Diffraction tensors for imperfectly conducting edges

T. B. A. Senior

Radiation Laboratory, The University of Michigan

Ann Arbor, Michigan 48104

The imperfectly conducting edges considered are those of nonmetallic, resistive and "conductive" half planes illuminated by a plane electromagnetic wave at arbitrary (oblique) incidence. By expressing the boundary conditions at the surface in terms of the normal components of the field and their normal derivatives, each problem is reduced to a combination of two scalar ones analogous to those for a plane wave incident in a plane perpendicular to the edge. From the explicit and exact solutions thus obtained, the edge diffracted fields are derived and expressed in terms of edge diffraction tensors. The tensors are relatively simple and compact. Their computation is discussed and data are presented to illustrate the behavior of the field.

1. INTRODUCTION

With the growing interest in the way that the material properties of a body affect its electromagnetic scattering, it is desirable to investigate those simple nonmetallic bodies for which the exact solution of the scattering problem can be found. An important problem in this category is the imperfectly conducting half plane and a knowledge of its edge diffraction coefficient is a necessary part of any GTD analysis of edged structures. In a recent paper [Senior, 1975] we considered the coefficient for nonmetallic, resistive and "conductive" half planes when the plane wave is incident in a plane perpendicular to the edge, and we now extend this treatment to the more general case of oblique (or skew) incidence.

In contrast to the perfectly conducting case where the problem of oblique incidence is only a simple extension of that for normal incidence, the nonmetallic half plane is made difficult by the boundary conditions imposed. It is not obvious that for oblique incidence the problem is other than an essentially vector one, and an attempt to duplicate the Wiener-Hopf analysis of Senior [1952] for normal incidence leads to coupled integral equations for the components of the electric and magnetic currents induced in the half plane. Nevertheless, these equations can be decoupled by considering certain combinations of the currents and their derivatives [Senior, 1959], and from the resulting solutions, the complete expressions for the field components are obtained. As Williams [1960] observed, this entire process can be simplified by concentrating on the corresponding components of the field from the outset, since these are the ones for which the boundary conditions separate. Though not explicitly pointed out by Williams, the required components are those of the electric and magnetic vectors normal to the surface.

For planar surfaces parallel to a coordinate surface in a Cartesian coordinate system, the impedance boundary condition can be reduced to two normal derivative conditions each involving the normal component of either the electric or magnetic field, thereby converting the vector problem to two scalar ones for these components. With this simplification, the case of oblique incidence on a nonmetallic half plane is not substantially more difficult than normal incidence, and this property has been exploited by Bucci and Franceschetti [1975] in extending Maliuzhinets' [1959] treatment of a half plane with different face impedances to oblique incidence to study the surface wave excitation.

In considering here the special case in which the face impedances are the same, the aim is to obtain the edge diffraction tensor not only for a nonmetallic surface but also for resistive and conductive sheets, and to reduce the expressions for the tensor elements to forms amenable to computation. Although the first part of the analysis could be shortened by picking up the necessary expressions from Senior [1959], it is in fact simpler and more instructive to do the problem from the beginning.

2. FORMULATION

An impedance (or Leontovich) boundary condition is widely used to approximate a two-region electromagnetic problem. Its validity has been discussed by Senior [1960] and in particular, for an imperfectly conducting or nonmetallic surface of large conductivity,

$$\underline{E} - (\hat{n}\cdot\underline{E})\hat{n} = \eta Z \hat{n} \wedge \underline{H} \tag{1}$$

where $\hat{n}$ is the unit vector outward normal and η is the surface impedance relative to the intrinsic impedance Z of free space.

The condition (1) is an essentially vector one connecting perpendicular components of the electric and magnetic fields at the surface, but an alternative form of the condition is

$$\frac{\partial E_n}{\partial n} + ik\eta E_n = 0 \quad , \qquad \frac{\partial H_n}{\partial n} + \frac{ik}{\eta} H_n = 0 \tag{2}$$

where a time factor $e^{-i\omega t}$ has been assumed and suppressed. Each (scalar) equation now involves only a single component of the electromagnetic field and in this sense the conditions (2) have the advantage of being scalar. The equivalence of the normal derivative and impedance boundary conditions (2) and (1) is implicit in the work of Grünberg [1943] and was demonstrated by Senior [1960] by a process of tangential differentiation applied to (1), but it should be noted that the equivalence holds only for parallel planar surfaces in a Cartesian coordinate system. For this reason, the diffraction of a plane wave at oblique incidence on a wedge is a very different and more difficult problem than the half plane one considered here.

The first of the half planes we shall treat is a nonmetallic one at each face of which the boundary condition is, by definition, (1). As pointed out by Senior [1975], such an infinitesimally thin structure is equivalent to a combination of electric and magnetic current sheets and if the half plane is defined by the equation $y = 0$, $x > 0$ where x, y, z are Cartesian coordinates, the total induced electric and magnetic currents $\underline{J}$ and $\underline{J}^*$ respectively are such that

$$\underline{J}(x, z) = -(\eta Z)^{-1} \hat{y} \wedge \left\{ \hat{y} \wedge \left[\underline{E}(x, +0, z) + \underline{E}(x, -0, z) \right] \right\} \tag{3}$$

$$\underline{J}^*(x, z) = -\eta Z \hat{y} \wedge \left\{ \hat{y} \wedge \left[\underline{H}(x, +0, z) + \underline{H}(x, -0, z) \right] \right\} \tag{4}$$

These are the actual conditions employed by Senior in his analysis for a plane wave at normal incidence, but when the wave is incident obliquely, it is more convenient to turn to the normal derivative forms (2).

For a surface lying in the plane $y = 0$, the equations (2) imply

$$\frac{\partial E_y}{\partial y} \pm ik\eta E_y = 0 \quad , \qquad \frac{\partial H_y}{\partial y} \pm \frac{ik}{\eta} H_y = 0 \tag{5}$$

where the upper and lower signs refer to the upper and lower surfaces $y = \pm 0$ respectively. The existence of an electric current sheet produces discontinuities in $\partial H_y/\partial y$ and E_y across it and if

$$\left.\frac{\partial H_y}{\partial y}\right|_{y=-0}^{y=+0} = f_1(x, z) \ , \qquad \left. E_y \right|_{y=-0}^{y=+0} = f_2(x, z) \tag{6}$$

the addition and subtraction of the appropriate equations 5 give

$$\begin{aligned} f_1(x, z) &= -\frac{ik}{\eta}\left[H_y(x, +0, z) + H_y(x, -0, z)\right] \\ f_2(x, z) &= \frac{i}{k\eta}\left\{\left[\frac{\partial E_y}{\partial y}\right]_{y=+0} + \left[\frac{\partial E_y}{\partial y}\right]_{y=-0}\right\} \end{aligned} \tag{7}$$

These derived forms of the normal derivative conditions (5) are the equivalent of (3). In exactly the same way, a magnetic current sheet produces discontinuities in $\partial E_y/\partial y$ and H_y if

$$\left.\frac{\partial E_y}{\partial y}\right|_{y=-0}^{y=+0} = -f_1^*(x, z) \ , \qquad \left. H_y \right|_{y=-0}^{y=+0} = f_2^*(x, z) \tag{8}$$

we have

$$\begin{aligned} f_1^*(x, z) &= ik\eta\left[E_y(x, +0, z) + E_y(x, -0, z)\right] \\ f_2^*(x, z) &= \frac{i\eta}{k}\left\{\left[\frac{\partial H_y}{\partial y}\right]_{y=+0} + \left[\frac{\partial H_y}{\partial y}\right]_{y=-0}\right\} \end{aligned} \tag{9}$$

and these conditions are the equivalent of (4). Just as f_1 and f_2 are associated with the electric current, so f_1^* and f_2^* are related to the magnetic current induced in the half plane.

An electrically resistive half plane is characterized by a jump discontinuity in the tangential components of the magnetic field but no discontinuity in the tangential electric field. It therefore supports only a total electric current $\underline{J}$ and for a half plane having a constant surface impedance R ohms,

$$\underline{J}(x, z) = -\frac{1}{R}\hat{y}\wedge\left[\hat{y}\wedge\underline{E}(x, 0, z)\right]$$

$$\underline{J}^*(x, z) = -\hat{y}\wedge\left[\underline{E}(x, +0, z) - \underline{E}(x, -0, z)\right] = 0 \qquad (10)$$

The continuity of $\hat{y}\wedge\underline{E}$ implies that $\partial E_y/\partial y$ and H_y are continuous also, and hence $f_1^* = f_2^* = 0$, but since the electric current is identical to that in (3) with $\eta = 2R/Z$, it follows that f_1 and f_2 must be the same as those in (7) with this identification of η. It is therefore a trivial matter to deduce the solution for a resistive half plane from that for a nonmetallic one by suppressing all portions of the solution attributable to f_1^* and f_2^* and putting $\eta = 2R/Z$ in the rest.

The third type of half plane is a "magnetically conductive" one having a constant conductivity R^* mhos. This is the electromagnetic dual of a resistive half plane and creates a jump discontinuity in the tangential components of the electric field, but none in the tangential magnetic field. Since E_y and $\partial H_y/\partial y$ are now continuous through the surface, $f_1 = f_2 = 0$, whereas the functions f_1^* and f_2^* associated with the magnetic current are identical to those in (9) with $\eta = 1/(2R^*Z)$. It follows that a conductive half plane is also a special case of a nonmetallic one and it is therefore sufficient to devote most of our attention to the latter.

3. THE INTEGRAL EQUATIONS AND THEIR SOLUTIONS

We consider an incident plane wave having

$$E_y^i = e_y e^{ik\hat{i}\cdot\underline{r}} \quad , \qquad ZH_y^i = h_y e^{ik\hat{i}\cdot\underline{r}} \tag{11}$$

where the unit vector specifying the direction of propagation is

$$\hat{i} = -\hat{x}\sin\beta\cos\phi_0 - \hat{y}\sin\beta\sin\phi_0 + \hat{z}\cos\beta$$

In terms of the normal components (11),

$$E_x^i = (1 - \sin^2\beta\sin^2\phi_0)^{-1}(-\sin^2\beta\sin\phi_0\cos\phi_0 e_y + \cos\phi_0 h_y)$$

$$E_z^i = (1 - \sin^2\beta\sin^2\phi_0)^{-1}\sin\beta(\cos\beta\sin\phi_0 e_y + \cos\phi_0 h_y)$$

The angle β is a measure of the obliquity with $\beta = \pi/2$ corresponding to incidence in a plane perpendicular to the edge, and without loss of generality it can be assumed that $0 \leq \beta,\ \phi_0 \leq \pi$.

The wave is incident on a nonmetallic half plane having the constant surface impedance η and occupying the region $x \geq 0$ of the Cartesian coordinate plane $y = 0$. Since the surface and the boundary conditions are independent of the coordinate z, it follows that the entire field must have the same z dependence as the incident field (11). In particular

$$f_1(x,z) = f_1(x)\,e^{ikz\cos\beta} \quad , \qquad f_2(x,z) = f_2(x)\,e^{ikz\cos\beta}$$

and we shall henceforth omit the z dependent factor. If we also define

$$\kappa = k\sin\beta \tag{12}$$

application of the scalar Green's theorem to the component H_y gives

$$H_y(x,y) = H_y^i(x,y) - \frac{i}{4}\int_0^\infty \left[f_1(x') + f_2^*(x')\frac{\partial}{\partial y}\right] H_0^{(1)}\left\{\kappa\left[(x'-x)^2 + y^2\right]^{1/2}\right\} dx' \qquad (13)$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order zero.

We can derive an integral equation for $f_1(x)$ by taking the limits as $y \to 0$ from above and below and adding the results. The first of the boundary conditions (7) then shows that

$$-\eta \sin\beta \, Z f_1(x) = 2i\kappa h_y \exp(-i\kappa x \cos\phi_0) + \frac{\kappa Z}{2}\int_0^\infty f_1(x') H_0^{(1)}(\kappa|x'-x|)dx'$$

valid for $0 \leq x \leq \infty$, and this differs from the integral equation (11) of Senior [1952] only in the form of the inhomogeneous term and the replacement of k by κ and η by $\eta\sin\beta$. We can therefore obtain the solution by the Wiener-Hopf method and if

$$\bar{f}_1(\zeta) = \int_0^\infty f_1(x) e^{-i\zeta x} dx$$

is the Fourier transform of $f_1(x)$, zero for $x < 0$,

$$Z\bar{f}_1(\zeta) = -(2/\pi)^{1/2}\left[\frac{\kappa h_y}{\zeta + \kappa\cos\phi_0} K_+(-\kappa\cos\phi_0) + A\right] K_+(-\zeta) \qquad (14)$$

where $K_+(\zeta)$ is a "split" function analytic in an upper half ζ plane and satisfying

$$K_+(\zeta)K_+(-\zeta) = \left[\eta\sin\beta + \frac{\kappa}{(\kappa^2 - \zeta^2)^{1/2}}\right]^{-1} \qquad (15)$$

An expression for $K_+(\zeta)$ can be deduced from (21) of Senior [1952] by the same substitutions as before. In contrast to the situation for normal incidence, the analytic function appearing in the Wiener-Hopf analysis cannot be proven zero, and the term involving the constant A in (14) corresponds to a solution of the homogeneous integral equation.

To obtain an integral equation for $f_2^*(x)$, (13) is differentiated with respect to y prior to summing the limits as $y \to \pm 0$. The result is

$$Zf_2^*(x) = 2\eta \sin\beta \sin\phi_0 h_y \exp(-i\kappa x \cos\phi_0) + \frac{\eta \sin\beta\, Z}{\kappa}\left(\lim_{y\to +0} + \lim_{y\to -0}\right)$$

$$\int_0^\infty f_2^*(x') \frac{\partial^2}{\partial y^2} H_0^{(1)}\left\{\kappa\left[(x'-x)^2 + y^2\right]^{1/2}\right\} dx'$$

valid for $0 \leq x \leq \infty$, and this is directly analogous to (12) of Senior [1952]. Its solution is therefore

$$Z\bar{f}_2^*(\zeta) = -i(2/\pi)^{1/2} \frac{\eta \sin\beta}{\kappa} \left[\frac{\kappa h_y}{\zeta + \kappa\cos\phi_0} K_+(-\kappa\cos\phi_0) + B^*\right]\left[\frac{\kappa(1+\cos\phi_0)}{\kappa - \zeta}\right]^{1/2} K_+(-\zeta) \tag{16}$$

where B^* is a further constant.

The functions $f_1(x)$ and $f_2^*(x)$ can be found from (14) and (16) by applying the inverse Fourier transform, and H_y is then given by (13). An alternative and more convenient approach is to write the scattered field as

$$H_y^s(x,y) = -\frac{i}{4}(2/\pi)^{1/2} \int_{-\infty}^{\infty} \left[\bar{f}_1(\zeta) + i\frac{y}{|y|}\Gamma \bar{f}_2^*(\zeta)\right] \exp\left[i(x\zeta + |y|\Gamma)\right] \frac{d\zeta}{\Gamma}$$

where $\Gamma = (\kappa^2 - \zeta^2)^{1/2}$ and the path of integration is indented above the branch point at $\zeta = -\kappa$, but below the pole at $\zeta = -\kappa\cos\phi_0$ and the branch point at $\zeta = \kappa$. On inserting the expressions for $\bar{f}_1$ and $\bar{f}_2^*$ we have

$$Z H_y^s(x, y) = \frac{i}{2\pi} \int_{-\infty}^{\infty} F(\zeta) \exp\left[i(x\zeta + |y|\Gamma)\right] \frac{d\zeta}{\Gamma} \tag{17}$$

with

$$F(\zeta) = \left[\left[\frac{\kappa h_y}{\zeta + \kappa\cos\phi_0} \left\{1 - \frac{y}{|y|} \frac{\eta\sin\beta}{\kappa} \left[\kappa(\kappa+\zeta)(1 + \cos\phi_0)\right]^{1/2}\right\} K_+(-\kappa\cos\phi_0) \right.\right.$$
$$\left.\left. + A - \frac{y}{|y|} \frac{\eta\sin\beta}{\kappa} \left[\kappa(\kappa+\zeta)(1 + \cos\phi_0)\right]^{1/2} B^*\right]\right] K_+(-\zeta) \tag{18}$$

The functions $f_1^*(x)$ and $f_2(x)$ and, hence, the scattered electric field component E_y^s can be determined in a similar manner, but we can also obtain them directly from the above by invoking the duality of the electromagnetic field. If η is replaced by $1/\eta$ and all electric or unstarred (magnetic or starred) quantities by the corresponding magnetic or starred (minus electric or unstarred) quantities, (14) and (16) give

$$\bar{f}_1^*(\zeta) = (2/\pi)^{1/2} \left[\frac{\kappa e_y}{\zeta + \kappa\cos\phi_0} L_+(-\kappa\cos\phi_0) - A^*\right] L_+(-\zeta) \tag{19}$$

$$\bar{f}_2(\zeta) = -i(2/\pi)^{1/2} \frac{\sin\beta}{\eta\kappa} \left[\frac{\kappa e_y}{\zeta + \kappa\cos\phi_0} L_+(-\kappa\cos\phi_0) + B\right] \left[\frac{\kappa(1 + \cos\phi_0)}{\kappa - \zeta}\right]^{1/2} L_+(-\zeta) \tag{20}$$

where $L_+(\zeta)$ differs from $K_+(\zeta)$ only in having η replaced by $1/\eta$. It is therefore the dual of $K_+(\zeta)$ and A^* and B are likewise the duals of the constants A and B^*. In terms of $\bar{f}_1^*$ and $\bar{f}_2$,

$$E_y^s(x,y) = \frac{i}{4}(2/\pi)^{1/2}\int_{-\infty}^{\infty}\left[\bar{f}_1^*(\zeta) - i\frac{y}{|y|}\Gamma\bar{f}_2(\zeta)\right]\exp\left[i(x\zeta + |y|\Gamma)\right]d\zeta/\Gamma$$

which can be written as

$$E_y^s(x,y) = -\frac{i}{2\pi}\int_{-\infty}^{\infty}F^*(\zeta)\exp\left[i(x\zeta + |y|\Gamma)\right]d\zeta/\Gamma \tag{21}$$

where

$$F^*(\zeta) = \Bigg[-\frac{\kappa e_y}{\zeta+\kappa\cos\phi_0}\left\{1 - \frac{y}{|y|}\frac{\sin\beta}{\eta\kappa}\left[\kappa(\kappa+\zeta)(1+\cos\phi_0)\right]^{1/2}\right\}L_+(-\kappa\cos\phi_0)$$

$$+ A^* + \frac{y}{|y|}\frac{\sin\beta}{\eta\kappa}\left[\kappa(\kappa+\zeta)(1+\cos\phi_0)\right]^{1/2}B\Bigg]L_+(-\zeta) \tag{22}$$

and is the dual of $F(\zeta)$.

Equations (17) and (21) are sufficient to determine all of the scattered field components for a nonmetallic half plane and they also yield the solutions for resistive and conductive sheets. If the half plane is a resistive one, no magnetic current is supported: $\bar{f}_1^*$ and $\bar{f}_2^*$ are then zero, and $\bar{f}_1$ and $\bar{f}_2$ are as shown in (14) and (20) with $\eta = 2R/Z$. If the half plane is conductive, $\bar{f}_1$ and $\bar{f}_2$ are zero, and $\bar{f}_1^*$ and $\bar{f}_2^*$ are as shown in (19) and (16) with $\eta = (2R^* Z)^{-1}$.

4. THE FIELD COMPONENTS AND THE CONSTANTS

The tangential components of the scattered field can be expressed in terms of the normal components using Maxwell's equations. Since

$$\begin{aligned}\left(k^2+\frac{\partial^2}{\partial y^2}\right)E_x^s(x,y) &= k^2\cos\beta\, Z H_y^s(x,y)+\frac{\partial^2}{\partial x\partial y}E_y^s(x,y)\\ \left(k^2+\frac{\partial^2}{\partial y^2}\right)E_z^s(x,y) &= ik\left[Z\frac{\partial}{\partial x}H_y^s(x,y)+\cos\beta\frac{\partial}{\partial y}E_y^s(x,y)\right]\end{aligned} \tag{23}$$

with similar formulas for the magnetic field, insertion of the expressions (17) and (21) gives

$$E_x^s(x,y)=\frac{i}{2\pi}\int_{-\infty}^{\infty}\left[k^2\cos\beta\,F(\zeta)+\frac{y}{|y|}\zeta\Gamma F^*(\zeta)\right]\frac{\exp\left[i(x\zeta+|y|\Gamma)\right]}{\zeta^2+\kappa^2\cot^2\beta}\frac{d\zeta}{\Gamma} \tag{24}$$

$$E_z^s(x,y)=\frac{i}{2\pi}\int_{-\infty}^{\infty}k\left[-\zeta F(\zeta)+\frac{y}{|y|}\Gamma F^*(\zeta)\right]\frac{\exp\left[i(x\zeta+|y|\Gamma)\right]}{\zeta^2+\kappa^2\cot^2\beta}\frac{d\zeta}{\Gamma} \tag{25}$$

The differentiation and integration necessary to obtain these has not increased the edge singularity but it has introduced a factor $\zeta^2+\kappa^2\cot^2\beta$ into the denominator of the integrands. In the subsequent evaluation of the integrals one or other of the poles at $\zeta=\pm i\cot\beta$ may be captured, and since these give rise to surface waves which grow exponentially as $x\to\mp\infty$ and thereby violate the radiation condition, it is necessary that their residues be zero. On enforcing this requirement, (24) and (25) individually demand

$$F(i\kappa\cot\beta) + i\frac{y}{|y|}F^*(i\kappa\cot\beta) = 0 \tag{26}$$

$$F(-i\kappa\cot\beta) - i\frac{y}{|y|}F^*(-i\kappa\cot\beta) = 0 \tag{27}$$

and these are also the equations that result from a consideration of the magnetic field components H_x^s and H_z^s.

Equations (26) and (27) relate the four constants A, B, A^* and B^*, but since they must be satisfied for y negative as well as positive, they are sufficient to specify them all. After some rather tedious algebra we find

$$A = -\frac{\sin\beta\cos\beta}{1-\sin^2\beta\sin^2\phi_0}\Bigg\{\left[\tan\beta\cos\phi_0 - \tan\left(\frac{\pi}{4}-\frac{\beta}{2}+\gamma\right)\right]K_+(-\kappa\cos\phi_0)h_y$$

$$+\left[\frac{1}{\eta}\sin\beta(1+\cos\phi_0)\right]^{1/2}\sec\left(\frac{\pi}{4}-\frac{\beta}{2}+\gamma\right)L_+(-\kappa\cos\phi_0)e_y\Bigg\} \tag{28}$$

$$B = -\frac{\sin\beta\cos\beta}{1-\sin^2\beta\sin^2\phi_0}\Bigg\{\left[\tan\beta\cos\phi_0 + \tan\left(\frac{\pi}{4}-\frac{\beta}{2}+\gamma\right)\right]L_+(-\kappa\cos\phi_0)e_y$$

$$-\left[\frac{1}{\eta}\sin\beta(1+\cos\phi_0)\right]^{-1/2}\sec\left(\frac{\pi}{4}-\frac{\beta}{2}+\gamma\right)K_+(-\kappa\cos\phi_0)h_y\Bigg\} \tag{29}$$

where we have written

$$\eta^{1/2}\frac{K_+(i\kappa\cot\beta)}{L_+(i\kappa\cot\beta)} = e^{i\gamma} \tag{30}$$

The expressions for A^* and B^* follow from the above using the duality relations $e_y \to h_y$, $h_y \to -e_y$ and $\eta \to 1/\eta$ implying $\gamma \to -\gamma$. Of course, for a resistive half plane the constants A^* and B^* do not occur but A and B are still given by (28) and (29) with $\eta = 2R/Z$; and for a conductive half plane we have only A^* and B^*.

The four constants specified above are equivalent to the A, B, C, D given in equations 29, 30, 34 and 35 of Senior [1959] when the erroneous multiplying radical is deleted from the right hand side of (35). Each is related to the normal components of the incident electric and magnetic fields and the corresponding terms are proportional to the same split functions that characterize the scattering when the plane wave is incident in a plane perpendicular to the edge. The remaining factors are almost entirely trigonometric, but these are complicated by the presence of the function $\gamma = \gamma(\beta, \eta)$, and we come back to this in Section 6.

5. EDGE DIFFRACTION TENSOR

An expression for the normal component of the scattered magnetic field is given in (17) and in terms of the cylindrical polar coordinates ρ, ϕ with $x = \rho\cos\phi$ $y = \rho\sin\phi$, the substitution $\zeta = \kappa\cos\alpha$ shows

$$Z\,H_y^s(\rho,\phi) = \frac{i}{2\pi}\int_S F(\kappa\cos\alpha)\exp\left[i\kappa\rho\cos(\alpha \mp \phi)\right]d\alpha$$

($\mp$ for $y \gtrless 0$) where S is a path on which $\cos\alpha$ runs from ∞ to $-\infty$. At large distances from the edge the integral can be evaluated by the stationary phase method provided ϕ is bounded away from the incident and reflected wave boundaries $\pi \pm \phi_0$ and from any surface wave poles of $F(\kappa\cos\alpha)$, and if we retain only those terms which are attributable to the edge

$$Z\,H_y^s(\rho,\phi) \sim (2/\pi\kappa\rho)^{1/2}\exp\left[i(\kappa\rho - \frac{\pi}{4})\right]P_H^y(\phi, \phi_0, \beta) \tag{31}$$

where P_H^y can be obtained from (18) by making the appropriate substitution. The result, valid for the entire range $0 \le \phi \le 2\pi$ apart from the regions excluded above, is

$$P_H^y(\phi, \phi_0, \beta) = \frac{i}{2}\left\{\frac{h_y}{\cos\phi + \cos\phi_0}\left[1 - 2\eta\sin\beta\cos(\phi/2)\cos(\phi_0/2)\right]K_+(-\kappa\cos\phi_0) + \left[A - 2\eta\sin\beta\cos(\phi/2)\cos(\phi_0/2)B^*\right]\right\}K_+(-\kappa\cos\phi)$$

On inserting the expressions for A and B^*, we have

$$P_H^y(\phi, \phi_0, \beta) = \frac{i}{2}(1 - \sin^2\beta\sin^2\phi_0)^{-1}\left[U(\phi, \phi_0, \eta)\, h_y + V(\phi, \phi_0, \eta)\, e_y\right] \tag{32}$$

where

$$U(\phi, \phi_0, \eta) = \left\{\frac{\cos^2\beta - \sin^2\beta\cos\phi\cos\phi_0}{\cos\phi + \cos\phi_0}\left[1 - 2\eta\sin\beta\cos(\phi/2)\cos(\phi_0/2)\right]\right.$$
$$+ \frac{\sin\beta\cos\beta}{\sin\beta + \cos 2\gamma}\left[\cos\beta - \sin 2\gamma\right.$$
$$\left.\left. + 2\eta\sin\beta\cos(\phi/2)\cos(\phi_0/2)(\cos\beta + \sin 2\gamma)\right]\right\} K_+(-\kappa\cos\phi)K_+(-\kappa\cos\phi_0) \tag{33}$$

$$V(\phi, \phi_0, \eta) = \sin\beta\cos\beta(2\sin\beta)^{1/2}\left[\frac{\eta^{1/2}\cos(\phi/2)}{\cos(\frac{\pi}{4} - \frac{\beta}{2} - \gamma)}\right.$$
$$\left. - \frac{\eta^{-1/2}\cos(\phi_0/2)}{\cos(\frac{\pi}{4} - \frac{\beta}{2} + \gamma)}\right] K_+(-\kappa\cos\phi)L_+(-\kappa\cos\phi_0) \tag{34}$$

We note that

$$U(\phi_0, \phi, \eta) = U(\phi, \phi_0, \eta) \quad , \qquad V(\phi_0, \phi, \eta) = -V(\phi, \phi_0, 1/\eta) \tag{35}$$

A similar analysis can be applied to the expression (21) for the normal component of the scattered electric field, but the solution is more immediately obtained using duality. Thus, if

$$E^s_y(\rho,\phi) \sim (2/\pi\kappa\rho)^{1/2} \exp\left[i(\kappa\rho - \frac{\pi}{4})\right] P^y_E(\phi,\phi_0,\beta) \tag{36}$$

the edge diffraction coefficient is

$$P^y_E(\phi,\phi_0,\beta) = \frac{i}{2}(1-\sin^2\beta\sin^2\phi_0)^{-1}\left[U(\phi,\phi_0,1/\eta)e_y - V(\phi,\phi_0,1/\eta)h_y\right] \tag{37}$$

where $U(\phi,\phi_0,1/\eta)$ and $V(\phi,\phi_0,1/\eta)$ can be found from (33) and (34) by replacing η by $1/\eta$, implying $K_+ \Longleftrightarrow L_+$ and $\gamma \Longrightarrow -\gamma$. For the other components of the edge diffracted field the equations 23 give

$$P^x_E = -(1-\sin^2\beta\sin^2\phi)^{-1}(\sin^2\beta\sin\phi\cos\phi\, P^y_E - \cos\beta\, P^y_H)$$

$$P^z_E = -(1-\sin^2\beta\sin^2\phi)^{-1}\sin\beta(\cos\beta\sin\phi\, P^y_E + \cos\phi\, P^y_H)$$

$$P^x_H = -(1-\sin^2\beta\sin^2\phi)^{-1}(\sin^2\beta\sin\phi\cos\phi\, P^y_H + \cos\beta\, P^y_E)$$

$$P^z_H = -(1-\sin^2\beta\sin^2\phi)^{-1}\sin\beta(\cos\beta\sin\phi\, P^y_H - \cos\phi\, P^y_E)$$

each of which is expressible as the sum of two contributions proportional to e_y and h_y respectively and involving the functions U and V. Knowledge of U and V is therefore sufficient to determine the entire edge diffracted field, and the above results can be written as

$$\underline{P}_E(\phi, \phi_0, \beta) = -\frac{i}{2}(1 - \sin^2\beta \sin^2\phi)^{-1}(1 - \sin^2\beta \sin^2\phi_0)^{-1} \left\{ \left[U(1/\eta)e_y - V(1/\eta)h_y\right]\hat{s}\wedge(\hat{s}\wedge\hat{y}) + \left[U(\eta)h_y + V(\eta)e_y\right]\hat{s}\wedge\hat{y} \right\} \quad (38)$$

where $\hat{s} = \hat{x}\sin\beta\cos\phi + \hat{y}\sin\beta\sin\phi + \hat{z}\cos\beta$

is a unit vector in the scattering direction. Similarly

$$\underline{P}_H(\phi, \phi_0, \beta) = -\frac{i}{2}(1 - \sin^2\beta \sin^2\phi)^{-1}(1 - \sin^2\beta \sin^2\phi_0)^{-1} \left\{ \left[U(\eta)h_y + V(\eta)e_y\right]\hat{s}\wedge(\hat{s}\wedge\hat{y}) - \left[U(1/\eta)e_y - V(1/\eta)h_y\right]\hat{s}\wedge\hat{y} \right\} \quad (39)$$

and using (35) it can be verified that the reciprocity theorem concerning the interchangeability of receiver and transmitter is satisfied.

The diffraction tensors implicit in (38) and (39) are based on the components of the incident electric and magnetic fields perpendicular to the half plane. As Clemmow [1953] has remarked, the preferred polarizations for an imperfectly conducting surface are those based on the normal components, whereas for a perfectly conducting edge at least it is more natural to select the components parallel to it. It would seem that for an imperfectly conducting edge the material effects dominate, and the diffraction tensors expressed in terms of e_z and h_z, or using ray rather than edge coordinates [Kouyoumjian and Pathak, 1974], are more complicated than those which we have given here. We also note that the physical interpretation of diffraction by a perfectly conducting edge that Deschamps [1973] has developed does not appear to have any analogue in the present case.

If the half plane is resistive or conductive, the diffraction tensors can be found by suppressing the necessary terms in the expressions for U and V and identifying η with $2R/Z$ or $(2R^* Z)^{-1}$ in the others. Thus, for a resistive half plane,

$$U(\phi, \phi_0, \eta) = \left(\frac{\cos^2\beta - \sin^2\beta \cos\phi \cos\phi_0}{\cos\phi + \cos\phi_0} + \sin\beta\cos\beta \, \frac{\cos\beta - \sin 2\gamma}{\sin\beta + \cos 2\gamma} \right) \cdot K_+(-\kappa\cos\phi) K_+(-\kappa\cos\phi_0)$$

$$V(\phi, \phi_0, \eta) = -\left(\frac{2}{\eta}\sin\beta\right)^{1/2} \sin\beta\cos\beta\cos(\phi_0/2)\sec(\pi/4 - \beta/2 + \gamma) K_+(-\kappa\cos\phi) L_+(-\kappa\cos\phi_0)$$

$$\tag{40}$$

$$U(\phi, \phi_0, 1/\eta) = -\frac{2}{\eta}\sin\beta\cos(\phi/2)\cos(\phi_0/2)\left(\frac{\cos^2\beta - \sin^2\beta\cos\phi\cos\phi_0}{\cos\phi + \cos\phi_0} + \sin\beta\cos\beta\,\frac{\cos\beta + \sin 2\gamma}{\sin\beta + \cos 2\gamma} \right) L_+(-\kappa\cos\phi) L_+(-\kappa\cos\phi_0)$$

$$V(\phi, \phi_0), 1/\eta) = \left(\frac{2}{\eta}\sin\beta\right)^{1/2} \sin\beta\cos\beta\cos(\phi/2)\sec(\pi/4 - \beta/2 + \gamma) L_+(-\kappa\cos\phi) K_+(-\kappa\cos\phi_0)$$

with $\eta = 2R/Z$. The second pair of functions are now unrelated to the first and the symmetry which characterizes the solution for the nonmetallic half plane has been lost.

6. NUMERICAL RESULTS

Even in the special cases of resistive and conductive half planes, the functions U and V still involve the quantity $\gamma = \gamma(\beta, \eta)$ defined in (30) and we must first consider its evaluation.

Using the identity (15) and the corresponding one for L_+, the definition of (30) can be written as

$$e^{i\gamma} = \eta^{-1/2} \frac{L_+(-i\kappa\cot\beta)}{K_+(-i\kappa\cot\beta)} \tag{41}$$

from which we have

$$\gamma(\pi - \beta, \eta) = -\gamma(\beta, \eta) \tag{42}$$

implying $\gamma(\pi/2, \eta) = 0$. It also follows trivially that

$$\gamma(\beta, 1/\eta) = -\gamma(\beta, \eta) \tag{43}$$

implying $\gamma(\beta, 1) = 0$, and it is therefore sufficient to restrict attention to $0 \leq \beta \leq \pi/2$ and $0 \leq \eta \leq 1$.

The split functions K_+ and L_+ can be expressed in terms of the function $f(\theta)$ defined in (19) of Senior [1975]. If $\cos\theta = i\cot\beta$, the solution which is $\pi/2$ when $\beta = \pi/2$ is $\theta = \frac{\pi}{2} + i\,\ell n(\tan\frac{\beta}{2})$ and

$$K_+(-ik\cos\beta) = (2/\eta_1)^{1/2}\sin(\frac{\pi}{4} - i\frac{\tau}{2})\, f(\frac{\pi}{2} + i\tau + \chi_1)\, f(\frac{\pi}{2} + i\tau - \chi_1)$$

where

$$\tau = -\ell n(\tan\frac{\beta}{2}) \qquad \text{and} \qquad \eta_1 = \eta\sin\beta = \sec\chi_1$$

Similarly

$$L_+(-ik\cos\beta) = (2/\eta_2)^{1/2}\sin(\frac{\pi}{4} - i\frac{\tau}{2})\, f(\frac{\pi}{2} + i\tau + \chi_2)\, f(\frac{\pi}{2} + i\tau - \chi_2)$$

with

$$\eta_2 = \sin(\beta/\eta) = \sec\chi_2$$

and thus

$$e^{i\gamma} = \eta^{1/2}\,\frac{f(\frac{\pi}{2} + i\tau + \chi_2)\, f(\frac{\pi}{2} + i\tau - \chi_2)}{f(\frac{\pi}{2} + i\tau + \chi_1)\, f(\frac{\pi}{2} + i\tau - \chi_1)}$$

The function $f(\theta)$ is related to the meromorphic function $\psi_\pi(\theta)$ introduced by Maliuzhinets [1959], and from one of its several forms that are available [Bowman, 1967]

$$\psi_\pi\left(\frac{\pi}{2}-\theta\right) = \left[\frac{2^{1/2}\cos(\pi/4-\theta/2)+1}{2^{1/2}+1}\right]^{1/2} \exp\left[\frac{1}{8\pi}\int_0^{\frac{\pi}{2}-\theta} \frac{\pi\sin v - 2v}{\cos v} dv\right]$$

A change in the variable of integration allows part of the resulting integral to be evaluated analytically, and after some manipulation we have

$$\frac{\psi_\pi\left(\frac{\pi}{2}-\theta\right)}{\psi_\pi(\pi/2)} = 2^{-1/2}\cos(\theta/2)^{-1/4}\left[2^{1/2}\cos(\pi/4-\theta/2)+1\right]^{1/2} \exp\left[-\frac{1}{4\pi}\int_0^\theta \frac{u\,du}{\sin u}\right]$$

It follows that

$$f\left(\frac{\pi}{2}-\theta\right) = \left[2^{1/2}\cos(\theta/2)\right]^{-1/2} \exp\left[-\frac{1}{2\pi}\int_0^\theta \frac{u\,du}{\sin u}\right]$$

and hence

$$\gamma(\beta,\eta) = \frac{1}{2\pi}\left(\int_{-\tau-i\chi_2}^{\tau-i\chi_2} - \int_{-\tau-i\chi_1}^{\tau-i\chi_1}\right)\frac{t\,dt}{\sinh t} \qquad (44)$$

For real η the limits of integration in the second integral are real for all η, but those in the first integral are real only for $\eta \geq \sin\beta$. If $\eta < \sin\beta$, χ_2 is itself real and it is then convenient to rewrite (44) as

$$\gamma(\beta,\eta) = \frac{1}{\pi}\int_0^{\tau} \frac{\cosh y\,\chi_2 \sin\chi_2 + y\sinh y\cos\chi_2}{\cosh^2 y - \cos^2\chi_2}\,dy - \frac{1}{2\pi}\int_{-\tau - i\chi_1}^{\tau - i\chi_1} \frac{t\,dt}{\sinh t} \tag{45}$$

From (44) and (45) it is evident that γ is a real function for all β and η. Since $\pm\tau - i\chi_1 \to \infty$ as $\eta \to 0$, (45) also shows that $\gamma(\beta, 0) = \pi/4 - \beta/2$, and from the small β behavior of the limits of integration in (44)

$$\gamma(0,\eta) = \frac{1}{\pi}\int_0^{\ell n\, 1/\eta} \frac{t\,dt}{\sinh t}$$

A program has been written to compute $\gamma(\beta,\eta)$ for real η by numerical evaluation of the integrals in (44) and (45), and selected data are presented in Fig. 1. We observe that γ is a decreasing function of β and η, and the curves for constant η are rather uniform functions of β bounded by the straight lines appropriate to $\eta = 1$ and 0. Equations (42) and (43) give the extensions to $\pi/\eta < \beta \leq \pi$ and $\eta > 1$.

Having determined γ, the only portions of the general expressions (33) and (34) for U and V which are other than straightforward are the products of the split functions, but these are insignificantly different from the ones for incidence in a plane perpendicular to the edge, and can be computed using the procedures described by Senior [1975]. If, for example, $\eta_1 \geq 1$ so that χ_1 is real,

$$K_+(-\kappa\cos\phi) = \sin(\phi/2)(4\eta_1'\eta_1'')^{1/4}\left[G(\pi,\pi,\eta_1')/G(\pi,\pi,0)\right]^{1/4}\left[G(\pi,\pi,\eta_1'')/G(\pi,\pi,0)\right]^{1/4} \tag{46}$$

where

$$\eta_1' = \sec(\pi - \phi + \chi_1)\ , \qquad \eta_1'' = \sec(\pi - \phi - \chi_1)$$

and the ratios of the G functions can be deduced from Fig. 1 of Senior [1975]. On the other hand, if χ_1 is not real, an analysis similar to that leading from (44) to (45) gives

$$K_+(-\kappa\cos\phi) = \left[\frac{\cos\chi_1(1-\cos\phi)}{\cos\chi_1+\sin\phi}\right]^{1/2} \exp\left[-\frac{1}{\pi}\int_0^{\phi-\pi/2} \frac{\cos y\chi_1 \sin\chi_1 - y\sin y\cos\chi_1}{\cos^2 y - \cos^2\chi_1}\, dy\right] \tag{47}$$

which is rather easily computed. Similar formulas obtain for $L_+(-\kappa\cos\phi)$, and we note in passing that the Maliuzhinets function ψ_π has been tabulated for a range of complex arguments by Bucci [1974].

To illustrate the behavior of the edge diffracted field, consider the simple case of a nonmetallic sheet for edge-on incidence and observation, i.e., $\phi_0 = \phi = \pi$. From (33) and (34)

$$U(\pi,\pi,\eta) = \left[\frac{1}{2} - \frac{\cos\beta\cos(\beta-2\gamma)}{\sin\beta+\cos 2\gamma}\right]\left[K_+(\kappa)\right]^2$$

$$V(\pi,\pi,\eta) = 0$$

and using the data in Fig. 1 and Fig. 1 of Senior [1975], $U(\pi,\pi,\eta)$ has been computed as a function of β. Since U is even about $\beta = \pi/2$ it is sufficient to restrict attention to $0 \le \beta \le \pi/2$, and the curves for five different values of η are shown in Fig. 2. As η increases, the values of the function for $\beta = \pi/2$ (incidence perpendicular to the edge) are indicative of the scattering for an increasing range of angles about $\pi/2$, a fact which is important in the practical application of materials for the reduction of edge scattering.

Acknowledgments. The author is indebted to S.R. Laxpati of the University of Illinois, Chicago Circle, for his assistance. The work described was supported by the United States Air Force Office of Scientific Research under Grant 72-2262.

References

Bowman, J.J. (1967), High-frequency backscattering from an absorbing infinite strip with arbitrary face impedances, Can. J. Phys. 45, 2409-2430.

Bucci, O.M. (1974), On a function occurring in the theory of scattering from an impedance half-plane, Laboratorio di Onde Elettromagnetiche, Istituto di Elettrotecnica, Napoli, Italy.

Bucci, O.M. and G. Franceschetti (1975), Electromagnetic scattering by half-plane with two face impedances,

Clemmow, P.C. (1953), Radio propagation over a flat earth across a boundary separating two different media, Phil. Trans. Roy. Soc. (London) 246(A), 1-55.

Deschamps, G.A. (1973), Diffraction of an arbitrary electromagnetic wave by a half-plane, IEEE Trans. Antennas Propagat., AP-21, 126-127.

Grünberg, G. (1943), Suggestions for a theory of coastal refraction, Phys. Rev. 63, 185-189.

Kouyoumjian, R.G. and P.H. Pathak (1974), A uniform geometrical theory of diffraction for an edge in a perfectly conducting surface, Proc. IEEE 62, 1448-1461.

Maliuzhinets, G.D. (1959), Excitation, reflection and emission of surface waves from a wedge with given face impedances, Sov. Phys.-Dokl. 3, 752-755.

Senior, T.B.A. (1952), Diffraction by a semi-infinite metallic sheet, Proc. Roy. Soc. (London), 213(A), 436-458.

Senior, T.B.A. (1959), Diffraction by an imperfectly conducting half-plane at oblique incidence, Appl. Sci. Res. 8(B), 35-61.

Senior, T.B.A. (1960), Impedance boundary conditions for imperfectly conducting surfaces, Appl. Sci. Res. 8(B), 418-436.

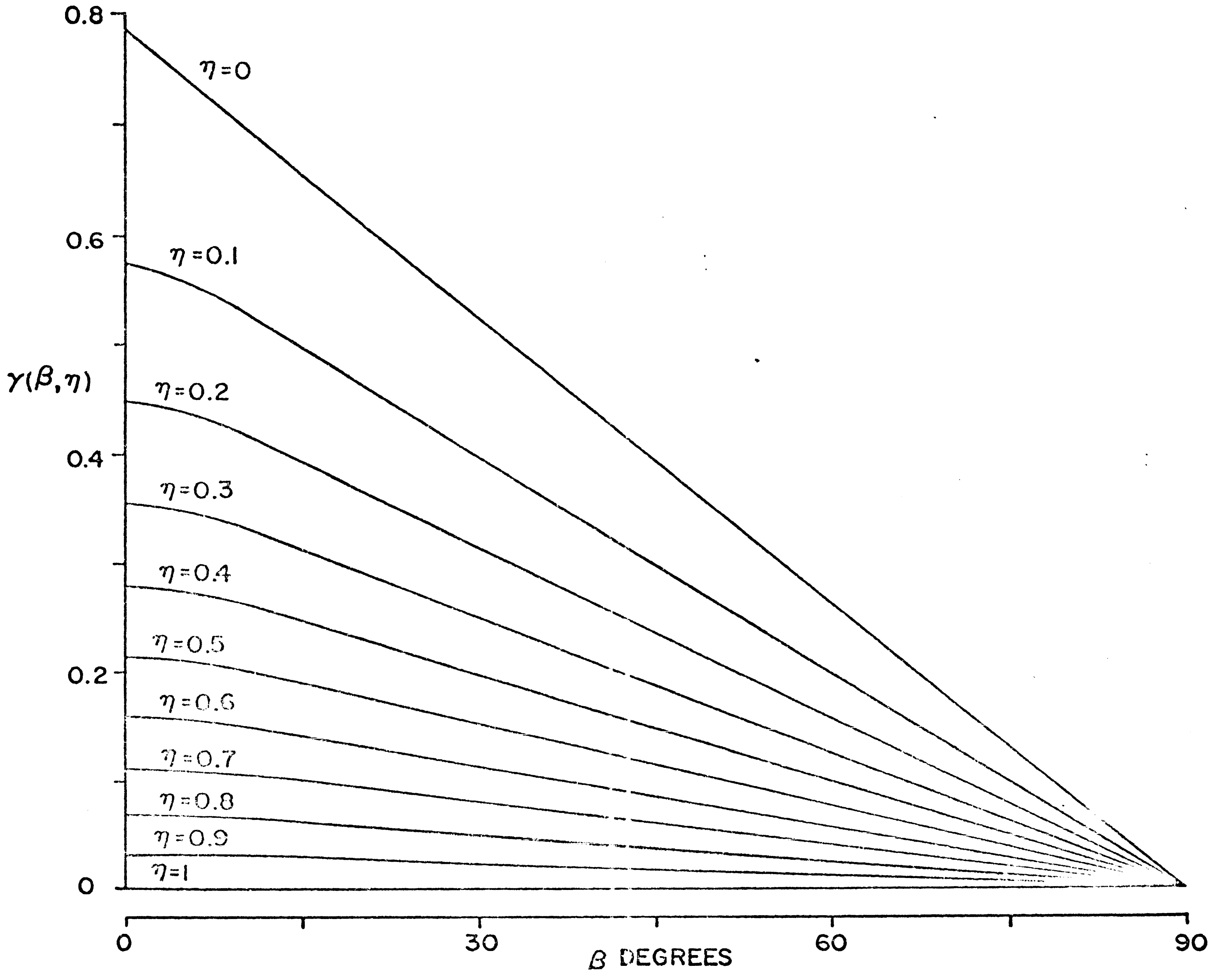
Senior, T.B.A. (1975), Half plane edge diffraction, Radio Sci. 10,

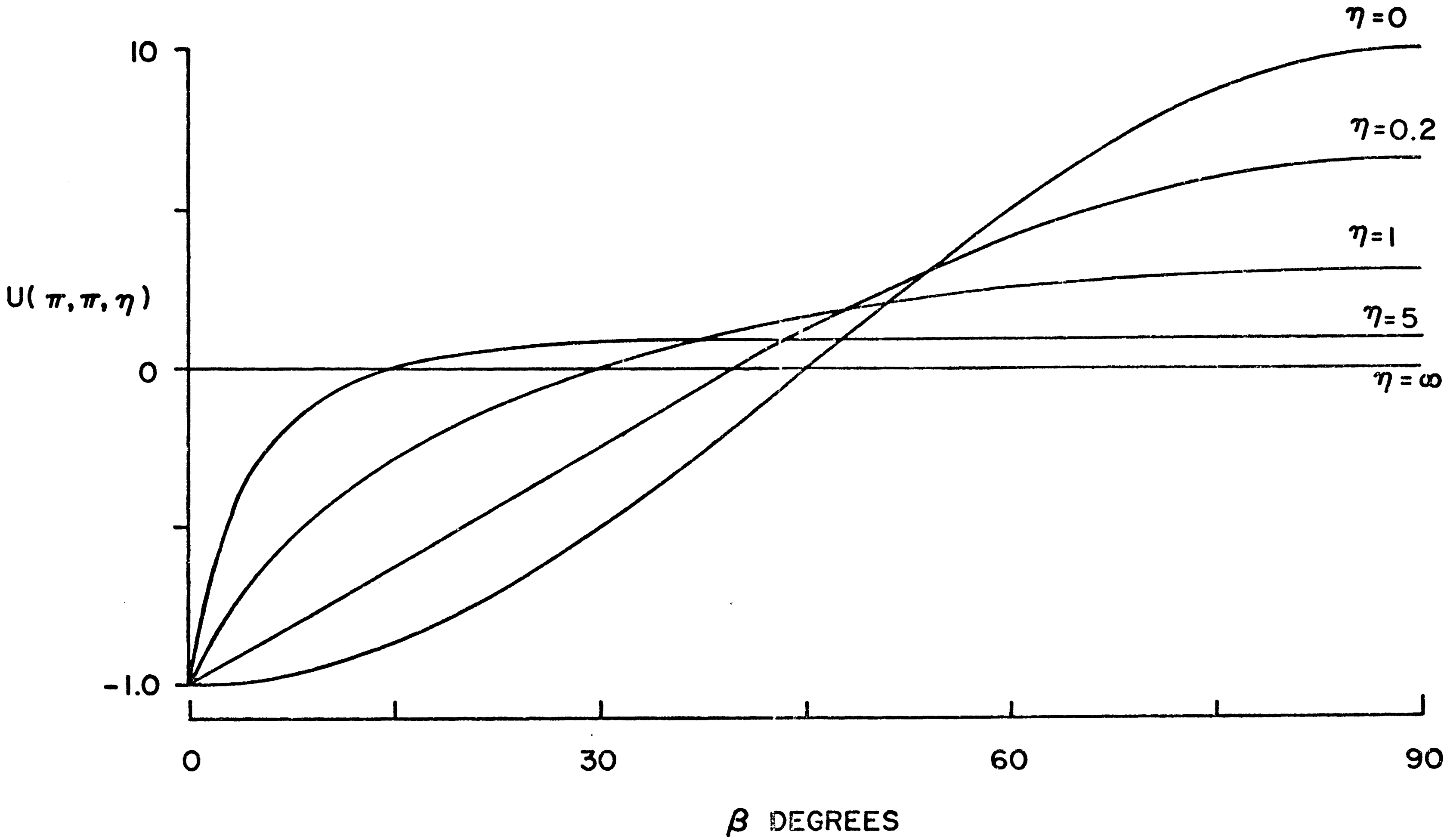
Williams, W.E. (1960), Diffraction of an electromagnetic plane wave by a metallic sheet, Proc. Roy. Soc. (London), 257(A), 413-419.

Figure Captions

Fig. 1. $\gamma(\beta, \eta)$ as a function of β for selected real η.

Fig. 2. The tensor element U for a nonmetallic half plane with edge-on incidence and observation.


0.8
0.6
0.4
0.2
0
γ(β,η)
η=0
η=0.1
η=0.2
η=0.3
η=0.4
η=0.5
η=0.6
η=0.7
η=0.8
η=0.9
η=1
0
30
60
90
β DEGREES



10
U(π, π, η)
0
-1.0
0
30
60
90
β DEGREES
η = 0
η = 0.2
η = 1
η = 5
η = ∞