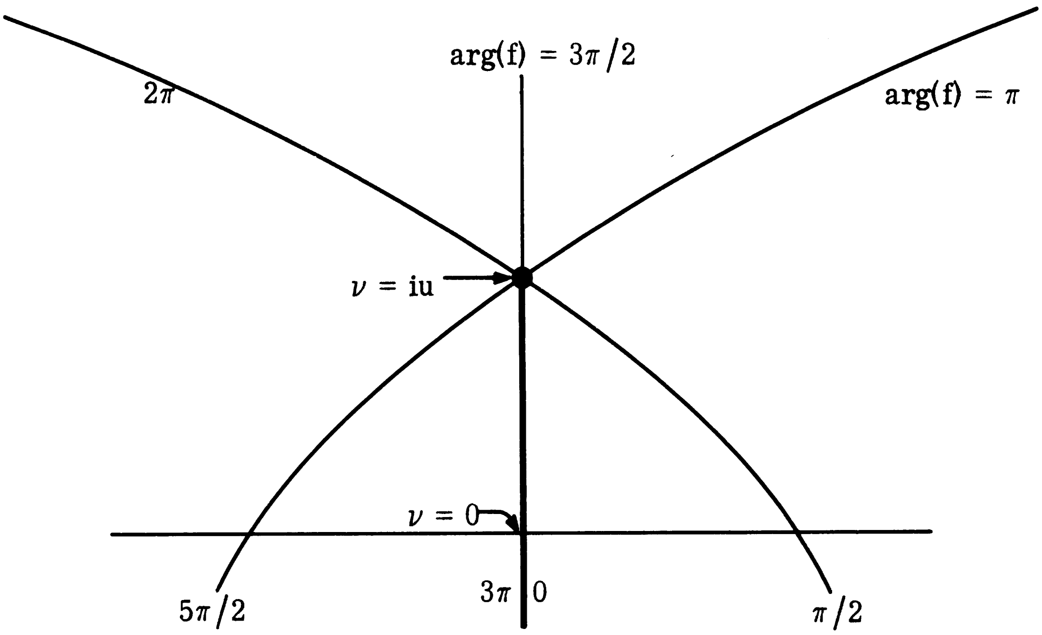

FIG. B-4: THE FUNCTION f.

at ν = iu and goes to infinity along any of the lines such that the argument of f is constant. There is a branch point at ν = iu with a branch line along the imaginary axis extending from $\nu = -\infty$ to ν = iu.

The functions (B.7) and (B.8) can now be extended into the upper half ν-plane, across the imaginary axis above the point ν = iu, into the right half ν-plane, and then down to the real axis in a manner such that $J_\nu(-iu)$ and $H_\nu^{(2)}(-iu)$ are continuous and so that on the positive real axis they agree with (B.5) and (B.6). For example, just above the real axis $J_\nu(-iu)$ from (B.7) may be written

$$J_\nu(-iu) = \frac{1}{\sqrt{2\pi u \cos W_o}} \left\{ i e^{-i\nu\pi} e^{-f} - (e^{i\nu\pi} - e^{-i\nu\pi}) e^{f} \right\} . \tag{B.12}$$

As the imaginary part of ν becomes more positive, $e^{i\nu\pi} \ll e^{-i\nu\pi}$ so that the third term of (B.12) may be dropped without destroying the continuity of this asymptotic function. In this region

$$J_\nu(-iu) = \frac{e^{-i\nu\pi}}{\sqrt{2\pi u \cos W_o}} \left\{ ie^{-f} + e^{f} \right\} . \tag{B.13}$$

Next, as the line where the argument of f equals 2π is approached, still avoiding a region about ν = iu, $e^{-f} \ll e^{f}$, and the first term of (B.13) may be dropped. In this next region

$$J_\nu(-iu) = \frac{e^{-i\nu\pi} e^{f}}{\sqrt{2\pi u \cos W_o}} . \tag{B.14}$$

This expression agrees with (B.5) obtained for ν on the positive real axis. The Hankel function may be handled in the same way.

In the lower half ν-plane let

$$g = u\left[\cos W_o + (W_o - \frac{\pi}{2}) \sin W_o\right] \tag{B.15}$$

Its behavior is shown in Fig. B-5. The expressions (B.9) and (B.10) can be extended down into the lower half ν-plane, across the imaginary axis, and then up

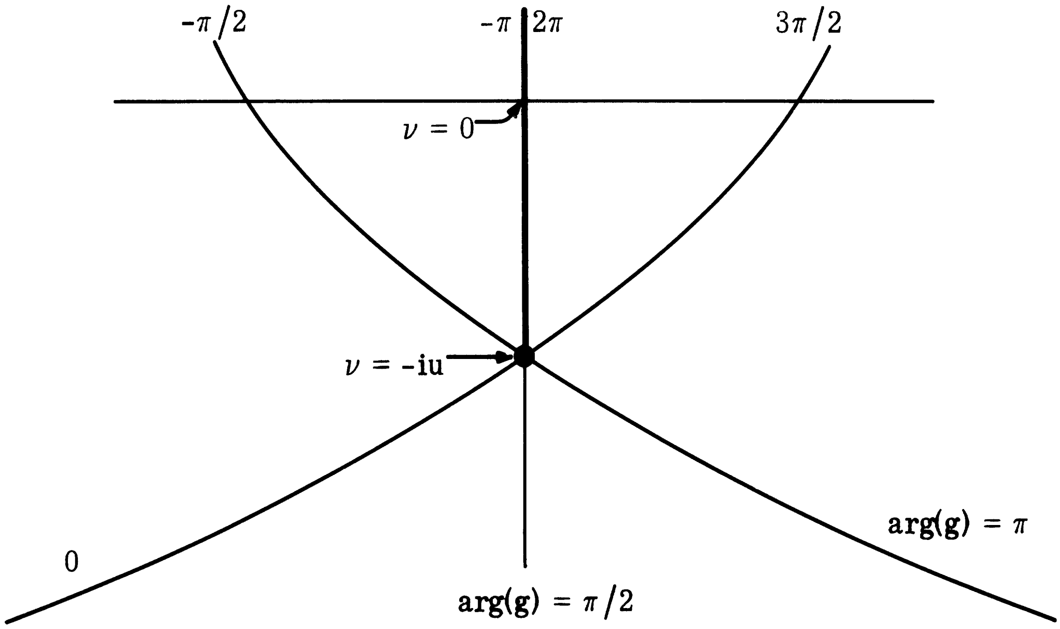

FIG. B-5: THE FUNCTION g.

to the positive real axis in a similar fashion. The result of doing this is summarized at the end of this appendix.

Finally about the point $\nu = iu$ asymptotically convergent solutions to the Bessel's equation are represented by expression involving Airy functions of $f^{2/3}$ (Langer, 1932; Olver, 1954). Of these expressions the one which, as f becomes large, agrees with the result previously obtained by the saddle point technique is chosen. About the point $\nu = -iu$ expressions involving the Airy function of $g^{2/3}$ occur. The correct one is chosen to agree with the previous results.

The table below summarizes the result of this discussion giving appropriate asymptotic expressions for the different regions of the ν-plane designated by the letters A through I bounded by the dashed lines and the branch cuts (heavy lines) shown in Fig. B-6. The region D includes the negative real ν axis. The symbol

$$\mathrm{Ai}\left\{u\right\}$$

designates the Airy function of u (see for example, Olver, 1954).

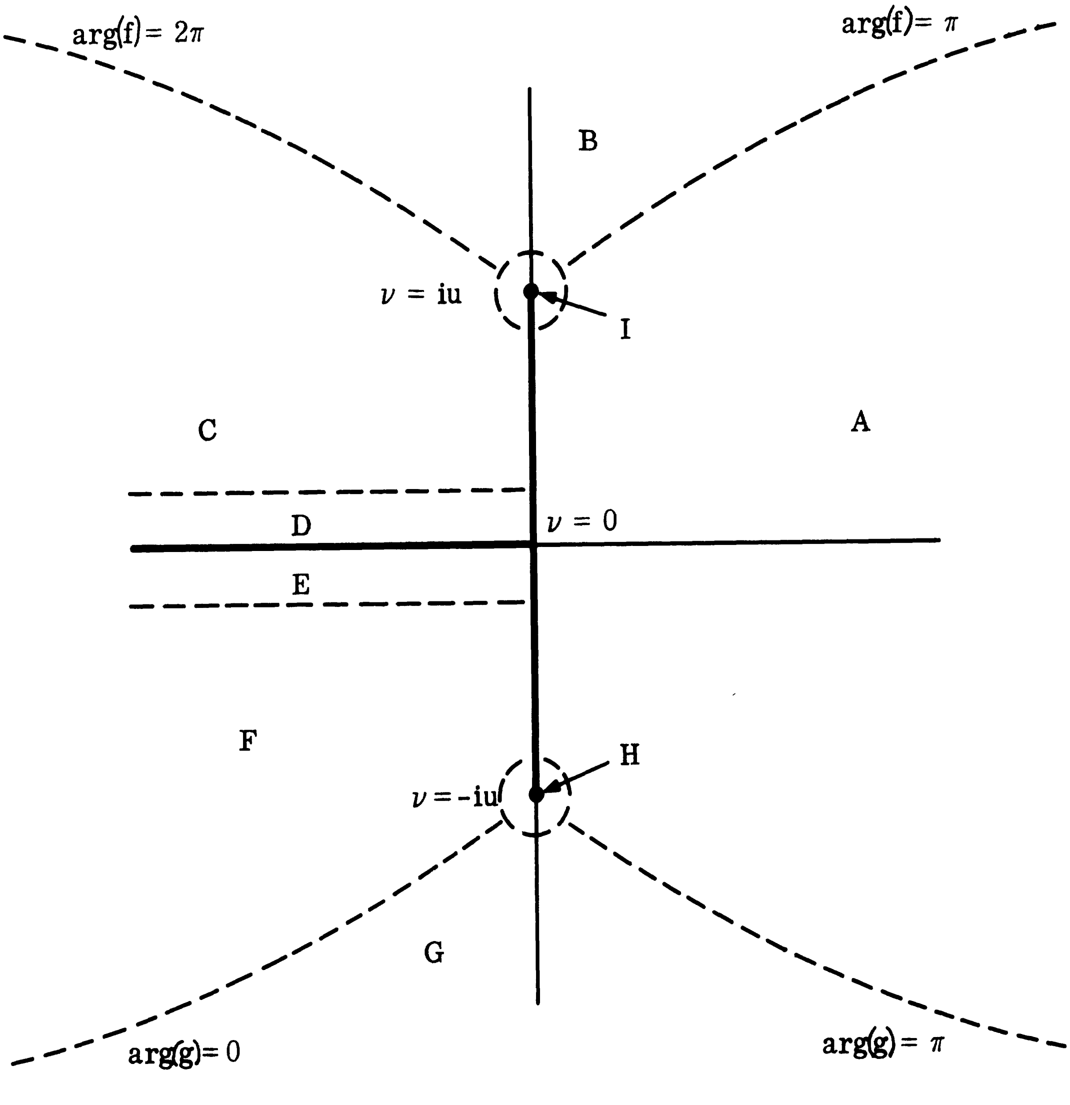

FIG. B-6: THE REGIONS A, B, C, D, E, F, G, H, AND I IN THE ν-PLANE.

TABLE B-I: THE ASYMPTOTIC EXPRESSIONS FOR THE BESSEL FUNCTIONS

Region	$\sqrt{2\pi u \cos W_o} \cdot J_\nu(-iu)$	$\frac{1}{2}\sqrt{2\pi u \cos W_o} \cdot H_\nu^{(2)}(-iu)$
A	$e^{-i\nu\pi} e^{f}$	$i e^{i\nu\pi} e^{-f}$
B	$e^{-i\nu\pi}$	$e^{i\nu\pi}(ie^{-f} - e^{f})$
C	$e^{i\nu\pi}(e^{f} + ie^{-f})$	$-e^{i\nu\pi} e^{f}$
D	$i(e^{-i\nu\pi} e^{-f} - 2\sin\nu\pi\, e^{f})$	$-e^{i\nu\pi} e^{f}$
E	$(1 - e^{-i\nu 2\pi})e^{g} - ie^{-g}$	e^{g}
F	$e^{g} - ie^{-g}$	e^{g}
G	$e^{-i\nu\pi} e^{f}$	$e^{g} + ie^{-g}$
H	$2\sqrt{\pi}\left(\frac{3}{2} g e^{-i\pi}\right)^{1/6} Ai\left\{\left[\left(\frac{3}{2} g e^{-i\pi}\right)^{2/3}\right]\right\}$	$2\sqrt{\pi}\left(\frac{3}{2} g e^{-i\pi}\right)^{1/6} Ai\left\{\left[\left(\frac{3}{2} g e^{-i\pi}\right)^{2/3}\right]\right\}$
I	$2\sqrt{\pi}\, e^{-i\nu\pi}\left(\frac{3}{2} f e^{-i\pi}\right)^{1/6} Ai\left\{\left[\left(\frac{3}{2} f e^{-i\pi}\right)^{2/3}\right]\right\}$	$i2\sqrt{\pi}\, e^{i\nu\pi}\left(\frac{3}{2} f\right)^{1/6} Ai\left\{\left[\left(\frac{3}{2} f\right)^{2/3}\right]\right\}$