THE DIPOLE MOMENTS OF A DIELECTRIC CUBE+

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Summary

As a part of a general study of the low frequency scattering properties of atmospheric particles of various shapes, the dipole moments of homogeneous dielectric rectangular parallelepipeds have been examined and data obtained for the special case of a cube. The results are presented and differ substantially from those previously reported in the literature.

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A knowledge of the scattering of visible and infrared radiation by high altitude clouds and other aerosol-laden atmospheres is vital in considering the radiation balance of the earth and is also important at microwave frequencies in connection with radar meteorology from space. This has led to a study aimed at developing techniques for computing the scattering from small dielectric particles of shapes known to occur in practice, and in the low frequency or Rayleigh region where the wavelength is much greater than the maximum dimensions of the particle the computations are relatively easy to perform. Among the shapes which have been examined is the rectangular parallelepiped, and we consider here the scattering of a low frequency electromagnetic wave by a single dielectric particle of this configuration, with particular emphasis on the special case of a cube.

A homogeneous isotropic dielectric body of finite extent is immersed in free space and illuminated by a linearly polarized plane electromagnetic wave whose electric and magnetic field vectors are taken as

$$\underline{E}^{i} = \hat{a}\, e^{ik_0 \hat{k}\cdot \underline{r}}, \qquad \underline{H}^{i} = Y_0 \hat{b}\, e^{ik_0 \hat{k}\cdot \underline{r}} \tag{1}$$

where $\hat{k}$, $\hat{a}$ and $\hat{b}$ are mutually perpendicular unit vectors in the directions of incidence, the electric field and the magnetic field respectively. The permittivity and permeability of free space are ϵ_0 and μ_0 respectively, k_0 is the free space propagation constant and Y_0 is the intrinsic admittance. Mks units are employed and a time factor $e^{-i\omega t}$ suppressed. For sufficiently small k_0 all of the field quantities can be expanded as power series in k_0, and as regards the scattered field it is well known [1] that in the far zone the leading term can be attributed to induced electric and magnetic dipoles, of moments $\underline{p}$ and $\underline{m}$ respectively, located at the origin of coordinates in the vicinity of the body.

We can also write

$$\underline{p} = \epsilon_0 \overline{\overline{P}} \cdot \hat{a} \ , \quad \underline{\underline{m}} = - Y_0 \overline{\overline{M}} \cdot \hat{b} \tag{2}$$

where $\overline{\overline{P}}$ and $\overline{\overline{M}}$ are the electric and magnetic polarizability tensors [2, 3]. These are independent of the incident field, but functions respectively of the relative permittivity and permeability ϵ_r and μ_r of the dielectric, and of the geometry of the body. For any given incident plane wave, $\overline{\overline{P}}$ and $\overline{\overline{M}}$ specify the dipole moments and, hence, the far zone scattered field in the Rayleigh region.

The tensor elements P_{ij} and M_{ij} are expressible as weighted surface integrals of electrostatic and magnetostatic potentials, and from the boundary conditions on the potentials at the surface of the body, it is found [3] that $\overline{\overline{P}}$ and $\overline{\overline{M}}$ are merely special cases of a general polarizability tensor $\overline{\overline{X}}(\tau)$ in terms of which

$$\overline{\overline{P}} = \overline{\overline{X}}(\epsilon_r) \ , \quad \overline{\overline{M}} = - \overline{\overline{X}}(\mu_r) \ . \tag{3}$$

We remark that for a perfectly conducting body $\epsilon_r = \infty$ and $\mu_r = 0$, and the $\overline{\overline{P}}$ and $\overline{\overline{M}}$ computed by Kleinman and Senior [4, 5] for a number of metallic shapes therefore yield $\overline{\overline{X}}(\tau)$ for these two extreme values of τ.

The general polarizability tensor is a function only of the geometry and material parameter τ of the body, and for real τ the tensor is real and symmetric, having at most six independent elements. If x_i, $i = 1, 2, 3$ are rectangular Cartesian coordinates, the tensor elements are

$$X_{ij} = (1 - \tau) \int_B \hat{n} \cdot \hat{x}_i \ \Phi_j \ d S \tag{4}$$

where $\hat{n}$ is the outward unit vector normal to the (closed) surface B of the body and Φ_j is a total exterior potential satisfying the integral equation [3, 6]

$$\Phi_j(\underline{r}) = \frac{-2}{1+\tau} X_j + \frac{1-\tau}{1+\tau} \frac{1}{2\pi} \int_B \Phi_j(\underline{r}') \frac{\partial}{\partial n'}\left(\frac{1}{|\underline{r}' - \underline{r}|}\right) dS' \tag{5}$$

where $\underline{r}$ is a position vector terminating on B and the integration is with respect to the primed variables. The tensor elements X_{ij} can also be expressed in terms of the normal derivative of the potential and an integral equation derived for the normal derivatives, but this formulation proves less desirable for numerical purposes.

Whereas (5) is in general a weakly singular integral equation, in the special case of a rectangular parallelepiped occupying (say) the region $|x_1| \leq a$, $|x_2| \leq b$, $|x_3| \leq c$, the kernel in (5) is identically zero when the points $\underline{r}$ and $\underline{r}'$ lie on the same face of the body, therefore removing the singularity at $\underline{r} = \underline{r}'$. In addition, the symmetry about the three perpendicular planes $x_i = 0$ produces corresponding symmetries in the potentials which result in diagonalization of the tensor and also enable us to limit the integration in (5) to a portion of the total surface B. If S_m, m = 1, 2, 3 are the surfaces S_1: $x_1 = a$, $0 \leq x_2 \leq b$, $0 \leq x_3 \leq c$; S_2: $0 \leq x_1 \leq a$, $x_2 = b$, $0 \leq x_3 \leq c$; S_3: $0 \leq x_1 \leq a$, $0 \leq x_2 \leq b$, $x_3 = c$, and $\underline{r}_m$ is the position vector of a point on S_m, the integral equation (5) can be written as

$$\frac{1+\tau}{1-\tau} \Psi_i(\underline{r}) = -2x_i + \frac{1}{2\pi} \sum_{m=1}^{3} \int_{S_m} \Psi_i(\underline{r}_m) K_{im}(\underline{r}, \underline{r}_m) dS_m \tag{6}$$

where

$$\Psi_i = (1-\tau)\,\Phi_i \quad . \tag{7}$$

The tensor elements are then

$$X_{ii} = 8\int_{S_i} \Psi_i\,(\underline{r}_i)\,d\,S_i \tag{8}$$

with $\qquad X_{ij} = 0\,, \qquad i \neq j\ .$

Because the kernels K_{ij} are non-singular it follows that the numerical solution of (6) by the moment method requires no special treatment of the self cells, and (6) is easily converted to a simultaneous system of linear algebraic equations in the sampled values of Ψ_i . For a parallelepiped having a square cross section $(a = b)$, $X_{22} = X_{11}$, and for a cube $(a = b = c)$, $X_{33} = X_{22} = X_{11}$, so that in the latter case the tensor is now specified by a single diagonal element.

A computer program has been written to solve (6) numerically and compute the tensor elements for a parallelepiped of square cross section, and we present here the results obtained in the particular case of a cube. Most data were computed using 25 or 36 square cells on each of the three quarter-faces S_m of the cube and the results are believed accurate to better than one percent. The normalized tensor element $X_{11}(\tau)/V$, where V is the volume of the cube, is shown in Figure 1 as a function of τ for $10^{-2} \leq \tau \leq 10^{3}$. It is seen that $X_{11}(\tau)/V$ is a slowly increasing function of τ, ranging from -1.654 at $\tau = 0$ to 3.542 at $\tau = \infty$ and equal to zero at $\tau = 1$. We remark that X_{11}/V is proportional to the dipole moment per unit volume, and at the extreme values $\tau = \infty$ and $\tau = 0$ yields respectively the electric and magnetic dipole moments per unit volume for a perfectly conducting cube.

Figure 1 also shows the dipole moments computed by Edwards and Van Bladel [1, 6] for this same geometry, and a substantial difference is seen between their data and ours. All tests of our formulation and programming support the validity of the present data. Although our formulation is almost identical to that used by Edwards and Van Bladel [7], we have noted that their results show a non-physical distribution of the surface potentials. From the comparisons which we have been able to carry out, there appears to be an error in their solution of the system of linear equations in Ψ_i by the matrix inversion method. We also remark that for a perfectly conducting body of arbitrary convex shape, it has been proved by Schiffer [8] that

$$\frac{1}{3}\sum_{i=1}^{3}\frac{X_{ii}(\infty)}{V} \geqslant 3 \tag{9}$$

with the equality obtained in case of a sphere. For a dielectric cube this inequality becomes $X_{ii}(\infty)/V > 3$ and is evidently violated by the data of Edwards and Van Bladel.

The values of X_{11}/V for a dielectric sphere have been computed using (9) and are included in Figure 1. It is evident that the new values of the dipole moments per unit volume for a cube are quite close to those for a sphere having the same material parameter τ, and in Figure 2 we have plotted the cube data normalized to those for the corresponding sphere. The maximum difference is no more than 18 percent, and for many purposes it may be sufficient to use a sphere to approximate the dipole moments for a dielectric cube. This approximation is commonly made in studying the scattering of atmospheric particles, but we caution that for geometries other than the cube the dipole moments can differ substantially from the values for a sphere.

References

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Legends for figures

Fig. 1. The normalized tensor element $X_{11}(\tau)/V$ for a sphere (———) and a cube: present data (x x x), Edwards and Van Bladel [6] (- - -).

Fig. 2. The tensor element $X_{11}(\tau)$ normalized to that for a sphere of the same volume and material parameter τ .

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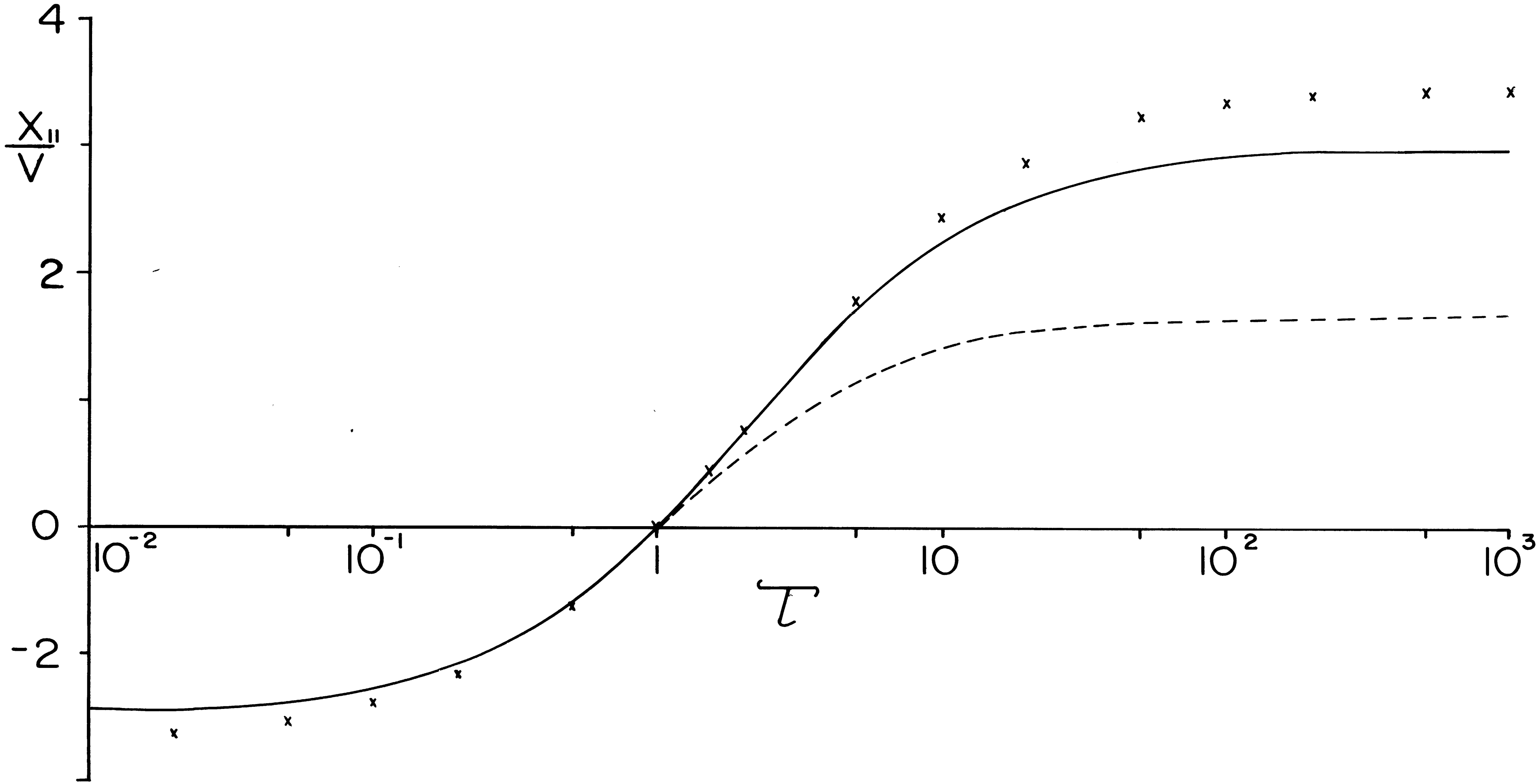

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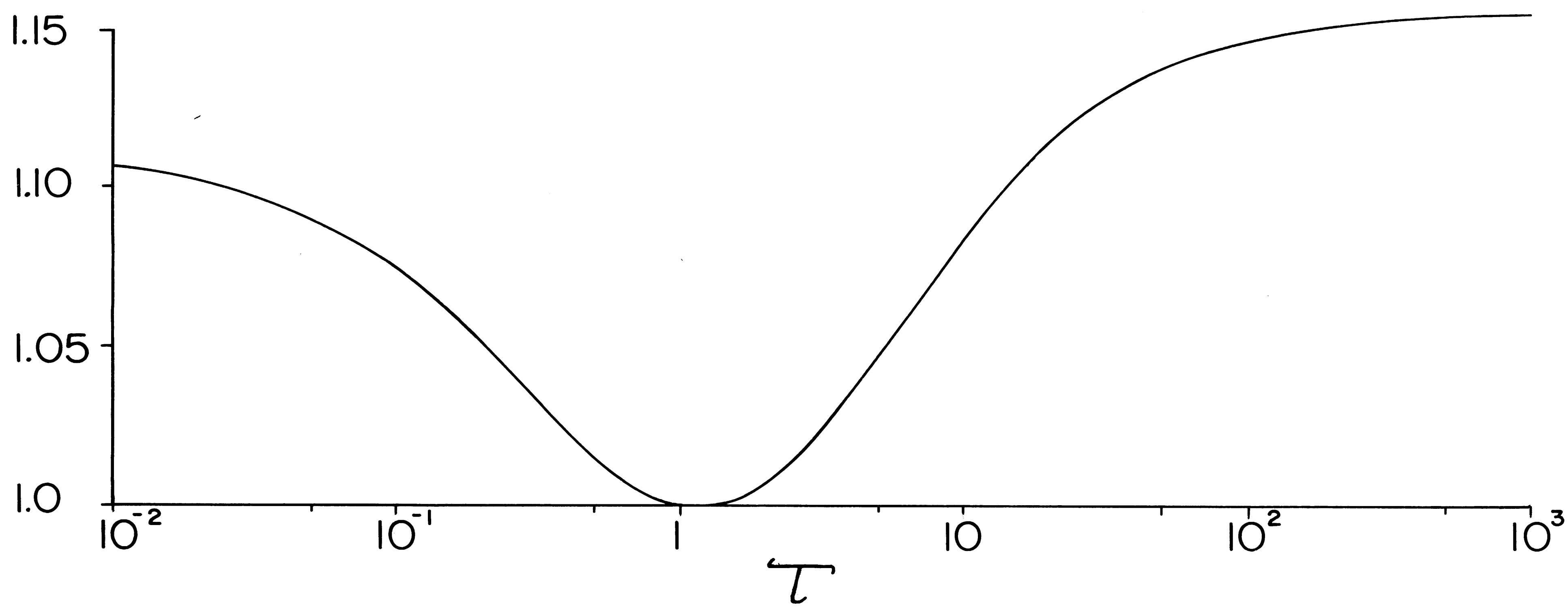
JRL AP ISSUE July KB 25 FIG 1

30 % OF ORIG ______ DENSITOMETER

30%



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JRL AP ISSUE July KB 25 FIG 2
30 % OF ORIG _____ DENSITOMETER

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