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THE ELECTROMAGNETIC THEORY OF THREE-DIMENSIONAL INHOMOGENEOUS LENSES AND THE DYADIC GREEN'S FUNCTIONS FOR CAVITIES

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ABSTRACT

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In this thesis the dyadic Green's functions of a number of cavities have been derived and the characteristics of some inhomogeneous lenses have been investigated.

To facilitate the treatment of problems involving cavities we have found the expressions for the dyadic Green's functions pertaining to rectangular, cylindrical and spherical cavities. Expressions for the electric and magnetic field involving the Green's functions are presented. An example of the application of the dyadic Green's function technique to the computations of the input admittance of the rectangular cavity is given.

The lenses covered in our work include: the Luneburg, Eaton-Lippmann and Eaton. The dyadic Green's functions for electric and magnetic dipoles in the presence of these lenses are found. The expressions for the electric field of an Huygens source in the presence of an inhomogeneous lens are constructed. Radiation patterns and the bistatic scattering cross sections for the small-diameter lenses and the directivity and the distribution of the energy around the geometrical focus of the Luneburg lenses are examined in detail.

To my parents

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I

INTRODUCTION

The aim of the present work is the application of the dyadic Green's function technique to two classes of problems of electromagnetic theory: the interior problem and the exterior problem.

The interior problem is represented by the study of electromagnetic fields in the interior of the cavity when the cavity is excited by some source of energy either inside or at an aperture. Three types of cavities are considered: rectangular, cylindrical and spherical.

The problem of the cavity is treated in almost every book on electromagnetic theory with varying degrees of mathematical difficulty. In many of them the cavity is treated as a particular case of waveguide. However, from the mathematical point of view, the problems of a waveguide and a cavity are quite distinct, because in the latter the spectrum of the vector wave operator is completely discrete while in the former it is not.

The exterior problem is represented by the study of propagation through and scattering by inhomogeneous spherical lenses. The Luneburg, Eaton-Lippmann and Eaton are investigated. All of their properties are known from geometrical optics, and brief reviews of these properties and published works related to the lenses are found in their respective sections. However, the exact electromagnetic formulation of the properties of these lenses is far from being complete.

The Luneburg lens is the most studied among all the lenses; the reason for this is that it allows us to scan the beam in a 360^{0} region without any distortion. The Eaton-Lippmann lens has also been thoroughly investigated because, from geometrical optics, it should work like an isotropic reflector.

Among less studied lenses is the Eaton lens, which should behave like a divergent lens according to geometrical optics.

In the process of studying propagation of electromagnetic waves in inhomogeneous media, we are faced with the problem of solving sometimes rather difficult differential equations. The difficulty lies in finding some convenient change of variables which allows us to cast them in the form of some standard differential equation. In this respect works by Heading (1965), Westcott (1968a, b) and Sharaf (1969a, b) deserve our attention. The basic idea of these works is to start from standard differential equations, and find all the possible relative permittivity distributions which can be fitted into them by means of suitable change of variables. A review of other techniques used in the electromagnetic formulation of the scattering by the inhomogeneous lenses can be found in Kerker's (1969) book.

The outline of the present work is as follows. In Chapter II dyadic Green's functions for the electric dipole excitation for the cavities are constructed. The method used is the one described recently by Tai (1973). However, it has not been applied before to find the dyadic Green's function in the cavities. Besides the dyadic Green's functions for the electric dipole excitation introduced previously by Tai, a new type of Green's functions is introduced here. These are the dyadic Green's functions for the magnetic dipole excitation and they are used mainly in the formulation involving a Huygens source. The application of the dyadic Green's functions technique to cavities is illustrated by an example.

In Chapter III, it is shown that the duality between the free-space dyadic Green's functions of electric and magnetic types for the electric and magnetic dipole excitation does not hold in an inhomogeneous medium.

By the method of scattering superposition, the dyadic Green's functions for the electric and magnetic dipole excitations in the presence of the lens are constructed. Electric fields of the electric and magnetic dipoles in the presence of the lens are found. Dyadic Green's functions for the magnetic dipole excitation as well as the corresponding electric and magnetic fields in the presence of the lens have not been previously discussed by Tai (1971) in his book. For completeness, we present here as an extension of his work.

In Chapter IV, we present a review and a more detailed discussion of the theory of the Luneburg and Eaton lenses originally discussed by Tai (1971). The exact electromagnetic theory of the Eaton-Lippmann lens is worked out. The numerical computations of the radiation patterns, the bistatic scattering cross-sections of all lenses, and the directivity and distribution ot ehe energy around the geometrical optics focus of the Luneburg lenses are included. This represents one of the main contributions of this thesis.

In Chapter V, the conclusions and the recommendations for future work are presented. In conclusion, we summarize below the important results contained in this thesis: for small Luneburg lenses we found that the location of the focal point of the lenses is displaced from the rim. As the radius of the lens increases the focal point tends to move to the rim. For the Eaton-Lippmann lens the backscattering cross-section of the lens approaches its geometrical cross-section as the radius of the lens increases and, finally, the radiation patterns and the bistatic scattering cross-sections of the Eaton lens are presented to show some of its characteristics from the electromagnetic theory point of view.

II

CAVITIES

2.1 Introduction

One method of attacking the interior problem is by expanding the electromagnetic field in the cavity in terms of a complete set of orthogonal functions so that the problem of solving the Maxwell's equations in the cavity is reduced to the determination of the expansion coefficients.

Slater (1950) expanded the electric field in terms of solenoidal and irrotational modes while the magnetic field was expanded in terms of solenoidal modes only. The reason for this is that the magnetic field is itself solenoidal. Using these expansions Slater determined the input impedance of the cavity. The expression for the impedance obtained by Slater was later found out to be incomplete.

Teichmann and Wigner (1953) found that the set of solenoidal modes of a completely closed cavity used in the expansion of the magnetic field in the cavities excited through holes is incomplete. They completed the set by adding an irrotational term to the set of solenoidal modes. This irrotational term corresponds to zero eigenfrequency of the solenoidal modes of a completely closed cavity and it contributes a term inversely proportional to the frequency in the expression for the input admittance.

Kurokawa (1958) presented a method which is more convenient for a general discussion. He started with a set of functions which is known to be complete for a cavity with holes. This set is composed of solenoidal modes which are the same as that of completely closed cavity and a set of irrotational modes. He expanded the electric and the magnetic fields in terms of these solenoidal and irrotational modes. Expanding all the terms which appear in the Maxwell's equations, Kurokawa was able to find the expansion coefficients and consequently the input admittance of the cavity. He gave an example in which if the irrotational terms are not considered in the expansion of the magnetic field, the result would be wrong.

As it will be shown in this thesis, the use of the dyadic Green's function technique allows us to determine the expansion of the electric and magnetic fields in a cavity in a direct and elegant manner; the irrotational terms which appeared in the previous works are inherently contained in the expansion of the magnetic field based on the present method.

In order to solve the vector wave equations in a cavity, two dyadic Green's functions are introduced, one of them, designated by $\bar{\bar{G}}_m$, is solenoidal and the other, designated by $\bar{\bar{G}}_e$, is non-solenoidal. The function $\bar{\bar{G}}_m$ is expanded in terms of solenoidal vector wave functions while $\bar{\bar{G}}_e$ is expanded in terms of irrotational as well as solenoidal vector wave functions. The electric and magnetic fields are given by integrals involving the dyadic Green's functions and the source functions. An example using the dyadic Green's function technique will be given and the result thus obtained agrees with that of Kurokawa.

2.2 Electric and Magnetic Fields for Electric and Magnetic Dipole Sources

The electromagnetic field in the interior of a cavity with current source $\bar{J}$ is described by the Maxwell's equations

$$\begin{aligned} \nabla \times \bar{E} &= i\omega\mu_0 \bar{H} \\ \nabla \times \bar{H} &= \bar{J} - i\omega\epsilon_0 \bar{E} \end{aligned} \tag{2.1}$$

where the time dependence of the type $e^{-i\omega t}$ is assumed. By eliminating $\bar{E}$ or $\bar{H}$ from these two equations, we get

$$\nabla \times \nabla \times \bar{E} - k^2 \bar{E} = i\omega\mu \bar{J} \tag{2.2}$$

and

$$\nabla \times \nabla \times \bar{H} - k^2 \bar{H} = \nabla \times \bar{J} \tag{2.3}$$

In order to integrate these equations, we introduce two types of dyadic Green's functions: $\bar{\bar{G}}_e$, (electric) and $\bar{\bar{G}}_m$, (magnetic). They satisfy the dyadic version of Maxwell's equations excited by the infinitesimal current sources

$$\nabla \times \bar{\bar{G}}_e = \bar{\bar{G}}_m \tag{2.4}$$

and

$$\nabla \times \bar{\bar{G}}_m = \bar{\bar{I}}\delta(\bar{R} - \bar{R}') + k^2 \bar{\bar{G}}_e \tag{2.5}$$

where $\bar{\bar{I}}$ is the idemfactor and $\delta(\bar{R} - \bar{R}')$ is the three-dimensional Dirac delta function.

The above equations are coupled in the sense that $\bar{\bar{G}}_m$ and $\bar{\bar{G}}_e$ appear in both of them. By taking the curl of every one of them and substituting one in another, we obtain the uncoupled equations, namely

$$\nabla \times \nabla \times \bar{\bar{G}}_e - k^2 \bar{\bar{G}}_e = \bar{\bar{I}}\delta(\bar{R} - \bar{R}') \tag{2.6}$$

and

$$\nabla \times \nabla \times \bar{\bar{G}}_m - k^2 \bar{\bar{G}}_m = \nabla \times \left[\bar{\bar{I}}\delta(\bar{R}-\bar{R}')\right] \quad . \tag{2.7}$$

Now, the reason for introducing beforehand two different dyadic Green's functions, $\bar{\bar{G}}_e$ and $\bar{\bar{G}}_m$, is apparent: the equations which $\bar{\bar{G}}_e$ and $\bar{\bar{G}}_m$ satisfy differ by the inhomogeneous term. Also,

$$\nabla \cdot \bar{\bar{G}}_e = -\frac{1}{k^2} \nabla \cdot \bar{\bar{I}}\delta(\bar{R} - \bar{R}') = -\frac{1}{k^2} \nabla \delta(\bar{R} - \bar{R}')$$

and

$$\nabla \cdot \bar{\bar{G}}_m = 0.$$

Therefore, $\bar{\bar{G}}_m$ is solenoidal dyadic while $\bar{\bar{G}}_e$ is non-solenoidal.

In Tai's (1971) book, dyadic Green's functions of the first and second kinds are defined only for the electric type; equation (2.7) was not mentioned. In a recent report (Tai, 1973), a more refined classification of various dyadic Green's functions is given. In this work, we follow closely the revised version of Tai's treatment.

The dyadic Green's functions are classified according to the boundary conditions which they satisfy. In this chapter we will be using two kinds of the dyadic Green's functions. The first satisfies the Dirichlet boundary conditions

$$\hat{n} \times \bar{\bar{G}}_{e1} = 0 \tag{2.8}$$

$$\hat{n} \times \bar{\bar{G}}_{m1} = 0 \tag{2.9}$$

on the interface of two media where $\hat{n}$ is the normal to the surface, the subscript indicating the kind of dyadic Green's function. The second satisfies the Neumann boundary conditions, i.e.,

$$\hat{n} \times \nabla \times \bar{\bar{G}}_{e2} = 0 \tag{2.10}$$

$$\hat{n} \times \nabla \times \bar{\bar{G}}_{m2} = 0 \quad . \tag{2.11}$$

These dyadic Green's functions are related to perfectly conducting bodies and are of interest in the problem involving cavities. In the problems involving inhomogeneous lenses, we will be studying two media, neither of which is perfectly conducting. We will then define dyadic Green's functions of a third kind.

There is an interdependence between $\bar{\bar{G}}_{e1}$ and $\bar{\bar{G}}_{m2}$ and between $\bar{\bar{G}}_{e2}$ and $\bar{\bar{G}}_{m1}$, because they satisfy (2.4) and (2.5). Specifically

$$\nabla \times \bar{\bar{G}}_{e1} = \bar{\bar{G}}_{m2} \tag{2.12}$$

$$\nabla \times \bar{\bar{G}}_{m2} = \bar{\bar{I}}\,\delta(\bar{R} - \bar{R}') + k^2\,\bar{\bar{G}}_{e1} \tag{2.13}$$

$$\nabla \times \bar{\bar{G}}_{e2} = \bar{\bar{G}}_{m1} \tag{2.14}$$

$$\nabla \times \bar{\bar{G}}_{m1} = \bar{\bar{I}}\,\delta(\bar{R} - \bar{R}') + k^2\,\bar{\bar{G}}_{e2} \tag{2.15}$$

The integration of equations (2.2) and (2.3) is made, now, with the help of the dyadic identity (Van Bladel, 1964).

$$\int_V \left[(\nabla \times \nabla \times \bar{a}) \cdot \bar{\bar{A}} - \bar{a} \cdot \nabla \times \nabla \times \bar{\bar{A}}\right] dv = \int_S \left[(\hat{n} \times \bar{a}) \cdot \nabla \times \bar{\bar{A}} + (\hat{n} \times \nabla \times \bar{a}) \cdot \bar{\bar{A}}\right] ds \quad , \tag{2.16}$$

where $\hat{n}$ is normal to the surface.

Identify $\bar{a}$ with $\bar{E}$ and $\bar{\bar{A}}$ with $\bar{\bar{G}}_{e1}$. Then,

$$\int_V \left[(\nabla \times \nabla \times \bar{E}) \cdot \bar{\bar{G}}_{e1} - \bar{E} \cdot \nabla \times \nabla \times \bar{\bar{G}}_{e1}\right] dv = \int_S \left[(\hat{n} \times \bar{E}) \cdot \nabla \times \bar{\bar{G}}_{e1} + (\hat{n} \times \nabla \times \bar{E}) \cdot \bar{\bar{G}}_{e1}\right] ds \tag{2.17}$$

The second term in the surface integral can be written as

$$\int_S (\hat{n} \times \nabla \times \bar{E}) \cdot \bar{\bar{G}}_{e1} \, ds = -\int_S (\nabla \times \bar{E}) \cdot (\hat{n} \times \bar{\bar{G}}_{e1}) \, ds$$

and it vanishes due to the boundary conditions (2.8) on the surface. On the other hand, the volume integral, with the help of (2.6) and (2.2), can be put in the form

$$\int_V \left[(\nabla \times \nabla \times \bar{E}) \cdot \bar{\bar{G}}_{e1} - \bar{E} \cdot \nabla \times \nabla \times \bar{\bar{G}}_{e1}\right] dv = k^2 \int_V \bar{E} \cdot \bar{\bar{G}}_{e1} dv + i\omega\mu_o \int_V \bar{J} \cdot \bar{\bar{G}}_{e1} dv - k^2 \int_V \bar{E} \cdot \bar{\bar{G}}_{e1} dv -$$

$$- \int_V \bar{E} \cdot \bar{\bar{I}} \delta(\bar{R} - \bar{R}') \, dv.$$

Cancelling out similar terms and substituting in (2.17), we obtain

$$-\int_V \bar{E} \cdot \bar{\bar{I}} \delta(\bar{R} - \bar{R}') \, dv + i \omega \mu_o \int_V \bar{J} \cdot \bar{\bar{G}}_{e1} \, dv = \int_S (\hat{n} \times \bar{E}) \cdot \nabla \times \bar{\bar{G}}_{e1} \, ds.$$

The first volume integral due to properties of the delta function gives us

$$\bar{E}(\bar{R}') = i\omega\mu_o \int_V \bar{J} \cdot \bar{\bar{G}}_{e1}\, dv - \int_S (\hat{n} \times \bar{E}) \cdot \nabla \times \bar{\bar{G}}_{e1}\, ds \quad .$$

By interchanging the variables $\bar{R}$ and $\bar{R}'$ and making use of the symmetrical property of $\bar{\bar{G}}_{e1}$

$$\tilde{\bar{\bar{G}}}_{e1}(\bar{R}'|\bar{R}) = \bar{\bar{G}}_{e1}(\bar{R}|\bar{R}')$$

and

$$\widetilde{\nabla' \times \bar{\bar{G}}_{e1}}(\bar{R}'|\bar{R}) = \nabla \times \bar{\bar{G}}_{e2}(\bar{R}|\bar{R}')$$

where "$\sim$" denote the transpose of the dyadic, we obtain

$$\bar{E}(\bar{R}) = i\omega\mu_o \int_V \bar{\bar{G}}_{e1}(\bar{R}|\bar{R}') \cdot \bar{J}(\bar{R}')\, dv' - \int_S \nabla \times \bar{\bar{G}}_{e2}(\bar{R}|\bar{R}') \cdot (\hat{n} \times \bar{E}(\bar{R}'))\, ds' \quad .$$

Recalling (2.14) we finally have

$$\bar{E}(\bar{R}) = i\omega\mu_o \int_V \bar{\bar{G}}_{e1}(\bar{R}|\bar{R}') \cdot \bar{J}(\bar{R}')\, dv' - \int_S \bar{\bar{G}}_{m1}(\bar{R}|\bar{R}') \cdot (\hat{n} \times \bar{E}(\bar{R}'))\, ds' \, . \tag{2.18}$$

Identifying, now $\bar{a}$ with $\bar{H}$ and $\bar{\bar{A}}$ with $\bar{\bar{G}}_{e2}$ in (2.16) we have

$$\int_V \left[(\nabla \times \nabla \times \bar{H}) \cdot \bar{\bar{G}}_{e2} - \bar{H} \cdot \nabla \times \nabla \times \bar{\bar{G}}_{e2} \right] dv = \int_S \left[(\hat{n} \times \bar{H}) \cdot \nabla \times \bar{\bar{G}}_{e2} + (\hat{n} \times \nabla \times \bar{H}) \cdot \bar{\bar{G}}_{e2} \right] ds \tag{2.19}$$

This time the first term in the surface integral can be written as

$$\int_S \hat{n} \times \bar{H} \cdot \nabla \times \bar{\bar{G}}_{e2}\, ds = - \int_S \bar{H} \cdot (\hat{n} \times \nabla \times \bar{\bar{G}}_{e2})\, ds$$

and it vanishes due to the boundary condition(2.10) on the surface.

The volume integral due to (2.6) and (2.3) becomes

$$\int_V \left[(\nabla x\, \nabla x\, \bar{H})\cdot \bar{\bar{G}}_{e2} - \bar{H}\cdot \nabla x \nabla x\, \bar{\bar{G}}_{e2}\right] dv = k^2 \int_V \bar{H}\cdot \bar{\bar{G}}_{e2}\, dv + \int_V \nabla x\, \bar{J}\cdot \bar{\bar{G}}_{e2}\, dv -$$

$$- \int_V \bar{H}\cdot \bar{\bar{I}}\,\delta(\bar{R} - \bar{R}')\, dv - k^2 \int_V \bar{H}\cdot \bar{\bar{G}}_{e2}\, dv \quad .$$

Canceling out similar terms and substituting in (2.19) we obtain

$$- \bar{H}(\bar{R}') = - \int_V \nabla x\, \bar{J}\cdot \bar{\bar{G}}_{e2}\, dv + \int_S (\hat{n}\, x\, \nabla x\, \bar{H})\cdot \bar{\bar{G}}_{e2}\, ds$$

where

$$\int_V \bar{H}\cdot \bar{\bar{I}}\,\delta(\bar{R} - \bar{R}')\, dv = \bar{H}(\bar{R}') \quad .$$

Recalling that

$$\nabla x\, \bar{H} = \bar{J} - i\omega\epsilon_o \bar{E} \,,$$

the expression for $\bar{H}(\bar{R}')$ becomes

$$- \bar{H}(\bar{R}') = - \int_V \nabla x\, \bar{J}\cdot \bar{\bar{G}}_{e2}\, dv + \int_S \hat{n}\, x\, \bar{J}\cdot \bar{\bar{G}}_{e2}\, ds - i\,\omega\epsilon_o \int_S \hat{n}\, x\, \bar{E}\cdot \bar{\bar{G}}_{e2}\, ds \quad .$$

It can be proved (see Appendix A-3) that

$$- \int_V \nabla x\, \bar{J}\cdot \bar{\bar{G}}_{e2}\, dv + \int_S \hat{n}\, x\, \bar{J}\cdot \bar{\bar{G}}_{e2}\, ds = - \int_V \bar{J}\cdot \nabla x\, \bar{\bar{G}}_{e2}\, dv$$

so that

$$\bar{H}(\bar{R}') = \int_V \bar{J}\cdot \nabla x\, \bar{\bar{G}}_{e2}\, dv + i\,\omega\,\epsilon_o \int_S \hat{n}\, x\, \bar{E}\cdot \bar{\bar{G}}_{e2}\, ds \quad .$$

By interchanging, again, the variables $\bar{R}$ and $\bar{R}'$ and making use of the symmetrical properties of $\bar{\bar{G}}_{e2}$

$$\tilde{\bar{\bar{G}}}_{e2}(\bar{R}'|\bar{R}) = \bar{\bar{G}}_{e2}(\bar{R}|\bar{R}')$$

and

$$\widetilde{\nabla' \times \bar{\bar{G}}_{e2}}(\bar{R}'|\bar{R}) = \nabla \times \bar{\bar{G}}_{e1}(\bar{R}|\bar{R}') \; ,$$

we have

$$\bar{H}(\bar{R}) = \int_V \nabla \times \bar{\bar{G}}_{e1}(\bar{R}|\bar{R}') \cdot \bar{J}(\bar{R}')\, dv' + i\omega\epsilon_o \int_S \bar{\bar{G}}_{e2}(\bar{R}|\bar{R}') \cdot \hat{n} \times \bar{E}(\bar{R}')\, ds'$$

which, in view of (2. 12) becomes

$$\bar{H}(\bar{R}) = \int_V \bar{\bar{G}}_{m2}(\bar{R}|\bar{R}') \cdot \bar{J}(\bar{R}')\, dv' + i\omega\epsilon_o \int_S \bar{\bar{G}}_{e2}(\bar{R}|\bar{R}') \cdot (\hat{n} \times \bar{E}(\bar{R}'))\, ds' . \qquad (2.20)$$

For sake of completeness of the discussion, and because we will be using the infinitesimal magnetic dipole excitation in the problem of inhomogeneous lenses, we will find the dyadic Green's function for this case. This version of dyadics was not previously introduced by Tai.

Consider Maxwell's equation with an infinitesimal magnetic current $\bar{J}_m$ as a source

$$\nabla \times \bar{E}_m = i\,\omega\,\mu_o \bar{H}_m - \bar{J}_m$$

$$\nabla \times \bar{H}_m = -\, i\omega\epsilon_o \bar{E}_m \qquad (2.21)$$

and introduce again two types of dyadic Green's functions: $\bar{\bar{G}}^*_e$, the new electric type and $\bar{\bar{G}}^*_m$, the new magnetic type. (The asterisk does not mean complex conjugation).

The dyadic versions of Maxwell's equations in this case are

$$\nabla \times \bar{\bar{G}}_e^* = k^2 \bar{\bar{G}}_m^* + \bar{\bar{I}} \delta (\bar{R} - \bar{R}') \tag{2.22}$$

and

$$\nabla \times \bar{\bar{G}}_m^* = \bar{\bar{G}}_e^* \tag{2.23}$$

and the decoupled equations become

$$\nabla \times \nabla \times \bar{\bar{G}}_m^* - k^2 \bar{\bar{G}}_m^* = \bar{\bar{I}} \delta(\bar{R} - \bar{R}') \tag{2.24}$$

and

$$\nabla \times \nabla \times \bar{\bar{G}}_e^* - k^2 \bar{\bar{G}}_e^* = \nabla \times \left[\bar{\bar{I}}_\delta (\bar{R} - \bar{R}')\right] \quad . \tag{2.25}$$

The dyadics of the first kind satisfy the boundary conditions

$$\hat{n} \times \bar{\bar{G}}_{e1}^* = 0$$

$$\hat{n} \times \bar{\bar{G}}_{m1}^* = 0$$

while those of the second kind satisfy the boundary condition

$$\hat{n} \times \nabla \times \bar{\bar{G}}_{e2}^* = 0$$

and

$$\hat{n} \times \nabla \times \bar{\bar{G}}_{m2}^* = 0$$

The relationship between $\bar{\bar{G}}_{e1}^*$ and $\bar{\bar{G}}_{m2}^*$ and $\bar{\bar{G}}_{e2}^*$ and $\bar{\bar{G}}_{m1}^*$ in this case is

$$\nabla \times \bar{\bar{G}}_{m1}^* = \bar{\bar{G}}_{e2}^*$$

$$\nabla \times \bar{\bar{G}}_{e2}^* = k^2 \bar{\bar{G}}_{m1}^* + \bar{\bar{I}} \delta(\bar{R} - \bar{R}')$$

$$\nabla \times \bar{\bar{G}}_{m2}^* = \bar{\bar{G}}_{e1}^*$$

$$\nabla \times \bar{\bar{G}}_{e1}^* = k^2 \bar{\bar{G}}_{m2}^* + \bar{\bar{I}} \delta (\bar{R} - \bar{R}') \quad .$$

The vector wave equations for electric and magnetic fields are

$$\nabla \times \nabla \times \bar{E}_m - k^2 \bar{E}_m = - \nabla \times \bar{J}_m$$

$$\nabla \times \nabla \times \bar{H}_m - k^2 \bar{H}_m = i\omega\epsilon_o \bar{J}_m \quad .$$

Proceeding in exactly the same way as in the case of electric current excitation we find

$$\bar{H}_m(\bar{R}) = i\omega\epsilon_o \int_V \bar{\bar{G}}^*_{m1}(\bar{R}|\bar{R}') \cdot \bar{J}_m(\bar{R}')\, dv' - \int_S \bar{\bar{G}}^*_{e1}(\bar{R}|\bar{R}') \cdot \hat{n} \times \bar{H}_m(\bar{R}')\, ds'$$

and

$$\bar{E}_m(\bar{R}) = - \int_V \bar{\bar{G}}^*_{e2}(\bar{R}|\bar{R}') \cdot \bar{J}_m(\bar{R}')\, dv' - i\omega\mu_o \int_S \bar{\bar{G}}^*_{m2}(\bar{R}|\bar{R}') \cdot \hat{n} \times \bar{H}_m(\bar{R}')\, ds' \quad .$$

Because $\bar{\bar{G}}^*_m$ and $\bar{\bar{G}}_e$ satisfy the same differential equations and $\bar{\bar{G}}^*_{m1}$ and $\bar{\bar{G}}_{e1}$ obey the same boundary conditions, they are dual. The same can be said of $\bar{\bar{G}}^*_{m2}$ and $\bar{\bar{G}}_{e2}$. Also, there is duality between $\bar{\bar{G}}^*_{e1}$ and $\bar{\bar{G}}_{m1}$ and $\bar{\bar{G}}^*_{e2}$ and $\bar{\bar{G}}_{m2}$. Therefore the expressions for the electric and magnetic fields generated by the magnetic current $\bar{J}_m$ and the tangential component of $\bar{H}$ on the surface can be stated as

$$\bar{H}_m(\bar{R}) = i\omega\epsilon_o \int_V \bar{\bar{G}}_{e1}(\bar{R}|\bar{R}') \cdot \bar{J}_m(\bar{R}')dv' - \int_S \bar{\bar{G}}_{m1}(\bar{R}|\bar{R}') \cdot \hat{n} \times \bar{H}_m(\bar{R}')\, ds' \tag{2.26}$$

and

$$\bar{E}_m(\bar{R}) = - \int_V \bar{\bar{G}}_{m2}(\bar{R}|\bar{R}') \cdot \bar{J}_m(\bar{R}')dv' - i\omega\mu_o \int_S \bar{\bar{G}}_{e2} \cdot \hat{n} \times \bar{H}_m\, ds' \quad . \tag{2.27}$$

Now that we have the expressions of electric and magnetic fields in terms of the appropriate dyadic Green's functions, we will go on to derive the expressions for these Green's functions.

Because the dyadic $\nabla \times \left[\bar{\bar{I}}\, \delta(\bar{R} - \bar{R}')\right]$ is solenoidal, it can be expanded in terms of two solenoidal vector wave functions; $\bar{M}$ and $\bar{N}$, defined by

$$\bar{M}(K) = \nabla \times \left[\psi\, \hat{c}\right] \tag{2.28}$$

and

$$\bar{N}(K) = \frac{1}{K}\, \nabla \times \nabla \times \left[\psi\, \hat{c}\right], \tag{2.29}$$

which are the eigenfunctions of the homogeneous vector wave equation

$$\nabla \times \nabla \times \bar{F} - K^2 \bar{F} = 0 \ .$$

ψ , the generating function, is the solution of the scalar wave equation

$$\nabla^2 \psi + K^2 \psi = 0 \quad . \tag{2.30}$$

The piloting vector, $\hat{c}$, is a vector such as $\hat{x}$, $\hat{y}$, $\hat{z}$, or $\bar{R}$, depending on the type of problem to be considered. It is easy to show that $\bar{M}(K)$ and $\bar{N}(K)$ are related by

$$\nabla \times \bar{M}(K) = K\, \bar{N}(K) \tag{2.31}$$

and

$$\nabla \times \bar{N}(K) = K\, \bar{M}(K) \ . \tag{2.32}$$

Because the dyadic $\bar{\bar{I}}\, \delta(\bar{R} - \bar{R}')$ is not solenoidal, it has to be expanded in terms of three sets of vector wave functions; the previously defined $\bar{M}$ and $\bar{N}$ functions plus a set of irrotational vector wave functions, denoted by $\bar{L}$ and defined by

$$\bar{L} = \nabla \psi \quad . \tag{2.33}$$

The boundary conditions for $\bar{L}$ - functions are:

$\psi = 0$ on the surface if the dyadic Green's function is of the first kind,

$\frac{\partial\psi}{\partial n} = 0$ on the surface if the dyadic Green's function is of the second kind.

Once the expansion of the exciting function is completed, the expansion of $\bar{\bar{G}}_m$ or $\bar{\bar{G}}_e$ is done in terms of the same functions used to expand the exciting functions in such a way that the equations (2.6) and (2.7) are satisfied. This completes the general discussion of the derivation of the expression of the dyadic Green's functions.

In what follows, we will construct the dyadic Green's functions with the electric source of excitation for the rectangular, the cylindrical and the spherical cavities.

2.3 Dyadic Green's Functions for Rectangular Cavities

The rectangular cavity is shown in Figure 1.

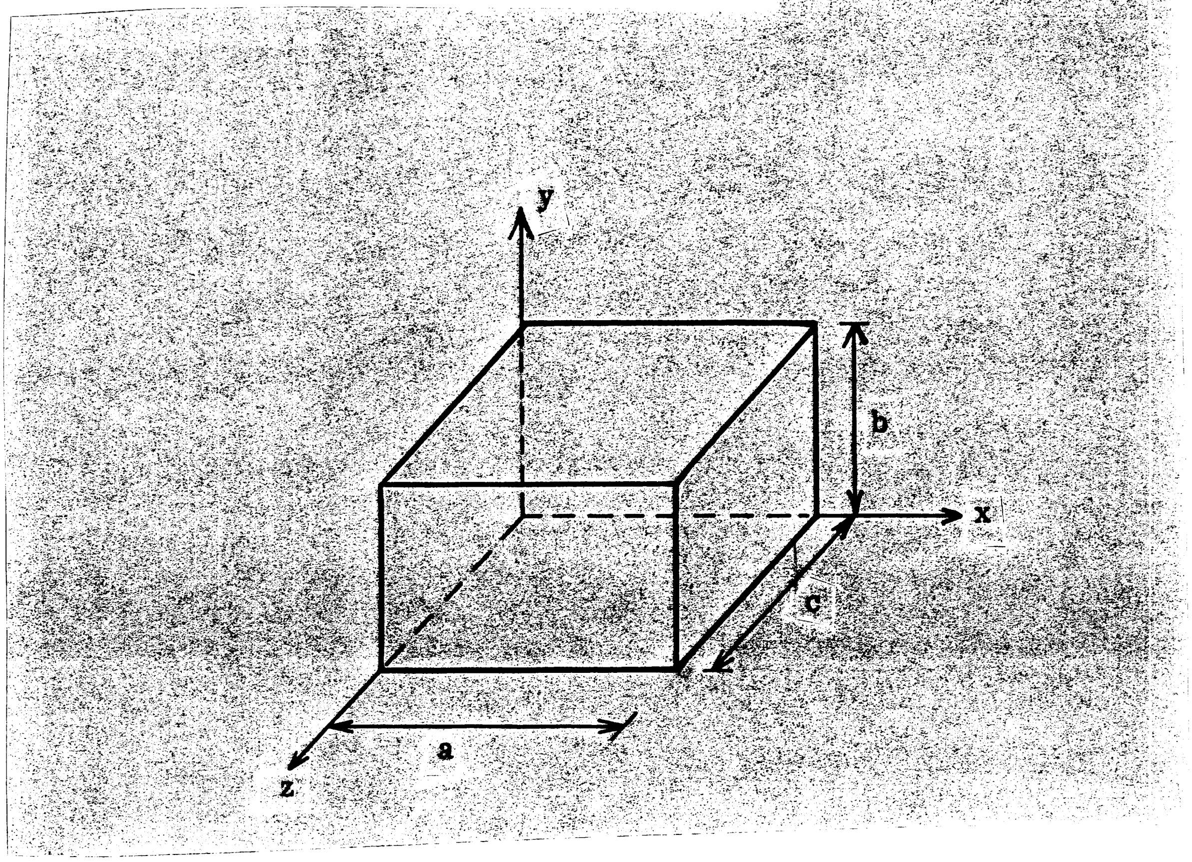

Figure 1: Rectangular Cavity.

The piloting vector in this case can most conveniently be chosen as $\hat{z}$. The partial differential equation (2.30) for the generating function, when solved in the rectangular coordinate system by the method of separation of variables, gives

$$\psi = (A\cos k_x x + B\sin k_x x)(C\cos k_y y + D\sin k_y y)(E\cos k_z z + F\sin k_z z) \tag{2.34}$$

where the constants A, B, C, D, E, F and separation constants k_x, k_y, k_z are determined by the excitation conditions and the boundary conditions and

$$K^2 = k_x^2 + k_y^2 + k_z^2 \ .$$

The expressions for $\bar{M}$, $\bar{N}$ and $\bar{L}$ functions, become

$$\begin{aligned}\bar{M} = \nabla\times(\psi\hat{z}) &= k_y(A\cos k_x x + B\sin k_x x)(-C\sin k_y y + D\cos k_y y)(E\cos k_z z + F\sin k_z z)\,\hat{x}\\ &- k_x(-A\sin k_x x + B\cos k_x x)(C\cos k_y y + D\sin k_y y)(E\cos k_z z + F\sin k_z z)\,\hat{y}\end{aligned} \tag{2.35}$$

$$\begin{aligned}\bar{N} = \frac{1}{K}\nabla\times\nabla\times(\psi\hat{z}) = \frac{1}{K}\Big\{ & k_x k_y(-A\sin k_x x + B\cos k_x x)\\ &(C\cos k_y y + D\sin k_y y)(-E\sin k_z z + F\cos k_z z)\,\hat{x}\\ &+ k_z k_y(A\cos k_x x + B\sin k_x x)(-C\sin k_y y + D\cos k_y y)(-E\sin k_z z + F\cos k_z z)\,\hat{y}\\ &+ (k_x^2 + k_y^2)(A\cos k_x x + B\sin k_x x)(C\cos k_y y + D\sin k_y y)(E\cos k_z z + F\sin k_z z)\,\hat{z}\Big\}\end{aligned} \tag{2.36}$$

$$\begin{aligned}\bar{L} = \nabla\psi &= k_x(-A\sin k_x x + B\cos k_x x)(C\cos k_y y + D\sin k_y y)(E\cos k_z z + F\sin K_z z)\,\hat{x}\\ &+ k_y(A\cos k_x x + B\sin k_x x)(-C\sin k_y y + D\cos k_y y)(E\cos k_z z + F\sin k_z z)\,\hat{y}\\ &+ k_z(A\cos k_x x + B\sin k_x x)(C\cos k_y y + D\sin k_y y)(-E\sin k_z z + F\cos k_z z)\,\hat{z}\end{aligned} \tag{2.37}$$

The expansion of $\bar{\bar{G}}_{m2}$ is made in terms of $\bar{M}$ and $\bar{N}$-function only, because $\bar{\bar{G}}_{m2}$ is solenoidal.

Using the boundary conditions (2.11) at x = 0, a ; y = 0, b and z = 0, c we find that A = C = F = 0 and $k_x = \frac{\ell\pi}{a}$; $k_y = \frac{m\pi}{b}$ and $k_z = \frac{n\pi}{c}$ where the running indices, ℓ, m and n assume the values 0, 1, 2,.... Therefore,

$$\bar{M}_{oe}(K) = \frac{m\pi}{b} \sin\frac{\ell\pi}{a} x \cos\frac{m\pi}{b} \cos\frac{n\pi}{c} z \hat{x} - \frac{\ell\pi}{a} \cos\frac{\ell\pi}{a} x \sin\frac{m\pi}{b} y \cos\frac{n\pi}{c} z \hat{y} \tag{2.38}$$

and

$$\bar{N}_{eo}(K) = \frac{1}{K}\left\{ -\frac{\ell\pi}{a}\frac{n\pi}{c} \sin\frac{\ell\pi}{a} x \cos\frac{m\pi}{b} y \cos\frac{n\pi}{c} \hat{x} - \frac{m\pi}{b}\frac{n\pi}{c} \cos\frac{\ell\pi}{a} x \sin\frac{m\pi}{b} y \cos\frac{n\pi}{c} z \hat{y} + \left[\left(\frac{\ell\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right] \cos\frac{\ell\pi}{a} x \cos\frac{n\pi}{b} y \sin\frac{n\pi}{c} z \hat{z} \right\} . \tag{2.39}$$

In the expressions of $\bar{M}$ and $\bar{N}$ functions, the first subscript represents the function used in the transversal direction of the generating function ψ, while the second subscript represents the function used in the longitudinal direction; o stands for odd and e stands for even function.

The functions $\bar{M}_{oe}(K)$ and $\bar{N}_{eo}(K)$ are orthogonal

$$\int_V \bar{M}_{eo}(K) \cdot \bar{N}_{eo}(K)\, dv = 0$$

as can be easily checked because of the trigonometric functions involved. The normalization constants are

$$\int_V \bar{M}_{oe}(K) \cdot \bar{M}_{oe}(K')\, dv = \begin{cases} 0 & K \neq K' \\ \dfrac{abc}{8}\left[\left(\dfrac{\ell n}{a}\right)^2 + \left(\dfrac{m\pi}{6}\right)^2\right](1+\delta_{on}) = \dfrac{1}{C_N} & K = K' \end{cases} \quad (2.40)$$

and

$$\int_V \bar{N}_{eo}(K) \cdot \bar{N}_{eo}(K')\, dv = \begin{cases} 0 & K \neq K' \\ \dfrac{abc}{8}\left[\left(\dfrac{\ell\pi}{a}\right)^2 + \left(\dfrac{m\pi}{b}\right)^2\right](1+\delta_{om})(1+\delta_{o\ell}) = \dfrac{1}{C_M} & K = K' \end{cases} \quad (2.41)$$

where

$$\delta_{o\alpha} = \begin{cases} 1 & \alpha = 0 \\ 0 & \alpha = 0 \end{cases} \quad \text{for } \alpha = \ell,\ m,\ n\ .$$

The expansion of $\nabla x \left[\bar{\bar{I}}\delta(\bar{R}-\bar{R}')\right]$ takes the form

$$\nabla x\left[\bar{\bar{I}}\,\delta(\bar{R}-\bar{R}')\right] = \sum \bar{M}_{oe}(K)\,\bar{A} + \sum \bar{N}_{eo}(K)\,\bar{B}\ . \quad (2.42)$$

The set of unknown vectors $\bar{A}$ and $\bar{B}$ is found by pre-multiplying (2.42) by $\bar{M}_{oe}(K)$ and $\bar{N}_{eo}(K)$ respectively and integrating through the volume of the cavity. Due to the orthogonality properties stated above, we find that

$$\bar{A} = C_N\, K\, \bar{N}'_{oe} \quad \text{and} \quad \bar{B} = C_M\, K\, \bar{M}'_{eo}$$

where the prime means that the functions are computed at point R' and

$$\int_V \bar{M}_{oe}(K) \cdot \nabla x\left[\bar{\bar{I}}\,\delta(\bar{R}-\bar{R}')\right] dv = \nabla' x\, \bar{M}'_{oe}(K) = K\, \bar{N}'_{oe}(K).$$

The final form of the expansion of $\nabla x \left[\bar{\bar{I}}\,\delta(\bar{R}-\bar{R}')\right]$ becomes

$$\nabla x\left[\bar{\bar{I}}\,\delta(\bar{R}-\bar{R}')\right] = \sum C_N\, K\, \bar{M}_{oe}\, \bar{N}'_{oe} + \sum C_M\, K\, \bar{N}_{eo}\, \bar{M}'_{eo}\ .$$

The expansion of the $\bar{\bar{G}}_{m2}$ can be written in the form

$$\bar{\bar{G}}_{m2}(\bar{R}|\bar{R}') = \sum \alpha C_N K \bar{M}_{oe} \bar{N}'_{oe} + \sum \beta C_M K \bar{N}_{eo} \bar{M}'_{eo}$$

where α and β are unknown coefficients.

Substituting the expansion of $\bar{\bar{G}}_{m2}$ and $\nabla \times \left[\bar{\bar{I}}\, \delta(\bar{R} - \bar{R}')\right]$ in equation (2.7) we find that

$$\alpha = \beta = \frac{1}{K^2 - k^2} \quad .$$

Therefore

$$\bar{\bar{G}}_{m2}(\bar{R}|\bar{R}') = \sum \frac{C_N K}{K^2 - k^2} \bar{M}_{oe} \bar{N}'_{oe} + \sum \frac{C_M K}{K^2 - k^2} \bar{N}_{eo} \bar{M}'_{eo} \quad . \tag{2.43}$$

The expansion of $\bar{\bar{G}}_{e1}$ involves the three functions $\bar{M}$, $\bar{N}$ and $\bar{L}$ because $\bar{\bar{G}}_{e1}$ is non-solenoidal.

Applying the boundary conditions (2.8) at $x = 0,\ a;\ y = 0,\ b;\ z = 0,\ c$ we get $B = D = E = 0$ in the expressions of $\bar{M}$ and $\bar{N}$. Therefore

$$\bar{M}_{eo}(K) = -\frac{m\pi}{b} \cos\frac{\ell\pi}{a} x \sin\frac{m\pi}{b} y \sin\frac{n\pi}{c} z\, \hat{x} + \frac{\ell\pi}{a} \sin\frac{\ell\pi}{a} x \cos\frac{m\pi}{b} y \sin\frac{n\pi}{c} z\, \hat{y} \tag{2.44}$$

and

$$\bar{N}_{oe}(K) = \frac{1}{K}\left\{-\frac{\ell\pi}{a}\frac{n\pi}{c}\cos\frac{\ell\pi}{a}x\sin\frac{m\pi}{b}y\sin\frac{n\pi}{c}z\,\hat{x} - \frac{m\pi}{b}\frac{n\pi}{c}\sin\frac{\ell\pi}{a}x\cos\frac{m\pi}{b}y\sin\frac{n\pi}{c}z\,\hat{y} + \left[\left(\frac{\ell\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right]\sin\frac{\ell\pi}{a}x\sin\frac{m\pi}{b}y\cos\frac{n\pi}{c}z\,\hat{z}\right\} \tag{2.45}$$

where this time the subscripts are reversed as compared to the case of $\bar{\bar{G}}_{m2}$. Again, the functions $\bar{M}_{eo}(K)$ and $\bar{N}_{oe}(K)$ are orthogonal,

$$\int_V \overline{M}_{eo}(K) \cdot \overline{N}_{oe}(K)\, dv = 0 \ , \tag{2.46}$$

due to the trigonometric function involved. The normalization constants are

$$\int_V \overline{M}_{eo}(K) \cdot \overline{M}_{eo}(K')\, dv = \begin{cases} 0 & K \neq K' \\ \frac{1}{C_M} & K = K' \end{cases} \tag{2.47}$$

and

$$\int_V \overline{N}_{oe}(K) \cdot \overline{N}_{oe}(K')\, dv = \begin{cases} 0 & K \neq K' \\ \frac{1}{C_N} & K = K' \end{cases} \tag{2.48}$$

where C_M and C_N are defined by (2.40) and (2.41). The boundary conditions for the $\overline{L}$-function is $\psi = 0$ on S. Therefore $A = C = E = 0$ and all the elementary functions in the generating function are odd and the expression of the $\overline{L}$-function becomes:

$$\overline{L}_{oo}(K) = \frac{\ell\pi}{a}\cos\frac{\ell\pi}{a}x \sin\frac{m\pi}{b}y \sin\frac{n\pi}{c}z\,\hat{x} + \frac{m\pi}{b}\sin\frac{\ell\pi}{a}x \cos\frac{m\pi}{b}y \sin\frac{n\pi}{c}z\,\hat{y} +$$

$$+ \frac{n\pi}{c}\sin\frac{\ell\pi}{a}x \sin\frac{m\pi}{b}y \cos\frac{n\pi}{c}z\,\hat{z} \ . \tag{2.49}$$

The $\overline{L}_{oo}(K)$-function is orthogonal to $\overline{M}_{eo}(K)$ and $\overline{N}_{oe}(K)$,

$$\int_V \overline{L}_{oo}(K) \cdot \overline{M}_{eo}(K)\, dv = 0 \tag{2.50}$$

$$\int_V \overline{L}_{oo}(K) \cdot \overline{N}_{oe}(K)\, dv = 0 \ , \tag{2.51}$$

due to trigonometric functions involved. The normalization constant is

$$\int_V \bar{L}_{oo}(K) \cdot \bar{L}_{oo}(K')\, dv = \begin{cases} 0 & K \neq K' \\ \frac{abc}{8} K^2 = \frac{1}{C_{Lo}} & K = K' \quad . \end{cases} \tag{2.52}$$

Now, we can expand $\bar{\bar{I}}\, \delta(\bar{R} - \bar{R}')$ in terms of the eigenfunctions as follows:

$$\bar{\bar{I}}\delta(\bar{R} - \bar{R}') = \sum \bar{L}_{oo}(K)\, \bar{A} + \sum \bar{M}_{eo}(K)\, \bar{B} + \sum \bar{N}_{oe}(K)\ \bar{C} \ .$$

Using the orthogonality relations (2.46) through (2.52) we determine the unknown vectors $\bar{A}$, $\bar{B}$, $\bar{C}$.

$$\bar{A} = C_{Lo}\, \bar{L}'_{eo} \ , \quad \bar{B} = C_M\, \bar{M}'_{eo} \ \text{ and } \ \bar{C} = C_N\, \bar{N}_{oe}$$

or

$$\bar{\bar{I}}\delta(\bar{R} - \bar{R}') = \sum C_{Lo} \bar{L}_{oo}(K)\bar{L}'_{oo}(K) + \sum C_M \bar{M}_{eo}(K)\bar{M}'_{eo}(K) + \sum C_N \bar{N}_{oe}(K)\bar{N}'_{oe}(K) \tag{2.53}$$

and the expansion of $\bar{\bar{G}}_{e1}$ has the form

$$\bar{\bar{G}}_{e1} = \sum \alpha\, C_{Lo}\, \bar{L}_{oo}(K)\, \bar{L}'_{oo}(K) + \sum \beta C_M \bar{M}_{eo}(K)\bar{M}'_{eo}(K) + \sum \alpha\, C_N \bar{N}_{oe}(K)\bar{N}'_{oe}(K).$$

Substituting the expansions of $\bar{\bar{G}}_{e1}$ and $\bar{\bar{I}}\, \delta(\bar{R}-\bar{R}')$ in equation (2.6) and remembering that $\bar{L}_{oo}$ is irrotational function, we determine the unknown coefficients α, β, and γ:

$$\alpha = -\frac{1}{k^2} \ \text{and} \ \ \beta = \gamma = \frac{1}{K^2 - k^2} \quad .$$

Finally,

$$\bar{\bar{G}}_{e1}(\bar{R}|\bar{R}') = -\sum \frac{C_{Lo}}{k^2} \bar{L}_{oo}(K)\bar{L}'_{oo}(K) + \sum \frac{C_M}{K^2-k^2} \bar{M}_{eo}(K)\bar{M}'_{eo}(K) + \sum \frac{C_N}{K^2-k^2} \bar{N}_{oe}(K)\bar{N}'_{oe}(K). \tag{2.54}$$

By eliminating $\bar{L}_{oo}(K)\,\bar{L}'_{oo}(K)$ between (2.53) and (2.54), and collecting similar terms, we have

$$\bar{\bar{G}}_{e1}(\bar{R}|\bar{R}') = -\frac{1}{k^2}\bar{\bar{I}}\delta(\bar{R}-\bar{R}') + \sum \frac{C_M K^2}{k^2(K^2-k^2)}\bar{M}_{eo}(K)\,\bar{M}'_{eo}(K) +$$

$$+\sum \frac{C_N K^2}{k^2(K^2-k^2)}\bar{N}_{oe}(K)\,\bar{N}'_{oe}(K) \quad . \qquad (2.55)$$

Let us direct our attention to the function $\bar{\bar{G}}_{m1}(\bar{R}|\bar{R}')$. Because $\bar{\bar{G}}_{e1}$ and $\bar{\bar{G}}_{m1}$ satisfy the same boundary conditions (see equations (2.8) and (2.9)), the vector wave function which are used in this case are $\bar{M}_{eo}(K)$ and $\bar{N}_{oe}(K)$, i.e., the same used in expansion of $\bar{\bar{G}}_{e1}(\bar{R}|\bar{R}')$. Therefore

$$\nabla \times \left[\bar{\bar{I}}\,\delta(\bar{R}-\bar{R}')\right] = \sum \bar{M}_{eo}(K)\,\bar{A} + \sum \bar{N}_{oe}(K)\,\bar{B} \, .$$

Pre-multiplying the above equation by $\bar{M}_{eo}(K)$ and $\bar{N}_{oe}(K)$, integrating through the volume of the cavity, and making use of orthogonality relations (2.46) through (2.48) we get

$$\bar{A} = C_M K\,\bar{N}'_{eo} \quad \text{and} \quad \bar{B} = C_N K\,\bar{M}'_{oe}$$

where

$$\int_V \bar{M}_{eo}\cdot\nabla \times \left[\bar{\bar{I}}\,\delta(\bar{R}-\bar{R}')\right] dv = K\,\bar{N}'_{eo}$$

and

$$\int_V \bar{N}_{oe}\cdot\nabla \times \left[\bar{\bar{I}}\,\delta(\bar{R}-\bar{R}')\right] dv = K\,\bar{M}'_{oe} \; .$$

Thus,

$$\nabla \times \left[\bar{\bar{I}}\,\delta(\bar{R}-\bar{R}')\right] = \sum C_M K\,\bar{M}_{eo}\,\bar{N}'_{eo} + \sum C_N K\,\bar{N}_{oe}\,\bar{M}'_{oe} \, .$$

The expansion of $\bar{\bar{G}}_{m1}$ can be written as

$$\bar{\bar{G}}_{m1} = \sum \alpha\, C_M\, K\, \bar{M}_{eo}\, \bar{N}'_{eo} + \sum \beta\, C_N K\, \bar{N}_{oe}\, \bar{M}'_{oe}$$

where α and β are constants yet to be determined.

Substituting the expressions for $\bar{\bar{G}}_{m1}$ and $\nabla \times \left[\bar{\bar{I}}\, \delta(\bar{R} - \bar{R}')\right]$ in equation (2.7), we find that

$$\alpha = \beta = \frac{1}{K^2 - k^2} \quad .$$

Finally,

$$\bar{\bar{G}}_{m1}(\bar{R}|\bar{R}') = \sum \frac{C_M K}{K^2-k^2}\, \bar{M}_{eo}(K)\, \bar{N}'_{eo}(K) + \sum \frac{C_N K}{K^2-k^2}\, \bar{N}_{oe}(K)\, \bar{M}'_{oe}(K) \; . \tag{2.56}$$

Let us write an expression of $\bar{\bar{G}}_{e2}(\bar{R}|\bar{R}')$ which involves the $\bar{L}$-function. The boundary condition for the $\bar{L}$-function is $\frac{\partial\psi}{\partial n} = 0$ on the surface, where, the derivative is taken in the direction normal to the surface S. This boundary condition makes B = D = F = 0 in expression (2.34) and, therefore, the generating function ψ is formed by the even functions only. Whence the expression for the $\bar{L}$-function is

$$\bar{L}_{ee} = -\frac{\ell\pi}{a} \sin\frac{\ell\pi}{a}x \cos\frac{m\pi}{b}y \cos\frac{n\pi}{c}z\, \hat{x} - \frac{m\pi}{b} \cos\frac{\ell\pi}{a}x \sin\frac{m\pi}{b}y \cos\frac{n\pi}{c}z\, \hat{y} -$$

$$- \frac{n\pi}{c} \cos\frac{\ell\pi}{a}x \cos\frac{m\pi}{b}y \sin\frac{n\pi}{c}z\, \hat{z} \; . \tag{2.57}$$

The solenoidal vector wave functions in this case are $\bar{M}_{oe}(K)$ and $\bar{N}_{eo}(K)$ because of the boundary conditions for $\bar{\bar{G}}_{e2}$. $\bar{L}_{ee}(K)$ is orthogonal to $\bar{M}_{oe}(K)$ and $\bar{N}_{eo}(K)$.

$$\int_V \bar{L}_{ee}(K) \cdot \bar{M}_{ee}(K)\, dv = 0$$

$$\int_V \bar{N}_{eo}(K) \cdot \bar{L}_{ee}(K)\, dv = 0$$

because of the trigonometric functions involved. The normalization constant is

$$\int_V \bar{L}_{ee}(K)\cdot\bar{L}_{ee}(K')\,dv = \begin{cases} 0 & K \neq K' \\ \frac{abc}{8}K^2(1+\delta_{o\ell})(1+\delta_{on})(1+\delta_{om}) = \frac{1}{C_{Le}} & K = K' \end{cases} \tag{2.58}$$

and the expansion of $\bar{\bar{I}}\delta(\bar{R}-\bar{R}')$ becomes

$$\bar{\bar{I}}\delta(\bar{R}-\bar{R}') = \sum \bar{L}_{ee}\bar{A} + \sum \bar{M}_{oe}\bar{B} + \sum \bar{N}_{eo}\bar{C} \ .$$

Pre-multiplying this expansion by $\bar{L}_{ee}$, $\bar{M}_{oe}$, $\bar{N}_{eo}$ and integrating through the volume of the cavity with the help of orthogonality relations (2.40), (2.41) and (2.58), we determine the unknown $\bar{A}$, $\bar{B}$ and $\bar{C}$ respectively. Thus,

$$\bar{A} = C_{Le}\bar{L}'_{ee}(K), \quad \bar{B} = C_M\bar{M}'_{oe}(K), \quad \bar{C} = C_N\bar{N}'_{eo}(K)$$

and

$$\bar{\bar{I}}\delta(\bar{R}-\bar{R}') = \sum C_{Le}\bar{L}_{ee}\bar{L}'_{ee} + \sum C_N\bar{M}_{oe}\bar{M}'_{oe} + \sum C_M\bar{N}_{eo}\bar{N}'_{eo} \ ,$$

whence the expansion of $\bar{\bar{G}}_{e2}(\bar{R}|\bar{R}')$ can be written as

$$\bar{\bar{G}}_{e2}(\bar{R}|\bar{R}') = \sum \alpha\, C_{Le}\bar{L}_{ee}\bar{L}'_{ee} + \sum \beta\, C_N\bar{M}_{oe}\bar{M}'_{oe} + \sum \gamma\, C_M\bar{N}_{eo}\bar{N}'_{eo}$$

where α, β and γ are the unknown coefficients.

Substituting the expansions of $\bar{\bar{G}}_{e2}(\bar{R}|\bar{R}')$ and $\bar{\bar{I}}\delta(\bar{R}|\bar{R}')$ in equation (2.6), we find that

$$\alpha = -\frac{1}{k^2}, \quad \beta = \gamma = \frac{1}{K^2-k^2}$$

or

$$\bar{\bar{G}}_{e2}(\bar{R}|\bar{R}') = -\sum \frac{C_{Le}\bar{L}_{ee}(K)\bar{L}'_{ee}(K)}{k^2} + \sum \frac{C_N}{K^2-k^2}\bar{M}_{oe}(K)\bar{M}'_{oe}(K) + \sum \frac{C_M}{K^2-k^2}\bar{N}_{eo}(K)\bar{N}'_{eo}(K) \ . \tag{2.59}$$

Eliminating $\bar{L}_{ee}$ $\bar{L}'_{ee}$ between (2.59) and the expansion of $\bar{\bar{I}}\,\delta(\bar{R} - \bar{R}')$ we find that

$$\bar{\bar{G}}_{e2} = -\frac{1}{k^2}\,\bar{\bar{I}}\,\delta(\bar{R} - \bar{R}') + \sum \frac{C_N K^2}{k^2(K^2-k^2)}\,\bar{M}_{oe}(K)\,\bar{M}'_{oe}(K) + \sum \frac{C_M K^2}{k^2(K^2-k^2)}\,\bar{N}_{eo}(K)\,\bar{N}'_{eo}(K)\,. \qquad (2.60)$$

The results of this section are summarized in Table I.

2.4 Dyadic Green's Functions for the Cylindrical Cavity

Figure 2 shows the geometry of the cylindrical cavity under consideration.

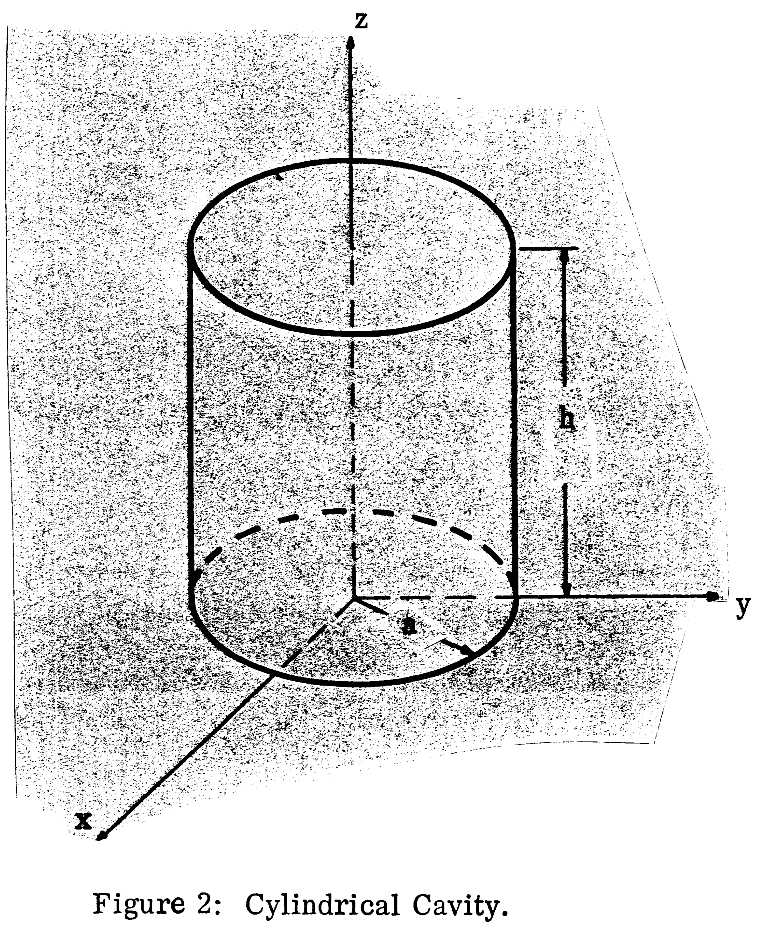

Figure 2: Cylindrical Cavity.

Table I: Dyadic Green's Functions for Rectangular Cavity.

$\bar{\bar{G}}_{e1}(\bar{R}\|\bar{R}')$	Eqn. (2.54)
$\bar{\bar{G}}_{e1}(\bar{R}\|\bar{R}')$	Eqn. (2.55)
$\bar{\bar{G}}_{e2}(\bar{R}\|\bar{R}')$	Eqn. (2.59)
$\bar{\bar{G}}_{e2}(\bar{R}\|\bar{R}')$	Eqn. (2.60)
$\bar{\bar{G}}_{m1}(\bar{R}\|\bar{R}')$	Eqn. (2.56)
$\bar{\bar{G}}_{m2}(\bar{R}\|\bar{R}')$	Eqn. (2.43)
K^2	$\left(\frac{\ell\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{c}\right)^2$
C_M^{-1}	$\frac{abc}{8}\left[\left(\frac{\ell\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right](1+\delta_{om})(1+\delta_{o\ell})$
C_N^{-1}	$\frac{abc}{8}\left[\left(\frac{\ell\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right](1+\delta_{on})$
C_{Lo}^{-1}	$\frac{abc}{8} K^2$
C_{Le}^{-1}	$\frac{abc}{8} K^2 (1+\delta_{o\ell})(1+\delta_{om})(1+\delta_{on})$
$\bar{M}_{oe}(K)$	Eqn. (2.38)
$\bar{M}_{eo}(K)$	Eqn. (2.44)
$\bar{N}_{oe}(K)$	Eqn. (2.45)
$\bar{N}_{eo}(K)$	Eqn. (2.39)
$\bar{L}_{oo}(K)$	Eqn. (2.49)
$\bar{L}_{ee}(K)$	Eqn. (2.57)

The differential equation for the generating function ψ (2.30) in cylindrical coordinate system is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} + K^2 \psi = 0 \quad . \qquad (2.61)$$

The method of separation of variables can be used in this system of coordinates. Thus, let $\psi = R\,\Phi\,Z$; substituting into (2.61) and dividing by $R\,\Phi\,Z$, we get

$$\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{rR}\frac{dR}{dr} + \frac{1}{r^2\Phi}\frac{d^2\Phi}{d\phi^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} + K^2 = 0 \quad .$$

Solving for Z component, we obtain

$$Z = A\cos\sqrt{K^2 - k_r^2}\,z + B\sin\sqrt{K^2 - k_r^2}\,z \ ,$$

where k_r is a separation constant still to be determined. The equation we are left with is of the form

$$\frac{r^2 d^2R}{R\,dr^2} + \frac{r^2}{rR}\frac{dR}{dr} + k_r^2 r^2 + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0$$

and allows us to solve for the ϕ component,

$$\Phi = C\cos k_\phi \phi + D\sin k_\phi \phi \ .$$

In order to ensure the uniqueness of the Φ function, the function must be single-valued for $0 \leq \phi \leq 2\pi$. This condition gives a value of k_ϕ which cannot be arbitrary but has to be an integer and therefore

$$k_\phi = n \qquad n = 0,\ 1,\ 2,\ \ldots.$$

whence

$$\Phi = C\cos n\phi + D\sin n\phi \ .$$

The differential equation for the r-component,

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(k_r^2 - \frac{n^2}{r^2}\right) R = 0 \ ,$$

has for its solution the cylindrical Bessel functions $J_n(k_r r)$ and the Neumann functions $Y_n(k_r r)$. The second solution is rejected because we need a finite solution at the origin.

Finally, the generating function has the form

$$\psi = J_n(k_r r)\,(C\cos n\phi + D\sin n\phi)(A\cos\sqrt{K^2 - k_r^2}\,z + B\sin\sqrt{K^2 - k_r^2}\,z).$$

The piloting vector in this case is again chosen to be $\hat{z}$. The vector wave function for the cylindrical cavity are

$$\bar{M} = \nabla\times(\psi\,\hat{z}) = \frac{n}{r} J_n(k_r r)(-C\sin n\phi + D\cos n\phi)(A\cos\sqrt{K^2-k_r^2}\,z + B\sin\sqrt{K^2-k_r^2}\,z)\,\hat{r}$$

$$- \frac{\partial J_n(k_r r)}{\partial r}(C\cos n\phi + D\sin n\phi)(A\cos\sqrt{K^2-k_r^2}\,z + B\sin\sqrt{K^2-k_r^2}\,z)\,\hat{\phi},$$

$$\bar{N} = \frac{1}{K}\nabla\times\nabla\times(\psi\,\hat{z}) = \frac{1}{K}\Bigg\{\sqrt{K^2-k_r^2}\,\frac{\partial J_n(k_r r)}{\partial r}(C\cos n\phi + D\sin n\phi)$$

$$(-A\sin\sqrt{K^2-k_r^2}\,z + B\cos\sqrt{K^2 - k_r^2}\,z)\,\hat{r} +$$

$$+ \sqrt{K^2-k_r^2}\,\frac{n}{r} J_n(k_r r)(-C\sin n\phi + D\cos n\phi)$$

$$(-A\sin\sqrt{K^2-k_r^2}\,z + B\cos\sqrt{K^2-k_r^2}\,z)\,\hat{\phi}$$

$$+ k_r^2 J_n(k_r r)(C\cos n\phi + D\sin n\phi)(A\cos\sqrt{K^2-k_r^2}\,z + B\sin\sqrt{K^2-k_r^2}\,z)\,\hat{z}\Bigg\}$$

and

$$\bar{L} = \nabla\psi = \frac{\partial J_n(k_r r)}{\partial r}(C\cos n\phi + D\sin n\phi)(A\cos\sqrt{K^2-k_r^2}\,z + B\sin\sqrt{K^2-k_r^2}\,z)\,\hat{r}$$

$$+ \frac{n}{r}J_n(k_r r)(-C\sin n\phi + D\cos n\phi)(A\cos\sqrt{K^2-k_r^2}\,z + B\sin\sqrt{K^2-k_r^2}\,z)\,\hat{\phi}$$

$$+\sqrt{K^2-k_r^2}\,J_n(k_r r)(C\cos n\phi + D\sin n\phi)(-A\sin\sqrt{K^2-k_r^2}\,z + B\cos\sqrt{K^2-k_r^2}\,z)\,\hat{z}\,.$$

Starting again with the $\bar{\bar{G}}_{m2}(\bar{R}|\bar{R}')$, and using the boundary conditions (2.11), we find that k_r for the $\bar{M}$-function is equal to λ, where $\lambda = p_{n\zeta}/a$ and $p_{n\zeta}$ is the root of the equation $J_n(x) = 0$. $p_{n\zeta}$ represents the ζ^{th} root of the cylindrical Bessel function of the n^{th} order. The eigenvalues in the z-direction are represented by $\frac{\ell\pi}{h}$, $\ell = 0, 1, 2, \ldots$ and $B = 0$.

The $\bar{M}$-functions for $\bar{\bar{G}}_{m2}$ are, therefore,

$$\bar{M}_{ee}(K_\lambda) = -\frac{n}{r}J_n(\lambda r)\sin n\phi\cos\frac{\ell\pi}{h}z\,\hat{r} - \frac{\partial J_n(\lambda r)}{\partial r}\cos n\phi\cos\frac{\ell\pi}{h}z\,\hat{\phi} \tag{2.62}$$

and

$$\bar{M}_{oe}(K_\lambda) = \frac{n}{r}J_n(\lambda r)\cos n\phi\cos\frac{\ell\pi}{h}z\,\hat{r} - \frac{\partial J_n(\lambda r)}{\partial r}\sin n\phi\cos\frac{\ell\pi}{h}z\,\hat{\phi} \tag{2.63}$$

where $K_\lambda^2 = \lambda^2 + \left(\frac{\ell\pi}{h}\right)^2$, the first subscript represents the ϕ-component of the generating function, and the second subscript represents the z-component of ψ.

Applying the boundary conditions (2.11), this time for the $\bar{N}$-function, we find that k_r is equal to μ where $\mu = q_{n\xi}/a$ and $q_{n\xi}$ is the root of the equation $\frac{dJ_n(x)}{dx} = 0$. $q_{n\xi}$ represents the ξth root of the derivative of the cylindrical Bessel function of the n^{th} order. The eigenvalue in the z-direction is represented by $\frac{m\pi}{h}$, $m = 0, 1, 2, \ldots$ and $A = 0$. The $\bar{N}$-functions for the $\bar{\bar{G}}_{m2}$ are, therefore,

$$\bar{N}_{eo}(K_\mu) = \frac{1}{K_\mu}\left[\frac{m\pi}{h}\frac{\partial J_n(\mu_r)}{\partial r}\cos n\phi \cos\frac{m\pi}{h}z\,\hat{r} - \frac{m\pi}{h}\frac{n}{r}J_n(\mu_r)\sin n\phi\cos\frac{m\pi}{h}z\,\hat{\phi} + \right.$$

$$\left. + \mu^2 J_n(\mu_r)\cos n\phi \sin\frac{m\pi}{h} z\,\hat{z}\right] \tag{2.64}$$

and

$$\bar{N}_{oo}(K_\mu) = \frac{1}{K_\mu}\left[\frac{m\pi}{h}\frac{\partial J_n(\mu_r)}{\partial r}\sin n\phi \cos\frac{m\pi}{h}z\,\hat{r} + \frac{m\pi}{h}\frac{n}{r}J_n(\mu_r)\cos n\phi\cos\frac{m\pi}{h}z\,\hat{\phi} + \right.$$

$$\left. + \mu^2 J_n(\mu_r)\sin n\phi \sin\frac{m\pi}{h} z\,\hat{z}\right]. \tag{2.65}$$

where $K_\mu^2 = \mu^2 + \left(\frac{m\pi}{h}\right)^2$.

The orthogonality of the $\bar{M}$- and $\bar{N}$-functions is shown as follows.

$$\int_V \bar{M}_{ee}(K_\lambda)\cdot\bar{M}_{oe}(K_\lambda)\,dv = 0 \tag{2.66}$$

because of trigonometric functions of ϕ ;

$$\int_V \bar{M}_{ee}(K_\lambda)\cdot\bar{N}_{eo}(K_\mu)\,dv = 0 \tag{2.67}$$

by the same reason.

$$\int_V \bar{M}_{ee}(K_\lambda)\cdot\bar{N}_{oo}(K_\mu)\,dv = 0$$

because of trigonometric functions of z, if $\ell \neq m$. If $\ell = m$, we have

$$\int_V \bar{M}_{ee}(K_\lambda)\cdot\bar{N}_{oo}(K_\mu)\,dv = \frac{1}{K_\mu}\left[\int_V -\frac{n}{r}\frac{m\pi}{h} J_n(\lambda r)\frac{\partial J_n(\mu_r)}{\partial r}\sin^2\phi\cos^2\frac{m\pi}{h}z - \right.$$

$$\left. -\frac{m\pi}{h}\frac{n}{r} J_n(\mu_r)\frac{\partial J_n(\lambda r)}{\partial r}\cos^2\phi\cos^2\frac{\ell\pi}{h}z\right] dv =$$

$$= -\frac{1}{K_\mu}\frac{m\pi}{2}^2\left[\int_0^a \frac{n}{r}\left(J_n(\lambda r)\frac{\partial J_n(\mu r)}{\partial r} + J_n(\mu r)\frac{\partial J_n(\lambda r)}{\partial r}\right) r\,dr\right] =$$

$$= -\frac{1}{K_\mu}\frac{m\,n\,\pi^2}{2} J_n(\lambda r)\,J_n(\mu r)\Big|_0^a = 0 \qquad (2.68)$$

because $J_n(\lambda a) = 0$.

$$\int_V \bar{N}_{eo}(K_\mu)\cdot\bar{N}_{oo}(K_\mu)\,dv = 0 \qquad (2.69)$$

because of trigonometric functions of ϕ.

$$\int \bar{M}_{oe}(K_\lambda)\cdot\bar{N}_{eo}(K_\mu)\,dv = 0 \qquad (2.70)$$

because of trigonometric functions of z, if $\ell \neq m$. If $\ell = m$, the proof runs as for (2.68). The normalization constants are:

$$\int_V \bar{M}_{ee}(K_\lambda)\cdot\bar{M}_{ee}(K_\lambda)\,dv = (1+\delta_{on})(1+\delta_{o\ell})\frac{\pi h}{2}\lambda^2 I_\lambda = 1/C_{Ne} \qquad (2.71)$$

where

$$I_\lambda = \int_0^a J_n^2(\lambda r)\,r\,dr = \frac{a^2}{2\lambda^2}\left[\frac{\partial J_n(\lambda r)}{\partial r}\right]^2\Bigg|_{r=a}; \qquad (2.72)$$

and

$$\int_V \bar{M}_{oe}(K_\lambda)\cdot\bar{M}_{oe}(K_\lambda)\,dv = (1+\delta_{o\ell})\frac{\pi h}{2}\lambda^2 I_\lambda = 1/C_{No}. \qquad (2.73)$$

$$\int_V \bar{N}_{eo}(K_\mu) \cdot \bar{N}_{eo}(K_\mu)\, dv = (1 + \delta_{on}) \frac{\pi h}{2} \mu^2 I_\mu = 1/C_{Me} \qquad (2.74)$$

where

$$I_\mu = \int_0^a J_n^2(\mu r)\, r\, dr = \frac{a^2}{2\mu^2} \left(\mu^2 - \frac{n^2}{a^2}\right) J_n^2(\mu_a) \quad ; \qquad (2.75)$$

and

$$\int_V \bar{N}_{oo}(K_\mu) \cdot \bar{N}_{oo}(K_\mu)\, dv = \frac{\pi h}{2} \mu^2 I_\mu = 1/C_{Mo} \quad . \qquad (2.76)$$

$$\int_V \bar{M}_{ee}(K_\lambda) \cdot \bar{M}_{ee}(K_{\lambda'})\, dv = 0 \qquad (2.77)$$

and

$$\int_V \bar{M}_{oe}(K_\lambda) \cdot \bar{M}_{oe}(K_{\lambda'})\, dv = 0 \qquad (2.78)$$

because of the orthogonality of the cylindrical Bessel functions, namely

$$\int_0^a J_n(\lambda r) \cdot J_n(\lambda' r)\, r dr = 0 \,. \qquad (2.79)$$

$$\int_0^a \bar{N}_{eo}(K_\mu) \cdot \bar{N}_{eo}(K_{\mu'})\, dv = 0 \qquad (2.80)$$

and

$$\int_V \bar{N}_{oo}(K_\mu) \cdot \bar{N}_{oo}(K_{\mu'})\, dv = 0 \qquad (2.81)$$

because of the orthogonality of the cylindrical Bessel functions, namely

$$\int_0^a J_n(\mu r) \cdot J_n(\mu' R)\, r dr = 0 \quad . \qquad (2.82)$$

The expansion of $\nabla \times \left[\bar{\bar{I}}\, \delta(\bar{R} - \bar{R}')\right]$ is written as

$$\nabla \times \left[\bar{\bar{I}}\delta(\bar{R}-\bar{R}')\right] = \sum \bar{M}_{ee}(K_\lambda)\bar{A} + \sum \bar{M}_{oe}(K_\lambda)\bar{B} + \sum \bar{N}_{eo}(K_\mu)\bar{C} + \sum \bar{N}_{oo}(K_\mu)\,\bar{D}\,,$$

where $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ are four sets of unknown vectors. Pre-multiplying this expansion by $\bar{M}_{ee}(K_\lambda)$, $\bar{M}_{oe}(K_\lambda)$, $\bar{N}_{eo}(K_\mu)$ and $\bar{N}_{oo}(K_\mu)$, respectively, integrating through the volume of the cavity, and using the orthogonality relations (2.66) through (2.80), we find

$$\bar{A} = C_{Ne}\, K_\lambda\, \bar{N}'_{ee}\,(K_\lambda)\,, \quad \bar{B} = C_{No}\, K_\lambda\, \bar{N}'_{oe}\,(K_\lambda)$$

and

$$\bar{C} = C_{Me} K_\mu\, \bar{M}'_{eo}\,(K_\mu)\,, \quad \bar{D} = C_{Mo}\, K_\mu\, \bar{M}'_{oo}\,(K_\mu)$$

where

$$\int_V \bar{M}_{ee}\,(K_\lambda) \cdot \nabla \times \left[\bar{\bar{I}}\delta(\bar{R} - \bar{R}')\right] dv = K_\lambda\, \bar{N}'_{ee}\,(K_\lambda)$$

$$\int_V \bar{M}_{oe}(K_\lambda) \cdot \nabla \times \left[\bar{\bar{I}}\, \delta(\bar{R} - \bar{R}')\right] dv = K_\lambda\, \bar{N}'_{oe}\,(K_\lambda)$$

$$\int_V \bar{N}_{eo}\,(K_\mu) \cdot \nabla \times \left[\bar{\bar{I}}\delta(\bar{R} - \bar{R}')\right] dv = K_\mu\, \bar{M}'_{eo}\,(K_\mu)$$

and

$$\int_V \bar{N}_{oo}\,(K_\mu) \cdot \nabla \times \left[\bar{\bar{I}}\, \delta(\bar{R} - \bar{R}')\right] dv = K_\mu\, \bar{M}'_{oo}\,(K_\mu)$$

was used. Therefore,

$$\nabla \times \left[\bar{\bar{I}}\, \delta(\bar{R}-\bar{R}')\right] = \sum C_{Ne}\, K_\lambda \bar{M}_{ee}(K_\lambda)\bar{N}'_{ee}(K_\lambda) + \sum C_{No} K_\lambda \bar{M}_{oe}(K_\lambda)\bar{N}'_{oe}(K_\lambda)$$

$$+ \sum C_{Me} K_\mu \bar{N}_{eo}(K_\mu)\bar{M}'_{eo}(K_\mu) + \sum C_{mo} K_\mu \bar{N}_{oo}(K_\mu)\bar{M}'_{oo}(K_\mu)\,, \quad (2.83)$$

whence, the expansion of $\bar{\bar{G}}_{m2}(\bar{R}|\bar{R}')$ can be written as

$$\bar{\bar{G}}_{m2}(\bar{R}|\bar{R}') = \sum \alpha\, C_{Ne} K_\lambda \bar{M}_{ee}(K_\lambda)\bar{N}'_{ee}(K_\lambda) + \sum \beta\, C_{No} K_\lambda \bar{M}_{oe}(K_\lambda)\bar{N}'_{oe}(K_\lambda)$$

$$+ \sum \gamma\, C_{Me} K_\mu \bar{N}_{eo}(K_\mu)\bar{M}'_{eo}(K_\mu) + \sum \delta\, C_{Mo} K_\mu \bar{N}_{oo}(K_\mu)\bar{M}'_{oo}(K_\mu).$$

Substituting the expansion of $\bar{\bar{G}}_{m2}(\bar{R}|\bar{R}')$ and (2.83) in (2.7), we find the unknown coefficients α, β, γ and δ:

$$\alpha = \beta = \frac{1}{K_\lambda^2 - k^2} \quad \text{and} \quad \gamma = \delta = \frac{1}{K_\mu^2 - k^2} \; .$$

Hence,

$$\bar{\bar{G}}_{m2}(\bar{R}|\bar{R}') = \sum \frac{C_{Ne} K_\lambda}{K_\lambda^2 - k^2} \bar{M}_{ee}(K_\lambda)\, \bar{N}'_{ee}(K_\lambda) + \sum \frac{C_{No} K_\lambda}{K_\lambda^2 - k^2} \bar{M}_{oe}(K_\lambda)\, \bar{N}'_{oe}(K_\lambda) +$$

$$+ \sum \frac{C_{Me} K_\mu}{K_\mu^2 - k^2} \bar{N}_{eo}(K_\mu)\, \bar{M}'_{eo}(K_\mu) + \sum \frac{C_{Mo} K_\mu}{K_\mu^2 - k^2} \bar{N}_{oo}(K_\mu)\bar{M}'_{eo}(K_\mu) \; . \tag{2.84}$$

Boundary conditions (2.8) for $\bar{\bar{G}}_{e1}(\bar{R}|\bar{R}')$ require the following sets of functions

$$\bar{M}_{eo}(K_\mu) = -\frac{n}{r} J_n(\mu r) \sin n\phi \sin \frac{m\pi}{h} z\, \hat{r} - \frac{\partial J_n(\mu r)}{\partial r} \cos n\phi \sin \frac{m\pi}{h} z\, \hat{\phi} \tag{2.85}$$

$$\bar{M}_{oo}(K_\mu) = \frac{n}{r} J_n(\mu r) \cos n\phi \sin \frac{m\pi}{h} z\, \hat{r} - \frac{\partial J_n(\mu r)}{\partial r} \sin n\phi \sin \frac{m\pi}{h} z\, \hat{\phi} \tag{2.86}$$

$$\bar{N}_{ee}(K_\lambda) = \frac{1}{K_\lambda} \Big\{ - \frac{\ell\pi}{h} \frac{\partial J_n(\lambda r)}{\partial r} \cos n\phi \sin \frac{\ell\pi}{h} z\, \hat{r} + \frac{\ell\pi n}{h r} J_n(\lambda r) \sin n\phi \sin \frac{\ell\pi}{h} z\, \hat{\phi} +$$

$$+ \lambda^2 J_n(\lambda r) \cos n\phi \cos \frac{\ell\pi}{h} z\, \hat{z} \Big\} \tag{2.87}$$

$$\bar{N}_{oe}(K_\lambda) = \frac{1}{K_\lambda}\left\{ -\frac{\ell\pi}{h}\frac{\partial J_n(\lambda r)}{\partial r} \sin n\phi \sin\frac{\ell\pi}{h} z\,\hat{r} - \frac{\ell\pi}{h}\frac{n}{r} J_n(\lambda r)\cos n\phi \sin\frac{\ell\pi}{h} z\,\hat{\phi} + \right.$$

$$\left. + \lambda^2 J_n(\lambda r) \sin n\phi \cos\frac{\ell\pi}{h} z\,\hat{z} \right\} \tag{2.88}$$

$$\bar{L}_{eo}(K_\lambda) = \frac{\partial J_n(\lambda r)}{\partial r}\cos n\phi \sin\frac{\ell\pi}{h} z\,\hat{r} - \frac{n}{r} J_n(\lambda r) \sin n\phi \sin\frac{\ell\pi}{h} z\,\hat{\phi} +$$

$$+ \frac{\ell\pi}{h} J_n(\lambda r) \cos n\phi \cos\frac{\ell\pi}{h} z\,\hat{z} \tag{2.89}$$

$$\bar{L}_{oo}(K_\lambda) = \frac{\partial J_n(\lambda r)}{\partial r}\sin n\phi \sin\frac{\ell\pi}{h} z\,\hat{r} + \frac{n}{r} J_n(\lambda r) \cos n\phi \sin\frac{\ell\pi}{h} z\,\hat{\phi} +$$

$$+ \frac{\ell\pi}{h} J_n(\lambda r) \sin n\phi \cos\frac{\ell\pi}{h} z\,\hat{z} \quad . \tag{2.90}$$

The orthogonality relations are checked as follows:

$$\int_V \bar{M}_{eo}(K_\mu)\cdot\bar{M}_{oo}(K_\mu)\,dv = 0$$

$$\int_V \bar{M}_{eo}(K_\mu)\cdot\bar{N}_{ee}(K_\lambda)\,dv = 0$$

$$\int_V \bar{M}_{eo}(K_\mu)\cdot\bar{L}_{eo}(K_\lambda)\,dv = 0$$

$$\int_V \bar{M}_{oo}(K_\mu)\cdot\bar{N}_{oe}(K_\lambda)\,dv = 0 \tag{2.91}$$

$$\int_V \bar{M}_{oo}(K_\mu)\cdot\bar{L}_{oo}(K_\lambda)\,dv = 0$$

$$\int_V \bar{N}_{ee}(K_\lambda) \cdot \bar{N}_{oe}(K_\lambda) dv = 0$$

$$\int_V \bar{N}_{ee}(K_\lambda) \cdot \bar{L}_{oo}(K_\lambda) dv = 0$$

$$\int_V \bar{N}_{oe}(K_\lambda) \cdot \bar{L}_{eo}(K_\lambda) dv = 0$$

$$\int_V \bar{L}_{eo}(K_\lambda) \cdot \bar{L}_{oo}(K_\lambda) dv = 0$$

because of trigonometric functions of ϕ, and

$$\int_V \bar{M}_{eo}(K_\mu) \cdot \bar{N}_{oe}(K_\lambda) dv = 0$$

$$\int_V \bar{M}_{eo}(K_\mu) \cdot \bar{L}_{oo}(K_\lambda) dv = 0 \qquad (2.92)$$

$$\int_V \bar{M}_{oo}(K_\mu) \cdot \bar{N}_{ee}(K_\lambda) dv = 0$$

$$\int_V \bar{M}_{oo}(K_\mu) \cdot \bar{L}_{eo}(K_\lambda) dv = 0$$

which can be proved as (2.68). The proof of

$$\int_V \bar{N}_{ee}(K_\lambda) \cdot \bar{L}_{eo}(K_\lambda) dv = 0 \qquad (2.93)$$

is slightly different. We have

$$\int_V \bar{N}_{ee}(K_\lambda)\cdot \bar{L}_{eo}(K_\lambda)dv = \frac{1}{K_\lambda}\frac{\ell\pi}{h}(1+\delta_{on})\left\{-\frac{\pi h}{2}\int_0^a\left(\frac{\partial J_n(\lambda r)}{.\partial r}\right)^2 rdr - \frac{\pi h}{2}\int_0^a (\frac{n}{r})^2 J_n^2(\lambda r)\, rdr + \right.$$

$$\left. + \lambda^2 \frac{\pi h}{2}\int_0^a J_n^2(\lambda r)\, rdr\right\} =$$

$$= \frac{(1+\delta_{on})\,\ell\pi^2}{2K_\lambda}\left\{-\int_0^a\left[\left(\frac{\partial J_n(\lambda r)}{\partial r}\right)^2 + (\frac{n}{r})^2 J_n^2(\lambda r)\right] rdr + \lambda^2\int_0^a J_n^2(\lambda r)\, rdr\right\}$$

$$= 0$$

because $\int_0^a \left[\left(\frac{\partial J_n(\lambda r)}{\partial r}\right)^2 + \frac{n^2 J_n^2(\lambda r)}{r^2}\right] rdr = \int_0^a \lambda^2 J_n^2(\lambda r)\, rdr$ performing the integration by parts.

$$\int_V \bar{N}_{oe}(K_\lambda)\cdot \bar{L}_{oo}(K_\lambda)\, dv = 0 \tag{2.94}$$

is proved by the same token. The normalization constants are:

$$\int_V \bar{M}_{eo}(K_\mu)\cdot \bar{M}_{eo}(K_\mu)dv = \frac{1}{C_{Me}} \tag{2.95}$$

$$\int_V \bar{M}_{oo}(K_\mu)\cdot \bar{M}_{oo}(K_\mu)dv = \frac{1}{C_{Mo}} \tag{2.96}$$

$$\int_V \bar{N}_{ee}(K_\lambda)\cdot \bar{N}_{ee}(K_\lambda)\, dv = \frac{1}{C_{Ne}} \tag{2.97}$$

$$\int_V \bar{N}_{ee}(K_\lambda)\cdot \bar{N}_{oe}(K_\lambda)\, dv = \frac{1}{C_{No}} \tag{2.98}$$

where C_{Me} is defined by expression (2.74), C_{Mo} by (2.76), C_{Ne} by (2.71) and C_{No} by (2.73). The normalization constants for the $\bar{L}$ functions are given by

$$\int_V \bar{L}_{eo}(K_\lambda) \cdot \bar{L}_{eo}(K_\lambda)\, dv = (1 + \delta_{on}) \frac{\pi h}{2} K_\lambda^2 I_\lambda = 1/C_{L\lambda e} \quad (2.99)$$

and

$$\int_V \bar{L}_{oo}(K_\lambda) \cdot \bar{L}_{oo}(K_\lambda)\, dv = \frac{\pi h}{2} K_\lambda^2 I_\lambda = 1/C_{L\lambda o} \quad . \quad (2.100)$$

$$\begin{aligned}
&\int_V \bar{M}_{eo}(K_\mu) \cdot \bar{M}_{eo}(K_{\mu'})\, dv = 0 \\
&\int_V \bar{M}_{oo}(K_\mu) \cdot \bar{M}_{oo}(K_{\mu'})\, dv = 0 \\
&\int_V \bar{N}_{ee}(K_\lambda) \cdot \bar{N}_{ee}(K_{\lambda'})\, dv = 0 \\
&\int_V \bar{N}_{oe}(K_\lambda) \cdot \bar{N}_{oe}(K_{\lambda'})\, dv = 0 \\
&\int_V \bar{L}_{eo}(K_\lambda) \cdot \bar{L}_{eo}(K_{\lambda'})\, dv = 0 \\
&\int_V \bar{L}_{oo}(K_\lambda) \cdot \bar{L}_{oo}(K_{\lambda'})\, dv = 0
\end{aligned} \quad (2.101)$$

due to the orthogonality properties of the cylindrical Bessel functions (2.79) and (2.82). The expansion of $\bar{\bar{I}}\,\delta(\bar{R} - \bar{R}')$ is written as

$$\bar{\bar{I}}\,\delta(\bar{R} - \bar{R}') = \sum \bar{L}_{eo}(K_\lambda)\,\bar{A} + \sum \bar{L}_{oo}(K_\lambda)\,\bar{B} + \sum \bar{M}_{eo}(K_\mu)\,\bar{C} + \sum \bar{M}_{oo}(K_\mu)\,\bar{D} +$$

$$+ \sum \bar{N}_{ee}(K_\lambda)\,\bar{E} + \sum \bar{N}_{oe}(K_\lambda)\,\bar{F} \ .$$

The unknown vectors $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$, $\bar{E}$ and $\bar{F}$ are found by pre-multiplying the above expansion by $\bar{L}_{eo}(K_\lambda)$, $\bar{L}_{oo}(K_\lambda)$, $\bar{M}_{eo}(K_\mu)$, $\bar{M}_{oo}(K_\mu)$, $\bar{N}_{ee}(K_\lambda)$ and $\bar{N}_{oe}(K_\lambda)$, respectively, integrating through the volume of the cavity, and making use of the orthogonality relations (2.91) through (2.101). Thus

$$\bar{A} = C_{L\lambda e}\,\bar{L}'_{eo}(K_\lambda),\quad \bar{B} = C_{L\lambda o}\,\bar{L}_{oo}(K_\lambda),\quad \bar{C} = C_{Me}\,\bar{M}'_{eo}(K_\mu)$$

$$\bar{D} = C_{Mo}\,\bar{M}_{oo}(K_\mu),\quad \bar{E} = C_{Ne}\,\bar{N}'_{ee}(K_\lambda),\quad \bar{F} = C_{No}\,\bar{N}'_{oe}(K_\lambda)$$

and

$$\bar{\bar{I}}\delta(\bar{R}-\bar{R}') = \sum C_{L\lambda e}\bar{L}_{eo}(K_\lambda)\bar{L}'_{eo}(K_\lambda) + \sum C_{L\lambda o}\bar{L}_{oo}(K_\lambda)\bar{L}'_{oo}(K_\lambda) + \sum C_{Me}\bar{M}_{eo}(K_\mu)\bar{M}'_{eo}(K_\mu) +$$
$$+ \sum C_{Mo}\bar{M}_{oo}(K_\mu)\bar{M}'_{oo}(K_\mu) + \sum C_{Ne}\bar{N}_{ee}(K_\lambda)\bar{N}'_{ee}(K_\lambda) + \sum C_{No}\bar{N}_{oe}(K_\lambda)\bar{N}'_{oe}(K_\lambda)\,. \tag{2.102}$$

Write the expansion of $\bar{\bar{G}}_{e1}(\bar{R}|\bar{R}')$ as

$$\bar{\bar{G}}_{e1}(\bar{R}|\bar{R}') = \sum \alpha\, C_{L\lambda e}\,\bar{L}_{eo}(K_\lambda)\,\bar{L}'_{eo}(K_\lambda) + \sum \beta C_{L\lambda o}\,\bar{L}_{oo}(K_\lambda)\,\bar{L}'_{oo}(K_\lambda) +$$
$$+ \sum \gamma\, C_{Me}\bar{M}_{eo}(K_\mu)\,\bar{M}'_{eo}(K_\mu) + \sum \delta\, C_{Mo}\bar{M}_{oo}(K_\mu)\,\bar{M}_{oo}(K_\mu) +$$
$$+ \sum \epsilon\, C_{Ne}\bar{N}_{ee}(K_\lambda)\,\bar{N}'_{ee}(K_\lambda) + \sum \zeta C_{No}\bar{N}_{oe}(K_\lambda)\bar{N}'_{oe}(K_\lambda)\quad.$$

Substituting this expansion and (2.102) into (2.6), we find that

$$\alpha = \beta = -\frac{1}{k^2}\,,\quad \gamma = \delta = \frac{1}{K_\mu^2 - k^2}\quad \text{and}\quad \epsilon = \zeta = \frac{1}{K_\lambda^2 - k^2}\quad.$$

Therefore,

$$\bar{\bar{G}}_{e1}(\bar{R}|\bar{R}') = -\sum \frac{C_{L\lambda e}}{k^2} \bar{L}_{eo}(K_\lambda)\, \bar{L}'_{eo}(K_\lambda) - \sum \frac{C_{L\lambda o}}{k^2} \bar{L}_{oo}(K_\lambda)\, \bar{L}'_{oo}(K_\lambda) +$$

$$+\sum \frac{C_{Me}}{K_\mu^2 - k^2} \bar{M}_{eo}(K_\mu)\bar{M}'_{eo}(K_\mu) + \sum \frac{C_{Mo}}{K_\mu^2 - k^2} \bar{M}_{oo}(K_\mu)\bar{M}'_{oo}(K_\mu) +$$

$$+\sum \frac{C_{Ne}}{K_\lambda^2 - k^2} \bar{N}_{ee}(K_\lambda)\bar{N}'_{ee}(K_\lambda) + \sum \frac{C_{No}}{K_\lambda^2 - k^2} \bar{N}_{oe}(K_\lambda)\, \bar{N}'_{ee}(K_\lambda) \quad . \qquad (2.103)$$

By eliminating $\sum C_{L\lambda e}\, \bar{L}_{eo}(K_\lambda)\, \bar{L}'_{eo}(K_\lambda) + \sum C_{L\lambda o}(K_\lambda)\bar{L}_{oo}(K_\lambda)\bar{L}'_{oo}(K_\lambda)$ between (2.103) and (2.102), we get the second form of expansion of $\bar{\bar{G}}_{e1}(\bar{R}|\bar{R}')$:

$$\bar{\bar{G}}_{e1}(\bar{R}|\bar{R}') = -\frac{1}{k^2}\bar{\bar{I}}\delta(\bar{R}-\bar{R}') + \sum \frac{C_{Me}K_\mu^2}{k^2(K_\mu^2 - k^2)} \bar{M}_{eo}(K_\mu)\, \bar{M}'_{eo}(K_\mu) +$$

$$+\sum \frac{C_{Mo}K_\mu^2}{k^2(K_\mu^2 - k^2)} \bar{M}_{oo}(K_\mu)\bar{M}'_{oo}(K_\mu) + \sum \frac{C_{Ne}K_\lambda^2}{k^2(K_\lambda^2 - k^2)} \bar{N}_{ee}(K_\lambda)\bar{N}'_{ee}(K_\lambda) +$$

$$+\sum \frac{C_{No}K_\lambda^2}{k^2(K_\lambda^2 - k^2)} \bar{N}_{oe}(K_\lambda)\bar{N}'_{oe}(K_\lambda) \quad . \qquad (2.104)$$

In a similar manner, we find that

$$\bar{\bar{G}}_{m1}(\bar{R}|\bar{R}') = \sum \frac{C_{Me}K_\mu}{K_\mu^2 - k^2} \bar{M}_{eo}(K_\mu)\bar{N}'_{eo}(K_\mu) + \sum \frac{C_{Mo}K_\mu}{K_\mu^2 - k^2} \bar{M}_{oo}(K_\mu)\, \bar{N}'_{oo}(K_\mu) +$$

$$+\sum \frac{C_{Ne}K_\lambda}{K_\lambda^2 - k^2} \bar{N}_{ee}(K_\lambda)\bar{M}'_{ee}(K_\lambda) + \sum \frac{C_{No}K_\lambda}{K_\lambda^2 - k^2} \bar{N}_{oe}(K_\lambda)\bar{M}'_{oe}(K_\lambda) \qquad (2.105)$$

and

$$\bar{\bar{G}}_{e2}(\bar{R}|\bar{R}') = -\sum \frac{C_{L\mu e}}{k^2} \bar{L}_{ee}(K_\mu)\bar{L}'_{ee}(K_\mu) - \sum \frac{C_{L\mu o}}{k^2} \bar{L}_{oe}(K_\mu) L'_{oe}(K_\mu) +$$

$$+\sum \frac{C_{Ne}}{K_\lambda^2 - k^2} \bar{M}_{ee}(K_\lambda)\bar{M}'_{ee}(K_\lambda) + \sum \frac{C_{No}}{K_\lambda^2 - k^2} \bar{M}_{oe}(K_\lambda)\bar{M}'_{oe}(K_\lambda) +$$

$$+\sum \frac{C_{Me}}{K_\mu^2 - k^2} \bar{N}_{eo}(K_\mu)\bar{N}'_{eo}(K_\mu) + \sum \frac{C_{Mo}}{K_\mu^2 - k^2} \bar{N}_{oo}(K_\mu)\bar{N}'_{oo}(K_\mu) \tag{2.106}$$

where

$$\bar{L}_{ee}(K_\mu) = \frac{\partial J_n(\mu r)}{\partial r} \cos n\phi \cos \frac{m\pi}{h} z \hat{r} - \frac{n}{r} J_n(\mu r) \sin n\phi \cos \frac{m\pi}{h} z \hat{\phi} -$$

$$- \frac{m\pi}{h} J_n(\mu r) \cos n\phi \sin \frac{m\pi}{h} z \hat{z} \tag{2.107}$$

and

$$\bar{L}_{oe}(K_\mu) = \frac{\partial J_n(\mu r)}{\partial r} \sin n\phi \cos \frac{m\pi}{h} z \hat{r} + \frac{n}{r} J_n(\mu r) \cos n\phi \cos \frac{m\pi}{h} z \hat{\phi} -$$

$$- \frac{m\pi}{h} J_n(\mu r) \sin n\phi \sin \frac{m\pi}{h} z \hat{z} \quad . \tag{2.108}$$

The orthogonality relations between $\bar{L}_{ee}(K_\mu)$ and $\bar{L}_{oe}(K_\mu)$ and other functions involved run as follows.

$$\int_V \bar{M}_{ee}(K_\lambda) \cdot \bar{L}_{ee}(K_\mu)\, dv = 0$$

$$\int_V \bar{M}_{oe}(K_\lambda) \cdot \bar{L}_{oe}(K_\mu)\, dv = 0$$

$$\int_V \bar{N}_{eo}(K_\mu) \cdot \bar{L}_{oe}(K_\mu)\, dv = 0 \tag{2.109}$$

$$\int_V \bar{N}_{oo}(K_\mu) \cdot \bar{L}_{ee}(K_\mu)\, dv = 0$$

$$\int_V \bar{L}_{ee}(K_\mu) \cdot \bar{L}_{oe}(K_\mu)\, dv = 0$$

because of the trigonometric functions of ϕ .

$$\int_V \bar{M}_{ee}(K_\lambda) \cdot \bar{L}_{oe}(K_\mu)\, dv = 0$$

(2.110)

$$\int_V \bar{M}_{oe}(K_\lambda) \cdot \bar{L}_{ee}(K_\mu)\, dv = 0$$

are proved in the same way as (2.68).

$$\int_V \bar{N}_{eo}(K_\mu) \cdot \bar{L}_{ee}(K_\mu)\, dv = 0$$

(2.111)

$$\int_V \bar{N}_{oo}(K_\mu) \cdot \bar{L}_{oe}(K_\mu)\, dv = 0$$

follow the proof of relation (2.93).

The normalization constants are

$$\int_V \bar{L}_{ee}(K_\mu) \cdot \bar{L}_{ee}(K_\mu)\, dv = (1 + \delta_{on}) \frac{\pi h}{2} K_\mu^2 I_\mu = 1/C_{L\mu e} \tag{2.112}$$

and

$$\int_V \bar{L}_{oe}(K_\mu) \cdot \bar{L}_{oe}(K_\mu)\, dv = \frac{\pi h}{2} K_\mu^2 I_\mu = 1/C_{L\mu o} \quad . \tag{2.113}$$

$$\int_V \bar{L}_{ee}(K_\mu) \cdot \bar{L}_{ee}(K_{\mu'})\, dv = 0$$

and

$$\int_V \bar{L}_{oe}(K_\mu) \cdot \bar{L}_{oe}(K_{\mu'})\, dv = 0$$

because of the orthogonality properties of the cylindrical Bessel functions (2.79) and (2.82).

The second form of the expansion of $\bar{\bar{G}}_{e2}(\bar{R}|\bar{R}')$ is given by

$$\bar{\bar{G}}_{e2}(\bar{R}|\bar{R}') = -\frac{1}{k^2}\bar{\bar{I}}\delta(\bar{R}-\bar{R}') + \sum \frac{C_{Ne}K_\lambda^2}{k^2(K_\lambda^2-k^2)} \bar{M}_{ee}(K_\lambda)\bar{M}'_{ee}(K_\lambda) +$$

$$+\sum \frac{C_{No}K_\lambda^2}{k^2(K_\lambda^2-k^2)} \bar{M}_{oe}(K_\lambda)\bar{M}'_{oe}(K_\lambda) + \sum \frac{C_{Me}K_\mu^2}{k^2(K_\mu^2-k^2)} \bar{N}_{eo}(K_\mu)\bar{N}'_{eo}(K_\mu) +$$

$$+\sum \frac{C_{Mo}K_\mu^2}{k^2(K_\mu^2-k^2)} \bar{N}_{oo}(K_\mu)\bar{N}'_{oo}(K_\mu) \qquad (2.114)$$

where the constants C_{Ne}, C_{No}, C_{Me}, and C_{Mo} are given by (2.71), (2.73), (2.74), and (2.76), respectively.

The results of this section are summarized in Table II.

2.5 Dyadic Green's Functions for Spherical Cavity

The spherical cavity of interest is shown in Figure 3. The origin of coordinates coincide with the center of the sphere. The radius of the sphere is a. The generating function of ψ (equation (2.30)) in spherical coordinates is

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial\psi}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} + K^2\psi = 0 \ . \qquad (2.115)$$

Table II: Dyadic Green's Functions for Cylindrical Cavity

$\bar{\bar{G}}_{e1}(\bar{R}\|\bar{R}')$	Eqn. (2.103)
$\bar{\bar{G}}_{e1}(\bar{R}\|\bar{R}')$	Eqn. (2.104)
$\bar{\bar{G}}_{e2}(\bar{R}\|\bar{R}')$	Eqn. (2.106)
$\bar{\bar{G}}_{e2}(\bar{R}\|\bar{R}')$	Eqn. (2.114)
$\bar{\bar{G}}_{m1}(\bar{R}\|\bar{R}')$	Eqn. (2.105)
$\bar{\bar{G}}_{m2}(\bar{R}\|\bar{R}')$	Eqn. (2.84)
K_λ^2	$\lambda^2 + \left(\frac{\ell\pi}{h}\right)^2$
K_μ^2	$\mu^2 + \left(\frac{m\pi}{h}\right)^2$
C_{Me}^{-1}	$(1 + \delta_{on}) \frac{\pi h}{2} \mu^2 I_\mu$
C_{Mo}^{-1}	$\frac{\pi h}{2} \mu^2 I_\mu$
C_{Ne}^{-1}	$(1+\delta_{on})(1+\delta_{o\ell}) \frac{\pi h}{2} \lambda^2 I_\lambda$
C_{No}^{-1}	$(1+\delta_{o\ell}) \frac{\pi h}{2} \lambda^2 I_\lambda$
$C_{L\lambda e}^{-1}$	$(1+\delta_{on}) \frac{\pi h}{2} K_\lambda^2 I_\lambda$
$C_{L\lambda o}^{-1}$	$\frac{\pi h}{2} K_\lambda^2 I_\lambda$
$C_{L\mu e}^{-1}$	$(1+\delta_{on}) \frac{\pi h}{2} K_\mu^2 I_\mu$
$C_{L\mu o}^{-1}$	$\frac{\pi h}{2} K_\mu^2 I_\mu$

Table II: continued

$\overline{M}_{ee}(K_\lambda)$	Eqn. (2.62)
$\overline{M}_{oe}(K_\lambda)$	Eqn. (2.63)
$\overline{M}_{eo}(K_\mu)$	Eqn. (2.85)
$\overline{M}_{oo}(K_\mu)$	Eqn. (2.86)
$\overline{N}_{eo}(K_\mu)$	Eqn. (2.64)
$\overline{N}_{oo}(K_\mu)$	Eqn. (2.65)
$\overline{N}_{ee}(K_\lambda)$	Eqn. (2.87)
$\overline{N}_{oe}(K_\lambda)$	Eqn. (2.88)
$\overline{L}_{eo}(K_\lambda)$	Eqn. (2.89)
$\overline{L}_{oo}(K_\lambda)$	Eqn. (2.90)
$\overline{L}_{ee}(K_\mu)$	Eqn. (2.107)
$\overline{L}_{oe}(K_\mu)$	Eqn. (2.108)

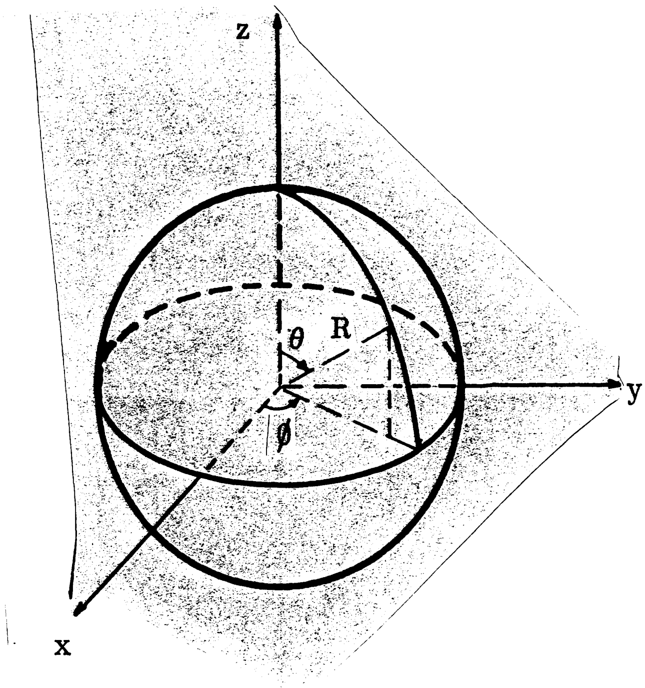

Figure 3: Spherical Cavity.

Let

$$\psi = F_1 \, F_2 \, F_3$$

and we will make a separation of variables. Equation (2.115) becomes

$$\frac{1}{F_1} \frac{\partial}{\partial R} (R^2 \frac{\partial F_1}{\partial R}) + \frac{1}{\sin\theta \, F_2} \frac{\partial}{\partial \theta} (\sin\theta \frac{\partial F_2}{\partial \theta}) + \frac{1}{\sin^2\theta \, F_3} \frac{\partial^2 F_3}{\partial \phi^2} + K^2 R^2 = 0 \ . \tag{2.116}$$

The radial part of (2.116) is

$$\frac{d}{dR}(R^2 \frac{dF_1}{dR}) + (K^2 R^2 - k_R^2) F_1 = 0 \tag{2.117}$$

where k_R^2 is a separation constant. This equation can be reduced to the standard form of Bessel equation if we make

$$F_1 = R^{-1/2}\,\bar{F}_1 \;;$$

then the differential equation for $\bar{F}_1$ is

$$\bar{F}_1'' + \frac{1}{R}\bar{F}_1' + \left(K^2 - \frac{(k_R^2 - 1/4)}{R^2}\right)\bar{F}_1 = 0$$

with the solutions $J_{\sqrt{k_R^2+1/4}}(KR)$ and $Y_{\sqrt{k_R^2+1/4}}(KR)$. The second solution is rejected because we require a solution which is finite at the origin. Therefore

$$F_1 = \frac{J_{\sqrt{k_R^2+1/4}}(KR)}{\sqrt{R}} .$$

The angular part of equation (2.116) is

$$\frac{\sin\theta}{F_2}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F_2}{\partial\theta}\right) + k_R^2 \sin^2\theta + \frac{1}{F_3}\frac{\partial^2 F_3}{\partial\phi^2} = 0 ,$$

and after a new separation of variables we obtain two differential equations, in ϕ and in θ. The first one,

$$\frac{d^2F_3}{d\phi^2} + k_\phi^2 F_3 = 0 ,$$

has solutions $\sin k_\phi \phi$ and $\cos k_\phi \phi$ and the condition of periodicity determines the value of the separation constant k_ϕ as equal to an integer m, m = 0, 1, 2,.. The second one,

$$\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{dF_3}{d\theta}\right) + (K_R^2\sin^2\theta - m^2)F_2 = 0 ,$$

after change of variable $\cos\theta = t$, takes the form

$$\frac{d^2F_2}{dt^2} - 2t\frac{dF_2}{dt} + (k_R^2 - \frac{m^2}{1-t^2}) F_2 = 0$$

of a Legendre equation if $k_R^2 = n(n+1)$, with two solutions $P_n^m(t)$ and $Q_n^m(t)$. The second solution is rejected because it is infinite at $t = \pm 1(\theta=0, \pi)$.

Finally, the generating function ψ is written as

$$\psi_{\substack{e\\o}mn} = j_n(KR)\, P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix} m\phi , \qquad (2.118)$$

where

$$j_n(KR) = \sqrt{\frac{\pi}{2KR}}\, J_{n+1/2}(KR)$$

is the so-called spherical Bessel function and $P_n^m(\cos\theta)$ is the associated Legendre function of the first kind. The above notation is a compact form of writing the even and odd ψ-functions

$$\psi_e = j_n(KR)\, P_n^m(\cos\theta) \cos m\phi$$

and

$$\psi_o = j_n(KR)\, P_n^m(\cos\theta) \sin m\phi .$$

The piloting vector, unlike in the previous cases, is chosen to be $\bar{R}$. The expressions of $\bar{M}$, $\bar{N}$ and $\bar{L}$ for the spherical cavity are:

$$\bar{M}_{\substack{e\\o}mn}(K) = \nabla \times (\psi_{\substack{e\\o}mn} \bar{R}) = \mp \frac{m}{\sin\theta} j_n(KR)\, P_n^m(\cos\theta) \begin{matrix}\sin\\ \cos\end{matrix} m\phi\, \hat{\theta} -$$

$$- j_n(KR) \frac{\partial P_n^m(\cos\theta)}{\partial\theta} \begin{matrix}\cos\\ \sin\end{matrix} m\phi\, \hat{\phi} , \qquad (2.119)$$

$$\bar{N}_{\substack{e\\o}mn}(K) = \frac{1}{K}\nabla x\ \nabla x\ (\psi_{\substack{e\\o}mn}\bar{R}) = \frac{n(n+1)}{KR}\ j_n(KR)\ P_n^m(\cos\theta)\ {\cos \atop \sin}\ m\ \phi\ \hat{R}\ +$$

$$+\frac{1}{KR}\ \frac{\partial}{\partial R}\left[R\ j_n(KR)\right]\left[\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\ {\cos \atop \sin}\ m\ \phi\ \hat{\theta} \mp \frac{m}{\sin\theta}\ P_n^m(\cos\theta)\ {\sin \atop \cos}\ m\phi\hat{\phi}\right], \tag{2.120}$$

and

$$\bar{L}_{\substack{e\\o}mn}(K) = \nabla\psi_{\substack{e\\o}mn} = \frac{\partial j_n(KR)}{\partial r} P_n^m(\cos\theta)\ {\cos \atop \sin}\ m\ \phi\ \hat{R}\ +$$

$$+\frac{j_n(KR)}{R}\ \frac{\partial P_n^m(\cos\theta)}{\partial\theta}\ {\cos \atop \sin}\ m\phi\hat{\theta} \mp \frac{m}{R\sin\theta}\ j_n(KR)P_n^m(\cos\theta)\ {\sin \atop \cos}\ m\ \phi\ \hat{\phi} \tag{2.121}$$

which is the compact form of writing the even and odd $\bar{M}$, $\bar{N}$ and $\bar{L}$-functions. Starting with the dyadic Green's function $\bar{\bar{G}}_{m2}$ and using boundary conditions (2.11), we find for the $\bar{M}$-function that $K = \mu$, where μ is the root of the equation $\frac{\partial}{\partial R}\ (R\ j_n(\mu R))\Big|_{R=a} = 0$. For the $\bar{N}$-function, $K = \lambda$ where λ is the root of the equation $j_n(\lambda a) = 0$. Therefore, the expressions for $\bar{M}$ and $\bar{N}$-functions are

$$\bar{M}_{\substack{e\\o}mn}(\mu) = \mp\frac{m}{\sin\theta}\ j_n\ (\mu R)\ P_n^m(\cos\theta)\ {\sin \atop \cos}\ m\ \phi\ \hat{\theta}\ -$$

$$-\ j_n\ (\mu R)\ \frac{\partial P_n^m\ (\cos\theta}{\partial\theta}\ {\cos \atop \sin}\ m\ \phi\ \hat{\phi} \tag{2.122}$$

and

$$\bar{N}_{\substack{e\\o}mn}(\lambda) = \frac{n(n+1)}{\partial R}\ j_n(\lambda R)\ P_n^m(\cos\theta)\ {\cos \atop \sin}\ m\ \phi\ \hat{R}\ +$$

$$+\frac{1}{\lambda R}\ \frac{\partial}{\partial\theta}\ \left[R\ j_n(\lambda R)\right]\left[\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\ {\cos \atop \sin}\ m\ \phi\ \hat{\theta}\ \mp\right.$$

$$\left.\mp\ \frac{m}{\sin\theta}\ P_n^m\ (\cos\theta)\ {\sin \atop \cos}\ m\ \phi\ \hat{\phi}\right]\ . \tag{2.123}$$

Before studying the orthogonality properties of M and N functions, it is necessary to study the orthogonality properties of the spherical Bessel function. The differential equation for the spherical Bessel function is

$$\frac{d}{dR}\left[R^2 \frac{dj_n(KR)}{dR}\right] + \left[K^2 R^2 - n(n+1)\right] j_n(KR) = 0 \;,$$

w hich can also be written as

$$\frac{d}{dR}\left[R \frac{d(Rj_n(KR))}{dR} - Rj_n(KR)\right] + \left[K^2 R^2 - n(n+1)\right] j_n(KR) = 0$$

because

$$\frac{d}{dR}\left[R^2 \frac{dj_n(KR)}{dR}\right] = \frac{d}{dR}\left[R \frac{d(Rj_n(KR))}{dR} - Rj_n(KR)\right] \;.$$

Consider two spherical Bessel functions $j_n(\alpha r)$ and $j_n(\beta r)$ which satisfy, respectively, the differential equations:

$$\frac{1}{R^2}\frac{d}{dR}\left[R \frac{d(Rj_n(\alpha R))}{dR} - R\,j_n(\alpha R)\right] + \left[\alpha^2 - \frac{n(n+1)}{R^2}\right] j_n(\alpha R) = 0$$

and

$$\frac{1}{R^2}\frac{d}{dR}\left[R \frac{d(Rj_n(\beta R))}{dR} - R\,j_n(\beta R)\right] + \left[\beta^2 - \frac{n(n+1)}{R^2}\right] j_n(\beta R) = 0 \;.$$

Multiply the first of these equations by $j_n(\beta R)$ and the second by $j_n(\alpha R)$, and subtract the second from the first. We obtain

$$(\alpha^2 - \beta^2)\, j_n(\alpha R)\, j_n(\beta R) = \frac{1}{R^2}\left\{ j_n(\alpha R) \frac{d}{dR}\left[R \frac{d(Rj_n(\beta R))}{dR} - R\,j_n(\beta R)\right] - \right.$$

$$- j_n(\beta R)\frac{d}{dR}\left[R\,\frac{d(Rj_n(\alpha R))}{dR} - R\,j_n(\alpha R)\right]\Bigg\}.$$

Integrate with respect to R^2dR from zero to a:

$$(\alpha^2-\beta^2)\int_0^a j_n(\alpha R)\,j_n(\beta R)\,R^2dR = \int_0^a \Bigg\{ j_n(\alpha R)\frac{d}{dR}\left[R\,\frac{d(Rj_n(\beta R))}{dR} - R\,j_n(\beta R)\right] -$$

$$- j_n(\beta R)\frac{d}{dR}\left[R\,\frac{d(R\,j_n(\alpha R))}{dR} - R\,j_n(\alpha R)\right]\Bigg\}dR\ .$$

The right hand side of this expression is the derivative of

$$\left[R\,j_n(\alpha R)\,\frac{d(R\,j_n(\beta R))}{dR} - R\,j_n(\beta R)\,\frac{d(R\,j_n(\alpha R))}{dR}\right]\ ,$$

and therefore

$$(\alpha^2-\beta^2)\int_0^a j_n(\alpha R)\,j_n(\beta R)\,R^2dR = R\left[j_n(\alpha R)\,\frac{d(Rj_n(\beta R))}{dR} - j_n(\beta R)\,\frac{d(R\,j_n(\alpha R))}{dR}\right]\Bigg|_0^a\ . \tag{2.124}$$

The right hand side of the above expression is always zero for the lower limit of integration. The value of the expression at its upper limit is also zero if $\alpha = \lambda$ and $\beta = \lambda'$ for $j_n(\lambda a) = j_n(\lambda' a) = 0$, or if $\alpha = \mu$ and $\beta = \mu'$ for

$$\frac{d\left[R\,j_n(\mu R)\right]}{dR}\Bigg|_{R=a} = \frac{d\left[R\,j_n(\mu' R)\right]}{dR}\Bigg|_{R=a} = 0.$$

Therefore, for $\alpha \neq \beta$

$$\int_0^a j_n(\lambda R)\,j_n(\lambda' R)\,R^2\,dR = 0 \tag{2.125}$$

and

$$\int_0^a j_n(\mu R)\, j_n(\mu' R)\, R^2\, dR = 0 \ . \qquad (2.126)$$

Consider, now α and $\beta = \alpha + \Delta$ for $\alpha \rightarrow 0$. In this case

$$I_\alpha = \int_0^a j_n(\alpha R)\, R^2\, dR = \lim_{\Delta \to 0} \int_0^a j_n(\alpha R)\, j_n\left[(\alpha + \Delta)R\right] R^2\, dR ,$$

which by (2.124) is

$$I_\alpha = \lim_{\Delta \to 0} \frac{R\left[j_n(\alpha R)\dfrac{d(Rj_n(\beta R))}{dR} - j_n(\beta R)\dfrac{d(Rj_n(\alpha R))}{dR}\right]}{\alpha^2 - \beta^2}\Bigg|_0^a .$$

But $\alpha^2 - \beta^2 = \alpha^2 - (\alpha+\Delta)^2 = \alpha^2 - \alpha^2 - \beta\,\Delta d - \Delta^2 \cong - 2\,\Delta\alpha$, where Δ^2 was neglected; consequently

$$I\alpha = -\frac{1}{2\alpha} \lim_{\Delta \to 0} \frac{1}{\Delta}\left\{R\left[j_n(\alpha R)\frac{d\left[Rj_n((\alpha+\Delta)R)\right]}{dR} - j_n((\alpha+\Delta)R)\frac{d\left[Rj_n(\alpha R)\right]}{dR}\right]\right\}\Bigg|_0^a$$

Taking the limit, we get

$$I\alpha = -\frac{1}{2\alpha}\left\{R\left[j_n(\alpha R)\frac{\partial^2(Rj_n(\alpha R))}{\partial\alpha\, \partial R} - \frac{\partial j_n(\alpha r)}{\partial\alpha}\frac{\partial(Rj_n(\alpha R))}{\partial R}\right]\right\}\Bigg|_0^a .$$

I_α is zero at the lower limit of integration,

$$I_\alpha = -\frac{a}{2\alpha}\left[j_n(\alpha R)\frac{\partial^2(R\, j_n(\alpha R))}{\partial\alpha\, \partial R} - \frac{\partial j_n(\alpha R)}{\partial\alpha}\frac{\partial(Rj_n(\alpha R))}{\partial R}\right]\Bigg|_{R=a}$$

In the brackets, we have

$$\frac{\partial j_n(\alpha R)}{\partial\alpha} = \frac{\partial(R\, j_n(\alpha R))}{\partial(\alpha R)}$$

and

$$\frac{\partial^2(R\, j_n(\alpha R))}{\partial\alpha\,\partial R} = \frac{\partial}{\partial\alpha}\left[\alpha\,\frac{\partial(Rj_n(\alpha R))}{\partial(\alpha R)}\right] = \frac{\partial(Rj_n(\alpha R))}{\partial(\alpha R)} + \alpha R\,\frac{\partial^2(R\, j_n(\alpha R))}{\partial(\alpha R)^2} \qquad (2.127)$$

From differential equation for spherical Bessel function, we have

$$\frac{d}{dR}\left[R\,\frac{d(R\, j_n(\alpha R))}{dR} - R\, j_n(\alpha R)\right] + \left[\alpha^2 R^2 - n\,(n+1)\right]\, j_n\,(\alpha R) = 0\;,$$

or

$$R\,\frac{d^2\,(Rj_n(\alpha R))}{dR^2} + \left[\alpha^2 R^2 - n\,(n+1)\right]\, j_n\,(\alpha R) = 0\;,$$

which can be written as

$$\alpha^2 R\,\frac{d^2(Rj_n(\alpha R))}{d(\alpha R)^2} = -\left[\alpha^2 R^2 - n(n+1)\right]\, j_n(\alpha R)\;.$$

Therefore

$$\alpha R\,\frac{d^2(Rj_n(\alpha R))}{d(\alpha R)^2} = -\,\frac{\left[\alpha^2 R^2 - n(n+1)\right]}{\alpha}\; j_n\,(\alpha R)\;,$$

and (2.127) becomes

$$\frac{\partial^2\,(R\, j_n(\alpha R))}{\partial\alpha\;\partial R} = \frac{\partial(R\, j_n(\alpha R))}{\partial\,(\alpha R)} - \frac{\left[\alpha^2 R^2 - n(n+1)\right]}{\alpha}\; j_n\,(\alpha R)\;.$$

Also,

$$\frac{\partial\,(R\, j_n(\alpha R))}{\partial(\alpha R)} = \frac{1}{\alpha}\;\frac{\partial(Rj_n(\alpha R))}{\partial R}\;,$$

so the expression for I_α becomes

$$I_\alpha = -\frac{a}{2\alpha}\left\{ j_n(\alpha R)\left[\frac{\partial(Rj_n(\alpha R))}{\alpha\,\partial R} - \frac{\left[\alpha^2 R^2 - n(n+1)\right]}{\alpha}\, j_n(\alpha R)\right] - \frac{1}{\alpha}\left[\frac{\partial(Rj_n(\alpha R))}{\partial R}\right]^2\right\}\Bigg|_{R=a}$$

When $\alpha = \lambda$,

$$I_\lambda = \int_0^a j_n^2(\lambda R)\, R^2\, dR = \frac{a}{2\lambda^2}\left[\frac{\partial(R\, j_n(\alpha R))}{\partial R}\right]^2\Bigg|_{R=a}, \tag{2.128}$$

and when $\alpha = \mu$

$$I_\mu = \int_0^a j_n^2(\mu R)\, R^2\, dR = \frac{a}{2\mu^2}\left[\mu^2 a^2 - n(n+1)\right] j_n^2(\mu a) . \tag{2.129}$$

The orthogonality of $\bar{M}$ and $\bar{N}$-functions for $\bar{\bar{G}}_{m2}$ runs as follows:

$$\int_V \bar{M}_{\substack{e\\o}mn}(\mu) \cdot \bar{N}_{\substack{e\\o}m'n'}(\lambda)\, dv = 0 \tag{2.130}$$

because of the trigonometric functions if we are considering the integration of the functions corresponding to the upper subscripts or to the lower subscripts. The integral is zero if we are considering the product of the functions corresponding to the upper subscript of one of them and lower subscript of the other because $P_n^m(1) = P_n^m(-1) = 0 (m \neq 0)$. The normalization constants are

$$\int_V \bar{M}_{\substack{e\\o}mn}(\mu) \cdot \bar{M}_{\substack{e\\o}m'n'}(\mu')\, dv = \begin{cases} 0 & m \neq m',\ n \neq n',\ \mu \neq \mu' \\ (1+\delta_{om})\pi \dfrac{2n(n+1)}{2n+1}\dfrac{(n+m)!}{(n-m)!} I_\mu = \dfrac{1}{C_\mu} & m = m',\ n = n',\ \mu = \mu' \end{cases} \tag{2.131}$$

and

$$\int_V \bar{N}_{\substack{e\\o}mn}(\lambda) \cdot \bar{N}_{\substack{e\\o}m'n'}(\lambda')\, dv = \begin{cases} 0 & m \neq m',\ n \neq n',\ \lambda \neq \lambda' \\ (1+\delta_{om})\pi \dfrac{2n(n+1)}{2n+1}\dfrac{(n+m)!}{(n-m)!} I_\lambda = \dfrac{1}{C_\lambda} & m = m',\ n = n',\ \mu = \mu' \end{cases} \tag{2.132}$$

In the computation of the normalization constants it is understood that the integration involve the functions corresponding to the upper subscripts or to the lower subscripts. The integration involving the product of the functions corresponding to the upper subscript of one of them and the lower subscript of the other is always zero because of the trigonometric functions. In the above, the following relations were used:

$$\int_0^{\pi} \left[\left(\frac{dP_n^m}{d\theta} \right)^2 + \left(\frac{mP_n^m}{\sin\theta} \right)^2 \right] \sin\theta \; d\theta = \frac{2n(n+1)}{2n+1} \frac{(n+m)!}{(n-m)!} ,$$

$$\int_0^{\pi} \left[P_n^m(\cos\theta) \right]^2 \sin\theta \, d\theta = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} ,$$

and

$$\frac{d}{dx} \left[x \, j_n(x) \right] = \frac{x}{2n+1} \left[(n+1) \, j_{n-1}(x) - n \, j_{n+1}(x) \right] ,$$

$$j_n(x) = \frac{x}{2n+1} \left[j_{n-1}(x) + j_{n+1}(x) \right] .$$

The expansion of $\nabla \times \left[\bar{\bar{I}} \, \delta(\bar{R} - \bar{R}') \right]$ is written as

$$\nabla \times \left[\bar{\bar{I}} \delta(\bar{R} - \bar{R}') \right] = \sum \bar{M}_{\substack{e \\ o}mn}(\mu) \, \bar{A} + \sum \bar{N}_{\substack{e \\ o}mn}(\lambda) \, \bar{B} .$$

Pre-multiplying this expression by $\bar{M}_{\substack{e \\ o}mn}(\mu)$ and $\bar{N}_{\substack{e \\ o}mn}(\lambda)$, integrating through the volume of the sphere and using the orthogonality relations (2.130) through (2.132), we find that

$$\bar{A} = C_{\mu} \, \mu \, \bar{N}'_{\substack{e \\ o}mn}(\mu) \quad \text{and} \quad \bar{B} = C_{\lambda} \, \lambda \, \bar{M}'_{\substack{e \\ o}mn}(\lambda) .$$

In the above

$$\int_V \bar{M}_{\substack{e \\ o}mn}(\mu) \cdot \nabla \times \left[\bar{\bar{I}}\,\delta(\bar{R}-\bar{R}')\right] dv = \mu\, \bar{N}'_{\substack{e \\ o}mn}(\mu)$$

and

$$\int_V \bar{N}_{\substack{e \\ o}mn}(\lambda) \cdot \nabla \times \left[\bar{\bar{I}}\,\delta(\bar{R}-\bar{R}')\right] dv = \lambda\, \bar{M}'_{\substack{e \\ o}mn}(\lambda)$$

was used. Therefore,

$$\nabla \times \left[\bar{\bar{I}}\,\delta(\bar{R}-\bar{R}')\right] = \sum C_\mu\, \mu\, \bar{M}_{\substack{e \\ o}mn}(\mu)\, \bar{N}'_{\substack{e \\ o}mn}(\mu) + \sum C_\lambda\, \lambda\, \bar{N}_{\substack{e \\ o}mn}(\lambda)\, \bar{M}'_{\substack{e \\ o}mn}(\lambda)$$

and the expansion of $\bar{\bar{G}}_{m2}(\bar{R}|\bar{R}')$ is written as

$$\bar{\bar{G}}_{m2} = \sum \alpha\, C_\mu\, \mu\, \bar{M}_{\substack{e \\ o}mn}(\mu)\, \bar{N}'_{\substack{e \\ o}mn}(\mu) + \sum \beta\, C_\lambda\, \lambda\, \bar{N}_{\substack{e \\ o}mn}(\lambda)\, \bar{M}'_{\substack{e \\ o}mn}(\lambda) .$$

Substituting the expansions of $\bar{\bar{G}}_{m2}$ and $\nabla \times \left[\bar{\bar{I}}\,\delta(\bar{R}-\bar{R}')\right]$ into (2.7), we get

$$\alpha = \frac{1}{\mu^2 - k^2} \quad \text{and} \quad \beta = \frac{1}{\lambda^2 - k^2} .$$

Finally,

$$\bar{\bar{G}}_{m2}(\bar{R}|\bar{R}') = \sum \frac{C_\mu\, \mu}{\mu^2 - k^2}\, \bar{M}_{\substack{e \\ o}mn}(\mu)\, \bar{N}'_{\substack{e \\ o}mn}(\mu) + \sum \frac{C_\lambda\, \lambda}{\lambda^2 - k^2}\, \bar{N}_{\substack{e \\ o}mn}(\lambda)\, \bar{M}'_{\substack{e \\ o}mn}(\lambda) . \tag{2.133}$$

For $\bar{\bar{G}}_{e1}(\bar{R}|\bar{R}')$, we use the $\bar{M}$, $\bar{N}$ and $\bar{L}$ functions

$$\bar{M}_{\substack{e \\ o}mn}(\lambda) = \mp \frac{m}{\sin\theta}\, j_n(\lambda R)\, P_n^m(\cos\theta) \begin{matrix}\sin \\ \cos\end{matrix}\, m\phi\, \hat{\theta} - j_n(\lambda R)\, \frac{\partial P_n^m(\cos\theta)}{\partial\theta} \begin{matrix}\cos \\ \sin\end{matrix}\, m\phi\, \hat{\phi} , \tag{2.134}$$

$$\bar{N}_{\substack{e\\o}mn}(\mu) = \frac{n(n+1)}{\mu R}\, j_n(\mu R)\, P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix} m\phi\, \hat{R} +$$

$$+\frac{1}{\mu R}\frac{\partial}{\partial R}(R\, j_n(\mu R))\left[\frac{\partial P_n^m(\cos\theta)}{\partial\theta} \begin{matrix}\cos\\ \sin\end{matrix} m\phi\, \hat{\theta} \mp \frac{m}{\sin\theta} P_n^m(\cos\theta) \begin{matrix}\sin\\ \cos\end{matrix} m\phi\, \hat{\phi}\right], \tag{2.135}$$

and

$$\bar{L}_{\substack{e\\o}mn}(\lambda) = \frac{\partial j_n(L\lambda)}{\partial R}\, P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix} m\phi\, \hat{R} + \frac{j_n(L\lambda)}{R}\, \frac{\partial P_n^m(\cos\theta)}{\partial R} \begin{matrix}\cos\\ \sin\end{matrix} m\phi\, \hat{\theta}$$

$$\mp \frac{m}{R\sin\theta}\, j_n(\lambda R)\, P_n^m(\cos\theta) \begin{matrix}\sin\\ \cos\end{matrix} m\phi\, \hat{\phi}\;. \tag{2.136}$$

The orthogonality relations are

$$\int_V \bar{M}_{\substack{e\\o}mn}(\lambda)\cdot\bar{N}_{\substack{e\\o}m'n'}(\mu)\, dv = 0\;,$$
$$\int_V \bar{M}_{\substack{e\\o}mn}(\lambda)\cdot\bar{L}_{\substack{e\\o}m'n'}(\lambda')\, dv = 0\;, \tag{2.137}$$

because of the observations valid for (2.130) and the orthogonality relations (2.125) of the spherical Bessel functions. The relation

$$\int_V \bar{N}_{\substack{e\\o}mn}(\mu)\cdot\bar{L}_{\substack{e\\o}m'n'}(\lambda)\, dv = 0 \tag{2.138}$$

needs a more detailed analysis which is given below.

$$\int_V \bar{N}_{\substack{e\\o}mn}(\mu)\cdot \bar{L}_{\substack{e\\o}m'n'}(\lambda)\,dv = \int_0^a\int_0^\pi\int_0^{2\pi}\left[\frac{n(n+1)}{\mu R}\, j_n(\mu R)\, P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix}\, m\phi\right.$$

$$\frac{\partial\, j_{n'}(\lambda R)}{\partial R}\, P_{n'}^{m'}(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix}\, m'\phi + \frac{1}{\mu R}\,\frac{\partial}{\partial R}(R\, j_n(\mu R))\,\frac{\partial P_n^m(\cos\theta)}{\partial\theta} \begin{matrix}\cos\\ \sin\end{matrix}\, m\phi$$

$$\frac{j_n{}'(\lambda R)}{R}\,\frac{\partial P_{n'}^{m'}(\cos\theta)}{\partial\theta} \begin{matrix}\cos\\ \sin\end{matrix}\, m\phi + \frac{1}{\mu R}\,\frac{\partial}{\partial R}(R\, j_n(\mu R))\,\frac{m}{\sin\theta}\, P_n^m(\cos\theta) \begin{matrix}\sin\\ \cos\end{matrix} m\phi$$

$$\left.\frac{m'}{R\sin\theta}\, j_{n1}(\lambda R)\, P_{n'}^{m1}(\cos\theta) \begin{matrix}\sin\\ \cos\end{matrix} m'\phi\right] R^2 \sin\theta\, dR\, d\theta\, d\phi\ .$$

The integrations of the trigonometric functions and associated Legendre functions are straightforward, and after performing them we find that m=m', n=n' and the expression takes the form:

$$\int_V \bar{N}_{\substack{e\\o}mn}(\mu)\cdot \bar{L}_{\substack{e\\o}m'n'}(\lambda)\,dv = \frac{2n(n+1)}{2n+1}\,\frac{(n+m)!}{(n-m)!}(1+\delta_{om})\pi\left[\int_0^a \frac{j_n(\mu R)}{\mu R}\,\frac{\partial j_n(\lambda R)}{\partial R} R^2\, dR + \right.$$

$$\left. + \int_0^a \frac{1}{\mu R}\,\frac{\partial}{\partial R}(R\, j_n(\mu R))\,\frac{j_n(\lambda R)}{R}\, R^2\, dR\right].$$

The integral on the right hand side can be written as

$$\frac{1}{\mu}\int_0^a \frac{j_n(\mu R)}{R}\,\frac{\partial j_n(\lambda R)}{\partial R}\, R^2\, dR + \frac{1}{\mu}\int_0^a \frac{\partial}{\partial R}(R\, j_n(\mu R))\,\frac{j_n(\lambda R)}{R^2}\, R^2 dR =$$

$$= \frac{1}{\mu}\left[\int_0^a R\, j_n(\mu R)\,\frac{\partial j_n(\lambda R)}{\partial R} + j_n(\lambda R)\,\frac{\partial}{\partial R}(R\, j_n(\mu R))\right] dR\ ,$$

and its integrand is equal to

$$\frac{d}{dR}\left[R\, j_n(\mu R)\, j_n(\lambda R)\right] .$$

Therefore,

$$\frac{1}{\mu}\int_0^a \frac{j_{n(\mu R)}}{R}\,\frac{\partial j_n(\lambda R)}{\partial R}\,R^2 dR + \frac{1}{\mu}\int_0^a \frac{\partial}{\partial R}\,(R\, j_n(\mu R))\,\frac{j_n(LR)}{R^2}\,R^2\, dR =$$

$$= \frac{1}{\mu}\int_0^a \frac{d}{dR}\left[R\, j_n(\mu R)\, j_n(LR)\right] dR = \frac{1}{\mu} R\, j_n(\mu R)\, j_n(\lambda R)\Bigg|_0^a = 0$$

because $j_n(\lambda a) = 0$.

The normalization constants are

$$\int_V \bar{M}_{\substack{e\\o}mn}(\lambda)\cdot\bar{M}_{\substack{e\\o}m'n'}(\lambda')\,dv = \begin{cases} 0 & m \neq m',\ n \neq n',\ \lambda \neq \lambda' \\ \frac{1}{C_\lambda} & n=m,\ n=n',\ \lambda=\lambda' \end{cases} \tag{2.139}$$

$$\int_V \bar{N}_{\substack{e\\o}mn}(\mu)\cdot\bar{N}_{\substack{e\\o}m'n'}(\mu')\,dv = \begin{cases} 0 & m \neq m',\ n \neq n',\ \lambda \neq \lambda' \\ \frac{1}{C_\mu} & m=m',\ n=n',\ \lambda=\lambda' \end{cases} \tag{2.140}$$

and

$$\int_V \bar{L}_{\substack{e\\o}mn}(\lambda)\cdot\bar{L}_{\substack{e\\o}m'n'}(\lambda')\,dv = \begin{cases} 0 & m \neq m',\ n \neq n',\ \lambda \neq \lambda' \\ (1+\delta_{om})\pi\lambda^2\,\frac{2}{2n+1}\,\frac{(n+m)!}{(n-m)!}\,I_\lambda = \frac{1}{C_{L\lambda}} & m=m',\ n=n',\ \lambda=\lambda' \end{cases} \tag{2.141}$$

where C_λ and C_μ are defined by (2.131) and (2.132).

The expansion of $\bar{\bar{I}}\left[\delta(\bar{R} - \bar{R}')\right]$, in this case, becomes

$$\bar{\bar{I}}\,\delta(\bar{R}-\bar{R}') = \sum \bar{L}_{\substack{e\\o}mn}(\lambda)\,\bar{A} + \sum \bar{M}_{\substack{e\\o}mn}(\lambda)\,\bar{B} + \sum \bar{N}_{\substack{e\\o}mn}(\mu)\,\bar{C} \,.$$

Pre-multiplying the above expression by $\bar{L}_{\substack{e\\o}mn}(\lambda)$, $\bar{M}_{\substack{e\\o}mn}(\lambda)$ and $\bar{N}_{\substack{e\\o}mn}(\mu)$, integrating through the volume of the cavity, and using the orthogonality relation (2.137) through (2.141), we get

$$\bar{A} = C_{L\lambda}\,\bar{L}'_{\substack{e\\o}mn}(\lambda)\,, \quad \bar{B} = C_{\lambda}\bar{M}'_{\substack{e\\o}mn}(\lambda), \text{ and } \bar{C} = C_{\mu}\bar{N}'_{\substack{e\\o}mn}(\mu)$$

or

$$\bar{\bar{I}}\delta(\bar{R}-\bar{R}') = \sum C_{L\lambda}\bar{L}_{\substack{e\\o}mn}(\lambda)\,\bar{L}'_{\substack{e\\o}mn}(\lambda) + \sum C_{\lambda}\bar{M}_{\substack{e\\o}mn}(\lambda)\,\bar{M}'_{\substack{e\\o}mn}(\lambda) + \sum C_{\mu}\bar{N}_{\substack{e\\o}mn}(\mu)\,\bar{N}'_{\substack{e\\o}mn}(\mu)\,.$$

The expansion of $\bar{\bar{G}}_{e1}(\bar{R}|\bar{R}')$ then can be written as

$$\bar{\bar{G}}_{e1}(\bar{R}|\bar{R}') = \sum \alpha\, C_{L\lambda}\,\bar{L}_{\substack{e\\o}mn}(\lambda)\,\bar{L}'_{\substack{e\\o}mn}(\lambda) + \sum \beta\, C_{\lambda}\bar{M}_{\substack{e\\o}mn}(\lambda)\,\bar{M}'_{\substack{e\\o}mn}(\lambda) +$$

$$+ \sum \gamma\, C_{\mu}\,\bar{N}_{\substack{e\\o}mn}(\mu)\,\bar{N}'_{\substack{e\\o}mn}(\mu)\,.$$

Substituting the expansions of $\bar{\bar{I}}\delta(\bar{R}-\bar{R}')$ and $\bar{\bar{G}}_{e1}(\bar{R}|\bar{R}')$ in (2.6), we obtain

$$\alpha = -\frac{1}{k^2}\,, \quad \beta = \frac{1}{\lambda^2 - k^2}\,, \quad \gamma = \frac{1}{\mu^2 - k^2}\;;$$

therefore

$$\bar{\bar{G}}_{e1}(\bar{R}|\bar{R}') = -\sum \frac{C_{L\lambda}}{k^2} \bar{L}_{\substack{e\\o}mn}(\lambda)\,\bar{L}'_{\substack{e\\o}mn}(\lambda) + \sum \frac{C_\lambda}{\lambda^2-k^2} \bar{M}_{\substack{e\\o}mn}(\lambda)\,\bar{M}'_{\substack{e\\o}mn}(\lambda) +$$

$$+\sum \frac{C_\mu}{\mu^2-k^2} \bar{N}_{\substack{e\\o}mn}(\mu)\,\bar{N}'_{\substack{e\\o}mn}(\mu) \ . \qquad (2.142)$$

Eliminating $\bar{L}_{\substack{e\\o}mn}(\lambda)\,\bar{L}'_{\substack{e\\o}mn}(\lambda)$ between the expansion of $\bar{\bar{I}}\delta(\bar{R}-\bar{R}')$ and (2.142) we obtain the second form of $\bar{\bar{G}}_{e1}(\bar{R}|\bar{R}')$,

$$\bar{\bar{G}}_{e1}(\bar{R}|\bar{R}') = -\frac{1}{k^2}\bar{\bar{I}}\,\delta(\bar{R}-\bar{R}') + \sum \frac{C_\lambda \lambda^2}{k^2(\lambda^2-k^2)} \bar{M}_{\substack{e\\o}mn}(\lambda)\,\bar{M}'_{\substack{e\\o}mn}(\lambda) +$$

$$+\sum \frac{C_\mu \mu^2}{k^2(\mu^2-k^2)} \bar{N}_{\substack{e\\o}mn}(\mu)\,\bar{N}'_{\substack{e\\o}mn}(\mu) \ . \qquad (2.143)$$

In a similar manner, we find that

$$\bar{\bar{G}}_{m1}(\bar{R}|\bar{R}') = \sum \frac{C_\lambda \lambda}{\lambda^2-k^2} \bar{M}_{\substack{e\\o}mn}(\lambda)\,\bar{N}'_{\substack{e\\o}mn}(\lambda) + \sum \frac{C_\mu \mu}{\mu^2-k^2} \bar{N}_{\substack{e\\o}mn}(\mu)\,\bar{M}'_{\substack{e\\o}mn}(\mu) \qquad (2.144)$$

and

$$\bar{\bar{G}}_{e2}(\bar{R}|\bar{R}') = -\sum \frac{C_{L\mu}}{k^2} \bar{L}_{\substack{e\\o}mn}(\mu)\,\bar{L}'_{\substack{e\\o}mn}(\mu) + \sum \frac{C_\mu}{\mu^2-k^2} \bar{M}_{\substack{e\\o}mn}(\mu)\,\bar{M}'_{\substack{e\\o}mn}(\mu) +$$

$$+\sum \frac{C_\lambda}{\lambda^2-k^2} \bar{N}_{\substack{e\\o}mn}(\lambda)\,\bar{N}'_{\substack{e\\o}mn}(\lambda) \ . \qquad (2.145)$$

where $\bar{L}_{\substack{e\\o}mn}(\mu)$ is given by

$$\bar{L}_{\substack{e\\o}mn}(\mu) = \frac{\partial j_n(\mu R)}{\partial R} P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix} m\phi \hat{R} + \frac{j_n(\mu R)}{R} \frac{\partial P_n^m(\cos\theta)}{\partial\theta} \begin{matrix}\cos\\ \sin\end{matrix} m\phi \hat{\theta}$$

$$\mp \frac{m}{R \sin\theta} j_n(\mu R) P_n^m(\cos\theta) \begin{matrix}\sin\\ \cos\end{matrix} m\phi \hat{\phi} . \tag{2.146}$$

The orthogonality relations in this case are

$$\int_V \bar{M}_{\substack{e\\o}mn}(\mu) \cdot \bar{L}_{\substack{e\\o}m'n'}(\mu')\, dv = 0 \tag{2.147}$$

because of the observations valid for (2.130) and the orthogonality relations (2.126) .

$$\int_V \bar{N}_{\substack{e\\o}mn}(\lambda) \cdot \bar{L}_{\substack{e\\o}m'n'}(\mu)\, dv = 0 . \tag{2.148}$$

The proof follows exactly the same steps as in the case of

$$\int_V \bar{N}_{\substack{e\\o}mn}(\mu) . \bar{L}_{\substack{e\\o}mn}(\lambda)\, dv .$$

The normalization constant is

$$\int_V \bar{L}_{\substack{e\\o}mn}(\mu) \cdot \bar{L}_{\substack{e\\o}m'n'}(\mu') dv = \begin{cases} 0 & m \neq m',\ n \neq n',\ \mu \neq \mu' \\ (1+\delta_{om})\pi \mu^2 \dfrac{2}{2n+1} \dfrac{(n+m)!}{(n-m)!} I_\mu = \dfrac{1}{C_{L\mu}} & m = m',\ n = n',\ \mu = \mu' \end{cases} \tag{2.149}$$

The second form of expansion of $\bar{\bar{G}}_{e2}(\bar{R}|\bar{R}')$ is

$$\bar{\bar{G}}_{e2}(\bar{R}|\bar{R}') = -\frac{1}{k^2}\bar{\bar{I}}\delta(\bar{R}-\bar{R}') + \sum \frac{C_\mu \mu^2}{k^2(\mu^2-k^2)} \bar{M}_{\substack{e\\o}mn}(\mu)\bar{M}'_{\substack{e\\o}mn}(\mu) +$$

$$+ \sum \frac{C_\lambda \lambda^2}{k^2(\lambda^2-k^2)} \bar{N}_{\substack{e\\o}mn}(\lambda) \bar{N}'_{\substack{e\\o}mn}(\lambda) . \tag{2.150}$$

Table III summarizes the results of this section.

Table III: Dyadic Green's Functions for the Spherical Cavity

$\bar{\bar{G}}_{e1}(\bar{R}\|\bar{R}')$	Eqn. (2.142)
$\bar{\bar{G}}_{e1}(\bar{R}\|\bar{R}')$	Eqn. (2.143)
$\bar{\bar{G}}_{e2}(\bar{R}\|\bar{R}')$	Eqn. (2.145)
$\bar{\bar{G}}_{e2}(\bar{R}\|\bar{R}')$	Eqn. (2.150)
$\bar{\bar{G}}_{m1}(\bar{R}\|\bar{R}')$	Eqn. (2.144)
$\bar{\bar{G}}_{m2}(\bar{R}\|\bar{R}')$	Eqn. (2.133)
K	λ
K	μ
C_{μ}^{-1}	$(1+\delta_{om})\,\pi\,\frac{2n(n+1)}{2n+1}\,\frac{(n+m)!}{(n-m)!}\,I_{\mu}$
C_{λ}^{-1}	$(1+\delta_{om})\,\pi\,\frac{2n(n+1)}{2n+1}\,\frac{(n+m)!}{(n-m)!}\,I_{\lambda}$
$C_{L\lambda}^{-1}$	$(1+\delta_{om})\,\pi\,\frac{2}{2n+1}\,\frac{(n+m)!}{(n-m)!}\,\lambda^2\,I_{\lambda}$
$C_{L\mu}^{-1}$	$(1+\delta_{om})\,\pi\,\frac{2}{2n+1}\,\frac{(n+m)!}{(n-m)!}\,\mu^2\,I_{\mu}$
$\bar{M}_{\substack{e\\o}mn}(\mu)$	Eqn. (2.122)
$\bar{M}_{\substack{e\\o}mn}(\lambda)$	Eqn. (2.134)
$\bar{N}_{\substack{e\\o}mn}(\mu)$	Eqn. (2.135)
$\bar{N}_{\substack{e\\o}mn}(\lambda)$	Eqn. (2.123)
$\bar{L}_{\substack{e\\o}mn}(\lambda)$	Eqn. (2.136)
$\bar{L}_{\substack{e\\o}mn}(\mu)$	Eqn. (2.146)

2.6 Application

Let us now compute the input admittance of a rectangular cavity formed by a short-circuited rectangular waveguide of dimensions a x b at a distance c from the shorted end. We do this by imposing an arbitrary electric field at the input and finding the transverse magnetic field generated by the electric field there. The ratio of the tangent component of the magnetic field to the imposed electric field will give the admittance.

Assume the cavity is excited by TE_{10} mode in the waveguide. The tangential component E_y of this mode is

$$E_y = \sqrt{\frac{2}{ab}} \sin\frac{\pi x}{a}\hat{y} .$$

It is assumed that all the current sources are zero. The magnetic field is given by (2.20) which in this case is

$$\bar{H}(\bar{R}) = i\omega\epsilon_o \int_S \bar{\bar{G}}_{e2}(\bar{R}|\bar{R}') \cdot \hat{n} \times \bar{E}(\bar{R}')\, ds' ,$$

or using explicit expressions for $\bar{\bar{G}}_{e2}(\bar{R}|\bar{R}')$, we get

$$\bar{H}(\bar{R}) = i\omega\epsilon_o \Bigg[-\sum \frac{C_{Le}}{k^2} \bar{L}_{ee} \int_S \bar{L}'_{ee} \cdot \hat{n} \times \bar{E}(\bar{R}')\, ds' + \sum \frac{C_N}{K^2 - k^2} \bar{M}_{oe} \int_S \bar{M}'_{oe} \cdot \hat{n} \times \bar{E}(\bar{R}')\, ds'$$

$$+ \sum \frac{C_M}{K^2 - k^2} \bar{N}_{eo} \int_S \bar{N}'_{eo} \cdot \hat{n} \times \bar{E}(\bar{R}')\, ds' \Bigg] .$$

In this case, the normal to the surface is $\hat{n} = +\hat{z}$, whence

$$\hat{n} \times \bar{E} = -\sqrt{\frac{2}{ab}} \sin\frac{\pi}{a} \times \hat{x} .$$

Let us perform every one of the integrals in the expression of $\bar{H}(\bar{R})$ separately.

$$\int_0^a \int_0^b \bar{M}'_{oe} \cdot \hat{n} \times \bar{E} \, dx dy =$$

$$= \int_0^a \int_0^b \frac{m\pi}{b} \sin \frac{\ell\pi}{a} x \cos \frac{m\pi}{b} y \cos \frac{n\pi}{c} z \, (-1) \sqrt{\frac{2}{ab}} \sin \frac{\pi}{a} x \, dx \, dy = 0$$

because in this case $\ell = 1$ and $m = 0$.

$$\int_0^a \int_0^b \bar{N}'_{eo} \cdot \hat{n} \times \bar{E} \, dx \, dy =$$

$$= \int_0^a \int_0^b (-1) \frac{1}{K} \frac{\ell\pi}{a} \frac{n\pi}{c} \sin \frac{\ell\pi}{a} x \cos \frac{m\pi}{b} y \cos \frac{n\pi}{c} z \, (-1) \sqrt{\frac{2}{ab}} \sin \frac{\pi}{a} x \, dx \, dy =$$

$$= \frac{1}{K} \frac{\pi}{a} \frac{n\pi}{c} \sqrt{\frac{ab}{2}} \cos \frac{n\pi}{c} z$$

with $\ell = 1$, $m = 0$ and $K^2 = (\frac{\pi}{a})^2 + (\frac{n\pi}{c})^2$.

$$\int_0^a \int_0^b \bar{L}_{ee} \cdot \hat{n} \times \bar{E} \, dx \, dy =$$

$$= \int_0^a \int_0^b (-1) \frac{\ell\pi}{a} \sin \frac{\ell\pi}{a} x \cos \frac{m\pi}{b} y \cos \frac{n\pi}{c} z \, (-1) \sqrt{\frac{2}{ab}} \sin \frac{\pi}{a} x \, dx \, dy =$$

$$= \frac{\pi}{a} \sqrt{\frac{ab}{2}} \cos \frac{n\pi}{c} z \, .$$

Again, in this integration $\ell = 1$ and $m = 0$. At the distance $z = c$ from the shorted end, the magnetic field is

$$\bar{H}(\bar{R}) = i\omega\epsilon_o \left[- \sum_{n=0,1}^{\infty} \frac{C_{Le}}{k^2} (-1) \frac{\pi}{a} \sin\frac{\pi}{a} x \frac{\pi}{a} \sqrt{\frac{ab}{2}} + \right.$$

$$\left. + \sum_{n=1,2}^{\infty} \frac{C_M}{K^2 - k^2} \frac{1}{K} (-1) \frac{\pi}{a} \frac{n\pi}{c} \sin\frac{\pi}{a} x \sqrt{\frac{ab}{2}} \right] \hat{x} .$$

Substituting the expressions for C_{Le} and C_M in the above expression, separating the term corresponding to $n = 0$ in the first summation and grouping the two summations into one, we get

$$\bar{H}(\bar{R}) = i\omega\epsilon_o \left[\frac{2}{abc} \frac{1}{k^2} \frac{(\pi/a)^2}{(\pi/a)^2} + \sum_{n=1,2}^{\infty} \frac{4}{abc} \left(\frac{(\pi/a)^2}{k^2 \; K^2} - \frac{(n\pi/c)^2}{K^2 (K^2 - k^2)} \right) \right] \sqrt{\frac{ab}{2}} \sin\frac{\pi}{a} x \hat{x} .$$

Simplifying the expression for $\bar{H}$, it becomes

$$\bar{H}(\bar{R}) = i\omega\epsilon_o \sqrt{\frac{2}{ab}} \frac{1}{ck^2} \left[1 + 2 \sum_{n=1}^{\infty} \frac{(\pi/a)^2 - k^2}{K^2 - k^2} \right] \sin\frac{\pi}{a} x \hat{x} .$$

Let $(\pi/a)^2 = k_c^2$, the cutoff wavenumber of the waveguide. Then

$$\bar{H}(\bar{R}) = - \frac{1}{i\omega\mu_o c} \sqrt{\frac{2}{ab}} \sin\frac{\pi}{a} x \left[1 + 2 \sum_{n=1} \frac{k_c^2 - k^2}{K^2 - k^2} \right] \hat{x} .$$

Finally, the expression of the input admittance becomes

$$Y = \frac{H_x}{E_y} = \frac{1}{i\omega\mu_o c} \left[1 + 2 \sum_{1}^{\infty} \frac{k_c^2 - k^2}{K^2 - k^2} \right] = - \frac{1}{i\omega\mu_o c} \left[1 + 2 \sum_{n=1}^{\infty} \frac{\omega^2 - \omega_c^2}{\omega^2 - \omega_n^2} \right]$$

(2.151)

where $k^2 = \omega^2 \epsilon_o \mu_o$; $k_c^2 = \omega_c^2 \epsilon_o \mu_o$ and $K^2 = \omega_n^2 \epsilon_o \mu_o$ was used.

Consider, now, a short-circuited waveguide of length c, propagating a TE_{10} mode. Let the waveguide be short-circuited at z = 0. The y-component of the electric field in this waveguide is

$$E_y = E_o \sin \frac{\pi x}{a} \left[e^{-i\beta_g z} - e^{i\beta_g z} \right]$$

where β_g is constant of propagation in the waveguide,

$$\beta_g = \sqrt{\mu_o \epsilon_o} \sqrt{\omega^2 - \omega_c^2} \; .$$

The x-component of the magnetic field is given by

$$H_x = \frac{1}{i\omega\mu_o} \frac{\partial E_y}{\partial z} = \frac{-i\beta_g}{i\omega\mu_o} E_o \sin \frac{\pi x}{a} \left[e^{-i\beta_g z} + e^{i\beta_g z} \right] \; .$$

Therefore, the admittance of the wave at z = c is

$$Y = \frac{H_x}{E_y} = \frac{-i\beta_g}{i\omega\mu_o} \frac{\left[e^{-i\beta_g c} + e^{-\beta_g c} \right]}{\left[e^{-i\beta_g c} - e^{-i\beta_g c} \right]} = -\sqrt{\frac{\epsilon_o}{\mu_o}} \frac{\sqrt{\omega^2 - \omega_c^2}}{i\omega} \cot \beta_g c \; . \qquad (2.152)$$

From Abramovitz and Stegun (1968) we know that the cotangent function can be expanded as

$$\cot z = \frac{1}{z} + 2z \sum_{k=1}^{\infty} \frac{1}{z^2 - k^2 \pi^2} \qquad (z \neq 0, \pm\pi, \pm 2\pi, \ldots)$$

which in our case becomes

$$\cot \beta_g c = \frac{1}{\beta_g c} + \frac{2}{\beta_g c} \sum_{n=1}^{\infty} \frac{1}{1 - \left(\frac{n\pi}{\beta_g c}\right)^2} \; .$$

Substituting in (2.152), we get

$$Y = -\sqrt{\frac{\epsilon_o}{\mu_o}} \frac{\sqrt{\omega^2-\omega_c^2}}{i\omega c} \frac{1}{\sqrt{\mu_o\epsilon_o}\sqrt{\omega^2-\omega_c^2}} \left[1 + 2 \sum_1^\infty \frac{1}{1 - \left(\frac{n\pi}{c\sqrt{\mu_o\epsilon_o}\sqrt{\omega^2-\omega_c^2}} \right)^2} \right],$$

and after some simplifications, we obtain

$$Y = -\frac{1}{i\omega\mu_o c} \left[1 + 2 \sum_{n=1}^{\infty} \frac{\mu_o\epsilon_o(\omega^2-\omega_c^2)}{\mu_o\epsilon_o(\omega^2-\omega_c^2) - (\frac{n\pi}{c})^2} \right]$$

or

$$Y = -\frac{1}{i\omega\mu_o c} \left[1 + 2 \sum_{n=1}^{\infty} \frac{\omega^2 - \omega_c^2}{\omega^2 - \omega_n^2} \right]$$

where $\omega_n^2 = \omega_c^2 + \mu_o\epsilon_o(\frac{n\pi}{c})^2$, which is the same as (2.151), obtained by using dyadic Green's functions. This result thus found is identical to that of Kurokawa (1953) who derived it by an entirely different approach.

The dyadic Green's functions derived herein for the cavities can be used to formulate various boundary value problems involving these cavities.

III

DYADIC GREEN'S FUNCTIONS IN THE PRESENCE OF A SPHERICALLY INHOMOGENEOUS SCATTERER

3.1 Introduction

We consider now the exterior problem, i.e., the scattering of the electromagnetic field by inhomogeneous spherical lenses. The geometry of the problem will always be the same and it is represented in Figure 4. It is assumed that the medium I is always free space and medium II is the interior of the lens whose relative permittivity varies only in the radial direction. The relative permeability of the lens is assumed constant and equal to 1. The center of the lens is at the origin of the spherical coordinates and the radius is a.

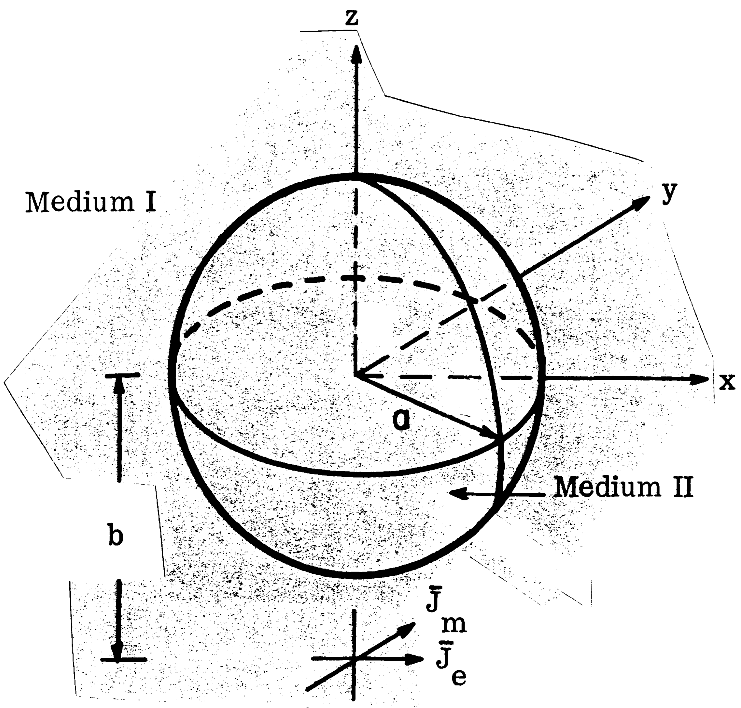

Figure 4: Geometry of Electric and Magnetic Dipoles in Front of an Inhomogeneous Lens.

Following the recent work of Tai (1973), we shall review the eigenfunction expansion of the free space Dyadic Green's functions in terms of the spherical vector wave functions because the original expression found in his book (Tai, 1971), has a missing singular term for $\bar{\bar{G}}_{eo}$.

The electric and magnetic dipoles which are referred to in this work are always infinitesimal.

3.2 Duality Between the Free Space Dyadic Green's Functions

To facilitate the discussion of the field originated by a Huygens source we would like to elaborate the duality principle from the point of view of dyadic Green's function technique.

We recall equations (2.1) and (2.21) which describe the Maxwell's equation for electric and magnetic dipoles as a source:

$$\nabla \times \bar{E}_e = i\omega\mu_o \bar{H}_e$$

$$\nabla \times \bar{H}_e = \bar{J}_e - i\omega\epsilon_o \bar{E}_e$$

and

$$\nabla \times \bar{E}_m = i\omega\mu_o \bar{H}_m - \bar{J}_m$$

$$\nabla \times \bar{H}_m = - i\omega\epsilon_o \bar{E}_m$$

where the subscripts e and m stand for the electric and magnetic sources respectively.

Two types of dyadic Green's functions, defined by (2.4), (2.5) and (2.22) and (2.23), this time take the form

$$\nabla \times \bar{\bar{G}}_{eo} = \bar{\bar{G}}_{mo}$$

$$\nabla \times \bar{\bar{G}}_{mo} = \bar{\bar{I}}\delta(\bar{R} - \bar{R}') + k_o^2 \bar{\bar{G}}_{eo}$$

$$\nabla \times \bar{\bar{G}}^*_{eo} = \bar{\bar{I}}\delta(\bar{R} - \bar{R}') + k_o^2 \bar{\bar{G}}^*_{mo}$$

$$\nabla \times \bar{\bar{G}}^*_{mo} = \bar{\bar{G}}^*_{eo}$$

where the additional subscript o is used to indicate that they are related to free space and the asterisk is used here to distinguish the functions pertaining to a magnetic current source from that of electric current source. It should not be confused with the notation normally used for complex conjugation. The differential equations for these functions are written as:

$$\nabla \times \nabla \times \bar{\bar{G}}_{eo} - k_o^2 \bar{\bar{G}}_{eo} = \bar{\bar{I}}\delta(\bar{R} - \bar{R}')$$

$$\nabla \times \nabla \times \bar{\bar{G}}_{mo} - k_o^2 \bar{\bar{G}}_{mo} = \nabla \times \left[\bar{\bar{I}}\delta(\bar{R} - \bar{R}')\right]$$

$$\nabla \times \nabla \times \bar{\bar{G}}^*_{mo} - k_o^2 \bar{\bar{G}}^*_{mo} = \bar{\bar{I}}\delta(\bar{R} - \bar{R}')$$

$$\nabla \times \nabla \times \bar{\bar{G}}^*_{eo} - k_o^2 \bar{\bar{G}}^*_{eo} = \nabla \times \left[\bar{\bar{I}}\delta(\bar{R} - \bar{R}')\right]$$

All of these dyadic Green's functions satisfy the radiation condition, namely

$$\lim_{R \to \infty} R\left[\nabla \times \bar{\bar{G}}_o(\bar{R}|\bar{R}') - ik\hat{R} \times \bar{\bar{G}}_o(\bar{R}|\bar{R}')\right] = 0$$

where $\bar{\bar{G}}_o$ stands for $\bar{\bar{G}}_{eo}$, $\bar{\bar{G}}_{mo}$, $\bar{\bar{G}}^*_{eo}$ and $\bar{\bar{G}}^*_{mo}$.

From the above, we infer that $\bar{\bar{G}}^*_{mo}$ is dual to $\bar{\bar{G}}_{eo}$ while $\bar{\bar{G}}_{mo}$ is dual to $\bar{\bar{G}}^*_{eo}$. It is necessary, therefore, to compute only one set of these dyadics for free space, the other set following by duality. Start with the $\bar{\bar{G}}_{mo}$, which is solenoidal and needs only two functions, $\bar{M}$ and $\bar{N}$, in its expansion. From section 2.5 we know that the expressions for these functions are

$$\bar{M}_{\substack{e\\o}mn}(K) = j_n(KR)\,\bar{m}_{\substack{e\\o}mn} \tag{3.1}$$

and

$$\bar{N}_{\substack{e\\o}mn}(K)=\frac{n(n+1)}{KR}j_n(KR)P_n^m(\cos\theta)\,{\cos \atop \sin}\,m\phi\hat{R}+\frac{1}{KR}\frac{\partial}{\partial R}\Big[R\,j_n(KR)\Big]\bar{n}_{\substack{e\\o}mn}\,, \tag{3.2}$$

where

$$\bar{m}_{\substack{e\\o}mn}=\mp\frac{m\,P_n^m(\cos\theta)}{\sin\theta}\,{\sin \atop \cos}\,m\phi\,\hat{\theta}-\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\,{\cos \atop \sin}\,m\phi\,\hat{\phi} \tag{3.3}$$

and

$$\bar{n}_{\substack{e\\o}mn}=\hat{R}\times\bar{m}_{\substack{e\\o}mn}\,. \tag{3.4}$$

The orthogonal properties in this case are

$$\int_V \bar{M}_{\substack{e\\o}mn}(K).\bar{N}_{\substack{e\\o}m'n'}(K')\,dv=0 \tag{3.5}$$

because of trigonometric functions of ϕ ,

$$\int_V \bar{M}_{\substack{e\\o}mn}(K)\,\bar{M}_{\substack{e\\o}m'n'}(K')dv=\begin{cases}0 & m\neq m',\ n\neq n'\\[2ex] \dfrac{(1+\delta_{om})\pi^2 n(n+1)(n+m)!}{K^2(2n+1)\,(n-m)!}\,\delta(K-K') & m=m',\ n=n'\end{cases} \tag{3.6}$$

and

$$\int_V \bar{N}_{\substack{e\\o}mn}(K)\,\bar{N}_{\substack{e\\o}m'n'}(K')dv=\begin{cases}0 & m\neq m',\ n\neq n'\\[2ex] \dfrac{(1+\delta_{om})\pi^2 n(n+1)(n+m)!}{K^2(2n+1)\,(n-m)!}\,\delta(K-K') & m=m',\ n=n'\end{cases} \tag{3.7}$$

where $\int_0^\infty R^2\,j_n(KR)\,j_n(K'R)\,dR=\frac{\pi}{2}\,\frac{\delta(K-K')}{K^2}$ was used.

Eventually, the expression for the dyadic $\nabla \times \left[\bar{\bar{I}}\,\delta(\bar{R} - \bar{R}')\right]$ in terms of $\bar{M}$ and $\bar{N}$ functions becomes

$$\nabla \times \left[\bar{\bar{I}}\,\delta(\bar{R} - \bar{R}')\right] = \int_0^\infty dK \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left[\bar{M}_{\substack{e \\ o}mn}(K)\,\bar{A} + \bar{N}_{\substack{e \\ o}mn}(K)\,\bar{B}\right] \tag{3.8}$$

where $\bar{A}$ and $\bar{B}$ are the set of vectors yet unknown. By taking the anterior scalar product of (3.8) with $\bar{M}_{\substack{e \\ o}mn}(K')$ and integrating it through the entire space, we obtain, due to (3.5) through (3.7)

$$\bar{A}_{\substack{e \\ o}mn} = \frac{2 - \delta_0}{2\pi^2} \frac{(2n+1)\,(n-m)!}{n(n+1)(n+m)!} K^3\, \bar{N}'_{\substack{e \\ o}mn}(K) .$$

In the same way, using $\bar{N}_{\substack{e \\ o}mn}(K')$ instead of $\bar{M}_{\substack{e \\ o}mn}(K')$ we obtain $\bar{B}_{\substack{e \\ o}mn}$:

$$\bar{B}_{\substack{e \\ o}mn} = \frac{2 - \delta_0}{2\pi^2} \frac{(2n+1)\,(n-m)\,!}{n(n+1)\,(n+m)!} K^3\, \bar{M}'_{\substack{e \\ o}mn}(K) ;$$

therefore

$$\nabla \times \left[\bar{\bar{I}}\,\delta(\bar{R} - \bar{R}')\right] = \int_0^\infty dK\, K^3 \sum_{n=1}^{\infty} \sum_{m=0}^{n} C_{mn} \left[\bar{M}_{\substack{e \\ o}mn}(K)\, \bar{N}'_{\substack{e \\ o}mn}(K) + \bar{N}_{\substack{e \\ o}mn}(K)\,\bar{M}'_{\substack{e \\ o}mn}(K)\right] \tag{3.9}$$

where

$$C_{mn} = \frac{2 - \delta_0}{2\pi^2} \frac{(2n+1)\,(n-m)!}{n(n+1)\,(n+m)!} .$$

The dyadic Green's function $\bar{\bar{G}}_{mo}$ is written, then, as

$$\bar{\bar{G}}_{mo}(\bar{R}|\bar{R}') = \int_0^\infty K^3\, dK \sum_{n=1}^{\infty} \sum_{m=0}^{n} C_{mn} \left[a(K)\,\bar{M}_{\substack{e \\ o}mn}(K)\, \bar{N}'_{\substack{e \\ o}mn}(K) + \right.$$

$$\left. + \, b(K)\, \bar{N}_{\substack{e \\ o}mn}(K)\, \bar{M}'_{\substack{e \\ o}mn}(K)\right]$$

and the unknown coefficients a (K) and b (K) are determined as in the case of the spherical cavity:

$$a(K) = b(K) = \frac{1}{K^2 - k_o^2} \quad .$$

The expression for the free space dyadic Green's function $\bar{\bar{G}}_{mo}(\bar{R} \mid \bar{R}')$ becomes:

$$\bar{\bar{G}}_{mo}(\bar{R}|\bar{R}') = \int_0^{\infty} \frac{K^3}{K^2 - k_o^2} dK \sum_{n=1}^{\infty} \sum_{m=0}^{n} C_{mn} \Big[\bar{M}_{\substack{e\\o}mn}(K) \bar{N}'_{\substack{e\\o}mn}(K) +$$

$$+ \bar{N}_{\substack{e\\o}mn}(K) \bar{M}'_{\substack{e\\o}mn}(K) \Big] \tag{3.10}$$

In order to perform above integration, we recall that

$$\int_0^{\infty} \frac{K^3 \, dK}{K^2 - k_o^2} j_n(KR) \frac{1}{KR'} \frac{\partial}{\partial R'} \left[R' j_n(KR') \right] = \frac{i\pi k_o}{2} \begin{cases} h_n^{(1)}(k_o R) \frac{1}{R} \frac{\partial}{\partial R} \left[R' j_n(k_o R') \right] & R > R' \\ j_n(k_o R) \frac{1}{R} \frac{\partial}{\partial R'} \left[R' h_n^{(1)}(k_o R') \right] & R < R', \end{cases}$$

or by writing the dyadic $\bar{M}_{\substack{e\\o}mn}(K) \bar{N}'_{\substack{e\\o}mn}(K)$ in an operational form

$$\bar{M}_{\substack{e\\o}mn}(K) \bar{N}'_{\substack{e\\o}mn}(K) = \bar{\bar{T}}_{\substack{e\\o}mn} \left[j_n(KR) \frac{1}{KR'} \frac{\partial}{\partial R'} (j_n(KR')) \right]$$

we have

$$\int_0^\infty \frac{K^3 dk}{K^2 - k_o^2} \bar{\bar{T}}_{\substack{e\\o}mn}\left[j_n(KR)\frac{1}{KR'}\frac{\partial}{\partial R'}(j_n(KR'))\right] = \frac{i\pi k_o}{2}\begin{cases}\bar{M}^{(1)}_{\substack{e\\o}mn}(k_o)\,\bar{N}'_{\substack{e\\o}mn}(k_o) & R > R' \\ \bar{M}_{\substack{e\\o}mn}(k_o)\,\bar{N}'^{(1)}_{\substack{e\\o}mn}(k_o) & R < R'\end{cases}$$

where

$$\bar{M}^{(1)}_{\substack{e\\o}mn}(k_o) = \nabla x \left[h_n^{(1)}(k_o R)\, P_n^m(\cos\theta)\cos m\phi\, \bar{R}\right]$$

and

$$\bar{N}^{(1)}_{\substack{e\\o}mn}(k_o) = \frac{1}{k_o}\nabla x\, M^{(1)}_{\substack{e\\o}mn}(k_o) \ .$$

Finally, the expression for $\bar{\bar{G}}_{mo}(\bar{R}|\bar{R}')$ becomes

$$\bar{\bar{G}}_{mo}(\bar{R}|\bar{R}') = \frac{ik_0}{4\pi}\sum_{n=1}^{\infty}\sum_{m=0}^{n}(2-\delta_o)\frac{(2n+1)(n-m)!}{n(n+1)(n+m)!}\begin{cases}\bar{M}^{(1)}_{\substack{e\\o}mn}(k_o)\bar{N}'_{\substack{e\\o}mn}(k_o) + \\ \bar{M}_{\substack{e\\o}mn}(k_o)\bar{N}'^{(1)}_{\substack{e\\o}mn}(k_o) +\end{cases}$$

$$+ \ \bar{N}^{(1)}_{\substack{e\\o}mn}(k_o)\,\bar{M}'_{\substack{e\\o}mn}(k_o) \qquad R > R'$$

$$+ \ \bar{N}_{\substack{e\\o}mn}(k_o)\,\bar{M}'^{(1)}_{\substack{e\\o}mn}(k_o) \qquad R < R' \quad . \tag{3.11}$$

Because of duality, the expression for $\bar{\bar{G}}^*_{eo}(\bar{R}|\bar{R}')$ is the same. Our next step will be to find $\bar{\bar{G}}_{eo}$. In order to do this we take the curl of equation (3.10)

$$\nabla\ x\,\bar{\bar{G}}_{mo}(\bar{R}|\bar{R}') = \int_0^\infty \sum_{n=1}^{\infty}\sum_{m=0}^{n} C_{mn}\frac{K^4 dK}{K^2 - k_o^2}\left[\bar{M}_{\substack{e\\o}mn}(K)\,\bar{M}'_{\substack{e\\o}mn}(K) + \right.$$

$$\left. + \ \bar{N}_{\substack{e\\o}mn}(K)\,\bar{N}'_{\substack{e\\o}mn}(K)\right] \quad . \tag{3.12}$$

This expression has a singular part in the integration with respect to K represented by

$$\bar{\bar{S}}_1(\bar{R}|\bar{R}') = \int_0^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{n} C_{mn} K^2 \left[\bar{M}_{\substack{e\\o}mn}(K)\, \bar{M}'_{\substack{e\\o}mn}(K) \right] dK. \tag{3.13}$$

Because

$$\int_0^{\infty} K^2 \bar{j}_n(KR)\, j_n(KR')\, dK = \frac{\pi\, \delta(R - R')}{2R^2} ,$$

(3.13) becomes

$$\bar{\bar{S}}_1(\bar{R}|\bar{R}') = \frac{\pi\, \delta(R - R')}{2R^2} \sum_{n=1}^{\infty} \sum_{m=0}^{n} C_{mn}\, \bar{m}_{\substack{e\\o}mn}\, \bar{m}'_{\substack{e\\o}mn} .$$

Having recognized the singular part of (3.12), we can evaluate the remaining part by contour integration. Thus, we obtain

$$\nabla x\, \bar{\bar{G}}_{mo}(\bar{R}|\bar{R}') = \int_0^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{n} C_{mn} K^2 \left[\bar{M}_{\substack{e\\o}mn}(K)\, \bar{M}'_{\substack{e\\o}mn}(K) \right] dK +$$

$$+ \int_0^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{n} C_{mn} K^2 \left[\frac{k_o^2}{K^2 - k_o^2} \bar{M}_{\substack{e\\o}mn}(K) \bar{M}'_{\substack{e\\o}mn}(K) + \frac{K^2}{K^2 - k_o^2} \bar{N}_{\substack{e\\o}mn}(K) \bar{N}'_{\substack{e\\o}mn}(K) \right] dK$$

or

$$\nabla x\, \bar{\bar{G}}_{mo}(\bar{R}|\bar{R}') = \bar{\bar{S}}_1(\bar{R}|\bar{R}') + \frac{i\pi k_o^3}{2} \sum_{n=1}^{\infty} \sum_{m=0}^{n} Cmn$$

$$\begin{cases} \bar{M}^{(1)}_{\substack{e\\o}mn}(k_o) \bar{M}'_{\substack{e\\o}mn}(k_o) + \bar{N}^{(1)}_{\substack{e\\o}mn}(k_o)\, \bar{N}'_{\substack{e\\o}mn}(k_o) & R > R' \\ \bar{M}_{\substack{e\\o}mn}(k_o) \bar{M}'^{(1)}_{\substack{e\\o}mn}(k_o) + \bar{N}_{\substack{e\\o}mn}(k_o)\, \bar{N}'^{(1)}_{\substack{e\\o}mn}(k_o) & R < R' . \end{cases}$$

Finally, recalling that

$$\bar{\bar{G}}_{eo} = \frac{1}{k_o^2}\left[\nabla \times \bar{\bar{G}}_{mo} - \bar{\bar{I}}\,\delta(\bar{R} - \bar{R}')\right],$$

we have

$$\bar{\bar{G}}_{eo}(\bar{R}|\bar{R}') = -\frac{1}{k_o^2}\left[\bar{\bar{I}}\,\delta(\bar{R} - \bar{R}') - \bar{\bar{S}}_1(\bar{R}|\bar{R}')\right] + \frac{i\pi k_o}{2}\sum_{n=1}^{\infty}\sum_{m=0}^{n} C_{mn}$$

$$\begin{cases} \bar{M}^{(1)}_{\substack{e\\o}mn}(k_o)\,\bar{M}'_{\substack{e\\o}mn}(k_o) + \bar{N}^{(1)}_{\substack{e\\o}mn}(k_o)\,\bar{N}'_{\substack{e\\o}mn}(k_o) & R > R' \\ \bar{M}_{\substack{e\\o}mn}(k_o)\,\bar{M}'^{(1)}_{\substack{e\\o}mn}(k_o) + \bar{N}_{\substack{e\\o}mn}(k_o)\,N'^{(1)}_{\substack{e\\o}mn}(k_o) & R < R' \end{cases} \tag{3.14}$$

and $\bar{\bar{G}}^*_{mo}(\bar{R}|\bar{R}') = \bar{\bar{G}}_{eo}(\bar{R}|\bar{R}')$ by duality .

3.3 Vector Wave Functions for a Spherically Inhomogeneous Medium

The Maxwell's equations for a harmonically oscillating field in a spherically inhomogeneous medium are

$$\nabla \times \bar{E}_e = i\omega\mu_o \bar{H}_e \tag{3.15}$$

$$\nabla \times \bar{H}_e = \bar{J}_e - i\omega\epsilon_o\epsilon_r(R)\,\bar{E}_e$$

for electric dipole excitations, and

$$\nabla \times \bar{E}_m = i\omega\mu_o \bar{H}_m - \bar{J}_m$$

$$\nabla \times \bar{H}_m = -i\omega\epsilon_o\epsilon_r(R)\,\bar{E}_m \tag{3.16}$$

for magnetic dipole excitation, where $\epsilon_r(R)$ is the relative permittivity of the medium.

The dyadic Green's functions in this case satisfy the equations

$$\nabla \times \bar{\bar{G}}_{einh} = \bar{\bar{G}}_{minh} \tag{3.17}$$

$$\nabla \times \bar{\bar{G}}_{minh} = \bar{\bar{I}}\delta(\bar{R} - \bar{R}') + k_o^2 \epsilon_r(R)\, \bar{\bar{G}}_{einh} \tag{3.18}$$

for electric dipole excitation, and

$$\nabla \times \bar{\bar{G}}^*_{einh} = k_o^2 \bar{\bar{G}}^*_{minh} + \bar{\bar{I}}\delta(\bar{R} - \bar{R}') \tag{3.19}$$

$$\nabla \times \bar{\bar{G}}^*_{minh} = \epsilon_r(R)\, \bar{\bar{G}}^*_{einh} \tag{3.20}$$

for magnetic dipole excitation, where the subscript "inh" means that the medium is inhomogeneous.

Again, by eliminating $\bar{\bar{G}}_{minh}$ or $\bar{\bar{G}}_{einh}$ between (3.17) and (3.18) and $\bar{\bar{G}}^*_{minh}$ or $\bar{\bar{G}}^*_{einh}$ between (3.19) and (3.20) we get

$$\nabla \times \nabla \times \bar{\bar{G}}_{einh} - k_o^2 \epsilon_r(R)\, \bar{\bar{G}}_{einh} = \bar{\bar{I}}\delta(\bar{R} - \bar{R}') \tag{3.21}$$

$$\nabla \times \left[\frac{1}{\epsilon_r(R)} \nabla \times \bar{\bar{G}}_{minh}\right] - k_o^2 \bar{\bar{G}}_{minh} = \nabla \times \left[\frac{1}{\epsilon_r(R)} \bar{\bar{I}}\delta(\bar{R} - \bar{R}')\right] \tag{3.22}$$

and

$$\nabla \times \left[\frac{1}{\epsilon_r(R)} \nabla \times \bar{\bar{G}}^*_{minh}\right] - k_o^2 \bar{\bar{G}}^*_{minh} = \bar{\bar{I}}\delta(\bar{R} - \bar{R}') \tag{3.23}$$

$$\nabla \times \nabla \times \bar{\bar{G}}^*_{einh} - k_o^2 \epsilon_r(R)\, \bar{\bar{G}}^*_{einh} = \nabla \times \left[\bar{\bar{I}}\delta(\bar{R} - \bar{R}')\right] \quad . \tag{3.24}$$

This time there is no duality between $\bar{\bar{G}}_{einh}$ and $\bar{\bar{G}}^*_{minh}$ and between $\bar{\bar{G}}_{minh}$ and $\bar{\bar{G}}^*_{einh}$ as in the case of free space dyadics. However, $\nabla \times \left[\frac{1}{\epsilon_r(R)} \bar{\bar{I}}\delta(\bar{R}-\bar{R}')\right]$ is still solenoidal and it can be expanded in terms of the eigenfunctions of the homogeneous vector wave equation

$$\nabla \times \left[\frac{1}{\epsilon_r(R)} \nabla \times \bar{F}\right] - k_o^2 \bar{F} = 0 \ .$$

These are the inhomogeneous spherical vector wave functions $\bar{M}^{(e)}_{\substack{e \\ o}mn}(k_o)$ and $\bar{N}^{(m)}_{\substack{e \\ o}mn}(k_o)$ defined by

$$\bar{M}^{(e)}_{\substack{e \\ o}mn}(k_o) = \nabla \times (\Phi \bar{R})$$

and

$$\bar{N}^{(m)}_{\substack{e \\ o}mn}(k_o) = \frac{1}{k_o} \nabla \times \nabla \times (\psi \bar{R}) .$$

$\nabla \times \left[\bar{\bar{I}}\delta(\bar{R} - \bar{R}')\right]$ is solenoidal too and can be expanded in terms of the eigen-functions of the homogeneous vector wave equation

$$\nabla \times \nabla \times \bar{F} - k_o^2 \epsilon_r(R) \bar{F} = 0.$$

These are the inhomogeneous spherical vector wave functions $\bar{M}^{(m)}_{\substack{e \\ o}mn}(k_o)$ and $\bar{N}^{(e)}_{\substack{e \\ o}mn}(k_o)$ defined by

$$\bar{M}^{(m)}_{\substack{e \\ o}mn}(k_o) = \nabla \times (\psi \bar{R})$$

and

$$\bar{N}^{(e)}_{\substack{e \\ o}mn}(k_o) = \frac{1}{k_o \epsilon_r(R)} \nabla \times \nabla \times (\Phi \bar{R}) .$$

ψ and Φ satisfy the scalar equations

$$\nabla^2 \psi + k_o^2 \epsilon_r(R) \psi = 0 \qquad (3.25)$$

and

$$\nabla^2 \Phi - \frac{1}{\epsilon_r(R)} \frac{d\epsilon_r(R)}{dR} \frac{1}{R} \frac{\partial (R\, \Phi)}{\partial R} + k_o^2\, \epsilon_r(R)\, \Phi = 0 \;. \tag{3.26}$$

The functions $\overline{M}^{(m)}_{\substack{e\\o}mn}$, $\overline{N}^{(m)}_{\substack{e\\o}mn}$, $\overline{M}^{(e)}_{\substack{e\\o}mn}$ and $\overline{N}^{(e)}_{\substack{e\\o}mn}$ satisfy the following symmetrical relations:

$$\overline{N}^{(e)}_{\substack{e\\o}mn}(k_o) = \frac{1}{k_o \epsilon_r(R)} \nabla \times \overline{M}^{(e)}_{\substack{e\\o}mn}(k_o)$$

$$\overline{M}^{(e)}_{\substack{e\\o}mn}(k_o) = \frac{1}{k_o} \nabla \times \overline{N}^{(e)}_{\substack{e\\o}mn}(k_o)$$

and (3.27)

$$\overline{N}^{(m)}_{\substack{e\\o}mn}(k_o) = \frac{1}{k_o} \nabla \times \overline{M}^{(m)}_{\substack{e\\o}mn}(k_o)$$

$$\overline{M}^{(m)}_{\substack{e\\o}mn}(k_o) = \frac{1}{k_o \epsilon_r(R)} \nabla \times \overline{N}^{(m)}_{\substack{e\\o}mn}(k_o) \;.$$

Consider equation (3.25) in spherical coordinates

$$\frac{1}{R^2} \frac{\partial}{\partial R}\left(R^2 \frac{\partial \psi}{\partial R}\right) + \frac{1}{R^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{R^2 \sin^2\theta} \frac{\partial^2 \psi}{\partial \phi^2} +$$

$$+ k_o^2\, \epsilon_r(R)\, \psi = 0 \;.$$

Let

$$\psi = F_1\, F_2\, F_3$$

and we will perform a separation of variables. The above equation becomes

$$\frac{1}{R^2 F_1} \frac{\partial}{\partial R}\left(R^2 \frac{\partial F_1}{\partial R}\right) + \frac{1}{R^2 \sin\theta} \frac{1}{F_2} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial F_2}{\partial \theta}\right) +$$

$$+ \frac{1}{R^2 \sin^2\theta} \frac{\partial^2 F_3}{\partial \phi^2} + k_o^2\, \epsilon_r(R) = 0$$

or

$$\frac{1}{F_1}\frac{\partial}{\partial R}\left(R^2\frac{\partial F_1}{\partial R}\right)+k_o^2\epsilon_r(R)R^2=-\left[\frac{1}{\sin\theta}\frac{1}{F_2}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F_2}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{1}{F_3}\frac{\partial^2 F_3}{\partial\phi^2}\right].$$

The angular part is the same as in the case of spherical cavity (see sec. 2.5). Therefore, its solution is

$$P_n^m(\cos\theta)\,{\cos \atop \sin}\, m\,\phi\ .$$

The radial part of this equation becomes

$$\frac{1}{F_1}\frac{\partial}{\partial R}\left(R^2\frac{\partial F_1}{\partial R}\right)+k_o^2\epsilon_r(R)R^2=n\,(n+1)$$

or

$$R^2\frac{d^2F_1}{dR^2}+2R\frac{dF_1}{dR}+\left[k_o^2\epsilon_r(R)\,R^2-n\,(n+1)\right]F_1=0\ .$$

Let

$$F_1=R^\gamma S_n(k_oR)$$

where γ is an arbitrary constant and $S_n(k_oR)$ is a new dependent variable. Making the change of dependent variable, we get

$$R^2S_n''(k_oR)+2(\gamma+1)RS_n'(k_oR)+\left[\gamma^2+\gamma+k_o^2\epsilon_r(R)R^2-n(n+1)\right]S_n(k_oR)=0\ .$$

Taking $\gamma=-1$, the above differential equation assumes its normal form (Rainville, 1964)

$$S_n''(k_oR)+\left[k_o^2\epsilon_r(R)-\frac{n\,(n+1)}{R^2}\right]S_n(k_oR)=0 \quad . \tag{3.28}$$

Finally, the solution of equation (3.25) can be written as

$$\psi_{\substack{emn\\o}}(k_o) = \frac{1}{R}\, S_n(k_oR)\, P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix} m\phi \tag{3.29}$$

where $S_n(k_oR)$ satisfies equation (3.28).

Consider, now, equation (3.26). In spherical coordinates it has the form

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial\Phi}{\partial R}\right) - \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) +$$

$$+\frac{1}{R^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} - \frac{1}{\epsilon_r(R)}\frac{d\epsilon_r(R)}{dR}\frac{1}{R}\frac{\partial(R\Phi)}{\partial R} + k_o^2\,\epsilon_r(R)\Phi = 0.$$

Let

$$\Phi = F_1\, F_2\, F_3\ ,$$

and the separation of variables yields

$$\frac{1}{F_1R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial F_1}{\partial R}\right) + \frac{1}{R^2\sin\theta\, F_2}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F_2}{\partial\theta}\right) + \frac{1}{F_3R^2\sin\theta}\frac{\partial^2F_3}{\partial\phi^2} - \frac{1}{F_1\epsilon_r(R)}$$

$$\frac{d\epsilon_r(R)}{dR}\frac{\partial F_1}{\partial R} + \left[k_o^2\,\epsilon_r(R) - \frac{1}{\epsilon_r(R)}\frac{1}{R}\frac{d\epsilon_r(R)}{dR}\right] = 0$$

or

$$\frac{1}{F_1}\frac{\partial}{\partial R}\left(R^2\frac{\partial F_1}{\partial R}\right) - \frac{R^2}{F_1\epsilon_r(R)}\frac{d\epsilon_r(R)}{dR}\frac{\partial F_1}{\partial R} + \left[k_o^2R^2\epsilon_r(R) - \frac{R}{\epsilon_r(R)}\frac{d\epsilon_r(R)}{dR}\right] =$$

$$= -\left[\frac{1}{F_2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F_2}{\partial\theta}\right) + \frac{1}{F_3\sin^2\theta}\frac{\partial^2F_3}{\partial\phi^2}\right].$$

The angular part gives us, again, the spherical harmonics $P_n^m(\cos\theta)\begin{matrix}\cos\\ \sin\end{matrix} m\phi$.

The radial part becomes

$$\frac{1}{F_1}\frac{\partial}{\partial R}\left(R^2\frac{\partial F_1}{\partial R}\right) - \frac{R^2}{F_1\epsilon_r(R)}\frac{d\epsilon_r(R)}{dR}\frac{\partial F_1}{\partial R} + \left[k_o^2R^2\epsilon_r(R) - \frac{R}{\epsilon_r(R)}\frac{d\epsilon_r(R)}{dR}\right] = n(n+1).$$

Let

$$F_1 = R^{\gamma}T_n(k_oR)$$

where γ is an arbitrary constant and $T_n(k_oR)$ is a new dependent variable. The above equation then becomes

$$R^2T_n''(k_oR) + \left[2(\gamma+1)R - \frac{R^2}{\epsilon_r(R)}\frac{d\epsilon_r(R)}{dR}\right]T_n'(k_oR) + \left[\gamma(\gamma+1) - \frac{\gamma R}{\epsilon_r(R)}\frac{d\epsilon_r(R)}{dR} - \right.$$

$$\left. - \frac{R}{\epsilon_r(R)}\frac{d\epsilon_r(R)}{dR} + k_o^2\epsilon_r(R)R^2 - n(n+1)\right]T_n(k_oR) = 0\ .$$

Taking $\gamma = -1$, the equation becomes

$$R^2\,T_n''(k_oR) - \frac{R^2}{\epsilon_r(R)}\frac{d\epsilon_r(R)}{dR}\,T_n'(k_oR) + \left[k_o^2\,\epsilon_r(R)\,R^2 - n(n+1)\right]T_n(k_oR) = 0$$

or

$$T_n''(k_oR) - \frac{1}{\epsilon_r(R)}\frac{d\epsilon_r(R)}{dR}T_n'(k_oR) + \left[k_o^2\epsilon_r(R) - \frac{n(n+1)}{R^2}\right]T_n(k_oR) = 0\ . \tag{3.30}$$

Therefore, the solution of equation (3.26) is

$$\Phi_{\substack{e\\o}mn}(k_o) = \frac{1}{R}\,T_n(k_oR)\,P_n^m(\cos\theta)\,{\cos \atop \sin}\,m\phi \tag{3.31}$$

where $T_n(k_oR)$ satisfies equation (3.30).

Instead of continuing with the usual process of determination of the dyadic Green's functions in an inhomogeneous medium, we will use a short cut to find out the dyadics $\bar{\bar{G}}_{e3}^{(11)}$ and $\bar{\bar{G}}_{m3}^{*(11)}$, which are necessary to determine the electric and magnetic fields of electric and magnetic dipoles in the presence of inhomogeneous lenses. This method is explained in the next section.

3.4 Construction of the Dyadic Green's Functions in the Presence of a Lens

In the case of the presence of two dielectric media, we have to define a new kind of dyadic Green's functions which we will call the dyadic Green's functions of the third kind. They solve equations (2.6) and (2.7) with the boundary conditions

$$(\hat{n} \times \bar{\bar{G}}_{e3})_{s-} = (\hat{n} \times \bar{\bar{G}}_{e3})_{s+}$$

$$\frac{1}{\mu_1} (\hat{n} \times \nabla \times \bar{\bar{G}}_{e3})_{s-} = \frac{1}{\mu_2} (\hat{n} \times \nabla \times \bar{\bar{G}}_{e3})_{s+}$$

$$\frac{1}{\mu_1} (\hat{n} \times \bar{\bar{G}}_{m3})_{s-} = \frac{1}{\mu_2} (\hat{n} \times \bar{\bar{G}}_{m3})_{s+}$$

$$\frac{1}{k_1^2} (\hat{n} \times \nabla \times \bar{\bar{G}}_{m3})_{s-} = \frac{1}{k_2^2} (\hat{n} \times \nabla \times \bar{\bar{G}}_{m3})_{s+}$$

for electric dipole excitation, and equation (2.24) and (2.25) with the boundary conditions

$$(\hat{n} \times \bar{\bar{G}}^{*}_{m3})_{s-} = (\hat{n} \times \bar{\bar{G}}^{*}_{m3})_{s+}$$

$$\frac{1}{\epsilon_1} (\hat{n} \times \nabla \times \bar{\bar{G}}^{*}_{m3})_{s-} = \frac{1}{\epsilon_2} (\hat{n} \times \nabla \times \bar{\bar{G}}^{*}_{m3})_{s+}$$

$$\frac{1}{\epsilon_1} (\hat{n} \times \bar{\bar{G}}^{*}_{e3})_{s-} = \frac{1}{\epsilon_2} (\hat{n} \times \bar{\bar{G}}^{*}_{e3})_{s+}$$

$$\frac{1}{k_1^2} (\hat{n} \times \nabla \times \bar{\bar{G}}^{*}_{e3})_{s-} = \frac{1}{k_2^2} (\hat{n} \times \nabla \times \bar{\bar{G}}^{*}_{m3})_{s+}$$

for magnetic dipole excitation. Here s- and s+ denote, respectively, the surface approached from opposite sides of a boundary separating the isotropic media with constitutive constants, μ_1, ϵ_1 and μ_2, ϵ_2 .

Each of the dyadic Green's functions of the third kind is made up of two components: the dyadic Green's function interior to the scatterer, indicated by superscript (21), and the dyadic Green's function exterior to the scatterer, indicated by superscript (11). The first superscript in this notation indicates the medium in which the observer is located and the second indicates the medium in which the source is located. The equations relating the dyadics of electric and magnetic types are

$$\nabla \times \bar{\bar{G}}_{m3}^{(11)} = \bar{\bar{I}}\delta(\bar{R} - \bar{R}') + k_o^2 \bar{\bar{G}}_{e3}^{(11)}$$

$$\nabla \times \bar{\bar{G}}_{e3}^{(11)} = \bar{\bar{G}}_{m3}^{(11)}$$

$$\nabla \times \bar{\bar{G}}_{m3}^{(21)} = k_o^2 \epsilon_r(R) \bar{\bar{G}}_{e3}^{(21)}$$

$$\nabla \times \bar{\bar{G}}_{e3}^{(21)} = \bar{\bar{G}}_{m3}^{(21)}$$

for electric dipole excitation and

$$\nabla \times \bar{\bar{G}}_{e3}^{*(11)} = \bar{\bar{I}}\delta(\bar{R} - \bar{R}') + k_o^2 \bar{\bar{G}}_{m3}^{*(11)}$$

$$\nabla \times \bar{\bar{G}}_{m3}^{*(11)} = \bar{\bar{G}}_{e3}^{*(11)}$$

$$\nabla \times \bar{\bar{G}}_{m3}^{*(21)} = k_o^2 \bar{\bar{G}}_{m3}^{*(21)}$$

$$\nabla \times \bar{\bar{G}}_{e3}^{*(21)} = \epsilon_r(R) \bar{\bar{G}}_{e3}^{*(21)}$$

for magnetic dipole excitation.

In what follows we will be interested only in $\bar{\bar{G}}_{e3}$ and $\bar{\bar{G}}^{*}_{m3}$, so only their expressions will be stated explicitly. The method used to construct the dyadic Green's functions in the presence of the lens is that of scattering superposition, which says that the dyadic Green's functions exterior to the scatterer comprise two parts: one corresponding to the dyadic Green's function in free space, and the other corresponding to the scattered dyadic Green's function. Stated explicitly:

$$\bar{\bar{G}}^{(11)}_{e3}(\bar{R}|\bar{R}') = \bar{\bar{G}}_{eo}(\bar{R}|\bar{R}') + \bar{\bar{G}}^{(11)}_{e3s}(\bar{R}|\bar{R}') \qquad R \geq a$$

$$\bar{\bar{G}}^{(21)}_{e3}(\bar{R}|\bar{R}') = \bar{\bar{G}}^{(21)}_{e3s}(\bar{R}|\bar{R}') \qquad R \leq a$$

and

$$\bar{\bar{G}}^{*(11)}_{m3}(\bar{R}|\bar{R}') = \bar{\bar{G}}^{*}_{mo}(\bar{R}|\bar{R}') + \bar{\bar{G}}^{*(11)}_{m3s}(\bar{R}|\bar{R}') \qquad R \geq a$$

$$\bar{\bar{G}}^{*(21)}_{m3}(\bar{R}|\bar{R}') = \bar{\bar{G}}^{*(21)}_{m3s}(\bar{R}|\bar{R}') , \qquad R \leq a$$

where the subscript s means 'scattered' and o refers to free space.

The construction of the scattered and transmitted dyadic Green's function is governed by the following rules:

1) $\bar{\bar{G}}^{(11)}_{e3s}$ and $\bar{\bar{G}}^{(21)}_{e3s}$ ($\bar{\bar{G}}^{*(11)}_{m3s}$ and $\bar{\bar{G}}^{*(21)}_{m3s}$) have similar forms as $\bar{\bar{G}}_{eo}(\bar{R}|\bar{R}')$ ($\bar{\bar{G}}^{*}_{mo}(\bar{R}|\bar{R}')$) for $\bar{R} < \bar{R}'$ with two unknown constants which are given by the boundary conditions.

2) By definition, any dyadic $\bar{\bar{D}}$ is formed by two vectors, namely $\bar{\bar{D}} = \bar{A}\bar{B}$, where $\bar{A}$ is called the anterior element and $\bar{B}$ the posterior element. In the case of a dyadic Green's function, the anterior element represents the field and the posterior element represents the source. Therefore, in the constructions of the scattered dyadic Green's function, only the anterior element (the field) differs from the anterior element of the corresponding free space dyadic Green's function, while the posterior element (the source)

is the same as that of the corresponding free space dyadic Green's function.

3) The choice of the anterior element in the scattered dyadic Green's function ($\bar{\bar{G}}^{(11)}_{e3s}(\bar{R}|\bar{R}')$ or $\bar{\bar{G}}^{*(11)}_{m3s}(\bar{R}|\bar{R}')$) is governed by the radiation condition. The choice of the anterior element in the interior dyadic ($\bar{\bar{G}}^{(21)}_{e3s}(\bar{R}|\bar{R}')$ or $\bar{\bar{G}}^{*(21)}_{m3s}(\bar{R}|\bar{R}')$) is governed by the condition of the finiteness of the dyadic at the origin and involves the inhomogeneous vector wave functions.

Having these rules in mind, the expressions for $\bar{\bar{G}}^{(11)}_{e3}(\bar{R}|\bar{R}')$ ($\bar{\bar{G}}^{*(11)}_{m3}(\bar{R}|\bar{R}')$) and $\bar{\bar{G}}^{(21)}_{e3}(\bar{R}|\bar{R}')$ ($\bar{\bar{G}}^{*(21)}_{m3}(\bar{R}|\bar{R}')$) become

$$\bar{\bar{G}}^{(11)}_{e3}(\bar{R}|\bar{R}') = \frac{ik_o}{4\pi}\sum_{n=1}^{\infty}\sum_{m=0}^{n}(2-\delta_o)\frac{2n+1}{n(n+1)}\frac{(n-m)!}{(n+m)!}\Big[\Big(\bar{M}_{\substack{e\\o}mn}(k_o)+A_{\substack{e\\o}n}\bar{M}^{(1)}_{\substack{e\\o}mn}(k_o)\Big)\bar{M}'^{(1)}_{\substack{e\\o}mn}(k_o)+$$

$$+\Big(\bar{N}_{\substack{e\\o}mn}(k_o)+B_{\substack{e\\o}n}\bar{N}^{(1)}_{\substack{e\\o}mn}(k_o)\Big)\bar{N}'^{(1)}_{\substack{e\\o}mn}(k_o)\Big] - \frac{1}{k_o}\Big[\bar{\bar{I}}\delta(\bar{R}-\bar{R}') - \bar{\bar{S}}_1(\bar{R}|\bar{R}')\Big] \tag{3.32}$$

$$\bar{\bar{G}}^{(21)}_{e3}(\bar{R}|\bar{R}') = \frac{ik_o}{4\pi}\sum_{n=1}^{\infty}\sum_{m=0}^{n}(2-\delta_o)\frac{2n+1}{n(n+1)}\frac{(n-m)!}{(n+m)!}\Big[C_{\substack{e\\o}n}\bar{M}^{(m)}_{\substack{e\\o}mn}(k_o)\bar{M}'^{(1)}_{\substack{e\\o}mn}(k_o)+$$

$$+ D_{\substack{e\\o}n}\bar{N}^{(e)}_{\substack{e\\o}mn}(k_o)\bar{N}'^{(1)}_{\substack{e\\o}mn}(k_o)\Big]$$

and

$$\bar{\bar{G}}^{*(11)}_{m3}(\bar{R}|\bar{R}') = \frac{ik_o}{4\pi}\sum_{n=1}^{\infty}\sum_{m=0}^{n}(2-\delta_o)\frac{2n+1}{n(n+1)}\frac{(n-m)!}{(n+m)!}\Big[\Big(\bar{M}_{\substack{e\\o}mn}(k_o)+E_{\substack{e\\o}n}\bar{M}^{(1)}_{\substack{e\\o}mn}(k_o)\Big)\bar{M}'^{(1)}_{\substack{e\\o}mn}(k_o)+$$

$$+\Big(\bar{N}_{\substack{e\\o}mn}(k_o)+F_{\substack{e\\o}n}\bar{N}^{(1)}_{\substack{e\\o}mn}(k_o)\Big)\bar{N}'^{(1)}_{\substack{e\\o}mn}(k_o)\Big] - \frac{1}{k_o^2}\Big[\bar{\bar{I}}\delta(\bar{R}-\bar{R}') - \bar{\bar{S}}_1(\bar{R}|\bar{R}')\Big] \tag{3.33}$$

$$\bar{\bar{G}}^{*(21)}_{m3}(\bar{R}|\bar{R}') = \frac{ik_o}{4\pi}\sum_{n=1}^{\infty}\sum_{m=0}^{n}(2-\delta_o)\frac{2n+1}{n(n+1)}\frac{(n-m)!}{(n+m)!}\Big[G_{\substack{e\\o}n}\bar{M}^{(e)}_{\substack{e\\o}mn}(k_o)\bar{M}'^{(1)}_{\substack{e\\o}mn}(k_o)+$$

$$+ H_{\substack{e\\o}n}\bar{N}^{(m)}_{\substack{e\\o}mn}(k_o)\bar{N}'^{(1)}_{\substack{e\\o}mn}(k_o)\Big] \quad .$$

The boundary conditions for $\bar{\bar{G}}_{e3}$ are

$$(\hat{R} \times \bar{\bar{G}}_{e3}^{(11)})_{s-} = (\hat{R} \times \bar{\bar{G}}_{e3}^{(21)})_{s+}$$

$$(\hat{R} \times \nabla \times \bar{\bar{G}}_{e3}^{(11)})_{s-} = (\hat{R} \times \nabla \times \bar{\bar{G}}_{e3}^{(21)})_{s+}$$

and for $\bar{\bar{G}}^{*}_{m3}$,

$$(\hat{R} \times \bar{\bar{G}}_{m3}^{*(11)})_{s-} = (\hat{R} \times \bar{\bar{G}}_{m3}^{*(21)})_{s+}$$

$$\epsilon_r (\hat{R} \times \nabla \times \bar{\bar{G}}_{m3}^{*(11)})_{s-} = (\hat{R} \times \nabla \times \bar{\bar{G}}_{m3}^{*(21)})_{s+}$$

with $\mu_1 = \mu_2 = 1$ and $\epsilon_r = \epsilon_2/\epsilon_1$.

Applying the first boundary conditions to (3.32) at R = a (i.e., on the rim of the lens) and assuming that the source is located at a distance b > a, we have for the $\bar{M}'^{(1)}_{\substack{e\\o}mn}(k_o)$ component

$$\left\{ \hat{R} \times \left[\bar{M}_{\substack{e\\o}mn}(k_o) + A_{\substack{e\\o}n} \bar{M}^{(1)}_{\substack{e\\o}mn}(k_o) \right] = \hat{R} \times \left[C_{\substack{e\\o}n} \bar{M}^{(m)}_{\substack{e\\o}mn}(k_o) \right] \right\} \Bigg|_{R=a} ,$$

which simplifies to

$$j_n(\rho_a) + A_{\substack{e\\o}n} h_n^{(1)}(\rho_a) = C_{\substack{e\\o}n} \frac{S_n(\rho_a)}{a}$$

where $\rho_a = k_o a$. The $\bar{N}'^{(1)}_{\substack{e\\o}mn}(k_o)$ component gives

$$\left\{ \hat{R} \times \left[\bar{N}_{\substack{e\\o}mn}(k_o) + B_{\substack{e\\o}n} \bar{N}^{(1)}_{\substack{e\\o}mn}(k_o) \right] = \hat{R} \times \left[D_{\substack{e\\o}n} \bar{N}^{(e)}_{\substack{e\\o}mn}(k_o) \right] \right\} \Bigg|_{R=a} ,$$

which after simplification becomes

$$\left[\rho_a j_n(\rho_a)\right]' + B_{\substack{e\\o}n} \left[\rho_a h_n^{(1)}(\rho_a)\right]' = D_{\substack{e\\o}n} \frac{k_o}{\epsilon_r(a)} T_n'(\rho_a)$$

where the prime over the function indicates the derivative

$$T'_n(\rho_a) = \left.\frac{\partial\, T_n(k_o R)}{\partial (k_o R)}\right|_{R=a} .$$

The second boundary condition applied to the $\bar{M}^{(1)}_{\substack{e\\o}mn}(k_o)$ component gives

$$\left\{ \hat{R} \times \nabla\times \left[\bar{M}_{\substack{e\\o}mn}(k_o) + A_{\substack{e\\o}n}\, \bar{M}^{(1)}_{\substack{e\\o}mn}(k_o) \right] = \hat{R} \times \nabla\times \left[C_{\substack{e\\o}n}\, \bar{M}^{(m)}_{\substack{e\\o}mn}(k_o) \right] \right\}\Bigg|_{R=a}$$

which, after some simplifications, becomes

$$\left[\rho_a\, j_n(\rho_a)\right]' + A_{\substack{e\\o}n} \left[\rho_a\, h^{(1)}_n(\rho_a)\right]' = C_{\substack{e\\o}n}\, k_o\, S'_n(\rho_a)$$

where

$$S'_n(\rho_a) = \left.\frac{\partial\, S_n(k_o R)}{\partial (k_o R)}\right|_{R=a} .$$

Finally, for the $\bar{N}'^{(1)}_{\substack{e\\o}mn}(k_o)$ component,

$$j_n(\rho_a) + B_{\substack{e\\o}n}\, h^{(1)}_n(\rho_a) = D_{\substack{e\\o}n} \frac{T_n(\rho_a)}{a} .$$

Thus, the unknowns A, B, C and D are determined from the system

$$-A_{\substack{e\\o}n}\, h^{(1)}_n(\rho_a) + C_{\substack{e\\o}n} \frac{S_n(\rho_a)}{a} = j_n(\rho_a)$$

$$-A_{\substack{e\\o}n} \left[\rho_a\, h^{(1)}_n(\rho_a)\right]' + C_{\substack{e\\o}n}\, k_o\, S'_n(\rho_a) = \left[\rho_a\, j_n(\rho_a)\right]'$$

$$-B_{\substack{e\\o}n}\, h^{(1)}_n(\rho_a) + D_{\substack{e\\o}n} \frac{T_n(\rho_a)}{a} = j_n(\rho_a)$$

$$-B_{\substack{e\\o}n}\left[\rho_a h_n^{(1)}(\rho_a)\right]' + D_{\substack{e\\o}n}\frac{k_o}{\epsilon_r(a)}\, T'_a(\rho_a) = \left[\rho_a j_n(\rho_a)\right]'$$

and are given by

$$A_{\substack{e\\o}n} = \frac{PS' - P'S}{Q'S - QS'}$$

$$B_{\substack{e\\o}n} = \frac{PT' - \epsilon_r(a)\, P'T}{\epsilon_r(a)\, Q'T - QT'}$$

$$C_{\substack{e\\o}n} = \frac{i}{k_o\left[Q'S - QS'\right]}$$

and

$$D_{\substack{e\\o}n} = \frac{1}{k_o}\;\frac{i\,\epsilon_r(a)}{\epsilon_r(a)\, Q'T - QT'}$$

where $P = \rho_a\, j_n(\rho_a)$; $P' = \left.\dfrac{d\left[k_o R\, j_n(k_o R)\right]}{d(k_o R)}\right|_{R=a}$

and $Q = \rho_a\, h_n^{(1)}(\rho_a)$; $Q' = \left.\dfrac{d\left[k_o R\, h_n^{(1)}(k_o R)\right]}{d(k_o R)}\right|_{R=a}$.

In the above relation $PQ' - P'Q = i$ was used (see Appendix A1).

Proceeding in the same fashion we find that the unknown coefficients for dyadic Green's function of the third kind with magnetic dipole excitation $\bar{\bar{G}}^{*}_{m3}(\bar{R}|\bar{R}')$ are

$$E_{\substack{e\\o}n} = \frac{PT' - \epsilon_r(a)\, P'T}{\epsilon_r(a)\, Q'T - QT'}$$

$$F_{\substack{e\\o}n} = \frac{P'S - PS'}{QS' - Q'S}$$

$$G_{\substack{e\\o}n} = \frac{1}{k_o}\;\frac{i\,\epsilon_r(a)}{\epsilon_r(a)\, Q'T - QT'}$$

$$H_{\substack{e\\o}n} = \frac{i}{k_o (Q'S - QS')}$$

where P, Q, T and S and their derivatives were defined previously.

Except the missing singular term contained in $\bar{\bar{G}}_{eo}$ the formulation for $\bar{\bar{G}}_{e3}$ contained in this section follows closely the work of Tai (1971). We review it here mainly for completeness. The formulation for $\bar{\bar{G}}^{*}_{m3}$, however, is not found there.

3.5 Electric Field of the Electric and Magnetic Dipoles in the Presence of a Lens

3.5.1 Electric Field of an Electric Dipole in the Presence of a Lens

Consider an infinitesimal electric dipole with current moment c_e oriented in the + x-direction at a distance b from the origin (see Fig. 4). The expression for the current of the dipole in spherical coordinates is

$$\bar{J}_e(\bar{R}') = - c_e \frac{\delta(R'-b)\,\delta(\theta'-\pi)\,\delta(\phi')}{b^2 \sin\theta'} \hat{\theta} .$$

The expression for the electric field radiated by such a dipole is

$$\bar{E}_e(\bar{R}) = i\omega\mu_o \int_V \bar{\bar{G}}^{(11)}_{e3}(\bar{R}|\bar{R}') \cdot \bar{J}_e(\bar{R}')\, dv' .$$

The dyadic Green's function in this case is given by

$$\bar{\bar{G}}^{(11)}_{e3}(\bar{R}|\bar{R}') = \bar{\bar{G}}_{eo}(\bar{R} > \bar{R}') + \bar{\bar{G}}^{(11)}_{e3s}(\bar{R}|\bar{R}') ,$$

where $\bar{\bar{G}}_{eo}(\bar{R} > \bar{R}')$ means that we should use the expression for the free-space dyadic Green's function which corresponds to $\bar{R} > \bar{R}'$. In the present work, our concern is only with the region exterior to the source. Therefore, the singular term which appears in the expression for $\bar{\bar{G}}^{(11)}_{e3}$ and which is important only in the source region, will be dropped from the expression for the Green's function. Substituting the expression for the dyadic in the expression for the field, we have

$$\bar{E}_e(\bar{R}) = \frac{-\omega\mu_o k_o \pi}{2} \sum_{n=1}^{\infty} \sum_{m=0}^{n} C_{mn} \Big\{ \bar{M}^{(1)}_{\substack{e\\o}mn}(k_o) \int_V \Big[\bar{M}'_{\substack{e\\o}mn}(k_o) + A_{\substack{e\\o}n} \bar{M}'^{(1)}_{\substack{e\\o}mn}(k_o)\Big) \cdot$$

$$\cdot \bar{J}_e(\bar{R}')\, dv' + \bar{N}^{(1)}_{\substack{e\\o}mn}(k_o) \int_V \Big(\bar{N}'_{\substack{e\\o}mn}(k_o) + B_{\substack{e\\o}n} \bar{N}'^{(1)}_{\substack{e\\o}mn}(k_o)\Big] \cdot \bar{J}_e(\bar{R})\, dv' \Big\} .$$

Let us compute each component of the electric field separately.

$$\int_V \bar{M}'_{\substack{e\\o}mn}(k_o) \cdot \bar{J}_e(\bar{R}')dv' = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \Big(\mp \frac{m}{\sin\theta} j_n(k_o R') P_n^m(\cos\theta') \begin{matrix}\sin\\ \cos\end{matrix} m\phi' \hat{\theta} -$$

$$- j_n(k_o R') \frac{\partial P_n^m(\cos\theta')}{\partial \theta'} \begin{matrix}\cos\\ \sin\end{matrix} m\phi' \hat{\phi} \Big) \cdot$$

$$\cdot \left[- c_e \frac{\delta(R'-b)\,\delta(\theta'-\pi)\,\delta(\phi')}{b^2 \sin\theta'} \right] R'^2 \sin\theta'\, dR'\, d\theta'\, d\phi' \quad .$$

After performing the integrations and simplifications, we find that

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \bar{M}'_{\substack{e\\o}mn}(k_o) \cdot \bar{J}_e(\bar{R}')\, dv' = - c_e j_n(k_o b) (-1)^{n+1} \frac{n(n+1)}{2} \quad , \quad (3.34)$$

where

$$\frac{P_n^1(\cos\pi)}{\sin\pi} = (-1)^{n+1} \frac{n(n+1)}{2}$$

was used (see Appendix A2). By replacing $j_n(k_o R)$ by $h_n^{(1)}(k_o R)$ in (3.34), we have

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \bar{M}'^{(1)}_{\substack{e\\o}mn}(k_o)\cdot\bar{J}_e(\bar{R}')\,dv' = -c_e\,h_n^{(1)}(k_ob)\,(-1)^{n+1}\,\frac{n(n+1)}{2}\,. \tag{3.35}$$

For the $\bar{N}^{(1)}$ component, we have

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \bar{N}'_{\substack{e\\o}mn}(k_o)\cdot\bar{J}_e(\bar{R}')\,dv' = \frac{c_e}{k_ob}\left[k_ob\,j_n(k_ob)\right]'\,(-1)^{n+1}\,\frac{n(n+1)}{2}\,, \tag{3.36}$$

where

$$\left.\frac{\partial P_n^1(\cos\theta)}{\partial\theta}\right|_{\theta=\pi} = (-1)^n\,\frac{n(n+1)}{2}$$

was used (see Appendix A2). By replacing $j_n(k_oR)$ by $h_n^{(1)}(k_oR)$ in (3.36), we have

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \bar{N}'^{(1)}_{\substack{e\\o}mn}(k_o)\cdot\bar{J}_e(\bar{R}')\,dv' = \frac{c_e}{k_ob}\left[k_ob\,h_n^{(1)}(k_ob)\right]'\,(-1)^{n+1}\,\frac{n(n+1)}{2}\,. \tag{3.37}$$

Substituting expressions from (3.34) through (3.37) in the expression for the electric field we obtain:

$$\bar{E}_e(\bar{R}) = \frac{\omega\mu_o k_o c_e}{4\pi}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}(-1)^{n+1}\Bigg\{\bar{M}^{(1)}_{o1n}(k_o)\left[j_n(k_ob)+A_{en}h_n^{(1)}(k_ob)\right] - \bar{N}^{(1)}_{e1n}(k_o)\,\frac{\left[k_ob\,j_n(k_ob)\right]' + B_{en}\left[k_ob\,h_n^{(1)}(k_ob)\right]'}{k_ob}\Bigg\}. \tag{3.38}$$

By bringing the dipole to the rim of the lens (i.e., b → a), some simplifications are possible:

$$j_n(\rho_a) + A_{en}\,h_n^{(1)}(\rho_a) = j_n(\rho_a) + \frac{PS' - PS'}{Q'S - QS'}\,h_n^{(1)}(\rho_a) = \frac{iS}{\rho_a\,(Q'S - QS')}$$

and

$$\left[\rho_a j_n(\rho_a)\right]' + B_{en}\left[\rho_a h_n^{(1)}(\rho_a)\right]' = P' + \frac{1/\epsilon_r(a)\, PT' - P'T}{Q'T - 1/\epsilon_r(a)\, QT'}\, Q' =$$

$$= \frac{iT'}{\epsilon_r(a)\, Q'T - QT'} \quad .$$

Therefore, (3.38) becomes

$$E_e(R) = \frac{\omega\mu_o k_o c_e}{4\pi} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} (-1)^{n+1} \left\{ M_{oln}^{(1)}(k_o) \frac{i}{\rho_a} \frac{S}{Q'S - QS'} - N_{eln}^{(1)}(k_o) \frac{i}{\rho_a} \frac{T'}{\epsilon_r(a)\, Q'T - QT'} \right\} .$$

Simplifying again we obtain

$$E_e(R) = \frac{i\omega\mu_o c_e}{4\pi a} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} (-1)^{n+1} \left\{ \frac{S}{Q'S - QS'} M_{oln}^{(1)}(k_o) + \frac{T'}{\epsilon_r(a)\, Q'T - QT'} N_{eln}^{(1)}(k_o) \right\} . \tag{3.39}$$

3.5.2 Electric Field of a Magnetic Dipole in the Presence of a Lens

The expression for the electric field of an electric dipole elaborated in the previous section is found in Tai's (1971) book and was presented in this work mainly for completeness because our goal is to find the expression for the electric field of the Huygens source. The expression for the electric field of a magnetic dipole which will be constructed in this section is not found in Tai's book.

Knowing the expression for $\bar{\bar{G}}_{m3}^{*(11)}$ from section 3.4, we can compute the magnetic field of an infinitesimal magnetic dipole located at a distance $b > a$ from the origin. We assume that the dipole is oriented in the + y-direction (see Fig. 4), and the expression of the magnetic current of this dipole in spherical coordinates, takes the form,

$$\bar{J}_m(R') = c_m \frac{\delta(R'-b)\,\delta(\theta'-\pi)\,\delta(\phi')}{b^2 \sin\theta'}\,\hat{\phi}$$

where c_m is the moment of the magnetic dipole.

The magnetic field radiated by the dipole is given by

$$\bar{H}_m(\bar{R}) = i\omega\epsilon_o \int_V \bar{\bar{G}}^{*(11)}_{m3} (\bar{R}|\bar{R}') \cdot \bar{J}_m (\bar{R}') \; dv' \; .$$

The expression for the dyadic Green's function in this case is

$$\bar{\bar{G}}^{*(11)}_{m3}(\bar{R}|\bar{R}') = \bar{\bar{G}}^{*}_{mo} (\bar{R} > \bar{R}') + \bar{\bar{G}}^{*(11)}_{m3s} (\bar{R}|\bar{R}') \; .$$

As in the previous section, because we are not interested in the source region, the singular term in the expression for $\bar{\bar{G}}^{*(11)}_{m3}$ will be dropped. Substituting $\bar{\bar{G}}^{*(11)}_{m3}$ in the expression for the magnetic field, we obtain

$$\bar{H}_m(\bar{R}) = -\frac{\omega\mu_o k_o \pi}{2} \sum_{n=1}^{\infty} \sum_{m=0}^{n} C_{mn} \Bigg\{ \bar{M}^{(1)}_{\substack{e\\o}mn}(k_o) \int_V (\bar{M}'_{\substack{e\\o}mn}(k_o) + E_{\substack{e\\o}n} \bar{M}'^{(1)}_{\substack{e\\o}mn}(k_o)) \cdot \bar{J}_m dv'$$

$$+ \bar{N}^{(1)}_{\substack{e\\o}mn}(k_o) \int_V (\bar{N}'_{\substack{e\\o}mn}(k_o) + F_{\substack{e\\o}n} \bar{N}'^{(1)}_{\substack{e\\o}mn}(k_o)) \cdot \bar{J}_m (\bar{R}') \, dv' \Bigg\} \; .$$

Integrating each component of the above expression separately, we get

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \bar{M}'_{\substack{e\\o}mn}(k_o) \cdot \bar{J}_m(\bar{R}') \, dv' = - j_n(k_o b) (-1)^n \frac{n(n+1)}{2} \; , \tag{3.40}$$

and replacing $j_n(k_o R)$ by $h^{(1)}_n(k_o R)$ in the above,

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \bar{M}'^{(1)}_{\substack{e\\o}mn}(k_o) \cdot \bar{J}_m(\bar{R}') \, dv' = -h^{(1)}_n(k_o b) (-1)^n \frac{n(n+1)}{2} \; . \tag{3.41}$$

The $N^{(1)}_{\substack{e\\o}mn}$ component of the expression for $\bar{H}(\bar{R})$ gives

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \bar{N}'_{\substack{e\\o}mn}(k_o) \cdot \bar{J}_m(\bar{R}') \, dv' = \frac{1}{k_o b} \left[k_o b \; j_n(k_o b) \right]' (-1)^{n+1} \frac{n(n+1)}{2} \tag{3.42}$$

and

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \bar{N}'^{(1)}_{\substack{e\\o}mn}(k_o) \cdot \bar{J}_m(\bar{R}')\, dv' = \frac{1}{k_o b}\left[k_o b\, h_n^{(1)}(k_o b)\right]' (-1)^{n+1} \frac{n(n+1)}{2} . \tag{3.43}$$

Substituting (3.40) through (3.43) in the expression for the magnetic field, we obtain

$$\bar{H}_m(\bar{R}) = -\frac{\omega\epsilon_o k_o c_m}{4\pi} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} (-1)^{n+1} \left\{ \bar{M}^{(1)}_{e1n}(k_o)\left[j_n(k_o b) + E_{en} h_n^{(1)}(k_o b)\right] + \right.$$

$$\left. + \bar{N}^{(1)}_{o1n}\left[\frac{(k_o b\, j_n(k_o b))' + F_{on}(k_o b\, h_n^{(1)}(k_o b))'}{k_o b}\right]\right\} .$$

We can now make more simplifications for the magnetic dipole on the rim of the lens. Thus

$$P + E_{en} Q = P + \frac{PT' - \epsilon_r(a) P'T}{\epsilon_r(a) Q'T - QT'} Q = \frac{i\epsilon_r(a) T}{\epsilon_r(a) Q'T - QT'}$$

and

$$P' + F_{on} Q' = P' + \frac{P'S - PS'}{QS' - Q'S} Q' = \frac{-iS'}{QS' - Q'S} ,$$

and considering these simplifications the expression for the magnetic field becomes

$$\bar{H}_m(\bar{R}) = \frac{-i\omega\epsilon_o c_m}{4\pi a} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+1}{n(n+1)} \left[\frac{\epsilon_r(a) T_n \bar{M}^{(1)}_{e1n}(k_o)}{\epsilon_r(a) Q_n' T_n - Q_n T_n'} + \frac{S_n' \bar{N}^{(1)}_{o1n}}{Q_n' S_n - Q_n S_n'}\right] . \tag{3.44}$$

Now, because

$$\bar{E}_m(\bar{R}) = \frac{i}{\omega\epsilon_o} \nabla \times \bar{H}_m(R) ,$$

we can calculate the electric field radiated by the magnetic dipole. By taking the curl of (3.44) we get

$$\bar{E}_m(\bar{R}) = \frac{k_o c_m}{4\pi a} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+1}{n(n+1)} \left[\frac{S'}{Q'S-QS'} \bar{M}^{(1)}_{o1n}(k_o) + \frac{\epsilon_r(a)T}{\epsilon_r(a)Q'T-QT'} \bar{N}^{(1)}_{e1n}(k_o) \right] . \tag{3.45}$$

Therefore, the final expression for the electric field radiated by a magnetic dipole with the moment given by $c_m = \eta_o c_e$ is

$$\bar{E}_m(\bar{R}) = \frac{\mu_o \omega c_e}{4\pi a} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+1}{n(n+1)} \left[\frac{S'}{Q'S-QS'} \bar{M}^{(1)}_{o1n}(k_o) + \frac{\epsilon_r(a)T}{\epsilon_r(a)Q'T-QT'} \bar{N}^{(1)}_{e1n}(k_o) \right] \tag{3.46}$$

where

$$\eta_o = \sqrt{\frac{\mu_o}{\epsilon_o}} .$$

The case in which the moment of the magnetic dipole and the moment of the electric dipole are related by

$$c_m = \eta_o c_e , \tag{3.47}$$

representing a Huygen's source, will be considered in the next section.

3.5.3 Huygens Source

Sometimes it is desirable to study the radiation of the aperture of an open-ended waveguide illuminating a microwave lens. A rigorous study of such a setup is very difficult and so we have to consider an approximate model for it. One such model is Huygens source.

Huygens source is formed by crossing two dipoles, one electric and the other magnetic, at 90^o in space, with the moment of the magnetic dipole related to the moment of the electric dipole by (3.47). The far-zone electric field radiated by such a device is given by Jordan and Balmain (1968).

$$\bar{E}_H = \frac{-i\omega\mu_o c_e}{4\pi} \frac{e^{ikr}}{R} (\cos\phi\, \hat{\theta} + \sin\phi\, \hat{\phi})(1+\cos\theta) .$$

The radiation field of such a source has a cardioid pattern. It has a maximum in the forward ($\theta = 0^o$) and a zero in the backward ($\theta = 180^o$) direction. This model could represent, in first approximation, the radiation pattern of an open-ended waveguide.

In order to study the effect of a microwave lens on the radiation pattern of an open-ended waveguide, we will combine the expression (3.39) and (3.46) for electric fields radiated by electric and magnetic dipoles presented in the previous sections. The expression for the electric field radiated by Huygens source, here denoted by $\bar{E}_H$, is

$$\bar{E}_H(\bar{R}) = \bar{E}_e(\bar{R}) + \bar{E}_m(\bar{R}) = \frac{\omega\mu_o c_e}{4\pi a} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+1}{n(n+1)} \left[\frac{i\, S+S'}{Q'S-QS'} \bar{M}^{(1)}_{o1n}(k_o) + \right.$$

$$\left. + \frac{\epsilon_r(a)\, T - i\, T'}{\epsilon_r(a)\, Q'T - QT'} \bar{N}^{(1)}_{e1n}(k_o) \right] . \qquad (3.48)$$

In the far-zone field the following approximations (Stratton, 1941) are valid:

$$\bar{M}^{(1)}_{o1n} \cong (-i)^{n+1} \frac{e^{ikR}}{kR} \bar{m}_{o1n}$$

and

$$\bar{N}^{(1)}_{e1n} \cong (-i)^n \frac{e^{ikR}}{kR} \bar{n}_{e1n}$$

where

$$\bar{m}_{o1n} = \frac{P^1_n(\cos\theta)}{\sin\theta} \cos\phi\, \hat{\theta} - \frac{\partial P^1_n(\cos\theta)}{\partial\theta} \sin\phi\, \hat{\phi}$$

and

$$\bar{n}_{e1n} = \frac{\partial P^1_n(\cos\theta)}{\partial\theta} \cos\phi\, \hat{\theta} - \frac{P^1_n(\cos\theta)}{\sin\theta} \sin\phi\, \hat{\phi} .$$

Thus, the expression for the electric field becomes

$$\bar{E}_H(\bar{R}) = \frac{\omega\mu_o c_e}{4\pi a}\,\frac{e^{ik_oR}}{k_oR}\sum_{n=1}^{\infty}(i)^{n+1}\frac{2n+1}{n(n+1)}\left[\frac{iS+S'}{Q'S-QS'}\,\bar{m}_{oln}+\frac{T'+i\epsilon_r(a)T}{\epsilon_r(a)Q'T-QT'}\,\bar{n}_{eln}\right]. \tag{3.49}$$

Up to this point, it was assumed that the relative permittivity of the lens at the rim has some constant value $\epsilon_r(a)$, which means that the lens is not matched to free-space. Henceforth, all the lenses we will consider will have $\epsilon_r(a) = 1$, i.e., the lenses are matched to free-space, and we can simplify expression (3.49) for the field:

$$\bar{E}_H(\bar{R}) = \frac{\omega\mu_o c_e}{4\pi a}\,\frac{e^{ik_oR}}{k_oR}\sum_{n=1}^{\infty}(i)^{n+1}\frac{2n+1}{n(n+1)}\left[\frac{iS+S'}{Q'S-QS'}\,\bar{m}_{oln}+\frac{T'+iT}{Q'T-QT'}\,\bar{n}_{eln}\right]. \tag{3.50}$$

Substituting the values of $\bar{m}_{oln}$ and $\bar{n}_{eln}$ into (3.50) yields

$$\bar{E}_H(\bar{R}) = \frac{\omega\mu_o c_e}{4\pi a}\,\frac{e^{ik_oR}}{k_oR}\sum_{n=1}^{\infty}(i)^{n+1}\frac{2n+1}{n(n+1)}\left[\left(\frac{iS+S'}{Q'S-QS'}\,\frac{P_n^1(\cos\theta)}{\sin\theta}+\frac{T'+iT}{Q'T-QT'}\,\frac{\partial P_n^1(\cos\theta)}{\partial\theta}\right)\cos\phi\,\hat{\theta}-\left(\frac{iS+S'}{Q'S-QS'}\,\frac{\partial P_n^1(\cos\theta)}{\partial\theta}+\frac{T'+iT}{Q'T-QT'}\,\frac{P_n^1(\cos\theta)}{\sin\theta}\right)\sin\phi\,\hat{\phi}\right]. \tag{3.51}$$

The expressions for the field in the vertical (ϕ=0) and horizontal (ϕ=π/2) planes become

$$\bar{E}\Big|_{\phi=0} = \frac{\omega\mu_o c_e}{4\pi a}\,\frac{e^{ik_oR}}{k_oR}\sum_{n=1}^{\infty}(i)^{n+1}\frac{2n+1}{n(n+1)}\left(\frac{iS+S'}{Q'S-QS'}\,\frac{P_n^1(\cos\theta)}{\sin\theta}+\frac{T'+iT}{Q'T-QT'}\,\frac{\partial P_n^1(\cos\theta)}{\partial\theta}\right) \tag{3.52}$$

and

$$\bar{E}\Big|_{\phi=\pi/2} = \frac{-\omega\mu_o c_e}{4\pi a}\,\frac{e^{ik_oR}}{k_oR}\sum_{n=1}^{\infty}(i)^{n+1}\frac{2n+1}{n(n+1)}\left(\frac{iS+S'}{Q'S-QS'}\,\frac{\partial P_n^1(\cos\theta)}{\partial\theta}+\frac{T'+iT}{Q'T-QT'}\,\frac{P_n^1(\cos\theta)}{\sin\theta}\right). \tag{3.53}$$

In order to make computations easier, we will transform the above expressions into a more convenient form. By grouping the terms and using the relation

$$h_n^{(1)'}(\rho_a) = h_{n-1}^{(1)}(\rho_a) - \frac{n+1}{\rho_a} h_n^{(1)}(\rho_a)$$

we get, finally

$$\bar{E}(\bar{R})\Big|_{\phi=0} = \frac{\omega\mu_o c_e}{4\pi a\rho_a} \frac{e^{ik_oR}}{k_oR} \sum_{n=1}^{\infty} (i)^{n+1} \frac{2n+1}{n(n+1)} \left[\frac{i + S'/S}{h_{n-1}^{(1)}(\rho_a) - \left(\frac{S'}{S} + \frac{n}{\rho_a}\right) h_n^{(1)}} \frac{P_n^1(\cos\theta)}{\sin\theta} + \right.$$

$$\left. + \frac{i + T'/T}{h_{n-1}^{(1)}(\rho_a) - \left(\frac{T'}{T} + \frac{n}{\rho_a}\right) h_n^{(1)}} \frac{\partial P_n^1(\cos\theta)}{\partial\theta} \right] \qquad (3.54)$$

an d

$$\bar{E}(\bar{R})\Big|_{\phi=\pi/2} = -\frac{\omega\mu_o c_e}{4\pi a \rho_a} \frac{e^{ik_oR}}{k_oR} \sum_{n=1}^{\infty} (i)^{n+1} \frac{2n+1}{n(n+1)} \left[\frac{i + S'/S}{h_{n-1}^{(1)}(\rho_a) - \left(\frac{S'}{S} + \frac{n}{\rho_a}\right) h_n^{(1)}(\rho_a)} \right.$$

$$\left. \frac{\partial P_n^1(\cos\theta)}{\partial\theta} + \frac{i + T'/T}{h_{n-1}^{(1)}(\rho_a) - \left(\frac{T'}{T} + \frac{n}{\rho_a}\right) h_n^{(1)}(\rho_a)} \frac{P_n^1(\cos\theta)}{\sin\theta} \right] . \qquad (3.55)$$

Expressions (3.54) and (3.55) will be used in section 4.5 to numerically compute the radiation patterns of the lenses whose effect is built into the terms S'/S and T'/T.

3.5.4 Plane Wave Incident on an Inhomogeneous Lens

In order to study the behavior of an inhomogeneous lens as a scatterer, plane wave incidence will now be considered. The method of computation of plane wave scattering by an inhomogeneous lens will be that of calculating the scattered field of an electric dipole in the presence of a lens when the dipole recedes to infinity.

Starting with the dyadic Green's function of the third kind, we take $\bar{\bar{G}}_{eo}(\bar{R} < \bar{R}')$ in the expression of $\bar{\bar{G}}_{e3}^{(11)}(\bar{R}|\bar{R}')$ and neglect in it the singular term. Thus,

$$\bar{\bar{G}}_{e3}^{(11)}(\bar{R}|\bar{R}') = \frac{ik_o}{4\pi} \sum_{n=1}^{\infty} \sum_{m=0}^{n} (2-\delta_o) \frac{2n+1}{n(n+1)} \frac{(n-m)!}{(n+m)!} \Big[(\bar{M}_{\substack{e\\o}mn}(k_o) + A_{\substack{e\\o}n} \bar{M}^{(1)}_{\substack{e\\o}mn}(k_o)) \cdot \bar{M}'^{(1)}_{\substack{e\\o}mn}(k_o) + (\bar{N}_{\substack{e\\o}mn}(k_o) + B_{\substack{e\\o}n} \bar{N}^{(1)}_{\substack{e\\o}mn}(k_o)) \bar{N}'^{(1)}_{\substack{e\\o}mn}(k_o) \Big] , \tag{3.56}$$

and the expression for the electric field becomes

$$\bar{E}(\bar{R}) = i\omega\mu_o \int_V \bar{\bar{G}}_{e3}^{(11)}(\bar{R}|\bar{R}') \cdot \bar{J}_e(\bar{R}')\, dv' = \frac{-\omega\mu_o k_o}{4\pi} \sum_{n=1}^{\infty} \sum_{m=0}^{n} (2-\delta_o) \frac{2n+1}{n(n+1)} \frac{(n-m)!}{(n+m)!}$$

$$\Big[\Big(\bar{M}_{\substack{e\\o}mn}(k_o) + A_{\substack{e\\o}n} \bar{M}^{(1)}_{\substack{e\\o}mn}(k_o)\Big) \int_V \bar{M}'^{(1)}_{\substack{e\\o}mn}(k_o) \cdot \bar{J}_e(\bar{R}')\, dv' + \Big(\bar{N}_{\substack{e\\o}mn}(k_o) + B_{\substack{e\\o}n} \bar{N}^{(1)}_{\substack{e\\o}mn}(k_o)\Big) \int_V \bar{N}'^{(1)}_{\substack{e\\o}mn}(k_o) \cdot \bar{J}_e(\bar{R}')\, dv' \Big] .$$

Performing the integration as before, we find

$$\bar{E}(\bar{R}) = \frac{-\omega\mu_o k_o c_e}{4\pi k_o b} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} (-1)^{n+1} \Big[-(\bar{M}_{o1n}(k_o) + A_{on} \bar{M}^{(1)}_{o1n}(k_o)) Q(k_o b) + (\bar{N}_{e1n}(k_o) + B_{en} \bar{N}^{(1)}_{e1n}(k_o)) Q'(k_o b) \Big] .$$

Taking the limit as $b \to \infty$ and remembering that

$$\lim_{b \to \infty} Q(k_o b) = (-i)^{n+1} e^{ik_o b}$$

and

$$\lim_{b \to \infty} Q'(k_o b) = (-i)^{n} e^{ik_o b} ,$$

the expression for the field becomes

$$\bar{E}(\bar{R}) = \frac{-\omega\mu_o k_o c_e}{4\pi} \frac{e^{ik_o b}}{k_o b} \sum_{n=1}^{\infty} (i)^{n+1} \frac{2n+1}{n(n+1)} \Big[i\,(\bar{N}_{e1n}(k_o) + B_{en} \bar{N}^{(1)}_{e1n}(k_o)) -$$

$$- (\bar{M}_{o1n}(k_o) + A_{on} \bar{M}^{(1)}_{o1n}(k_o)) \Big] .$$

After some simplifications, we obtain

$$\bar{E}(\bar{R}) = E_o \sum_{n=1}^{\infty} (i)^{n+1} \frac{2n+1}{n(n+1)} \Big[- i\,(\bar{M}_{o1n}(k_o) + A_{on} \bar{M}^{(1)}_{o1n}(k_o)) -$$

$$- (\bar{N}_{e1n}(k_o) + B_{en} \bar{N}^{(1)}_{e1n}(k_o)) \Big] \tag{3.57}$$

where $E_o = \frac{i\omega\mu_o c_e}{4\pi} \frac{e^{ik_o b}}{b}$ is called the amplitude of the plane wave. The expression for the electric field in two principal planes now become

$$\bar{E}(\bar{R})\Big|_{\phi=0} = -E_o \sum_{n=1}^{\infty} (i)^{n+1} \frac{2n+1}{n(n+1)} \Bigg\{ \Bigg[\frac{1}{k_o R} \frac{\partial}{\partial R} \Big[R(j_n(k_o R) + B_{en} h_n^{(1)}(k_o R)) \Big] \frac{\partial P_n^1(\cos\theta)}{\partial\theta}$$

$$+ i\,(j_n(k_o R) + A_{on} h_n^{(1)}(k_o R)) \cdot \frac{P_n^1(\cos\theta)}{\sin\theta} \Bigg] \hat{\theta} +$$

$$+\left[\frac{n(n+1)}{k_o R}\left(j_n(k_o R) + B_{en} h_n^{(1)}(k_o R)\right) P_n^1(\cos\theta)\right]\hat{R}\Bigg\} \tag{3.58}$$

and

$$\bar{E}(\bar{R})\Big|_{\phi=\pi/2} = E_o \sum_{n=1}^{\infty} (i)^{n+1} \frac{2n+1}{n(n+1)} \Bigg\{ i\left(j_n(k_o R) + A_{on} h_n^{(1)}(k_o R)\right) \frac{\partial P_n^1(\cos\theta)}{\partial\theta} +$$

$$+\frac{1}{k_o R}\frac{\partial}{\partial R}\left[R\left(j_n(k_o R) + B_{en} h_n^{(1)}(k_o R)\right)\right]\frac{P_n^1(\cos\theta)}{\sin\theta}\Bigg\}\hat{\phi}\ . \tag{3.59}$$

Equations (3.58) and (3.59) allow us to derive the expressions for the bistatic scattering cross-section of the lenses in the next section and that for the field around the geometrical optics focus of the Luneburg lens in section 4.1.5.

3.5.5 Bistatic Scattering Cross-Sections

From (3.58) and (3.59), the expressions for the scattered field of a plane wave incident on a lens are

$$\bar{E}^s(\bar{R})\Big|_{\phi=0} = -E_o \sum_{n=1}^{\infty} (i)^{n+1} \frac{2n+1}{n(n+1)} \left[\frac{B_{en}}{k_o R}\frac{\partial}{\partial R}\left(R\, h_n^{(1)}(k_o R)\right)\frac{\partial P_n^1(\cos\theta)}{\partial\theta} + \right.$$

$$\left. + i A_{on} h_n^{(1)}(k_o R) \cdot \frac{P_n^1(\cos\theta)}{\sin\theta}\right]\hat{\theta}$$

and

$$\bar{E}^s(\bar{R})\Big|_{\phi=\pi/2} = E_o \sum_{n=1}^{\infty} (i)^{n+1} \frac{2n+1}{n(n+1)} \left[i A_{on} h_n^{(1)}(k_o R) \frac{\partial P_n^1(\cos\theta)}{\partial\theta} + \right.$$

$$\left. + \frac{B_{en}}{k_o R}\frac{\partial}{\partial R}\left(R\, h_n^{(1)}(k_o R)\right)\frac{P_n^1(\cos\theta)}{\sin\theta}\right]\hat{\phi}$$

where the superscript s means the scattered field. Recalling that

$$h_n^{(1)}(k_oR) = (-i)^{n+1}\,\frac{e^{ik_oR}}{k_oR}$$

and

$$\frac{1}{k_oR}\,\frac{\partial\left[k_oR\,h_n^{(1)}(k_oR)\right]}{\partial(k_oR)} = (-i)^n\,\frac{e^{ik_oR}}{k_oR}$$

as R ⟶ ∞, the expressions for the far-zone scattered field in the principal planes are

$$\bar{E}^s(\bar{R})\Big|_{\phi=0} = -iE_o\,\frac{e^{ik_oR}}{k_oR}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\left[A_{on}\,\frac{P_n^1(\cos\theta)}{\sin\theta} + B_{en}\,\frac{\partial P_n^1(\cos\theta)}{\partial\theta}\right] \tag{3.60}$$

and

$$\bar{E}^s(\bar{R})\Big|_{\phi=\pi/2} = iE_o\,\frac{e^{ik_oR}}{k_oR}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\left[A_{on}\,\frac{\partial P_n^1(\cos\theta)}{\partial\theta} + B_{en}\,\frac{P_n^1(\cos\theta)}{\sin\theta}\right]. \tag{3.61}$$

Bistatic scattering cross-sections in the principal planes are defined as

$$\sigma_E(\theta, 0) = \lim_{R\to\infty} 4\pi R^2\,\frac{\left|E_\theta^s(\theta, 0)\right|^2}{\left|E_o\right|^2}$$

and

$$\sigma_H(\theta, \pi/2) = \lim_{R\to\infty} 4\pi R^2\,\frac{\left|E_\phi^s(\theta, \pi/2)\right|^2}{\left|E_o\right|^2}$$

Substituting (3.60) and (3.61) in the above definitions, we have

$$\sigma_E(\theta, 0) = \frac{4\pi}{k_o^2}\left|\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\left[A_{on}\,\frac{P_n^1(\cos\theta)}{\sin\theta} + B_{en}\,\frac{\partial P_n^1(\cos\theta)}{\partial\theta}\right]\right|^2 \tag{3.62}$$

and

$$\sigma_H(\theta, \pi/2) = \frac{4\pi}{k_o^2} \left| \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left[A_{on} \frac{\partial P_n^1(\cos\theta)}{\partial\theta} + B_{en} \frac{P_n^1(\cos\theta)}{\sin\theta} \right] \right|^2 . \tag{3.63}$$

The bistatic scattering cross-sections normalized to the geometrical cross-sections of the lenses are

$$\frac{\sigma_E(\theta, 0)}{\pi a^2} = \frac{4}{\rho_a^2} \left| \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left[A_{on} \frac{P_n^1(\cos\theta)}{\sin\theta} + B_{en} \frac{\partial P_n^1(\cos\theta)}{\partial\theta} \right] \right|^2 \tag{3.64}$$

and

$$\frac{\sigma_H(\theta, \pi/2)}{\pi a^2} = \frac{4}{\rho_a^2} \left| \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left[A_{on} \frac{\partial P_n^1(\cos\theta)}{\partial\theta} + B_{en} \frac{P_n^1(\cos\theta)}{\sin\theta} \right] \right|^2 . \tag{3.65}$$

These expressions will be used in the numerical computations of the bistatic scattering cross-sections of the lenses.

IV

LENSES

4.1 Luneburg Lens

4.1.1 Introduction and Review of the Literature

The Luneburg lens is a dielectric lens with relative permittivity varying according to the law

$$\epsilon_r(R) = 2 - R^2/a^2 \quad , \qquad 0 \leq R \leq a, \tag{4.1}$$

where a is the radius of the lens and R is the radial distance measured from the center of the lens. R. K. Luneburg (1944), using geometrical optics, showed that such a lens transforms the rays of a point source on the rim of the lens into a collimated beam, as shown in Figure 5.

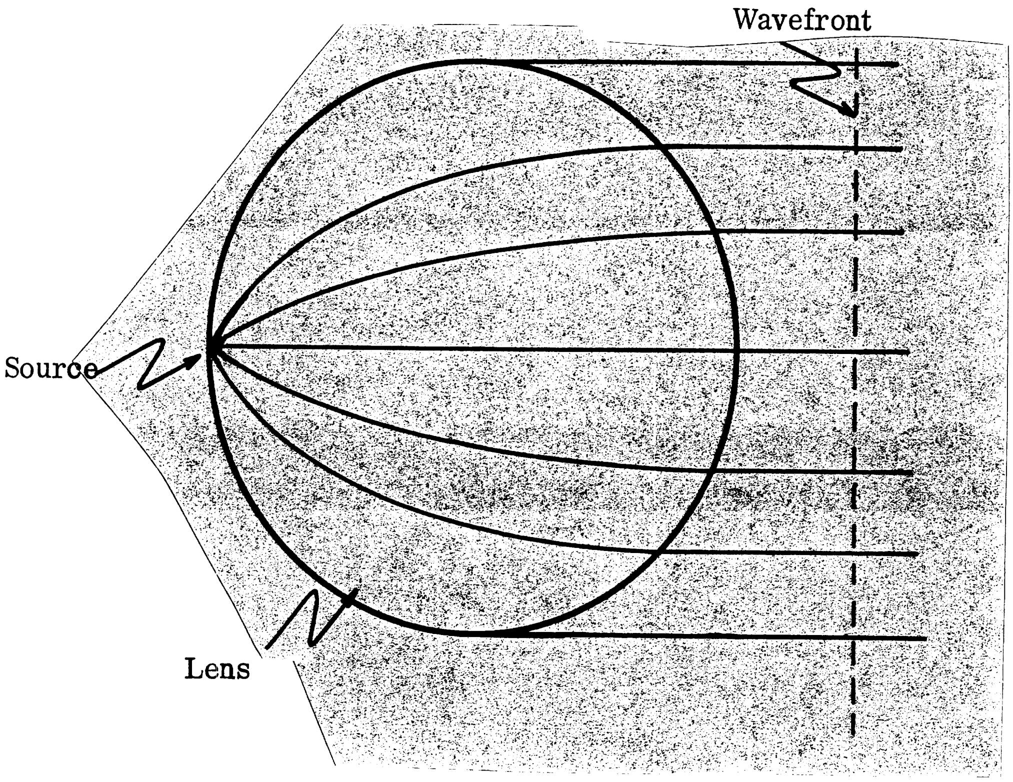

Figure 5: Luneburg Lens.

Several possible modifications of Luneburg's work, the so-called generalized Luneburg lenses, have been reported. Morgan (1958, 1959), Kay (1956, 1959), Brown (1953), Gutman (1954), Huynen (1958) and Uslenghi (1969) describe the geometrical optics behavior of generalized Luneburg lenses when the source is located inside and outside the lens.

Robinson (1954), Peeler and Coleman (1958) and Horst (1963) describe the method of fabrication of the Luneburg lens and present the measured radiation patterns of the prototypes. All of those lenses are of large diameter. Schrank (1967) discusses the method of fabrication of precision Luneburg lenses for microwave antennas.

Gunderson and Holmes (1968) and Gunderson and Kauffman (1968) present the radiation patterns of two and three dimensional Luneburg lenses made from foam glass, which has the advantage of bigger power-handling capability than the plastic lenses described earlier. Webster (1958) presents the radiation patterns of large diameter Luneburg lenses when fed by different sources, and Jones (1966) reported the behavior of a two-dimensional wire grid Luneburg Lens. Peeler and Coleman (1958), and Mikulski and Murphy (1963) discuss the effect of discontinuity in the refractive index, which occurs in the fabrication of the Luneburg lenses, on the radiation patterns. Lerner (1964) by an approximation method, computed the temperature distribution within the lens and found that the hottest point is at the feed.

Garbacz (1962) by means of the method of surface impedance, determined the bistatic scattering cross-section of the Luneburg lens of small diameter. This method requires that the end condition, i.e., a given surface impedance (or admittance) at a given radius, be specified. Depending on the type of the variation of the permittivity, Garbacz used different approximat expressions for the end condition. This method fails if the permittivity has a zero in the interval of the sphere radius. The functions which represent the imedance and admittance as a function of the radial distance become very large in the region of resonance and anti-resonance and the numerical computations of these functions become very difficult. The transformation from impedance function to admittance function and vice-versa is made so that this difficulty is overcome. A more convenient method from the computational point of view is described by Shafai (1972). This is the method of amplitude and phase functions. The scattered field is described by the phase functions and they were found to be relatively smooth and well behaved. Therefore, their computation is more efficient than that of the impedance and admittance functions used by Garbacz. This method is also approximate because the initial phase shift must be assumed. Hizal and Tosun (1973) used the state-space formulation for the same purpose. This formulation was applied to a model of the Luneburg distribution represented by a spherical shell with interior radius of 0.2 where $\epsilon_r = 0$ and exterior radius of 3, 5, $6\lambda_o$. In between, ϵ_r varies according to the Luneburg lens law. Their results are in very good agreement with those of Shafai.

The comparison of the radiation patterns of a Luneburg lens and a homogeneous sphere of small diameter using the geometrical optics is reported by Ryan and Cain (1971). They found that the homogeneous lens compares favorably with the Luneburg lens when comparing their directivities, beamwidths and sidelobes.

The exact electromagnetic treatment of the two-dimensional Luneburg lens is given by Jasik (1954) and the three-dimensional Luneburg lens is treated by Tai (1958a) and Sharaf (1962). Sharaf defined S- and T-functions which are proportional to those of Tai. The final results of both works are the same.

A survey of applications of the Luneburg lens is given by Rudduck and Walter (1960) while Rudduck and Walter (1962) and Dörge (1971) describe the use of the lens in communication satellites.

Olaofe and Levin (1967) made an exact electromagnetic formulation of the scattering of plane waves by a lens which is a generalization of Luneburg lens. Its relative permittivity varies according to the parabolic law:

$$n^2(R) = m^2(1 - 2\epsilon R^2/a^2)$$

where m and ϵ are parameters and Luneburg distribution holds for $m^2 = 2$ and $\epsilon = 1/4$. Our treatment of the Luneburg lens is that of Tai, though our method of solving the differential equations involved is different.

4.1.2 S-Function for the Luneburg Lens

In order to compute the field of the Huygens source in the presence of the lens, we have to find the expressions for the vector wave functions involving the S- and T-functions, which in turn depend on the permittivity of each lens.

Recalling equation (3.28) and substituting in it equation (4.1), we have

$$\frac{d^2 S_n(k_oR)}{dR^2} + \left[k_o^2\left(2 - \frac{R^2}{a^2}\right) - \frac{n(n+1)}{R^2}\right] S_n(k_oR) = 0 .$$

By changing the variable $\rho = k_oR$ and defining the constant $\rho_a = k_oa$, this differential equation is transformed into

$$\frac{d^2 S_n(\rho)}{d\rho^2} + \left[2 - \frac{\rho^2}{\rho_a^2} - \frac{n(n+1)}{\rho^2}\right] S_n(\rho) = 0. \tag{4.2}$$

This is the so-called normal form of the differential equation (Rainville, 1964) and its solution can be found by the polynomial method of Sommerfeld (Kemble, 1937). The method consists in finding three functions such that their product is the exact solution of the differential equation. These functions are chosen in such a way that one of them describes the behavior of the solution around the origin, another determines the behavior at infinity and the third determines the nature of the solution in the intermediate region.

When $\rho \rightarrow \infty$, the differential equation (4.2) can be approximated by

$$\frac{d^2 S_n}{d\rho^2} - \frac{\rho^2}{\rho_a^2} S_n = 0$$

which has as an asymptotic solution function proportional to $e^{\pm\rho^2/2\rho_a}$, i.e.,

$$S_n \propto e^{\pm\rho^2/2\rho_a} .$$

We choose the negative sign to ensure that the function vanishes at infinity. At the origin, the Frobenius method (Rainville, 1964) gives that the solution is a polynomial with exponent (n + 1) or -n. We use the first exponent because we need a solution finite at the origin. Whence, we can write the solution of (4.2) as

$$S_n = \rho^{n+1} e^{-\rho^2/2\rho_a} U_n(\rho) \tag{4.3}$$

where $U_n(\rho)$ describes the behavior of S_n in the intermediate region. Differentiating (4.3) twice and substituting it into (4.2), we find that $U_n(\rho)$ satisfies differential equation

$$U_n''(\rho) + 2\left(\frac{n+1}{\rho} - \frac{\rho}{\rho_a}\right) U_n'(\rho) + \left(2 - \frac{2n+3}{\rho_a}\right) U_n(\rho) = 0.$$

By making change of variable $z = \rho^2/\rho_a$, we find that $U_n(z)$ satisfies a standard differential equation, namely, the confluent hypergeometric differential equation

$$z \frac{d^2 U_n(z)}{dz^2} + \left[n+3/2 - z\right] \frac{dU_n(z)}{dz} - \frac{1}{2}\left[n + 3/2 - \rho_a\right] U_n(z) = 0.$$

The solution of this differential equation is the confluent hypergeometric (Kummer's) function, i.e.,

$$U_n(z) = {}_1F_1(\alpha, \nu; z)$$

where

$$\alpha = 1/2\,(n + 3/2 - \rho_a)$$

$$\nu = n + 3/2$$

and

$$z = \rho^2/\rho_a \quad .$$

[The second solution of the confluent hypergeometric equation $z^{1-\nu}\,{}_1F_1$ $(2 - \nu+1,\ 2-\nu; z)$ is rejected because it is not finite at the origin.]

The confluent hypergeometric function (Abramovitz and Stegun, 1968) is given by the series

$${}_1F_1(\alpha,\ \nu;\ z) = 1 + \frac{\alpha}{\nu} z + \frac{(\alpha)_2}{(\nu)_2 2!} z + \ldots\ldots + \frac{(\alpha)_n}{(\nu)_n\, n!} z^n + \ldots$$

where $(\alpha)_n = \alpha\,(\alpha+1)(\alpha+2)\ldots\ldots(\alpha+n-1)$ and $(\alpha)_o = 1$. If $\nu \neq -n$ and $\alpha \neq -m$ where m and n are positive integers, ${}_1F_1(\alpha, \nu; z)$ is a convergent series for all values of α, ν and z. In our case $\alpha = 1/2\,(n+3/2 - \rho_a)$ and $\nu = n + 3/2$ with n = 1, 2, 3.... Therefore, $\nu \neq -n$, but α can assume negative integer values. When α is negative the function exhibits an oscillatory behavior and when α is a negative integer the series terminates and becomes an oscillatory polynomial. When α is positive, ${}_1F_1(\alpha,\ \nu;\ z)$ increases monotonically. Therefore, the confluent hypergeometric function is always well-behaved in our case. The derivative of the confluent hypergeometric function is given by

$$\frac{d\,{}_1F_1(\alpha,\ \nu; z)}{dz} = \frac{\alpha}{\nu}\,{}_1F_1(\alpha + 1,\ \nu + 1;\ z)$$

[More about confluent hypergeometric functions can be found in Slater's (1960) work]. Finally, the expression of the S-function is

$$S_n(\rho) = \rho^{n+1} e^{-\rho^2/2\rho_a} \, {}_1F_1(\alpha, \nu; \rho^2/\rho_a) \tag{4.4}$$

4.1.3 T-Function for the Luneburg Lens

Recalling equation (3.30) and substituting in it (4.1), we get

$$\frac{d^2T_n(k_oR)}{dR^2} + \frac{2R}{(2a^2 - R^2)} \frac{dT_n(k_oR)}{dR} + \left[k_o^2(2 - R^2/a^2) - \frac{n(n+1)}{R^2}\right] T_n(k_oR) = 0,$$

which, after making a change of variable $\rho = k_oR$ and defining the constant $\rho_a = k_oa$, takes the form

$$\frac{d^2T_n(\rho)}{d\rho^2} + \frac{2\rho}{2\rho_a^2 - \rho^2} \frac{dT_n(\rho)}{d\rho} + \left[2 - \frac{\rho^2}{\rho_a^2} - \frac{n(n+1)}{\rho^2}\right] T_n(\rho) = 0 \; .$$

We can transform it into the normal form by making the following change of dependent variable (Rainville, 1964).

$$T_n = W_n \exp\left(-\frac{1}{2}\int p \, d\rho\right),$$

where p is the coefficient of the first derivative in the differential equation. In this case $p = \frac{2\rho}{2\rho_a^2 - \rho^2}$, and after performing the integration and taking the exponential we find that

$$T_n(\rho) = \sqrt{2\rho_a^2 - \rho^2} \, W_n(\rho) \; .$$

The differential equation for the new dependent variable $W_n(\rho)$ becomes

$$W_n'' + \left(2 - \frac{\rho^2}{\rho_a^2} - \frac{n(n+1)}{\rho^2} - \frac{2(\rho_a^2 + \rho^2)}{(2\rho_a^2 - \rho^2)^2}\right) W_n = 0 ,$$

which is the normal form of the differential equation. Applying again the polynomial method of Sommerfeld (Kemble, 1937), we find that

$$W_n(\rho) = e^{-\rho^2/2\rho_a} \rho^{n+1} V_n(\rho)$$

where V_n satisfies the differential equation

$$V_n''(\rho) + 2\left(\frac{n+1}{\rho} - \frac{\rho}{\rho_a}\right) V_n' + \left[2 - \frac{2n+3}{\rho_a} - \frac{2(\rho_a^2 + \rho^2)}{(2\rho_a^2 - \rho^2)^2}\right] V_n = 0.$$

The change of independent variable $z = \rho^2/\rho_a$ reduces it to

$$\frac{d^2 V_n(z)}{dz^2} + \left[\frac{\gamma}{z} - 1\right] \frac{dV_n(z)}{dz} - \left[\frac{\alpha_1}{z} + \frac{2\alpha_2}{z - a_2} - \frac{2a_2\alpha_3}{(z - a_2)^2}\right] V_n(z) = 0$$

where

$\gamma = n + 3/2$

$\alpha_1 = 1/2\,(n + 3/2 - \rho_a + 1/(4\rho_a))$

$\alpha_2 = -1/(16\rho_a)$

$\alpha_3 = -3/(16\rho_a)$

$a_2 = 2\rho_a$

and this equation cannot be cast in the form of any standard differential equation.

Tai (1958a) called it a "generalized" confluent hypergeometric equation. The only possible solution of this equation is the series solution. We expand the solution around zero which has exponents 0 and $-(n+1/2)$. Then two possible series solutions around zero are (Rainville, 1964),

$$V_n = \sum_{m=0}^{\infty} A_m z^m$$

and

$$V_n = z^{-(n+1/2)} + \sum_{m=1}^{\infty} A'_m z^{m-(n+1/2)} .$$

The second solution is rejected because it is not finite at the origin. Substituting the first solution in the differential equation and equating the coefficients of the terms of the same power to zero, we find a four term recurrence relation between the coefficients. This recurrence relation can be further reduced to the relation between the first coefficient, which is arbitrary and any subsequent term. The relations are of the following form:

$$\frac{A_1}{A_0} = \frac{\alpha_1}{\gamma}$$

$$\frac{A_2}{A_0} = \frac{1}{2}\,\frac{\alpha_1(\alpha_1+1)}{\gamma(\gamma+1)} - \frac{\alpha_2+\alpha_3}{a_2(\gamma+1)}$$

$$\frac{A_3}{A_0} = \frac{1}{3!}\,\frac{\alpha_1(\alpha_1+1)\,(\alpha_2+2)}{\gamma(\gamma+1)\quad(\gamma+2)} - \frac{1}{3(\gamma+2)}\left[\frac{2\alpha_1(\alpha_2+\alpha_3)}{a_2\gamma} + \frac{(2+\alpha_1)(\alpha_2+\alpha_3)}{a_2(\gamma+1)} + \frac{2(\alpha_2+2\alpha_3)}{a_2^2}\right] .$$

For $m \geq 3$, the relation between the first coefficient and subsequent coefficients becomes very cumbersome, so it is more convenient to use the recurrence relation,

$$a_0^2 (m+1)(\gamma+m) A_{m+1} - a_2 \left[a_2(m+\alpha_1)+2m(m+\gamma-1)\right] A_m +$$

$$+\left[(m-1)(\gamma+m-2)+2a_2(\alpha_1+\alpha_2+\alpha_3+m-1)\right] A_{m-1} - (\alpha_1+2\alpha_2+m-2)A_{m-2} = 0.$$

In our computations we adopted $A_0 = 1$. The series solution for V_n converges absolutely and uniformly for $z < a_2$ or $\rho < \sqrt{2}\,\rho_a$. Because our lens is of radius ρ_a, the series for V_n always converges, including on the rim of the lens.

It is readily apparent that the differential equation for V_n is very similar to that for the confluent hypergeometric function except that it has another regular singular point at $z = a_2$. When ρ_a is very large, the coefficient of the pole at $z = a_2$ becomes very small and the effect of the pole on the behavior of the "generalized" confluent hypergeometric function in the region $0 \leq \rho \leq \rho_a$ is negligible. The coefficient α_1 of the pole at $z = 0$ of this function is nearly the same as the coefficient α of the pole at $z = 0$ of the confluent hypergeometric function. Consequently the "generalized" confluent hypergeometric function becomes very similar to the confluent hypergeometric function in this region. The same does not happen when ρ_a is very small. In this case the coefficient of the pole at $z = a_2$ of the "generalized" confluent hypergeometric function becomes very large and its effect on the behavior of this function is not negligible in the region $0 \leq \rho \leq \rho_a$. Also, the dominant term in α_1 is $1/(4\rho_a)$. α_1 and α are, then, completely different and the behavior of the "generalized" confluent hypergeometric function differs considerably from that of confluent hypergeometric function.

Returning now to the T-function, we see that it has the form

$$T_n(\rho) = \sqrt{2\rho_a^2 - \rho^2}\; \rho^{n+1}\, e^{-\rho^2/2\rho_a} \sum_{m=0}^{\infty} A_m \left(\frac{\rho^2}{\rho_a}\right)^m. \qquad (4.5)$$

Looking back at the equations (3.54) and (3.55), we see that the expressions for the electric field contain terms $\left.\frac{S'}{S}\right|_{\rho=\rho_a}$ and $\left.\frac{T'}{T}\right|_{\rho=\rho_a}$, and we can compute them readily. Thus, from (4.4) we have

$$S_n(\rho_a) = \rho_a^{n+1} e^{-\rho_a/2} {}_1F_1(\alpha, \nu; \rho_a)$$

and

$$S'_n(\rho_a) = e^{-\rho_a/2} \rho_a^n \left[(n+1-\rho_a) {}_1F_1(\alpha, \nu; \rho_a) + 2\alpha/\nu \, \rho_a \, {}_1F_1(\alpha+1, \nu+1; \rho_a)\right],$$

where the relation

$$\frac{d \, {}_1F_1(\alpha, \nu; \rho^2/\rho_a)}{d(\rho^2/\rho_a)} = \frac{\alpha}{\nu} \, {}_1F_1(\alpha+1, \nu+1; \rho^2/\rho_a)$$

was used.

Consequently,

$$\left.\frac{S'_n}{S_n}\right|_{\rho=\rho_a} = \frac{n+1}{\rho_a} - 1 + 2\frac{\alpha}{\nu} \frac{{}_1F_1(\alpha+1, \nu+1; \rho_a)}{{}_1F_1(\alpha, \nu; \rho_a)}. \tag{4.6}$$

From (4.5), for the T-function, we have

$$T_n(\rho_a) = \rho_a^{n+2} e^{-\rho_a/2} \sum_{m=0}^{\infty} A_m \rho_a^m$$

and

$$T'_n(\rho_a) = (n-\rho_a)\rho_a^{n+1} e^{-\rho_a/2} \sum_{m=0}^{\infty} A_m \rho_a^m + 2\rho_a^{n+2} e^{-\rho_a/2} \sum_{m=0}^{\infty} m A_m \rho_a^{m-1};$$

therefore,

$$\left.\frac{T_n'}{T_n}\right|_{\rho=\rho_a} = \frac{n}{\rho_a} - 1 + 2\,\frac{\sum_{m=0}^{\infty} m A_m \rho_a^{m-1}}{\sum_{m=0}^{\infty} A_m \rho_a^{m}} . \qquad (4.7)$$

Expressions (4.6) and (4.7) are needed for all numerical computations which involve the Luneburg lenses.

4.1.4 Directivity

One means of comparing the performance of the Luneburg lens with other lenses is to compare their directivities. The directivity in the forward direction relative to an isotropic source is defined by

$$D = \frac{4\pi \text{ (radiation intensity in } \theta = 0^{\circ} \text{ direction)}}{\text{total power radiated .}}$$

Equation (3.50) for the electric field of the Huygens source can be written as

$$\bar{E}_H(\bar{R}) = \frac{\omega\mu_o c_e}{4\pi a}\,\frac{e^{ik_oR}}{k_oR}\sum_{n=1}^{\infty}(i)^{n+1}\bar{A}_n$$

where

$$\bar{A}_n = \frac{2n+1}{n(n+1)}\left\{\frac{iS+S'}{Q'S-QS'}\,\bar{m}_{o1n} + \frac{T'+iT}{Q'T-QT'}\,\bar{n}_{e1n}\right\} .$$

First we compute the total power radiated by the source:

$$W = \frac{1}{2\eta_o}\int_0^{2\pi}\!\!\int_0^{\pi} \bar{E}_H \cdot \bar{E}_H^{*}\, R^2 \sin\theta \, d\theta \, d\phi$$

where the asterisk indicates the complex conjugation. Thus,

$$W = \frac{1}{2\eta_o}\left(\frac{\omega\mu_o c_e}{4\pi k a_o}\right)^2 \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (i)^{n+1}(-i)^{m+1}\int_0^{2\pi}\int_0^{\pi} \bar{A}_n \bar{A}_m^* \sin\theta \, d\theta \, d\phi .$$

The double integral can be performed separately.

$$I = \int_0^{2\pi}\int_0^{\pi} \bar{A}_n \bar{A}_m^* \sin\theta \, d\theta \, d\phi = \frac{(2n+1)}{n(n+1)}\frac{(2m+1)}{m(m+1)}\int_0^{2\pi}\int_0^{\pi}\left[\alpha_n \bar{m}_{oln} + \beta_n \bar{n}_{eln}\right] \cdot$$

$$\cdot \left[\alpha_m^* \bar{m}_{olm} + \beta_m^* \bar{n}_{elm}\right] \sin\theta \, d\theta \, d\phi$$

where

$$\alpha_n = \frac{i\,S + S'}{Q'S-QS'} \quad \text{and} \quad \beta_n = \frac{T' + i\,T}{Q'T-QT'} \quad .$$

After substituting the values of $\bar{m}_{oln}$ and $\bar{n}_{eln}$ and performing the scalar product, the integral becomes

$$I = \frac{(2n+1)}{n(n+1)}\frac{(2m+1)}{m(m+1)}\int_0^{2\pi}\int_0^{\pi}\left[\left(\alpha_n \frac{P_n^1(\cos\theta)}{\sin\theta} + \beta_n \frac{\partial P_n^1(\cos\theta)}{\partial\theta}\right) \cdot\right.$$

$$\cdot \left(\alpha_m^* \frac{P_m^1(\cos\theta)}{\sin\theta} + \beta_m^* \frac{\partial P_m^1(\cos\theta)}{\partial\theta}\right)\cos^2\phi +$$

$$+ \left(\alpha_n \frac{\partial P_n^1(\cos\theta)}{\partial\theta} + \beta_n \frac{P_n^1(\cos\theta)}{\sin\theta}\right) \cdot$$

$$\left.\cdot \left(\alpha_m^* \frac{\partial P_m^1(\cos\theta)}{\partial\theta} + \beta_m^* \frac{P_m^1(\cos\theta)}{\sin\theta}\right)\sin^2\phi\right] \sin\theta \, d\theta \, d\phi;$$

the ϕ integration is immediate, the expression then becomes

$$I=\frac{(2n+1)}{n(n+1)}\frac{(2m+1)}{m(m+1)}\pi\left[\int_0^\pi\left(\alpha_n\frac{P_n^1(\cos\theta)}{\sin\theta}+\beta_n\frac{\partial P_n^1(\cos\theta)}{\partial\theta}\right)\left(\alpha_m^*\frac{P_m^1(\cos\theta)}{\sin\theta}+\beta_m^*\frac{\partial P_m^1(\cos\theta)}{\partial\theta}\right)\sin\theta d\theta+\right.$$

$$\left.+\int_0^\pi\left(\alpha_n\frac{\partial P_n^1(\cos\theta)}{\partial\theta}+\beta_n\frac{P_n^1(\cos\theta)}{\sin\theta}\right)\left(\alpha_m^*\frac{\partial P_m^1(\cos\theta)}{\partial\theta}+\beta_m^*\frac{P_m^1(\cos\theta)}{\sin\theta}\right)\sin\theta\, d\theta\right].$$

Performing the multiplication and rearranging the terms we get

$$I=\frac{(2n+1)}{n(n+1)}\frac{(2m+1)}{m(m+1)}\pi\left[(\alpha_n\alpha_m^*+\beta_n\beta_m^*)\int_0^\pi\left(\frac{\partial P_n^1(\cos\theta)}{\partial\theta}\frac{\partial P_m^1(\cos\theta)}{\partial\theta}+\frac{P_n^1(\cos\theta)}{\sin\theta}\frac{P_m^1(\cos\theta)}{\sin\theta}\right)\sin\theta d\theta+\right.$$

$$\left.+(\alpha_n\beta_m^*+\beta_n\alpha_m^*)\int_0^\pi\left(\frac{\partial P_n^1(\cos\theta)}{\partial\theta}\frac{P_m^1(\cos\theta)}{\sin\theta}+\frac{\partial P_n^1(\cos\theta)}{\partial\theta}\frac{P_m^1(\cos\theta)}{\sin\theta}\right)\sin\theta\, d\theta\right].$$

It is known (Stratton, 1941) that

$$\int_0^\pi\left(\frac{\partial P_n^1(\cos\theta)}{\partial\theta}\frac{\partial P_m^1(\cos\theta)}{\partial\theta}+\frac{P_n^1(\cos\theta)\,P_m^1(\cos\theta)}{\sin^2\theta}\right)\sin\theta\, d\theta=\begin{cases}0 & n\neq m\\ \frac{2}{2n+1}n^2(n+1)^2 & n=m\end{cases}$$

and

$$\int_0^\pi\left(\frac{\partial P_n^1(\cos\theta)}{\partial\theta}P_m^1(\cos\theta)+\frac{\partial P_m^1(\cos\theta)}{\partial\theta}P_n^1(\cos\theta)\right)d\theta=0\ .$$

Using these relations, we find

$$\int_0^{2\pi}\int_0^\pi\bar{A}_n\cdot\bar{A}_m^*\sin\theta\, d\theta\, d\phi=\begin{cases}0 & n\neq m\\ 2\pi\,(2n+1)(\alpha_n\alpha_m^*+\beta_n\beta_m^*) & n=m\ .\end{cases}$$

The total power radiated by the source is

$$W = \frac{1}{2\eta_o}\left(\frac{\omega\mu_o c_e}{4\pi k_o a}\right)^2 \sum_{n=1}^{\infty} 2\pi\,(2n+1)\left(|\alpha_n|^2 + |\beta_n|^2\right)$$

or

$$W = \frac{\pi}{\eta_o}\left(\frac{\omega\mu_o c_e}{4\pi k_o a}\right)^2 \sum_{n=1}^{\infty} (2n+1)\left(|\alpha_n|^2 + |\beta_n|^2\right). \qquad (4.8)$$

The radiation intensity in the $\theta = 0^o$ direction is given by $\frac{R^2}{2\eta_o}\bar{E}_H\cdot\bar{E}_H^*\Big|_{\theta=0}$,

$$\bar{E}_H\Big|_{\theta=0} = \frac{\omega\mu_o c_e}{4\pi a}\,\frac{e^{ik_oR}}{k_oR}\sum_{n=1}^{\infty}(i)^{n+1}\frac{2n+1}{n(n+1)}\left[\alpha_n\,\bar{m}_{o1n}\Big|_{\theta=0} + \beta_n\,\bar{n}_{e1n}\Big|_{\theta=0}\right].$$

But

$$\bar{m}_{o1n}\Big|_{\theta=0} = \frac{n(n+1)}{2}\left(\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}\right)$$

and

$$\bar{n}_{e1n}\Big|_{\theta=0} = \frac{n(n+1)}{2}\left(\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}\right).$$

Thus, the field in the $\theta = 0^o$ direction is

$$\bar{E}_H\Big|_{\theta=0} = \frac{1}{2}\,\frac{\omega\mu_o c_e}{4\pi a}\,\frac{e^{ik_oR}}{k_oR}\sum_{n=1}^{\infty}(i)^{n+1}(2n+1)(\alpha_n+\beta_n)\left[\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}\right]$$

and the expression for the radiation intensity in the $\theta = 0^o$ direction becomes

$$\frac{R^2}{2\eta_o}\bar{E}_H\cdot\bar{E}_H^*\Big|_{\theta=0} = \frac{R^2}{2\eta_o}\,\frac{1}{4}\left(\frac{\omega\mu_o c_e}{4\pi a}\right)^2\frac{1}{k_o^2R^2}\left|\sum_{n=1}^{\infty}(i)^{n+1}(2n+1)(\alpha_n+\beta_n)\right|^2(\cos^2\phi+\sin^2\phi)$$

or

$$\frac{R^2}{2\eta_o} \bar{E}_H \cdot \bar{E}_H^* \Big|_{\theta=0} = \frac{1}{8\eta_o} \left(\frac{\omega\mu_o c_e}{4\pi k_o a} \right)^2 \left| \sum_{n=1}^{\infty} (i)^{n+1} (2n+1) (\alpha_n + \beta_n) \right|^2 . \tag{4.9}$$

Dividing 4π times (4.8) by (4.9) we obtain the directivity

$$D = \frac{1}{2} \frac{\left| \sum_{n=1}^{\infty} (i)^{n+1} (2n+1) (\alpha_n + \beta_n) \right|^2}{\sum_{n=1}^{\infty} (2n+1) (|\alpha_n|^2 + |\beta_n|^2)} ,$$

which expressed in dB becomes

$$D\Big|_{dB} = 10 \log \left[\frac{1}{2} \frac{\left| \sum_{n=1}^{\infty} (i)^{n+1} (2n+1) (\alpha_n + \beta_n) \right|^2}{\sum_{m=1}^{\infty} (2n+1) (|\alpha_n|^2 + |\beta_n|^2)} \right] . \tag{4.10}$$

The directivity of the Huygens source without a lens is known to be equal to 3 or 4.77dB. The numerical calculation based on (4.10) will be compared with this figure in evaluating the focussing capability of a Luneburg lens.

4.1.5 Electric Field Around the Geometric Optics Focus

4.1.5.1 Field Exterior to the Lens

From the expressions (3.58) and (3.59) we can write the square of the absolute value of the field near and on the rim of the lens in two principal planes.

Vertical plane:

$$\frac{\left|\bar{E}(\bar{R})\right|^2\Big|_{\phi=0}}{\left|E_o\right|^2} = \left|E_R\right|^2 + \left|E_\theta\right|^2 = \left|\sum_{n=1}^{\infty} (i)^{n+1} \frac{2n+1}{n(n+1)} \left[\frac{1}{k_oR}\frac{\partial}{\partial R}(R(j_n(k_oR) + B_{en} h_n^{(1)}(k_oR)))\frac{\partial P_n^1(\cos\theta)}{\partial\theta} + i(j_n(k_oR) + A_{on} h_n^{(1)}(k_oR))\frac{P_n^1(\cos\theta)}{\sin\theta}\right]\right|^2 + \left|\sum_{n=1}^{\infty} (i)^{n+1} \frac{2n+1}{k_oR}(j_n(k_oR) + B_{en} h_n^{(1)}(k_oR))P_n^1(\cos\theta)\right|^2$$

Horizontal plane:

$$\frac{\left|\bar{E}(\bar{R})\right|^2\Big|_{\phi=\pi/2}}{\left|E_o\right|^2} = \left|E_\phi\right|^2 = \left|\sum_{n=1}^{\infty} (i)^{n+1} \frac{2n+1}{n(n+1)} \left[i(j_n(k_oR) + A_{on} h_n^{(1)}(k_oR))\frac{\partial P_n^1(\cos\theta)}{\partial\theta} + \frac{1}{k_oR}\frac{\partial}{\partial R}\left(R(j_n(k_oR) + B_{en} h_n^{(1)}(k_oR))\right)\frac{P_n^1(\cos\theta)}{\sin\theta}\right]\right|^2 .$$

After some simplifications, we obtain

$$\frac{\left|\bar{E}(\bar{R})\right|^2\Big|_{\phi=0}}{\left|E_o\right|^2} = \left|\sum_{n=1}^{\infty} (i)^{n+1} \frac{2n+1}{n(n+1)} \left[\left(j_{n-1}(k_oR) + B_{en} h_{n-1}^{(1)}(k_oR) - \frac{n}{k_oR}(j_n(k_oR) + B_{en} h_n^{(1)}(k_oR)\right)\frac{\partial P_n^1(\cos\theta)}{\partial\theta} + i\left(j_n(k_oR) + A_{en} h_n^{(1)}(k_oR)\right)\frac{P_n^1(\cos\theta)}{\sin\theta}\right]\right|^2 + \left|\sum_{n=1}^{\infty} (i)^{n+1} \frac{2n+1}{k_oR}(j_n(k_oR) + B_{en} h_n^{(1)}(k_oR))P_n^1(\cos\theta)\right|^2 \quad (4.11)$$

and

$$\frac{\left|\bar{E}(\bar{R})\right|^2\Big|_{\phi=\pi/2}}{\left|E_o\right|^2} = \left|\sum_{n=1}^{\infty} (i)^{n+1} \frac{2n+1}{n(n+1)} \left[i\left(j_n(k_oR)+A_{on}\, h_n^{(1)}(k_oR)\right) \frac{\partial P_n^1(\cos\theta)}{\partial\theta} + \right.\right.$$

$$\left.\left. + \left(j_{n-1}(k_oR)+B_{en} h_{n-1}^{(1)}(k_oR) - \frac{n}{k_oR}\left(j_n(k_oR)+B_{en} h_n^{(1)}(k_oR)\right)\right) \frac{P_n^1(\cos\theta)}{\sin\theta}\right]\right|^2 \tag{4.12}$$

where A_{on} and B_{en} are given by

$$A_{on} = \frac{\left(\frac{S'_n}{S_n} + \frac{n}{\rho_a}\right) j_n - j_{n-1}}{h_{n-1}^{(1)} - \left(\frac{S'_n}{S_n} + \frac{n}{\rho_a}\right) h_n^{(1)}}$$

and

$$B_{en} = \frac{\left(\frac{T'_n}{T_n} + \frac{n}{\rho_a}\right) j_n - j_{n-1}}{h_{n-1}^{(1)} + \left(\frac{T'_n}{T_n} + \frac{n}{\rho_a}\right) h_n^{(1)}} \; .$$

This is a more convenient form for computation.

4.1.5.2 Field Interior to the Lens

The interior field is given by

$$\bar{E}(\bar{R}) = i\,\omega\mu_o \int_V \bar{\bar{G}}_{e3}^{(21)}(\bar{R}|\bar{R}') \cdot \bar{J}_e(\bar{R}')\, dv' \tag{4.13}$$

where

$$\bar{\bar{G}}^{(21)}_{e3}(\bar{R}|\bar{R}') = -\frac{1}{4\pi}\sum_{n=1}^{\infty}\sum_{m=0}^{n}(2-\delta_o)\frac{2n+1}{n(n+1)}\frac{(n-m)!}{(n+m)!}\left[\frac{\bar{M}^{(m)}_{\substack{e\\o}mn}(k_o)\,\bar{M}'^{(1)}_{\substack{e\\o}mn}(k_o)}{Q'(\rho_a)\,S(\rho_a)-Q(\rho_a)S'(\rho_a)}\right. +$$

$$\left. + \frac{\bar{N}^{(e)}_{\substack{e\\o}mn}(k_o)\,\bar{N}'^{(1)}_{\substack{e\\o}mn}(k_o)}{Q'(\rho_a)\,T(\rho_a) - Q(\rho_a)\,T'(\rho_a)}\right]$$

and

$$\bar{J}_e(\bar{R}) = -c_e\frac{\delta(R'-b)\,\delta(\theta'-\pi)\,\delta(\phi')}{b^2\sin\theta'}\,\hat{\theta}\,.$$

Substituting in (4.13) and integrating, we find

$$\bar{E}(\bar{R}) = \frac{i\omega\mu_o c_e}{4\pi k_o b}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}(-1)^{n+1}\left[\frac{Q(k_o b)}{Q'(\rho_a)S(\rho_a)-Q(\rho_a)S'(\rho_a)}\,\bar{M}^{(m)}_{o1n}(k_o)\right. -$$

$$\left. - \frac{Q'(k_o b)}{Q'(\rho_a)T(\rho_a)-Q(\rho_a)T'(\rho_a)}\,\bar{N}^{(e)}_{e1n}(k_o)\right]\,.$$

Taking again the limit when $b \to \infty$, we have

$$\bar{E}(\bar{R}) = \frac{E_o}{k_o}\sum_{n=1}^{\infty}(i)^{n+1}\frac{2n+1}{n(n+1)}\left[\frac{\bar{M}^{(m)}_{o1n}(k_o)}{Q'(\rho_a)S(\rho_a)-Q(\rho_a)S'(\rho_a)} - \frac{i\,\bar{N}^{(e)}_{e1n}(k_o)}{Q'(\rho_a)T(\rho_a)-Q(\rho_a)T'(\rho_a)}\right] \tag{4.14}$$

where

$$\bar{M}^{(m)}_{o1n}(k_o) = \frac{S_n(k_o R)}{R}\,\frac{P^1_n(\cos\theta)}{\sin\theta}\cos\phi\,\hat{\theta} - \frac{S_n(k_o R)}{R}\,\frac{\partial P^1_n(\cos\theta)}{\partial\theta}\sin\phi\,\hat{\phi}$$

and

$$\bar{N}^{(e)}_{e1n}(k_o) = \frac{1}{k_o \epsilon_r(R)} \left[\frac{T_n(k_oR)}{R^2} n(n+1) P^1_n(\cos\theta) \cos\phi \hat{R} + \right.$$

$$\left. + \frac{1}{R} \frac{\partial T_n(k_oR)}{\partial R} \left(\frac{\partial P^1_n(\cos\theta)}{\partial\theta} \cos\phi \hat{\theta} - \frac{P^1_n(\cos\theta)}{\sin\theta} \sin\phi \hat{\phi} \right) \right].$$

Now substituting the above expressions in (4.14) and taking $\phi = 0$ and $\phi = \pi/2$, we have the expressions for the field in the principal planes:

$$\bar{E}(\bar{R})\Big|_{\phi=0} = E_o \sum_{n=1}^{\infty} (i)^{n+1} \frac{2n+1}{n(n+1)} \left[\frac{-i}{Q'(\rho_a)T(\rho_a) - Q(\rho_a)T'(\rho_a)} \frac{1}{\epsilon_r(R)} \frac{T_n(k_oR)}{(k_oR)^2} n(n+1) P^1_n(\cos\theta) \hat{R} + \right.$$

$$+ \left(\frac{1}{Q'(\rho_a)\, S(\rho_a) - Q(\rho_a)\, S'(\rho_a)} \frac{S_n(k_oR)}{k_oR} \frac{P^1_n(\cos\theta)}{\sin\theta} - \right.$$

$$\left. \left. - \frac{i}{Q'(\rho_a)T(\rho_a) - Q(\rho_a)T'(\rho_a)} \frac{1}{\epsilon_r(R)\, k_oR} \frac{\partial T_n(k_oR)}{\partial(k_oR)} \frac{\partial P^1_n}{\partial\theta} \right) \hat{\theta} \right] \tag{4.15}$$

and

$$\bar{E}(\bar{R})\Big|_{\phi=\pi/2} = -E_o \sum_{n=1}^{\infty} (i)^{n+1} \frac{2n+1}{n(n+1)} \left[\frac{1}{Q'(\rho_a)S(\rho_a) - Q(\rho_a)S'(\rho_a)} \frac{S_n(k_oR)}{k_oR} \frac{\partial P^1_n(\cos\theta)}{\partial\theta} - \right.$$

$$\left. - \frac{i}{Q'(\rho_a)T(\rho_a) - Q(\rho_a)T'(\rho_a)} \frac{1}{\epsilon_r(R)\, k_oR} \frac{\partial T_n(k_oR)}{\partial(k_oR)} \frac{P^1_n}{\sin\theta} \right] \hat{\phi} . \tag{4.16}$$

Obviously, these expressions check against those of the external field at R=a.

The intensity of the interior field is given by

$$\frac{|\bar{E}|^2\Big|_{\phi=0}}{|E_o|^2} = \left| \sum_{n=1}^{\infty} \frac{(i)^n(2n+1)}{\epsilon_r(R)\,(Q'T-QT')} \frac{T_n(k_oR)}{(k_oR)^2} P_n^1(\cos\theta) \right|^2 +$$

$$+ \left| \sum_{n=1}^{\infty} (i)^{n+1} \frac{2n+1}{n(n+1)} \left(\frac{1}{Q'S-QS'} \frac{S_n(k_oR)}{k_oR} \frac{P_n^1}{\sin\theta} - \frac{i}{\epsilon_r(R)k_oR(Q'T-QT')} \frac{\partial T_n(k_oR)}{\partial(k_oR)} \frac{\partial P_n^1}{\sin\theta} \right) \right|^2 \tag{4.17}$$

and

$$\frac{|E|^2\Big|_{\phi=\pi/2}}{|E_o|^2} = \left| \sum_{n=1}^{\infty} (i)^{n+1} \frac{2n+1}{n(n+1)} \left[\frac{1}{Q'S-QS'} \frac{S_n(k_oR)}{k_oR} \frac{\partial P_n^1}{\partial\theta} - \frac{i}{\epsilon_r(R)k_oR(Q'T-QT')} \frac{\partial T_n(k_oR)}{\partial(k_oR)} \frac{P_n^1}{\sin\theta} \right] \right|^2 . \tag{4.18}$$

Expressions (4.17), (4.18) together with (4.11) and (4.12) will be numerically computed for the Luneburg lenses of diameters $1.27\lambda_o$ and $4.12\lambda_o$.

4.2 Eaton-Lippmann Lens

4.2.1 Introduction

The Eaton-Lippmann lens is also known as the Eaton lens or isotropic lens. It has relative permittivity given by

$$\epsilon_r(R) = (2a - R)/R \qquad 0 \leq R \leq a \tag{4.19}$$

where R is the radial distance from the origin and a is the radius of the lens. By using geometrical optics, Eaton (1952) showed that the path of the rays inside the lens is elliptical and that the emerging wavefront is planar (see Figure 6).

It can be proved that this lens has the property of imaging an incoming plane wave into an outgoing plane wave with a 180^o reversal of direction of propagation. Lippmann (1954), as cited by Rheinstein (1962), prodicted that this property of the lens should make it an excellent omnidirectional radar reflector. Kay (1958), again as cited by Rheinstein (1962), observed that in prior considerations of the Eaton-Lippmann lens as a reflector, scalar theory

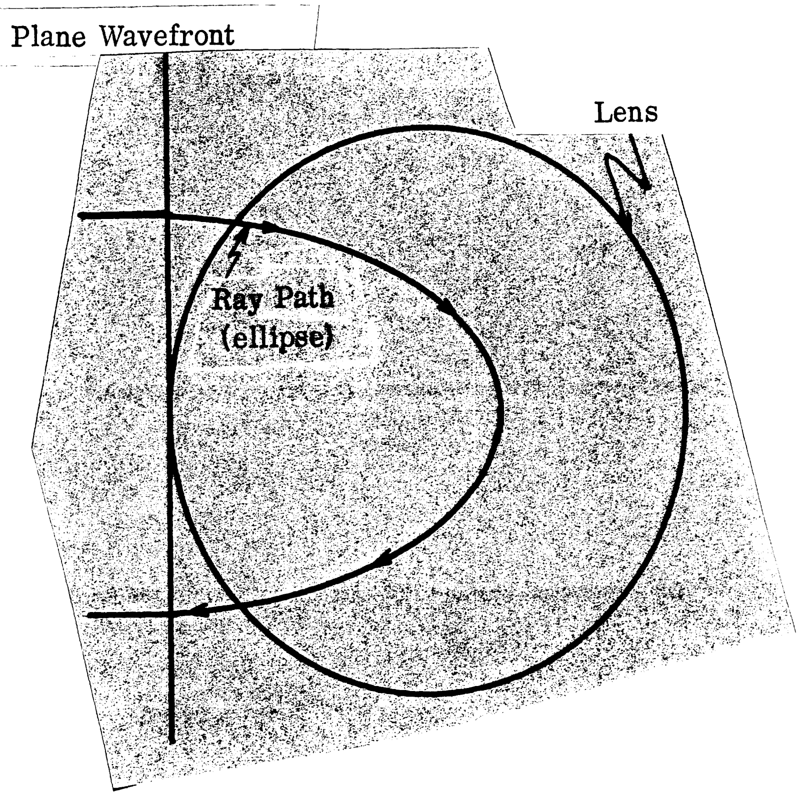

Figure 6: Eaton-Lippmann Lens.

only had been employed, Taking the vector nature of the electromagnetic field into account, Kay (1958) showed, by using geometrical optics, that the backscatter from a spherical Eaton-Lippmann lens should actually be zero. A simple proof of this effect is given, for example, in Appendix C of Rheinstein's (1962) work.

Garbacz (1962) used the surface impedance technique to compute the bistatic scattering cross-sections of an Eaton-Lippmann lens of $\rho_a=5$. In this case he used the initial surface impedance of value zero which corresponds to the perfectly conducting core. This core has a radius equal to one tenth of the radius of the lens. Rheinstein (1962) described the effect of the discontinuity in refractive index on the scattering cross-section of the lens. He considered perfectly conducting and dielectric cores at the center of large diameter spheres, and noted that the effect of different cores is unnoticeable on forward scattering, but appears for scattering at other angles.

Shafai (1972) used the method of amplitude and phase function in order to compute the scattering cross-sections of the Eaton-Lippmann lenses. He considered a conducting core of radius 0.05 for the lenses of radius 3 and 5. Hizal and Tosun (1973) used the state-space formulation with the same purpose. They assumed zero permittivity for the distance from the center up to 0.2, logarithmic variation of the permittivity with the distance up to 0.4, and the Eaton-Lippmann law thereafter. They studied lenses of radii 3, 5 and 7 λ_o. Their results agree well with those of Shafai, with the exception that they found smaller backscattering. However, their results for backscattering agreed well with those of Mikulski and Murphy (1963). Mikulski and Murphy used a layered model for the Eaton-Lippmann lenses with perfectly conducting cores occupying the innermost 5 percent of the lenses.

There is so far no report in the literature on the exact electromagnetic formulation of the properties of the Eaton-Lippmann lenses. This is undertaken in the present work.

4.2.2 S and T-Functions for the Eaton-Lippmann Lens

Recalling the differential equation (3.28) for the S-function and substituting the expression for ϵ_r in it, we have

$$\frac{d^2 S_n(k_o R)}{d R^2} + \left[k_o^2 \frac{2a-R}{R} - \frac{n(n+1)}{R^2} \right] S_n(k_o R) = 0 \quad .$$

By changing the variable $\rho = k_o R$ and defining $\rho_a = k_o a$, the above differential equation becomes

$$\frac{d^2 S_n(\rho)}{d\rho^2} + \left[-1 + \frac{2\rho_a}{\rho} - \frac{n(n+1)}{\rho^2} \right] S_n(\rho) = 0 \quad .$$

Applying the polynomial method of Sommerfeld (Kemble, 1937), we find that

$$S_n(\rho) = e^{-\rho} \rho^{n+1} V_n(2\rho)$$

where V_n satisfies the following differential equation

$$\rho V_n''(2\rho) + (n+1-\rho) V_n'(2\rho) - \left(\frac{n+1}{2} - \frac{\rho_a}{2}\right) V_n(2\rho) = 0 \ .$$

Calling $x = 2\rho$, the differential equation turns out to be

$$x V_n''(x) + \left[2(n+1)-x\right] V_n'(x) - \left[n+1-\rho_a\right] V_n(x) = 0 \ ,$$

which is the confluent hypergeometric equation with the solution

$$V_n(x) = {}_1F_1(n+1-\rho_a, 2(n+1), x)$$

finite at the origin. The expression for the S-function becomes

$$S_n(\rho) = e^{-\rho} \rho^{n+1} {}_1F_1(\alpha, \nu; 2\rho) \tag{4.20}$$

where

$$\alpha = n + 1 - \rho_a$$

and

$$\nu = 2\,(n+1) \ .$$

Substituting the expression for $\epsilon_r(R)$ in differential equation (3.30) for T, we have

$$\frac{d^2 T_n(k_o R)}{dR^2} + \frac{2a}{R(2a-R)} \frac{dT_n(k_o R)}{dR} + \left[k_o^2 \frac{2a-R}{R} - \frac{n(n+1)}{R^2}\right] T_n(k_o R) = 0 \ .$$

By changing the independent variable $\rho = k_o R$ and taking $\rho_a = k_o a$, we have

$$\frac{d^2T_n(\rho)}{d\rho^2} + \left(\frac{1}{\rho} + \frac{1}{2\rho_a-\rho}\right)\frac{dT_n(\rho)}{d\rho} + \left[-1 + \frac{2\rho_a}{\rho} - \frac{n(n+1)}{\rho^2}\right]T_n(\rho) = 0 \ ,$$

and by changing the dependent variable

$$T_n(\rho) = W_n(\rho)\sqrt{\frac{2\rho_a-\rho}{\rho}} \ ,$$

we obtain the differential equation in the normal form

$$\frac{d^2W_n(\rho)}{d\rho^2} + \left(-1 + \frac{2\rho_a - \frac{1}{4\rho_a}}{\rho} - \frac{\frac{1}{4\rho_a}}{2\rho_a-\rho} + \frac{\frac{1}{4}-\alpha^2}{\rho^2} - \frac{\frac{3}{4}}{(2\rho_a-\rho)^2}\right)W_n(\rho) = 0$$

where $\alpha^2 = n(n+1)$. Applying again the polynomial method of Sommerfeld (Kemkle, 1937), we have

$$W_n(\rho) = e^{-\rho}\,\rho^{1/2+\alpha}\,V_n(2\rho) \ ,$$

where $V_n(2\rho)$ satisfies the differential equation

$$\frac{d^2V_n(x)}{dx^2} + \left[\frac{2\left(\alpha+\frac{1}{2}\right)}{x} - 1\right]\frac{dV_n(x)}{dx} + \left[\frac{\frac{1}{2}\left(2\rho_a - \frac{1}{4\rho_a} - 2\left(\alpha+\frac{1}{2}\right)\right)}{x} - \frac{\frac{1}{8\rho_a}}{4\rho_a-x} - \frac{\frac{3}{4}}{(4\rho_a-x)^2}\right]V_n(x) = 0 \qquad (4.21)$$

with $x = 2\rho$. Therefore,

$$T_n(\rho) = \sqrt{2\rho_a - \rho}\; e^{-\rho}\,\rho^{\alpha}\,V_n(2\rho) \ . \qquad (4.22)$$

Equation (4.21) can be rewritten as

$$\frac{d^2 V_n(x)}{dx^2} + \left(\frac{\gamma}{x} - 1\right) \frac{dV_n(x)}{dx} - \left[\frac{\alpha_1}{x} + \frac{2\alpha_2}{x-a_2} - \frac{2a_2\,\alpha_3}{(x-a_2)^2}\right] V_n(x) = 0 ,$$

where

$$\gamma = 2\,(\alpha + 1/2)$$

$$\alpha_1 = 1/2\,(2(\alpha+1/2) + \frac{1}{4\rho_a} - 2\rho_a)$$

$$a_2 = 4\rho_a$$

$$\alpha_2 = -\,1/(16\rho_a)$$

$$\alpha_3 = -\,3/(32\rho_a) \quad .$$

We see that we are again in the presence of the "generalized" confluent hypergeometric equation which we met before in the study of the T-function of the Luneburg lens. Thus, the solution of this equation is represented by the same power series described in the study of the T-function of the Luneburg lens.

From (4.20) the expression for $S_n(\rho)$ and $S'_n(\rho)$ at $\rho = \rho_a$ are

$$S_n(\rho_a) = e^{-\rho_a}\,\rho_a^{n+1}\;{}_1F_1(\alpha, \nu; 2\rho_a)$$

and

$$S'_n(\rho_a) = (n+1-\rho_a)\,e^{-\rho_a}\,\rho_a^{n}\;{}_1F_1(\alpha, \nu; 2\rho_a) + 2\,\alpha/\nu\;e^{-\rho_a}\,\rho_a^{n+1}\;{}_1F_1(\alpha+1, \nu+1; 2\rho_a) ;$$

thus,

$$\frac{S_n'(\rho_a)}{S_n(\rho_a)} = \frac{n+1}{\rho_a} - 1 + 2\frac{\alpha}{\nu}\,\frac{{}_1F_1(\alpha+1,\ \nu+1;\ 2\rho_a)}{{}_1F_1(\alpha,\ \nu;\ 2\rho_a)} \quad . \tag{4.23}$$

From (4.22), the expressions for $T_n(\rho)$ and $T_n'(\rho)$ at $\rho = \rho_a$ are

$$T_n(\rho_a) = e^{-\rho_a}\,\rho_a^{\alpha+1/2}\,V_n(2\rho_a)$$

and

$$T_n'(\rho_a) = (\alpha + \rho_a - 1/2)\, e^{-\rho_a}\,\rho_a^{\alpha-1/2} V_n(2\rho_a) + 2e^{-\rho_a}\rho_a^{\alpha+1/2}\left.\frac{dV_n(2\rho)}{d(2\rho)}\right|_{\rho=\rho_a} .$$

Therefore,

$$\frac{T_n'(\rho_a)}{T_n(\rho_a)} = \frac{\alpha-1/2}{\rho_a} - 1 + 2\,\frac{\left.\dfrac{dV_n(2\rho)}{d(2\rho)}\right|_{\rho=\rho_a}}{V_n(2\rho_a)} \quad . \tag{4.24}$$

(4.23) and (4.23) are fundamental in numerical computation involving the Eaton-Lippmann lenses.

4.3 Eaton Lens

4.3.1 Introduction

The relative permittivity of the Eaton lens is given by

$$\epsilon_r(R) = R^2/a^2 \qquad 0 \leq R \leq a \quad . \tag{4.25}$$

This lens may be considered as a particular case of Nomura-Takaku (1955, 1956) distribution because the relative permittivity of the latter is

$$\epsilon_r(R) = R^{2q}/a^{2q}$$

whence we see that by taking $q = 1$ we have permittivity corresponding to the Eaton lens. Eaton (1952) showed, using geometrical optics, that the emerging phase front of such a lens is cylindrical and the ray path through the lens is described by an equilateral hyperbola. The lens, therefore, corresponds to a divergent lens in optics. The exact electromagnetic formulation of this lens is found in Tai's (1971) work, but apart from these, there are no studies of the properties of this lens in the literature.

4.3.2 S- and T-Functions for the Eaton Lens

Substituting (4.25) into the differential equation for the S-function, we have

$$\frac{d^2 S_n(k_o R)}{dR^2} + \left[k_o^2 \frac{R^2}{a^2} - \frac{n(n+1)}{R^2}\right] S_n(k_o R) = 0 \; .$$

Making the usual change of variable (i.e., $\rho = k_o R$ and $\rho_a = k_o a$), we obtain

$$\frac{d^2 S_n(\rho)}{d\rho^2} + \left[\frac{\rho^2}{\rho_a^2} - \frac{n(n+1)}{\rho^2}\right] S_n(\rho) = 0 \; .$$

We know (Jahnke-Emde, 1945) that the solution to this equation is

$$S_n(\rho) = \sqrt{\rho}\; J_p\left(\frac{\rho^2}{2\rho_a}\right) ,$$

with $p = \frac{n+1/2}{2}$. The equation for the T-function in this case is

$$\frac{d^2 T_n(k_o R)}{dR^2} - \frac{2}{R}\frac{d T_n(k_o R)}{dR} + \left[k_o^2 \frac{R^2}{a^2} - \frac{n(n+1)}{R^2}\right] T_n(k_o R) = 0$$

and making the same change of variable, we get

$$\frac{d^2T_n(\rho)}{d\rho^2} - \frac{2}{\rho}\frac{dT_n(\rho)}{d\rho} + \left[\frac{\rho^2}{\rho_a^2} - \frac{n(n+1)}{\rho^2}\right] T_n(\rho) = 0 \ ,$$

which solution is again known (Jahnke-Emde, 1945)

$$T_n(\rho) = \rho^{3/2} J_{p'}\left(\frac{\rho^2}{2\rho_a}\right)$$

with $p' = \frac{\sqrt{(n+1/2)^2 + 2}}{2}$.

The expressions for the S- and T-functions, and their derivatives at the rim of the lens are

$$S_n(\rho_a) = \sqrt{\rho_a}\, J_p\left(\frac{\rho_a}{2}\right)$$

$$S'_n(\rho_a) = \frac{1}{2\sqrt{\rho_a}} J_p\left(\frac{\rho_a}{2}\right) + \sqrt{\rho_a}\, \left.\frac{dJ_p\left(\frac{\rho^2}{2\rho_a}\right)}{d\left(\frac{\rho^2}{2\rho_a}\right)}\right|_{\rho=\rho_a} .$$

and

$$T_n(\rho_a) = \rho_a^{3/2} J_{p'}\left(\frac{\rho_a}{2}\right)$$

$$T'_n(\rho_a) = \frac{3}{2}\sqrt{\rho_a}\, J_{p'}\left(\frac{\rho_a}{2}\right) + \rho_a^{3/2} \left.\frac{dJ_{p'}\left(\frac{\rho^2}{2\rho_a}\right)}{d\left(\frac{\rho^2}{2\rho_a}\right)}\right|_{\rho=\rho_a} .$$

From Dettman (1969), or NBS (1948), we have

$$J'_\nu(x) = J_{\nu-1}(x) - \frac{\nu}{x} J_\nu(x) \ ,$$

which in our case takes the form

$$\left.\frac{d\,J_p\left(\frac{\rho^2}{2\rho_a}\right)}{d\left(\frac{\rho^2}{2\rho_a}\right)}\right|_{\rho=\rho_a} = J_{p-1}\left(\frac{\rho_a}{2}\right) - \frac{2p}{\rho_a}\,J_p\left(\frac{\rho_a}{2}\right)$$

and

$$\left.\frac{d\,J_{p'}\left(\frac{\rho^2}{2\rho_a}\right)}{d\left(\frac{\rho^2}{2\rho_a}\right)}\right|_{\rho=\rho_a} = J_{p'-1}\left(\frac{\rho_a}{2}\right) - \frac{2p'}{\rho_a}\,J_{p'}\left(\frac{\rho_a}{2}\right) .$$

Now, $S'_n(\rho_a)$ becomes

$$S'_n(\rho_a) = \frac{1}{\sqrt{\rho_a}}(1/2 - 2p)\,J_p\left(\frac{\rho_a}{2}\right) + \sqrt{\rho_a}\,J_{p-1}\left(\frac{\rho_a}{2}\right)$$

and

$$T'_n(\rho_a) = \sqrt{\rho_a}\,(3/2 - 2p')\,J_{p'}\left(\frac{\rho_a}{2}\right) + \sqrt{\rho_a^3}\,J_{p'-1}\left(\frac{\rho_a}{2}\right)$$

and finally

$$\frac{S'_n(\rho_a)}{S_n(\rho_a)} = \frac{1}{2\rho_a} - \frac{2p}{\rho_a} + \frac{J_{p-1}\left(\frac{\rho_a}{2}\right)}{J_p\left(\frac{\rho_a}{2}\right)} \tag{4.26}$$

and

$$\frac{T'_n(\rho_a)}{T_n(\rho_a)} = \frac{3}{2\rho_a} - \frac{2p'}{\rho_a} + \frac{J_{p'-1}\left(\frac{\rho_a}{2}\right)}{J_{p'}\left(\frac{\rho_a}{2}\right)} \tag{4.27}$$

which will be used in numerical computations of the Eaton lenses.

4.4 Numerical Computations and Results

The expressions for the far-zone electric field radiated by the Huygens source in two principal planes are given by equations (3.54) and (3.55). The expressions for S'/S and T'/T for all lenses are summarized in Table IV. The expressions (3.54) and (3.55) are valid for any kind of lens, the effect of the lens being built into the S- and T-functions and their derivatives.

The radiation patterns which appear in the following figures corresponds to the sum of the terms of equations (3.54) and (3.55). The constant term which appears in front of the summation, is not taken into account because the radiation patterns are given in dB above the value at 0°.

The problem of infinite summation is solved by the following considerations. The behavior of the spherical Bessel function is oscillatory for order of the function less than its argument, and monotonic for order of the function greater than the argument, so, the number of terms of the summation should

Table IV: S'/S and T'/T for all the Lenses

Lenses	S'/S	T'/T
Luneburg	$\frac{n+1}{\rho_a} -1+2\frac{\alpha}{\nu}\frac{{}_1F_1(\alpha+1,\nu+1;\rho_a)}{{}_1F_1(\alpha,\nu;\rho_a)}$	$\frac{n}{\rho_a} - 1+2\frac{\sum_{m=0} mA_m\rho_a^{m-1}}{\sum_{m=0} A_m\rho_a^m}$
Eaton-Lippmann	$\frac{n+1}{\rho_a} -1+2\frac{\alpha}{\nu}\frac{{}_1F_1(\alpha+1,\nu+1;2\rho_a)}{{}_1F_1(\alpha,\nu;2\rho_a)}$	$\frac{\alpha-1/2}{\rho_a}-1+2\frac{\sum_{m=0} mA_m(2\rho_a)^{m-1}}{\sum_{m=0} A_m(2\rho_a)^m}$
Eaton	$\frac{1}{2\rho_a} - \frac{2p}{\rho_a} + \frac{J_{p-1}(\rho_a/2)}{J_p(\rho_a/2)}$	$\frac{3}{2\rho_a} - \frac{2p'}{\rho_a} + \frac{J_{p'-1}(\rho_a/2)}{J_{p'}(\rho_a/2)}$

be at least of the order of the argument of the Bessel function, ρ_a. The behavior of the confluent hypergeometric function is similar, as explained in section 4.1.2. The behavior of the "generalized" confluent hypergeometric function is of the same sort. So, by taking the summation up to two times ρ_a, we are sure of being in the region in which all the functions involved are monotonic and the error committed by neglecting the terms of higher order is always of the same sign. Also, all the functions which appear as S- and T-functions for the lenses which we are studying, are dominated by the spherical Bessel functions which have large imaginary part and therefore, the error of truncation of the series is small.

A program has been written which computes the radiation pattern of the Huygens source in the presence of the lenses. In this program only the calculations of the S- and T-functions change from lens to lens, and the radiation patterns are given in 2^{o} steps. The S- and T-functions are calculated in subroutines by summing the respective series and the computation stops when the absolute value of the nth term over the absolute value of the sum of the series becomes less than 10^{-7}.

The subroutine for the Bessel fun ctions was checked against NBS(1948). The subroutine for the confluent hypergeometric function was checked against tables by McDonald (1949), Webb and Airey(1918), Nath (1951) and Rushton and Lang (1954). The values of the "generalized" confluent hypergeometric function were compared with those of the confluent hypergeometric function and it was found that in the monotonic region they compare quite closely. The subroutine for associated Legendre functions was checked against NBS (1945).

The values for the diameters of the lenses were chosen equal to those used in Mason's (1972) work, which deal with homogeneous lenses, in order to make a comparison of homogeneous and inhomogeneous lenses of the same physical dimensions.

Figure 7 shows the relative permittivities of all the lenses discussed here. The range of the permittivities is fairly representstive, giving one

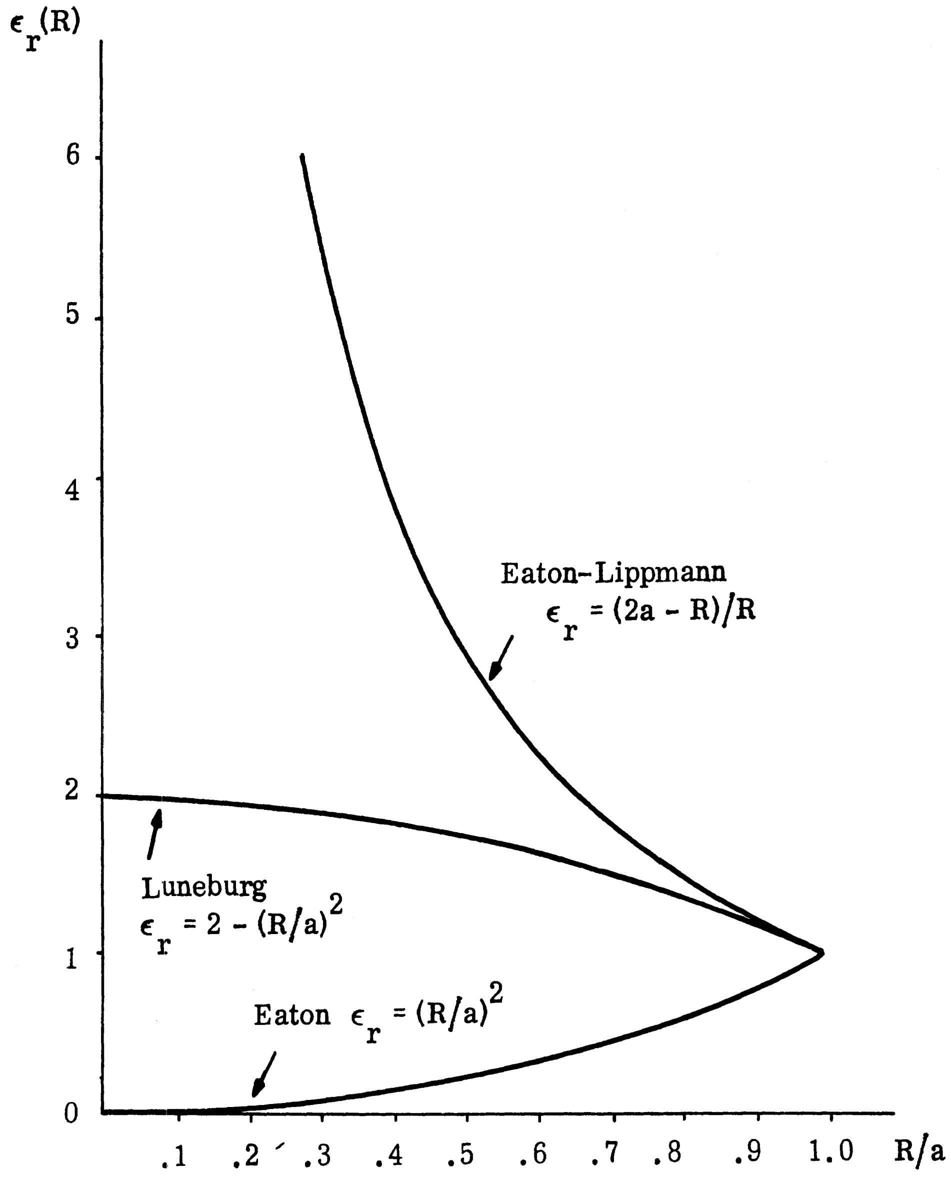

Figure 7: Relative Permittivities of the Lenses

permittivity which is finite at the origin and two which present extremal behavior of the origin. Figures 8 through 11 show the radiation pattern of Luneburg lenses in two principal planes. Their main features are shown in Table V.

Table V: Characteristics of the Radiation Patterns of the Luneburg Lenses

Diameter (λ_o)	3dB bandwidth E-plane	3dB bandwidth H-plane	Side lobe level (dB) E-plane	Side lobe level (dB) H-plane	Backward power (dB)
1.27	25^o	25^o	-	-	-25.5
2.12	14^o	14^o	-14	-14.5	-30.8
3.39	9^o	9^o	-16	-16	-56.3
4.23	8^o	7^o	-16.5	-16.5	-46.0

We see that the pattern becomes more and more directive as the radius of the sphere increases, which is predicted by geometrical optics. The radiation in the backward direction decreases.

Figures 12 through 15 show the radiation patterns of the Eaton-Lippmann lenses. Their characteristics are listed in Table VI.

Table VI: Characteristics of the Radiation Patterns of the Eaton-Lippmann Lenses

Diameter(λ_o)	Direction of the broadside lobe E-plane	Direction of the broadside lobe H-plane	Broadside power (dB) E-plane	Broadside power (dB) H-plane	Backward power (dB)
1.27	102^o	102^o	-4.0	-3.0	-12
2.12	118^o	116^o	-3.5	-3.0	-17
3.39	128^o	142^o	-3.0	-2.2	-24
4.23	132^o	148^o	-3.5	-2.8	-21

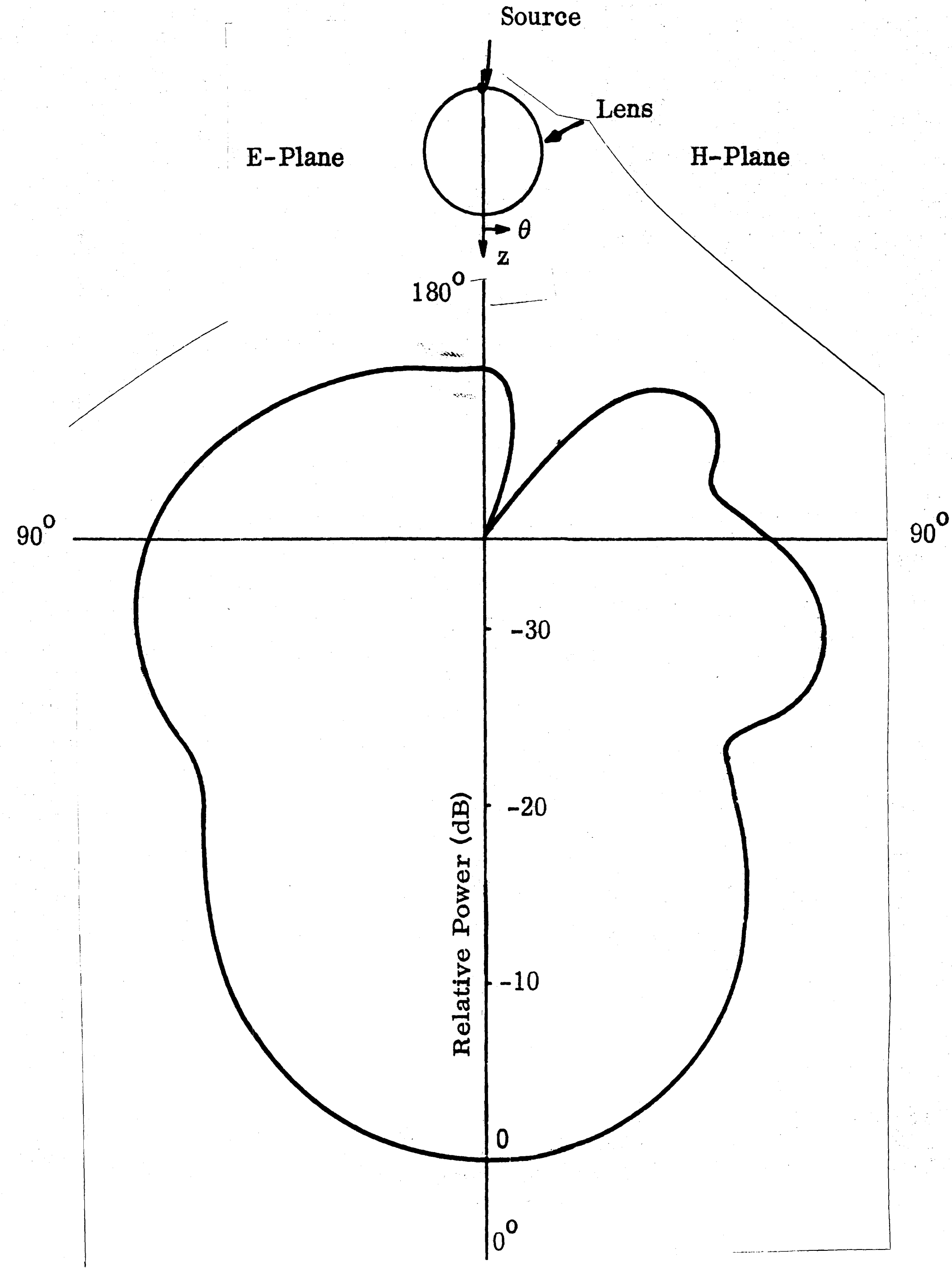

Figure 8: Radiation Pattern for Luneburg Lens ($D = 1.27\ \lambda_o$).

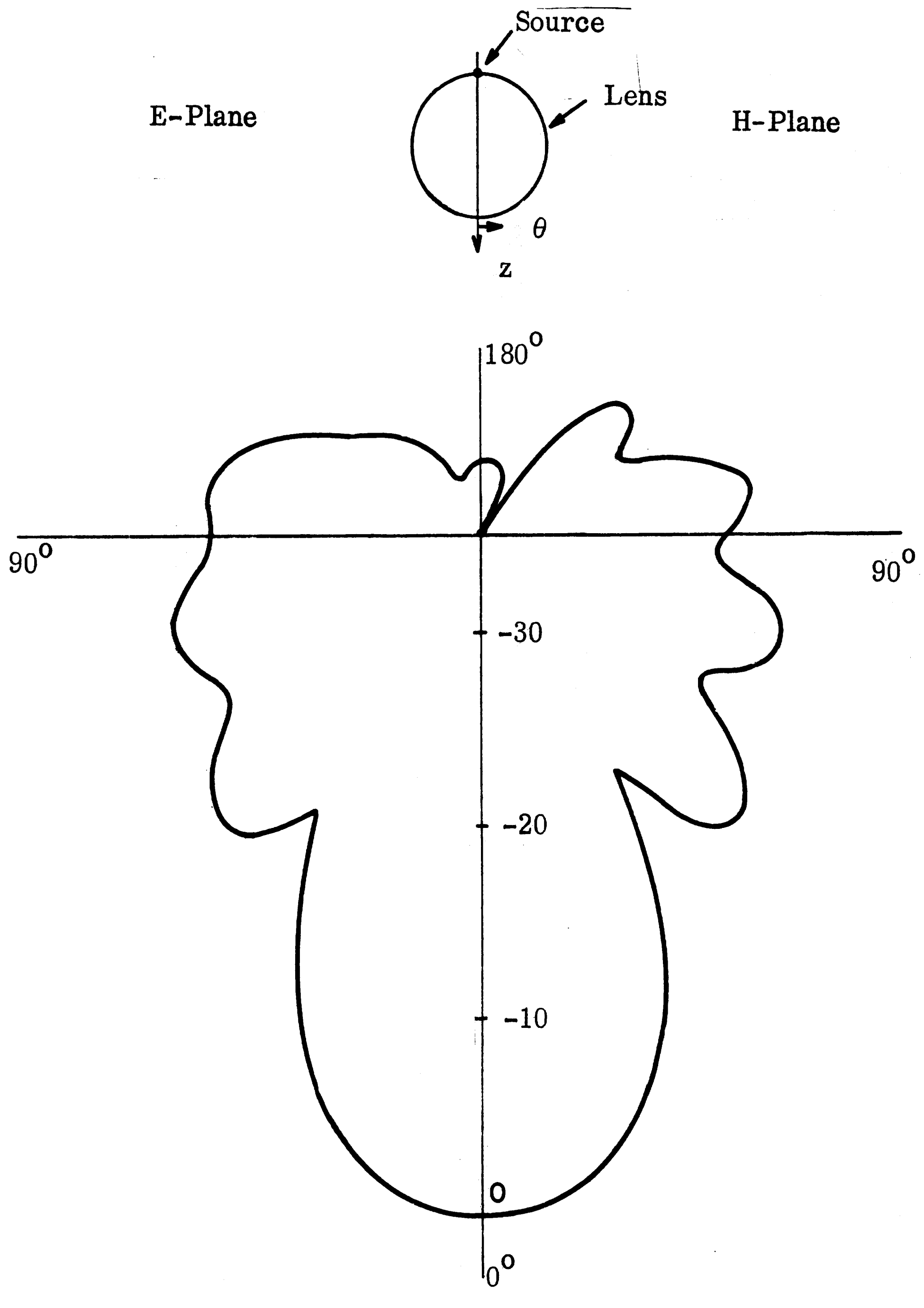

Figure 9: Radiation Pattern for Luneburg Lens ($D = 2.12\ \lambda_o$)

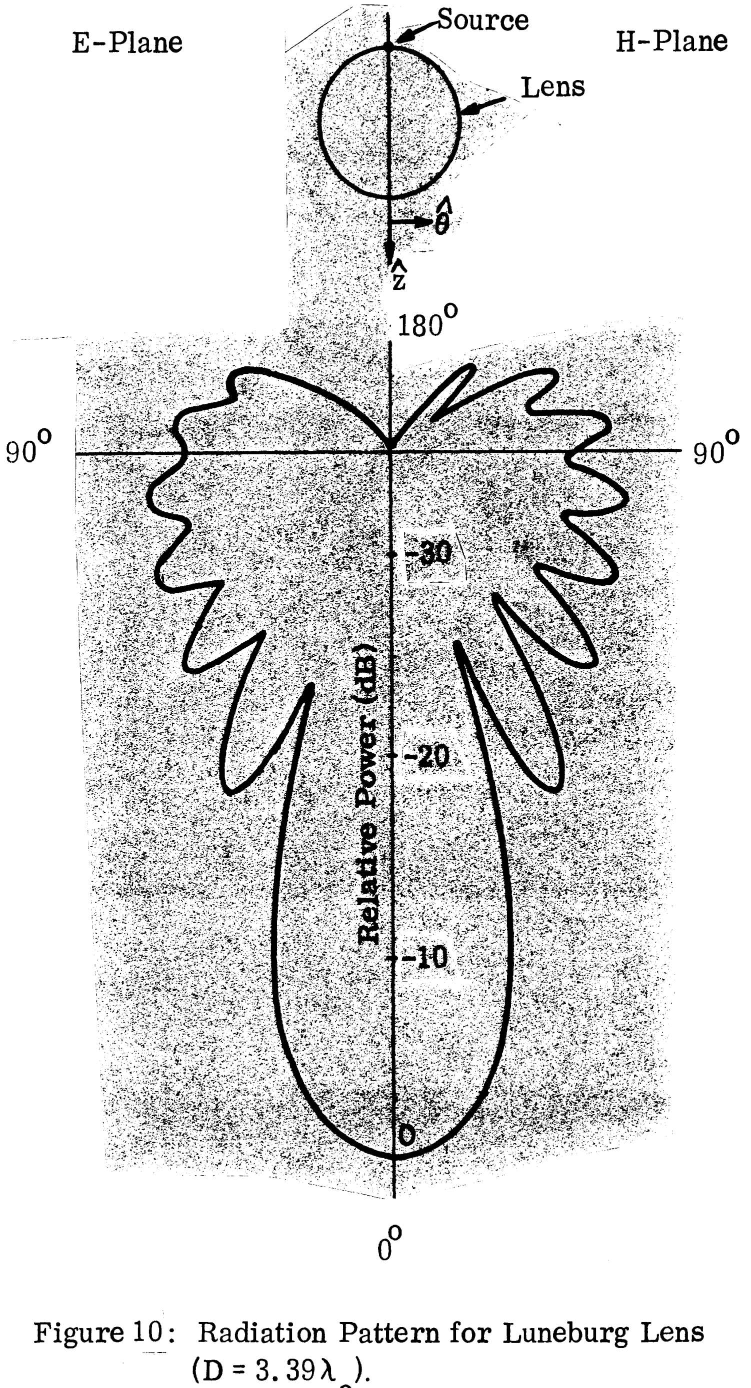

Figure 10: Radiation Pattern for Luneburg Lens ($D = 3.39\lambda_o$).

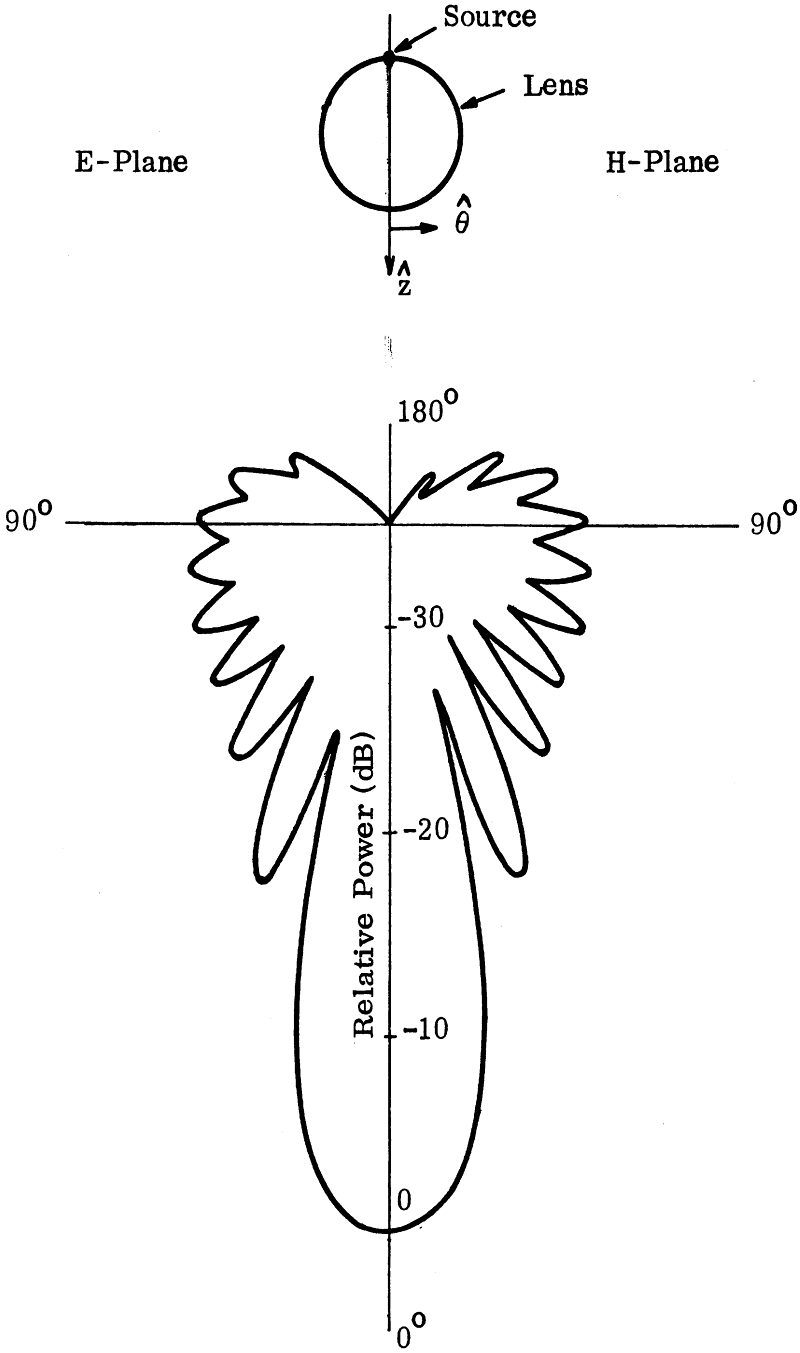

Figure 11: Radiation Pattern for Luneburg Lens ($D = 4.23\ \lambda_o$).

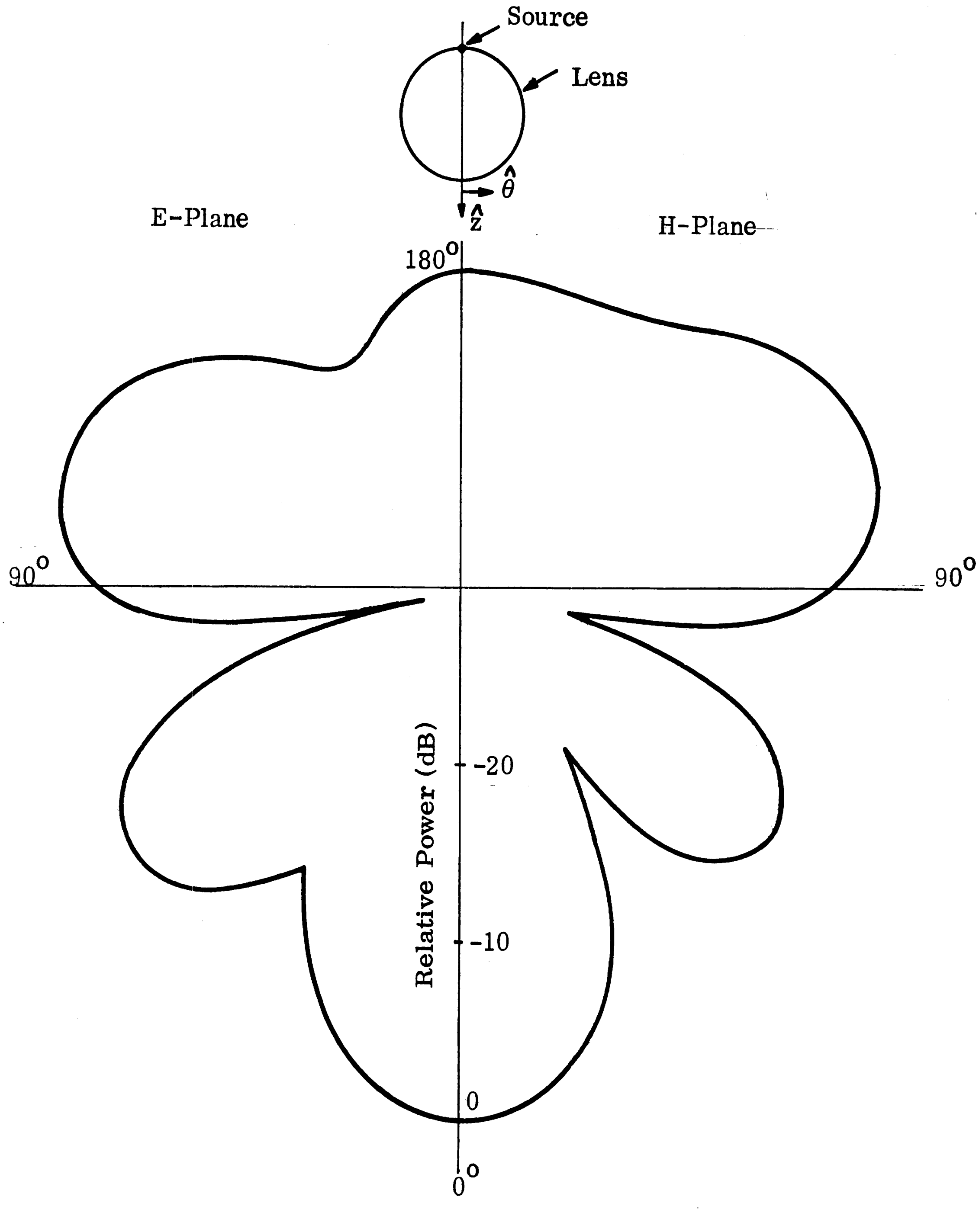

Figure 12: Radiation Pattern for Eaton-Lippmann Lens ($D = 1.27\ \lambda_o$).

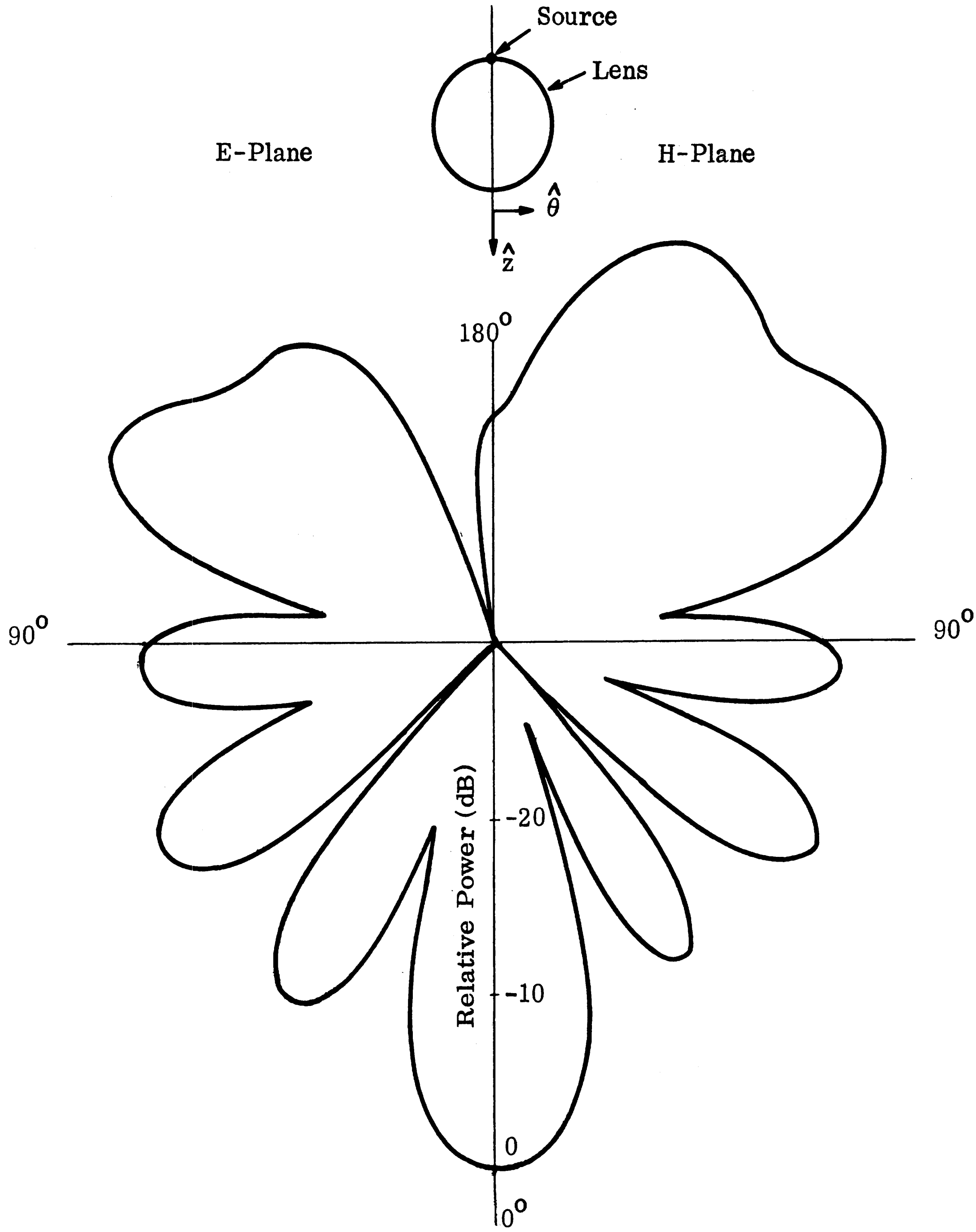

Figure 13: Radiation Pattern for Eaton-Lippmann Lens ($D = 2.12\lambda_o$).

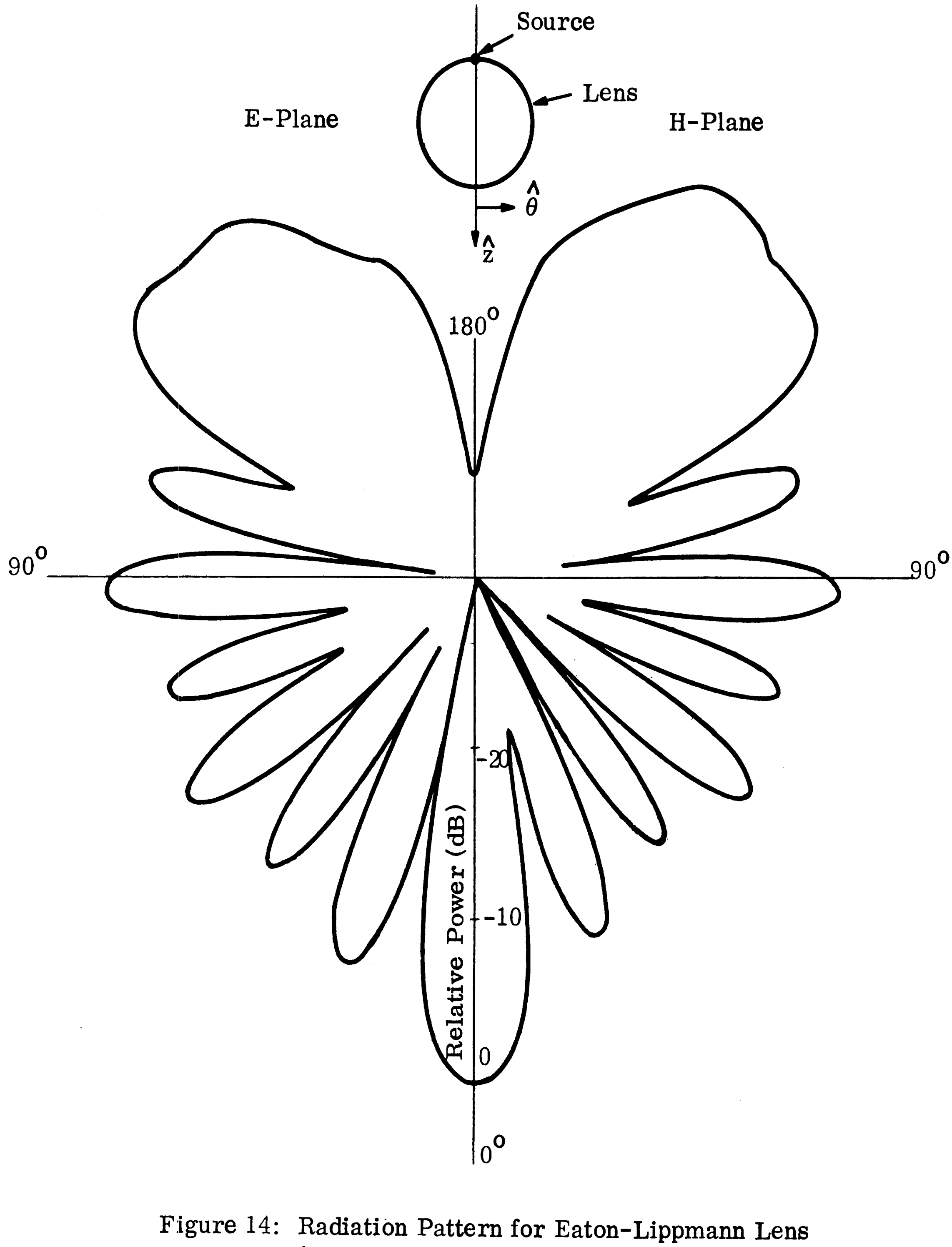

Figure 14: Radiation Pattern for Eaton-Lippmann Lens ($D = 3.39\,\lambda_o$).

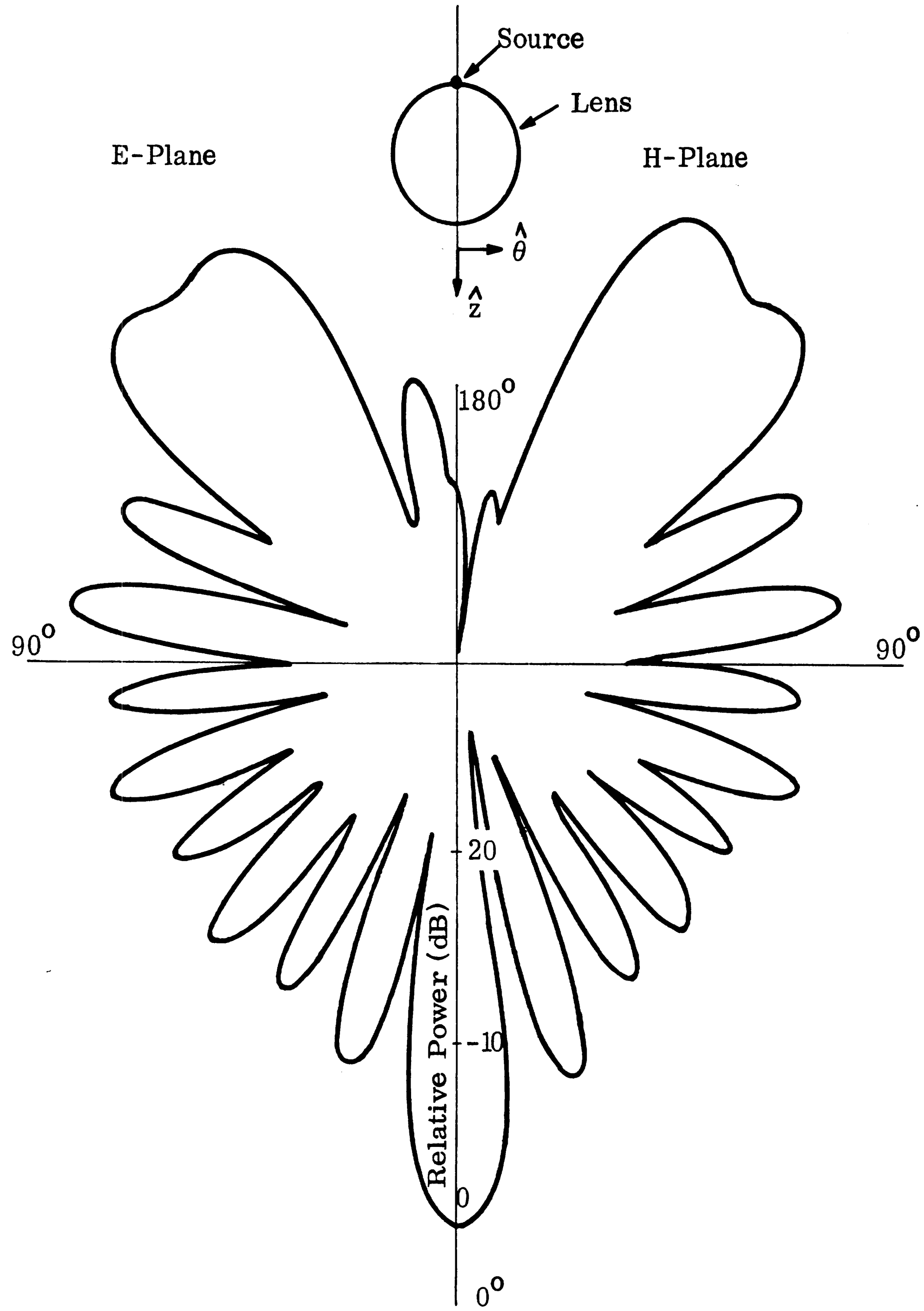

Figure 15: Radiation Pattern for Eaton-Lippmann Lens ($D = 4.23\ \lambda_o$).

We see that there are broadside lobes which tend to the backward direction when the diameter of the lens increases. Figures 16 through 19 show the radiation patterns of the Eaton lens, whose characteristics are summarized in Table VII. The main lobes are broadside oriented in this type of the lens, which agrees with the behavior predicted by geometrical optics.

Table VII: Characteristics of the Radiation Patterns of the Eaton Lenses

Diameter (λ_0)	Direction of the main lobe E-plane	Direction of the main lobe H-plane	Forward power (dB)	Backward power (dB)
1.27	98^o	86^o	-7.0	-13.0
2.12	100^o	88^o	-14.0	-15.0
3.39	94^o	90^o	-20.5	-24.0
4.23	94^o	90^o	-27.0	-22.5

The bistatic scattering cross-sections of all the lenses normalized to their geometrical cross-section, as given by expressions (3.64) and (3.65), are shown in Figures 20 and 21. The ρ_a of the lenses are 5 and 10. The bistatic scattering cross-sections of the Luneburg lens with $\rho_a = 5$, as computed here, agree very well with those computed by Garbacz (1962), while those of the Eaton-Lippmann lens with $\rho_a = 5$ agree with those of Garbacz (1962) only in the forward direction. This is explained by the fact that in Garbacz' (1962) work, the central portion of the lens was substituted by a perfectly conducting core with radius equal to one tenth of the radius of the lens. The influence of the core on the bistatic scattering cross-sections was also observed by Rheinstein (1962).

The characteristics of the bistatic scattering cross-sections of the lenses with ρ_a equals to 5 and 10, are summed up in Table VIII.

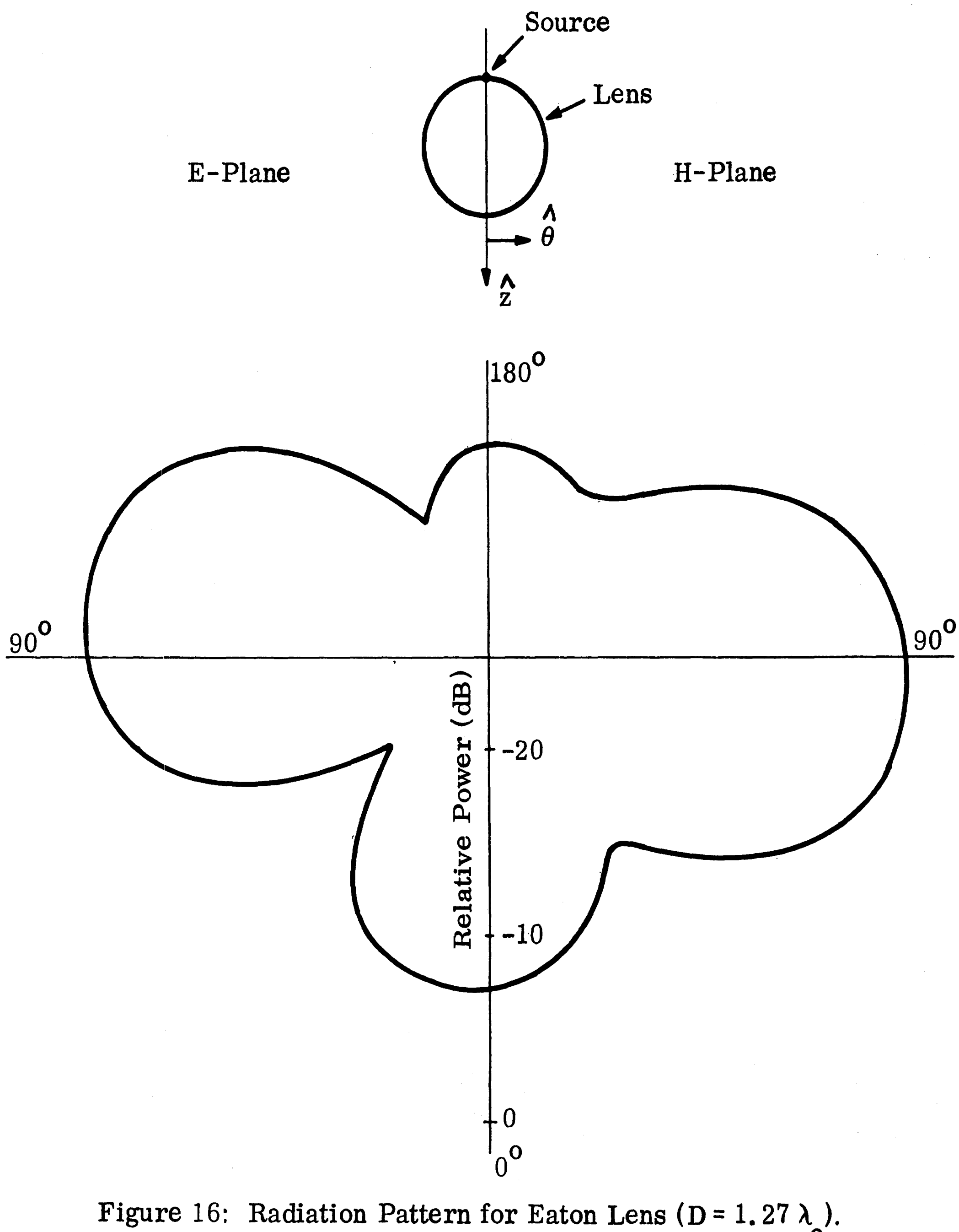

Figure 16: Radiation Pattern for Eaton Lens ($D = 1.27\,\lambda_o$).

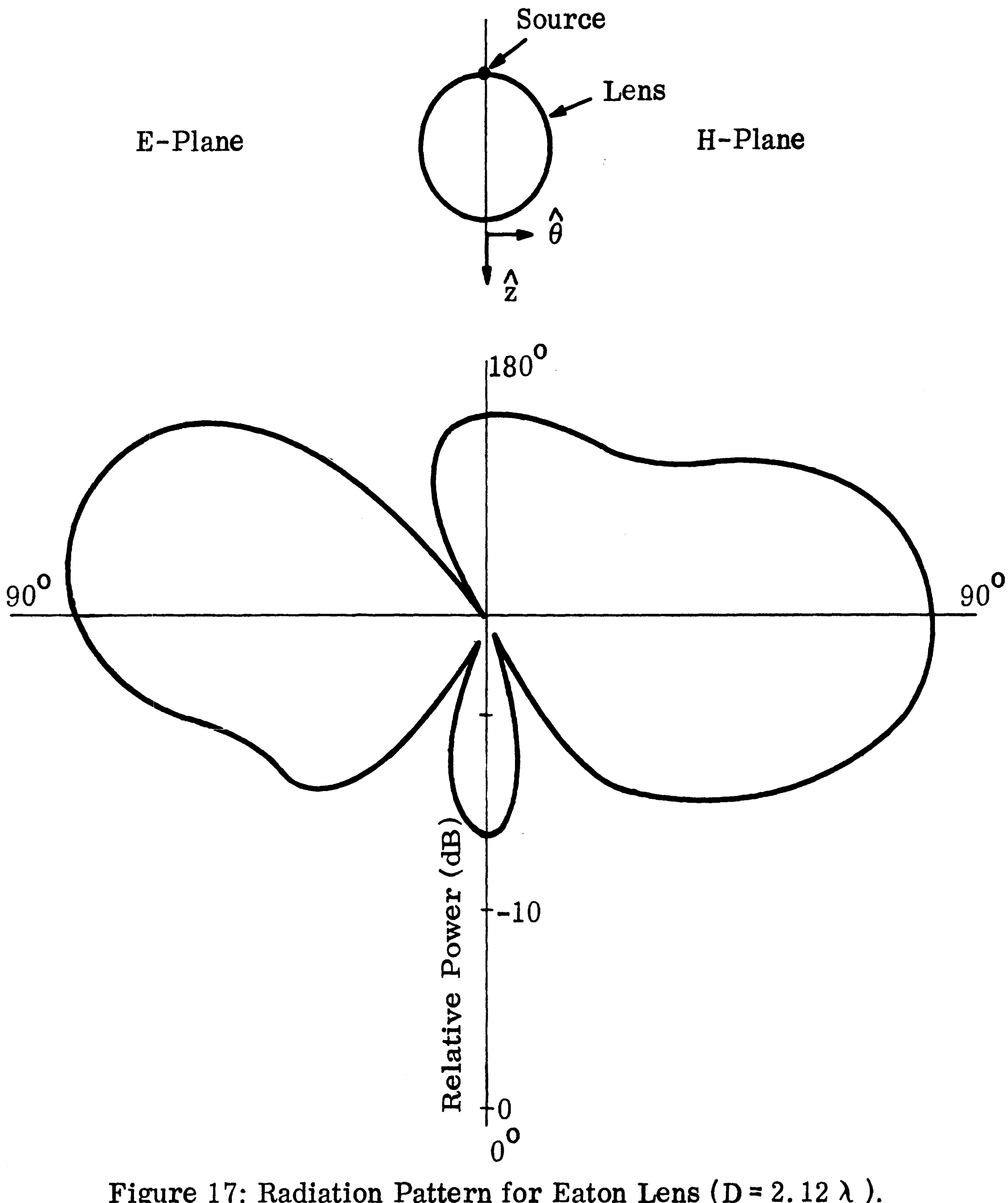

Figure 17: Radiation Pattern for Eaton Lens ($D = 2.12\ \lambda_o$).

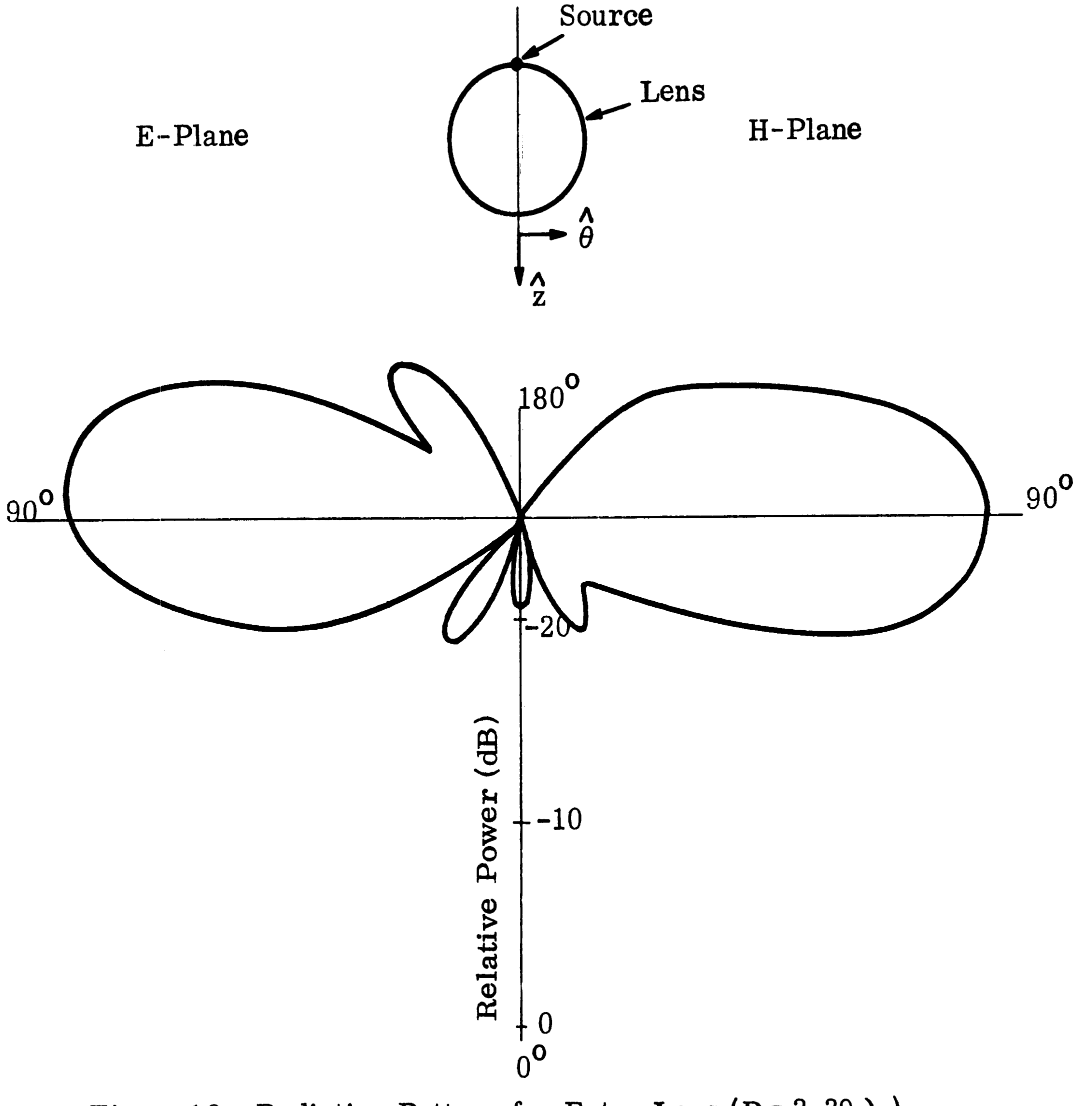

Figure 18: Radiation Pattern for Eaton Lens (D = 3.39 λ_o).

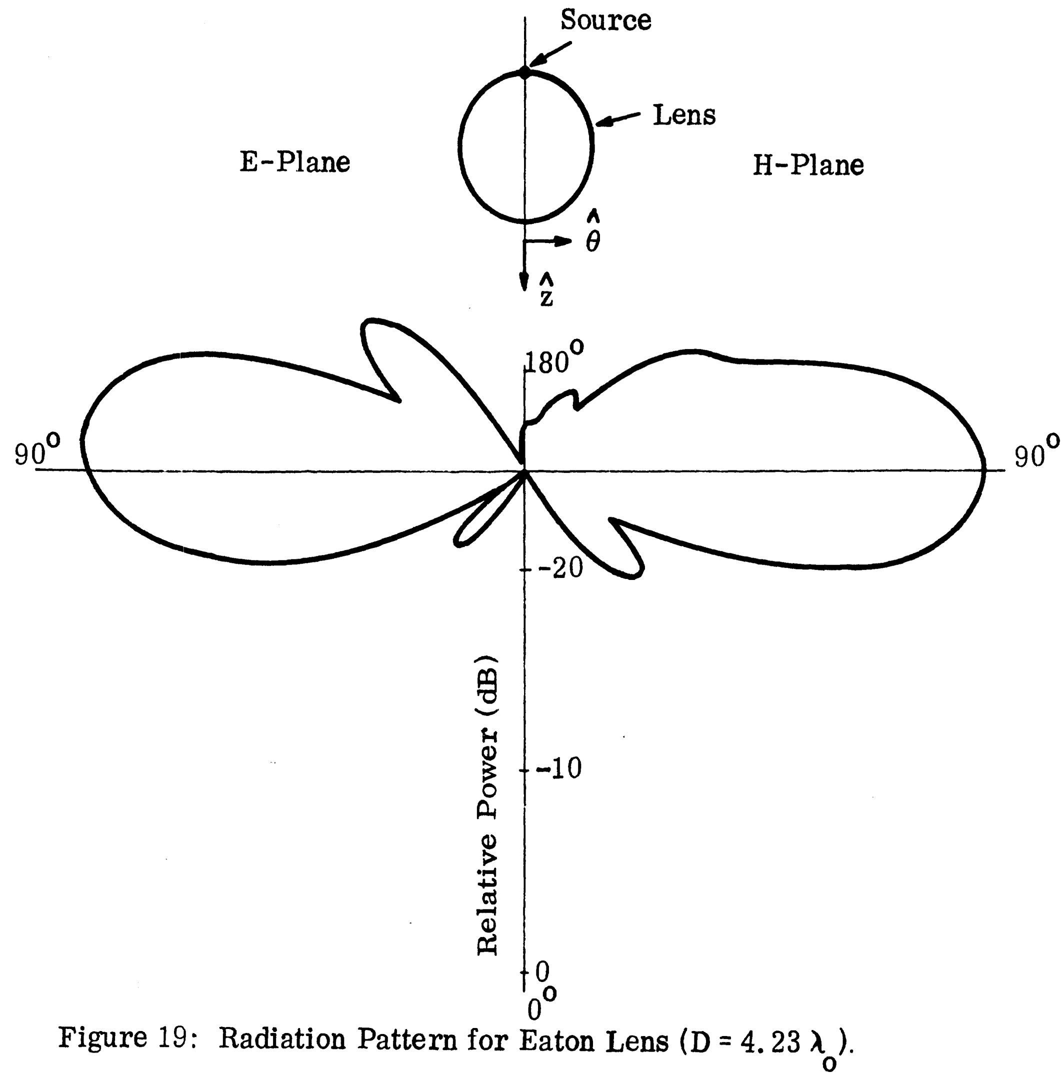

Figure 19: Radiation Pattern for Eaton Lens ($D = 4.23\,\lambda_o$).

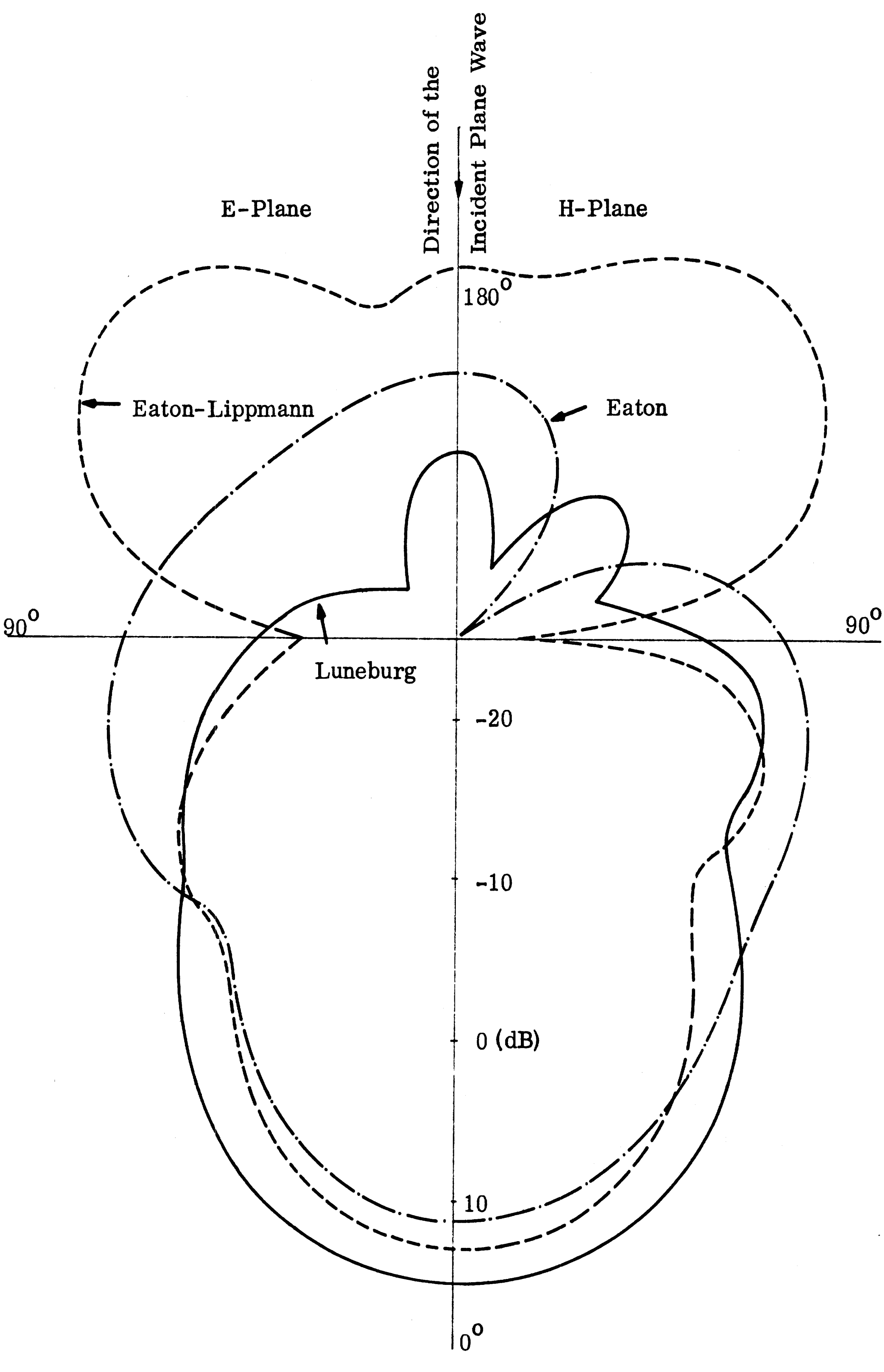

Figure 20: Bistatic Scattering Cross Section of the Lenses Normalized to their Geometrical Cross-Sections ($\rho_a = 5$).

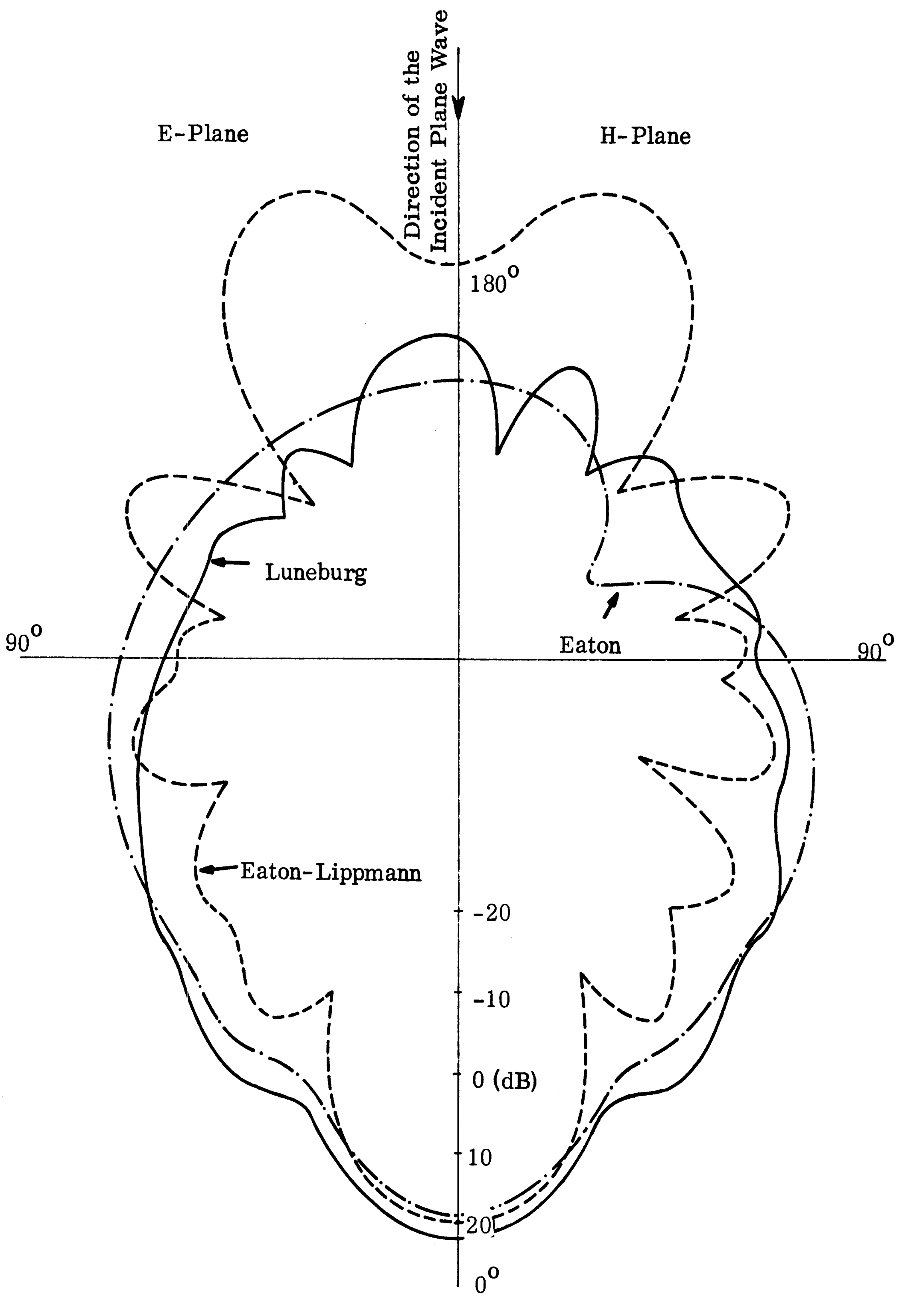

Figure 21: Bistatic Scattering Cross-Section of the Lenses Normalized to their Geometrical Cross-Sections ($\rho_a = 10$).

Table VIII: Characteristics of the Bistatic Scattering Cross-Sections of the Lenses

Lenses	ρ_a=5 Forward Scattering (dB)	ρ_a=5 Backward Scattering (dB)	ρ_a=10 Forward Scattering (dB)	ρ_a=10 Backward Scattering (dB)
Luneburg	15.3	-13.3	20.8	-10.5
Eaton-Lippmann	13.0	-1.8	19.4	-.84
Eaton	11.2	-8.3	18.5	-15.1

The Luneburg lens has the highest and the two Eaton lenses the lowest forward bistatic scattering cross section. The backward bistatic scattering cross-section of the Luneburg lens for $\rho_a = 5$ and Eaton for $\rho_a = 10$ are the lowest followed by the Eaton-Lippmann lens, in this order. The Eaton-Lippmann lens has the highest backscattering cross-section and it tends to the value of its geometric cross-section as the diameter of the lens increases. The forward bistatic scattering cross-sections of all lenses becomes nearly equal and the scattering at the angles close to the backward direction in the case of the Eaton-Lippmann lens increases, as the diameter of the lenses increases.

The directivity of the Luneburg lens as given by (4.10) is shown in Figure 22, along with the directivities of two homogeneous spheres with $\epsilon_e = 3.00$ and $\epsilon_r = 1.667$, as computed using Mason's (1972) program. The directivities are computed in $0.1\lambda_0$ steps. All the curves in Figure 22 start from point 4.77dB, the directivity of the Huygens source without any lens. It is seen that the directivity of the Luneburg lens of small diameter is not better than that of the homogeneous spheres. However, the Luneburg lens does not present the problem of resonance which occurs in homogeneous lenses.

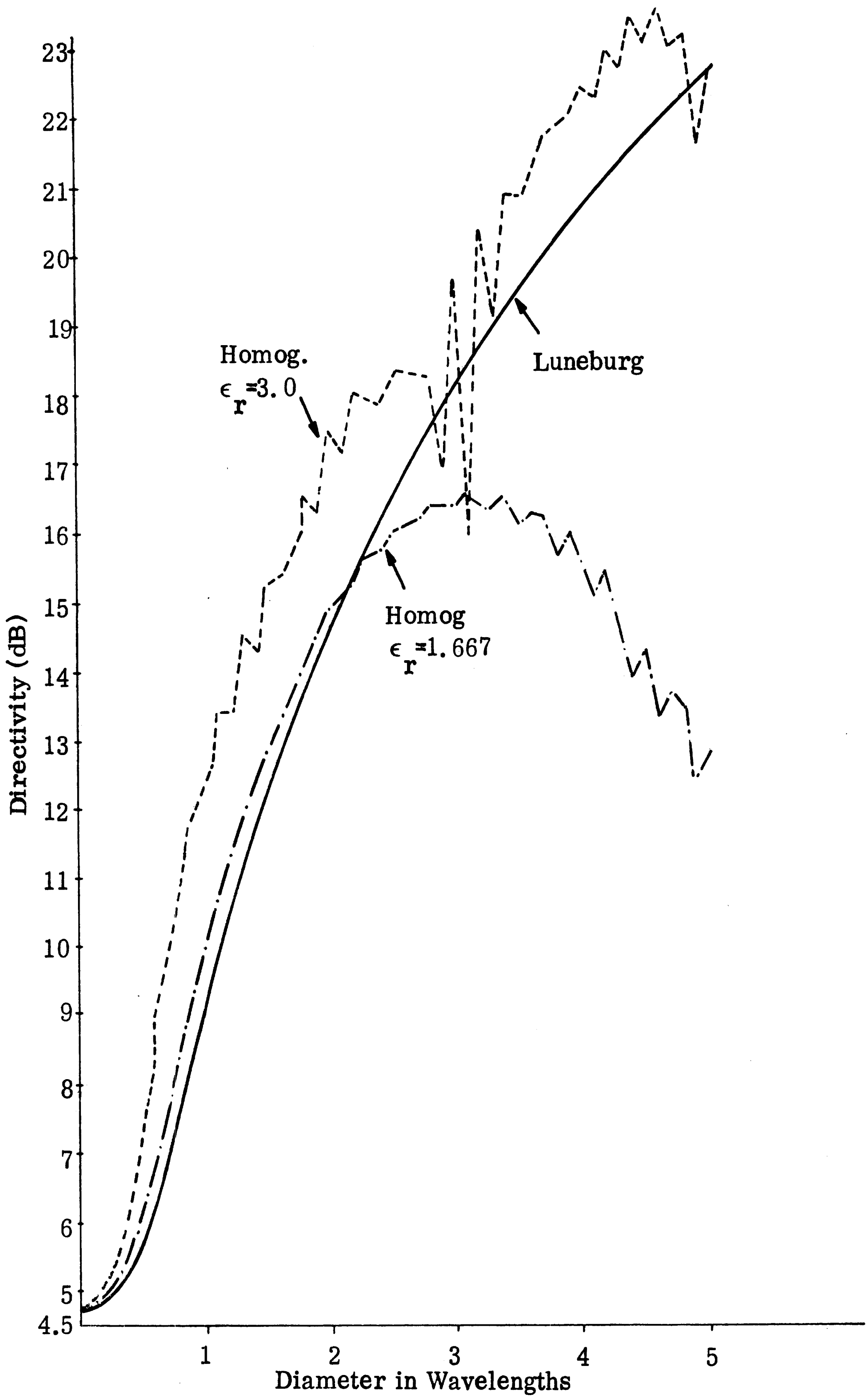

Figure 22: Directivity of the Luneburg and Homogeneous Lenses.

In order to see how well the Luneburg lens performs as a focusing device, the distribution of the energy around its geometrical optics focus, was computed. The expressions (4.11) and (4.12) for lenses with diameters $1.27\lambda_o$ and $4.23\lambda_o$ were computed and are plotted on Figures 23 and 24, respectively. The square of the absolute value of the field was normalized to the square of the absolute value of the field at point $R = a$, $\theta = 0$, $\phi = 0$ which is the geometric optics focus of the lens. It was observed that the concentration of the energy around the geometrical optics focus increases as the diameter of the lens increases. For the $1.27\lambda_o$ lens the 3-dB contour line is located as far as 41^o in the E-plane from the focus while for the $4.23\lambda_o$ lens the same line is only 12^o from the focus.

The plot of the field on the interior of the lens given by (4.17) and (4.18) for $1.27\lambda_o$ lens is shown in Figure 28, and an enlarged view of it is given in Figure 25. We see from the figures that, actually, the focus is located inside the lens and not on the rim, because the intensity of the field inside the lens is greater than at the focus. It was found that for the lens of diameter $1.27\lambda_o$ the focus is located at $0.12\lambda_o$ inside the lens and $0.11\lambda_o$ inside for that of diameter $2.12\lambda_o$; the lens of diameter $3.39\lambda_o$ has its focus located $0.10\lambda_o$ inside and that of diameter $4.23\lambda_o$ has it $0.09\lambda_o$ inside. The location of the focus moves to the rim of the lens as its diameter increases. This could account for the lower directivity of the Luneburg lens when compared to the homogeneous lens when the diameter is small. In other words, when we put a source on the rim of the Luneburg lens of small diameter, we do not put it at the focus of the lens, losing in the process some of its effectiveness.

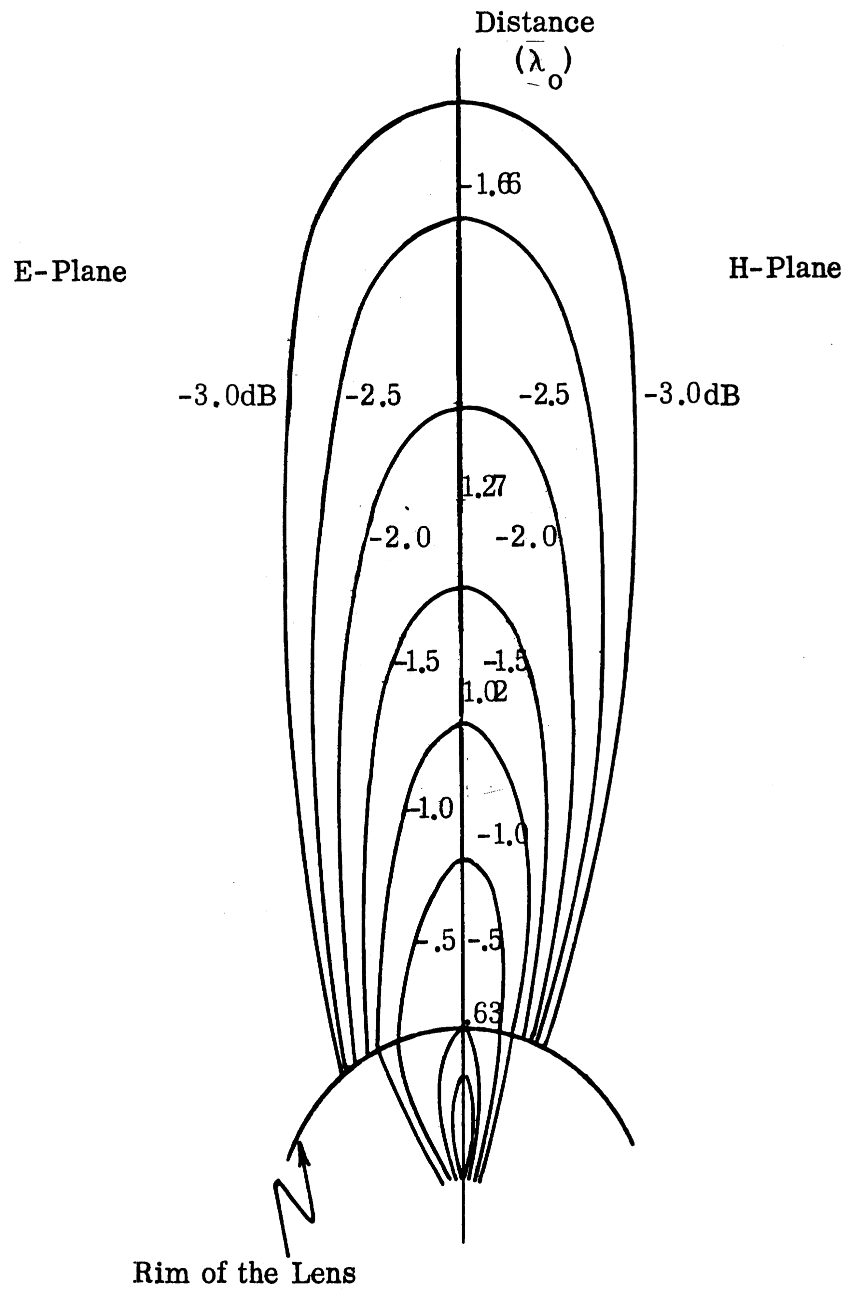

Figure 23: Field Around Focus of the Luneburg Lens ($D = 1.27\,\lambda_o$).

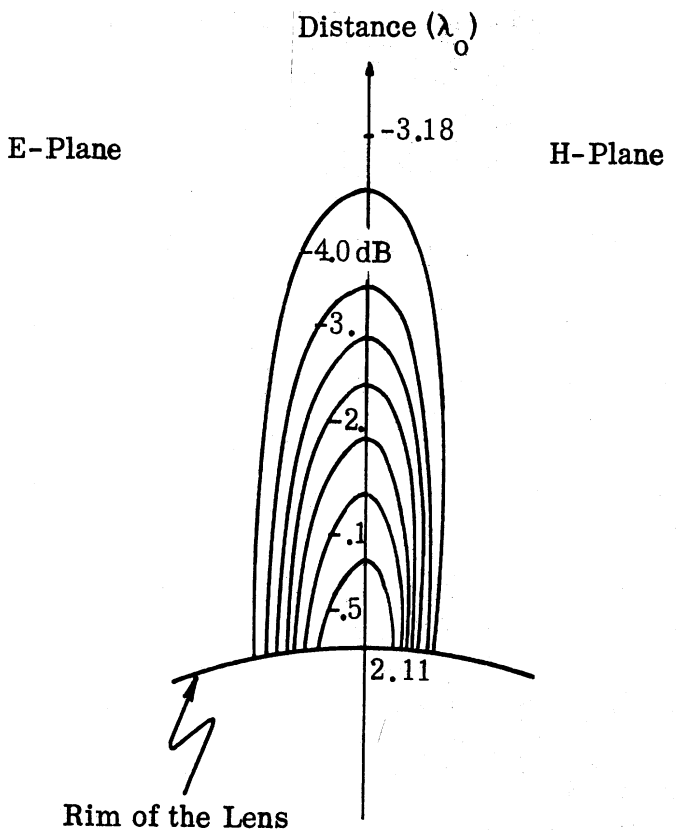

Figure 24: Field Around Focus of the Luneburg Lens ($D = 4.23\ \lambda_o$)

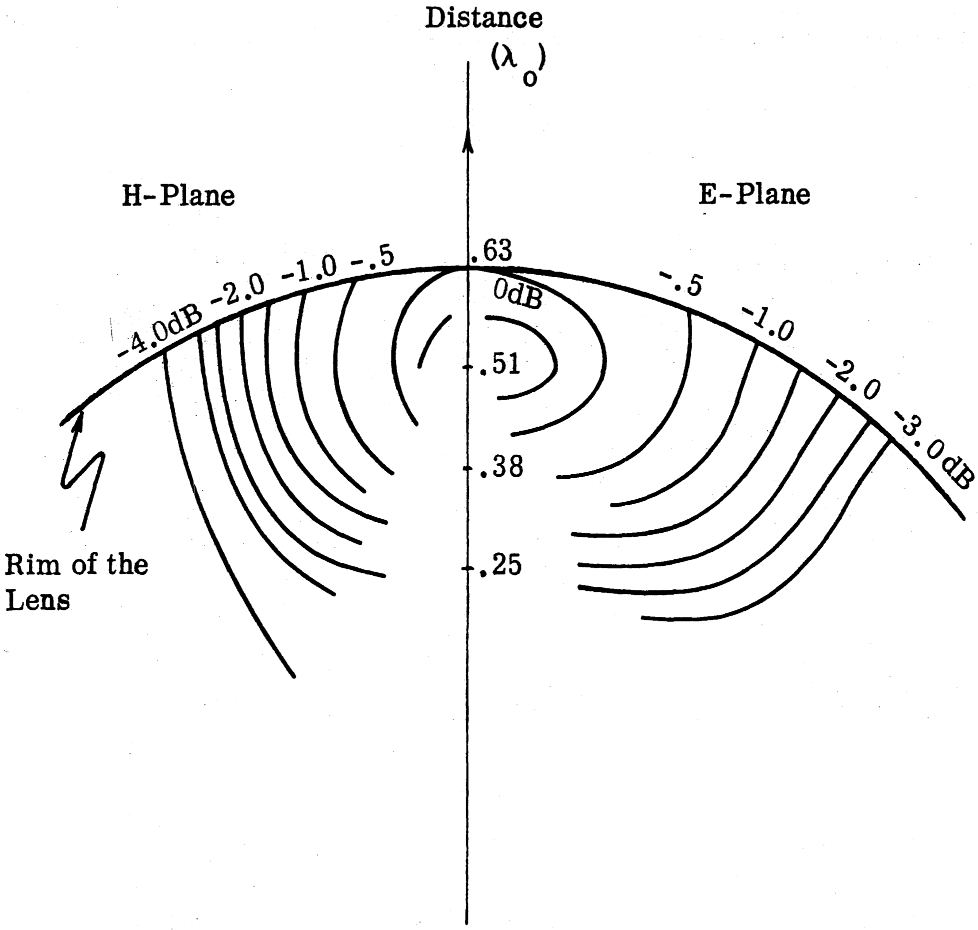

Figure 25: Interior Field Around Focus of the Luneburg Lens (D = 1.27 λ_o)

V

CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE WORK

In the first part of this work, dyadic Green's functions for electric dipole excitation for rectangular, cylindrical and spherical cavities are constructed. They can be used to formulate various boundary value problems involving these cavities. The dyadic Green's functions for magnetic dipole excitation are introduced later. The construction of solenoidal dyadic Green's function involves the solenoidal vector wave functions, while the construction of non-solenoidal dyadic Green's functions involves irrotational as well as solenoidal vector wave functions. Expressions for the electric and magnetic fields inside the cavity using dyadic Green's functions are presented and an example of the application of the formulas for the case of input admittance of the rectangular waveguide is shown.

In the second part of the work, we deal with inhomogeneous spherical lenses, including the Luneburg, Eaton-Lippmann and the Eaton lenses. It is shown that the duality between the dyadic Green's functions of the electric and magnetic types for the electric and magnetic dipole excitations which exists in free-space does not hold in an inhomogeneous medium. By the method of scattering superposition, the dyadic Green's functions for electric and magnetic dipole excitation in the presence of an inhomogeneous scatterer are constructed. An expression for the electric field generated by the Huygens source in the presence of the lenses is found. The complete electromagnetic theory of the Eaton-Lippmann lens is worked out. It is shown that the S- and T-functions for the Eaton-Lippmann lens, like those of the Luneburg lens, are represented by the confluent hypergeometric and the "generalized" confluent hypergeometric functions. Some of the properties of these functions are carefully examined.

The radiation patterns for small-diameter lenses excited by a Huygens source are computed and plotted. Bistatic scattering cross-sections of the lenses with plane wave incidence are shown for $\rho_a = 5$ and $\rho_a = 10$.

It is observed that as the diameter of the lenses increases the forward bistatic scattering cross-section of the different lenses tends to the same value. The backward bistatic scattering cross-section of the Eaton-Lippmann lens tends to its geometric cross-section.

An investigation of the field around the geometrical optics focus of the Luneburg lens is made. It is found that for small-diameter lenses, the focal point is inside the lens and it tends to the rim of the lens as its diameter increases.

The directivity of the Luneburg lens is computed and compared to the directivity of the homogeneous lens. It is found that the directivity of small-diameter Luneburg lenses is in general lower than that of the homogeneous lenses. The directivity of the homogeneous lenses deteriorates while the directivity of the Luneburg lenses improves as their diameter increases. Also, the Luneburg lens does not present the phenomenon of resonance which is present in the homogeneous lenses and, therefore, the Luneburg lens is a more frequency-independent antenna than the homogeneous lens.

The low directivity of the Luneburg lens as compared to the homogeneous sphere could be accounted for by the fact that its "focus" is located inside the lens and not on the rim. Thus, when the source is located away from the focal point the lens certainly will not function properly from the point of view of geometrical optics.

Our recommendations for the future work would be:

1. Find the dyadic Green's functions in a spherically inhomogeneous medium;

2. Study the characteristics, including the directivity of the generalized small-diameter Luneburg lens designed to have its focus outside the lens. The source should be located on the rim;

3. Study the characteristics, including the directivity, of the small-diameter Luneburg lens with the source located inside the lens;

4. Develop the asymptotic theory of Luneburg lens.

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APPENDIX

A.1 Computation of PQ' - QP'

$PQ' - QP' = \rho_a j_n(\rho_a)\left[\rho_a h_n^{(1)}(\rho_a)\right]' - \rho_a h_n^{(1)}(\rho_a)\left[\rho_a j_n(\rho_a)\right]'$. Performing the differentiation, recalling that $h_n^{(1)}(\rho_a) = j_n(\rho_a) + i y_n(\rho_a)$ and collecting the terms, we have

$$PQ' - QP' = i\rho_a^2\left[j_n(\rho_a)\, y_n'(\rho_a) - y_n(\rho_a)\, j_n'(\rho_a)\right].$$

But

$$j_n(\rho_a) = \sqrt{\frac{\pi}{2\rho_a}}\, J_{n+1/2}(\rho_a)$$

and

$$y_n(\rho_a) = \sqrt{\frac{\pi}{2\pi_a}}\, Y_{n+1/2}(\rho_a) .$$

Therefore,

$$PQ'-QP' = i\rho_a^2\left[\sqrt{\frac{\pi}{2\rho_a}}\, J_{n+1/2}(\rho_a)\left(\sqrt{\frac{\pi}{2\rho_a}}\, Y_{n+1/2}(\rho_a)\right)' - \sqrt{\frac{\pi}{2\rho_a}}\, Y_{n+1/2}(\rho_a)\left(\sqrt{\frac{\pi}{2\rho_a}}\, J_{n+1/2}(\rho_a)\right)'\right] .$$

Performing a new differentiation and collecting the terms, we have

$$PQ'-QP' = \frac{i\pi}{2}\rho_a\left[J_{n+1/2}(\rho_a)\, Y'_{n+1/2}(\rho_a) - Y_{n+1/2}(\rho_a)\, J'_{n+1/2}(\rho_a)\right] .$$

From Dettman (1969), we have

$$x\left[J'_m(x)\, Y_m(x) - Y'_m(x)\, J_m(x)\right] = -\frac{2}{\pi}$$

which, when applied to our case, gives

$PQ' - QP' = i$

A.2 Legendre Functions at $\theta = 0$ and $\theta = \pi$

From Jahnke and Emde (1945), we have

$$P_n^m(x) = \frac{(n+m)!}{2^m\, m!\, (n-m)!} (1 - x^2)^{m/2} \left\{ 1 - \frac{(n-m)\,(n+m+1)}{1\,(m+1)} \left(\frac{1-x}{2}\right) + \right.$$

$$\left. + \frac{(n-m)\,(n-m-1)\,(n+m+1)\,(n+m+2)}{1,\ 2,\ (m+1)\,(m+2)} \left(\frac{1-x}{2}\right)^2 + \ldots\ldots \right\}$$

where $x = \cos\theta$.

For $m = 1$, we obtain

$$P_n^1(x) = \frac{(n+1)!}{2.\ 1.\ (n-1)!} (1-x^2)^{1/2} \left\{ 1 - \frac{(n-1)\,(n+2)}{1.2} \left(\frac{1-x}{2}\right) + \right.$$

$$\left. + \frac{(n-1)\,(n-2)\,(n+2)\,(n+3)}{1.\ 2.\ 2.\ 3} \left(\frac{1-x}{2}\right)^2 + \ldots\ldots \right\} .$$

For $\theta = 0$, $x = \cos\theta = 1$; therefore,

$$\left.\frac{P_n^1(x)}{(1-x^2)^{1/2}}\right|_{x=1} = \left.\frac{P_n^1(\cos\theta)}{\sin\theta}\right|_{\theta=0} = \frac{n(n+1)}{2} \left\{ 1 - \frac{(n-1)(n+2)}{2}\, 0 + \right.$$

$$\left. + \frac{(n-1)\,(n-2)\,(n+2)\,(n+3)}{1,\ 2,\ 2,\ 3}\, 0 + \ldots\ldots \right\}$$

or

$$\left.\frac{P_n^1(\cos\theta)}{\sin\theta}\right|_{\theta=0} = \frac{n(n+1)}{2} . \qquad (A.1)$$

Also, we know (see Jahnke and Emde, 1945) that

$$P_n^m(-x) = (-1)^{n+m}\, P_n^m(x)$$

which in the case of $m = 1$, becomes

$$P_n^1(-x) = (-1)^{n+1} P_n^1(x) .$$

Therefore,

$$P_n^1(\cos 180) = (-1)^{n+1} P_n^1(\cos\theta)$$

or

$$\left. \frac{P_n^1(\cos\theta)}{\sin\theta} \right|_{\theta=\pi} = (-1)^{n+1} \left. \frac{P_n^1(\cos\theta)}{\sin\theta} \right|_{\theta=0} .$$

Substituting in it (A. 1), we finally get

$$\left. \frac{P_n^1(\cos\theta)}{\sin\theta} \right|_{\theta=\pi} = (-1)^{n+1} \frac{n(n+1)}{2} . \qquad \text{(A. 2)}$$

From Stratton (1941), we have

$$(1-x^2) \frac{dP_n^m(x)}{dx} = (n+1) x P_n^m(x) - (n - m + 1) P_{n+1}^m(x)$$

where $x = \cos\theta$;

or

$$\sin^2\theta \frac{dP_n^m(\cos\theta)}{d(\cos\theta)} = (n+1) \cos\theta P_n^m(\cos\theta) - (n - m + 1) P_{n+1}^m(\cos\theta) .$$

For $m = 1$,

$$\sin^2\theta \frac{dP_n^1(\cos\theta)}{d(\cos\theta)} = (n+1) \cos\theta P_n^1(\cos\theta) - n P_{n+1}^1(\cos\theta) ,$$

or

$$- \sin\theta \frac{dP_n^1(\cos\theta)}{d\theta} = (n+1)\cos\theta\, P_n^1(\cos\theta) - n\, P_{n+1}^1(\cos\theta) \quad .$$

Thus,

$$\frac{d\, P_n^1(\cos\theta)}{d\theta} = n\, \frac{\overline{P}_{n+1}^1(\cos\theta)}{\sin\theta} - (n+1)\cos\theta\, \frac{P_n^1(\cos\theta)}{\sin\theta} \quad .$$

At $\pi = 0$, we have

$$\left.\frac{dP_n^1(\cos\theta)}{d\theta}\right|_{\theta=0} = n \left.\frac{P_{n+1}^1(\cos\theta)}{\sin\theta}\right|_{\theta=0} - (n+1) \left.\frac{P_n^1(\cos)\theta)}{\sin\theta}\right|_{\theta=0} \quad .$$

Substituting (A. 1) in the above, we get

$$\left.\frac{dP_n^1}{d\theta}\right|_{\theta=0} = \frac{n(n+1)(n+2)}{2} - \frac{(n+1)(n+1)\, n}{2}$$

or, finally,

$$\left.\frac{dP_n^1(\cos\theta)}{d\theta}\right|_{\theta=0} = \frac{n(n+1)}{2} \quad . \qquad \text{(A. 3)}$$

At $\theta = \pi$, we have

$$\left.\frac{dP_n^1(\cos\theta)}{d\theta}\right|_{\theta=\pi} = n \left.\frac{P_{n+1}^1(\cos\theta)}{\sin\theta}\right|_{\theta=\pi} - (n+1)(-1) \left.\frac{P_n^1(\cos\theta)}{\sin\theta}\right|_{\theta=\pi}$$

and substituting (A. 2) in it, we get

$$\left.\frac{dP_n^1(\cos\theta)}{d\theta}\right|_{\theta=\pi} = (-1)^n \, \frac{n(n+1)}{2} \, . \qquad \text{(A.4)}$$

A.3 Simplification of $-\int_V \nabla \times \bar{J} \cdot \bar{\bar{G}}_{e2} \, dv + \int_S \hat{n} \times \bar{J} \cdot \bar{\bar{G}}_{e2} \, dS$.

To prove

$$-\int_V \nabla \times \bar{J} \cdot \bar{\bar{G}}_{e2} dv + \int_S \hat{n} \times \bar{J} \cdot \bar{\bar{G}}_{e2} \, ds = -\int_V \bar{J} \cdot \nabla \times \bar{\bar{G}}_{e2} \, dv \, .$$

Consider the surface integral in the above. The mixed product can be changed (see Van Bladel, 1964) such that

$$\int_S \hat{n} \times \bar{J} \cdot \bar{\bar{G}}_{e2} \, ds = \int_S \hat{n} \cdot (\bar{J} \times \bar{\bar{G}}_{e2}) \, ds \, .$$

The right hand side of this equality can be transformed into a volume integral (Van Bladel, 1964)

$$\int_S \hat{n} \cdot (\bar{J} \times \bar{\bar{G}}_{e2}) \, ds = \int_V \nabla \cdot (\bar{J} \times \bar{\bar{G}}_{e2}) \, dv$$

or

$$\int_S \hat{n} \times \bar{J} \cdot \bar{\bar{G}}_{e2} \, ds = \int_V \nabla \cdot (\bar{J} \times \bar{\bar{G}}_{e2}) \, dv \quad .$$

Therefore,

$$-\int_V \nabla \times \bar{J} \cdot \bar{\bar{G}}_{e2} dv + \int_S \hat{n} \times \bar{J} \cdot \bar{\bar{G}}_{e2} \, ds = -\int_V \nabla \times \bar{J} \cdot \bar{\bar{G}}_{e2} dv + \int_V \nabla \cdot (\bar{J} \times \bar{\bar{G}}_{e2}) \, dv \, .$$

But,

$$\int_V \nabla \cdot (\bar{J} \times \bar{\bar{G}}_{e2})\, dv = \int_V \nabla \times \bar{J} \cdot \bar{\bar{G}}_{e2}\, dv - \int_V \bar{J} \cdot \nabla \times \bar{\bar{G}}_{e2}\, dv$$

and substituting in the above, we have

$$-\int_V \nabla \times \bar{J} \cdot \bar{\bar{G}}_{e2} dv + \int_S \hat{n} \times \bar{J} \cdot \bar{\bar{G}}_{e2} ds = -\int_V \nabla \times \bar{J} \cdot \bar{\bar{G}}_{e2} dv + \int_V \nabla \times \bar{J} \cdot \bar{\bar{G}}_{e2} dv -$$

$$- \int_V \bar{J} \cdot \nabla \times \bar{\bar{G}}_{e2}\, dv \quad .$$

Cancelling out similar terms, we finally get

$$- \int_V \nabla \times \bar{J} \cdot \bar{\bar{G}}_{e2}\, dv + \int_S \hat{n} \times \bar{J} \cdot \bar{\bar{G}}_{e2}\, ds = - \int_V \bar{J} \cdot \nabla \times \bar{\bar{G}}_{e2}\, dv \ .$$