RAYLEIGH SCATTERING CROSS SECTIONS

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Abstract

The significance of the electric and magnetic polarizability tensors in low frequency scattering is emphasized. In the particular case of perfectly conducting, rotationally symmetric bodies with plane wave illumination, it is shown how the entire Rayleigh scattered field can be expressed in terms of just three tensor elements, functions only of the geometry of the body. Inequalities satisfied by these elements are used to establish optimum lower bounds on the scattering cross sections and, in addition, the elements themselves are examined analytically and computationally for a variety of shapes. Some of the implications of these results are discussed.

Introduction

The usefulness of the electric and magnetic polarizability tensors has long been recognized in statics, but it has only recently been pointed out (Keller et al, 1972) that they are also applicable in scattering. For arbitrary plane wave illumination, these same tensors specify the Rayleigh scattered field in its entirety, and since they are characteristic of the body, they provide a convenient means for describing low frequency scattering. Moreover, for a metallic body of revolution all but six of the tensor elements are zero and only three are independent (Keller et al, 1972).

In the present paper the radar cross sections of perfectly conducting bodies of revolution are considered in the light of these findings. Inequalities satisfied by the three independent tensor elements are used to generate lower bounds on the cross sections in such cases as backscattering for arbitrary incidence, bistatic scattering for axial incidence and total (integrated) scattering. Each bound is an optimum in the sense that at least one body is known for which the bound is achieved. The tensor elements themselves are examined analytically and computationally for a variety of shapes, and results are presented for spheroids (prolate and oblate), ogives, lenses and spherically-capped (or rounded) cones. Particular attention is given to the dependence of the elements on the length-to-width ratio of the body, and from this it is apparent that Rayleigh cross section measurements can be used to deduce more about the geometry of a body than its volume alone.

Dipole Moments

Consider a finite, closed, perfectly conducting body illuminated by a linearly polarized, plane electromagnetic wave. The incident electric and magnetic fields can be written as

$$\underline{E}^i = \hat{a}\, e^{ik\hat{k}\cdot\underline{r}}$$
$$\underline{H}^i = Y\hat{b}\, e^{ik\hat{k}\cdot\underline{r}} \qquad (1)$$

where $\hat{k}$, $\hat{a}$, $\hat{b}$ are unit vectors specifying the direction of incidence, the electric field direction (or polarization) and the magnetic field direction, respectively. All three are mutually perpendicular and

$$\hat{b} = \hat{k}\wedge\hat{a}, \qquad \hat{a}\cdot\hat{k} = 0 \ .$$

The propagation constant is k and $Y = Z^{-1}$ is the intrinsic admittance of the free space medium (whose permittivity is ϵ) exterior to the body. Mks units are employed and a time factor $e^{-i\omega t}$ is suppressed.

For k sufficiently small, the field vectors of the incident and scattered fields can be expanded as power series in k. As regards the scattered field, it is well known that the leading term can be attributed to the field of electric and magnetic dipoles located at the origin of coordinates (within the body), and in the far zone ($r \to \infty$)

$$\underline{E}^s \sim -\frac{e^{ikr}}{4\pi r} k^2 \left\{ \frac{1}{\epsilon} \hat{r} \wedge (\hat{r} \wedge \underline{p}) + Z(\hat{r} \wedge \underline{m}) \right\} ,$$

$$\underline{H}^s \sim \frac{e^{ikr}}{4\pi r} k^2 \left\{ \frac{Y}{\epsilon} (\hat{r} \wedge \underline{p}) - \hat{r} \wedge (\hat{r} \wedge \hat{\underline{m}}) \right\} , \tag{2}$$

where $\underline{p}$ and $\underline{m}$ are the electric and magnetic dipole moments respectively, and $\hat{r}$ is a unit vector in the direction of the point of observation.

The moment $\underline{p}$ is a function of the polarization vector $\hat{a}$ as well as of the geometry of the body. As shown by Keller et al (1972), however,

$$\underline{p} = \epsilon \bar{\bar{P}} \cdot \hat{a} \tag{3}$$

where $\bar{\bar{P}}$ is a real symmetric tensor of rank 2 whose elements P_{ij} are independent of $\hat{a}$. $\bar{\bar{P}}$ is the electric polarizability tensor and is related to the polarization tensor $\bar{\bar{Q}}$ for an isolated body (Schiffer and Szegö, 1949) by the equation

$$\bar{\bar{P}} = \bar{\bar{Q}} + V\bar{\bar{I}} \tag{4}$$

where V is the volume of the body and $\bar{\bar{I}}$ is the identity tensor. In like manner,

$$\underline{m} = -Y\bar{\bar{M}} \cdot \hat{b} \tag{5}$$

where $\bar{\bar{M}}$ is also a real symmetric tensor of rank 2 whose elements M_{ij} are independent of $\hat{b}$. $\bar{\bar{M}}$ is the magnetic polarizability tensor and can be written as

$$\bar{\bar{M}} = \bar{\bar{W}} + V\bar{\bar{I}} \tag{6}$$

where $\overline{\overline{W}}$ is equivalent to the added or virtual mass tensor (Schiffer and Szegö, 1949; Payne, 1967) for the irrotational flow of an incompressible inviscid fluid past a rigid body.

The elements P_{ij} and M_{ij} (or Q_{ij} and W_{ij}) can be expressed as either volume or surface integrals over exterior potential functions (Keller et al, 1972) whose determination is usually required for the solution of low frequency scattering problems. There are, however, certain inequalities which these elements satisfy. Thus

$$\begin{aligned} Q_{ij}, W_{ij} &\geq 0 \\ Q_{ii}Q_{jj} &\geq Q_{ij}^2 \\ W_{ii}W_{jj} &\geq W_{ij}^2 \\ Q_{ii}W_{ii} &\geq V^2 \end{aligned} \tag{7}$$

(Schiffer and Szegö, 1949), where $i, j = 1, 2, 3,$ and repeated suffices do not imply summation. Also

$$\sum_{i=1}^{3} Q_{ii} \sum_{j=1}^{3} W_{jj} \geq 9V^2 \tag{8}$$

and still other inequalities are quoted by Payne (1967), but more striking are the results that obtain when the body has an axis of symmetry.

For bodies which are roll symmetric about (say) the x_3 (= z) axis of a Cartesian coordinate system (x_1, x_2, x_3) it can be shown that

$$\begin{aligned} P_{ij} &= 0 = M_{ij}, \qquad i \neq j \\ P_{22} &= P_{11} \\ M_{22} &= M_{11} \\ \text{and} \qquad M_{33} &= \frac{1}{2} P_{11} \end{aligned} \tag{9}$$

(Keller et al, 1972), provided the body is not ring-shaped (toroidal). We are now left with just three independent quantities, namely, the axial component of the electric dipole moment and a transverse component of each of the electric and magnetic dipole moments, and these are sufficient to specify the Rayleigh scattering behavior of the body in its entirety.

In addition, there are certain limitations on these three tensor elements that follow immediately from the above. Since Q_{ii} and W_{ii} are non-negative,

$$P_{33},\ M_{11},\ M_{33} \geq V \tag{10}$$

and hence, from the last of (9),

$$P_{11} \geq 2V\ . \tag{11}$$

Also, from the last of (7),

$$\begin{aligned} (P_{11}-V)(M_{11}-V) &\geq V^2\ , \\ (P_{33}-V)(M_{33}-V) &\geq V^2\ , \end{aligned} \tag{12}$$

which serve to establish lower bounds on M_{11} and P_{33} once P_{11} is determined; and other inequalities which can be deduced are (Payne, 1956):

$$\begin{aligned} 2P_{11}+P_{33} &\geq 9V\ , \\ 2M_{11}+M_{33} &\geq \frac{9V}{2}\ , \\ (P_{11}-V)(P_{11}+2P_{33}-3V) &\geq 12V^2\ , \\ (M_{11}-V)(M_{11}+2M_{33}-3V) &\geq \frac{3}{4}V^2\ . \end{aligned} \tag{13}$$

All of these are optimum in the sense that equality holds for at least one body (a sphere), and they are required for the deduction of lower bounds on the radar cross section.

There are two more inequalities which are important in the sequel. Consideration of $(P_{11} \pm M_{11})^2$ shows that

$$(P_{11} + M_{11})^2 \geq 4P_{11}M_{11} \tag{14}$$

which, in conjunction with the first of (12), yields

$$P_{11} + M_{11} \geq 4V \ . \tag{15}$$

Similarly,

$$P_{33} + M_{33} \geq 4V \tag{16}$$

and though these bounds are optimum, the optimum shape is no longer a sphere. In (16) equality obtains for an oblate spheroid having $\ell/w = 0.5$ where ℓ is the body length in the direction of the symmetry axis and w is the maximum dimension in a perpendicular direction; whereas in (14) and (15) equality is approached by a prolate spheroid as $\ell/w \to \infty$.

Radar Cross Section Bounds

For axially symmetric bodies it is a trivial matter to express the scattered fields (2) in terms of the remaining tensor elements, and the results in various special cases have been listed by Keller et al (1972). We now examine these and, by making use of the inequalities (10) through (16), deduce lower bounds on the radar cross sections. In almost all instances the bounds are optimal since bodies are known for which the limits are achieved.

From equations (2), (3), (5) and (9) the scattered electric field in the far zone is

$$\underline{E}^s \sim -\frac{e^{ikr}}{4\pi r} k^2 \Big\{ P_{11}\, \hat{r} \wedge (\hat{r} \wedge \hat{a}) + (P_{33} - P_{11})(\hat{a} \cdot \hat{z})\, \hat{r} \wedge (\hat{r} \wedge \hat{z}) - M_{11}\, \hat{r} \wedge \hat{b} - (M_{33} - M_{11})(\hat{b} \cdot \hat{z})\, \hat{r} \wedge \hat{z} \Big\} \tag{17}$$

in terms of which the radar cross section σ is

$$\sigma = 4\pi r^2 \left|E^s\right|^2 \quad . \tag{18}$$

For axial incidence the cross section is obtained by putting $\hat{k} = -\hat{z}$ (implying $\hat{a}\cdot\hat{z} = \hat{b}\cdot\hat{z} = 0$) and is

$$\sigma = \frac{k^4}{4\pi}\left\{P_{11}^2\left[1-(\hat{r}\cdot\hat{a})^2\right]+2P_{11}M_{11}\,\hat{r}\cdot\hat{z}+M_{11}^2\left[1-(\hat{r}\cdot\hat{b})^2\right]\right\} \quad . \tag{19}$$

When $\hat{r} = \hat{z}$ (backscattering):

$$\sigma = \frac{k^4}{4\pi}\left(P_{11}+M_{11}\right)^2 \tag{20}$$

and (15) now shows that

$$\sigma \geq \frac{4}{\pi}\,k^4V^2 \tag{21}$$

which is an optimum bound approached, for example, by a prolate spheroid as $\ell/w \to \infty$. When $\hat{r} = -\hat{z}$ (forward scattering):

$$\sigma = \frac{k^4}{4\pi}\left(P_{11}-M_{11}\right)^2 \tag{22}$$

and here the optimum bound is simply

$$\sigma \geq 0 \ , \tag{23}$$

attained by the above-mentioned shape. For reception perpendicular to the axis of symmetry, there are two cases to consider depending on the polarization. If $\hat{r} = \hat{a}$,

$$\sigma = \frac{k^4}{4\pi}\,M_{11}^2 \tag{24}$$

which, on using (10), yields

$$\sigma \geq \frac{1}{4\pi}\,k^4V^2 \ ; \tag{25}$$

while if $\hat{r} = \hat{b}$,

$$\sigma = \frac{k^4}{4\pi} P_{11}^2 \tag{26}$$

and on using (11),

$$\sigma \geq \frac{1}{\pi} k^4 V^2 \ . \tag{27}$$

The bounds in (25) and (27) are both optimal and are approached, respectively, by an oblate spheroid as $\ell/w \rightarrow 0$, i.e. a disk, and a prolate spheroid as $\ell/w \rightarrow \infty$.

In the case of backscattering for arbitrary incidence, the radar cross section is obtained from eqs. (17) and (18) by putting $\hat{r} = -\hat{k}$ (so that $\hat{a}\cdot\hat{r} = \hat{b}\cdot\hat{r} = 0$) and is

$$\sigma = \frac{k^4}{4\pi}\left\{\left[P_{11}+M_{11}+(P_{33}-P_{11})(\hat{a}\cdot\hat{z})^2+(M_{33}-M_{11})(\hat{b}\cdot\hat{z})^2\right]^2 + (P_{33}+M_{11}-P_{11}-M_{33})^2(\hat{a}\cdot\hat{z})^2(\hat{b}\cdot\hat{z})^2\right\} \ . \tag{28}$$

When $\hat{a}\cdot\hat{z} = 0$ (vertical polarization):

$$\sigma = \frac{k^4}{4\pi}\left\{P_{11}+M_{11}+(M_{33}-M_{11})(\hat{b}\cdot\hat{z})^2\right\}^2 \ . \tag{29}$$

If $M_{33}-M_{11} \geq 0$, then, from (15),

$$\sigma \geq \frac{k^4}{4\pi}(P_{11}+M_{11})^2 \geq \frac{4}{\pi} k^4 V^2 \tag{30}$$

which is the same as the optimum bound (21) for axial incidence. On the other hand, if $M_{33}-M_{11} < 0$ (which can occur with long thin bodies), a lower bound is obtained by taking $\hat{b}\cdot\hat{z} = 1$ in (29), giving

$$\sigma \geq \frac{k^4}{4\pi}\left(\frac{3}{2} P_{11}\right)^2 \ . \tag{31}$$

This is an equality at broadside incidence: $\hat{k}\cdot\hat{z} = 0$. Using (11), it now follows that

$$\sigma \geq \frac{9}{4\pi} k^4 V^2 \tag{32}$$

which is an optimum bound achieved by a vanishingly thin prolate spheroid viewed broadside.

When $\hat{b}\cdot\hat{z} = 0$ (horizontal polarization):

$$\sigma = \frac{k^4}{4\pi}\left\{P_{11} + M_{11} + (P_{33} - P_{11})(\hat{a}\cdot\hat{z})^2\right\}. \tag{33}$$

Since $P_{33} - P_{11}$ may vary in sign, the procedure is the same as before. If $P_{33} - P_{11} \geq 0$, then

$$\sigma \geq \frac{k^4}{4\pi}(P_{11} + M_{11})^2 \geq \frac{4}{\pi} k^4 V^2 , \tag{34}$$

c.f. (30), whereas if $P_{33} - P_{11} < 0$,

$$\sigma \geq \frac{k^4}{4\pi}(P_{33} + M_{11})^2 . \tag{35}$$

Equality obtains at broadside incidence and hence, from (10), an optimum bound is

$$\sigma \geq \frac{1}{\pi} k^4 V^2 \tag{36}$$

achieved also by a vanishingly thin prolate spheroid.

The total scattering cross section σ_T is

$$\sigma_T = \frac{1}{4\pi}\int \sigma d\Omega \tag{37}$$

where $d\Omega$ is an element of solid angle. By inserting the expression for σ deduced from eq. (17) and carrying out the integration, we find

$$\sigma_T = \frac{k^4}{6\pi}\left\{P_{11}^2 + M_{11}^2 - (P_{11}^2 - P_{33}^2)(\hat{a}\cdot\hat{z})^2 - (M_{11}^2 - M_{33}^2)(\hat{b}\cdot\hat{z})^2\right\} \tag{38}$$

valid for all angles of incidence. Two cases are of special interest. The first is vertical polarization, $\hat{a}\cdot\hat{z} = 0$, for which

$$\sigma_T = \frac{k^4}{6\pi}\left\{P_{11}^2 + M_{11}^2\left[1 - (\hat{b}\cdot\hat{z})^2\right] + M_{33}^2(\hat{b}\cdot\hat{z})^2\right\} . \tag{39}$$

Hence, using (10) and (11),

$$\sigma_T \geq \frac{5}{6\pi} k^4 V^2 \tag{40}$$

which is an optimum bound achieved by a vanishingly thin prolate spheroid when $\hat{b}\cdot\hat{z} = 1$, implying broadside incidence. Similarly, for horizontal polarization, $\hat{b}\cdot\hat{z} = 0$, eq. (38) becomes

$$\sigma_T = \frac{k^4}{6\pi}\left\{P_{11}^2\left[1 - (\hat{a}\cdot\hat{z})^2\right] + P_{33}^2(\hat{a}\cdot\hat{z})^2 + M_{11}^2\right\} \tag{41}$$

which yields

$$\sigma_T \geq \frac{1}{6\pi}\left\{5 - 3(\hat{a}\cdot\hat{z})^2\right\} k^4 V^2 . \tag{42}$$

Although this is not an optimum for all $\hat{a}\cdot\hat{z}$, nevertheless, when $\hat{a}\cdot\hat{z} = 1$ (implying broadside incidence), (42) reduces to

$$\sigma_T \geq \frac{1}{3\pi} k^4 V^2 \tag{43}$$

which is an optimum bound achieved by a disk.

In our earlier discussion of back and forward scattering, the only cases considered were those for which there is no cross polarized component of the scattered field. In general, however, some depolarization will occur, and in the back scattering direction the cross polarized cross section deduced from (17) and (18) is

$$\sigma_{cr} = \frac{k^4}{4\pi}\left\{(\hat{a}\cdot\hat{z})(\hat{b}\cdot\hat{z})(P_{33}+M_{11}-\frac{3}{2}P_{11})\right\}^2 \tag{44}$$

As expected, this is zero for $\hat{a}\cdot\hat{z} = 0$ and $\hat{b}\cdot\hat{z} = 0$, corresponding to vertical and horizontal polarization respectively, but in addition σ_{cr} will be zero if

$$\Gamma = P_{33}+M_{11}-\frac{3}{2}P_{11} \tag{45}$$

is zero. The possibility of using Γ as a shape discriminant for a radar target is just one of the questions that we now explore.

Tensor Elements

Exact analytical expressions for P_{11}, P_{33} and M_{11} are known for only a restricted class of rotationally symmetric bodies, and some of these results are listed by Schiffer and Szegö (1949). In particular, for a spheroid, prolate or oblate, the tensor elements can be expressed as ratios of Legendre functions of the first and second kinds. To get some feeling for how the elements vary with the axial ratio of the spheroid, Figures 1 through 3 show P_{11}/V, P_{33}/V and M_{11}/V plotted as functions of ℓ/w. The length-to-width ratio ℓ/w therefore varies from zero for a disk, through unity for a sphere, to infinity for a vanishingly thin prolate spheroid, i.e. a "rod". Because of the normalization with respect to volume, a normalized tensor element can be infinite, and indeed, $P_{11}/V \rightarrow \infty$ as the (oblate) spheroid approaches a disk, whereas $P_{33}/V \rightarrow \infty$ as the (prolate) spheroid approaches a rod.

The spheroid is peculiar in that equality holds in both of (12). Thus, there is only one independent tensor element for a prolate or oblate spheroid, and since $P_{11} \geq 2V$ it follows that $V \leq M_{11} \leq 2V$. This is evident in Figure 3, but such a restriction on M_{11} seems more generally valid. An example is provided by the rounded cone consisting of the intersection of a cone of (interior) half angle θ ($\leq \pi/2$) with a sphere centered on the apex of the cone. Numerical data for the dipole moments were obtained by Senior (1971) using a mode matching method, and the resulting values of the normalized tensor elements are plotted as functions of $\ell/w = \frac{1}{2}\operatorname{cosec}\theta$ in Figures 1 through 3. Here again $P_{33}/V \to \infty$ as $\ell/w \to \infty$ and $M_{11} \leq 2V$ and it would appear that for all rotationally symmetric bodies:

a) $\frac{P_{33}}{V} \to \infty$, $\frac{P_{11}}{V}$, $\frac{M_{11}}{V} \to 2$ (implying $\frac{M_{33}}{V} \to 1$)

as $\ell/w \to \infty$;

b) $\frac{P_{11}}{V} \to \infty$, $\frac{P_{33}}{V}$, $\frac{M_{11}}{V} \to 1$ (implying $\frac{M_{33}}{V} \to \infty$)

as $\ell/w \to 0$;

and c) $V \leq M_{11} \leq 2V$.

Most of these properties are only conjectural at the moment. In particular, it has not yet been proved that $M_{11} \leq 2V$ though it can be shown that for convex surfaces with equation $r = r(\theta)$ in spherical polar coordinates, M_{11}/V does have an upper bound in contrast to the other normalized tensor quantities. From the definition of M_{11} (Keller et al, 1972),

$$M_{11} = \left| \int_S \psi \hat{n} \cdot \hat{x} \, dS \right| .$$

where ψ is an exterior Neumann potential which is bounded on S and has the same ϕ dependence as $\hat{n} \cdot \hat{x}$. Using these facts and the explicit formulae for the outward normal $\hat{n}$ and the surface element dS, it follows that there exists a finite constant A such that

$$M_{11} \leq A \int_0^{\pi} r \sin\theta \left| r \sin\theta - \frac{dr}{d\theta} \cos\theta \right| d\theta . \qquad (46)$$

For convex surfaces, however,

$$\left| r \sin\theta - \frac{dr}{d\theta} \cos\theta \right| < cr^2$$

where c is a further finite constant, and since

$$V = \frac{2\pi}{3} \int_0^{\pi} r^3 \sin\theta \, d\theta ,$$

the boundedness of M_{11}/V is verified.

To test the conjectures a) through c) when the body is not convex, the normalized tensor elements were computed for a rounded cone with half angle $\theta > \pi/2$, i.e. a sphere with a conical part removed. Instead of the mode matching method previously employed, the integral equations for the appropriate potential functions were now solved numerically using the moment method and, in some cases, independently by an iterative procedure. The computed results* are included in Figures 1 through 3, and it will be observed that the conjectures are still supported. We also note the reasonably close agreement between the normalized tensor elements for the rounded cone and those for a spheroid having the same value of ℓ/w.

The above computer programs have been used to calculate the tensor elements for a wide variety of rotationally symmetric shapes with special emphasis on missile-like geometries. In all cases the normalized elements are similar in value to those for a spheroid with the same ℓ/w, the agreement being particularly close for bodies having a plane of symmetry perpendicular to the axis of rotation. This is illustrated by the results for an ogive ($\ell/w > 1$) and its analogue for $\ell/w < 1$, namely, a lens, which are included in Figures 1 through 3. In view of this agreement, it is hardly surprising that a method such as that of Siegel (1959), which

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* The discontinuity in slope of the curves when $\ell/w = 1/2$, i.e. when the cone is a hemisphere, stems from the definition of ℓ/w: for $\theta \geq \pi/2$, $\ell/w = \frac{1}{2}(1 - \cos\theta)$.

uses the spheroid as a model, has proved so effective in estimating the Rayleigh scattering cross sections of rotationally symmetric targets.

It is also of interest to examine the element combination Γ (eq. 45), which is a measure of the cross polarized contribution to the back scattered return. As evident from the Figures, Γ is zero for a sphere ($\ell/w = 1$), but it is also zero for a rounded cone of half angle $\theta \simeq 29^0$ ($\ell/w \simeq 1.03$). Moreover, for all shapes that have been examined so far, it has been found that $\Gamma > 0$ if $\ell/w \gtrsim 1$, but $\Gamma < 0$ if $\ell/w \lesssim 1$, suggesting that the magnitude of the cross polarized return could serve as a means of estimating the value of ℓ/w for a target. This is similar to the situation at high frequencies where the cross polarized component is also a potential source of width information (Knott and Senior, 1971).

Conclusions

For any rotationally symmetric, perfectly conducting body there are just three independent tensor elements that serve to specify the Rayleigh scattering in its entirety, and since these are functions only of the geometry of the body, they provide a convenient means for categorizing low frequency scattering. When normalized to the volume of the body even these elements are subject to constraints from which lower bounds on the scattering cross sections have been deduced. With this normalization, no finite upper bounds are possible, nor has any other normalization yet been found for which such bounds do exist.

The behavior of the normalized tensor elements has been examined as a function of the length-to-width ratio, ℓ/w, of the body, and for all of the geometries considered, a wide degree of uniformity is observed. Nevertheless, V and ℓ/w are insufficient to specify the body uniquely, and to better understand the remaining variations in the tensor elements, their dependence on other geometrical parameters such as surface area is now under investigation.

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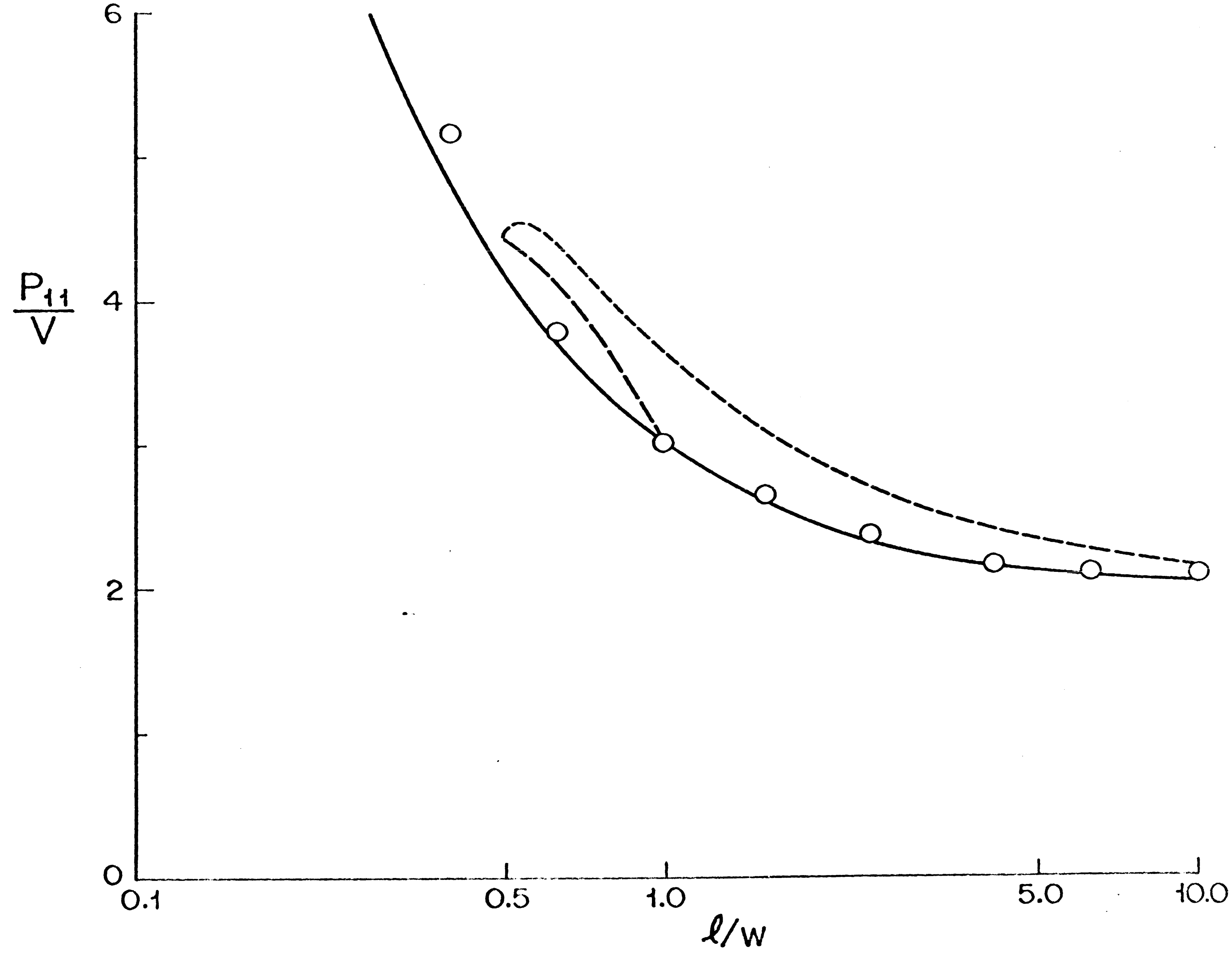
Legends for Figures

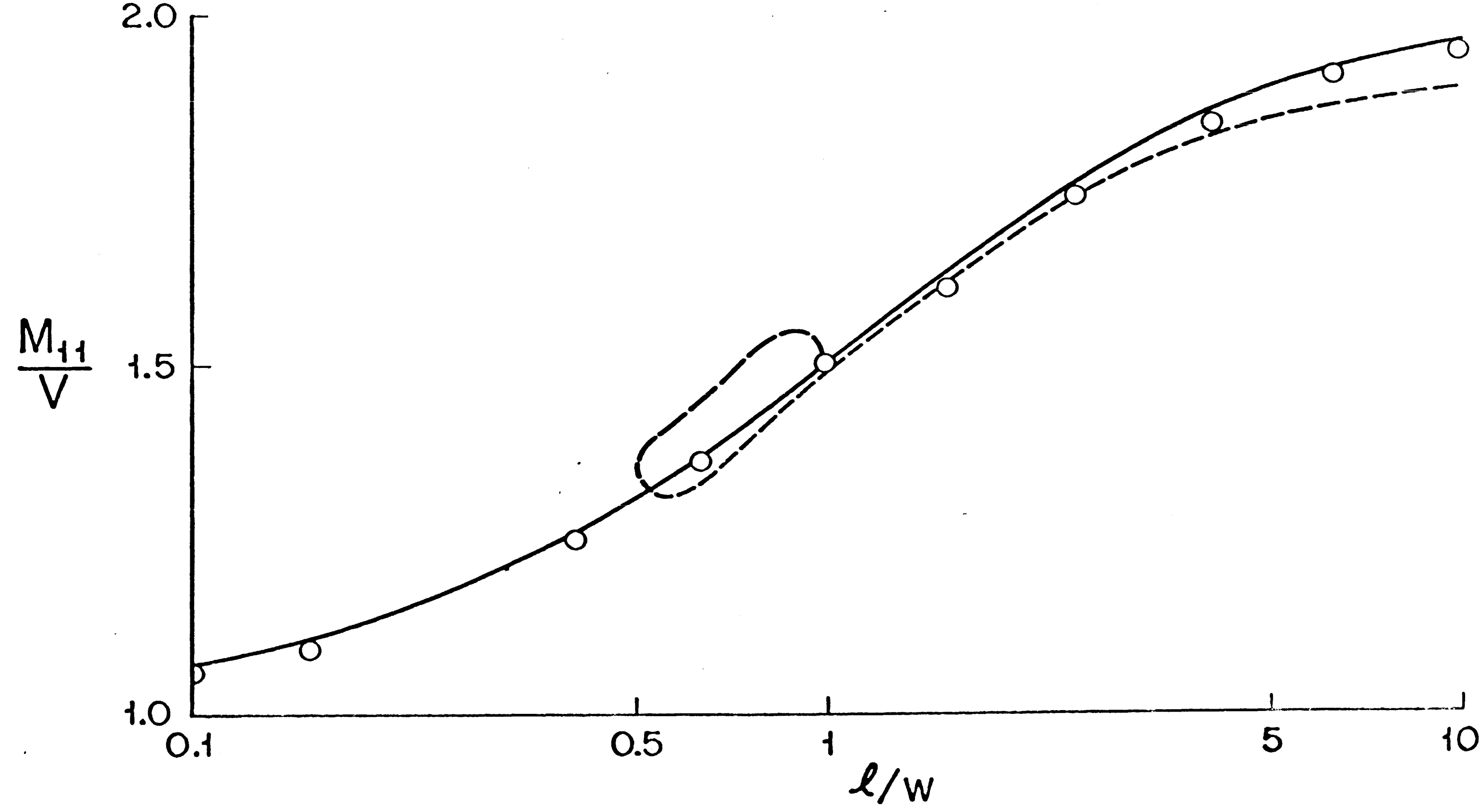
Fig. 1: Normalized tensor element P_{11}/V as a function of the length-to-width ratio of the body for prolate and oblate spheroids (——); rounded cones of half angle $\theta < \pi/\sigma$ (---) and $\theta > \pi/2$ (— —), and ogives and lenses (o o o).

Fig. 2: Normalized tensor element P_{33}/V as a function of the length-to-width ratio of the body for prolate and oblate spheroids (——); rounded cones of half angle $\theta < \pi/2$ (---) and $\theta > \pi/2$ (— —), and ogives and lenses (ooo).

Fig. 3: Normalized tensor element M_{11}/V as a function of the length-to-width ratio of the body for prolate and oblate spheroids (——); rounded cones of half angle $\theta < \pi/2$ (---) and $\theta > \pi/2$ (— —), and ogives and lenses (o oo).



6
4
2
0
P11/V
0.1
0.5
1.0
5.0
10.0
ℓ/w



M11 / V
2.0
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1.0
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5
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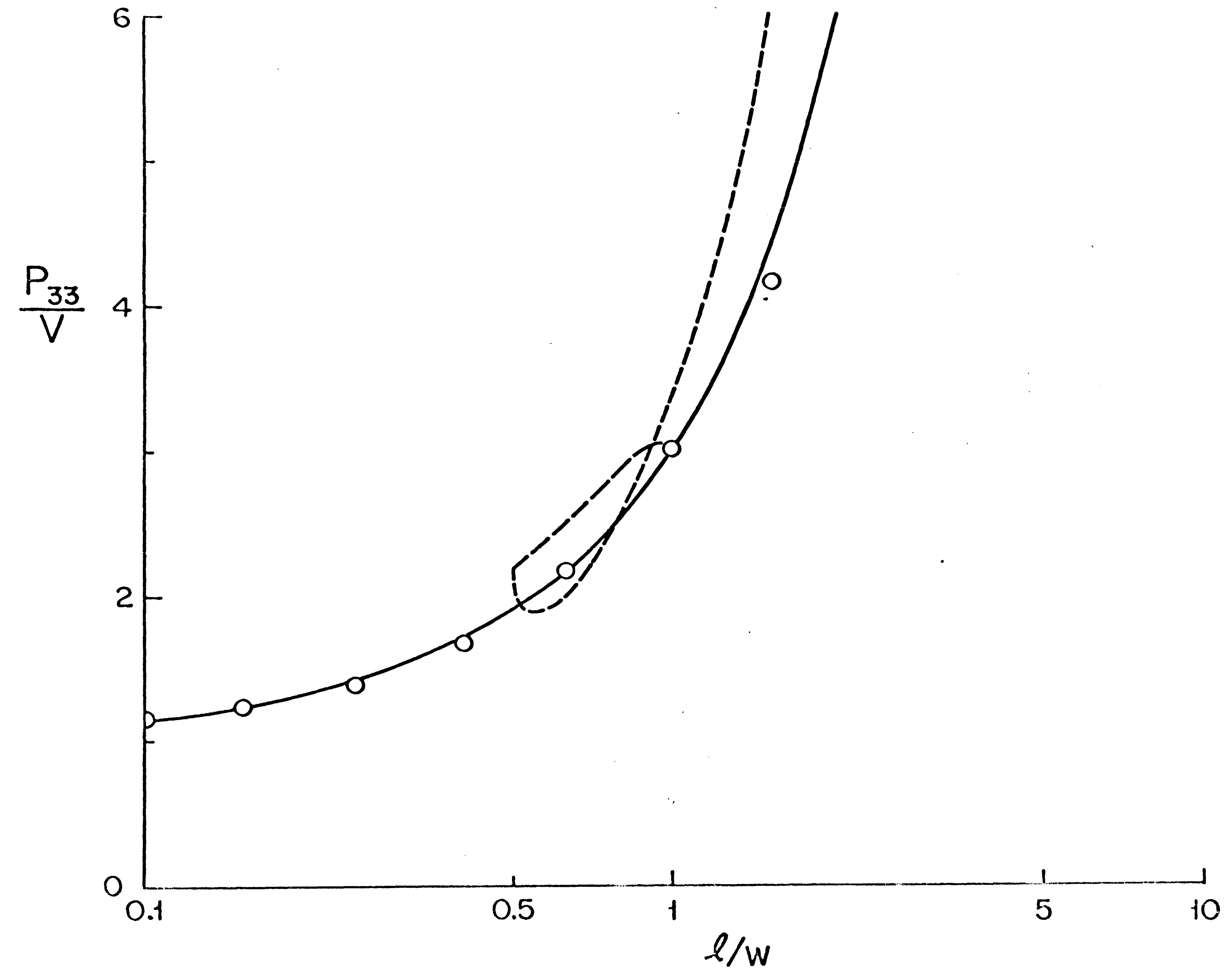
ℓ/W



$\frac{P_{33}}{V}$
6
4
2
0
0.1
0.5
1
5
10
ℓ/w