ELECTROMAGNETIC PLANE WAVE SCATTERING BY A PERFECTLY CONDUCTING DISK

by

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ABSTRACT

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In this research, the phenomenon of plane wave electromagnetic scattering by a perfectly conducting disk is studied for both low and high frequencies. Consideration is limited to only the far-zone backscattered fields for incident plane waves having either the electric or magnetic vector parallel to the plane of the disk.

For frequencies near or below resonance, a method based upon Flammer's (1953) exact solutions to the plane wave scattering problems is developed for computing the far-zone backscattered fields for both incident plane waves. This method, which involves the computation of various oblate spheroidal wave functions and has been programmed for use on an IBM 360 computer, is used to compute direct and cross-polarized radar cross sections for integer values of the disk ka product (k = free space wave number, a = disk radius) that range from one to seven. These computed radar cross sections are compared with the same cross sections obtained experimentally.

The high frequency investigation is based upon an approximate method proposed and applied by Ufimtsev (1958), which has as its foundation the approximation that for a perfectly conducting flat plate or disk of large radius of curvature the edge behaves locally like a half-plane. Known solutions to plane wave electromagnetic scattering problems for the half-plane are used to obtain explicit, though approximate, expressions for the surface current densities on the disk, which, in turn, are used to find approximate expressions for the far-zone backscattered fields. These expressions, which are very similar to those

obtained by Ufimtsev for the disk scattering problems, represent a formal extension of his results to one greater inverse power of (ka) and to greater aspect angles for the case of backscattering.

Several critical comparisons are made in order to test the exact and approximate solutions of the plane wave scattering problems. These comparisons are between the exact solutions and low frequency solutions to the disk scattering problems due to Eggimann (1961) for a ka-product of one half, between the exact and high-frequency solutions for an intermediate value of ka, and between the high frequency solutions and solutions due to the Geometrical Theory of Diffraction. The last comparison is not a quantitative one but is concerned with the forms of the two approximate solutions. By employing the approximate results obtained in this research, an analysis patterned after that of Ross (1967) is used to find new expressions for disk and cone backscattering from expressions due to the Geometrical Theory of Diffraction.

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TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF APPENDICES

Chapter I

INTRODUCTION

1.1 General Discussion

Exact solutions to problems of scattering of plane electromagnetic waves by perfectly conducting bodies of finite dimensions are few in number. For instance, if the class of body shapes known as oblate spheroids is considered, it is found that exact analytical solutions are known only for the two limiting shapes, the ideal disk and the sphere. The solution to the sphere scattering problem is the simpler of the two and is given in terms of the Mie series, which has proven amenable to many types of analysis. The nature of the disk scattering problem has resulted in several different formulations for the exact solution. Solutions have been obtained by Meixner and Andrejewski (1950) in terms of Hertz vectors, by Flammer (1953) in terms of oblate spheroidal wave functions, by Nomura and Katsura (1955) in terms of hypergeometric polynomials, and Luré (1960) using sets of paired integral equations.

Except for the special case of the direction of incidence normal to the plane of the disk, few calculations have been done using any of the above solutions. During the course of this work a means of carrying out some calculations of radar cross sections using one of the exact solutions to the disk scattering problem became desirable. The solution as formulated by Flammer was chosen for this work because it is given directly in terms of oblate spheroidal wave functions. Chapter II is concerned with this computational effort and provides the theoretical development necessary to calculate the scattering cross sections of the disk as a function of aspect angle. Actual computations, however, are carried out only for back-scattering.

The exact solution considered above is useful only for frequencies near or below resonance due to convergence properties of the functions involved, limitations of the existing tables of oblate spheroidal functions, and computer time limitations. Consequently, it is desirable to obtain a solution or approximation that is valid and easily applied at high frequencies. Hence an asymptotic expansion of the solution is desirable.

Jones (1965) (Heins and Jones (1967)) has developed and applied to the problem of electromagnetic scattering by a disk of a plane wave incident at normal incidence a systematic process which yields as many terms of the asymptotic development of the solution for high frequencies as one is willing to calculate. An intention to apply the process to the problem of oblique incidence has been indicated. Other asymptotic solutions have also been considered. These generally are based upon the principle that the edge of the disk behaves locally like a half-plane. Different degrees of approximation can be assumed. In the Geometrical Theory of Diffraction an edge diffraction mechanism, in which each point on the edge diffracts a cone of rays with the cone half angle equal to the smaller angle between the direction of incidence and the tangent to the edge, is assumed. This means that the scattered field at any point in space will generally be the superposition of the scattered fields from a finite number of points on the disk edge. Situations where all points on the disk edge contribute must be considered separately. An exposition of the Geometrical Theory of Diffraction has been given by Keller (1962).

Another approach has been suggested and investigated by Ufimtsev (1958). This method approximates the local disk edge currents by those that would be found on a half-plane tangential to the disk at the given edge point. The scattered fields are then found from the resulting current distribution. The method, of course, is approximate as it fails to account for perturbations in the assumed current distribution that arise because of the finite dimensions of the disk. Ufimtsev has found only the first term in each of the asymptotic expansions for the far-zone scattered fields. Also, his results are valid only for small angles. An investigation of this method is undertaken in Chapter III. The solution is formally extended to obtain another term in each asymptotic series and to improve the description of the dependence on aspect angle for the case of backscattering.

Mention should be made of the fact that solutions for scattering problems involving the disk may be applied to scattering by a circular aperture by proper application of the rigorous form of Babinet's principle (Bouwkamp (1954)). Consequently, an exact solution to plane wave scattering by a circular aperture may be found. Conversely, approximate solutions to problems of scattering by a circular aperture may be applied to problems of disk scattering. In any case, only disk scattering will be considered here.

1.2 The Scattering Problem

Figure 1-1 defines the geometry of the problem to be considered. A plane wave $\overline{F}_i$ is incident in the yz-plane with an angle θ between the negative z-axis and the direction of incidence. $\overline{F}_i$, which may be either the incident electric or magnetic field, can be resolved into components in the θ-direction and the ϕ-direction. Hence any incident plane wave $\overline{F}_i$ can be expressed in terms of the following two incident fields:

$$\overline{E}_1^i = \hat{x} E_0 e^{-ik(y \sin\theta + z\cos\theta)} \tag{1.1}$$

$$\overline{H}_2^i = \hat{x} H_0 e^{-ik(y \sin\theta + z\cos\theta)} \tag{1.2}$$

$$E_0 = \eta_0 H_0$$

The constants η_0 and k appearing in these equations are the characteristic impedance of free space and the wave number of the incident field respectively. The unit vector $\hat{x}$ has been used in place of $-\hat{\phi}$ to indicate that the direction of incidence is confined to the yz-plane. An $e^{-i\omega t}$ time-dependence has been suppressed here, and will continue to be suppressed throughout this work.

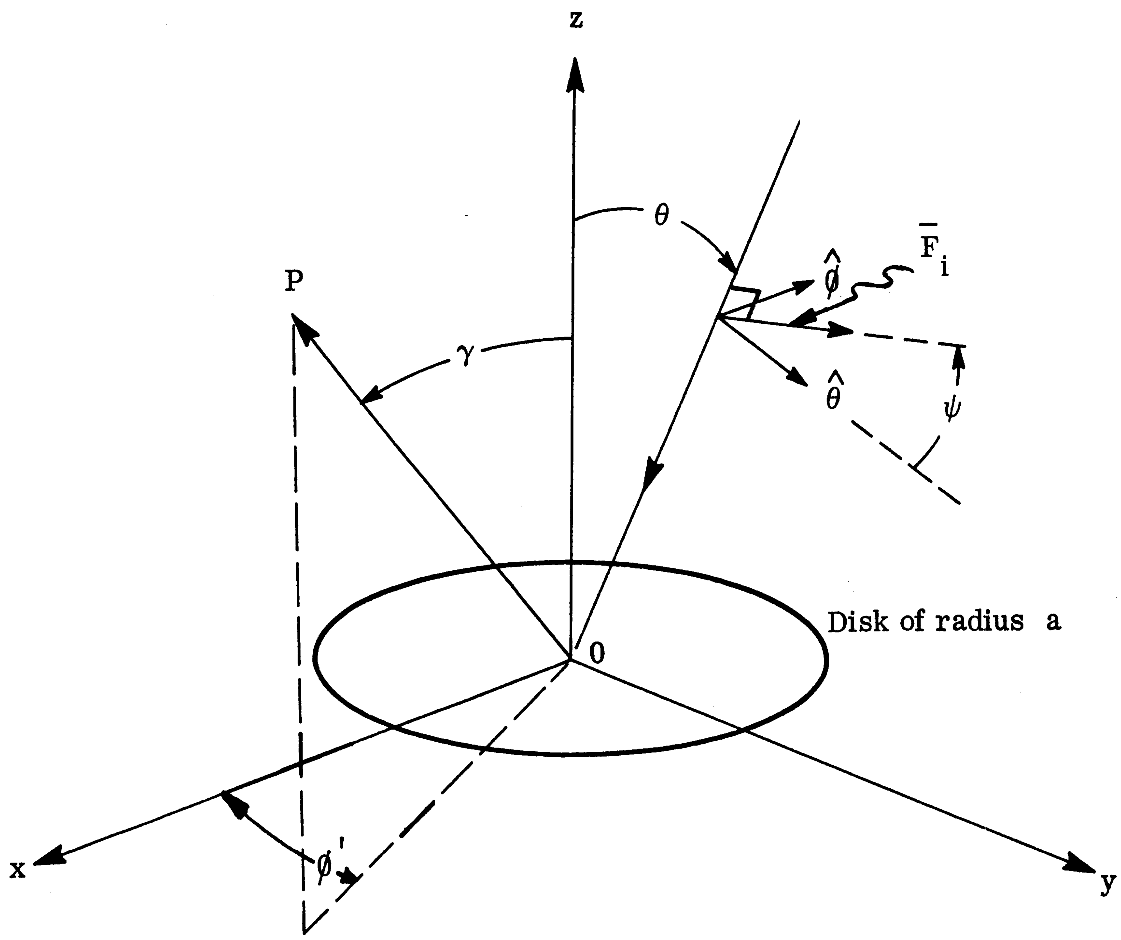

FIG. 1-1: The Geometry of the General Disk Scattering Problem. P is the Point of Observation.

The incident field of equation (1.1), which has the electric field parallel to the plane of the disk is said to be E-polarized. Similarly, the incident field of equation (1.2) is said to be H-polarized.

In order to keep the computational effort tractable only the far-zone scattered field will be considered. Also, consideration will be further restricted to only the backscattered far-zone field in order to make comparisons with experimental data, even though several of the techniques used can be applied to the case of bistatic-scattering in the far-zone. The behavior of both direct and cross-polarized components of the far-zone backscattered field will be investigated. It can be shown that the cross-polarized component of the backscattered field will depend on the polarization angle ψ as sin 2ψ when ψ varies and all other aspects of the incident field remain unchanged. In particular, when ψ is equal to zero or ninety degrees there will be no cross-polarized component of the backscattered field. Because of this, the cross-polarized component of the backscattered field can be expressed in terms of the backscattered fields due to the incident fields of equations (1.1) and (1.2). Since the cross-polarized component of the backscattered field attains its greatest value for a polarization angle of forty-five degrees, all measurements and theoretical calculations involving that component will be done only for this polarization angle.

Calculation of the cross-polarized component of the far-zone backscattered field is straightforward for that polarization angle. Let the incident electric field be written as

$$\overline{E}_i = -\frac{E_0}{\sqrt{2}}\hat{\theta} + \frac{E_0}{\sqrt{2}}\hat{\phi} \; . \qquad (1.3)$$

The total backscattered field is just the vector sum of the scattered fields due to each component of the incident field. Neither component of equation

(1.3) will give rise to a cross-polarized term. In any case the backscattered field may be written as

$$\overline{E}^s = \frac{1}{\sqrt{2}} \left(E^s_\theta \hat{\theta} + E^s_\phi \hat{\phi} \right) . \tag{1.4}$$

Then the magnitude of the cross-polarized component of the backscattered field may be found from

$$\left| E^s_{cr} \right| = \hat{E}^i_\perp \cdot \overline{E}^s , \tag{1.5}$$

where

$$\hat{E}^i_\perp = \pm \frac{\left(\hat{\theta} - \hat{\phi} \right)}{\sqrt{2}} . \tag{1.6}$$

If either sign is taken in equation (1.6), equation (1.5) yields

$$\left| E^s_{cr} \right| = \frac{1}{2} \left| E^s_\theta - E^s_\phi \right| . \tag{1.7}$$

This simple result will be used in Chapters II and III to find expressions for the cross-polarized radar cross section of a disk.

Chapter II

EXACT DISK SCATTERING

2.1 The Curvilinear Coordinate System

An ideal conducting disk of radius a may be modeled by the focal circle, $\xi = 0$, of the oblate coordinate system of which a cut in the yz-plane is shown in Fig. 2-1. The system of confocal hyperbolae and ellipses shown is rotationally symmetric about the z-axis. Only two oblate coordinates, the angular coordinate η and the radial coordinate ξ are indicated, as the third oblate coordinate ν is simply equal to cos ϕ, ϕ being the angle of rotation about the z-axis measured in the xy-plane relative to the x-axis. For this reason it is convenient to adopt a hybrid coordinate system and use ϕ instead of ν. Relationships between Cartesian coordinates and this hybrid coordinate system are given in Appendix 2-A. Also given are the metrical coefficients h_ξ, h_η, and h_ϕ.

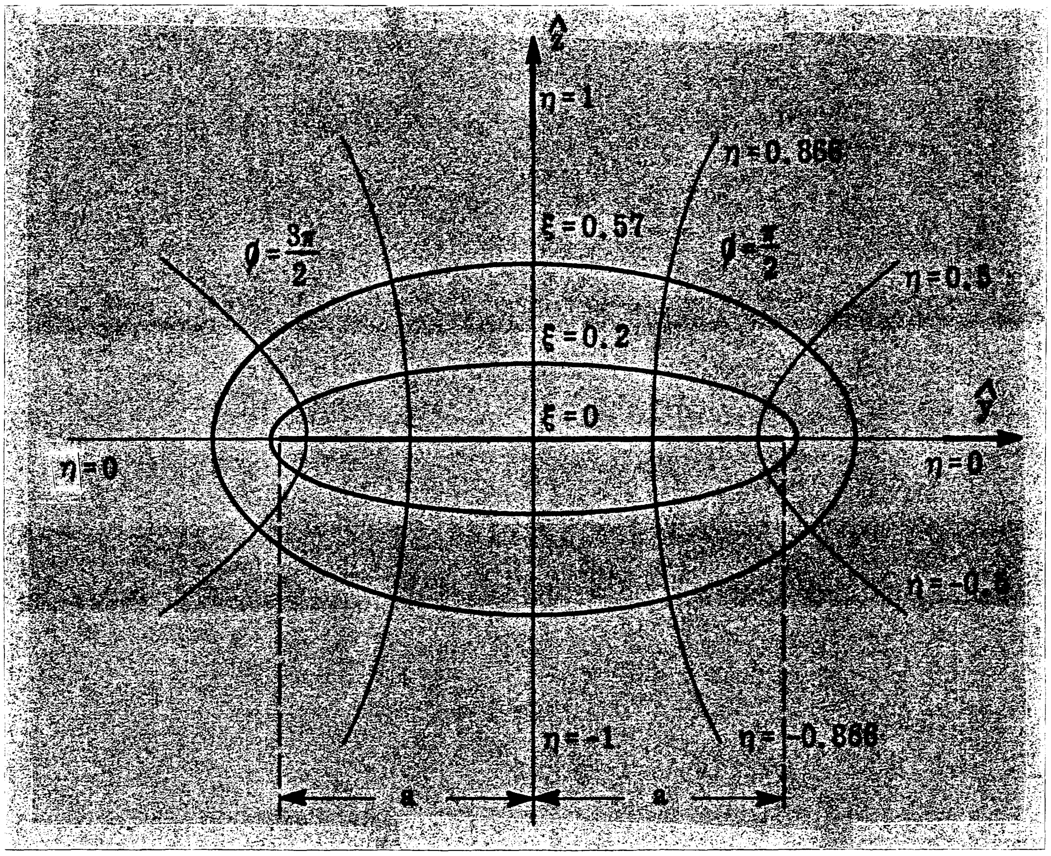

FIG. 2-1: THE OBLATE SPHEROIDAL COORDINATE SYSTEM

The scalar Helmholz equation,

$$(\nabla^2 + k^2)\psi = 0 , \tag{2.1}$$

is separable in oblate spheroidal coordinates, and its eigenfunctions are expressible as

$$\psi_{\substack{e\\o}\,m\ell}^{(i)}(\eta, \xi, \phi) = S_{m\ell}^{(1)}(-ic, \eta)\, R_{m\ell}^{(i)}(-ic, i\xi) \begin{Bmatrix} \cos m\phi \\ \sin m\phi \end{Bmatrix} . \tag{2.2}$$

The functions $S_{m\ell}^{(1)}(-ic, \eta)$ and $R_{m\ell}^{(i)}(-ic, i\xi)$ are respectively the angular oblate spheroidal functions of the first kind and the radial functions of the ith kind, $i = 1, 2, 3, 4$, following the definition used by Flammer (1953) except that the subscript ℓ used here is his subscript n. The quantity c is equal to the product of the wave number $k = 2\pi/\lambda$ and the radius of the disk ($c = ka$).

The angular functions $S_{m\ell}^{(1)}(-ic, \eta)$ can be expressed in terms of associated Legendre functions by

$$S_{m\ell}^{(1)}(-ic, \eta) = \sum_{n=0,1}^{\infty}{}' d_n^{m\ell} P_{m+n}^{m}(\eta) . \tag{2.3}$$

The prime indicates the sum is to be taken over even or odd values of n according as $(\ell - m)$ is even or odd. Quite extensive tables of the $d_n^{m\ell}$ have been published by Stratton, et al (1956).

The angular functions are orthogonal over the interval $(-1, 1)$ in η. That is,

$$N_{m\ell}\, \delta_{\ell r} = \int_{-1}^{1} S_{mr}^{(1)}(-ic, \eta)\, S_{m\ell}^{(1)}(-ic, \eta)\, d\eta , \tag{2.4}$$

with

$$N_{m\ell} = 2 \sum_{n=0,1}^{\infty}{}' \frac{(n + 2m)!\, (d_n^{m\ell})^2}{n!\, (2n + 2m + 1)} . \tag{2.5}$$

Further properties of the angular and radial functions will be developed as needed. In general, the notation used here will be that used by Flammer in Spheroidal Wave Functions (1957), which gives a comprehensive discussion of the properties of the oblate spheroidal functions. Numerical values for the functions are given for some values of c, η, and ξ; and tables for the coefficients in the series expansions of both angular and radial functions are listed.

2.2 Solution to the Scattering Problem

If the field $\bar{F}^i$ in Fig. 1-1 is either the electric field $\bar{E}^i$ or magnetic field $\bar{H}^i$, then both $\bar{F}^i$ and the scattered field $\bar{F}^s$ satisfy the wave equation,

$$\nabla \times \nabla \times \bar{F} - k^2 \bar{F} = 0 \quad , \tag{2.6}$$

which is a special case of the vector Helmholz equation. The boundary condition on the total field $\bar{F}^t = \bar{F}^i + \bar{F}^s$ on the disk surface is then either the Dirichlet ($\hat{n} \times \bar{F}^t = 0$) or Neumann ($\hat{n} \times (\nabla \times \bar{F}^t) = 0$) condition for $\bar{F}^i$ equal to the incident electric or magnetic field respectively. The scattered field $\bar{F}^s$ must, in either case, satisfy Sommerfeld's radiation condition as the distance from the disk becomes infinite ($\xi \rightarrow \infty$).

No vector function that is a solution of equation (2.6) and that also satisfies either the Dirichlet or Neumann condition on the disk is known, or is one likely to be found (Morse and Feshbach; Sec. 13.1, 1953). In spite of this, Flammer (1953) has shown that it is possible to solve exactly the problem of scattering of plane electromagnetic waves by a perfectly conducting disk in terms of even wave functions of the $\bar{M}$ and $\bar{N}$ type given by

$$\bar{M}_{em\ell}^{u(i)}(\eta, \xi, \phi) = \nabla\psi_{em\ell}^{(i)}(\eta, \xi, \phi - \tfrac{\pi}{2}) \times \hat{u}, \tag{2.7a}$$

$$\bar{N}_{em\ell}^{u(i)}(\eta, \xi, \phi) = k^{-1} \nabla \times \bar{M}_{em\ell}^{u(i)}(\eta, \xi, \phi), \tag{2.7b}$$

$u = x, y, z,$

where $\psi_{em\ell}^{(i)}(\eta, \xi, \phi)$ is given by the part of equation (2.2) that is even with respect to the angle ϕ.

The solutions to the scattering problems as given by Flammer are completely general and give the scattered fields everywhere in space for any angles of incidence and observation in Fig. 1-1. However, considerable simplification results if the point of observation is restricted to the far-zone and only terms of order $(R)^{-1}$ are retained. This will be done here, and calculations will be carried out only for the case of backscattering at arbitrary angles of incidence in order to compare with experimental data and various approximate methods for describing the far-zone fields of the disk for plane wave incidence. This restriction is not as severe as it might appear for all the essential features of the computational problem are preserved, the major differences being that fewer values of the radial functions need be found and that fewer terms are needed. All of the quantities dependent on c and θ computed for this special scattering problem are also needed for the general one. Expressions for far-zone bistatic scattering by the disk for the two incident fields $\bar{F}^i$ of Fig. 1-1 given by equations (1.1) and (1.2) have been relegated to Appendix 2-B. Only backscattering will be considered in the following, and the far-zone scattered fields will be given in terms of spherical coordinates.

The far-zone backscattered electric field due to the incident field of equation (1.1) has a component only in the ϕ-direction, $E^s_{\phi_E}$, given by

$$E^s_{\phi_E} = -\eta_0 H^s_E \sim \frac{E_0 e^{ikR}}{ikR} \sum_{m=0}^{\infty} (-1)^m \Bigg\{ 2(2-\delta_{0m})\beta^E_m(c,\theta) \sum_{\ell=m}^{\infty}{}' \frac{J_{m\ell}(c)}{N_{m\ell}} \left(S^{(1)}_{m\ell}(-ic, \cos\theta)\right)^2 +$$

$$+ 2\alpha^E_m(c,\theta) \Bigg[\sum_{\ell=m+2}^{\infty}{}' \frac{J'_{m+1,\ell}(c)}{N_{m+1,\ell}} \left(S^{(1)}_{m+1,\ell}(-ic, \cos\theta)\right)^2 +$$

$$+ (1-\delta_{0m}) \sum_{\ell=m}^{\infty}{}' \frac{J'_{m-1,\ell}(c)}{N_{m-1,\ell}} \left(S^{(1)}_{m-1,\ell}(-ic, \cos\theta)\right)^2 \Bigg] \Bigg\}, \qquad (2.8)$$

where

$$\delta_{0m} = \begin{cases} 0, & m \neq 0 \\ 1, & m = 0 \end{cases} = \text{Kroneker delta function,}$$

$c = ka,$

$\eta_0 = \sqrt{\mu_0/\epsilon_0}$ = impedance of free space,

and R is the distance from the disk at which the fields are measured.

The angular function $S_{m\ell}^{(1)}(-ic, \cos\theta)$ and normalization constant $N_{m\ell}$ have been discussed in section 2.1. The $J_{m\ell}(c)$ and $J'_{m\ell}(c)$ are "joining" factors defined in terms of values of radial functions (or derivatives of) of first and third kinds on the surface of the disk. The $\alpha_m^E(c,\theta)$ and $\beta_m^E(c,\theta)$ are "weighting" factors that determine the contributions of terms in the various vector wave functions to the scattered fields. A prime on a summation sign in this chapter indicates that the sum is to be taken over alternate values of the index.

The joining factors $J_{m\ell}(c)$ and $J'_{m\ell}(c)$ are defined by:

$$J_{m\ell}(c) = \begin{cases} \dfrac{R_{m\ell}^{(1)}(-ic, i0)}{R_{m\ell}^{(3)}(-ic, i0)} & (\ell-m) \text{ even} \\ 0 & (\ell-m) \text{ odd} \end{cases} \tag{2.9a}$$

$$J'_{m\ell}(c) = \begin{cases} 0 & (\ell-m) \text{ even} \\ \dfrac{R_{m\ell}^{(1)'}(-ic, i0)}{R_{m\ell}^{(3)'}(-ic, i0)} & (\ell-m) \text{ odd} \end{cases} \tag{2.9b}$$

It is also necessary to consider four other joining factors that relate the angular functions for $\cos\theta = 0$ to the radial functions of the first and third kinds. These are defined by the following expressions:

$$K^{(1)}_{m\ell}(c) \equiv \begin{cases} \dfrac{S^{(1)}_{m\ell}(-ic,0)}{R^{(1)}_{m\ell}(-ic,i0)} & (\ell-m) \text{ even} \\ 0 & (\ell-m) \text{ odd} \end{cases} \tag{2.10a}$$

$$K^{(1)'}_{m\ell}(c) \equiv \begin{cases} \dfrac{S^{(1)'}_{m\ell}(-ic,0)}{R^{(1)'}_{m\ell}(-ic,i0)} & (\ell-m) \text{ odd} \\ 0 & (\ell-m) \text{ even} \end{cases} \tag{2.10b}$$

$$K^{(3)}_{m\ell}(c) \equiv \begin{cases} \dfrac{S^{(1)}_{m\ell}(-ic,0)}{R^{(3)}_{m\ell}(-ic,i0)} & (\ell-m) \text{ even} \\ 0 & (\ell-m) \text{ odd} \end{cases} \tag{2.11a}$$

$$K^{(3)'}_{m\ell}(c) \equiv \begin{cases} 0 & (\ell-m) \text{ even} \\ \dfrac{S^{(1)'}_{m\ell}(-ic,0)}{R^{(3)'}_{m\ell}(-ic,i0)} & (\ell-m) \text{ odd} \end{cases} \tag{2.11b}$$

The primes on the angular and radial functions indicate differentiation with respect to the angular (η) and radial (ξ) variables respectively. The various kinds of joining factors are not independent, but are related by

$$K^{(3)}_{m\ell}(c) = J_{m\ell}(c)\, K^{(1)}_{m\ell}(c), \tag{2.12a}$$

$$K^{(3)'}_{m\ell}(c) = J'_{m\ell}(c)\, K^{(1)'}_{m\ell}(c). \tag{2.12b}$$

The weighting factors $\alpha_m^E(c,\theta)$ and $\beta_m^E(c,\theta)$ have not yet been considered in detail. These are complex functions of θ and c given by

$$\beta_m^E(c,\theta) = 1 - \alpha_m^E(c,\theta) \, , \tag{2.13a}$$

$$\alpha_0^E(c,\theta) = \frac{\sum\limits_{n=0}^{\infty}{}' b_{0n}(c,\theta)\, K_{0n}^{(3)}(c)}{\sum\limits_{n=0}^{\infty}{}' b_{0n}(c,\theta)\, K_{0n}^{(3)}(c) - \frac{1}{2}\sum\limits_{n=2}^{\infty}{}' a_{1n}(c,\theta)\, K_{1n}^{(3)\prime}(c)} \, , \tag{2.13b}$$

$$\alpha_1^E(c,\theta) = \frac{\sum\limits_{n=1}^{\infty}{}' b_{1n}(c,\theta)\, K_{1n}^{(3)}(c)}{\sum\limits_{n=1}^{\infty}{}' b_{1n}(c,\theta) K_{1n}^{(3)}(c) - \sum\limits_{n=1}^{\infty}{}' a_{0n}(c,\theta) K_{0n}^{(3)\prime}(c) - \frac{1}{2}\sum\limits_{n=3}^{\infty}{}' a_{2n}(c,\theta) K_{2n}^{(3)\prime}(c)} \, , \tag{2.13c}$$

$$\alpha_m^E(c,\theta) = \frac{\sum\limits_{n=m}^{\infty}{}' b_{mn}(c,\theta) K_{mn}^{(3)}(c)}{\sum\limits_{n=m}^{\infty}{}' b_{mn}(c,\theta) K_{mn}^{(3)}(c) - \frac{1}{2}\sum\limits_{n=m+2}^{\infty}{}' a_{m+1,n}(c,\theta) K_{m+1,n}^{(3)\prime}(c) - \frac{1}{2}\sum\limits_{n=m}^{\infty}{}' a_{m-1,n}(c,\theta) K_{m-1,n}^{(3)\prime}(c)} \, ,$$

$$m > 1 \, , \tag{2.13d}$$

where

$$b_{mn}(c,\theta) = 2(2-\delta_{0m}) i^{n-1} N_{mn}^{-1} (\sin\theta)^{-1} S_{mn}^{(1)}(-ic, \cos\theta) \, , \tag{2.14a}$$

$$a_{mn}(c,\theta) = -2(2-\delta_{0m}) i^{n-1} N_{mn}^{-1} (\cos\theta)^{-1} S_{mn}^{(1)}(-ic, \cos\theta) \, . \tag{2.14b}$$

This completes the expressions for far-zone backscattering for E-polarization. For H-polarization the situation is entirely similar. The incident magnetic field is now that of equation (1.2), for which the far-zone backscattered electric field will have only a θ-component.

It is

$$E^s_{\theta_H} = \eta_0 H^s_H \sim \frac{-E_0 e^{ikR}}{ikR} \sum_{m=0}^{\infty} (-1)^m \left\{ 2(2-\delta_{0m}) \beta^H_m(c,\theta) \sum_{\ell=m+1}^{\infty}{}' \frac{J'_{m\ell}(c)}{N_{m\ell}} \left(S^{(1)}_{m\ell}(-ic, \cos\theta) \right)^2 \right.$$

$$+ 2\,\alpha^H_m(c,\theta) \left[\sum_{\ell=m+1}^{\infty}{}' \frac{J_{m+1,\ell}(c)}{N_{m+1,\ell}} \left(S^{(1)}_{m+1,\ell}(-ic, \cos\theta) \right)^2 \right.$$

$$\left.\left. + (1-\delta_{0m}) \sum_{\ell=m-1}^{\infty}{}' \frac{J_{m-1,\ell}(c)}{N_{m-1,\ell}} \left(S^{(1)}_{m-1,\ell}(-ic, \cos\theta) \right)^2 \right] \right\}. \qquad (2.15)$$

All quantities are as defined before,except for the weighting factors, which are given by

$$\beta^H_m(c,\theta) = 1 - \alpha^H_m(c,\theta) \;, \qquad (2.16a)$$

$$\alpha^H_0 = \frac{\sum_{n=1}^{\infty}{}' b_{0n}(c,\theta) K^{(3)\prime}_{0n}(c)}{\sum_{n=1}^{\infty}{}' b_{0n}(c,\theta) K^{(3)\prime}_{0n}(c) + \frac{1}{2} \sum_{n=1}^{\infty}{}' a_{1n}(c,\theta)\, K^{(3)}_{1n}(c)} \;, \qquad (2.16b)$$

$$\alpha^H_1 = \frac{\sum_{n=2}^{\infty}{}' b_{1n}(c,\theta) K^{(3)\prime}_{1n}(c)}{\sum_{n=2}^{\infty}{}' b_{1n}(c,\theta) K^{(3)\prime}_{1n}(c) + \sum_{n=0}^{\infty}{}' a_{0n}(c,\theta) K^{(3)}_{0n}(c) + \frac{1}{2} \sum_{n=2}^{\infty}{}' a_{2n}(c,\theta) K^{(3)}_{2n}(c)} \;, \qquad (2.16c)$$

$$\alpha_m^H = \frac{\sum\limits_{n=m+1}^{\infty}{}' b_{mn}(c,\theta) K_{mn}^{(3)'}(c)}{\sum\limits_{n=m+1}^{\infty}{}' b_{mn}(c,\theta) K_{mn}^{(3)'}(c) + \frac{1}{2}\sum\limits_{n=m-1}^{\infty}{}' a_{m-1,n}(c,\theta) K_{m-1,n}^{(3)}(c) + \frac{1}{2}\sum\limits_{n=m+1}^{\infty}{}' a_{m+1,n}(c,\theta) K_{m+1,n}^{(3)}(c)},$$

$$m > 1, \qquad (2.16d)$$

with $b_{mn}(c,\theta)$ and $a_{mn}(c,\theta)$ as given by equations (2.14a, b).

2.3 Calculations Necessary to Implement Equations (2.8) and (2.15)

In principle, equations (2.8) and (2.15) allow computation of the far-zone backscattered fields for an arbitrary incident plane wave for any disk. Actual computations, however, are limited by the tables used and by convergence properties of the various series as c is varied. For this work the tables of Stratton, et al (1956) have been used. These contain the coefficients $d_n^{m\ell}$ for $0 \le m, \ell \le 8$ and varying orders of n up to 23 for a range of c (g in the tables) from .1 to 8 in increments in c of .1 or .2. The coefficients are given to seven significant digits, which is less than modern computers can utilize, but which is ample for the computations done here. The tables do not contain values of the normalization constants or of the joining factors, so these must be calculated from known properties of the oblate spheroidal functions.

Equations (2.3) and (2.5) allow computation of the angular functions and normalization constants in a straightforward manner given a program to calculate the associated Legendre functions needed in equation (2.3). Calculation of the joining factors presents the greatest challenge and requires some further development of the properties of the angular functions. The angular functions may be expressed in power series expansions of $(1 - \eta^2)$:

$$S_{m\ell}^{(1)}(-ic,\eta) = (1-\eta^2)^{\frac{m}{2}} \sum_{k=0}^{\infty} C_{2k}^{m\ell}(1-\eta^2)^k, \quad (\ell - m) \text{ even} \qquad (2.17a)$$

$$S^{(1)}_{m\ell}(-ic,\eta) = \eta(1-\eta^2)^{\frac{m}{2}} \sum_{k=0}^{\infty} C^{m\ell}_{2k}(1-\eta^2)^k \ , \quad (\ell - m) \text{ odd} \qquad (2.17b)$$

The $C^{m\ell}_{2k}$'s are related to the $d^{m\ell}_n$ coefficients by

$$C^{m\ell}_{2k} = \frac{1}{2^m k!(m+k)!} \sum_{r=k}^{\infty} \frac{(2m+2r)!}{2r!}(-r)_k (m+r+\tfrac{1}{2})_k \, d^{m\ell}_{2r} \ , \quad (\ell - m) \text{ even,} \qquad (2.18a)$$

$$C^{m\ell}_{2k} = \frac{1}{2^m k!(m+k)!} \sum_{r=k}^{\infty} \frac{(2m+2r+1)!}{(2r+1)!}(-r)_k (m+r+\tfrac{3}{2})_k \, d^{m\ell}_{2r+1}, \quad (\ell - m) \text{ odd,} \qquad (2.18b)$$

where

$$(n)_0 = 1, \qquad (n)_k = n(n+1)\ldots.(n+k-1) \ .$$

The $C^{m\ell}_{2k}$ enter into the calculation of all the joining factors. For the $J_{m\ell}(c)$ and $J'_{m\ell}(c)$ the equations are

$$J_{m\ell}(c) = \frac{1}{1 - \frac{i\pi}{2} Q^*_{m\ell}(-ic)} \ , \quad (\ell - m) \text{ even,} \qquad (2.19a)$$

$$J'_{m\ell}(c) = \frac{1}{1 - \frac{i\pi}{2} Q^*_{m\ell}(-ic)} \ , \quad (\ell - m) \text{ odd} \ , \qquad (2.19b)$$

where the function $Q^*_{m\ell}(-ic)$ has different expressions when $(\ell - m)$ is even and odd. Briefly,

$$Q^*_{m\ell}(-ic) = \frac{(-1)^m \left[K^{(1)}_{m\ell}(-ic)\right]^2}{c} \sum_{r=0}^{m} \alpha^{m\ell}_r(-ic) \frac{(2m-2r)!}{r!\left[2^{m-r}(m-r)!\right]^2}, \quad (\ell - m) \text{ even,} \qquad (2.20a)$$

$$Q^{*}_{m\ell}(-ic) = \frac{(-1)^m \left[K^{(1)'}_{m\ell}(-ic)\right]^2}{c} \sum_{r=0}^{m} \alpha^{m\ell}_r(-ic) \frac{(2m-2r+1)!}{r! \left[2^{m-r}(m-r)!\right]^2}, \quad (\ell-m) \text{ odd}, \tag{2.20b}$$

and $\alpha^{m\ell}_r(-ic)$ is defined as

$$\alpha^{m\ell}_r(-ic) \equiv \left\{ \frac{d^r}{dx^r} \left[\left(\sum_{k=0}^{\infty} C^{m\ell}_{2k} x^k \right)^{-2} \right] \right\} \Bigg|_{x=0} . \tag{2.21}$$

The calculation of $\alpha^{m\ell}_r(-ic)$ is straightforward, but requires tedious differentiation. The series for $\alpha^{m\ell}_r(-ic)$ in terms of the $C^{m\ell}_{2k}$ will consist of a finite number of terms and is given for $0 \le r \le 8$ in Appendix 2-C in terms of the normalized coefficients $C'^{m\ell}_{2k}$ defined by

$$C'^{m\ell}_{2k} \equiv C^{m\ell}_{2k} \Big/ C^{m\ell}_{0} . \tag{2.22}$$

By virtue of equations (2.20a, b) and (2.12a, b) it is only necessary to determine a means of finding values for the $K^{(1)}_{m\ell}(c)$ and $K^{(1)'}_{m\ell}(c)$ in order to be able to calculate all of the joining factors. Reference to equations (2.10a, b) indicates that only angular functions and radial functions of the first kind are involved. The radial function of the first kind and its derivative at $\xi = 0$ are easily found from

$$R^{(1)}_{m\ell}(-ic, i0) = (-1)^{\frac{\ell-m}{2}} c^m \frac{(\ell-m)! \quad m!\, 2^m}{(\ell+m)!\,(2m+1)} d^{m\ell}_0(-ic), \quad (\ell-m) \text{ even}, \tag{2.23a}$$

$$R^{(1)'}_{m\ell}(-ic, i0) = (-1)^{\frac{\ell-m-1}{2}} c^{m+1} \frac{(\ell-m)!\ m!\, 2^m}{(\ell+m)!\ (2m+3)} d_1^{m\ell}(-ic), \quad (\ell-m) \text{ odd.} \qquad (2.23b)$$

Values of the angular function of the first kind and its derivative at $\eta = 0$ may be obtained either from equation (2.3) or equations (2.17a, b). For the computations done here it was decided to use the latter equations, since the $C_{2k}^{m\ell}$'s must be found in any case, and since the resulting expressions are quite simple. For the angular function and its derivative at $\eta = 0$, equations (2.17a, b) yield

$$S^{(1)}_{m\ell}(-ic, 0) = C_0^{m\ell} \sum_{k=0}^{\infty} {}' C_{2k}^{m\ell} , \qquad (\ell-m) \text{ even,} \qquad (2.24a)$$

$$S^{(1)'}_{m\ell}(-ic, 0) = C_0^{m\ell} \sum_{k=0}^{\infty} {}' C_{2k}^{m\ell} . \qquad (\ell-m) \text{ odd.} \qquad (2.24b)$$

Test values of the angular function at $\eta = 0$ computed from equations (2.24a) and (2.3) were found to be in excellent agreement. The series in equations (2.24a, b), however, sometimes converge rather slowly to small values, especially for large values of c. This results in an effective loss in the number of significant digits in the summations given by equations (2.24a, b) compared with the input data. For the range of c considered, this loss is considered to be no more than 3 digits. In any case, computation of the $C_{2k}^{m\ell}$'s was done as carefully as possible. This effort appears to have been successful in keeping propagation of round-off and other errors to a minimum.

2.4 Numerical Computations and Comparison with Experimental Data

Let the scattered fields of equations (2.8) and (2.15) be written as

$$E^s_{\phi_E} = -\eta_0 H^s_\theta \sim \frac{E_0 e^{ikR}}{ikR} P^E(c, \theta) , \qquad (2.8')$$

$$E^s_{\theta_H} = \eta_0 H^s_\phi \sim \frac{-E_0 e^{ikR}}{ikR} \, P^H(c,\theta) \, , \qquad (2.15')$$

where $P^E(c,\theta)$ and $P^H(c,\theta)$ are complex quantities equal to the entire summations over m in equations (2.8) and (2.15) respectively. In terms of these quantities the radar cross-section (RCS-normalized to a square wavelength) can be found by means of the expression

$$\frac{\sigma_j}{\lambda^2} = \lim_{R \to \infty} \frac{4\pi R^2}{\lambda^2} \frac{|E^s_j|^2}{|E_0|^2} \, . \qquad (2.25)$$

For the incident field of equation (1.1) the radar cross-section in dB is,

$$10 \log_{10}\left(\frac{\sigma_E}{\lambda^2}\right) = 20 \log_{10}\left(\frac{1}{\sqrt{\pi}} \left|P^E(c,\theta)\right|\right). \qquad (2.26)$$

Similarly, for the incident field given by equation (1.2) the result is

$$10 \log_{10}\left(\frac{\sigma_H}{\lambda^2}\right) = 20 \log_{10}\left(\frac{1}{\sqrt{\pi}} \left|P^H(c,\theta)\right|\right). \qquad (2.27)$$

Finally, for the cross-polarized scattered field for a polarization angle $\psi = 45^\circ$ the radar cross-section in dB is found from equation (1.7) to be

$$10 \log_{10}\left(\frac{\sigma_{(C)}}{\lambda^2}\right) = 20 \log_{10}\left(\frac{1}{2\sqrt{\pi}} \left|P^H(c,\theta) - P^E(c,\theta)\right|\right) \, . \qquad (2.28)$$

A program has been written to compute the direct and cross-polarized radar cross-sections for any value of c for which input data are available for an angular range in θ of $0 < \theta < \frac{\pi}{2}$. The restriction on θ is necessary because the program would attempt to divide by 0 at either 0 or $\frac{\pi}{2}$, even though the actual solutions to the scattering problems have well behaved limits at these points. The input data consist of the value of c, a list of the

$d_n^{m\ell}(-ic)$'s , and the angular range to be considered. In addition to the various radar cross-sections, the program also returns the real and imaginary parts of $P^E(c,\theta)$ and $P^H(c,\theta)$, the normalization constants $N_{m\ell}$, $J_{m\ell}(c)$ and $J'_{m\ell}(c)$, $K_{m\ell}(c)$ and $K'_{m\ell}(c)$, and the $C_{2k}^{m\ell}$'s . A complete listing and explanation of the program is given in Appendix D. Also included are illustrative tables of the $N_{m\ell}$, $K_{m\ell}^{(1)}$, $K_{m\ell}^{(1)'}$, $J_{m\ell}(c)$, $J'_{m\ell}(c)$, and $C_{2k}^{m\ell}$ for $c = 4.0$.

Values of computed direct and cross-polarized radar cross sections given by equations (2.26) through (2.28) are compared with experimental results obtained at the University of Michigan Radiation Laboratory experimental facility at Willow Run for values of c of 1 (1) 7 and values of θ of $2^o(2^o)\ 88^o$ in Figs. (2-2a) through (2-8c). The experimental data are given by two sets of solid lines which reproduce the data obtained as the incident field was swept through a 180^o scan from edge-on incidence through normal incidence ($\theta = 0^o$) and on to edge-on incidence. The two 90^o segments were then plotted one upon the other as a function of θ. The computed values are indicated as points every two degrees. On the whole, agreement between the experimental and computed data is quite good for the values of c that are used. There are, however, several types of discrepancies between the experimental and computed data that require explanation. For instance, the cross-polarized radar cross sections tend to show greater differences than the corresponding direct radar cross sections. This behavior is not very surprising in light of the fact that the cross polarized return is generally much weaker than the two direct returns. Indeed, a difference in peak magnitudes between the two types of cross sections of 20dB or more occurs several times. Consequently, the cross-polarized return is more likely to be affected by stray returns and equipment limitations. The stray returns would generally be small and would manifest themselves principally by displacing the locations and changing the amplitudes of the nulls in the experimental patterns. Evidence of this type of behavior is found in all returns for c = 1.0 where the experimental data are consistently higher than the computed values. Again, in Fig. 2-3a similar behavior is found for the E-polarized return for θ near 90°. The behavior of the computed return is more consistent with the observed behavior of the cross-polarized return than the experimental data are. Both cases cited are for low values of c, and consequently low amplitude returns, which are, of course, just the situations where stray returns would cause the greatest errors.

Because of the symmetry inherent in a disk, the cross-polarized return for a plane wave at normal (θ=0) incidence should be zero. In practice, however, it proves very difficult to obtain this condition with the equipment available. The major limitation is the degree of isolation available in the receiving antenna for signals ninety degrees out of space phase with respect to the desired signal. The direct return at normal incidence is so strong that the receiving antenna will pick up a slight error signal. This error signal will increase with increasing c, an effect that is actually observed in the strength of the cross-polarized return at zero aspect angle.

For large values of c and θ there occur large discrepancies between the measured and computed returns for E-polarization and cross-polarization. These are believed to reflect errors in the computed returns due to poor convergence of the various series used in computing the radar cross sections. This effect must be observed eventually, because of the limitations inherent in the tables of Stratton, et al (1956) that were used to compute the oblate spheroidal functions. The actual differences do become worse with increasing c, which is consistent with this view. The given data then can be used to determine in a qualitative way the limits of convergence of the exact solution as programmed.

The success of the programming effort as outlined in this section implies that there now exists a standard, the computed returns, against which both experimental results and results due to approximate theories can be judged. Such a procedure will, for instance, be used in Chapter IV.

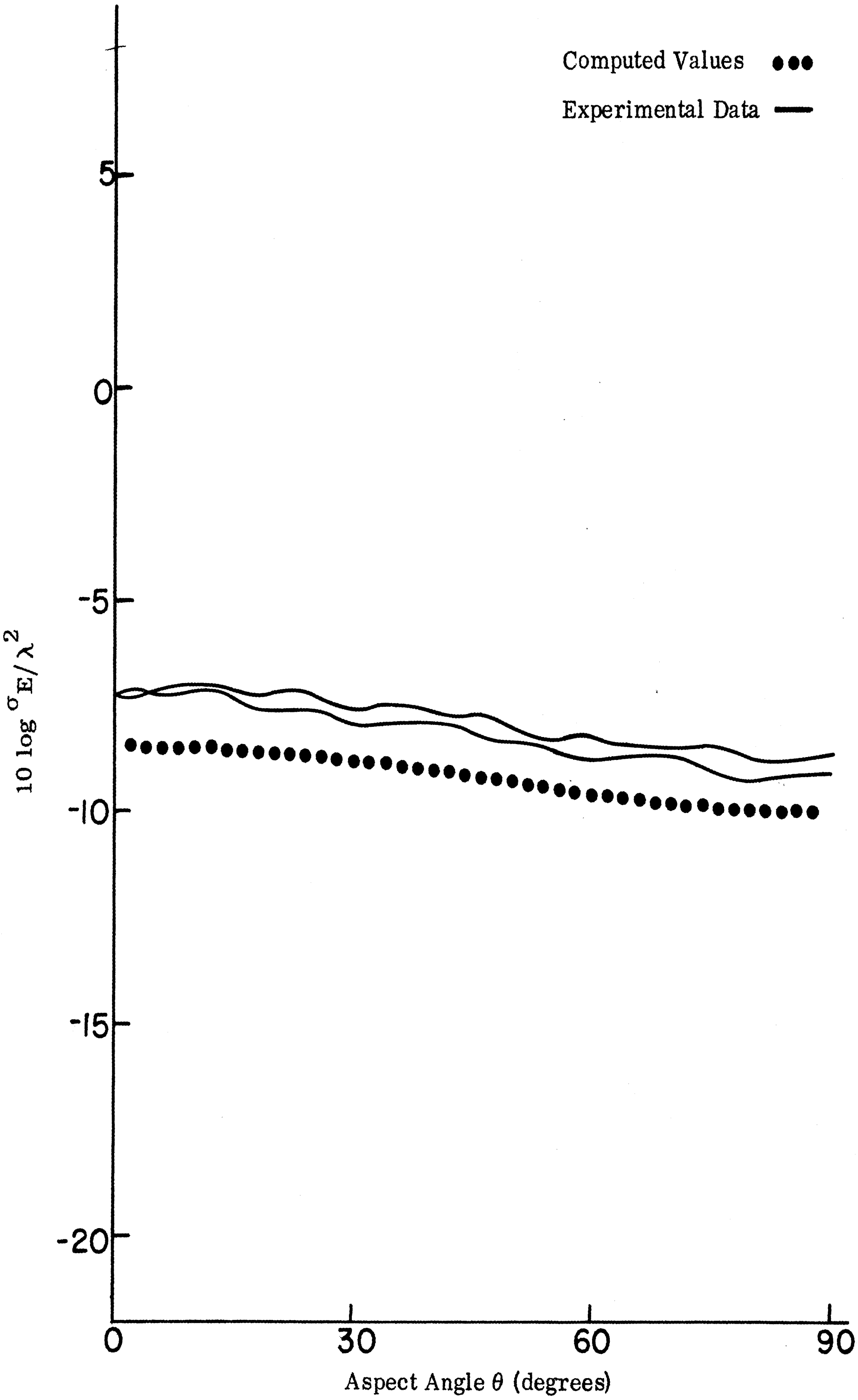

FIG. 2-2a: RCS of a Disk for E-Polarization; c = 1.0.

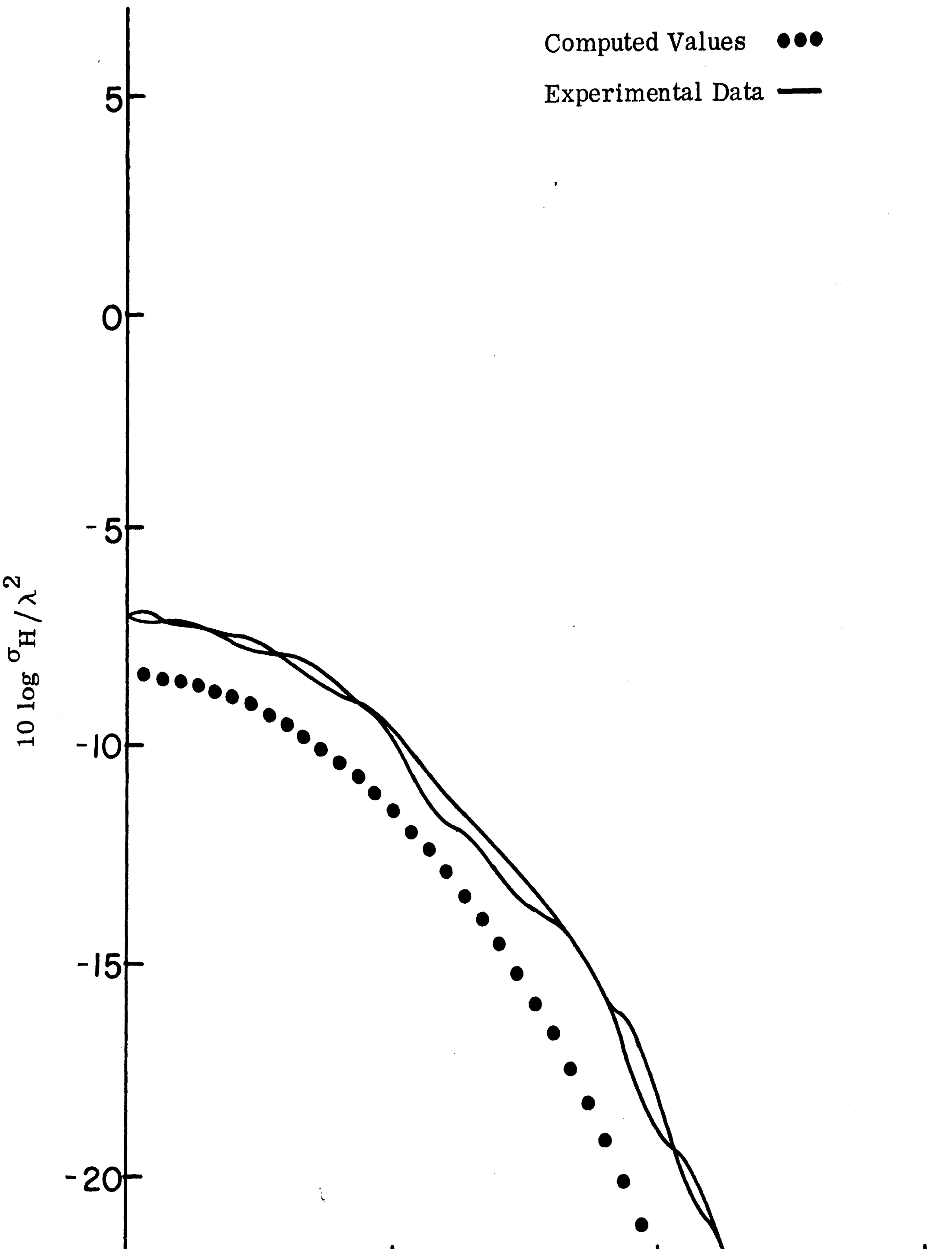

FIG. 2-2b: RCS of a Disk for H-Polarization; c = 1.0.

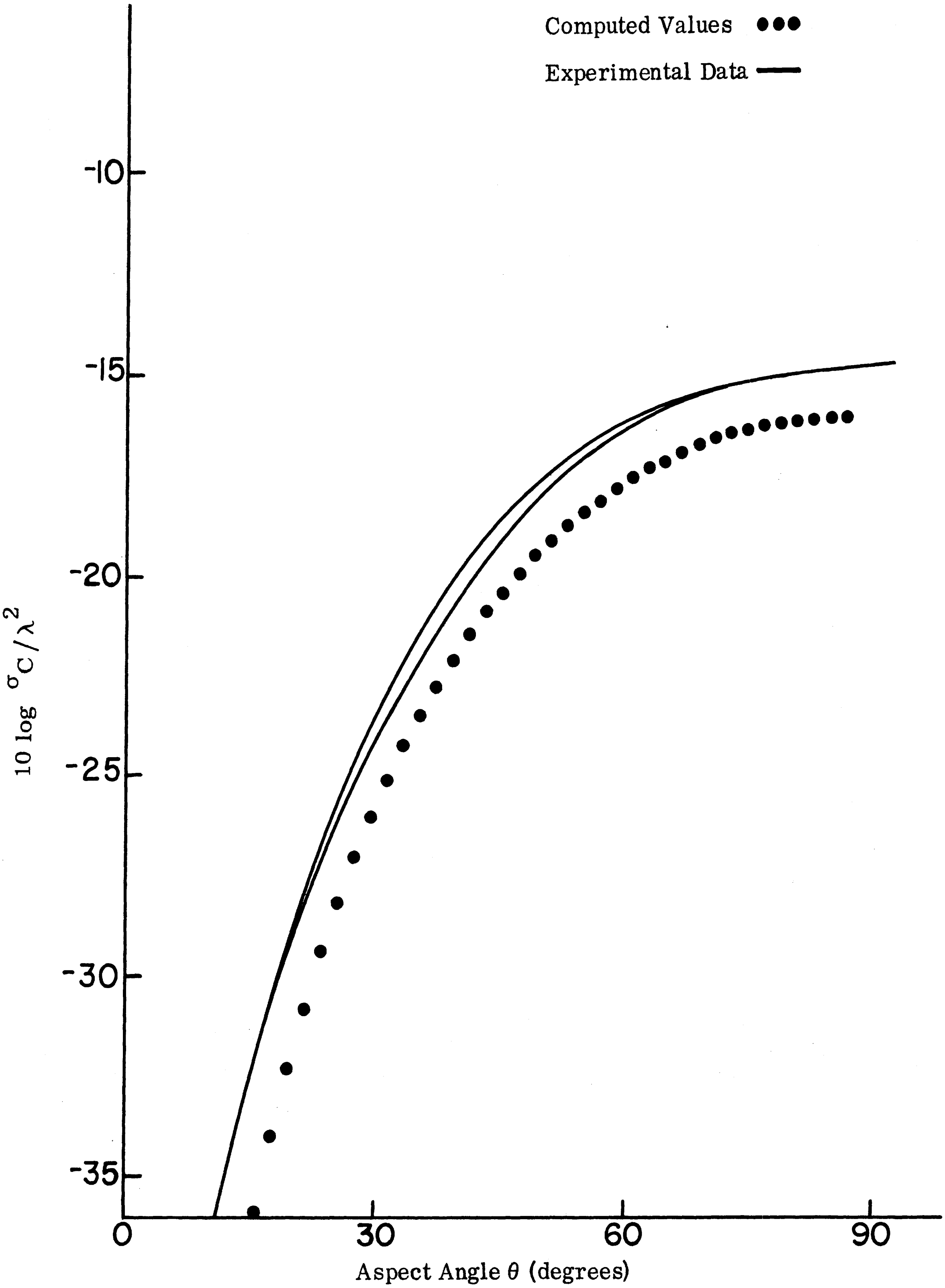

FIG: 2-2c: Cross-Polarized RCS of a Disk; c = 1.0.

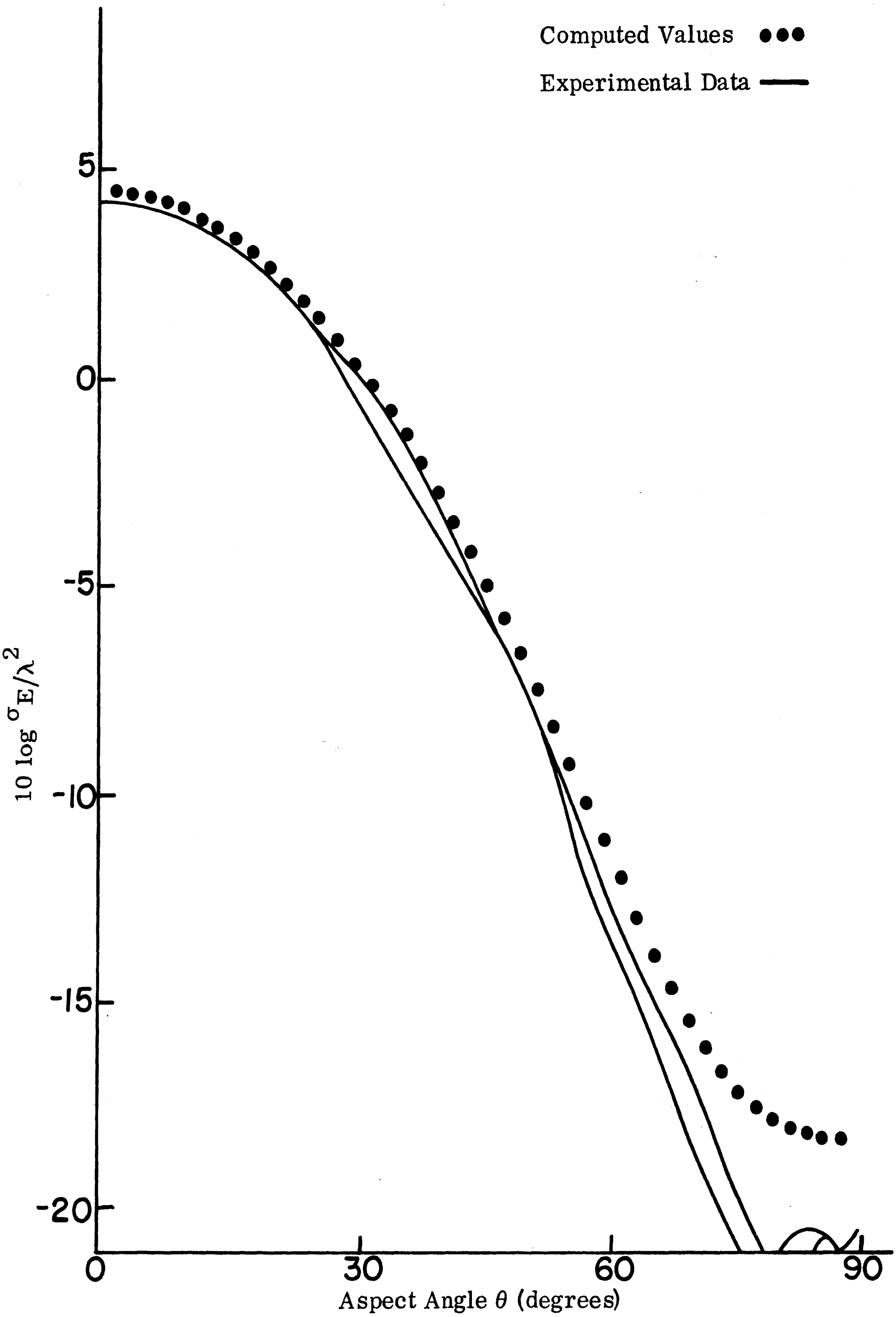

FIG. 2-3a: RCS of a Disk for E-Polarization; c = 2.0.

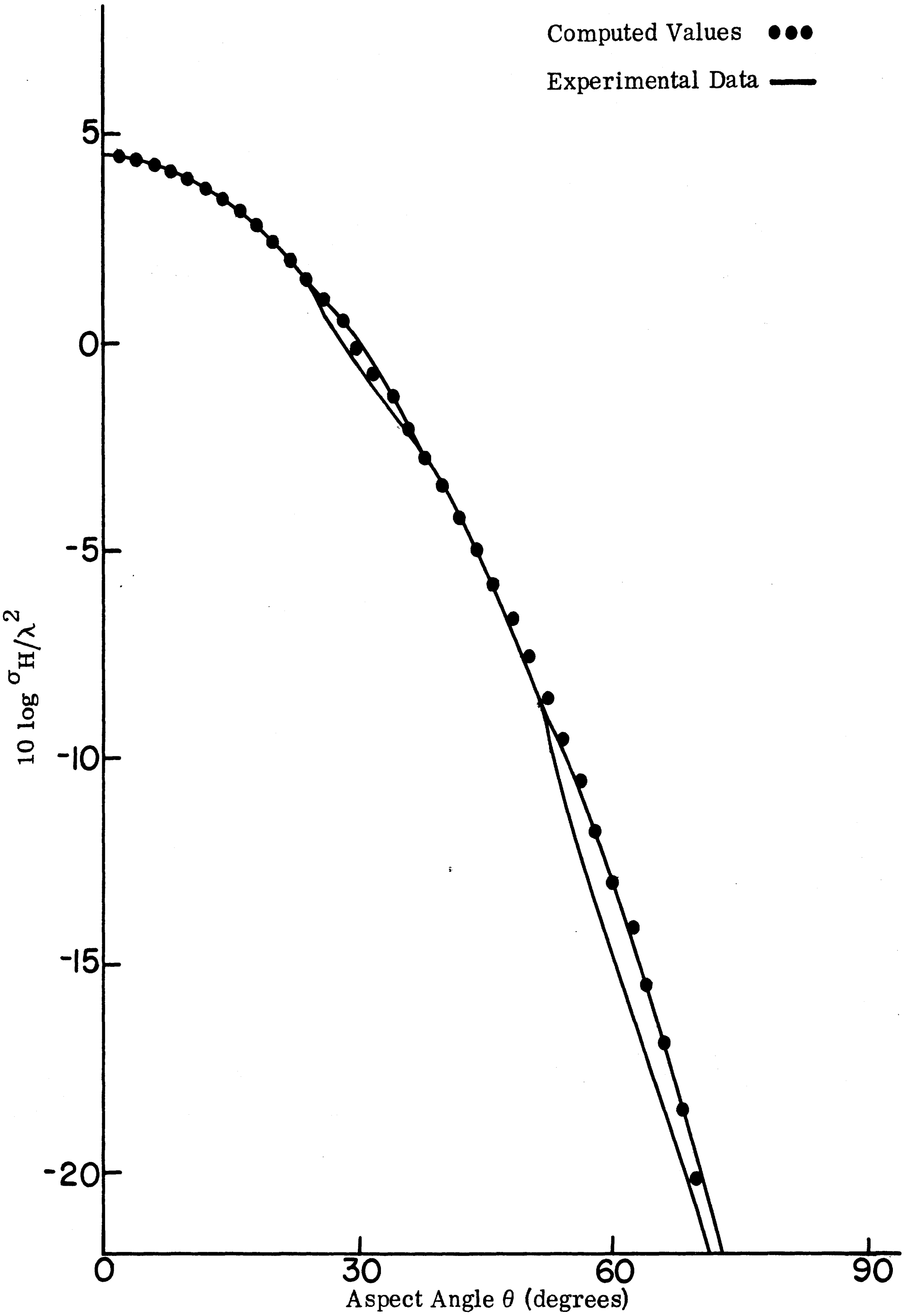

FIG. 2-3b: RCS of a Disk for H-Polarization; c = 2.0.

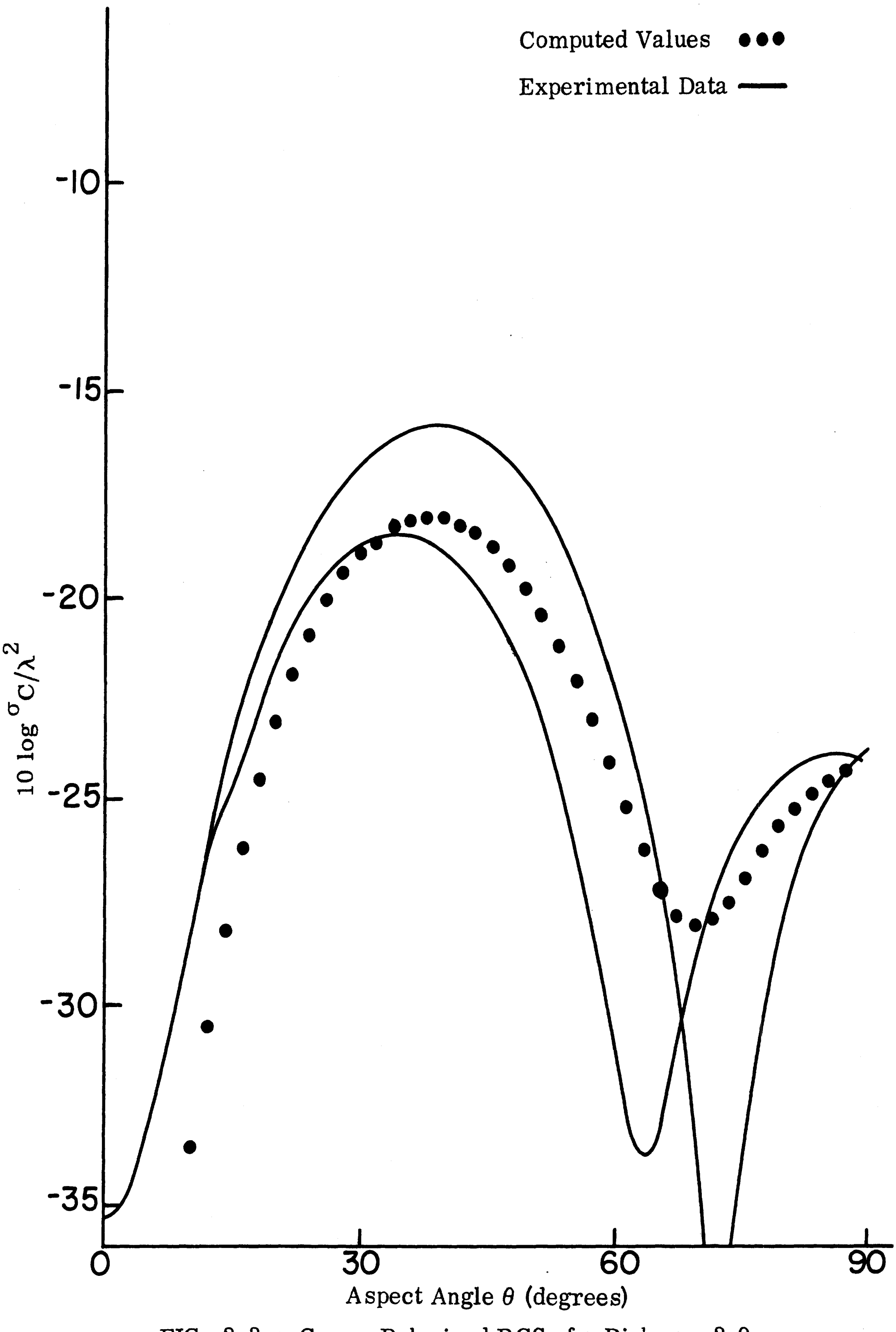

FIG. 2-3c: Cross-Polarized RCS of a Disk; c = 2.0.

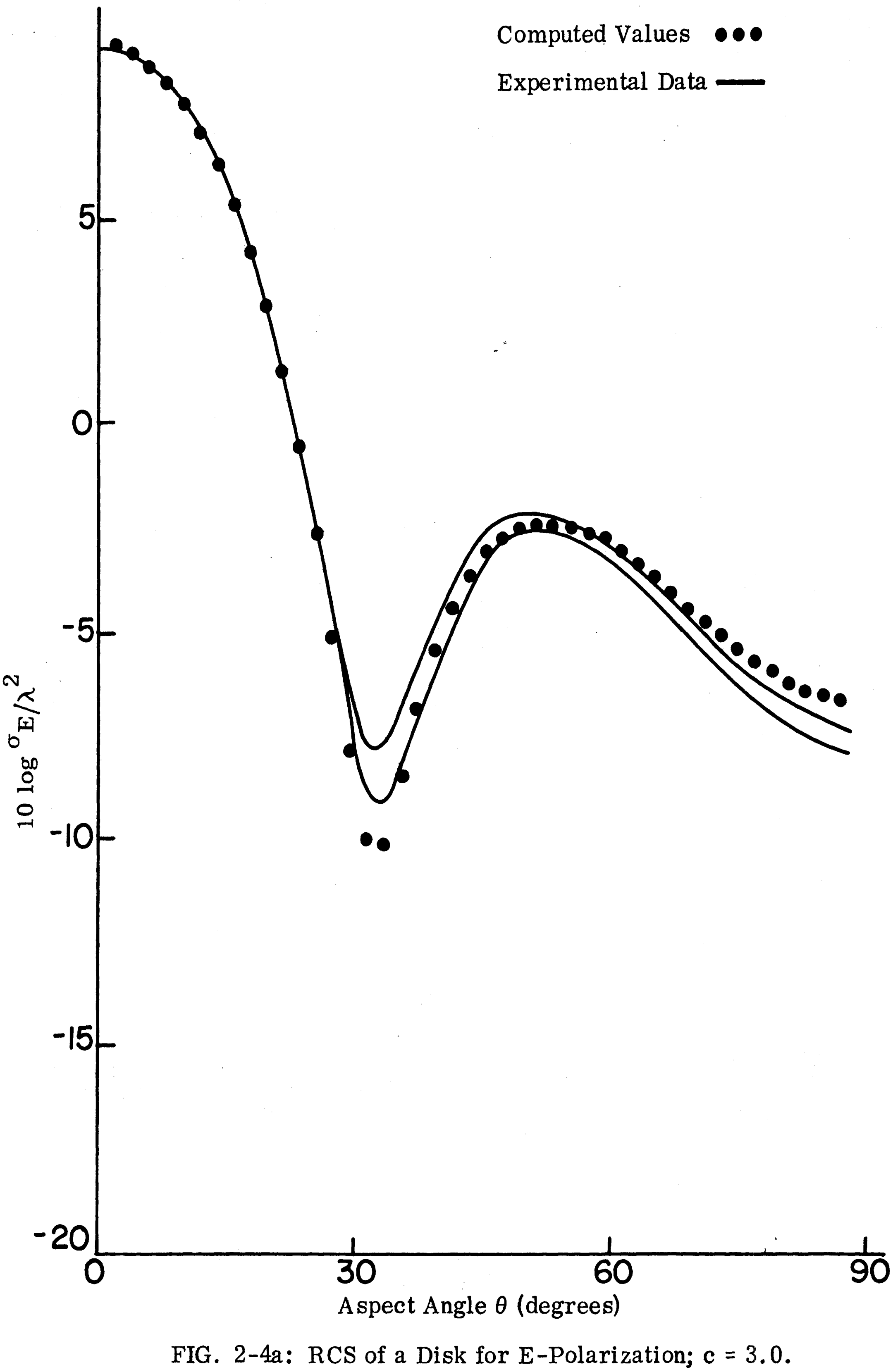

FIG. 2-4a: RCS of a Disk for E-Polarization; c = 3.0.

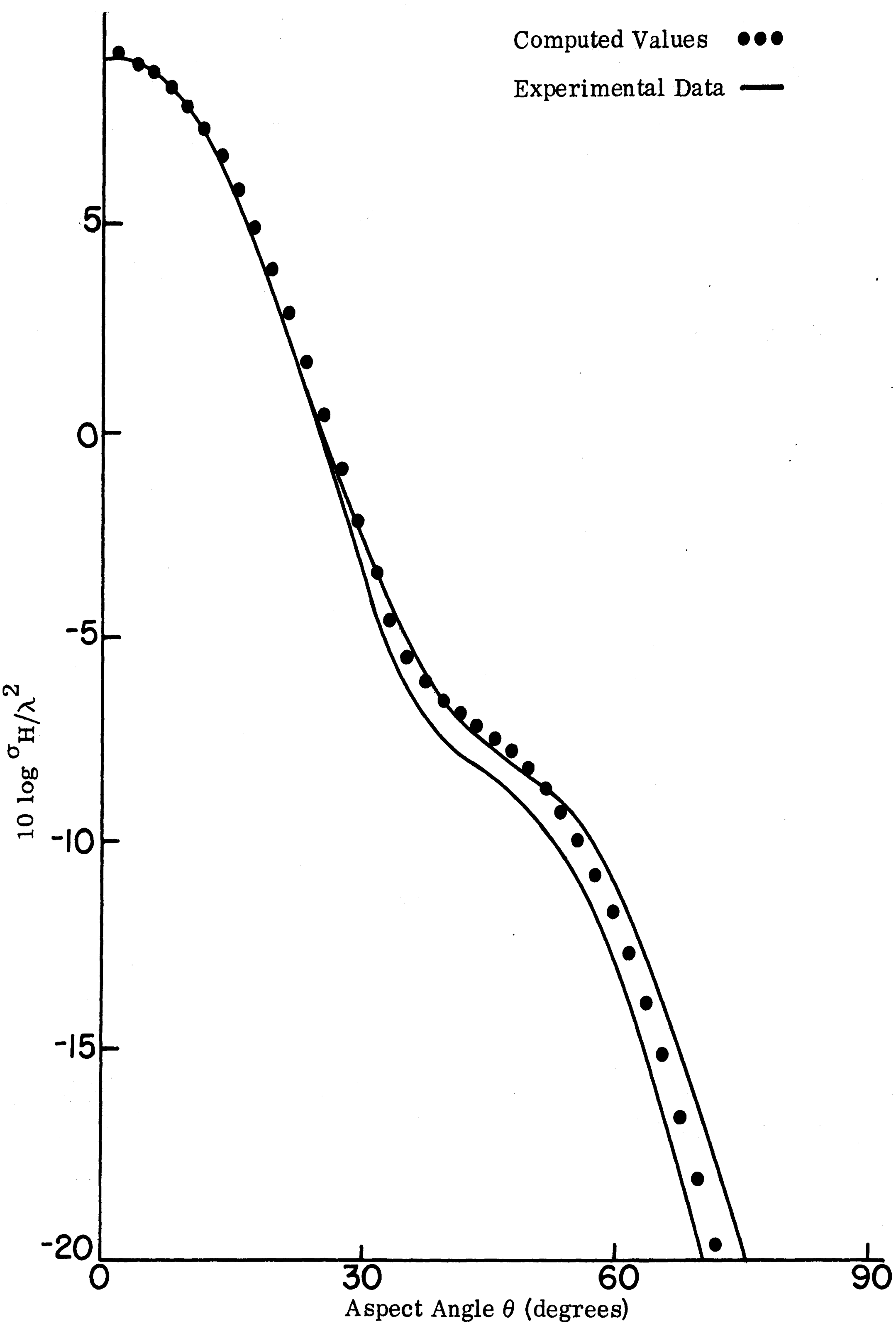

FIG. 2-4b: RCS of a Disk for H-Polarization; c = 3.0.

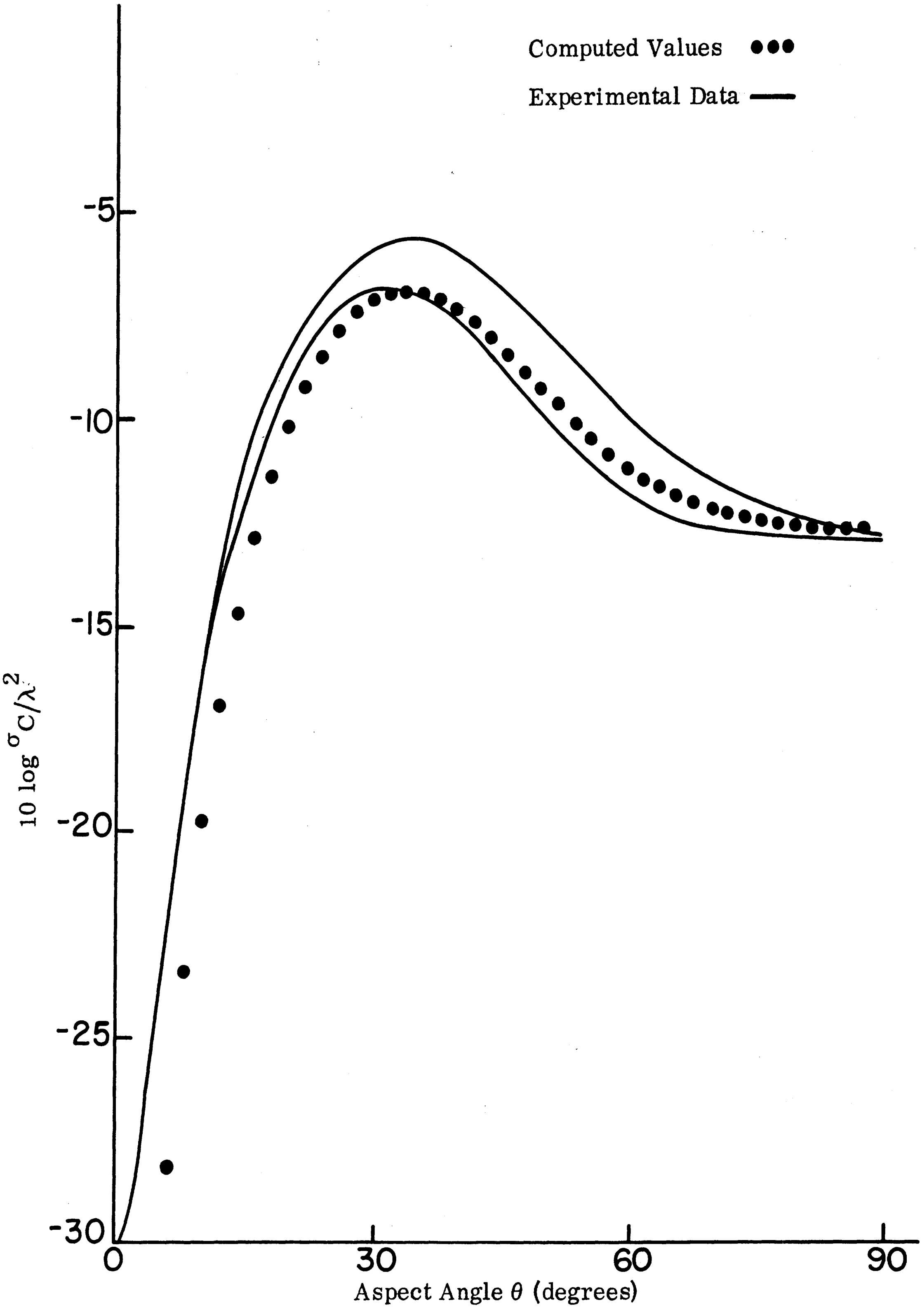

FIG. 2-4c: Cross-Polarized RCS of a Disk; c = 3.0.

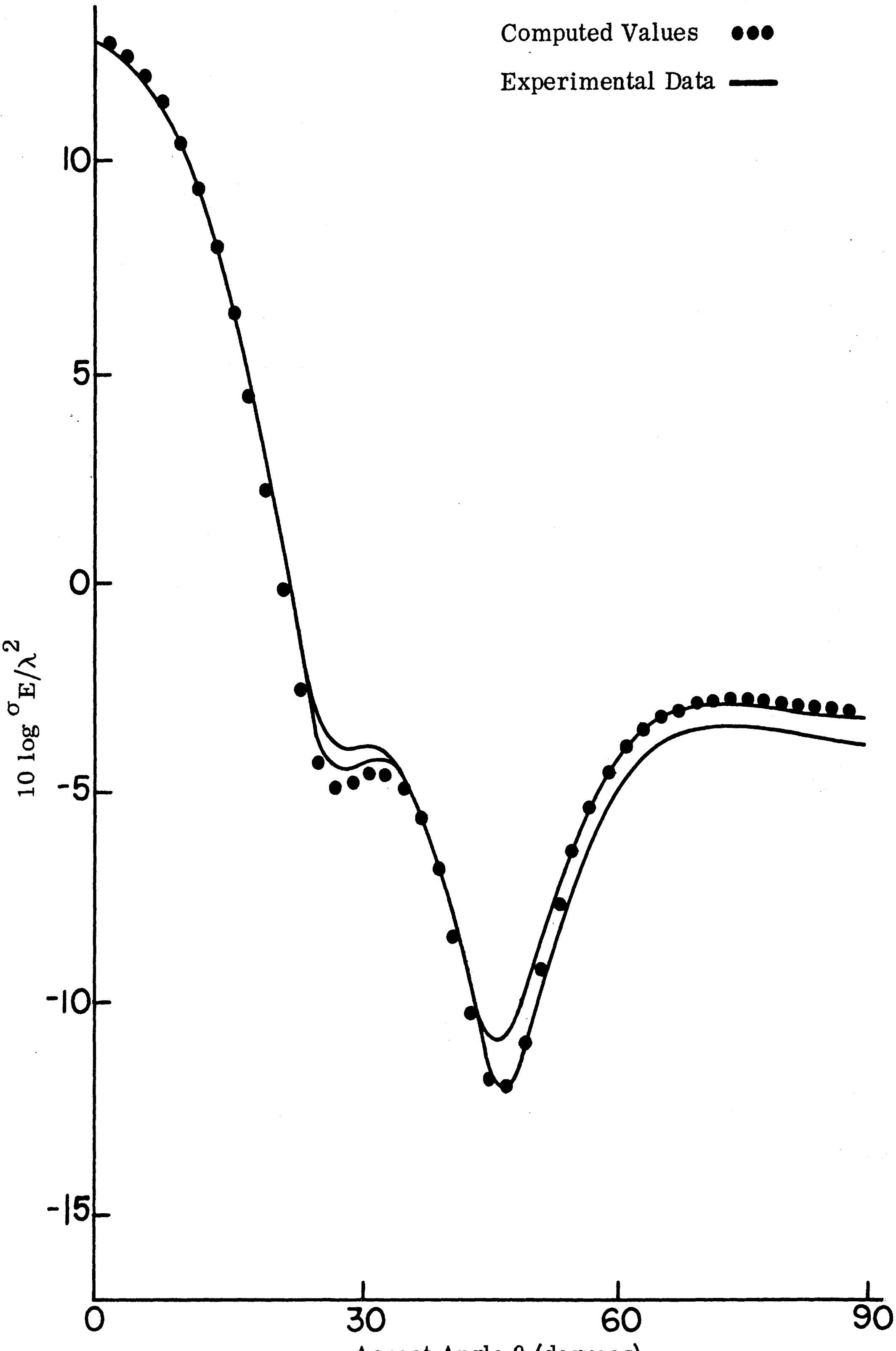

FIG. 2-5a: RCS of a Disk for E-Polarization; c = 4.0.

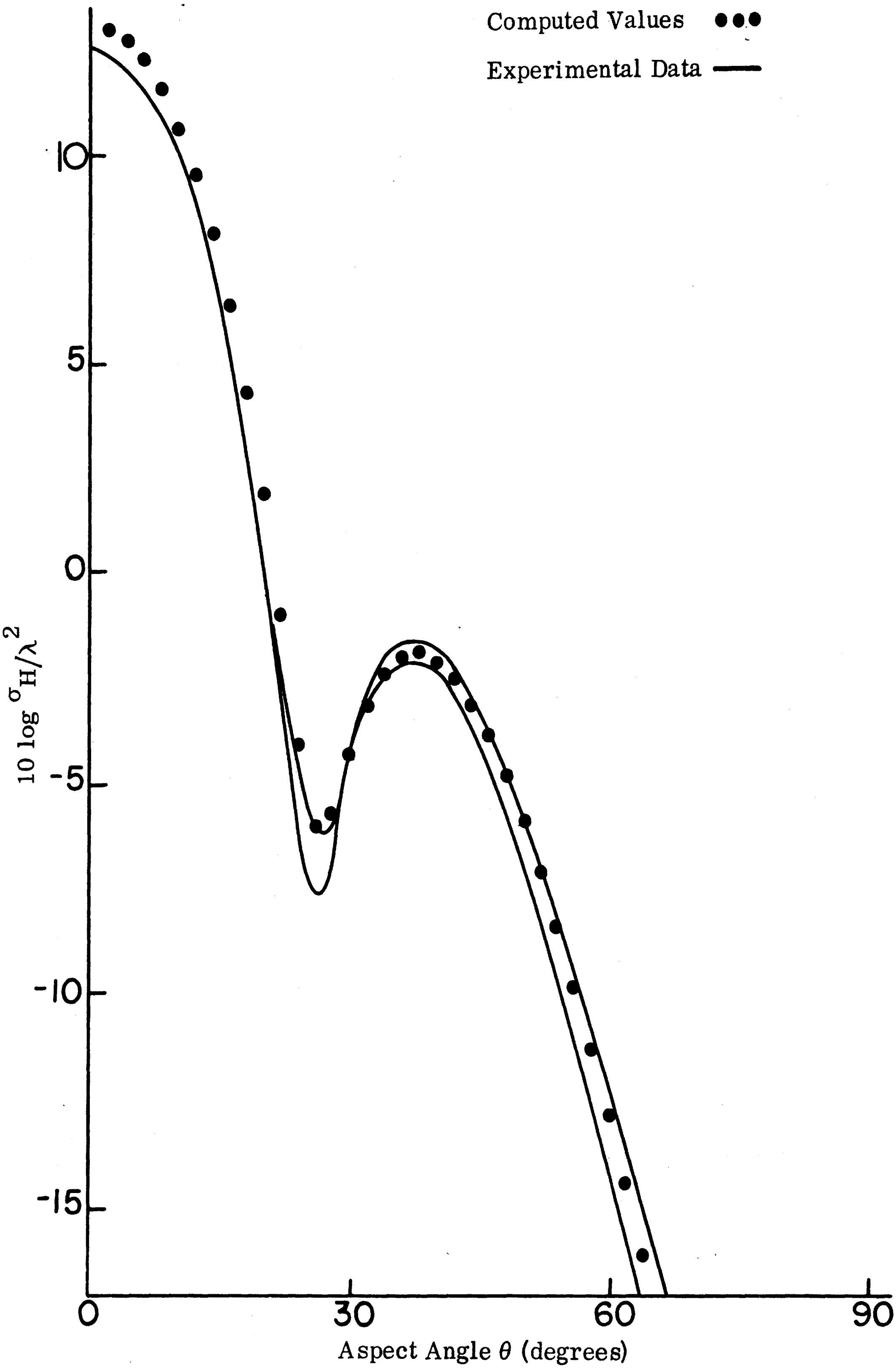

FIG. 2-5b: RCS of a Disk for H-Polarization; c = 4.0.

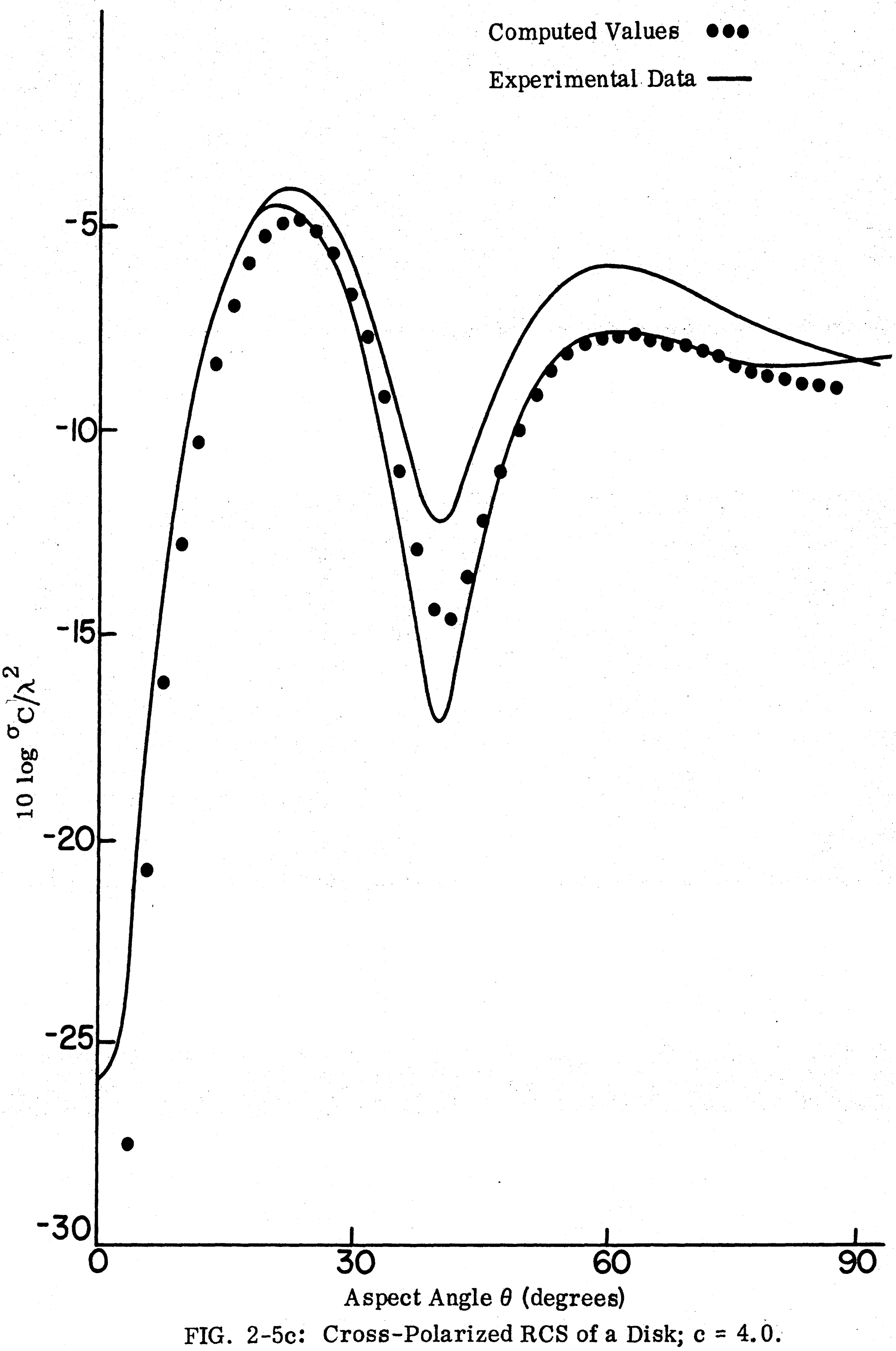

FIG. 2-5c: Cross-Polarized RCS of a Disk; c = 4.0.

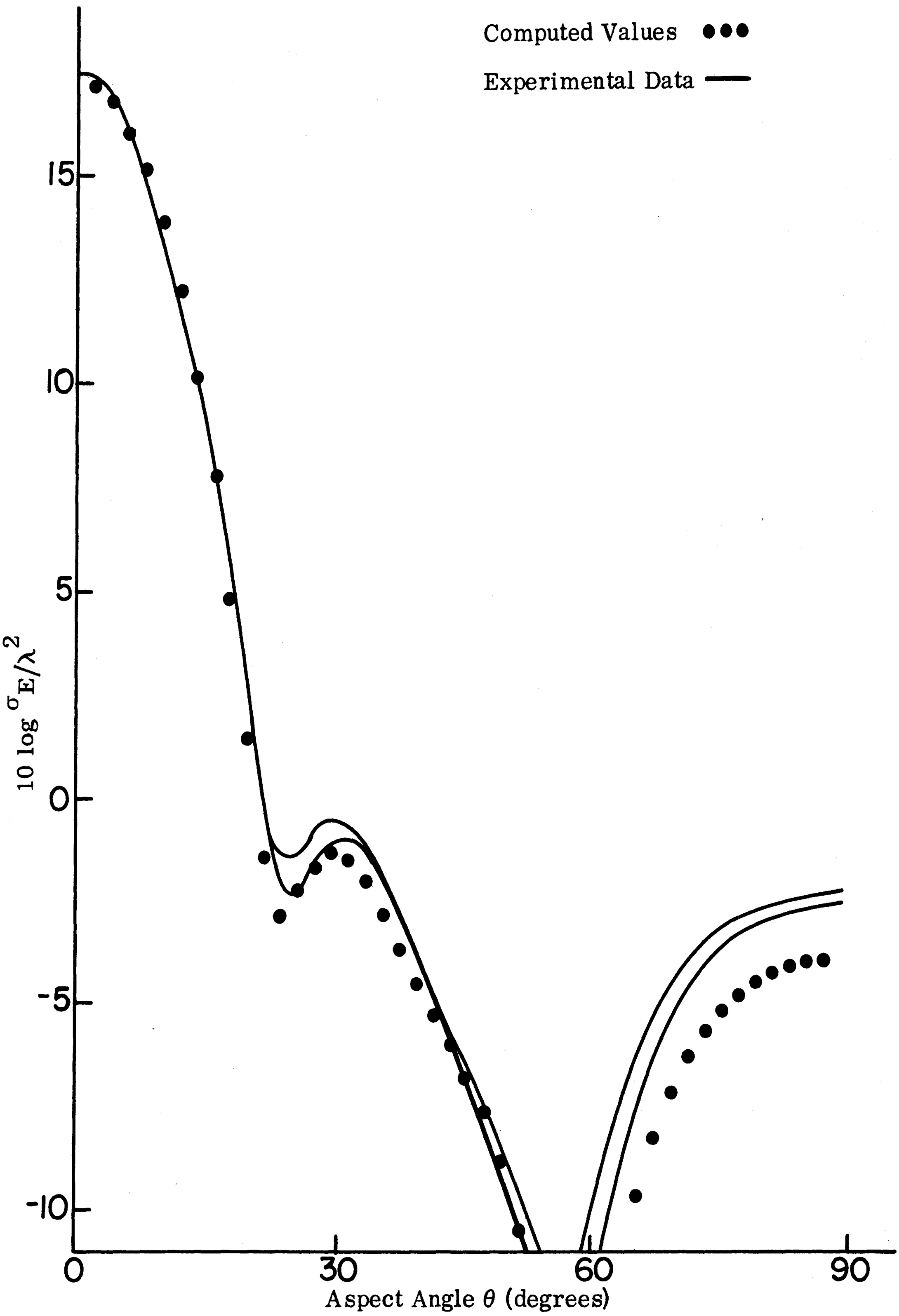

FIG. 2-6a: RCS of a Disk for E-Polarization; c = 5.0.

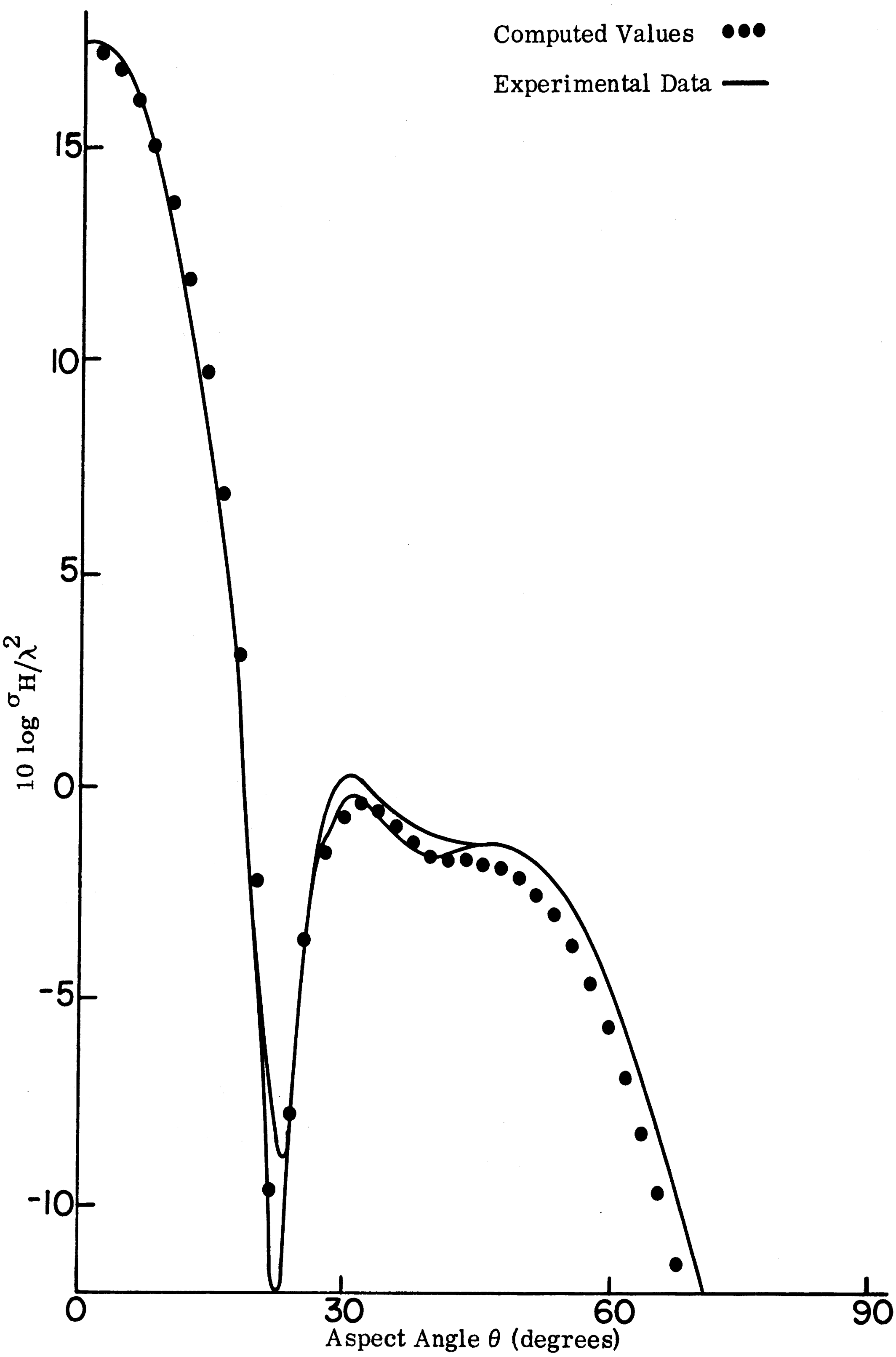

FIG. 2-6b: RCS of a Disk for H-Polarization; c = 5.0.

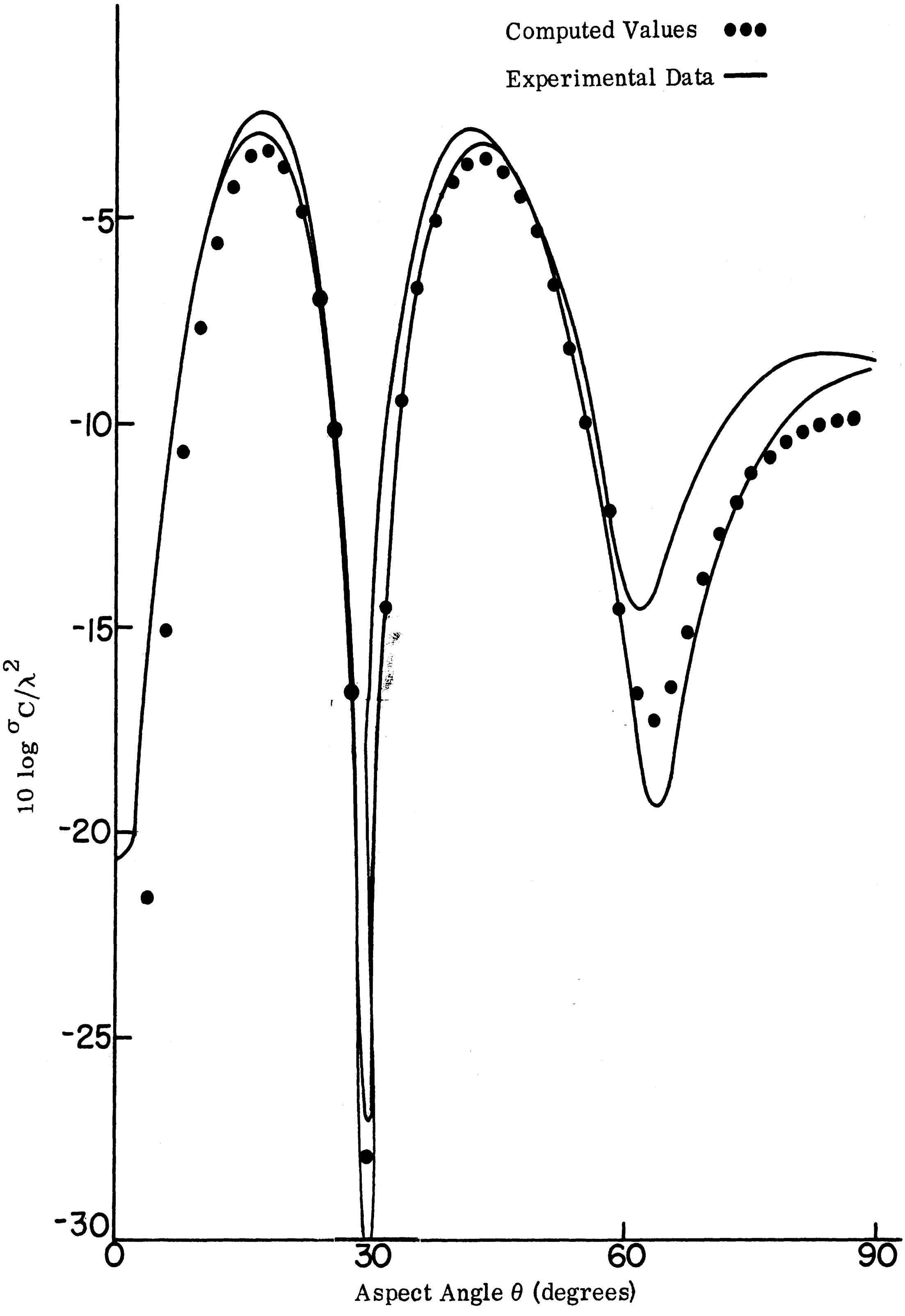

FIG. 2-6c: Cross-Polarized RCS of a Disk; c = 5.0.

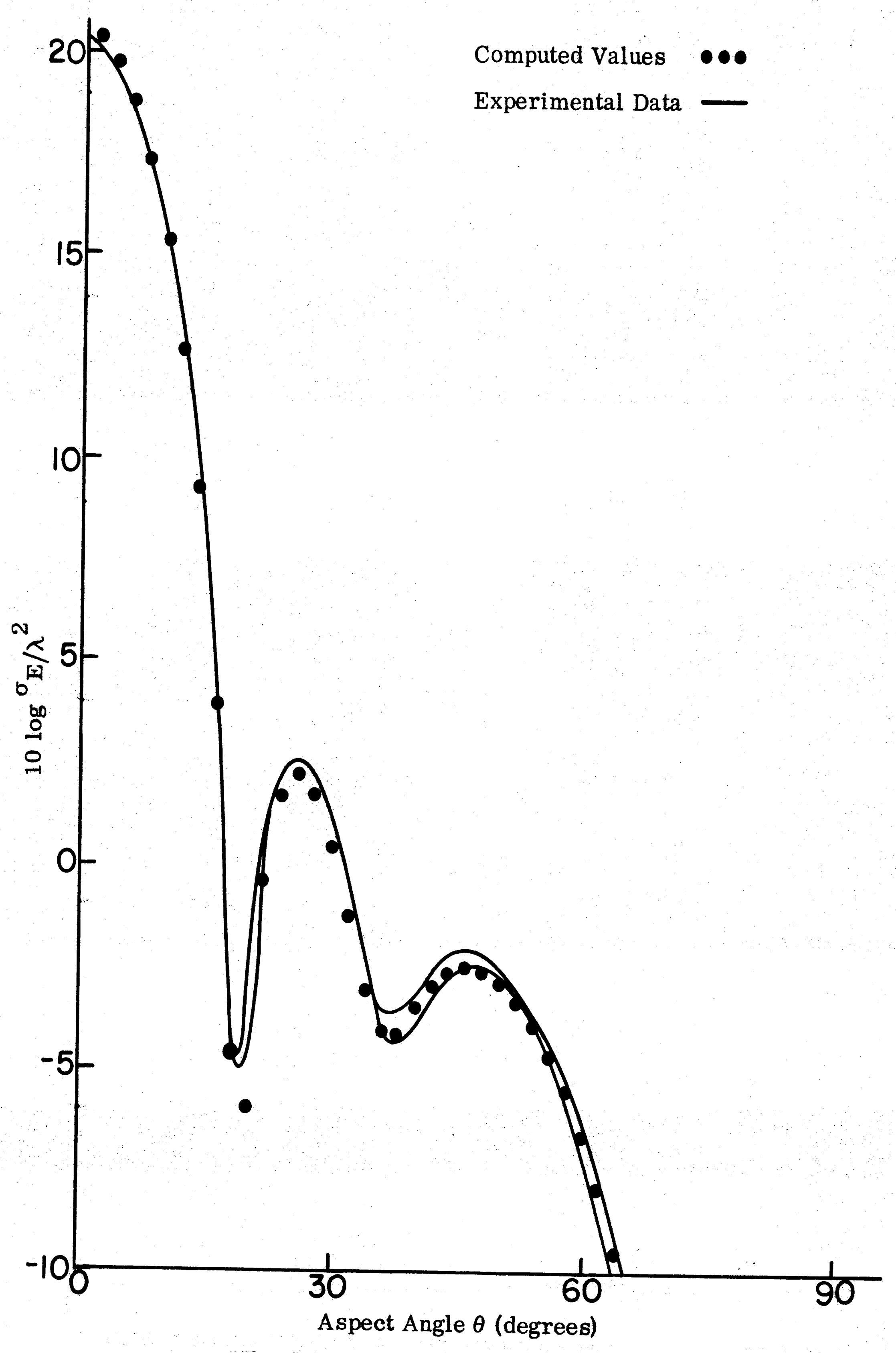

FIG. 2-7a: RCS of a Disk for E-Polarization; c = 6.0.

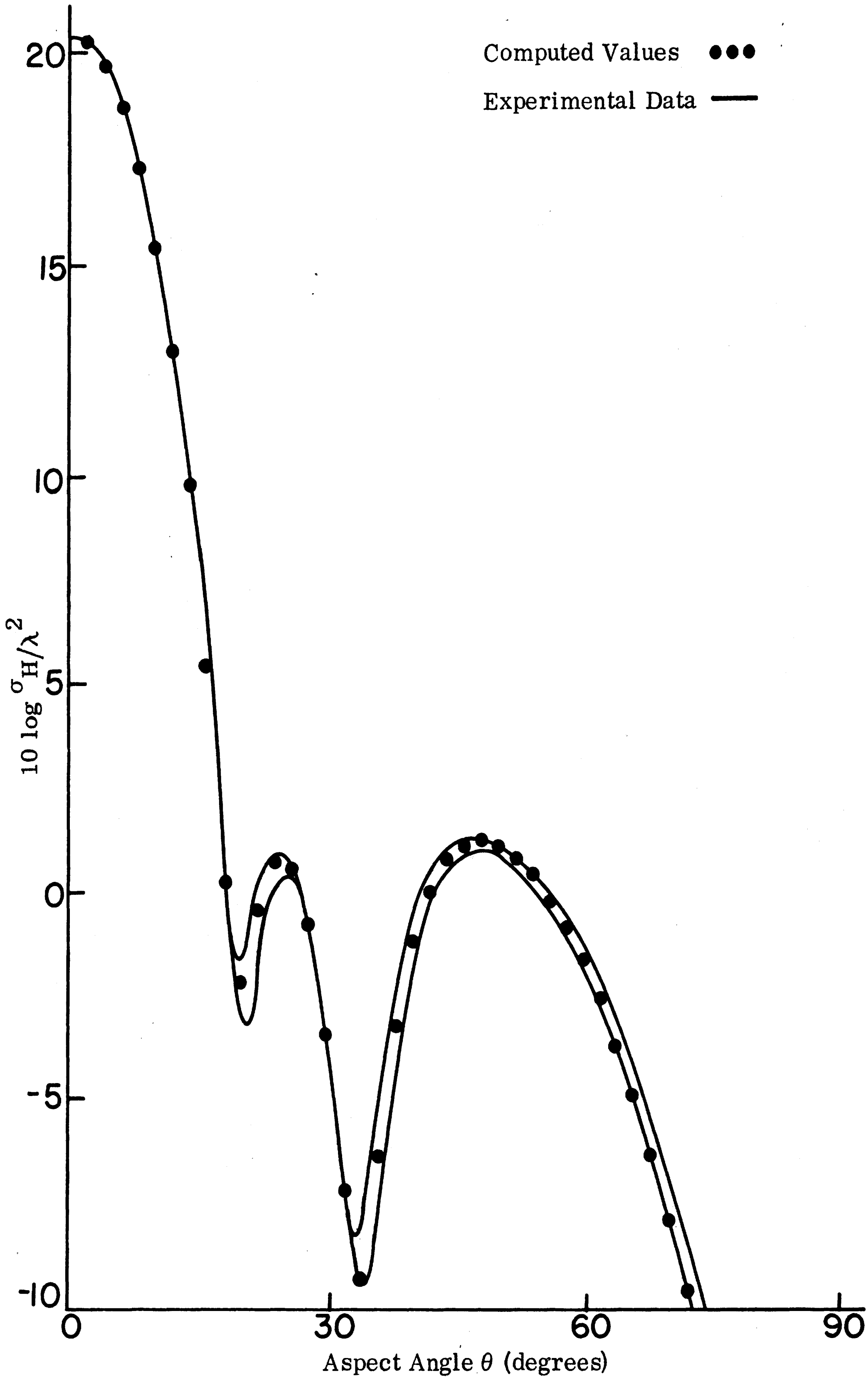

FIG. 2-7b: RCS of a Disk for H-Polarization; c = 6.0.

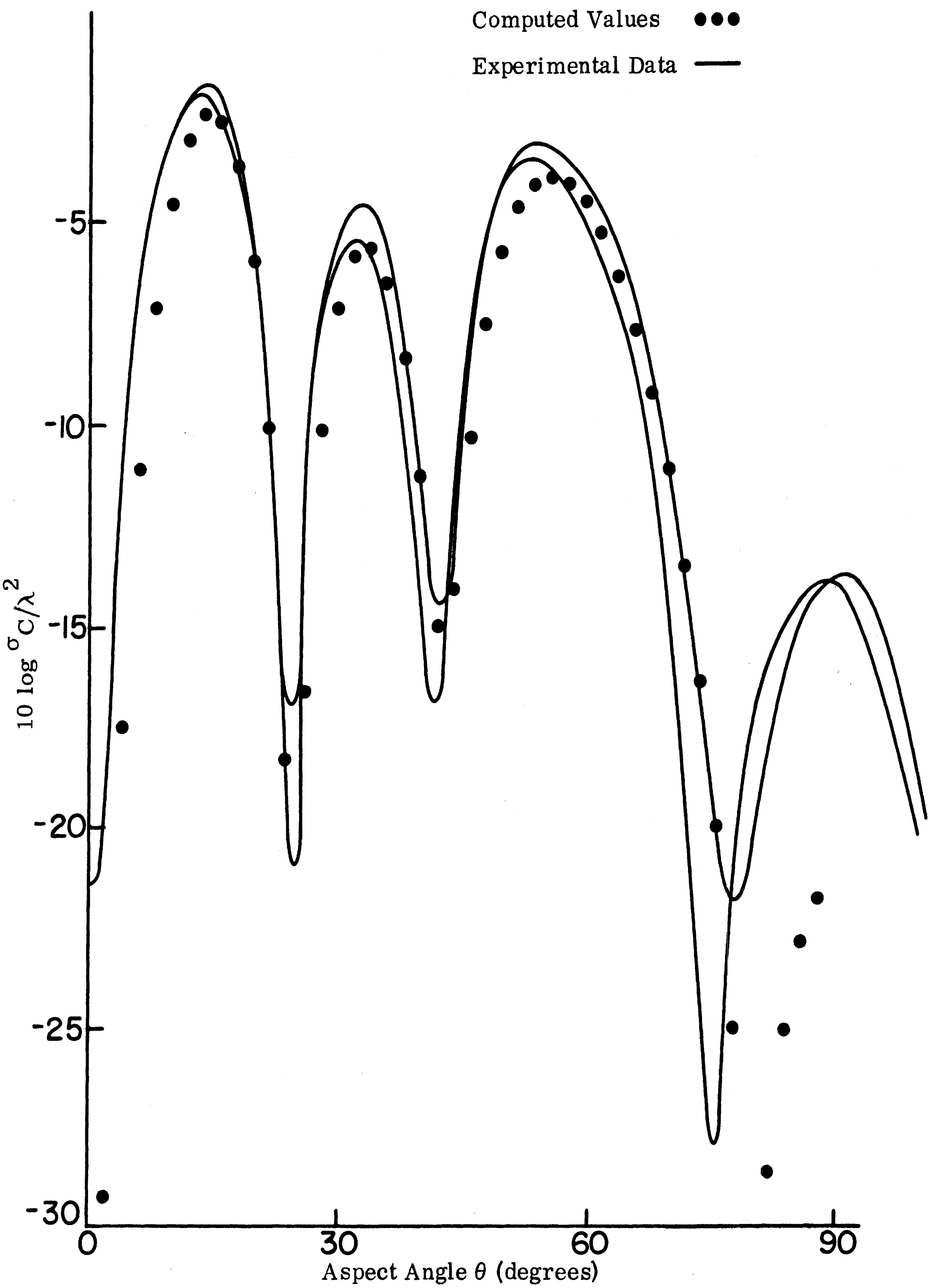

FIG. 2-7c: Cross-Polarized RCS of a Disk; c = 6.0.

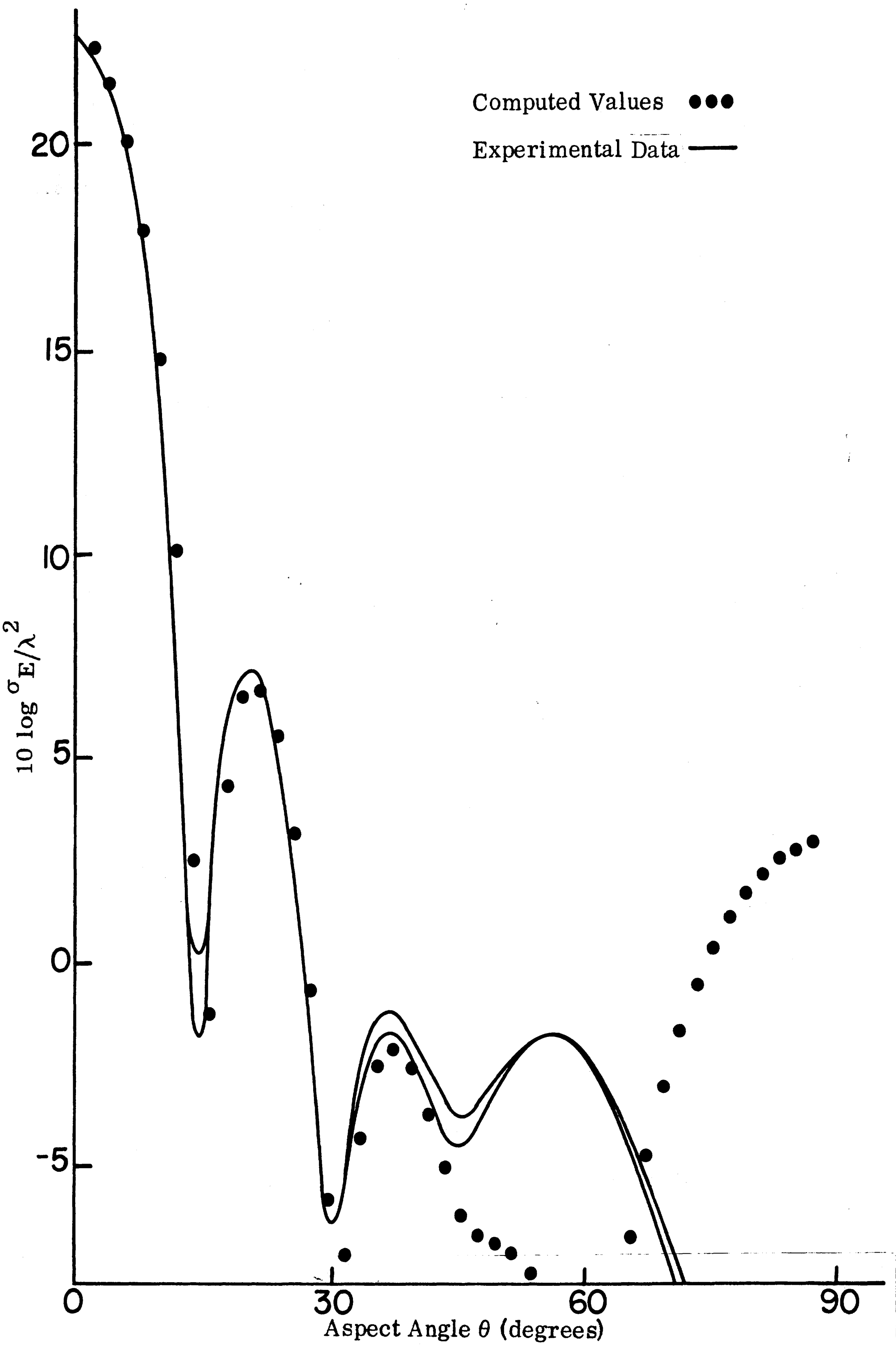

FIG. 2-8a: RCS of a Disk for E-Polarization; c = 7.0.

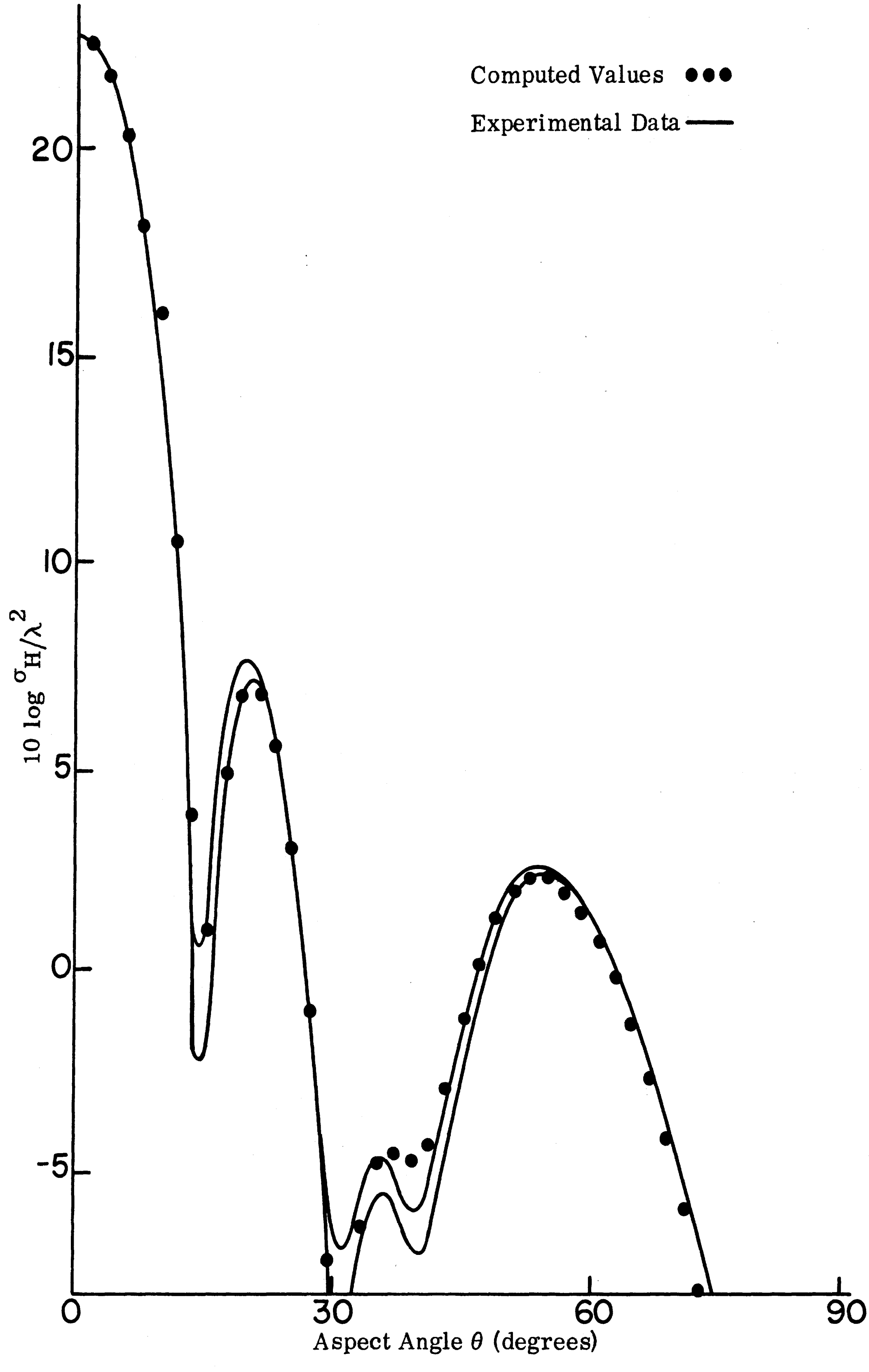

FIG. 2-8b: RCS of a Disk for H-Polarization; c = 7.0.

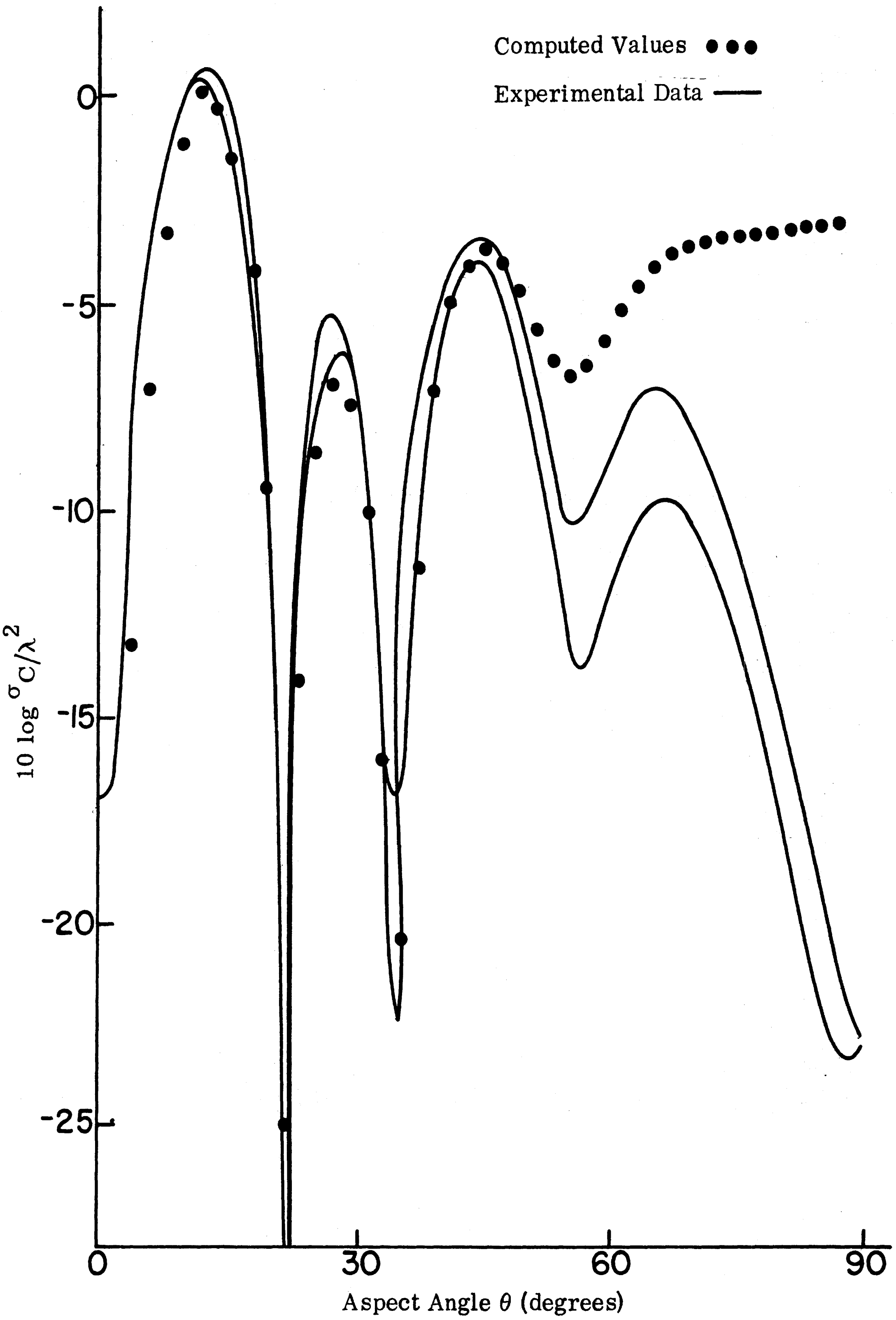

FIG. 2-8c: Cross-Polarized RCS of a Disk; c = 7.0.

Chapter III

HALF PLANE APPROXIMATION TO DISK CURRENTS

3.1 The Basic Assumptions

One usually assumes for a first order approximation that the surface currents on a disk are given by those of physical optics; namely $2\hat{n} \times \bar{H}^i$ on the top surface and zero on the bottom surface. This approximation gives reasonable results only for large disks for near normal incidence (the angle θ in Fig. 1-1 small) and can predict no cross-polarized return for any value of θ. The presence of edges on the diffracting body is neglected and edge conditions (Bouwkamp (1954)) for the surface current density are violated. These edge conditions require that the component of surface current density normal to the edge remain finite at the edge while the component tangential to the edge become finite as the reciprocal of the square root of the distance to the edge. A better approximation that can satisfy the edge conditions is based upon the observation that for a disk of radius much larger than a wavelength the edge appears locally to have the properties of a half-plane tangential to the edge at the given edge point. We may then add to the physical optics current density a perturbation term that gives at the edge of the disk the current density that would exist on the tangential half-plane. Since for distances greater than a wavelength from the edge the half-plane current density approaches that of physical optics, the perturbation current density along a ray proceeding from the given edge point through the center of the disk may be taken to be the difference of the associated half-plane current density and that of physical optics. This approach was followed by Ufimstev (1958) and will be used here to extend his results formally to greater values of θ and inverse powers of (ka).

3.2 The Half-Plane Currents

Figure 3-1 gives the statement of the half-plane scattering problem. The incident field is a plane wave directed obliquely to the half-plane as shown. Because of our interest in finding the surface current density along a ray normal to the edge, the observation point P is restricted to lie in the x'y'-plane. The incident field $\bar{F}_i$ will be restricted to lie in a plane parallel to the half-plane and may either be the incident electric field or the incident magnetic field. For $\bar{F}_i = \bar{H}^i_H$ (H-polarization), the incident fields may be written

$$\bar{H}^i_H = \frac{-H_0}{\Gamma(\theta_0, \alpha)} \left(\cos\theta_0 \hat{x}' - \cos\alpha \sin\theta_0 \hat{z}' \right) e^{-ikS} , \tag{3.1a}$$

$$\bar{E}^i_H = \frac{+E_0}{\Gamma(\theta_0, \alpha)} \Big[\sin\alpha \cos\alpha \sin^2\theta_0 \hat{x}' - (\cos^2\theta_0 + \cos^2\alpha \sin^2\theta_0) \hat{y}' + $$
$$+ \sin\alpha \cos\theta_0 \sin\theta_0 \hat{z}' \Big] e^{-ikS} . \tag{3.1b}$$

Similarly, for $\bar{F}_i = \bar{E}^i_E$ (E-polarization) the incident fields are given by

$$\bar{E}^i_E = \frac{-E_0}{\Gamma(\theta_0, \alpha)} (\cos\theta_0 \hat{x}' - \cos\alpha \sin\theta_0 \hat{z}') e^{-ikS} , \tag{3.2a}$$

$$\bar{H}^i_E = \frac{-H_0}{\Gamma(\theta_0, \alpha)} \Big[\sin\alpha \cos\alpha \sin^2\theta_0 \hat{x}' - (\cos^2\theta_0 + \cos^2\alpha \sin^2\theta_0) \hat{y}' + $$
$$+ \sin\alpha \cos\theta_0 \sin\theta_0 \hat{z}' \Big] e^{-ikS} , \tag{3.2b}$$

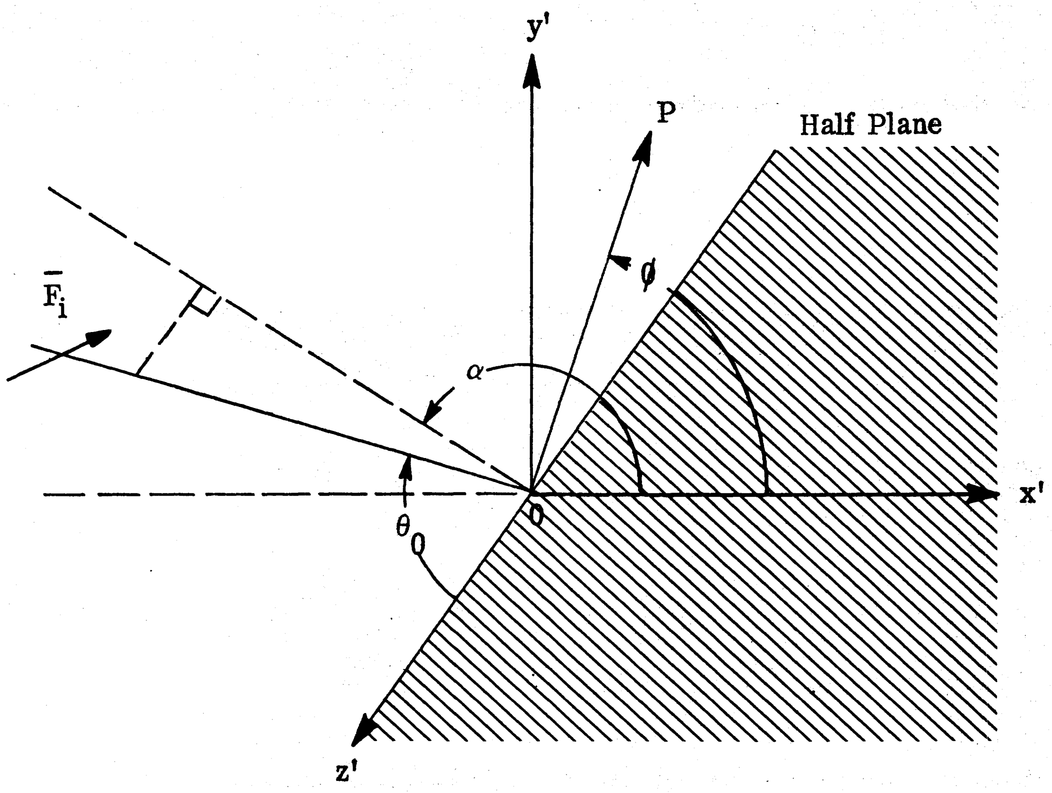

FIG. 3-1: The Half-Plane Scattering Problem.

where in the above equations

$$\Gamma(\theta_{0,}\alpha) = \sqrt{\cos^2\theta_0 + \cos^2\alpha\,\sin^2\theta_0},$$

$$S = x'\cos\alpha\sin\theta_0 + y'\sin\alpha\sin\theta_0 + z'\cos\theta_0,$$

$$E_0 = \eta_0 H_0.$$

The exact solutions to the three-dimensional diffraction problems for a half-plane may be obtained in a straightforward manner from those for the two-dimensional diffraction problems (Born and Wolf (1964)). The three dimensional solutions are given in terms of the following incident unit plane waves:

$$\hat{E}_1^i = (-\cos\alpha\cos\theta_0\,\hat{x}' - \sin\alpha\cos\theta_0\,\hat{y}' + \sin\theta_0\,\hat{z}')\,e^{-ikS} \qquad (3.3a)$$

$$\hat{H}_1^i = (-\sin\alpha\,\hat{x}' + \cos\alpha\,\hat{y}')\,e^{-ikS} \qquad (3.3b)$$

$$\hat{E}_2^i = (\sin\alpha\,\hat{x}' - \cos\alpha\,\hat{y}')\,e^{-ikS} \qquad (3.4a)$$

$$\hat{H}_2^i = (-\cos\alpha\cos\theta_0\,\hat{x}' - \sin\alpha\cos\theta_0\,\hat{y}' + \sin\theta_0\,\hat{z}')\,e^{-ikS} \qquad (3.4b)$$

The fields of equations (3.1a,b) and (3.2a,b) can be expressed in terms of those of equations (3.3a,b) and (3.4a,b). For H-polarization we have:

$$\bar{H}_H^i = \frac{+H_0}{\Gamma(\theta_{0,}\alpha)}\left[\sin\alpha\cos\theta_0\,\hat{H}_1^i + \cos\alpha\,\hat{H}_2^i\right] \qquad (3.5a)$$

$$\bar{E}_H^i = \frac{+E_0}{\Gamma(\theta_{0,}\alpha)}\left[\sin\alpha\cos\theta_0\,\hat{E}_1^i + \cos\alpha\,\hat{E}_2^i\right] \qquad (3.5b)$$

For E-polarization the expressions are:

$$\bar{E}^{i}_{E} = \frac{+E_0}{\Gamma(\theta_0, \alpha)} \left[\cos\alpha\, \hat{E}^{i}_{1} - \sin\alpha\cos\theta_0\, \hat{E}^{i}_{2} \right] \tag{3.6a}$$

$$\bar{H}^{i}_{E} = \frac{+H_0}{\Gamma(\theta_0, \alpha)} \left[\cos\alpha\, \hat{H}^{i}_{1} - \sin\alpha\cos\theta_0 \hat{H}^{i}_{2} \right] . \tag{3.6b}$$

The solution to the two-dimensional diffraction problem corresponding to $\hat{E}^{i}_{1}$ is given by

$$\bar{E} = V\hat{z}' = \left[U(2\sqrt{\frac{kr}{\pi}}\cos\frac{\psi_1}{2}) - U(2\sqrt{\frac{kr}{\pi}}\cos\frac{\psi_2}{2}) \right] \hat{z}' , \tag{3.7}$$

where

$$U(2\sqrt{\frac{kr}{\pi}}\cos\frac{\psi}{2}) = e^{-ikr\cos\psi} \int_{-\infty}^{2\sqrt{\frac{kr}{\pi}}\cos\frac{\psi}{2}} \frac{(1-i)}{2} e^{\frac{i\pi}{2}\tau^2} d\tau , \tag{3.8}$$

$$\psi_{1,2} = \phi \mp \alpha ,$$

and r is the distance of the point of observation from the edge. Of interest here is the tangential value of $\bar{H}_1$ along the half-plane for arbitrary values of θ_0. This value is found by forming

$$V' = V(k\sin\theta_0)\, e^{-ikz'\cos\theta_0} , \tag{3.9}$$

where the function $V(k\sin\theta_0)$ is obtained by replacing the wave number k in V of equation (3.7) by $k\sin\theta_0$. Utilizing equation (11.6.4) of Born and Wolf (1964) the desired value of $\bar{H}_1$ is given by

$$\bar{H}^{(1)} = \frac{-i}{k}\left(\frac{\partial V'}{\partial y'}\hat{x}' - \frac{\partial V'}{\partial x'}\hat{y}' \right)\Bigg|_{\phi = 0,\ 2\pi;\ z' = 0} . \tag{3.10}$$

The components of $\overline{H}^{(1)}$ that are tangential to the half-plane surface are

$$H^{(1)}_{z'}\Big|_{\phi = 0,\ 2\pi} = 0 \quad , \tag{3.11a}$$

$$H^{(1)}_{x'}\Big|_{\phi = 0} = -2\sin\alpha F\left(kx'\sin\theta_0, \frac{\alpha}{2}\right)e^{-ikx'\sin\theta_0\cos\alpha} - \sqrt{\frac{2}{\pi kx'\sin\theta_0}}\ \sin\frac{\alpha}{2}\, e^{i(kx'\sin\theta_0 + \frac{\pi}{4})} \quad , \tag{3.11b}$$

$$H^{(1)}_{x'}\Big|_{\phi = 2\pi} = -\left\{2\sin\alpha\, F\left(kx'\sin\theta_0, \pi - \frac{\alpha}{2}\right)e^{-ikx'\sin\theta_0\cos\alpha} - \sqrt{\frac{2}{\pi kx'\sin\theta_0}}\ \sin\frac{\alpha}{2}\, e^{i(kx'\sin\theta_0 + \frac{\pi}{4})}\right\} . \tag{3.11c}$$

The function $F(kr\sin\theta_0, \alpha)$ is a Fresnel integral defined as follows:

$$F(kr\sin\theta_0, \alpha) = \int_{-\infty}^{2\sqrt{\frac{kr\sin\theta_0}{\pi}}\cos\alpha} \left(\frac{1-i}{2}\right) e^{\frac{i\pi}{2}\tau^2}\, d\tau = \frac{1}{2} + \int_{0}^{2\sqrt{\frac{kr\sin\theta_0}{\pi}}\cos\alpha} \left(\frac{1-i}{2}\right) e^{\frac{i\pi}{2}\tau^2}\, d\tau \tag{3.12}$$

The solution to the two-dimensional diffraction problem corresponding to the incident magnetic field $\hat{H}^i_2$ is given by

$$\overline{H} = W\hat{z}' = \left[U\left(2\sqrt{\frac{kr}{\pi}}\cos\frac{\psi_1}{2}\right) + U\left(2\sqrt{\frac{kr}{\pi}}\cos\frac{\psi_2}{2}\right)\right]\hat{z}' \quad , \tag{3.13}$$

where the function $U(2\sqrt{\frac{kr}{\pi}}\cos\frac{\psi}{2})$ and $\psi_{1,2}$ are as given in equation (3.8).

In a fashion analogous to equation (3.9), we form

$$W' = W(k\sin\theta_0)\, e^{-ikz'\cos\theta_0} \quad . \tag{3.14}$$

The desired solution to the three dimensional diffraction problem due to the incident magnetic field $\hat{H}^i_2$ may be written with the help of equation (11.6.5) of Born and Wolf (1964) as

$$\overline{H}^{(2)} = -\frac{i}{k}\cos\theta_0\left[\frac{\partial W'}{\partial x'}\hat{x}' + \frac{\partial W'}{\partial y'}\hat{y}'\right] + \sin^2\theta_0\left[W'\right]\hat{z}'\Bigg|_{\phi = 0,\ 2\pi;\ z' = 0} . \tag{3.15}$$

Only the x'- and z'-components of $\overline{H}^{(2)}$ along the x'-axis are needed. They are:

$$H_{z'}^{(2)}\Big|_{\phi=0} = 2\sin\theta_0\, e^{-ik x'\cos\alpha\sin\theta_0}\left(F(kx'\sin\theta_0, \frac{\alpha}{2})\right) \tag{3.16a}$$

$$H_{z'}^{(2)}\Big|_{\phi=2\pi} = 2\sin\theta_0\, e^{-ik x'\cos\alpha\sin\theta_0}\left(F(kx'\sin\theta_0, \pi - \frac{\alpha}{2})\right) \tag{3.16b}$$

$$H_{x'}^{(2)}\Big|_{\phi=0} = -2\cos\alpha\cos\theta_0\, e^{-ikx'\cos\alpha\sin\theta_0}\left(F(kx'\sin\theta_0, \frac{\alpha}{2})\right) - \sqrt{\frac{2}{\pi kx'\sin\theta_0}}\cos\theta_0\cos\frac{\alpha}{2}\, e^{i(kx'\sin\theta_0 + \frac{\pi}{4})} \tag{3.16c}$$

$$H_{x'}^{(2)}\Big|_{\phi = 2\pi} = -\cos\theta_0\left\{2\cos\alpha\, e^{-ikx'\cos\alpha\sin\theta_0}\left(F(kx'\sin\theta_0, \pi - \frac{\alpha}{2})\right) - \sqrt{\frac{2}{\pi kx'\sin\theta_0}}\cos\frac{\alpha}{2}\, e^{i(kx'\sin\theta_0 + \frac{\pi}{4})}\right\} \tag{3.16d}$$

The total surface current densities induced on the half-plane along the x'-axis by the incident fields of equations (3.3) and (3.4) may easily be found from equations (3.11) and (3.16) respectively. We wish, however, to find those non-uniform current densities that are defined as the difference between the total current densities and the current densities assumed by the method of physical optics for the two types of incident fields. The physical optics current density is simply given by $2\hat{n} \times \overline{H}^i$ on the illuminated side of the half plane and by $\underline{0}$ on the shadow side. If we limit α to the range $0 \leq \alpha \leq \pi$, $\hat{n}$ becomes $\hat{y}'$, and the two physical optics current densities are given by

$$\bar{K}^{(1)}_{p.o.} = 2\hat{y}' x \hat{H}^{i}_{1} = 2\hat{z}' \sin\alpha\, e^{-ikx'\cos\alpha\sin\theta_0}, \tag{3.17}$$

$$\bar{K}^{(2)}_{p.o.} = 2\hat{y}' x \hat{H}^{i}_{2} = 2\,(\sin\theta_0\hat{x}' + \cos\alpha\cos\theta_0\hat{z}')\, e^{-ikx'\cos\alpha\sin\theta_0}, \tag{3.18}$$

for $\phi = 0$, and by zero for $\phi = 2\pi$.

Subtracting these current densities from the total current densities obtained from equations (3.11) and (3.16) one easily finds the desired non-uniform current densities to be

$$\bar{K}^{(1)}_{N}\Big|_{\phi=0} = \bar{K}^{(1)}_{N}\Big|_{\phi=2\pi} = \left\{ 2\sin\alpha\, e^{-ikx'\cos\alpha\sin\theta_0}\left[F(kx'\sin\theta_0, \frac{\alpha}{2}) - 1\right] + \right.$$

$$\left. + \sqrt{\frac{2}{\pi kx'\sin\theta_0}}\,\sin\frac{\alpha}{2}\, e^{i(kx'\sin\theta_0 + \frac{\pi}{4})}\right\}\hat{z}', \tag{3.19}$$

$$\bar{K}^{(2)}_{N}\Big|_{\phi=0} = \bar{K}^{(2)}_{N}\Big|_{\phi=2\pi} = 2\, e^{-ikx'\cos\alpha\sin\theta_0}\left[F(kx'\sin\theta_0, \frac{\alpha}{2}) - 1\right] x$$

$$x \left[\sin\theta_0\hat{x}' + \cos\alpha\cos\theta_0\hat{z}'\right] + \left\{\sqrt{\frac{2}{\pi kx'\sin\theta_0}}\,\cos\theta_0\,\cos\frac{\alpha}{2}\, e^{i(kx'\sin\theta_0 + \frac{\pi}{4})}\right\}\hat{z}', \tag{3.20}$$

where use was made of the easily proven relationship:

$$F(kr\sin\theta_0, \pi \pm \alpha) = 1 - F(kr\sin\theta_0, \alpha) \cdot$$

The non-uniform surface current densities, unlike the total surface current densities, are the same on both sides of the half-plane. Since an approximation to the non-uniform surface current densities on the disk is going to be obtained from equations (3.19) and (3.20) by purely geometrical means, the same can be said of the approximate non-uniform surface current densities on the disk. Also, since an idealized half-plane has no thickness, it may be replaced formally by an open current sheet having a total surface current density of $\bar{K}_{p.o.} + 2\bar{K}_N$ without changing any field quantities. The same may also be said for a disk with respect to the surface current densities obtained from the approximations used here. Such a replacement will be implicit in all that follows for both the half-plane and the disk.

We are now in a position to write down the half-plane non-uniform surface current densities that result for the incident plane waves of equations (3.1) and (3.2). According to equations (3.5a, b) and (3.6a, b) these are

$$\bar{K}_H = \frac{2H_0}{\Gamma(\theta_0, \alpha)} \left[\sin\alpha \cos\theta_0 \, \bar{K}_N^{(1)} + \cos\alpha \, \bar{K}_N^{(2)} \right] , \tag{3.21}$$

$$\bar{K}_E = \frac{+2H_0}{\Gamma(\theta_0, \alpha)} \left[\cos\alpha \, \bar{K}_N^{(1)} - \sin\alpha \cos\theta_0 \, \bar{K}_N^{(2)} \right] . \tag{3.22}$$

3.3 The Non-Uniform Currents on the Disk

Figure 3-2 shows how the half-plane geometry of Fig. 3-1 is to be used in approximating the non-uniform currents for the disk. The y'-axis is parallel to the z-axis of the disk. The x'-axis is in the plane of the disk in the $-\hat{\rho}$ - direction, and the z'-axis is in the plane of the disk in the $\hat{\phi}$-direction. Hence, we may immediately write $\hat{y}' = \hat{z}$, $\hat{x}' = -\hat{\rho}$, $\hat{z}' = \hat{\phi}$, and $x' = a - \rho$. The half-plane angles θ_0 and α are related to the angles θ and ϕ of the unprimed coordinate system by

$$\cos\theta_0 = \sin\theta \cos\phi, \tag{3.23a}$$

$$\sin\theta_0 = \Gamma(\theta, \phi + \tfrac{\pi}{2}) = \sqrt{\cos^2\theta + \sin^2\theta \sin^2\phi} \quad , \tag{3.23b}$$

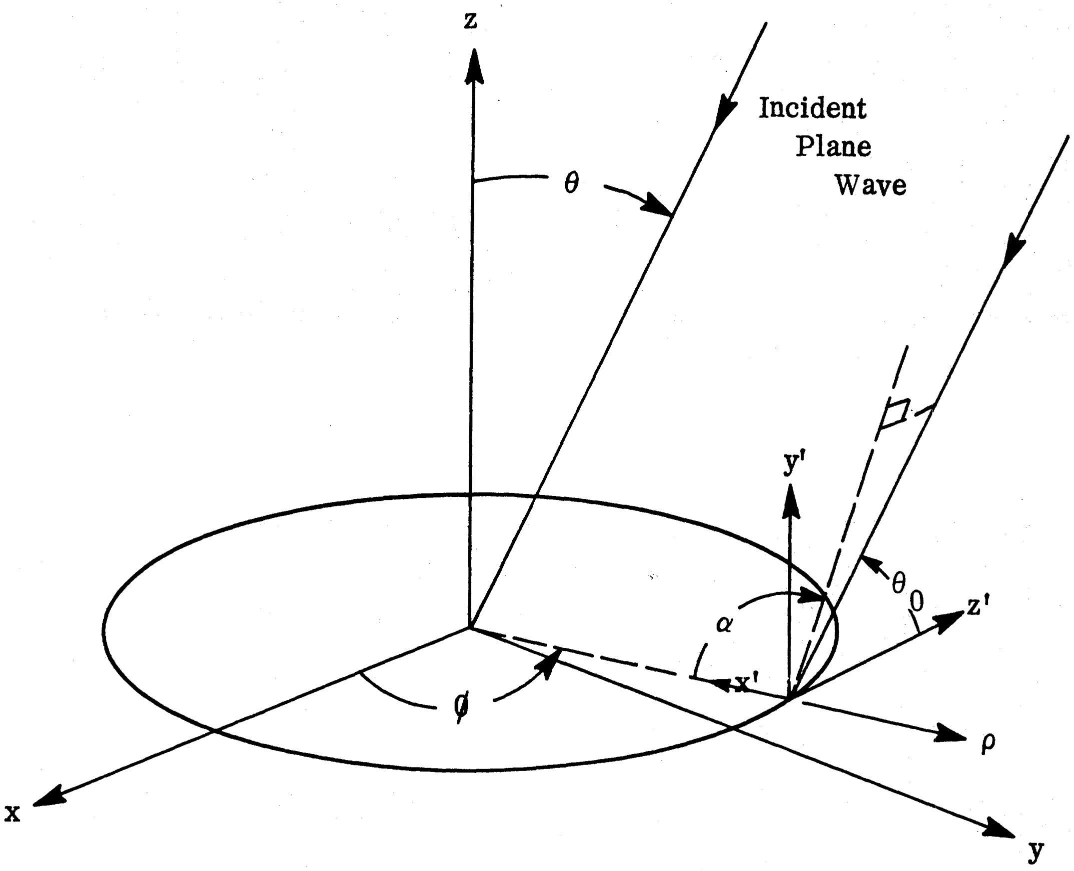

FIG. 3-2: The Edge Geometry Used to Approximate the Disk Surface Current Density.

$$\cos\alpha = \frac{-\sin\theta\,\sin\phi}{\Gamma(\theta,\phi+\frac{\pi}{2})} \quad , \tag{3.24a}$$

$$\sin\alpha = \frac{\cos\theta}{\Gamma(\theta,\phi+\frac{\pi}{2})} \quad . \tag{3.24b}$$

Equations (3.19) and (3.20) also involve $\cos\frac{\alpha}{2}$ and $\sin\frac{\alpha}{2}$, which can be expressed directly in terms of θ and ϕ. It proves convenient, however, for later usage to express $\sin\frac{\alpha}{2}$ and $\cos\frac{\alpha}{2}$ in terms of two functions $T(\theta,\phi)$ and $Q(\theta,\phi)$ which are defined by

$$T(\theta,\phi) = \Gamma(\theta,\phi+\frac{\pi}{2}) - \sin\theta\,\sin\phi \quad , \tag{3.25a}$$

$$Q(\theta,\phi) = \Gamma(\theta,\phi+\frac{\pi}{2}) + \sin\theta\,\sin\phi \quad . \tag{3.25b}$$

Then $\cos\frac{\alpha}{2}$ and $\sin\frac{\alpha}{2}$ are

$$\cos\frac{\alpha}{2} = \left[\frac{T(\theta,\phi)}{2\Gamma(\theta,\phi+\frac{\pi}{2})}\right]^{1/2} \quad , \tag{3.26a}$$

$$\sin\frac{\alpha}{2} = \left[\frac{Q(\theta,\phi)}{2\Gamma(\theta,\phi+\frac{\pi}{2})}\right]^{1/2} \quad . \tag{3.26b}$$

The phase reference for the disk diffraction problem will be taken to be the center of the disk while the phase center for the half-plane diffraction problem has been taken to be the origin of the primed coordinate system, a point that is on the edge of the disk. Hence, the phase of (3.21) and (3.22) must be shifted by $[ka\sin\theta\sin\phi]$ radians before the equations can be used to approximate the non-uniform currents on the disk. With this addition and the substitutions indicated by equations (3.23a, b), (3.24a, b) and (3.26a, b) equations (3.21) and (3.22) give the following for the disk non-uniform currents:

$$\bar{K}_H = \frac{2H_0}{\Gamma(\theta,\phi+\frac{\pi}{2})}\left[\cos\theta\cos\phi\,\bar{K}_N^{(1)}(\theta,\phi) - \sin\phi\,\bar{K}_N^{(2)}(\theta,\phi)\right] e^{-ika\sin\theta\sin\phi} , \tag{3.27}$$

$$\bar{K}_E = \frac{-2H_0}{\Gamma(\theta,\phi+\frac{\pi}{2})}\left[\sin\phi\,\bar{K}_N^{(1)}(\theta,\phi) + \cos\theta\cos\phi\,\bar{K}_N^{(2)}(\theta,\phi)\right] e^{-ika\sin\theta\sin\phi} . \tag{3.28}$$

Finally, application of equations (3.19) and (3.20) allows these complex appearing relations to be rewritten after some manipulation as

$$\bar{K}_H = 2H_0\left\{2\hat{y}\left[F\left(\frac{k(a-\rho)\,T(\theta,\phi)}{2},0\right)-1\right]e^{-ik\rho\sin\theta\sin\phi} + \right.$$
$$\left. + \hat{\phi}\,\frac{\cos\phi}{\Gamma(\theta,\phi+\frac{\pi}{2})}\sqrt{\frac{T(\theta,\phi)}{\pi k(a-\rho)}}\; e^{\,i\left[ka\,T(\theta,\phi) - k\rho\Gamma(\theta,\phi+\frac{\pi}{2}) + \frac{\pi}{4}\right]}\right\} , \tag{3.29}$$

$$\bar{K}_E = -2H_0\left\{-2\hat{x}\cos\theta\left[F\left(\frac{k(a-\rho)\,T(\theta,\phi)}{2},0\right)-1\right]e^{-ik\rho\sin\theta\sin\phi} + \right.$$
$$\left. + \hat{\phi}\left(\frac{\sin\phi + \sin\theta\,\Gamma(\theta,\phi+\frac{\pi}{2})}{\sqrt{\pi k(a-\rho)\,Q(\theta,\phi)}\;\Gamma(\theta,\phi+\frac{\pi}{2})}\right) e^{\,i\left[ka T(\theta,\phi) - k\rho\Gamma(\theta,\phi+\frac{\pi}{2}) + \frac{\pi}{4}\right]}\right\} . \tag{3.30}$$

Note that since ρ cannot be negative these choices of surface current densities require that the half-plane approximations be arbitrarily terminated at the center of the disk. These expressions complete the approximate descriptions of the non-uniform surface current densities on the disk.

Because of the restrictions placed on the half-plane problem these current densities are the non-uniform surface current densities on the disk current sheet that result when the incident electromagnetic field is a plane wave with either the electric vector or the magnetic vector constrained to be parallel to the plane of the disk. However, these are just the incident fields given by equations (1.1) and (1.2) respectively. The total surface current density on the disk current sheet may then be written in terms of the above surface current densities and the corresponding physical optics current densities as

$$\bar{K}_T^E = \bar{K}_{p.o.}^E + \bar{K}_E , \tag{3.31}$$

for the incident field of equation (1.1) and ,

$$\bar{K}_T^H = \bar{K}_{p.o.}^H + \bar{K}_H , \tag{3.32}$$

for the incident field of equation (1.2). These surface current densities will be used in the next section to find the far-zone backscattered fields for the two problems.

3.4 The Far-Zone Backscattered Fields

The previous sections of this chapter were devoted entirely to finding an approximate description of the surface current densities induced on a disk by the incident fields of equations (1.1) and (1.2). With such a description available the task then becomes that of finding the scattered fields. The scattered electric and magnetic fields in free space are related by Maxwell's equations, so only one need be considered here. A natural choice is the scattered magnetic field, which for the scattering problems of equations (1.1) and (1.2) determined by the equation,

$$\bar{H}^S = \frac{1}{4\pi} \int_S (\nabla' \Phi \times \bar{K}_T^{E,H}) \, dS' , \tag{3.33}$$

where the surface of integration S is the open surface bounded by the disk rim that has a unit normal vector $\hat{n} = \hat{z}$. The surface current densities $\bar{K}_T^E$ and $\bar{K}_T^H$ are given by equations (3.31) and (3.32) and are used for the incident fields of equations (1.1) and (1.2) respectively. The function Φ is the free space Green's function given by

$$\Phi = \frac{e^{ikr}}{r}, \quad r = \left| \bar{R} - \bar{R}' \right| , \tag{3.34}$$

where $\bar{R}$ and $\bar{R}'$ are respectively position vectors from the center of the disk to the point of observation ($\overline{OP}$ of Fig. 1-1) and the point of integration. The prime over the ∇ - operator in equation (3.33) denotes that the operation is to be taken with respect to the coordinates of integration.

Equation (3.33) gives the magnetic field anywhere in space. Only the far-zone backscattered field is of interest here. For this case $r \simeq |\bar{R}| = R$, and the integral of equation (3.33) may be written in terms of the ρ'- and ϕ'-components of the surface current density as

$$\bar{H}^S = \frac{-ik\, e^{ikR}}{4\pi R} \int_0^{2\pi} \int_0^a e^{-ik\rho \sin\theta \sin\phi} \left\{ \cos\theta \left[K_{T\rho}^{E,H} \hat{\phi} - K_{T\phi}^{E,H} \hat{\rho} \right] \right.$$

$$\left. + \sin\theta \left[\sin\phi\, K_{T\phi}^{E,H} - \cos\phi\, K_{T\rho}^{E,H} \right] \hat{z} \right\} \rho d\rho d\phi , \tag{3.35}$$

where the primes have been dropped for clarity. The contributions of the physical optics surface current densities to equation (3.35) will be considered first. The two physical optics current densities are defined as $2\hat{n} \times \bar{H}^{i,E,H}$ and are:

$$\bar{K}_{p.o.}^H = \hat{y}\, 2H_0\, e^{-iky \sin\theta} \tag{3.36a}$$

$$\overline{K}^{E}_{p.o.} = \hat{x}\, 2H_0 \cos\theta\, e^{-iky\sin\theta} \qquad (3.36b)$$

Determination of the two scattered magnetic fields from equation (3.35) is straightforward. The results, expressed in spherical coordinates, are

$$\overline{H}^{S,H}_{p.o.} \sim -\hat{\phi}\, \frac{iaH_0 \cos\theta\, e^{ikR}}{2R}\, \frac{J_1(2ka\sin\theta)}{\sin\theta} , \qquad (3.37a)$$

and

$$\overline{H}^{S,E}_{p.o.} \sim -\hat{\theta}\, \frac{ia\, H_0 \cos\theta\, e^{ikR}}{2R}\, \frac{J_1(2k\, a\sin\theta)}{\sin\theta} . \qquad (3.37b)$$

Both fields have the same θ-dependence, a result that is not unexpected for the scattered physical optics fields from a regular body. Also, both fields have been found in closed form. This is a fortuitous state of affairs that will not be repeated in finding the scattered far-zone magnetic fields due to the non-uniform components of current. Instead an attempt will be made to expand the integrals for these fields asymptotically in inverse powers of the wave number k. This will be the primary approximation. A secondary approximation will consist, where appropriate, of expanding the resulting functions in ascending powers of $\sin\theta$. The objective is to seek a solution that is most accurate for aspect angles near normal incidence.

When the non-uniform surface current densities given by equations (3.29) and (3.30) are substituted into equation (3.35) the resulting expressions may be written as

$$\overline{H}^{S}_{HN} \sim \frac{ikH_0 e^{ikR}}{2\pi R} \int_0^{2\pi}\!\!\int_0^{a} \left\{ 2\hat{x}\cos\theta\, e^{-i2k\rho\sin\theta\sin\phi} \left[F\left(\frac{k(a-\rho)\, T(\theta,\phi)}{2}, 0 \right) - 1 \right] + \right.$$

$$+ \frac{\cos\phi}{\Gamma(\theta,\phi+\frac{\pi}{2})} \sqrt{\frac{T(\theta,\phi)}{\pi k(a-\rho)}}\, e^{i\left[kaT(\theta,\phi) - k\rho Q(\theta,\phi) + \frac{\pi}{4}\right]} \times$$

$$\times \left[\cos\theta\,(\cos\phi\,\hat{x} + \sin\phi\,\hat{y}) - \sin\theta\sin\phi\,\hat{z}\right]\Bigg\}\rho d\rho d\phi \;, \qquad (3.38)$$

$$\bar{H}^{S}_{EN} \sim \frac{ikH_0 e^{ikR}}{2\pi R}\int_0^{2\pi}\int_0^{a}\Bigg\{-2\hat{\theta}\cos\theta\, e^{-i2k\rho\sin\theta\sin\phi}\left[F\left(\frac{k(a-\rho)\,T(\theta,\phi)}{2},0\right)-1\right]$$

$$+ \frac{\sin\phi + \sin\theta\,\Gamma(\theta,\phi+\frac{\pi}{2})}{\sqrt{\pi k(a-\rho)\,Q(\theta,\phi)}\,\Gamma(\theta,\phi+\frac{\pi}{2})}\, e^{i\left[kaT(\theta,\phi) - k\rho Q(\theta,\phi) + \frac{\pi}{4}\right]} \times$$

$$\times\left[-\cos\theta\,(\cos\phi\,\hat{x} + \sin\phi\,\hat{y}) + \sin\theta\sin\phi\,\hat{z}\right]\Bigg\}\rho d\rho d\phi \,. \qquad (3.39)$$

The unit vectors $\hat{x}$, $\hat{y}$, $\hat{z}$, and $\hat{\theta}$ are all constant with respect to the variables of integration. This mixed system was chosen to make similarities between the two equations evident and to keep the equations as compact as possible. Both equations contain a term that is bounded for every value of ρ and an unbounded term that becomes infinite as $(a-\rho)^{-1/2}$. Furthermore, the bounded terms will give scattered fields of equal magnitude for the two polarizations, thereby never contributing to the cross polarized scattered far-zone fields. The unbounded terms have the same ρ-dependence for the two polarizations but different θ- and ϕ-dependencies, so that they will contribute to the cross-polarized scattered far-zone fields. Then, since the physical optics scattered fields given in equations (3.37a, b) cannot give any cross-polarized component, it is only necessary to compute the contributions to the far-zone scattered fields in equations (3.38) and (3.39) from the unbounded terms if only the cross-polarized far-zone backscattered field is desired. Otherwise all the terms must be included.

3.5 The Far-Zone Backscattered Fields Due to the Unbounded Component

The contributions to the far-zone fields from the unbounded terms in equations (3.38) and (3.39) will be considered first. Both have the same ρ-dependence, which may be written

$$I_1(ka,\ \theta,\phi) = \int_0^a e^{-ik\rho Q(\theta,\phi)} \frac{\rho\, d\rho}{\sqrt{a-\rho}} \quad . \tag{3.40}$$

This may be evaluated in terms of Fresnel integrals. Let $t = a-\rho$. Then,

$$I_1(ka,\theta,\phi) = e^{-ikaQ(\theta,\phi)} \left\{ \int_0^a e^{iktQ(\theta,\phi)} \frac{a\,dt}{\sqrt{t}} - \int_0^a e^{iktQ(\theta,\phi)} \sqrt{t}\, dt \right\} \quad . \tag{3.41}$$

Integrating the second integral by parts to bring it into the same form as the first integral and recognizing that the result is a Fresnel integral we obtain

$$I_1(ka,\theta,\phi) = \frac{-\sqrt{a}}{ikQ(\theta,\phi)} + e^{-ikaQ(\theta,\phi)} \left(a + \frac{1}{i2kQ(\theta,\phi)} \right) \sqrt{\frac{2\pi}{kQ(\theta,\phi)}}\ f\left(\sqrt{\frac{2kaQ(\theta,\phi)}{\pi}} \right), \tag{3.42}$$

where the Fresnel integral is given by

$$f(\sqrt{W}) = \frac{1}{2} \int_0^W e^{\frac{i\pi}{2}\tau} \frac{d\tau}{\sqrt{\tau}} = \int_0^{\sqrt{W}} e^{\frac{i\pi}{2}\tau^2} d\tau \quad . \tag{3.43}$$

Inspection of equations (3.23b) and (3.25b) reveals that $Q(\theta,\phi)$ is non-zero for all values of ϕ as long as θ is not equal to ninety degrees. This means that the argument of the Fresnel integral in equation (3.42) will be of the order of (ka).

Hence the asymptotic expansion of the Fresnel integral may be used in this angular range of θ. The first few terms of that asymptotic expansion are

$$f(\sqrt{W}) = \frac{1+i}{2} + \frac{e^{\frac{i\pi}{2}W}}{i\pi\sqrt{W}}\left(1 + \frac{1}{i\pi W} - \frac{3}{\pi^2 W^2}\right) + \frac{15}{i\pi^3}\int_{\sqrt{W}}^{\infty} e^{\frac{i\pi}{2}\tau^2}\frac{d\tau}{\tau^6} \;. \tag{3.44}$$

The remaining integral in equation (3.44) is of the order of $(W)^{-\frac{5}{2}}$, which is the order of the last term in the series. Substituting equation (3.44) into (3.42), that equation can be written

$$I_1(ka,\theta,\phi) = a^{3/2}\left\{\frac{-1}{ikaQ(\theta,\phi)} + e^{\frac{i\pi}{4}}\sqrt{\frac{\pi}{kaQ(\theta,\phi)}}\left(1 + \frac{1}{i2kaQ(\theta,\phi)}\right)e^{-ikaQ(\theta,\phi)} + \right.$$

$$+ \frac{1}{ikaQ(\theta,\phi)}\left(1 + \frac{1}{i2kaQ(\theta,\phi)}\right)^2 -$$

$$\left. - \frac{3}{4i\pi(kaQ(\theta,\phi))^3}\left(1 + \frac{1}{i2kaQ(\theta,\phi)}\right) + O\left[(ka)^{-3}\right]\right\}, \; (\theta < \tfrac{\pi}{2}) \, . \tag{3.45}$$

Inspection of this equation reveals that the first term is canceled by another, so that $I_1(ka,\theta,\phi)$ may be written as

$$I_1(ka,\theta,\phi) = a^{3/2} e^{\frac{i\pi}{4}}\sqrt{\frac{\pi}{kaQ(\theta,\phi)}}\left(1 + \frac{1}{i2kaQ(\theta,\phi)}\right)e^{-ikaQ(\theta,\phi)} +$$

$$+ O((ka)^{-2}), \; (\theta < \tfrac{\pi}{2}) \, . \tag{3.46}$$

Keeping terms to $(ka)^{-3/2}$ the expressions for the components of the far-zone backscattered magnetic fields for the two polarizations due to the unbounded surface current density are given by

$$\bar{H}^S_{HN_1} \sim \frac{-aH_0 e^{ikR}}{2\pi R} \int_0^{2\pi} \frac{\cos\theta \cos\phi}{Q(\theta,\phi)\,\Gamma(\theta,\phi+\frac{\pi}{2})} \left(1 + \frac{1}{i2kaQ(\theta,\phi)}\right) e^{-i2ka \sin\theta \sin\phi} \times$$

$$\times \left[\cos\theta\,(\cos\phi\,\hat{x} + \sin\phi\,\hat{y}) - \sin\theta \sin\phi\,\hat{z}\right] d\phi\,, \quad (\theta < \frac{\pi}{2})\,, \tag{3.47}$$

$$\bar{H}^S_{EN_1} \sim \frac{-aH_0 e^{ikR}}{2\pi R} \int_0^{2\pi} \frac{\sin\phi + \sin\theta\,\Gamma(\theta,\phi+\frac{\pi}{2})}{Q(\theta,\phi)\,\Gamma(\theta,\phi+\frac{\pi}{2})} \left(1 + \frac{1}{i2kaQ(\theta,\phi)}\right) e^{-i2ka \sin\theta \sin\phi} \times$$

$$\times \left[-\cos\theta\,(\cos\phi\,\hat{x} + \sin\phi\,\hat{y}) + \sin\theta \sin\phi\,\hat{z}\right] d\phi\,, \quad (\theta < \frac{\pi}{2})\,, \tag{3.48}$$

where use has been made of the easily proven relationships

$$\sqrt{T(\theta,\phi)\,Q(\theta,\phi)} = \cos\theta, \tag{3.49}$$

$$Q(\theta,\phi) - T(\theta,\phi) = 2\sin\theta\sin\phi\,. \tag{3.50}$$

Both integrals are well behaved for the allowed ranges of θ and ϕ. In order to obtain an approximate solution to the integrals, the integrands will be expanded in ascending powers of $\sin\theta$. Two functions need to be considered, $\Gamma(\theta,\phi+\frac{\pi}{2})$ and $Q(\theta,\phi)$. $\Gamma(\theta,\phi+\frac{\pi}{2})$ is given by equation (3.23b), which can be rewritten in terms of only $\sin\theta$ by replacing $\cos^2\theta$ by $1-\sin^2\theta$. When this is done $(\Gamma(\theta,\phi+\frac{\pi}{2}))^{-1}$ becomes

$$\left(\Gamma(\theta,\phi+\frac{\pi}{2})\right)^{-1} = (1 - \sin^2\theta\cos^2\phi)^{-1/2}$$

$$= 1 + \frac{\sin^2\theta\cos^2\phi}{2} + \frac{3}{8}\sin^4\theta\cos^4\phi + \ldots\,,$$

$$(\theta < \frac{\pi}{2})\,. \tag{3.51}$$

The function $(Q(\theta,\phi))^{-1}$ may also be expanded in powers of $\sin\theta$. The result is

$$\left[Q(\theta,\phi)\right]^{-1} = 1 - \sin\theta\sin\phi + \frac{\sin^2\theta}{2}(1+\sin^2\phi) - \sin^3\theta\sin\phi + \ldots \; , \qquad (\theta < \tfrac{\pi}{2}) \; . \tag{3.52}$$

The limit on θ follows from the fact that $1 - \sin\theta \leq Q(\theta,\phi) \leq 1 + \sin\theta$ for any value of ϕ.

When the expansions given by equations (3.51) and (3.52) are used to find the scattered magnetic fields from equations (3.47) and (3.48), some terms will give zero contributions when integrated. These are of the form

$$\int_0^{2\pi} e^{-i2ka\sin\theta\sin\phi}(\sin\phi)^{\ell}\cos\phi\, d\phi \equiv 0 \; , \tag{3.53}$$

which is easily shown to be 0 for any positive integer value of ℓ. Since equations (3.51) and (3.52) as given either are, or may be, written entirely in terms of powers of $\sin\phi$ as well as $\sin\theta$, only the x-component of $\bar{H}^S_{HN1}$ and the y- and z-components of $\bar{H}^S_{EN1}$ will be non-zero. In fact, since $\hat{\theta} = \cos\theta\,\hat{y} - \sin\theta\,\hat{z}$, $\bar{H}^S_{EN1}$ will have a component only in the θ-direction. By virtue of the above observations the far-zone scattered magnetic fields may be written in powers of $\sin\theta$ to the order of $\sin^2\theta$ as

$$\bar{H}^S_{HN_1} \sim \frac{-aH_0e^{ikR}}{2\pi R}\cos^2\theta\,\hat{x}\int_0^{2\pi} e^{-i2ka\sin\theta\sin\phi}\left\{\cos^2\phi\,(1+\sin^2\theta - \sin\theta\sin\phi) + \frac{1}{i2ka}\left[\cos^2\phi\left(1+\frac{3}{2}\sin^2\theta - 2\sin\theta\sin\phi + \frac{3}{2}\sin^2\theta\sin^2\phi\right)\right]\right\} d\phi \; , \tag{3.54}$$

$$\bar{H}^{S}_{EN1} \sim \frac{aH_0 e^{ikR}}{2\pi R}\, \hat{\theta} \int_0^{2\pi} e^{-i2ka \sin\theta \sin\phi} \sin\phi \Big\{ \sin\phi + \sin\theta\,(1-\sin^2\phi) +$$

$$+ \frac{1}{i2ka}\left[\sin\phi + \sin\theta\,(1-2\sin^2\phi) + \frac{\sin^2\theta \sin\phi}{2}\,(-1+3\sin^2\phi)\right]\Big\}\, d\phi \,. \tag{3.55}$$

All of the integrals over ϕ are of the form

$$I_\phi = \int_0^{2\pi} e^{-i2ka \sin\theta \sin\phi} (\sin\phi)^{\ell}\, d\phi \,. \tag{3.56a}$$

If $\ell = 0$, the value of the integral is a Bessel function:

$$2\pi\, J_0(x) = \int_0^{2\pi} e^{-ix \sin\phi}\, d\phi \,. \tag{3.56b}$$

Both sides of this equation may be differentiated with respect to x to give

$$2\pi\, J_0'(x) = -2\pi\, J_1(x) = -i\int_0^{2\pi} e^{-ix\sin\phi} \sin\phi\, d\phi \,, \tag{3.57a}$$

as both integrands are continuous within the given limits. Continued differentiation followed by application of the recursion formulas for Bessel functions yields

$$\pi\left[J_0(x) - J_2(x)\right] = \int_0^{2\pi} e^{-ix\sin\phi} \sin^2\phi\, d\phi \,, \tag{3.57b}$$

$$\pi\left[\frac{3}{2} J_1(x) - \frac{1}{2} J_3(x)\right] = i\int_0^{2\pi} e^{-ix\sin\phi} \sin^3\phi\, d\phi \,, \tag{3.57c}$$

$$\pi\left[\frac{3}{4} J_0(x) - J_2(x) + \frac{1}{4} J_4(x)\right] = +\int_0^{2\pi} e^{-ix \sin \phi} \sin^4 \phi \, d\phi \,. \tag{3.57d}$$

The components of the far-zone scattered magnetic fields of equations (3.47) and (3.48) are then given in terms of Bessel functions of the argument 2ka sin θ by

$$\bar{H}^S_{HN1} \sim \frac{-aH_0 e^{ikR}}{2R} \cos^2\theta \left\{(1 + \sin^2\theta) \left[J_0(2ka \sin \theta) + J_2 (2ka \sin \theta)\right] + \right.$$

$$+ i \frac{J_2(2ka \sin \theta)}{ka} + \frac{1}{2ika} \left[(1 + \frac{15}{8} \sin^2\theta) J_0(2ka \sin \theta) + \right.$$

$$+ (1 + \frac{3}{2} \sin^2 \theta) J_2(2ka \sin \theta) - \frac{3}{8} \sin^2\theta \; J_4(2ka \sin \theta) +$$

$$\left.\left. + \frac{i2J_2(2ka \sin \theta)}{ka}\right]\right\}\hat{x} \quad , \tag{3.58}$$

$$\bar{H}^S_{EN1} \sim \frac{aH_0 e^{ikR}}{2R} \left\{J_0(2ka \sin \theta) - J_2(2ka \sin \theta) - \right.$$

$$- i \frac{J_2(2ka \sin \theta)}{ka} + \frac{1}{2 ika} \left[(1 + \frac{5}{8} \sin^2\theta) J_0(2ka \sin \theta) - \right.$$

$$- (1 + \sin^2\theta) J_2(2ka \sin \theta) + \frac{3}{8} \sin^2 \theta \; J_4(2ka \sin \theta) +$$

$$\left.\left. + i \sin \theta \left[J_1(2ka \sin \theta) - J_3(2ka \sin \theta)\right]\right]\right\}\hat{\theta} \quad . \tag{3.59}$$

While use of the recursion relations for the Bessel functions has resulted in the inclusion in equations (3.58) and (3.59) of terms that are of the order of $(ka)^{-2}$, no claim can be made that the equations are accurate to that order, for the approximations that were made earlier to the asymptotic series involved only the first two orders.

The bounded integrals of equations (3.38) and (3.39) must now be considered. Again, the integrals over ρ are identical for both polarizations, so only one need be considered. It is

$$I_2(ka,\theta,\phi) = \int_0^a e^{-i2k\rho \sin\theta \sin\phi} \left[F\left(\frac{k(a-\rho)\,T(\theta,\phi)}{2}, 0 \right) - 1 \right] \rho d\rho. \tag{3.60}$$

Comparison of equations (3,12) and (3.43) reveals that this may be rewritten as

$$I_2\,(ka,\theta,\phi) = \int_0^a e^{-i2k\rho \sin\theta \sin\phi} \left[\left(\frac{1-i}{2} \right) f\left(\sqrt{\frac{2k(a-\rho)\,T(\theta,\phi)}{\pi}} \right) - \frac{1}{2} \right] \rho d\rho\,, \tag{3.61}$$

which is the form that will be considered here. This integral can also be solved exactly in terms of Fresnel integrals. The solution, however, is complicated and expansion of it in terms of inverse powers of (ka) and ascending powers of $\sin\theta$ is at best very difficult. Direct expansion of the integral in inverse powers of (ka) appears to be a better procedure. As before, the integral must be transformed. To this end let a variable W be defined by

$$W \equiv \frac{2k\,(a-\rho)\,T\,(\theta,\phi)}{\pi} \quad .$$

Then in terms of W, $I_2\,(ka,\,\theta,\phi)$ may be written as

$$I_2(ka,\theta,\phi) = \frac{\pi e^{-i2ka \sin\theta \sin\phi}}{2kT(\theta,\phi)} \int_0^{\frac{2kaT(\theta,\phi)}{\pi}} e^{\frac{i\pi W \sin\theta \sin\phi}{T(\theta,\phi)}} \times$$

$$\times \left[\left(\frac{1-i}{2} \right) f\left(\sqrt{W} \right) - \frac{1}{2} \right] \left(a - \frac{\pi W}{2kT(\theta,\phi)} \right) dW \quad . \tag{3.62}$$

This will be written as the sum of two integrals, the first and simplest of which is

$$I_a(ka,\theta,\phi)=\frac{(\pi a)e^{-i2ka\sin\theta\sin\phi}}{2kT(\theta,\phi)}\int_0^{\frac{2kaT(\theta,\phi)}{\pi}} e^{\frac{i\pi W\sin\theta\sin\phi}{T(\theta,\phi)}}\left[\left(\frac{1-i}{2}\right)f(\sqrt{W})-\frac{1}{2}\right]dW$$

$$=\frac{(\pi a)e^{-i2ka\sin\theta\sin\phi}}{2kT(\theta,\phi)}\left\{\int_0^{\infty} e^{\frac{i\pi W\sin\theta\sin\phi}{T(\theta,\phi)}}\left[\left(\frac{1-i}{2}\right)f(\sqrt{W})-\frac{1}{2}\right]dW\;-\right.$$

$$\left.-\int_{\frac{2kaT(\theta,\phi)}{\pi}}^{\infty} e^{\frac{i\pi W\sin\theta\sin\phi}{T(\theta,\phi)}}\left[\left(\frac{1-i}{2}\right)f(\sqrt{W})-\frac{1}{2}\right]dW\right\}. \tag{3.63}$$

When the integral from zero to infinity in equation (3.63) is integrated by parts the result is

$$\int_0^{\infty} e^{\frac{i\pi W\sin\theta\sin\phi}{T(\theta,\phi)}}\left[\left(\frac{1-i}{2}\right)f(\sqrt{W})-\frac{1}{2}\right]dW=\frac{T(\theta,\phi)}{i\pi\sin\theta\sin\phi}\;\times$$

$$\times\left\{e^{\frac{i\pi W\sin\theta\sin\phi}{T(\theta,\phi)}}\left[\left(\frac{1-i}{2}\right)f(\sqrt{W})-\frac{1}{2}\right]\Bigg|_0^{\infty}-\frac{1}{2}\int_0^{\infty}\left(\frac{1-i}{2}\right)e^{\frac{i\pi WQ(\theta,\phi)}{2T(\theta,\phi)}}\frac{dW}{\sqrt{W}}\right\}. \tag{3.64}$$

By virtue of equations (3.43), (3.44), and (3.50) this is expressible as

$$\int_0^{\infty} e^{\frac{i\pi W \sin\theta \sin\phi}{T(\theta,\phi)}} \left[\left(\frac{1-i}{2}\right) f(\sqrt{W}) - \frac{1}{2}\right] dW = \frac{2T(\theta,\phi)}{i\pi\,(Q(\theta,\phi) - T(\theta,\phi))} \times$$

$$\times \left\{\frac{1}{2} - \frac{1}{2}\sqrt{\frac{T(\theta,\phi)}{Q(\theta,\phi)}}\right\} = + \frac{T(\theta,\phi)}{i\pi Q(\theta,\phi)} \left(1 + \sqrt{\frac{T(\theta,\phi)}{Q(\theta,\phi)}}\right)^{-1} . \tag{3.65}$$

Evaluation of the second integral in equation (3.63) can be done asymptotically by applying equation (3.44). This yields

$$\int_{\frac{2kaT(\theta,\phi)}{\pi}}^{\infty} e^{\frac{i\pi W \sin\theta \sin\phi}{T(\theta,\phi)}} \left[\left(\frac{1-i}{2}\right) f(\sqrt{W}) - \frac{1}{2}\right] dW$$

$$= \int_{\frac{2kaT(\theta,\phi)}{\pi}}^{\infty} \left(\frac{1-i}{2}\right) \frac{e^{\frac{i\pi Q(\theta,\phi) W}{2T(\theta,\phi)}}}{i\pi\sqrt{W}} \left(1 + \frac{1}{i\pi W} - \frac{3}{\pi^2 W^2}\right) dW +$$

$$+ \left(\frac{1-i}{2}\right) \frac{15}{i\pi^3} \int_{\frac{2kaT(\theta,\phi)}{\pi}}^{\infty} e^{\frac{i\pi W \sin\theta \sin\phi}{T(\theta,\phi)}} \times \left(\int_{\sqrt{W}}^{\infty} e^{\frac{i\pi}{2}\tau^2} \frac{d\tau}{\tau^6}\right) dW, \tag{3.66}$$

where the second integral, which will be neglected, is of the order of $(ka)^{-3/2}$. Asymptotic evaluation of the first integral is straightforward. The lowest order terms of the resulting series may be written as

$$\int_{\frac{2kaT(\theta,\phi)}{\pi}}^{\infty} e^{\frac{i\pi W \sin\theta \sin\phi}{T(\theta,\phi)}} \left[\left(\frac{1-i}{2}\right) f(\sqrt{W}) - \frac{1}{2}\right] dW =$$

$$= \frac{e^{-i\pi/4}}{\pi Q(\theta,\phi)} \sqrt{\frac{T(\theta,\phi)}{\pi ka}}\; e^{ikaQ(\theta,\phi)} \left[1 + \left(1 + \frac{T(\theta,\phi)}{Q(\theta,\phi)}\right) \left[i2kaT(\theta,\phi)\right]^{-1} \right]$$

$$+ O\left[(ka)^{-\frac{3}{2}}\right] . \qquad (3.67)$$

Then $I_a(ka,\theta,\phi)$ is given by

$$I_a(ka,\theta,\phi) = -\frac{a\,e^{-i2ka\sin\theta\sin\phi}}{2kQ(\theta,\phi)} \left[i\left(1 + \frac{\cos\theta}{Q(\theta,\phi)}\right)^{-1} + \right.$$

$$\left. + \frac{e^{-i\pi/4}}{\sqrt{\pi kaT(\theta,\phi)}}\; e^{ikaQ(\theta,\phi)} \right] + 0\left[(ka)^{-5/2}\right] . \qquad (3.68)$$

The second part of equation (3.62) must now be considered. This is an integral given by

$$I_b(ka,\theta,\phi) = e^{-i2ka\sin\theta\sin\phi} \left(\frac{\pi}{2kT(\theta,\phi)} \right)^2 \int_0^{\frac{2kaT(\theta,\phi)}{\pi}} e^{\frac{i\pi W\sin\theta\sin\phi}{T(\theta,\phi)}} \times$$

$$\times \left[\left(\frac{1-i}{2}\right) f(\sqrt{W}) - \frac{1}{2} \right] W\,dW . \qquad (3.69)$$

This integral does not converge when the upper limit becomes infinite. Hence, it must be evaluated by somewhat different means than $I_a(ka, \theta, \phi)$. Integrating the exponential term by parts transforms this integral to a sum of integrals that can be expanded asymptotically. The resulting expression is

$$I_b(ka, \theta, \phi) = \frac{\pi e^{-i2ka \sin\theta \sin\phi}}{i2k^2 \, T(\theta,\phi)\,(Q(\theta,\phi) - T(\theta,\phi))} \times$$

$$\times \left\{ e^{\frac{i\pi W \sin\theta \sin\phi}{T(\theta,\phi)}} \left[\left(\frac{1-i}{2}\right) f(\sqrt{W}) - \frac{1}{2} \right] W \Bigg|_0^{\frac{2ka\, T(\theta,\phi)}{\pi}} - \right.$$

$$- \int_0^{\frac{2ka\, T(\theta,\phi)}{\pi}} \left(\frac{1-i}{4}\right) e^{\frac{i\pi W Q(\theta,\phi)}{2T(\theta,\phi)}} \sqrt{W}\, dW -$$

$$\left. - \int_0^{\frac{2ka T(\theta,\phi)}{\pi}} e^{\frac{i\pi W \sin\theta \sin\phi}{T(\theta,\phi)}} \left[\left(\frac{1-i}{2}\right) f(\sqrt{W}) - \frac{1}{2} \right] dW \right\} , \qquad (3.70)$$

where use has been made of equation (3.50). Of the two remaining integrals the first is further reducible to a Fresnel integral while the second is proportional to $I_a(ka,\theta,\phi)$. So $I_b(ka,\theta,\phi)$ reduces to

$$I_b(ka,\theta,\phi) = \frac{\pi e^{-i2ka \sin\theta \sin\phi}}{i2k^2 T(\theta,\phi)(Q(\theta,\phi)-T(\theta,\phi))} \left\{ e^{i2ka \sin\theta \sin\phi} \left[\left(\frac{1-i}{2}\right) f \sqrt{\frac{2kaT(\theta,\phi)}{\pi}} - \frac{1}{2} \right] \times \right.$$

$$\times \frac{2kaT(\theta,\phi)}{\pi} - \left(\frac{1-i}{4}\right) \frac{2T(\theta,\phi)}{i\pi Q(\theta,\phi)} \sqrt{\frac{2kaT(\theta,\phi)}{\pi}}\, e^{ikaQ(\theta,\phi)} +$$

$$\left. + \left(\frac{1-i}{2}\right) \frac{T(\theta,\phi)}{i\pi Q(\theta,\phi)} \sqrt{\frac{T(\theta,\phi)}{Q(\theta,\phi)}}\, f\left(\sqrt{\frac{2kaQ(\theta,\phi)}{\pi}} \right) \right\}$$

$$- \frac{I_a(ka,\theta,\phi)}{ika\,(Q(\theta,\phi) - T(\theta,\phi))} \, . \qquad (3.71)$$

The asymptotic expansions of all of the functions contained in this equation have been considered before. Substitution of the appropriate expansions allows $I_b(ka,\theta,\phi)$ to be written as

$$I_b(ka,\theta,\phi) = e^{-i2ka\,\sin\theta\,\sin\phi}\left\{-\frac{a\,e^{\frac{-i\pi}{4}}}{2kQ(\theta,\phi)}\;\frac{1}{\sqrt{\pi ka\;T(\theta,\phi)}}\;e^{\,ika\,Q(\theta,\phi)}\;+\right.$$

$$\left. +\;\frac{1+\dfrac{\cos\theta}{2\,Q\,(\theta,\phi)}}{2k^2\left[Q(\theta,\phi)\right]^2\left(1+\dfrac{\cos\theta}{Q(\theta,\phi)}\right)^2}\right\}+O\left[(ka)^{-5/2}\right]. \qquad (3.72).$$

The first term in this equation is identical to the second term in equation (3.68). Therefore, when the two equations are combined to give $I_2(ka,\ \theta,\ \phi)$, those terms will cancel. From equation (3.62) $I_2(ka,\theta,\phi)$ is found to be

$$I_2(ka,\theta,\phi) = I_a(ka,\theta,\phi) - I_b(ka,\theta,\phi) = -\,\frac{a\,e^{-i2ka\,\sin\theta\,\sin\phi}}{2k}\;\times$$

$$\times\left\{\frac{i}{\left[Q(\theta,\phi)+\cos\theta\right]}+\frac{1+\dfrac{\cos\theta}{2Q(\theta,\phi)}}{ka\left[Q(\theta,\phi)+\cos\theta\right]^2}\right\}+O\left[(ka)^{-5/2}\right] \qquad (3.73)$$

This is the desired expression for the integral over ρ of the bounded term in equations (3.38) and (3.39). In order to carry out the indicated integrations over ϕ in those equations, it is necessary to expand $I_2(ka,\theta,\phi)$ in powers of $\sin\theta$. Both terms in equation (3.73) contain inverse powers of the expression $G(\theta,\phi) = Q(\theta,\phi) + \cos\theta$. If θ is required to lie in the interval containing zero and ninety degrees, it is easy to verify that the maximum and minimum values of $G(\theta,\phi)$ occur for $\phi = \frac{\pi}{2}$ and $\frac{3\pi}{2}$ respectively and are

$$G(\theta,\tfrac{\pi}{2}) = 1+\cos\theta+\sin\theta = 2\left(1+\frac{\cos\theta+\sin\theta-1}{2}\right), \qquad (3.74a)$$

$$G(\theta,\tfrac{3\pi}{2}) = 1+\cos\theta-\sin\theta = 2\left(1+\frac{\cos\theta-\sin\theta-1}{2}\right). \qquad (3.74b)$$

The particular form of these expressions was chosen to correspond to a general form which will be used in obtaining expansions for the two inverse powers of $\bar{G}(\theta,\phi)$ in question. The function $\bar{G}(\theta,\phi)$ may be written explicitly as

$$G(\theta,\phi) = \sin\theta\sin\phi + \sqrt{1-\sin^2\theta\cos^2\phi} + \sqrt{1-\sin^2\theta} \quad . \tag{3.75}$$

Both radicals may be replaced by their power series expansions for $\theta < \frac{\pi}{2}$. Hence,

$$\begin{aligned} G(\theta,\phi) &= 2 + \sin\theta\sin\phi - \frac{1}{2}\sin^2\theta\,(2-\sin^2\phi) - \frac{1}{8}\sin^4\theta(1+(1-\sin^2\phi)^2) + \dots \\ &= 2\left(1 + \frac{\sin\theta\sin\phi}{2} - \frac{1}{4}\sin^2\theta\,(2-\sin^2\phi) - \frac{1}{16}\sin^4\theta\,(1+(1-\sin^2\phi)^2) + \dots\right), \quad \theta < \frac{\pi}{2} \quad . \end{aligned} \tag{3.76}$$

This has the form $2(1+\alpha(\theta,\phi))$. By virtue of equations (3.74a, b) and the condition on θ, $|\alpha(\theta,\phi)|$ is always less than unity. Therefore, convergent expansions for $[G(\theta,\phi)]^{-1}$ and $[G(\theta,\phi)]^{-2}$ can be found. They are

$$[G(\theta,\phi)]^{-1} = \frac{1}{2}\left[1 - \frac{\sin\theta\sin\phi}{2} + \frac{\sin^2\theta}{2} - \frac{\sin^3\theta\sin\phi\,(1+3\cos^2\phi)}{8} + \dots\right], \quad \theta < \frac{\pi}{2} \quad , \tag{3.77a}$$

and

$$[G(\theta,\phi)]^{-2} = \frac{1}{4}\left[1 - \sin\theta\sin\phi + \sin^2\theta\left(1 + \frac{\sin^2\phi}{4}\right) - \frac{3\sin^3\theta\sin\phi}{4}(1+\cos^2\phi) + \dots\right], \quad \theta < \frac{\pi}{2} \quad . \tag{3.77b}$$

An expansion is also needed for $1 + \frac{\cos\theta}{2Q(\theta,\phi)}$, the determination of which is straightforward, since $(Q(\theta,\phi))^{-1}$ has been found before and is given by equation (3.52). If $\cos\theta$ is replaced by its power series expansion in powers of $\sin\theta$ and the two series multiplied, the desired expansion is found to be

$$1+\frac{\cos\theta}{2Q(\theta,\phi)} = \frac{3}{2} - \frac{\sin\theta\ \sin\phi}{2} + \frac{\sin^2\theta\ \sin^2\phi}{4} - \frac{\sin^3\theta\ \sin\phi}{4} + \ldots, \quad \theta < \frac{\pi}{2} \ . \qquad (3.78)$$

Straightforward multiplication yields the desired expansion for $I_2(ka,\theta,\phi)$, which is just the integrand of the remaining integral over ϕ of the bounded terms in equations (3.38) and (3.39). Taking terms only to $\sin^2\theta$ the desired approximation to that integral becomes

$$I_3(ka,\theta) = \frac{-a}{2k}\int_0^{2\pi} e^{-i2ka\ \sin\theta\ \sin\phi}\left\{\frac{i}{2}\left[1-\frac{\sin\theta\ \sin\phi}{2}+\frac{\sin^2\theta}{2}\right] + \right.$$

$$\left. + \frac{1}{8ka}\left[3 - 4\sin\theta\ \sin\phi + 3\sin^2\theta\ (1+\frac{3}{4}\sin^2\phi)\right]\right\} d\phi,$$

$$\theta < \frac{\pi}{2} \ . \qquad (3.79)$$

Unlike the integrals over ϕ in equations (3.47) and (3.48) every term in the integrand will give a non-zero contribution to the integral, as none of them has the form of the integral in equation (3.53). Indeed, all terms are of the form of equation (3.56a), and hence, the integral can be evaluated as combinations of Bessel functions. For this integral, only equations (3.57a, b) and (3.56b) apply. The result is

$$I_3(ka,\theta) = -\frac{a\pi}{2k}\left\{ i\ (1+\frac{\sin^2\theta}{2})\ J_0(2ka\ \sin\theta) - \frac{\sin\theta}{2}\ J_1(2ka\ \sin\theta) + \right.$$

$$+ \frac{1}{ka}\left[(\frac{3}{4}+\frac{33}{32}\sin^2\theta)\ J_0(2ka\ \sin\theta) - \right.$$

$$\left.\left. - \frac{9}{32}\sin^2\theta\ J_2(2ka\ \sin\theta) + i\ \sin\theta\ J_1(2ka\ \sin\theta)\right]\right\}, \qquad (3.80)$$

$$\theta < \frac{\pi}{2} \ .$$

The components of the far-zone scattered magnetic field due to the bounded components in equations (3.38) and (3.39) are expressible in terms of $I_3(ka,\theta)$ as

$$\bar{H}^S_{HN_2} \sim -\hat{\phi}\,\frac{ikH_0 e^{ikR}}{\pi R}\cos\theta\, I_3(ka,\theta), \tag{3.81}$$

$$\bar{H}^S_{EN_2} \sim -\hat{\theta}\,\frac{ikH_0 e^{ikR}}{\pi R}\cos\theta\, I_3(ka,\theta)\,. \tag{3.82}$$

When these scattered fields are added to the physical optics scattered fields of equations (3.37a) and (3.37b) respectively and the scattered fields due to the unbounded currents of equations (3.58) and (3.59) respectively, the desired approximations to the two total backscattered far-zone magnetic fields are obtained. The resulting expressions can be simplified somewhat by making use of the series expansion of $\cos\theta$ in powers of $\sin\theta$ where appropriate and by applying the recursion relations for the Bessel functions in order to combine terms. The resulting approximate expressions, again valid to the orders of $\sin^2\theta$ and $(ka)^{-1}$, may be written as

$$\begin{aligned}\bar{H}^S_H \sim \hat{\phi}\,\frac{H_0\, a\, e^{ikR}}{2R}\cos\theta \Bigg\{ &-\frac{i\,J_1(2ka\sin\theta)}{\sin\theta} + \left(1+\frac{\sin^2\theta}{2}\right)J_2(2ka\sin\theta) + \\ &+\frac{i\sin\theta}{2}J_3(2ka\sin\theta) + \frac{1}{2ika}\Bigg[-\left(\frac{1}{2}+\frac{11}{16}\sin^2\theta\right)J_0(2ka\sin\theta) + \\ &+\left(1+\frac{25}{16}\sin^2\theta\right)J_2(2ka\sin\theta) - \frac{3}{8}\sin^2\theta\, J_4(2ka\sin\theta) - \\ &- i\sin\theta\Big[J_1(2ka\sin\theta) - J_3(2ka\sin\theta)\Big]\Bigg]\Bigg\}\,, \quad \theta<\frac{\pi}{2}\,,\end{aligned} \tag{3.83}$$

$$\bar{H}_E^S \sim -\hat{\theta}\,\frac{aH_0 e^{ikR}}{2R}\left\{ + \frac{i\cos\theta\, J_1(2ka\sin\theta)}{\sin\theta} + J_2(2ka\sin\theta) + \right.$$

$$+ i\left[\frac{J_2(2ka\sin\theta)}{ka} + \frac{\sin\theta}{2} J_1(2ka\sin\theta)\right] +$$

$$+ \frac{1}{2ika}\left[\left(\frac{1}{2} + \frac{11}{16}\sin^2\theta\right) J_0(2ka\sin\theta) + \right.$$

$$+ \left(1 + \frac{7}{16}\sin^2\theta\right) J_2(2ka\sin\theta) - \frac{3}{8}\sin^2\theta\, J_4(2ka\sin\theta) +$$

$$\left.\left. + \frac{2i}{ka} J_2(2ka\sin\theta)\right]\right\}, \qquad \theta < \frac{\pi}{2} \quad . \tag{3.84}$$

3.6 A Note on Extending Equations (3.83) and (3.84)

It may, at some time, become desirable to extend equations (3.83) and (3.84) to higher powers of $\sin\theta$ and $(ka)^{-1}$ by using the methods considered in this chapter. The extension to higher powers of $\sin\theta$ is straightforward and requires nothing fundamentally different than was done here. The extension to higher powers of $(ka)^{-1}$, however, will introduce an integral over ϕ that has not been considered heretofore. This integral arises when the asymptotic expansions of the various integrals over ρ are taken to greater powers of $(ka)^{-1}$ than was done in deriving equations (3.83) and (3.84). It has the general form

$$I_c(ka,\theta) = \int_0^{2\pi} e^{ika\,T(\theta,\phi)} \cos^m\phi \sin^\ell\phi \, d\phi \, , \tag{3.85}$$

where m may be restricted to be either zero or one with no loss in generality and ℓ may assume any non-negative integer value (consider for instance the consequence of retaining more terms in expanding equation (3.42)).

When m = 1 this expression, like equation (3.53), can be shown to be identically zero. When m = 0, however, the integral is in general non-zero and is a function of both θ and (ka).

Because of the nature of the solutions to the scattering problems that were obtained in this chapter it would be desirable to expand I_c (ka, θ) in a series of Bessel functions of the argument (2ka sin θ) and inverse powers of (ka). It is not obvious how to proceed, nor has this been done in this work, since, fortunately, the integral did not arise in the derivations of equations (3.83) and (3.84). Any attempt to extend the results of this chapter to higher orders of $(ka)^{-1}$ will necessitate either an exact or approximate evaluation of $I_c(ka, \theta)$.

Chapter IV

SOME FURTHER CONSIDERATIONS

4.1 A Comparison Involving the Exact Solution

In Chapter II it was found that for $c = ka = 1.0$ all computed radar cross sections were consistently lower than the corresponding ones obtained experimentally. The regularity of this phenomenon indicates that some effect other than stray radar returns may be predominant. In particular, the computed values may be in error. Consequently, an independent verification of the accuracy of the programmed formulation of the exact solution for small values of c would be most helpful. This can be done by applying a solution to the problem of electromagnetic scattering by a disk of a unit incident plane wave for small values of c that was developed by Eggimann (1961). His solutions to the far-zone scattered fields are given as power series in c. For the case of back-scattering they may be written as

$$E^s_{\phi_E} = -\eta_0 H^s_{\theta_E} \sim \frac{e^{ikR}}{kR}\left(\frac{2c^3}{3\pi}\right)\left\{2 + \sin^2\theta + \frac{c^2}{15}\left[16 - 15\sin^2\theta - 5\sin^4\theta\right]\right\}, \tag{4.1}$$

$$E^s_{\theta_H} = \eta_0 H^s_{\phi_H} \sim \frac{e^{ikR}}{kR}\left(\frac{2c^3\cos^2\theta}{3\pi}\right)\left\{2 + \frac{c^2}{15}\left[16 - 9\sin^2\theta\right]\right\}. \tag{4.2}$$

The dependence on c in these two expressions is quite simple, for even though the solutions have been carried out to the order of c^5 only two terms in each of the resulting power series have non-zero coefficients. For small c both backscattered fields behave like c^3, which implies that the measurement problem will become quite acute for small values of c, in accordance with actual experience.

In order to compare equations (4.1) and (4.2) with the exact solutions considered in Chapter II it is necessary to choose a suitable value of c for which to compute the various radar cross sections. What is a reasonable choice depends on the anticipated effect of those terms in the power series expansions for the far-zone backscattered fields that have not been taken into account in equations (4.1) and (4.2). Clearly, the choice of $c = 1.0$ would not be expected to yield good results. Choice of $c = 0.5$ might be acceptable, however, as the first missing term in the series, which is of the order of c^6, would be of the order of 1/8 as large as the first term. On this basis it was decided to calculate the three radar cross sections as a function of aspect angle using the exact solution and equations (4.1) and (4.2) for $c = 0.5$. The agreement between the two solutions was extremely good, as is shown by the comparisons given in Table 4-1. The slight differences in cross section encountered in that table can be attributed to neglect of terms of the order of c^6 or higher in equations (4.1) and (4.2).

On the strength of Table 4-1 it is reasonable to conclude that the discrepancies found in Chapter II between the experimental and calculated cross sections for $c = 1.0$ reflect effects other than errors in the computational effort. A possible effect not mentioned before would be differences in the measured scattering cross sections due to the finite thickness of the actual disk.

4.2 Comparison Between the Exact and Approximate Solutions

In this section some radar cross sections for the disk as predicted by equations (3.83) and (3.84) will be compared with the same cross sections as predicted by the exact solution. Since the exact solution gives best results for low frequencies (low values of $c = ka$), while equations (3.83) and (3.84) are by nature high frequency solutions, a compromise in the choice of c must be effected so that both solutions can be expected to give reasonable results. Choice of $c = 6$ appears to be good, for the exact solution agrees very well with the experimental data for aspect angles as large as seventy degrees. Also, the terms of highest order in $(ka)^{-1}$ in equations (3.83) and (3.84) will be quite small compared to the leading terms (except perhaps near minima of the scattered fields) so that these equations may also be expected

TABLE 4-1

Some Radar Cross Sections for $c = 0.5$ as Computed Using Eggimann's Solution and the Exact Solution due to Flammer (in dB per square wavelength).

	Radar Cross Sections From The Exact Solution (dB/λ^2)			Radar Cross Sections From Eqs. (4.1) and (4.2) (dB/λ^2)		
θ	E-Pol.	H-Pol.	X-Pol.	E-Pol.	H-Pol.	X-Pol.
2^o	-29.32	-29.33	-90.78	-29.39	-29.40	-90.79
10^o	-29.24	-29.61	-62.91	-29.30	-29.67	-62.94
20^o	-29.01	-30.47	-51.19	-29.06	-30.54	-51.22
30^o	-28.67	-31.98	-44.68	-28.72	-32.03	-44.71
40^o	-28.29	-34.21	-40.43	-28.33	-34.26	-40.45
50^o	-27.91	-37.37	-37.50	-27.94	-37.41	-37.52
60^o	-27.58	-41.84	-35.48	-27.61	-41.87	-35.50
70^o	-27.33	-48.52	-34.15	-27.36	-48.55	-34.17
80^o	-27.18	-60.35	-33.39	-27.20	-60.38	-33.42
88^o	-27.13	-88.24	-33.16	-27.15	-88.27	-33.18

to perform well, particularly for small aspect angles. Computed radar cross sections using both methods are shown in Figs. 4-1a through 4-1c for E-polarization, H-polarization, and cross-polarization respectively. Qualitative agreement is good throughout, and quantitative agreement is generally good for aspect angles that are less than thirty degrees. Oddly, agreement is best for the cross-polarized case with the two methods predicting practically the same radar cross section for aspect angles as large as forty-two degrees. The case of E-polarization shows the best agreement of the two direct returns with equation (3.84) and the exact solution predicting very nearly the same radar cross section for aspect angles as large as thirty-four degrees. Only the minimum at nineteen degrees shows any sizeable discrepancy between the two methods for this angular range. Finally, equation (3.83) and the exact solution agree well only to aspect angles of twenty degrees for H-polarization. While it is expected that equations (3.83) and (3.84) will perform poorly for large aspect angles because of the approximations made in their derivations, it is odd that equation (3.83) would fail for such low aspect angles. Since Fig. 2-7b indicates that the exact solution is valid in this case, equation (3.83) must be in error, or must fail to account for some effect.

Some deliberation reveals that both equations (3.83) and (3.84) were derived without taking into account the possibility of multiple diffraction by the disk. Furthermore, the effects of multiple diffraction will be greater for H-polarization than for E-polarization. This view is consistent with the actual behavior of equations (3.83) and (3.84). Introduction of the effects of multiple diffraction, which will not be considered in this work, is one means by which one can seek to improve the performance of the approximate solution. Another would be to keep still higher powers of $\sin \theta$ in equations (3.83) and (3.84). It would be extremely difficult to predict how successful these undertakings would be, but there is certainly reason to expect at least partial success, as equations (3.83) and (3.84) already predict the proper qualitative behavior for the radar cross-sections for large aspect angles.

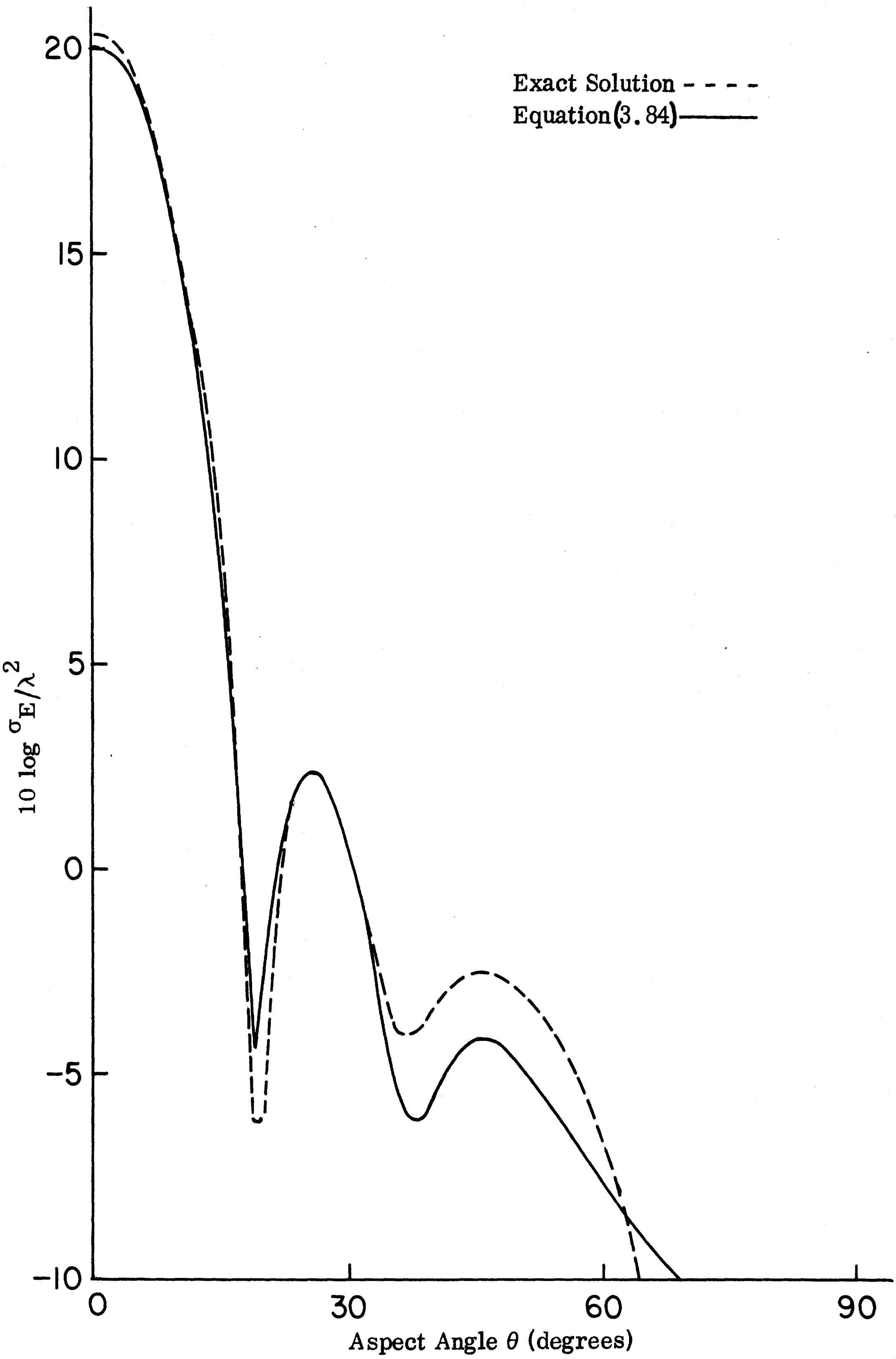

FIG. 4-1a: Computed RCS of a Disk for E-Polarization, c = 6.0.

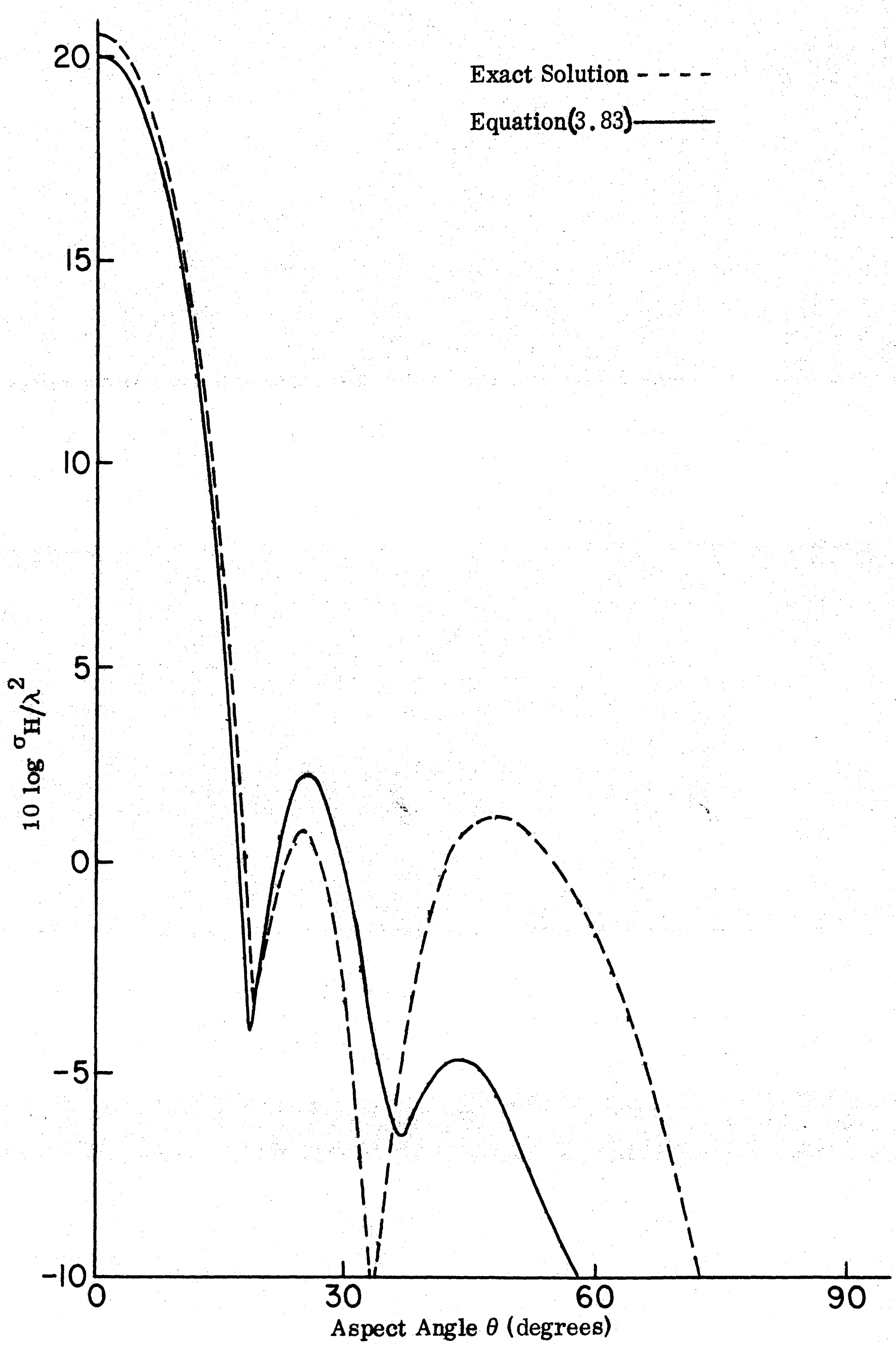

FIG. 4-1b: Computed RCS of a Disk for H-Polarization, $c = 6.0$.

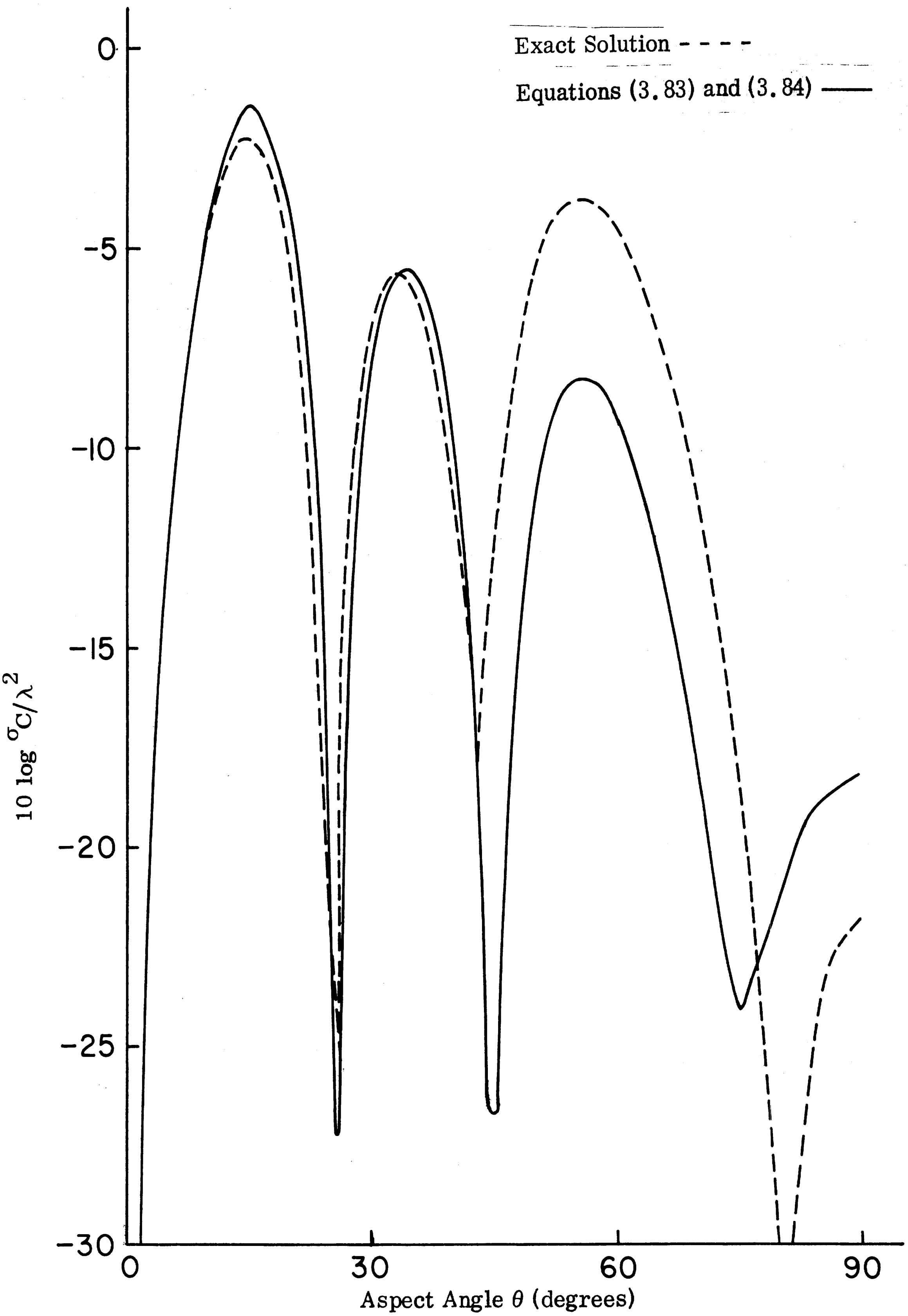

FIG. 4-1c: Computed Cross-Polarized RCS of a Disk; c = 6.0.

4.3 An Application of the Geometrical Theory of Diffraction in Light of the Results of Chapter III

The important question of how the approximate solutions developed in this work agree with or differ from other approximate solutions to the problem of backscattering from a disk will be considered in this section. The approximate solutions that will form the basis for discussion will be those obtained by application of the Geometrical Theory of Diffraction, which was mentioned in Chapter I, and which can be expected to give reasonable results for disks of large c for aspect angles away from normal incidence. Since equations (3.83) and (3.84) purport to be most accurate for normal incidence, a non-rigorous procedure will be developed to obtain a continuation to normal incidence of the range of validity of the solutions obtained from the Geometrical Theory of Diffraction.

Different arguments have been advanced in order to continue the results of the Geometrical Theory of Diffraction into the caustic region that occurs in the disk or cone backscattering problems for θ equal to zero. The argument which is probably of most use here is that given by R.A. Ross in "Investigation of Scattering Center Theory" (1967), as his treatment attempts to account for depolarizing effects. Ross considers backscattering by a perfectly conducting flat-backed cone of arbitrary cone half angle, which problem, in principle, includes the disk problem, for the disk can be considered to be the limiting case of a flat backed cone as the cone half angle approaches ninety degrees.

As it happens, Ross' results are at variance with equations (3.83) and (3.84). In fact his results, given by his equation (B-10), become infinite for the limiting case of the disk. This not very satisfactory state of affairs may be eliminated by modifying Ross' analysis, which is not valid for the case of the disk. His analysis begins with his equations (B-3) and (B-4), which are reproduced below.

$$\mu_{e1} = \frac{\sin \pi/n}{2n\,R}\sqrt{\frac{a}{\pi\, k \sin\theta}}\; A\, e^{ikR}\left\{\left[\cos\frac{\pi}{n} - \cos\frac{3\pi - 2\theta}{n}\right]^{-1} \mp \left[\cos\frac{\pi}{n} - 1\right]^{-1}\right\} e^{\,i\,(kR + \frac{\pi}{4} - 2ka\sin\theta)} \tag{4.3}$$

$$\mu_{e2} = \frac{\sin \pi/n}{2n\,R}\sqrt{\frac{a}{\pi\, k \sin\theta}}\; A\, e^{ikR}\left\{\left[\cos\frac{\pi}{n} - \cos\frac{3\pi + 2\theta}{n}\right]^{-1} \mp \left[\cos\frac{\pi}{n} - 1\right]^{-1}\right\} e^{\,i\,(kR - \frac{\pi}{4} + 2ka\sin\theta)} \; . \tag{4.4}$$

These are the fields singly diffracted from two edge scattering centers on the base of the cone. The upper signs are to be used for E-polarization and the lower ones for H-polarization. Also ,

$A\,e^{ikR}$ represents the incident plane wave by a different convention than used previously,

$n = \frac{3}{2} + \frac{\gamma}{2}$, where γ is the cone half angle,

k is the wave number.

The total far-zone backscattered fields for $\theta < \gamma$ are just the algebraic sum of μ_{e1} and μ_{e2} and may be written as

$$\mu_{e1} + \mu_{e2} = \frac{\sin \pi/n}{2nR}\sqrt{\frac{a}{\pi\, k\sin\theta}}\; A e^{i2kR}\left\{\left[\frac{e^{\,i(2ka\sin\theta - \frac{\pi}{4})}}{(\cos\frac{\pi}{n} - \cos\frac{3\pi + 2\theta}{n})} + \frac{e^{-i\,(2ka\sin\theta - \pi/4)}}{(\cos\frac{\pi}{n} - \cos\frac{3\pi - 2\theta}{n})}\right] \mp \frac{e^{\,i(2ka\sin\theta - \pi/4)} + e^{-i\,(2ka\sin\theta - \pi/4)}}{(\cos\frac{\pi}{n} - 1)}\right\} . \tag{4.5}$$

Ross seeks continuations of these expressions that are valid for values of θ near zero by expanding $(\cos\frac{\pi}{n} - \cos\frac{3\pi \pm 2\theta}{n})^{-1}$ for θ near zero and $n \neq 2$, which makes his results invalid for the disk. One could also take the point of view that since these expressions are valid for large θ, continuations could be found by expanding $(\cos\frac{\pi}{n} - \cos\frac{3\pi \pm 2\theta}{n})^{-1}$ for $\sin\theta$ large and n close to two. Actually, either expansion encounters difficulties since both terms in $(\cos\frac{\pi}{n} - \cos\frac{3\pi \pm 2\theta}{n})^{-1}$ go to zero for certain values of n and θ. Consequently, an alternate approach which requires no troublesome expansions will be used here. The total backscattered fields may be rewritten as

$$\mu_{e1} + \mu_{e2} = \frac{a A \sin\frac{\pi}{n} e^{i2kR}}{n R} \times$$

$$\times \left\{ \left[\frac{(\cos\frac{\pi}{n} - \cos\frac{3\pi}{n}\cos\frac{2\theta}{n}) C_1 - i \sin\frac{3\pi}{n}\sin\frac{2\theta}{n} C_2}{\cos^2\left(\frac{\pi}{n}\right) - 2\cos\frac{\pi}{n}\cos\frac{3\pi}{n}\cos\frac{2\theta}{n} + \cos^2\left(\frac{3\pi}{n}\right) - \sin^2\left(\frac{2\theta}{n}\right)} \right] \mp \frac{C_3}{(\cos\frac{\pi}{n} - 1)} \right\}, \quad (4.6)$$

where

$$C_1 = C_3 = \sqrt{\frac{2}{2\pi\, ka \sin\theta}} \left[\frac{e^{i(2ka \sin\theta - \frac{\pi}{4})} + e^{-i(2ka \sin\theta - \frac{\pi}{4})}}{2} \right],$$

$$C_2 = \sqrt{\frac{2}{2\pi ka \sin\theta}} \left[\frac{e^{i(2ka \sin\theta - \frac{\pi}{4})} - e^{-i(2ka \sin\theta - \frac{\pi}{4})}}{2i} \right]. \quad (4.7)$$

The term C_1 can be recognized as the large argument expansion of the Bessel function $(-1)^{\ell} J_{2\ell}(2ka \sin\theta)$, where $\ell = 0, 1, 2 \ldots$ Similarly, C_2 is the large argument expansion of $(-1)^{\ell} J_{1+2\ell}(2ka \sin\theta)$. The actual choice of the Bessel functions must be made such that the resulting expressions for the backscattered fields be well behaved as functions of θ for each value of ℓ. Consider first the coefficient multiplying C_1. For any permissible value of n it can be shown to be finite. Hence the lowest order Bessel function corresponding to C_1, $J_0(2ka \sin\theta)$, is the appropriate choice for that term. Note that for n = 2 the coefficient of C_1 is identically zero and that the term disappears from consideration, a feature that will also appear in the final expressions for the back-scattered fields. Next consider the coefficient multiplying C_2. The lowest order Bessel function corresponding to C_2, $J_1(2ka \sin\theta)$, will suffice to keep its contribution finite. Indeed, the forms of equations (3.83) and (3.84) dictate this choice. Finally, the Bessel function corresponding to C_3 is taken to be $-J_2(2ka \sin\theta)$ so that the polarization dependent term will behave in agreement with equations (3.83) and (3.84). The final form for the total backscattered fields is

$$\mu_{e1} + \mu_{e2} = \frac{a A \sin\frac{\pi}{n} e^{i2kR}}{n R} \times$$

$$\times \left\{ \left[\frac{\left(\cos\frac{\pi}{n} - \cos\frac{3\pi}{n}\cos\frac{2\theta}{n}\right) J_0(2ka\sin\theta) - i \sin\frac{3\pi}{n}\sin\frac{2\theta}{n} J_1(2ka\sin\theta)}{\cos^2\left(\frac{\pi}{n}\right) - 2\cos\frac{\pi}{n}\cos\frac{3\pi}{n}\cos\frac{2\theta}{n} + \cos^2\left(\frac{3\pi}{n}\right) - \sin^2\left(\frac{2\theta}{n}\right)} \right] \right.$$

$$\left. \pm \left(\cos\frac{\pi}{n} - 1\right)^{-1} J_2(2ka\sin\theta) \right\} , \qquad (4.8)$$

where again the upper sign is to be used for E-polarization and the lower sign is to be used for H-polarization. This equation is valid only for θ less than the cone half angle γ. For n = 2 the coefficient of $J_0(2ka \sin\theta)$ becomes identically equal to zero and the expression reduces to a form which is in agreement with the leading terms of equations (3.83) and (3.84) for θ near zero, the desired result. The method of approximation used in Chapter III is seen to be in agreement with the Geometrical Theory of Diffraction. In fact, consideration of the results of Chapter III has resulted in a new form for expressions for backscattering by a finite cone for aspect angles near nose-on which differs radically from that obtained by Ross. The simplicity of this result suggests the possibility of applying the method of Chapter III to cone scattering in the same manner as was done for the disk. Such an undertaking certainly bears consideration.

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APPENDIX A

SOME PROPERTIES OF THE OBLATE SPHEROIDAL COORDINATE SYSTEM

The variables x, y, z of the Cartesian coordinate system are expressible in terms of ξ, η, and ϕ of the oblate spheroidal coordinate system of Fig. 2-1 by

$$x = a \sqrt{(1 - \eta^2)(1 + \xi^2)} \cos \phi, \tag{A.1.a}$$

$$y = a \sqrt{(1 - \eta^2)(1 + \xi^2)} \sin \phi, \tag{A.1.b}$$

$$z = a \eta \xi . \tag{A.1.c}$$

The gradient in the oblate spheroidal coordinate system can be written

$$\nabla \psi = (h_\xi)^{-1} \frac{\partial \psi}{\partial \xi} \hat{\xi} + (h_\eta)^{-1} \frac{\partial \psi}{\partial \eta} \hat{\eta} + (h_\phi)^{-1} \frac{\partial \psi}{\partial \phi} \hat{\phi} , \tag{A.2}$$

where h_ξ, h_η, and h_ϕ are the metrical coefficients given by

$$h_\xi = a \sqrt{\frac{\xi^2 + \eta^2}{1 + \xi^2}} , \tag{A.3.a}$$

$$h_\eta = a \sqrt{\frac{\xi^2 + \eta^2}{1 - \eta^2}} , \tag{A.3.b}$$

$$h_\phi = a \sqrt{(1 + \xi^2)(1 - \eta^2)} . \tag{A.3.c}$$

Expressions relating the unit vectors in the Cartesian and oblate spheroidal coordinate systems can easily be obtained with the help of equation (A. 2) by taking the gradient of both sides of equations (A. 1a, b, c). The results are

$$\hat{x} = -\sqrt{\frac{1+\xi^2}{\xi^2+\eta^2}}\,\eta\cos\phi\,\hat{\eta} + \xi\sqrt{\frac{1-\eta^2}{\xi^2+\eta^2}}\,\cos\phi\,\hat{\xi} - \sin\phi\,\hat{\phi}\,, \tag{A.4.a}$$

$$\hat{y} = -\eta\sqrt{\frac{1+\xi^2}{\xi^2+\eta^2}}\,\sin\phi\,\hat{\eta} + \xi\sqrt{\frac{1-\eta^2}{\xi^2+\eta^2}}\,\sin\phi\,\hat{\xi} + \cos\phi\,\hat{\phi}\,, \tag{A.4.b}$$

$$\hat{z} = \xi\sqrt{\frac{1-\eta^2}{\xi^2+\eta^2}}\,\hat{\eta} + \eta\sqrt{\frac{1+\xi^2}{\xi^2+\eta^2}}\,\hat{\xi}. \tag{A.4.c}$$

As ξ becomes large, the surfaces of constant ξ approach spheres and the hyperbolae of constant η asymptotically approach lines of constant θ, where θ is the spherical angle defined with respect to the z-axis. We may write,

$$a\,\xi \sim R\,, \tag{A.5.a}$$

$$\eta \sim \cos\theta\,. \tag{A.5.b}$$

where $\sim$ is to indicate "in the limit as ξ becomes infinite". Because of this asymptotic behavior of the coordinate system for large values of the radial argument, the radial functions behave asymptotically like the corresponding spherical Bessel or Hankel functions. For instance, for the normalization used by Flammer (1957) or Stratton, et al (1956), the radial function of the third kind behaves identically with the spherical Hankel function of the first kind for very large ξ. That is,

$$R^{(3)}_{mn}(-ic,\ i\xi) \sim \frac{1}{kR}\, e^{i\left(kR - \left(\frac{n+1}{2}\right)\pi\right)} . \tag{A.6}$$

The components of the curl operator in the oblate spheroidal coordinate system are given in terms of the components of a vector $\bar{A}$ by

$$(\nabla \times \bar{A})_\xi = \frac{1}{h_\eta h_\phi}\left[\frac{\partial(h_\eta A_\eta)}{\partial\phi} - \frac{\partial(h_\phi A_\phi)}{\partial\eta}\right], \tag{A.7.a}$$

$$(\nabla \times \bar{A})_\eta = \frac{1}{h_\xi h_\phi}\left[\frac{\partial(h_\phi A_\phi)}{\partial\xi} - \frac{\partial(h_\xi A_\xi)}{\partial\phi}\right], \tag{A.7.b}$$

$$(\nabla \times \bar{A})_\phi = \frac{1}{h_\xi h_\eta}\left[\frac{\partial(h_\xi A_\xi)}{\partial\eta} - \frac{\partial(h_\eta A_\eta)}{\partial\xi}\right]. \tag{A.7.c}$$

APPENDIX B

THE FAR-ZONE BISTATIC SCATTERED FIELDS

For the incident field $\bar{F}_i$ of Fig. 1-1 given by equation (1.1) the far-zone scattered fields may be written:

$$E^S_{\phi_E} = -\eta_0 H^S_{\theta_E} \sim \frac{E_0 e^{ikR}}{ikR} \sum_{m=0}^{\infty} \cos m\left(\phi + \frac{\pi}{2}\right) \left\{ \frac{\beta^E_m(c,\theta)\sin\gamma}{\sin\theta} \sum_{\ell=m}^{\infty}{}' 2(2-\delta_{0m}) \times \right.$$

$$\times \frac{J_{m\ell}}{N_{m\ell}} S^{(1)}_{m\ell}(-ic, \cos\gamma)\, S^{(1)}_{m\ell}(-ic, \cos\theta) +$$

$$+ \frac{2\alpha^E_m(c,\theta)\cos\gamma}{\cos\theta} \left[\sum_{\ell=m+2}^{\infty}{}' \frac{J'_{m+1,\ell}}{N_{m+1,\ell}} S^{(1)}_{m+1,\ell}(-ic, \cos\gamma)\, S^{(1)}_{m+1,\ell}(-ic, \cos\theta) + \right.$$

$$\left.\left. + (1-\delta_{0m}) \sum_{\ell=m}^{\infty}{}' \frac{J'_{m-1,\ell}}{N_{m-1,\ell}} S^{(1)}_{m-1,\ell}(-ic, \cos\gamma)\, S^{(1)}_{m-1,\ell}(-ic, \cos\theta) \right] \right\} \tag{B.1}$$

$$E^S_{\theta_E} = \eta_0 H^S_{\phi_E} \sim -\frac{E_0 e^{ikR}}{ikR} \sum_{m=0}^{\infty} \frac{2\alpha^E_m(c,\theta)}{\cos\theta} \sin m\left(\phi + \frac{\pi}{2}\right) \times$$

$$\times \left\{ \sum_{\ell=m+2}^{\infty}{}' \frac{J'_{m+1,\ell}}{N_{m+1,\ell}} S^{(1)}_{m+1,\ell}(-ic, \cos\gamma)\, S^{(1)}_{m+1,\ell}(-ic, \cos\theta) - \right.$$

$$\left. - (1-\delta_{0m}) \sum_{\ell=m}^{\infty}{}' \frac{J'_{m-1,\ell}}{N_{m-1,\ell}} S^{(1)}_{m-1,\ell}(-ic, \cos\gamma)\, S^{(1)}_{m-1,\ell}(-ic, \cos\theta) \right\} \tag{B.2}$$

For the incident field $\bar{F}_i$ given by equation (1.2) the far-zone scattered fields are

$$E^S_{\theta_H} = \eta_0 H^S_{\phi_H} \sim \frac{-E_0 e^{ik\pi}}{ikR} \sum_{m=0}^{\infty} \cos m\left(\phi + \frac{\pi}{2}\right) \Bigg\{ \frac{\beta^H_m(c,\theta)\sin\gamma}{\sin\theta} \sum_{\ell=m+1}^{\infty}{}' \, 2(2-\delta_{0m}) \times$$

$$\times \frac{J'_{m\ell}}{N_{m\ell}} S^{(1)}_{m\ell}(-ic, \cos\gamma)\, S^{(1)}_{m\ell}(-ic, \cos\theta) \; +$$

$$+ 2\,\frac{\alpha^H_m(c,\theta)\cos\gamma}{\cos\theta}\Bigg[\sum_{\ell=m+1}^{\infty}{}' \frac{J_{m+1,\ell}}{N_{m+1,\ell}} S^{(1)}_{m+1,\ell}(-ic, \cos\gamma)\, S^{(1)}_{m+1,\ell}(-ic, \cos\theta) +$$

$$+ (1-\delta_{0m}) \sum_{\ell=m-1}^{\infty}{}' \frac{J_{m-1,\ell}}{N_{m-1,\ell}} S^{(1)}_{m-1,\ell}(-ic, \cos\gamma)\, S^{(1)}_{m-1,\ell}(-ic, \cos\theta)\Bigg]\Bigg\} \tag{B.3}$$

$$E^S_{\phi_H} = -\eta_0 H^S_{\theta_H} \sim + \frac{E_0 e^{ikR}}{ikR} \sum_{m=0}^{\infty} \frac{2\alpha^H_m(c,\theta)}{\cos\theta} \sin m\left(\phi + \frac{\pi}{2}\right) \times$$

$$\times \Bigg\{ \sum_{\ell=m+1}^{\infty}{}' \frac{J_{m+1,\ell}}{N_{m+1,\ell}} S^{(1)}_{m+1,\ell}(-ic, \cos\gamma)\, S^{(1)}_{m+1,\ell}(-ic, \cos\theta) -$$

$$- (1-\delta_{0m}) \sum_{\ell=m-1}^{\infty}{}' \frac{J_{m-1,\ell}}{N_{m-1,\ell}} S^{(1)}_{m-1,\ell}(-ic, \cos\gamma)\, S^{(1)}_{m-1,\ell}(-ic, \cos\theta)\Bigg\} \tag{B.4}$$

All quantities in equations (B.1) through (B.4) have been discussed and defined in Chapter II. Equations (B.4) and (B.2) give field components that are orthogonal to the respective incident fields. These cross polarized components become identically zero when $\overline{OP}$ lies in the yz-plane ($\phi = \pm\frac{\pi}{2}$), and, for the special case of backscattering, equations (B. 1) and (B. 2) reduce to equations (2. 2. 3) and (2. 2. 10) respectively with (B. 2) and (B. 4) both giving zero contributions.

The program for calculating the backscattered fields can easily be extended to give these bistatic fields. It is only necessary to compute products of angular functions for two different values of the argument instead of for single values of the argument and to introduce the ϕ dependence in the sum over m in equations (B. 1) and (B. 3). Computation of equations (B. 2) and (B. 4) will then be a trivial extension, as the summations over ℓ in them are also found in the respective direct returns. The apparent singularities due to $\cos\theta$ and $\sin\theta$ in equations (B. 1 - B. 4) can be eliminated by considering appropriate limits. In some cases ratios of the weighting factors and the singular function will be well behaved, while in other cases the angular functions will cancel the singularity.

APPENDIX C

THE $\alpha_r^{m\ell}(-ic)$

Series representations of the coefficients $\alpha_r^{m\ell}(-ic)$ as defined in equation (2.3.5) are given below in terms of $C_0^{m\ell}$ and the normalized coefficients $C'^{m\ell}_{2b} = C^{m\ell}_{2b}/C^{m\ell}_0$ for a range in the index r of $0 \le r \le 8$. Series for the $\alpha_r^{m\ell}(-ic)$ for $0 \le r \le 4$ have also been derived by Flammer (1957) and agree with those given here with a slight change in notation. The argument $(-ic)$ has been omitted here for brevity.

$$\alpha_0^{m\ell} = (C_0^{m\ell})^{-2}$$

$$\alpha_1^{m\ell} = -2(C_0^{m\ell})^{-2} C'^{m\ell}_2$$

$$\alpha_2^{m\ell} = 2!\,(C_0^{m\ell})^{-2}\left[3(C'^{m\ell}_2)^2 - 2C'^{m\ell}_4\right]$$

$$\alpha_3^{m\ell} = -2\,(3!)\,(C_0^{m\ell})^{-2}\left[2(C'^{m\ell}_2)^3 - 3C'^{m\ell}_4 C'^{m\ell}_2 + C'^{m\ell}_6\right]$$

$$\alpha_4^{m\ell} = 4!\,(C_0^{m\ell})^{-2}\left[5(C'^{m\ell}_2)^4 - 12\,C'^{m\ell}_4 (C'^{m\ell}_2)^2 + 3(C'^{m\ell}_4)^2 + 6C'^{m\ell}_6 C'^{m\ell}_2 - 2C'^{m\ell}_8\right]$$

$$\alpha_5^{m\ell} = -2(5!)(C_0^{m\ell})^{-2}\left[3(C'^{m\ell}_2)^5 - 10C'^{m\ell}_4 (C'^{m\ell}_2)^3 + 6(C'^{m\ell}_4)^2 C'^{m\ell}_2 + \right.$$
$$\left. + 6C'^{m\ell}_6 (C'^{m\ell}_2)^2 - 3C'^{m\ell}_6 C'^{m\ell}_4 - 3C'^{m\ell}_8 C'^{m\ell}_2 + C'^{m\ell}_{10}\right]$$

$$\alpha_6^{m\ell} = 6!\,(C_0^{m\ell})^{-2}\left[7(C'^{m\ell}_2)^6 - 30C'^{m\ell}_4 (C'^{m\ell}_2)^4 + 30(C'^{m\ell}_4 C'^{m\ell}_2)^2 - 4(C'^{m\ell}_4)^3 + \right.$$
$$+ 20C'^{m\ell}_6 (C'^{m\ell}_2)^3 - 24C'^{m\ell}_6 C'^{m\ell}_4 C'^{m\ell}_2 + 3\,(C'^{m\ell}_6)^2 -$$
$$\left. - 12C'^{m\ell}_8 (C'^{m\ell}_2)^2 + 6C'^{m\ell}_8 C'^{m\ell}_4 + 6C'^{m\ell}_{10} C'^{m\ell}_2 - 2C'^{m\ell}_{12}\right]$$

$$\alpha_7^{m\ell} = -2(7!)(C_0^{m\ell})^{-2}\Big[4(C_2'^{m\ell})^7 - 21C_4'^{m\ell}(C_2'^{m\ell})^5 + 30(C_4'^{m\ell})^2(C_2'^{m\ell})^3 -$$

$$- 10(C_4'^{m\ell})^3 C_2'^{m\ell} + 15C_6'^{m\ell}(C_2'^{m\ell})^4 - 30C_6'^{m\ell}C_4'^{m\ell}(C_2'^{m\ell})^2 +$$

$$+ 6C_6'^{m\ell}(C_4'^{m\ell})^2 + 6(C_6'^{m\ell})^2(C_2'^{m\ell}) - 10C_8'^{m\ell}(C_2'^{m\ell})^3 +$$

$$+ 12C_8'^{m\ell}C_4'^{m\ell}C_2'^{m\ell} - 3C_8'^{m\ell}C_6'^{m\ell} + 6C_{10}'^{m\ell}(C_2'^{m\ell})^2 -$$

$$- 3C_{10}'^{m\ell}C_4'^{m\ell} - 3C_{12}'^{m\ell}C_2'^{m\ell} + C_{14}'^{m\ell}\Big]$$

$$\alpha_8^{m\ell} = 8!(C_0^{m\ell})^{-2}\Big[9(C_2'^{m\ell})^8 - 56C_4'^{m\ell}(C_2'^{m\ell})^6 + 105(C_4'^{m\ell})^2(C_2'^{m\ell})^4 - 60(C_4'^{m\ell})^3(C_2'^{m\ell})^2 +$$

$$+ 5(C_4'^{m\ell})^4 + 42C_6'^{m\ell}(C_2'^{m\ell})^5 - 120C_6'^{m\ell}C_4'^{m\ell}(C_2'^{m\ell})^3 +$$

$$+ 60C_6'^{m\ell}(C_4'^{m\ell})^2 C_2'^{m\ell} + 30(C_6'^{m\ell})^2(C_2'^{m\ell})^2 - 12(C_6'^{m\ell})^2 C_4'^{m\ell} -$$

$$- 30C_8'^{m\ell}(C_2'^{m\ell})^4 + 60C_8'^{m\ell}C_4'^{m\ell}(C_2'^{m\ell})^2 - 12C_8'^{m\ell}(C_4'^{m\ell})^2 -$$

$$- 24C_8'^{m\ell}C_6'^{m\ell}C_2'^{m\ell} + 3(C_8'^{m\ell})^2 + 20C_{10}'^{m\ell}(C_2'^{m\ell})^3 - 24C_{10}'^{m\ell}C_4'^{m\ell}C_2'^{m\ell} +$$

$$+ 6C_{10}'^{m\ell}C_6'^{m\ell} - 12C_{12}'^{m\ell}(C_2'^{m\ell})^2 + 6C_{12}'^{m\ell}C_4'^{m\ell} + 6C_{14}'^{m\ell}C_2'^{m\ell} - 2C_{16}'^{m\ell}\Big].$$

APPENDIX D

```
C **********************************************************************DISK0001
C *                                                                    *DISK0002
C *           D I S K    S C A T T E R I N G    P R O G R A M          *DISK0003
C *                                                                    *DISK0004
C *                                                                    *DISK0005
C *    PURPOSE:                                                        *DISK0006
C *       TO COMPUTE THE FAR ZONE BACKSCATTERING FROM A DISK.          *DISK0007
C *                                                                    *DISK0008
C *    METHOD:                                                         *DISK0009
C *       THE EXACT SOLUTION OF FLAMMER.                               *DISK0010
C *                                                                    *DISK0011
C *    LANGUAGE:                                                       *DISK0012
C *       FORTRAN IV G-LEVEL.  THE PROGRAM HAS BEEN SUCCESSFULLY RUN ON *DISK0013
C *       AN IBM 360/67 UNDER THE UNIVERSITY OF MICHIGAN TERMINAL      *DISK0014
C *       SYSTEM (MTS).                                                *DISK0015
C *                                                                    *DISK0016
C *    SUBROUTINES REQUIRED:                                           *DISK0017
C *       THE ONLY SUBROUTINES REQUIRED IN ADDITION TO THOSE LISTED HERE*DISK0018
C *       ARE STANDARD FORTRAN IV LIBRARY ROUTINES (E.G., SIN, COS).   *DISK0019
C *                                                                    *DISK0020
C **********************************************************************DISK0021
```

The description of the main program is followed by descriptions of the required subroutines in alphabetical order.

In all descriptions, standard FORTRAN IV variable naming conventions are used, unless otherwise indicated.

The program as listed was tested on an IBM 360/67 under the University of Michigan Terminal System (MTS), but as no system or machine specific subroutines are required, it should be directly transferable to any computer with a standard FORTRAN IV G-level compiler.

MAIN PROGRAM

The main program reads in from I/0 unit 4 the namelist IN, containing the variables:

X1 the double precision value of the initial angle in degrees (default is 2.^{o}),

X2 the double precision value of the final angle in degrees (default is 88.^{o}), and

DX the double precision value of the step size (default is 2.^{o}).

The program then calls on the routine RCS to compute and print the values of the radar cross section for the indicated range of incidence angles, for a value of ka read in (together with the Stratton-Chu coefficients $d_n^{m\ell}(-ic)$) from I/0 unit 5.

The program then writes on I/0 unit 6 some of the auxiliary quantities which have been computed: the (unnormalized) $C_{2k}^{m\ell}(-ic)$, the $N_{m\ell}$, the $K_{m\ell}^{(1)}(c)$ and $K_{m\ell}^{(1)\prime}(c)$, and the $J_{m\ell}(c)$ and $J'_{m\ell}(c)$.

This process is repeated for each succeeding set of data, until an end of file is encountered.

If it is desired to dispense with the printing out of these auxiliary quantities, the main program as listed may be replaced by the following shorter program:

```
      DOUBLE PRECISION X1, X2, DX
      NAMELIST /IN/ X1,X2,DX
1     READ(4,IN,END=500)
      CALL RCS(X1,X2,DX)
      GO TO 1
500   STOP
      END
```

```
C      MAIN PROGRAM                                                    DISK0022
C                                                                      DISK0023
       IMPLICIT COMPLEX(Z)                                             DISK0024
       REAL K,KA                                                       DISK0025
       COMMON /ARRAYC/ C(9,9,12)                                       DISK0026
       COMMON /ARRAYN/ NMAX(9,9)                                       DISK0027
       COMMON /ARRAYJ/ ZJ(9,9)                                         DISK0028
       COMMON /ARRAYK/ K(9,9)                                          DISK0029
       COMMON /ARRAYO/ ORTH(9,9)                                       DISK0030
       DIMENSION M1(9)                                                 DISK0031
       DATA M1 /0,1,2,3,4,5,6,7,8/                                     DISK0032
       REAL*8 X1,X2,DX                                                 DISK0033
       NAMELIST /IN/ X1,X2,DX                                          DISK0034
1      X1 = 2.D0                                                       DISK0035
       X2 = 88.D0                                                      DISK0036
       DX = X1                                                         DISK0037
       READ(4,IN,END=500)                                              DISK0038
       CALL RCS(X1,X2,DX)                                              DISK0039
       WRITE(6,901)                                                    DISK0040
       DO 20 M = 1, 9                                                  DISK0041
       DO 20 L = M, 9                                                  DISK0042
       L1 = L - 1                                                      DISK0043
       LIM = (NMAX(M,L) + 1) / 2                                       DISK0044
       DO 15 KK = 2, LIM                                               DISK0045
15     C(M,L,KK) = C(M,L,1) * C(M,L,KK)                                DISK0046
20     WRITE(6,900) M1(M),L1,(C(M,L,KK), KK = 1, LIM)                  DISK0047
       WRITE(6,902)                                                    DISK0048
       WRITE(6,903) ((M1(M), (ORTH(M,L), L = M, 9)), M = 1, 9)         DISK0049
       WRITE(6,904)                                                    DISK0050
       WRITE(6,903) ((M1(M), (  K(M,L), L = M, 9)), M = 1, 9)          DISK0051
       WRITE(6,905)                                                    DISK0052
       DO 40 M = 1, 9                                                  DISK0053
       DO 40 L = M, 9                                                  DISK0054
```

```
C     LISTING OF MAIN PROGRAM, CONTINUED.                               DISK0000
C                                                                       DISK0000
      L1 = L - 1                                                        DISK0055
40    WRITE(6,906) M1(M),L1,ZJ(M,L)                                     DISK0056
      GO TO 1                                                           DISK0057
500   STOP                                                              DISK0058
900   FORMAT('OM =',I3,5X,'L =',I3,5X,1P6E15.5/23X,6E15.5)             DISK0059
901   FORMAT('1THE (UNNORMALIZED) CONSTANTS C(M,L,2K)...')             DISK0060
902   FORMAT('1THE NORMALIZATION CONSTANTS NML...')                     DISK0061
903   FORMAT(/I4,1P9E14.5//I4,14X,8E14.5//I4,28X,7E14.5//I4,42X,6E14.5//DISK0062
     &I4,56X,5E14.5//I4,70X,4E14.5//I4,84X,3E14.5//I4,98X,2E14.5//      DISK0063
     &I4,112X,E14.5)                                                    DISK0064
904   FORMAT('1THE JOINING FACTORS KML(1) AND KML(1)''...')             DISK0065
905   FORMAT('1THE JOINING FACTORS JML(1) AND JML(1)''...')             DISK0066
906   FORMAT('OM =',I2,5X,'L =',I2,5X,1P2E15.5)                         DISK0067
      END                                                               DISK0068
```

ROUTINE: ABMEH

USE:

CALL ABMEH (M, KA, X, ZAE, ZAH, ZBE, ZBH)

where

M is m + 1,

KA is the real value of c = KA,

ZAE is the returned complex value of $\alpha_m^E(c,X)$,

ZAH is the returned complex value of $\alpha_m^H(c,X)$,

ZBE is the returned complex value of $\beta_m^E(c,X)$,

ZBH is the returned complex value of $\beta_m^H(c,X)$.

COMMENTS:

These quantities are described by equations (2.13a, b, c, d) and (2.16a, b, c, d).

```
      SUBROUTINE ABMEH(M,KA,X,ZAE,ZAH,ZBE,ZBH)                          DISK0069
C                                                                       DISK0070
      IMPLICIT COMPLEX (Z)                                              DISK0071
      REAL*4 KA                                                         DISK0072
      REAL*8 X                                                          DISK0073
      DATA ZERO/(0.,0.)/, ZHALF/(0.5,0.0)/, ZONE/(1.0,0.0)/,            DISK0074
     & ZTWO/(2.0,0.0)/                                                  DISK0075
      Z1E = ZERO                                                        DISK0076
      Z1H = ZERO                                                        DISK0077
      DO 20 N = M, 9                                                    DISK0078
      CALL BMN(M,N,X,ZAB)                                               DISK0079
      CALL KML3(M,N,KA,ZK)                                              DISK0080
      IF(MOD(N-M,2)) 21,24,21                                           DISK0081
21    ZTERMH = ZAB * ZK                                                 DISK0082
      Z1H = Z1H + ZTERMH                                                DISK0083
      GO TO 20                                                          DISK0084
24    ZTERME = ZAB * ZK                                                 DISK0085
      Z1E = Z1E + ZTERME                                                DISK0086
20    CONTINUE                                                          DISK0087
      Z2E = ZERO                                                        DISK0088
      Z2H = ZERO                                                        DISK0089
      IF(M .EQ. 1) GO TO 40                                             DISK0090
      M1 = M - 1                                                        DISK0091
      DO 30 N = M1, 9                                                   DISK0092
      CALL AMN(M1,N,X,ZAB)                                              DISK0093
      CALL KML3(M1,N,KA,ZK)                                             DISK0094
      IF(MOD(N-M1,2)) 31,34,31                                          DISK0095
31    ZTERME = ZAB * ZK                                                 DISK0096
      Z2E = Z2E + ZTERME                                                DISK0097
      GO TO 30                                                          DISK0098
34    ZTERMH = ZAB * ZK                                                 DISK0099
      Z2H = Z2H + ZTERMH                                                DISK0100
30    CONTINUE                                                          DISK0101
```

```
C     LISTING OF ROUTINE "ABMEH", CONTINUED.                          DISK0102
C                                                                     DISK0103
40    Z3E = ZERO                                                      DISK0104
      Z3H = ZERO                                                      DISK0105
      IF(M .GE. 9) GO TO 60                                           DISK0106
      MP1 = M + 1                                                     DISK0107
      DO 50 N = MP1, 9                                                DISK0108
      CALL AMN(MP1,N,X,ZAB)                                           DISK0109
      CALL KML3(MP1,N,KA,ZK)                                          DISK0110
      IF(MOD(N-MP1,2)) 51,54,51                                       DISK0111
51    ZTERME = ZAB * ZK                                               DISK0112
      Z3E = Z3E + ZTERME                                              DISK0113
      GO TO 50                                                        DISK0114
54    ZTERMH = ZAB * ZK                                               DISK0115
      Z3H = Z3H + ZTERMH                                              DISK0116
50    CONTINUE                                                        DISK0117
      IF(M .NE. 2) GO TO 60                                           DISK0118
      Z2E = ZTWO * Z2E                                                DISK0119
      Z2H = ZTWO * Z2H                                                DISK0120
60    ZAE = Z1E / (Z1E - ZHALF * (Z2E + Z3E))                         DISK0121
      ZAH = Z1H / (Z1H + ZHALF * (Z2H + Z3H))                         DISK0122
      ZBE = ZONE - ZAE                                                DISK0123
      ZBH = ZONE - ZAH                                                DISK0124
      RETURN                                                          DISK0125
      END                                                             DISK0126
```

ROUTINE: ABMN

USE:

```
CALL AMN(M, N, X, ZA), or
CALL BMN(M, N, X, ZA)
```

where

M	is $m + 1$,
N	is $n + 1$,
X	is a double precision angle in degrees,
ZA	is the returned complex value of $a_{mn}(KA, X)$ or $b_{mn}(KA, X)$ as AMN or BMN is called.

COMMENTS:

These quantities are described by equations (2.14a, b).

```
      SUBROUTINE ABMN                                                   DISK0127
C                                                                       DISK0128
      COMMON /ARRAYO/ ORTH(9,9)                                         DISK0129
      COMMON /ARRAYS/ S(9,9)                                            DISK0130
      COMPLEX ZA,Z                                                      DISK0131
      REAL*8 X                                                          DISK0132
      DATA RAD /.1745329E-1/                                            DISK0133
      ENTRY AMN(M,N,X,ZA)                                               DISK0134
      XC = RAD * SNGL(X)                                                DISK0135
      XC = -1. / COS(XC)                                                DISK0136
      GO TO 1                                                           DISK0137
      ENTRY BMN(M,N,X,ZA)                                               DISK0138
      XC = RAD * SNGL(X)                                                DISK0139
      XC = 1. / SIN(XC)                                                 DISK0140
 1    IF(M-1) 5,10,5                                                    DISK0141
 10   DEL = 2.0                                                         DISK0142
      GO TO 15                                                          DISK0143
 5    DEL = 4.0                                                         DISK0144
 15   IPOWR = MOD(N+2,4)+ 1                                             DISK0145
      GO TO (20,21,22,23), IPOWR                                        DISK0146
 20   Z = CMPLX(1.0,0.0)                                                DISK0147
      GO TO 30                                                          DISK0148
 21   Z = CMPLX(0.0,1.0)                                                DISK0149
      GO TO 30                                                          DISK0150
 22   Z = CMPLX(-1.0,0.0)                                               DISK0151
      GO TO 30                                                          DISK0152
 23   Z = CMPLX(0.0,-1.0)                                               DISK0153
 30   ZA = Z * CMPLX(XC * S(M,N) * DEL /ORTH(M,N), 0.0)                 DISK0154
      RETURN                                                            DISK0155
      END                                                               DISK0156
```

ROUTINE: ALGNDR

USE:

```
CALL ALGNDR(N, M, X, P)
```

where

- N is n + 1,
- M is m + 1,
- X is a double precision value in degrees,
- P is the returned single precision value of the associated Legendre function, $P_n^m(\eta)$, where $\eta = \cos X$.

COMMENTS:

Intermediate calculations are in double precision; the returned value P is single precision.

```
      SUBROUTINE ALGNDR(N,M,X,P)                                        DISK0157
C                                                                       DISK0158
      IMPLICIT REAL*8(A-H,O-Z)                                          DISK0159
      REAL*4 P                                                          DISK0160
      COMMON /DFCTRL/ FACT(57)                                          DISK0161
      ETA = DCOS(X * .17453292519943296D-1)                             DISK0162
      ABSETA = DABS(ETA)                                                DISK0163
      IF(ABSETA .LE. 1.D-12) GO TO 201                                  DISK0164
      IF(ABSETA .GE. .999999999999) GO TO 250                           DISK0165
      M1 = M + 1                                                        DISK0166
      NMO2 = (N + M1) / 2                                               DISK0167
      SUM = 0.D0                                                        DISK0168
      DO 90 K = NMO2, N                                                 DISK0169
      K2 = 2 * K                                                        DISK0170
      TERM=FACT(K2-1)/FACT(K)*ETA**(K2-N-M)/(FACT(N-K+1)*FACT(K2-N-M+1))DISK0171
      IF(MOD(K,2) .EQ. 0) TERM = -TERM                                  DISK0172
   90 SUM = SUM + TERM                                                  DISK0173
      A=(1.D0-ETA*ETA)**(DFLOAT(M-1)/2.D0)/2.D0**(N-1)                  DISK0174
      IF(MOD(N,2) .EQ. 0) A = -A                                        DISK0175
      P = SNGL(A*SUM)                                                   DISK0176
  500 RETURN                                                            DISK0177
  201 IF(MOD(N-M,2)) 202,203,202                                        DISK0178
  202 P = 0.                                                            DISK0179
      GO TO 500                                                         DISK0180
  203 K = (N+M) / 2                                                     DISK0181
      A = (FACT(N+M-1)/FACT(K))/(FACT(N-K+1)*2.**(N-1))                 DISK0182
      IF(MOD(N+K,2) .NE. 0) A = -A                                      DISK0183
      P = A                                                             DISK0184
      GO TO 500                                                         DISK0185
  250 IF(M .EQ. 1) GO TO 260                                            DISK0186
      P = 0.                                                            DISK0187
      GO TO 500                                                         DISK0188
  260 P = 1.                                                            DISK0189
      GO TO 500                                                         DISK0190
      END                                                               DISK0191
```

ROUTINE: ALPHA

USE:

```
CALL ALPHA(M, L)
```

where

M is $m + 1$,

L is $\ell + 1$.

COMMENTS:

This routine returns the values

$$\alpha_r^{m\ell}(-ic) / r!$$

for $r = 0, 1, \ldots, m$ in the common array

```
COMMON /ARRAYA/ ALF(9).
```

These quantities are described in equation (2.21).

```
      SUBROUTINE ALPHA(M,L)                                              DISK0192
C                                                                        DISK0193
      COMMON /ARRAYA/ ALF(9)                                             DISK0194
      COMMON /ARRAYC/ C(9,9,12)                                          DISK0195
      COMMON /ARRAYN/ NMAX(9,9)                                          DISK0196
      COMMON /FCTRL/ FACT(57)                                            DISK0197
      CO = C(M,L,1)                                                      DISK0198
      C2 = C(M,L,2)                                                      DISK0199
      C4 = C(M,L,3)                                                      DISK0200
      C6 = C(M,L,4)                                                      DISK0201
      C8 = C(M,L,5)                                                      DISK0202
      C10 = C(M,L,6)                                                     DISK0203
      C12 = C(M,L,7)                                                     DISK0204
      C14 = C(M,L,8)                                                     DISK0205
      C16 = C(M,L,9)                                                     DISK0206
      CSQR = 1. / (CO * CO)                                              DISK0207
      ALF(1) = CSQR                                                      DISK0208
      IF(M .EQ. 1) GO TO 500                                             DISK0209
      ALF(2) = -2. * CSQR * C2                                           DISK0210
      IF(M .EQ. 2) GO TO 500                                             DISK0211
      C2P2 = C2 * C2                                                     DISK0212
      ALF(3) = CSQR * (3.*C2P2 - 2.*C4)                                  DISK0213
      IF(M .EQ. 3) GO TO 500                                             DISK0214
      C2P3 = C2P2 * C2                                                   DISK0215
      ALF(4) = -2. * CSQR * (2.*C2P3 - 3.*C4*C2 + C6)                    DISK0216
      IF(M .EQ. 4) GO TO 500                                             DISK0217
      C2P4 = C2P3 * C2                                                   DISK0218
      C4P2 = C4 * C4                                                     DISK0219
      ALF(5) = CSQR * (5.*C2P4 - 12.*C4*C2P2 + 3.*C4P2 + 6.*C6*C2        DISK0220
     &  - 2.*C8)                                                         DISK0221
      IF(M .EQ. 5) GO TO 500                                             DISK0222
      C2P5 = C2P4 * C2                                                   DISK0223
      ALF(6) = -2. * CSQR * (3.*C2P5 - 10.*C4*C2P3 + 6.*C4P2*C2          DISK0224
     &  + 6.*C6*C2P2 - 3.*C6*C4 - 3.*C8*C2 + C10)                        DISK0225
```

```
C     LISTING OF ROUTINE "ALPHA", CONTINUED.                              DISK0226
C                                                                         DISK0227
      IF(M .EQ. 6) GO TO 500                                              DISK0228
      C2P6 = C2P5 * C2                                                    DISK0229
      C4P3 = C4P2 * C4                                                    DISK0230
      C6P2 = C6 * C6                                                      DISK0231
      ALF(7) = CSQR * (7.*C2P6 - 30.*C4*C2P4 + 30.*C4P2*C2P2 - 4.*C4P3    DISK0232
     &  + 20.*C6*C2P3 - 24.*C6*C4*C2 + 3.*C6P2 - 12.*C8*C2P2 + 6.*C8*C4   DISK0233
     &  + 6.*C10*C2 - 2.*C12)                                             DISK0234
      IF(M .EQ. 7) GO TO 500                                              DISK0235
      C2P7 = C2P6 * C2                                                    DISK0236
      ALF(8) = -2. * CSQR * (4.*C2P7 - 21.*C4*C2P5 + 30.*C4P2*C2P3        DISK0237
     &  - 10.*C4P3*C2 + 15.*C6*C2P4 - 30.*C6*C4*C2P2 + 6.*C6*C4P2         DISK0238
     &  + 6.*C6P2*C2 - 10.*C8*C2P3 + 12.*C8*C4*C2 - 3.*C8*C6              DISK0239
     &  + 6.*C10*C2P2 - 3.*C10*C4 - 3.*C12*C2 + C14)                      DISK0240
      IF(M .EQ. 8) GO TO 500                                              DISK0241
      C2P8 = C2P7 * C2                                                    DISK0242
      C4P4 = C4P3 * C4                                                    DISK0243
      C8P2 = C8 * C8                                                      DISK0244
      ALF(9) = CSQR * (9.*C2P8 - 56.*C4*C2P6 + 105.*C4P2*C2P4             DISK0245
     &  - 60.*C4P3*C2P2 + 5.*C4P4 + 42.*C6*C2P5 - 120.*C6*C4*C2P3         DISK0246
     &  + 60.*C6*C4P2*C2 + 30.*C6P2*C2P2 - 12.*C6P2*C4 - 30.*C8*C2P4      DISK0247
     &  + 60.*C8*C4*C2P2 - 12.*C8*C4P2 - 24.*C8*C6*C2 + 3.*C8P2           DISK0248
     &  + 20.*C10*C2P3 - 24.*C10*C4*C2 + 6.*C10*C6 - 12.*C12*C2P2         DISK0249
     &  + 6.*C12*C4 + 6.*C14*C2 - 2.*C16)                                 DISK0250
500   RETURN                                                              DISK0251
      END                                                                 DISK0252
```

ROUTINE: CMLK

USE:

CALL CMLK(M, L, K, ANS)

where

M is $m + 1$,

L is $\ell + 1$,

K is $k + 1$,

ANS is the returned double precision value of $C_{2k}^{m\ell}|$.

COMMENTS:

The computations are described by equations (2.18a, b).

```
      SUBROUTINE CMLK(M,L,K,ANS)                                       DISK0253
C                                                                      DISK0254
      IMPLICIT REAL*8(A-H,O-Z)                                         DISK0255
      REAL*4 D                                                         DISK0256
      COMMON /ARRAYN/ NMAX(9,9)                                        DISK0257
      COMMON /ARRAYD/ D(9,9,12)                                        DISK0258
      COMMON /DFCTRL/ FACT(57)                                         DISK0259
      M2 = 2*M                                                         DISK0260
      M1 = M - 1                                                       DISK0261
      LIM = (NMAX(M,L) + 1) / 2                                        DISK0262
      IFLG = MOD(L-M,2)                                                DISK0263
      IF(IFLG) 11,12,11                                                DISK0264
   11 HALVES = -0.5D0                                                  DISK0265
      GO TO 13                                                         DISK0266
   12 HALVES = -1.5D0                                                  DISK0267
   13 KK = K - 2                                                       DISK0268
      SUM = 0.D0                                                       DISK0269
      DO 10 N = K, LIM                                                 DISK0270
      N2 = 2 * N                                                       DISK0271
      IF(KK) 2,3,4                                                     DISK0272
    2 T1 = 1.D0                                                        DISK0273
      T2 = 1.D0                                                        DISK0274
      GO TO 6                                                          DISK0275
    3 T1 = DFLOAT(M+N) + HALVES                                        DISK0276
      T2 = 1.D0 - DFLOAT(N)                                            DISK0277
      GO TO 6                                                          DISK0278
    4 T1 = DFLOAT(M+N) + HALVES                                        DISK0279
      F1 = T1                                                          DISK0280
      T2 = 1.D0 - DFLOAT(N)                                            DISK0281
      F2 = T2                                                          DISK0282
      DO 5 J = 1, KK                                                   DISK0283
      F1 = F1 + 1.D0                                                   DISK0284
      F2 = F2 + 1.D0                                                   DISK0285
      T1 = T1 * F1                                                     DISK0286
```

```
C     LISTING OF ROUTINE "CMLK", CONTINUED.                            DISK0287
C                                                                      DISK0288
5     T2 = T2 * F2                                                     DISK0289
6     IF(IFLG) 8,7,8                                                   DISK0290
7     T3 = FACT(M2+N2-3)/FACT(N2-1)                                    DISK0291
      GO TO 9                                                          DISK0292
8     T3 = FACT(M2+N2-2) / FACT(N2)                                    DISK0293
9     SUM = SUM + T1*T2*T3*DBLE(D(M,L,N))                              DISK0294
10    CONTINUE                                                         DISK0295
      ANS = SUM / (2.D0**M1 * FACT(K) * FACT(M1+K) )                   DISK0296
      RETURN                                                           DISK0297
      END                                                              DISK0298
```

ROUTINES: FILLJO, FILLK, and FILLS

USE:

```
CALL FILLJO(KA)
CALL FILLK(KA)
CALL FILLS(X)
```

where

KA is the real value of $c = KA$,

X is the double precision value of an angle in degrees.

COMMENTS:

These three utility subroutines fill the common arrays

```
COMMON /ARRAYJ/ ZJ(9,9),
COMMON /ARRAYO/ ORTH(9,9),
COMMON /ARRAYK/ XK(9,9), and
COMMON /ARRAYS/ S(9,9)
```

with the quantities $J_{m\ell}(c)$ or $J'_{m\ell}(c)$, $N_{m\ell}(c)$, $K^{(1)}_{m\ell}(c)$ or $K^{(1)\prime}_{m\ell}(c)$, and $S_{m\ell}(-ic, \eta)$, where $\eta = \cos X$, by calling the routines JML, NML, KML, and SML, respectively.

Since, for a given combination of m and ℓ, either $J_{m\ell}$ or $J'_{m\ell}$ is zero, computer storage is conserved by storing both quantities in the same array; a similar statement holds for $K^{(1)}_{m\ell}$ and $K^{(1)\prime}_{m\ell}$.

```
      SUBROUTINE FILLJO(KA)                                         DISK0299
      COMMON /ARRAYJ/ ZJ(9,9)                                       DISK0300
      COMMON /ARRAYO/ ORTH(9,9)                                     DISK0301
      COMPLEX ZJ                                                    DISK0302
      REAL*4 KA                                                     DISK0303
      DO 100 M = 1, 9                                               DISK0304
      DO 100 L = M, 9                                               DISK0305
      CALL JML(M,L,KA,ZJ(M,L))                                      DISK0306
  100 CALL NML(M,L,ORTH(M,L))                                       DISK0307
      RETURN                                                        DISK0308
      END                                                           DISK0309


      SUBROUTINE FILLK(KA)                                          DISK0310
      COMMON /ARRAYK/ XK(9,9)                                       DISK0311
      REAL*4 KA                                                     DISK0312
      DO 100 M = 1, 9                                               DISK0313
      DO 100 L = M, 9                                               DISK0314
  100 CALL KML(M,L,KA,XK(M,L))                                      DISK0315
      RETURN                                                        DISK0316
      END                                                           DISK0317


      SUBROUTINE FILLS(X)                                           DISK0318
      COMMON /ARRAYS/ S(9,9)                                        DISK0319
      REAL*8 X                                                      DISK0320
      DO 100 M = 1, 9                                               DISK0321
      DO 100 L = M, 9                                               DISK0322
  100 CALL SML(M,L,X,S(M,L),TERM)                                   DISK0323
      RETURN                                                        DISK0324
      END                                                           DISK0325
```

ROUTINE: GETCN

USE:

CALL GETCN

COMMENTS:

This routine obtains the coefficients $C_{2k}^{m\ell}$ by calling on the routine CMLK. The coefficients are then normalized to give $C_0^{m\ell}$, $C_2^{m\ell}/C_0^{m\ell}$, $C_4^{m\ell}/C_0^{m\ell}, \ldots\ldots C_{2k}^{m\ell}/C_0^{m\ell}$. These normalized values are then converted to single precision.

The (normalized) value of $C_{2k}^{m\ell}$ is returned in the $(m+1, \ell+1, k+1)^{st}$ element of the common array

COMMON /ARRAYC/ C(9,9,12).

```
      SUBROUTINE GETCN                                                  DISK0326
C                                                                       DISK0327
      REAL*8 DC                                                         DISK0328
      DIMENSION DC(12)                                                  DISK0329
      COMMON /ARRAYC/ C(9,9,12)                                         DISK0330
      COMMON /ARRAYN/ NMAX(9,9)                                         DISK0331
      DO 100 M = 1, 9                                                   DISK0332
      DO 100 L = M, 9                                                   DISK0333
      LIM = (NMAX(M,L) + 1) / 2                                         DISK0334
      LIM1 = LIM + 1                                                    DISK0335
      DO 20 K = 1, LIM                                                  DISK0336
20    CALL CMLK(M,L,K,DC(K))                                            DISK0337
      DO 30 K = 2, LIM                                                  DISK0338
30    C(M,L,K) = SNGL(DC(K)/DC(1))                                      DISK0339
      C(M,L,1) = SNGL(DC(1))                                            DISK0340
      IF(LIM1 .GT. 12) GO TO 100                                        DISK0341
      DO 40 K = LIM1, 12                                                DISK0342
40    C(M,L,K) = 0.                                                     DISK0343
100   CONTINUE                                                          DISK0344
      RETURN                                                            DISK0345
      END                                                               DISK0346
```

ROUTINE: GETD

USE:

```
CALL GETD(KA)
```

where

KA is the returned real (single precision) value of $c = KA$.

COMMENTS:

This routine reads the coefficients $d_n^{m\ell}(-ic)$ from I/0 unit 5, and returns them in the common array

```
COMMON /ARRAYD/ D(9,9,12).
```

The value of $d_n^{m\ell}(-ic)$ is stored in the array element $D(M, L, [(n+2)/2])$, where $M = m + 1$, $L = \ell + 1$, and $[x]$ denotes the greatest integer less than or equal to x.

The input format is described on the following page, "INPUT FORMAT FOR THE SUBROUTINE GETD".

INPUT FORMAT FOR THE SUBROUTINE GETD

For each value of $c = KA$ there will be one group of cards, as follows. The first card of each group should contain the floating point value of c (corresponding to g in the Stratton-Chu tables) in card columns 1-10. Columns 11-80 may be used for comments. The last card of each group should contain a 9 in card column 1. Columns 15-80 may be used for comments.

The body of each group is composed of one set of cards for each combination of m and ℓ. The first card of each such set should contain the value of m in column 1, and that of ℓ in column 2, where $0 \leq m \leq \ell \leq 8$. In columns 3 and 4 should be punched the maximum value of n, with leading zero, if any, as read directly from the Stratton-Chu tables. In columns 5-14, the floating point value of $c = KA$ may be punched; if punched, it is checked against the value of c given on the first card as an aid in detecting out of order cards. Columns 15-80 may be used for comments.

The remaining cards in each set should contain the values of the coefficients, which will be read by the format "(6E12.7)". Columns 73-80 may be used for comments or identification. The sets may be arranged in any order within the group.

```
      SUBROUTINE GETD(C)                                                DISK0347
C                                                                       DISK0348
      COMMON /ARRAYD/ D(9,9,12)                                         DISK0349
      COMMON /ARRAYN/ NMAX(9,9)                                         DISK0350
      DO 100 I = 1, 9                                                   DISK0351
      DO 100 J = 1, 9                                                   DISK0352
  100 NMAX(I,J) = 0                                                     DISK0353
      READ(5,900,END=500) C                                             DISK0354
  900 FORMAT(F10.0)                                                     DISK0355
    1 READ(5,901) M1, L1, N1, C1                                        DISK0356
  901 FORMAT(2I1, I2, F10.0)                                            DISK0357
      IF(M1 .EQ. 9) GO TO 500                                           DISK0358
      IF(C1) 3,4,3                                                      DISK0359
    3 IF(ABS(C1 - C) .GE. .001) GO TO 501                               DISK0360
    4 IF(MOD(L1-M1,2) - MOD(N1,2)) 501,7,501                            DISK0361
    7 M = M1 + 1                                                        DISK0362
      L = L1 + 1                                                        DISK0363
      NLIM = (N1 + 2) / 2                                               DISK0364
      NMAX(M,L) = N1 + 1                                                DISK0365
      READ(5,902) ( D(M,L,I), I = 1, NLIM)                              DISK0366
  902 FORMAT(6E12.7)                                                    DISK0367
      GO TO 1                                                           DISK0368
  501 WRITE(6,903) C,M1,L1,N1,C1                                        DISK0369
  903 FORMAT('-****ERROR****    C = ', F5.2, ' LAST CARD READ WAS:'  /  DISK0370
     &  1X, 2I1,I2,F10.2)                                               DISK0371
      STOP                                                              DISK0372
  500 RETURN                                                            DISK0373
      END                                                               DISK0374
```

ROUTINE: JML

USE:

```
CALL JML(M, L, KA, ZJ)
```

where

M	is $m + 1$,
L	is $\ell + 1$,
KA	is the real value of c = KA,
ZJ	is the returned complex value of $J_{m\ell}(c)$ or $J'_{m\ell}(c)$ as (L-M) is even or odd.

COMMENTS:

These quantities are described in equations (2.9a, b).

```
      SUBROUTINE JML(M,L,KA,ZANS)                                       DISK0375
C                                                                       DISK0376
      COMPLEX ZANS                                                      DISK0377
      REAL KA                                                           DISK0378
      DATA PIO2 /1.570796/                                              DISK0379
      CALL QSTAR(M,L,KA,Q)                                              DISK0380
      ZANS = CMPLX(1.,-PIO2*Q)                                          DISK0381
      ZANS = CMPLX(1.,0.) / ZANS                                        DISK0382
      RETURN                                                            DISK0383
      END                                                               DISK0384
```

ROUTINES: KML and KML3

USE:

CALL KML(M, L, KA, ANS) and

CALL KML3(M, L, KA, ZK3)

where

M is m + 1,

L is $\ell + 1$,

KA is the real value of c = KA,

ANS is the returned value of $K_{m\ell}^{(1)}(c)$ or $K_{m\ell}^{(1)}{}'(c)$ as (L-M) is even or odd,

ZK3 is the returned value of $K_{m\ell}^{(3)}(c)$ or $K_{m\ell}^{(3)}{}'(c)$ as (L-M) is even or odd.

COMMENTS:

These quantities are described in equations (2.10a, b) and (2.11a, b).

```
      SUBROUTINE KML(M,L,KA,ANS)                                        DISK0385
      COMMON /ARRAYC/ C(9,9,12)                                         DISK0386
      COMMON /ARRAYN/ NMAX(9,9)                                         DISK0387
      REAL*4 KA                                                         DISK0388
      LIM = (NMAX(M,L) + 1) / 2                                         DISK0389
      SUM = 0.                                                          DISK0390
      DO 20 K = 2, LIM                                                  DISK0391
   20 SUM = SUM + C(M,L,K)                                              DISK0392
      CALL RML(M,L,KA,R)                                                DISK0393
      ANS = C(M,L,1) * (SUM + 1.0) / R                                  DISK0394
      RETURN                                                            DISK0395
      END                                                               DISK0396

      SUBROUTINE KML3(M,L,KA,ZK3)                                       DISK0397
      COMMON /ARRAYK/ K(9,9)                                            DISK0398
      COMMON /ARRAYJ/ ZJ(9,9)                                           DISK0399
      REAL K, KA                                                        DISK0400
      COMPLEX ZK3,ZJ                                                    DISK0401
      ZK3 = CMPLX(K(M,L), 0.0) * ZJ(M,L)                                DISK0402
      RETURN                                                            DISK0403
      END                                                               DISK0404
```

ROUTINE: NML

USE:

```
CALL NML(M, L, ANS)
```

where

- M is $m + 1$,
- L is $\ell + 1$,
- ANS is the returned value of the normalization or orthogonality constant, $N_{m\ell}(c)$.

COMMENTS:

These constants are given by equation (2.5).

```
      SUBROUTINE NML(M,L,ANS)                                           DISK0405
C                                                                       DISK0406
      COMMON /ARRAYD/ D(9,9,12)                                         DISK0407
      COMMON /ARRAYN/ NMAX(9,9)                                         DISK0408
      COMMON /FCTRL/ FACT(57)                                           DISK0409
      N1 = 1+ MOD(L-M,2)                                                DISK0410
      LIM = NMAX(M,L)                                                   DISK0411
      I = 0                                                             DISK0412
      SUM = 0.                                                          DISK0413
      M23 = 2*M - 3                                                     DISK0414
      M22 = 2*M - 2                                                     DISK0415
      DO 10 N = N1, LIM, 2                                              DISK0416
      N2 = 2*N                                                          DISK0417
      I = I + 1                                                         DISK0418
      D1 = D(M,L,I)                                                     DISK0419
   10 SUM = SUM + (FACT(N+M22)/FACT(N)) * D1*D1 / FLOAT(N2 + M23)       DISK0420
      ANS = 2. * SUM                                                    DISK0421
      RETURN                                                            DISK0422
      END                                                               DISK0423
```

ROUTINE: PEH

USE:

```
CALL PEH(KA, X, ZPE, ZPH)
```

where

KA is the real value of $c = KA$,

X is the double precision value of an angle in degrees,

ZPE is the returned complex value of $P_E(c, X)$,

ZPH is the returned complex value of $P_H(c, X)$.

COMMENTS:

The returned quantities are given by the sum over m in equations (2.8) and (2.15).

All otherwise undefined quantities (e.g., $J_{-1,-1}(-ic)$) are set to zero.

```
      SUBROUTINE PEH(KA,X,ZPE,ZPH)                                      DISK0424
C                                                                       DISK0425
      IMPLICIT COMPLEX (Z)                                              DISK0426
      REAL*4 KA                                                         DISK0427
      REAL*8 X                                                          DISK0428
      COMMON /ARRAYJ/ ZJ(9,9)                                           DISK0429
      COMMON /ARRAYO/ XN(9,9)                                           DISK0430
      COMMON /ARRAYS/ S(9,9)                                            DISK0431
      DATA ZERO /(0.,0.)/, ZTWO /(2.,0.)/, ZFOUR /(4.,0.)/              DISK0432
      ZPE = ZERO                                                        DISK0433
      ZPH = ZERO                                                        DISK0434
      DO 100 M = 1, 9                                                   DISK0440
      M1 = M - 1                                                        DISK0441
      IF(M1) 5,5,10                                                     DISK0442
5     ZDEL = ZTWO                                                       DISK0443
      ZDEL3 = ZERO                                                      DISK0444
      GO TO 15                                                          DISK0445
10    ZDEL = ZFOUR                                                      DISK0446
      ZDEL3 = ZTWO                                                      DISK0447
15    CALL ABMEH(M,KA,X,ZAE,ZAH,ZBE,ZBH)                                DISK0448
      Z1E = ZERO                                                        DISK0449
      Z1H = ZERO                                                        DISK0450
      DO 20 L = M, 9                                                    DISK0451
      S2 = S(M,L)                                                       DISK0452
      S2 = S2 * S2                                                      DISK0453
      IF(MOD(L-M,2)) 21,24,21                                           DISK0454
21    Z1H = Z1H + ZJ(M,L) * CMPLX(S2/XN(M,L), 0.0)                      DISK0455
      GO TO 20                                                          DISK0456
24    Z1E = Z1E + ZJ(M,L) * CMPLX(S2/XN(M,L), 0.0)                      DISK0457
20    CONTINUE                                                          DISK0458
      Z2E = ZERO                                                        DISK0459
      Z2H = ZERO                                                        DISK0460
      IF(M .GE. 9) GO TO 40                                             DISK0461
```

```
C       LISTING OF ROUTINE "PEH", CONTINUED.                                DISK0462
C                                                                           DISK0463
        MP1 = M + 1                                                         DISK0464
        DO 30 L = MP1, 9                                                    DISK0465
        S2 = S(MP1,L)                                                       DISK0466
        S2 = S2 * S2                                                        DISK0467
        IF(MOD(L-MP1,2)) 31,34,31                                           DISK0468
31      Z2E = Z2E + ZJ(MP1,L) * CMPLX(S2/XN(MP1,L), 0.0)                    DISK0469
        GO TO 30                                                            DISK0470
34      Z2H = Z2H + ZJ(MP1,L) * CMPLX(S2/XN(MP1,L), 0.0)                    DISK0471
30      CONTINUE                                                            DISK0472
40      Z3E = ZERO                                                          DISK0473
        Z3H = ZERO                                                          DISK0474
        IF(M .LE. 1) GO TO 65                                               DISK0475
        DO 50 L = M1, 9                                                     DISK0476
        S2 = S(M1,L)                                                        DISK0477
        S2 = S2 * S2                                                        DISK0478
        IF(MOD(L-M1,2)) 51,54,51                                            DISK0479
51      Z3E = Z3E + ZJ(M1,L) * CMPLX(S2/XN(M1,L), 0.0)                      DISK0480
        GO TO 50                                                            DISK0481
54      Z3H = Z3H + ZJ(M1,L) * CMPLX(S2/XN(M1,L), 0.0)                      DISK0482
50      CONTINUE                                                            DISK0483
65      ZTERME = ZDEL*ZBE*Z1E + ZAE * (ZTWO*Z2E+ZDEL3*Z3E)                  DISK0484
        ZTERMH = ZDEL*ZBH*Z1H + ZAH * (ZTWO*Z2H+ZDEL3*Z3H)                  DISK0485
        SIGN = 1.0                                                          DISK0486
        IF(MOD(M,2) .EQ. 0) SIGN = -1.0                                     DISK0487
        ZPE = ZPE + SIGN * ZTERME                                           DISK0488
        ZPH = ZPH + SIGN * ZTERMH                                           DISK0489
100     CONTINUE                                                            DISK0490
        END                                                                 DISK0493
```

ROUTINE: QSTAR

USE:

```
CALL QSTAR(M, L, KA, Q)
```

where

M	is $m + 1$,
L	is $\ell + 1$,
KA	is the real (single precision) value of c = KA,
Q	is the returned value of $Q^*_{m\ell}(-ic)$.

COMMENTS:

The $Q^*_{m\ell}(-ic)$ are described in equations (2.20a, b).

```
      SUBROUTINE QSTAR(M,L,KA,Q)                                        DISK0494
C                                                                       DISK0495
      COMMON /FCTRL/ FACT(57)                                           DISK0496
      COMMON /ARRAYA/ ALF(9)                                            DISK0497
      COMMON /ARRAYK/ K(9,9)                                            DISK0498
      REAL KA,K                                                         DISK0499
      INTEGER R                                                         DISK0500
      CALL ALPHA(M,L)                                                   DISK0501
      SUM = 0.                                                          DISK0502
      IF(MOD(L-M,2)) 21,10,21                                           DISK0503
10    DO 15 R = 1, M                                                    DISK0504
15    SUM = SUM + ALF(R) * FACT(2*M-2*R+1) / ((2**(M-R) * FACT(M-R+1))  DISK0505
     &    **2)                                                          DISK0506
      GO TO 100                                                         DISK0507
21    DO 25 R = 1, M                                                    DISK0508
25    SUM = SUM + ALF(R) * FACT(2*M-2*R+2) / ((2**(M-R) * FACT(M-R+1))  DISK0509
     &    **2)                                                          DISK0510
100   XK2 = K(M,L)                                                      DISK0511
      XK2 = XK2 * XK2                                                   DISK0512
      Q = XK2 * SUM / KA                                                DISK0513
      IF(MOD(L-M,2) .NE. 0) Q = -Q                                      DISK0514
      RETURN                                                            DISK0515
      END                                                               DISK0516
```

ROUTINE: RCS

USE:

CALL RCS(X1, X2, DX)

where

X1 is the double precision value of the initial angle in degrees,

X2 is the double precision value of the final angle in degrees,

DX is the double precision value of the step size in degrees.

COMMENTS:

RCS is the main routine of the disk scattering program. It first performs all necessary initialization, and then computes the radar cross sections

$$10 \log_{10}\left[\left(\frac{\sigma_E}{\lambda^2}\right)\right], \quad 10 \log_{10}\left[\left(\frac{\sigma_H}{\lambda^2}\right)\right] \quad \text{and } 10 \log_{10}\left[\left(\frac{\sigma_C}{\lambda^2}\right)\right],$$

and writes these values on I/0 unit 6.

The angle X1 must be greater than 0.° The program has been tested only for values of X2 less than 90.°

```
      SUBROUTINE RCS(X1,X2,DX)                                          DISK0517
C                                                                       DISK0518
      IMPLICIT COMPLEX (Z)                                              DISK0519
      REAL*4 KA                                                         DISK0520
      REAL*8 X1,X2,DX,X                                                 DISK0521
      DATA SQTPIR /0.5641896/, HSPR/0.2820948/                          DISK0522
      DATA ISET /0/                                                     DISK0523
      IF(ISET) 150,100,150                                              DISK0524
100   CALL SET                                                          DISK0525
      ISET = 1                                                          DISK0526
150   CALL GETD(KA)                                                     DISK0527
      CALL GETCN                                                        DISK0528
      CALL FILLK(KA)                                                    DISK0529
      CALL FILLJO(KA)                                                   DISK0530
      WRITE(6,901) X1,X2,DX,KA                                          DISK0531
      X = X1                                                            DISK0532
      LINE = 0                                                          DISK0533
      WRITE(6,900)                                                      DISK0534
1     CALL FILLS(X)                                                     DISK0535
      CALL PEH(KA,X,ZPE,ZPH)                                            DISK0536
      PE = CABS(ZPE)                                                    DISK0537
      PH = CABS(ZPH)                                                    DISK0538
      ZPCR = ZPE - ZPH                                                  DISK0539
      PCR = CABS(ZPCR)                                                  DISK0540
      E = 20. * ALOG10(SQTPIR * PE)                                     DISK0541
      H = 20. * ALOG10(SQTPIR * PH)                                     DISK0542
      CR = 20. * ALOG10(HSPR * PCR)                                     DISK0543
      IF(MOD(LINE,5)) 15,10,15                                          DISK0544
10    WRITE(6,902)                                                      DISK0545
15    WRITE(6,903) X,E,H,CR,ZPE,ZPH,ZPCR                                DISK0546
903   FORMAT(F6.2,3F9.2, 3(3X,2G13.6))                                  DISK0547
      LINE = LINE + 1                                                   DISK0548
      X = X + DX                                                        DISK0549
      IF(X-X2 .LE. 1.D-2) GO TO 1                                       DISK0550
```

```
C      LISTING OF ROUTINE "RCS", CONTINUED.                             DISK0551
C                                                                       DISK0552
       RETURN                                                           DISK0553
 900   FORMAT('1ANGLE     E-POL     H-POL    CR-POL      RE(PE)       IM(PE)DISK0554
      &           RE(PH)         IM(PH)          RE(PE-PH)     IM(PE-PH)')  DISK0555
 901   FORMAT('1',/////// '-RADAR CROSS SECTION OF A DISK' // 6X,'INPUT PADISK0556
      &RAMETERS:' / 10X, 'INITIAL ANGLE IS', F7.2, ' DEGREES' / 10X,    DISK0557
      &'FINAL   ANGLE IS', F7.2, ' DEGREES' / 10X, 'IN INCREMENTS OF',  DISK0558
      &F7.2, ' DEGREES' / 10X, 'VALUE OF C IS    ',  F7.2)              DISK0559
 902   FORMAT(' ')                                                      DISK0560
       END                                                              DISK0561
```

ROUTINE: RML

USE:

```
CALL RML(M, L, KA, R)
```

where

- M is $m + 1$,
- L is $\ell + 1$,
- KA is the real (single precision) value of $c = KA$,
- R is the returned value of $R^{(1)}_{m\ell}(-ic, 0)$ or $R^{(1)\prime}_{m\ell}(-ic, 0)$, depending on whether (L-M) is even or odd.

COMMENTS:

These quantities are described in equations (2.23a, b).

```
      SUBROUTINE RML(M,L,C,ANS)                                         DISK0562
C                                                                       DISK0563
      COMMON /FCTRL/ FACT(57)                                           DISK0564
      COMMON /ARRAYD/ D(9,9,12)                                         DISK0565
      X = FACT(L-M+1) * FACT(M) / FACT(L+M-1) * 2**(M-1) * D(M,L,1)     DISK0566
      IF(MOD(L-M,2)) 21,10,21                                           DISK0567
   10 ANS = C**(M-1) * X / FLOAT(2*M-1)                                 DISK0568
      IF(MOD((L-M)/2,2) .NE. 0) ANS = -ANS                              DISK0569
      GO TO 500                                                         DISK0570
   21 ANS = C**M * X / FLOAT(2*M+1)                                     DISK0571
      IF(MOD((L-M-1)/2,2) .NE. 0) ANS = -ANS                            DISK0572
  500 RETURN                                                            DISK0573
      END                                                               DISK0574
```

ROUTINE: SET

USE:

```
CALL SET
```

COMMENTS:

This routine calculates factorials for use in other routines. Single precision factorials are returned in the common array

```
COMMON /FCTRL/ FACT(57)
```

and double precision factorials are returned in the common array

```
COMMON /DFCTRL/ DFACT(57).
```

In both cases, the value of n! is contained in the n+1 st array element. The value of n is restricted to $0 \leq n \leq 56$.

```
      SUBROUTINE SET                                                    DISK0575
C                                                                       DISK0576
      COMMON /DFCTRL/ DFACT                                             DISK0577
      COMMON /FCTRL/ FACT                                               DISK0578
      REAL*8 DFACT(57), X                                               DISK0579
      REAL*4 FACT(57), Y                                                DISK0580
      DFACT(1) = 1.0D0                                                  DISK0581
      FACT(1) = 1.0                                                     DISK0582
      X = 0.0D0                                                         DISK0583
      Y = 0.0                                                           DISK0584
      DO 10 I = 2, 57                                                   DISK0585
      X = X + 1.0D0                                                     DISK0586
      Y = Y + 1.0                                                       DISK0587
      DFACT(I) = DFACT(I-1) * X                                         DISK0588
10    FACT(I) = FACT(I-1) * Y                                           DISK0589
      RETURN                                                            DISK0590
      END                                                               DISK0591
```

ROUTINE: SML

USE:

```
CALL SML(M, L, X, S, TERM)
```

where

M is $m + 1$,

L is $\ell + 1$,

X is a double precision value in degrees,

S is the returned value of $S_{m\ell}(-ic, \eta)$, where $\eta = \cos X$,

TERM is the returned value of the last term in the truncated infinite series for $S_{m\ell}(-ic, \eta)$.

COMMENTS:

These quantities are described in equation (2.3).

```
      SUBROUTINE SML(M,L,X,S,TERM)                                      DISK0592
C                                                                       DISK0593
      COMMON /ARRAYN/ NMAX                                              DISK0594
      COMMON /ARRAYD/ D                                                 DISK0595
      DIMENSION D(9,9,12),NMAX(9,9)                                     DISK0596
      M1 = M - 1                                                        DISK0597
      N1 = 1 + MOD(L-M,2)                                               DISK0598
      SUM = 0.                                                          DISK0599
      I = 0                                                             DISK0600
      LIM = NMAX(M,L)                                                   DISK0601
      DO 10 N = N1, LIM, 2                                              DISK0602
      I = I + 1                                                         DISK0603
      CALL ALGNDR(N+M1,M,X,P)                                           DISK0604
      A = D(M,L,I) * P                                                  DISK0605
   10 SUM = SUM + A                                                     DISK0606
      S = SUM                                                           DISK0607
      TERM = A                                                          DISK0608
      RETURN                                                            DISK0609
      END                                                               DISK0610
```

TABLE D.1: Computed Values of the $N_{m\ell}$; c = 4.0.

m	ℓ=m	ℓ=m+1	ℓ=m+2	ℓ=m+3	ℓ=m+4	ℓ=m+5	ℓ=m+6	ℓ=m+7	ℓ=m+8	m
0	3.01944 -01	2.86300 -01	5.35684 -01	2.78851 -01	2.28613 -01	1.83212 -01	1.54446 -01	1.33615 -01	1.17794 -01	0
1	1.35184 -01	7.93609 -01	2.31898 +00	3.17825 +00	4.32010 +00	5.43325 +00	6.53625 +00	7.62515 +00		1
2	1.31711 +00	1.13096 +01	4.16236 +01	9.55735 +01	1.82917 +02	3.09306 +02	4.80918 +02			2
3	4.02492 +01	3.99638 +02	1.82122 +03	5.55766 +03	1.35837 +04	2.86467 +04				3
4	2.29307 +03	2.58174 +04	1.40912 +05	5.25775 +05	1.55329 +06					4
5	2.05663 +05	2.61883 +06	1.66664 +07	7.30644 +07						5
6	2.66495 +07	3.81786 +08	2.77795 +09							6
7	4.71717 +09	7.54895 +10								7
8	1.09507 +12									8

TABLE D.2: Computed Values of the $J_{m\ell}$; c = 4.0. When (ℓ - m) is Odd $J_{m\ell}$ is Identically Equal to Zero.

m		ℓ=m	ℓ=m+2	ℓ=m+4	ℓ=m+6	ℓ=m+8		m
0	Re	9.99211 -01	3.42953 -01	8.77859 -05	1.23782 -10	1.31852 -17	Re	0
	Im	2.80788 -02	4.74696 -01	9.36900 -03	1.11258 -05	3.63114 -09	Im	
1	Re	9.36914 -01	2.02982 -02	2.03551 -07	6.67737 -14		Re	1
	Im	2.43120 -01	1.41018 -01	4.51166 -04	2.58406 -07		Im	
2	Re	5.56934 -01	2.96440 -04	3.48214 -10	3.08583 -17		Re	2
	Im	4.96748 -01	1.72149 -02	1.86605 -05	5.55503 -09		Im	
3	Re	1.47108 -01	2.16126 -06	4.50728 -13			Re	3
	Im	3.54213 -01	1.47012 -03	6.71363 -07			Im	
4	Re	1.57472 -02	9.88357 -09	4.40904 -16			Re	4
	Im	1.24496 -01	9.94162 -05	2.09977 -08			Im	
5	Re	6.47630 -04	2.96546 -11				Re	5
	Im	2.54403 -02	5.44561 -06				Im	
6	Re	1.08692 -05	5.97020 -14				Re	6
	Im	3.29682 -03	2.44340 -07				Im	
7	Re	8.71379 -08					Re	7
	Im	2.95191 -04					Im	
8	Re	3.82533 -10					Re	8
	Im	1.95585 -05					Im	

TABLE D.3: Computed Values of the $J'_{m\ell}$; c = 4.0. When (ℓ - m) is Even $J'_{m\ell}$ is Identically Equal to Zero.

m		ℓ=m+1	ℓ=m+3	ℓ=m+5	ℓ=m+7	ℓ=m+9		m
0	Re	9.99117 -01	2.34236 -02	1.54044 -07	5.29698 -14		Re	0
	Im	-2.97216 -02	-1.51244 -01	-3.92484 -04	-2.30152 -07		Im	
1	Re	8.60676 -01	1.23395 -04	1.59806 -10	1.62796 -17		Re	1
	Im	-3.46285 -01	-1.11076 -02	-1.26414 -05	-4.03480 -09		Im	
2	Re	8.90760 -02	4.82366 -07	1.35059 -13			Re	2
	Im	-2.84854 -01	-6.94526 -04	-3.67504 -07			Im	
3	Re	1.71770 -03	1.33709 -09	9.14819 -17			Re	3
	Im	-4.14096 -02	-3.65663 -05	-9.56462 -09			Im	
4	Re	2.17955 -05	2.62702 -12				Re	4
	Im	-4.66851 -03	-1.62081 -06				Im	
5	Re	1.69702 -07	3.68423 -15				Re	5
	Im	-4.11950 -04	-6.06979 -08				Im	
6	Re	7.97290 -10					Re	6
	Im	-2.82363 -05					Im	
7	Re	2.33184 -12					Re	7
	Im	-1.52704 -06					Im	
8	Re						Re	8
	Im						Im	

TABLE D. 4: Computed Values of the $K_{m\ell}^{(1)}$; c = 4.0. When (ℓ - m) is Odd $K_{m\ell}^{(1)}$ is Identically Equal to Zero.

m	ℓ=m	ℓ=m+2	ℓ=m+4	ℓ=m+6	ℓ=m+8	m
0	2.67504 -01	-1.87742 +00	1.64856 +01	-4.78416 +02	2.64819 +04	0
1	4.49342 -01	-1.09092 +01	3.63493 +02	-2.19328 +04		1
2	2.24319 +00	-1.59099 +02	1.26205 +04	-1.36436 +06		2
3	2.17631 +01	-4.01265 +03	6.14827 +05			3
4	3.58824 +02	-1.47948 +05	3.95432 +07			4
5	9.09732 +03	-7.40904 +06				5
6	3.29894 +05	-4.82454 +08				6
7	1.62300 +07					7
8	1.04112 +09					8

TABLE D.5: Computed Values of the $K^{(1)\prime}_{m\ell}$; c = 4.0. When (ℓ - m) is Even $K^{(1)\prime}_{m\ell}$ is Identically Equal to Zero.

m	ℓ=m+1	ℓ=m+3	ℓ=m+5	ℓ=m+7	ℓ=m+9	m
0	2.75232 -01	-4.05493 +00	8.05489 +01	-3.32631 +03		0
1	1.61564 +00	-5.65928 +01	2.65734 +03	-2.01991 +05		1
2	1.76146 +01	-1.36485 +03	1.26106 +05			2
3	3.11178 +02	-4.92092 +04	7.99710 +06			3
4	8.22116 +03	-2.44114 +06				4
5	3.05946 +05	-1.58419 +08				5
6	1.53127 +07					6
7	9.94077 +08					7
8						8

TABLE D.6: Computed Values of the $C_{2k}^{m\ell}$ for $0 \leq \ell, m \leq 8$; $c = 4.0$.

2k	m=0, ℓ=0	m=0, ℓ=1	m=0, ℓ=2	m=0, ℓ=3	m=0, ℓ=4	m=0, ℓ=5	2k
0	1.00000 +00	1.00000 +00	1.00000 +00	1.00000 +00	1.00000 +00	1.00000 +00	0
2	-1.71230 +00	-1.24357 +00	-4.05535 +00	-4.58477 +00	-7.09304 +00	-9.05448 +00	2
4	1.09088 +00	6.09385 -01	3.59071 +00	3.38955 +00	1.09179 +01	1.58369 +01	4
6	-3.62524 -01	-1.62934 -01	-1.42549 +00	-1.12806 +00	-5.69152 +00	-7.63939 +00	6
8	7.36102 -02	2.75326 -02	3.23501 -01	2.18829 -01	1.51755 +00	1.83664 +00	8
10	-1.00450 -02	-3.20476 -03	-4.76286 -02	-2.80492 -02	-2.48568 -01	-2.71226 -01	10
12	9.81254 -04	2.68510 -04	4.91521 -03	2.56038 -03	2.77103 -02	2.73837 -02	12
14	-7.07229 -05	-1.45616 -05	-3.68945 -04	-1.72903 -04	-2.24284 -03	-1.99184 -03	14
16	3.33636 -06		1.78998 -05	7.77808 -06	1.35936 -04	9.43467 -05	16
18					-5.49739 -06		18

TABLE D.6 continued

2k	m=0, ℓ=6	m=0, ℓ=7	m=0, ℓ=8	m=1, ℓ=1	m=1, ℓ=2	m=1, ℓ=3	2k
0	1.00000 +00	1.00000 +00	1.00000 +00	1.00000 +00	3.00000 +00	6.00000 +00	0
2	-1.25381 +01	-1.55281 +01	-2.00216 +01	-1.38635 +00	-3.04880 +00	-1.45544 +01	2
4	3.55994 +01	5.15756 +01	9.37084 +01	7.29676 +00	1.25433 +00	9.70406 +00	4
6	-3.53896 +01	-5.57729 +01	-1.65304 +02	-2.05070 -01	-2.87974 -01	-3.11402 +00	6
8	1.34079 +01	1.99636 +01	1.21800 +02	3.59428 -02	4.25530 -02	5.94224 -01	8
10	-2.73303 +00	-3.75558 +00	-3.62982 +01	-4.30613 -03	-4.40015 -03	-7.55256 -02	10
12	3.53856 -01	4.47602 -01	5.99303 +00	3.74505 -04	3.36444 -04	6.85953 -03	12
14	-3.19344 -02	-3.72988 -02	-6.41861 -01	-2.43340 -05	-1.95707 -05	-4.60427 -04	14
16	2.09928 -03	2.28105 -03	4.87320 -02	1.05028 -06	7.79559 -07	2.03285 -05	16
18	-8.98435 -05	-9.26557 -05	-2.73616 -03				18
20			1.01829 -04				20

TABLE D. 6 continued

2k	m=1, ℓ=4	m=1, ℓ=5	m=1, ℓ=6	m=1, ℓ=7	m=1, ℓ=8	m=2, ℓ=2	2k
0	1.00000 +01	1.50000 +01	2.10000 +01	2.80000 +01	3.60000 +01	3.00000 +00	0
2	-2.82859 +01	-6.83366 +01	-1.16343 +02	-2.17850 +02	-3.33854 +02	-3.25690 +00	2
4	1.68363 +01	8.53019 +01	1.60985 +02	4.92877 +02	8.61278 +02	1.40103 +00	4
6	-4.73857 +00	-3.77888 +01	-6.66883 +01	-4.24231 +02	-7.96539 +02	-3.32090 -01	6
8	7.98281 -01	8.76867 +00	1.41252 +01	1.42286 +02	2.53801 +02	5.02592 -02	8
10	-9.04938 -02	-1.27136 +00	-1.86682 +00	-2.60232 +01	-4.31565 +01	-5.29413 -03	10
12	7.40692 -03	1.27120 -01	1.70659 -01	3.05508 +00	4.69673 +00	4.10690 -04	12
14	-4.53708 -04	-9.32721 -03	-1.13503 -02	-2.52183 -01	-3.60232 -01	-2.41062 -05	14
16	1.87494 -05	5.17384 -04	4.96890 -04	1.52846 -02	2.04182 -02	9.54314 -07	16
18		-1.93699 -05		-6.08990 -04	-7.74821 -04		18

TABLE D.6 continued

2k	m=2, ℓ=3	m=2, ℓ=4	m=2, ℓ=5	m=2, ℓ=6	m=2, ℓ=7	m=2, ℓ=8	2k
0	1.50000 +01	4.50000 +01	1.05000 +02	2.10000 +02	3.78000 +02	6.30000 +02	0
2	-1.25576 +01	-8.87741 +01	-2.37376 +02	-7.84548 +02	-1.65802 +03	-3.91131 +03	2
4	4.37868 +00	4.93351 +01	1.20217 +02	8.60895 +02	1.98358 +03	7.70994 +03	4
6	-8.70801 -01	-1.35373 +01	-2.94968 +01	-3.35926 +02	-7.27629 +02	-5.99037 +03	6
8	1.13381 -01	2.25384 +00	4.40556 +00	6.94083 +01	1.38348 +02	1.81546 +03	8
10	-1.04721 -02	-2.53927 -01	-4.48538 -01	-9.05733 +00	-1.65881 +01	-3.02084 +02	10
12	7.23226 -04	2.07066 -02	3.33182 -02	8.22766 -01	1.38907 +00	3.24980 +01	12
14	-3.83750 -05	-1.26225 -03	-1.86934 -03	-5.52937 -02	-8.64613 -02	-2.47456 +00	14
16	1.41088 -06	5.13217 -05	7.15361 -05	2.83096 -03	4.12951 -03	1.39257 -01	16
18				-9.88037 -05	-1.36890 -04	-5.19657 -03	18

TABLE D. 6 continued

2k	m=3, ℓ=3	m=3, ℓ=4	m=3, ℓ=5	m=3, ℓ=6	m=3, ℓ=7	m=3, ℓ=8	2k
0	1.50000 +01	1.05000 +02	4.20000 +02	1.26000 +03	3.15000 +03	6.93000 +03	0
2	-1.30307 +01	-7.38281 +01	-7.43359 +02	-2.49614 +03	-1.04929 +04	-2.64307 +04	2
4	4.66416 +00	2.21587 +01	3.59758 +02	1.10913 +03	1.05193 +04	2.85573 +04	4
6	-9.45793 -01	-3.86504 +00	-8.68360 +01	-2.41940 +02	-3.69014 +03	-9.45236 +03	6
8	1.25005 -01	4.47923 -01	1.28748 +01	3.25024 +01	6.88871 +02	1.63408 +03	8
10	-1.16838 -02	-3.72622 -02	-1.30608 +00	-3.00571 +00	-8.18291 +01	-1.79496 +02	10
12	8.14537 -04	2.34055 -03	9.68166 -02	2.04486 -01	6.81560 +00	1.38630 +01	12
14	-4.34587 -05	-1.13952 -04	-5.41369 -03	-1.05881 -02	-4.22739 -01	-8.00562 -01	14
16	1.58623 -06	3.88676 -06	2.04320 -04	3.77607 -04	2.01050 -02	3.56744 -02	16
18					-6.57678 -04	-1.11166 -03	18

TABLE D.6 continued

2k	m=4, ℓ=4	m=4, ℓ=5	m=4, ℓ=6	m=4, ℓ=7	m=4, ℓ=8	m=5, ℓ=5	2k
0	1.05000 +02	9.45000 +02	4.72500 +03	1.73250 +04	5.19750 +04	9.45000 +02	0
2	-7.54405 +01	-5.69636 +02	-7.78369 +03	-3.13929 +04	-1.60504 +05	-5.77352 +02	2
4	2.30075 +01	1.49518 +02	3.35014 +03	1.24721 +04	1.50282 +05	1.53116 +02	4
6	-4.06271 +00	-2.31626 +01	-7.22717 +02	-2.45176 +03	-4.80370 +04	-2.39146 +01	6
8	4.75435 -01	2.41355 +00	9.66326 +01	2.99425 +02	8.18784 +03	2.50855 +00	8
10	-3.98638 -02	-1.82213 -01	-8.91706 +00	-2.53696 +01	-8.93013 +02	-1.90571 -01	10
12	2.51984 -03	1.03527 -02	6.05958 -01	1.59218 +00	6.86917 +01	1.10122 -02	12
14	-1.23097 -04	-4.01232 -04	-3.12994 -02	-7.65497 -02	-3.95639 +00	-4.95899 -04	14
16	4.16688 -06		1.10284 -03	2.55736 -03	1.75704 -01	1.56525 -05	16
18					-5.41116 -03		18

TABLE D. 6 continued

2k	m=5, ℓ=6	m=5, ℓ=7	m=5, ℓ=8	m=6, ℓ=6	m=6, ℓ=7	m=6, ℓ=8	2k
0	1.03950 +04	6.23700 +04	2.70270 +05	1.03950 +04	1.35135 +05	9.45945 +05	0
2	-5.46951 +03	-9.74069 +04	-4.58623 +05	-5.51829 +03	-6.30032 +04	-1.41645 +06	2
4	1.27328 +03	3.77923 +04	1.65061 +05	1.29368 +03	1.31642 +04	5.00458 +05	4
6	-1.77169 +02	-7.36992 +03	-2.95423 +04	-1.81034 +02	-1.66128 +03	-8.90266 +04	6
8	1.67528 +01	8.97122 +02	3.30751 +03	1.71994 +01	1.43703 +02	9.94320 +03	8
10	-1.15762 +00	-7.59066 +01	-2.58567 +02	-1.19401 +00	-9.15021 +00	-7.76576 +02	10
12	6.06797 -02	4.76071 +00	1.50595 +01	6.35247 -02	4.45032 -01	4.52105 +01	12
14	-2.19290 -03	-2.28451 -01	-6.75736 -01	-2.65276 -03	-1.50679 -02	-2.02547 +00	14
16		7.55073 -03	2.12398 -02	7.84514 -05		6.30543 -02	16

TABLE D.6 continued

2k	m=7, ℓ=7	m=7, ℓ=8	m=8, ℓ=8	2k
0	1.35135 +05	2.02702 +06	2.02702 +06	0
2	-6.33916 +04	-8.47796 +05	-8.51555 +05	2
4	1.33114 +04	1.60596 +05	1.61899 +05	4
6	-1.68678 +03	-1.85338 +04	-1.87415 +04	6
8	1.46411 +02	1.47683 +03	1.49728 +03	8
10	-9.34866 +00	-8.71681 +01	-8.85625 +01	10
12	4.55080 -01	3.95357 +00	4.01891 +00	12
14	-1.52873 -02	-1.25937 -01	-1.27080 -01	14